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Chapter 1

Fundamental Concepts

In this introductory chapter, we present fundamental concepts of semigroup, ideal in semigroups, bi-ideal, generalized bi-ideal, quasi-ideal, interior-ideal, prime and semi-prime ideals in semigroups and review some of the background material that will be of value for our subsequent chapters. For undefined terms and notions we refer to [11, 17, 18].

1.1 Semigroup: Basic Definitions and Examples

A non-empty set S together with an associative binary operation $*$ is called a semigroup. We shall write xy for $x * y$ and usually refer to the binary operation as multiplication "$\cdot$" on S. If a semigroup S has the property that, for all x, y in S, $xy = yx$ we shall say that S is a commutative semigroup. If a semigroup S contains an element 1 with the property that, for all x in S, $x \cdot 1 = 1 \cdot x = x$ for all $x \in S$, we say that 1 is an identity element of S, and that S is a semigroup with identity or (more usually) a monoid. A semigroup has at most one identity element.

If a semigroup S has no identity element, then it is easy to adjoin an extra element 1 to S to form a monoid. We define $1 \cdot s = s \cdot 1 = s$, for all $s \in S$ and $1 \cdot 1 = 1$. Then $S \cup \{1\}$ becomes a monoid. We shall use the notation S^1 with the following meanings

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element} \\ S \cup \{1\} & \text{otherwise} \end{cases}$$

and call S^1 the semigroup obtained from S by adjoining an identity element.

If a semigroup S with at least two elements contains an element 0 with the property that, for all x in S,

$$x \cdot 0 = 0 \cdot x = 0$$

we say that 0 is a zero element of S, and that S is a semigroup with zero.

If a semigroup has no zero element, then it is easy to adjoin an extra element 0 to S by defining $0 \cdot s = s \cdot 0 = 0$ and $0 \cdot 0 = 0$ for all s in S. This makes the set $S \cup \{0\}$ a semigroup with zero element 0. We shall use the notation S^0 with the following meanings

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero element} \\ S \cup \{0\} & \text{otherwise} \end{cases}$$

and call S^0 the semigroup obtained from S by adjoining a zero element.

An element e of a semigroup S is called an idempotent if $e \cdot e = e$. If all elements of a semigroup S are idempotents, then S is called an idempotent semigroup.

Example 1 *Let S be a non-empty set and define a binary operation $*$ on S by $a*b = a$ for all $a, b \in S$. Then $(S, *)$ is a semigroup, called the left zero semigroup. Similarly, if we define the binary operation $\circ$ on S as $a \circ b = b$ for all $a, b \in S$. Then $(S, \circ)$ is a semigroup, called the right zero semigroup. In these semigroups each element is idempotent.*

If A and B are non-empty subsets of a semigroup S, then we define $AB = \{ab : a \in A \text{ and } b \in B\}$.

A non-empty subset T of a semigroup S is called a subsemigroup of S if it is closed with respect to the induced operation of S, that is, $xy \in T$ for all $x, y \in T$ or $TT \subseteq T$.

The following Proposition is well known.

Proposition 2 *The intersection of any family of subsemigroups of a semigroup S is either empty or a subsemigroup of S.*

1.2 Ideals in Semigroups

A non-empty subset A of a semigroup S is called a left (right) ideal of S if $SA \subseteq A$ ($AS \subseteq A$). The non-empty subset A of S is called a (two-sided) ideal if it is both a left and a right ideal of S. Evidently every ideal (whether right, left or two-sided) is a subsemigroup, but the converse is not true. Clearly every semigroup S is an ideal of itself, and if S is a semigroup with zero 0, then $\{0\}$ is an ideal of S. $\{0\}$ and S are called improper ideals of S. All other ideals of S are called proper ideals of S.

The following results are well known.

Proposition 3 *The intersection of any family of left (right) ideals of a semigroup S is either empty or a left (right) ideal of S.*

Corollary 4 *The intersection of any family of ideals of a semigroup S is an ideal of S.*

Proposition 5 *The union of any family of left (right) ideals of a semigroup S is a left (right) ideal of S.*

Corollary 6 *The union of any family of ideals of a semigroup S is an ideal of S.*

Proposition 7 *Let L and R be left and right ideals of a semigroup S, respectively. Then $RL \subseteq R \cap L$.*

Definition 8 *A non-empty subset A of a semigroup S is called an interior-ideal of S if $SAS \subseteq A$.*

Obviously, every ideal of a semigroup S is an interior-ideal of S but the converse is not true.

Example 9 *Consider the semigroup $S = \{a, b, c, d\}$ with the following multiplication table:*

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

The subset $A = \{a, c\}$ of S is an interior-ideal of S but neither a left nor a right ideal of S.

Definition 10 *A subsemigroup A of a semigroup S is called a bi-ideal of S if $ASA \subseteq A$.*

A non-empty subset A of a semigroup S is called a generalized bi-ideal of S if $ASA \subseteq A$.

Obviously every one-sided ideal of a semigroup S is a bi-ideal of S and every bi-ideal of S is a generalized bi-ideal of S but the converse is not true.

Example 11 *Consider the semigroup S of Example 9. The subset $A = \{a, c\}$ of S is a bi-ideal of S but neither a left nor a right ideal of S.*

Example 12 *Consider the semigroup $S = \{a, b, c, d\}$ with the following multiplication table:*

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

The subset $A = \{a, c\}$ of S is a generalized bi-ideal of S but not a bi-ideal of S.

Proposition 13 *The intersection of any family of bi-ideals of a semigroup S is either empty or a bi-ideal of S.*

Corollary 14 *The intersection of any family of bi-ideals of a semigroup S with* 0 *is a bi-ideal of S.*

Proposition 15 *The intersection of a bi-ideal B and a subsemigroup T of a semigroup S is a bi-ideal of the semigroup T if it is non-empty.*

Proposition 16 *Let T be a non-empty subset and B be a bi-ideal of a semigroup S. Then the products BT and TB are bi-ideals of S.*

Definition 17 *A non-empty subset Q of a semigroup S is called a quasi-ideal of S if $QS \cap SQ \subseteq Q$.*

Obviously every quasi-ideal of a semigroup is a bi-ideal and every one sided ideal is a quasi-ideal but the converse is not true.

Example 18 *Let S be the set of all 2×2 matrices with entries from the set of non-negative integers. Then S is a semigroup with respect to the usual multiplication of matrices. Consider the subset*

$$A = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : \ a \text{ is a non-negative integer} \right\}.$$

Then A is a quasi-ideal of S but neither a right nor a left ideal of S.

Proposition 19 *The intersection of any family of quasi-ideals of a semigroup S is either empty or a quasi-ideal of S.*

1.3 Regular, Intra-regular and Semisimple Semigroups

An element a of a semigroup S is called regular if $a = axa$ for some $x \in S$, that is $a \in aSa$. The semigroup S is called regular if every element of S is regular. An element a of a semigroup S is called intra-regular if there exist elements x and y in S such that $a = xa^2y$. A semigroup S is called intra-regular if every element of S is intra-regular.

Definition 20 *A semigroup S is called semisimple if every two sided ideal of S is idempotent. It is clear that a semigroup S is semisimple if and only if $a \in (SaS)(SaS)$ for every $a \in S$, that is there exist $x, y, z, t \in S$ such that $a = (xay)(taz)$.*

Neither a regular semigroup is intra-regular nor an intra-regular semigroup is regular. However, if a semigroup is commutative then both the concepts coincide.

Example 21 *Let X be a countably infinite set and let S be the set of one-one maps $\alpha : X \to X$ with the property that $X - \alpha(X)$ is infinite. Then S is a semigroup with respect to the composition of functions and is called Baer Levi semigroup (cf. [18]).*

This semigroup S is a right cancellative, right simple semigroup without idempotents (cf. [18], Th. 8.2). Thus S is not regular but intra-regular.

Example 22 *Let $S = \{a, b, c, d, e\}$ be the semigroup with the multiplication table:*

	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	b	c
c	a	b	c	a	a
d	a	a	a	d	e
e	a	d	e	a	a

It is easy to check that S is a regular semigroup but not an intra-regular semigroup. The following theorem are well known:

Theorem 23 *The following conditions on a semigroup S are equivalent.*

(1) S is regular.
(2) $RL = R \cap L$ for every right ideal R and left ideal L of S.
(3) $A = ASA$ for every quasi-ideal A of S.

Theorem 24 *For a semigroup S, the following conditions are equivalent.*

(1) S is regular.
(2) $Q \cap L \subseteq QL$ for every quasi-ideal Q and every left ideal L of S.
(3) $B \cap L \subseteq BL$ for every bi-ideal B and every left ideal L of S.
(4) $G \cap L \subseteq GL$ for every generalized bi-ideal G and every left ideal L of S.

Theorem 25 *For a semigroup S, the following conditions are equivalent.*

(1) S is regular.
(2) $Q \cap R \subseteq RQ$ for every quasi-ideal Q and every right ideal R of S.
(3) $B \cap R \subseteq RB$ for every bi-ideal B and every right ideal R of S.
(4) $G \cap R \subseteq RG$ for every generalized bi-ideal G and every right ideal R of S.

Theorem 26 *The following conditions on a semigroup S are equivalent.*

(1) S is intra-regular.
(2) $LR \supseteq R \cap L$ for every right ideal R and left ideal L of S.
The following is the example of a semigroup which is both regular and intra-regular.

Example 27 *Consider the semigroup $S = \{0, a, b\}$ with the multiplication table:*

$\cdot$	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

It is evident that S is both regular and intra-regular.
The following theorem is well known that

Theorem 28 *The following assertions for a semigroup S are equivalent.*

(1) S is both regular and intra-regular.
(2) Every quasi-ideal of S is idempotent.
(3) Every bi-ideal of S is idempotent.

1.4 Fuzzy Sets

A fuzzy subset f of a set X is a function from X into the unit closed interval $[0, 1]$, that is $f : X \to [0, 1]$. If f and g are fuzzy subsets of X, then $f \leq g$ means that $f(x) \leq g(x)$ for all $x \in X$. The fuzzy subsets $f \wedge g$ and $f \vee g$ of X are defined as $(f \wedge g)(x) = f(x) \wedge g(x)$ and $(f \vee g)(x) = f(x) \vee g(x)$ for all $x \in X$. If $\{f_i\}_{i \in I}$ is a family of fuzzy subsets of X, then $\bigwedge_{i\in I} f_i$ and $\bigvee_{i\in I} f_i$ are fuzzy subsets of X defined by $\left(\bigwedge_{i\in I} f_i\right)(x) = \inf\{f_i(x)\}_{i\in I}$ and $\left(\bigvee_{i\in I} f_i\right)(x) = \sup\{f_i(x)\}_{i\in I}$ for all $x \in X$.

Let f be a fuzzy subset of X and $t \in [0,1]$. Then

$$U(f;t) = \{x \in S : f(x) \geq t\}$$

is called the level subset of f.

For a fuzzy subset f of X the crisp set

$$S_0 = \{x \in S : f(x) > 0\}$$

is called the support of f.

Let f and g be two fuzzy subsets of a semigroup S. Then the product $f \circ g$ is defined by

$$(f \circ g)(x) = \begin{cases} \bigvee_{x=yz} \{f(y) \wedge g(z)\} & \text{if } \exists\ y, z \in S, \text{ such that } x = yz \\ 0 & \text{otherwise} \end{cases}$$

A fuzzy subset f in a universe X of the form

$$f(y) = \begin{cases} t \in (0,1] & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t. For a fuzzy point x_t and a fuzzy set f in a set X, Pu and Liu [12] gave meaning to the symbol $x_t \alpha f$, when $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to (resp. be quasi-coincident with) a fuzzy set f written $x_t \in f$ (resp. $x_t q f$) if $f(x) \geq t$ (resp. $f(x) + t > 1$) and in this case, $x_t \in \vee q f$ (resp. $x_t \in \wedge q f$) means that $x_t \in f$ or $x_t q f$ (resp. $x_t \in f$ and $x_t q f$). To say that $x_t \overline{\alpha} f$ means that $x_t \alpha f$ does not hold.

For a fuzzy subset f of S and $t \in (0,1]$, the q-set and $(\in \vee q)$-set with respect to t (briefly, t-q-set and t-$(\in \vee q)$-set, respectively) are defined as follows:

$$S_q^t = \{x \in S \mid x_t\, q f\} \text{ and } S_{(\in \vee q)}^t = \{x \in S \mid x_t\, (\in \vee q)\, f\} = U(f;t) \cup S_q^t.$$

Note that, for any $t, r \in (0,1]$, if $t \geq r$ then $S_q^r \subseteq S_q^t$.

1.5 Fuzzy Subsemigroups and Fuzzy Ideals

A fuzzy subset f of a semigroup S is said to be a fuzzy subsemigroup of S if

$$f(xy) \geq \min\{f(x), f(y)\} \tag{1.1}$$

for all $x, y \in S$.

A fuzzy subset f of a semigroup S is said to be a *fuzzy* left (resp. right) ideal of S if

$$f(xy) \geq f(y) \text{ (resp. } f(xy) \geq f(x)) \tag{1.2}$$

for all $x, y \in S$.

A fuzzy subset f of a semigroup S is said to be a *fuzzy* ideal of S if it is both a fuzzy left and fuzzy right ideal of S.

A fuzzy subset f of a semigroup S is said to be a fuzzy generalized bi-ideal of S if

$$f(xyz) \geq \min\{f(x), f(z)\} \tag{1.3}$$

for all $x, y, z \in S$.

A fuzzy subset f of a semigroup S is said to be a fuzzy bi-ideal of S if it is a fuzzy subsemigroup and fuzzy generalized bi-ideal of S.

A fuzzy subset f of a semigroup S is said to be a fuzzy quasi-ideal of S if

$$f(x) \geq \min\{(f \circ \mathcal{S})(x), (\mathcal{S} \circ f)(x)\}$$

for all $x \in S$, where $\mathcal{S}$ is the fuzzy subset of S mapping every element of S on 1.

A fuzzy subset f of a semigroup S is said to be a fuzzy interior-ideal of S if

$$f(xyz) \geq f(y) \tag{1.4}$$

for all $x, y, z \in S$.

Obviously, every fuzzy ideal of a semigroup S is a fuzzy interior-ideal of S, every one sided fuzzy ideal of S is a fuzzy quasi-ideal, every fuzzy quasi-ideal is a fuzzy bi-ideal and every fuzzy bi-ideal is a fuzzy generalized bi-ideal of S but the converse is not true.

1.6 (α, β)-fuzzy Ideals

A fuzzy subset f of a semigroup S is said to be an (α, β)-fuzzy subsemigroup of S, where $\alpha \neq \in \wedge q$, if it satisfies the following condition:

$$x_t \alpha f \text{ and } y_r \alpha f \text{ implies } (xy)_{\min\{t,r\}} \beta f \tag{1.5}$$

for all $x, y \in S$ and $t, r \in (0, 1]$.

Let f be a fuzzy subset of S such that $f(x) \leq 0.5$ for all $x \in S$. Let $x \in S$ and $t \in (0, 1]$ be such that $x_t \in \wedge qf$. Then $f(x) \geq t$ and $f(x) + t > 1$. It follows that $2f(x) = f(x) + f(x) \geq f(x) + t > 1$. This implies that $f(x) > 0.5$. This means that $\{x_t : x_t \in \wedge qf\} = \emptyset$. Therefore, the case $\alpha = \in \wedge q$ in the above definition is not included.

A fuzzy subset f of a semigroup S is said to be an (α, β)-fuzzy left (resp. right) ideal of S, where $\alpha \neq \in \wedge q$, if it satisfies:

$$y_t \alpha f \text{ (resp. } x_t \alpha f\text{) implies } (xy)_t \beta f \tag{1.6}$$

for all $x, y \in S$ and $t \in (0, 1]$.

A fuzzy subset f of a semigroup S is said to be an (α, β)-fuzzy two sided ideal or simply an (α, β)-fuzzy ideal of S if it is an (α, β)-fuzzy left ideal and an (α, β)-fuzzy right ideal of S.

A fuzzy subset f of a semigroup S is said to be an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S, if it satisfies:

$$y_t \in f \text{ (resp. } x_t \in f\text{) implies } (xy)_t \in \vee qf \tag{1.7}$$

for all $x, y \in S$ and $t \in (0, 1]$.

A fuzzy subset f of a semigroup S is said to be an $(\in, \in \vee q)$-fuzzy two sided ideal or simply an $(\in, \in \vee q)$-fuzzy ideal of S if it is an $(\in, \in \vee q)$-fuzzy left ideal and an $(\in, \in \vee q)$-fuzzy right ideal of S.

A fuzzy subset f of a semigroup S is called an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S if it satisfies

$$x_t \in f \text{ and } z_r \in f \Rightarrow (xyz)_{\min\{t,r\}} \in \vee qf$$

for all $x, y, z \in S$ and $t, r \in (0, 1]$.

A fuzzy subset f of a semigroup S is called an $(\in, \in \vee q)$-fuzzy bi-ideal of S if it satisfies

$$\begin{aligned} x_t &\in f \text{ and } y_r \in f \Rightarrow (xy)_{\min\{t,r\}} \in \vee qf \\ x_t &\in f \text{ and } z_r \in f \Rightarrow (xyz)_{\min\{t,r\}} \in \vee qf \end{aligned}$$

for all $x, y, z \in S$ and $t, r \in (0, 1]$.

Let f be a fuzzy subset of a semigroup S. We define the upper part f^{+} and the lower part f^{-} of f as follows, $f^{+}(x) = f(x) \vee 0.5$ and $f^{-}(x) = f(x) \wedge 0.5$.

Theorem 29 *[43] For a semigroup S the following conditions are equivalent.*

(1) S *is regular.*

(2) $(f \wedge g)^{-} = (f \circ g)^{-}$ *for every* $(\in, \in \vee q)$-fuzzy right ideal f *and every* $(\in, \in \vee q)$-fuzzy left ideal g *of* S.

Theorem 30 *[43] For a semigroup S, the following conditions are equivalent.*

(1) S is intra-regular.

(2) $(f \wedge g)^{-} \leq (f \circ g)^{-}$ for every $(\in, \in \vee q)$-fuzzy left ideal f and every $(\in, \in \vee q)$-fuzzy right ideal g of S.

Theorem 31 *[43] For a semigroup S, the following conditions are equivalent.*

(1) S is both regular and intra-regular.

(2) $(f \circ f)^{-} = f^{-}$ for every $(\in, \in \vee q)$-fuzzy quasi-ideal f of S.

(3) $(f \circ f)^{-} = f^{-}$ for every $(\in, \in \vee q)$-fuzzy bi-ideal f of S.

(4) $(f \wedge g)^{-} \leq (f \circ g)^{-}$ for all $(\in, \in \vee q)$-fuzzy quasi-ideals f, g of S.

(5) $(f \wedge g)^{-} \leq (f \circ g)^{-}$ for every $(\in, \in \vee q)$-fuzzy quasi-ideal f and every $(\in, \in \vee q)$-fuzzy bi-ideal g of S.

(6) $(f \wedge g)^{-} \leq (f \circ g)^{-}$ for all $(\in, \in \vee q)$-fuzzy bi-ideals f, g of S.

Chapter 2

Characterizations of Semigroups by $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy Ideals

In this chapter we study $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals, generalized bi-ideals and quasi-ideals of a semigroup and characterize regular, intra-regular and semisimple semigroups by the properties of these ideals.

2.1 $(\overline{\alpha}, \overline{\beta})$-fuzzy Subsemigroups

In this section, we study $(\overline{\alpha}, \overline{\beta})$-fuzzy subsemigroups of semigroups, specially $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subsemigroups. Throughout this chapter S will denote a semigroup and $\overline{\alpha}, \overline{\beta}$ are any two of $\{\overline{\in}, \overline{q}, \overline{\in} \vee \overline{q}, \overline{\in} \wedge \overline{q}\}$ unless otherwise mentioned.

We start this section with the following definition.

Definition 32 *A fuzzy subset f of S is said to be an $(\overline{\alpha}, \overline{\beta})$-fuzzy subsemigroup of S, where $\overline{\alpha} \neq \overline{\in} \wedge \overline{q}$, if it satisfies the following condition:*

$$(xy)_{\min\{t,r\}} \overline{\alpha} f \text{ implies } x_t \overline{\beta} f \text{ or } y_r \overline{\beta} f \tag{2.1}$$

for all $x, y \in S$ and $t, r \in (0, 1]$.

Let f be a fuzzy suset of S such that $f(x) \geq 0.5$ for all $x \in S$. Let $x \in S$ and $t \in (0, 1]$ be such that $x_t \overline{\in} \wedge \overline{q} f$. Then $f(x) < t$ and $f(x) + t \leq 1$. It follows that $2f(x) = f(x) + f(x) < f(x) + t \leq 1$. This implies that $f(x) < 0.5$. This means that $\{x_t : x_t \overline{\in} \wedge \overline{q} f\} = \emptyset$. Therefore, the case $\overline{\alpha} = \overline{\in} \wedge \overline{q}$ in the above definition is not included.

Theorem 33 *A fuzzy subset f of S is a fuzzy subsemigroup of S if and only if f is an $(\overline{\in}, \overline{\in})$-fuzzy subsemigroup of S.*

Proof. Let f be a fuzzy subsemigroup of S. Let $x, y \in S$ and $t, r \in (0,1]$ be such that $(xy)_{\min\{t,r\}} \overline{\in} f$. Then $f(xy) < \min\{t, r\}$. As f is a fuzzy subsemigroup of S, we have

$$\min\{f(x), f(y)\} \leq f(xy) < \min\{t, r\}.$$

This implies that $f(x) < t$ or $f(y) < r$, that is, $x_t \overline{\in} f$ or $y_r \overline{\in} f$. Hence f is an $(\overline{\in}, \overline{\in})$-fuzzy subsemigroup of S.

Conversely, assume that f is an $(\overline{\in}, \overline{\in})$-fuzzy subsemigroup of S. Let x, $y \in S$ be such that

$$f(xy) < \min\{f(x), f(y)\}.$$

Then we can choose $t \in (0, 1]$ such that

$$f(xy) < t \leq \min\{f(x), f(y)\}.$$

Thus $(xy)_t \overline{\in} f$ but $x_t \in f$ and $y_t \in f$, which is a contradiction. Hence $f(xy) \geq \min\{f(x), f(y)\}$. ■

Theorem 34 *(i) Every $(\overline{\in}, \overline{\in})$-fuzzy subsemigroup of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subsemigroup of S.*

(ii) Every $(\overline{\in}, \overline{q})$-fuzzy subsemigroup of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subsemigroup of S.

Proof. Straightforward. ■

The converse of above Theorem is not true in general.

Example 35 *Consider the semigroup $S = \{0, 1, -1, \iota, -\iota\}$ with multiplication table given below:*

	0	1	-1	ι	$-\iota$
0	0	0	0	0	0
1	0	1	-1	ι	$-\iota$
-1	0	-1	1	$-\iota$	ι
ι	0	ι	$-\iota$	-1	1
$-\iota$	0	$-\iota$	ι	1	-1

Let f be the fuzzy subset of S defined by $f(0) = 0.6$, $f(1) = 0.2$, $f(-1) = 0.2$, $f(\iota) = 0.3$, and $f(-\iota) = 0.4$. Then f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subsemigroup of S but not an $(\overline{\in}, \overline{\in})$-fuzzy subsemigroup of S. Because $f(-1) = f(\iota\iota) = 0.2 < 0.25$ that is $(\iota\iota)_{0.25} \overline{\in} f$ whereas $f(\iota) \not< 0.25$, that is $\iota_{0.25} \in f$. Similarly $f(-1) = f(-\iota. - \iota) = 0.2 < 0.7$ that is $(-\iota. - \iota)_{0.7} \overline{\in} f$ whereas $f(-\iota) + 0.7 = 1.1 \not< 1$, that is $(-\iota)_{0.7} q f$. Hence f is not an $(\overline{\in}, \overline{q})$-fuzzy subsemigroup of S.

Theorem 36 *Every $(\overline{\in}\vee\overline{q},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S.*

Proof. Straightforward. ■

There are twelve different types of (α,β)-fuzzy subsemigroups of a semigroup S, they are $(\overline{\in},\overline{\in})$, $(\overline{\in},\overline{q})$, $(\overline{\in},\overline{\in}\wedge\overline{q})$, $(\overline{\in},\overline{\in}\vee\overline{q})$, $(\overline{q},\overline{\in})$, $(\overline{q},\overline{q})$, $(\overline{q},\overline{\in}\wedge\overline{q})$, $(\overline{q},\overline{\in}\vee\overline{q})$, $(\overline{\in}\vee\overline{q},\overline{\in})$, $(\overline{\in}\vee\overline{q},\overline{q})$, $(\overline{\in}\vee\overline{q},\overline{\in}\wedge\overline{q})$, and $(\overline{\in}\vee\overline{q},\overline{\in}\vee\overline{q})$. The following thorem gives the relations between them.

Theorem 37 *Let S be a semigroup. Then the following are true:*

(*i*) Every $(\overline{\in},\overline{\in}\wedge\overline{q})$-fuzzy subsemigroup of S is an $(\overline{\in},\overline{\in})$-fuzzy subsemigroup of S.

(*ii*) Every $(\overline{\in},\overline{\in}\wedge\overline{q})$-fuzzy subsemigroup of S is an $(\overline{\in},\overline{q})$-fuzzy subsemigroup of S.

(*iii*) Every $(\overline{\in},\overline{\in}\wedge\overline{q})$-fuzzy subsemigroup of S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S.

(*iv*) Every $(\overline{\in},\overline{\in})$-fuzzy subsemigroup of S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S.

(*v*) Every $(\overline{\in},\overline{q})$-fuzzy subsemigroup of S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S.

(*vi*) Every $(\overline{\in}\vee\overline{q},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S.

(*vii*) Every $(\overline{\in}\vee\overline{q},\overline{\in})$-fuzzy subsemigroup of S is an $(\overline{\in}\vee\overline{q},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S.

(*viii*) Every $(\overline{\in}\vee\overline{q},\overline{q})$-fuzzy subsemigroup of S is an $(\overline{\in}\vee\overline{q},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S.

(*ix*) Every $(\overline{\in}\vee\overline{q},\overline{\in}\wedge\overline{q})$-fuzzy subsemigroup of S is an $(\overline{\in}\vee\overline{q},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S.

(*x*) Every $(\overline{\in}\vee\overline{q},\overline{\in}\wedge\overline{q})$-fuzzy subsemigroup of S is an $(\overline{\in}\vee\overline{q},\overline{\in})$-fuzzy subsemigroup of S.

(*xi*) Every $(\overline{\in}\vee\overline{q},\overline{\in}\wedge\overline{q})$-fuzzy subsemigroup of S is an $(\overline{\in}\vee\overline{q},\overline{q})$-fuzzy subsemigroup of S.

(*xii*) Every $(\overline{q},\overline{\in}\wedge\overline{q})$-fuzzy subsemigroup of S is an $(\overline{q},\overline{\in})$-fuzzy subsemigroup of S.

(*xiii*) Every $(\overline{q},\overline{\in}\wedge\overline{q})$-fuzzy subsemigroup of S is an $(\overline{q},\overline{q})$-fuzzy subsemigroup of S.

(*xiv*) Every $(\overline{q},\overline{\in}\wedge\overline{q})$-fuzzy subsemigroup of S is an $(\overline{q},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S.

(xv) Every $(\overline{q},\overline{\in})$-fuzzy subsemigroup of S is an $(\overline{q},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S.

(xvi) Every $(\overline{q},\overline{q})$-fuzzy subsemigroup of S is an $(\overline{q},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S.

Proof. Straightforward. ■

Recall that for a fuzzy set f in S, we denote $S_0 := \{x \in S : f(x) > 0\}$.

Theorem 38 *If f is one of the following:*

(i) an $(\overline{\in},\overline{\in})$-fuzzy subsemigroup of S;
(ii) an $(\overline{\in},\overline{q})$-fuzzy subsemigroup of S;
(iii) a $(\overline{q},\overline{\in})$-fuzzy subsemigroup of S;
(iv) a $(\overline{q},\overline{q})$-fuzzy subsemigroup of S;
(v) an $(\overline{\in},\overline{\in}\wedge\overline{q})$-fuzzy subsemigroup of S;
(vi) an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S;
(vii) an $(\overline{\in}\vee\overline{q},\overline{q})$-fuzzy subsemigroup of S;
$(viii)$ an $(\overline{\in}\vee\overline{q},\overline{\in})$-fuzzy subsemigroup of S;
(ix) an $(\overline{\in}\vee\overline{q},\overline{\in}\wedge\overline{q})$-fuzzy subsemigroup of S;
(x) a $(\overline{q},\overline{\in}\wedge\overline{q})$-fuzzy subsemigroup of S;
(xi) a $(\overline{q},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S;
(xii) an $(\overline{\in}\vee\overline{q},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S;

then the support of f, S_0 is a subsemigroup of S.

Proof. (i) Straightforward.

(ii) Let $x,\ y \in S_0$. Then $f(x) > 0$, $f(y) > 0$. If $f(xy) = 0$, then $(xy)_{\min\{1,1\}}\overline{\in}f$ but x_1qf and y_1qf, which is a contradiction. Thus $f(xy) > 0$, and so $xy \in S_0$. Therefore S_0 is a subsemigroup of S.

(iii) Let $x, y \in S_0$. Then $f(x) > 0$, $f(y) > 0$. If $f(xy) = 0$, then $(xy)_{\min\{f(x),f(y)\}}\overline{q}f$ but $x_{f(x)} \in f$ and $y_{f(y)} \in f$, which is a contradiction. It follows that $f(xy) > 0$ so that $xy \in S_0$. Therefore S_0 is a subsemigroup of S.

(iv) Let $x,\ y \in S_0$. Then $f(x) > 0$, $f(y) > 0$. Suppose that $xy \notin S_0$. Then $f(xy) = 0$. Note that $(xy)_{\min\{1,1\}}\overline{q}f$ but x_1qf and y_1qf, because $f(x) + 1 > 1$ and $f(y) + 1 > 1$. This is a contradiction, and thus $f(xy) > 0$, which shows that $xy \in S_0$. Consequently S_0 is a subsemigroup of S.

Similarly we can prove the remaining parts. ■

The following is a characterization of $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup.

Theorem 39 *Let f be a fuzzy subset of a semigroup S, $x, y \in S$ and $t, r \in (0,1]$. Then the following conditions are equivalent*

(2.1a) $(xy)_{\min\{t,r\}}\overline{\in}f \Rightarrow x_t\overline{\in}\vee\overline{q}f$ or $y_r\overline{\in}\vee\overline{q}f$.

(2.1b) $\max\{f(xy),0.5\}\geq\min\{f(x),f(y\}$.

Proof. (2.1a) $\Rightarrow$ (2.1b) Suppose there exist $x,y\in S$ such that $\max\{f(xy),0.5\}<\min\{f(x),f(y\}$. Then we can choose $t\in(0.5,1]$ such that $\max\{f(xy),0.5\}<t=\min\{f(x),f(y\}$. Then $(xy)_t\overline{\in}f$ but $x_t\in\wedge qf$ and $y_t\in\wedge qf$, which is a contradiction. Hence $\max\{f(xy),0.5\}\geq\min\{f(x),f(y\}$.

(2.1b) $\Rightarrow$ (2.1a) Let $(xy)_{\min\{t,r\}}\overline{\in}f$. Then $f(xy)<\min\{t,r\}$. If $\max\{f(xy),0.5\}=f(xy)$, then $\min\{f(x),f(y\}\leq f(xy)<\min\{t,r\}$ and consequently, $f(x)<t$ or $f(y)<r$. It follows that $x_t\overline{\in}f$ or $y_r\overline{\in}f$. Thus $x_t\overline{\in}\vee\overline{q}f$ or $y_r\overline{\in}\vee\overline{q}f$.

If $\max\{f(xy),0.5\}=0.5$, then $\min\{f(x),f(y\}\leq 0.5$. Suppose $x_t\in f$ and $y_r\in f$, then $t\leq f(x)\leq 0.5$ or $r\leq f(y)\leq 0.5$. It follows that $x_t\overline{q}f$ or $y_r\overline{q}f$. Thus $x_t\overline{\in}\vee\overline{q}f$ or $y_r\overline{\in}\vee\overline{q}f$. ■

Corollary 40 *A fuzzy subset f of a semigroup S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S if it satisfies the condition* (2.1b)*.*

Theorem 41 *A fuzzy subset f of a semigroup S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S if and only if $U(f;t)$ $(\neq\phi)$ is a subsemigroup of S for all $t\in(0.5,1]$.*

Proof. Suppose f is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S and $x,y\in U(f;t)$ for some $t\in(0.5,1]$. Then $f(x)\geq t$ and $f(y)\geq t$. Hence $0.5<t\leq\min\{f(x),f(y)\}\leq\max\{f(xy),0.5\}$. Thus $f(xy)\geq t$ and so $xy\in U(f;t)$. Consequently $U(f;t)$ is a subsemigroup of S.

Conversely, assume that $U(f;t)$ $(\neq\phi)$ is a subsemigroup of S for all $t\in(0.5,1]$. Suppose that there exist $x,y\in S$ such that $\max\{f(xy),0.5\}<\min\{f(x),f(y\}=t$. Then $t\in(0.5,1]$ and $x,y\in U(f;t)$ but $xy\notin U(f;t)$. This contradicts our hypothesis. Hence $\max\{f(xy),0.5\}\geq\min\{f(x),f(y\}$ and so f is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S. ■

Corollary 42 *A non-empty subset A of S is a subsemigroup of S if and only if the characteristic function C_A of A is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S.*

Theorem 43 *Let $\{f_i\mid i\in\Lambda\}$ be a family of $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subsemigroups of S. Then $f:=\bigcap_{i\in\Lambda}f_i$ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S.*

Proof. Straightforward. ■

Theorem 44 *If f is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy subsemigroup of S, then the non-empty t-q-set S_q^t is a subsemigroup of S for all $t\in(0,0.5]$.*

Proof. Let $t \in (0, 0.5]$ and $x, y \in S_q^t$. Then $x_t q f$ and $y_t q f$, that is, $f(x) + t > 1$ and $f(y) + t > 1$. It follows from Theorem 39 that

$$\begin{aligned} \max\{f(xy), \ 0.5\} + t &\geq \min\{f(x), f(y)\} + t \\ &= \min\{f(x) + t, f(y) + t\} \\ &> 1. \end{aligned}$$

So $(xy)_t q f$. Hence $xy \in S_q^t$, and therefore S_q^t is a subsemigroup of S. ■

2.2 $(\overline{\alpha}, \overline{\beta})$-fuzzy Ideals

In this section we study $(\overline{\alpha}, \overline{\beta})$-fuzzy ideals of semigroups, specialy $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals, where $\overline{\alpha}, \overline{\beta}$ are any two of $\{\overline{\in}, \overline{q}, \overline{\in} \vee \overline{q}, \overline{\in} \wedge \overline{q}\}$ unless otherwise mentioned.

We start this section with the following definition.

Definition 45 *A fuzzy subset f of a semigroup S is said to be an $(\overline{\alpha}, \overline{\beta})$-fuzzy left (resp. right) ideal of S, where $\overline{\alpha} \neq \overline{\in} \wedge \overline{q}$, if it satisfies:*

$$(xy)_t \, \overline{\alpha} f \text{ implies } y_t \overline{\beta} f \ \left(\text{resp. } x_t \overline{\beta} f\right) \tag{2.2}$$

for all $x, y \in S$ and $t \in (0, 1]$.

A fuzzy subset f of a semigroup S is said to be an $(\overline{\alpha}, \overline{\beta})$-fuzzy two sided ideal or simply an $(\overline{\alpha}, \overline{\beta})$-fuzzy ideal of S if it is an $(\overline{\alpha}, \overline{\beta})$-fuzzy left ideal and an $(\overline{\alpha}, \overline{\beta})$-fuzzy right ideal of S.

Theorem 46 *A fuzzy subset f of a semigroup S is a fuzzy left (resp. right) ideal of S if and only if it is an $(\overline{\in}, \overline{\in})$-fuzzy left (resp. right) ideal of S.*

Proof. Straightforward. ■

Theorem 47 (i) *Every $(\overline{\in}, \overline{\in})$-fuzzy left (right) ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S.*

(ii) Every $(\overline{\in}, \overline{q})$-fuzzy left (right) ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S.

Proof. Straightforward. ■

The converse of above Theorem is not true in general.

Example 48 *Consider the semigroup $S = \{1, 2, 3, 4, 5\}$ given by the following table:*

$*$	1	2	3	4	5
1	1	4	1	4	4
2	1	2	1	4	4
3	1	4	3	4	5
4	1	4	1	4	4
5	1	4	3	4	5

Let f be the fuzzy subset of S defined by $f(1) = 0.6$, $f(2) = 0.45$, $f(3) = 0.2$, $f(4) = 0.3$, and $f(5) = 0.4$. Then f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S but not an $(\overline{\in}, \overline{\in})$-fuzzy left ideal of S. Because $f(4) = f(1 * 2) = 0.3 < 0.4$ that is $(1 * 2)_{0.4} \overline{\in} f$ whereas $f(2) = 0.45 \geq 0.4$, that is $2_{0.4} \in f$. Similarly $f(4) = f(1 * 2) = 0.3 < 0.7$ that is $(1 * 2)_{0.7} \overline{\in} f$ whereas $f(2) + 0.7 = 1.15 > 1$, that is $(2)_{0.7} q f$. Hence f is not an $(\overline{\in}, \overline{q})$-fuzzy left ideal of S.

Theorem 49 *Every $(\overline{\in} \vee \overline{q}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S.*

Proof. Straightforward. ■

There are twelve different types of (α, β)-fuzzy left (right) ideals of a semigroup S, they are $(\overline{\in}, \overline{\in})$, $(\overline{\in}, \overline{q})$, $(\overline{\in}, \overline{\in} \wedge \overline{q})$, $(\overline{\in}, \overline{\in} \vee \overline{q})$, $(\overline{q}, \overline{\in})$, $(\overline{q}, \overline{q})$, $(\overline{q}, \overline{\in} \wedge \overline{q})$, $(\overline{q}, \overline{\in} \vee \overline{q})$, $(\overline{\in} \vee \overline{q}, \overline{\in})$, $(\overline{\in} \vee \overline{q}, \overline{q})$, $(\overline{\in} \vee \overline{q}, \overline{\in} \wedge \overline{q})$, and $(\overline{\in} \vee \overline{q}, \overline{\in} \vee \overline{q})$. The following thorem gives the relations between them.

Theorem 50 *Let S be a semigroup. Then the following are true:*

(i) Every $(\overline{\in}, \overline{\in} \wedge \overline{q})$-fuzzy left (right) ideal of S is an $(\overline{\in}, \overline{\in})$-fuzzy left (right) ideal of S.

(ii) Every $(\overline{\in}, \overline{\in} \wedge \overline{q})$-fuzzy left (right) ideal of S is an $(\overline{\in}, \overline{q})$-fuzzy left (right) ideal of S.

(iii) Every $(\overline{\in}, \overline{\in} \wedge \overline{q})$-fuzzy left (right) ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S.

(iv) Every $(\overline{\in}, \overline{\in})$-fuzzy left (right) ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S.

(v) Every $(\overline{\in}, \overline{q})$-fuzzy left (right) ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S.

(vi) Every $(\overline{\in} \vee \overline{q}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S.

(vii) Every $(\overline{\in} \vee \overline{q}, \overline{\in})$-fuzzy left (right) ideal of S is an $(\overline{\in} \vee \overline{q}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S.

(viii) Every $(\in \vee \overline{q}, \overline{q})$-fuzzy left (right) ideal of S is an $(\overline{\in} \vee \overline{q}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S.

(ix) Every $(\in \vee \overline{q}, \overline{\in} \wedge \overline{q})$-fuzzy left (right) ideal of S is an $(\overline{\in} \vee \overline{q}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S.

(x) Every $(\in \vee \overline{q}, \overline{\in} \wedge \overline{q})$-fuzzy left (right) ideal of S is an $(\overline{\in} \vee \overline{q}, \overline{\in})$-fuzzy left (right) ideal of S.

(xi) Every $(\in \vee \overline{q}, \overline{\in} \wedge \overline{q})$-fuzzy left (right) ideal of S is an $(\overline{\in} \vee \overline{q}, \overline{q})$-fuzzy left (right) ideal of S.

(xii) Every $(\overline{q}, \overline{\in} \wedge \overline{q})$-fuzzy left (right) ideal of S is an $(\overline{q}, \overline{\in})$-fuzzy left (right) ideal of S.

(xiii) Every $(\overline{q}, \overline{\in} \wedge \overline{q})$-fuzzy left (right) ideal of S is an $(\overline{q}, \overline{q})$-fuzzy left (right) ideal of S.

(xiv) Every $(\overline{q}, \overline{\in} \wedge \overline{q})$-fuzzy left (right) ideal of S is an $(\overline{q}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S.

(xv) Every $(\overline{q}, \overline{\in})$-fuzzy left (right) ideal of S is an $(\overline{q}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S.

(xvi) Every $(\overline{q}, \overline{q})$-fuzzy left (right) ideal of S is an $(\overline{q}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S.

Proof. Straightforward. ■

Theorem 51 *Let f be a fuzzy subset of S. If f is one of the following:*

(i) an $(\overline{\in}, \overline{\in})$-fuzzy left (resp. right) ideal of S;
(ii) an $(\overline{\in}, \overline{q})$-fuzzy left (resp. right) ideal of S;
(iii) a $(\overline{q}, \overline{\in})$-fuzzy left (resp. right) ideal of S;
(iv) a $(\overline{q}, \overline{q})$-fuzzy left (resp. right) ideal of S;
(v) an $(\overline{\in}, \overline{\in} \wedge \overline{q})$-fuzzy left (resp. right) ideal of S;
(vi) an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right) ideal of S;
(vii) an $(\in \vee \overline{q}, \overline{q})$-fuzzy left (resp. right) ideal of S;
(viii) an $(\in \vee \overline{q}, \overline{\in})$-fuzzy left (resp. right) ideal of S;
(ix) an $(\in \vee \overline{q}, \overline{\in} \wedge \overline{q})$-fuzzy left (resp. right) ideal of S;
(x) a $(\overline{q}, \overline{\in} \wedge \overline{q})$-fuzzy left (resp. right) ideal of S;
(xi) a $(\overline{q}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right) ideal of S;
(xii) a $(\overline{\in} \vee \overline{q}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right) ideal of S;

then the set S_0 is a left (resp. right, lateral) ideal of S.

Proof. The proof is similar to the proof of Theorem 38. ■

The following theorem is a characterization of left (right) ideal of S.

Theorem 52 *Let f be a fuzzy subset of a semigroup S, $x, y \in S$ and $t \in (0,1]$. Then the following conditions are equivalent*

(2.2a) $(xy)_t \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q} f$.
(2.2b) $\max\{f(xy), 0.5\} \geq f(x)$.
(resp. (2.3a) $(xy)_t \overline{\in} f \Rightarrow y_t \overline{\in} \vee \overline{q} f$.
(2.3b) $\max\{f(xy), 0.5\} \geq f(y)$).

Proof. (2.2a) $\Rightarrow$ (2.2b) Suppose there exist $x, y \in S$ such that $\max\{f(xy), 0.5\} < f(x)$. Then we can choose $t \in (0.5, 1]$ such that $\max\{f(xy), 0.5\} < t = f(x)$. Then $(xy)_t \overline{\in} f$ but $x_t \in \wedge q f$, which is a contradiction. Hence $\max\{f(xy), 0.5\} \geq f(x)$.

(2.2b) $\Rightarrow$ (2.2a) Let $(xy)_t \overline{\in} f$. Then $f(xy) < t$. If $\max\{f(xy), 0.5\} = f(xy)$, then $f(x) \leq f(xy) < t$ and consequently, $f(x) < t$. It follows that $x_t \overline{\in} f$. Thus $x_t \overline{\in} \vee \overline{q} f$.

If $\max\{f(xy), 0.5\} = 0.5$, then $f(x) \leq 0.5$. Suppose $x_t \in f$, then $t \leq f(x) < 0.5$. It follows that $x_t \overline{q} f$. Thus $x_t \overline{\in} \vee \overline{q} f$. ■

Corollary 53 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right (resp. left) ideal of S if it satisfies condition* (2.2b) *(resp.* (2.3b)*).*

Corollary 54 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S if it satisfies conditions* (2.2b) *and* (2.3b)*.*

Theorem 55 *A non-empty subset A of a semigroup S is a left (resp. right) ideal of S if and only if the fuzzy subset f of S defined by:*

$$f(x) = \begin{cases} \leq 0.5 & \text{for all } x \in S - A, \\ 1 & \text{for all } x \in A, \end{cases}$$

is an $(\overline{\alpha}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right) ideal of S, where $\overline{\alpha} \in \{\overline{\in}, \overline{q}, \overline{\in} \vee \overline{q}\}$.

Proof. Let A be a left ideal of S.

(a) We show that f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S. Let $x, y \in S$ and $t \in (0,1]$ be such that $(xy)_t \overline{\in} f$. Then $f(xy) < t$, this implies $xy \notin A$. Thus $y \notin A$. If $t > 0.5$, then $f(y) \leq 0.5 < t$, implies that $f(y) < t$. Hence $y_t \overline{\in} f$. If $t \leq 0.5$, then $f(y) + t \leq 0.5 + 0.5 = 1$, so $y_t \overline{q} f$. Thus $y_t \overline{\in} \vee \overline{q} f$. Hence f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S.

(b) Next, we show that f is a $(\overline{q}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S. Let $x, y \in S$ and $t \in (0,1]$ be such that $(xy)_t \overline{q} f$. Then $f(xy) + t \leq 1$, so $xy \notin A$. Therefore $y \notin A$. If $t > 0.5$, then $f(y) \leq 0.5 < t$. Hence $y_t \overline{\in} f$. If $t \leq 0.5$, then $f(y) + t \leq 0.5 + 0.5 = 1$, so $y_t \overline{q} f$. Thus $y_t \overline{\in} \vee \overline{q} f$. Hence f is a $(\overline{q}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S.

(c) Now, we show that f is an $(\overline{\in} \vee \overline{q}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S. Let $x, y \in S$ and $t \in (0,1]$ be such that $(xy)_t \overline{\in} \vee \overline{q} f$. Then $(xy)_t \overline{\in} f$ or $(xy)_t \overline{q} f$. The rest of the proof is a consequence of (a) and (b).

Conversely, assume that f is an $(\overline{\alpha},\overline{\in}\vee\overline{q})$-fuzzy left ideal of S. Suppose $\overline{\alpha}=\overline{\in}$. Let $y\in A$. Then $f(y)=1$. Since f is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal of S, so $f(xy)\vee 0.5\geq f(y)=1$. This implies that $f(xy)=1$. Hence $xy\in A$. Similarly, we can prove for $\overline{\alpha}=\overline{q}$ and $\overline{\alpha}=\overline{\in}\vee\overline{q}$. ■

Corollary 56 *A non-empty subset A of a semigroup S is a left (resp. right) ideal of S if and only if the characteristic function C_A of A is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left (resp. right) ideal of S.*

Theorem 57 *A fuzzy subset f of a semigroup S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right (resp. left) ideal of S if and only if $U(f;t)$ $(\neq\phi)$ is a right (resp. left) ideal of S for all $t\in(0.5,1]$.*

Proof. Suppose f is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal of S and $x\in U(f;t)$ and $y\in S$, for some $t\in(0.5,1]$. Then $f(x)\geq t$. Hence $0.5<t\leq f(x)\leq\max\{f(xy),0.5\}$. Thus $f(xy)\geq t$ and so $xy\in U(f;t)$. Consequently $U(f;t)$ is a right ideal of S.

Conversely, assume that $U(f;t)$ $(\neq\phi)$ is a right ideal of S for all $t\in(0.5,1]$. Suppose that there exist $x,y\in S$ such that $\max\{f(xy),0.5\}<f(x)=t$. Then $t\in(0.5,1]$ and $x\in U(f;t)$ but $xy\notin U(f;t)$. Which contradicts our hypothesis. Hence $\max\{f(xy),0.5\}\geq f(x)$ and so f is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal of S. ■

Corollary 58 *A fuzzy subset f of a semigroup S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideal of S if and only if $U(f;t)$ $(\neq\phi)$ is an ideal of S for all $t\in(0.5,1]$.*

Theorem 59 *Let f be an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal and g an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal of a semigroup S. Then $f\circ g$ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy two-sided ideal of S.*

Proof. Let $x,y\in S$. Then

$$(f\circ g)(y)=\bigvee_{y=pq}\{f(p)\wedge g(q)\}.$$

(If $y=pq$, then $xy=x(pq)=(xp)q$. Since f is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal, so by Theorem 52 $f(xp)\vee 0.5\geq f(p)$).

Thus

$$\begin{aligned}
(f \circ g)(y) &= \bigvee_{y=pq} \{f(p) \wedge g(q)\} \\
&\leq \bigvee_{xy=(xp)q} \{(f(xp) \vee 0.5) \wedge g(q)\} \\
&\leq \bigvee_{xy=ab} \{(f(a) \vee 0.5) \wedge g(b)\} \\
&= \bigvee_{xy=ab} \{(f(a) \wedge g(b)) \vee 0.5\} \\
&= \left(\bigvee_{xy=ab} \{f(a) \wedge g(b)\}\right) \vee 0.5 \\
&= (f \circ g)(xy) \vee 0.5
\end{aligned}$$

So

$$(f \circ g)(y) \leq (f \circ g)(xy) \vee 0.5.$$

If $(f \circ g)(y) = 0$, then $(f \circ g)(y) \leq (f \circ g)(xy) \vee 0.5$. Thus $f \circ g$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S.

Similarly we can show that $f \circ g$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal of S.

Thus $f \circ g$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy *ideal of* S. ■

Lemma 60 *The intersection of any family of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right) ideals of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right) ideal of S.*

Proof. Let $\{f_i\}_{i \in I}$ be a family of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideals of S and $x, y \in S$.

Since each f_i is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S, so $f_i(xy) \vee 0.5 \geq f_i(y)$ for all $i \in I$. Thus

$$\begin{aligned}
\left(\left(\bigwedge_{i \in I} f_i\right)(xy)\right) \vee 0.5 &= \left(\bigwedge_{i \in I} (f_i(xy))\right) \vee 0.5 \\
&= \bigwedge_{i \in I} (f_i(xy) \vee 0.5) \\
&\geq \bigwedge_{i \in I} (f_i(y)) \\
&= \left(\bigwedge_{i \in I} f_i\right)(y).
\end{aligned}$$

Hence $\bigwedge_{i \in I} f_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S. ■

Similarly we can prove that intersection of $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideals of a semigroup S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal of S. Thus intersection of $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideals of a semigroup S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideal of S.

Now we show that if f and g are $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideals of a semigroup S, then $f\circ g\nleq f\wedge g$.

Example 61 *Consider the semigroup $S=\{a,b,c,d\}$ of Example 12.*

One can easily check that $\{a\},\{a,b\},\{a,b,c\},\{a,b,d\}$ and $\{a,b,c,d\}$ are all ideals of S.

Define the fuzzy subsets f,g of S by

$$f(a)=0.7,\quad f(b)=0.3,\quad f(c)=0.4,\quad f(d)=0$$
$$g(a)=0.8,\quad g(b)=0.3,\quad g(c)=0.4,\quad g(d)=0.2$$

Then we have

$$U(f;t)=\begin{cases}\{a,b,c\} & \text{if } 0<t\leq 0.3\\ \{a,c\} & \text{if } 0.3<t\leq 0.4\\ \{a\} & \text{if } 0.4<t\leq 0.7\\ \phi & \text{if } 0.7<t\leq 1\end{cases}$$
$$U(g;t)=\begin{cases}\{a,b,c,d\} & \text{if } 0<t\leq 0.2\\ \{a,b,c\} & \text{if } 0.2<t\leq 0.3\\ \{a,c\} & \text{if } 0.3<t\leq 0.4\\ \{a\} & \text{if } 0.4<t\leq 0.8\\ \phi & \text{if } 0.8<t\leq 1\end{cases}$$

Thus by Theorem 57, f,g are $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideals of S.

Now

$$\begin{aligned}(f\circ g)(b) &= \bigvee_{b=xy}\{f(x)\wedge g(y)\}\\ &= \bigvee\{0.4,0,0\}\\ &= 0.4\nleq (f\wedge g)(b)=0.3\end{aligned}$$

Hence $f\circ g\nleq f\wedge g$ in general.

Theorem 62 *The union of any family of $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left (resp. right) ideals of a semigroup S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left (resp. right) ideal of S.*

Proof. Straightforward. ■

Definition 63 *A fuzzy subset f of a semigroup S is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S if it satisfies* (2.1a) *and*

(2.4a) $(xyz)_{\min\{t,r\}} \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q} f$ or $z_r \overline{\in} \vee \overline{q} f$ for all $x, y, z \in S$ and $t, r \in (0, 1]$.

Theorem 64 *Let f be a fuzzy subset of a semigroup S, $x, y, z \in S$ and $t, r \in (0, 1]$. Then the following conditions are equivalent*

(2.4a) $(xyz)_{\min\{t,r\}} \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q} f$ or $z_r \overline{\in} \vee \overline{q} f$.
(2.4b) $\max\{f(xyz), 0.5\} \geq \min\{f(x), f(z\}$.
Proof. Similar to the proof of Theorem 52. ■

Corollary 65 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S if it satisfies condition* (2.1b) *and* (2.4b)*.*

Theorem 66 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S if and only if $U(f; t)$ $(\neq \phi)$ is a bi-ideal of S for all $t \in (0.5, 1]$.*

Proof. Similar to the proof of Theorem 57. ■

Lemma 67 *The intersection of any family of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S.*

Proof. Straightforward. ■

Lemma 68 *Let f and g be $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals of a semigroup S. Then $f \circ g$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S.*

Proof. Similar to Theorem 59. ■

Next we show that the union of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy *bi-ideals* of a semigroup S need not be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy *bi-ideal* of S.

Example 69 *Consider the semigroup $S = \{0, 1, 2, 3, 4, 5\}$ with the following multiplication table:*

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

Bi-ideals in S are: $\{0\}$, $\{0,1\}$, $\{0,1,2\}$, $\{0,1,3\}$, $\{0,1,4\}$, $\{0,1,5\}$, $\{0,1,2,4\}$, $\{0,1,3,5\}$, $\{0,1,2,3\}$, $\{0,1,4,5\}$, and S.

Define fuzzy subsets f, g of S by

$f(0) = 0.7$, $f(1) = 0.6 = f(3)$, $f(2) = 0.2 = f(4) = f(5)$

$g(0) = 0.75$, $g(1) = 0.62 = g(4)$, $g(2) = 0.25 = g(3) = g(5)$

$$U(f;t) = \begin{cases} \{0,1,2,3,4,5\} & \text{if } 0 < t \leq 0.2 \\ \{0,1,3\} & \text{if } 0.2 < t \leq 0.6 \\ \{0\} & \text{if } 0.6 < t \leq 0.7 \\ \phi & \text{if } 0.7 < t \leq 1 \end{cases}$$

$$U(g;t) = \begin{cases} \{0,1,2,3,4,5\} & \text{if } 0 < t \leq 0.25 \\ \{0,1,4\} & \text{if } 0.25 < t \leq 0.62 \\ \{0\} & \text{if } 0.62 < t \leq 0.75 \\ \phi & \text{if } 0.75 < t \leq 1 \end{cases}$$

Then by Theorem 66 f, g are $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals of S.

As $U(f \vee g; t) = \{0,1,2,3,4\}$ for $0.2 < t \leq 0.6$, which is not a bi-ideal of S, so $f \vee g$ is not an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S.

Definition 70 *A fuzzy subset f of a semigroup S is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal of S if it satisfies* (2.4a) *or* (2.4b).

Theorem 71 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal of S if and only if $U(f;t)$ $(\neq \phi)$ is a generalized bi-ideal of S for all $t \in (0.5, 1]$.*

Proof. Similar to the proof of Theorem 57. ■

It is clear that every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal of S. The next example shows that the $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal of S is not necessarily a fuzzy bi-ideal of S.

Example 72 *Consider the semigroup $S = \{a, b, c, d\}$ of Example 12.*

One can easily check that $\{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}$ and $\{a,b,c,d\}$ are all generalized bi-ideals of S and $\{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, \{a,b,c,d\}$ are all bi-ideals of S.

Define a fuzzy subset f of S by

$$f(a) = 0.8, \quad f(b) = 0, \quad f(c) = 0.7, \quad f(d) = 0.$$

Then, we have

$$U(f;t) = \begin{cases} \{a,c\} & \text{if } 0 < t \leq 0.7 \\ \{a\} & \text{if } 0.7 < t \leq 0.8 \\ \phi & \text{if } 0.8 < t \leq 1 \end{cases}$$

Thus by Theorem 71, f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal of S but not an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S, because $U(f;0.6) = \{a,c\}$ is a generalized bi-ideal of S but not a bi-ideal of S.

Lemma 73 *Every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal of a regular semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S.*

Proof. Let S be a regular semigroup and f an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal of S. Let $a, b \in S$. Then there exists $x \in S$ such that $b = bxb$. Thus we have

$$\max\{f(ab), 0.5\} = \max\{f(a(bxb)), 0.5\} = \max\{f(a(bx)b), 0.5\} \geq \min\{f(a), f(b)\}.$$

This shows that f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subsemigroup of S and so f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S. ■

Definition 74 *A fuzzy subset f of a semigroup S is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S, if it satisfies,*

(2.5a) $x_t \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q}((f \circ \mathcal{S}) \wedge (\mathcal{S} \circ f))$.

Where $\mathcal{S}$ is the fuzzy subset of S mapping every element of S on 1.

Theorem 75 *Let f be a fuzzy subset of a semigroup S, $x \in S$ and $t \in (0,1]$. Then the following conditions are equivalent*

(2.5a) $x_t \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q}((f \circ \mathcal{S}) \wedge (\mathcal{S} \circ f))$.

(2.5b) $\max\{f(x), 0.5\} \geq \min\{(f \circ \mathcal{S})(x), (\mathcal{S} \circ f)(x)\}$.

Proof. The proof is similar to the proof of Theorem 52. ■

Theorem 76 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S if and only if $U(f;t)$ $(\neq \phi)$ is a quasi-ideal of S for all $t \in (0.5, 1]$.*

Proof. Similar to the proof of Theorem 57. ■

Lemma 77 *The intersection of any family of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideals of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S.*

Proof. Straightforward. ■

Lemma 78 *Let f and g be $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideals of a semigroup S. Then $f\circ g$ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal of S.*

Proof. Straightforward. ■

Lemma 79 *A non-empty subset Q of a semigroup S is a quasi-ideal of S if and only if the characteristic function C_Q is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal of S.*

Proof. Suppose Q is a quasi-ideal of S. Let C_Q be the characteristic function of Q. Let $x\in S$. If $x\notin Q$ then $x\notin SQ$ or $x\notin QS$. If $x\notin SQ$ then $(\mathcal{S}\circ C_Q)(x)=0$ and so $\min\{(C_Q\circ\mathcal{S})(x),(\mathcal{S}\circ C_Q)(x)\}=0\leq C_Q(x)\vee 0.5$. If $x\in Q$ then $C_Q(x)\vee 0.5=1\geq \min\{(C_Q\circ\mathcal{S})(x),(\mathcal{S}\circ C_Q)(x)\}$. Hence C_Q is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal *of* S.

Conversely, assume that C_Q is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal *of* S. Let $a\in QS\cap SQ$. Then there exist $b,c\in S$ and $x,y\in Q$ such that $a=xb$ and $a=cy$. Then

$$\begin{aligned}(C_Q\circ\mathcal{S})(a) &= \bigvee_{a=pq}\{C_Q(p)\wedge\mathcal{S}(q)\}\\ &\geq C_Q(x)\wedge\mathcal{S}(b)\\ &= 1\wedge 1\\ &= 1.\end{aligned}$$

So $(C_Q\circ\mathcal{S})(a)=1$. Similarly $(\mathcal{S}\circ C_Q)(a)=1$.

Since $C_Q(a)\vee 0.5\geq\min\{(C_Q\circ\mathcal{S})(a),(\mathcal{S}\circ C_Q)(a)\}=1$. Thus $C_Q(a)=1$, which implies that $a\in Q$. Hence $SQ\cap QS\subseteq Q$, this shows that Q is a quasi-ideal of S. ■

The proof of the following Lemma is similar to the proof of Lemma 79.

Lemma 80 *Let A be a non-empty subset of a semigroup S. Then the characteristic function C_A of A is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal (resp. generalized bi-ideal) of S if and only if A is a bi-ideal (resp. generalized bi-ideal) of S.*

Theorem 81 *Every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left (right) ideal of S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal of S.*

Proof. Let $x\in S$. Then

$$\begin{aligned}(\mathcal{S}\circ f)(x) &= \bigvee_{x=yz}\{\mathcal{S}(y)\wedge f(z)\}\\ &= \bigvee_{x=yz} f(z)\\ &\leq \bigvee_{x=yz}(f(yz)\vee 0.5)\quad \left(\begin{array}{l}\text{because } f \text{ is an } (\overline{\in},\overline{\in}\vee\overline{q})\text{-fuzzy}\\ \text{left ideal } of\ S.\end{array}\right)\\ &= f(x)\vee 0.5\end{aligned}$$

Thus $(\mathcal{S} \circ f)(x) \leq f(x) \vee 0.5$. Hence $f(x) \vee 0.5 \geq (\mathcal{S} \circ f)(x) \geq \min\{(f \circ \mathcal{S})(x), (\mathcal{S} \circ f)(x)\}$. Therefore f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal *of* S. ■

Lemma 82 *Every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S.*

Proof. Suppose f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of a semigroup S and $x, y, z \in S$. Then

$$\begin{aligned}
f(xy) \vee 0.5 &\geq (f \circ \mathcal{S})(xy) \wedge (\mathcal{S} \circ f)(xy) \\
&= \left(\bigvee_{xy=ab} \{\{f(a) \wedge \mathcal{S}(b)\}\}\right) \wedge \left(\bigvee_{xy=pq} \{\mathcal{S}(p) \wedge f(q)\}\right) \\
&\geq (f(x) \wedge \mathcal{S}(y)) \wedge (\mathcal{S}(x) \wedge f(y)) \\
&= (f(x) \wedge 1) \wedge (1 \wedge f(y)) \\
&= f(x) \wedge f(y)
\end{aligned}$$

So $f(xy) \vee 0.5 \geq \min\{f(x), f(y)\}$.
Also

$$\begin{aligned}
f(xyz) \vee 0.5 &\geq (f \circ \mathcal{S})(xyz) \wedge (\mathcal{S} \circ f)(xyz) \\
&= \left(\bigvee_{xyz=ab} \{f(a) \wedge \mathcal{S}(b)\}\right) \wedge \left(\bigvee_{xyz=pq} \{\mathcal{S}(p) \wedge f(q)\}\right) \\
&\geq (f(x) \wedge \mathcal{S}(yz)) \wedge (\mathcal{S}(xy) \wedge f(z)) \\
&= (f(x) \wedge 1) \wedge (1 \wedge f(z)) \\
&= f(x) \wedge f(z)
\end{aligned}$$

So $f(xyz) \vee 0.5 \geq \min\{f(x), f(z)\}$. Thus f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S. ■

Definition 83 *A fuzzy subset f of a semigroup S is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior-ideal of S if it satisfies*

(2.6a) $(xyz)_t \overline{\in} f \Rightarrow y_t \overline{\in} \vee \overline{q} f$ for all $x, y, z \in S$ and $t \in (0, 1]$.

Theorem 84 *Let f be a fuzzy subset of a semigroup S, $x, y, z \in S$ and $t \in (0, 1]$. Then the following conditions are equivalent*

(2.6a) $(xyz)_t \overline{\in} f \Rightarrow y_t \overline{\in} \vee \overline{q} f$.
(2.6b) $\max\{f(xyz), 0.5\} \geq f(y)$.
Proof. Similar to the proof of Theorem 52. ■

Corollary 85 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior-ideal of S if it satisfies condition* (2.6b).

Theorem 86 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior-ideal of S if and only if $U(f;t)$ $(\neq \phi)$ is an interior-ideal of S for all $t \in (0.5, 1]$.*

Proof. Similar to the proof of Theorem 57. ∎

Lemma 87 *The intersection of any family of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior-ideals of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior-ideal of S.*

The following example shows that the union of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior-ideals of a semigroup S need not be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior-ideal of S.

Example 88 *Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table:*

.	a	b	c	d
a	a	a	a	a
b	a	a	d	a
c	a	a	a	a
d	a	a	a	a

Then the interior-ideals of S are, $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, d\}$, $\{a, c, d\}$ and S. Define fuzzy subsets f, g of S as follows:

$f(a) = 0.8 = f(b)$, $f(c) = 0 = f(d)$ and $g(a) = 0.8 = g(c)$, $g(b) = 0 = g(d)$.

Then we have

$$U(f;t) = \begin{cases} \{a, b\} & \text{if } 0 < t \leq 0.8 \\ \phi & \text{if } 0.8 < t \leq 1 \end{cases}$$
$$U(g;t) = \begin{cases} \{a, c\} & \text{if } 0 < t \leq 0.8 \\ \phi & \text{if } 0.8 < t \leq 1 \end{cases}$$

Thus by Theorem 86 f, g are $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy *interior-ideals of S. But $U(f \vee g; t) =$* $\{a, b, c\}$ if $0 < t \leq 0.8$, which is not an interior-ideal of S, so $f \vee g$ is not $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy *interior-ideal of S.*

Lemma 89 *Every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior-ideal of S.*

Proof. Straightforward. ∎

The following example shows that the converse of the Lemma 89 does not hold in general.

Example 90 *Consider the semigroup $S = \{0, a, b, c\}$ with the following multiplication table:*

.	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Then $\{0\}$, $\{0, a\}$, $\{0, b\}$, $\{0, a, b\}$, and S are interior-ideals of S, but $\{0, b\}$ is not an ideal. Define a fuzzy subset f of S by

$f(0) = 0.6$, $f(a) = 0.4$, $f(b) = 0.6$ and $f(c) = 0.2$

Then we have

$$U(f;t) = \begin{cases} \{0, a, b, c\} & \text{if } 0 < t \leq 0.2 \\ \{0, a, b\} & \text{if } 0.2 < t \leq 0.4 \\ \{0, b\} & \text{if } 0.4 < t \leq 0.6 \\ \phi & \text{if } 0.6 < t \leq 1 \end{cases}$$

Thus by Theorem 86 f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy *interior-ideal of S.*

Since $\{0, b\}$ is not an ideal of S, so f is not an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S.

Lemma 91 *Every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior-ideal of a regular semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S.*

Proof. Let S be a regular semigroup and f an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior-ideal of S. Let $a, b \in S$. Then there exists $x \in S$ such that $a = axa$. Thus we have

$\max\{f(ab), 0.5\} = \max\{f((axa)b), 0.5\} = \max\{f((ax)a(b)), 0.5\} \geq f(a)$.

This shows that f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal of S.

Similarly $b = byb$ for some $y \in S$, thus we have

$\max\{f(ab), 0.5\} = \max\{f(a(byb)), 0.5\} = \max\{f((a)b(yb)), 0.5\} \geq f(b)$.

This shows that f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S. Hence f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S. ■

2.3 Regular Semigroups in Terms of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy Ideals

In this section we prove that if f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S, then f^+ is also an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S. We also characterize regular semigroups in terms of their $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal, quasi-ideals and bi-ideals.

Definition 92 [42] *Let f be a fuzzy subset of a semigroup S. We define the upper part f^+ and the lower part f^- of f as follows, $f^+(x) = f(x) \vee 0.5$ and $f^-(x) = f(x) \wedge 0.5$.*

Lemma 93 [42] *Let f and g be fuzzy subsets of a semigroup S. Then the following holds.*

(1) $(f \wedge g)^- = (f^- \wedge g^-)$
(2) $(f \vee g)^- = (f^- \vee g^-)$
(3) $(f \circ g)^- = (f^- \circ g^-)$
(4) $(f \wedge g)^+ = (f^+ \wedge g^+)$
(5) $(f \vee g)^+ = (f^+ \vee g^+)$
(6) $(f \circ g)^+ \geq (f^+ \circ g^+)$
If every element x of S is expressible as $x = bc$ for some $b, c \in S$, then $(f \circ g)^+ = (f^+ \circ g^+)$.

Lemma 94 *Let A and B be non-empty subsets of a semigroup S. Then the following holds.*

(1) $(C_A \wedge C_B)^+ = C^+_{A \cap B}$
(2) $(C_A \vee C_B)^+ = C^+_{A \cup B}$
(3) $(C_A \circ C_B)^+ = C^+_{AB}$
Where C_A is the characteristic function of A.

Proposition 95 *If f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right) ideal of S then f^+ is also an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right) ideal of S.*

Proof. Suppose f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S and $x, y \in S$. Then
$\max\{f^+(xy), 0.5\} = \max\{\max(f(xy), 0.5), 0.5\}$
$= \max\{f(xy), 0.5\} \geq f(y)$ because f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S.
Also $\max\{f^+(xy), 0.5\} \geq 0.5$. Thus $\max\{f^+(xy), 0.5\} \geq f(y) \vee 0.5 = f^+(y)$.
Hence f^+ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S. ■
Similarly we can prove the following proposition.

Proposition 96 *If f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal (resp. bi-ideal, generalized bi-ideal, interior-ideal) of S then f^+ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal (resp. bi-ideal, generalized bi-ideal, interior-ideal) of S.*

Lemma 97 *Let L be a non-empty subset of a semigroup S. Then L is a left (resp. right) ideal of S if and only if C^+_L, the upper part of characteristic function C_L is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right) ideal of S.*

Proof. Let L be a left ideal of S. Then by Lemma 94 and Proposition 95 C_L^+ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S.

Conversely, assume that C_L^+ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S. Let $x \in L$. Then $C_L^+(x) = 1$. Since C_L^+ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S, so $\max\{C_L^+(yx), 0.5\} \geq C_L^+(x) = 1$. Thus $C_L^+(yx) = 1$ and hence $yx \in L$. This implirs that L is a left ideal of S. ■

Similarly we can prove the following lemma.

Lemma 98 *A non-empty subset Q of a semigroup S is a quasi-ideal (resp. bi-ideal, generalized bi-ideal, interior-ideal) of S if and only if the upper part C_Q^+ of characteristic function C_Q, is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal (resp. bi-ideal, generalized bi-ideal, interior-ideal) of S.*

Next we show that if f is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right) ideal of S then f^+ is a fuzzy left (right) ideal of S.

The following example shows that every fuzzy left ideal of S is not of the form f^+ for some $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal f of S.

Example 99 *Consider the semigroup of Example 12. The fuzzy subset f of S defined by $f(a) = 0.4$, $f(b) = 0.3$, $f(c) = 0.2$, $f(d) = 0.1$ is a fuzzy left ideal of S but is not of the form g^+ for some $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal g of S.*

In Example 61 it is shown that $f \circ g \nleq f \wedge g$ for $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals f, g of S. Now we show that $(f \circ g)^+ \leq (f \wedge g)^+$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals f, g of S.

Proposition 100 *Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal and g an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S. Then $(f \circ g)^+ \leq (f \wedge g)^+$.*

Proof. Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal and g an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S. Then for all $a \in S$, we have

$$\begin{aligned}
(f \circ g)^+(a) &= (f \circ g)(a) \vee 0.5 \\
&= \left(\bigvee_{a=yz} \{f(y) \wedge g(z)\} \right) \vee 0.5 \\
&\leq \left(\bigvee_{a=yz} \{(f(yz) \vee 0.5) \wedge (g(yz) \vee 0.5)\} \right) \vee 0.5 \\
&= \{(f(a) \wedge g(a)) \vee 0.5\} \vee 0.5 \\
&= (f \wedge g)(a) \vee 0.5 \\
&= (f \wedge g)^+(a).
\end{aligned}$$

If a is not expressible as $a = yz$, then
$(f \circ g)^{+}(a) = (f \circ g)(a) \vee 0.5 = 0 \vee 0.5 = 0.5$
$\leq (f \wedge g)(a) \vee 0.5 \leq (f \wedge g)^{+}(a)$.
So $(f \circ g)^{+} \leq (f \wedge g)^{+}$. ■

Next we characterize regular semigroups by the properties of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals, bi-ideals and generalized bi-ideals.

Theorem 101 *For a semigroup S the following conditions are equivalent.*

(1) *S is regular.*

(2) $(f \wedge g)^{+} = (f \circ g)^{+}$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal f *and every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal g *of* S.

Proof. (1) $\Rightarrow$ (2) Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal and g be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S. Then by Proposition 100, $(f \circ g)^{+} \leq (f \wedge g)^{+}$.

Let $a \in S$. Then there exists $x \in S$ such that $a = axa$.

So

$$\begin{aligned}
(f \circ g)^{+}(a) &= (f \circ g)(a) \vee 0.5 \\
&= \left(\bigvee_{a=yz} \{f(y) \wedge g(z)\} \right) \vee 0.5 \\
&\geq \{f(ax) \wedge g(a)\} \vee 0.5 \\
&= \{(f(ax) \vee 0.5) \wedge (g(a) \vee 0.5)\} \\
&\geq f(a) \wedge (g(a) \vee 0.5) \\
&= (f(a) \wedge g(a)) \vee 0.5 \\
&= (f \wedge g)^{+}(a).
\end{aligned}$$

So $(f \circ g)^{+} \geq (f \wedge g)^{+}$. Thus $(f \wedge g)^{+} = (f \circ g)^{+}$.

(2) $\Rightarrow$ (1) Let R be a right ideal and L a left ideal of S. Then by Corolarry 56, C_R and C_L are $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right and left ideals of S, respectively. Thus we have

$$\begin{aligned}
C_{RL}^{+} &= (C_R \circ C_L)^{+} \\
&= (C_R \wedge C_L)^{+} \qquad \text{by } (2) \\
&= C_{R \cap L}^{+}
\end{aligned}$$

Thus $R \cap L = RL$. Hence it follows from Theorem 23 that S is regular. ■

Theorem 102 *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*

(2) $(h\wedge f\wedge g)^{+}\leq(h\circ f\circ g)^{+}$ *for every* $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal h, *every* $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy generalized bi-ideal f *and every* $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal g *of* S*.*

(3) $(h\wedge f\wedge g)^{+}\leq(h\circ f\circ g)^{+}$ *for every* $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal h, *every* $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal f*, and every* $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal g *of* S*.*

(4) $(h\wedge f\wedge g)^{+}\leq(h\circ f\circ g)^{+}$ *for every* $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal h*, every* $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal f*, and every* $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal g *of* S*.*

Proof. (1) $\Rightarrow$ (2) : Let h,f and g be $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal, generalized bi-ideal, and left ideal of S**,** respectively. Let a be any element of S**.** Since S is regular, so there exists $x\in S$ such that $a=axa$. Hence we have

$$\begin{aligned}
(h\circ f\circ g)^{+}(a) &= (h\circ f\circ g)(a)\vee 0.5=\left(\bigvee_{a=yz}\{h(y)\wedge(f\circ g)(z)\}\right)\vee 0.5\\
&\geq \{h(ax)\wedge(f\circ g)(a)\}\vee 0.5\\
&= (h(ax)\vee 0.5)\wedge((f\circ g)(a)\vee 0.5)\\
&\geq (h(a))\wedge\left(\left(\bigvee_{a=pq}\{f(p)\wedge g(q)\}\right)\vee 0.5\right)\\
&\geq (h(a))\wedge\{(f(a)\wedge g(xa))\vee 0.5\}\\
&= (h(a))\wedge\{(f(a)\vee 0.5)\wedge(g(xa)\vee 0.5)\}\\
&= (h(a))\wedge\{(f(a)\vee 0.5)\wedge(g(xa)\vee 0.5)\}\\
&\geq (h(a))\wedge\{(f(a)\vee 0.5)\wedge(g(a))\}\\
&= \{h(a)\wedge(f(a)\wedge g(a))\}\vee 0.5\\
&= (h\wedge f\wedge g)^{+}(a).
\end{aligned}$$

So (1) implies (2).

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) Straightforward, because every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal is $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy generalized bi-ideal and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal is $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal.

(4) $\Rightarrow$ (1) : Let h and g be any $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal and $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal of S, respectively. Since $\mathcal{S}$ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal of S**,** so by the

assumption, we have

$$\begin{aligned}
(h \wedge g)^{+}(a) &= (h \wedge g)(a) \vee 0.5 \\
&= (h \wedge \mathcal{S} \wedge g)(a) \vee 0.5 \\
&= (h \wedge \mathcal{S} \wedge g)^{+}(a) \\
&\leq (h \circ \mathcal{S} \circ g)^{+}(a) \\
&= (h \circ \mathcal{S} \circ g)(a) \vee 0.5 \\
&= \left(\bigvee_{a=bc} \{(h \circ \mathcal{S})(b) \wedge g(c)\} \right) \vee 0.5 \\
&= \left(\bigvee_{a=bc} \left\{ \left(\bigvee_{b=pq} \{h(p) \wedge \mathcal{S}(q)\} \right) \wedge g(c) \right\} \right) \vee 0.5 \\
&= \left(\bigvee_{a=bc} \left\{ \left(\bigvee_{b=pq} \{h(p) \wedge 1\} \right) \wedge g(c) \right\} \right) \vee 0.5 \\
&= \left(\bigvee_{a=bc} \left\{ \left(\bigvee_{b=pq} h(p) \right) \wedge g(c) \right\} \right) \vee 0.5 \\
&\leq \left(\bigvee_{a=bc} \left\{ \left(\bigvee_{b=pq} \{h(pq) \vee 0.5\} \right) \wedge g(c) \right\} \right) \vee 0.5 \\
&\leq \left(\bigvee_{a=bc} \{(h(b) \vee 0.5) \wedge (g(c))\} \right) \vee 0.5 \\
&= \left(\bigvee_{a=bc} \{(h(b) \wedge g(c)\} \vee 0.5 \right) \vee 0.5 \\
&= \left(\bigvee_{a=bc} \{(h(b) \wedge g(c)\} \right) \vee 0.5 \\
&= (h \circ g)(a) \vee 0.5 \\
&= (h \circ g)^{+}(a).
\end{aligned}$$

Thus it follows that $(h \wedge g)^{+} \leq (h \circ g)^{+}$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal h and every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal g of S. But $(h \circ g)^{+} \leq (h \wedge g)^{+}$ always. Thus $(h \circ g)^{+} = (h \wedge g)^{+}$. Hence by Theorem 101 S is regular. ■

Theorem 103 *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*

(2) $f^{+} = (f \circ \mathcal{S} \circ f)^{+}$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal f *of* S.

(3) $f^{+} = (f \circ \mathcal{S} \circ f)^{+}$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal f *of* S.

(4) $f^{\,+} = (f \circ \mathcal{S} \circ f)^{+}$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal f *of* S.

Proof. (1) $\Rightarrow$ (2) : Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal of S and let $a \in S$. Since S is regular, so there exists $x \in S$ such that $a = axa$. Hence we have

$$\begin{aligned}
(f \circ \mathcal{S} \circ f)^{+}(a) &= (f \circ \mathcal{S} \circ f)(a) \vee 0.5 \\
&= \left(\bigvee_{a=yz} \{(f \circ \mathcal{S})(y) \wedge f(z)\} \right) \vee 0.5 \\
&\geq \{(f \circ \mathcal{S})(ax) \wedge f(a)\} \vee 0.5 \\
&= \left\{ \left(\bigvee_{ax=pq} \{f(p) \wedge \mathcal{S}(q)\} \right) \wedge f(a) \right\} \vee 0.5 \\
&\geq \{(f(a) \wedge \mathcal{S}(x)) \wedge f(a)\} \vee 0.5 \\
&= \{(f(a) \wedge 1) \wedge f(a)\} \vee 0.5 \\
&= f(a) \vee 0.5 \\
&= f^{+}(a)
\end{aligned}$$

Thus $(f \circ \mathcal{S} \circ f)^{+} \geq f^{\,+}$.

Also

$$\begin{aligned}
(f \circ \mathcal{S} \circ f)^{+}(a) &= (f \circ \mathcal{S} \circ f)(a) \vee 0.5 \\
&= \left(\bigvee_{a=yz} \{(f \circ \mathcal{S})(y) \wedge f(z)\} \right) \vee 0.5 \\
&= \left(\bigvee_{a=yz} \left\{ \left(\bigvee_{y=pq} \{f(p) \wedge \mathcal{S}(q)\} \right) \wedge f(z) \right\} \right) \vee 0.5 \\
&= \left(\bigvee_{a=yz} \left\{ \left(\bigvee_{y=pq} \{f(p) \wedge 1\} \right) \wedge f(z) \right\} \right) \vee 0.5 \\
&= \left(\bigvee_{a=yz} \left\{ \bigvee_{y=pq} f(p) \wedge f(z) \right\} \right) \vee 0.5 \\
&\leq \bigvee_{a=(pq)z} \{f(pqz) \vee 0.5\} \vee 0.5 \quad \left(\begin{array}{l} \text{because } f \text{ is an } (\overline{\in}, \overline{\in} \vee \overline{q})\text{-fuzzy} \\ \text{generalized bi-ideal of } S. \end{array} \right) \\
&= f(a) \vee 0.5 \\
&= f^{+}(a).
\end{aligned}$$

So, $(f \circ \mathcal{S} \circ f)^{+} \leq f^{\,+}$. Thus $f^{\,+} = (f \circ \mathcal{S} \circ f)^{+}$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (1) Let A be any quasi-ideal of S. Since by Corollary 56 C_A is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S, so we have

$$\begin{aligned} C_A^+ &= (C_A \circ \mathcal{S} \circ C_A)^+ = C_{ASA}^+ \\ &\Rightarrow A = ASA. \end{aligned}$$

Hence it follows from Theorem 23 that S is regular. ■

Theorem 104 *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*

(2) $(f \wedge g)^+ = (f \circ g \circ f)^+$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal f *and every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior-ideal g *of* S.

(3) $(f \wedge g)^+ = (f \circ g \circ f)^+$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal f *and every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior-ideal g *of* S.

(4) $(f \wedge g)^+ = (f \circ g \circ f)^+$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal f *and every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior-ideal g *of* S.

(5) $(f \wedge g)^+ = (f \circ g \circ f)^+$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal f *and every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy two-sided ideal g *of* S.

(6) $(f \wedge g)^+ = (f \circ g \circ f)^+$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal f *and every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy two-sided ideal g *of* S.

(7) $(f \wedge g)^+ = (f \circ g \circ f)^+$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal f *and every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy two-sided ideal g *of* S.

Proof. (1) $\Rightarrow$ (2) : Let f be an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy generalized bi-ideal and g an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior-ideal of S. Then for all $a\in S$,

$$\begin{aligned}
(f\circ g\circ f)^{+}(a) &= (f\circ g\circ f)(a)\vee 0.5\\
&\leq (f\circ \mathcal{S}\circ f)(a)\vee 0.5\\
&= \left(\bigvee_{a=yz}\{(f\circ\mathcal{S})(y)\wedge f(z)\}\right)\vee 0.5\\
&= \left(\bigvee_{a=yz}\left\{\left(\bigvee_{y=pq}\{f(p)\wedge\mathcal{S}(q)\}\right)\wedge f(z)\right\}\right)\wedge 0.5\\
&= \left(\bigvee_{a=yz}\left\{\left(\bigvee_{y=pq}\{f(p)\wedge 1\}\right)\wedge f(z)\right\}\right)\vee 0.5\\
&= \bigvee_{a=yz}\left\{\left(\bigvee_{y=pq}f(p)\right)\wedge f(z)\right\}\vee 0.5\\
&= \bigvee_{a=yz}\left\{\bigvee_{y=pq}f(p)\wedge f(z)\right\}\vee 0.5\\
&= \bigvee_{a=(pq)z}\{f(p)\wedge f(z)\}\vee 0.5\\
&\leq \bigvee_{a=(pq)z}\{f(pqz)\vee 0.5\}\vee 0.5\\
&= f(a)\vee 0.5\\
&= f^{+}(a)
\end{aligned}$$

and

$$\begin{aligned}
(f \circ g \circ f)^{+}(a) &\leq (\mathcal{S} \circ g \circ \mathcal{S})^{+}(a) \\
&= (\mathcal{S} \circ g \circ \mathcal{S})(a) \vee 0.5 \\
&= \left(\bigvee_{a=yz} \{(\mathcal{S} \circ g)(y) \wedge \mathcal{S}(z)\} \right) \vee 0.5 \\
&= \left(\bigvee_{a=yz} \left\{ \left(\bigvee_{y=pq} \{\mathcal{S}(p) \wedge g(q)\} \right) \wedge \mathcal{S}(z) \right\} \right) \vee 0.5 \\
&= \left(\bigvee_{a=yz} \left\{ \left(\bigvee_{y=pq} \{1 \wedge g(q)\} \right) \wedge 1 \right\} \right) \vee 0.5 \\
&= \left(\bigvee_{a=yz} \left\{ \bigvee_{y=pq} g(q) \right\} \right) \vee 0.5 \\
&\leq \left(\bigvee_{a=(pq)z} g(pqz) \vee 0.5 \right) \vee 0.5 \\
&= g(a) \vee 0.5 \\
&= g^{+}(a).
\end{aligned}$$

Thus $(f \circ g \circ f)^{+} \leq (f^{+} \wedge g^{+}) = (f \wedge g)^{+}$.

Since S is regular, so there exists an element $x \in S$ such that $a = axa\,(= axaxa)$. Since g is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior-ideal of S, we have

$$\begin{aligned}
(f \circ g \circ f)^{+}(a) &= (f \circ g \circ f)(a) \vee 0.5 \\
&= \left(\bigvee_{a=yz} \{f(y) \wedge (g \circ f)(z)\} \right) \vee 0.5 \\
&\geq (f(a) \wedge (g \circ f)(xaxa)) \vee 0.5 \\
&= \left(f(a) \wedge \left(\bigvee_{xaxa=pq} \{g(p) \wedge f(q)\} \right) \right) \vee 0.5 \\
&\geq (f(a) \wedge (g(xax) \wedge f(a))) \vee 0.5 \\
&= (f(a) \vee 0.5) \wedge \{(g(xax) \wedge f(a)) \vee 0.5\} \\
&= (f(a) \vee 0.5) \wedge \{(g(xax) \vee 0.5) \wedge (f(a) \vee 0.5)\} \\
&= (f(a) \vee 0.5) \wedge \{(g(xax) \vee 0.5) \wedge (f(a) \vee 0.5)\} \\
&\geq (f(a) \vee 0.5) \wedge \{(g(a)) \wedge (f(a) \vee 0.5)\} \\
&= (f(a) \vee 0.5) \wedge (\{(g(a) \wedge (f(a)\} \vee 0.5) \\
&= (f(a) \wedge \{(g(a) \wedge (f(a)\} \vee 0.5) \\
&= (f \wedge g)(a) \vee 0.5 \\
&= (f \wedge g)^{+}(a).
\end{aligned}$$

So $(f \circ g \circ f)^{+} \geq (f \wedge g)^{+}$. Hence $(f \circ g \circ f)^{+} = (f \wedge g)^{+}$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (7) and (2) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are clear.

(7) $\Rightarrow$ (1) Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S. Then, since $\mathcal{S}$ itself is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S, we have

$$\begin{aligned} f^{+}(a) &= f(a) \vee 0.5 \\ &= (f \wedge \mathcal{S})(a) \vee 0.5 \\ &= (f \wedge \mathcal{S})^{+}(a) \\ &= (f \circ \mathcal{S} \circ f)^{+}(a) . \end{aligned}$$

Thus it follows from Theorem 57 that S is regular. ∎

Theorem 105 *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*

(2) $(f \wedge g)^{+} \leq (f \circ g)^{+}$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal f *and every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal g *of* S.

(3) $(f \wedge g)^{+} \leq (f \circ g)^{+}$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal f *and every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal g *of* S.

(4) $(f \wedge g)^{+} \leq (f \circ g)^{+}$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal f *and every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal g *of* S.

(5) $(f \wedge g)^{+} \leq (f \circ g)^{+}$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal g *and every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal f *of* S.

(6) $(f \wedge g)^{+} \leq (f \circ g)^{+}$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal g *and every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal f *of* S.

(7) $(f \wedge g)^{+} \leq (f \circ g)^{+}$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal g *and every* $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal f *of* S.

Proof. (1) $\Rightarrow$ (2) Let f and g be any $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal and any $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S, respectively. Let a be any element of S. Then there exists an element $x \in S$ such that $a = axa$. Thus we have

$$\begin{aligned}
(f\circ g)^{+}(a) &= (f\circ g)(a)\vee 0.5 \\
&= \left(\bigvee_{a=yz}\{f(y)\wedge g(z)\}\right)\vee 0.5 \\
&\geq (f(a)\wedge g(xa))\vee 0.5 \\
&= (f(a)\vee 0.5)\wedge(g(xa)\vee 0.5) \\
&= (f(a)\vee 0.5)\wedge(g(xa)\vee 0.5) \\
&\geq (f(a)\vee 0.5)\wedge(g(a)) \\
&= (f(a)\wedge g(a))\vee 0.5 \\
&= (f\wedge g)(a)\vee 0.5 \\
&= (f\wedge g)^{+}(a).
\end{aligned}$$

So $(f\circ g)^{+} \geq (f\wedge g)^{+}$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1) : Let f be an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal and g be an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal of S. Since every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal of S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal of S. So $(f\circ g)^{+} \geq (f\wedge g)^{+}$. By Proposition 100, $(f\circ g)^{+} \leq (f\wedge g)^{+}$. Hence $(f\circ g)^{+} = (f\wedge g)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal g of S. Thus by Theorem 101 S is regular.

Similarly we can show that (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (1). ■

2.4 Intra-regular Semigroups in Terms of $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy Ideals

In this section we characterize intra-regular and regular and intra-regular semigroups by the properties of their $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideals, quasi-ideals and bi-ideals.

Theorem 106 *For a semigroup S, the following conditions are equivalent.*

(1) S is intra-regular.

(2) $(f\wedge g)^{+} \leq (f\circ g)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal g of S.

Proof. (1) $\Rightarrow$ (2) Let f be an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal and g be an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal of S. Let $x \in S$. Then there exist $a, b \in S$ such that $x = axxb$. Thus

$(f\circ g)^{+}(x) = (f\circ g)(x)\vee 0.5$

$= (\bigvee_{a=yz}\{f(y)\wedge g(z)\})\vee 0.5$

$\geq (f(ax)\wedge g(xb))\vee 0.5$

$= \{(f(ax) \vee 0.5) \wedge (g(xb) \vee 0.5)\} \vee 0.5$
$\geq \{(f(x) \wedge g(x)\} \vee 0.5 = (f \wedge g)^{+}(x).$

(2) $\Rightarrow$ (1) Let R and L be right and left ideals of S, respectively. Then C_R and C_L are $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right and left ideals of S, respectively. Thus by hypothesis

$C^{+}_{LR} = (C_L \circ C_R)^{+} \geq (C_L \wedge C_R)^{+} = C^{+}_{L \cap R}.$

Thus $R \cap L \subseteq LR$. This implies that S is an intra-regular semigroup. ■

Theorem 107 *For a semigroup S, the following conditions are equivalent.*

(1) S is both regular and intra-regular.

(2) $(f \circ f)^{+} = f^{+}$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal f of S.

(3) $(f \circ f)^{+} = f^{+}$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal f of S.

(4) $(f \wedge g)^{+} \leq (f \circ g)^{+}$ for all $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideals f, g of S.

(5) $(f \wedge g)^{+} \leq (f \circ g)^{+}$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal g of S.

(6) $(f \wedge g)^{+} \leq (f \circ g)^{+}$ for all $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals f, g of S.

Proof. (1) $\Rightarrow$ (6) Let f, g be any $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals of S and $x \in S$. Then there exist $a, b, c \in S$ such that $x = xax$ and $x = bxxc$. Thus $x = xax = xaxax = xa\,(bxxc)\,ax = (xabx)\,(xcax)$. Therefore

$(f \circ g)^{+}(x) = (f \circ g)(x) \vee 0.5$
$= (\bigvee_{a=yz} \{f(y) \wedge g(z)\}) \vee 0.5$
$\geq (f(xabx) \wedge g(xcax)) \vee 0.5$
$= \{(f(xabx) \vee 0.5) \wedge (g(xcax) \vee 0.5)\} \vee 0.5$
$\geq \{(f(x) \wedge g(x)\} \vee 0.5 = (f \wedge g)^{+}(x).$

Thus $(f \wedge g)^{+} \leq (f \circ g)^{+}$ for all $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals f, g of S.

(6) $\Rightarrow$ (5) $\Rightarrow$ (4) are Obvious.

(4) $\Rightarrow$ (2) Take $f = g$ in (4). We have $f^{+} = (f \wedge f)^{+} \leq (f \circ f)^{+}$ but $(f \circ f)^{+} \leq f^{+}$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideals f of S. Hence $f^{+} = (f \circ f)^{+}$.

(6) $\Rightarrow$ (3) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (1) Let Q be any quasi-ideal of S. Then C_Q is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S. Hence by hypothesis $(C_Q \circ C_Q)^{+} = C^{+}_{Q}$, that is $QQ = Q$. Then by Theorem 28 S is both regular and intra-regular. ■

Theorem 108 *For a semigroup S, the following conditions are equivalent.*

(1) S is both regular and intra-regular.

(2) $(f \wedge g)^{+} \leq (f \circ g)^{+} \wedge (g \circ f)^{+}$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal g of S.

(3) $(f\wedge g)^{+}\leq(f\circ g)^{+}\wedge(g\circ f)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal g of S.

(4) $(f\wedge g)^{+}\leq(f\circ g)^{+}\wedge(g\circ f)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal g of S.

(5) $(f\wedge g)^{+}\leq(f\circ g)^{+}\wedge(g\circ f)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy generalized bi-ideal g of S.

(6) $(f\wedge g)^{+}\leq(f\circ g)^{+}\wedge(g\circ f)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal g of S.

(7) $(f\wedge g)^{+}\leq(f\circ g)^{+}\wedge(g\circ f)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal g of S.

(8) $(f\wedge g)^{+}\leq(f\circ g)^{+}\wedge(g\circ f)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy generalized bi-ideal g of S.

(9) $(f\wedge g)^{+}\leq(f\circ g)^{+}\wedge(g\circ f)^{+}$ for all $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideals f,g of S.

(10) $(f\wedge g)^{+}\leq(f\circ g)^{+}\wedge(g\circ f)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal g of S.

(11) $(f\wedge g)^{+}\leq(f\circ g)^{+}\wedge(g\circ f)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy generalized bi-ideal g of S.

(12) $(f\wedge g)^{+}\leq(f\circ g)^{+}\wedge(g\circ f)^{+}$ for all $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideals f,g of S.

(13) $(f\wedge g)^{+}\leq(f\circ g)^{+}\wedge(g\circ f)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy generalized bi-ideal g of S.

(14) $(f\wedge g)^{+}\leq(f\circ g)^{+}\wedge(g\circ f)^{+}$ for all $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy generalized bi-ideals f,g of S.

Proof. (1) $\Rightarrow$ (14) Let f,g be any $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy generalized bi-ideals of S and $x\in S$. Then there exist $a,b,c\in S$ such that $x=xax$ and $x=bxxc$. Thus $x=xax=xaxax=xa\,(bxxc)\,ax=(xabx)\,(xcax)$. Therefore

$$(f\circ g)^{+}(x)=(f\circ g)(x)\vee 0.5$$
$$=(\bigvee_{a=yz}\{f(y)\wedge g(z)\})\vee 0.5$$
$$\geq(f(xabx)\wedge g(xcax))\vee 0.5$$
$$=\{(f(xabx)\vee 0.5)\wedge(g(xcax)\vee 0.5)\}\vee 0.5$$
$$\geq\{(f(x)\wedge g(x)\}\vee 0.5=(f\wedge g)^{+}(x).$$

Similarly we can show that $(f\wedge g)^{+}\leq(g\circ f)^{+}$.

Thus $(f\wedge g)^{+}\leq(f\circ g)^{+}\wedge(g\circ f)^{+}$ for all $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy generalized bi-ideals f,g of S.

(14) $\Rightarrow$ (13) $\Rightarrow$ (12) $\Rightarrow$ (10) $\Rightarrow$ (9) $\Rightarrow$ (3) $\Rightarrow$ (2), (14) $\Rightarrow$ (11) $\Rightarrow$ (10), (14) $\Rightarrow$ (8) $\Rightarrow$ (7) $\Rightarrow$ (6) $\Rightarrow$ (2) and (14) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (1) Let f be an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right and g be an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal of S. Then by hypothesis

$(f\wedge g)^{+}\leq(f\circ g)^{+}\wedge(g\circ f)^{+}$ but $(f\wedge g)^{+}\geq(f\circ g)^{+}$ is always true. Hence $(f\wedge g)^{+}=(f\circ g)^{+}$ and $(f\wedge g)^{+}\leq(g\circ f)^{+}$, this shows that S is both regular and intra-regular. ■

2.5 Semisimple Semigroups in Terms of $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy Ideals

Recall that a semigroup S is semisimple if every two sided ideal of S is idempotent. It is clear that a semigroup S is semisimple if and only if $a\in(SaS)(SaS)$ for every $a\in S$, that is there exist $x,y,z,t\in S$ such that $a=(xay)(taz)$.

Theorem 109 *In a semisimple semigroup S, a fuzzy subset f of S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideal of S if and only if it is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior-ideal of S.*

Proof. Let S be a semisimple semigroup and f be an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior-ideal of S. Then for any $x,y\in S$ there exist $a,b,c,d\in S$ such that $x=(axb)(cxd)$. Thus we have

$f(xy)\vee 0.5=f((axbcxd)y)\vee 0.5=f((ax(bc))x(dy))\vee 0.5\geq f(x)$.

Similarly $f(xy)\vee 0.5\geq f(y)$. Hence f is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideal of S.

Conversely, assume that f is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideal of S. Then f is always an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior-ideal of S. ■

Theorem 110 *For a semigroup S the following assertions are equivalent*

(1) S is semisimple.

(2) $(f\circ f)^{+}=f^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideal f of S.

(3) $(f\circ f)^{+}=f^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior-ideal f of S.

(4) $(f\wedge g)^{+}=(f\circ g)^{+}$ for all $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideals f,g of S.

(5) $(f\wedge g)^{+}=(f\circ g)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior-ideal g of S.

(6) $(f\wedge g)^{+}=(f\circ g)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior-ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideal g of S.

(7) $(f\wedge g)^{+}=(f\circ g)^{+}$ for all $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior-ideals f,g of S.

Proof. (1) ⇒ (7) Let S be a semisimple semigroup and f,g be $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior-ideals of S. Let $x\in S$. Then there exist $a,b,c,d\in S$ such that $x=(axb)(cxd)$. Thus we have

$$(f\circ g)^{+}(x)=\left(\left(\bigvee_{x=yz}(f(y)\wedge g(z))\right)\vee 0.5\right)$$
$$\geq((f(axb)\wedge g(cxd))\vee 0.5)$$

$= ((f(axb) \vee 0.5) \wedge (g(cxd) \vee 0.5))$
$\geq f(x) \wedge g(x) \vee 0.5$
$= ((f \wedge g)(x)) \vee 0.5$
$= (f \wedge g)^{+}(x).$

Thus $(f \circ g)^{+} \geq (f \wedge g)^{+}$. Since every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior-ideal of S in a semisimple semigroup is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S, so$(f \circ g)^{+} \leq (f \wedge g)^{+}$. Hence $(f \circ g)^{+} = (f \wedge g)^{+}$.

(7) $\Rightarrow$ (6) $\Rightarrow$ (4) $\Rightarrow$ (2), (7) $\Rightarrow$ (3) $\Rightarrow$ (2) and (7) $\Rightarrow$ (5) $\Rightarrow$ (4) are obvious.

(2) $\Rightarrow$ (1) Let A be any ideal of S. Then C_A is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideal of S. Thus by hypothesis $(C_A \circ C_A)^{+} = (C_A)^{+}$, that is $AA = A$. Hence S is a semisimple semigroup. ■

Chapter 3

Semigroups Characterized by the Properties of their $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy Ideals

3.1 Introduction

Generalizing the concept of the quasi-coincidence of a fuzzy point with a fuzzy set, Jun [21] defined $(\in, \in \vee q_k)$-fuzzy subalgebras in BCK/BCI-algebras. In [43] Shabir et al. characterized different classes of semigroups by the properties of $(\in, \in \vee q_k)$-fuzzy ideals. In [44] Shabir and Mahmood characterized different classes of hemirings by $(\in, \in \vee q_k)$-fuzzy ideals. In this chapter we initiate the study of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy idelas, generalized bi-ideals and quasi-ideals of a semigroup and characterize semigroups by the properties of these fuzzy ideals.

3.2 $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals

In what follows, let S denote a semigroup and k be an arbitrary element of $[0, 1]$ unless otherwise specified. Generalizing the concept of $x_t q f$, Jun [21] defined $x_t q_k f$, where f is a fuzzy subset of S as $x_t q_k f$ if $f(x) + t + k > 1$.

Definition 111 *A fuzzy subset f of a semigroup S is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S if for all $x, y \in S$ and $t, r \in (0, 1]$ the following condition holds*

(3.1a) $(xy)_{\min\{t,r\}} \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$ or $y_r \overline{\in} \vee \overline{q_k} f$.

Theorem 112 *Let f be a fuzzy subset of a semigroup S, $x, y \in S$ and $t, r \in (0, 1]$. Then the following conditions are equivalent*

(3.1a) $(xy)_{\min\{t,r\}}\overline{\in}f \Rightarrow x_t\overline{\in} \vee \overline{q_k}f$ or $y_r\overline{\in} \vee \overline{q_k}f$.

(3.1b) $\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}$.

Proof. (3.1a) $\Rightarrow$ (3.1b) Suppose there exist $x, y \in S$ such that $\max\{f(xy), \frac{1-k}{2}\} < \min\{f(x), f(y)\}$. Then we can choose $t \in (\frac{1-k}{2}, 1]$ such that $\max\{f(xy), \frac{1-k}{2}\} < t = \min\{f(x), f(y)\}$. Then $(xy)_t\overline{\in}f$ but $x_t \in \wedge q_k f$ and $y_t \in \wedge q_k f$, which is a contradiction. Hence $\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}$.

(3.1b) $\Rightarrow$ (3.1a) Let $(xy)_{\min\{t,r\}}\overline{\in}f$. Then $f(xy) < \min\{t, r\}$. If $\max\{f(xy), \frac{1-k}{2}\} = f(xy)$, then $\min\{f(x), f(y)\} \leq f(xy) < \min\{t, r\}$ and consequently, $f(x) < t$ or $f(y) < r$. It follows that $x_t\overline{\in}f$ or $y_r\overline{\in}f$. Thus $x_t\overline{\in} \vee \overline{q_k}f$ or $y_r\overline{\in} \vee \overline{q_k}f$.

If $\max\{f(xy), \frac{1-k}{2}\} = \frac{1-k}{2}$, then $\min\{f(x), f(y)\} \leq \frac{1-k}{2}$. Suppose $x_t \in f$ and $y_r \in f$, then $t \leq f(x) < \frac{1-k}{2}$ or $r \leq f(y) < \frac{1-k}{2}$. It follows that $x_t\overline{q_k}f$ or $y_r\overline{q_k}f$. ■

Corollary 113 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S if it satisfies condition* (3.1b).

Theorem 114 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S if and only if $U(f;t)(\neq \phi)$ is a subsemigroup of S for all $t \in (\frac{1-k}{2}, 1]$.*

Proof. Suppose f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S and $x, y \in U(f;t)$ for some $t \in (\frac{1-k}{2}, 1]$. Then $f(x) \geq t$ and $f(y) \geq t$. Hence $\frac{1-k}{2} < t \leq \min\{f(x), f(y)\} \leq \max\{f(xy), \frac{1-k}{2}\}$. Thus $f(xy) \geq t$ and so $xy \in U(f;t)$. Consequently, $U(f;t)$ is a subsemigroup of S.

Conversely, assume that $U(f;t)(\neq \phi)$ is a subsemigroup of S for all $t \in (\frac{1-k}{2}, 1]$. Suppose that there exist $x, y \in S$ such that $\max\{f(xy), \frac{1-k}{2}\} < \min\{f(x), f(y)\} = t$. Then $t \in (\frac{1-k}{2}, 1]$ and $x, y \in U(f;t)$ but $xy \notin U(f;t)$. This contradicts our hypothesis. Hence $\max\{f(xy), \frac{1-k}{2}\} \geq \min\{f(x), f(y)\}$ and so f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S. ■

Theorem 115 *Let A be a non-empty subset of a semigroup S and define a fuzzy subset f of S by*

$$f(x) = \begin{cases} 1 & if\ x \in A \\ \leq \frac{1-k}{2} & \text{otherwise.} \end{cases}$$

Then A is a subsemigroup of S if and only if f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S.

Proof. Let A be a subsemigroup of S and $x, y \in S$. If $x, y \in A$ then $f(x) = f(y) = 1$. As A is a subsemigroup of S, so $xy \in A$. This implies $f(xy) = 1$. Hence $\max\left\{f(xy), \frac{1-k}{2}\right\} = 1 = \min\{f(x), f(y)\}$. If $x \notin A$ or $y \notin A$ then $f(x) \leq \frac{1-k}{2}$ or $f(y) \leq \frac{1-k}{2}$. Thus $\min\{f(x), f(y)\} \leq \frac{1-k}{2} \leq \max\left\{f(xy), \frac{1-k}{2}\right\}$. Hence in any case $\max\left\{f(xy), \frac{1-k}{2}\right\} \geq \min\{f(x), f(y)\}$. This shows that f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S.

Conversely, assume that f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S and $x, y \in A$. Then $f(x) = f(y) = 1$. By hypothesis $\max\left\{f(xy), \frac{1-k}{2}\right\} \geq \min\{f(x), f(y)\} = 1$. This implies $f(xy) = 1$, that is $xy \in A$. Hence A is a subsemigroup of S. ■

Corollary 116 *A non-empty subset A of a semigroup S is a subsemigroup of S if and only if C_A, the characteristic function of A is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S.*

Definition 117 *A fuzzy subset f of a semigroup S is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right (resp. left) ideal of S if for all $x, y \in S$ and $t \in (0, 1]$ the following condition holds*

$(3.2a)$ $(xy)_t \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$ (resp. $(3.3a)$ $(xy)_t \overline{\in} f \Rightarrow y_t \overline{\in} \vee \overline{q_k} f$).

A fuzzy subset f of a semigroup S is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of S if it is both an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S.

Theorem 118 *Let f be a fuzzy subset of a semigroup S, $x, y \in S$ and $t \in (0, 1]$. Then the following conditions are equivalent*

$(3.2a)$ $(xy)_t \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$.
$(3.2b)$ $\max\{f(xy), \frac{1-k}{2}\} \geq f(x)$.

$$\left(\begin{array}{l} \text{resp. } (3.3a)(xy)_t \overline{\in} f \Rightarrow y_t \overline{\in} \vee \overline{q_k} f. \\ (3.3b) \max\{f(xy), \frac{1-k}{2}\} \geq f(y) \end{array} \right).$$

Proof. $(3.2a) \Rightarrow (3.2b)$ Suppose there exist $x, y \in S$ such that $\max\{f(xy), \frac{1-k}{2}\} < f(x)$. Then we can choose $t \in (\frac{1-k}{2}, 1]$ such that $\max\{f(xy), \frac{1-k}{2}\} < t = f(x)$. Then $(xy)_t \overline{\in} f$ but $x_t \in \wedge q_k f$, which is a contradiction. Hence $\max\{f(xy), \frac{1-k}{2}\} \geq f(x)$.

$(3.2b) \Rightarrow (3.2a)$ Let $(xy)_t \overline{\in} f$. Then $f(xy) < t$. If $\max\{f(xy), \frac{1-k}{2}\} = f(xy)$, then $f(x) \leq f(xy) < t$ and consequently, $f(x) < t$. It follows that $x_t \overline{\in} f$. Thus $x_t \overline{\in} \vee \overline{q_k} f$. If $\max\{f(xy), \frac{1-k}{2}\} = \frac{1-k}{2}$, then $f(x) \leq \frac{1-k}{2}$. Suppose $x_t \in f$, then $t \leq f(x) < \frac{1-k}{2}$. It follows that $x_t \overline{q_k} f$. Thus $x_t \overline{\in} \vee \overline{q_k} f$. ■

Corollary 119 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right (resp. left) ideal of S if it satisfies condition $(3.2b)$ (resp. $(3.3b)$).*

Corollary 120 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of S if it satisfies conditions $(3.2b)$ and $(3.3b)$.*

Theorem 121

A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right (resp. left) ideal of S if and only if $U(f;t)(\neq \emptyset)$ is a right (resp. left) ideal of S for all $t \in (\frac{1-k}{2}, 1]$.

Proof. Suppose f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal of S and $x \in U(f;t)(\neq \emptyset)$ and $y \in S$, for some $t \in (\frac{1-k}{2}, 1]$. Then $f(x) \geq t$. Hence $\frac{1-k}{2} < t \leq f(x) \leq \max\{f(xy), \frac{1-k}{2}\}$. Thus $f(xy) \geq t$ and so $xy \in U(f;t)$. Consequently, $U(f;t)$ is a right ideal of S.

Conversely, assume that $U(f;t)(\neq \phi)$ is a right ideal of S for all $t \in (\frac{1-k}{2}, 1]$. Suppose that there exist $x, y \in S$ such that $\max\{f(xy), \frac{1-k}{2}\} < f(x) = t$. Then $t \in (\frac{1-k}{2}, 1]$ and $x \in U(f;t)$ but $xy \notin U(f;t)$. Which contradicts our hypothsis. Hence $\max\{f(xy), \frac{1-k}{2}\} \geq f(x)$ and so f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal of S. ■

Corollary 122 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of S if and only if $U(f;t)(\neq \phi)$ is an ideal of S for all $t \in (\frac{1-k}{2}, 1]$.*

Theorem 123 *Let A be a non-empty subset of a semigroup S and define a fuzzy subset f of S by*

$$f(x) = \begin{cases} 1 & if x \in A \\ \leq \frac{1-k}{2} & \text{otherwise.} \end{cases}$$

Then A is a right (left) ideal of S if and only if f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right (left) ideal of S.

Proof. The proof is similar to the proof of Theorem 115. ■

Corollary 124 *A non-empty subset A of a semigroup S is a right (left) ideal of S if and only if C_A, the characteristic function of A, is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right (left) ideal of S.*

Theorem 125 *Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S and g be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal of S, then $f \circ g$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy two-sided ideal of S.*

Proof. Let $x, y \in S$. Then $(f \circ g)(y) = \bigvee_{y=pq} \{f(p) \wedge g(q)\}$.

(If $y = pq$, then $xy = x(pq) = (xp)q$. Since f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal, so by Theorem 118, $f(xp) \vee \frac{1-k}{2} \geq f(p)$.)

Thus

$$\begin{aligned}
(f \circ g)(y) &= \bigvee_{y=pq} \{f(p) \wedge g(q)\} \\
&\leq \bigvee_{y=pq} \{f(xp) \vee \tfrac{1-k}{2}) \wedge g(q)\} \\
&\leq \bigvee_{xy=ab} \{(f(a) \vee \tfrac{1-k}{2}) \wedge g(b)\} \\
&= \bigvee_{xy=ab} \{(f(a) \wedge g(b)) \vee \tfrac{1-k}{2}\}
\end{aligned}$$

$= (\bigvee_{xy=ab} \{(f(a) \wedge g(b)\}) \vee \frac{1-k}{2}$
$= (f \circ g)(xy) \vee \frac{1-k}{2}$.
So $(f \circ g)(y) \leq (f \circ g)(xy) \vee \frac{1-k}{2}$.

If $(f \circ g)(y) = 0$, then $(f \circ g)(y) \leq (f \circ g)(xy) \vee \frac{1-k}{2}$. Thus $f \circ g$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S.

Similarly we can show that $f \circ g$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal of S. Thus $f \circ g$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of S. ■

Lemma 126 *The intersection of any family of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right) ideals of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right) ideal of S.*

Proof. Let $\{f_i\}_{i \in I}$ be a family of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideals of S and $x, y \in S$. Since each f_i is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S, so $f_i(xy) \vee \frac{1-k}{2} \geq f_i(y)$ for all $i \in I$. Thus

$((\bigwedge_{i \in I} f_i)(xy)) \vee \frac{1-k}{2} = (\bigwedge_{i \in I} (f_i(xy))) \vee \frac{1-k}{2}$
$= \bigwedge_{i \in I} (f_i(xy) \vee \frac{1-k}{2})$
$\geq \bigwedge_{i \in I} (f_i(y))$
$= (\bigwedge_{i \in I} f_i)(y)$.

Hence $\bigwedge_{i \in I} f_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S. ■

Similarly, we can prove that the intersection of any family of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideals of S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal of S. Thus the intersection of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of S.

Now we show that if f and g are $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals of a semigroup S, then $f \circ g \not\leq f \wedge g$.

Example 127 *Consider the semigroup $S = \{a, b, c, d\}$ of Example 12.*

One can easily check that $\{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}$ and $\{a, b, c, d\}$ are all ideals of S.

Define fuzzy sets f, g of S by

$$\begin{aligned} f(a) &= 0.7,\ f(b) = 0.3,\ f(c) = 0.4,\ f(d) = 0, \\ g(a) &= 0.8,\ g(b) = 0.3,\ g(c) = 0.4,\ g(d) = 0.2. \end{aligned}$$

Then

$$U(f;t) = \begin{cases} \{a,b,c\} & if\ 0 < t \leq 0.3 \\ \{a,c\} & if\ 0.3 < t \leq 0.4 \\ \{a\} & if\ 0.4 < t \leq 0.7 \\ \phi & if\ 0.7 < t \leq 1 \end{cases}$$

$$U(g;t) = \begin{cases} \{a,b,c,d\} & if\ 0 < t \leq 0.2 \\ \{a,b,c\} & if\ 0.2 < t \leq 0.3 \\ \{a,c\} & if\ 0.3 < t \leq 0.4 \\ \{a\} & if\ 0.4 < t \leq 0.8 \\ \phi & if\ 0.8 < t \leq 1 \end{cases}$$

Then by Theorem 121, f and g are $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals of S for $k = 0.2$. Now,

$$\begin{aligned} (f \circ g)(b) &= \bigvee_{b=xy} \{f(x) \wedge g(y)\} \\ &= \vee\{0.4, 0, 0\} = 0.4 \nleq (f \wedge g)(b) = 0.3. \end{aligned}$$

Hence $f \circ g \nleq f \wedge g$ in general.

Theorem 128 *The union of any family of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right) ideals of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right) ideal of S.*

Proof. Let $\{f_i\}_{i \in I}$ be a family of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideals of S and $x, y \in S$. Since each f_i is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S, so $f_i(xy) \vee \frac{1-k}{2} \geq f_i(y)$ for all $i \in I$. Thus

$$\begin{aligned} &((\bigvee_{i \in I} f_i)(xy)) \vee \tfrac{1-k}{2} = (\bigvee_{i \in I} (f_i(xy))) \vee \tfrac{1-k}{2} \\ &= \bigvee_{i \in I} (f_i(xy) \vee \tfrac{1-k}{2}) \\ &\geq \bigvee_{i \in I} (f_i(y)) \\ &= (\bigvee_{i \in I} f_i)(y). \end{aligned}$$

Hence $\bigvee_{i \in I} f_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S. ■

Definition 129 *A fuzzy subset f of a semigroup S is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S if it satisfies* (3.1*a*) *and* (3.4*a*)*, where*

(3.4a) $(xyz)_{\min\{t,r\}}\overline{\in}f \Rightarrow x_t\overline{\in} \vee \overline{q_k}f$ or $z_r\overline{\in} \vee \overline{q_k}f$ for all $x, y, z \in S$ and $t, r \in (0, 1]$.

Theorem 130 *Let f be a fuzzy subset of a semigroup S, $x, y, z \in S$ and $t, r \in (0, 1]$.*

Then the following conditions are equivalent
(3.4a) $(xyz)_{\min\{t,r\}}\overline{\in}f \Rightarrow x_t\overline{\in} \vee \overline{q_k}f$ or $z_r\overline{\in} \vee \overline{q_k}f$.
(3.4b) $\max\{f(xyz), \frac{1-k}{2}\} \geq \min\{f(x), f(z)\}$.
Proof. The proof is similar to the proof of Theorem 118. ■

Corollary 131 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi- ideal of S if it satisfies* (3.1b) *and* (3.4b).

Theorem 132

A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi- ideal of S if and only if $U(f; t)(\neq \emptyset)$ is a bi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$.
Proof. The proof is similar to the proof of Theorem 121. ■

Theorem 133 *Let A be a non-empty subset of a semigroup S and define a fuzzy subset f of S by*

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ \leq \frac{1-k}{2} & \text{otherwise.} \end{cases}$$

Then A is a bi-ideal of S if and only if f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S.
Proof. The proof is similar to the proof of Theorem 115. ■

Corollary 134 *A non-empty subset A of a semigroup S is a bi-ideal of S if and only if C_A, the characteristic function of A, is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S.*

Lemma 135 *The intersection of any family of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S.*

Proof. The proof is similar to the proof of Lemma 126. ■

Lemma 136 *Let f and g be $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals of a semigroup S. Then $f \circ g$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S.*

Proof. The proof is similar to the proof of Theorem 125. ■

Next we show that the union of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals of a semigroup S need not be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S.

Example 137 *Consider the semigroup* $S = \{0, 1, 2, 3, 4, 5\}$.

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

One can easily check that the bi-ideals in S are: $\{0\}$, $\{0,1\}$, $\{0,1,2\}$, $\{0,1,3\}$, $\{0,1,4\}$, $\{0,1,5\}$, $\{0,1,2,4\}$, $\{0,1,3,5\}$, $\{0,1,2,3\}$, $\{0,1,4,5\}$, $\{0,1,2,3\}$, $\{0,1,4,5\}$ and S.

Define fuzzy subsets f, g of S by

$$f(0) = 0.7,\ f(1) = 0.6 = f(3),\ f(2) = 0.2 = f(4) = f(5)$$
$$g(0) = 0.75,\ g(1) = 0.62 = g(4),\ g(2) = 0.25 = g(3) = g(5).$$

Then

$$U(f;t) = \begin{cases} \{0,1,2,3,4,5\} & \text{if } 0 < t \leq 0.2 \\ \{0,1,3\} & \text{if } 0.2 < t \leq 0.6 \\ \{0\} & \text{if } 0.6 < t \leq 0.7 \\ \phi & \text{if } 0.7 < t \leq 1 \end{cases}$$

$$U(g;t) = \begin{cases} \{0,1,2,3,4,5\} & \text{if } 0 < t \leq 0.25 \\ \{0,1,4\} & \text{if } 0.25 < t \leq 0.62 \\ \{0\} & \text{if } 0.62 < t \leq 0.75 \\ \phi & \text{if } 0.75 < t \leq 1 \end{cases}$$

Thus by Theorem 132, f and g are $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals of S for $k = 0.2$.

As $U(f \vee g; t) = \{0,1,3,4\}$ for $0.25 < t \leq 0.6$, which is not a bi-ideal of S. So $f \vee g$ is not an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S.

Definition 138 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal of S if it satisfies* (3.4a) *or* (3.4b).

Theorem 139 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal of S if and only if $U(f;t)(\neq \phi)$ is a generalized bi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$.*

Proof. The proof is similar to the proof of Theorem 121. ■

Theorem 140 *Let A be a non-empty subset of a semigroup S and define a fuzzy subset f of S by*

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ \leq \frac{1-k}{2} & \text{otherwise.} \end{cases}$$

Then A is a generalized bi-ideal of S if and only if f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal of S.

Proof. The proof is similar to the proof of Theorem 115. ■

Corollary 141 *A non-empty subset A of a semigroup S is a generalized bi-ideal of S if and only if C_A, the characteristic function of A, is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal of S.*

It is clear that every $(\overline{\in}, \overline{\in}\vee\overline{q_k})$-fuzzy bi- ideal of a semigroup S is an $(\overline{\in}, \overline{\in}\vee\overline{q_k})$-fuzzy generalized bi-ideal S. The next example shows that the fuzzy generalized bi-ideal of S is not necessarily a fuzzy bi-ideal of S.

Example 142 *Consider the semigroup $S = \{a, b, c, d\}$ of Example 12.*

One can easily check that $\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}$ and $\{a, b, c, d\}$ are all generalized bi-ideals of S and $\{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}$ and $\{a, b, c, d\}$ are all bi-ideals of S.

Define a fuzzy subset f of S by

$$f(a) = 0.8,\ f(b) = 0,\ f(c) = 0.7,\ f(d) = 0.$$

Then

$$U(f;t) = \begin{cases} \{a, c\} & \text{if } 0 < t \leq 0.7 \\ \{a\} & \text{if } 0.7 < t \leq 0.8 \\ \phi & \text{if } 0.8 < t \leq 1 \end{cases}$$

Thus by Theorem 139, f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal of S but not an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S, because $U(f; 0.6) = \{a, c\}$ is a generalized bi-ideal of S but not a bi-ideal of S for $k = 0.2$.

Lemma 143 *Every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal of a regular semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S.*

Proof. Let S be a regular semigroup and f an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal of S. Let $a, b \in S$, then there exists $x \in S$ such that $b = bxb$. Thus we have $\max\{f(ab), \frac{1-k}{2}\} = \max\{f(a(bxb)), \frac{1-k}{2}\} = \max\{f(a(bx)b), \frac{1-k}{2}\} \geq \min\{f(a), f(b)\}$.

This shows that f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy subsemigroup of S and so f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of a S. ■

Definition 144 *A fuzzy subset f of a semigroup S is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S if it satisfies*

(3.5a) $\max\{f(x), \frac{1-k}{2}\} \geq \min\{(f \circ \mathcal{S})(x), (\mathcal{S} \circ f)(x)\}$.

Where $\mathcal{S}$ is the fuzzy subset of S mapping every element of S on 1.

Theorem 145 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S if and only if $U(f;t)(\neq \emptyset)$ is a quasi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$.*

Proof. Suppose f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. Let $x \in U(f;t)S \cap SU(f;t)$. Then there exist $a, b \in U(f;t)$ and $u, v \in S$ such that $x = au$ and $x = vb$. Now,

$(f \circ \mathcal{S})(x) = \bigvee_{x=yz} \{f(y) \wedge \mathcal{S}(z)\} \geq f(a) \wedge \mathcal{S}(u) = f(a) \geq t.$

Similarly $(\mathcal{S} \circ f)(x) \geq t$. Thus by hypothesis

$\max\{f(x), \frac{1-k}{2}\} \geq \min\{(f \circ \mathcal{S})(x), (\mathcal{S} \circ f)(x)\} \geq t > \frac{1-k}{2}$.

This implies $f(x) \geq t$, that is $x \in U(f;t)$. Hence $U(f;t)$ is a quasi-ideal of S.

Conversely, assume that $U(f;t)(\neq \emptyset)$ is a quasi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$. Let $x \in S$ be such that

$\max\{f(x), \frac{1-k}{2}\} < \min\{(f \circ \mathcal{S})(x), (\mathcal{S} \circ f)(x)\}$.

Select $t \in (\frac{1-k}{2}, 1]$ such that

$\max\{f(x), \frac{1-k}{2}\} < t \leq \min\{(f \circ \mathcal{S})(x), (\mathcal{S} \circ f)(x)\}$.

This implies $f(x) < t$ and $(f \circ \mathcal{S})(x) \geq t, (\mathcal{S} \circ f)(x) \geq t$. If $(f \circ \mathcal{S})(x) \geq t$, then

$(f \circ \mathcal{S})(x) = \bigvee_{x=yz} \{f(y) \wedge \mathcal{S}(z)\} = \bigvee_{x=yz} f(y) \geq t \Rightarrow$ there exists $y \in U(f;t)$ such that $x = yz$ for some $z \in S$. Similarly $(\mathcal{S} \circ f)(x) \geq t$ implies $a \in U(f;t)$ such that $x = ba$ for some $b \in S$. Thus $x \in U(f;t)S \cap SU(f;t) \subseteq U(f;t)$. A contradiction because $f(x) < t$. Hence

$\max\{f(x), \frac{1-k}{2}\} \geq \min\{(f \circ \mathcal{S})(x), (\mathcal{S} \circ f)(x)\}$.

This shows that f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. ■

Theorem 146 *Let A be a non-empty subset of a semigroup S and define a fuzzy subset f of S by*

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ \leq \frac{1-k}{2} & \text{otherwise.} \end{cases}$$

Then A is a quasi-ideal of S if and only if f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S.

Proof. Suppose A is a quasi-ideal of S and $x \in S$. If $x \in A$, then $f(x) = 1$. This implies $\max\{f(x), \frac{1-k}{2}\} = 1 \geq \min\{(f \circ \mathcal{S})(x), (\mathcal{S} \circ f)(x)\}$. If $x \notin A$, then $x \notin SA \cap AS$. So $f(x) \leq \frac{1-k}{2}$. Since $x \notin SA \cap AS$, therefore, $\min\{(f \circ \mathcal{S})(x), (\mathcal{S} \circ f)(x)\} \neq 1$. This implies $\min\{(f \circ \mathcal{S})(x), (\mathcal{S} \circ f)(x)\} \leq \frac{1-k}{2} = \max\{f(x), \frac{1-k}{2}\}$.

Hence in any case $\max\{f(x), \frac{1-k}{2}\} \geq \min\{(f \circ \mathcal{S})(x), (\mathcal{S} \circ f)(x)\}$, that is f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S.

Conversely, assume that f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S and $x \in SA \cap AS$. Then there exist $a, b \in A$ and $r, s \in S$ such that $x = ra = bs$. Now

$(f \circ \mathcal{S})(x) = \bigvee_{x=yz} \{f(y) \wedge \mathcal{S}(z)\} \geq f(b) \wedge \mathcal{S}(s) = 1 \Rightarrow (f \circ \mathcal{S})(x) = 1$.

Similarly, $(\mathcal{S} \circ f)(x) = 1$. Thus by hypothesis $\max\{f(x), \frac{1-k}{2}\} \geq \min\{(f \circ \mathcal{S})(x), (\mathcal{S} \circ f)(x)\} = 1$.

This implies $f(x) = 1$, that is $x \in A$. This shows that $SA \cap AS \subseteq A$, that is A is a quasi-ideal of S. ■

Corollary 147 *A non-empty subset A of a semigroup S is a quasi-ideal of S if and only if C_A, the characteristic function of A, is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S.*

Lemma 148 *The intersection of any family of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S.*

Proof. Straightforward. ■

Theorem 149 *Every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right) ideal of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S.*

Proof. The proof follows from Theorem 121 and Theorem 145. ■

Theorem 150 *Every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S.*

Proof. The proof follows from Theorem 132 and Theorem 145. ■

Definition 151 *A fuzzy subset f of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideal of S if it satisfies*

(3.6a) $(xyz)_t \overline{\in} f \Rightarrow y_t \overline{\in} \vee \overline{q_k} f$ for all $x, y, z \in S$ and $t \in (0, 1]$.

Theorem 152 *Let f be a fuzzy subset of a semigroup S, $x,y,z\in S$ and $t\in(0,1]$. Then the following conditions are equivalent*

(3.6a) $(xyz)_t\overline{\in}f\Rightarrow y_t\overline{\in}\vee\overline{q_k}f$.
(3.6b) $\max\{f(xyz),\frac{1-k}{2}\}\geq f(y)$.
Proof. The proof is similar to the proof of Theorem 118. ■

Corollary 153 *A fuzzy subset f of a semigroup S is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of S if it satisfies* (3.1b) *and* (3.6b)*.*

Theorem 154 *A fuzzy subset f of a semigroup S is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of S if and only if $U(f;t)(\neq\phi)$ is an interior-ideal of S for all $t\in(\frac{1-k}{2},1]$.*

Proof. The proof is similar to the proof of Theorem 121. ■

Theorem 155 *Let A be a non-empty subset of a semigroup S and define a fuzzy subset f of S by*

$$f(x)=\begin{cases}1 & \text{if } x\in A\\ \leq\dfrac{1-k}{2} & \text{otherwise.}\end{cases}$$

Then A is an interior-ideal of S if and only if f is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of S.

Proof. The proof is similar to the proof of Theorem 115. ■

Corollary 156 *A non-empty subset A of a semigroup S is an interior-ideal of S if and only if C_A, the characteristic function of A, is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of S.*

Lemma 157 *The intersection of any family of $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideals of a semigroup S is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of S.*

Proof. Straightforward. ■

The following example shows that the union of $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideals of a semigroup S need not be an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of S.

Example 158 *Consider the semigroup $S=\{a,b,c,d\}$ with the following multiplication table*

.	a	b	c	d
a	a	a	a	a
b	a	a	d	a
c	a	a	a	a
d	a	a	a	a

One can easily check that $\{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,d\}, \{a,c,d\}$ and S are all interior-ideals of S.

Define fuzzy subsets f, g of S by

$$\begin{aligned} f(a) &= 0.8 = f(b),\ f(c) = 0 = f(d), \\ g(a) &= 0.8 = g(c),\ g(b) = 0 = g(d) \end{aligned}$$

Then

$$U(f;t) = \begin{cases} \{a,b\} & \text{if } 0<t \leq 0.8 \\ \phi & \text{if } 0.8<t \leq 1 \end{cases}$$

$$U(g;t) = \begin{cases} \{a,c\} & \text{if } 0<t \leq 0.8 \\ \phi & \text{if } 0.8<t \leq 1 \end{cases}$$

Then by Theorem 154, f and g are $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideals of S for $k = 0.2$.

But $U(\ f \vee g; t) = \{a, b, c\}$ if $0 < t \leq 0.8$, which is not an interior-ideal of S, so $f \vee g$ is not an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideal of S.

Lemma 159 *Every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of a semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideal of S.*

Proof. Straightforward. ■

The following example shows that the converse of the above Lemma does not hold in general.

Example 160 *Consider the semigroup $S = \{0, a, b, c\}$.*

.	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Then $\{0\}, \{0,a\}, \{0,b\}, \{0,a,b\}$ and S are all interior-ideals of S, but $\{0,b\}$ is not an ideal of S.

Define a fuzzy subset f of S by

$f(0) = 0.6$, $f(a) = 0.4$, $f(b) = 0.6$, and $f(c) = 0.2$. Then

$$U(f;t) = \begin{cases} \{0,a,b,c\} & \text{if } 0<t \leq 0.2 \\ \{0,a,b\} & \text{if } 0.2<t \leq 0.4 \\ \{0,b\} & \text{if } 0.4<t \leq 0.6 \\ \phi & \text{if } 0.6<t \leq 1 \end{cases}$$

Thus by Theorem 154 f is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of S for $k = 0.2$. Since $\{0,b\}$ is not an ideal of S, so f is not an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy ideal of S for $k = 0.2$.

Lemma 161 *Every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of a regular semigroup S is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy ideal of S.*

Proof. Let S be a regular semigroup and f an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy interior-ideal of S. Let $a,b \in S$. Then there exists $x \in S$ such that $a = axa$. Thus we have $\max\{f(ab), \frac{1-k}{2}\} = \max\{f((axa)b), \frac{1-k}{2}\} = \max\{f((ax)a(b)), \frac{1-k}{2}\} \geq f(a)$. This shows that f is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right ideal of S. Similarly we can show that f is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left ideal of S. Hence f is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy ideal of S. ■

3.3 Regular Semigroups in Terms of $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy Ideals

In this section we characterize regular semigroups by the properties of their $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy ideals, quasi-ideals and bi-ideals.

Definition 162 *Let f,g be fuzzy subsets of a semigroup S. We define the fuzzy subsets $f^k, f\wedge^k g, f\vee^k g$ and $f\circ^k g$ as follows:*

$$\begin{aligned} f^k(x) &= f(x) \vee \frac{1-k}{2} \\ \left(f \wedge^k g\right)(x) &= (f \wedge g)(x) \vee \frac{1-k}{2} \\ \left(f \vee^k g\right)(x) &= (f \vee g)(x) \vee \frac{1-k}{2} \\ \left(f \circ^k g\right)(x) &= (f \circ g)(x) \vee \frac{1-k}{2} \end{aligned}$$

for all $x \in S$.

Lemma 163 *Let f and g be fuzzy subsets of a semigroup S. Then the following hold.*

(1) $(f \wedge^k g) = (f^k \wedge g^k)$
(2) $(f \vee^k g) = (f^k \vee g^k)$
(3) $(f \circ^k g) \geq (f^k \circ g^k)$
If every element x of S is expressible as $x = bc$, then $(f \circ^k g) = (f^k \circ g^k)$.
Proof. Straightforward. ■

Lemma 164 *Let A and B be non-empty subsets of a semigroup S. Then the following hold.*

(1) $(C_A \wedge^k C_B) = C^k_{A\cap B}$
(2) $(C_A \vee^k C_B) = C^k_{A\cup B}$
(3) $(C_A \circ^k C_B) = C^k_{AB}$.
Proof. Straightforward. ■

Proposition 165 *If f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right) ideal of S, then f^k is also an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right) ideal of S.*

Proof. Suppose f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S and $x, y \in S$. Then $\max\{f^k(xy), \frac{1-k}{2}\} = \max\{\max(f(xy), \frac{1-k}{2}), \frac{1-k}{2}\} \geq \max\{f(y), \frac{1-k}{2}\} = f^k(y)$. Thus f^k is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S. ■

Similarly we can show that:

Proposition 166 *If f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal (bi-ideal, generalized bi-ideal, interior-ideal) of S, then f^k is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal (bi-ideal, generalized bi-ideal, interior-ideal) of S.*

Lemma 167 *Let A be a non-empty subset of a semigroup S. Then A is a left (resp. right) ideal of S if and only if C^k_A is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right) ideal of S.*

Proof. Straightforward. ■

Similarly we can prove that:

Lemma 168 *A non-empty subset Q of a semigroup S is a quasi-ideal (bi-ideal, generalized bi-ideal, interior-ideal) of S if and only if C_Q^k is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal (bi-ideal, generalized bi-ideal, interior-ideal) of S.*

In example 61, it is shown that if f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal and g is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal of S, then $f \circ g \nleq f \wedge g$. Now we show that $(f \circ^k g) \leq (f \wedge^k g)$.

Proposition 169 *Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal and g an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S. Then $f \circ^k g \leq f \wedge^k g$.*

Proof. Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal and g be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S. Then for all $a \in S$, we have

$(f \circ^k g)(a) = (f \circ g)(a) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{f(y) \wedge g(z)\}) \vee \frac{1-k}{2}$
$\leq (\bigvee_{a=yz} \{(f(yz) \vee \frac{1-k}{2}) \wedge (g(yz) \vee \frac{1-k}{2})\}) \vee \frac{1-k}{2}$
$= \{(f(a) \wedge g(a)) \vee \frac{1-k}{2}\} \vee \frac{1-k}{2}$
$= (f \wedge g)(a) \vee \frac{1-k}{2}$
$= (f \wedge^k g)(a).$

If a is not expressible as $a = yz$, then
$(f \circ^k g)(a) = (f \circ g)(a) \vee \frac{1-k}{2} = 0 \vee \frac{1-k}{2} = \frac{1-k}{2}$
$\leq (f \wedge g)(a) \vee \frac{1-k}{2} = (f \wedge^k g)(a).$

Thus $(f \circ^k g) \leq (f \wedge^k g)$. ■

Next we characterize regular semigroups by the properties of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals, bi-ideals and generalized bi-ideals.

Theorem 170 *For a semigroup S the following conditions are equivalent.*

(1) S is regular.

(2) $f \wedge^k g = f \circ^k g$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal g of S.

Proof. (1) $\Rightarrow$ (2) Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal and g be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S. Then by Proposition 169, $f \circ^k g \leq f \wedge^k g$.

Let $a \in S$. Then there exists $x \in S$ such that $a = axa$. So

$(f \circ^k g)(a) = (f \circ g)(a) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{f(y) \wedge g(z)\}) \vee \frac{1-k}{2}$

$\geq (f(ax) \wedge g(a)) \vee \frac{1-k}{2}$
$= (f(ax) \vee \frac{1-k}{2}) \wedge (g(a) \vee \frac{1-k}{2})$
$\geq f(a) \wedge (g(a) \vee \frac{1-k}{2})$
$= (f(a) \wedge g(a)) \vee \frac{1-k}{2} = (f \wedge^k g)(a).$
So $(f \circ^k g) \geq (f \wedge^k g)$. Thus $(f \circ^k g) = (f \wedge^k g)$.

(2) $\Rightarrow$ (1) Let R be a right ideal and L be a left ideal of S. Then C_R and C_L are $(\in, \in \vee \overline{q_k})$-fuzzy right and left ideals of S, respectively. Since
$C^k_{RL} = (C_R \circ^k C_L) = (C_R \wedge^k C_L) = C^k_{R\cap L}$
we have $R \cap L = RL$. This shows that S is regular. ■

Theorem 171 *For a semigroup S, the following conditions are equivalent.*

(1) S is regular.

(2) $(h \wedge^k f \wedge^k g) \leq (h \circ^k f \circ^k g)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal h, every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal g of S.

(3) $(h \wedge^k f \wedge^k g) \leq (h \circ^k f \circ^k g)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal h, every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal g of S.

(4) $(h \wedge^k f \wedge^k g) \leq (h \circ^k f \circ^k g)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal h, every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal g of S.

Proof. (1) $\Rightarrow$ (2) Let h, f and g be $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal, generalized bi-ideal, and left ideal of S, respectively. Let a be any element of S. Since S is regular, so there exists $x \in S$ such that $a = axa$. Hence we have $(h \circ^k f \circ^k g)(a) = (h \circ f \circ g)(a) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{h(y) \wedge (f \circ g)(z)\}) \vee \frac{1-k}{2}$
$\geq \{h(ax) \wedge (f \circ g)(a)\} \vee \frac{1-k}{2}$
$= (h(ax) \vee \frac{1-k}{2}) \wedge ((f \circ g)(a) \vee \frac{1-k}{2})$
$\geq (h(a)) \wedge ((\bigvee_{a=pq} \{f(p) \wedge g(q)\}) \vee \frac{1-k}{2})$
$\geq (h(a)) \wedge \{f(a) \wedge g(xa)) \vee \frac{1-k}{2}\}$
$= (h(a)) \wedge \{(f(a) \vee \frac{1-k}{2}) \wedge (g(xa)) \vee \frac{1-k}{2})\}$
$\geq (h(a)) \wedge \{(f(a) \vee \frac{1-k}{2}) \wedge (g(a))\}$
$= \{h(a) \wedge (f(a) \wedge g(a))\} \vee \frac{1-k}{2} = (h \wedge^k f \wedge^k g)(a).$

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) Straight forward, because every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal is $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal is $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal.

(4) $\Rightarrow$ (1) Let h and g be any $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S, respectively. Since $\mathcal{S}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, so by the assumption, we have
$(h \wedge^k g)(a) = (h \wedge g)(a) \vee \frac{1-k}{2}$
$= (h \wedge \mathcal{S} \wedge g)(a) \vee \frac{1-k}{2}$

$= (h \wedge^k \mathcal{S} \wedge^k g)(a)$
$\leq (h \circ^k \mathcal{S} \circ^k g)(a)$
$= (h \circ \mathcal{S} \circ g)(a) \vee \frac{1-k}{2}$
$= (\bigvee_{a=bc} \{(h \circ \mathcal{S})(b) \wedge g(c)\}) \vee \frac{1-k}{2}$
$= (\bigvee_{a=bc} \{(\bigvee_{b=pq} \{h(p) \wedge \mathcal{S}(q)\}) \wedge g(c)\}) \vee \frac{1-k}{2}$
$= (\bigvee_{a=bc} \{(\bigvee_{b=pq} \{h(p) \wedge 1\}) \wedge g(c)\}) \vee \frac{1-k}{2}$
$= (\bigvee_{a=bc} \{(\bigvee_{b=pq} h(p)) \wedge g(c)\}) \vee \frac{1-k}{2}$
$\leq (\bigvee_{a=bc} \{(\bigvee_{b=pq} \{h(pq) \vee \frac{1-k}{2}\}) \wedge g(c)\}) \vee \frac{1-k}{2}$
$\leq (\bigvee_{a=bc} \{(h(b) \vee \frac{1-k}{2}\}) \wedge g(c)\}) \vee \frac{1-k}{2}$
$= (\bigvee_{a=bc} \{(h(b) \wedge g(c)\}) \vee \frac{1-k}{2}) \vee \frac{1-k}{2}$
$= (\bigvee_{a=bc} \{(h(b) \wedge g(c)\}) \vee \frac{1-k}{2}$
$= (h \circ g)(a) \vee \frac{1-k}{2} = (h \circ^k g)(a).$

Thus it follows that $(h \wedge^k g) \leq (h \circ^k g)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal h and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal g of S. But $(h \circ^k g) \leq (h \wedge^k g)$ always holds. Thus $(h \circ^k g) = (h \wedge^k g)$. Hence by Theorem 170, S is regular. ■

Theorem 172 *For a semigroup S, the following conditions are equivalent.*

(1) S is regular.
(2) $f^k = (f \circ^k \mathcal{S} \circ^k f)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal f of S.
(3) $f^k = (f \circ^k \mathcal{S} \circ^k f)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal f of S.
(4) $f^k = (f \circ^k \mathcal{S} \circ^k f)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi ideal f of S.

Proof. (1) ⇒ (2) Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal of S and a be any element of S. Since S is regular, so there exists $x \in S$ such that $a = axa$. Hence we have

$(f \circ^k \mathcal{S} \circ^k f)(a) = (f \circ \mathcal{S} \circ f)(a) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{(f \circ \mathcal{S})(y) \wedge f(z)\}) \vee \frac{1-k}{2}$
$\geq \{(f \circ \mathcal{S})(ax) \wedge f(a)\} \vee \frac{1-k}{2}$
$= \{(\bigvee_{ax=pq} \{f(p) \wedge \mathcal{S}(q)\}) \wedge f(a)\} \vee \frac{1-k}{2}$
$\geq \{(f(a) \wedge \mathcal{S}(x)) \wedge f(a)\} \vee \frac{1-k}{2}$
$= \{(f(a) \wedge 1) \wedge f(a)\} \vee \frac{1-k}{2}$
$= f(a) \vee \frac{1-k}{2} = f^k(a).$

Thus $(f \circ^k \mathcal{S} \circ^k f)(a) \geq f^k$.

Also

$(f \circ^k \mathcal{S} \circ^k f)(a) = (f \circ \mathcal{S} \circ f)(a) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{(f \circ \mathcal{S})(y) \wedge f(z)\}) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{(\bigvee_{y=pq} \{f(p) \wedge \mathcal{S}(q)\}) \wedge f(z)\}) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{(\bigvee_{y=pq} \{f(p) \wedge 1\}) \wedge f(z)\}) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{ \bigvee_{y=pq} \{f(p) \wedge f(z)\}) \vee \frac{1-k}{2}$
$\leq \bigvee_{a=(pq)z} \{f(pqz) \vee \frac{1-k}{2}\} \vee \frac{1-k}{2}$
$= f(a) \vee \frac{1-k}{2} = f^k(a)$.

So, $(f \circ^k \mathcal{S} \circ^k f) \leq f^k$. Thus $f^k = (f \circ^k \mathcal{S} \circ^k f)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) obvious.

(4) $\Rightarrow$ (1) Let A be any quasi-ideal of S. Then C_A is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi ideal of S, so we have

$C_A^k = (C_A \circ^k \mathcal{S} \circ^k C_A) = C_{ASA}^k \Rightarrow A = ASA$.

Hence it follows from Theorem 23, that S is regular. ■

Theorem 173 *For a semigroup S, the following conditions are equivalent.*

(1) S is regular.

(2) $(f \wedge^k g) = (f \circ^k g \circ^k f)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideal g of S.

(3) $(f \wedge^k g) = (f \circ^k g \circ^k f)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideal g of S.

(4) $(f \wedge^k g) = (f \circ^k g \circ^k f)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideal g of S.

(5) $(f \wedge^k g) = (f \circ^k g \circ^k f)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy two-sided ideal g of S.

(6) $(f \wedge^k g) = (f \circ^k g \circ^k f)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy two-sided ideal g of S.

(7) $(f \wedge^k g) = (f \circ^k g \circ^k f)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy two-sided ideal g of S.

Proof. (1) $\Rightarrow$ (2) Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal and g an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideal of S. Then for all $a \in S$,

$(f \circ^k g \circ^k f)(a) = (f \circ g \circ f)(a) \vee \frac{1-k}{2}$
$\leq (f \circ \mathcal{S} \circ f)(a) \vee \frac{1-k}{2}$

$= (\bigvee_{a=yz} \{(f \circ \mathcal{S})(y) \wedge f(z)\}) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{(\bigvee_{y=pq} \{f(p) \wedge \mathcal{S}(q)\}) \wedge f(z)\}) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{(\bigvee_{y=pq} \{f(p) \wedge 1\}) \wedge f(z)\}) \vee \frac{1-k}{2}$
$= \bigvee_{a=yz} \{(\bigvee_{y=pq} f(p)) \wedge f(z)\} \vee \frac{1-k}{2}$
$= \bigvee_{a=yz} \{ \bigvee_{y=pq} f(p) \wedge f(z)\} \vee \frac{1-k}{2}$
$= \bigvee_{a=(pq)z} \{f(p) \wedge f(z)\} \vee \frac{1-k}{2}$
$\leq \bigvee_{a=(pq)z} \{f(pqz) \vee \frac{1-k}{2}\} \vee \frac{1-k}{2}$
$= f(a) \vee \frac{1-k}{2} = f^k(a)$
and
$(f \circ^k g \circ^k f)(a) \leq (\mathcal{S} \circ^k g \circ^k \mathcal{S})(a) = (\mathcal{S} \circ g \circ \mathcal{S})(a) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{(\mathcal{S} \circ g)(y) \wedge \mathcal{S}(z)\}) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{(\bigvee_{y=pq} \{\mathcal{S}(p) \wedge g(q)\}) \wedge \mathcal{S}(z)\}) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{(\bigvee_{y=pq} \{1 \wedge g(q)\}) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{ \bigvee_{y=pq} g(q)\}) \wedge \frac{1-k}{2}$
$\leq (\bigvee_{a=(pq)z} g(pqz) \vee \frac{1-k}{2}) \vee \frac{1-k}{2}$
$= g(a) \vee \frac{1-k}{2} = g^k(a).$
Thus $(f \circ^k g \circ^k f) \leq (f^k \wedge g^k) = (f \wedge^k g)$.

Since S is regular, so there exists an element $x \in S$ such that $a = axa(= axaxa)$. Since g is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideal of S, we have
$(f \circ^k g \circ^k f)(a) = (f \circ g \circ f)(a) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{f(y) \wedge (g \circ f)(z)\}) \vee \frac{1-k}{2}$
$\geq (f(a) \wedge (g \circ f)(xaxa)) \vee \frac{1-k}{2}$
$= (f(a) \wedge (\bigvee_{xaxa=pq} \{g(p) \wedge f(q)) \vee \frac{1-k}{2}$
$\geq (f(a) \wedge (g(xax) \wedge f(a))) \vee \frac{1-k}{2}$
$= (f(a) \vee \frac{1-k}{2}) \wedge \{(g(xax) \wedge f(a)) \vee \frac{1-k}{2}\}$
$= (f(a) \vee \frac{1-k}{2}) \wedge \{(g(xax) \vee \frac{1-k}{2}) \wedge (f(a)) \vee \frac{1-k}{2})\}$
$\geq (f(a) \vee \frac{1-k}{2}) \wedge \{(g(a)) \wedge (f(a)) \vee \frac{1-k}{2})\}$
$= (f(a) \vee \frac{1-k}{2}) \wedge (\{(g(a) \wedge (f(a)\} \vee \frac{1-k}{2})$

$= (f(a) \wedge \{(g(a) \wedge (f(a)\} \vee \frac{1-k}{2})$
$= (f \wedge g)(a) \vee \frac{1-k}{2} = (f \wedge^k g)(a).$
So $(f \circ^k g \circ^k f) \geq (f \wedge^k g)$. Hence $(f \circ^k g \circ^k f) = (f \wedge^k g)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (7) and (2) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are clear.

(7) $\Rightarrow$ (1) Let f be an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal of S. Since $\mathcal{S}$ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy ideal of S, we have for each $a \in S$

$f^k(a) = f(a) \vee \frac{1-k}{2} = (f \wedge \mathcal{S})(a) \vee \frac{1-k}{2} = (f \wedge^k S)(a) = (f \circ^k S \circ^k f)(a).$

Thus it follows from Theorem 172 that S is regular. ■

Theorem 174 *For a semigroup S, the following conditions are equivalent.*

(1) S is regular.

(2) $(f \wedge^k g) \leq (f \circ^k g)$ for every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy generalized bi-ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left ideal g of S.

(3) $(f \wedge^k g) \leq (f \circ^k g)$ for every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy bi-ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left ideal g of S.

(4) $(f \wedge^k g) \leq (f \circ^k g)$ for every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal f and every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left ideal g of S.

(5) $(f \wedge^k g) \leq (f \circ^k g)$ for every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy generalized bi-ideal g and every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right ideal f of S.

(6) $(f \wedge^k g) \leq (f \circ^k g)$ for every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy bi-ideal g and every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right ideal f of S.

(7) $(f \wedge^k g) \leq (f \circ^k g)$ for every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal g and every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right ideal f of S.

Proof. (1) $\Rightarrow$ (2) Let f and g be any $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy generalized bi-ideal and any $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left ideal of S, respectively. Let a be any element of S. Then there exists $x \in S$ such that $a = axa$. Thus we have

$(f \circ^k g)(a) = (f \circ g)(a) \vee \frac{1-k}{2}$
$= (\bigvee_{a=yz} \{f(y) \wedge g(z)\}) \vee \frac{1-k}{2}$
$\geq (f(a) \wedge g(xa)) \vee \frac{1-k}{2}$
$= (f(a) \vee \frac{1-k}{2}) \wedge (g(xa) \vee \frac{1-k}{2})$
$\geq (f(a) \vee \frac{1-k}{2}) \wedge (g(a))$
$= (f(a) \wedge g(a)) \vee \frac{1-k}{2}$
$= (f \wedge g)(a) \vee \frac{1-k}{2} = (f \wedge^k g)(a).$
So $(f \circ^k g) \geq (f \wedge^k g)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are Obviuos.

(4) $\Rightarrow$ (1) Let f be an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right-ideal and g be an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left ideal of S. Since every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right-ideal is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi

ideal of S, we have $(f \circ^k g) \geq (f \wedge^k g)$. But $(f \circ^k g) \leq (f \wedge^k g)$ always holds. Hence $(f \circ^k g) = (f \wedge^k g)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal g of S. Thus by Theorem 170 S is regular.

Similarly we can show that (1) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (1). ■

3.4 Intra-Regular Semigroups in Terms of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy Ideals

In this section we characterize intra-regular and regular and intra-regular semigroups by the properties of their $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals, quasi-ideals and bi-ideals.

Theorem 175 *For a semigroup S, the following conditions are equivalent.*

(1) S is intra-regular.

(2) $(f \wedge^k g) \leq (f \circ^k g)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal g of S.

Proof. (1) $\Rightarrow$ (2) Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal and g be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal of S. Let $x \in S$. Then there exist $a, b \in S$ such that $x = axxb$. Thus

$$(f \circ^k g)(x) = (f \circ g)(x) \vee \tfrac{1-k}{2}$$
$$= \Big(\bigvee_{a=yz} \{f(y) \wedge g(z)\}\Big) \vee \tfrac{1-k}{2}$$
$$\geq (f(ax) \wedge g(xb)) \vee \tfrac{1-k}{2}$$
$$= \big\{(f(ax) \vee \tfrac{1-k}{2}) \wedge (g(xb) \vee \tfrac{1-k}{2})\big\} \vee \tfrac{1-k}{2}$$
$$\geq \{(f(x) \wedge g(x)\} \vee \tfrac{1-k}{2} = \big(f \wedge^k g\big)(x).$$

(2) $\Rightarrow$ (1) Let R and L be right and left ideals of S, respectively. Then C_R and C_L are $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right and left ideals of S, respectively. Thus by hypothesis

$$C^k_{LR} = \big(C_L \circ^k C_R\big) \geq \big(C_L \wedge^k C_R\big) = C^k_{L \cap R}.$$

Thus $R \cap L \subseteq LR$. This implies that S is an intra-regular semigroup. ■

Theorem 176 *For a semigroup S, the following conditions are equivalent.*

(1) S is both regular and intra-regular.

(2) $f \circ^k f = f^k$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal f of S.

(3) $f \circ^k f = f^k$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal f of S.

(4) $(f \wedge^k g) \leq (f \circ^k g)$ for all $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals f, g of S.

(5) $(f \wedge^k g) \leq (f \circ^k g)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal g of S.

(6) $(f \wedge^k g) \leq (f \circ^k g)$ for all $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals f, g of S.

Proof. (1) ⇒ (6) Let f, g be any $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals of S and $x \in S$. Then there exist $a, b, c \in S$ such that $x = xax$ and $x = bxxc$. Thus $x = xax = xaxax = xa\,(bxxc)\,ax = (xabx)\,(xcax)$. Therefore

$$(f \circ^k g)(x) = (f \circ g)(x) \vee \tfrac{1-k}{2}$$
$$= \Big(\bigvee_{a=yz} \{f(y) \wedge g(z)\}\Big) \vee \tfrac{1-k}{2}$$
$$\geq (f(xabx) \wedge g(xcax)) \vee \tfrac{1-k}{2}$$
$$= \big\{(f(xabx) \vee \tfrac{1-k}{2}) \wedge (g(xcax) \vee \tfrac{1-k}{2})\big\} \vee \tfrac{1-k}{2}$$
$$\geq \{(f(x) \wedge g(x)\} \vee \tfrac{1-k}{2} = \big(f \wedge^k g\big)(x).$$

Thus $(f \wedge^k g) \leq (f \circ^k g)$ for all $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals f, g of S.

(6) ⇒ (5) ⇒ (4) are Obvious.

(4) ⇒ (2) Take $f = g$ in (4). We have $f^k = (f \wedge^k f) \leq (f \circ^k f)$ but $(f \circ^k f) \leq f^k$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals f of S. Hence $f^k = (f \circ^k f)$.

(6) ⇒ (3) ⇒ (2) Obvious.

(2) ⇒ (1) Let Q be any quasi-ideal of S. Then C_Q is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. Hence by hypothesis $C_Q \circ^k C_Q = C_Q^k$, that is $QQ = Q$. Then by Theorem 28 S is both regular and intra regular. ■

Theorem 177 *For a semigroup S, the following conditions are equivalent.*

(1) S is both regular and intra-regular.

(2) $\big(f \wedge^k g\big) \leq \big(f \circ^k g\big) \wedge \big(g \circ^k f\big)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal g of S.

(3) $\big(f \wedge^k g\big) \leq \big(f \circ^k g\big) \wedge \big(g \circ^k f\big)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal g of S.

(4) $\big(f \wedge^k g\big) \leq \big(f \circ^k g\big) \wedge \big(g \circ^k f\big)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal g of S.

(5) $\big(f \wedge^k g\big) \leq \big(f \circ^k g\big) \wedge \big(g \circ^k f\big)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal g of S.

(6) $\big(f \wedge^k g\big) \leq \big(f \circ^k g\big) \wedge \big(g \circ^k f\big)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal g of S.

(7) $\big(f \wedge^k g\big) \leq \big(f \circ^k g\big) \wedge \big(g \circ^k f\big)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal g of S.

(8) $\big(f \wedge^k g\big) \leq \big(f \circ^k g\big) \wedge \big(g \circ^k f\big)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal g of S.

(9) $\big(f \wedge^k g\big) \leq \big(f \circ^k g\big) \wedge \big(g \circ^k f\big)$ for all $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals f, g of S.

(10) $\big(f \wedge^k g\big) \leq \big(f \circ^k g\big) \wedge \big(g \circ^k f\big)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal g of S.

(11) $\left(f \wedge^k g\right) \leq \left(f \circ^k g\right) \wedge \left(g \circ^k f\right)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal g of S.

(12) $\left(f \wedge^k g\right) \leq \left(f \circ^k g\right) \wedge \left(g \circ^k f\right)$ for all $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals f, g of S.

(13) $\left(f \wedge^k g\right) \leq \left(f \circ^k g\right) \wedge \left(g \circ^k f\right)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal g of S.

(14) $\left(f \wedge^k g\right) \leq \left(f \circ^k g\right) \wedge \left(g \circ^k f\right)$ for all $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideals f, g of S.

Proof. (1) $\Rightarrow$ (14) Let f, g be any $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideals of S and $x \in S$. Then there exist $a, b, c \in S$ such that $x = xax$ and $x = bxxc$. Thus $x = xax = xaxax = xa\,(bxxc)\,ax = (xabx)\,(xcax)$. Therefore

$$\begin{aligned}
&(f \circ^k g)(x) = (f \circ g)(x) \vee \tfrac{1-k}{2} \\
&= \Big(\bigvee_{a=yz} \{f(y) \wedge g(z)\}\Big) \vee \tfrac{1-k}{2} \\
&\geq (f(xabx) \wedge g(xcax)) \vee \tfrac{1-k}{2} \\
&= \left\{(f(xabx) \vee \tfrac{1-k}{2}) \wedge (g(xcax) \vee \tfrac{1-k}{2})\right\} \vee \tfrac{1-k}{2} \\
&\geq \{(f(x) \wedge g(x)\} \vee \tfrac{1-k}{2} = \left(f \wedge^k g\right)(x).
\end{aligned}$$

Similarly, we can show that $\left(f \wedge^k g\right) \leq \left(g \circ^k f\right)$.

Thus $\left(f \wedge^k g\right) \leq \left(f \circ^k g\right) \wedge \left(g \circ^k f\right)$ for all $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideals f, g of S.

(14) $\Rightarrow$ (13) $\Rightarrow$ (12) $\Rightarrow$ (10) $\Rightarrow$ (9) $\Rightarrow$ (3) $\Rightarrow$ (2), (14) $\Rightarrow$ (11) $\Rightarrow$ (10), (14) $\Rightarrow$ (8) $\Rightarrow$ (7) $\Rightarrow$ (6) $\Rightarrow$ (2) and (14) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (1) Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right and g be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S. Then by hypothesis

$\left(f \wedge^k g\right) \leq \left(f \circ^k g\right) \wedge \left(g \circ^k f\right)$ but $\left(f \wedge^k g\right) \geq \left(f \circ^k g\right)$ is always true. Hence $\left(f \wedge^k g\right) = \left(f \circ^k g\right)$ and $\left(f \wedge^k g\right) \leq \left(g \circ^k f\right)$, this shows that S is both regular and intra regular. ■

3.5 Semisimple Semigroups in terms of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals

In this section we characterize Semisimple semigroups by the properties of their $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals, quasi-ideals and bi-ideals.

Theorem 178 *In a semisimple semigroup S, a fuzzy subset f of S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of S if and only if it is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideal of S.*

Proof. Let S be a semisimple semigroup and f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideal of S. Then for any $x, y \in S$ there exist $a, b, c, d \in S$ such that $x = (axb)\,(cxd)$. Thus we have

$f(xy) \vee \frac{1-k}{2} = f((axbcxd)y) \vee \frac{1-k}{2} = f((ax(bc))x(dy)) \vee \frac{1-k}{2} \geq f(x)$.

Similarly $f(xy) \vee \frac{1-k}{2} \geq f(y)$. Hence f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of S.

Conversely, assume that f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of S. Then f is always an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideal of S. ■

Theorem 179 *For a semigroup S the following assertions are equivalent*

(1) S is semisimple.

(2) $\left(f \circ^k f\right) = f^k$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal f of S.

(3) $\left(f \circ^k f\right) = f^k$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideal f of S.

(4) $\left(f \wedge^k g\right) = \left(f \circ^k g\right)$ for all $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals f, g of S.

(5) $\left(f \wedge^k g\right) = \left(f \circ^k g\right)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideal g of S.

(6) $\left(f \wedge^k g\right) = \left(f \circ^k g\right)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal g of S.

(7) $\left(f \wedge^k g\right) = \left(f \circ^k g\right)$for all $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideals f, g of S.

Proof. (1) $\Rightarrow$ (7) Let S be a semisimple semigroup and f, g be $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideals of S. Let $x \in S$. Then there exist $a, b, c, d \in S$ such that $x = (axb)(cxd)$. Thus we have

$$\begin{aligned}
\left(f \circ^k g\right)(x) &= \left(\left(\bigvee_{x=yz} (f(y) \wedge g(z))\right) \vee \tfrac{1-k}{2}\right) \\
&\geq \left((f(axb) \wedge g(cxd)) \vee \tfrac{1-k}{2}\right) \\
&= \left(\left(f(axb) \vee \tfrac{1-k}{2}\right) \wedge \left(g(cxd) \vee \tfrac{1-k}{2}\right)\right) \\
&\geq f(x) \wedge g(x) \vee \tfrac{1-k}{2} \\
&= ((f \wedge g)(x)) \vee \tfrac{1-k}{2} \\
&= \left(f \wedge^k g\right)(x).
\end{aligned}$$

Thus $\left(f \circ^k g\right) \geq \left(f \wedge^k g\right)$. Since every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy interior-ideal of S in a semisimple semigroup is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of S, so$\left(f \circ^k g\right) \leq \left(f \wedge^k g\right)$. Hence $\left(f \circ^k g\right) = \left(f \wedge^k g\right)$.

(7) $\Rightarrow$ (6) $\Rightarrow$ (4) $\Rightarrow$ (2), (7) $\Rightarrow$ (3) $\Rightarrow$ (2) and (7) $\Rightarrow$ (5) $\Rightarrow$ (4) are obvious.

(2) $\Rightarrow$ (1) Let A be any ideal of S. Then C_A is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of S. Thus by hypothesis $\left(C_A \circ^k C_A\right) = \left(C_A^k\right)$, that is $AA = A$. Hence S is a semisimple semigroup. ■

Chapter 4

Characterizations of Semigroups by the Properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Ideals

In this chapter, generalizing the notions of $(\in, \in \vee q)$-fuzzy left (right) ideal, $(\in, \in \vee q)$-fuzzy quasi-ideal, and $(\in, \in \vee q)$-fuzzy bi-ideal, the notions of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of semigroup are defined. Regular, intra-regular, and semisimple semigroups are characterized by the properties of these fuzzy ideals.

4.1 (Φ, Ψ) - fuzzy ideals

Let $\gamma, \delta \in [0,1]$ be such that $\gamma < \delta$. For a fuzzy point x_t and a fuzzy subset f of X, we say that:

(1) $x_t \in_\gamma f$ if $f(x) \geq t > \gamma$.

(2) $x_t q_\delta f$ if $f(x) + t > 2\delta$.

(3) $x_t \in_\gamma \vee q_\delta f$ if $x_t \in_\gamma f$ or $x_t q_\delta f$.

(4) $x_t \in_\gamma \wedge q_\delta f$ if $x_t \in_\gamma f$ and $x_t q_\delta f$.

(5) $x_t \overline{\Phi} f$ if $x_t \Phi f$ does not hold for $\Phi \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$.

Throughout this chapter $\gamma, \delta \in [0,1]$, where $\gamma < \delta$. $\Phi, \Psi \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$ and $\Phi \neq \in_\gamma \wedge q_\delta$.

Let f be a fuzzy subset of a semigroup S such that $f(x) \leq \delta$. Let $x \in S$ and $t \in [0,1]$ be such that $x_t \in_\gamma \wedge q_\delta$. Then $f(x) \geq t > \gamma$ and $f(x) + t > 2\delta$. It follows that $2\delta < f(x) + t \leq f(x) + f(x) = 2f(x)$, that is $f(x) > \delta$. This means that $\{x_t : x_t \in_\gamma \wedge q_\delta f\} = \emptyset$. Therefore we are not taking $\Phi = \in_\gamma \wedge q_\delta$.

Definition 180 *A fuzzy subset f of a semigroup S is called an (Φ, Ψ)-fuzzy subsemigroup of S, if it satisfies*

$(F1)$ $x_t \Phi f$ and $y_r \Phi f \Rightarrow (xy)_{\min\{t,r\}} \Psi f$ for all $x, y \in S$ and $t, r \in (\gamma, 1]$.

Definition 181 *A fuzzy subset f of a semigroup S is called an (Φ, Ψ)-fuzzy left (right) ideal of S, if it satisfies*

$(F2)$ $x_t \Phi f \Rightarrow (yx)_t \Psi f$ (resp. $(xy)_t \Psi f$) for all $x, y \in S$ and $t \in (\gamma, 1]$.

A fuzzy subset f of a semigroup S is called an (Φ, Ψ)-fuzzy ideal of S, if it is both (Φ, Ψ)-fuzzy left ideal and (Φ, Ψ)-fuzzy right ideal of S.

Definition 182 *A fuzzy subset f of a semigroup S is called an (Φ, Ψ)-fuzzy interior-ideal of S, if it satisfies*

$(F3)$ $x_t \Phi f \Rightarrow (yxz)_t \Psi f$ for all $x, y, z \in S$ and $t \in (\gamma, 1]$.

Definition 183 *A fuzzy subset f of a semigroup S is called an (Φ, Ψ)-fuzzy generalized bi-ideal of S, if it satisfies*

$(F4)$ $x_t \Phi f$ and $y_r \Phi f \Rightarrow (xzy)_{\min\{t,r\}} \Psi f$ for all $x, y, z \in S$ and $t, r \in (\gamma, 1]$.

Definition 184 *A fuzzy subset f of a semigroup S is called an (Φ, Ψ)-fuzzy bi-ideal of S, if it satisfies conditions $(F1)$ and $(F4)$.*

Theorem 185 *Let $2\delta = 1 + \gamma$ and f be an (Φ, Ψ)-fuzzy subsemigroup of S. Then $f_\gamma = \{x \in S : f(x) > \gamma\}$ is a subsemigroup of S.*

Proof. Let $x, y \in f_\gamma$. Then $f(x) > \gamma$ and $f(y) > \gamma$. Suppose that $f(xy) \leq \gamma$. If $\Phi \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$, then $x_{f(x)} \Phi f$ and $y_{f(x)} \Phi f$ but $(xy)_{\min\{f(x), f(y)\}} \overline{\Psi} f$ for every $\Psi \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$ (because $f(xy) \leq \gamma < \min\ \{f(x), f(y)\}$, so $(xy)_{\min\{f(x), f(y)\}} \overline{\in} f$ and $f(xy) + \min\{f(x), f(y)\} \leq \gamma + \min\{f(x), f(y)\} \leq \gamma + 1 = 2\delta$, so $(xy)_{\min\{f(x), f(y)\}} \overline{q_\delta} f$), a contradiction. Hence $f(xy) > \gamma$, that is $xy \in f_\gamma$. If $\Phi = q_\delta$ then $x_1 q_\delta f$ and $y_1 q_\delta f$ (because $f(x) + 1 > 1 + \gamma = 2\delta$ and $f(y) + 1 > 1 + \gamma = 2\delta$). But $(xy)_1 \overline{\Psi} f$ for every $\Psi \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$ (because $f(xy) \leq \gamma$, so $(xy)_1 \overline{\in} f$ and $f(xy) + 1 \leq \gamma + 1 = 2\delta$, so $(xy)_1 \overline{q_\delta} f$), a contradiction. Hence $f(xy) > \gamma$, that is $xy \in f_\gamma$. This shows that f_γ is a subsemigroup of S. ■

Theorem 186 *Let $2\delta = 1 + \gamma$ and f be an (Φ, Ψ)-fuzzy left (right) ideal of S. Then $f_\gamma = \{x \in S : f(x) > \gamma\}$ is a left (right) ideal of S.*

Proof. Let f be an (Φ,Ψ)-fuzzy left ideal of S and $x \in f_\gamma$. Suppose there exists $y \in S$ such that $f(yx) \leq \gamma$. If $\Phi \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$, then $x_{f(x)}\Phi f$, but $f(yx) \leq \gamma < f(x)$, so $(yx)_{f(x)}\overline{\in_\gamma} f$. Also $f(yx) + f(x) \leq \gamma + f(x) \leq \gamma + 1 = 2\delta$, so $(yx)_{f(x)}\overline{q_\delta} f$. This shows that $(yx)_{f(x)}\overline{\Psi} f$ for every $\Psi \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, which is a contradiction. Hence $f(yx) > \gamma$, that is $yx \in f_\gamma$. If $\Phi = q_\delta$ then $x_1 q_\delta f$. But $f(yx) \leq \gamma$, so $(yx)_1 \overline{\in}_\gamma f$ and $f(yx) + 1 \leq \gamma + 1 = 2\delta$, so $(yx)_1 \overline{q}_\gamma f$. Thus $(yx)_1 \overline{\Psi} f$ for every$\Psi \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, which is a contradiction. Hence $f(yx) > \gamma$, that is $yx \in f_\gamma$. This shows that f_γ is a left ideal of S. ∎

Theorem 187 *(1) Let $2\delta = 1 + \gamma$ and f be an (Φ,Ψ)-fuzzy generalized bi-ideal of S. Then $f_\gamma$is a generalized bi-ideal of S.*

(2) Let $2\delta = 1 + \gamma$ and f be an (Φ,Ψ)-fuzzy bi-ideal of S. Then f_γ is a bi-ideal of S.

(3) Let $2\delta = 1 + \gamma$ and f be an (Φ,Ψ)-fuzzy interior-ideal of S. Then f_γ is an interior-ideal of S.

Proof. The proof is similar to the proof of Theorem 185. ∎

Theorem 188 *Let $2\delta = 1 + \gamma$ and A be a non-empty subset of S. Then A is a subsemigroup of S if and only if the fuzzy subset f of S defined by*

$$f(x) = \begin{cases} \geq \delta & \text{if } x \in A \\ \leq \gamma & \text{if } x \notin A \end{cases}$$

is an $(\Phi, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

Proof. Let A be a subsemigroup of S.

(1) Let $x, y \in S$ and $t, r \in (\gamma, 1]$ be such that $x_t \in_\gamma f, y_r \in_\gamma f$. Then $f(x) \geq t > \gamma$ and $f(y) \geq r > \gamma$. Thus $x, y \in A$ and so $xy \in A$, that is $f(xy) \geq \delta$. If $\min\{t, r\} \leq \delta$, then $f(xy) \geq \delta \geq \min\{t, r\} > \gamma$. This implies $(xy)_{\min\{t,r\}} \in_\gamma f$. If $\min\{t, r\} > \delta$, then $f(xy) + \min\{t, r\} > \delta + \delta = 2\delta$. This implies $(xy)_{\min\{t,r\}} q_\delta f$. Hence f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

(2) Let $x, y \in S$ and $t, r \in (\gamma, 1]$ be such that $x_t q_\delta f, y_r q_\delta f$. Then $f(x) + t > 2\delta$ and $f(y) + r > 2\delta$. This implies $f(x) > 2\delta - t \geq 2\delta - 1 = \gamma$ and $f(y) > 2\delta - r \geq 2\delta - 1 = \gamma$. Thus $x, y \in A$ and so $xy \in A$. This implies $f(xy) \geq \delta$. Now if $\min\{t, r\} \leq \delta$, then $f(xy) \geq \delta \geq \min\{t, r\} > \gamma$, so $(xy)_{\min\{t,r\}} \in f$. If $\min\{t, r\} > \delta$, then $f(xy) + \min\{t, r\} > \delta + \delta = 2\delta$. Thus $(xy)_{\min\{t,r\}} q_\delta f$. Hence f is a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

(3) Let $x, y \in S$ and $t, r \in (\gamma, 1]$ be such that $x_t \in_\gamma f$ and $y_r q_\delta f$. Then $f(x) \geq t > \gamma$ and $f(y) + r > 2\delta$. Thus $f(y) + r > 2\delta \Rightarrow f(y) > 2\delta - r \geq 2\delta - 1 = \gamma$. This implies

$x, y \in A$ and so $xy \in A$. Analogous to (1) and (2) we obtain $(xy)_{\min\{t,r\}} \in_\gamma \vee q_\delta f$, that is f is an $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

Conversely, assume that f is an $(\Phi, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Then $A = f_\gamma$. It follows from Theorem 185 that A is a subsemigroup of S. ■

Corollary 189 *Let $2\delta = 1 + \gamma$ and A be a non-empty subset of S. Then A is a subsemigroup of S if and only if C_A, the characteristic function of A is an $(\Phi, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.*

Similarly we can prove the following theorem.

Theorem 190 *Let $2\delta = 1 + \gamma$ and A be a non-empty subset of S. Define a fuzzy subset f of S as*

$$f(x) = \begin{cases} \geq \delta & \text{if } x \in A \\ \leq \gamma & \text{if } x \notin A \end{cases}.$$

Then

(1) f is an $(\Phi, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S if and only if A is a left (right) ideal of S.

(2) f is an $(\Phi, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S if and only if A is a generalized bi-ideal (bi-ideal) of S.

(3) f is an $(\Phi, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S if and only if A is an interior-ideal of S.

Corollary 191 *(1) Let $2\delta = 1 + \gamma$ and A be a non-empty subset of S. Then A is a left (right) ideal of S if and only if C_A, the characteristic function of A is an $(\Phi, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S.*

(2) Let $2\delta = 1 + \gamma$ and A be a non-empty subset of S. Then A is a generalized bi-ideal (bi-ideal) of S if and only if C_A, the characteristic function of A is an $(\Phi, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S.

(3) Let $2\delta = 1 + \gamma$ and A be a non-empty subset of S. Then A is an interior-ideal of S if and only if C_A, the characteristic function of A is an $(\Phi, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S.

It is easy to see that each (Φ, Ψ) - fuzzy subsemigroup (left ideal, right ideal, generalized bi-ideal, bi-ideal, interior-ideal) of S is an $(\Phi, \in \vee q)$-fuzzy subsemigroup (left ideal, right ideal, generalized bi-ideal, bi-ideal, interior-ideal) of S.

The following example shows that the converse is not true.

Example 192 *Consider the semigroup $S = \{a, b, c, d\}$ with the following multiplication table:*

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define a fuzzy subset f of S as following
$f(a) = 0.5$, $f(b) = 0.4$, $f(c) = 0.6$ and $f(d) = 0.3$.
Thus

$$U(f;t) = \begin{cases} S & \text{if } 0 < t \leq 0.3 \\ \{a, b, c\} & \text{if } 0.3 < t \leq 0.4 \\ \{a, c\} & \text{if } 0.4 < t \leq 0.5 \\ \{c\} & \text{if } 0.5 < t \leq 0.6 \\ \emptyset & \text{if } 0.6 < t \end{cases}.$$

Then
(1) f is an $(\in_0, \in_0 \vee q_{0.4})$-fuzzy ideal of S.
(2) f is not an $(\in_0, \in_0)$-fuzzy ideal of S, because $c_{0.5} \in_0 f$ but $(cc)_{0.5} \overline{\in_0} f$.
(3) f is not an $(\in_0, q_{0.4})$-fuzzy ideal of S, because $c_{0.25} \in_0 f$ but $(cc)_{0.25} \overline{q_{0.4}} f$.

Theorem 193 (1) *Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.*

(2) Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S.

(3) Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S.

(4) Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S.

Proof. Proof follows from the fact that if $x_t \in_\gamma f$ then $x_t \in_\gamma \vee q_\delta f$. ■

Theorem 194 (1) *Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.*

(2) Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S.

(3) Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S.

(4) Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S.

Proof. We prove only (1). Proofs of $(2), (3)$ and (4) are similar to the proof of (1).

Let f be a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Let $x, y \in S$ and $t, r \in (\gamma, 1]$ be such that $x_t \in_\gamma f, y_r \in_\gamma f$. Then $f(x) \geq t > \gamma$ and $f(y) \geq r > \gamma$. Suppose $(xy)_{\min\{t,r\}}\overline{\in_\gamma \vee q_\delta}$ then $f(xy) < \min\{t, r\}$ and $f(xy) + \min\{t, r\} \leq 2\delta \Rightarrow f(xy) < \delta$. Now $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$. Then select an $s \in (\gamma, 1]$ such that

$2\delta - \max\{f(xy), \gamma\} > s \geq 2\delta - \min\{f(x), f(y), \delta\}$

$\Rightarrow 2\delta - f(xy) \geq 2\delta - \max\{f(xy), \gamma\} > s \geq \min\{2\delta - f(x), 2\delta - f(y), \delta\}$

$\Rightarrow f(x) + s \geq 2\delta,\ f(y) + s \geq 2\delta$

and $f(xy) + s < 2\delta$ and $f(xy) < \delta < s$. Hence $x_s q_\delta f, y_s q_\delta f$ but $(xy)_s\overline{\in_\gamma \vee q_\delta} f$. This is a contradiction. Hence $(xy)_{\min\{t,r\}} \in_\gamma \vee q_\delta f$, that is f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. ■

The above discussion shows that every (α, β)-fuzzy subsemigroup (left ideal, right ideal, generalized bi-ideal, bi-ideal, interior-ideal) of a semigroup S is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup (left ideal, right ideal, generalized bi-ideal, bi-ideal, interior-ideal) of S. Also every $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup (left ideal, right ideal, generalized bi-ideal, bi-ideal, interior-ideal) of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup (left ideal, right ideal, generalized bi-ideal, bi-ideal, interior-ideal) of S. Thus in the theory of (α, β)-fuzzy subsemigroups (left ideals, right ideals, generalized bi-ideals, bi-ideals, interior-ideals) of S, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroups (left ideals, right ideals, generalized bi-ideals, bi-ideals, interior-ideals) play central role.

4.2 $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals

We start this section with the following theorem.

Theorem 195 *For any fuzzy subset f of a semigroup S and for all $x, y, z \in R$ and $t, r \in (\gamma, 1]$ $(4.1a)$ is equivalent to $(4.1b)$, $(4.2a)$ is equivalent to $(4.2b)$, $(4.3a)$ is equivalent to $(4.3b)$ and $(4.4a)$ is equivalent to $(4.4b)$, where*

$(4.1a)$ $x_t,\ y_r \in_\gamma f \Rightarrow (xy)_{\min\{t,r\}} \in_\gamma \vee q_\delta f$.

$(4.1b)$ $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$.

$(4.2a)$ $x_t \in_\gamma f \Rightarrow (yx)_t \in_\gamma \vee q_\delta f \qquad ((xy)_t \in_\gamma \vee q_\delta f)$.

$(4.2b)$ $\max\{f(yx), \gamma\} \geq \min\{f(x), \delta\} \qquad (\max\{f(xy), \gamma\} \geq \min\{f(x), \delta\}$.

$(4.3a)$ $x_t,\ y_r \in_\gamma f \Rightarrow (xzy)_{\min\{t,r\}} \in_\gamma \vee q_\delta f$.

(4.3b) $\max\{f(xzy), \gamma\} \geq \min\{f(x), f(y), \delta\}$.

(4.4a) $x_t \in_\gamma f \Rightarrow (yxz)_t \in_\gamma \vee q_\delta f$.

(4.4b) $\max\{f(yxz), \gamma\} \geq \min\{f(x), \delta\}$.

Proof. We prove only (4.1a) $\Leftrightarrow$ (4.1b). Proofs of the remaining parts are similar to this.

(4.1a) $\Rightarrow$ (4.1b) Let f be a fuzzy subset of S which satisfies (4.1a). Let $x, y \in S$ be such that $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$. Select $t \in (\gamma, 1]$ such that $\max\{f(xy), \gamma\} < t \leq \min\{f(x), f(y), \delta\}$. Then $f(x) \geq t > \gamma, f(y) \geq t > \gamma, f(xy) < t$ and $f(xy) + t < \delta + \delta = 2\delta$, that is $x_t \in_\gamma f, y_t \in_\gamma f$ but $(xy)_t \overline{\in_\gamma \vee q_\delta} f$. Which is a contradiction. Hence $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$.

(4.1b) $\Rightarrow$ (4.1a) Let f be a fuzzy subset of S which satisfies (4.1b). Let $x, y \in S$ and $t, r \in (\gamma, 1]$ be such that $x_t \in_\gamma f, y_r \in_\gamma f$ but $(xy)_{\min\{t,r\}} \overline{\in_\gamma \vee q_\delta} f$. Then

$f(x) \geq t > \gamma$ (1)

$f(y) \geq r > \gamma$ (2)

$f(xy) < \min\{t, r\}$ (3)

and $f(xy) + \min\{t, r\} \leq 2\delta$ (4).

It follows from (3) and (4) that $f(xy) < \delta$. Now $\max\{f(xy), \gamma\} < \delta$ and $\max\{f(xy), \gamma\} < \min\{f(x), f(y)\}$. Thus $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$. Which is a contradiction. Hence $(xy)_{\min\{t,r\}} \in_\gamma \vee q_\delta f$. ■

From the above theorem we deduce the following definition.

Definition 196 *A fuzzy subset f of a semigroup S is called an*

- $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S if it satisfies (4.1b).
- $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S if it satisfies (4.2b).
- $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S if it satisfies (4.3b).
- $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if it satisfies (4.1b) and (4.3b).
- $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S if it satisfies (4.4b).

Definition 197 *Let f be a fuzzy subset of a semigroup S. We define*

$f_r = \{x \in S : x_r \in_\gamma f\} = \{x \in S : f(x) \geq r > \gamma\} = U(f; r)$.

$f_r^\delta = \{x \in S : x_r q_\delta f\} = \{x \in S : f(x) + r > 2\delta\}$.

$[f]_r^\delta = \{x \in S : x_r \in_\gamma \vee q_\delta f\} = f_r \cup f_r^\delta$ for all $r \in (\gamma, 1]$.

Theorem 198 *Let f be a fuzzy subset of a semigroup S. Then*

(1) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S if and only if $U(f;t)(\neq \emptyset)$ is a subsemigroup of S for all $t \in (\gamma, \delta]$.

(2) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S if and only if $U(f;t)(\neq \emptyset)$ is a left (right) ideal of S for all $t \in (\gamma, \delta]$.

(3) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S if and only if $U(f;t)(\neq \emptyset)$ is a generalized bi-ideal (bi-ideal) of S for all $t \in (\gamma, \delta]$.

(4) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S if and only if $U(f;t)(\neq \emptyset)$ is a interior-ideal of S for all $t \in (\gamma, \delta]$.

Proof. (1) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S and $x, y \in U(f;t)$ for some $t \in (\gamma, \delta]$. Then $f(x) \geq t$ and $f(y) \geq t$. By hypothesis $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\} \geq \min\{t, \delta\} = t \Rightarrow f(xy) \geq t$. Hence $xy \in U(f, t)$, that is $U(f;t)$ is a subsemigroup of S.

Conversely, assume that $U(f;t) \neq \emptyset$ is a subsemigroup of S for all $t \in (\gamma, \delta]$. Suppose that there exist $x, y \in S$ such that $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$. Choose $t \in (\gamma, \delta]$ such that $\max\{f(xy), \gamma\} < t \leq \min\{f(x), f(y), \delta\}$. This implies $f(x) \geq t$, $f(y) \geq t$ and $f(xy) < t$, that is $x, y \in U(f;t)$ but $xy \notin U(f;t)$. Which is a contradiction. Hence $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$, that is f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

Similarly we can prove (2), (3) and (4). ∎

From the above Theorem it follows that

(1) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S.

(2) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal ideal of S.

(3) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S.

Theorem 199 *Let f be a fuzzy subset of a semigroup S and $2\delta = 1 + \gamma$. Then*

(1) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S if and only if $f_r^\delta(\neq \emptyset)$ is a subsemigroup of S for all $r \in (\delta, 1]$

(2) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left(right) ideal of S if and only if $f_r^\delta(\neq \emptyset)$ is a left(right) ideal of S for all $r \in (\delta, 1]$

(3) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S if and only if $f_r^\delta(\neq \emptyset)$ is a generalized bi-ideal (bi-ideal) of S for all $r \in (\delta, 1]$

(4) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S if and only if $f_r^\delta(\neq \emptyset)$ is a interior-ideal of S for all $r \in (\delta, 1]$

Proof. (1) Suppose f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S and $x, y \in f_r^\delta$. Then $x_r, y_r q_\delta f$, that is $f(x) + r > 2\delta$ and $f(y) + r > 2\delta \Rightarrow f(x) > 2\delta - r \geq 2\delta - 1 = \gamma$

and similarly $f(y) > \gamma$. By hypothesis $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$

$\Rightarrow f(xy) \geq \min\{f(x), f(y), \delta\}$

$\Rightarrow f(xy) > \min\{2\delta - r, 2\delta - r, \delta\}$.

Since $r \in (\delta, 1]$ so $\delta < r \leq 1 \Rightarrow 2\delta - r < \delta$. Thus $f(xy) > 2\delta - r \Rightarrow f(xy) + r > 2\delta \Rightarrow xy \in f_r^\delta$. Hence f_r^δ is a subsemigroup of S.

Conversely, assume that $f_r^\delta (\neq \emptyset)$ is a subsemigroup of S for all $r \in (\delta, 1]$. Let $x, y \in S$ be such that $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\} \Rightarrow 2\delta - \min\{f(x), f(y), \delta\} < 2\delta - \max\{f(xy), \gamma\}$

$\Rightarrow \max\{2\delta - f(x), 2\delta - f(y), \delta\} < \min\{2\delta - f(xy), 2\delta - \gamma\}$.

Take $r \in (\delta, 1]$ such that $\max\{2\delta - f(x), 2\delta - f(y), \delta\} < r \leq \min\{2\delta - f(xy), 2\delta - \gamma\}$. Then $2\delta - f(x) < r, 2\delta - f(y) < r$ and $r \leq 2\delta - f(xy) \Rightarrow f(x) + r > 2\delta$ and $f(y) + r > 2\delta$ but $f(xy) + r \leq 2\delta$, that is $x_r q_\delta f, y_r q_\delta f$ but $(xy)_r \overline{q}_\delta f$. Which is a contradiction. Hence $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$.

Similarly, we can prove the parts (2), (3) and (4). ■

Theorem 200 *Let f be a fuzzy subset of a semigroup S and $2\delta = 1 + \gamma$. Then*

(1) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S if and only if $[f]_r^\delta (\neq \emptyset)$ is a subsemigroup of S for all $r \in (\gamma, 1]$.

(2) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left(right) ideal of S if and only if $[f]_r^\delta (\neq \emptyset)$ is a left(right) ideal of S for all $r \in (\gamma, 1]$.

(3) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S if and only if $[f]_r^\delta (\neq \emptyset)$ is a generalized bi-ideal (bi-ideal) of S for all $r \in (\gamma, 1]$.

(4) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S if and only if $[f]_r^\delta (\neq \emptyset)$ is a interior-ideal of S for all $r \in (\gamma, 1]$.

Proof. (1) Suppose f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S and $x, y \in [f]_r^\delta$. Then $x_r \in_\gamma \vee q_\delta f$ and $y_r \in_\gamma \vee q_\delta f$, that is $f(x) \geq r > \gamma$ or $f(x) + r > 2\delta$ and $f(y) \geq r > \gamma$ or $f(y) + r > 2\delta$. Thus $f(x) \geq r > \gamma$ or $f(x) > 2\delta - r \geq 2\delta - 1 = \gamma$ and $f(y) \geq r > \gamma$ or $f(y) > 2\delta - r \geq 2\delta - 1 = \gamma$. If $r \in (\gamma, \delta]$, then $\gamma < r \leq \delta$. This implies $2\delta - r \geq \delta \geq r$. Then it follows from the above that $f(x) \geq r$ and $f(y) \geq r$. By hypothesis

$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$

$\Rightarrow f(xy) \geq \min\{f(x), f(y), \delta\} \geq \min\{r, r, r\} = r$

and so $(xy)_r \in_\gamma f$. Thus $xy \in [f]_r^\delta$.

If $r \in (\delta, 1]$, then $\delta < r \leq 1$. This implies $2\delta - r < \delta < r$. Then it follows that $f(x) > 2\delta - r$ and $f(y) > 2\delta - r$.

Now by hypothesis $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$

$\Rightarrow f(xy) \geq \min\{f(x), f(y), \delta\} > \min\{2\delta - r, 2\delta - r, 2\delta - r\} = 2\delta - r$

$\Rightarrow f(xy) + r > 2\delta \Rightarrow (xy)_r q_\delta f$.

This implies $xy \in [f]_r^\delta$. Thus $[f]_r^\delta$ is a subsemigroup of S.

Conversely, assume that $[f]_r^\delta$ is a subsemigroup of S for all $r \in (\gamma, 1]$. Let $x, y \in S$ be such that $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$. Select $r \in (\gamma, 1]$ such that $\max\{f(xy), \gamma\} < r \leq \min\{f(x), f(y), \delta\}$. Then $x_r \in_\gamma f, y_r \in_\gamma f$ but $(xy)_r \overline{\in_\gamma \vee q_\delta} f$. Which contradicts our hypothesis. Hence $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$, that is f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

Similarly, we can prove the parts (2), (3) and (4). ■

Theorem 201 *(1) The intersection of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroups of S is again an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.*

(2) The intersection of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals of S is again an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S.

(3) The intersection of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S is again an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S.

(4) The intersection of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideals of S is again an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S.

Proof. (1) Let $\{f_i\}_{i\in I}$ be a family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroups of S and $x, y \in S$. Then $((\bigwedge_{i\in I} f_i)(xy)) \vee \gamma = (\bigwedge_{i\in I} f_i(xy)) \vee \gamma = (\bigwedge_{i\in I}((f_i(xy)) \vee \gamma)) \geq (\bigwedge_{i\in I}(\min\{f_i(x), f_i(y), \delta\})$

$= (\bigwedge_{i\in I} f_i(x)) \wedge (\bigwedge_{i\in I} f_i(y)) \wedge \delta = ((\bigwedge_{i\in I} f_i)(x)) \wedge ((\bigwedge_{i\in I} f_i)(y)) \wedge \delta$.

Thus $\bigwedge_{i\in I} f_i$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroups of S.

Similarly, we can prove the parts (2), (3) and (4). ■

Theorem 202 *The union of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals of S is again an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S.*

Proof. Let $\{f_i\}_{i\in I}$ be a family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals of S and $x, y \in S$. Then $((\bigvee_{i\in I} f_i)(xy)) \vee \gamma = (\bigvee_{i\in I} f_i(xy)) \vee \gamma = \bigvee_{i\in I}((f_i(xy)) \vee \gamma) \geq \bigvee_{i\in I}(f_i(x) \wedge \delta\}) = (\bigvee_{i\in I} f_i(x)) \wedge \delta = (\bigvee_{i\in I} f_i)(x)) \wedge \delta$. Thus $\bigvee_{i\in I} f_i$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. ■

Proposition 203 *Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S. Then fg is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S.*

Proof. Let f, g be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left and right ideal of S, respectively. Let $x, y \in S$. Then $(fg)(y) \wedge \delta = (\bigvee_{y=ab}(f(a) \wedge g(b)) \wedge \delta = \bigvee_{y=ab}(f(a) \wedge g(b) \wedge \delta) = \bigvee_{y=ab}((f(a) \wedge \delta) \wedge g(b)) \leq \bigvee_{y=ab}((f(xa) \vee \gamma) \wedge g(b))$

$= \bigvee_{y=ab} ((f(xa) \wedge g(b)) \vee \gamma \leq \bigvee_{xy=cd} (f(c) \wedge g(d)) \vee \gamma = (fg)(xy) \vee \gamma.$

Similarly, we can show that $(fg)(x) \wedge \delta \leq (fg)(xy) \vee \gamma$. This shows that fg is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S. ■

Next we show that if f and g are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of S, then $fg \nleq f \wedge g$.

Example 204 *Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table*

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define fuzzy subsets f and g of S by $f(a) = 0.6,\ f(b) = 0.3,\ f(c) = 0.4,\ f(d) = 0.1,$ $g(a) = 0.65,\ \ g(b) = 0.3,\ \ g(c) = 0.4,\ \ g(d) = 0.2.$

Then

$$U(f;t) = \begin{cases} \{a,b,c,d\} & 0 < t \leq 0.1 \\ \{a,b,c\} & 0.1 < t \leq 0.3 \\ \{a,c\} & 0.3 < t \leq 0.4 \\ \{a\} & 0.4 < t \leq 0.6 \\ \emptyset & 0.6 < t \end{cases}$$

$$U(g;t) = \begin{cases} \{a,b,c,d\} & 0 < t \leq 0.2 \\ \{a,b,c\} & 0.2 < t \leq 0.3 \\ \{a,c\} & 0.3 < t \leq 0.4 \\ \{a\} & 0.4 < t \leq 0.65 \\ \emptyset & 0.65 < t \end{cases}$$

By Theorem 198, f and g are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of S for $\gamma = 0$ and $\delta = 0.3$. But $fg(b) = \bigvee_{b=xy} \{f(x) \wedge g(y)\} = \bigvee\{0.4, 0.1, 0.1\} = 0.4 \nleq (f \wedge g)(b) = 0.3$. Hence $fg \nleq f \wedge g$ in general.

Definition 205 *Let f, g be fuzzy subsets of a semigroup S. We define the fuzzy subsets $f^*, f \wedge^* g, f \vee^* g$ and $f * g$ of S as follows:*

$f^*(x) = (f(x) \vee \gamma) \wedge \delta$
$(f \wedge^* g)(x) = (((f \wedge g)(x)) \vee \gamma) \wedge \delta$
$(f \vee^* g)(x) = (((f \vee g)(x)) \vee \gamma) \wedge \delta$
$(f * g)(x) = (((fg)(x)) \vee \gamma) \wedge \delta$
for all $x \in S$.

Lemma 206 *Let f and g be fuzzy subsets of a semigroup S. Then the following hold:*

(1) $f \wedge^* g = f^* \wedge g^*$
(2) $f \vee^* g = f^* \vee g^*$
(3) $f * g \geq f^* g^*$
(4) $f \overset{*}{\wedge} g \geq f * g$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal g of S.

Proof. Let $x \in S$.

(1) $(f \wedge^* g)(x) = (((f \wedge g)(x)) \vee \gamma) \wedge \delta = ((f(x) \wedge g(x)) \vee \gamma) \wedge \delta$
$= ((f(x) \vee \gamma) \wedge (g(x)) \vee \gamma)) \wedge \delta = ((f(x) \vee \gamma) \wedge \delta) \wedge ((g(x) \vee \gamma) \wedge \delta)$
$= f^*(x) \wedge g^*(x) = (f^* \wedge g^*)(x)$.

(2) The proof is similar to the proof of part (1).

(3) If x is not expressible as $x = yz$ for all $y, z \in S$, then $(f*g)(x) = (((fg)(x)) \vee \gamma) \wedge \delta = \gamma \wedge \delta \geq 0 = f^* g^*(x)$. Otherwise $(f*g)(x) = ((fg(x)) \vee \gamma) \wedge \delta = \left(\left(\bigvee_{x=yz} fg(x)\right) \vee \gamma\right) \wedge \delta = \left(\bigvee_{x=yz} ((f(y) \vee \gamma) \wedge ((g(z) \vee \gamma)))\right) \wedge \delta$

$= \left(\bigvee_{x=yz} ([(f(y) \vee \gamma) \wedge \delta] \wedge ([(g(z) \vee \gamma) \wedge \delta])\right) = \bigvee_{x=yz} (f^*(y) \wedge g^*(z)) = (f^* g^*)(x)$.

(4) $(f*g)(x) = ((fg(x)) \vee \gamma) \wedge \delta = \left(\left(\bigvee_{x=yz} fg(x)\right) \vee \gamma\right) \wedge \delta = \left(\bigvee_{x=yz} ((f(y) \vee \gamma) \wedge ((g(z) \vee \gamma)))\right) \wedge \delta$

$= \left(\left(\bigvee_{x=yz} ([f(y) \wedge \delta] \wedge ([g(z) \wedge \delta])\right) \vee \gamma\right) \wedge \delta \leq \left(\left(\bigvee_{x=yz} ([f(yz) \vee \gamma] \wedge ([g(yz) \vee \gamma])\right) \vee \gamma\right) \wedge \delta$

$= \left(\left(\bigvee_{x=yz} ([f(x) \vee \gamma] \wedge ([g(x) \vee \gamma])\right) \vee \gamma\right) \wedge \delta = ((f(x) \wedge g(x)) \vee \gamma) \wedge \delta = \left(f \overset{*}{\wedge} g\right)(x)$.

■

Lemma 207 *Let A, B be non-empty subsets of a semigroup S. Then the following hold.*

(1) $C_A \overset{*}{\wedge} C_B = \overset{*}{C}_{A \cap B}$.
(2) $C_A * C_B = \overset{*}{C}_{AB}$.

Theorem 208 *(1) A fuzzy subset f of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S if and only if $f * f \leq f^*$.*

(2) A fuzzy subset f of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S if and only if $f * \mathcal{S} * f \leq f^*$.

(3) A fuzzy subset f of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S if and only if $\mathcal{S} * f \leq f^*$. $(f * \mathcal{S} \leq f^*)$.

(4) A fuzzy subset f of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if and only if $f * f \leq f^*$ and $f * \mathcal{S} * f \leq f^*$.

Where $\mathcal{S}$ is a fuzzy subset of S mapping every element of S on 1.

Proof. (1) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S and $x \in S$. If $(f * f)(x) = \gamma \wedge \delta$, then $(f * f)(x) = \gamma \wedge \delta \leq (f(x) \vee \gamma) \wedge \delta$. Otherwise

$$(f * f)(x) = ((\bigvee_{x=ab} \{f(a) \wedge f(b)\}) \vee \gamma) \wedge \delta = ((\bigvee_{x=ab} \{f(a) \wedge f(b) \wedge \delta\}) \vee \gamma) \wedge \delta$$
$$\leq ((\bigvee_{x=ab} \{f(ab) \vee \gamma\}) \vee \gamma) \wedge \delta = (f(x) \vee \gamma) \wedge \delta = f^*(x).$$

Thus $f * f \leq f^*$.

Conversely, assume that $f * f \leq f^*$ and $x, y \in S$. Then

$$f(xy) \vee \gamma \geq (f(xy) \vee \gamma) \wedge \delta = f^*(xy) \geq f * f(xy)$$
$$= ((\bigvee_{xy=ab} \{f(a) \wedge f(b)\}) \vee \gamma) \wedge \delta \geq (\{f(x) \wedge f(y)\}) \vee \gamma) \wedge \delta \geq f(x) \wedge f(y) \wedge \delta.$$

Thus f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

(2) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S and $x \in S$. If $f * \mathcal{S} * f(x) = \gamma \wedge \delta$, then $f * \mathcal{S} * f(x) = \gamma \wedge \delta \leq f(x) \vee (\gamma \wedge \delta) = (f(x) \vee \gamma) \wedge \delta = f^*(x)$.

Otherwise

$$f * \mathcal{S} * f(x) = (\bigvee_{x=ab} \{f(a) \wedge (\mathcal{S} * f)(b)\} \vee \gamma) \wedge \delta = (\bigvee_{x=ab} \{f(a) \wedge [((\bigvee_{b=cd} \{\mathcal{S}(c) \wedge f(d)\}) \vee \gamma) \wedge \delta]\} \vee \gamma) \wedge \delta$$
$$= (\bigvee_{x=ab} \{f(a) \wedge [((\bigvee_{b=cd} f(d)\}) \vee \gamma) \wedge \delta]\} \vee \gamma) \wedge \delta = ((\bigvee_{x=ab} \{\bigvee_{b=cd} [f(a) \wedge f(d)] \vee \gamma\} \wedge \delta) \vee \gamma) \wedge \delta$$
$$= ((\bigvee_{x=ab} \{\bigvee_{b=cd} [f(a) \wedge f(d) \wedge \delta]\} \vee \gamma\} \vee \gamma) \wedge \delta \leq (\bigvee_{x=ab} \{\bigvee_{b=cd} \{f(acd) \vee \gamma\}) \vee \gamma \wedge \delta = f(x) \vee \gamma \wedge \delta = f^*(x).$$

Thus $f * \mathcal{S} * f \leq f^*$.

Conversely, assume that $f * \mathcal{S} * f \leq f^*$ and $x, y, z \in S$. Then

$$f(xyz) \vee \gamma \geq (f(xyz) \vee \gamma) \wedge \delta \geq (f * \mathcal{S} * f)(xyz)$$
$$= ((\bigvee_{xyz=ab} \{f(a) \wedge (\mathcal{S} * f)(b)\}) \vee \gamma) \wedge \delta \geq ((f(x) \wedge (\mathcal{S} * f)(yz)) \vee \gamma) \wedge \delta$$
$$= ((f(x) \wedge [(\bigvee_{yz=cd} \{\mathcal{S}(c) \wedge f(d)\}) \vee \gamma) \wedge \delta]) \vee \gamma) \wedge \delta$$
$$\geq ((f(x) \wedge ([f(z) \vee \gamma) \wedge \delta]) \vee \gamma) \wedge \delta \geq ((f(x) \wedge f(z) \wedge \delta) \vee \gamma) \wedge \delta \geq ((f(x) \wedge f(z) \wedge \delta) \wedge \delta.$$

Thus f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S.

(3) The proof is similar to the proof of (2).

(4) Follows from (1) and (3). ■

Definition 209 *A fuzzy subset f of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S if and only if $f * \mathcal{S} \wedge \mathcal{S} * f \leq f^*$.*

Lemma 210 *A non-empty subset A of a semigroup S is a quasi-ideal of S if and only if the characteristic function C_A of A is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S.*

Proof. Straightforward. ■

4.3 Regular Semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Ideals

In this section we characterize regular semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals.

Proposition 211 *In a regular semigroup S, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S.*

Proof. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S and $x, y \in S$. Then there exist $a, b \in S$ such that $x = xax$ and $y = yby$. Now

$\max\{f(xy), \gamma\} = \max\{f((xa)xy), \gamma\}$
$\geq \min\{f(x), \delta\}$.

Also

$\max\{f(xy), \gamma\} = \max\{f(xy(by)), \gamma\}$
$\geq \min\{f(y), \delta\}$.

This shows that f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S. ■

Proposition 212 *In a regular semigroup S, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S.*

Proof. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S and $x, y \in S$. Then there exists $a \in S$ such that $x = xax$. Now

$\max\{f(xy), \gamma\} = \max\{f(x(ax)y), \gamma\}$
$\geq \min\{f(x), f(y), \delta\}$.

This shows that f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. ■

Theorem 213 *The following assertions are equivalent for a semigroup S:*

(1) S is regular.

(2) $f \overset{*}{\wedge} g = f * g$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal g of S.

Proof. (1) $\Rightarrow$ (2) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. Then by Lemma 206, $f * g \leq f \overset{*}{\wedge} g$.

Let $x \in S$. Then there exists $a \in S$ such that $x = xax$. Now

$$(f * g)(x) = \left(\left(\bigvee_{x=yz} (f(y) \wedge g(z))\right) \vee \gamma\right) \wedge \delta$$
$$\geq ((f(x) \wedge g(ax)) \vee \gamma) \wedge \delta$$
$$= ((f(x) \wedge (g(ax) \vee \gamma)) \vee \gamma) \wedge \delta$$

$\geq ((f(x) \wedge (g(x) \wedge \delta)) \vee \gamma) \wedge \delta$
$= ((f(x) \wedge g(x)) \vee \gamma) \wedge \delta$
$= \left(f \overset{*}{\wedge} g\right)(x).$

Thus $f * g \geq f \overset{*}{\wedge} g$. Hence $f * g = f \overset{*}{\wedge} g$.

(2) $\Rightarrow$ (1) Let R be a right ideal and L a left ideal of S. Then C_R and C_L are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals of S, respectively. By hypothesis $C_R * C_L = C_R \overset{*}{\wedge} C_L$. By Lemma 207, this implies that $\overset{*}{C}_{RL} = \overset{*}{C}_{R\cap L} \Rightarrow RL = R \cap L$. Hence S is a regular semigroup. ■

Theorem 214 *The following assertions are equivalent for a semigroup S:*

(1) S is regular.

(2) $f \overset{*}{\wedge} g \overset{*}{\wedge} h \leq f * g * h$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal g and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal h of S.

(3) $f \overset{*}{\wedge} g \overset{*}{\wedge} h \leq f * g * h$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal g and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal h of S.

(4) $f \overset{*}{\wedge} g \overset{*}{\wedge} h \leq f * g * h$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal g and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal h of S.

Proof. (1) $\Rightarrow$ (2) Let f, g, h be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S, respectively. Let $x \in S$. Then there exists $a \in S$ such that $x = xax$. Thus we have

$(f * g * h)(x) = \left[\left(\left(\bigvee_{x=yz} (f(y) \wedge (g * h)(z))\right) \vee \gamma\right) \wedge \delta\right]$
$\geq [((f(xa) \wedge (g * h)(x)) \vee \gamma) \wedge \delta]$
$= [(((f(xa) \vee \gamma) \wedge (g * h)(x)) \vee \gamma) \wedge \delta]$
$\geq [(((f(x) \wedge \delta) \wedge (g * h)(x)) \vee \gamma) \wedge \delta]$
$= \left[\left(\left(f(x) \wedge \left(\left(\bigvee_{x=yz} (g(y) \wedge h(z))\right) \vee \gamma\right) \wedge \delta\right) \vee \gamma\right) \wedge \delta\right]$
$\geq [((f(x) \wedge ((g(x) \wedge h(ax)) \vee \gamma) \wedge \delta) \vee \gamma) \wedge \delta]$
$= [((f(x) \wedge ((g(x) \wedge (h(ax) \vee \gamma))) \wedge \delta) \vee \gamma) \wedge \delta]$
$\geq [((f(x) \wedge ((g(x) \wedge (h(x) \wedge \delta))) \wedge \delta) \vee \gamma) \wedge \delta]$
$= [((f(x) \wedge g(x) \wedge h(x)) \vee \gamma) \wedge \delta]$
$= \left(f \overset{*}{\wedge} g \overset{*}{\wedge} h\right)(x).$

Thus $f \overset{*}{\wedge} g \overset{*}{\wedge} h \leq f * g * h$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are straightforward.

(4) $\Rightarrow$ (1) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. Since $\mathcal{S}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S, so by hypothesis we have

$\left(f \overset{*}{\wedge} g\right)(x) = \left(f \overset{*}{\wedge} \mathcal{S} \overset{*}{\wedge} g\right)(x) \leq (f * \mathcal{S} * g)(x) \leq (f * g)(x).$

But $\left(f \overset{*}{\wedge} g\right)(x) \geq (f * g)(x)$ always holds. Thus $f \overset{*}{\wedge} g = f * g$. Hence by Theorem 213, S is regular. ■

Theorem 215 *The following assertions are equivalent for a semigroup S:*

(1) S is regular.
(2) $\overset{*}{f} = f * \mathcal{S} * f$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f of S.
(3) $\overset{*}{f} = f * \mathcal{S} * f$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f of S.
(4) $\overset{*}{f} = f * \mathcal{S} * f$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f of S.

Proof. (1) ⇒ (2) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S and $x \in S$. Then there exists $a \in S$ such that $x = xax$. Thus we have

$$\begin{aligned}
(f * \mathcal{S} * f)(x) &= \left(\left(\bigvee_{x=yz} ((f * \mathcal{S})(y) \wedge f(z))\right) \vee \gamma\right) \wedge \delta \\
&\geq (((f * \mathcal{S})(xa) \wedge f(x)) \vee \gamma) \wedge \delta \\
&= \left(\left(\left[\left(\left(\bigvee_{xa=bc} ((f)(b) \wedge \mathcal{S}(c))\right) \vee \gamma\right) \wedge \delta\right] \wedge f(x)\right) \vee \gamma\right) \wedge \delta \\
&\geq (([(((f)(x) \wedge \mathcal{S}(a)) \vee \gamma) \wedge \delta] \wedge f(x)) \vee \gamma) \wedge \delta \\
&= ((f(x) \wedge f(x)) \vee \gamma) \wedge \delta \\
&= \overset{*}{f}(x).
\end{aligned}$$

Thus $f * \mathcal{S} * f \geq \overset{*}{f}$.

As f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S, so we have

$$\begin{aligned}
(f * \mathcal{S} * f)(x) &= \left(\left(\bigvee_{x=yz} ((f * \mathcal{S})(y) \wedge f(z))\right) \vee \gamma\right) \wedge \delta \\
&= \left(\left(\bigvee_{x=yz} \left(\left[\left(\left(\bigvee_{y=bc} ((f)(b) \wedge \mathcal{S}(c))\right) \vee \gamma\right) \wedge \delta\right] \wedge f(z)\right)\right) \vee \gamma\right) \wedge \delta \\
&= \left(\left(\bigvee_{x=yz} \left(\left(\bigvee_{y=bc} ((f)(b) \wedge 1)\right) \wedge f(z)\right)\right) \vee \gamma\right) \wedge \delta \\
&= \left(\left(\bigvee_{x=yz} \left(\bigvee_{y=bc} ((f)(b) \wedge f(z))\right)\right) \vee \gamma\right) \wedge \delta \\
&= \left(\left(\bigvee_{x=yz} \left(\bigvee_{y=bc} (((f)(b) \wedge f(z)) \vee \gamma)\right)\right) \vee \gamma\right) \wedge \delta \\
&\leq \left(\left(\bigvee_{x=yz} \left(\bigvee_{y=bc} (f(bcz) \wedge \delta)\right)\right) \vee \gamma\right) \wedge \delta \\
&= \left(\left(\bigvee_{x=yz} \left(\bigvee_{y=bc} f(x)\right)\right) \vee \gamma\right) \wedge \delta \\
&= \overset{*}{f}(x).
\end{aligned}$$

Thus $f * \mathcal{S} * f \leq \overset{*}{f}$. Hence $f * \mathcal{S} * f = \overset{*}{f}$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1) Let Q be any quasi-ideal of S. Then C_Q is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S. Hence by hypothesis $\overset{*}{C}_Q = C_Q * \mathcal{S} * C_Q = \overset{*}{C}_{QSQ}$. This implies that $Q = QSQ$. So by Theorem 23, S is regular. ■

Theorem 216 *The following assertions are equivalent for a semigroup S:*

(1) S is regular.

(2) $f \overset{*}{\wedge} g = f*g*f$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal g of S.

(3) $f \overset{*}{\wedge} g = f*g*f$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal g of S.

(4) $f \overset{*}{\wedge} g = f * g * f$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal g of S.

(5) $f \overset{*}{\wedge} g = f * g * f$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal g of S.

(6) $f \overset{*}{\wedge} g = f * g * f$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal g of S.

(7) $f \overset{*}{\wedge} g = f * g * f$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal g of S.

Proof. (1) $\Rightarrow$ (7) Let f and g be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S, respectively. Then for any $x \in S$

$$(f * g * f)(x) \leq (f * \mathcal{S} * f)(x) = \left(\left(\bigvee_{x=yz} ((f * \mathcal{S})(y) \wedge f(z))\right) \vee \gamma\right) \wedge \delta$$

$$= \left(\left(\bigvee_{x=yz} \left(\left[\left(\left(\bigvee_{y=bc} ((f)(b) \wedge \mathcal{S}(c))\right) \vee \gamma\right) \wedge \delta\right] \wedge f(z)\right)\right) \vee \gamma\right) \wedge \delta$$

$$= \left(\left(\bigvee_{x=yz} \left(\left(\bigvee_{y=bc} ((f)(b) \wedge 1)\right) \wedge f(z)\right)\right) \vee \gamma\right) \wedge \delta$$

$$= \left(\left(\bigvee_{x=yz} \left(\bigvee_{y=bc} ((f)(b) \wedge f(z))\right)\right) \vee \gamma\right) \wedge \delta$$

$$= \left(\left(\bigvee_{x=yz} \left(\bigvee_{y=bc} (((f)(b) \wedge f(z)) \vee \gamma)\right)\right) \vee \gamma\right) \wedge \delta$$

$$\leq \left(\left(\bigvee_{x=yz} \left(\bigvee_{y=bc} (f(bcz) \wedge \delta)\right)\right) \vee \gamma\right) \wedge \delta$$

$$= \left(\left(\bigvee_{x=yz} \left(\bigvee_{y=bc} f(x)\right)\right) \vee \gamma\right) \wedge \delta$$

$$= \overset{*}{f}(x).$$

Thus $f * \mathcal{S} * f \leq \overset{*}{f}$.

Also

$$(f*g*f)(x) \leq (\mathcal{S}*g*\mathcal{S})(x) = \left(\left(\bigvee_{x=yz} ((\mathcal{S}*g)(y) \wedge \mathcal{S}(z))\right) \vee \gamma\right) \wedge \delta$$

$$= \left(\left(\bigvee_{x=yz}\left(\left[\left(\left(\bigvee_{y=bc} (\mathcal{S}(b) \wedge g(c))\right) \vee \gamma\right) \wedge \delta\right] \wedge \mathcal{S}(z)\right)\right) \vee \gamma\right) \wedge \delta$$

$$= \left(\left(\bigvee_{x=yz}\left(\left(\bigvee_{y=bc} (\mathcal{S}(b) \wedge g(c)\right) \wedge \mathcal{S}(z)\right)\right) \vee \gamma\right) \wedge \delta$$

$$= \left(\left(\bigvee_{x=yz}\left(\bigvee_{y=bc} g(c)\right)\right) \vee \gamma\right) \wedge \delta$$

$$= \left(\left(\bigvee_{x=yz}\left(\bigvee_{y=bc} (g(c) \vee \gamma)\right)\right) \vee \gamma\right) \wedge \delta$$

$$\leq \left(\left(\bigvee_{x=yz}\left(\bigvee_{y=bc} (g(bcz) \wedge \delta)\right)\right) \vee \gamma\right) \wedge \delta$$

$$= \left(\left(\bigvee_{x=yz}\left(\bigvee_{y=bc} g(x)\right)\right) \vee \gamma\right) \wedge \delta$$

$$= \overset{*}{g}(x).$$

Thus $f*g*f \leq f \overset{*}{\wedge} g$.

Now let $x \in S$. Then there exists $a \in S$ such that $x = xax = xaxax$.

$$(f*g*f)(x) = \left[\left(\left(\bigvee_{x=yz} (f(y) \wedge (g*f)(z))\right) \vee \gamma\right) \wedge \delta\right]$$

$$\geq [((f(x) \wedge (g*h)(axax)) \vee \gamma) \wedge \delta]$$

$$= \left[\left(\left(f(x) \wedge \left(\left(\bigvee_{axax=yz} (g(y) \wedge f(z))\right) \vee \gamma\right) \wedge \delta\right) \vee \gamma\right) \wedge \delta\right]$$

$$\geq [(f(x) \wedge ((g(axa) \wedge f(x)) \vee \gamma) \wedge \delta)]$$

$$= [(f(x) \wedge (([g(axa) \wedge \delta] \wedge f(x)) \vee \gamma) \wedge \delta)]$$

$$\geq [(f(x) \wedge (([g(x) \vee \gamma] \wedge f(x)) \vee \gamma) \wedge \delta)]$$

$$= [(f(x) \wedge ((g(x) \wedge f(x)) \vee \gamma) \wedge \delta)]$$

$$= \left(f \overset{*}{\wedge} g\right)(x).$$

This shows that $f*g*f \geq f \overset{*}{\wedge} g$. Hence $f*g*f = f \overset{*}{\wedge} g$.

(7) ⇒ (5) ⇒ (3) ⇒ (2) and (7) ⇒ (6) ⇒ (4) ⇒ (2) are clear.

(2) ⇒ (1) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S. Then $\overset{*}{f} = f \overset{*}{\wedge} \mathcal{S} = f*\mathcal{S}*f$.

Thus it follows from Theorem 215, that S is regular. ■

Theorem 217 *The following assertions are equivalent for a semigroup S:*

(1) S is regular.

(2) $f \overset{*}{\wedge} g \leq f*g$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal g of S.

(3) $f \overset{*}{\wedge} g \leq f * g$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal g of S.

(4) $f \overset{*}{\wedge} g \leq f * g$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal g of S.

Proof. (1) $\Rightarrow$ (4) Let f and g be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S, respectively. Let $x \in S$. Then there exists $a \in S$ such that $x = xax$. Thus we have

$$(f * g)(x) = \left(\left(\bigvee_{x=yz} (f(y) \wedge g(z)) \right) \vee \gamma \right) \wedge \delta$$
$$\geq ((f(x) \wedge g(ax)) \vee \gamma) \wedge \delta$$
$$= ((f(x) \wedge (g(ax) \vee \gamma)) \vee \gamma) \wedge \delta$$
$$\geq ((f(x) \wedge (g(x) \wedge \delta)) \vee \gamma) \wedge \delta$$
$$= ((f(x) \wedge g(x)) \vee \gamma) \wedge \delta$$
$$= \left(f \overset{*}{\wedge} g \right)(x).$$

Thus $f * g \geq f \overset{*}{\wedge} g$.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (1) Let f and g be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S, respectively. Then by hypothesis $f * g \geq f \overset{*}{\wedge} g$. But $f * g \leq f \overset{*}{\wedge} g$ always holds. Thus $f * g = f \overset{*}{\wedge} g$. Hence by Theorem 213, S is regular. ■

4.4 Intra-Regular Semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Ideals

In this section we characterize intra-regular semigroups by the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals.

Theorem 218 *The following assertions are equivalent for a semigroup S:*

(1) S is intra-regular.

(2) $f \overset{*}{\wedge} g \leq f * g$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal g of S.

Proof. (1) $\Rightarrow$ (2) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S. Let $x \in S$. Then there exist $a, b \in S$ such that $x = axxb$. Now

$$(f * g)(x) = \left(\left(\bigvee_{x=yz} (f(y) \wedge g(z)) \right) \vee \gamma \right) \wedge \delta$$
$$\geq ((f(ax) \wedge g(xb)) \vee \gamma) \wedge \delta$$
$$= (((f(ax) \vee \gamma) \wedge (g(xb) \vee \gamma)) \vee \gamma) \wedge \delta$$

$\geq (((f(x) \wedge \delta) \wedge (g(x) \wedge \delta)) \vee \gamma) \wedge \delta$
$= ((f(x) \wedge g(x)) \vee \gamma) \wedge \delta$
$= \left(f \overset{*}{\wedge} g\right)(x).$

Thus $f * g \geq f \overset{*}{\wedge} g$.

(2) $\Rightarrow$ (1) Let R be any right ideal and L be any left ideal of S. Then C_R and C_L are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals of S, respectively. By hypothesis $C_L * C_R \geq C_L \overset{*}{\wedge} C_R$. By Lemma 207, this implies that $\overset{*}{C}_{LR} \geq \overset{*}{C}_{L\cap R} \Rightarrow LR \supseteq L \cap R$. Hence S is an intra-regular semigroup. ■

Theorem 219 *The following assertions are equivalent for a semigroup S:*

(1) S is both regular and intra-regular.

(2) $\overset{*}{f} = f * f$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f of S.

(3) $\overset{*}{f} = f * f$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f of S.

(4) $f \overset{*}{\wedge} g \leq f * g$ for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals f, g of S.

(5) $f \overset{*}{\wedge} g \leq f * g$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal g of S.

(6) $f \overset{*}{\wedge} g \leq f * g$ for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals f, g of S.

Proof. (1) $\Rightarrow$ (6) Let f, g be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals of S. Let $x \in S$. Then there exist $a, b, c \in S$ such that $x = axxb$ and $x = xcx$. Thus $x = xcx = xcxcx = xc(axxb)cx = (xcax)(xbcx)$. Thus we have

$(f * g)(x) = \left(\left(\bigvee_{x=yz} (f(y) \wedge g(z))\right) \vee \gamma\right) \wedge \delta$
$\geq ((f(xcax) \wedge g(xbcx)) \vee \gamma) \wedge \delta$
$= (((f(xcax) \vee \gamma) \wedge (g(xbcx) \vee \gamma)) \vee \gamma) \wedge \delta$
$\geq (((f(x) \wedge \delta) \wedge (g(x) \wedge \delta)) \vee \gamma) \wedge \delta$
$= ((f(x) \wedge g(x)) \vee \gamma) \wedge \delta$
$= \left(f \overset{*}{\wedge} g\right)(x).$

Thus $f * g \geq f \overset{*}{\wedge} g$.

(6) $\Rightarrow$ (5) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (2) Take $f = g$ in (4), we get $f * f \geq \overset{*}{f}$. Since every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S, so $f * f \leq \overset{*}{f}$. Thus $f * f = \overset{*}{f}$.

(6) $\Rightarrow$ (3) Take $f = g$ in (6), we get $f * f \geq \overset{*}{f}$. Since every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S, so $f * f \leq \overset{*}{f}$. Thus $f * f = \overset{*}{f}$.

(3) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (1) Let Q be a quasi-ideal of S. Then C_Q is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S. Hence by hypothesis $C_Q * C_Q = \overset{*}{C}_Q$. This implies that $\overset{*}{C}_{QQ} = \overset{*}{C}_Q$, that is $QQ = Q$. Hence by Theorem 28, S is both regular and intra-regular. ■

Theorem 220 *The following assertions are equivalent for a semigroup S:*

(1) S is both regular and intra-regular.

(2) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal g of S.

(3) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal g of S.

(4) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal g of S.

(5) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal g of S.

(6) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal g of S.

(7) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal g of S.

(8) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal g of S.

(9) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f, g of S.

(10) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal g of S.

(11) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal g of S.

(12) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals f, g of S.

(13) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal g of S.

(14) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f, g of S.

Proof. (1) $\Rightarrow$ (14) Let f, g be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideals of S. Let $x \in S$. Then there exist $a, b, c \in S$ such that $x = axxb$ and $x = xcx$. Thus $x = xcx = xcxcx = xc(axxb)cx = (xcax)(xbcx)$. Thus we have

$$
\begin{aligned}
(f * g)(x) &= \left(\left(\bigvee_{x=yz} (f(y) \wedge g(z))\right) \vee \gamma\right) \wedge \delta \\
&\geq ((f(xcax) \wedge g(xbcx)) \vee \gamma) \wedge \delta \\
&= (((f(xcax) \vee \gamma) \wedge (g(xbcx) \vee \gamma)) \vee \gamma) \wedge \delta \\
&\geq (((f(x) \wedge \delta) \wedge (g(x) \wedge \delta)) \vee \gamma) \wedge \delta
\end{aligned}
$$

$= ((f(x) \wedge g(x)) \vee \gamma) \wedge \delta$
$= \left(f \overset{*}{\wedge} g\right)(x).$

Thus $f * g \geq f \overset{*}{\wedge} g$. Similarly we can show that $g * f \geq f \overset{*}{\wedge} g$. Hence $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$.

(14) ⇒ (13) ⇒ (12) ⇒ (10) ⇒ (9) ⇒ (3) ⇒ (2)

(14) ⇒ (11) ⇒ (10)

(14) ⇒ (8) ⇒ (7) ⇒ (6) ⇒ (2) and

(14) ⇒ (5) ⇒ (4) ⇒ (3) ⇒ (2) are obvious.

(2) ⇒ (1) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. Then by hypothesis we have $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$, that is $f \overset{*}{\wedge} g \leq f * g$ and $f \overset{*}{\wedge} g \leq g * f$. But $f \overset{*}{\wedge} g \geq f * g$ always hold. Hence $f \overset{*}{\wedge} g = f * g$ and $f \overset{*}{\wedge} g \leq g * f$. Thus by Theorems 213 and 218, S is both regular and intra-regular. ■

4.5 Semisimple Semigroups

In this section we characterize semisimple semigroups by the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideals.

Theorem 221 *In a semisimple semigroup S, a fuzzy subset f of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S if and only if it is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S.*

Proof. Let S be a semisimple semigroup and f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S. Then for any $x, y \in S$ there exist $a, b, c, d \in S$ such that $x = (axb)(cxd)$. Thus we have

$f(xy) \vee \gamma = f((axbcxd)\,y) \vee \gamma = f((ax\,(bc))\,x\,(dy)) \vee \gamma \geq f(x) \wedge \delta.$

Similarly $f(xy) \vee \gamma \geq f(y) \wedge \delta$. Hence f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S.

Conversely assume that f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S. Then f is always an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S. ■

Theorem 222 *For a semigroup S the following assertions are equivalent*

(1) S is semisimple.

(2) $f * f = \overset{*}{f}$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal f of S.

(3) $f * f = \overset{*}{f}$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal f of S.

(4) $f \overset{*}{\wedge} g = f * g$ for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals f, g of S.

(5) $f \overset{*}{\wedge} g = f * g$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal g of S.

(6) $f \overset{*}{\wedge} g = f * g$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal g of S.

(7) $f \overset{*}{\wedge} g = f * g$ for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideals f, g of S.

Proof. (1) $\Rightarrow$ (7) Let S be a semisimple semigroup and f, g be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideals of S. Let $x \in S$. Then there exist $a, b, c, d \in S$ such that $x = (axb)(cxd)$. Thus we have

$$(f * g)(x) = \left(\left(\bigvee_{x=yz} (f(y) \wedge g(z))\right) \vee \gamma\right) \wedge \delta$$
$$\geq ((f(axb) \wedge g(cxd)) \vee \gamma) \wedge \delta$$
$$= (((f(axb) \vee \gamma) \wedge (g(cxd) \vee \gamma)) \vee \gamma) \wedge \delta$$
$$\geq (((f(x) \wedge \delta) \wedge (g(x) \wedge \delta)) \vee \gamma) \wedge \delta$$
$$= ((f(x) \wedge g(x)) \vee \gamma) \wedge \delta$$
$$= \left(f \overset{*}{\wedge} g\right)(x).$$

Thus $f * g \geq f \overset{*}{\wedge} g$. Since every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior-ideal of S in a semisimple semigroup is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S, so $f * g \leq f \overset{*}{\wedge} g$. Hence $f * g = f \overset{*}{\wedge} g$.

(7) $\Rightarrow$ (6) $\Rightarrow$ (4) $\Rightarrow$ (2), (7) $\Rightarrow$ (3) $\Rightarrow$ (2) and (7) $\Rightarrow$ (5) $\Rightarrow$ (4) are obvious.

(2) $\Rightarrow$ (1) Let A be any ideal of S. Then C_A is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S. Thus by hypothesis $C_A * C_A = \overset{*}{C}_A$, that is $AA = A$. Hence S is a semisimple semigroup. ■

Chapter 5

Characterizations of Semigroups by the Properties of their $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy Ideals

In this chapter, we initiate the study of $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal, $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal and $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of semigroup. Regular, intra-regular, and semisimple semigroups are characterized by the properties of these fuzzy ideals. If we take $\gamma = 0$ and $\delta = 0.5$ then $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right, quasi-ideal, bi-ideal) becomes $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right, quasi-ideal, bi-ideal). If we take $\gamma = 0$ and $\delta = \frac{1-k}{2}$ then $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right, quasi-ideal, bi-ideal) becomes $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (right, quasi-ideal, bi-ideal). Thus this chapter is a generalization of Chapter 2 as well as of Chapter 3.

5.1 $(\overline{\Phi}, \overline{\Psi})$ - fuzzy subsemigroups

Throughout this chapter $\gamma, \delta \in [0,1]$, where $\gamma < \delta$. $\overline{\Phi}, \overline{\Psi} \in \{\overline{\in_\gamma}, \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta}\}$ and $\overline{\Phi} \neq \overline{\in_\gamma} \wedge \overline{q_\delta}$.

Let f be a fuzzy subset of a semigroup S such that $f(x) \leq \delta$. Let $x \in S$ and $t \in [0,1]$ be such that $x_t \overline{\in_\gamma} \wedge \overline{q_\delta}$. Then $f(x) < t$ and $f(x) + t \leq 2\delta$. It follows that $2\delta \geq f(x) + t > f(x) + f(x) = 2f(x)$, that is $f(x) < \delta$. This means that $\{x_t : x_t \overline{\in_\gamma} \wedge \overline{q_\delta} f\} = \emptyset$. Therefore we are not taking $\overline{\Phi} = \overline{\in_\gamma} \wedge \overline{q_\delta}$.

Definition 223 *A fuzzy subset f of a semigroup S is called a $(\overline{\Phi}, \overline{\Psi})$-fuzzy subsemigroup of S, if it satisfies*

(5.1) $(xy)_{\min\{t,r\}} \overline{\Phi} f \Rightarrow x_t \overline{\Psi} f$ or $y_r \overline{\Psi} f$ for all $x, y \in S$ and $t, r \in (\gamma, 1]$.

There are twelve different types of $(\overline{\Phi}, \overline{\Psi})$-fuzzy subsemigroups of a semigroup S, they are $(\overline{\in_\gamma}, \overline{\in_\gamma})$, $(\overline{\in_\gamma}, \overline{q_\delta})$, $(\overline{\in_\gamma}, \overline{\in_\gamma} \wedge \overline{q_\delta})$, $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$, $(\overline{q_\delta}, \overline{\in_\gamma})$, $(\overline{q_\delta}, \overline{q_\delta})$, $(\overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$, $(\overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$, $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma})$, $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{q_\delta})$, $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$, and $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$. The following thorem gives the relations between them.

Theorem 224 *Let $2\delta = 1 + \gamma$ and S be a semigroup. Then the following are true:*

(i) Every $(\overline{\in_\gamma}, \overline{\in_\gamma})$-fuzzy subsemigroup of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

(ii) Every $(\overline{\in_\gamma}, \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

(iii) Every $(\overline{\in_\gamma}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma})$-fuzzy subsemigroup of S.

(iv) Every $(\overline{\in_\gamma}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{\in_\gamma}, \overline{q_\delta})$-fuzzy subsemigroup of S.

(v) Every $(\overline{\in_\gamma}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

(vi) Every $(\overline{q_\delta}, \overline{\in_\gamma})$-fuzzy subsemigroup of S is an $(\overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

(vii) Every $(\overline{q_\delta}, \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

(viii) Every $(\overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{q_\delta}, \overline{\in_\gamma})$-fuzzy subsemigroup of S.

(xi) Every $(\overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{q_\delta}, \overline{q_\delta})$-fuzzy subsemigroup of S.

(x) Every $(\overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

(xi) Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma})$-fuzzy subsemigroup of S is an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

(xii) Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

(xiii) Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

(xvi) Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

(xv) Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{\in_\gamma} \wedge \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

(xv) Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

(xvi) Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma})$-fuzzy subsemigroup of S.

$(xvii)$Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{\in} \vee \overline{q_\delta}, \overline{q_\delta})$-fuzzy subsemigroup of S.

Proof. Straightforward. ■

Theorem 225 *Let $2\delta = 1 + \gamma$. If f is one of the following:*

(i) an $(\overline{\in_\gamma}, \overline{\in_\gamma})$-fuzzy subsemigroup of S;
(ii) an $(\overline{\in_\gamma}, \overline{q_\delta})$-fuzzy subsemigroup of S;
(iii) a $(\overline{q_\delta}, \overline{\in_\gamma})$-fuzzy subsemigroup of S;
(iv) a $(\overline{q_\delta}, \overline{q_\delta})$-fuzzy subsemigroup of S;
(v) an $(\overline{\in_\gamma}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy subsemigroup of S;
(vi) an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S;
(vii) an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{q_\delta})$-fuzzy subsemigroup of S;
$(viii)$ an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma})$-fuzzy subsemigroup of S;
(ix) an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy subsemigroup of S;
(x) a $(\overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy subsemigroup of S;
(xi) a $(\overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S;
(xii) an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S;

then $f_\gamma = \{x \in S : f(x) > \gamma\}$ is a subsemigroup of S.

Proof. (i) Let $x, y \in f_\gamma$. Then $f(x) > \gamma$, $f(y) > \gamma$. If $f(xy) \leq \gamma$, then $(xy)_{\min\{f(x), f(y)\}} \overline{\in_\gamma} f$ but $x_{f(x)} \in_\gamma f$ and $y_{f(y)} \in_\gamma f$, which is a contradiction. Thus $f(xy) > \gamma$, and so $xy \in f_\gamma$.

(ii) Let x, $y \in f_\gamma$. Then $f(x) > \gamma$, $f(y) > \gamma$. If $f(xy) \leq \gamma$, then $(xy)_{\min\{1,1\}} \overline{\in_\gamma} f$ but $x_1 q_\delta f$ and $y_1 q_\delta f$, which is a contradiction. Thus $f(xy) > \gamma$, and so $xy \in f_\gamma$.

(iii) Let x, $y \in f_\gamma$. Then $f(x) > \gamma$, $f(y) > \gamma$. If $f(xy) \leq \gamma$, then $(xy)_{\min\{f(x), f(y)\}} \overline{q_\delta} f$ but $x_{f(x)} \in f$ and $y_{f(y)} \in f$, which is a contradiction. It follows that $f(xy) > \gamma$ so that $xy \in f_\gamma$.

(iv) Let x, $y \in f_\gamma$. Then $f(x) > \gamma$, $f(y) > \gamma$. Suppose that $xy \notin f_\gamma$. Then $f(xy) \leq \gamma$. Note that $(xy)_{\min\{1,1\}} \overline{q_\delta} f$ but $x_1 q_\delta f$ and $y_1 q_\delta f$, because $f(x) + 1 > 1 + \gamma = 2\delta$ and $f(y) + 1 > 1 + \gamma = 2\delta$. This is a contradiction, and thus $f(xy) > \gamma$, which shows that $xy \in f_\gamma$.

Similarly we can prove the remaining parts. ■

Theorem 226 *Let $2\delta = 1 + \gamma$ and A be a non-empty subset of S. Then A is a subsemigroup of S if and only if the fuzzy subset f of S defined by*

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ \leq \gamma & \text{if } x \notin A \end{cases}$$

is an $(\overline{\Phi}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

Proof. Let A be a subsemigroup of S.

(1) When $\overline{\Phi} = \overline{\in_\gamma}$. Let $x, y \in S$ and $t, r \in (\gamma, 1]$ be such that $(xy)_{\min\{t,r\}} \overline{\in_\gamma} f$. Then $f(xy) < \min\{t, r\}$. This implies that $xy \notin A$. Since A is a subsemigroup of S, we have $x \notin A$ or $y \notin A$. If $x \notin A$, then $f(x) \leq \gamma$. Thus $f(x) + t \leq \gamma + 1 = 2\delta$, that is $x_t \overline{q_\delta} f$. Similarly if $y \notin A$, then $y_r \overline{q_\delta} f$. This shows that f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

(2) When $\overline{\Phi} = \overline{q_\delta}$. Let $x, y \in S$ and $t, r \in (\gamma, 1]$ be such that $(xy)_{\min\{t,r\}} \overline{q_\delta} f$. Then $f(xy) + \min\{t, r\} \leq 2\delta = 1 + \gamma$. Since $\min\{t, r\} > 1$, we have $f(xy) < 1$. This implies that $xy \notin A$. Since A is a subsemigroup of S, we have $x \notin A$ or $y \notin A$. If $x \notin A$, then $f(x) \leq \gamma$. Thus $f(x) + t \leq \gamma + 1 = 2\delta$, that is $x_t \overline{q_\delta} f$. Similarly if $y \notin A$, then $y_r \overline{q_\delta} f$. This shows that f is an $(\overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

(3) Follows from parts (1) and (2).

Conversely, assume that f is an $(\overline{\Phi}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S. Then $A = f_\gamma$. It follows from Theorem 225 that A is a subsemigroup of S. ■

Corollary 227 *Let $2\delta = 1 + \gamma$ and A be a non-empty subset of S. Then A is a subsemigroup of S if and only if C_A, the characteristic function of A is an $(\overline{\Phi}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.*

5.2 $(\overline{\Phi}, \overline{\Psi})$ - fuzzy ideals

In this section we define $(\overline{\Phi}, \overline{\Psi})$ - fuzzy left (right) ideal of a semigroup and prove some relations between them.

Definition 228 *A fuzzy subset f of a semigroup S is called a $(\overline{\Phi}, \overline{\Psi})$-fuzzy left (right) ideal of S, if it satisfies*

(5.2) $(yx)_t \overline{\Phi} f \Rightarrow x_t \overline{\Psi} f$ ($y_t \overline{\Psi} f$) for all $x, y \in S$ and $t \in (\gamma, 1]$.

A fuzzy subset f of a semigroup S is called a $(\overline{\Phi}, \overline{\Psi})$-fuzzy ideal of S, if it is both $(\overline{\Phi}, \overline{\Psi})$-fuzzy left ideal and $(\overline{\Phi}, \overline{\Psi})$-fuzzy right ideal of S.

There are twelve different types of $(\overline{\Phi}, \overline{\Psi})$-fuzzy left (right) ideals of a semigroup S, they are $(\overline{\in_\gamma}, \overline{\in_\gamma})$, $(\overline{\in_\gamma}, \overline{q_\delta})$, $(\overline{\in_\gamma}, \overline{\in_\gamma} \wedge \overline{q_\delta})$, $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$, $(\overline{q_\delta}, \overline{\in_\gamma})$, $(\overline{q_\delta}, \overline{q_\delta})$, $(\overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$, $(\overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$, $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma})$, $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{q_\delta})$, $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$, and $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$. The following thorem gives the relations between them.

Theorem 229 *Let $2\delta = 1 + \gamma$ and S be a semigroup. Then the following are true:*

(i) Every $(\overline{\in_\gamma}, \overline{\in_\gamma})$-fuzzy left (right) ideal of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.

(ii) Every $(\overline{\in_\gamma}, \overline{q_\delta})$-fuzzy left (right) ideal of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.

(iii) Every $(\overline{\in_\gamma}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy left (right) ideal of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma})$-fuzzy left (right) ideal of S.

(iv) Every $(\overline{\in_\gamma}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy left (right) ideal of S is an $(\overline{\in_\gamma}, \overline{q_\delta})$-fuzzy left (right) ideal of S.

(v) Every $(\overline{\in_\gamma}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy left (right) ideal of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.

(vi) Every $(\overline{q_\delta}, \overline{\in_\gamma})$-fuzzy left (right) ideal of S is an $(\overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.

(vii) Every $(\overline{q_\delta}, \overline{q_\delta})$-fuzzy left (right) ideal of S is an $(\overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.

$(viii)$ Every $(\overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy left (right) ideal of S is an $(\overline{q_\delta}, \overline{\in_\gamma})$-fuzzy left (right) ideal of S.

(xi) Every $(\overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy left (right) ideal of S is an $(\overline{q_\delta}, \overline{q_\delta})$-fuzzy left (right) ideal of S.

(x) Every $(\overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy left (right) ideal of S is an $(\overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.

(xi) Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma})$-fuzzy left (right) ideal of S is an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.

(xii) Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{q_\delta})$-fuzzy left (right) ideal of S is an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.

$(xiii)$ Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.

(xvi) Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S is an $(\overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.

(xv) Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S is an $(\overline{\in_\gamma} \wedge \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.

(xv) Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy left (right) ideal of S is an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.

(xvi) Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy left (right) ideal of S is an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma})$-fuzzy left (right) ideal of S.

$(xvii)$Every $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy left (right) ideal of S is an $(\overline{\in} \vee \overline{q_\delta}, \overline{q_\delta})$-fuzzy left (right) ideal of S.

Proof. Straightforward. ■

Theorem 230 *Let $2\delta = 1 + \gamma$. If f is one of the following:*

(i) an $(\overline{\in_\gamma}, \overline{\in_\gamma})$-fuzzy left (right) ideal of S;

(ii) an $(\overline{\in_\gamma}, \overline{q_\delta})$-fuzzy left (right) ideal of S;
(iii) a $(\overline{q_\delta}, \overline{\in_\gamma})$-fuzzy left (right) ideal of S;
(iv) a $(\overline{q_\delta}, \overline{q_\delta})$-fuzzy left (right) ideal of S;
(v) an $(\overline{\in_\gamma}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy left (right) ideal of S;
(vi) an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S;
(vii) an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{q_\delta})$-fuzzy left (right) ideal of S;
$(viii)$ an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma})$-fuzzy left (right) ideal of S;
(ix) an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy left (right) ideal of S;
(x) a $(\overline{q_\delta}, \overline{\in_\gamma} \wedge \overline{q_\delta})$-fuzzy left (right) ideal of S;
(xi) a $(\overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S;
(xii) an $(\overline{\in_\gamma} \vee \overline{q_\delta}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S;
then $f_\gamma = \{x \in S : f(x) > \gamma\}$ is a left (right) ideal of S.

Proof. The proof is similar to the proof of Theorem 225. ■

Theorem 231 *Let $2\delta = 1 + \gamma$ and A be a non-empty subset of S. Then A is a left (right) ideal of S if and only if the fuzzy subset f of S defined by*

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ \leq \gamma & \text{if } x \notin A \end{cases}$$

is an $(\overline{\Phi}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.

Proof. The proof is similar to the proo of Theorem 226. ■

Corollary 232 *Let $2\delta = 1 + \gamma$ and A be a non-empty subset of S. Then A is a left (right) ideal of S if and only if C_A, the characteristic function of A is an $(\overline{\Phi}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.*

5.3 $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroups and ideals

We start this section with the following theorem.

Theorem 233 *For any fuzzy subset f of a semigroup S and for all $x, y, z \in R$ and $t, r \in (\gamma, 1]$ $(5.1a)$ is equivalent to $(5.1b)$, $(5.2a)$ is equivalent to $(5.2b)$, $(5.3a)$ is equivalent to $(5.3b)$ and $(5.4a)$ is equivalent to $(5.4b)$, where*

$(5.1a)$ $x_t, y_r \overline{\in_\gamma} f \Rightarrow (xy)_{\min\{t,r\}} \overline{\in_\gamma} \vee \overline{q_\delta} f$.
$(5.1b)$ $\max\{f(xy), \delta\} \geq \min\{f(x), f(y)\}$.
$(5.2a)$ $x_t \overline{\in_\gamma} f \Rightarrow (yx)_t \overline{\in_\gamma} \vee \overline{q_\delta} f$ $\quad ((xy)_t \overline{\in_\gamma} \vee \overline{q_\delta} f)$.
$(5.2b)$ $\max\{f(yx), \delta\} \geq f(x)$ $\quad (\max\{f(xy), \delta\} \geq f(x)$.

(5.3a) $x_t,\, y_r \overline{\in_\gamma} f \Rightarrow (xzy)_{\min\{t,r\}} \overline{\in_\gamma} \vee \overline{q_\delta} f$.

(5.3b) $\max\{f(xzy), \delta\} \geq \min\{f(x), f(y)\}$.

(5.4a) $x_t \overline{\in_\gamma} f \Rightarrow (yxz)_t \overline{\in_\gamma} \vee \overline{q_\delta} f$.

(5.4b) $\max\{f(yxz), \delta\} \geq f(x)$.

Proof. We prove only (5.1a) if and only if (5.1b). Proofs of the remaining parts are similar to this.

(5.1a) $\Rightarrow$ (5.1b) Let f be a fuzzy subset of S which satisfies (5.1a). Let $x, y \in S$ be such that $\max\{f(xy), \delta\} < \min\{f(x), f(y)\}$. Select $t \in (\gamma, 1]$ such that $\max\{f(xy), \delta\} < t \leq \min\{f(x), f(y)\}$. Then $f(xy) < t$ and $f(x) \geq t > \gamma, f(y) \geq t > \gamma$, $f(x) + t > \delta + \delta = 2\delta, f(y) + t > \delta + \delta = 2\delta$ that is $(xy)_t \overline{\in_\gamma} f$ but $x_t\, (\in_\gamma \wedge q_\delta)\, f$ and $y_t\, (\in_\gamma \wedge q_\delta)\, f$, which is a contradiction. Hence $\max\{f(xy), \delta\} \geq \min\{f(x), f(y)\}$.

(5.1b) $\Rightarrow$ (5.1a) Let f be a fuzzy subset of S which satisfies (5.1b). Let $x, y \in S$ and $t, r \in (\gamma, 1]$ be such that $(xy)_{\min\{t,r\}} \overline{\in_\gamma} f$ but $x_t\, (\in_\gamma \wedge q_\delta)\, f$ and $y_r\, (\in_\gamma \wedge q_\delta)\, f$. Then $f(xy) < \min\{t, r\}$. Now $f(x) \geq t > \gamma$ and $f(x) + t > 2\delta$, also $f(y) \geq r > \gamma$ and $f(y) + r > 2\delta$. Thus $\min\{f(x), f(y)\} \geq \min\{t, r\} > f(xy)$ and $\min\{f(x), f(y)\} \geq \min\{2\delta - t, 2\delta - r\}$

$f(xy) < \min\{t, r\}$ (3)

and $f(xy) + \min\{t, r\} \leq 2\delta$ (4).

It follows from (3) and (4) that $f(xy) < \delta$. Now $\max\{f(xy), \gamma\} < \delta$ and $\max\{f(xy), \gamma\} < \min\{f(x), f(y)\}$. Thus $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$. Which is a contradiction. Hence $(xy)_{\min\{t,r\}} \overline{\in_\gamma} \vee \overline{q_\delta} f$. ■

From the above theorem we deduce that

Definition 234 *A fuzzy subset f of a semigroup S is called an*

- $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S if it satisfies (5.1a or 5.1b).
- $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S if it satisfies (5.2a or 5.2b).
- $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal of S if it satisfies (5.3a or 5.3b).
- $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S if it satisfies (5.1a and 5.3a) or (5.1b and 5.3b).
- $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal of S if it satisfies (5.4a or 5.4b).

Definition 235 *Let f be a fuzzy subset of a semigroup S. We define*

$f_r = \{x \in S : x_r \in_\gamma f\} = \{x \in S : f(x) \geq r > \gamma\} = U(f; r)$.
$f_r^\delta = \{x \in S : x_r q_\delta f\} = \{x \in S : f(x) + r > 2\delta\}$.

Theorem 236 *Let f be a fuzzy subset of a semigroup S. Then*

(1) f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S if and only if $U(f;t)(\neq \emptyset)$ is a subsemigroup of S for all $t \in (\delta, 1]$.

(2) f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S if and only if $U(f;t)(\neq \emptyset)$ is a left (right) ideal of S for all $t \in (\delta, 1]$.

(3) f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal (bi-ideal) of S if and only if $U(f;t)(\neq \emptyset)$ is a generalized bi-ideal (bi-ideal) of S for all $t \in (\delta, 1]$.

(4) f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal of S if and only if $U(f;t)(\neq \emptyset)$ is a interior-ideal of S for all $t \in (\delta, 1]$.

Proof. (1) Let f be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S and $x, y \in U(f;t)$ for some $t \in (\delta, 1]$. Then $f(x) \geq t$ and $f(y) \geq t$. By hypothesis $\max\{f(xy), \delta\} \geq \min\{f(x), f(y)\} = t \Rightarrow f(xy) \geq t$. Hence $xy \in U(f;t)$, that is $U(f;t)$ is a subsemigroup of S.

Conversely, assume that $U(f;t) \neq \emptyset$ is a subsemigroup of S for all $t \in (\delta, 1]$. Suppose that there exist $x, y \in S$ such that $\max\{f(xy), \delta\} < \min\{f(x), f(y)\}$. Choose $t \in (\delta, 1]$ such that $\max\{f(xy), \delta\} < t \leq \min\{f(x), f(y)\}$. This implies $f(x) \geq t$, $f(y) \geq t$ and $f(xy) < t$, that is $x, y \in U(f;t)$ but $xy \notin U(f;t)$, which is a contradiction. Hence $\max\{f(xy), \delta\} \geq \min\{f(x), f(y)\}$, that is f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

Similarly we can prove (2), (3) and (4). ■

From the above Theorem it follows that

(1) Every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal of a semigroup S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal of S.

(2) Every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of a semigroup S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal ideal of S.

(3) Every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of a semigroup S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal of S.

Theorem 237 *Let f be a fuzzy subset of a semigroup S and $2\delta = 1 + \gamma$. Then*

(1) f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S if and only if $f_r^\delta(\neq \emptyset)$ is a subsemigroup of S for all $r \in [0, \delta)$

(2) f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left(right) ideal of S if and only if $f_r^\delta(\neq \emptyset)$ is a left(right) ideal of S for all $r \in [0, \delta)$

(3) f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal (bi-ideal) of S if and only if $f_r^\delta(\neq \emptyset)$ is a generalized bi-ideal (bi-ideal) of S for all $r \in [0, \delta)$

(4) f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal of S if and only if $f_r^\delta(\neq \emptyset)$ is a interior-ideal of S for all $r \in [0, \delta)$.

Proof. (1) Suppose f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S and $x, y \in f_r^\delta$. Then $f(x) + r > 2\delta$ and $f(y) + r > 2\delta \Rightarrow f(x) > 2\delta - r > 2\delta - \delta = \delta$ (because

$r < \delta$, so $2\delta - r > 2\delta - \delta = \delta$) and similarly $f(y) > \delta$. By hypothesis $\max\{f(xy), \delta\} \geq \min\{f(x), f(y)\} > \delta$

$\Rightarrow f(xy) \geq \min\{f(x), f(y)\}$

$\Rightarrow f(xy) > \min\{2\delta - r, 2\delta - r\} = 2\delta - r$.

Thus $f(xy) > 2\delta - r \Rightarrow f(xy) + r > 2\delta \Rightarrow xy \in f_r^\delta$. Hence f_r^δ is a subsemigroup of S.

Conversely, assume that $f_r^\delta(\neq \emptyset)$ is a subsemigroup of S for all $r \in [0, \delta)$. Let $x, y \in S$ be such that $\max\{f(xy), \delta\} < \min\{f(x), f(y)\} \Rightarrow 2\delta - \min\{f(x), f(y)\} < 2\delta - \max\{f(xy), \delta\}$

$\Rightarrow \max\{2\delta - f(x), 2\delta - f(y)\} < \min\{2\delta - f(xy), 2\delta - \delta\}$.

Take $r \in [0, \delta)$ such that $\max\{2\delta - f(x), 2\delta - f(y)\} < r \leq \min\{2\delta - f(xy), \delta\}$. Then $2\delta - f(x) < r, 2\delta - f(y) < r$ and $r \leq 2\delta - f(xy) \Rightarrow f(x) + r > 2\delta$ and $f(y) + r > 2\delta$ but $f(xy) + r \leq 2\delta$, that is $x \in f_r^\delta, y \in f_r^\delta$ but $(xy) \notin f_r^\delta$, which is a contradiction. Hence $\max\{f(xy), \delta\} \geq \min\{f(x), f(y)\}$.

Similarly, we can prove the parts (2), (3) and (4). ■

Theorem 238 *(1) The intersection of any family of $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroups of S is again an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.*

(2) The intersection of any family of $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideals of S is again an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.

(3) The intersection of any family of $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal (bi-ideal) of S is again an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal (bi-ideal) of S.

(4) The intersection of any family of $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideals of S is again an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal of S.

Proof. (1) Let $\{f_i\}_{i \in I}$ be a family of $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroups of S and $x, y \in S$. Then $((\bigwedge_{i \in I} f_i)(xy)) \vee \delta = (\bigwedge_{i \in I} f_i(xy)) \vee \delta = (\bigwedge_{i \in I} ((f_i(xy)) \vee \delta)) \geq (\bigwedge_{i \in I} (\min\{f_i(x), f_i(y)\})$

$= (\bigwedge_{i \in I} f_i(x)) \wedge (\bigwedge_{i \in I} f_i(y)) = ((\bigwedge_{i \in I} f_i)(x)) \wedge ((\bigwedge_{i \in I} f_i)(y))$.

Thus $\bigwedge_{i \in I} f_i$ is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroups of S.

Similarly, we can prove the parts (2), (3) and (4). ■

Theorem 239 *The union of any family of $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideals of S is again an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S.*

Proof. Let $\{f_i\}_{i \in I}$ be a family of $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideals of S and $x, y \in S$. Then $((\bigvee_{i \in I} f_i)(xy)) \vee \delta = (\bigvee_{i \in I} f_i(xy)) \vee \delta = \bigvee_{i \in I} ((f_i(xy)) \vee \delta) \geq \bigvee_{i \in I} (f_i(y)) = (\bigvee_{i \in I} f_i)(x)$. Thus $\bigvee_{i \in I} f_i$ is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal of S. ■

Proposition 240 *Let f be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal of S and g be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal of S. Then fg is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal of S.*

Proof. Let f, g be $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left and right ideal of S, respectively. Let $x, y \in S$. Then $(fg)(y) = (\bigvee_{y=ab} (f(a) \wedge g(b)) = \bigvee_{y=ab} (f(a) \wedge g(b)) \leq \bigvee_{y=ab} ((f(xa) \vee \delta) \wedge g(b))$ $= \bigvee_{y=ab} ((f(xa) \wedge g(b)) \vee \delta \leq \bigvee_{xy=cd} (f(c) \wedge g(d)) \vee \delta = (fg)(xy) \vee \delta$.

Similarly, we can show that $(fg)(x) \leq (fg)(xy) \vee \delta$. This shows that fg is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal of S. ■

Next we show that if f and g are $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideals of S, then $fg \not\leq f \wedge g$.

Example 241 *Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table*

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define fuzzy subset f, g of S by $f(a) = 0.9,\ f(b) = 0.8,\ f(c) = 0.7,\ f(d) = 0.7,$ $g(a) = 0.95,\ g(b) = 0.3,\ g(c) = 0.7,\ g(d) = 0.6$.

Then

$$U(f;t) = \begin{cases} \{a,b,c,d\} & 0 < t \leq 0.7 \\ \{a,b\} & 0.7 < t \leq 0.8 \\ \{a\} & 0.8 < t \leq 0.9 \\ \emptyset & 0.9 < t \end{cases}$$

$$U(g;t) = \begin{cases} \{a,b,c,d\} & 0 < t \leq 0.3 \\ \{a,c,d\} & 0.3 < t \leq 0.6 \\ \{a,c\} & 0.6 < t \leq 0.7 \\ \{a\} & 0.7 < t \leq 0.95 \\ \emptyset & 0.95 < t \end{cases}$$

By Theorem 236, f and g are $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideals of S for $\gamma = 0.6$ and $\delta = 0.8$. But $fg(b) = \bigvee_{b=xy} \{f(x) \wedge g(y)\} = \bigvee\{0.7, 0.7, 0.6\} = 0.7 \not\leq (f \wedge g)(b) = 0.3$.
Hence $fg \not\leq f \wedge g$ in general.

Definition 242 *Let f, g be fuzzy subsets of a semigroup S. We define the fuzzy subsets $\overset{\diamond}{f}, f \wedge^\diamond g, f \vee^\diamond g$ and $f \diamond g$ of S as follows:*

$\overset{\diamond}{f}(x) = f(x) \vee \delta$

$(f \wedge^{\diamond} g)(x) = ((f \wedge g)(x)) \vee \delta)$
$(f \vee^{\diamond} g)(x) = ((f \vee g)(x)) \vee \delta$
$(f \diamond g)(x) = ((fg)(x)) \vee \delta$
for all $x \in S$.

Lemma 243 *Let f, g be fuzzy subsets of a semigroup S. Then the following hold:*

(1) $f \wedge^{\diamond} g = \overset{\diamond}{f} \wedge \overset{\diamond}{g}$
(2) $f \vee^{\diamond} g = \overset{\diamond}{f} \vee \overset{\diamond}{g}$
(3) $f \diamond g \geq \overset{\diamond}{f}\overset{\diamond}{g}$.
Proof. Let $x \in S$.
(1) $(f \wedge^{\diamond} g)(x) = ((f \wedge g)(x)) \vee \delta = (f(x) \wedge g(x)) \vee \delta$
$= (f(x) \vee \delta) \wedge (g(x) \vee \delta)) = \overset{\diamond}{f}(x) \wedge \overset{\diamond}{g}(x) = (\overset{\diamond}{f} \wedge \overset{\diamond}{g})(x)$.
(2) $(f \vee^{\diamond} g)(x) = ((f \vee g)(x)) \vee \delta = (f(x) \vee g(x)) \vee \delta$
$= (f(x) \vee \delta) \vee (g(x) \vee \delta)) = \overset{\diamond}{f}(x) \vee \overset{\diamond}{g}(x) = (\overset{\diamond}{f} \vee \overset{\diamond}{g})(x)$.
(3) If x is not expressible as $x = yz$ for all $y, z \in S$, then $(f \diamond g)(x) = ((fg)(x)) \vee \delta = \delta \geq 0 = \overset{\diamond}{f}\overset{\diamond}{g}(x)$. Otherwise $(f \diamond g)(x) = (fg)(x) \vee \delta = \left(\bigvee_{x=yz} fg(x)\right) \vee \delta = \left(\bigvee_{x=yz} (f(y) \wedge (g(z))\right) \vee \delta$

$$= \left(\bigvee_{x=yz} ([f(y) \vee \delta] \wedge [g(z) \vee \delta])\right) = \bigvee_{x=yz} \left(\overset{\diamond}{f}(y) \wedge \overset{\diamond}{g}(z)\right) = \left(\overset{\diamond}{f}\overset{\diamond}{g}\right)(x).$$ ■

Lemma 244 *Let f, g be $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right and $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideals of S, respectively. Then $f \diamond g \leq f \wedge^{\diamond} g$.*

Proof. Straightforward. ■

Lemma 245 *Let A, B be non-empty subsets of a semigroup S. Then the following hold.*

(1) $C_A \wedge^{\diamond} C_B = C^{\diamond}_{A \cap B}$.
(2) $C_A \vee^{\diamond} C_B = C^{\diamond}_{A \cup B}$.
(2) $C_A \diamond C_B = C^{\diamond}_{AB}$.
Where C_A is the characteristic function of A.
Proof. Straightforward. ■

Theorem 246 *(1) A fuzzy subset f of a semigroup S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S if and only if $f \diamond f \leq \overset{\diamond}{f}$.*

(2) A fuzzy subset f of a semigroup S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal of S if and only if $f \diamond \mathcal{S} \diamond f \leq \overset{\diamond}{f}$.

(3) A fuzzy subset f of a semigroup S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left (right) ideal of S if and only if $\mathcal{S} \diamond f \leq \overset{\diamond}{f}$ $(f \diamond \mathcal{S} \leq \overset{\diamond}{f})$.

(4) A fuzzy subset f of a semigroup S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S if and only if $f \diamond f \leq \overset{\diamond}{f}$ and $f \diamond \mathcal{S} \diamond f \leq \overset{\diamond}{f}$.

Where $\mathcal{S}$ is a fuzzy subset of S mapping every element of S on 1.

Proof. (1) Let f be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S and $x \in S$. If $(f \diamond f)(x) = \delta$, then $(f \diamond f)(x) = \delta \leq f(x) \vee \delta$. Otherwise

$$(f \diamond f)(x) = \left(\bigvee_{x=ab} \{f(a) \wedge f(b)\} \right) \vee \delta$$
$$\leq \left(\bigvee_{x=ab} (f(ab) \vee \delta) \right) \vee \delta = f(x) \vee \delta = \overset{\diamond}{f}(x).$$

Thus $f \diamond f \leq \overset{\diamond}{f}$.

Conversely, assume that $f \diamond f \leq \overset{\diamond}{f}$ and $x, y \in S$. Then

$$f(xy) \vee \delta = \overset{\diamond}{f}(xy) \geq f \diamond f(xy)$$
$$= \left(\bigvee_{xy=ab} \{f(a) \wedge f(b)\} \right) \vee \delta \geq (f(x) \wedge f(y)) \vee \delta \geq (f(x) \wedge f(y)).$$

Thus f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S.

(2) Let f be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal of S and $x \in S$. If $(f \diamond \mathcal{S} \diamond f)(x) = \delta$, then $(f \diamond \mathcal{S} \diamond f)(x) = \delta \leq f(x) \vee \delta = \overset{\diamond}{f}(x)$.

Otherwise

$(f \diamond \mathcal{S} \diamond f)(x) = (\bigvee_{x=ab} \{f(a) \wedge (\mathcal{S} \diamond f)(b)\}) \vee \delta = (\bigvee_{x=ab} \{f(a) \wedge [(\bigvee_{b=cd} \{\mathcal{S}(c) \wedge f(d)\}) \vee \delta]\} \vee \delta)$

$$= (\bigvee_{x=ab} \{f(a) \wedge [(\bigvee_{b=cd} f(d)) \vee \delta]\} \vee \delta) = ((\bigvee_{x=ab} \{\bigvee_{b=cd} [f(a) \wedge f(d)] \vee \delta\} \vee \delta)$$
$$= \bigvee_{x=ab} \{\bigvee_{b=cd} [f(a) \wedge f(d)]\} \vee \delta \leq (\bigvee_{x=ab} \{\bigvee_{b=cd} \{f(acd) \vee \delta\}) \vee \delta = f(x) \vee \delta = \overset{\diamond}{f}(x).$$

Thus $f \diamond \mathcal{S} \diamond f \leq \overset{\diamond}{f}$.

Conversely, assume that $f \diamond \mathcal{S} \diamond f \leq \overset{\diamond}{f}$ and $x, y, z \in S$. Then

$$f(xyz) \vee \delta = \overset{\diamond}{f}(xyz) \geq (f \diamond \mathcal{S} \diamond f)(xyz)$$
$$= (\bigvee_{xyz=ab} \{f(a) \wedge (\mathcal{S} \diamond f)(b)\}) \vee \delta \geq (f(x) \wedge (\mathcal{S} \diamond f)(yz)) \vee \delta$$
$$= (f(x) \wedge [(\bigvee_{yz=cd} \{\mathcal{S}(c) \wedge f(d)\}) \vee \delta]) \vee \delta$$
$$\geq ((f(x) \wedge (f(z) \vee \delta)) \vee \delta \geq ((f(x) \wedge f(z)) \vee \delta \geq f(x) \wedge f(z).$$

Thus f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal of S.

(3) The proof is similar to the proof of (2).

(4) Follows from (1) and (3). ■

Definition 247 *A fuzzy subset f of a semigroup S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal of S if and only if $(f \diamond \mathcal{S}) \wedge (\mathcal{S} \diamond f) \leq \overset{\diamond}{f}$.*

Lemma 248 *A non-empty subset A of a semigroup S is a quasi-ideal of S if and only if the characteristic function C_A of A is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal of S.*

Proof. Straightforward. ■

5.4 Regular Semigroups in Terms of $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy Ideals

In this section we characterize regular semigroups in terms of $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideals, $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideals and $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideals.

Proposition 249 *In a regular semigroup S, every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal of S.*

Proof. Let f be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal of S and $x, y \in S$. Then there exist $a, b \in S$ such that $x = xax$ and $y = yby$. Now

$\max\{f(xy), \delta\} = \max\{f((xa)xy), \delta\} \geq f(x)$.

Also

$\max\{f(xy), \delta\} = \max\{f(xy(by)), \delta\} \geq f(y)$.

This shows that f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal of S. ■

Proposition 250 *In a regular semigroup S, every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S.*

Proof. Let f be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal of S and $x, y \in S$. Then there exists $a \in S$ such that $x = xax$. Now

$\max\{f(xy), \delta\} = \max\{f(x(ax)y), \delta\} \geq \min\{f(x), f(y)\}$.

This shows that f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S. ■

Theorem 251 *The following assertions are equivalent for a semigroup S:*

(1) S is regular.

(2) $f \overset{\diamond}{\wedge} g = f \diamond g$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal g of S.

Proof. (1) $\Rightarrow$ (2) Let f be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal and g be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal of S. Then by Lemma 244, $f \diamond g \leq f \overset{\diamond}{\wedge} g$.

Let $x \in S$. Then there exists $a \in S$ such that $x = xax$. Now

$$\begin{aligned}
(f \diamond g)(x) &= \left(\bigvee_{x=yz} (f(y) \wedge g(z)) \right) \vee \delta \\
&\geq (f(x) \wedge g(ax)) \vee \delta \\
&= (f(x) \wedge (g(ax) \vee \delta)) \vee \delta \\
&\geq (f(x) \wedge g(x)) \vee \delta \\
&= (f(x) \wedge g(x)) \vee \delta \\
&= \left(f \overset{\diamond}{\wedge} g \right)(x).
\end{aligned}$$

Thus $f \diamond g \geq f \overset{\diamond}{\wedge} g$. Hence $f \diamond g = f \overset{\diamond}{\wedge} g$.

(2) $\Rightarrow$ (1) Let R be a right ideal and L a left ideal of S. Then C_R and C_L are $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right and $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideals of S, respectively. By hypothesis $C_R \diamond C_L = C_R \overset{\diamond}{\wedge} C_L$. By Lemma 245, this implies that $\overset{\diamond}{C}_{RL} = \overset{\diamond}{C}_{R\cap L} \Rightarrow RL = R \cap L$. Hence S is a regular semigroup. ■

Theorem 252 *The following assertions are equivalent for a semigroup S:*

(1) S is regular.

(2) $f \overset{\diamond}{\wedge} g \overset{\diamond}{\wedge} h \leq f \diamond g \diamond h$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal f, every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal g and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal h of S.

(3) $f \overset{\diamond}{\wedge} g \overset{\diamond}{\wedge} h \leq f \diamond g \diamond h$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal f, every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal g and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal h of S.

(4) $f \overset{\diamond}{\wedge} g \overset{\diamond}{\wedge} h \leq f \diamond g \diamond h$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal f, every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal g and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal h of S.

Proof. (1) $\Rightarrow$ (2) Let f, g, h be any $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal, $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal and $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal of S, respectively. Let $x \in S$. Then there exists $a \in S$ such that $x = xax$. Thus we have

$$\begin{aligned}
(f \diamond g \diamond h)(x) &= \left(\bigvee_{x=yz} (f(y) \wedge (g \diamond h)(z)) \right) \vee \delta \\
&\geq (f(xa) \wedge (g \diamond h)(x)) \vee \delta \\
&= (((f(xa) \vee \delta) \wedge (g \diamond h)(x)) \vee \delta) \\
&\geq (f(x) \wedge (g \diamond h)(x)) \vee \delta \\
&= \left(f(x) \wedge \left(\left(\bigvee_{x=yz} (g(y) \wedge h(z)) \right) \vee \delta \right) \vee \delta \right) \\
&\geq (f(x) \wedge (g(x) \wedge h(ax)) \vee \delta) \vee \delta \\
&= [f(x) \wedge ((g(x) \wedge (h(ax) \vee \delta))) \vee \delta] \\
&\geq (f(x) \wedge ((g(x) \wedge h(x))) \vee \delta) \\
&= (f(x) \wedge g(x) \wedge h(x)) \vee \delta \\
&= \left(f \overset{\diamond}{\wedge} g \overset{\diamond}{\wedge} h \right)(x).
\end{aligned}$$

Thus $f \overset{\diamond}{\wedge} g \overset{\diamond}{\wedge} h \leq f \diamond g \diamond h$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are straightforward.

(4) $\Rightarrow$ (1) Let f be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal and g be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal of S. Since $\mathcal{S}$ is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal of S, so by hypothesis we have

$$\left(f \overset{\diamond}{\wedge} g\right)(x) = \left(f \overset{\diamond}{\wedge} \mathcal{S} \overset{\diamond}{\wedge} g\right)(x) \leq (f \diamond \mathcal{S} \diamond g)(x) \leq (f \diamond g)(x).$$

But $\left(f \overset{\diamond}{\wedge} g\right)(x) \geq (f \diamond g)(x)$ always holds. Thus $f \overset{\diamond}{\wedge} g = f \diamond g$. Hence by Theorem 251, S is regular. ■

Theorem 253 *The following assertions are equivalent for a semigroup S:*

(1) S is regular.

(2) $\overset{\diamond}{f} = f \diamond \mathcal{S} \diamond f$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal f of S.

(3) $\overset{\diamond}{f} = f \diamond \mathcal{S} \diamond f$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal f of S.

(4) $\overset{\diamond}{f} = f \diamond \mathcal{S} \diamond f$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal f of S.

Proof. (1) $\Rightarrow$ (2) Let f be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal of S and $x \in S$. Then there exists $a \in S$ such that $x = xax$. Thus we have

$$\begin{aligned}
(f \diamond \mathcal{S} \diamond f)(x) &= \left(\bigvee_{x=yz} ((f \diamond \mathcal{S})(y) \wedge f(z))\right) \vee \delta \\
&\geq ((f \diamond \mathcal{S})(xa) \wedge f(x)) \vee \delta \\
&= \left(\left[\left(\bigvee_{xa=bc} ((f)(b) \wedge \mathcal{S}(c))\right) \vee \delta\right] \wedge f(x)\right) \vee \delta \\
&\geq ([((f)(x) \wedge \mathcal{S}(a)) \vee \delta] \wedge f(x)) \vee \delta \\
&= (f(x) \wedge f(x)) \vee \delta \\
&= \overset{\diamond}{f}(x).
\end{aligned}$$

Thus $f \diamond \mathcal{S} \diamond f \geq \overset{\diamond}{f}$.

As f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal of S, so we have

$$\begin{aligned}
(f \diamond \mathcal{S} \diamond f)(x) &= \left(\bigvee_{x=yz} ((f \diamond \mathcal{S})(y) \wedge f(z))\right) \vee \delta \\
&= \left(\bigvee_{x=yz} \left(\left[\left(\bigvee_{y=bc} ((f)(b) \wedge \mathcal{S}(c))\right) \vee \delta\right] \wedge f(z)\right)\right) \vee \delta \\
&= \left(\bigvee_{x=yz} \left(\left(\bigvee_{y=bc} ((f)(b) \wedge 1)\right) \wedge f(z)\right)\right) \vee \delta \\
&= \left(\bigvee_{x=yz} \left(\bigvee_{y=bc} ((f)(b) \wedge f(z))\right)\right) \vee \delta \\
&= \left(\bigvee_{x=yz} \left(\bigvee_{y=bc} ((f)(b) \wedge f(z))\right)\right) \vee \delta \\
&\leq \left(\bigvee_{x=yz} \left(\bigvee_{y=bc} (f(bcz) \vee \delta)\right)\right) \vee \delta
\end{aligned}$$

$$= \left(\bigvee_{x=yz} \left(\bigvee_{y=bc} f(x) \right) \right) \vee \delta$$
$$= \overset{\diamond}{f}(x).$$

Thus $f \diamond \mathcal{S} \diamond f \leq \overset{\diamond}{f}$. Hence $f \diamond \mathcal{S} \diamond f = \overset{\diamond}{f}$.

(2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (1) Let Q be any quasi-ideal of S. Then C_Q is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal of S. Hence by hypothesis $\overset{\diamond}{C}_Q = C_Q \diamond \mathcal{S} \diamond C_Q = \overset{\diamond}{C}_{QSQ}$. This implies that $Q = QSQ$. So by Theorem 23, S is regular. ■

Theorem 254 *The following assertions are equivalent for a semigroup S:*

(1) S is regular.

(2) $f \overset{\diamond}{\wedge} g = f \diamond g \diamond f$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal g of S.

(3) $f \overset{\diamond}{\wedge} g = f \diamond g \diamond f$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal g of S.

(4) $f \overset{\diamond}{\wedge} g = f \diamond g \diamond f$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal g of S.

(5) $f \overset{\diamond}{\wedge} g = f \diamond g \diamond f$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal g of S.

(6) $f \overset{\diamond}{\wedge} g = f \diamond g \diamond f$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal g of S.

(7) $f \overset{\diamond}{\wedge} g = f \diamond g \diamond f$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal g of S.

Proof. (1) ⇒ (7) Let f and g be any $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal and $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal of S, respectively. Then for any $x \in S$

$$(f \diamond g \diamond f)(x) \leq (f \diamond \mathcal{S} \diamond f)(x) = \left(\bigvee_{x=yz} ((f \diamond \mathcal{S})(y) \wedge f(z)) \right) \vee \delta$$
$$= \left(\bigvee_{x=yz} \left(\left[\left(\bigvee_{y=bc} ((f)(b) \wedge \mathcal{S}(c)) \right) \vee \delta \right] \wedge f(z) \right) \right) \vee \delta$$
$$= \left(\bigvee_{x=yz} \left(\left(\bigvee_{y=bc} ((f)(b) \wedge 1) \right) \wedge f(z) \right) \right) \vee \delta$$
$$= \left(\bigvee_{x=yz} \left(\bigvee_{y=bc} ((f)(b) \wedge f(z)) \right) \right) \vee \delta$$
$$= \bigvee_{x=yz} \bigvee_{y=bc} ((f)(b) \wedge f(z)) \vee \delta$$
$$\leq \left(\bigvee_{x=yz} \left(\bigvee_{y=bc} (f(bcz) \vee \delta) \right) \right) \vee \delta$$

$= \left(\bigvee_{x=yz} \left(\bigvee_{y=bc} f(x) \right) \right) \vee \delta$
$= \overset{\diamond}{f}(x).$
Thus $f \diamond \mathcal{S} \diamond f \leq \overset{\diamond}{f}$.
Also
$(f \diamond g \diamond f)(x) \leq (\mathcal{S} \diamond g \diamond \mathcal{S})(x) = \left(\bigvee_{x=yz} ((\mathcal{S} \diamond g)(y) \wedge \mathcal{S}(z)) \right) \vee \delta$
$= \left(\bigvee_{x=yz} \left(\left[\left(\bigvee_{y=bc} (\mathcal{S}(b) \wedge g(c)) \right) \vee \delta \right] \wedge \mathcal{S}(z) \right) \right) \vee \delta$
$= \left(\bigvee_{x=yz} \left(\left(\bigvee_{y=bc} (\mathcal{S}(b) \wedge g(c) \right) \wedge \mathcal{S}(z) \right) \right) \vee \delta$
$= \left(\bigvee_{x=yz} \left(\bigvee_{y=bc} g(c) \right) \right) \vee \delta$
$\leq \left(\bigvee_{x=yz} \bigvee_{y=bc} (g(bcz) \vee \delta) \right) \vee \delta$
$= \left(\bigvee_{x=yz} \left(\bigvee_{y=bc} g(x) \right) \right) \vee \delta$
$= \overset{\diamond}{g}(x).$
Thus $f \diamond g \diamond f \leq f \overset{\diamond}{\wedge} g$.
Now let $x \in S$. Then there exists $a \in S$ such that $x = xax = xaxax$.
$(f \diamond g \diamond f)(x) = \left[\left(\bigvee_{x=yz} (f(y) \wedge (g \diamond f)(z)) \right) \vee \delta \right]$
$\geq [(f(x) \wedge (g \diamond h)(axax)) \vee \delta]$
$= \left[\left(f(x) \wedge \left(\bigvee_{axax=yz} (g(y) \wedge f(z)) \right) \vee \delta \right) \vee \delta \right]$
$\geq [(f(x) \wedge (g(axa) \wedge f(x)) \vee \delta)]$
$= [(f(x) \wedge ([g(axa) \vee \delta] \wedge f(x)) \vee \delta)]$
$\geq [(f(x) \wedge ([g(x)] \wedge f(x)) \vee \delta)]$
$= [(f(x) \wedge (g(x) \wedge f(x)) \vee \delta)]$
$= \left(f \overset{\diamond}{\wedge} g \right)(x).$

This shows that $f \diamond g \diamond f \geq f \overset{\diamond}{\wedge} g$. Hence $f \diamond g \diamond f = f \overset{\diamond}{\wedge} g$.
(7) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (2) and (7) $\Rightarrow$ (6) $\Rightarrow$ (4) $\Rightarrow$ (2) are clear.
(2) $\Rightarrow$ (1) Let f be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal of S. Then
$\overset{\diamond}{f} = f \overset{\diamond}{\wedge} \mathcal{S} = f \diamond \mathcal{S} \diamond f$.
Thus it follows from Theorem 253, that S is regular. ■

Theorem 255 *The following assertions are equivalent for a semigroup S:*

(1) S is regular.

(2) $f \overset{\diamond}{\wedge} g \leq f \diamond g$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal g of S.

(3) $f \overset{\diamond}{\wedge} g \leq f \diamond g$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal g of S.

(4) $f \overset{\diamond}{\wedge} g \leq f \diamond g$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal g of S.

Proof. (1) $\Rightarrow$ (4) Let f and g be any $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal and $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal of S, respectively. Let $x \in S$. Then there exists $a \in S$ such that $x = xax$. Thus we have

$$\begin{aligned}(f \diamond g)(x) &= \left(\bigvee_{x=yz} (f(y) \wedge g(z)) \right) \vee \delta \\ &\geq (f(x) \wedge g(ax)) \vee \delta \\ &= f(x) \wedge g(ax) \vee \delta \\ &\geq f(x) \wedge g(x)) \vee \delta \\ &= \left(f \overset{\diamond}{\wedge} g \right)(x).\end{aligned}$$

Thus $f \diamond g \geq f \overset{\diamond}{\wedge} g$.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (1) Let f and g be any $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal and $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal of S, respectively. Then by hypothesis $f \diamond g \geq f \overset{\diamond}{\wedge} g$. But $f \diamond g \leq f \overset{\diamond}{\wedge} g$ always holds. Thus $f \diamond g = f \overset{\diamond}{\wedge} g$. Hence by Theorem 251, S is regular. ■

5.5 Intra-Regular Semigroups in Terms of $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy Ideals

In this section we characterize intra-regular semigroups by the properties of their $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideals, $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideals and $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideals.

Theorem 256 *The following assertions are equivalent for a semigroup S:*

(1) S is intra regular.

(2) $f \overset{\diamond}{\wedge} g \leq f \diamond g$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal g of S.

Proof. (1) $\Rightarrow$ (2) Let f be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal and g be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal of S. Let $x \in S$. Then there exist $a, b \in S$ such that $x = axxb$. Now

$$\begin{aligned}(f \diamond g)(x) &= \left(\bigvee_{x=yz} (f(y) \wedge g(z)) \right) \vee \delta \\ &\geq (f(ax) \wedge g(xb)) \vee \delta\end{aligned}$$

$= ((f(ax) \vee \delta) \wedge (g(xb) \vee \delta)) \vee \delta$
$\geq (f(x) \wedge g(x)) \vee \delta$
$= \left(f \overset{\diamond}{\wedge} g\right)(x).$

Thus $f \diamond g \geq f \overset{\diamond}{\wedge} g$.

(2) $\Rightarrow$ (1) Let R be any right ideal and L be any left ideal of S. Then C_R and C_L are $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right and $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideals of S, respectively. By hypothesis $C_L \diamond C_R \geq C_L \overset{\diamond}{\wedge} C_R$. By Lemma ?, this implies that $\overset{\diamond}{C}_{LR} \geq \overset{\diamond}{C}_{L\cap R} \Rightarrow LR \supseteq L \cap R$. Hence S is an intra regular semigroup. ■

Theorem 257 *The following assertions are equivalent for a semigroup S:*

(1) S is both regular and intra regular.
(2) $\overset{\diamond}{f} = f \diamond f$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal f of S.
(3) $\overset{\diamond}{f} = f \diamond f$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal f of S.
(4) $f \overset{\diamond}{\wedge} g \leq f \diamond g$ for all $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideals f, g of S.
(5) $f \overset{\diamond}{\wedge} g \leq f \diamond g$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal g of S.
(6) $f \overset{\diamond}{\wedge} g \leq f \diamond g$ for all $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideals f, g of S.

Proof. (1) $\Rightarrow$ (6) Let f, g be $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideals of S and $x \in S$. Then there exist $a, b, c \in S$ such that $x = axxb$ and $x = xcx$. Thus $x = xcx = xcxcx = xc(axxb)cx = (xcax)(xbcx)$. Thus we have

$(f \diamond g)(x) = \left(\bigvee_{x=yz} (f(y) \wedge g(z))\right) \vee \delta$
$\geq (f(xcax) \wedge g(xbcx)) \vee \delta$
$= ((f(xcax) \vee \delta) \wedge (g(xbcx) \vee \delta)) \vee \delta$
$\geq (f(x) \wedge g(x)) \vee \delta$
$= \left(f \overset{\diamond}{\wedge} g\right)(x).$

Thus $f \diamond g \geq f \overset{\diamond}{\wedge} g$.

(6) $\Rightarrow$ (5) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (2) Take $f = g$ in (4), we get $f \diamond f \geq \overset{\diamond}{f}$. Since every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S, so $f \diamond f \leq \overset{\diamond}{f}$. Thus $f \diamond f = \overset{\diamond}{f}$.

(6) $\Rightarrow$ (3) Take $f = g$ in (6), we get $f \diamond f \geq \overset{\diamond}{f}$. Since every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy subsemigroup of S, so $f \diamond f \leq \overset{\diamond}{f}$. Thus $f \diamond f = \overset{\diamond}{f}$.

(3) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (1) Let Q be a quasi-ideal of S. Then C_Q is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal of S. Hence by hypothesis $C_Q \diamond C_Q = \overset{\diamond}{C}_Q$. This implies that $\overset{\diamond}{C}_{QQ} = \overset{\diamond}{C}_Q$, that is

$QQ = Q$. Hence by Theorem 28, S is both regular and intra regular. ■

Theorem 258 *The following assertions are equivalent for a semigroup S:*

(1) S is both regular and intra regular.

(2) $f \overset{\diamond}{\wedge} g \leq (f \diamond g) \wedge (g \diamond f)$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal g of S.

(3) $f \overset{\diamond}{\wedge} g \leq (f \diamond g) \wedge (g \diamond f)$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal g of S.

(4) $f \overset{\diamond}{\wedge} g \leq (f \diamond g) \wedge (g \diamond f)$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal g of S.

(5) $f \overset{\diamond}{\wedge} g \leq (f \diamond g) \wedge (g \diamond f)$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal g of S.

(6) $f \overset{\diamond}{\wedge} g \leq (f \diamond g) \wedge (g \diamond f)$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal g of S.

(7) $f \overset{\diamond}{\wedge} g \leq (f \diamond g) \wedge (g \diamond f)$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal g of S.

(8) $f \overset{\diamond}{\wedge} g \leq (f \diamond g) \wedge (g \diamond f)$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal g of S.

(9) $f \overset{\diamond}{\wedge} g \leq (f \diamond g) \wedge (g \diamond f)$ for all $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal f, g of S.

(10) $f \overset{\diamond}{\wedge} g \leq (f \diamond g) \wedge (g \diamond f)$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal g of S.

(11) $f \overset{\diamond}{\wedge} g \leq (f \diamond g) \wedge (g \diamond f)$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy quasi-ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal g of S.

(12) $f \overset{\diamond}{\wedge} g \leq (f \diamond g) \wedge (g \diamond f)$ for all $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideals f, g of S.

(13) $f \overset{\diamond}{\wedge} g \leq (f \diamond g) \wedge (g \diamond f)$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy bi-ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal g of S.

(14) $f \overset{\diamond}{\wedge} g \leq (f \diamond g) \wedge (g \diamond f)$ for all $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideal f, g of S.

Proof. (1) ⇒ (14) Let f, g be $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy generalized bi-ideals of S. Let $x \in S$. Then there exist $a, b, c \in S$ such that $x = axxb$ and $x = xcx$. Thus $x = xcx = xcxcx = xc(axxb)cx = (xcax)(xbcx)$. Thus we have

$$\begin{aligned}
(f \diamond g)(x) &= \left(\bigvee_{x=yz} (f(y) \wedge g(z)) \right) \vee \delta \\
&\geq (f(xcax) \wedge g(xbcx)) \vee \delta \\
&= ((f(xcax) \vee \delta) \wedge (g(xbcx) \vee \delta)) \vee \delta \\
&\geq (f(x) \wedge g(x)) \vee \delta \\
&= \left(f \overset{\diamond}{\wedge} g \right)(x).
\end{aligned}$$

Thus $f \diamond g \geq f \overset{\diamond}{\wedge} g$. Similarly we can show that $g \diamond f \geq f \overset{\diamond}{\wedge} g$. Hence $f \overset{\diamond}{\wedge} g \leq (f \diamond g) \wedge (g \diamond f)$.

(14) $\Rightarrow$ (13) $\Rightarrow$ (12) $\Rightarrow$ (10) $\Rightarrow$ (9) $\Rightarrow$ (3) $\Rightarrow$ (2)

(14) $\Rightarrow$ (11) $\Rightarrow$ (10)

(14) $\Rightarrow$ (8) $\Rightarrow$ (7) $\Rightarrow$ (6) $\Rightarrow$ (2) and

(14) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (1) Let f be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy right ideal and g be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy left ideal of S. Then by hypothesis we have $f \overset{\diamond}{\wedge} g \leq (f \diamond g) \wedge (g \diamond f)$, that is $f \overset{\diamond}{\wedge} g \leq f \diamond g$ and $f \overset{\diamond}{\wedge} g \leq g \diamond f$. But $f \overset{\diamond}{\wedge} g \geq f \diamond g$ always hold. Hence $f \overset{\diamond}{\wedge} g = f \diamond g$ and $f \overset{\diamond}{\wedge} g \leq g \diamond f$. Thus by Theorems 251 and 257, S is both regular and intra regular. ■

5.6 Semisimple Semigroups

In this section we characterize semisimple semigroups by the properties of their $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideals and $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideals.

Theorem 259 *In a semisimple semigroup S, a fuzzy subset f of S is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal of S if and only if it is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal of S.*

Proof. Let S be a semisimple semigroup and f be an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal of S. Then for any $x, y \in S$ there exist $a, b, c, d \in S$ such that $x = (axb)(cxd)$. Thus we have

$f(xy) \vee \delta = f((axbcxd)y) \vee \delta = f((ax(bc))x(dy)) \vee \delta \geq f(x)$.

Similarly $f(xy) \vee \delta \geq f(y)$. Hence f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal of S.

Conversely, assume that f is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal of S. Then f is always an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal of S. ■

Theorem 260 *For a semigroup S the following assertions are equivalent*

(1) S is semisimple.

(2) $f \diamond f = \overset{\diamond}{f}$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal f of S.

(3) $f \diamond f = \overset{\diamond}{f}$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal f of S.

(4) $f \overset{\diamond}{\wedge} g = f \diamond g$ for all $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideals f, g of S.

(5) $f \overset{\diamond}{\wedge} g = f \diamond g$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal g of S.

(6) $f \overset{\diamond}{\wedge} g = f \diamond g$ for every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal f and every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal g of S.

(7) $f \overset{\diamond}{\wedge} g = f \diamond g$ for all $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideals f, g of S.

Proof. (1) ⇒ (7) Let S be a semisimple semigroup and f, g be $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideals of S. Let $x \in S$. Then there exist $a, b, c, d \in S$ such that $x = (axb)(cxd)$. Thus we have

$$\begin{aligned}(f \diamond g)(x) &= \left(\bigvee_{x=yz} (f(y) \wedge g(z)) \right) \vee \delta \\ &\geq (f(axb) \wedge g(cxd)) \vee \delta \\ &= ((f(axb) \vee \delta) \wedge (g(cxd) \vee \delta)) \vee \delta \\ &\geq (f(x) \wedge g(x)) \vee \delta \\ &= \left(f \overset{\diamond}{\wedge} g \right)(x).\end{aligned}$$

Thus $f \diamond g \geq f \overset{\diamond}{\wedge} g$. Since every $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy interior-ideal of S in a semisimple semigroup is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal of S, so $f \diamond g \leq f \overset{\diamond}{\wedge} g$. Hence $f \diamond g = f \overset{\diamond}{\wedge} g$.

(7) ⇒ (6) ⇒ (4) ⇒ (2), (7) ⇒ (3) ⇒ (2) and (7) ⇒ (5) ⇒ (4) are obvious.

(2) ⇒ (1) Let A be any ideal of S. Then C_A is an $(\overline{\in_\gamma}, \overline{\in_\gamma} \vee \overline{q_\delta})$-fuzzy ideal of S. Thus by hypothesis $C_A \diamond C_A = \overset{\diamond}{C}_A$, that is $AA = A$. Hence S is a semisimple semigroup. ■

Chapter 6

Implication-based Fuzzy Ideals in Semigroups

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\wedge, \vee$, $\neg$, $\rightarrow$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$. For a universe of discourse U, we display the fuzzy logical and corresponding set-theoretical notations used in this chapter

$$[x \in f] = f(x), \tag{6.1}$$

$$[\Phi \wedge \Psi] = \min\{[\Phi], \ [\Psi]\}, \tag{6.2}$$

$$[\Phi \rightarrow \Psi] = \min\{1, \ 1 - [\Phi] + [\Psi]\}, \tag{6.3}$$

$$[\forall x \ \Phi(x)] = \inf_{x \in U} [\Phi(x)], \tag{6.4}$$

$$\models \Phi \text{ if and only if } [\Phi] = 1 \text{ for all valuations.} \tag{6.5}$$

The truth valuation rules given in (6.3) are those in the Łuckasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a section of them in the following

(a) Gaines-Rescher implication operator (I_{GR}):

$$I_{GR}(a, b) = \begin{cases} 1 & \text{if } a \leq b, \\ 0 & \text{otherwise,} \end{cases}$$

(b) Gödel implication operator (I_G):

$$I_G(a, b) = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise,} \end{cases}$$

(c) The contraposition of Gödel implication operator (I_{cG}):

$$I_{cG}(a,b) = \begin{cases} 1 & \text{if } a \leq b, \\ 1-a & \text{otherwise,} \end{cases}$$

(d) The Łuckasiewicz implication operator (I_{LI}):

$$I_{LI}(a,b) = \begin{cases} 1 & \text{if } a \leq b, \\ 1-a+b & \text{otherwise,} \end{cases}$$

for all $a, b \in [0,1]$. Ying [48] introduced the concept of fuzzifying topology. We can expand his/her idea to semigroups, and we define fuzzifying left (resp. right) ideal as follows:

Definition 261 *A fuzzy subset f of S is called a fuzzifying left (resp. right) ideal of S if it satisfies the following condition:*

$$\models [x \in f] \to [yx \in f] \ \ (resp.\ \ [xy \in f]\,) \tag{6.6}$$

for all $x, y \in S$.

Obviously, condition (6.6) is equivalent to (1). Therefore a fuzzifying left (resp. right) ideal is an ordinary fuzzy left (resp. right) ideal. In [48] the concept of t-tautology is introduced, that is,

$$\models_t \Phi \text{ if and only if } [\Phi] \geq t \text{ for all valuations.}$$

Now we extend the concept of implication-based fuzzy left (resp. right) ideal in the following way:

Definition 262 *Let f be a fuzzy subset of S and $t \in (0,1]$. Then f is called a t-implication-based fuzzy left (resp. right) ideal of S if and only if it satisfies:*

$$\models_t [x \in f] \to [yx \in f] \ \ (resp.\ \ [xy \in f]\,) \tag{6.7}$$

for all $x, y \in S$.

Let I be an implication operator. Clearly, f is a t-implication-based fuzzy left (resp. right) ideal of S if and only if it satisfies

$$I(f(x), f(yx)) \geq t \ \ (\text{resp. } I(f(x), f(xy)) \geq t)$$

for all $x, y \in S$

Note that if $t_1, t_2 \in (0,1]$ with $t_1 > t_2$, then every t_1-implication-based fuzzy left ideal of S is a t_1-implication-based fuzzy left ideal of S.

Lemma 263 *A fuzzy subset f of S is an $(\in, \in \vee q)$-fuzzy left ideal of S if and only if it satisfies:*

$$(\forall x, y \in S)\,(f(xy) \geq \min\{f(y)\ \ 0.5\})\,. \tag{6.8}$$

Theorem 264 *For any fuzzy set f in S, if $I = I_G$ and f is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S, then f is a t-implication-based fuzzy left (resp. right) ideal of S for all $t \in (0, 0.5]$.*

Proof. Let $t \in (0, 0.5]$ and assume that f is an $(\in, \in \vee q)$-fuzzy left ideal of S. Then

$$f(xy) \geq \min\{\ f(y)\,,\ 0.5\}$$

for all $x, y \in S$. If $f(y) \leq 0.5$, then

$$f(xyz) \geq\ f(y)$$

and so

$$I_G\ \ (f(y)\,, f(xy)) = 1 \geq t.$$

Now suppose that $f(y) > 0.5$. Then $f(xy) \geq 0.5$, and either $f(xy) \geq f(y)$ or $f(xy) < f(y)$. If $f(xy) \geq f(y)$, then

$$I_G\,(f(y)\,, f(xy)) = 1 \geq t.$$

If $f(xy) < f(y)$, then

$$I_G\,(f(y)\,,\ f(xy)) = f(xy) \geq 0.5 \geq t.$$

Therefore f is a t-implication-based fuzzy left ideal of S for all $t \in (0, 0.5]$. ■

Corollary 265 *For any fuzzy subset f of S if the level set*

$$U(f; t) := \{x \in S \mid f(x) \geq t\}$$

is a left (resp. right) ideal of S, then f is a t-implication-based fuzzy left (resp. right) ideal of S for all $t \in (0, 0.5]$ under the Gödel implication operator.

Proof. Straightforward. ■

Theorem 266 *For any fuzzy subset f of S and $I = I_G$, if there exists $t \in [0.5, 1]$ such that f is a t-implication-based fuzzy left (resp. right) ideal of S, then f is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S.*

Proof. Let $t \in [0.5, 1]$ be such that f is a t-implication-based fuzzy left ideal of S. Then

$$I_G(f(y), f(xy)) \geq t$$

for all $x, y \in S$, and so either $I_G(f(y), f(xy)) = 1$, that is,

$$f(xy) \geq f(y)$$

or $I_G(f(y), f(xy)) = f(xy) \geq t \geq 0.5$. Hence

$$f(xy) \geq \min\{f(y), 0.5\}.$$

Using Lemma 263, we know that f is an $(\in, \in \vee q)$-fuzzy left ideal of S. ■

Corollary 267 *For any $t \in [0.5, 1]$, if f is a t-implication-based fuzzy left (resp. right) ideal of S under the Gödel implication operator I_G, then f is a fuzzy left (resp. right) ideal of S with thresholds $\gamma = 0$ and $\delta \in (0, 0.5]$.*

Proof. Straightforward. ■

Corollary 268 *For any $t \in [0.5, 1]$, if f is a t-implication-based fuzzy left (resp. right) ideal of S under the Gödel implication operator I_G, then the level set*

$$U(f; k) := \{x \in S \mid f(x) \geq k\}$$

is a left (resp. right) ideal of S for all $k \in (0, 0.5]$.

Proof. Straightforward. ■

Combining Theorems 264 and 266 we have the following corollary.

Corollary 269 *For any fuzzy subset f of S, if $I = I_G$, then f is a 0.5-implication-based fuzzy left (resp. right) ideal of S if and only if f is a fuzzy left (resp. right) ideal of S with thresholds $\gamma = 0$ and $\delta = 0.5$, that is, f is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S.*

Theorem 270 *Consider $I = I_{cG}$ and let $t \in [0.5, 1]$. If f is a t-implication-based fuzzy left (resp. right) ideal of S, then f is a fuzzy left (resp. right) ideal with thresholds $\gamma = t$ and δ, where $\delta = \sup_{x \in S} f(x)$.*

Proof. Let $t \in [0.5, 1]$ and assume that f is a t-implication-based fuzzy left ideal of S. Then

$$I_{cG}(f(y), f(xy)) \geq t$$

for all $x, y \in S$, and so either $I_{cG}(f(y), f(xy)) = 1$, that is,

$$f(y) \leq f(xy)$$

or $1 - f(y) = I_{cG} f((y), f(xy)) \geq t$, that is,

$$f(y) \leq 1 - t \leq t$$

since $t \in [0.5, 1]$. It follows that

$$\max\{f(xy), t\} \geq f(y) = \min\{f(y), \ \delta\}.$$

Therefore f is a fuzzy left ideal of S with thresholds $\gamma = t$ and $\delta = \sup_{x \in S} f(x)$. ■

Now we prove:

Theorem 271 *Consider $I = I_{cG}$ and let f be a fuzzy subset of S. For every $t \in (0, 0.5]$, if f is a t-implication-based fuzzy left (resp. right) ideal of S, then f is a fuzzy left (resp. right) ideal with thresholds $\gamma = 1 - t$ and $\delta = \sup_{x \in S} f(x)$.*

Proof. Assume that f is a t-implication-based fuzzy left ideal of S for $t \in (0, 0.5]$. Then

$$I_{cG}(f(y), f(xy)) \geq t$$

for all $x, y \in S$, which implies that either $f(y) \leq f(xy)$ or

$$1 - f(y) = I_{cG}(f(y), f(xy)) \geq t$$

and so $f(y) \leq 1 - t$. It follows that

$$\max\{f(xy), 1 - t\} \geq f(y) = \{f(y), \ \delta\}.$$

Therefore f is a fuzzy left ideal of S with thresholds $\gamma = 1 - t$ and $\delta = \sup_{x \in S} f(x)$.
■

Corollary 272 *For every $t \in (0, 0.5]$, if f is a t-implication-based fuzzy left (resp. right) ideal of S under the contraposition of Gödel implication operator I_{cG}, then f is a fuzzy left (resp. right) ideal with thresholds $\gamma = 1 - t$ and $\delta = 1$.*

For the converse of Theorem 270, we have the following theorem.

Theorem 273 *Consider $I = I_{cG}$ and let f be a fuzzy set in S. For every $t \in (0, 0.5]$, if f is a fuzzy left (resp. right) ideal of S with thresholds $\gamma = t$ and $\delta = \sup_{x \in S} f(x)$, then f is a t-implication-based fuzzy left (resp. right) ideal of S.*

Proof. Let $t \in (0, 0.5]$ and assume that f is a fuzzy left ideal of S with thresholds $\gamma = t$ and $\delta = \sup_{x \in S} f(x)$. Then, for all $x, y \in S$, we have

$$\max\{f(xy), t\} \geq \min\{f(y),\ \delta\} = f(y).$$

If $f(xy) \geq t$, then $f(xy) \geq f(y)$, and so

$$I_{cG}(f(y), f(xy)) = 1 \geq t.$$

If $f(xy) < t$, then $f(y) \leq t$. Hence if $f(y) \leq f(xyz)$, then

$$I_{cG}(f(y), f(xy)) = 1 \geq t.$$

If $f(y) > f(xy)$, then

$$I_{cG}(f(y), f(xy)) = 1 - f(y) \geq 1 - t \geq t.$$

Consequently f is a t-implication-based fuzzy left ideal of S for every $t \in (0, 0.5]$. ■

Corollary 274 *For every $t \in (0, 0.5]$, if f is a fuzzy left (resp. right) ideal of S with thresholds $\gamma = t$ and $\delta = 1$, then f is a t-implication-based fuzzy left (resp. right) ideal of S under the contraposition of Gödel implication operator I_{cG}.*

Combining Corollaries 272 and 274, we have the following corollary.

Corollary 275 *For any fuzzy subset f of S, if $I = I_{cG}$, then f is a 0.5-implication-based fuzzy left (resp. right) ideal of S if and only if f is a fuzzy left (resp. right) ideal of S with thresholds $\gamma = t$ and $\delta = 1$.*

Theorem 276 *Consider $I = I_{GR}$ and let $t \in (0, 1]$. If f is a t-implication-based fuzzy left (resp. right) ideal of S, then f is a fuzzy left (resp. right) ideal of S.*

Proof. Let $t \in (0, 1]$ be such that f is a t-implication-based fuzzy left ideal of S under the Gaines-Rescher implication operator I_{GR}. Then

$$I_{GR}(f(y), f(xy)) \geq t.$$

Since $t \neq (0, 1]$, it follows that $I_{GR}(f(y),\ f(xy)) = 1$ and so that $f(xy) \geq f(y)$. Therefore f is a fuzzy left ideal of S. ■

Corollary 277 *For any $t \in (0, 1]$, if f is a t-implication-based fuzzy left (resp. right) ideal of S under the Gaines-Rescher implication operator I_{GR}, then the set*

$$U(f; t) := \{x \in S \mid f(x) \geq t\}$$

is a left (resp. right) ideal of S.

Proof. Straightforward. ■

Theorem 278 *Every fuzzy left (resp. right) ideal of S is a t-implication-based fuzzy left (resp. right) ideal for all $t \in (0,1]$ under the Gaines-Rescher implication operator I_{GR}.*

Proof. Straightforward. ■

The following corollary is by Theorems 276 and 278.

Corollary 279 *A fuzzy subset of S is a 0.5-implication-based fuzzy left (resp. right) ideal of S under the Gaines-Rescher implication operator I_{GR} if and only if it is a fuzzy left (resp. right) ideal of S.*

Theorem 280 *Every fuzzy left (resp. right) ideal of S is a t-implication-based fuzzy left (resp. right) ideal for all $t \in (0,1]$ under the Łuckasiewicz implication operator I_{LI}.*

Proof. Straightforward. ■

We provide conditions for an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S to be a t-implication-based fuzzy left (resp. right) ideal of S under the Łuckasiewicz implication operator I_{LI}.

Theorem 281 *Let f be an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S. If there exist $x, y, z \in S$ such that $f(y) > f(xy)$, and let $\omega = 1 - f(y)$. Then f is a t-implication-based fuzzy left ideal of S for all $t \in (0, \omega]$.*

Proof. Assume that f is an $(\in, \in \vee q)$-fuzzy left ideal of S. Then

$$f(xy) \geq \min \ \{f(y), \ 0.5\}$$

for all $x, y \in S$. Suppose that $f(y) \leq 0.5$. Then $f(xy) \geq f(y)$, and so

$$I_{LI}(f(y), \ f(xy)) = 1 \geq t$$

for all $t \in (0, \omega]$. Assume that $f(y) > 0.5$ for all $x, y \in S$. Then $f(xy) \geq 0.5$. Thus we have two cases:

(1) $f(xy) \geq f(y)$

(2) $f(xy) < f(y)$.

First case implies that

$$I_{LI}(\ f(y), \ f(xy)) = 1 \geq t$$

for all $t \in (0, \omega]$. The second case induces

$$\begin{aligned} I_{LI}(\ f(y),\ f(xy)) &= 1 - \ f(y) + f(xy) \\ &= \omega \geq t \end{aligned}$$

for all $t \in (0, \omega]$. Therefore f is a t-implication-based fuzzy left ideal of S for all $t \in (0, \omega]$ under the Łuckasiewicz implication operator I_{LI}. ■

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