COMPACT MOVING MESH METHOD FOR GAS DYNAMICS EQUATIONS

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This is to certify that the following dissertation is submitted to the Department of Mathematics, University of Engineering and Technology Lahore Pakistan, in partial fulfillment of the requirements for the Degree of **Doctor of Philosophy (Ph.D.) in Mathematics**

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2011

In the name of Allah

The Most Beneficent and the Most Merciful

Read: In the Name of Your
Lord Who Created,
Created Man From A Clot.

Read: And Your Lord is the
Most Bounteous. Who Taught
by the Pen. Taught Man that
which he did not know.

Al-Quran

DEDICATION

Dedicated to my sons
Salman Maqbool
and
Umair Maqbool

I certify that I have read this thesis and that, in my opinion, it is fully adequate in scope and quality, as thesis for the Degree of Doctor of Philosophy

Prof. Dr. Muhammad Ozair Ahmad
(Internal Examiner)

I certify that I have read this thesis and that, in my opinion, it is fully adequate in scope and quality, as thesis for the Degree of Doctor of Philosophy.

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Maqbool Ahmad Chaudhry

ABSTRACT

The objective of this dissertation is to develop fourth order compact moving mesh method. This method is a hybrid of compact method and moving mesh method. The usual second order method becomes less suitable due to the increase of number of grid points to achieve the required accuracy. In this method, only three nodes are needed to yield a fourth order accuracy whereas other methods would have required five nodes and more computational effort to achieve the same accuracy. Compact moving mesh method is used to solve one dimensional gas dynamics equations. Compact method and moving mesh method are also reviewed in this dissertation. The results obtained by using compact moving mesh method are compared with the results determined by using second order finite difference method for gas dynamics equations.

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Chapter - 1

INTRODUCTION

1.1 ONE DIMENSIONAL GAS DYNAMICS EQUATIONS

In this chapter, one dimensional gas dynamics equations are considered. These equations represent the laws of conservation of mass, momentum and energy along with the equation of state of gas. We will use this problem as a case study because gas dynamics equations display most of the aspects of hyperbolic (non-linear) conservation laws. This problem represents the next level of complexity after Burger's equations.

1.2 DERIVATION OF GAS DYNAMICS EQUATIONS

The Navier-Stokes Equations in three – dimensional – unsteady flow are given by:

Continuity equation : $$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

x-momentum : $$\frac{\partial(\rho u)`}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho u v)}{\partial y}+\frac{\partial(\rho u w)}{\partial z}$$

$$= -\frac{\partial P}{\partial x}+\frac{1}{\mathrm{Re}}\left[\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}\right]$$

y-momentum : $$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u v)}{\partial x}+\frac{\partial(\rho v^2)}{\partial y}+\frac{\partial(\rho v w)}{\partial z}$$

$$= -\frac{\partial P}{\partial y}+\frac{1}{\mathrm{Re}}\left[\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}\right]$$

z-momentum : $$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho u w)}{\partial x}+\frac{\partial(\rho v w)}{\partial y}+\frac{\partial(\rho w^2)}{\partial z}$$

$$= -\frac{\partial P}{\partial y}+\frac{1}{\text{Re}}\left[\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right]$$

Energy Equation:

$$\frac{\partial(E)}{\partial t}+\frac{\partial(uE)}{\partial x}+\frac{\partial(vE)}{\partial y}+\frac{\partial(wE)}{\partial z}=\frac{\partial(uP)}{\partial x}+\frac{\partial(vP)}{\partial y}$$

$$-\frac{\partial(wP)}{\partial z}-\frac{1}{\text{Re Pr}}\left[\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right]$$

$$+\frac{1}{\text{Re}}\left[\frac{\partial}{\partial x}(u\tau_{xx}+v\tau_{xy}+w\tau_{xz})+\frac{\partial}{\partial y}(u\tau_{xy}+v\tau_{yy}+w\tau_{yz})\right.$$

$$\left.+\frac{\partial}{\partial z}(u\tau_{xz}+v\tau_{yz}+w\tau_{zz}\right]$$

where t denotes Time, ρ *denotes* Density, *E* denotes Total Energy, *Re* denotes Reynolds Number.

Coordinates : (*x*, *y*, *z*), Stress : τ , Heat Flux : *q*, Prandtl Number : *Pr*

Velocity Components : (*u*, *v*, *w*), Pressure : *P*.

The corresponding equations in One-dimension by neglecting the sheared stress, are;

Continuity equation : $$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x} = 0$$

x-momentum : $$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}= -\frac{\partial P}{\partial x}$$

Energy equation : $$\frac{\partial E}{\partial t}+\frac{\partial(uE)}{\partial x}= -\frac{\partial(uP)}{\partial x}$$

where, heat flow is ideal i.e., stream line flow.

Now, replacing (ρu) by m, in (i), (ii) and (iii), we get,

$$\frac{\partial \rho}{\partial t} + \frac{\partial m}{\partial x} = 0 \quad , \qquad \rho_t + m_x = 0$$

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial x}\left[\frac{\rho^2 u^2}{\rho}\right] + \frac{\partial P}{\partial x} = 0$$

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial x}\left[\frac{m^2}{\rho}\right] + \frac{\partial P}{\partial x} = 0$$

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial x}\left[\frac{m^2}{\rho} + P\right] = 0$$

$$m_t + \left[\frac{m^2}{\rho} + P\right]_x = 0$$

and $$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[\frac{\rho u E}{\rho}\right] + \frac{\partial}{\partial x}\left[\frac{\rho u P}{\rho}\right] = 0$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[\frac{m\ E}{\rho}\right] + \frac{\partial}{\partial x}\left[\frac{m\ P}{\rho}\right] = 0$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[\frac{m}{\rho}(E+P)\right] = 0$$

$$E_t + \left[\frac{m}{\rho}(E+P)\right]_x = 0$$

1.3 FORMULATION

The one dimensional equations of gas dynamics can be written in conservation form as follows:

$$\rho_t + m_x = 0 \tag{1.1}$$

$$m_t + \left[\frac{m^2}{\rho} + P\right]_x = 0 \tag{1.2}$$

$$E_t + \left[\frac{m}{\rho}\,(E + P)\right]_x = 0 \tag{1.3}$$

where ρ stands for density, *m* for momentum, *P* for pressure and *E* is the total energy per unit volume.

The equation of state for an ideal gas to express the internal energy per unit mass '*e*' as a function of pressure *P* and density ρ is also used.

$$e = \frac{P}{\rho\,(\gamma - 1)} \qquad \gamma > 1 \tag{1.4}$$

where γ is ratio of specific heats. We shall take here γ = 1.4 corresponding to a diatomic gas.

From Bernoullie's equations we have

$$E = e\rho + \frac{m^2}{2\rho}$$

Substituting the values of e from equation (1.4) in this equation, we get

$$E = \frac{P}{\gamma - 1} + \frac{m^2}{2\rho}$$

or $$P = (\gamma - 1)\left[E - \frac{m^2}{2\rho}\right] \tag{1.5}$$

Thus we have four equations (1.1), (1.2), (1.3) and (1.5) involving three independent variables ρ, *m* and *E*, where '*t*' and '*x*' are time and position co-ordinates. Equations (1.1), (1.2) and (1.3) can be written in vector form as:

$$u_t + [F(u)]_x = 0 \tag{1.6}$$

where

$$u = \begin{bmatrix} \rho \\ m \\ E \end{bmatrix}, \quad F(u) = \begin{bmatrix} m \\ \dfrac{m^2}{\rho} + P \\ \dfrac{m}{\rho}(E + P) \end{bmatrix}$$

The following initial and boundary conditions will be used:

$$\rho(x,0) = \begin{cases} 1.00 & x < 0 \\ 0.5625 & x = 0 \\ 0.1250 & x > 0 \end{cases} \tag{1.7}$$

$$m(x, 0) = 0 \qquad \textit{for all } x \tag{1.8}$$

$$E(x,0) = \begin{cases} 2.5000 & x < 0 \\ 1.3750 & x = 0 \\ 0.2500 & x > 0 \end{cases} \tag{1.9}$$

$$\rho(-0.5, t) = 1.0000 \tag{1.10}$$

$$m(-0.5, t) = 0.0000 \tag{1.11}$$

$$E(-0.5, t) = 2.5000 \qquad \text{for all } t \tag{1.12}$$

1.4 ADDITION OF ARTIFICIAL VISCOSITY

If we try to convert $u_t + [F(u)]_x = 0$ to a form suitable for use of the subroutine LSODE directly, replacing u_x by the appropriate linear combination of u values as an approximation, we get a system of first order differential equations. When we solve the system by using LSODE, the approximate solution has discontinuities under the given initial and boundary conditions. Thus, In order to make the problem simpler, we use the classical technique of adding artificial viscosity to the hyperbolic system by adding a term proportion to u_{xx}. The system of equations (1.1), (1.2) and (1.3) then becomes as:

$$\rho_t + m_x = \lambda \rho_{xx} \tag{1.13}$$

$$m_t + \left[\frac{m^2}{\rho} + P \right]_x = \lambda m_{xx} \tag{1.14}$$

and $$E_t + \left[\frac{m}{\rho} (E + P) \right]_x = \lambda E_{xx} \tag{1.15}$$

or in vector form

$$u_t + [F(u)]_x = \lambda u_{xx} \tag{1.16}$$

The following boundary conditions are used:

$$\rho(-0.5, t) = 1.000, \quad \rho(0.5, t) = 0.1250 \tag{1.17}$$

$$m(-0.5, t) = 0.000 = m(0.5, t) \tag{1.18}$$

$$E(-0.5, t) = 2.5000, \ \textit{and} \ \ E(0.5, t) = 0.2500 \tag{1.19}$$

Hyperbolic conservation laws even for smooth initial conditions can produce solutions which ultimately become discontinuous. Hence when we speak here of a solution of equations (1.6), we will mean a weak solution. Lax [1] proves that if the solution $u(x, t; \lambda)$ of (1.16) converges to a limit u(x, t) as $\lambda \to 0$ then $u(x, t)$ is a weak solution of equation (1.6). Further Lax [1] proves that u (x, t) is the correct physically realizable weak solution of equation (1.6). Finally Foy [2] proves that solutions of equation (1.16) do indeed converge if the original shocks are weak enough. Therefore the addition of artificial viscosity, while simple, will not destroy the essential character of the hyperbolic equations (1.1), (1.12) and (1.3). We take $\lambda = 5\times 10^{-4}$, since λ is arbitrary.

1.5 SECOND ORDER FIFNITE DIFFERENCE METHOD (SOFDM)

In the classical Second Order Finite Difference Method, the *x* domain

$[-0.5, 0.5]$ is replaced by a discrete set of points $x_i = ih$ for $i=0,1,2,n$ where $h=\frac{1}{n}$ and central differences for the first, and second spatial (x) derivatives are used. The following system of ODEs for the discretized variables ρ, m and E is found. For the equation (1.13) the following set of ODEs are obtained:

$$\frac{\partial \rho_1}{\partial t} = \frac{\lambda}{h^2}(1-2\rho_1+\rho_2)-\frac{1}{2h}m_2 \tag{1.20}$$

$$\frac{\partial \rho_i}{\partial t} = \frac{\lambda}{h^2}(\rho_{i-1}-2\rho_i+\rho_{i+1})-\frac{1}{2h}(m_{i+1}-m_{i-1}) \tag{1.21}$$

$$\frac{\partial \rho_{n-1}}{\partial t} = \frac{\lambda}{h^2}(\rho_{n-2}-2\rho_{n-1}+0.125)-\frac{1}{2h}m_{n-2}$$

For the equation (1.14), the following set of ODEs are obtained

$$\frac{\partial m_1}{\partial t} = \frac{\lambda}{h^2}(-2m_1+m_2)-\frac{1}{2h}\left[\left(\frac{m_2^2}{\rho_2}+P_2\right)-P_0\right] \tag{1.23}$$

$$\frac{\partial m_i}{\partial t} = \frac{\lambda}{h^2}(m_{i-1}-2m_i+m_{i+1})-\frac{1}{2h}$$

$$\left[\left(\frac{m_{i+1}^2}{\rho_{i+1}}+P_{i+1}\right)-\left(\frac{m_{i-1}^2}{\rho_{i-1}}+P_{i-1}\right)\right] \quad i=2,\ldots,n-2 \tag{1.24}$$

$$\frac{\partial m_{n-1}}{\partial t} = \frac{\lambda}{h^2}(m_{n-2}-2m_{n-1})-\frac{1}{2h}\left[P_n-\left(\frac{m_{n-2}^2}{\rho_{n-2}}+P_{n-2}\right)\right] \tag{1.25}$$

For the equation (1.15), the following set of ODEs are obtained.

$$\frac{\partial E_1}{\partial t} = \frac{\lambda}{h^2}[2.5-2E_1+E_2]-\frac{1}{2h}\left[\frac{m_2}{\rho_2}(E_2+P_2)\right] \tag{1.26}$$

$$\frac{\partial E_i}{\partial t} = \frac{\lambda}{h^2}(E_{i-1}-2E_i+E_{i+1})-\frac{1}{2h}$$

$$\left[\frac{m_{i+1}}{\rho_{i+1}}(E_{i+1}+P_{i+1})-\frac{m_{i-1}}{\rho_{i-1}}(E_{i-1}+P_{i-1})\right], \qquad i=2,\ldots,n-2 \tag{1.27}$$

$$\frac{\partial E_{n-1}}{\partial t} = \frac{\lambda}{h^2}[E_{n-2}-2E_{n-1}+0.25]-\frac{1}{2h}\left[\frac{m_{n-2}}{\rho_{n-2}}(E_{n-2}+P_{n-2})\right] \tag{1.28}$$

where P is given by (1.5).

The above equations along with appropriate boundary and initial conditions completely define the solution of equation (1.16).

Chapter - 2

COMPACT METHOD

2.1 INTRODUCTION

In this chapter fourth order compact method [3] is derived for one dimensional gas dynamics equations. The salient feature of compact method is that it yields fourth order accuracy by using only three nodes, whereas five nodes are normally needed for the same accuracy. This is accomplished by using differencing technique which considers the function and all necessary derivatives as unknowns. The relations for these derivatives yield a simple tridiagonal system of equations which then can be solved easily. The results obtained by fourth order compact method and those by second order finite difference method are compared at the end. We find that the accuracy achieved by fourth order compact method is significantly better than that of by second order finite difference method.

2.2 DERIVATION OF THE COMPACT METHOD

In this section fourth order compact schemes are derived and used to solve the one dimensional gas dynamics equations:

Continuity equation : $\rho_t + m_x = \lambda \rho_{xx}$

Momentum equation : $m_t + \left[\frac{m^2}{\rho} + P\right]_x = \lambda m_{xx}$

Energy equation : $E_t + \left[\frac{m}{\rho}(E+P)\right]_x = \lambda E_{xx}$

As derived in the previous chapter

First we consider $\rho_t + m_x = \lambda \rho_{xx}$ (2.1)

The compact method approximates the given equation by two difference equations of fourth order using only three grid points say x_{i-1}, x_i, x_{i+1}. In order to derive the compact schemes we introduce new variables for the derivatives.

Let us denote first and second derivatives of ρ w.r.t., *x* by *F* and *S* respectively,

$$\rho_x = F \tag{2.2}$$

$$\rho_{xx} = S$$

so $F_x = S$

Integrating both sides of equation (2.2) from "x_{i-1}" to "x_{i+1}" we get

$$\rho_{i+1}^n - \rho_{i-1} = \int_{x_{i-1}}^{x_{i+1}} F(\xi, l)\, d\xi$$

or $$\rho_{i+1}^n = \rho_{i-1}^n + \int_{x_{i-1}}^{x_{i+1}} F(\xi, l)\, d\xi \tag{2.3}$$

Approximating the integral by Simpson's rule:

$$\rho_{i+1}^n = \rho_{i-1}^n + \frac{h}{3}\left[F_{i-1}^n + 4F_i^n + F_{i+1}^n\right] - \frac{1}{90} h^5 F^{(4)}(\xi, l)$$

$$F_{i-1}^n + 4F_i^n + F_{i+1}^n + \frac{3}{h}\left(\rho_{i-1}^n - \rho_{i+1}^n\right) = \frac{1}{30} h^4 F^{(4)}(\xi, l)$$

Thus to Fourth Order we have:

$$F_{i-1}^n + 4F_i^n + F_{i+1}^n + \frac{3}{h}\left(\rho_{i-1}^n - \rho_{i+1}^n\right) = 0 \tag{2.4}$$

Thus this expression represents a relationship between ρ and *F*. This is the first difference equation. In order to obtain the second equation, we start by evaluating (2.1) at the mid- point *i*.

Suppose

$$m_x = U$$

Equation (2.1) gives $\rho_t|_i^n = \lambda S_i^n - U_i^n$ (2.5)

To find an expression for S_i^n, expressing ρ_{i+1}^n, ρ_{i-1}^n in the Taylor's expansion about the point (i,n)), we get:

$$\rho_{i+1}^n = \rho_i^n + h\rho_x|_i^n + \frac{h^2}{2!}\rho_{xx}|_i^n + \frac{h^3}{3!}\rho_{xxx}|_i^n + \frac{h^4}{4!}\rho_{xxxx}|_i^n +$$

$$\frac{h^5}{5!}\rho_{xxxxx} + |_i^n + \frac{h^6}{6!}\rho_{xxxxxx}|_i^n + \quad (2.6)$$

$$\rho_{i-1}^n = \rho_i^n - h\rho_x|_i^n + \frac{h^2}{2!}\rho_{xx}|_i^n - \frac{h^3}{3!}\rho_{xxx}|_i^n + \frac{h^4}{4!}\rho_{xxxx}|_i^n -$$

$$\frac{h^5}{5!}\rho_{xxxxx} + |_i^n + \frac{h^6}{6!}\rho_{xxxxxx}|_i^n + \ldots \quad (2.7)$$

Adding equation (2.6) and equation (2.7)

$$\therefore \quad \rho_{i+1}^n + \rho_{i-1}^n = 2\rho_i^n + h^2\rho_{xx}|_i^n + \frac{h^4}{12}\rho_i^{n(4)} + \frac{h^6}{360}\rho_i^{n(6)}(\xi, l)$$

$$\because \quad \rho_{xx} = S$$

$$\rho_{i+1}^n + \rho_{i-1}^n = 2\rho_i^n + h^2 S|_i^n + \frac{h^4}{12}\rho_i^{n(4)} + \frac{h^6}{360}\rho_i^{n(6)}(\xi, l) \quad (2.8)$$

Similarly:

$$F_{i+1}^n = F_i^n + hF_x|_i^n + \frac{h^2}{2!}F_{xx}|_i^n + \frac{h^3}{3!}F_{xxx}|_i^n + \frac{h^4}{4!}F_{xxxx}|_i^n +$$

$$\frac{h^5}{5!}F_{xxxxx} + |_i^n + \frac{h^6}{6!}F_{xxxxxx}|_i^n + \ldots \quad (2.9)$$

$$F_{i-1}^{n}=F_i^n-hF_x\,|_i^n+\frac{h^2}{2!}F_{xx}\,|_i^n-\frac{h^3}{3!}F_{xxx}\,|_i^n+\frac{h^4}{4!}F_{xxxx}\,|_i^n-$$

$$\frac{h^5}{5!}F_{xxxxx}+|_i^n+\frac{h^6}{6!}F_{xxxxxx}\,|_i^n+\ldots \qquad (2.10)$$

Subtracting equation (2.10) from equation (2.9)

$$\therefore\quad F_{i+1}^n-F_{i-1}^n=2hF_x\,|_i^n+2\frac{h^3}{3!}F_{xxx}\,|_i^n+\frac{2h^5}{5!}F_{xxxxx}(\xi,l)$$

Since

$$F_x=S,\;\;F_{xxx}\,|_i^n=\rho_i^{n(4)},\;\;F_i^{n(5)}=\rho_i^{n(6)}$$

Therefore

$$F_{i+1}^n-F_{i-1}^n=2hS_i^n+\frac{h^3}{3}\rho_i^{n(4)}+\frac{h^5}{60}\rho_i^{n(6)}(\xi,l) \qquad (2.11)$$

Now we eliminate $\rho_i^{n(4)}$ from equations (2.8) and (2.11)

Multiplying equation (2.11) by $\frac{h}{4}$, we get

$$\frac{h}{4}(F_{i+1}^n-F_{i-1}^n)=\frac{h^2}{2}S_i^n+\frac{h^4}{12}\rho_i^{n(4)}+\frac{h^6}{240}\rho_i^{n(6)}(\xi,l) \qquad (2.12)$$

Subtracting equation (2.12) from equation (2.8), we get

$$(\rho_{i+1}^n+\rho_{i-1}^n)-\frac{h}{4}(F_{i+1}^n-F_{i-1}^n)=2\rho_i^n+\frac{h^2}{2}S_i^n-\frac{h^6}{720}\rho_i^{n(6)}(\xi,l)$$

$$\frac{h^2}{2}S_i^n=\rho_{i+1}^n-2\rho_i^n+\rho_{i-1}^n-\frac{h}{4}(F_{i+1}^n-F_{i-1}^n)+\frac{h^6}{720}\rho_i^{n(6)}(\xi,l)$$

$$S_i^n=\frac{2}{h^2}(\rho_{i+1}^n-2\rho_i^n+\rho_{i-1}^n)-\frac{1}{2h}(F_{i+1}^n-F_{i-1}^n)+\frac{h^4}{360}\rho_i^{n(6)}(\xi,l) \qquad (2.13)$$

Similarly

$$S_{i-1}^n=\frac{1}{2h^2}(-23\rho_{i-1}^n+16\rho_i^n+7\rho_{i+1}^n)-\frac{1}{h}(6F_{i-1}^n+8F_i^n+F_{i+1}^n)+$$

$$\frac{h^4}{90}\rho_i^{n(6)}(\xi,l) \qquad (2.14)$$

$$S_{i+1}^n = \frac{1}{2h^2}(7\rho_{i-1}^n + 16\rho_i^n - 23\rho_{i+1}^n) + \frac{1}{h}(F_{i-1}^n + 8F_i^n + 6F_{i+1}^n) +$$

$$\frac{h^4}{90}\rho_i^{n(6)}(\xi,l) \qquad (2.15)$$

From equation (2.13) substituting the value of S_i^n in equation (2.5), we get

$$\rho_t|_i^n = \frac{2\lambda}{h^2}(\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n) - \frac{\lambda}{2h}(F_{i+1}^n - F_{i-1}^n) - U_i^n \qquad (2.16)$$

The differential equation (2.1) is replaced by the two difference equations (2.4) and (2.16)

Now, we consider the left boundary condition i.e., at $x = -0.5$ and denote the points

$$x = -0.5, -0.5 + h, -0.5 + 2h \quad by \quad 0, 1, 2$$

The first difference equation obtained from the boundary condition is:

$$\rho_0^n = 1 \qquad (2.17)$$

In order to get the second equation using equation (2.5) at the points 0 and 1

$$\rho_t|_0^n = \lambda S_0^n - U_0^n \qquad (2.18)$$

$$\rho_t|_1^n = \lambda S_1^n - U_1^n \qquad (2.19)$$

For $i = 1$

Equation (2.14) implies, $S_0^n = \frac{1}{2h^2}(-23\rho_0^n + 16\rho_1^n + 7\rho_2^n) - \frac{1}{h}(6F_0^n + 8F_1^n + F_2^n)$ (2.20)

Equation (2.13) implies $S_1^n = \frac{2}{h^2}(\rho_0^n - 2\rho_1^n + \rho_2^n) - \frac{1}{2h}(F_2^n - F_0^n)$ (2.21)

Equation (2.4) implies $F_0^n + 4F_1^n + F_2^n + \frac{3}{h}(\rho_0^n - \rho_2^n) = 0$ (2.22)

Eliminate ρ_2^n, S_0^n, S_1^n, and F_2^n from these five equations from equation (2.18) to equation (2.22).

From equations (2.18) and (2.20) we get

$$\rho_t|_0^n = -\frac{23\lambda}{2h^2}\rho_0^n + \frac{8\lambda}{h^2}\rho_1^n + \frac{7\lambda}{2h^2}\rho_2^n - \frac{6\lambda}{h}F_0^n - \frac{8\lambda}{h}F_1^n - \frac{\lambda}{h}F_2^n - U_0^n \quad (2.23)$$

From equations (2.19) and (2.21) we get

$$\rho_t|_1^n = \frac{2\lambda}{h^2}\rho_0^n - \frac{4\lambda}{h^2}\rho_1^n + \frac{2\lambda}{h^2}\rho_2^n - \frac{\lambda}{2h}F_2^n + \frac{\lambda}{2h}F_0^n - U_1^n \quad (2.24)$$

Equation (2.22) gives $\rho_2^n = \rho_0^n + \frac{h}{3}F_0^n + \frac{4h}{3}F_1^n + \frac{h}{3}F_2^n$ (2.25)

Substituting this value of ρ_2^n in equation (2.23) we get

$$\rho_t|_0^n = -\frac{23\lambda}{2h^2}\rho_0^n + \frac{8\lambda}{h^2}\rho_1^n + \frac{7\lambda}{2h^2}(\rho_0^n + \frac{h}{3}F_0^n + \frac{4h}{3}F_1^n + \frac{h}{3}F_2^n) -$$

$$\frac{6\lambda}{h}F_0^n - \frac{8\lambda}{h}F_1^n - \frac{\lambda}{h}F_2^n - U_0^n$$

$$\rho_t|_0^n = -\frac{8\lambda}{h^2}\rho_0^n + \frac{8\lambda}{h^2}\rho_1^n - \frac{29\lambda}{6h^2}F_0^n - \frac{10\lambda}{3h}F_1^n + \frac{\lambda}{6h}F_2^n - U_0^n \quad (2.26)$$

Substituting the value of ρ_2^n from equation (2.5) in equation (2.24) we get

$$\rho_t|_1^n = \frac{2\lambda}{h^2}\rho_0^n - \frac{4\lambda}{h^2}\rho_1^n + \frac{2\lambda}{h^2}(\rho_0^n + \frac{h}{3}F_0^n + \frac{4h}{3}F_1^n + \frac{h}{3}F_2^n) -$$

$$\frac{\lambda}{2h}F_2^n + \frac{\lambda}{2h}F_0^n - U_1^n$$

$$\rho_t\,|_1^n = \frac{4\lambda}{h^2}\rho_0^n - \frac{4\lambda}{h^2}\rho_1^n + \frac{7\lambda}{6h}F_0^n + \frac{8\lambda}{3h}F_1^n + \frac{\lambda}{6h}F_2^n - U_0^n \tag{2.27}$$

Subtracting equation (2.26) from equation (2.27) we get

$$\frac{12\lambda}{h^2}\rho_0^n - \frac{12\lambda}{h^2}\rho_1^n + \frac{6\lambda}{h}F_0^n + \frac{6\lambda}{h}F_1^n - U_1^n + U_0^n = \rho_t\,|_1^n - \rho_t\,|_0^n \tag{2.28}$$

Equation (2.28) is the second difference equation valid at $x = -0.5$.

In a similar manner, we can derive he following two difference equations for ρ and F at the right boundary point x = 0.5 and denote the points x = 0.5, 0.5–h, 0.5–2h by a, a–1, a–2. Therefore we have

$$\rho_a^n = 0.125 \tag{2.29}$$

$$\frac{12\lambda}{h^2}\rho_{a-1}^n - \frac{12\lambda}{h^2}\rho_a^n + \frac{6\lambda}{h}F_{a-1}^n + \frac{6\lambda}{h}F_a^n - U_a^n + U_{a-1}^n = \rho_t\,|_a^n - \rho_t\,|_{a-1}^n \tag{2.30}$$

Thus for each point we have two difference equations. If we write them all together, we have the following fourth order compact scheme for equation (2.1).

$$\rho_0^n = 1$$

$$\frac{12\lambda}{h^2}\rho_0^n - \frac{12\lambda}{h^2}\rho_1^n + \frac{6\lambda}{h}F_0^n + \frac{6\lambda}{h}F_1^n - U_1^n + U_0^n = \rho_t\,|_1^n - \rho_t\,|_0^n$$

$$F_{i-1}^n + 4F_i^n + F_{i+1}^n + \frac{3}{h}(\rho_{i-1}^n - \rho_{i+1}^n) = 0$$

$$\frac{2\lambda}{h^2}(\rho_{i+1}^n - 2\rho_i^n + 2\rho_{i-1}^n) - \frac{\lambda}{2h}(F_{i+1}^n - F_{i-1}^n) - U_1^n = \rho_t\,|_i^n$$

$$\rho_a^n = 0.125$$

$$\frac{12\lambda}{h^2}\rho_{a-1}^n - \frac{12\lambda}{h^2}\rho_a^n + \frac{6\lambda}{h}F_{a-1}^n + \frac{6\lambda}{h}F_a^n - U_a^n + U_{a-1}^n = \rho_t|_a^n - \rho_t|_{a-1}^n$$

The superscript n is used to denote the time grid line.

Now consider $m_t + \left[\frac{m^2}{\rho} + P\right]_x = \lambda m_{xx}$ (2.31)

Denote first and second derivatives of m w.r.t., x by U and V respectively, i.e.,

$$m_x = U \quad (3.32)$$

$$m_{xx} = V$$

$$U_x = V$$

Integrating both sides of equation (2.32) from "x_{i-1}" to "x_{i+1}" we get

$$m_{i+1}^n - m_{i-1}^n = \int_{x_{i-1}}^{x_{i+1}} U(\xi, l)\, d\xi$$

$$m_{i+1}^n = m_{i-1}^n + \int_{x_{i-1}}^{x_{i+1}} U(\xi, l)\, d\xi \quad (2.33)$$

Approximating the integral by Simpson's Rule:

$$m_{i+1}^n = m_{i-1}^n + \frac{h}{3}[U_{i-1}^n + 4U_i^n + U_{i+1}^n] + \frac{1}{90}h^5 U^{(4)}(\xi, l)$$

$$U_{i-1}^n + 4U_i^n + U_{i+1}^n + \frac{3}{h}(m_{i-1}^n - m_{i+1}^n) = -\frac{1}{30}h^4 U^{(4)}(\xi, l)$$

Thus to Fourth Order we have

$$U_{i-1}^n + 4U_i^n + U_{i+1}^n + \frac{3}{h}(m_{i-1}^n - m_{i+1}^n) = 0 \quad (3.34)$$

Thus a relationship between *m* and *U* is derived. This is the first difference equation. In order to obtain the second difference equation, evaluate (2.31) at the mid point *i*.

Suppose $\left[\frac{m^2}{\rho} + P\right]_x = W$

Equation (2.31) implies $m_t \,|_i^n = \lambda V_i^n - W_i^n$ (2.35)

To find an expression for V_i^n, expressing m_{i+1}^n, m_{i-1}^n in the Taylor's expansion about the point (*i*, *n*), we get

$$m_{i+1}^n = m_i^n + h\, m_x \,|_i^n + \frac{h^2}{2!} m_{xx} \,|_i^n + \frac{h^3}{3!} m_{xxx} \,|_i^n + \frac{h^4}{4!} m_{xxxx} \,|_i^n + \frac{h^5}{5!} m_{xxxxx} \,|_i^n + \frac{h^6}{6!} m_{xxxxxx} \,|_i^n + ... \quad (2.36)$$

$$m_{i-1}^n = m_i^n - h\, m_x \,|_i^n + \frac{h^2}{2!} m_{xx} \,|_i^n - \frac{h^3}{3!} m_{xxx} \,|_i^n + \frac{h^4}{4!} m_{xxxx} \,|_i^n - \frac{h^5}{5!} m_{xxxxx} \,|_i^n + \frac{h^6}{6!} m_{xxxxxx} \,|_i^n + ... \quad (2.37)$$

$$\therefore \quad m_{i+1}^n + m_{i-1}^n = 2\, m_i^n + h^2\, m_{xx} \,|_i^n + \frac{h^4}{12} m_i^{n(4)} + \frac{h^6}{360} m_i^{n(6)}(\xi, l)$$

Since $m_{xx} = V$

$$m_{i+1}^n + m_{i-1}^n = 2\, m_i^n + h^2 V \,|_i^n + \frac{h^4}{12} m_i^{n(4)} + \frac{h^6}{360} m_i^{n(6)}(\xi, l) \quad (2.38)$$

Taking

$$U_{i+1}^n = U_i^n + hU_x|_i^n + \frac{h^2}{2!}U_{xx}|_i^n + \frac{h^3}{3!}U_{xxx}|_i^n + \frac{h^4}{4!}U_{xxxx}|_i^n +$$

$$\frac{h^5}{5!}U_{xxxxx}|_i^n + \frac{h^6}{6!}U_{xxxxxx}|_i^n + ... \tag{2.39}$$

$$U_{i-1}^n = U_i^n - hU_x|_i^n + \frac{h^2}{2!}U_{xx}|_i^n - \frac{h^3}{3!}U_{xxx}|_i^n + \frac{h^4}{4!}U_{xxxx}|_i^n -$$

$$\frac{h^5}{5!}U_{xxxxx}|_i^n + \frac{h^6}{6!}U_{xxxxxx}|_i^n + \tag{2.40}$$

$$\therefore \quad U_{i+1}^n + U_{i-1}^n = 2hU_x|_i^n + 2\frac{h^3}{3!}U_{xxx}|_i^n + \frac{2h^5}{5!}U_{xxxxx}^n(\xi, l)$$

Since

$$U_x = V, U_{xxx}|_i^n = m_i^{n(4)}, u_{xxxxx}^n = m_i^{n(6)}$$

We have $U_{i+1}^n + U_{i-1}^n = 2hV_i^n + \frac{h^3}{3}m_i^{n(4)} + \frac{h^5}{60}m_i^{n(6)}(\xi, l)$ (2.41)

Now eliminate $m_i^{n(4)}$ from equations (2.38) and (2.41).

Multiplying equation (2.41) by $\frac{h}{4}$,

$$\frac{h}{4}(U_{i+1}^n - U_{i-1}^n) = \frac{h^2}{2}V_i^n + \frac{h^4}{12}m_i^{n(4)} + \frac{h^6}{240}m_i^{n(6)}(\xi, l) \tag{2.42}$$

Subtracting equation (2.42) from equation (2.38), we get

$$(m_{i+1}^n + m_{i-1}^n) - \frac{h}{4}(U_{i+1}^n - U_{i-1}^n) = 2m_i^n + \frac{h^2}{2}V_i^n - \frac{h^6}{720}\ m_i^{n(6)}(\xi, l)$$

$$\frac{h^2}{2}V_i^n = m_{i+1}^n - 2m_i^n + m_{i-1}^n - \frac{h}{4}(U_{i+1}^n - U_{i-1}^n) + \frac{h^6}{720}\ m_i^{n(6)}(\xi, l)$$

$$V_i^n = \frac{2}{h^2}(m_{i+1}^n - 2m_i^n + m_{i-1}^n) - \frac{1}{2h}(U_{i+1}^n - U_{i-1}^n) + \frac{h^4}{360}\, m_i^{n(6)}(\xi, l) \tag{2.43}$$

Similarly

$$V_{i-1}^n = \frac{1}{2h^2}(-23m_{i-1}^n + 16m_i^n + 7m_{i+1}^n) - \frac{1}{h}(6U_{i-1}^n + 8U_i^n + U_{i+1}^n) + \frac{h^4}{90}\, m_i^{n(6)}(\xi, l) \tag{2.44}$$

$$V_{i+1}^n = \frac{1}{2h^2}(7m_{i-1}^n + 16m_i^n - 23m_{i+1}^n) + \frac{1}{h}(U_{i-1}^n + 8FU + 6U_{i+1}^n) + \frac{h^4}{90}\, m_i^{n(6)}(\xi, l) \tag{2.45}$$

From equation (2.43) substituting the value of V_i^n in equation (2.35),

$$m_t\,|_i^n = \frac{2\lambda}{h^2}(m_{i+1}^n - 2m_i^n + m_{i-1}^n) - \frac{\lambda}{2h}(U_{i+1}^n - U_{i-1}^n) - W_i^n \tag{2.46}$$

So the differential equation (2.31) is replaced by two difference equations (2.34) and (2.46).

Now, consider the left boundary condition i.e., at $x = 0$ and denote the points

$$x = -0.5, -0.5 + h, -0.5 + 2h \quad by \quad 0, 1, 2$$

The first difference equation obtained using the boundary condition is:

$$m_0^n = 0 \tag{2.47}$$

In order to get the second equation we consider the differential equation (2.35) at the points 0 and 1.

$$m_t\ |_0^n = \lambda V_0^n - W_0^n \tag{2.48}$$

$$m_t\ |_i^n = \lambda V_1^n - W_1^n \tag{2.49}$$

For $i=1$

Eqn.(2.44) implies $V_0^n = \frac{1}{2h^2}(-23m_0^n + 16m_1^n + 7m_2^n) - \frac{1}{h}(6U_0^n + 8U_1^n + U_2^n)$ (2.50)

Equation (2.43) implies $V_1^n = \frac{2}{h^2}(m_0^n - 2m_1^n + m_2^n) - \frac{1}{2h}(U_2^n - U_0^n)$ (2.51)

Equation (2.34) implies $U_0^n + 4U_1^n + U_2^n + \frac{3}{h}(m_0^n - m_2^n) = 0$ (2.52)

Eliminate m_2^n, V_0^n, V_1^n and U_2^n from five equations from (2.48) to (2.62)

From equations (2,48) and (2.50), we get:

$$m_t\ |_0^n = -\frac{23\lambda}{2h^2}m_0^n + \frac{8\lambda}{h^2}m_1^n + \frac{7\lambda}{2h^2}m_2^n - \frac{6\lambda}{h}U_0^n - \frac{8\lambda}{h}U_1^n - \frac{\lambda}{h}U_2^n - W_0^n \tag{2.53}$$

From equations (2.49) and (2.51) we get

$$m_t\ |_1^n = \frac{2\lambda}{h^2}m_0^n - \frac{4\lambda}{h^2}m_1^n + \frac{2\lambda}{h^2}m_2^n - \frac{\lambda}{2h}U_2^n + \frac{\lambda}{2h}U_0^n - W_1^n \tag{2.54}$$

Equation (2.52) implies $m_2^n = m_0^n + \frac{h}{3}U_0^n + \frac{4h}{3}U_1^n + \frac{h}{3}U_2^n$ (2.55)

Substituting this value of m_2^n in equation (2.53), we get

$$m_t\ |_0^n = -\frac{23\lambda}{2h^2}m_0^n + \frac{8\lambda}{h^2}m_1^n + \frac{7\lambda}{2h^2}(m_0^n + \frac{h}{3}U_0^n + \frac{4h}{3}U_1^n + \frac{h}{3}U_2^n) -$$

$$\frac{6\lambda}{h}U_0^n - \frac{8\lambda}{h}U_1^n - \frac{\lambda}{h}U_2^n - W_0^n$$

$$m_t\,|_0^n = -\frac{8\lambda}{h^2}m_0^n + \frac{8\lambda}{h^2}m_1^n - \frac{29\lambda}{6h}U_0^n - \frac{10\lambda}{3h}U_1^n + \frac{\lambda}{6h}U_2^n - W_0^n \tag{2.56}$$

Substituting the value of m_2^n from equation (2.55) in equation (2.54),

$$m_t\,|_1^n = \frac{4\lambda}{h^2}m_0^n - \frac{4\lambda}{h^2}m_1^n + \frac{7\lambda}{6h}U_0^n + \frac{8\lambda}{3h}U_1^n + \frac{\lambda}{6h}U_2^n - W_1^n \tag{2.57}$$

Subtracting equation (2.56) from equation (2.57)

$$\frac{12\lambda}{h^2}m_0^n - \frac{12\lambda}{h^2}m_1^n + \frac{6\lambda}{h}U_0^n + \frac{6\lambda}{h}U_1^n - W_1^n + W_0^n = m_t\,|_1^n - m_t\,|_0^n \tag{2.58}$$

Equation (2.58) is the second difference equation valid at $x = -0.5$.

In a similar manner, we can derive the following two difference equations for *m* and *U* at the right boundary point *x*=0.5 and denote the points *x* = 0.5, 0.5-*h*, 0.5–2*h* by *b*, *b*-1, *b*-2. Therefore we have

$$m_b^n = 0 \tag{2.59}$$

$$\frac{12\lambda}{h^2}m_{b-1}^n - \frac{12\lambda}{h^2}m_b^n + \frac{6\lambda}{h}U_{b-1}^n + \frac{6\lambda}{h}U_b^n - W_b^n + W_{b-1}^n = m_t\,|_b^n - m_t\,|_{b-1}^n \tag{2.60}$$

Thus for each point we have two difference equations. If we write them all together, we have the following fourth order compact scheme for equation (2.31)..

$$m_0^n = 0$$

$$\frac{12\lambda}{h^2}m_0^n - \frac{12\lambda}{h^2}m_1^n + \frac{6\lambda}{h}U_0^n + \frac{6\lambda}{h}U_1^n - W_1^n + W_0^n = m_t\,|_1^n - m_t\,|_0^n$$

$$U_{i-1}^n + 4U_i^n + U_{i+1}^n + \frac{3}{h}(m_{i-1}^n - m_{i+1}^n) = 0$$

$$\frac{2\lambda}{h^2}(m_{i+1}^n - 2m_i^n + m_{i-1}^n) - \frac{\lambda}{2h}(U_{i+1}^n - U_{i-1}^n) - W_i^n = m_t\,|_i^n$$

$$m_b^n = 0$$

$$\frac{12\lambda}{h^2}m_{b-1}^n - \frac{12\lambda}{h^2}m_b^n + \frac{6\lambda}{h}U_{b-1}^n + \frac{6\lambda}{h}U_b^n - W_b^n + W_{b-1}^n = m_t\,|_b^n - m_t\,|_{b-1}^n$$

The superscript *n* is used to denote the time grid line.

Finally consider $$E_t + \left[\frac{m}{\rho}(E+P)\right]_x = \lambda E_{xx} \qquad (2.61)$$

Denote first and second derivatives of *E* w.r.t. *x* by *Y* and *Z* respectively, i.e.,

$$E_x = Y \qquad (2.62)$$

$$E_{xx} = Z$$

$$\therefore \quad Y_x = Z$$

Integrating both sides of equation (2.62) from "x_{i-1}" to "x_{i+1}" we get

$$E_{i+1}^n - E_{i-1}^n = \int_{x_{i-1}}^{x_{i+1}} Y(\xi, l)\, d\xi$$

$$\text{or} \quad E_{i+1}^n = E_{i-1}^n + \int_{x_{i-1}}^{x_{i+1}} Y(\xi, l)\, d\xi \qquad (2.63)$$

Approximating the integral by Simpson's Rule:

$$E_{i+1}^n = E_{i-1}^n + \frac{h}{3}[Y_{i-1}^n + 4Y_i^n + Y_{i+1}^n] + \frac{1}{90}h^5\, Y^{(4)}(\xi, l)$$

$$Y_{i-1}^n + 4Y_i^n + Y_{i+1}^n + \frac{3}{h}(E_{i-1}^n - E_{i+1}^n) = -\frac{1}{30}h^4 Y^{(4)}(\xi.l)$$

Thus to fourth order

$$Y_{i-1}^{n} + 4Y_{i}^{n} + Y_{i+1}^{n} + \frac{3}{h}(E_{i-1}^{n} - E_{i+1}^{n}) = 0 \tag{2.64}$$

Thus a relationship between E and Y is obtained. This is the first difference equation. In order to obtain the second equation, we evaluate equation (2.61) at the mid point i.

Suppose $\left[\frac{m}{\rho}(E+P)\right]_x = D$

Equation (2.61) implies $E_t \,|_i^n = \lambda Z_i^n - D_i^n$ (2.65)

To find an expression for Z_i^n, expressing E_{i+1}^n, E_{i-1}^n in the Taylor's expansion about the point (i, n), we get

$$E_{i+1}^{n} = E_{i}^{n} + h\,E_x\,|_i^n + \frac{h^2}{2!}E_{xx}\,|_i^n + \frac{h^3}{3!}E_{xxx}\,|_i^n + \frac{h^4}{4!}E_{xxxx}\,|_i^n + \frac{h^5}{5!}E_{xxxxx}\,|_i^n + \frac{h^6}{6!}E_{xxxxxx}\,|_i^n + \ldots \tag{2.66}$$

$$E_{i-1}^{n} = E_{i}^{n} - +h\,E_x\,|_i^n + \frac{h^2}{2!}E_{xx}\,|_i^n - \frac{h^3}{3!}E_{xxx}\,|_i^n + \frac{h^4}{4!}E_{xxxx}\,|_i^n - \frac{h^5}{5!}E_{xxxxx}\,|_i^n + \frac{h^6}{6!}E_{xxxxxx}\,|_i^n + \ldots \tag{2.67}$$

Adding equations (2.66) and (2.67)

$$\therefore \quad E_{i+1}^{n} + E_{i-1}^{n} = 2E_{i}^{n} + h^2\,E_{xx}\,|_i^n + \frac{h^2}{12}E_i^{n(4)} + \frac{h^6}{360}E_i^{n(6)}(\xi,\, l)$$

Since $E_{xx} = Z$

$$E_{i+1}^{n} + E_{i-1}^{n} = 2E_i^n + h^2 Z_i^n + \frac{h^2}{12} E_i^{n(4)} + \frac{h^6}{360} E_i^{n(6)}(\xi,\, l) \tag{2.68}$$

Taking

$$Y_{i+1}^{n} = Y_i^n + hY_x|_i^n + \frac{h^2}{2!} Y_{xx}|_i^n + \frac{h^3}{3!} Y_{xxx}|_i^n + \frac{h^4}{4!} Y_{xxxx}|_i^n +$$

$$\frac{h^5}{5!} Y_{xxxxx}|_i^n + \frac{h^6}{6!} Y_{xxxxxx}|_i^n + ... \tag{2.69}$$

$$Y_{i+1}^{n} = Y_i^n - hY_x|_i^n + \frac{h^2}{2!} Y_{xx}|_i^n - \frac{h^3}{3!} Y_{xxx}|_i^n + \frac{h^4}{4!} Y_{xxxx}|_i^n -$$

$$\frac{h^5}{5!} Y_{xxxxx}|_i^n + \frac{h^6}{6!} Y_{xxxxxx}|_i^n + ... \tag{2.70}$$

$$\therefore \quad Y_{i+1}^{n} - Y_i^n = hY_x|_i^n + \frac{h^3}{3!} Y_{xxx}|_i^n + \frac{2h^5}{5!} Y_{xxxxx}(\xi,\, l)$$

Since $Y_x = Z,\ Y_{xxx}|_i^n = E_i^{n(4)},\ Y_i^{n(5)} = E_i^{n(6)}$

$$\therefore \quad Y_{i+1}^{n} - Y_i^n = 2hZ_i^n + \frac{h^3}{3} E_i^{n(4)} + \frac{h^5}{60} E_i^{n(6)}(\xi,\, l) \tag{2.71}$$

Now we eliminate $E_i^{n(4)}$ from equations (2.68) and (2.71)

Multiplying equation (2.71) by $\frac{h}{4}$, we get

$$\frac{h}{4}(Y_{i+1}^n - Y_{i-1}^n) = \frac{h^2}{2} Z_i^n + \frac{h^4}{12} E_i^{n(4)} + \frac{h^6}{240} E_i^{n(6)}(\xi,\, l) \tag{2.72}$$

Subtracting equation (2.72) from equation (2.68),

$$(E_{i+1}^n + E_{i-1}^n) - \frac{h}{4}(Y_{i+1}^n - Y_{i-1}^n) = 2E_i^n + \frac{h^2}{2} Z_i^n - \frac{h^6}{720} E_i^{n(6)}(\xi,\, l)$$

$$\frac{h^2}{2} Z_i^n = E_{i+1}^n - 2E_i^n + E_{i-1}^n - \frac{h}{4}(Y_{i+1}^n - Y_{i-1}^n) + \frac{h^6}{720} E_i^{n(6)}(\xi,\, l)$$

$$Z_i^n = \frac{2}{h^2}(E_{i+1}^n - 2E_i^n + E_{i-1}^n) - \frac{1}{2h}(Y_{i+1}^n - Y_{i-1}^n) + \frac{h^4}{360} E_i^{n(6)}(\xi,\ l) \qquad (2.73)$$

Similarly

$$Z_{i-1}^n = \frac{1}{2h^2}(-23E_{i-1}^n + 16E_i^n + 7E_{i+1}^n) - \frac{1}{h}(6Y_{i-1}^n + 8Y_i^n + Y_{i+1}^n) + \frac{h^4}{90} E_i^{n(6)}(\xi,\ l) \qquad (2.74)$$

$$Z_{i+1}^n = \frac{1}{2h^2}(7E_{i-1}^n + 16E_i^n - 23E_{i+1}^n) + \frac{1}{h}(Y_{i-1}^n + 8Y_i^n + 6Y_{i+1}^n) + \frac{h^4}{90} E_i^{n(6)}(\xi,\ l) \qquad (2.75)$$

From equation (2.73) substituting the value of Z_i^n in equation (2.65),

$$E_t\ |_i^n = \frac{2\lambda}{h^2}(E_{i+1}^n - 2E_i^n + E_{i-1}^n) - \frac{\lambda}{2h}(Y_{i+1}^n - Y_{i-1}^n) - D_i^n \qquad (2.76)$$

So the differential equation (2.61) is replaced by two difference equations (2.64) and (2.76).

Now, consider the left boundary condition i.e., at $x = -0.5$ and denote the points

$$x = -0.5, -0.5 + h, -0.5 + 2h \quad by \quad 0, 1, 2$$

The first difference equation obtained using the boundary condition is:

$$E_0^n = 2.5 \qquad (2.77)$$

In order to get the second equation we consider the differential equation (2.65) at the points 0 and 1.

$$E_t\ |_0^n = \lambda Z_0^n - D_0^n \qquad (2.78)$$

$$E_t\ |_1^n = \lambda Z_1^n - D_1^n \qquad (2.79)$$

For $i = 1$

Equation (2.74) implies $Z_0^n = \frac{1}{2h^2}(-23E_0^n + 16E_1^n + 7E_2^n) - \frac{1}{h}(6Y_0^n + 8Y_1^n + Y_2^n)$ (2.80)

Equation (2.73) implies $Z_1^n = \frac{2}{h^2}(E_0^n - 2E_1^n + E_2^n) - \frac{1}{2h}(Y_2^n - Y_0^n)$ (2.81)

Equation (2.64) implies $Y_0^n + 4Y_1^n + Y_2^n + \frac{3}{h}(E_0^n - E_2^n) = 0$ (2.82)

Eliminate E_2^n, Z_0^n, Z_1^n and Y_2^n from five equations from (2.78) to (2.82)

From equations (2.78) and (2.80) we get:

$$E_t |_0^n = -\frac{23\lambda}{2h^2} E_0^n + \frac{8\lambda}{h^2} E_1^n + \frac{7\lambda}{2h^2} E_2^n - \frac{6\lambda}{h} Y_0^n - \frac{8\lambda}{h} Y_1^n - \frac{\lambda}{h} Y_2^n - D_0^n \quad (2.83)$$

From equations (2.79) and (2.81) we get

$$E_t |_1^n = \frac{2\lambda}{h^2} E_0^n - \frac{4\lambda}{h^2} E_1^n + \frac{2\lambda}{h^2} E_2^n - \frac{\lambda}{2h} Y_2^n + \frac{\lambda}{2h} Y_0^n - D_1^n \quad (2.84)$$

Equation (2.82) gives $E_2^n = E_0^n + \frac{h}{3} Y_0^n + \frac{4h}{3} Y_1^n + \frac{h}{3} Y_2^n)$ (2.85)

Substituting this value of E_2^n, in equation (2.83), we get

$$E_t |_0^n = -\frac{23\lambda}{2h^2} E_0^n + \frac{8\lambda}{h^2} E_1^n + \frac{7\lambda}{2h^2} (E_0^n + \frac{h}{3} Y_0^n + \frac{4h}{3} Y_1^n + \frac{h}{3} Y_2^n)$$

$$-\frac{6\lambda}{h} Y_0^n - \frac{8\lambda}{h} Y_1^n - \frac{\lambda}{h} Y_2^n - D_0^n$$

$$E_t |_0^n = -\frac{8\lambda}{h^2} E_0^n + \frac{8\lambda}{h^2} E_1^n - \frac{29\lambda}{6h} Y_0^n - \frac{10\lambda}{3h} Y_1^n + \frac{\lambda}{6h} Y_2^n - D_0^n \quad (2.86)$$

Substituting the value of E_2^n from equation (2.85) in equation (2.84),

$$E_t |_1^n = \frac{4\lambda}{h^2} E_0^n - \frac{4\lambda}{h^2} E_1^n + \frac{7\lambda}{6h} Y_0^n + \frac{8\lambda}{3h} Y_1^n + \frac{\lambda}{6h} Y_2^n - D_0^n \quad (2.87)$$

Subtracting equation (2.86) from equation (2.87)

$$\frac{12\lambda}{h^2} E_0^n - \frac{12\lambda}{h^2} E_1^n + \frac{6\lambda}{h} Y_0^n + \frac{6\lambda}{h} Y_1^n - D_1^n + D_0^n = E_t |_1^n - E_t |_0^n \quad (2.88)$$

Equation (2.88) is the second difference equation valid at $x = -0.5$.

In a similar manner, we can derive the following two difference equations for *m* and *U* at the right boundary point $x = 0.5$ and denote the points $x = 0.5$, 0.5-h, 0.5-2h by c, c–1, c–2. Therefore we have

$$E_c^n = 0.25 \tag{2.89}$$

$$\frac{12\lambda}{h^2}E_{c-1}^n - \frac{12\lambda}{h^2}E_c^n + \frac{6\lambda}{h}Y_{c-1}^n + \frac{6\lambda}{h}Y_c^n - D_c^n + D_{c-1}^n = E_t\,|_c^n - E_t\,|_{c-1}^n \tag{2.90}$$

Thus for each point we have two difference equations. If we write them all together, we have the following fourth order compact scheme for equation (2.61).

$$E_0^n = 2.5$$

$$\frac{12\lambda}{h^2}E_0^n - \frac{12\lambda}{h^2}E_1^n + \frac{6\lambda}{h}Y_0^n + \frac{6\lambda}{h}Y_1^n - D_1^n + D_0^n = E_t\,|_1^n - E_t\,|_0^n$$

$$Y_{i-1}^n + 4y_i^n + Y_{i+1}^n + \frac{3}{h}(E_{i-1}^n - E_{i+1}^n) = 0$$

$$\frac{2\lambda}{h^2}(E_{i+1}^n - 2E_i^n + E_{i-1}^n) - \frac{\lambda}{2h}(Y_{i+1}^n - Y_{i-1}^n) - D_i^n = E_t\,|_i^n$$

$$E_c^n = 0.25$$

$$\frac{12\lambda}{h^2}E_{c-1}^n - \frac{12\lambda}{h^2}E_c^n + \frac{6\lambda}{h}Y_{c-1}^n + \frac{6\lambda}{h}Y_c^n - D_c^n + D_{c-1}^n = E_t\,|_c^n - E_t\,|_{c-1}^n$$

The superscript n is used to denote the time grid line.

2.3 RESULTS

We have derived fourth order compact method for one dimensional gas dynamics equations in this chapter. The values of density, velocity, pressure and internal energy of the gas are obtained at $t = 0.15$ sec by using fourth order compact method and by using second order finite difference method for one dimensional gas dynamics equations are plotted together in Figures 2.1 to 2.4. It is observed that results of fourth order compact method are more accurate than the results of second order finite difference method.

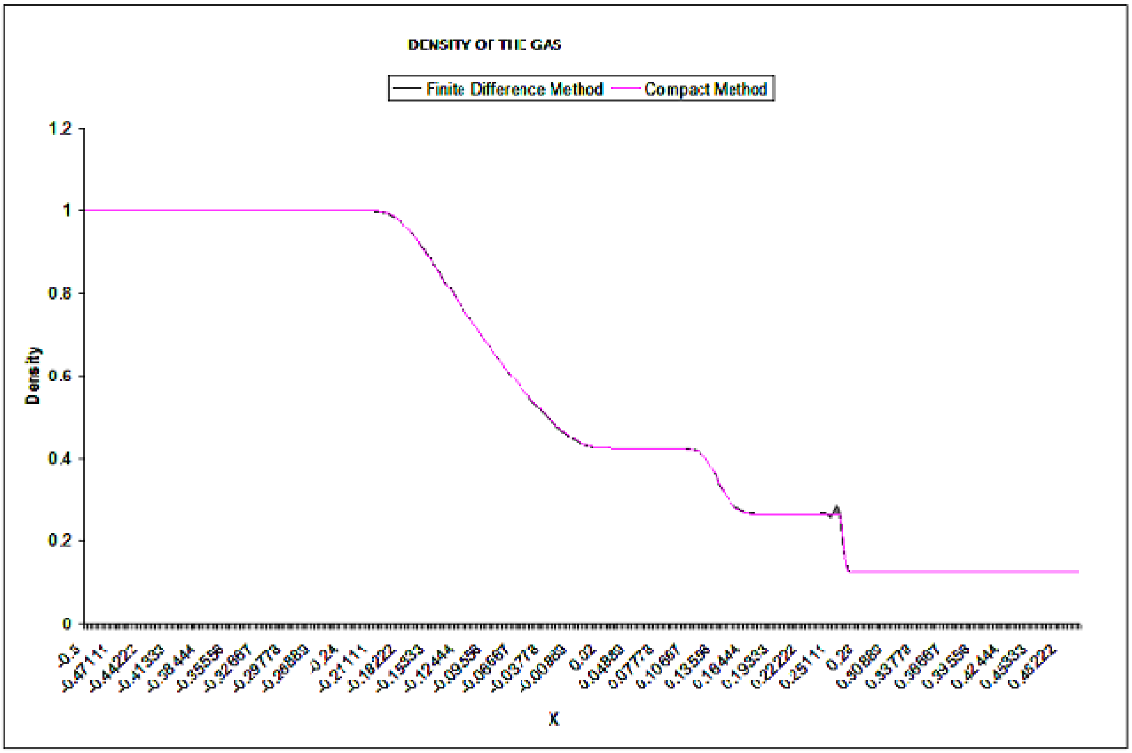

Figure 2.1

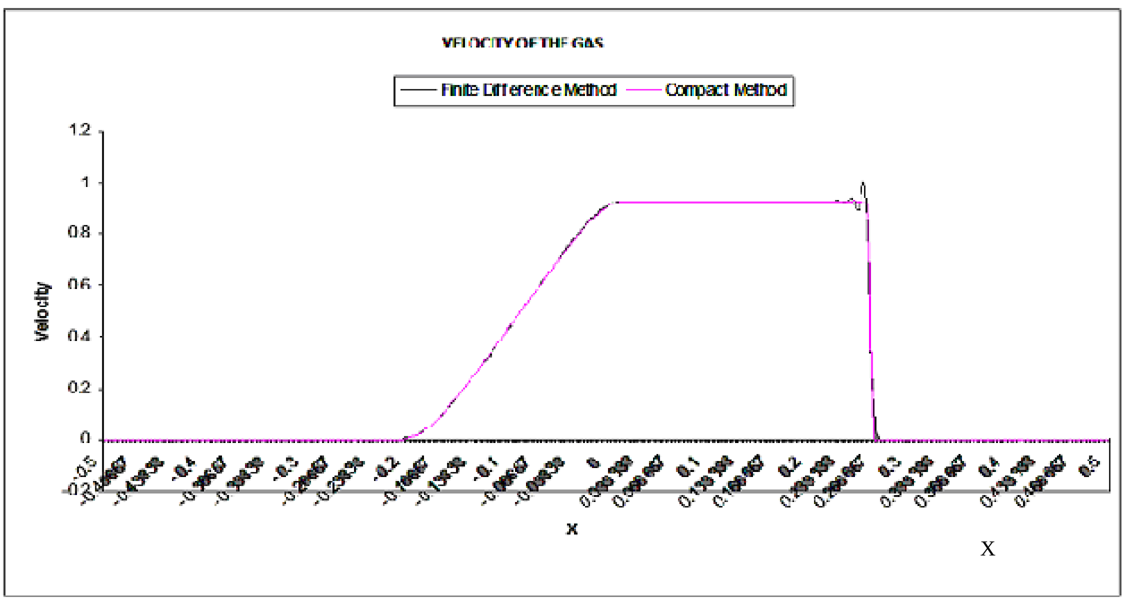

Figure 2.2

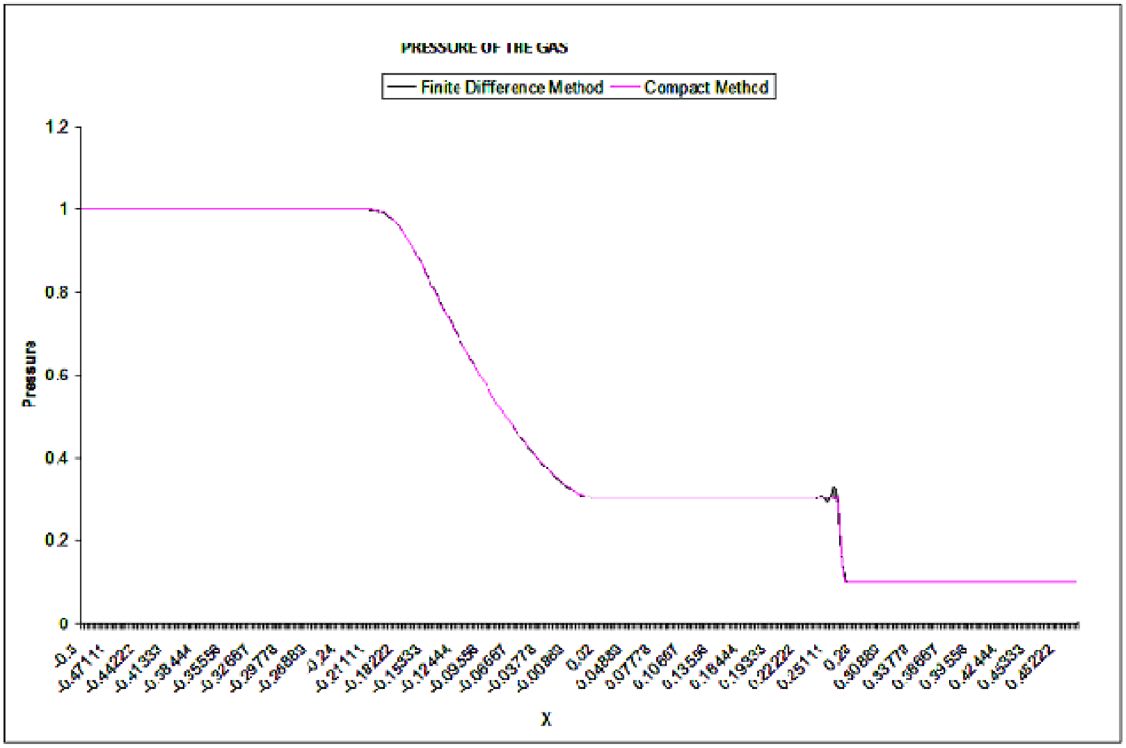

Figure 2.3

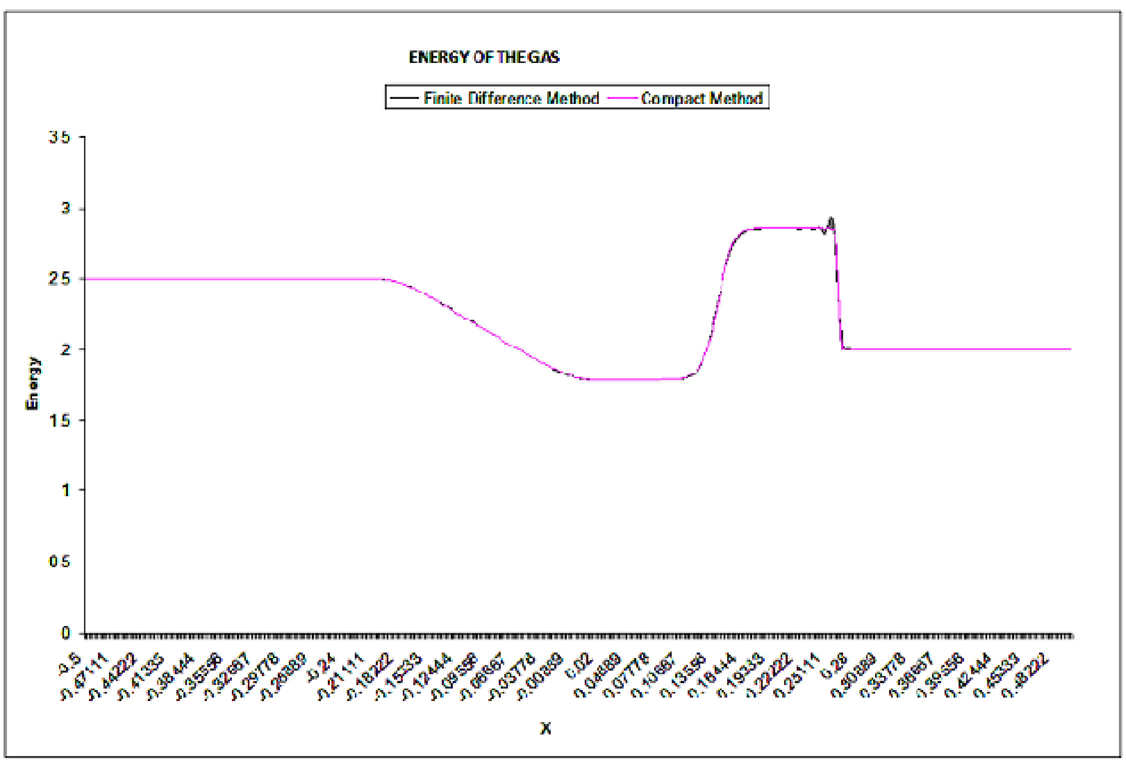

Figure 2.4

TABLE 2.1- DENSITY OF THE GAS

X	Finite Difference	Compact Method	Absolute Error
-0.500000	1.00000000	1.00000000	0.00000000
-0.400000	1.00000000	1.00000000	0.00000000
-0.300000	1.00000000	1.00000000	0.00000000
-0.200000	0.90522825	0.97566875	0.0704405
-0.100000	0.59331406	0.61387751	0.02056345
0.000000	0.43400938	0.43562232	0.0161294
0.100000	0.42314568	0.42394730	0.00080162
0.200000	0.24555786	0.42293783	0.17737997
0.300000	0.24500000	0.26350210	0.01850210
0.400000	0.12500000	0.12500000	0.00000000
0.500000	0.12500000	0.12500000	0.00000000

TABLE 2.2- VELOCITY OF THE GAS

X	Finite Difference	Compact Method	Absolute Error
-0.500000	0.00000000	0.00000000	0.00000000
-0.400000	0.00000000	0.00000000	0.00000000
-0.300000	0.00000000	0.00000000	0.00000000
-0.200000	0.00565540	0.00455840	0.0.107140
-0.100000	0.42905106	0.41804510	0.01100596
0.000000	0.90723119	0.73451415	0.17271704
0.100000	0.92746010	0.91145450	0.0159956
0.200000	0.92742284	0.92745038	0.00002754
0.300000	0.00000000	0.00004860	0.00004860
0.400000	0.00000000	0.00000000	0.00000000
0.500000	0.00000000	0.00000000	0.00000000

TABLE 2.3- PRESSURE OF THE GAS

X	Finite Difference	Compact Method	Absolute Error
-0.500000	1.00000000	1.00000000	0.00000000
-0.400000	1.00000000	1.00000000	0.00000000
-0.300000	1.00000000	1.00000000	0.00000000
-0.200000	0.99332728	0.99605518	0.0027279
-0.100000	0.52936241	0.59736538	0.06800297
0.000000	0.31182478	0.31254275	0.00071797
0.100000	0.30312357	0.30311860	0.00000497
0.200000	0.30311914	0.30312286	0.00000372
0.300000	0.10000000	0.10000000	0.00000000
0.400000	0.10000000	0.10000000	0.00000000
0.500000	0.10000000	0.10000000	0.00000000

TABLE 2.4- ENERGY OF THE GAS

X	Finite Difference	Compact Method	Absolute Error
-0.500000	2.50000000	2.50000000	0.00000000
-0.400000	2.50000000	2.50000000	0.00000000
-0.300000	2.50000000	2.50000000	0.00000000
-0.200000	2.49522477	2.50000000	0.00477523
-0.100000	2.16122261	2.16258570	0.00136309
0.000000	1.79618690	1.85535465	0.05916775
0.100000	1.79089372	1.86234784	0.07145412
0.200000	1.85360731	1.8360620	0.01754531
0.300000	2.00000000	2.00000000	0.00000000
0.400000	2.00000000	2.00000000	0.00000000
0.500000	2.00000000	2.00000000	0.00000000

Chapter – 3

MOVING MESH METHOD

3.1 INTRODUCTION

In this chapter moving mesh method [4] is discussed and is used to solve one dimensional gas dynamics equations. Adaptive mesh methods are much more efficient than uniform mesh methods for solving time-dependent partial differential equations with large gradients such as shock waves, propagated boundary layers, etc. The adaptive mesh methods can be applied in three different ways.

(1) The h-refinement methods, which add or delete mesh points according to the profile of the solution and control the mesh points by the estimated local errors of the solution.

(2) The p-refinement methods, which alter the order of the numerical method to fit the local solution characteristics.

(3) The moving mesh methods, in which a fixed number of mesh points move automatically to minimize the estimated errors of the solution.

We have derived some moving mesh partial differential equations (MMPDEs) for solving one-dimensional initial value problems.

3.2 EQUIDISTRIBUTION PRINCIPLE (EP)

The Equidistribution Principle (EP) is discussed in this section. Suppose x and ξ represent the physical and computational coordinates respectively, whose domains are assumed to be the interval [0, 1], then a one-to-one coordinate transformation between these domains can be written as:

$$x = x(\xi, t) \qquad x \in [0,1]$$

$$x(0,t) = 0, \qquad x(1,t) = 1 \tag{3.1}$$

where t denotes time. For a given uniform mesh on the computational domain,

$$\xi_i = \frac{i}{n}, \qquad i = 0, 1, 2, \ldots, n \tag{3.2}$$

where n is a positive integer, the corresponding mesh in x is denoted by:

$$\{x_o, x_1, x_2, \ldots, x_n\} \tag{3.3}$$

Thus on this computational mesh, the values of any arbitrary function f can be written as

$$f_i = f(\xi_i, t) \tag{3.4}$$

To develop the moving mesh method an approach which is directly based on the equidistribution principle is used. The equidistribution principle, which was first introduced by de Boor [23] for solving boundary value problems for ordinary differential equations, is based upon the simple idea, that if some measure of the error $M(x, t)$ (monitor function) is available, then we select the mesh points

$$0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$$

such that the contributions to the solution error over each subinterval are equalized (or "distributed equally"). Mathematically, this is the goal of finding moving meshes.

$$0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1 \tag{3.5}$$

which equally distribute error for all values of t, such that

$$\int_{x_{i-1}}^{x_i} M(\bar{x}, t)\, d\bar{x} = \frac{1}{n} \int_0^1 M(\bar{x}, t)\, d\bar{x} \tag{3.6}$$

This equidistribution equation (3.6) can be written as

$$\int_0^{x_i} M(\bar{x},t)\,d\bar{x} = \frac{i}{n}\int_0^1 M(\bar{x},t)\,d\bar{x}$$

$i = 0,1,2,\ldots,n$ (3.7)

A continuous version of equation (3.7) is

$$\int_0^{x(\xi,t)} M(\bar{x},t)\,d\bar{x} = \xi\,\theta(t) \tag{3.8}$$

where

$$\theta(t) = \int_0^1 M(\bar{x},t)\,d\bar{x} \tag{3.9}$$

Equation (3.8) is the one-dimensional equidistribution principle in the integral form. Differentiating equation (3.8) with respect to ξ**,** we obtain

$$M(x(\xi,t)t)\frac{\partial}{\partial\xi}x(\xi,t) = \theta(t) \tag{3.10}$$

Differentiating equation (3.10) with respect to ξ**,** we obtain

$$\frac{\partial}{\partial\xi}\left\{M(x(\xi,t)t)\frac{\partial}{\partial\xi}\,x(\xi,t)\right\} = 0 \tag{3.11}$$

These are two differential forms of equation (3.8). Since none these equations i.e. (3.8), (3.10) and (3.11) contain the node speed $\dot{x}$ **(**ξ**,** *t*), they are called quasi-static equidistribution principles (QSEPs).

Here are some examples of monitor functions commonly used:

Arc length:

$$M\,(x,t) = (1+(u')^2)^{1/2}$$

Local truncation error:

$$M\,(x,t) = (1+(u''')^2)^{1/4}$$

Single-step error:

$$M\,(x,t) = (1+(u''')^2)^{1/6}$$

3.3 MOVING MESH PARTIAL DIFFERENTIAL EQUATION (MMPDEs)

Differentiating equation (3.8) with respect to t.

$$M(x(\xi,t)t)\,\dot{x}\,(\xi,t) + \int_0^{x(\xi,t)} M(\bar{x},t)\,d\bar{x} = \xi\dot{\theta}(t) \qquad (3.12)$$

Here $\dot{x}$ denotes $\left.\frac{\partial x}{\partial t}\right|_{\xi\ fixed}$

Now differentiating equation (3.12) with respect to ξ we have

$$M\,\frac{\partial \dot{x}}{\partial \xi} + \frac{\partial M}{\partial \xi}\,\dot{x} + \frac{\partial M}{\partial t}\,\frac{\partial x}{\partial \xi} = \dot{\theta}(t) \qquad (3.13)$$

Differentiating equation (3.13) with respect to ξ we have the following MMPDE1:

$$\frac{\partial}{\partial \xi}\left[M\,\frac{\partial \dot{x}}{\partial \xi}\right] + \frac{\partial}{\partial \xi}\left[M\,\frac{\partial M}{\partial \xi}\,\dot{x}\right] = -\frac{\partial}{\partial \xi}\left[\frac{\partial M}{\partial t}\,\frac{\partial x}{\partial \xi}\right] \qquad (3.14)$$

Now differentiating equation (3.10), with respect to t.

$$\frac{\partial}{\partial \xi}\,(M\,x) + \frac{\partial M}{\partial t}\,\frac{\partial x}{\partial \xi} = \dot{\theta}(t) \qquad (3.15)$$

Differentiate equation (3.15) with respect to ξ to get the following MMPDE2:

$$\frac{\partial^2}{\partial \xi^2}\,(M\,\dot{x}) = \frac{-\partial}{\partial \xi}\left[\frac{\partial M}{\partial t}\;\frac{\partial x}{\partial \xi}\right] \tag{3.16}$$

To derive MMPDE from the differential form of the equation (3.11), write (3.11) at a later time, $t+\Im$ $(0\le\Im\le 1)$

$$\frac{\partial}{\partial \xi}\left\{M\,(x(\xi,t+\Im),t+\Im)\frac{\partial}{\partial \xi}\;x\;(\xi,t+\Im)\;\right\}=0 \tag{3.17}$$

Equation (3.17) is regarded as a condition to regularize the mesh movement. Applying Taylor's expansion,

$$\frac{\partial}{\partial \xi}x(\xi,t+\Im)=\frac{\partial}{\partial \xi}\;x\,(\xi,t)+\Im\frac{\partial}{\partial \xi}\dot{x}(\xi,t)+0(\Im^2)$$

$$M\,(x(\xi,t+\Im),t+\Im)=M\,(x(\xi,t),t)+\Im\,\dot{x}\frac{\partial}{\partial x}M\,(x(\xi,t),t)$$

$$\Im\frac{\partial}{\partial x}M\,(x(\xi,t),t)+0(\Im^2) \tag{3.18}$$

In equation (3.17),

$$\frac{\partial}{\partial \xi}\left\{\left[M+\Im\,\dot{x}\frac{\partial M}{\partial x}+\Im\;\frac{\partial M}{\partial t}\right]\left[\frac{\partial x}{\partial \xi}+\Im\frac{\partial \dot{x}}{\partial \xi}\right]\right\}=0$$

After simplification, this equation takes the form

$$\frac{\partial}{\partial \xi}\left\{M\;\frac{\partial x}{\partial \xi}+M\Im\frac{\partial \dot{x}}{\partial \xi}+\Im\,\dot{x}\frac{\partial M}{\partial \xi}+\Im^2\;\dot{x}\frac{\partial M}{\partial x}\;\frac{\partial \dot{x}}{\partial \xi}\;+\Im\frac{\partial M}{\partial t}\;\frac{\partial x}{\partial \xi}+\Im^2\;\frac{\partial M}{\partial t}\;\frac{\partial \dot{x}}{\partial \xi}\right\}=0$$

Therefore, after dropping the higher order terms,

$$\frac{\partial}{\partial \xi}\left[M\;\frac{\partial x}{\partial \xi}\right]+\frac{\partial}{\partial \xi}\left[M\;\Im\frac{\partial \dot{x}}{\partial \xi}\right]+\frac{\partial}{\partial \xi}\left[\Im\,\dot{x}\frac{\partial M}{\partial \xi}\right]+\frac{\partial}{\partial \xi}\left[\;\Im\frac{\partial M}{\partial t}\;\frac{\partial x}{\partial \xi}\right]=0$$

$$\frac{\partial}{\partial \xi}\left[M \frac{\partial \dot{x}}{\partial \xi}\right]+\frac{\partial}{\partial \xi}\left[\frac{\partial M}{\partial \xi}\ \dot{x}\right]$$

$$= -\frac{\partial}{\partial \xi}\ \left[\frac{\partial M}{\partial t}\ \frac{\partial x}{\partial \xi}\right] - \frac{1}{\Im}\ \frac{\partial}{\partial \xi}\left[M \frac{\partial x}{\partial \xi}\right] \qquad (3.19)$$

Equation (3.19) is written as MMPDE3.

The MMPDE (3.19) contains the term $-\frac{1}{\Im}\frac{\partial}{\partial \xi}\left(M \frac{\partial x}{\partial \xi}\right)$, which measures how closely the mesh *x* (ξ, t) satisfies the QSEP. When *x* (ξ, *t*) is not equidistributed, then the MMPDE given by equation (3.19) moves the mesh toward equidistribution even when *M*(*x*, *t*) is independent of *t*. The term $\frac{\partial M}{\partial t}$ which is usually difficult to calculate, is a relatively unimportant term. Therefore one can argue that it is reasonable to drop the term $\frac{\partial x}{\partial \xi}\frac{\partial M}{\partial t}$ or both $\frac{\partial x}{\partial \xi}\frac{\partial M}{\partial t}$ and $\dot{x}\ \frac{\partial M}{\partial \xi}$ in MMPDE (3.19). This leads to the simplified MMPDE.

$$\frac{\partial}{\partial \xi}\left[M \frac{\partial \dot{x}}{\partial \xi}\right]+\frac{\partial}{\partial \xi}\left[\frac{\partial M}{\partial \xi}\dot{x}\right]= -\frac{1}{\Im}\ \frac{\partial}{\partial \xi}\left[\ M\ \frac{\partial x}{\partial \xi}\right] \qquad (3.20)$$

This equation can also be written as MMPDE 4:

$$\frac{\partial}{\partial \xi}\left[M \frac{\partial \dot{x}}{\partial \xi}+\frac{\partial M}{\partial \xi}\dot{x}\right]= -\frac{1}{\Im}\ \frac{\partial}{\partial \xi}\left[\ M\ \frac{\partial x}{\partial \xi}\right]$$

$$\frac{\partial}{\partial \xi}\left[\ \frac{\partial x}{\partial \xi}(M\ \dot{x})\right] =-\frac{1}{\Im}\ \frac{\partial}{\partial \xi}\left[\ M\ \frac{\partial x}{\partial \xi}\right]$$

$$\frac{\partial^2}{\partial \xi^2}\ (M\ \dot{x}) =\ -\frac{1}{\Im}\ \frac{\partial}{\partial \xi}\left[\ M\ \frac{\partial x}{\partial \xi}\right] \qquad (3.21)$$

If we drop both $\frac{\partial x}{\partial \xi}\frac{\partial M}{\partial \xi}$ and $\dot{x}\ \frac{\partial M}{\partial \xi}$ from MMPDE3 (3.19) we have

MMPDE5:

$$\frac{\partial}{\partial \xi}\left[M \frac{\partial \dot{x}}{\partial \xi}\right] = \frac{-1}{\Im} \frac{\partial}{\partial \xi}\left[M \frac{\partial x}{\partial \xi}\right] \tag{3.22}$$

The above formulation, which is based directly on the equidistribution principle and uses the correction term $\frac{-1}{\Im} \frac{\partial}{\partial \xi}\left[\frac{\partial x}{\partial \xi}\right]$ is very useful since it is quite simple. In principle, the approach can be directly extended to higher space dimensions, if a formula for an equidistributions principles is available.

3.4 MMPDES BASED ON ATTRACTION AND REPULSION PSEUDO FORCES

In this section, some moving mesh methods based on attraction and repulsion pseudo-forces between nodes will be reviewed. A node attracts others when a measure of the truncation error at this point is larger than average. If the measure is smaller than average, the neighbouring nodes are repelled. Methods considered here compute node speed in response to deviation in an error measure from some average value. An error measure, denoted by W, is generally related to some error function.

In particular, the error measure is usually expressed by

$$w_i = \int_{x_i}^{x_{i+1}} M(\bar{x}, t)\, d\bar{x} \tag{3.23}$$

where M is a certain error function. Its discrete form is

$$w = M \frac{\partial x}{\partial \xi} \tag{3.24}$$

although the function M here may be slightly motivated, e.g., by taking a simple approximation for (3.23) such as the midpoint rule,

$$w_i = M_{i+1/2}(x_{i+1} - x_i) \tag{3.25}$$

The error functions are often chosen to be proportional to the first and/or second derivatives of the physical solutions. Probably the most common choices in practice are the arc-length and curvature monitor function. Anderson [5] computes the node speed by

$$\dot{x} = \frac{1}{\Im} \frac{\partial}{\partial \xi}\left[M \frac{\partial x}{\partial \xi} \right] \tag{3.26}$$

where $\Im$ is a positive constant. Equation (3.26) is MMPDE6. Regarding W as an error indicator, one observes from equation (3.25) that MMPDE6 moves the nodes towards regions where the error is large. It also forces the mesh to have zero speed whenever the mesh is equidistributed.

The node speed is determined by

$$\dot{x}_{i+1} - \dot{x}_i = -\lambda (w_i - \overline{w}) \tag{3.27}$$

where λ is a positive parameter, w_i is an error indicator on the subinterval (x_i, x_{i+1}), and $\overline{w}$ is the average of the w_i value. From equation (3.27),

$$\dot{x}_i - \dot{x}_{i-1} = -\lambda (w_{i-1} - \overline{w}) \tag{3.28}$$

subtracting equation (3.28) from equation (3.27)

$$\dot{x}_{i+1} - 2\dot{x}_i + \dot{x}_{i-1} = -\lambda (w_i - w_{i-1}) \tag{3.29}$$

Denoting λ by $\frac{1}{\Im}$,

$$\dot{x}_{i+1} - 2\dot{x}_i + \dot{x}_{i-1} = -\frac{1}{\Im} (w_i - w_{i-1}) \tag{3.30}$$

If we use equation (3.24), we can view equation (3.30) as a centered finite difference approximation of the following MMPDE, which is called MIMIPDE 7:

$$\frac{\partial \dot{x}}{\partial \xi^2} = \frac{1}{\Im} \frac{\partial}{\partial \xi}\left[M \frac{\partial x}{\partial \xi} \right] \tag{3.31}$$

3.5 DESCRIPTION OF THE METHOD

Now a complete description of the moving mesh method is given. Consider a time dependent problem of the form

$$u_t = f(u, u_x, u_{xx}), \qquad 0 < x < 1,\ t > 0 \tag{3.32}$$

subject to appropriate boundary and initial conditions, where *f* represents a differential operator involving only spatial derivatives. Using the coordinate transformation (3.1), equation (3.32) can be rewritten in quasi-Lagrangian form

$$\dot{u} - u_x\, \dot{x} = f(u, u_x, u_{xx}), \qquad 0 < x < 1,\ t > 0 \tag{3.33}$$

Next, discretizing by using a central difference scheme for spatial derivatives., the following equation is obtained.

$$\dot{u}_i - \frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i-1}}\dot{x}_i = f_i, \qquad i = 1, 2, \ldots, n-1 \tag{3.34}$$

where f_i is the discrete approximation of $f(u,\, u_x,\, u_{xx})$.

For a given monitor function $M(x, t)$, we have to solve the coupled system of equations (3.34), one of the moving mesh partial differential equations (MMPDEs), and the corresponding boundary and initial conditions for the mesh *x* and solution u. We use the method of lines (MOL) to convert partial differential equations into a system of ordinary differential equations (ODEs) and solve the system of (ODEs) using ODE solver LSODE. It is a subroutine for solving initial value problems in ordinary differential equations.

3.6 APPLICATION

As a case study one-dimensional gas dynamics equations are solved by using moving mesh method in this section.

To apply a moving mesh method, the gas dynamics equation (1.16) can be written as:

$$u_t = G \tag{3.35}$$

Where

$$G = \lambda u_{xx} - [F(u)]_x \tag{3.36}$$

The vector u is given by

$$\begin{bmatrix} \rho \\ m \\ E \end{bmatrix}$$

and F(u) is given by:

$$\begin{bmatrix} m \\ \dfrac{m^2}{\rho} + P \\ \dfrac{m}{\rho}(E + P) \end{bmatrix}$$

Take $\lambda = 5 \times 10^{-4}$

Finite difference approximations for first and second derivatives on a moving grid are given by:

$$u_x = \frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i-1}} \tag{3.37}$$

$$u_{xx} = 2\left[\frac{u_{i+1}(x_i - x_{i-1}) - u_i(x_{i+1} - x_{i-1}) + u_{i-1}(x_{i+1} - x_i)}{(x_{i+1} - x_i)(x_i - x_{i-1})(x_{i+1} - x_{i-1})}\right] \tag{3.38}$$

Now using the transformation (3.1) equation (3.35) is expressed in the quasi-Langrangian form as:

$$\dot{u} - u_x \ \dot{x} = G \tag{3.39}$$

Using central differences, equation (3.39) becomes

$$\dot{u}_i - \frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i-1}} \dot{x}_i = G_i , \; i = 0,1,2,..., N \tag{3.40}$$

where G_1 is the discrete approximation of G. Monitors involving high derivatives of $u(x)$ can be extremely complicated to implement. White [6] recommends using arc length. Here we also choose M to be the arc length monitor function.

$$M\,(x,t) = [1 + (u_x)^2]^{1/2}$$

We will be using MMPDE6, because it gives the best result. We discretize MMPDE6 i.e. equation (3.26) in space with centered finite differences on the uniform mesh (3.2) and use the method of lines. We have the following discrete approximation of MMPDE6;

$$\dot{x}_i = \frac{E_i}{\mathfrak{I}} \tag{3.41}$$

Here

E_i is the discrete approximation of

$$\frac{\partial}{\partial \xi}\left(M \,\frac{\partial x}{\partial \xi}\right) at \;\; \xi = \xi_i \;\; \text{given by}$$

$$E_i = \frac{\overline{M}_{i+1} + \overline{M}_i}{2\left(\frac{1}{n}\right)^2}(x_{i+1} - x_i) - \frac{\overline{M}_i + \overline{M}_{i-1}}{2\left(\frac{1}{n}\right)^2}(x_i - x_{i-1}) \tag{3.42}$$

Thus equation (3.41) takes the form

$$\dot{x}_i = \frac{n^2}{2\mathfrak{I}}[(\overline{M}_i + \overline{M}_{i+1})\,x_{i+1} - (\overline{M}_{i-1} + 2\overline{M}_i + \overline{M}_{i+1})\,x_i$$

$$+ (\overline{M}_{i-1} + \overline{M}_i)\,x_{i-1}] \tag{3.43}$$

$$i = 1, 2, 3, \ldots\ldots, \; n-1$$

$$\dot{x}_1 = \frac{n^2}{2\mathfrak{I}}[(\overline{M}_i + \overline{M}_2)\,x_2 - (\overline{M}_0 + 2\overline{M}_1 + \overline{M}_2)\,x_1 - 0.5(\overline{M}_0 + \overline{M}_1)] \tag{3.44}$$

$$\dot{x}_i = \frac{n^2}{2\Im}[(\overline{M}_i + \overline{M}_{i+1})\, x_{i+1} - (\overline{M}_{i-1} + 2\overline{M}_i + \overline{M}_{i+1})\, x_i$$

$$+ (\overline{M}_{i-1} + \overline{M}_i)\, x_{i-1}] \tag{3.45}$$

$$i = 2, 3, 4, \ldots\ldots,\ n-2$$

$$\dot{x}_{n-1} = \frac{n^2}{2\Im}[0.5(\overline{M}_{n-1} + \overline{M}_n) - (\overline{M}_{n-2} + 2\overline{M}_{n-1} + \overline{M}_n)\, x_{n-1}$$

$$+ (\overline{M}_{n-2} + \overline{M}_{n-1})\, x_{n-2}] \tag{3.46}$$

where $\overline{M}_i$ is the smoothed form of

$$M_i = \left[1 + \left(\frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i+1}} \right)^2 \right]^{1/2} \tag{3.47}$$

and

$$\overline{M}_i = \left(\frac{M_i *}{S_i *} \right)^{1/2} \tag{3.48}$$

where

$$M_i * = \sum_{k=i-j}^{i+j} (M_k)^2 \left(\frac{\eta}{1+\eta} \right)^{|k-i|} \tag{3.49}$$

$$S_i * = \sum_{k=i-j}^{i+j} \left(\frac{\eta}{1+\eta} \right)^{|k-i|} \tag{3.50}$$

where $\eta > 0$ is the smoothing parameter, and j a non-negative integer, is the smoothing index. The summation is understood to contain only elements with indices between zero and n. Thus the problem is reduced to solving two sets of equations (3.40) and (3.41). The initial conditions for x_i is a uniform mesh, i.e.,

$$x_i\,(0) = \frac{i}{n} \qquad\qquad i = 0, 1, 2, \,.\,.\,.\,,\; n$$

with the boundary conditions $\dot{x}(0) = 0$, $\dot{x}(n) = 0$

As mentioned before, the systems of ordinary differential equations are solved using ordinary differential equation solver LSODE. For calculations, a relative and absolute tolerance of 10^{-8} is assumed. After testing various values and combination for the parameters η, j and $\Im$, following values have been chosen since they give the most accurate result:

$$\eta = 2 \qquad j = 2 \quad \text{and} \quad \Im = 10^{-3}$$

One-dimensional gas dynamics equation is solved using moving mesh method with n = 100 at t = 0.15. Other moving mesh formulations have also been considered. However, the best results were obtained by using MMPDE6.

3.7 MOVING MESH FORM MONITOR FUNCTION

$$M_i = \left[1 + \left(\frac{\rho_{i+1} - \rho_{i-1}}{x_{i+1} - x_{i-1}} \right)^2 + \left(\frac{m_{i+1} - m_{i-1}}{x_{i+1} - x_{i-1}} \right)^2 + \left(\frac{E_{i+1} - E_{i-1}}{x_{i+1} - x_{i-1}} \right)^2 \right]^{1/2}$$

For left boundary point, we use Forward difference i.e.

$$M_i = \left[1 + \left(\frac{-3u_i + 4u_{i+1} - u_{i+2}}{4(x_{i+1} - x_i) - (x_{i+2} - x_i)} \right)^2 \right]^{1/2}$$

$$M_i = \left[1 + \left(\frac{-3\rho_i + 4\rho_{i+1} - \rho_{i+2}}{4(x_{i+1} - x_i) - (x_{i+2} - x_i)} \right)^2 + \left(\frac{-3m_i + 4m_{i+1} - m_{i+2}}{4(x_{i+1} - x_i) - (x_{i+2} - x_i)} \right)^2 \right.$$

$$\left. + \left(\frac{-3E_i + 4E_{i+1} - 4E_{i+2}}{4(x_{i+1} - x_i) - (x_{i+2} - x_i)} \right)^2 \right]^{1/2}$$

For right boundary point, we use backward difference i.e.,

$$M_i = \left[1+\left(\frac{3u_i - 4u_{i-1} + 4u_{i-2}}{(x_{i-2}-x_i)-4(x_{i-2}-x_i)}\right)^2\right]^{1/2}$$

i.e.,

$$M_i = \left[1+\left(\frac{3\rho_i - 4\rho_{i-1} + \rho_{i-2}}{(x_{i+2}-x_i)-4(x_{i-1}-x_i)}\right)^2 + \left(\frac{3m_i - 4m_{i-1} + m_{i-2}}{(x_{i-2}-x_i)-4(x_{i-1}-x_i)}\right)^2\right.$$

$$\left. + \left(\frac{3E_i - 4E_{i-1} + 4E_{i-2}}{(x_{i-2}-x_i)-4(x_{i-1}-x_i)}\right)^2\right]^{1/2}$$

Thus we have the following equations for the monitor functions.

$$M_0 = \left[1+\frac{(-3+4\rho_1-\rho_2)^2+(4m_1-m_2)^2+(-7.5+4E_1-E_2)^2}{\left(4(x_1+0.5)-(x_2+0.5)\right)^2}\right]^{1/2}$$

$$M_1 = \left[1+\frac{(\rho_2-1.0)^2+m_2^2+(E_2-2.5)^2}{(x_2+0.5)^2}\right]^{1/2}$$

$$M_i = \left[1+\frac{(\rho_{1+1}-\rho_{i-1})^2+(m_{i+1}-m_{i-1})^2+(E_{i+1}-E_{i-1})^2}{(x_{i+1}-x_{i-1})^2}\right]^{1/2}$$

$$i = 2, \ldots, n-2$$

$$M_{n-1} = \left[1+\frac{(0.125-\rho_{n-2})^2+m_{n-2}^2+(0.25-E_{n-2})^2}{(0.5-x_{n-2})^2}\right]^{1/2}$$

$$M_n = \left[1+\frac{(0.375+4\rho_{n-1}+\rho_{n-2})^2+(-4m_{n-1}+m_{n-2})^2+(0.75-4E_{n-1}+E_{n-2})^2}{(x_{n-2}-0.5)-4(x_{n-1}-0.5)^2}\right]^{1/2}$$

3.8 MOVING MESH FORM FOR EQUATION (3.36)

Equation (3.36) is

$$G = \lambda u_{xx} - [F(u)]_x$$

Let

$$V = \lambda \rho_{xx} - m_x$$

$$V_1 = 2\lambda \left[\frac{\rho_2 (x_1 + 0.5) - \rho_1 (x_2 + 0.5)^2 + (x_2 - x_1)}{(x_2 - x_1)(x_1 + 0.5)(x_2 + 0.5)} \right] - \frac{m_2}{x_2 + 0.5}$$

$$V_i = 2\lambda \left[\frac{\rho_{i+1}(x_i - x_{i-1}) - \rho_i (x_{i+1} - x_{i-1}) + \rho_{i-1}(x_{i+1} - x_i)}{(x_{i+1} - x_i)(x_i - x_{i-1})(x_{i+1} - x_{i-1})} \right]$$

$$- \frac{m_{i+1} - m_{i-1}}{x_{i+1} - x_{i-1}} \qquad i = 2, \ldots, n-2$$

$$V_{n-1} = 2\lambda \left[\frac{0.125(x_{n-1} - x_{n-2}) - \rho_{n-1}(0.5 - x_{n-2}) + \rho_{n-2}(0.5 - x_{n-1})}{(0.5 - x_{n-1})(x_{n-1} - x_{n-2})(0.5 - x_{n-2})} \right]$$

$$+ \frac{m_{n-2}}{0.5 - x_{n-2}}$$

Let

$$W = \lambda m_{xx} - \left(\frac{m^2}{\rho} + P \right)_x$$

$$W_1 = 2\lambda \left[\frac{m_2 (x_1 + 0.5) - m_1 (x_2 + 0.5)}{(x_2 - x_1)(x_1 + 0.5)(x_2 + 0.5} \right] - \left[\frac{\left(\frac{m_2^2}{\rho_2} + P_2 \right) - 1}{x_2 + 0.5} \right]$$

$$W_i = 2\lambda\left[\frac{m_{i+1}(x_i - x_{i-1}) - m_i(x_{i+1} - x_{i-1}) + m_{i-1}(x_{i+1} - x_i)}{(x_{i+1} - x_i)(x_i - x_{i-1})(x_{i+1} - x_{i-1})}\right]$$

$$-\left[\frac{\left(\frac{m_{i+1}}{\rho_{i+1}} + P_{i+1}\right) - \left(\frac{m_{i-1}^2}{\rho_{i-1}} + P_{i-1}\right)}{x_{i+1} - x_{i-1}}\right] \qquad i = 2,\dots,n-2$$

$$W_{n-1} = 2\lambda\left[\frac{-m_{n-1}(0.5 - x_{n-2}) + m_{n-2}(0.5 - x_{n-1})}{(0.5 - x_{n-1})(x_{n-1} - x_{n-2})(0.5 - x_{n-2})}\right]$$

$$-\left[\frac{0.1 - \left(\frac{m_{n-2}^2}{\rho_{i+1}} + P_{n-2}\right)}{0.5 - x_{n-2}}\right]$$

Let $$Y = \lambda E_{xx} - \left(\frac{m}{\rho}(E + P)\right)_x$$

$$Y_1 = 2\lambda\left[\frac{E_2(x_1 + 0.5) - E_1(x_2 + 0.5) + 2.5(x_2 - x_1)}{(x_2 - x_1)(x_1 + 0.5)(x_2 + 0.5)}\right] - \frac{\frac{m_2}{\rho_2}(E_2 + P_2)}{x_2 + 0.5}$$

$$Y_i = 2\lambda\left[\frac{E_{i+1}(x_i - x_{i-1}) - E_i(x_{i+1} - x_{i-1}) + E_{i-1}(x_{i+1} - x_i)}{(x_{i+1} - x_i)(x_i - x_{i-1})(x_{i+1} - x_{i-1})}\right]$$

$$= -\left[\frac{\frac{m_{i+1}}{\rho_{i+1}}(E_{i+1} + P_{i+1}) - \frac{m_{i-1}}{\rho_{i-1}}(E_{i-1} + P_{i-1})}{x_{i+1} - x_{i-1}}\right] \qquad i = 2,\dots,n-2$$

$$Y_{n-1} = 2\lambda\left[\frac{0.25(x_{n-1} - x_{n-2}) - E_{n-1}(0.5 - x_{n-2}) + E_{n-2}(0.5 - x_{n-1})}{(0.5 - x_{n-1})(x_{n-1} - x_{n-2})(0.5 - x_{n-2})}\right]$$

$$+\frac{\frac{m_{n-2}}{\rho_{n-2}}(E_{n-2}+P_{n-2})}{(0.5-x_{n-2})}$$

3.9 MOVING MESH FORM OF MMPDE 6

$$\dot{\rho}_i = \left(\frac{\rho_{i+1}-\rho_{i-1}}{x_{i+1}-x_{i-1}}\right)\dot{x}_1 + V_i$$

$$\dot{\rho}_1 = \left(\frac{\rho_2 - 1}{x_2 + 0.5}\right)\dot{x}_1 + V_1$$

$$\dot{\rho}_i = \left(\frac{\rho_{i+1}-\rho_{i-1}}{x_{i+1}-x_{i-1}}\right)\dot{x}_i + V_i \qquad i = 2, \ldots, n-2$$

$$\dot{\rho}_{n-1} = \left(\frac{0.125-\rho_{n-2}}{0.5-x_{n-2}}\right)\dot{x}_{n-1} + V_{n-1}$$

$$\dot{m}_i = \left(\frac{m_{i+1}-m_{i-1}}{x_{i+1}-x_{i-1}}\right)\dot{x}_i + W_i$$

$$\dot{m}_1 = \left(\frac{m_2}{x_2 - 0.5}\right)\dot{x}_i + W_1$$

$$\dot{m}_i = \left(\frac{m_{i+1}-m_{i-1}}{x_{i+1}-x_{i-1}}\right)\dot{x}_i + W_i\,, \qquad i = 2, \ldots, n-2$$

$$\dot{m}_{n-1} = -\left(\frac{m_{n-2}}{0.5-x_{n-2}}\right)\dot{x}_{n-1} + W_{n-1}$$

$$\dot{E}_i = \left(\frac{E_{i+1}-E_{i-1}}{x_{i+1}-x_{i-1}}\right)\dot{x}_i + Y_i$$

$$\dot{E}_1 = \left(\frac{E_2 - 2.5}{x_2 + 0.5}\right)\dot{x}_1 + Y_i$$

$$\dot{E}_i = \left(\frac{E_{i+1} - E_{i-1}}{x_{i+1} - x_{i-1}} \right) \dot{x}_i + Y_i , \qquad i = 2, \ldots, n-2$$

$$\dot{E}_{n-1} = \left(\frac{0.25 - E_{n-2}}{0.5 - x_{n-2}} \right) \dot{x}_{n-1} + Y_{n-1}$$

3.10 RESULTS

We have derived in this chapter, the moving mesh partial differential equations are derived for one dimensional gas dynamics equations based on the Equidistribution Principle. The one dimensional gas dynamics equations are solved by using moving mesh method. We have considered different dynamic mesh formulations to solve the equations, but the best results were obtained by using equation (3.26).

The values of the density, velocity, pressure and internal energy of the gas obtained at t = 0.15 sec by using moving mesh method and uniform mesh method are plotted together in Figures 3.1 to 3.4. It is observed that the results obtained by using moving mesh method yields equally accurate results as results of uniform mesh method for significantly smaller number of points for the gas dynamics equations being considered in this chapter.

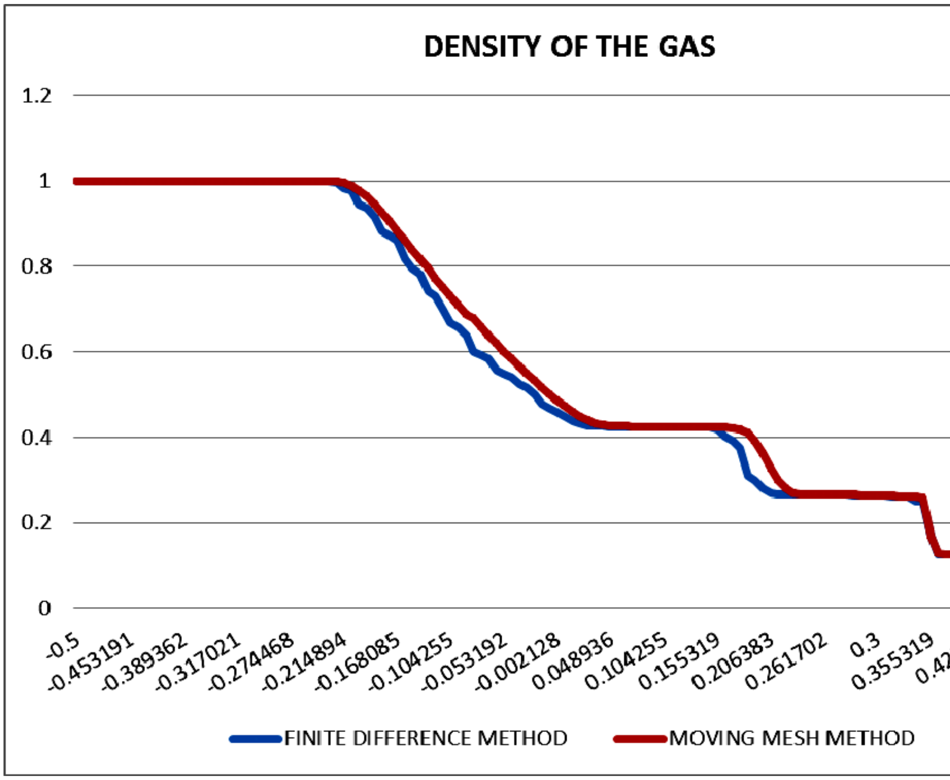

Figure 3.1

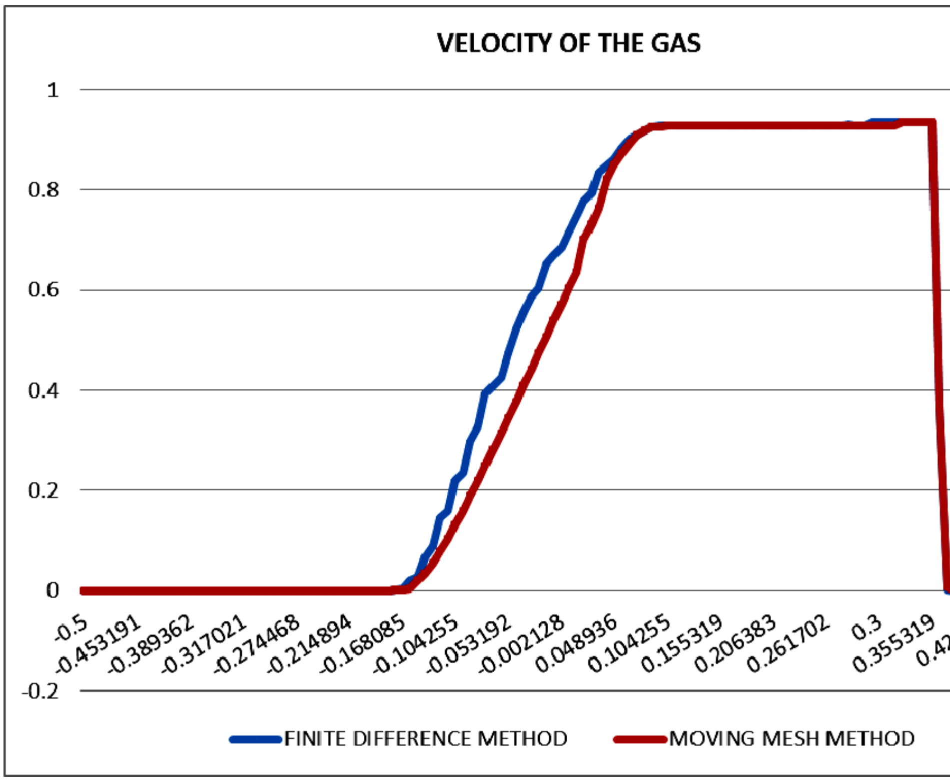

Figure 3.2

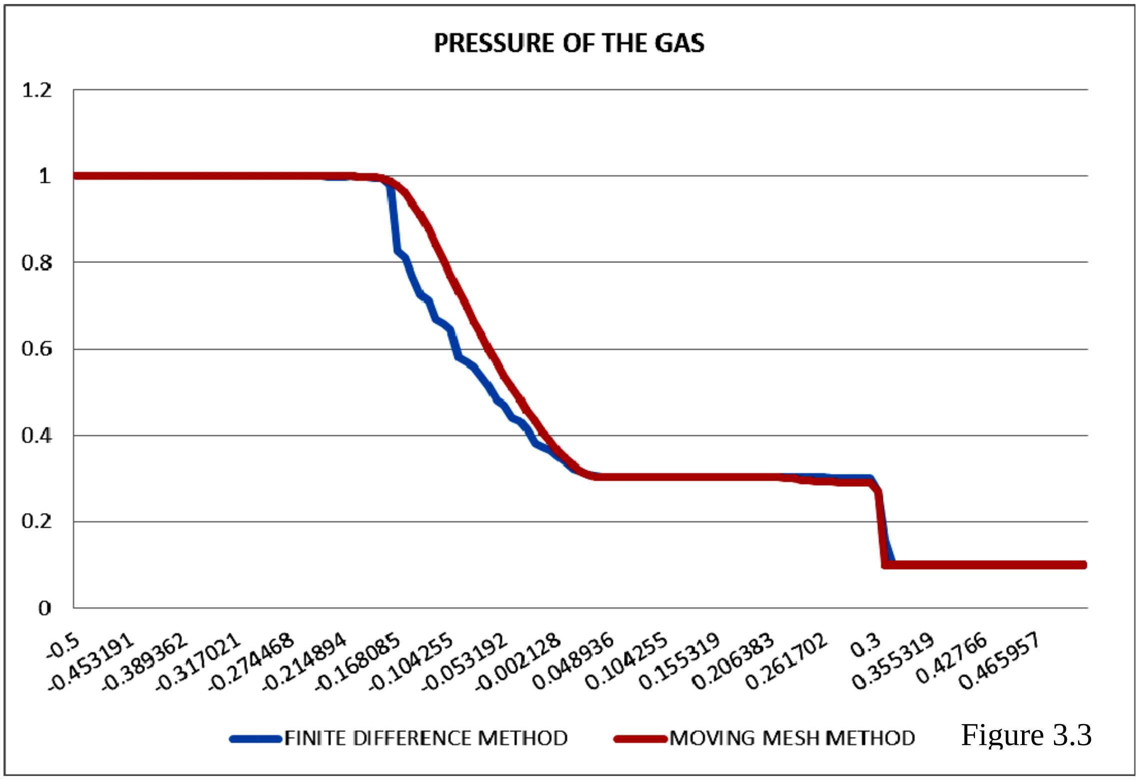

Figure 3.3

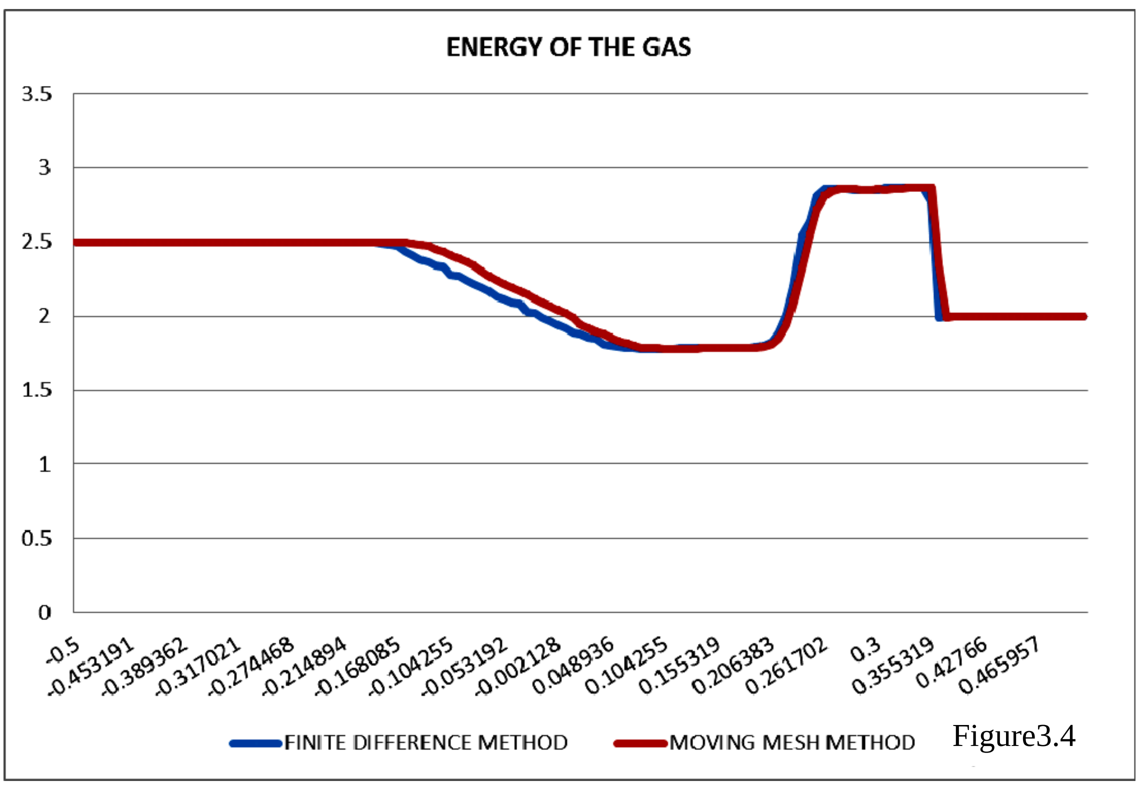

Figure3.4

TABLE 3.1- DENSITY OF THE GAS

X	Finite Difference	Moving Mesh	Absolute Error
-0.500000	1.00000000	1.00000000	0.00000000
-0.400000	1.00000000	1.00000000	0.00000000
-0.300000	1.00000000	1.00000000	0.00000000
-0.200000	0.90522825	0.91599216	0.01076391
-0.100000	0.59331406	0.60146611	0.00815205
0.000000	0.43400938	0.43642349	0.00241411
0.100000	0.42314568	0.42465446	0.00150878
0.200000	0.24555786	0.26980234	0.02424448
0.300000	0.24500000	0.26450688	0.01950688
0.400000	0.12500000	0.12500000	0.00000000
0.500000	0.12500000	0.12500000	0.00000000

TABLE 3.2- VELOCITY OF THE GAS

X	Finite Difference	Moving Mesh	Absolute Error
-0.500000	0.00000000	0.00000000	0.00000000
-0.400000	0.00000000	0.00000000	0.00000000
-0.300000	0.00000000	0.00000000	0.00000000
-0.200000	0.00565540	0.00000983	0.00564557
-0.100000	0.42905106	0.32891547	0.10013559
0.000000	0.90723119	0.74714813	0.16008306
0.100000	0.92746010	0.92765451	0.00019441
0.200000	0.92742284	0.92746329	0.00004045
0.300000	0.00000000	0.93505693	0.93505693
0.400000	0.00000000	0.00000003	0.00000003
0.500000	0.00000000	0.00000000	0.00000000

TABLE 3.3- PRESSURE OF THE GAS

X	Finite Difference	Moving Mesh	Absolute Error
-0.500000	1.00000000	1.00000000	0.00000000
-0.400000	1.00000000	1.00000000	0.00000000
-0.300000	1.00000000	1.00000000	0.00000000
-0.200000	0.99332728	0.99694026	0.00361298
-0.100000	0.52936241	0.55750428	0.02814187
0.000000	0.31182478	0.3220423	0.01021752
0.100000	0.30312357	0.30311585	0.00000772
0.200000	0.30311914	0.30312362	0.00000448
0.300000	0.10000000	0.10000000	0.00000000
0.400000	0.10000000	0.10000000	0.00000000
0.500000	0.10000000	0.10000000	0.00000000

TABLE 3.4- ENERGY OF THE GAS

X	Finite Difference	Moving Mesh	Absolute Error
-0.500000	2.50000000	2.50000000	0.00000000
-0.400000	2.50000000	2.50000000	0.00000000
-0.300000	2.50000000	2.50000000	0.00000000
-0.200000	2.49522477	2.49854773	0.00332296
-0.100000	2.16122261	2.21779172	0.05656911
0.000000	1.79618690	1.88824636	0.09205946
0.100000	1.79089372	1.78172459	0.00916913
0.200000	1.85360731	1.82745774	0.02614957
0.300000	2.00000000	2.72168210	0.72168210
0.400000	1.99999999	2.00000000	0.00000001
0.500000	2.00000000	2.00000000	0.00000000

Chapter - 4

FOURTH ORDER COMPACT MOVING MESH METHOD

4.1 INTRODUCTION

In this chapter fourth order compact moving mesh method is derived and is used to solve the one dimensional gas dynamics equations. This method is a hybrid of compact method and moving mesh method. It needs only three nodes like compact method and requires fewer grid points like moving mesh method to yield a fourth order accuracy in results.

4.2 DERIVATION

One dimensional gas dynamics equations in conservation form using artificial viscosity are given by equations (1.13), (1.14) and (1.15) as under

$$\rho_t + m_x = \lambda \rho_{xx}$$

$$m_t + \left[\frac{m^2}{\rho} + P\right]_x = \lambda m_{xx}$$

$$E_t + \left[\frac{m}{\rho}(E+P)\right]_x = \lambda E_{xx}$$

where $P = (\gamma - 1)\left[e - \frac{m^2}{2\rho}\right]$

γ = 1.4 for diatomic gas

$\therefore$ P = $0.4\left[e - \frac{m^2}{2\rho}\right]$

Let $\rho_i = y_i$ (4.1)

$m_i = y_{i+\ell}$ (4.2)

$E_i = y_{i+2\ell}$ (4.3)

$x_i = y_{i+3\ell}$, $i = 1, 2, 3 \ldots, \ell$ (4.4)

Let $\rho_x = F$

Integrating w.r.t., 'x' from $x = x_{i-1}$ *to* $x = x_{i+1}$, we get

$$\rho_{i+1} - \rho_{i-1} = \int_{x_{i-1}}^{x_{i+1}} F(x)\,dx$$

$$y_{i+1} - y_{i-1} = \int_{x_{i-1}}^{x_{i+1}} F(x)\,dx$$

Using Simpson's rule, to the fourth order, we have

$$y_{i+1} - y_{i-1} = \frac{h}{3}(F_{i-1} + 4F_i + F_{i+1}) \tag{4.5}$$

$$\therefore \quad F_{i-1} + 4F_i + F_{i+1} = \frac{3}{h}(y_{i+1} - y_{i-1}) \tag{4.6}$$

Let $y_{xx} = S$

Therefore using equation (2.13), to the fourth order we get

$$S_i = \frac{2}{h^2}(y_{i-1} - 2y_i + y_{i+1}) - \frac{1}{2h}(F_{i+1} - F_{i-1}) \tag{4.7}$$

$\therefore$ Moving mesh form of equation (4.6) is

$$y_{i+1} - y_{i-1} = (x_{i+1} - x_{i-1})F_i + \frac{1}{2}\left[(x_{i+1} - x_i)^2 - (x_i - x_{i-1})^2\right]\left[\frac{F_{i+1} - F_{i-1}}{x_{i+1} - x_{i-1}}\right]$$

$$+\frac{1}{3}\left[(x_{i+1} - x_i^3) + (x_i - x_{i-1})^3\right]\frac{F_{i+1}(x_i - x_{i-1}) - F_i(x_{i+1} - x_{i-1}) + F_{i-1}(x_{i+1} - x_i)}{(x_{i+1} - x_{i-1})(x_i - x_{i-1})(x_{i+1} - x_i)} \tag{4.8}$$

and moving mesh form of equation (4.7) is

$$S_i = \frac{4[y_{i+1}(x_i - x_{i-1}) - y_i(x_{i+1} - x_{i-1}) + y_{i-1}(x_{i+1} - x_i)}{(x_{i+1} - x_i - 1)(x_{i+1} - x_i)(x_i - x_{i-1})} - \frac{F_{i+1} - F_{i-1}}{x_{i+1} - x_{i-1}} \tag{4.9}$$

Let $Z_1(i) = x_{i+1} - x_{i-1}$ (4.10)

$$Z_2(i) = x_{i+1} - x_i \tag{4.11}$$

$$Z_3(i) = x_i - x_{i-1} \tag{4.12}$$

$$\therefore \quad Z_1(i) = x_{i+1} - x_{i-1} = y_{i+3\ell+1} - y_{i+3\ell-1}, \qquad i = 1, 2, 3. . . , \ell \tag{4.13}$$

$$Z_2(i) = x_{i+1} - x_i = y_{i+3\ell+1} - y_{i+3\ell}, \quad i = 1, 2, 3. . . , \ell \tag{4.14}$$

$$Z_3(i) = x_i - x_{i-1} = y_{i+3\ell} - y_{i+3\ell-1}, \quad i = 1, 2, 3. . . , \ell \tag{4.15}$$

Therefore equation (4.8) becomes

$$y_{i+1} - y_{i-1} = Z_1(i)F_i + \frac{1}{2}[Z_2^2(i) - Z_3^2(i)]\frac{F_{i+1} - F_{i-1}}{Z_i(i)}$$

$$+\frac{1}{3}\left[Z_2^3(i) + Z_3^3(i)\right]\left[\frac{F_{i+1}Z_3(i) - F_iZ_1(i) + F_{i-1}Z_2(i)}{Z_1(i)\ Z_2(i)\, Z_3(i)}\right]$$

$$= \left[-\frac{1}{2}\ \frac{Z_2^2(i) - Z_3^2(i)}{Z_1(i)} + \frac{1}{3}\ \frac{\{Z_2^3(i) + Z_3^2(i)}{Z_1(i)Z_3(i)}\right]F_{i-1} + \left[Z_1(i) - \frac{1}{3}\ \frac{Z_2^3(i) + Z_3^3(i)}{Z_2(i)Z_3(i)}\right]F_i$$

$$+\frac{1}{2}\left[\frac{Z_2^2(i) - Z_3^2(i)}{Z_1(i)} + \frac{1}{3}\ \frac{Z_2^3(i) + Z_3^3(i)}{Z_1(i)Z_2(i)}\right]F_{i+1}$$

$$y_{i+1} - y_{i-1} = a_i\, F_{i-1} + b_i\ F_i + c_i\ F_{i+1}, \qquad i = 2, 3, ..., \ell - 1 \tag{4.16}$$

where $a_i = \left[-\frac{1}{2}\ \frac{Z_2^2(i) - Z_3^2(i)}{Z_1(i)} + \frac{1}{3}\ \frac{\{Z_2^3(i) + Z_3^2(i)}{Z_1(i)Z_3(i)}\right], b_i = \left[Z_1(i) - \frac{1}{3}\ \frac{Z_2^3(i) + Z_3^3(i)}{Z_2(i)Z_3(i)}\right],$

$$c_i = \frac{1}{2}\left[\frac{Z_2^2(i) - Z_3^2(i)}{Z_1(i)} + \frac{1}{3}\ \frac{Z_2^3(i) + Z_3^3(i)}{Z_1(i)Z_2(i)}\right]$$

Therefore, Equation for ρ is

$$a_i F_{i-1} + b_i F_i + c_i F_{i+1} = y_{i+1} - y_{i-1} \quad , \quad i = 2,3,\ldots,\ell-1 \tag{4.17}$$

$$For\ i=2,\ \ b_2 F_2 + c_2 F_3 = y_3 - 1, \qquad (F_1 = 0, \because \rho_1 = 1)$$

$$a_i F_{i-1} + b_i F_i + c_i F_{i+1} = y_{i+1} - y_{i-1} \qquad i = 3,4,\ldots,\ell-2$$

For $\quad i = \ell - 1$

$$a_{\ell-1} F_{\ell-2} + b_{\ell-1} F_{\ell-1} = 0.125 - y_{\ell-2}, \qquad (F_\ell = 0, \because \rho_\ell = 0.125)$$

Let $\quad m_x = M$

$\therefore$ Equation (4.17) implies

$$a_i M_{i-1} + b_i M_i + c_i M_{i+1} = m_{i+1} - m_{i-1}, \qquad i = 2,3,\ldots,\ell-1$$

$$a_i M_{i-1} + b_i M_i + c_i M_{i+1} = y_{i+\ell+1} - y_{i+\ell-1} \tag{4.18}$$

For $\quad i = 2$

$$b_2 M_2 + c_2 M_3 = y_{\ell+3}, \qquad (M_1 = 0,\ y_{\ell+1} = m_1 = 0)$$

$$a_i M_{i-1} + b_i M_i + c_i M_{i+1} = y_{i+\ell+1} - y_{i+\ell+1} \qquad i = 3,4,\ldots,\ell-2$$

For $\quad i = \ell - 1$

$$a_{\ell-1} M_{\ell-2} + b_{\ell-1} M_{\ell-1} = -y_{2\ell-2}, \qquad (M_\ell = 0,\ y_{2\ell} = m_\ell = 0)$$

Let $\quad E_x = H$

$\therefore$ Equation (4.17) implies

$$a_i H_{i-1} + b_i H_i + c_i H_{i+1} = E_{i+1} - E_{i-1}, \qquad i = 2,3,\ldots,\ell-1$$

$$a_i H_{i-1} + b_i H_i + c_i H_{i+1} = y_{i+2\ell+1} - y_{i+2\ell-1} \tag{4.19}$$

For $\quad i = 2$

$$b_2 H_2 + c_2 H_3 = y_{2\ell+3} - 2.5, \quad (H_1 = 0, y_{2\ell+1} = E_1 = 2.5)$$

$$a_i H_{i-1} + b_i H_i + c_i H_{i+1} = y_{i+2\ell+1} - y_{i+2\ell+1}, \qquad i = 3, 4, \ldots, \ell - 2$$

For $i = \ell - 1$

$$a_{\ell-1} H_{\ell-2} + b_{\ell-1} H_{\ell-1} = 0.25 - y_{3\ell-2}, \qquad (H_\ell = 0, y_{3\ell} = E_\ell = 0.25)$$

Let $q_x = D$ where $q = \dfrac{m^2}{\rho} + P$

$\therefore$ Equation (4.17) implies

$$a_i D_{i-1} + b_i D_i + c_i D_{i+1} = q_{i+1} - q_{i-1}, \qquad i = 2, 3, \ldots, \ell - 1 \tag{4.20}$$

For $i = 2$

$$b_2 D_2 + c_2 D_3 = q_3 - 1.0,$$

$$a_i D_{i-1} + b_i D_i + c_i D_{i+1} = q_{i+1} - q_{i-1} \qquad i = 3, 4, \ldots, \ell - 2$$

For $i = \ell - 1$

$$a_{\ell-1} D_{\ell-2} + b_{\ell-1} D_{\ell-1} + c_{\ell-1} D_\ell = 0.1 - q_{\ell-2}, \qquad (q_\ell = 0.1)$$

Let $u_x = V$ where $u = \dfrac{m}{\rho}(E + p)$

$\therefore$ Equation (4.17) implies

$$a_i V_{i-1} + b_i V_i + c_i V_{i+1} = u_{i+1} - u_{i-1}, \qquad i = 2, 3, \ldots, \ell - 1 \tag{4.21}$$

For $i = 2$

$$b_2 V_2 + c_2 V_3 = u_3, \qquad (V_1 = 0,\ u_1 = 0)$$

$$a_i V_{i-1} + b_i V_i + c_i V_{i+1} = u_{i+1} - u_{i-1}, \qquad i = 3 4, \ldots, \ell - 2$$

For $i = \ell - 1$

$$a_\ell V_{\ell-2} + b_{\ell-i} V_{\ell-1} = -u_{\ell-2}, \qquad (V_\ell = 0,\ u_\ell = 0)$$

As $$S_i = \frac{4[y_{i+1} Z_3(i) - y_i Z_1(i) + y_{i-1} Z_2(i)}{Z_1(i)\ Z_2(i)\ Z_3(i)} - \frac{F_{i+1} - F_{i-1}}{Z_1(i)}$$

Therefore we have

$$\rho_t = -m_x + \lambda\rho_{xx} = -M_i + 4\lambda \frac{y_{i+1}Z_3(i) - y_i Z_1(i) + y_{i-1}Z_2(i)}{Z_1(i)Z_2(i)\ Z_3(i)}$$

$$-\frac{\lambda(F_{i+1} - F_{i-1})}{Z_1(i)} \qquad (4.22)$$

$$m_t = -\left[\frac{m^2}{\rho} + P\right]_x + \lambda m_{xx} = -D_i + 4\lambda \frac{[m_{i+1}Z_3(i) - m_i Z_1(i) + m_{i-1}Z_2(i)]}{Z_1(i)Z_2(i)\ Z_3(i)}$$

$$-\frac{\lambda(M_{i+1} - M_{i-1})}{Z_1(i)} \qquad (4.23)$$

$$E_t = -\left[\frac{m}{\rho}(E+p)\right]_x + \lambda E_{xx} = -V_i + \frac{4\lambda[E_{i+1}Z_3(i) - E_i Z_1(i) + E_{i-1}Z_2(i)]}{Z_1(i)Z_2(i)\ Z_3(i)}$$

$$-\frac{\lambda(H_{i+1} - H_{i-1})}{Z_1(i)} \qquad (4.24)$$

From equation (4.22) we get

$$\left.\frac{dy}{dt}\right|_1 = 0$$

$$\left.\frac{dy}{dt}\right|_2 = -M_2 + \frac{4\lambda[y_3 Z_3(2) - y_2 Z_1(2) + Z_2(2)]}{Z_1(2)Z_2(2)\ Z_3(2)} - \lambda\frac{F_2}{Z_1(2)}$$

$$\left.\frac{dy}{dt}\right|_i = -M_i + \frac{4\lambda[y_{i+1}Z_3(i) - y_i Z_1(i) + y_{i-1}Z_2(i)]}{Z_1(i)Z_2(i)\ Z_3(i)} - \lambda\frac{F_{i+1} - F_{i-1}}{Z_1(i)}$$

$$i = 3, 4, \ldots, \ell - 2$$

$$\left.\frac{dy}{dt}\right|_{\ell-1} = -M_{\ell-1} + \frac{4\lambda[0.125Z_3(i) - y_{\ell-1}Z_1(i) + Z_{\ell-2}Z_2(i)]}{Z_1(\ell-1)Z_2(\ell-1)\ Z_3(\ell-1)} + \lambda\frac{F_{\ell-3}}{Z_1(\ell-1)}$$

$$\left.\frac{dy}{dt}\right|_{\ell} = 0$$

From equation (4.23) we get

$$\left.\frac{dy}{dt}\right|_{\ell+1} = 0$$

$$\left.\frac{dy}{dt}\right|_{\ell+2} = -D_1 + \frac{4\lambda[y_{\ell+3}Z_3(2) - y_{\ell+2}Z_1(2)]}{Z_1(2)\,Z_2(2)\;Z_3(2)} - \lambda\frac{M_2}{Z_1(2)}$$

$$\left.\frac{dy}{dt}\right|_{i+\ell} = -D_i + \frac{4\lambda[y_{i+\ell+1}Z_3(i) - y_{i+\ell}Z_1(i) + y_{i+\ell-1}Z_2(i)]}{Z_1(i)\,Z_2(i)\;Z_3(i)} - \lambda\frac{(M_{i+1} - M_{i-1})}{Z_1(i)}$$

$$i = 3, 4, \ldots, \ell - 2$$

$$\left.\frac{dy}{dt}\right|_{2\ell-1} = -D_{\ell-1} + \frac{4\lambda[-y_{2\ell-1}Z_1(\ell-1) + y_{2\ell-2}\;Z_2(\ell-1)]}{Z_1(\ell-1)\,Z_2(\ell-1)\;Z_3(\ell-1)} + \lambda\frac{M_{\ell-3}}{Z_1(\ell-1)}$$

$$\left.\frac{dy}{dt}\right|_{2\ell} = 0$$

From equation (4.24) we get

$$\left.\frac{dy}{dt}\right|_{2\ell+1} = 0$$

$$\left.\frac{dy}{dt}\right|_{2\ell+2} = -V_2 + \frac{4\lambda[y_{2\ell+3}Z_3(2) - y_{2\ell+2}Z_1(2) + 2.5Z_2(2)]}{Z_1(2)Z_2(2)\;Z_3(2)} - \lambda\frac{H_2}{Z_1(2)}$$

$$\left.\frac{dy}{dt}\right|_{i+2\ell} = -V_i + \frac{4\lambda[y_{i+2\ell+1}Z_3(i) - y_{i+2\ell}Z_1(i) + y_{i+2\ell-1}Z_2(i)]}{Z_1(i)\,Z_2(i)\;Z_3(i)} - \lambda\frac{(H_{i+1} - H_{i-1})}{Z_1(i)}$$

$$i = 3, 4, \ldots, \ell - 2$$

$$\left.\frac{dy}{dt}\right|_{3\ell-1} = -V_{\ell-1} + \frac{4\lambda[0.25Z_3(\ell-1) - y_{3\ell-1}Z_1(\ell-1) + y_{3\ell-2}Z_2(\ell-1)]}{Z_1(\ell-1)\,Z_2(\ell-1)\;Z_3(\ell-1)} + \frac{\lambda H_{\ell-3}}{Z_1(\ell-1)}$$

$$\left.\frac{dy}{dt}\right|_{3\ell} = 0$$

For the monitor function $R = [1 + u_x^2]^{1/2}$

Let u = g

Therefore, for the left boundary we use forward difference formula

$$g_i{}' = \frac{-3g_i + 4g_{i+1} - g_{i+2}}{[4(x_{i+1} - x_i) - (x_{i+2} - x_i)]} \tag{4.25}$$

$$\therefore g_1{}' = \frac{-3g_1 + 4g_2 - g_3}{[4(x_2 - x_i) - (x_3 - x_i)]}$$

Therefore, moving mesh monitor function for the system of equations is

$$R = 1 + \left[\frac{(-3y_1 + 4y_2 - y_3)^2 + (-3y_{\ell+1} + 4y_{\ell+2} - y_{\ell+3})^2 + (-3y_{2l+1} + 4y_{2l+2} - y_{2l+3})}{4(y_{3l+2} - y_{3l+1}) - (y_{3l+3} - y_{3l+1})} \right]$$

For the right boundary we use the backward difference formula

$$g_i{}' = \frac{3g_i - 4g_{i-1} + g_{i-2}}{(x_{i-2} - x_i) - 4(x_{i-1} - x_i)}$$

$$g_l{}' = \frac{3y_\ell - 4y_{\ell-1} + y_{\ell-2}}{[(x_{\ell-2} - x_\ell) - 4(x_{\ell-1} - x_\ell)]} = \frac{3y_\ell - 4y_{\ell-1} + y_{\ell-2}}{[(y_{3l-2} - y_{3l}) - 4(y_{3l-1} - y_{3l})]}$$

$$R_{3l+1} = 1 + \frac{(3y_\ell - 4y_{\ell-1} + y_{\ell-2})^2 + (3y_{2l} - 4y_{2l-1} + y_{2l-2})^2 + (3y_{3l} - 4y_{3l-1} + y_{3l-2})^2}{[(y_{3l-2} - y_{3l}) - 4(y_{3l-1} - y_{3l})]^2}$$

$$R(i) = \left[1 + \frac{(y_{i+1} - y_{i-1})^2 + (y_{i+\ell+1} - y_{i+\ell-1})^2 + (y_{i+2l+1} - y_{i+2l-1})^2}{Z_1^2(i)} \right]^{1/2}$$

$$R(2) = \left[1 + \frac{(y_3 - y_1)^2 + (y_{\ell+3} - y_{\ell+1})^2 + (y_{2l+3} - y_{2l+1})^2}{Z_1^2(2)} \right]^{1/2}$$

$$R(i) = \left[1+\frac{(y_{i+1}-y_{i-1})^2+(y_{i+\ell+1}-y_{i+\ell-1})^2+(y_{i+\ell+1}-y_{i+\ell-1})^2}{y_{i+3l+1}-y_{i+3l-1}}\right]^{1/2}$$

$$i = 3, 4, \ldots, \ell-2$$

$$R_{\ell-1} = 1+\frac{(y_\ell-y_{\ell-2})^2+(y_{2l}-y_{2l-2})^2+(y_{3l}-y_{3l-2})^2}{(0.5-y_{3l-2})^2}$$

$$R_\ell = 1+\frac{(3y_\ell-4y_{\ell-1}+y_{\ell-2})^2+(3y_{2l}-4y_{2l-1}+y_{2l-2})^2+(3y_{3l}+4y_{3l-1}+y_{3l-2})^2}{[(y_{3l-2}-y_{3l})-4(y_{3l-1}-y_{3l})]^2}$$

Since $$\dot{g}_i - \left[\frac{g_{i+1}-g_{i-1}}{x_{i+1}-x_{i-1}}\right]\dot{x}_i = f_i \qquad (4.26)$$

Therefore, moving mesh equation for the equation (4.26)

$$\dot{\rho}_i = \left[\frac{\rho_{i+1}-\rho_{i-1}}{x_{i+1}-x_{i-1}}\right]\dot{x}_i + V_1(i)$$

$$\therefore \quad \dot{\rho}_1 = 0 \qquad \text{gives } \dot{y}(1)=0$$

$$\dot{y}(2) = \left[\frac{y_3-1.00}{y_{3\ell+3}+0.5}\right]\dot{y}_{3\ell+2} + V_1(2)$$

$$\dot{y}_i = \left[\frac{y_{i+1}-y_{i-1}}{y_{i+3\ell+1}+y_{i+3\ell-1}}\right]\dot{y}_{i+3\ell} + V_1(i-1), \qquad i = 3, 4, \ldots, \ell-2$$

$$\dot{y}_{\ell-1} = \left[\frac{0.125-y_{\ell-2}}{0.5-y_{4\ell-2}}\right]\dot{y}_{4\ell-1} + V_1(\ell-2)$$

$$\dot{y}_\ell = 0$$

Since $$\dot{m}_i = \left[\frac{m_{i+1}-m_{i-1}}{x_{i+1}-x_{i-1}}\right]\dot{x}_i + W_1(i) \qquad (4.27)$$

$$\therefore\ \dot{m}_1 = 0 \qquad \text{gives } \dot{y}_{\ell+1} = 0$$

$$\dot{y}_{\ell+2} = \left[\frac{y_{\ell+3}}{y_{\ell+3}+0.5}\right] \dot{y}_{3\ell+2} + W_1(1)$$

$$\dot{y}_{i+\ell} = \left[\frac{y_{i+\ell+1}-y_{i+\ell-1}}{y_{i+3\ell+1}-y_{i+3\ell-1}}\right] \dot{y}_{i+3\ell} + W_1(i-1) \qquad i = 3,4,\ldots,\ell-2$$

$$\dot{y}_{2\ell-1} = \left[\frac{-y_{2\ell-2}}{0.5-y_{4\ell-2}}\right] \dot{y}_{4\ell-1} + W_1(\ell-2)$$

$$\dot{y}_{2\ell} = 0$$

Since

$$\dot{E}_i = \left[\frac{E_{i+1}-E_{i-1}}{x_{i+1}-x_{i-1}}\right] \dot{x}_i + g_i \tag{4.28}$$

$$\therefore\ \dot{E}_1 = 0 \qquad \text{gives } \dot{y}_{2\ell+1} = 0$$

$$\dot{y}_{2\ell+2} = \left[\frac{y_{2\ell+3}-2.5}{y_{3\ell+3}+0.5}\right] \dot{y}_{3\ell+2} + g_1$$

$$\dot{y}_{2\ell+i} = \left[\frac{y_{i+2\ell+1}-y_{i+2\ell-1}}{y_{i+3\ell+1}-y_{i+3\ell-1}}\right] \dot{y}_{i+3\ell} + g_{i-1}, \qquad i = 3,4,\ldots,\ \ell-2$$

$$\dot{y}_{3\ell-1} = \left[\frac{0.25-y_{3\ell-2}}{0.5-y_{4\ell-2}}\right] \dot{y}_{4\ell-1} + g_{\ell-2}$$

$$\dot{y}_{3\ell} = 0$$

Since

$$\dot{x}_i = \frac{n^2}{2\tau}[(\overline{M}_i + \overline{M}_{i+1})x_{i+1} - (\overline{M}_{i-1} + 2\overline{M}_i + \overline{M}_{i+1})x_i + (\overline{M}_i + \overline{M}_{i-1})x_{i-1}] \qquad (4.29)$$

Let $b_1 = \dfrac{n^2}{2\tau}$

$\therefore \quad \dot{x}_i = 0 \qquad$ gives $\dot{y}_{3\ell+1} = 0$

$$\dot{y}_{3\ell+2} = b_1[(\overline{M}_2 + \overline{M}_3)y_{3\ell+3} - (\overline{M}_1 + 2\overline{M}_2 + \overline{M}_3)\, y_{3\ell+2} - 0.5(\overline{M}_2 + \overline{M}_1)]$$

$$\dot{y}_{i+3\ell} = b_1[(\overline{M}_1 + \overline{M}_{i+1})y_{i+3\ell+1} - (\overline{M}_{i-1} + 2\overline{M}_i + \overline{M}_{i+1})\, y_{i+3\ell} + (\overline{M}_i + \overline{M}_{i-1})\, y_{i+3\ell-1}]$$

$$i = 3, 4, \ldots, \ell - 2$$

$$\dot{y}_{4\ell-1} = b_1[0.5(\overline{M}_{\ell-1} + \overline{M}_{\ell}) - (\overline{M}_{\ell-2} + 2\overline{M}_{\ell-1} + \overline{M}_{\ell})\, y_{4\ell-1} + (\overline{M}_{\ell-1} + \overline{M}_{\ell-2})y_{4\ell-2}]$$

$$\dot{y}_{4\ell} = 0$$

We utilize the method of lines (MOL). The MOL is a classical technique for converting PDE into a system of differential equations. It can be done in two ways. The first one consists of discretizing in time t and then solving the boundary value ODE problem by BVP codes such as COLSYS, COLCON and AUTO. This approach is called the transverse method of lines. The second scheme involves discretizing in the space – like independent variables and leaving the time variable t alone, and then solving an initial value problem for t by IVP codes such as LSODE, LSODI and DASSL. This approach is called the longitudinal method of line. We used this approach and the resulting ODE systems are solved using the double precision version of ODE solver LSODE.

4.3 RESULTS

In this chapter we have derived compact moving mesh method for gas dynamics equations. The values of density, velocity, pressure and internal energy of the gas are obtained at t = 0.15 sec by using compact moving mesh method. The values of density, velocity, pressure and internal energy obtained by using this method and second order finite difference method are given in Tables 4.1 to 4.4 and plotted together in Figures 4.1 to 4.4.

4.4 CONCLUSIONS

In the Previous two chapters the compact method and moving mesh method were reviewed. Solution of gas dynamics equation was obtained by using compact and moving mesh methods. It is verified that both of these methods give better results than finite difference methods. In this chapter fourth order compact moving mesh method is derived. It is hybrid of compact method and moving mesh method. It is an ultimate computational technique and is used to find the solution of gas dynamics equation. The values of density, velocity, pressure and internal energy of the gas obtained at t = 0.15 sec using compact moving mesh method and finite difference method are plotted together for comparison in Figures 4.1 to 4.4. It is observed that this technique is quit efficient and results obtained by using this method are more accurate than the results of second order finite difference method.

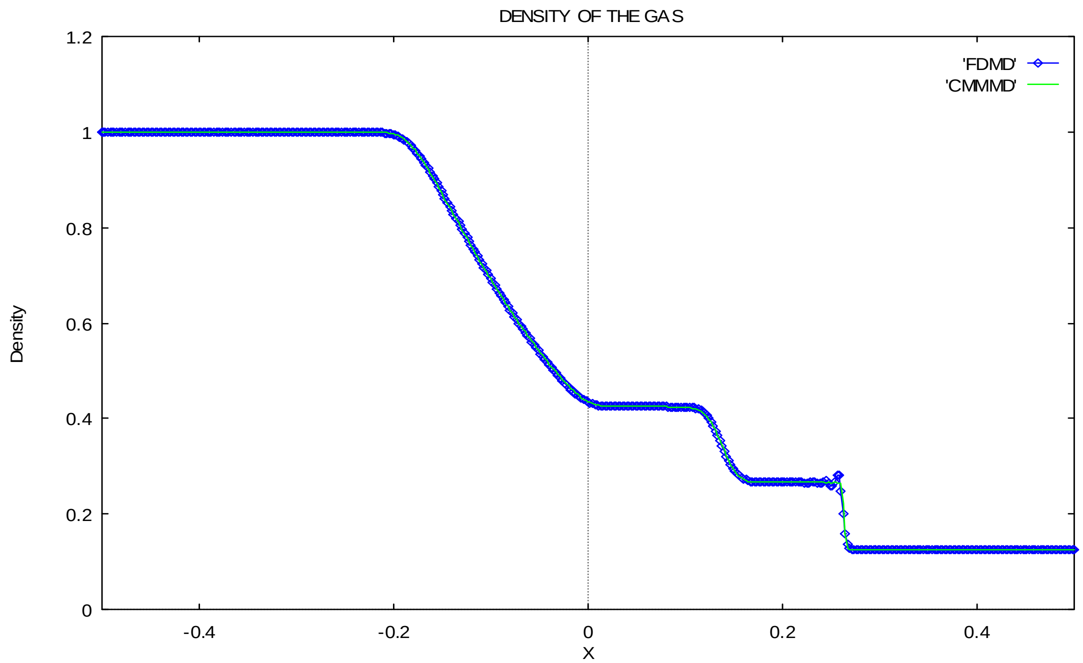

Figure 4.1

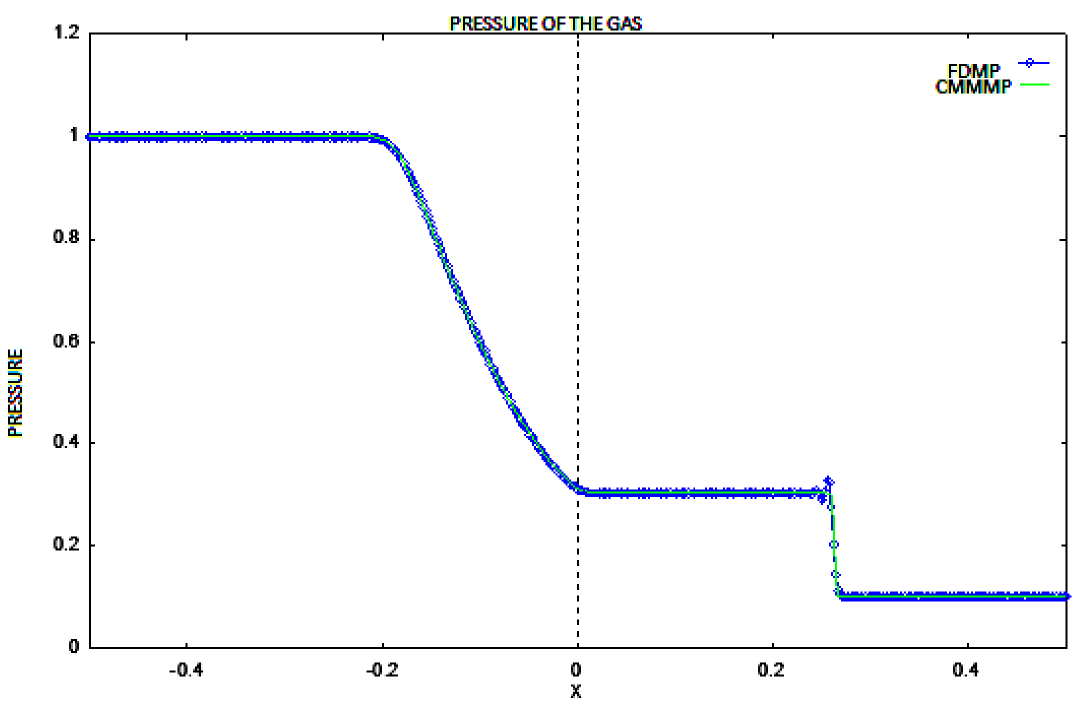

Figure 4.2

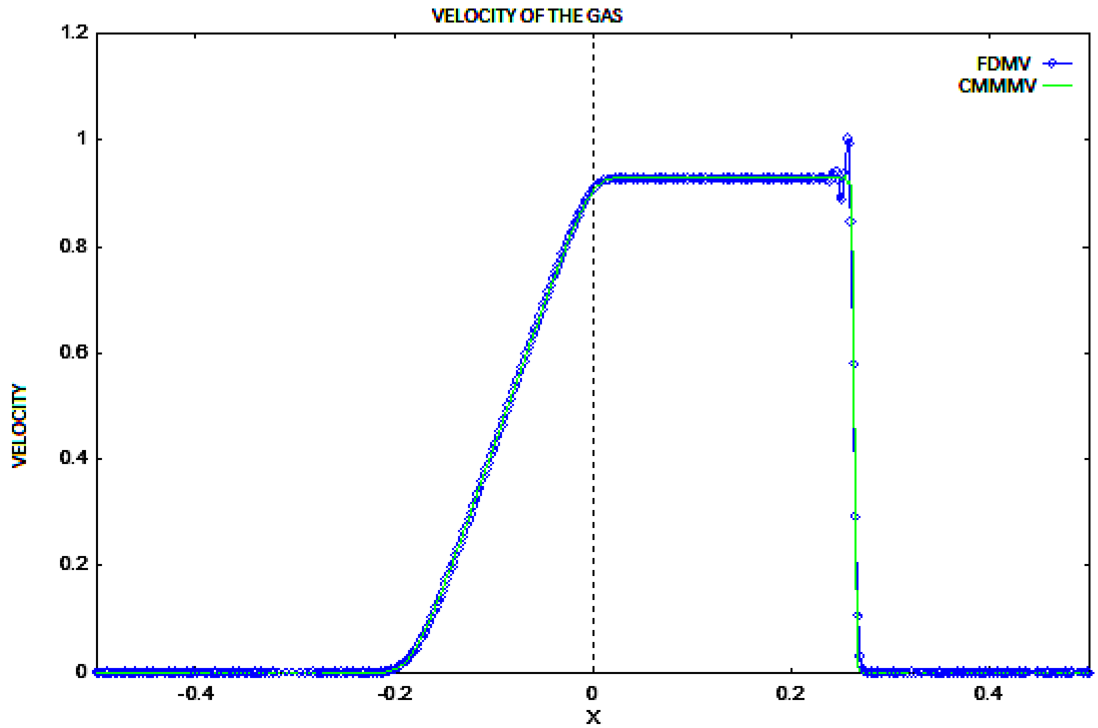

Figure 4.3

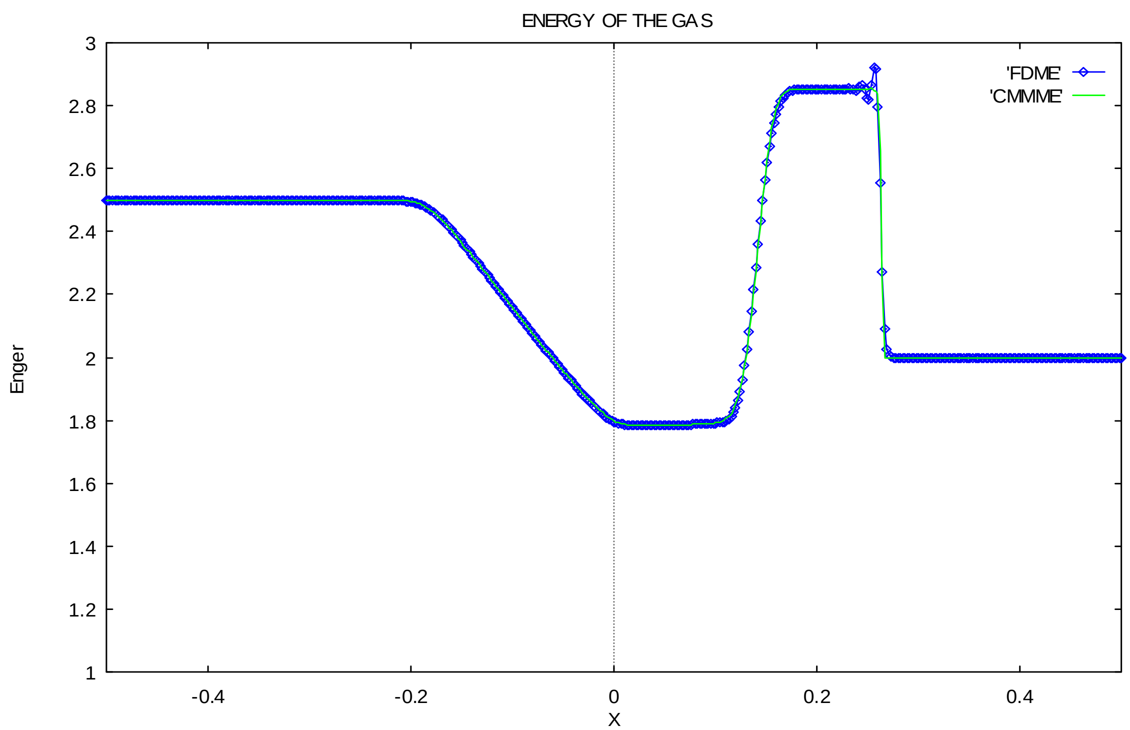

Figure 4.4

TABLE 4.1- DENSITY OF THE GAS

X	Finite Difference	Compact Moving Mesh	Absolute Error
-0.500000	1.00000000	1.00000000	0.00000000
-0.400000	1.00000000	1.00000000	0.00000000
-0.300000	1.00000000	1.00000000	0.00000000
-0.200000	0.90522825	0.99666877	0.09144052
-0.100000	0.59331406	0.62399951	0.03068545
0.000000	0.43400938	0.43562112	0.00161174
0.100000	0.42314568	0.42295793	0.00018775
0.200000	0.24555786	0.26555015	0.01999229
0.300000	0.24500000	0.25629110	0.01129110
0.400000	0.12500000	0.12500000	0.00000000
0.500000	0.12500000	0.12500000	0.00000000

TABLE 4.2- VELOCITY OF THE GAS

X	Finite Difference	Compact Moving Mesh	Absolute Error
-0.500000	0.00000000	0.00000000	0.00000000
-0.400000	0.00000000	0.00000000	0.00000000
-0.300000	0.00000000	0.00000000	0.00000000
-0.200000	0.00565540	0.00389752	0.00175788
-0.100000	0.42905106	0.41861584	0.01043522
0.000000	0.90723119	0.71330618	0.19392501
0.100000	0.92746010	0.90372680	0.02373330
0.200000	0.92742284	0.92745883	0.00003599
0.300000	0.00000000	0.00000000	0.00000000
0.400000	0.00000000	0.00000000	0.00000000
0.500000	0.00000000	0.00000000	0.00000000

TABLE 4.3- PRESSURE OF THE GAS

X	Finite Difference	Compact Moving Mesh	Absolute Error
-0.500000	1.00000000	1.00000000	0.00000000
-0.400000	1.00000000	1.00000000	0.00000000
-0.300000	1.00000000	1.00000000	0.00000000
-0.200000	0.99332728	0.99539736	0.00207008
-0.100000	0.52936241	0.59819054	0.06882813
0.000000	0.31182478	0.31335386	0.00152908
0.100000	0.30312357	0.30312390	0.00000033
0.200000	0.30311914	0.30312646	0.00000732
0.300000	0.10000000	0.10000000	0.00000000
0.400000	0.10000000	0.10000000	0.00000000
0.500000	0.10000000	0.10000000	0.00000000

TABLE 4.4- ENERGY OF THE GAS

X	Finite Difference	Compact Moving Mesh	Absolute Error
-0.500000	2.50000000	2.50000000	0.00000000
-0.400000	2.50000000	2.50000000	0.00000000
-0.300000	2.50000000	2.50000000	0.00000000
-0.200000	2.49522477	2.50000000	0.00477523
-0.100000	2.16122261	2.16279069	0.00156808
0.000000	1.79618690	1.86322400	0.06703710
0.100000	1.79089372	1.86632000	0.07542628
0.200000	1.85360731	2.70112430	0.84751699
0.300000	2.00000000	2.00000000	0.00000000
0.400000	1.99999999	2.00000000	0.00000001
0.500000	2.00000000	2.00000000	0.00000000

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