Asymptotic Behavior of Solutions for a Class of Semi-linear Differential Systems in Finite Dimensional Spaces

Since 1864

Name : Akbar Zada

Year of Admission : 2005

Registration No. : 15-GCU-Ph.D-SMS-2005

Abdus Salam School of Mathematical Sciences

GC University Lahore, Pakistan

Asymptotic Behavior of Solutions for a Class of Semi-linear Differential Systems in Finite Dimensional Spaces

Submitted to

Abdus Salam School of Mathematical Sciences

GC University Lahore, Pakistan

In the partial fulfillment of the requirements for the award of degree of

Doctor of Philosophy

in

Mathematics

By

Name : Akbar Zada

Year of Admission : 2005

Registration No. : 15-GCU-Ph.D-SMS-2005

Abdus Salam School of Mathematical Sciences

GC University Lahore, Pakistan

DECLARATION

I, Mr. **Akbar Zada** Registration No. **15-GCU-Ph.D-SMS-2005** student at **Abdus Salam School of Mathematical Sciences GC University** in the subject of **Mathematics, Year of Admission (2005)**, hereby declare that the matter printed in thesis titled

"Asymptotic Behavior of Solutions for a Class of Semi-linear Differential Systems in Finite Dimensional Spaces "

is my own work and that

(i) I am not registered for similar degree elsewhere contemporaneously.

(ii) No major work had already been done by me or anybody else on the topic; I worked on for the Ph.D degree.

(iii) The work I am submitting for the Ph.D degree has not already been submitted elsewhere and shall not in future be submitted by me for obtaining similar degree from any other institution.

Dated: ------------------------ --------------------------

Signature of Deponent

<u>RESEARCH COMPLETION CERTIFICATE</u>

Certified that the research work contained in this thesis titled

"Asymptotic Behavior of Solutions for a Class of Semi-linear Differential Systems in Finite Dimensional Spaces"

has been carried out and completed by **Mr. Akbar Zada** Registration No. **15-GCU-Ph.D-SMS-2005** under my supervision.

-----------------------------	-------------------------------
Date	Supervisor

Submitted Through

Prof. Dr. A. D. Raza Choudary	-------------------------------
Director General	Controller of Examination
Abdus Salam School of Mathematical Sciences	GC University
GC University Lahore, Pakistan.	Lahore, Pakistan.

To my parents

Table of Contents

Acknowledgements

At the out set, I would like to thank Almighty Allah who is the most Gracious, Merciful, Omniscient, Omnipotent and without the help of whom nothing could be completed.

I would like to express my sincere and deep gratitude to Prof. **Constantin Buşe**, my thesis supervisor, for his motivation, encouragement, expert guidance and continued assistance throughout this thesis . He has been a great role model as a teacher and as a human being for me.

I would also like to thank Prof. **Alla Ditta Raza Choudary** for providing me an opportunity to work and do research under the supervision of world class faculty. I also thank him for his invaluable guidance, help and encouragement through early years of chaos and confusion, as well as for his kindness and support during the past more than four years.

I appreciate very much, indeed the help of Prof. **Rozikove Utkir** . He really inspired and encouraged me. I am very much thankful to him, for his creative guidance, valuable suggestions and our combined research work in Dynamical Systems in Biology.

Also i would like to thank Prof. **Vasile Lupulescu** for his guidance and help. He sheared with me his knowledge of Impulsive Differential Equations on Time Scales. I am also very much thankful to him for our combined paper on time scales Impulsive

Differential Equations.

I had the pleasure of enrolling in indigenous PhD 5000 Fellowship Program launched by Higher Education Commission (HEC) of Pakistan. This scholarship and financial support by the Govt. of Punjab were crucial to the successful completion of this research.

I am also very thankful to Prof. **Viorel Radu** for his valuable suggestions and help.

I am especially grateful to the Abdus Salam School of Mathematical Sciences for the strong academic atmosphere and friendly environment, from which i have greatly benefited.

Of course, at this stage I can not forget my whole family especially my parents, my brothers and my sisters for their prayers, continuing support and love. Their prayers, have lightened up my spirit to finish this thesis.

Finally, I wish to thank All the foreign professors at ASSMS (for they made ASSMS, a real place of learning) and all the staff members especially; Ejaz Husain Malik, Shokat Ali, Numan Jaffery and Awais Naeem (for so many things). Lahore,

Pakistan Akbar Zada

List of publications arising from this thesis

- **Akbar Zada,** A characterization of dichotomy in terms of bounded-ness of solutions for some Cauchy problems, *Electronic Journal of Differential Equations*, Vol. **2008**(2008), No. 94, 1-5.
- [**ISI**-Journal] C. Buşe, **A. Zada**, Bounded-ness and exponential stability for periodic time dependent systems, *Electronic Journal of Qualitative Theory of Differential Equations*, **37**, pp. 1-9 (2009).
- [**ISI**-Proceedings] C. Buşe, **A. Zada**, Dichotomy and bounded-ness of solutions for some discrete Cauchy problems, *Proceedings of IWOTA- 2008, Operator Theory, Advances and Applications*, Vol. **203**, 165-174, (OT) Series, Birkhäuser Verlag, Eds: J.A. Ball, V. Bolotnikov, W. Helton, L. Rodman, T. Spitkovsky (2010).

Further publications produced during my PhD candidature

- [**ISI**-Journal] Rozikov U. A, **Zada A.** , On l-Volterra Quadratic Stochastic Operators, *Doklady Math* **79** no. 1, 32-34 (2009).
- [**ISI**-Journal] V. Lupulescu, **A. Zada** , On linear Impulsive dynamic systems on time scales. *Electronic Journal of Qualitative Theory of Differential Equations*, to appear.
- Rozikov U. A, **Zada A.**, On Dynamics of l-Volterra Quadratic Stochastic Operators, *International Journal of Biomathematics* , to appear.
- Rozikov U. A, **Zada A.**, l-Volterra Quadratic Stochastic Operators, Lyapunov Functions, Trajectories. *Applied mathematics and Information Sciences*, to appear.

Introduction

The natural tendency for abstraction and for generalization in the study of differential systems has led to the theory of linear operator groups and linear operator semi-groups. A first step was taken in 1888 by **Giuseppe Peano**, who had the idea of writing a system of scalar differential equations, briefly as one single matrix differential equation. Moreover, Peano wrote the formula of the variation of constants with the help of the exponential of a matrix, given, with respect to the operatorial norm, by:

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{t!} A^k.$$

The main goal of this dissertation is to study the asymptotic behavior of solutions of the semi linear differential systems, in the continuous case as well as in the discrete case. The concept of asymptotical stability is fundamental in the theory of ordinary and partial differential equations. In this way the stability theory leads to real world applications. The recent advances of the stability theory deeply interact with spectral theory, harmonic analysis, modern topics of complex functions theory and also with control theory.

The finding of necessary and (or) sufficient conditions for a system to be asymptotically stable, is justified by the existence of a vast field of applications, especially in the domain of equations of the mathematical-physics.

Let A be an $n \times n$- matrix having complex scalars as entries. It is well-known that the group of operators $\{e^{tA}\}_{t\in\mathbb{R}}$ is asymptotically stable (or equivalently exponentially stable), i.e. $\lim_{t\to\infty} ||e^{tA}|| = 0$, if and only if the spectrum of the matrix A is included in the open left half-plane of the complex plane. This classical result, obtained by **Lyapunov**, has already entered in the present mathematical folklore. In essence, the proof of this result is based on the *spectral mapping theorem* which states that

$$\sigma(e^{tA}) = e^{t\sigma(A)} \quad \text{for all } t \in \mathbb{R}$$

and operates under the assumptions exposed before. The concept of exponential dichotomy for linear differential systems was introduced by **Oscar Perron** in 1930. Perron also established a connection between the exponential dichotomy and the conditional stability of the system. Extensions of the Perron problem to the general framework of the infinite dimensional Banach spaces were obtained by **M. G. Krein, R. Bellman, J. L. Massera** and **J. J. Schäffer** in the period 1948-1966. This vast domain of research is far to exhausted, fact proved by the existence in the mathematical literature of the last four decades of an impressive number of papers and monographs dedicated to this interesting topics. We mention here only some of the names of prestigious authors who have signed such works: **W. Arendt, A.V. Balakrishnan, V. Barbu, B. Basit, A. G. Baskakov, C. J. K. Batty, C. Corduneanu, R. Datko, K. Engel, H.O. Fattorini, C. Foiaş, I. Gohberg, J. A. Goldstein, Aristide Halanay, E. Hille, F. L. Huang, A. Ichikawa, Yuri Latushkin, Yu. I. Lyubich, M. Megan, N. V. Minh, V. Müller, R. Nagel, F. Neubrander, Jan van Neerven, A. Pazy, A. Pogan, G. Da Prato, C. I. Preda, P. Preda, J. Prüss. M. Reghiş, A. L. Sasu, B. Sasu, R. Schnaubelt, Vu Quoc Phong, G. Weiss.** See [1], [2], [4], [5], [6], [7], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [30], [31], [32], [33], [35], [37], [38], [39], [40], [41], [42], [44],

[45], [46], [47], [48], [50], [51], [52], [43], [53].

In a particular case and using the most simple terms, Perron's result may be formulated as follows:

The system

$$\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{R} \tag{$A(t)$}$$

is exponentially stable if and only if it is admissible relative to the space of all continuous and bounded functions, that is that for each input f, *continuous and bounded function defined on the semi-axis of all nonnegative real numbers* ($\mathbb{R}_+$), *the output, i.e. the solution of the Cauchy Problem*

$$\dot{y}(t) = A(t)y(t) + f(t), \quad t \geq 0, \quad y(0) = 0, \tag{$A(t), f, 0$}$$

is bounded. A more general concept of admissibility can be defined as follows:

Let us denote the solution of $(A(t), f, 0)$ by $y_f(\cdot, 0)$ and let $\mathcal{X}_+$ and $\mathcal{Y}_+$ be two nonempty sets of functions defined on $\mathbb{R}_+$. The system $(A(t))$ is called $(\mathcal{X}_+, \mathcal{Y}_+)$-*admissible* if for all input $f \in \mathcal{X}_+$, the output $y_f(\cdot, 0)$ belongs to $\mathcal{Y}_+$.

The enunciation:

The system $(A(t))$ *is uniformly exponentially stable if and only if it is* $(\mathcal{X}_+, \mathcal{Y}_+)$-*admissible*, will be called (ad-hoc) *theorem of Perron's type.*

In particular, when $\mathcal{X}_+ = \mathcal{Y}_+$ is a certain normed space of functions, the corresponding theorem of Perron leads to a spectral mapping theorem for so called evolution semigroup associated to the system $(A(t))$.

For more details on this topic we refer to readers the monograph [19] by **Carmen Chicone** and **Yuri Latushkin**.

It is well-known that if a nonzero solution of the scalar differential equation $\dot{x}(t) = ax(t),\ t \in \mathbb{R}$ is asymptotically stable then each other solution has the same property and it happens if and only if for each real number μ and each complex number b the solution of the Cauchy Problem:

$$\dot{z}(t) = az(t) + e^{i\mu t}b, \quad t \geq 0, \quad z(0) = 0,$$

is bounded.

This result can be extended approximately with the same formulation to the case of bounded linear operators acting on a Banach space X, see [3]. The result can also be extended for strongly continuous bounded semigroups, see [10] , [15] or [16] , but cannot be extended to the case when a is an infinitesimal generator of a strongly continuous semigroup on an arbitrary complex Banach space X, see [16].

In the first chapter of this thesis we recall the Spectral Decomposition Theorem in the continuous case as well as in the discrete case, which will be very helpful in proving other results. In the first section of the second chapter we establish the connection between the stability and (or) dichotomy of the continuous system

$$\dot{x}(t) = Ax(t), \quad t \in \mathbb{R} \qquad (A)_c$$

with the bounded-ness of the solutions of the following Cauchy problems:

$$\dot{y}(t) = Ay(t) + e^{i\mu t}b, \quad t \geq 0, \quad \mu \in \mathbb{R}, \quad b \in X, \quad y(0) = 0. \qquad (A, \mu, b, 0)_c$$

In the second section of the same chapter we will discuss the analogous properties with respect to the following discrete Cauchy problems:

$$x_{n+1} = Ax_n, \quad n \in \mathbb{Z}_+ \qquad (A)_d$$

and

$$y_{n+1} = Ay_n + e^{i\mu n}b, \quad n \in \mathbb{Z}_+, \quad y_0 = 0. \qquad (A, \mu, b, 0)_d$$

Let X be a complex Banach space, A be a bounded linear operator acting on X and let $\mathcal{B}$ be a non-empty subset of X. Let us denote the set of all functions by $e^{i\mathbb{R}}\mathcal{B}$

$$t \mapsto e^{i\mu t}b : \mathbb{R}_+ \to X, \quad \mu \in \mathbb{R}, \quad b \in \mathcal{B}.$$

We are in the position to describe the main result of this thesis using the terms defined before. We proved that the continuous system $(A)_c$ is dichotomic, i.e. the spectrum of A does not intersect the imaginary axis $i\mathbb{R}$ if and only if there exists a projection P on X such that the system $(A)_c$ is $(e^{i\mathbb{R}}P(X), CB(\mathbb{R}_+, X))$-admissible and the opposite system $(-A)_c$ is $(e^{i\mathbb{R}}[(I-P)(X)], CB(\mathbb{R}_+, X))$-admissible. Here $CB(\mathbb{R}_+, X)$ denotes the Banach space of all continuous and bounded, X-valued functions defined on $\mathbb{R}_+$ with respect to the norm "sup". Analogous results in the discrete case are established as well. Of course these results are described using appropriate spaces of sequences. In the finite dimensional case, the proof of the above described results is given on a new and very elementary way, using only linear algebra settings.

The proof realize that if A is an invertible and non-dichotomic compact operator acting on a complex Banach space X, P is any projection that commutes with A, b is a certain nonzero vector in X and μ is a certain real number then the solution of at least one of the Cauchy problems

$$x(n+1) = Ax(n) + e^{i\mu n}Pb, \quad n \in \mathbb{Z}_+, \quad x(0) = 0$$

or

$$y(n+1) = A^{-1}y(n) + e^{i\mu n}(I-P)b, \quad n \in \mathbb{Z}_+, \quad y(0) = 0,$$

grows at the rate no slower than

$$\max\{||Pb||, ||(I-P)b||\} \times n.$$

Moreover, under such assumptions there exists a projection P commuting with A, such that for each nonzero vector b the solution of at least one of the above two discrete Cauchy problems grows at the rate exactly $(constant) \times n$. Analogous theorems in the continuous case are also pointed out.

All the results contained in the first and the second chapters of this thesis have been published see [17],[54] .

Let $t \to a(t) : \mathbb{R} \to \mathbb{C}$ be a continuous and 2-periodic function. Consider the non-autonomous scalar differential equations:

$$\dot{x}(t) = a(t)x(t), \quad t \in \mathbb{R} \qquad (a(t))$$

and the associated Cauchy problem

$$\dot{y}(t) = a(t)y(t) + e^{i\mu t}b, \quad t \in \mathbb{R}, \quad y(0) = 0. \qquad (a(t), \mu, b, 0)_c$$

Here μ is a given real number and b is a given complex scalar. Under the above assumptions, the solution of $(a(t), \mu, b, 0)_c$ is given by

$$y_f(t) = \int_0^t e^{i\mu u + \int_u^t a(s)ds} b du$$

and it is bounded for each $\mu \in \mathbb{R}$ and each $b \in \mathbb{C}$ if the real part of $a(t)$ is negative. The problem whether the converse of this statement is true or not, seems to be an open problem. In the last chapter of this thesis a related problem is risen and solved.

First section of the third chapter deals with some preliminary results. The most important of them is contained in Corollary 3.1.5 and says that the spectrum of a square matrix T of order n, lies in the interior of the circle of radius one if for each real number μ, the sequence of matrices $(I + e^{i\mu}T + \cdots + (e^{i\mu}T)^N)_{N \geq 1}$ is bounded. Based on this result, we will be able to prove in the second section of the same chapter, that the time dependent 2-periodic system:

$$\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{C}^n$$

is uniformly exponentially stable if and only if for each real number μ and each 2-periodic, $\mathbb{C}^n$-valued function f, the solution of the Cauchy Problem:

$$\dot{y}(t) = A(t)y(t) + e^{i\mu t} f(t), \quad t \in \mathbb{R}_+, \quad y(t) \in \mathbb{C}^n, \quad y(0) = 0,$$

is bounded. The result contained in the third chapter of this dissertation has been published, see [18].

Chapter 1

Preliminaries

In this chapter we give the Spectral Decomposition Theorem in the Continuous case as well as in the discrete case, which will be very helpful in proving our some important results.

We will use the following notation throughout this thesis.

$\mathbb{C}$:= The set of all complex numbers;

$\mathbb{R}$:= The set of all real numbers;

$\mathbb{R}_+$:= The set of all nonnegative real numbers;

$\mathbb{Z}$:= The set of all integer numbers;

$\mathbb{Z}_+$:= The set of all nonnegative integer numbers;

$\sigma(A)$:= The spectrum of a matrix A;

$\mathbb{C}^m$:= The space on m-tuples of complex numbers;

X := A Complex Banach space;

$\mathcal{L}(X)$:= The set of all bounded linear operators acting on X;

$\mathcal{M}(m, \mathbb{C})$:= The space of all $m \times m$ matrices with complex entries;

$\|x\|$:= The norm of x.

1.1 Spectral Decomposition Theorem in The Continuous Case

In this section first we give some well-known Lemmas from $m \times m$ matrix theory and then we give the Spectral Decomposition Theorem in the continuous case.

Let $A \in \mathcal{M}(m, \mathbb{C})$ and x_0 be a fixed vector in $\mathbb{C}^m$. Let us consider the following differential linear Cauchy problem in $\mathcal{M}(m, \mathbb{C})$:

$$\begin{cases} \dot{x}(t) = Ax(t), & t \in \mathbb{R} \\ x(0) = x_0. \end{cases} \qquad (A, 0, x_0)_c$$

We know that the Cauchy Problem $(A, 0, x_0)_c$ has a unique solution given by

$$\phi(t) = e^{tA} x_0, \quad t \in \mathbb{R}.$$

Here e^{tA} is the sum of the series

$$I + \frac{(tA)}{1!} + \cdots + \frac{(tA)^N}{N!} + \ldots$$

with respect to the operatorial norm.

Let $p = a_0 + a_1\lambda + \cdots + a_k\lambda^k \in \mathbb{C}[\lambda]$. Then by $p(A)$ we mean the matrix

$$p(A) = a_0 I + a_1 A + \cdots + a_k A^k.$$

Clearly if $p = 1$, then $p(A) = I$, and if $p = \lambda$ then $p(A) = A$ also if $p, q \in \mathbb{C}[\lambda]$ then $(pq)(A) = p(A)q(A)$.

Lemma 1.1.1. *Let $A \in \mathcal{M}(m, \mathbb{C})$. The polynomial $p(A)$ and the exponential e^{tA} commutes i.e.*

$$e^{tA}p(A) = p(A)e^{tA}.$$

Proof. Let

$$s_N(t, A) = I + \frac{tA}{1!} + \frac{(tA)^2}{2!} + \cdots + \frac{(tA)^N}{N!}.$$

Clearly

$$s_N(t, A)p(A) = p(A)s_N(t, A).$$

Passing to the *limit* for $N \to \infty$ and having in mind that the multiplicative operation in $\mathcal{M}(m, \mathbb{C})$ is continuous, obtain the conclusion. □

Lemma 1.1.2. *If p and q are two relative prime complex valued polynomials, then*

$$\ker[pq(A)] = \ker[p(A)] \oplus \ker[q(A)].$$

Proof. First we prove that $ker[p(A)] \cap ker[q(A)] = 0$. Let $v \in ker[p(A)] \cap ker[q(A)]$, then

$$\begin{cases} p(A)v = 0 \\ q(A)v = 0. \end{cases} \tag{1.1.1}$$

Given that p and q are relatively prime there exist $\mu, \nu \in \mathbb{C}[\lambda]$ such that $p\mu + q\nu = 1$. Which implies that

$$p(A)\mu(A) + q(A)\nu(A) = I,$$

and then for all $v \in \mathbb{C}^m$ one has

$$p(A)\mu(A)v + q(A)\nu(A)v = v.$$

Using (1.1.1) we get $v = 0$. It remains to prove that any vector in $\ker[(pq)(A)]$ may be written as sum of the vectors in $\ker[p(A)]$ and $\ker[q(A)]$. Let $y \in \ker[(pq)(A)]$ i.e. $p(A)q(A)y = 0$. We know that $p\mu + q\nu = 1$, and then

$$p(A)\mu(A)y + q(A)\nu(A)y = y.$$

Put $y_1 = p(A)\mu(A)y$ and $y_2 = q(A)\nu(A)y$. Follows

$$\begin{aligned} q(A)y_1 &= q(A)p(A)\mu(A)y \\ &= \mu(A)q(A)p(A)y = 0. \end{aligned}$$

Thus $y_1 \in \ker[q(A)]$. Similarly $y_2 \in \ker[p(A)]$. This completes the proof. □

We recall that a complex scalar λ is called an eigenvalue of the $m \times m$ matrix A, if there exists a non zero $x \in \mathbb{C}^m$ such that $Ax = \lambda x$. The set of all eigenvalues of a matrix A is called spectrum of A and is denoted by $\sigma(A)$. The polynomial of degree m defined by

$$P_A(\lambda) = \det(\lambda I - A) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_{m-1}\lambda + a_m$$

is called the characteristic polynomial associated to A. It is clear that the spectrum of A is the set of all roots of the polynomial P_A. Let $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, $k \le m$ the spectrum of A. From a well known theorem in Algebra there exist the integer numbers $m_1, m_2, \ldots, m_k \ge 1$ such that

$$P_A(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \ldots (\lambda - \lambda_k)^{m_k},$$

$$m_1 + m_2 + \cdots + m_k = m.$$

Note that the polynomials $p = (\lambda - \lambda_i)^{m_i}$ and $q = (\lambda - \lambda_j)^{m_j}$ are relative prime because $\lambda_i \ne \lambda_j$ when $i \ne j$. From Hamilton-Cayley theorem it follows that

$$P_A(A) = 0 = (A - \lambda_1 I)^{m_1}(A - \lambda_2 I)^{m_2} \ldots (A - \lambda_k I)^{m_k}.$$

Taking kernel of both sides

$$\ker(0) = \ker[(A - \lambda_1 I)^{m_1}(A - \lambda_2 I)^{m_2} \ldots (A - \lambda_k I)^{m_k}],$$

as $\ker(0) = \mathbb{C}^m$ thus

$$\mathbb{C}^m = \ker[(A - \lambda_1 I)^{m_1}(A - \lambda_2 I)^{m_2} \ldots (A - \lambda_k I)^{m_k}].$$

Applying Lemma 1.1.2, we obtain

$$\mathbb{C}^m = \ker(A - \lambda_1 I)^{m_1} \oplus \ker(A - \lambda_2 I)^{m_2} \oplus \cdots \oplus \ker(A - \lambda_k I)^{m_k}.$$

For each $j \in \{1, 2, \ldots, k\}$ let denote $W_j := \ker(A - \lambda_j I)^{m_j}$. Then

$$\mathbb{C}^m = W_1 \oplus W_2 \oplus \cdots \oplus W_k. \tag{1.1.2}$$

Lemma 1.1.3. *The subspace* $\ker(A - \lambda_j I)^{m_j}$ *is* e^{tA}*-invariant.*

Proof. For $w \in W_j$ we prove that $e^{tA}w \in W_j$. From Lemma 1.1.1

$$(A - \lambda I)^{m_j} e^{tA} w_j = e^{tA}(A - \lambda I)^{m_j} w_j.$$

But $w \in W_j$ and therefore $(A - \lambda I)^{m_j} w_j = 0$. Thus

$$(A - \lambda I)^{m_j} e^{tA} w_j = 0.$$

□

The following result shows that any solution of the Cauchy Problem $(A, 0, x_0)_c$ may be split in a sum of k solutions of the system $\dot{x} = Ax$. Moreover, each of such summands has a relatively simple structure described as follows.

Theorem 1.1.4. *Let $A \in \mathcal{M}(m, \mathbb{C})$. For each $x \in \mathbb{C}^m$ there exist $w_j \in W_j$ ($j \in \{1, 2, \ldots, k\}$) such that*

$$e^{tA}x = e^{tA}w_1 + e^{tA}w_2 + \cdots + e^{tA}w_k, \quad t \in \mathbb{R}.$$

Moreover, if $w_j(t) := e^{tA}w_j$ then $w_j(t) \in W_j$ for all $t \in \mathbb{R}$ and there exist a $\mathbb{C}^m$-valued polynomials $p_j(t)$ with $\deg(p_j) \leq m_j - 1$ such that

$$w_j(t) = e^{\lambda_j t}p_j(t), \quad t \in \mathbb{R}, \ j \in \{1, 2, \ldots, k\}.$$

Proof. Let $x \in \mathbb{C}^m$. Using (1.1.2) follows that for each $j \in \{1, 2, \ldots, k\}$ there exists a unique $w_j \in W_j$ such that

$$x = w_1 + w_2 + \cdots + w_k$$

and then

$$e^{tA}x = e^{tA}w_1 + e^{tA}w_2 + \cdots + e^{tA}w_k, \quad t \in \mathbb{R}.$$

Let $z_j(t) = e^{-\lambda_j t}w_j(t)$. Simple calculation gives

$$\frac{d^{m_j} z_j(t)}{dt} = e^{-\lambda_j t}(A - \lambda_j I)^{m_j} w_j(t) = 0.$$

The last equality follows from Lemma 1.1.2 using the fact that $w_j(t) \in W_j$ for all $t \in \mathbb{R}$. Then $z_j(t)$ is a $\mathbb{C}^m$-valued polynomial having degree less than m_j.

□

1.2 Spectral Decomposition Theorem in The Discrete Case.

In this section first we give two basic lemmas and then we give Spectral Decomposition Theorem in the discrete case.

In the discrete case the Cauchy Problem associated with matrix A is

$$\begin{cases} z_{n+1} = Az_n, \quad z_n \in \mathbb{C}^m, \quad n = 0, 1, 2, \dots \\ z_n(0) = z_0. \end{cases} \tag{$(A, 0, z_0)_d$}$$

As $z_1 = Az_0$, $z_2 = Az_1 = A^2 z_0$, finally we get $z_n = A^n z_0$, thus the solution of $(A, 0, z_0)_d$ is $z_n = A^n z_0$. We try to write the solution of $(A, 0, z_0)_d$ in a more simple form i.e. in terms of the eigenvalues of the matrix A.

Here we give a well-known lemma without proof, which is helpful in proving the next lemma.

Lemma 1.2.1. *The expression $E_k(n) = 1^k + 2^k + \cdots + n^k$, with k given natural number is a polynomial in n of degree $(k+1)$.*

Let us denote $z(n+1) - z(n)$ by $\Delta z(n)$.

Lemma 1.2.2. *If $\Delta^N q(n) = 0$ for all $n = 0, 1, 2 \dots$ and $N \geq 1$ is a natural number then q is a $\mathbb{C}^m$-valued polynomial of degree less than or equal to $N - 1$.*

Proof. We argue by mathematical induction. For $N = 1$, $\Delta q(n) = 0$, implies that

$$q(n+1) - q(n) = 0, \text{ for all } \ n = 0, 1, 2 \dots$$

and then $q(n)$ is a constant polynomial.

For $N \geq 2$ let us suppose that if $\Delta^{N-1} q(n) = 0$ then q is a polynomial of degree less than or equal to $N - 2$. We prove that the same result is true for N. If $\Delta^N q(n) = 0$ then $\Delta^{N-1}(\Delta q(n)) = 0$. Using the hypothesis we get that $\Delta q(n)$ is a polynomial of degree less than or equal to $N - 2$, i.e.

$$\Delta q(n) = q(n) - q(n-1) = b_{N-2} n^{N-2} + b_{N-3} n^{N-3} + \cdots + b_1 n + b_0 = P_{N-2}(n).$$

Similarly

$$\begin{aligned} q(n-1) - q(n-2) &= P_{N-2}(n-1), \\ q(n-2) - q(n-3) &= P_{N-2}(n-2) \end{aligned}$$

and finally we get

$$q(2) - q(1) = P_{N-2}(2).$$

These equalities yield :

$$q(n) = q(1) + P_{N-2}(2) + P_{N-2}(3) + \cdots + P_{N-2}(n).$$

Now Lemma 1.2.1 implies that $q(n)$ is a polynomial of degree $N-1$ and ends the proof. □

Theorem 1.2.3. *Let $A \in \mathcal{M}(m, \mathbb{C})$ be an invertible matrix. For each $z \in \mathbb{C}^m$ there exist $w_j \in W_j$ where $W_j := \ker(A - \lambda_j I)^{m_j}$, $(j \in \{1, 2, \ldots, k\})$ such that*

$$A^n z = A^n w_1 + A^n w_2 + \cdots + A^n w_k.$$

Moreover, if $w_j(n) := A^n w_j$ then $w_j(n) \in W_j$ for all $n = 0, 1, 2, \ldots$ and also there exist $\mathbb{C}^m$-valued polynomials $q_j(n)$ with $\deg(q_j) \leq m_j - 1$ such that

$$w_j(n) = \lambda_j^n q_j(n), \quad n = 0, 1, 2, \ldots\ , \ j \in \{1, 2, \ldots, k\}.$$

Proof. Let $z \in \mathbb{C}^m$. Using (1.1.2) follows that, for each $j \in \{1, 2, \ldots, k\}$ there exists unique $w_j \in W_j$ such that

$$z = w_1 + w_2 + \cdots + w_k$$

and then

$$A^n z = A^n w_1 + A^n w_2 + \cdots + A^n w_k, \quad n = 0, 1, 2, \ldots.$$

Let $q_j(n) = \lambda_j^{-n} w_j(n)$. Successively one has :

$$\begin{aligned}
\Delta q_j(n) &= \Delta(\lambda_j^{-n} w_j(n)) \\
&= \Delta(\lambda_j^{-n} A^n w_j) \\
&= \lambda_j^{-(n+1)} A^{n+1} w_j - \lambda_j^{-n} A^n w_j \\
&= \lambda_j^{-(n+1)} (A - \lambda_j I) A^n w_j.
\end{aligned}$$

Again taking Δ

$$\begin{aligned}
\Delta^2 q_j(n) &= \Delta[\Delta q_j(n)] \\
&= \Delta[\lambda_j^{-(n+1)} (A - \lambda_j I) A^n w_j] \\
&= \lambda_j^{-(n+2)} (A - \lambda_j I) A^{(n+1)} w_j - \lambda_j^{-(n+1)} (A - \lambda_j I) A^n w_j \\
&= \lambda_j^{-(n+2)} (A - \lambda_j I)^2 A^n w_j.
\end{aligned}$$

Continuing up to m_j, we get

$$\Delta^{m_j} q_j(n) = \lambda_j^{-(n+m_j)} (A - \lambda_j I)^{m_j} A^n w_j.$$

But $w_j(n)$ belongs to W_j for each $n = 0, 1, 2, \ldots$. Thus $\Delta^{m_j} q_j(n) = 0$. Using Lemma 1.2.2 we can say that the degree of polynomial $q_j(n)$ is less than or equal to $m_j - 1$. The proof is complete. □

Chapter 2

Dichotomic System on $\mathbb{C}^m$

In the first section of this chapter we study the stability and dichotomy of an $m \times m$ matrix A having complex entries in connection with bounded-ness of the solutions of the following semi-linear Cauchy problems:

$$\begin{cases} \dot{y}(t) = Ay(t) + e^{i\mu t}b, \quad t \geq 0, \\ y(0) = 0. \end{cases} \qquad (A, \mu, b, 0)_c$$

In the second section, we shall analyze the same problems in the framework of discrete systems:

$$\begin{cases} y_{n+1} = Ay_n + e^{i\mu n}b, \quad n \in \mathbb{Z}_+ \\ y_0 = 0. \end{cases} \qquad (A, \mu, b, 0)_d$$

The last section of this chapter deals with the study of the same problems but instead of the matrix A we take a bounded linear operator acting on an infinite dimensional Banach space.

2.1 Dichotomy and Bounded-ness in The Continuous Case

This section deals with the study of stability and dichotomy of a matrix A and their connections with the bounded-ness of solutions of the following Cauchy problems:

$$\begin{cases} \dot{y}(t) = Ay(t) + e^{i\mu t}b, \quad t \geq 0, \\ y(0) = 0. \end{cases} \qquad (A, \mu, b, 0)_c$$

It is known that if a nonzero solution of the scalar differential equation $\dot{x}(t) = ax(t)$, $t \geq 0$ is asymptotically stable, i.e. $\lim_{t\to\infty} |e^{(t-t_0)a}| = 0$ for all $t_0 \in \mathbb{R}$, then each other solution has the same property and it happens if and only if the real part of the complex number a is negative or if and only if for each real number μ and each complex number b the solution of the Cauchy problem:

$$\begin{cases} \dot{z}(t) = az(t) + e^{i\mu t}b, \quad t \geq 0, \\ z(0) = 0 \end{cases} \qquad (a, \mu, b, 0)_c$$

is bounded.

This result can be extended with approximatively the same formulation for the case of bounded linear operators acting on a Banach space X, see [3]. The result can also be extended for strongly continuous bounded semigroups, see [10, 16, 15, 40]. Under a slightly different assumption the result on stability is also preserved for any strongly continuous semigroups acting on complex Hilbert spaces, see for example [41, 43] and references therein.

Let us denote $\mathbb{C}_+ := \{z \in \mathbb{C} : Re(z) > 0\}$, $\mathbb{C}_- := \{z \in \mathbb{C} : Re(z) < 0\}$ and $i\mathbb{R} = \{i\eta : \eta \in \mathbb{R}\}$. Clearly $\mathbb{C} = \mathbb{C}_+ \cup \mathbb{C}_- \cup i\mathbb{R}$.

A matrix A is called:

(i) *Stable* if $\sigma(A)$ belongs to $\mathbb{C}_-$ or, equivalently, if there exist two positive constants N and ν such that $\|e^{tA}\| \leq Ne^{-\nu t}$ for all $t \geq 0$,

(ii) *Expansive* if $\sigma(A)$ belongs to $\mathbb{C}_+$ and

(iii) *Dichotomic* if $\sigma(A)$ does not intersect the set $i\mathbb{R}$.

Next theorem establishes a connection between the stability of the matrix A and bounded-ness of the solutions of the Cauchy problems $(A, \mu, b, 0)_c$:

Theorem 2.1.1. *The matrix A is stable if and only if for each $\mu \in \mathbb{R}$ and each $b \in \mathbb{C}^m$ the solutions of the Cauchy problems $(A, \mu, b, 0)_c$ is bounded.*

Proof. **Necessity:** Let $\mu \in \mathbb{R}$ and $b \in \mathbb{C}^m$ be given. The solution of $(A, \mu, b, 0)_c$ is

$$\psi_{\mu,b}(t) = \int_0^t e^{(t-s)A} e^{i\mu s} b \, ds \quad , \quad t \geq 0.$$

Then

$$\begin{aligned}
\|\psi_{\mu,b}(t)\| &\leq \int_0^t \|e^{(t-s)A} e^{i\mu s} b\| \, ds \\
&= \int_0^t \|e^{(t-s)A}\| \|b\| \, ds \\
&\leq \int_0^t N e^{-\nu(t-s)} \|b\| \, ds \\
&\leq \frac{N}{\nu} \|b\|.
\end{aligned}$$

Thus $\psi_{\mu,b}$ is bounded.

Sufficiency: The solution of $(A, \mu, b, 0)_c$ can be written as

$$\psi_{\mu,b}(t) = e^{i\mu t} \int_0^t e^{(-i\mu I + A)r} b \, dr, \quad t \geq 0.$$

It is clear that

$$\sigma(-i\mu I + A) = \{-i\mu + \lambda_1, -i\mu + \lambda_2, \ldots, -i\mu + \lambda_k\}.$$

Suppose for the contrary that the matrix A is not stable i.e. there exists $\nu \in \{1, 2, \ldots, k\}$ such that $Re(\lambda_\nu) \geq 0$. Then $Re(-i\mu + \lambda_\nu) \geq 0$. Denoting $B_\mu := (-i\mu I + A)$ may write

$$e^{B_\mu s} b = e^{B_\mu s} z_1 + e^{B_\mu s} z_2 + \cdots + e^{B_\mu s} z_k.$$

Let choose $b = z_\nu$ be a nonzero vector. Then $e^{(-i\mu I + A)s} b = e^{B_\mu s} z_\nu$ and by Theorem 1.1.4 follows that $e^{B_\mu s} z_\nu = e^{\mu_\nu s} p(s)$, where $\mu_\nu := -i\mu + \lambda_\nu$ and p is a $\mathbb{C}^m$-valued polynomial with $\deg(p) \leq m_\nu - 1$.

We are going to consider two cases.

Case 1: When $Re(\lambda_\nu) > 0$. Applying again Theorem 1.1.4, obtain

$$\psi_{\mu,b}(t) = e^{i\mu t} \int_0^t e^{\mu_\nu s} p(s)\, ds = e^{i\mu t} e^{\mu_\nu t} q(t), \quad t \geq 0.$$

Here $q(t)$ is a $\mathbb{C}^m$-valued, nonzero polynomial. Thus $\psi_{\mu,b}$ is an unbounded function and we arrived at a contradiction.

Case 2: When $Re(\lambda_\nu) = 0$. Let $\mu = \frac{\lambda_\nu}{i}$. Then $\mu_\nu t = 0$ and

$$\psi_{\mu,b}(t) = e^{i\mu t} \int_0^t p(s)\, ds = \begin{cases} e^{i\mu t} q(t) & \text{if } \deg(p) \geq 1 \\ t e^{i\mu t} & \text{if } \deg(p) = 0. \end{cases}$$

Here q is a polynomial of degree greater than 1. In this case $\psi_{\mu,b}$ is also an unbounded, which is a contradiction. □

Having in mind that A is expansive if and only if $(-A)$ is stable we obtained the following result.

Corollary 2.1.2. *The matrix A is expansive if and only if for each $\mu \in \mathbb{R}$ and each $b \in \mathbb{C}^m$ the solution of $(A, \mu, b, 0)$, with $-A$ instead of A, is bounded.*

A linear map P acting on $\mathbb{C}^m$ is called *projection* if $P^2 = P$.

The following theorem gives a relation between the dichotomy and bounded-ness of the solutions of the Cauchy problem $(A, \mu, b, 0)_c$.

Theorem 2.1.3. *The matrix A is dichotomic if and only if there exists a projection P having the property $e^{tA} P = P e^{tA}$ for all $t \geq 0$ such that for each μ and each $b \in \mathbb{C}^m$*

the following two inequalities hold

$$\sup_{t\geq 0} \| \int_0^t e^{(-i\mu+A)s} Pb\, ds\| < \infty, \tag{2.1.1}$$

$$\sup_{t\geq 0} \| \int_0^t e^{(-i\mu-A)s} (I-P)b\, ds\| < \infty\,. \tag{2.1.2}$$

Proof. **Necessity:** Working under the assumption that A is a dichotomic matrix we may suppose that there exists $\nu \in \{1, 2, \ldots, k\}$ such that

$$Re(\lambda_1) \leq Re(\lambda_2) \leq \cdots \leq Re(\lambda_\nu) < 0 < Re(\lambda_{\nu+1}) \leq \cdots \leq Re(\lambda_k).$$

Having in mind the decomposition of $\mathbb{C}^m$ given by (1.1.2). Let us consider

$$X_0 = W_1 \oplus W_2 \oplus \cdots \oplus W_\nu, \quad X_1 = W_{\nu+1} \oplus W_{\nu+2} \oplus \cdots \oplus W_k.$$

Then $\mathbb{C}^m = X_0 \oplus X_1$. Let us define $P : \mathbb{C}^m \to \mathbb{C}^m$ by $Px = x_0$, where $x = x_0 + x_1$, $x_0 \in X_0$ and $x_1 \in X_1$. It is clear that P is a projection. Moreover for all $x \in \mathbb{C}^m$ and all $t \geq 0$, this yields

$$Pe^{tA}x = P(e^{tA}(x_0 + x_1)) = P(e^{tA}(x_0) + e^{tA}(x_1)) = e^{tA}(x_0) = e^{tA}Px,$$

where the fact that X_0 is an $e^{tA}-$ invariant subspace, was used. Then $Pe^{tA} = e^{tA}P$. Now, we have

$$\begin{aligned} e^{s(-i\mu I+A)}Pb &= e^{-i\mu s}Pe^{sA}b \\ &= e^{-i\mu s}P\big(e^{\lambda_1 s}p_1(s) + \cdots + e^{\lambda_\nu s}p_\nu(s) + \cdots + e^{\lambda_k s}p_k(s)\big) \\ &= e^{-i\mu s}\big(e^{\lambda_1 s}p_1(s) + e^{\lambda_2 s}p_2(s) + \cdots + e^{\lambda_\nu s}p_\nu(s)\big), \end{aligned}$$

where $p_1, p_2, \ldots, p_\nu$ are polynomials as in Theorem 1.1.4. Now it is clear that the map $t \mapsto \int_0^t e^{(-i\mu+A)s}Pbds$ is bounded. The condition (2.1.2) can be verified in a similar manner.

Sufficiency: Suppose for a contradiction that A is not dichotomic. Then there exists $j \in \{1, 2, \ldots, k\}$ such that $\lambda_j = i\eta$ with $\eta \in \mathbb{R}$. Let us take $b = x_j \in W_j$, $x_j \neq 0$ and consider $x_{j0} =: Px_j$ and $x_{j1} =: (I-P)x_j$. We have

$$\psi_{\mu,Px_j}(t) = \int_0^t e^{-i\mu s}e^{sA}Px_j\, ds = \int_0^t e^{i(-\mu+\eta)s}p_j(s)\, ds.$$

First consider the case when $\deg(p_j) \geq 1$. If $x_{j0} \neq 0$ the map $t \mapsto \psi_{\mu,Px_j}(t)$ is clearly unbounded and if $x_{j1} \neq 0$ we may repeat the above argument in order to arrive at a contradiction that the map $t \mapsto \int_0^t e^{-i\mu s}e^{-sA}(I-P)x_j ds$ is unbounded.

Next we consider the case when $\deg(p_j) = 0$. If $x_{j0} \neq 0$ choose $\mu = \eta$ and then there exists $p_j \in W_j$, $p_j \neq 0$ such that $\psi_{\mu,Px_j}(t) = tp_j$ which is unbounded. If $x_{j0} = 0$ choose $\mu = -\eta$ and then

$$\int_0^t e^{-i\mu s} e^{-sA}(I-P)x_j\, ds = \int_0^t e^{-i\mu s} e^{-sA} x_{j1}\, ds = \int_0^t q_j ds = q_j t, \quad q_j \in W_j.$$

Here $q_j \neq 0$ because $x_{j1} \neq 0$. Then $\psi_{\mu,(I-P)x_j}$ is an unbounded map. This is a contradiction and the proof is complete. □

It is clear that the Theorem 2.1.1 is a particular case $(P = 1)$ of Theorem 2.1.3. But we have presented its proof because it is different from that given in [3].

2.2 Dichotomy and Bounded-ness in The Discrete Case

This section deals with the study of stability and dichotomy of a matrix A and their connection with bounded-ness of solutions of the discrete Cauchy problems:

$$\begin{cases} y_{n+1} = Ay_n + e^{i\mu n} b, \quad n \in \mathbb{Z}_+ \\ y_0 = 0. \end{cases} \qquad (A, \mu, b, 0)_d$$

It is clear that if a nonzero solution of the scalar difference equation

$$x_{n+1} = ax_n, \; n \in \mathbb{Z}_+ \qquad (a)$$

is asymptotically stable then each other solution has the same property and it happens if and only if $|a| < 1$ or if and only if for each real number μ and each complex number b the solution of the discrete Cauchy Problem:

$$\begin{cases} z_{n+1} = az_n + e^{i\mu n} b, \; n \in \mathbb{Z}_+ \\ z_0 = 0 \end{cases} \qquad (a, \mu, b, 0)_d$$

is bounded. The discrete Cauchy problem, associated with a square matrix A of order m is

$$\begin{cases} z(n+1) = Az(n), & z(n) \in \mathbb{C}^m, \quad n \in \mathbb{Z}_+ \\ z(0) = z_0. \end{cases} \qquad (A, 0, z_0)_d$$

Obviously the solution of $(A, 0, z_0)_d$ is given by $z_n = A^n z_0$, where $z(n)$ is denoted by z_n. If the eigenvalues of the matrix A are known then we could know much more about this solution, see Theorem 1.2.3.

Let us denote $\Gamma_1 = \{z \in \mathbb{C} : |z| = 1\}$, $\Gamma_i := \{z \in \mathbb{C} : |z| < 1\}$ and $\Gamma_e := \{z \in \mathbb{C} : |z| > 1\}$. Clearly $\mathbb{C} = \Gamma_1 \cup \Gamma_i \cup \Gamma_e$. A square matrix A of order m is called:

(i) *Stable* if $\sigma(A)$ is the subset of Γ_i or, equivalently, if there exist two positive constants N and ν such that $\|A^n\| \le Ne^{-\nu n}$ for all $n = 0, 1, 2 \ldots$,

(ii) *Expansive* if $\sigma(A)$ is the subset of Γ_e and

(iii) *Dichotomic* if $\sigma(A)$ does not intersect the set Γ_1.

It is clear that any expansive matrix A whose spectrum consists of $\lambda_1, \lambda_2, \ldots, \lambda_k$ is an invertible one and its inverse is stable, because

$$\sigma(A^{-1}) = \{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_k}\} \subset \Gamma_i.$$

The connection between stability and bounded-ness of the solutions of $(A, \mu, b, 0)_d$ can be seen in the following theorem.

Theorem 2.2.1. *The matrix A is stable if and only if for each $\mu \in \mathbb{R}$ and each $b \in \mathbb{C}^m$ the solution of the discrete Cauchy problem $(A, \mu, b, 0)_d$ is bounded.*

Proof. **Necessity:** Let $\mu \in \mathbb{R}$ and $b \in \mathbb{C}^m$. The solution of $(A, \mu, b, 0)_d$ is given by

$$y_n = [e^{i\mu(n-1)}I + e^{i\mu(n-2)}A + e^{i\mu(n-3)}A^2 + \cdots + e^{i\mu}A^{n-2} + A^{n-1}]b. \qquad (2.2.1)$$

But $e^{i\mu} \notin \sigma(A)$ and thus $(e^{i\mu}I - A)$ is an invertible matrix. So the equation (2.2.1) may be shortened to

$$y_n = (e^{i\mu}I - A)^{-1}[(e^{i\mu n}b - A^n b)]. \tag{2.2.2}$$

Passing to the norm in (2.2.2), we get

$$\|y_n\| \leq \|(e^{i\mu}I - A)^{-1}\|\|b\| + \|(e^{i\mu}I - A)^{-1}\|\|A^n b\|$$

and by applying Theorem 1.2.3, we obtain

$$A^n b = \lambda_1^n q_1(n) + \lambda_2^n q_2(n) + \cdots + \lambda_\nu^n q_\nu(n), \tag{2.2.3}$$

where $q_1, q_2, \ldots, q_\nu$ are some $\mathbb{C}^m$-valued polynomials. Thus (y_n) is bounded.

Remark. The decomposition (2.2.3) *holds for all* $n \geq m$ *even the matrix* A *is not invertible. Indeed, if* $\lambda_j = 0$ *then* $A^{m_j} w_j = 0$*, as* $m \geq m_j$ *thus* $A^m w_j = 0$*. Also as* $n \geq m$ *so* $A^n w_j = 0$*.*

Sufficiency: Suppose on contrary that the matrix A is not stable, i.e. there exists $\nu \in \{1, 2, \ldots, k\}$ such that $|\lambda_\nu| \geq 1$. We are going to consider two cases.

Case 1: When $\sigma(A) \cap \Gamma_1 \neq \emptyset$. Let $\lambda_j \in \sigma(A) \cap \Gamma_1$ and let choose $\mu \in \mathbb{R}$ such that $\lambda_j = e^{i\mu}$. For each eigenvector b associated to λ_j have that $A^n b = e^{i\mu n} b$. Thus the equation (2.2.1) yields :

$$y_n = [e^{i\mu(n-1)} + e^{i\mu(n-1)} + \cdots + e^{i\mu(n-1)}]b = n e^{i\mu(n-1)} b.$$

Therefore, (y_n) is an unbounded sequence and we arrive at a contradiction.

Case 2: When $\sigma(A)$ does not intersect Γ_1 but it intersects Γ_e. Let $\lambda_j \in \sigma(A) \cap \Gamma_e$. Having in mind that $\dim(W_j) \geq 1$, may choose $b = w_j \in W_j \setminus \{0\}$. By applying again Theorem 1.2.3, we obtain

$$A^n b = \lambda_j^n p_j(n), \quad n \in \mathbb{Z}_+,$$

p_j being a nonzero $\mathbb{C}^m$-valued polynomial of degree less than or equal to $m_j - 1$. The formula (2.2.2) still can be applied because $e^{i\mu} \notin \sigma(A)$, and thus, the solution can be written as

$$y_n = (e^{i\mu}I - A)^{-1} e^{i\mu} b - (e^{i\mu}I - A)^{-1} \lambda_j^n p_j(n), \quad n \in \mathbb{Z}_+.$$

This representation indicates that (y_n) is an unbounded sequence, it being a sum of the bounded sequence given by $z_n = (e^{i\mu}I - A)^{-1} e^{i\mu n} b$ and an unbounded one. Indeed,

$$\|(e^{i\mu}I - A)^{-1} \lambda_j^n p_j(n)\| = |\lambda_j^n| \|(e^{i\mu}I - A)^{-1} p_j(n)\| \to \infty \quad \text{when} \quad n \to \infty.$$

□

Corollary 2.2.2. *A square matrix A of order m is expansive if and only if it is invertible and for each $\mu \in \mathbb{R}$ and each $b \in \mathbb{C}^m$ the solution of the discrete Cauchy problem:*

$$\begin{cases} y_{n+1} = A^{-1}y_n + e^{i\mu n}b, & n \in \mathbb{Z}_+ \\ y_0 = 0, \end{cases} \qquad (A^{-1}, \mu, b, 0)_d$$

is bounded.

Proof. Apply the above Theorem 2.2.1 to the inverse of A. □

We recall that a linear map P acting on $\mathbb{C}^m$ (or a square size matrix of order m) is called a *projection* if $P^2 = P$. The following theorem gives a relation between the dichotomy and bounded-ness of the solutions of the Cauchy problem $(A, \mu, b, 0)_d$.

Theorem 2.2.3. *The matrix A is dichotomic if and only if there exists a projection P having the property $AP = PA$ such that for each $\mu \in \mathbb{R}$ and each vector $b \in \mathbb{C}^m$ the solutions of the following two discrete Cauchy problems are bounded.*

$$\begin{cases} x_{n+1} = Ax_n + e^{i\mu n}Pb, & n \in \mathbb{Z}_+ \\ x_0 = 0 \end{cases} \qquad (A, \mu, Pb, 0)_d$$

and

$$\begin{cases} y_{n+1} = A^{-1}y_n + e^{i\mu n}(I-P)b, & n \in \mathbb{Z}_+ \\ y_0 = 0\,. \end{cases} \qquad (A^{-1}, \mu, (I-P)b, 0)_d$$

Proof. **Necessity:** Working under the assumption that A is a dichotomic matrix we may suppose that there exists $\nu \in \{1, 2, \ldots, k\}$ such that

$$|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_\nu| < 1 < |\lambda_{\nu+1}| \le \cdots \le |\lambda_k|.$$

Having in mind the decomposition of $\mathbb{C}^m$ given in (1.1.2), consider

$$X_1 = W_1 \oplus W_2 \oplus \cdots \oplus W_\nu, \quad X_2 = W_{\nu+1} \oplus W_{\nu+2} \oplus \cdots \oplus W_k.$$

Then $\mathbb{C}^m = X_1 \oplus X_2$. Define $P : \mathbb{C}^m \to \mathbb{C}^m$, by $Px = x_1$, where $x = x_1 + x_2$, $x_1 \in X_1$ and $x_2 \in X_2$. It is clear that P is a projection. Moreover for all $x \in \mathbb{C}^m$ and all $n \in \mathbb{Z}_+$, may write

$$PA^n x = P(A^n(x_1 + x_2)) = P(A^n(x_1) + A^n(x_2)) = A^n(x_1) = A^n Px,$$

where the fact that X_1 is an A^n-invariant subspace, was used. Then $PA^n = A^nP$ for all $n \in \mathbb{Z}_+$. Now, we have

$$x_n = (e^{i\mu}I - A)^{-1}[(e^{i\mu n}Pb - PA^nb)].$$

Passing to the norm in both the sides of the previous equality, we get

$$\|x_n\| \leq \|(e^{i\mu}I - A)^{-1}\|\|Pb\| + \|(e^{i\mu}I - A)^{-1}\|\|PA^nb\|.$$

Now from Theorem 1.2.3 follows

$$PA^nb = \lambda_1^n q_1(n) + \lambda_2^n q_2(n) + \cdots + \lambda_\nu^n q_\nu(n),$$

where $q_1, q_2, \cdots q_\nu$ are polynomials. Then the sequence (x_n) is bounded. Our next goal is to prove that the solution of the second Cauchy problem is bounded. We have again

$$y_n = (e^{i\mu}I - A^{-1})^{-1}[(e^{i\mu n}(I - P)b - A^{-n}(I - P)b)].$$

again passing to the norm in both the sides of the previous equality, we get

$$\|y_n\| \leq \|(e^{i\mu}I - A^{-1})^{-1}\|\|(I - P)b\| + \|(e^{i\mu}I - A^{-1})^{-1}\|\|A^{-n}(I - P)b\|.$$

First we prove that $A^{-n}v_2 \to 0$ as $n \to \infty$ for any $v_2 \in X_2$. Since $(I - P)b \in X_2$ the assertion would follows. On the other hand

$$X_2 = W_{\nu+1} \oplus W_{\nu+2} \oplus \cdots \oplus W_k,$$

so each vector from X_2 can be represented as a sum of $k-\nu$ vectors $w_{\nu+1}, w_{\nu+2}, \ldots, w_k$. It would be sufficient to prove that $A^{-n}w_j \to 0$, for any $j \in \{\nu + 1, \ldots, k\}$. Let $W \in \{W_{\nu+1}, W_{\nu+2}, \ldots, W_k\}$, say instantly that $W = \ker(A - \lambda I)^\rho$, where $\rho \geq 1$ is an integer number and $|\lambda| > 1$. Consider $w_1' \in W \setminus \{0\}$ such that $(A - \lambda I)w_1' = 0$ and let $w_2', w_3', \ldots, w_\rho'$ given by $(A - \lambda I)w_j' = w_{j-1}', \quad j = 2, 3, \ldots, \rho$. Then $B := \{w_1', w_2', \ldots, w_\rho'\}$ is a basis in W, see for instance [34]. It is then sufficient to prove that $A^{-n}w_j' \to 0$ for any $j = 1, 2, \ldots, \rho$. For $j = 1$ we have that $A^{-n}w_1' = \frac{1}{\lambda^n}w_1' \to 0$. For $j = 2, 3, \ldots, \rho$ let denote $X_n = A^{-n}w_j'$. Then $(A - \lambda I)^\rho X_n = 0$, i.e.

$$X_n - C_\rho^1 X_{n-1}\alpha + C_\rho^2 X_{n-2}\alpha^2 + \cdots + C_\rho^\rho X_{n-\rho}\alpha^\rho = 0 \tag{2.2.4}$$

for all $n \geq \rho$, where $\alpha = \frac{1}{\lambda}$. Passing for instance at the components in (2.2.4) it results that there exists a $\mathbb{C}^m$-valued polynomial P_ρ having degree at most $\rho-1$ and such that $X_n = \alpha^n P_\rho(n)$. Thus $X_n \to 0$, when $n \to \infty$, i.e. $A^{-n}w_j' \to 0$ for any $j \in \{1, 2, \ldots, \rho\}$.

Sufficiency: Suppose on a contrary that the matrix A is not dichotomic. Then there exists $j \in \{1, 2, \ldots, k\}$ such that $|\lambda_j| = 1$. Let $b \in \mathbb{C}^m$ be a fixed nonzero eigenvector associated to λ_j. We are going to analyze two cases.

Case 1: When $Pb \neq 0$. Choose $\mu \in \mathbb{R}$ such that $\lambda_j = e^{i\mu}$. Then $APb = e^{i\mu}Pb$ and $A^n Pb = e^{i\mu n}Pb$, which yield

$$x_n = [e^{i\mu(n-1)} + e^{i\mu(n-1)} + \cdots + e^{i\mu(n-1)}]Pb = ne^{i\mu(n-1)}Pb.$$

Thus (x_n) is an unbounded sequence, in contradiction with the hypothesis.

Case 2: When $Pb = 0$. In this case $(I-P)b \neq 0$. Let $\mu \in \mathbb{R}$ such that $\lambda_j = e^{-i\mu}$. Then $A^{-1}(I-P)b = e^{i\mu}(I-P)b$ and $A^{-n}(I-P)b = e^{i\mu n}(I-P)b$, which produce

$$y_n = [e^{i\mu(n-1)} + e^{i\mu(n-1)} + \cdots + e^{i\mu(n-1)}](I-P)b = ne^{i\mu(n-1)}(I-P)b.$$

Thus (y_n) is an unbounded sequence. These completes the proof.

□

We are remarking that in the enunciation of Theorem 2.2.3 we did not imposed the condition that the matrix A to be invertible. In fact, if viewing A as a map acting on $\mathbb{C}^m$, then $A_{|X_2}$ is an injective map and we may work with the inverse of this restriction instead of the global inverse of A.

2.3 Dichotomy and Bounded-ness of Operators Acting on Banach Spaces

In this section instead of a square size matrix A we take a bounded linear operator acting on a complex Banach space. The connection between dichotomy and boundedness of solutions of the Cauchy problems:

$$\begin{cases} x_{n+1} = Ax_n + e^{i\mu n}Pb, \quad n \in \mathbb{Z}_+ \\ x_0 = 0 \end{cases} \qquad (A, \mu, Pb, 0)_d$$

and

$$\begin{cases} y_{n+1} = A^{-1}y_n + e^{i\mu n}(I-P)b, \quad n \in \mathbb{Z}_+ \\ y_0 = 0\,. \end{cases} \qquad (A^{-1}, \mu, (I-P)b, 0)_d$$

has made in Theorem 2.3.2, where P is a projection on a complex Banach space X.

With our notations, the result contained in [[12], Theorem 1] may be reformulated as follows.

Proposition 2.3.1. *Let A in $\mathcal{L}(X)$. The following three statements concerning on the operator A are equivalent:*
(i) *$A^n \to 0$ in the norm of $\mathcal{L}(X)$.*
(ii) *The spectral radius of A, i.e.*

$$r(A) := \lim_{n\to\infty} ||A^n||^{\frac{1}{n}} = \inf_{n\geq 1} ||A^n||^{\frac{1}{n}} = \sup\{|z| : z \in \sigma(A)\},$$

is less than one.
(iii) *For each $\mu \in \mathbb{R}$ and each $b \in X$, the solution of $(A, \mu, b, 0)_d$, is bounded.*

The equivalence between **(i)** and **(ii)** is well-known. Clearly, the second condition implies the third one. Now suppose that the statement **(iii)** is fulfilled. From (2.2.1), we have:

$$y_n = e^{i\mu(n-1)} \sum_{k=0}^{n-1} (e^{-i\mu} A)^k b.$$

Now, the bounded-ness of the solution (y_n) and the Principle of Bounded-ness, yield:

$$\sup_{n\geq 1} \left\| \sum_{k=0}^{n-1} (e^{-i\mu} A)^k \right\| < \infty.$$

The assertion in **(ii)** follows now directly from [[12], Lemma 1]. We are in the position

to state the last result of this chapter. It reads as follows:

Theorem 2.3.2. *A bounded linear operator A acting on the complex Banach space X is dichotomic if and only if there exists a projection P on X that commutes with A and such that for each real number μ and each vector $b \in X$ the solutions of the Cauchy problems $(A, \mu, Pb, 0)_d$ and $(A^{-1}, \mu, (I-P)b, 0)_d$ are bounded.*

Proof. Assume that A is dichotomic. Let $K_1 := \{\lambda \in \sigma(A) : \ |\lambda| \leq 1\}$ and $K_2 := \{\lambda \in \sigma(A) : \ |\lambda| \geq 1\}$. Clearly K_1 and K_2 are compact and disjoint sets.

Moreover, $\sigma(A) = K_1 \cup K_2$. It is known that $P_1 = \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1} d\lambda$, where Γ is contour in the resolvent set of A with K_1 in interior and separating K_1 from K_2, is a projection (called the Riesz projection corresponding to K_1). Analogously we can define the Riesz projection P_2 corresponding to K_2 such that $P_1 + P_2 = I$. If $X_1 = ImP_1$ and $X_2 = ImP_2$, then $X = X_1 \oplus X_2,\ AX_1 \subset X_1,\ AX_2 \subset X_2$ and $\sigma(A_1) = K_1,\ \sigma(A_2) = K_2$, where $A_1 = A|_{X_1},\ A_2 = A|_{X_2}$. We may apply successively Proposition 2.3.1 to the pairs (A_1, P_1) and respective (A_2^{-1}, P_2) in order to prove that the Cauchy problems $(A, \mu, Pb, 0)_0$ and $(A^{-1}, (I-P)b, y_0, 0)_0$, with $P =: P_1$, have bounded solutions. Conversely, if both above Cauchy problems have bounded solutions then Proposition 2.3.1 gives that the restriction of A to the range of P and the restriction of A^{-1} to the range of $I - P$ have spectral radius less than one. Hence A is dichotomic. □

Remark 2.3.1. Our result in finite dimensions, i.e. Theorem 2.2.3 above, is more informative than Theorem 2.3.2. By the proof of Theorem 2.2.3 follows that in the case when the matrix A is not dichotomic, for each projection P which commutes with matrix A and for a certain real number μ and a certain nonzero vector b, the solution of at least one of the Cauchy problems $(A, \mu, Pb, 0)_d$ or $(A^{-1}, \mu, (I - P)b, 0)_d$, with given real number μ, grows at the rate no slower than $\max\{||Pb||, ||(I-P)b||\} \times n$, and moreover there exists a projection P commuting with A such that for each nonzero vector b, the solution of at least one of the above two discrete Cauchy problems grows at the rate exactly (constant) $\times\, n$.

Remark 2.3.2. Let X be a Banach space and A be a compact linear operator acting on X. Since each nonzero $\lambda \in \sigma(A)$ is an isolated eigenvalues of finite multiplicity see [29], the statements from the previous remark remain true in this more general framework.

Chapter 3

Asymptotic Behavior of The Solutions for The time-dependent Periodic Systems

First section of this chapter deals with some preliminary results. In the second section of this chapter we prove that the time dependent 2-periodic system:

$$\dot{x}(t) = A(t)x(t), \; t \in \mathbb{R}, \quad x(t) \in \mathbb{C}^m \tag{$A(t)$}$$

is uniformly exponentially stable if and only if for each real number μ and each 2-periodic, $\mathbb{C}^m$-valued function f, the solution of the Cauchy problem:

$$\begin{cases} \dot{y}(t) = A(t)y(t) + e^{i\mu t}f(t), \quad t \in \mathbb{R}, \; y(t) \in \mathbb{C}^m \\ y(0) = 0 \end{cases}$$

is bounded. Also in this section some new characterizations for uniform exponential stability of $(A(t))$ in terms of the Datko type theorems are obtained as corollaries.

3.1 Some Preliminary Results

In this section we give some lemmas which will be useful in the second section.

Let $h_1, h_2 : [0,2] \to \mathbb{C}$ given by

$$h_1(u) = \begin{cases} u, \text{ u} \in [0,1) \\ 2-u, \text{ u} \in [1,2] \end{cases} \tag{3.1.1}$$

and

$$h_2(u) = u(2-u).$$

Lemma 3.1.1. *For each real number μ have that*

$$I_1(\mu) := \int_0^2 h_1(u)e^{i\mu u}\,du = \frac{1}{\mu^2}[2e^{i\mu} - e^{2i\mu} - 1], \tag{3.1.2}$$

and

$$I_2(\mu) := \int_0^2 h_2(u)e^{i\mu u}\,du = e^{2i\mu}(2-i\mu) - (2+i\mu). \tag{3.1.3}$$

Moreover, $I_1(\mu) \neq 0$ if and only if μ is in the set $\mathbb{C} \setminus \{2k\pi : k \in \mathbb{Z}\}$ and $I_2(\mu) \neq 0$ for all $\mu \in \{2k\pi : k \in \mathbb{Z}\}$.

Proof. After an obvious calculation we can see that the equality (3.1.2) is fulfilled and thus $I_1(\mu) = 0$ if and only if

$$\begin{cases} 2\sin\mu - \sin 2\mu = 0 \\ 2\cos\mu - \cos 2\mu - 1 = 0, \end{cases}$$

it happens if and only if $\mu \in \{2k\pi : k \in \mathbb{Z}\}$. Using (3.1.3) we get $I_2(\mu) \neq 0$ for all $\mu \in \{2k\pi : k \in \mathbb{Z}\}$. □

Lemma 3.1.2. *Let L be an $m \times m$ matrix of order $m \geq 1$ having complex entries. If*

$$\sup_{n \in \{1,2,3\dots\}} \|L^n\| = M < \infty$$

then each absolute value of the eigenvalue λ of the matrix L is less than or equals to 1.

Proof. Let λ be an eigenvalue of L. Suppose on contrary that $|\lambda| > 1$. Then there exists a nonzero vector $x \in \mathbb{C}^m$ such that $Lx = \lambda x$. Therefore $L^n x = \lambda^n x$ for all $n = 1, 2, \dots$ and then

$$M \geq \|L^n\| \geq \frac{\|L^n x\|}{\|x\|} = |\lambda|^n \to \infty \quad when \quad n \to \infty.$$

which is a contradiction which complete the proof. □

Lemma 3.1.3. *Let L be an $m \times m$ matrix of order $m \geq 1$ having complex entries. If*

$$\sup_{N \in \{1,2,3\dots\}} \|I + L + \cdots + L^N\| = K < \infty \tag{3.1.4}$$

then 1 is not an eigenvalue of L.

Proof. Suppose $1 \in \sigma(L)$. Then $Lx = x$ for some non zero vector x in $\mathbb{C}^m$ and $L^k x = x,$ for all $k = 1, 2, \dots N$. Therefore

$$\begin{aligned} \sup_{N \in \{1,2,3\dots\}} \|I + L + \cdots + L^N\| &= \sup_{N \in \{1,2,3\dots\}} \sup_{\xi \neq 0} \frac{\|(I + L + \cdots + L^N)(\xi)\|}{\|\xi\|} \\ &\geq \sup_{N \in \{1,2,3\dots\}} \frac{N\|\xi\|}{\|\xi\|} = \infty, \end{aligned}$$

which is a contradiction. This completes the proof. □

Corollary 3.1.4. *Let T be an $m \times m$ matrix of order $m > 1$ having complex entries. If for a real number μ, have that*

$$\sup_{N \in \{1,2,3\dots\}} \|I + e^{i\mu}T + \cdots + (e^{i\mu}T)^N\| = K(\mu) < \infty \tag{3.1.5}$$

then $e^{-i\mu}$ is not an eigenvalue of T.

Proof. We apply Lemma 3.1.3 for $L = e^{i\mu}T$. Have that $1 \in \rho(e^{i\mu}T)$ and then $I - e^{i\mu}T$ is an invertible matrix. Equivalently $e^{i\mu}(e^{-i\mu}I - T)$ is an invertible matrix i.e. $e^{-i\mu} \in \rho(T)$. □

Corollary 3.1.5. *Let T be an $m \times m$ matrix as above. If for each real number μ, the inequality (3.1.5) is fulfilled then the spectrum of the matrix T lies in the interior of the circle of radius one.*

Proof. We use the identity

$$(I - e^{i\mu}T)(I + e^{i\mu}T + \cdots + (e^{i\mu}T)^{N-1}) = I - (e^{i\mu}T)^N.$$

Passing to the norm we get :

$$\begin{aligned} \|(e^{i\mu}T)^N\| &\leq 1 + \|(I - e^{i\mu}T)\| \|(I + e^{i\mu}T + \cdots + (e^{i\mu}T)^{N-1})\| \\ &\leq 1 + (1 + \|T\|)K(\mu). \end{aligned}$$

From Lemma 3.1.2 follows that the absolute value of each eigenvalue λ of $e^{i\mu}T$ is less than or equal to one and from Lemma 3.1.3, $e^{-i\mu}$ is in the resolvent set of T. □

The infinite dimensional version of Corollary 3.1.5 has been stated in [13].

3.2 Bounded-ness and Exponential Stability

In this section we study the exponential stability for time dependent 2-periodic system

$$\dot{x}(t) = A(t)x(t), \; t \in \mathbb{R}, \quad x(t) \in \mathbb{C}^m. \tag{$A(t)$}$$

in terms of bounded-ness of solutions for some semi-linear non-autonomous Cauchy problems. where $A(t)$ is a 2-periodic continuous function, i.e. $A(t+2) = A(t)$ for all $t \in \mathbb{R}$. The choice of 2 as period is due to the method of the proof but the result may be preserved with arbitrary period $T > 0$ instead of 2. As a consequence we get some new characterizations for uniform exponential stability of the system $(A(t))$, in terms of the Datko type theorems.

It is well-known that the system $(A(t))$ is *uniformly exponentially stable*, i.e. there exist two positive constants N and ν such that

$$\|\Phi(t)\Phi^{-1}(s)\| \leq Ne^{-\nu(t-s)} \quad \text{for all} \quad t \geq s,$$

if and only if the spectrum of the matrix $V := \Phi(2)$ lies inside of the circle of radius one, e.g. see [15], where even the infinite dimensional version of this result is stated.

It is natural to ask, whether the negativeness of all the real parts of eigenvalues of the matrix $A(t)$ yields the exponential stability of the system $(A(t))$ or not. We give here a counterexample, adapted from [50], in order to justify that the answer of the previous question is NO. Let us denote the fundamental matrix of $(A(t))$ by $\Phi(t)$ i.e. the unique solution of the operatorial Cauchy problem:

$$\begin{cases} \dot{X}(t) = A(t)X(t) \\ X(0) = I. \end{cases} \tag{$A(t),0,I$}$$

Let us consider the matrices:

$$D(t) := \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad A = \begin{pmatrix} -1 & -5 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & \pi - 5 \\ -\pi & -1 \end{pmatrix}.$$

Define $A(t) = D(-\pi t)AD(\pi t)$ and $\Phi(t) = D(-\pi t)e^{tB}$. Then

$$\Phi(0) = I, \Phi'(t) = A(t)\Phi(t), \quad \sigma(A(t)) = \{-1\} \text{ for all } t \in \mathbb{R}$$

and $\sigma(\Phi(2)) = \{e^{2\lambda_1}, e^{2\lambda_2}\}$, where $\lambda_1 = \rho - 1, \lambda_2 = -\rho - 1$ and $\rho^2 = \pi(5 - \pi)$. This shows that the system $(A(t))$ is not uniformly exponentially stable because $e^{2\lambda_1}$ is a real number greater than one. As a consequence of the uniqueness of the solution of the Cauchy problem $(A(t), 0, I)$, have that $\Phi(2 + \tau) = \Phi(2)\Phi(\tau)$ for all $\tau \in \mathbb{R}$.

Let us consider also the vectorial non-homogenous Cauchy problem:

$$\begin{cases} \dot{y}(t) = A(t)y(t) + e^{i\mu t}f(t), \quad t \in \mathbb{R}_+ \\ y(0) = 0, \end{cases} \tag{$A(t),\mu,f(t),0$}$$

where f is some continuous function. With $P_{2,0}(\mathbb{R}_+, \mathbb{C}^m)$ we shall denote the space consisting of all continuous and 2-periodic functions g with the property that $g(0) = 0$. We endow this space with the norm "sup". For each $k \in \{1, 2\}$ let us consider the set $\mathcal{A}_k$ consisting by all functions $f \in P_{2,0}(\mathbb{R}_+, \mathbb{C}^m)$ given for $t \in [0, 2]$ by $f(t) = \Phi(t)h_k(t)$.

Theorem 3.2.1. *The following two statements hold true.*
(i) *If the system $(A(t))$ is uniformly exponentially stable then for each continuous and bounded function f and each real number μ the solution of $(A(t), \mu, f, 0)$ is bounded.*
(ii) *Let $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$. If for each $f \in \mathcal{A}$ and for each real number μ the solution of the Cauchy problem $(A, \mu, f, 0)$ is bounded then the system $(A(t))$ is uniformly exponentially stable.*

Proof. The solution ψ_f of $(A(t), \mu, f, 0)$ is given by

$$\psi_f(t) = \int_0^t \Phi(t)\Phi^{-1}(s)e^{i\mu s} f(s)\, ds.$$

The assertion **(i)** is now an easy consequence of the following estimates.

$$\begin{aligned} \|\psi_f(t)\| &\leq \int_0^t \|\Phi(t)\Phi^{-1}(s)\| \|f(s)\|\, ds \\ &\leq \int_0^t Ne^{-\nu(t-s)} \|f(s)\|\, ds \\ &\leq Ne^{-\nu t} \int_0^t e^{\nu s} \|f(s)\|\; ds\ . \end{aligned}$$

Let $\sup\limits_{\tau \in [0,2]} \|f(\tau)\| = M_f$. Then

$$\begin{aligned} \|\psi_f(t)\| &\leq Ne^{-\nu t} \int_0^t e^{\nu s} M_f ds \\ &= M_f \frac{N}{\nu}(1 - e^{-\nu t}) \\ &\leq \frac{N}{\nu} M_f. \end{aligned}$$

Thus ψ_f is bounded.

The argument for the second statement is a bit more difficult.
Let $b \in \mathbb{C}^m$ and $f_1 \in P_{2,0}(\mathbb{R}_+, \mathbb{C}^m)$ given on $[0, 2]$ by

$$f_1(\tau) = \begin{cases} \Phi(\tau)(\tau b), & \text{if } \tau \in [0, 1) \\ \Phi(\tau)(2 - \tau)b, & \text{if } \tau \in [1, 2]. \end{cases}$$

and h_1 defined in (3.1.1). Then for each $\tau \in \mathbb{R}$ have that $f_1(\tau) = \Phi(\tau)h_1(\tau)b$. For each natural number n, one has

$$\begin{aligned} \psi_{f_1}(2n) &= \int_0^{2n} \Phi(2n)\Phi^{-1}(s)e^{i\mu s} f_1(s)\, ds \\ &= \sum_{k=0}^{n-1} \int_{2k}^{2k+2} \Phi(2n)\Phi^{-1}(s)e^{i\mu s} f_1(s)\, ds\ . \end{aligned}$$

Put $s = 2k + \tau$, and using the fact that $\Phi^{-1}(2k+\tau) = \Phi^{-1}(2k)\Phi^{-1}(\tau)$, we get

$$\begin{aligned} \psi_{f_1}(2n) &= \sum_{k=0}^{n-1} \int_0^1 \Phi(2n)\Phi^{-1}(2k+\tau) e^{2i\mu k} e^{i\mu\tau} f_1(\tau)\, d\tau \\ &= \sum_{k=0}^{n-1} e^{2i\mu k} \Phi(2n-2k) b \int_0^2 e^{i\mu\tau} h_1(\tau)\, d\tau. \end{aligned}$$

Let us denote

$$A_1 = \mathbb{C}\backslash\{2k\pi : k \in \mathbb{Z}\} \quad \text{and} \quad M_1(\mu) = \int_0^2 e^{i\mu\tau} h_1(\tau)\, d\tau.$$

We know that $M_1(\mu) \neq 0$ for every $\mu \in A_1$ and thus

$$\psi_{f_1}(2n)(M_1(\mu))^{-1} = \sum_{k=0}^{n-1} e^{2i\mu k} \Phi(2n-2k) b, \quad \text{for all} \quad \mu \in A_1. \tag{3.2.1}$$

Consider also the function h_2 defined by

$$h_2(\tau) = \tau(1-\tau), \quad \tau \in [0,2]$$

and the function $f_2 \in P_{2,0}(\mathbb{R}_+, \mathbb{C}^m)$ given on $[0,2]$ by the formula

$$f_2(\tau) = \Phi(\tau) h_2(\tau) b.$$

By the same procedure, we obtain

$$\psi_{f_2}(2n)(M_2(\mu))^{-1} = \sum_{k=0}^{n-1} e^{2i\mu k} \Phi(2n-2k) b, \quad \mu \in \{2k\pi : k \in \mathbb{Z}\}, \tag{3.2.2}$$

where

$$M_2(\mu) = \int_0^2 e^{i\mu\tau} h_2(\tau)\, d\tau.$$

We know that ψ_{f_1} and ψ_{f_2} are bounded functions. Then there are two positive constants $K_1(\mu, f_1), K_2(\mu, f_2)$ such that

$$\|\psi_{f_1}(2n)\| \leq K_1(\mu, f_1) \quad \text{and} \quad \|\psi_{f_2}(2n)\| \leq K_2(\mu, f_2) \quad \text{for all} \quad n = 1, 2 \ldots.$$

From (3.2.1) follows that if $\mu \in A_1$ then

$$\left\| \sum_{k=0}^{n-1} e^{2i\mu k} \Phi(2n-2k) b \right\| \leq \frac{K_1(\mu, f_1)}{|M_1(\mu)|} = r_1(\mu, f_1)$$

and analogously using (3.2.2), if $\mu \in \{2k\pi : k \in \mathbb{Z}\}$ then we get

$$\left\|\sum_{k=0}^{n-1} e^{2i\mu k}\Phi(2n-2k)b\right\| \leq \frac{K_2(\mu, f_2)}{|M_2(\mu)|} = r_2(\mu, f_2).$$

Now for each real number μ and each $b \in \mathbb{C}^m$, the above inequalities yield

$$\left\|\sum_{k=0}^{n-1} e^{2i\mu k}\Phi(2n-2k)b\right\| \leq r_1(\mu, f_1) + r_2(\mu, f_2). \tag{3.2.3}$$

On the other hand replacing j by $n-k$, we get

$$\sum_{k=0}^{n-1} e^{2i\mu k}\Phi(2n-2k)b = e^{2i\mu n}\sum_{j=1}^{n} e^{-2i\mu j}\Phi(2j)b. \tag{3.2.4}$$

Following Uniform Bounded-ness Principle and using the relations (3.2.3) and (3.2.4) we can find a positive constant $L(\mu)$ such that

$$\left\|\sum_{j=1}^{n} e^{-2i\mu s}(\Phi(2))^j\right\| \leq L(\mu) < \infty.$$

Now we can apply Corollary 3.1.5 for $T = \Phi(2)$ and can say that the spectrum of $\Phi(2)$ lies in the interior of the circle of radius one, i.e. the system $(A(t))$ is uniformly exponentially stable. This completes the proof. □

Corollary 3.2.2. *The system $(A(t))$ is uniformly exponentially stable if and only if for each real number μ and each function f belonging to $P_{2,0}(\mathbb{R}_+, \mathbb{C}^m)$ the solution of $(A(t), \mu, f, 0)$ is bounded.*

Using the periodicity of Φ and f it is easy to see that the solution ψ_f is bounded if the sequence $(\psi_f(n))$ is also bounded. If come back to (3.2.1) and (3.2.2) we should be able to recapture the inequality (3.2.3) under the assumption that for each vector b the series $(\sum_{j\geq 0} ||\Phi(2j)b||)$ is convergent. Then the following Corollary of Datko type may be stated as well.

Corollary 3.2.3. *With the above notations we have that the system* $(A(t))$ *is uniformly exponentially stable if and only if for each vector* b *the following inequality holds true.*

$$\sum_{j=1}^{\infty} ||\Phi(2j)b|| < \infty. \tag{3.2.5}$$

It is not difficult to see that the requirement (3.2.5) may be replaced by an apparently weaker one, namely with the inequality

$$\sum_{j=1}^{\infty} |\langle \Phi(2j)b, b\rangle| < \infty, \quad \forall\, b \in \mathbb{C}^m.$$

3.3 Future research and further references

Our idea for future research leads to an extension of the results from this dissertation to the case of time varying and periodic coefficients. First we draw our attention about the following:

Theorem 3.3.1. *Let $N \geq 2$ be a natural number and let (A_n) be an N-periodic sequence of $m \times m$ matrices having complex numbers as entries. The following three statements are equivalent:*
(i) *The homogeneous discrete system*

$$x_{n+1} = A_n x_n, \quad n \in \mathbb{Z}_+ \tag{A_n}$$

is asymptotically stable, i.e. any its solution tends to 0 when $n \to \infty$.
(ii) *The spectrum of the matrix $L := A_{N-1}A_{N-2}\cdots A_0$ is included in the interior of the unit circle.*
(iii) *For each vector b and each real number μ, the solution of the following discrete Cauchy Problem*

$$\begin{cases} y_{n+1} = A_n y_n + e^{i\mu n} b, & n \in \mathbb{Z}_+ \\ y_0 = 0 \end{cases}$$

is bounded.

Looks like, that for the proof of the implication **(iii)** $\Rightarrow$ **(ii)** it is necessary to add new hypothesis to **(iii)**. Next we try to find a suitable formulation and find its proof for a similar result in the periodic non-autonomous case with that duet already for autonomous dichotomic systems. Also we are looking for an answer (positive or negative) to the question could the Remark 2.3.1 be extended from compact operators to bounded linear operators acting on an infinite dimensional Banach space. Also we have a question that could it be extended for the case when A_n are bounded linear operators acting on a general Banach space see(Theorem 3.3.1). We hope to get a positive answer of this question having in mind the following result which has been proved in [12].

Theorem 3.3.2. *Let $q > 1$ be a fixed integer number. The following three statements are equivalent:*
(i) *The spectral radius of the operator $T = U(q, 0)$ is less than 1.*
(ii) *$U(p, k)$ tends to 0 when $(p - k) \to \infty$ with respect to the operator norm in $\mathcal{L}(X)$.*
(iii) *For each real number μ and each q-periodic X-valued sequence (z_n) with $z_0 = 0$, the solution of the problem $(A_n, \mu, z_0, 0)$ is bounded.*

Here $U(n, k) := A_{n-1}A_{n-2} \cdots A_k$ if $k \leq n-1$ and $U(n, n) = I$. As a consequence of the previous theorem in [12] the discrete variant of the theorem of Datko is obtained. Details about the Datko type theorems can be found in [22]. The statement **(iii)** in Theorem 3.3.1 is true if and only if

$$\sup_{n\geq 1} \left\| \sum_{k=1}^{n} e^{i\mu k} U(n, j)b \right\| < \infty. \tag{SBC}$$

Thus a necessary and sufficient condition for the homogeneous system (A_n) to be asymptotically stable is that the function

$$n \mapsto \sum_{j=1}^{n} \|U(n, j)b\|$$

is bounded. This is the strong variant of a Barbashin Theorem. Similar strong variant of the Barbashin Theorem in the continuous case seems to be an open problem. See [11] and [14] where some progress in this direction were made. Having in mind that our system is a periodic one it is possible to put a more refined condition instead of (SBC).

It is known (see for example [8], [9], [18]) that the non-autonomous system

$$\dot{x}(t) = A(t)x(t) \tag{$A(t)$}$$

is uniformly exponentially stable if (and only if) for each $\mu \in \mathbb{R}$ and each function f belonging to a set of functions $\mathcal{F}$, which is given below, the solution of the Cauchy

Problem

$$\dot{y}(t) = A(t)y(t) + e^{i\mu t}f(t), \quad t \geq 0, \quad y(0) = 0,$$

is bounded on $\mathbb{R}_+$. In [9], (see also [8]), the result is proved when $A(t)$ is an unbounded operator for each $t \geq 0$, and the operator-valued function $A(\cdot)$ is continuous and periodic with period $q > 1$. The class of functions $\mathcal{F}$ consists of all continuous and q-periodic functions which decays to zero. In [18], $A(t)$ are matrices and the map $A(\cdot)$ is continuous and 2-periodic, but the class $\mathcal{F}$ is much smaller. This topic is widely represented in the mathematical literature. We recall here only few papers and books in a very large bibliography. Namely: [1], [13], [16],[19], [38], [40]. In all these works and in many others, the class of functions $\mathcal{F}$ is much more consistent and instead of bounded-ness condition, given above, a uniform ergodic one is assumed.

In the connection with the problems exposed before in the continuous case we will be interested to replace the class of functions $\mathcal{F}$ used in [18] with the class of all constant (n-vector)-valued functions. Also we shall be interested to built a formulation and its proof in the case when the coefficients $A(t)$ are bounded linear operators acting on a general Banach space. Also we can connect our future research with the papers [36] and [49].

Bibliography

[1] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, Vector-valued Laplace Transforms and Cauchy Problems, *Monographs in Mathematics*, vol. **96**, Birkhäuser, Basel, 2001.

[2] A. V. Balakrishnan, Fractional powers of closed operators and semigroups generated by them, *Pacific J. Math.*, **10**(1960), 419-439.

[3] Ştefan Balint, On the Perron-Bellman theorem for systems with constant coefficients, *Ann. Univ. Timisoara*, **21**, fasc 1-2 (1983), 3-8.

[4] V. Barbu, *Differential Equations*, Ed. Junimea, Iasi (1985).

[5] B. Basit, Some problems concerning different types of vector- valued almost periodic functions, *Dissertationes Math.* (Rozprawy Mat.) 338(1995).

[6] A. G. Baskakov, Some conditions for invertibility of linear differential and differnece operators, *Russian Acad. Sci. Dokl. Math.* **48**(1994), 498-501.

[7] C. J. K. Batty, R. Chill, Yuri Tomilov, Strong stability of bounded evolution families and semigroups, *Journal of Functional Analysis*, 193(2002), 116-139.

[8] Constantin Buşe, Asymptotic behavior of evolutionary processes, *Publishing House SEDONA, Timişoara*, (1998), 172 pages, ISBN 973-9345-14-X, (in Romanian);

[9] Constantin Buşe, Exponential Stability for Periodic Evolution Families of Bounded Linear Operators, *Rend. Sem. Matematico. Univers. Pol. Torino.,* Vol. **59**(1)(2001), 17-22.

[10] C. Buşe, D. Barbu, Some remarks about the Perron condition for strongly continious semigroups, *Analele Univ. Timisora*, fasc 1 (1997).

[11] C. Buşe, A. D. R. Choudary, S. S. Dragomir and M. S. Prajea, On Uniform Exponential Stability of Exponentially Bounded Evolution Families, *Integral Equation Operator Theory*, 61(2008), 325-340.

[12] C. Buşe, P. Cerone, S. S. Dragomir and A. Sofo, Uniform stability of periodic discrete system in Banach spaces,*J. Difference Equ. Appl.* **11**, No .12 (2005) 1081-1088.

[13] C. Buşe, S. S. Dragomir, V. Lupulescu, Characterizations of stability for strongly continuous semigroups by convolutions, *International Journal of Differential Equations and Applications*, Vol. **2**, No.1(2001), 103-109.

[14] C. Buşe, M. Megan, M. Prajea, P. Preda, The strong variant of a Barbashin's theorem on stability for non-autonomous differential equations in Banach spaces, *Integral Equations Operator Theory*, Vol. **59**(2007), 491-500.

[15] C. Buşe, M. S. Prajea, Asymptotic behavior of discrete and continuous semigroups on Hilbert spaces, *Bull. Math. Soc. Sci. Math. Roumanie, Tome* 51(99), No. 2, 2008, 123-135.

[16] M. Reghiş, C. Buşe, On the Perron-Bellman theorem for strongly continuous semigroups and periodic evolutionary processes in Banach spaces, *Italian Journal of Pure and Applied Mathematics* , No. 4 (1998), 155-166.

[17] C. Buşe, **Akbar Zada**, Dichotomy and bounded-ness of solutions for some discrete Cauchy problems, *Proceedings of IWOTA- 2008, Operator Theory, Advances and Applications*, Vol. **203**, 165-174, (OT) Series, Birkhäuser Verlag, Eds: J.A. Ball, V. Bolotnikov, W. Helton, L. Rodman, T. Spitkovsky 2010.

[18] C. Buşe, **Akbar Zada**, Bounded-ness and exponential stability for periodic time dependent systems, *Electronic Journal of Qualitative Theory of Differential Equations*, 37, pp. 1-9 (2009).

[19] C. Chicone, Y. Latushkin, Evolution Semigroups in Dynamical Systems and Differential Equations, *Amer. Math. Soc., Math. Surv. and Monographs*, No. 70(1999).

[20] S. Clark, Y. Latushkin, S. Montgomery-Smith, T. Randolph, Stability radius and internal versus external stability in Banach spaces: an evolution semigroup approach, *SIAM J. Control Optim.,* **38**(2000), 1757-1793.

[21] C. Corduneanu, Almost Periodic Functions, 2nd edition, *Chelsea, New-York*, 1989.

[22] R. Datko, Uniform asymptotic stability of evolutionary processes in Banach space, *SIAM J. Math. Analysis* **3** (1973), 428-445.

[23] Dragan, A. Halanay, *Stabilization of linear systems*, Ed. All (1994).

[24] Y. L. Daletkii, M. G. Krein, Stability of Solutions of Differential Equations in Banach Spaces, *Izd. Nauka, Moskwa*, 1970.

[25] K. J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolutions Equations, *Springer-Verlag*, 2000.

[26] Abdelkader Elkoutri and Mohamed Aziz Taoudi, Spectral inclusions and stability results for strongly continuous semigroups, *International Journal of Mathematics and Mathematical Sciences* (IJMMS), **37**(2003), 2379-2387.

[27] H. O. Fattorini, The Cauchy Problem, Addison Wesley, 1983.

[28] C. Foias and F. H. Vasilescu, Non-analytic local functional calculus, *Czech. Math. J. 24*(**99)**(1974), 270-283.

[29] I. Gohberg, S. Goldberg, Basic Operator Theory, *Birkhäuser, Boston-Basel*, 1981.

[30] J. A. Goldstein, Periodic and pseudo-periodic solutions of evolution equations, in: H. Brézis, M. G. Crandall, F. Kappel(Eds.), *Semigroups, Theory and applications I,* Longman, 1986, 142-146.

[31] F. L. Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, *Ann. Diff. Eq.*, **1**(1983), 43-56.

[32] E. Hille, R. S. Philips, Functional Analysis and Semigroups, *Amer. Math. Soc. Coll. Publ.* 31, Providence R.I., 1957.

[33] A. Ichikawa, Null controllability with vanishing energy for discrete-time systems in Hilbert space, *SIAM J. Control Optim.,* Vol. **46**, No. 2,(2007), pp. 683-693.

[34] P. D. Lax, Linear Algebra, Wiley-Interscience Publication, John Wiley and Sons Inc, (1996).

[35] Yu. I. Lyubich and Quôc Phong Vũ, Asymptotic stability of linear differential equations in Banach Spaces, *Studia Math.* **88**(1988), 37-42.

[36] V. Lupulescu, **A. Zada**, On linear Impulsive dynamical systems on time scales, *Electronic Journal of Qualitative Theory of Differential Equations*, to appear.

[37] V. Müller, Power bounded operators and supercyclic vectors, *Proceedings of the American Mathematical Society*, Volume **131**, Number 12, 3807-3812, (2003).

[38] Nguyen Van Minh, Frank Räbiger, and Roland Schnaubelt, Exponential Stability, Exponential Expansiveness, and Exponential Dichotomy of Evolution Equations on the half line, *Integral Equations Operator Theory*, **32**(1998), 332-353.

[39] M. Megan, A. L. Sasu, B. Sasu, On uniform exponential stability of periodic evolution operators in Banach spaces, *Acta Math. Univ Comenian.* **69** (2000), 97-106.

[40] Jan van Neerven, The Asymptotic Behaviour of Semigroups of Linear Operators, Birkhäuser, Basel, 1996.

[41] Jan van Neerven, Individual stability of strongly continuous semigroups with uniformly bounded local resolvent, *Semigroup Forum,* **53** (1996), 155-161.

[42] A. Pazy, Semigroups of linear operators and applications to partial differential equations, *Springer-Verlag*, 1983.

[43] Vu Quoc Phong, On stability of C_0-semigroups, *Proceedings of the American Mathematical Society*, Vol. *129*, No. 10, 2002, 2871-2879.

[44] P. Preda, M. Megan, Exponential dichotomy of evolutionary processes in Banach spaces, *Czehoslovak Math. Journal*, **35**(110), (1985), nr.2, 312-333.

[45] C. I. Preda, P. Preda, An extension of the admissibility-type conditions for the exponential dichotomy of C_0-semigroups, *Integral Equations and Operator Theory*, **63**(2009), no. 2, 249-261.

[46] J. Prüss, Evolutionary integral equations and applications, Monographs in Mathematics, Birkhäuser Verlog, Basel, 1993.

[47] G. Da Prato, E. Sinestrari, Non autonomous evolution operators of hyperbolic type, *Semigroup Forum* **45**(1992), 302-321.

[48] M. Reghis, About exponential dichotomy of linear autonomous differential equations, *Qualitative problems for differential equations and control theory*, 53-60, World Sci. Publ. River Edge, NJ, 1995.

[49] Rozikov U. A, **Zada A.**, On l-volterra Quadratic Stochastic Operators, *Doklady Math* **79** no. 1,32-34(2009).

[50] R. Schnaubelt, Well-posedness and asymptotic behavior of non-autonomous linear evolution equations, Evolution equations, *semigroups and functional analysi* (Milano, 2000), 311-338, *Progr. Nonlinear Differential Equations Appl.*, 50, Birkhäuser, Basel, 2002.

[51] A. L. Sasu, Integral equations on function spaces and dichotomy on the real line, *Integral Equations and Operator Theory*, **58**(2007), no.1, 133-152.

[52] A. L. Sasu, B. Sasu, Exponential Stability for linear skew-product flows, *Bull. Sci. Math.* **128**(2004), no. 9, 728-738.

[53] G. Weiss, Weak L^p-stability of linear semigroup on a Hilbert space implies exponential stability, *J. Diff. Eqns.,* **76**, (1988), 269-285.

[54] **Akbar Zada**, A characterization of dichotomy in terms of boundedness of solutions for some Cauchy problems, *Electronic Journal of Differential Equations*, Vol.**(2008)**, No. 94, 1-5.