Stanley Decompositions of Multigraded Modules and Reductions Modulo Regular Elements

Since 1864

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<u>DECLARATION</u>

I, <u>**Miss Asia Rauf**</u> Registration No. <u>**18-GCU-PHD-SMS-05**</u> student at **Abdus Salam School of Mathematical Sciences GC University** in the subject of **Mathematics**, **Year of Admission** (**2005)**, hereby declare that the matter printed in this thesis titled

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is my own work and that

(i) I am not registered for the similar degree elsewhere contemporaneously.

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(iii) The work, I am submitting for the Ph. D. degree has not already been submitted elsewhere and shall not in future be submitted by me for obtaining similar degree from any other institution.

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<u>RESEARCH COMPLETION CERTIFICATE</u>

Certified that the research work contained in this thesis titled

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has been carried out and completed by Miss Asia Rauf Registration No. **<u>18-GCU-PHD-SMS-2005</u>** under my supervision.

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To my dear parents

and

respected teachers

Contents

Abstract

In Chapter 1, some necessary definitions and results from commutative algebra are given along with a description on the progress towards the Stanley decompositions of multigraded S-modules, where $S = K[x_1, \dots, x_n]$ is a polynomial ring in n variables over a field K.

In Chapter 2, we study the behavior of Stanley decompositions and of pretty clean filtrations under reduction modulo a regular element. We prove that the Stanley depth of cyclic module drop by one under reduction modulo a regular element. We see that the cyclic module is pretty clean if and only if the reduction modulo a regular element is pretty clean. We also discuss the behavior of depth, Stanley depth and dimension on algebra tensor products.

In Chapter 3, it is shown that to what extend Stanley depth behaves like ordinary depth under reduction modulo an element and is clarified the different behavior of the two concepts by some examples. We discuss the behavior of depth, Stanley depth and dimension along short exact sequences of finitely generated multigraded S-modules.

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Lahore, Pakistan | Asia Rauf

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Preface

In commutative algebra graded rings and graded modules are of great importance, especially the graded local rings, that is, graded rings with just one graded maximal ideal. Graded local rings share many properties with local rings. For example, consider the polynomial ring $S = K[x_1, \ldots, x_n]$ in n variables over a field K. If $n > 0$ this has infinitely many maximal ideals but $(x_1, \ldots, x_n) \subset S$ is the only graded maximal ideal of S. This thesis is structured in three chapters. In the first chapter, I have discussed graded rings and modules and the related concepts. Here we discuss basic notions related to monomial ideals, dimension theory, primary decomposition of modules, regular sequences in the context of graded rings and modules. The Koszul complex and Koszul homology are described as much as they are used in the later chapters.

Prime filtrations are useful tools in commutative algebra and will also be used in my work. Every finitely generated module over a Noetherian ring has a prime filtration. Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over the field K. For finitely generated $\mathbb{Z}^n$-graded S-module M the prime filtration is defined as follows: let

$$\mathcal{F}: \ 0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

be a chain of $\mathbb{Z}^n$-graded submodules of M. Then $\mathcal{F}$ is called a prime filtration of M if $M_i/M_{i-1} \cong (S/P_i)(-a_i)$ where $a_i \in \mathbb{Z}^n$ and P_i is a monomial prime ideal for all i. We call the set $\operatorname{Supp} \mathcal{F} = \{P_1, \ldots, P_r\}$ the support of $\mathcal{F}$. It is well known that $\operatorname{Ass}(M) \subset \operatorname{Supp}(\mathcal{F}) \subset \operatorname{Supp}(M)$. Dress [11] defined clean modules as follows: a prime filtration $\mathcal{F}$ is called clean if $\operatorname{Supp}(\mathcal{F}) = \operatorname{Min}(M)$. The S-module M is called clean, if it has a clean filtration. As a generalization of clean modules, Herzog and Popescu, in [15], introduced pretty clean modules. A prime filtration $\mathcal{F}$ of M is called pretty clean, if for all $i < j$ with $P_i \subset P_j$, it follows that $P_i = P_j$. If M has a pretty clean filtration then M is called pretty clean. For the pretty clean filtration, one has $\operatorname{Supp}(\mathcal{F}) = \operatorname{Ass}(M)$. Note that any clean filtration is pretty clean.

Prime filtrations guarantee the existence of Stanley decompositions of multigraded S-modules. The Stanley decompositions of multigraded modules are described in detail in the beginning of Chapter 2. Let K be a field and $S = K[x_1, \ldots, x_n]$ a polynomial ring in n variables over K. Sturmfels and White proved existence of Stanley decompositions of finitely generated K-algebra in [33]. Let M be a finitely generated multigraded (i.e. $\mathbb{Z}^n$-graded) S-module. Let $m \in M$ be a $\mathbb{Z}^n$-homogeneous element in M and $Z \subseteq \{x_1, \ldots, x_n\}$. We denote by $mK[Z]$ the K-subspace of M generated by all elements mv, where v is a monomial in $K[Z]$. The multigraded K-subspace $mK[Z] \subset M$ is called Stanley space of dimension $|Z|$, if $mK[Z]$ is a free $K[Z]$-module. A Stanley decomposition of M is a presentation of the K-vector space M as a finite direct sum of Stanley spaces $\mathcal{D}: \ M = \bigoplus_{i=1}^{r} m_i K[Z_i]$. Set $\operatorname{sdepth} \mathcal{D} = \min\{|Z_i| : i = 1, \ldots, r\}$. The number

$$\operatorname{sdepth}(M) := \max\{\operatorname{sdepth}(\mathcal{D}) : \ \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

is called Stanley depth of M. R. P. Stanley [31, Conjecture 5.1] conjectured that $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$ for all finitely generated $\mathbb{Z}^n$-graded S-modules M. The conjecture is discussed in some special cases in [16], [2], [17], [4], [1], [25] [8], [9]. Actually Stanley's conjecture is somewhat more general than quoted here.

One of the main results of Chapter 2 is Theorem 2.5.1 which says that if $I \subset S$ is a monomial ideal and $u \in S$ is a monomial being regular on S/I, then S/I has a pretty clean filtration if and only if $S/(I, u)$ has a pretty clean filtration. This result implies that an ideal generated by a regular sequence of monomials is pretty clean. This fact was first proved in [16, Proposition 1.2] by a different method.

In the last section of Chapter 2, we prove that if $I \subset S_1 = K[x_1, \ldots, x_n]$, $J \subset S_2 = K[y_1, \ldots, y_m]$ are monomial ideals and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$, then the Stanley depth of the tensor product of S_1/I and S_2/J (over K) is greater than or equal to the sum of $\operatorname{sdepth}(S_1/I)$ and $\operatorname{sdepth}(S_2/J)$. This inequality could be strict as shows Example 2.6.2. It implies that Stanley depth behaves quite differently than the depth and dimension in this situation.

It is a well known that fact depth and dimension decrease by one under reduction modulo a regular element. What about Stanley depth? In Chapter 2 and Chapter 3, we study Stanley depth under reduction modulo an element. Let M be a finitely generated multigraded S-module and $f \in S$ a monomial which is regular on M. We show $\operatorname{sdepth}(M/fM) = \operatorname{sdepth}(M) - 1$, if $M = S/I$ where $I \subset S$ is a monomial ideal (see Theorem 2.4.1) or if M is almost clean and f is variable and see (Theorem 3.1.14). Recall that a finitely generated module M is called almost clean if there exists a prime filtration $\mathcal{F}$ of M such that $\operatorname{Supp}(\mathcal{F}) = \operatorname{Ass}(M)$. On the other hand for any multigraded S-module M and f a variable which is regular on M, in Proposition 3.1.8 we show that $\operatorname{sdepth}(M/fM) \leq \operatorname{sdepth}(M) - 1$. In fact this inequality can be strict as we observe in Example 3.1.6. As an application we get that Stanley's conjecture holds for M if it holds for the module M/x_kM (see Corollary 3.1.9). Moreover if M has a maximal regular sequence given by monomials then Stanley's conjecture holds for M (see Corollary 3.1.11).

In the last chapter, I have tried to extend some results from the second chapter to finitely generated multigraded S-modules. If M is a finitely generated module over a Noetherian local ring (R, m) and $x \in m$, then it is well-known that $\dim(M/xM) \geq \dim(M) - 1$. Our Proposition 3.1.1 shows that the above inequality is preserved for depth and sdepth when $M = S/I$ and $I \subset S$ is a monomial ideal and $x = x_k$ for any $k \in [n]$. If M is a general multigraded S-module, then we might have $\operatorname{depth}(M/x_kM) < \operatorname{depth}(M) - 1$ as shown in Example 3.1.4. Also we might have $\operatorname{sdepth}(M/x_kM) < \operatorname{sdepth}(M) - 1$ even if x_k is regular on M, as we see in Example 3.1.6.

Given a short exact sequence of finitely generated multigraded S-modules, the dimension of middle one is equal to maximum of the dimensions of ends and for the depth we have "Depth lemma". A natural question arises how does the Stanley depth behave on short exact sequence of multigraded S-modules. It is proved in Lemma 3.2.1 that the Stanley depth of the middle one is greater than or equal to the minimum of the Stanley depths of the ends. Several examples show that the

"Depth lemma" is mainly wrong in the frame of sdepth (see Examples 3.2.4 and 3.2.5). However, we prove in Lemma 3.2.6, that if I is any monomial complete intersection of S, then $\operatorname{sdepth} I$ is greater than or equal to $\operatorname{sdepth}(S/I)+1$. But in general for any monomial ideal this inequality is still an open question.

We acknowledge the support provided by the Computer Algebra Systems SINGULAR [13] for the extensive experiments which helped us to obtain some of the results in this thesis.

Chapter 1

Preliminaries

In this chapter we recall basic notions and some results that will be used later. Throughout this work all rings are considered to be commutative.

1.1 Graded rings and modules

In this section we recall some definitions and important results about graded rings and modules.

Definition 1.1.1. Let $(G, +)$ be an Abelian group. A *graded ring* or *G-graded ring* is a ring R, if there exists a direct sum decomposition $R = \bigoplus_{i \in G} R_i$ as $\mathbb{Z}$-modules such that $R_i R_j \subset R_{i+j}$ for all $i, j \in G$. If R is G-graded and M is an R-module, then M is called *G-graded*, if there exists a direct sum decomposition $M = \bigoplus_{i \in G} M_i$ as a $\mathbb{Z}$-module such that $R_i M_j \subset M_{i+j}$ for all $i, j \in G$.

An element $u \in M$ is *homogeneous*, if there exists $i \in G$ such that $u \in M_i$ and then i is called the *degree* of u, written as $\deg(u) = i$. Each M_i is called a *homogeneous component* of M of degree i, for $i \in G$. So every element $m \in M$ can be uniquely represented as $m = \sum_{i \in G} m_i$ where $m_i \in M_i$ with only finitely many $m_i \neq 0$ and are called *homogeneous components* of m.

A submodule $N \subset M$ is called a homogeneous or G-graded submodule if it can be generated by homogeneous elements with respect to G-grading. This condition is equivalent to each of the following two:

(i) For $m \in M$, if $m \in N$ then each homogeneous component of m belong to N;

(ii) $N = \sum_{i \in G}(N \cap M_i)$.

If $N \subset M$ is a homogeneous submodule of M and we set $N_i = M_i \cap N$ then $N = \bigoplus_{i \in G} N_i$. We have $M/N = \bigoplus_{i \in G} M_i/N_i$ is again a G-graded R-module.

Definition 1.1.2. Let R be a G-graded ring and M, N are G-graded R-modules. An R-linear map $\varphi : M \to N$ is graded of degree d for some $d \in G$, if $\varphi(M_i) \subset N_{i+d}$ for all $i \in G$. We call φ graded, if it is homogeneous of degree 0. The kernel and cokernel of the graded map φ are also naturally graded modules.

If G is $\mathbb{Z}$ or $\mathbb{Z}^n$, we say that R is a graded or a multigraded ring, respectively, and the R-module M is called graded or multigraded R-module. In both cases for any G-graded ring $R = \bigoplus_{i \in G} R_i$, then we can define another G-graded ring for a fixed $t \in G$ such that $R(t) = \bigoplus_{i \in G} R(t)_i$, where $R(t)_i := R_{t+i}$.

Examples 1.1.3. (a): Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and $J \subset I \subset S$ be graded ideals. If we set $\deg(x_i) = 1$ then S is a graded ring and I/J is graded S-modules. In particular, if $J = (0)$ then I is graded ideal and if $I = S$ then S/J is graded S-module. If we set $\deg(x_i) = \varepsilon_i$, where ε_i is i-th unit vector of $\mathbb{Z}^n$ and, J and I are monomial ideals then S is naturally multigraded ring and I/J is multigraded S-module.

(b): Let K be a field, and let $S = K[x_1, x_2, x_3]$ be a polynomial ring. We have a graded short exact sequence $0 \to \Omega^1 m \xrightarrow{f} Se_1 \oplus Se_2 \oplus Se_3 \xrightarrow{g} m \to 0$, where $S = K[x, y, z]$, $m = (x, y, z)$ and $\Omega^1 m$ is first syzygy module. Also f is a graded inclusion map and g is a graded surjection map such that $g(e_i) = x_i$ for all i. Then $Se_1 \oplus Se_2 \oplus Se_3$ is a multigraded S-module with $\deg(e_i)$ is i-th unit vector of $\mathbb{Z}^3$. And also $\Omega^1 m$ is a multigraded S-module, generated by multigraded elements $-x_2e_1 + x_1e_2$, $-x_3e_2 + x_2e_3$ and $-x_1e_3 + x_3e_1$ of degrees $(1, 1, 0)$, $(0, 1, 1)$ and $(1, 0, 1)$, respectively.

A graded ring R is called *positively graded* or $\mathbb{N}$-*graded* R_0-algebra if R is a graded R_0-algebra generated by elements of positive degree, that is, $R = \bigoplus_{i=0}^{\infty} R_i$. Let $I \subset R$ be an arbitrary ideal of R. Then the graded ideal I^* is defined to be ideal generated by homogeneous elements $u \in I$. If I is itself a graded ideal then $I = I^*$ by definition.

Now we will see which graded rings are Noetherian. Firstly we consider $\mathbb{N}$-graded rings.

Proposition 1.1.4. ([7]) *Let R be a $\mathbb{N}$-graded R_0-algebra, and $x_1, \ldots, x_n$ homogeneous elements of positive degree. Then the following statements are equivalent:*

(a) *$x_1, \ldots, x_n$ generate the ideal $m = \bigoplus_{i=1}^{\infty} R_i$;*

(b) *$x_1, \ldots, x_n$ generate R as an R_0-algebra.*

In particular, R is Noetherian if and only if R_0 is Noetherian and R is a finitely generated R_0-algebra.

In general, the last assertion of the above Proposition 1.1.4 holds for graded rings.

Definition 1.1.5. Let K be a field and let $R = \bigoplus_{i \geq 0} R_i$ be a graded K-algebra. If R is a finitely generated K-algebra and all its generators are of degree 1. Let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a finitely generated graded R-module. Then the numerical function

$$\mathrm{H}(M, -) : \mathbb{Z} \to \mathbb{Z} \quad \text{with} \quad \mathrm{H}(M, i) := \dim_K M_i \quad \text{for all} \quad i \in \mathbb{Z},$$

is called *Hilbert function* of M and

$$\mathrm{Hilb}_M(t) = \sum_{i \in \mathbb{Z}} \mathrm{H}(M, i) t^i$$

is called the *Hilbert series* of M.

The classical example of Hilbert function and Hilbert series is the following:

Example 1.1.6. If we consider the polynomial ring $S = K[x_1, \dots, x_n]$ in n variables over the field K and $\deg x_i = 1$ for $i = 1, \dots, n$, then we have

$$\mathrm{H}(S, i) = \dim_K S_i = \binom{i+n-1}{n-1},$$

and for $n \geq 1$,

$$\mathrm{Hilb}_S(t) = \sum_{i=0}^{\infty} \dim_K S_i t^i = \sum_{i=0}^{\infty} \binom{i+n-1}{n-1} t^i = \frac{1}{(1-t)^n}.$$

Associated to any graded R-module M there exists a unique up to isomorphism minimal graded free resolution

$$0 \to \bigoplus_j R(-j)^{\beta_{rj}} \to \cdots \to \bigoplus_j R(-j)^{\beta_{1j}} \to \bigoplus_j R(-j)^{\beta_{0j}} \to M \to 0.$$

The number β_{ij} is called the *graded Betti number* of M. There are two important invariants attached to a minimal graded free resolution of M.

Definition 1.1.7. The *projective dimension* of M is

$$\operatorname{proj\,dim}(M) = \max\{i : \beta_{ij} \neq 0, \text{for some}\, j\}.$$

Definition 1.1.8. The *regularity* of M is

$$\operatorname{reg}(M) = \max\{j - i : \text{for all}\, i,\, j\}.$$

Example 1.1.9. Let $I = (x^2, xy^2, yz) \subset S = K[x, y, z]$ be a monomial ideal of the polynomial ring S over the field K. Consider the graded module $M = S/I$. We have the graded minimal free resolution of M

$$0 \to S(-5) \to S^3(-4) \to S^2(-2) \oplus S(-3) \to S \to M \to 0.$$

In this resolution the graded Betti numbers are as follows $\beta_{00} = 1$, $\beta_{12} = 2$, $\beta_{13} = 1$, $\beta_{24} = 3$ and $\beta_{35} = 1$. Hence $\operatorname{proj\,dim}(M) = 3$ and $\operatorname{reg}(M) = 2$.

1.2 Monomial ideals

In this section, we recall the basis properties of monomial ideals and facts related to monomial ideals which are used later in this thesis. Our main reference for this section is [14].

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring in n variables over field K. Any product $x_1^{a_1} \cdots x_n^{a_n}$ with $a_i \in \mathbb{N}$ is called *monomial*. The set of monomials of S, which is denoted by $\mathrm{Mon}(S)$, forms a K-basis. Any polynomial $f \in S$ is a unique K-linear combination of monomials. An ideal $I \subset S$ is called monomial ideal if it is generated by monomials. The set of monomials belonging to I forms a K-basis of I. The residue classes of the monomials not belonging to I form a K-basis of the residue class ring S/I. We denote by $I^c \subset S$ the K-linear subspace of S generated by all monomials which do not belong to I. Then $S = I \bigoplus I^c$ and $S/I \cong I^c$ as K-linear spaces. By Hilbert's basis theorem, S is Noetherian. So any ideal of S is finitely generated. In particular, any monomial ideal $I \subset S$ is finitely generated and has a unique minimal set of generators which is denoted by $G(I)$.

There are some standard algebraic operations on ideals. The sum, products and intersection of monomial ideals are again monomial ideals. Moreover, if we have I and J are monomial ideals, then $G(I+J) \subset G(I) \cup G(J)$, $G(IJ) \subset G(I)G(J)$ and $I \cap J = (\{\mathrm{lcm}(u,v) : u \in G(I),\ v \in G(J)\})$, where $\mathrm{lcm}(u,v)$ is the least common multiple of u and v. The colon ideal $I : J$ of two monomial ideal is also a monomial ideal and

$$I : J = \bigcap_{v \in G(J)} I : (v).$$

Moreover, $\{u/\gcd(u,v) : u \in G(I)\}$ is a set of generators of $I : (v)$, where $\gcd(u,v)$ is greatest common divisor of the monomials u and v.

Example 1.2.1. Let $I = (x_1^2, x_1x_2^2, x_2x_3^2)$ and $J = (x_1^3x_2, x_2x_3)$ be monomial ideals in the polynomial ring $S = K[x_1, x_2, x_3]$ over the field K. Then

$$I + J = (x_1^2, x_1x_2^2, x_2x_3), \quad IJ = (x_1^5x_2, x_1^2x_2x_3, x_1^4x_2^3, x_1x_2^3x_3, x_1^3x_2^2x_3^2, x_2^2x_3^3),$$

$I \cap J = (x_1^3x_2, x_1^2x_2x_3, x_1x_2^2x_3, x_2x_3^2)$ and $I : J = (x_1^2, x_1x_2, x_3)$.

1.3 Primary decomposition of modules

Let $I \subset R$ be an ideal. The radical of I is

$$\mathrm{Rad}(I) = \{x \in R : x^n \in R \text{ for some } n > 0\}.$$

It is the intersection of all prime ideals containing I.

Definition 1.3.1. Let M be a module over a ring R, the set of *associated primes* of M, denoted by $\mathrm{Ass}_R(M)$, is the set of all prime ideals $P \subset R$ such that there is a monomorphism φ of R-modules:

$$R/P \overset{\varphi}{\hookrightarrow} M.$$

We note that $P = \mathrm{Ann}(\varphi(1))$.

Definition 1.3.2. Let M be an R-module. A submodule N of M is said to be a primary submodule if $\mathrm{Ass}(M/N) = \{P\}$. An ideal $Q \subset R$ is a primary ideal if $\mathrm{Ass}(R/Q) = \{P\}$. A submodule N of M is said to be irreducible if N cannot be written as an intersection of two submodules of M that properly contain N.

Proposition 1.3.3. *Let M be an R-module. If $N \neq M$ is an irreducible submodule of M, then N is a primary submodule.*

Definition 1.3.4. Let M be an R-module and N, $N_1, \ldots, N_r$ be submodules of M. A decomposition $N = N_1 \cap \ldots \cap N_r$ of N, is said to be irredundant if

$$N \neq N_1 \cap \ldots \cap N_{i-1} \cap N_{i+1} \ldots \cap N_r \quad \text{for all } i.$$

Proposition 1.3.5. ([12] or [34]) *If R is a Noetherian ring and I a proper ideal of R, then I has an irredundant primary decomposition*

$$I = Q_1 \cap \ldots \cap Q_r$$

such that Q_i is a P_i-primary ideal and $\mathrm{Ass}(R/I) = \{P_1, \ldots, P_r\}$.

Let M be R-module. An element $x \in R$ is a zero divisor of M if there is $0 \neq m \in M$ such that $xm = 0$. The set of zero divisors of M is denoted by $\mathrm{Z}(M)$.

Lemma 1.3.6. ([22] or [34]) *If M is an R-module, then*

$$\mathrm{Z}(M) = \bigcup_{P \in \mathrm{Ass}(M)} P.$$

Example 1.3.7. Let $I = (x_1^2x_3, x_1^2x_4, x_2^2, x_2x_3, x_2x_4)$ be a monomial ideal of the polynomial ring $S = K[x_1, x_2, x_3, x_4]$ over the field K. Then the primary decomposition of the ideal I is $(x_1^2, x_2) \cap (x_2^2, x_3, x_4)$. Let $M = S/I$ be a graded S-module. We obtain that $\mathrm{Ass}(M) = \{(x_1, x_2), (x_2, x_3, x_4)\}$. Hence $\mathrm{Z}(M) = (x_1, x_2) \cup (x_2, x_3, x_4)$.

1.4 Dimension theory

Let R be a commutative ring. We denote by $\operatorname{Spec} R$ the set of all prime ideals of R. Let $P \in \operatorname{Spec} R$. We define *height* of P as

$$\operatorname{height} P = \sup\{t \in \mathbb{N} : \ P_0 \subset P_1 \subset \cdots \subset P_t = P\}$$

where $P_i \in \operatorname{Spec} R$. The *height* of any ideal $I \subset R$ is

$$\operatorname{height} I = \inf\{\operatorname{height} P : P \in \operatorname{Spec} R, \ I \subset P\}.$$

Example 1.4.1. If $I = (x_1^2, x_1x_2^2, x_1x_3)$ is an ideal in the polynomial ring $S = K[x_1, x_2, x_3]$ over the field K. Then there are three prime ideals $P_1 = (x_1)$, $P_2 = (x_1, x_2)$ and $P_3 = (x_1, x_2, x_3)$ which contain the ideal I. We have the heights of P_1, P_2 and P_3 are 1 ,2 and 3, respectively. It follows that the height of the ideal I is 1.

Definition 1.4.2. The *(Krull) dimension* of a ring R is the supremum of the heights of its prime ideals.

Let M be a R-module. The annihilator of M is given by

$$\mathrm{Ann}_R(M) = \{x \in R : \ xM = 0\}.$$

If M is a finitely generated R-module then $\operatorname{Supp} M = \{P \in \operatorname{Spec} R : \ \operatorname{Ann} M \subset P\}$.

Definition 1.4.3. The *dimension* of the module M which is denoted by $\dim M$, is the supremum over the lengths t of strictly increasing chains

$$P_0 \subset P_1 \subset \cdots \subset P_t \quad \text{with} \quad P_i \in \operatorname{Supp} M,$$

that is, $\dim M = \dim(R/\operatorname{Ann} M)$.

Example 1.4.4. If we consider $M = S/I$, where $I = (x_1^2, x_1x_2^2, x_1x_3) \subset S = K[x_1, x_2, x_3]$ is the monomial ideal in the polynomial ring S over the field K. Then we obtain $\dim M = 2$.

Proposition 1.4.5. [34, Proposition 1.1.31] *If M is a finitely generated R-module, then*

$$\operatorname{Ass}(M) \subset \operatorname{Supp}(M)$$

and any minimal element of $\operatorname{Supp}(M)$ *is in* $\operatorname{Ass}(M)$.

Lemma 1.4.6. ([34]) *If* $0 \to U \to M \to N \to 0$ *is a short exact sequence of modules over a ring R, then*

$$\operatorname{Supp}(M) = \operatorname{Supp}(U) \cup \operatorname{Supp}(N).$$

Theorem 1.4.7. ([7]) *Let R be a graded ring.*

(a) *For every prime ideal P the ideal P^* is prime ideal.*

(b) *Let M be a graded R-module.*

  (i) *If $P \in \operatorname{Supp} M$, then $P^* \in \operatorname{Supp} M$.*

  (ii) *If $P \in \operatorname{Ass} M$, then P is graded; moreover P is the annihilator of a homogeneous element.*

Theorem 1.4.8. [7, Proposition A.4] *Let M be a finitely generated graded or multigraded R-module and $x_1, \ldots x_r$ belong to the graded maximal ideal of R. Then*

$$\dim(M/(x_1, \ldots x_r)M) \geq \dim M - r.$$

Proposition 1.4.9. [34, Proposition 1.1.32] *If*

$$0 \to U \to M \to N \to 0$$

is a short exact sequence of finitely generated R-modules, then

$$\dim(M) = \max\{\dim(U), \dim(N)\}.$$

1.5 Regular sequences and depth

Throughout this section R is a finitely generated graded (i.e. $\mathbb{N}$-graded) K-algebra with unique graded maximal ideal $\mathfrak{m} = \bigoplus_{i=1}^{\infty} R_i$ and all R-modules M are considered in this section, finitely generated graded. After dimension, depth is the most fundamental notion of commutative algebra. The depth is defined in terms of regular sequences.

Definition 1.5.1. An element $x \in \mathfrak{m}$ is an *M-regular* if $xz = 0$ for $z \in M$ implies $z = 0$. In other words, if x is nonzero divisor on M.

Examples 1.5.2. (a): Let K be a field. Let $S = K[x_1, x_2]$ be the polynomial ring in two variables over the field K and $M = S$. Since S is integral domain, so every monomial in $\operatorname{Mon}(S)$ is S-regular element.

(b): Let $S = K[x_1, x_2, x_3]$ be a polynomial ring, where K is a field. Let $M = S/(x_1^2x_2)$. Then x_3^3 is a M-regular element, since $x_3^3 \notin \mathrm{Z}(M)$.

Definition 1.5.3. A sequence $x = x_1, \ldots, x_r$ of homogeneous elements in the graded maximal ideal of R is called an *M-regular sequence* or simply an *M-sequence* if x_i is regular on $M/(x_1, \ldots, x_{i-1})M$ for all $1 \leq i \leq r$.

We say r is the *length* of the sequence x. If $M \neq 0$ a finitely generated R-module then by Nakayama's Lemma (see [5, Proposition 2.6]), we have $(x)M \neq M$.

Examples 1.5.4. (a): Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over the field K. Then $x = x_1, \ldots, x_n$ is a regular sequence on S.

(b): Let K be a field and $S = K[x_1, x_2, x_3, x_4]$ be the polynomial ring over the field K. Consider the module $M = S/I$, where $I = (x_1^2, x_2x_3) \subset S$ is a monomial ideal. Then $x = x_2 + x_3, x_4^2$ is a M-regular sequence, since $x_2 + x_3 \notin \mathrm{Z}(M) = (x_1, x_2) \cup (x_1, x_3)$ and $x_4^2 \notin \mathrm{Z}(M/(x_2 + x_3)M) = (x_1, x_2, x_3)$.

Proposition 1.5.5. ([7]) *Let R be a graded ring and M be a R-module. If x is a M-regular sequence, then an exact sequence*

$$N_2 \longrightarrow N_1 \longrightarrow N_0 \longrightarrow M \longrightarrow 0$$

of R-modules induces an exact sequence

$$N_2/(x)N_2 \longrightarrow N_1/(x)N_1 \longrightarrow N_0/(x)N_0 \longrightarrow M/(x)M \longrightarrow 0.$$

In general not any permutation of regular sequence in a ring is regular. But for a graded ring with graded maximal ideal $\mathfrak{m}$, we have the following result.

Proposition 1.5.6. ([7]) *Let R be a graded ring with graded maximal ideal $\mathfrak{m}$ and M be a R-module, and $x = x_1, \ldots, x_r$ an M-sequence. Then every permutation of x is an M-sequence too.*

The next result gives a criteria when an ideal I contains an M-regular element.

Proposition 1.5.7. ([7]) *Let R be a graded ring and $I \subset R$ be a graded ideal. Let M finitely generated graded R-module. Then I contains an M-regular element if and only if $\mathrm{Hom}_R(R/I, M) = (0)$.*

Let M be an R-module and $x = x_1, x_2, \ldots, x_n, \ldots$ be an M-sequence. Then we have the strictly ascending sequence of ideals

$$(x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, x_2, \ldots, x_n) \subset \cdots.$$

Since R is a Noetherian ring, this sequence must stop. Hence any M-sequence is finite. This leads to next definition.

Definition 1.5.8. A M-regular sequence $x = x_1, \ldots, x_n$ in an ideal $I \subset \mathfrak{m}$ is called *maximal M-sequence* if any element of I is a zero divisor on M/xM.

Theorem 1.5.9. *(*Rees [7]) Let R be a graded ring, M a finitely generated graded R-module with graded maximal ideal $\mathfrak{m}$, and $I \subset \mathfrak{m}$. Then all maximal M-sequences in I have the same length n given by

$$n = \min\{i : \operatorname{Ext}_R^i(R/I, M) \neq 0\}.$$

Definition 1.5.10. Let R be a graded ring with graded maximal ideal $\mathfrak{m}$, and $M \neq 0$ a finitely generated graded R-module. Then the common length of maximal regular sequence on M in the ideal $\mathfrak{m}$, is called the *depth of* M and it will be denoted by $\operatorname{depth}(M)$.

Example 1.5.11. In Example (a) of 1.5.4 $x = x_1, \ldots, x_n$ is a maximal regular sequence on $S = K[x_1, \ldots, x_n]$, since $S/(x) \cong K$. Hence $\operatorname{depth}(S) = n$. In Example ($b$) of 1.5.4 where $S = K[x_1, x_2, x_3, x_4]$ is the polynomial ring and $I = (x_1^2, x_2x_3) \subset S$ is monomial ideal, and $M = S/I$, the M-sequence $x = x_2 + x_3, x_4^2$ is a maximal M-sequence, since the graded maximal ideal (x_1, x_2, x_3, x_4) belong to the set $\operatorname{Ass}(M/xM)$.

The Theorem 1.5.9 has the following important consequence.

Theorem 1.5.12. ([7]) *Let R be a graded ring with graded maximal ideal $\mathfrak{m}$ and $k = R/\mathfrak{m}$ be a residue field, and $M \neq 0$ a finitely generated R-module. Then*

$$\operatorname{depth}(M) = \min\{i : \operatorname{Ext}_R^i(k, M) \neq 0\}.$$

Depth is a homological invariant while dimension is a geometrical invariant. The following result shows the relation between these.

Proposition 1.5.13. ([34]) *Let R be a graded ring with graded maximal ideal $\mathfrak{m}$ and $M \neq 0$ a finitely generated graded R-module. Then* $\operatorname{depth}(M) \leq \dim(R/P)$ *for all $P \in \operatorname{Ass}(M)$. In particular,* $\operatorname{depth}(M) \leq \dim(M)$.

Theorem 1.5.14. (Auslander-Buchsbaum [7]) *Let $S = K[x_1, \ldots, x_n]$ over the field K and M be a finitely generated graded or multigraded S-module. Then*

$$\operatorname{proj}\dim(M) + \operatorname{depth}(M) = n.$$

In particular, $\operatorname{proj}\dim(M) \leq n$.

The following "Depth Lemma" is well-known. It shows the behavior of depth along short exact sequence of modules.

Lemma 1.5.15. (Depth Lemma [34, Lemma 1.3.9]) *If*

$$0 \to U \to M \to N \to 0$$

is a short exact sequence of finitely generated graded modules over graded ring R, then

(a) *If* $\operatorname{depth} M < \operatorname{depth} N$, *then* $\operatorname{depth} U = \operatorname{depth} M$.

(b) *If* $\operatorname{depth} M = \operatorname{depth} N$, *then* $\operatorname{depth} U \geq \operatorname{depth} M$.

(c) *If* $\operatorname{depth} M > \operatorname{depth} N$, *then* $\operatorname{depth} U = \operatorname{depth} N + 1$.

Corollary 1.5.16. *Let*

$$0 \to U \to M \to N \to 0$$

be an exact sequence of finitely generated graded R-modules. Then

$$\operatorname{depth} M \geq \min\{\operatorname{depth} U, \operatorname{depth} N\}.$$

Lemma 1.5.17. ([34]) *If M is a graded module over the graded ring R and z is a homogeneous regular element on M in the graded maximal ideal, then*

(a) $\operatorname{depth}(M/zM) = \operatorname{depth}(M) - 1$,

(b) $\dim(M/zM) = \dim(M) - 1$.

Proof. **(a):** We have $\operatorname{depth}(M) > \operatorname{depth}(M/zM)$ and consider the short exact sequence

$$0 \to M \xrightarrow{z} M \to M/zM \to 0.$$

Applying the Depth Lemma to this short exact sequence, we obtain

$$\operatorname{depth}(M) = \operatorname{depth}(M/zM) + 1.$$

(b): We claim that $\dim(M/zM) < \dim(M)$. Indeed, otherwise we have an ascending chain of prime ideals

$$\operatorname{Ann}(M) \subset \operatorname{Ann}(M/zM) \subset P_0 \subset \cdots \subset P_d,$$

where d is dimension of M and P_0 is minimal over $\operatorname{Ann}(M)$. By Proposition 1.4.5 the ideal P_0 consists of zero divisors and $z \in \operatorname{Ann}(M/zM)$, hence $z \in P_0$, a contradiction. The reverse inequality $\dim(M/zM) \geq \dim(M) - 1$ follows from Theorem 1.4.8. ☐

Corollary 1.5.18. *Let $x = x_1, \ldots, x_r$ be a regular sequence on the graded R-module M in the graded maximal ideal of R. Then*

(a) $\operatorname{depth}(M/xM) = \operatorname{depth}(M) - r$*,*

(b) $\dim(M/xM) = \dim(M) - r$*.*

The proof of this Corollary follows by using induction on r.

Definition 1.5.19. Let M be a graded module over a graded ring R with graded maximal ideal $\mathfrak{m}$. Then we define the *socle of M* as $\operatorname{Soc} M = (0 :_M \mathfrak{m})$.

Definition 1.5.20. *Let R be a graded ring. A nonzero finitely generated graded R-module is called* Cohen-Macaulay *if* $\operatorname{depth}(M) = \dim(M)$*. If R is a Cohen-Macaulay R-module then R is called a Cohen-Macaulay ring.*

1.6 Koszul complex and Koszul homology

In this section, we recall the basic properties of Koszul homology that are used in this thesis.

Let R be any commutative ring with unit and $f = f_1, \ldots, f_r$ a sequence of elements of R. The *Koszul complex* $K(f; R)$ attached to the sequence f is defined as follows: let F be a free R-module with basis $e_1, \ldots, e_r$. Then we let $K_j(f; R)$ be the *jth* exterior power of F, that is, $K_j(f; R) = \bigwedge^j F$. A basis of the free R-module $K_j(f; R)$ is given by the wedge products $e_F = e_{i_1} \wedge \cdots \wedge e_{i_j}$, where $F = \{i_1 < i_2 \cdots < i_j\}$. In particular, it follows that rank $K_j(f; R) = \binom{r}{j}$. Thus we have Koszul complex $K(f; R)$ as follows

$$K(f; R):\ 0 \to K_r(f; R) \xrightarrow{\partial} K_{r-1}(f; R) \xrightarrow{\partial} \cdots \xrightarrow{\partial} K_1(f; R) \xrightarrow{\partial} K_0(f; R) \to 0$$

where the differentials $\partial : K_j(f; R) \to K_{j-1}(f; R)$ are defined by the formula

$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_j}) = \sum_{k=1}^{j} (-1)^{k+1} f_{i_k} e_{i_1} \wedge \cdots \wedge e_{k-1} \wedge e_{k+1} \wedge \cdots \wedge e_{i_j}.$$

Note that $K(f; R)$ is indeed a complex.

Example 1.6.1. Let $S = K[x_1, x_2]$ be the polynomial ring over the field K and $f = x_1^3, x_1^2x_2, x_2^2$ be a sequence of monomials in S. The Koszul complex associated to the sequence f is:

$$0 \to S(-8) \xrightarrow{\partial_3} S(-6) \oplus S(-5)^2 \xrightarrow{\partial_2} S(-3)^2 \oplus S(-2) \xrightarrow{\partial_1} S \to S/(f) \to 0$$

where the differentials are:

$\partial_1(e_i) = f_i$ for all $1 \le i \le 3$, that is

$$\partial_1 = \begin{pmatrix} x_1^3 & x_1^2x_2 & x_2^2 \end{pmatrix}.$$

$\partial_2(e_1 \wedge e_2) = x_1^3 e_2 - x_1^2x_2 e_1$, $\partial_2(e_2 \wedge e_3) = x_1^2x_2 e_3 - x_2^2 e_2$, $\partial_2(e_1 \wedge e_3) = x_1^3 e_3 - x_2^2 e_1$, that is

$$\begin{pmatrix} -x_1^2x_2 & 0 & -x_2^2 \\ x_1^3 & -x_2^2 & 0 \\ 0 & x_1^2x_2 & x_1^3 \end{pmatrix}$$

$\partial_3(e_1 \wedge e_2 \wedge e_3) = x_1^3(e_2 \wedge e_3) - x_1^2 x_2(e_1 \wedge e_3) + x_2^2(e_1 \wedge e_2)$, that is

$$\partial_3 = \begin{pmatrix} x_2^2 \\ x_1^3 \\ -x_1^2 x_2 \end{pmatrix}$$

Now let M be an R-module. We define the complex

$$K(f; M) := K(f; R) \otimes M.$$

We say $H_i(f; M) = H_i(K(f; M))$ is the *ith Koszul homology module of f* with respect to M.

Proposition 1.6.2. ([7]) *Let $I = (f)$ be the ideal of R generated by sequence $f = f_1, \dots, f_r$ of elements of R. Then*

$$H_0(f; M) = M/IM \quad \text{and} \quad H_r(f; M) = 0 :_M I = \{x \in M : Ix = 0\}.$$

The Koszul complex $K(f; R)$ is a graded R-algebra with multiplication wedge product.

We now consider two fundamental long exact sequences related to the Koszul homology which are very useful for computing the Koszul homology.

Theorem 1.6.3. ([7]) *Let $f = f_1, \dots, f_r$ be a sequence of elements in R, and denote by $g = f_1, \dots, f_{r-1}$. Furthermore let M be an R-module and $0 \to U \to M \to N \to 0$ a short exact sequence of R-modules. Then we get the following long exact sequences:*

$0 \to H_r(f; U) \to H_r(f; M) \to H_r(f; N) \to H_{r-1}(f; U) \to H_{r-1}(f; M) \to \cdots \to H_{i+1}(f; N) \to H_i(f; U) \to H_i(f; M) \to H_i(f; N) \to H_{i-1}(f; U) \to \cdots \to H_1(f; N) \to H_0(f; U) \to H_0(f; M) \to H_0(f; N) \to 0,$

and

$0 \to H_r(f; M) \to H_{r-1}(g; M) \to H_{r-1}(g; M) \to H_{r-1}(f; M) \to \cdots \to H_{i+1}(f; N) \to H_i(g; M) \to H_i(g; M) \to H_i(f; M) \to \cdots \to H_1(f; M) \to H_0(g; M) \to H_0(g; M) \to H_0(f; M) \to 0,$

where for all i, the map $H_i(g; M) \to H_i(g; M)$ is multiplication by $\pm f_r$.

Theorem 1.6.4. ([7]) *Let $f = f_1, \dots, f_r$ be a sequence of elements in R and M a R-module.*

(a) *If f is an M-sequence then $H_i(f; M) = 0$ for all $i > 0$.*

(b) *If R is a $\mathbb{N}$-graded K-algebra with maximal graded ideal $\mathfrak{m}$ and $(f) \subset \mathfrak{m}$ such that f is homogeneous sequence and M is finitely generated graded R-module, then we have: if $H_1(f; M) = 0$ then f is an M-sequence.*

1.7 Prime filtrations

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over the field K and M be a finitely generated graded or multigraded S-module. A basic fact in commutative algebra says that there exists a finite filtration

$$\mathcal{F}: \ 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

with cyclic quotients $M_i/M_{i-1} \cong S/P_i(-a_i)$ where $a_i \in \mathbb{Z}$ or $\mathbb{Z}^n$, respectively and P_i are graded or multigraded, respectively, prime ideals in the support of M. We call any such filtration of M a *prime filtration*. The set of prime ideals $\{P_1, \dots, P_r\}$ is denoted by $\mathrm{Supp}(\mathcal{F})$.

Example 1.7.1. Let $S = K[x_1, x_2, x_3, x_4]$ be a polynomial ring over the field K. Consider the ideal $I = (x_1x_2, x_3x_4) \subset S$, then a prime filtration of the module $M = S/I$ is the following:

$$\mathcal{F}: M_0 = (0) \subset M_1 = (x_1x_3)M \subset M_2 = M_1 + x_1M \subset M_3 = M_2 + x_3M \subset M_4 = M.$$

Note that $M_1 \cong S/(x_2, x_4)(-2) \cong K[x_1, x_3](-2)$, $M_2/M_1 \cong S/(x_2, x_3)(-1) \cong K[x_1, x_4](-1)$, $M_3/M_2 \cong S/(x_1, x_4)(-1) \cong K[x_2, x_3](-1)$, $M_4/M_3 \cong S/(x_1, x_3) \cong K[x_2, x_4]$.

For any prime filtration $\mathcal{F}$ one has $\mathrm{Ass}(M) \subset \mathrm{Supp}(\mathcal{F}) \subset \mathrm{Supp}(M)$. Let $\mathrm{Min}(M)$ denote the set of minimal prime ideals in $\mathrm{Supp}(M)$. Dress [11] calls a prime filtration

$\mathcal{F}$ of M *clean* if $\text{Supp}(\mathcal{F}) = \text{Min}(M)$. The S-module M is called clean if it admits a clean filtration. In Example 1.7.1, we note that $\mathcal{F}$ is a clean filtration of M, hence M is a clean module.

Herzog and Popescu introduced *pretty clean modules* as a generalization of the definition of clean modules.

Definition 1.7.2. A prime filtration $\mathcal{F}$

$$\mathcal{F}: \quad 0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

of M with $M_i/M_{i-1} \cong S/P_i(-a_i)$ is called *pretty clean*, if for all $i < j$ for which $P_i \subseteq P_j$, it follows that $P_i = P_j$. The module M is called *pretty clean*, if it has a pretty clean filtration.

We say that a monomial ideal $I \subset S$ is pretty clean if S/I is pretty clean. A prime filtration $\mathcal{F}$ which is pretty clean satisfies that $\text{Supp}(\mathcal{F}) = \text{Ass}(M)$, proved by Herzog and Popescu [15, Corollary 3.6]. Herzog, Vladoiu and Zheng (see [17]) introduced almost clean modules.

Definition 1.7.3. A module M is called *almost clean*, if it has a prime filtration $\mathcal{F}$ such that $\text{Supp}(\mathcal{F}) = \text{Ass}(M)$.

Thus we have the following implications:

$$\textit{clean} \Longrightarrow \textit{pretty clean} \Longrightarrow \textit{almost clean}.$$

Chapter 2

Stanley decompositions, pretty clean filtrations and reductions modulo regular elements

Let K be a field and $S = K[x_1, x_2, \ldots, x_n]$ be a polynomial ring in n variables over the field K. Let $I \subset S$ be a monomial ideal, and let $u \in S$ be a monomial such that u is regular on S/I. In this chapter, we discuss the Stanley decompositions and pretty clean filterations of $\mathbb{Z}^n$-graded modules of the form S/I. We investigate how the Stanley depth and the property of S/I to be pretty clean behaves when we pass from S/I to $S/(I, u)$, and vice versa.

2.1 Stanley decompositions of multigraded modules

Let K be a field and $S = K[x_1, \ldots, x_n]$ a polynomial ring in n variables over a field K. Let M be a finitely generated multigraded (i.e. $\mathbb{Z}^n$-graded) S-module. Let $m \in M$ be a homogeneous element in M and $Z \subseteq \{x_1, \ldots, x_n\}$. We denote by $mK[Z]$ the K-subspace of M generated by all elements mv where v is a monomial in $K[Z]$. The multigraded K-subspace $mK[Z] \subset M$ is called *Stanley space of dimension* $|Z|$, if

$mK[Z]$ is a free $K[Z]$-module. A *Stanley decomposition* of M is a presentation of the K-vector space M as a finite direct sum of Stanley spaces $\mathcal{D}:\ M = \bigoplus_{i=1}^{r} m_i K[Z_i]$. Set sdepth $\mathcal{D} = \min\{|Z_i| : i = 1, \ldots, r\}$. The number

$$\text{sdepth}(M) := \max\{\text{sdepth}(\mathcal{D}) :\ \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

is called *Stanley depth of M.*

Examples 2.1.1. (a): Consider the ideal $I = (x_1^2 x_2) \subset S = K[x_1, x_2]$. The following Figure 2.1 displays a Stanley decomposition of I and I^c. The gray area represents the ideal I. The Stanley decomposition of the ideal I is $I = x_1^2 x_2 K[x_1, x_2]$, hence sdepth$(I) = 2$ since S is polynomial ring in two variables and $I \subset S$ is a principal ideal. In the Figure 2.1, the fat lines represent Stanley spaces of dimension 1 and Stanley decomposition of the module S/I is

$$I^c = K[x_1] \oplus x_2 K[x_2] \oplus x_1 x_2 K[x_2].$$

This implies that $1 \leq \text{sdepth}(S/I) \leq 2$. If sdepth$(S/I) = 2$, then $I \cap I^c \neq \{0\}$, a contradiction. Hence sdepth$(S/I) = 1$.

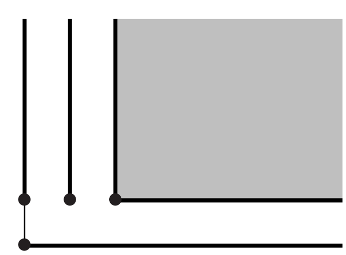

Figure 2.1:

(b): Let $I = (x_1x_2, x_1x_3)$ be a monomial ideal in the polynomial ring $S = K[x_1, x_2, x_3]$. Then a Stanley decomposition of S/I is $I^c = K[x_2, x_3] \oplus x_1 K[x_1]$. It follows that $1 \leq \text{sdepth}(S/I) \leq 3$. We obtain that sdepth$(S/I) = 1$, indeed we have one Stanley space $x_1 K[x_1]$ of dimension 1 and since x_1 with multiplication x_2 or x_3 belongs to I, so this Stanley space can not be contained in a Stanley space of dimension bigger than one. A Stanley decomposition of I is $I = x_1x_2K[x_1, x_2, x_3] \oplus x_1x_3K[x_1, x_3]$. Hence $\operatorname{sdepth}(I) = 2$, since I is not a principal ideal and we have $\operatorname{sdepth}(I) = 3$ if and only if I is a principal ideal.

2.2 Prime filtrations and Stanley decompositions

Let $I \subset S$ be a monomial ideal. Let

$$\mathcal{F}: \quad I = I_0 \subset I_1 \subset \cdots \subset I_r = S$$

be a $\mathbb{N}^n$-graded prime filtration of S/I with $I_i/I_{i-1} \cong S/P_{F_i}(-a_i)$ where $F_i \subset [n]$ and $P_{F_i} = (x_j : j \in F_i)$. It was proved in [15] that this prime filtration $\mathcal{F}$ of S/I induces the Stanley decomposition

$$S/I = \bigoplus_{i=1}^{r} u_i K[Z_{F_i^c}]$$

of S/I, where $Z_{F_i^c} = \{x_j : j \notin F_i\}$, and where $u_i = x^{a_i}$. For any finitely generated $\mathbb{Z}^n$-graded S-module, this procedure is generalized by Soleyman-Jahan [19].

Theorem 2.2.1. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. If*

$$(0) = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

is a prime filtration of M such that $M_i/M_{i-1} \cong S/P_{F_i}(-a_i)$, then

$$M \cong \bigoplus_{i=1}^{r} m_i K[Z_{F_i^c}]$$

is a Stanley decomposition of M where $m_i \in M_i$ is a homogeneous element of degree a_i such that $(M_{i-1} :_S m_i) = P_{F_i}$ and $Z_{F_i^c} = \{x_j : j \notin F_i\}$.

Example 2.2.2. Consider the Example 1.7.1 to find the Stanley decomposition. We have $I = (x_1x_2, x_3x_4) \subset S = K[x_1, x_2, x_3, x_4]$ monomial ideal of S, then a prime filtration of the module $M = S/I$ is the following:

$$\mathcal{F} : M_0 = (0) \subset M_1 = (x_1x_3)M \subset M_2 = M_1 + x_1M \subset M_3 = M_2 + x_3M \subset M_4 = M$$

where $M_1 \cong S/(x_2, x_4)(-2) \cong K[x_1, x_3](-2)$, $M_2/M_1 \cong S/(x_2, x_3)(-1) \cong K[x_1, x_4](-1)$, $M_3/M_2 \cong S/(x_1, x_4)(-1) \cong K[x_2, x_3](-1)$, $M_4/M_3 \cong S/(x_1, x_3) \cong K[x_2, x_4]$. So the Stanley decomposition of S/I arises from the filtration $\mathcal{F}$ is

$$S/I = x_1x_3K[x_1, x_3] \oplus x_1K[x_1, x_4] \oplus x_3K[x_2, x_3] \oplus K[x_2, x_4].$$

Thus corresponding to every prime filtration of the $\mathbb{Z}^n$-graded S-module there is a Stanley decomposition, but converse is not true. Not all Stanley decompositions arise from prime filtrations even the module is pretty clean. A counter example is given by firstly McLagan and Smith in [21]. Let $I = (x_1x_2x_3) \subset S = K[x_1, x_2, x_3]$ be the monomial ideal of S. Then

$$S/I = K \oplus x_1K[x_1, x_2] \oplus x_2K[x_2, x_3] \oplus x_3K[x_1, x_3]$$

is the Stanley decomposition of S/I which is not corresponding to any prime filtration of S/I. We note that S/I is pretty clean by [18, Theorem 1.10].

The next result shows the existence of Stanley decompositions of multigraded modules (see [17]).

Lemma 2.2.3. *Every finitely generated $\mathbb{Z}^n$-graded S-module M admits a Stanley decomposition.*

The proof of this lemma follows from the fact that for every module there is a prime filtration and every prime filtration induces a Stanley decomposition of the module.

2.3 Stanley's conjecture

In 1982, Richard P. Stanley [31, Conjecture 5.1] conjectured that

$$\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$$

for all finitely generated $\mathbb{Z}^n$-graded S-modules M. The conjecture is discussed in some special cases in [16], [2], [17], [1], [25] [8], [9], [23] and [26].

Definition 2.3.1. A monomial ideal I is called *Stanley ideal* if the module S/I satisfies Stanley's conjecture.

We know an upper bound for depth of the module M in Proposition 1.5.13, that is $\operatorname{depth}(M) \leq \dim(R/P)$ for all $P \in \operatorname{Ass}(M)$. Stanley depth behaves similar to ordinary depth with respect to this bound. The following result proved by Apel [3] in the case when $M = S/I$, where $I \subset S$ is a monomial ideal.

Proposition 2.3.2. (Soleyman-Jahan [19]) *If* $\dim_K M_a \leq 1$ *for all* $a \in \mathbb{Z}^n$. *Then*

$$\operatorname{sdepth}(M) \leq \min\{\dim S/P : P \in \operatorname{Ass}(M)\},$$

Let R be finitely generated $\mathbb{N}$-graded K-algebra where K is a field, and let M be a finitely generated graded R-module. The Hilbert series of M is defined to be $\operatorname{Hilb}(M) = \sum_{i\in\mathbb{Z}}(\dim_K M_i)t^i$. If $\dim(M) = d$, then there exists a $Q_M(t) \in \mathrm{Z}[t, t^{-1}]$ such that

$$\operatorname{Hilb}(M) = Q_M(t)/(1-t)^d$$

and $Q_M(1) \neq 0$. The multiplicity of M is $Q_M(1)$, denoted by $e(M)$.

The number of Stanley spaces of a given dimension in a Stanley decomposition may depend on this particular decomposition. For example, if $I = (x_1^2x_2) \subset S = K[x_1, x_2]$, then corresponding to every integer $k > 0$ and $l > 0$ we have the Stanley decomposition

$$S/I = x_1^l K[x_1] \oplus x_2^k K[x_2] \oplus (\oplus_{i=0}^{l-1} x_1^i K) \oplus (\oplus_{j=1}^{k-1} x_2^j K) \oplus x_1x_2K[x_2]$$

of S/I with as many Stanley spaces of dimension zero as we want, however only three Stanley spaces of dimension one in any Stanley decomposition. Indeed we have the following:

Proposition 2.3.3. (Soleyman-Jahan [18]) *Let M be a $\mathbb{Z}^n$-graded S-module of dimension d. Then the number of Stanley spaces of maximal dimension d is independent of the special Stanley decomposition of M. In fact, this number is equal to the multiplicity, $e(M)$, of M.*

Corollary 2.3.4. *Let $I \subset S$ be a monomial ideal such that S/I is Cohen-Macaulay. Then the following conditions are equivalent:*

(a) *Stanley's conjecture is true for S/I.*

(b) *There exists a Stanley decomposition $\mathcal{D}$ of S/I such that each Stanley space in $\mathcal{D}$ has dimension $d = \dim S/I$.*

(c) *There exists a Stanley decomposition $\mathcal{D}$ of S/I which has $e(S/I)$ summands.*

Theorem 2.3.5. *Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal. Then in the following cases Stanley's conjecture holds for S/I.*

(a) *If I is a monomial ideal of height $\geq n-1$* (Soleyman-Jahan [18])*;*

(b) *If $n \leq 3$* ([3, Theorem 4])*;*

(c) *If $n = 4$* (Anwar and Popescu, [2])*;*

(d) *If $n = 5$* (Popescu, [25])*;*

(e) *If S/I is Cohen-Macaulay of dimension $n-2$ then S/I has a pretty clean filtration and I is a Stanley ideal* (Herzog Soleyman and Yassemi, [16]).

Proposition 2.3.6. (Soleyman-Jahan [18]) *Let M be a $\mathbb{Z}^n$-graded S-module, and $\mathcal{F}$ is a prime filtration of M with $\operatorname{Supp}(\mathcal{F}) = \operatorname{Ass}(M)$. Then the Stanley decomposition of M which is induced by the prime filtration $\mathcal{F}$ satisfies Stanley's conjecture.*

Stanley depth of some multigraded S-module are known.

Theorem 2.3.7. (Cimpoeas [10]) *If M is a $\mathbb{Z}^n$-graded S-module with $\dim_K M_a \leq 1$ for all $a \in \mathbb{Z}^n$, then $\operatorname{sdepth}(M) = 0$ if and only if $\operatorname{depth}(M) = 0$.*

Theorem 2.3.8. ([6], Theorem 2.2) *Let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the maximal ideal in S. Then $\operatorname{sdepth}(\mathfrak{m}) = \lceil \frac{n}{2} \rceil$.*

Herzog, Vladoiu and Zheng computed the Stanley depth of complete intersection monomial ideal generated by three elements.

Proposition 2.3.9. [17, Proposition 3.8]*Let I be a complete intersection monomial ideal minimally generated by three elements. Then* $\operatorname{sdepth}(I) = n - 1$.

Theorem 2.3.10. [30, Theorem 2.3] *Let $I \subset S$ be a complete intersection monomial ideal minimally generated by m elements. Then* $\operatorname{sdepth}(I) = n - \lfloor \frac{m}{2} \rfloor$

Theorem 2.3.11. [24, Theorem 2.3] *For a monomial ideal $I \subset S$ with $|G(I)| = m$. Then*

$$\operatorname{sdepth}(I) \geq \max\{1, n - \lfloor \frac{m}{2} \rfloor\}.$$

2.4 Stanley decompositions and regular elements

The aim of this section is to show that the Stanley depth behaves like the ordinary depth (see 1.5.17) with respect to reduction modulo regular elements in the case of $M = S/I$. Indeed we have the following result:

Theorem 2.4.1. *Let $I \subset S$ be a monomial ideal of $S = K[x_1, \ldots, x_n]$ and $u \in S$ be a monomial regular on S/I. Then* $\operatorname{sdepth}(S/(I, u)) = \operatorname{sdepth}(S/I) - 1$. *In particular, I is a Stanley ideal if and only if (I, u) is a Stanley ideal.*

We first prove a special case of the theorem:

Lemma 2.4.2. *Let $m < n$ and $J \subset S' = K[x_1, \ldots, x_m]$ be a monomial ideal. Then for the monomial ideal $I = JS$ and for any x_k with $m < k \leq n$ we have*

$$\operatorname{sdepth}(S/(I, x_k)) = \operatorname{sdepth}(S/I) - 1.$$

Proof. Let $T = S'[x_{m+1}, \ldots, x_{k-1}, x_{k+1} \ldots, x_n]$ and $L \subset T$ be the monomial ideal such that $L = JT$. Then we have $S/(I, x_k) = T/L$. Let

$$\mathcal{D}: \ T/L = \bigoplus_{i=1}^{r} u_i K[Z_i]$$

be a Stanley decomposition of T/L such that sdepth $(\mathcal{D}) = \text{sdepth}(T/L)$. Then

$$\mathcal{D}_1 : S/I = (T/L)[x_k] = \bigoplus_{i=1}^{r} u_i K[Z_i][x_k] = \bigoplus_{i=1}^{r} u_i K[Z_i, x_k].$$

is a Stanley decomposition of S/I. It follows that

$$\text{sdepth}(\mathcal{D}_1) = \text{sdepth}(\mathcal{D}) + 1 = \text{sdepth}(T/L) + 1$$

and

$$\text{sdepth}(\mathcal{D}_1) \leq \text{sdepth}(S/I).$$

Hence

$$\text{sdepth}(T/L) + 1 \leq \text{sdepth}(S/I).$$

In order to prove the opposite inequality we consider a Stanley decomposition

$$\mathcal{D}_2 : \ S/I = \bigoplus_{i=1}^{s} v_i K[W_i]$$

of S/I with $\text{sdepth}(\mathcal{D}_2) = \text{sdepth}(S/I)$.

Let $\mathcal{I} = \{i \in [s] : \ v_i K[W_i] \cap T \neq \{0\}\}$. We claim that

$$\mathcal{D}_3 : \ T/L = L^c = \bigoplus_{i \in \mathcal{I}} v_i K[W_i] \cap T. \tag{2.1}$$

and $\bigoplus_{i \in \mathcal{I}} v_i K[W_i] \cap T$ is a direct sum decomposition of T/L.

In order to prove (2.1), choose a monomial $v \in L^c$. We want to show that there exists $i \in \mathcal{I}$ such that $v \in v_i K[W_i] \cap T$. Suppose on the contrary that $v \notin v_i K[W_i] \cap T$ for all $i \in \mathcal{I}$. Since $v \in T$, it implies that $v \notin v_i K[W_i]$, for all i. Hence we have $v \in I = JS$. Since $v \in T$ and $L = JT$, it follows that $v \in L$, a contradiction. Conversely, choose a monomial $w \in v_i K[W_i] \cap T$. This implies that $w \notin I = JS$ and since $L = JT \subset JS = I$, we see that $w \in L^c$.

Now we will show that $\mathcal{D}_3$ is a Stanley decomposition. Indeed, we have

$$v_i K[W_i] \cap T = \begin{cases} v_i K[W_i \setminus \{x_k\}], & \text{if } x_k \text{ does not divide } v_i \\ 0, & \text{otherwise.} \end{cases}$$

Comparing the Stanley decomposition $\mathcal{D}_2$ of S/I with the Stanley decomposition $\mathcal{D}_3$ of T/L we see that $\operatorname{sdepth}(\mathcal{D}_2) \leq \operatorname{sdepth}(\mathcal{D}_3) + 1$. Hence

$$\operatorname{sdepth}(S/I) = \operatorname{sdepth}(\mathcal{D}_2) \leq \operatorname{sdepth}(\mathcal{D}_3) + 1 \leq \operatorname{sdepth}(T/L) + 1.$$

□

For the proof of Theorem 2.4.1 we also need the following simple fact:

Lemma 2.4.3. *Let*

$$I = I_0 \subset I_1 \subset \ldots \subset I_r = S$$

be an ascending chain of monomial ideals of S such that each I_j/I_{j-1} is a cyclic module, and hence $I_j/I_{j-1} \cong S/L_j(-a_j)$ for some monomial ideal L_j and some $a_j \in \mathbb{Z}^n$. Then

$$\operatorname{sdepth}(S/I) \geq \min\{\operatorname{sdepth}(S/L_j) : \ j \in \{1, \ldots, r\}\}$$

Proof. We have the following decomposition of S/I as a K-vector space:

$$S/I = I_1/I_0 \oplus I_2/I_1 \oplus \cdots \oplus S/I_{r-1}.$$

Since each $I_j/I_{j-1} \cong S/L_j(-a_j)$ we get the isomorphism

$$S/I \cong S/L_1(-a_1) \oplus S/L_2(-a_2) \oplus \cdots \oplus S/L_r(-a_r). \tag{2.2}$$

For each j let $\mathcal{D}_j : S/L_j = \bigoplus_{k=1}^{r_j} u_{jk} K[Z_{jk}]$ be a Stanley decomposition of S/L_j such that $\operatorname{sdepth}(\mathcal{D}_j) = \operatorname{sdepth}(S/L_j)$. Then by the isomorphism (2.2) we obtain the following Stanley decomposition

$$S/I = \bigoplus_{j=1}^{r} \bigoplus_{k=1}^{r_j} u_j u_{jk} K[Z_{jk}],$$

of S/I, where $u_j = x^{a_j}$ for $j = 1, \ldots, r$. From this Stanley decomposition of S/I the desired inequality follows. □

Proof of Theorem 2.4.1. Without loss of generality we may assume that $I = JS$ where $J \subset S' = K[x_1, \dots, x_m]$ and that $u = x_{m+1}^{a_1} \dots x_n^{a_{n-m}}$. We consider an ascending chain of ideals of S between (I, u) and S where two successive members of the chain are of the form

$$(I, x_{m+1}^{b_1} \cdots x_k^{b_k} \cdots x_n^{b_{n-m}}) \subset (I, x_{m+1}^{b_1} \cdots x_k^{b_k-1} \cdots x_n^{b_{n-m}})$$

and where $b_i \leq a_i$ for all $i = 1, \dots, n-m$.

Observe that

$$(I, x_{m+1}^{b_1} \cdots x_k^{b_{k-1}} \cdots x_n^{b_{n-m}})/(I, x_{m+1}^{b_1} \cdots x_k^{b_k} \cdots x_n^{b_{n-m}}) \simeq S/(I, x_k).$$

Therefore Lemma 2.4.2 and Lemma 3.2.2 imply that

$$\operatorname{sdepth}(S/(I, u)) \geq \operatorname{sdepth}(S/(I, x_k)) = \operatorname{sdepth}(S/I) - 1.$$

In order to prove other inequality, we choose a Stanley decomposition

$$\mathcal{D}' : (I, u)^c = \bigoplus_{i=1}^{r} u_i K[Z_i']$$

of $S/(I, u)$ with $\operatorname{sdepth}(\mathcal{D}') = \operatorname{sdepth}(S/(I, u))$. We obtain a direct sum of K-vector subspaces $\bigoplus_{i=1}^{r} u_i K[Z_i'] \cap S'$ of S'. We observe that

$$J^c = \bigoplus_{i=1}^{r} u_i K[Z_i'] \cap S'$$

and that $\bigoplus_i u_i K[Z_i'] \cap S'$ is a Stanley decomposition of S'/J, where the sum is taken over those $i \in \{1, \dots, r\}$ for which $u_i K[Z_i'] \cap S' \neq \{0\}$, cf. proof of Lemma 2.4.2.

We have

$$u_i K[Z_i'] \cap S' = \begin{cases} u_i K[Z_i' \cap \{x_1, \dots, x_m\}], & \text{if } \operatorname{supp}(u_i) \subset \{x_1, \dots, x_m\} \\ 0, & \text{otherwise.} \end{cases}$$

Hence if we set $\Lambda = \{i : \operatorname{supp}(u_i) \subset \{x_1, \dots, x_m\}\}$, then

$$\mathcal{D} : S/I = \bigoplus_{i \in \Lambda} u_i K[Z_i]$$

is a Stanley decomposition of S/I, where $Z_i := \{Z'_i \cap \{x_1, \ldots, x_m\}\} \cup \{x_{m+1}, \ldots, x_n\}$.

We claim that $|Z_i| > |Z'_i|$. Indeed, otherwise $\{x_{m+1}, \ldots, x_n\} \subset Z'_i$, contradicting the fact that $(u) \cap u_i K[Z'_i] = \{0\}$. Therefore, $\operatorname{sdepth}(\mathcal{D}) \geq \operatorname{sdepth}(\mathcal{D}') + 1$.

Hence or final conclusion is that

$$\operatorname{sdepth}(S/(I, u)) = \operatorname{sdepth}(S/I) - 1.$$

□

As an immediate consequence of our theorem we obtain the following result first proved in [16, Proposition 1.2].

Corollary 2.4.4. *Let I be a monomial ideal generated by regular sequence of monomials. Then I is a Stanley ideal.*

Corollary 2.4.5. *If $u_1, \ldots, u_r \in Mon(S)$ is S/I-sequence, then*

$$\operatorname{sdepth}(S/(u_1, \ldots, u_r) + I) = \operatorname{sdepth}(S/I) - r.$$

2.5 Pretty clean filtrations and regular elements

In this section we observe that the property of being pretty clean is preserved under reduction modulo regular elements.

Theorem 2.5.1. *Let $S = K[x_1, x_2, ..., x_n]$ be a polynomial ring and $I \subset S$ be a monomial ideal and u a monomial in S such that u is regular on S/I. Then S/I is pretty clean if and only if $S/(I, u)$ is pretty clean.*

Proof. Suppose S/I is pretty clean and let

$$\mathcal{F}: \; I = I_0 \subset I_1 \subset \ldots \subset I_r = S$$

be a pretty clean filtration of S/I with $I_j/I_{j-1} \cong S/P_j$ for $j = 1, 2, ..., r$. It is known from [15, Corollary 3.6] that $\operatorname{Ass}(S/I) = \{P_1, \ldots, P_r\}$.

We have $I_j = (I_{j-1}, z_j)$ where z_j is a monomial in S. The prime filtration $\mathcal{F}$ induces the following filtration

$$\mathcal{G}: \ (I, u) \subset (I_1, u) \subset \ldots \subset (I_r, u) = S,$$

where

$$(I_j, u)/(I_{j-1}, u) = ((I_{j-1}, u), z_j)/(I_{j-1}, u) \cong S/(I_{j-1}, u) : z_j.$$

Since u is regular on S/I, it follows that u is regular on S/I_j for all j. Indeed, since S/I is pretty clean it follows that S/I_j is pretty clean. Hence $\text{Ass}(S/I_j) = \{P_{j+1}, \ldots, P_r\}$ which is contained in $\text{Ass}(S/I)$. Since $\gcd(u, z_j) = 1$ it follows that

$$(I_{j-1}, u) : z_j = ((I_{j-1} : z_j), u) = (P_j, u).$$

Hence

$$(I_j, u)/(I_{j-1}, u) \cong S/(P_j, u).$$

Suppose, without loss of generality, that

$$P_j = (x_1, ..., x_t) \quad \text{and} \quad u = \prod_{i=t+1}^{n} {x_i}^{a_i}.$$

Then $S/(P_j, u) \cong K[x_{t+1}, ..., x_n]/(u)K[x_{t+1}, ..., x_n]$, which is clean by [18]. Hence we see that $(I_j, u)/(I_{j-1}, u)$ is clean and

$$\text{Ass}((I_j, u)/(I_{j-1}, u)) = \{(P_j, x_i) : \ x_i \mid u\}.$$

Therefore our filtration $\mathcal{G}$ can be refined as follows

$$(I_{j-1}, u) = I_{j-1,0} \subset I_{j-1,1} \subset \ldots \subset I_{j-1,s_j} = (I_j, u)$$

where

$$I_{j-1,k}/I_{j-1,k-1} \cong S/P_{j-1,k}$$

with $P_{j-1,k} \in \{(P_j, x_i) : x_i \mid u\}$.

In the refined filtration of $\mathcal{G}$ if we have $I_{j,k} \subset I_{i,l}$, then either $j < i$ or $j = i$ and $k < l$. Suppose $j < i$ and $P_{j,k} \subset P_{i,l}$. We have $P_{j,k} = (P_{j+1}, x_r)$ for some r

and $P_{i,l} = (P_{i+1}, x_s)$ for some s. Since $u \notin \bigcup_{P \in \mathrm{Ass}(S/I)} P$ it follows that $x_s \notin P_{j+1}$. Therefore, $P_{j+1} \subseteq P_{i+1}$. However, since $\mathcal{F}$ is a pretty clean filtration it follows that $P_{j+1} = P_{i+1}$, and hence $P_{j,k} = P_{i,l}$.

Next suppose that $i = j$ and $k < l$ and suppose that $P_{i,k} \subseteq P_{i,l}$. Since $\mathrm{height}\, P_{i,k} = \mathrm{height}\, P_{i,l}$ we conclude that $P_{j,k} = P_{i,l}$, also in this case. Thus we have shown that the refinement of $\mathcal{G}$ is a pretty clean filtration of $S/(I,u)$, and hence $S/(I,u)$ is pretty clean.

Conversely, suppose that $S/(I,u)$ is pretty clean. Since u is regular on S/I, we may suppose that $I = JS$ where $J \subset S' = K[x_1, \ldots, x_m]$ for $m < n$ and $\mathrm{supp}(u) \subset \{x_{m+1}, \ldots, x_n\}$. Since $S/(I,u)$ is pretty clean there exist a pretty clean filtration

$$\mathcal{M}:\ (I,u) = I'_0 \subset I'_1 \subset \ldots \subset I'_r = S$$

such that $I'_j/I'_{j-1} \cong S/P_j$ where $P_j \in \mathrm{Ass}(S/(I,u))$. Recall that

$$\mathrm{Ass}(S/(I,u)) = \{(P', x_k) \colon P' \in \mathrm{Ass}(S'/J) \text{ and } x_k \mid u\}.$$

By taking the intersection of above filtration $\mathcal{M}$ with S', we get the filtration

$$\mathcal{N}:\ J_0 \subseteq J_1 \subseteq \ldots \subseteq J_r = S'$$

of S'/J_0 where $J_j = I'_j \cap S'$ for $j = 0, \ldots, r$. We claim that $J_0 = J$. Let I be generated by the monomials $u_1, \ldots, u_l$. Since $I = JS$ with $J \subset S'$ it follows that $u_i \in S'$ for all i. Choose a monomial $v \in J_0 = (I,u) \cap S'$. Then either $v = eu_i$ where $e \in S'$, or $v = fu$ where $f \in S'$. The second case cannot happen since $v \in S'$. This shows that $J_0 \subset J$. The other inclusion is obvious.

Take an ideal $I'_j \in \mathcal{M}$. Then $I'_j = (I'_{j-1}, w_j)$ where $w_j \in S$ and $(I'_{j-1} : w_j) = (P', x_k)$ for some $P' \in \mathrm{Ass}(S'/J)$ and some x_k such that $x_k \mid u$. Then we have $I'_{j-1} \cap S' = I'_j \cap S'$ if and only if $w_j \notin S'$.

Let $\{r_0, \ldots, r_k\}$ be the subset of $[r]$ for which we have J_{r_i} is properly contained in J_{r_i+1} in the filtration $\mathcal{N}$. Set $L_i = J_{r_i}$ for $i = 0, \ldots, k$ and $L_{k+1} = S'$. Then we obtain the filtration

$$\mathcal{L}:\ J = L_0 \subset L_1 \subset \ldots \subset L_{k+1} = S'.$$

We note that $L_i = (J, w_{r_0+1}, w_{r_1+1}, \ldots, w_{r_{i-1}+1})$ for $i = 0, \ldots, k+1$ with $w_{r_i+1} \in S'$ for all i.

Since $L_i = (L_{i-1}, w_{r_{i-1}+1})$, we have that $L_i/L_{i-1} \cong S'/(L_{i-1} :_{S'} w_{r_{i-1}+1})$ and also we have that $L_i = I'_{r_i} \cap S'$. So $(L_{i-1} :_{S'} w_{r_{i-1}+1}) = (I'_{r_{i-1}} \cap S' :_{S'} w_{r_{i-1}+1})$.

We claim that $(I'_{r_{i-1}} \cap S' :_{S'} w_{r_{i-1}+1}) = (I'_{r_{i-1}} :_S w_{r_{i-1}+1}) \cap S'$. In fact, the inclusion $(I'_{r_{i-1}} \cap S' :_{S'} w_{r_{i-1}+1}) \subset (I'_{r_{i-1}} :_S w_{r_{i-1}+1}) \cap S'$ is obvious. In order to prove the other inclusion we choose a monomial $v \in (I'_{r_{i-1}} :_S w_{r_{i-1}+1}) \cap S'$. Then we have that $v \in (I'_{r_{i-1}} :_S w_{r_{i-1}+1})$ and $v \in S'$. Hence $vw_{r_{i-1}+1} \in I'_{r_{i-1}}$ and $vw_{r_{i-1}+1} \in S'$, since $w_{r_{i-1}+1} \in S'$. Therefore $vw_{r_{i-1}+1} \in I'_{r_{i-1}} \cap S'$ which implies that $v \in (I'_{r_{i-1}} \cap S' :_{S'} w_{r_{i-1}+1})$, as desired.

Now we see that

$$\begin{aligned} (L_{i-1} :_{S'} w_{r_{i-1}+1}) &= (I'_{r_{i-1}} \cap S' :_{S'} w_{r_{i-1}+1}) \\ &= (I'_{r_{i-1}} :_S w_{r_{i-1}+1}) \cap S' = (P', x_k) \cap S' = P', \end{aligned}$$

where $(P', x_k) \in \mathrm{Ass}(S/(I, u))$.

This shows that $\mathcal{L}$ is a prime filtration with the property that the prime ideals in $\mathrm{Supp}(\mathcal{L})$ form a subsequence of $P_1, \ldots, P_r$. Therefore, since $\mathcal{M}$ is a pretty clean filtration, the filtration $\mathcal{L}$ is pretty clean as well. From this fact we will deduce that S/I is pretty clean. This then will complete the proof of the theorem.

Indeed, our filtration $\mathcal{L}$ induce the filtration

$$\mathcal{K} : I = JS = L_0 S \subset L_1 S \subset \cdots \subset L_{k+1} S = S.$$

with $L_i S/L_{i-1} S \cong S/P'S$ where $L_i/L_{i-1} \cong S'/P'$ for $i = 1, \ldots, k+1$. This holds because the extension $S' \to S$ is flat. Now, since $\mathcal{L}$ is a pretty clean filtration of S'/J, it is obvious that $\mathcal{K}$ is a pretty clean filtration of S/I. $\square$

As an immediate consequence we obtain the following result from [16, Proposition 1.2].

Corollary 2.5.2. *Let $u_1, \ldots, u_k$ be a regular sequence in the polynomial ring S. Then $S/(u_1, \ldots, u_k)$ is pretty clean.*

Proof. We use induction on k. For $k = 1$ the assertion follows from Theorem 2.5.1 applied to $I = (0)$, or from [18]. By induction hypothesis we may now assume that $S/(u_1, \ldots, u_{k-1})$ is pretty clean. Since u_k is regular on $S/(u_1, \ldots, u_{k-1})$ it follows again from Theorem 2.5.1 that $S/(u_1, \ldots, u_k)$ is pretty clean. □

2.6 Behavior of depth, Stanley depth and dimension on algebra tensor products

Let K be a field. Let $I \subset S_1 = K[x_1, \ldots, x_n]$, $J \subset S_2 = K[y_1, \ldots, y_m]$ be monomial ideals and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Then $S_1/I \otimes_K S_2/J \cong S/(IS, JS)$.

We know that $\operatorname{depth}(S/(IS, JS)) = \operatorname{depth}(S_1/I) + \operatorname{depth}(S_2/J)$ (see [34, Theorem 2.2.21]). This equality is also preserved for (Krull) dimension, that is,

$$\dim(S/(IS, JS)) = \dim(S_1/I) + \dim(S_2/J),$$

(see [34]). A natural question arises how Stanley depth behaves on algebra tensor product.

Theorem 2.6.1. *Let $I \subset S_1 = K[x_1, \ldots, x_n]$, $J \subset S_2 = K[y_1, \ldots, y_m]$ be monomial ideals and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Then* $\operatorname{sdepth}(S_1/I) + \operatorname{sdepth}(S_2/J) \leq \operatorname{sdepth}(S/(IS, JS))$. *In particular, if Stanley's conjecture hold for the modules S_1/I and S_2/J it holds also for $S/(IS, JS)$.*

Proof. Let

$$\mathcal{D}_1 : \ S_1/I = \bigoplus_{i=1}^{r} u_i K[Z_i]$$

be a Stanley decomposition of S_1/I such that $\operatorname{sdepth}\mathcal{D}_1 = \operatorname{sdepth} S_1/I$ and

$$\mathcal{D}_2 : \ S_2/J = \bigoplus_{j=1}^{s} v_j K[W_j]$$

be a Stanley decomposition of S_2/J such that $\operatorname{sdepth}\mathcal{D}_2 = \operatorname{sdepth} S_2/J$. Then we have

$$\begin{aligned} S/IS &= S_1[y_1,\ldots,y_m]/IS \\ &= (S_1/I)[y_1,\ldots,y_m] \\ &= \bigoplus_{i=1}^{r} u_i K[Z_i][y_1,\ldots,y_m] \\ &= \bigoplus_{i=1}^{r} u_i K[Z_i,y_1,\ldots,y_m] \end{aligned}$$

and

$$\begin{aligned} S/JS &= S_2[x_1,\ldots,x_n]/JS \\ &= (S_2/J)[x_1,\ldots,x_n] \\ &= \bigoplus_{j=1}^{s} v_j K[W_j][x_1,\ldots,x_n] \\ &= \bigoplus_{j=1}^{s} v_j K[W_j,x_1,\ldots,x_n]. \end{aligned}$$

We claim that

$$S/(IS,JS) = \bigoplus_{i,j} u_i v_j K[Z_i,W_j].$$

Let $w \in (IS,JS)^c = S/(IS,JS)$ be a monomial; that is, $w \in S$ and $w \notin (IS,JS)$. We have $w \notin IS$ and $w \notin JS$. It follows that $w \in (IS)^c$ and $w \in (JS)^c$. Hence there exist i and j such that $w \in u_iK[Z_i,y_1,\ldots,y_m]$ and $w \in v_jK[W_j,x_1,\ldots,x_n]$. So we have $w \in u_iK[Z_i,y_1,\ldots,y_m] \cap v_jK[W_j,x_1,\ldots,x_n]$ and $u_iK[Z_i,y_1,\ldots,y_m] \cap v_jK[W_j,x_1,\ldots,x_n] = u_iv_jK[Z_i,W_j]$, since $u_i \in S_1$ and $v_j \in S_2$.

In order to prove the opposite inclusion, consider a monomial $v \in u_iv_jK[Z_i,W_j]$. Then $v \in u_iK[Z_i,y_1,\ldots,y_n] \subset (IS)^c$ and similarly $v \in (JS)^c$. Thus $v \in (IS,JS)^c$. So $S/(IS,JS) = \sum_{i,j} u_iv_jK[Z_i,W_j]$.

Now we prove that this sum is direct. Let $i_1,\ i_2 \in [r]$ and $j_1,\ j_2 \in [s]$ be such that $(i_1, j_1) \neq (i_2, j_2)$, let us say $i_1 \neq i_2$. Then $u_{i_1}v_{j_1}K[Z_{i_1}, W_{j_1}] \cap u_{i_2}v_{j_2}K[Z_{i_2}, W_{j_2}] \subset u_{i_1}K[Z_{i_1}, y_1, \ldots, y_m] \cap u_{i_2}K[Z_{i_2}, y_1, \ldots, y_m] = \{0\}$, which shows our claim. It follows that $\operatorname{sdepth}(S_1/I) + \operatorname{sdepth}(S_2/J) \leq \operatorname{sdepth}(S/(IS, JS))$. $\square$

The following example shows that the inequality in the above theorem can be strict.

Example 2.6.2. Let $S = K[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$ be a polynomial ring over the field K. Let $I = (x_1x_3, x_1x_4, x_2x_3, x_2x_4) \subset S_1 = K[x_1, x_2, x_3, x_4]$ be the ideal of the polynomial ring S_1 and $J = (x_5x_7, x_5x_8, x_6x_7, x_6x_8) \subset S_2 = K[x_5, x_6, x_7, x_8]$ be the ideal of the polynomial ring S_2. Consider the ideal $(IS, JS) \subset S$, then a Stanley decomposition $\mathcal{D}$ of $S/(IS, JS)$ is

$\mathcal{D}:\ S/(IS, JS) = K[x_1, x_2, x_5] \oplus x_3K[x_3, x_5, x_6] \oplus x_4K[x_4, x_5, x_6] \oplus x_6K[x_1, x_2, x_6] \oplus x_7K[x_1, x_2, x_7] \oplus x_8K[x_1, x_2, x_8] \oplus x_3x_4K[x_3, x_4, x_5] \oplus x_3x_7K[x_3, x_7, x_8] \oplus x_3x_8K[x_3, x_4, x_8] \oplus x_4x_7K[x_3, x_4, x_7] \oplus x_4x_8K[x_4, x_7, x_8] \oplus x_5x_6K[x_1, x_5, x_6] \oplus x_7x_8K[x_1, x_7, x_8]$
$\oplus\, x_2x_5x_6K[x_1, x_2, x_5, x_6] \oplus x_3x_4x_6K[x_3, x_4, x_5, x_6] \oplus x_2x_7x_8K[x_1, x_2, x_7, x_8]$
$\oplus\, x_3x_4x_7x_8K[x_3, x_4, x_7, x_8]$,

hence $\operatorname{sdepth}\mathcal{D} = 3$. Note that $\operatorname{sdepth}(S_1/I) = 1$ with a Stanley decomposition $S_1/I = K[x_1, x_2] \oplus x_3K[x_3] \oplus x_4K[x_3, x_4]$. We observe that $\operatorname{sdepth}(S_1/I)$ can not be greater than one. Similarly we have $\operatorname{sdepth}(S_2/J) = 1$. Hence we obtain that $\operatorname{sdepth}(S_1/I) + \operatorname{sdepth}(S_2/J) < \operatorname{sdepth}(S/(IS, JS))$.

Now we see an example in which the equality in the Theorem 2.6.1 holds.

Example 2.6.3. Let $S_1 = K[x_1, x_2, x_3]$ and $S_2 = K[y_1, y_2]$ be the polynomial ring over the field K. Let $I = (x_1x_2, x_1x_3) \subset S_1$ and $J = ({y_1}^2, y_1y_2) \subset S_2$ be ideals of S_1 and S_2, respectively. Since $\operatorname{depth}(S_1/I) \neq 0$, we get $\operatorname{sdepth}(S_1/I) \geq 1$. We have $\operatorname{sdepth}(S_1/I) \leq \min\{\dim(S_1/P) : P \in \operatorname{Ass}(S_1/I)\}$, where $\operatorname{Ass}(S_1/I) = \{(x_1), (x_2, x_3)\}$. Hence $\operatorname{sdepth}(S_1/I) = 1$. It is clear that $\operatorname{sdepth}(S_2/J) = 0$. Let $S = K[x_1, x_2, x_3, y_1, y_2]$ be the polynomial ring over the field K. Since $\operatorname{Ass}(S/(IS, JS)) =$

$\{(x_1, y_1), (x_1, y_1, y_2), (x_2, x_3, y_1), (x_2, x_3, y_1, y_2)\}$ and $\operatorname{sdepth}(S/(IS, JS)) \leq \min \{\dim S/P : P \in \operatorname{Ass}(S/(IS, JS)\}$, we obtain that $\operatorname{sdepth}(S/(IS, JS)) = 1$.

The following corollary is a particular case of Theorem 2.4.1.

Corollary 2.6.4. *Let $I \subset S$ be a monomial ideal and $u \in S$ is a monomial, which is regular on S/I. Then* $\operatorname{sdepth}(S/(I, u)) \geq \operatorname{sdepth}(S/I) - 1$.

Proof. Renumbering $x_i \in \operatorname{supp}(u_j)$ for all $u_j \in G(I)$, we may suppose that I is generated by a monomial ideal $J \subset S_1 = K[x_1, \ldots, x_r]$ and $u \in S_2 = K[x_{r+1}, \ldots, x_n]$ for some $1 < r < n$. Then $\operatorname{sdepth}(S/(I, u)) \geq \operatorname{sdepth}(S_1/J) + \operatorname{sdepth}(S_2/(u))$, by Theorem 2.6.1. Since $\operatorname{sdepth}(S_2/(u)) = n-r-1$ and $\operatorname{sdepth}(S/I) = \operatorname{sdepth}(S_1/J) + n - r$, by Lemma 2.4.2, it follows that $\operatorname{sdepth}(S/(I, u)) \geq \operatorname{sdepth}(S/I) - 1$. $\square$

Chapter 3

Depth, Stanley depth and dimension of multigraded modules

3.1 Behavior of depth, Stanley depth and dimension under reduction modulo an element

If we consider reduction by a regular element, then the depth and dim decreases by one. But what happens if we take reduction by a non-regular element? The behavior of dimension is under reduction modulo element is known (see Theorem 1.4.8).

Proposition 3.1.1. *Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, $I \subset S$ a monomial ideal and $R = S/I$. Then*

(a) $\operatorname{depth}(R/x_nR) \geq \operatorname{depth}(R) - 1$,

(b) $\operatorname{sdepth}(R/x_nR) \geq \operatorname{sdepth}(R) - 1$.

Proof. **(a):** Let $\bar{R} = R/x_nR \simeq S/(I, x_n)$. We denote $\bar{S} = K[x_1, \ldots, x_{n-1}]$ and let $x' = \{x_1, \ldots, x_{n-1}\}$, $x = \{x_1, \ldots, x_n\}$.

Let φ be the canonical map from R to $\bar{R}$ and α be the composite map

$$\bar{S} \longrightarrow S \longrightarrow R = S/I,$$

where the first map is the canonical embedding and the second map is the canonical surjection. It is clear that $\ker(\alpha) = I \cap \bar{S}$. Let α_1 be the composite map

$$\bar{S} \longrightarrow S \longrightarrow R = S/I \longrightarrow S/(I, x_n).$$

It is clear that α_1 is surjective. We claim that $\ker(\alpha_1) = I \cap \bar{S}$. One inclusion is obvious. To prove other inclusion, we consider a monomial $v \in \ker(\alpha_1)$, that is, $v \in (I, x_n)$. Since $v \in \bar{S}$ and I is a monomial ideal, it follows that $v \in I$. Let $\ker(\alpha_1) = \bar{I}$ then $\bar{S}/\bar{I} \simeq S/(I, x_n)$. It follows that the composition $\bar{R} \to R \to \bar{R}$ of the natural maps is the identity. Therefore, the S-module $\bar{R}$ is a direct summand of the S-module R. This implies that the S-module $H_i(x'; \bar{R})$ is a direct summand of $H_i(x'; R)$ for all i, where $H_i(x'; \bar{R})$ and $H_i(x'; R)$ are the i-th Koszul homology modules of x' with respect to $\bar{R}$ and R respectively. In particular, if $H_i(x'; \bar{R}) \neq 0$, then $H_i(x'; R) \neq 0$. Let $k = \max\{i \mid H_i(x' : \bar{R}) \neq 0\}$. Then $\operatorname{depth} \bar{R} = n - 1 - k$, by [7, Theorem 1.6.17]. Since $H_k(x'; \bar{R}) \neq 0$, it follows that $H_k(x'; R) \neq 0$ which implies that $H_k(x; R) \neq 0$ by [7, Lemma 1.6.18]. Therefore applying again [7, Theorem 1.6.17] it follows that $\operatorname{depth} R \leq n - k = \operatorname{depth} \bar{R} + 1$.

(b): Considering again $\bar{S}$, composition map α_1 and $\bar{I}$ in (a). We have a Stanley decomposition

$$\mathcal{D}_1 : \ \ I^c \cong R = \bigoplus_{i=1}^{r} u_i K[Z_i]$$

of R with $\operatorname{sdepth}(\mathcal{D}_1) = \operatorname{sdepth}(R)$.

Let $\mathcal{I} = \{i \in [r] : u_i K[Z_i] \cap \bar{S} \neq \{0\}\}$. We claim that

$$\mathcal{D}_2 : \bar{S}/\bar{I} = \bar{I}^c = \bigoplus_{i \in \mathcal{I}} u_i K[Z_i] \cap \bar{S} \qquad (3.1)$$

and $\bigoplus_{i \in \mathcal{I}} u_i K[Z_i] \cap \bar{S}$ is a direct sum decomposition of $\bar{S}/\bar{I}$. In order to prove (3.1), choose a monomial $v \in \bar{I}^c$. We want to show that there exists $i \in \mathcal{I}$ such that $v \in u_i K[Z_i] \cap \bar{S}$. Suppose on contrary that $v \notin u_i K[Z_i] \cap \bar{S}$, for all $i \in \mathcal{I}$. Since $v \in \bar{S}$, it implies that $v \notin u_i K[Z_i]$, for all i. Hence we have $v \in I$. Since $v \in \bar{S}$, it follows that $v \in \bar{I} = I \cap \bar{S}$, a contradiction. To prove the other inclusion, we choose a monomial $v_1 \in u_i K[Z_i] \cap \bar{S}$. This implies that $v_1 \notin I$ and since $\bar{I} \subset I$, we have $v_1 \in \bar{I}^c$. Now we will show that $\mathcal{D}_2$ is a Stanley decomposition. Indeed, we have

$$u_i K[Z_i] \cap \bar{S} = \begin{cases} u_i K[Z_i \setminus \{x_n\}], & \text{if } x_n \text{ does not divide } u_i \\ 0, & \text{otherwise.} \end{cases}$$

Thus we obtain that $\operatorname{sdepth}(\mathcal{D}_1) \leq \operatorname{sdepth}(\mathcal{D}_2) + 1$. Hence

$$\operatorname{sdepth}(S/I) = \operatorname{sdepth}(\mathcal{D}_1) \leq \operatorname{sdepth}(\mathcal{D}_2) + 1 \leq \operatorname{sdepth}(\bar{S}/\bar{I}) + 1.$$

□

Corollary 3.1.2. *Let $I \subset S$ be a monomial ideal. Then* $\operatorname{depth}(S/(I : u)) \geq \operatorname{depth}(S/I)$ *for all monomials $u \notin I$.*

Proof. Since $(I : uv) = ((I : u) : v)$, where I is a monomial ideal and u and v are monomials, we may reduce to the case $u = x_n$, and apply recurrence. Then we have the exact sequence

$$0 \longrightarrow S/(I : x_n) \longrightarrow S/I \longrightarrow S/(I, x_n) \longrightarrow 0.$$

By Depth Lemma [34, Lemma 1.3.9] and Proposition 3.1.1, we obtain the required result. □

This Corollary does not hold (and so Proposition 3.1.1) statement (a), if u is not a monomial, as we have the following example:

Example 3.1.3. Let $S = K[x, y, z, t]$ and $I = (x, y) \cap (y, z) \cap (z, t)$ and $u = y + z$. Then $J := (I : u) = (x, y) \cap (z, t)$ and $\operatorname{depth}(S/J) = 1 < 2 = \operatorname{depth}(S/I)$.

The Proposition 3.1.1 is not true in general. If M is a finitely generated graded R-module and $x \in R_1$ then we might have

$$\operatorname{depth}(M/xM) < \operatorname{depth}(M) - 1,$$

as shows the following example:

Example 3.1.4. Let $S = K[x,y,z,t]$, $M = (x,y,z)/(xt)$. We have $\operatorname{depth} M = 2$ and $M/xM = (x,y,z)/(x^2,xy,xz,xt)$. Since the maximal ideal is an associated prime ideal of M/xM, we get $\operatorname{depth}(M/xM) = 0$. Hence $\operatorname{depth}(M/xM) < \operatorname{depth}(M) - 1$.

In Proposition 3.1.1 we might have

$$\operatorname{depth}(R/x_nR) > \operatorname{depth}(R) - 1,$$

as shows the following example:

Example 3.1.5. Let $I = (x_1^2, x_1x_2, \ldots, x_1x_n) \subset S = K[x_1,\ldots,x_n]$ be a monomial ideal of S and $R = S/I$. Then $\operatorname{depth}(R) = 0$ since the maximal ideal $(x_1,x_2,\ldots,x_n) \in \operatorname{Ass}(R)$. Since $R/x_1R = S/(x_1) \simeq K[x_2,\ldots,x_n]$, we get $\operatorname{depth}(R/x_1R) = n-1$. Hence $\operatorname{depth}(R/x_1R) > \operatorname{depth}(R) - 1$.

This Proposition 3.1.1 statement (b) can not be extended to general multigraded modules M as shows the following example, where the variable is even regular on M.

Example 3.1.6. Let $M = (x,y,z)$ be an ideal of $S = K[x,y,z]$. Consider a Stanley decomposition $M = zK[x,z] \oplus xK[x,y] \oplus yK[y,z] \oplus xyzK[x,y,z]$. Since $\operatorname{sdepth}(M) \leq \dim(S) = 3$ and M is not a principle ideal, it follows $\operatorname{sdepth}(M) = 2$. Note that x induces a non-zero element in the socle of M/xM which cannot be contained in any Stanley space of dimension greater or equal with one. Hence $\operatorname{sdepth}(M/xM) = 0$. Thus $\operatorname{sdepth}(M/xM) < \operatorname{sdepth}(M) - 1$.

However for the special case when M is *almost clean* (see [17]), that is there exists a prime filtration $\mathcal{F}$ of M such that $\operatorname{Supp}(\mathcal{F}) = \operatorname{Ass}(M)$, we have the following:

Lemma 3.1.7. *Let M be a finitely generated multigraded S-module. If M is almost clean and $x_k \in S$ is regular on M, then*

$$\operatorname{sdepth}(M/x_kM) \geq \operatorname{sdepth}(M) - 1.$$

Proof. Suppose that $\mathcal{F}$ is given by

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

with $M_i/M_{i-1} \cong S/P_i(-a_i)$ for some $a_i \in \mathbb{N}^n$ and some monomial prime ideals P_i. Since $\operatorname{Ass} M = \{P_1, \ldots, P_r\}$ and x_k is regular on M, we get $x_k \notin P_i$ and so x_k is regular on M_i/M_{i-1}. Set $\bar{M}_i = M_i/x_kM_i$. Then $\bar{M}_i \subset \bar{M}_{i+1}$ and $\{\bar{M}_i\}$ define a filtration $\bar{\mathcal{F}}$ of $\bar{M} = \bar{M}_r$ with $\bar{M}_i/\bar{M}_{i-1} \cong S/(P_i, x_k)(-a_i)$. Thus $\operatorname{sdepth}(\bar{M}) \geq \min_i \dim(S/(P_i, x_k)) = \operatorname{sdepth}(M) - 1$ (see Corollary 3.2.2). □

The above Example 3.1.6 hints that if x is a regular element on M, then $\operatorname{sdepth}(M/xM) \leq \operatorname{sdepth}(M) - 1$. This is the subject of our next proposition.

Proposition 3.1.8. *Let M be finitely generated $\mathbb{Z}^n$-graded S-module and let x_k be regular on M. If $\mathcal{D}_1 : \ M/x_kM = \bigoplus_{i=1}^r \bar{m}_i K[Z_i]$, is a Stanley decomposition of M/x_kM, where $m_i \in M$ is homogeneous and $\bar{m}_i = m_i + x_kM$. Then*

$$M = \bigoplus_{i=1}^{r} m_i K[Z_i, x_k] \tag{3.2}$$

is a Stanley decomposition of M. In particular

$$\operatorname{sdepth}(M/x_kM) \leq \operatorname{sdepth}(M) - 1.$$

Proof. Let $N = \sum_{i=1}^r m_i K[Z_i, x_k]$. Then $N \subseteq M$. Since $\mathcal{D}_1$ is a Stanley decomposition of M/x_kM it follows that $\psi(N) = M/x_kM$ where $\psi : \ M \to M/x_kM$ is the canonical epimorphism. This implies that $M = x_kM + N$ as $\mathbb{Z}^n$-graded K vector spaces. We show that $M = N$. First we observe that $M = x_k^dM + N$ for all d. This follows by induction on d, because if we have $M = x_k^{d-1}M + N$, then $M = x_k^{d-1}(x_kM + N) + N = x_k^dM + x_k^{d-1}N + N = x_k^dM + N$ since $x_k^{d-1}N \subset N$. This completes the induction. Since M is finitely generated there exists an integer c such that $\deg_{x_k}(m) \geq c$ for all homogeneous elements $m \in M$. Now let $m \in M$ be a homogeneous element with $\deg_{x_k}(m) = a$ and let $d > a - c$ be an integer. Since $M = x_k^dM + N$, there exist homogeneous elements $v \in M$ and $w \in N$ such that

$m = x_k^d v + w$, where $a = \deg_{x_k} v + d = \deg_{x_k} w$. It follows that $\deg_{x_k} v = a - d < c$, a contradiction. It implies that $v = 0$, hence $m = w \in N$.

Now we show that the sum $\sum_{i=1}^r m_i K[Z_i, x_k]$ is direct, that is

$$m_i K[Z_i, x_k] \cap \sum_{\substack{j=1 \\ j\neq i}}^r m_j K[Z_j, x_k] = (0).$$

Let $u = m_i q_i = \sum_{\substack{j=1 \\ j\neq i}}^r m_j q_j \in M$ be homogeneous for some q_j monomials in $K[Z_j, x_k]$ such that $\deg(u) = \deg(m_j q_j)$ for all j. Let p be the biggest power of x_k dividing q_i. If $p = 0$, then we have $\bar{u} = \bar{m}_i q_i \neq 0$ in $M/x_k M$ since $\bar{m}_i K[Z_i]$ is a Stanley space. It follows that $\bar{u} \in \bar{m}_i K[Z_i] \cap \sum_{\substack{j=1 \\ j\neq i}}^r \bar{m}_j K[Z_j]$, a contradiction. In the case of $p > 0$, then in $M/x_k M$ we get $\bar{u} = 0 = \sum_{\substack{j=1 \\ j\neq i}}^r \bar{m}_j \bar{q}_j$. It follows that $\bar{q}_j = 0$, since $\mathcal{D}_1$ is a Stanley decomposition of $M/x_k M$. Thus $q_j = x_k q_j'$ for some $q_j' \in K[Z_j, x_k]$ and we get $x_k(m_i q_i' - \sum_{\substack{j=1 \\ j\neq i}}^r m_j q_j') = 0$, which implies $m_i q_i' - \sum_{\substack{j=1 \\ j\neq i}}^r m_j q_j' = 0$ since x_k is regular on M. Applying the same argument by recurrence we get $q_j = x_k^p s_j$ for some $s_j \in K[Z_j, x_k]$, and $m_i s_i = \sum_{\substack{j=1 \\ j\neq i}}^r m_j s_j$. We set $v = m_i s_i$. Since $\bar{s}_i \neq 0$, we get $\bar{v} \neq 0$ because $\bar{m}_i K[Z_i]$ is a Stanley space. On the other hand $\bar{v} \in \bar{m}_i K[Z_i] \cap \sum_{\substack{j=1 \\ j\neq i}}^r \bar{m}_j K[Z_j]$. It implies that $\bar{v} = 0$, a contradiction.

Finally we show that each $m_i K[Z_i, x_k]$ is a Stanley space. Indeed, suppose that $m_i f = 0$ for some $f \in K[Z_i, x_k]$ where $f = \sum_{j=0}^a f_j x_k^j$ such that x_k does not divide f_j for all j then $\sum_{j=0}^a m_i f_j x_k^j = 0$ implies that $\bar{m}_i f_0 = 0$ in $M/x_k M$. We get $f_0 = 0$ since $\bar{m}_i K[Z_i]$ is a Stanley space. It follows that $f = x_k g$ where $g = \sum_{j=1}^a f_j x_k^{j-1}$ and from $x_k m_i g = m_i f = 0$ we get $m_i g = 0$, x_k being regular on M. Then induction on the degree of f concludes the proof since $\deg_{x_k} g < \deg_{x_k} f$. □

Corollary 3.1.9. *If Stanley's conjecture holds for the module $M/x_i M$, where $x_i \in S$ is regular on M, then it also holds for M.*

Corollary 3.1.10. *The equality holds in Proposition 3.1.1 (b), if x_n is regular on S/I.*

The proof follows from Proposition 3.1.1 (b) and Proposition 3.1.8 for $M = S/I$.

Corollary 3.1.11. *Let* $\operatorname{depth}(M) = t$. *If there exists* $u = u_1, \ldots, u_t \in Mon(S)$ *such that* u *is regular sequence on* M *then Stanley's conjecture holds for* M.

Proof. For any regular sequence $u = u_1, \ldots, u_t \in Mon(S)$ we may choose u such that $u_i = x_{i_j}$ for all $1 \leq i \leq t$, where $x_{i_j} \in \operatorname{supp}(u_i)$, since for any monomial $u_i \in S$ being regular on M implies that each $x_{i_j} \in \operatorname{supp}(u_i)$ is regular on M, because if x_{i_j} belong to the set of zero divisors of M then $x_{i_j} \in P$ for some $P \in \operatorname{Ass}(M)$, so $u_i \in P$, which is not true as u_i is regular on M. Since u is a maximal regular sequence on M, we have $\operatorname{depth} M/(u_1, \ldots, u_t)M = 0$. Applying Proposition 3.1.8 by recurrence we get $\operatorname{sdepth}(M) \geq \operatorname{sdepth}(M/(u_1, \ldots, u_t)M) + t \geq t = \operatorname{depth}(M)$. Hence Stanley's conjecture holds for M. □

Example 3.1.12. Let $S = K[x, y, z, t]$ and $M = (x, y, z)/(xy)$. Since $\operatorname{depth} M = 2$ and $\{z, t\}$ is a M-regular sequence, we may apply Corollary 2.4.1 to see that Stanley's conjecture holds for M.

In general it is possible that the $\mathbb{Z}^n$-graded module M does not have a monomial regular sequence as shown by the following example

Example 3.1.13. Let $I = (xyz) \subset S = K[x, y, z]$ be the monomial ideal in the polynomial ring S. Consider the module $M = S/I$, then no monomial is regular on M, since $\operatorname{Mon}(S) \subset \operatorname{Z}(M) = (x) \cup (y) \cup (z)$. Note that $\operatorname{depth}(M) = 2$.

Theorem 3.1.14. *Let* M *be a finitely generated multigraded* S*-module. If* M *is almost clean and* $x_k \in S$ *is regular on* M*, then*

$$\operatorname{sdepth}(M/x_kM) = \operatorname{sdepth}(M) - 1.$$

The proof follows from Lemma 3.1.7 and Proposition 3.1.8.

Theorem 3.1.15. *Let* M *be a finitely generated multigraded* S*-module. If* M *is almost clean and* $u \in S$ *is a monomial, which is regular on* M*, then* $\operatorname{sdepth}(M/uM) \geq \operatorname{sdepth}(M) - 1$.

Proof. Let $u = x_{i_1}^{a_1} \dots x_{i_t}^{a_t}$. Since u is regular on M, it follows that each $x_{i_k} \in \mathrm{supp}(u)$ is regular on M, where we denote by $\mathrm{supp}(u)$ the set of all variables x_j such that x_j divides the monomial u. We consider an ascending chain of submodules of M between uM and M where two successive members of the chain are of the form

$$x_{i_1}^{b_1} \cdots x_{i_k}^{b_k} \cdots x_{i_t}^{b_t} M \subset x_{i_1}^{b_1} \cdots x_{i_k}^{b_k - 1} \cdots x_{i_t}^{b_t} M,$$

and where $b_i \leq a_i$ for all $i = 1, \dots, t$.

We obtain

$$x_{i_1}^{b_1} \cdots x_{i_k}^{b_k - 1} \cdots x_{i_t}^{b_t} M / x_{i_1}^{b_1} \cdots x_{i_k}^{b_k} \cdots x_{i_t}^{b_t} M \simeq M / x_{i_k} M,$$

since each $x_{i_k} \in \mathrm{supp}(u)$ is regular on M. Therefore Lemma 3.1.7 and Corollary 3.2.2 imply that

$$\mathrm{sdepth}(M/uM) \geq \mathrm{sdepth}(M/x_{i_k}M) = \mathrm{sdepth}(M) - 1.$$

□

3.2 Behavior of depth, Stanley depth and dimension on short exact sequences

In literature we know that how depth and dim behave on short exact sequences of multigraded finitely generated S-modules. For depth we have Depth Lemma and for dimension we have a result Proposition 1.4.9. For sdepth there are still many open questions. We will show that most of the statements of the "Depth Lemma" are wrong if we replace depth by sdepth. We first observe

Lemma 3.2.1. *Let*

$$0 \to U \xrightarrow{f} M \xrightarrow{g} N \to 0$$

be an exact sequence of finitely generated $\mathbb{Z}^n$-graded S-modules. Then

$$\mathrm{sdepth}(M) \geq \min\{\mathrm{sdepth}(U), \mathrm{sdepth}(N)\}$$

Proof. Let $\mathcal{D}: \; U = \bigoplus_{i=1}^{r} u_i K[Z_i]$ be a Stanley decomposition of U with $\operatorname{sdepth}(\mathcal{D}) = \operatorname{sdepth}(U)$ and let $\mathcal{D}': \; N = \bigoplus_{j=1}^{s} n_j K[Z_j']$ be a Stanley decomposition of N with $\operatorname{sdepth}(\mathcal{D}') = \operatorname{sdepth}(N)$. Since f is injective map, we may suppose that f is an inclusion. Let $n_j' \in M$ be a $\mathbb{Z}^n$ homogeneous element such that $g(n_j') = n_j$. Clearly, $M = \sum_{i=1}^{r} u_i K[Z_i] + \sum_{j=1}^{s} n_j' K[Z_j']$. We prove that the sum $\sum_{i=1}^{r} u_i K[Z_i] + \sum_{j=1}^{s} n_j' K[Z_j']$ is direct. Set $V = \sum_j n_j' K[Z_j']$. Since the exact sequence splits as linear spaces we see that $U \cap V = \{0\}$. Clearly $\mathcal{D}$ is already a Stanley decomposition of U and remains to show only that if $y \in n_j' K[Z_j'] \cap \sum_{\substack{k=1 \\ k \neq j}}^{s} n_k' K[Z_k']$ then $y = 0$. As $g(y) \in n_j K[Z_j'] \cap \sum_{\substack{k=1 \\ k \neq j}}^{s} n_k K[Z_k'] = \{0\}$, we see that $y \in U$, that is $y \in U \cap V = \{0\}$. □

Corollary 3.2.2. *Let*

$$(0) = M_0 \subset M_1 \subset \ldots \subset M_{r-1} \subset M_r = M$$

be an ascending chain of $\mathbb{Z}^n$-graded submodules of M. Then

$$\operatorname{sdepth}(M) \geq \min\{\operatorname{sdepth}(M_i/M_{i-1}) : \; i \in \{1, \ldots, r\}\} \tag{3.3}$$

for all $i \in [r]$.

Proof. We consider the exact sequence of $\mathbb{Z}^n$-graded submodules of M such that

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0.$$

By Lemma 3.2.1, we get $\operatorname{sdepth}(M_i) \geq \min\{\operatorname{sdepth}(M_{i-1}), \operatorname{sdepth}(M_i/M_{i-1})\}$. We apply induction to prove the inequality (3.3). For $i = 1$ this holds clearly. We suppose (3.3) is true for $i = t$ then we have

$$\operatorname{sdepth}(M_t) \geq \min\{\operatorname{sdepth}(M_i/M_{i-1}) : \; i \in \{1, \ldots, t\}\}.$$

Let $i = t + 1$ then we have $\operatorname{sdepth}(M_{t+1}) \geq \min\{\operatorname{sdepth}(M_t), \operatorname{sdepth}(M_{t+1}/M_t)\}$, which is enough. □

The analogue of Lemma 1.5.15(a) only holds under an additional assumption.

Corollary 3.2.3. *In the hypothesis of Lemma 3.2.1 suppose that* $\operatorname{sdepth}(M) < \operatorname{sdepth}(N)$. *Then* $\operatorname{sdepth}(M) \geq \operatorname{sdepth}(U)$.

Proof. If $\operatorname{sdepth}(M) < \operatorname{sdepth}(U)$, we get $\operatorname{sdepth}(M) < \min\{\operatorname{sdepth}(U), \operatorname{sdepth}(N)\}$ contradicting Lemma 3.2.1. □

The analogue of 1.5.15(b) is wrong.

Example 3.2.4. Let $S = K[x, y, z]$, $M = (x, y, z)$. In the exact sequence $0 \to M \to S \to K \to 0$, we have $\operatorname{sdepth}(S) = 3 > \operatorname{sdepth}(K) = 0$ but $\operatorname{sdepth}(M) = 2 \neq \operatorname{sdepth}(K) + 1$.

Note that the case treated in Proposition 3.1.8, that is the short exact sequence $0 \to M \xrightarrow{x_k} M \to M/x_kM \to 0$, and Lemma 3.2.6 apparently hints that some analogue of (b) from "Depth Lemma" in the frame of sdepth might be true. Unfortunately, this is not the case as shows the following:

Example 3.2.5. We have a resolution $0 \to \Omega^1 m \to S^3 \to m \to 0$, where $S = K[x, y, z]$ and $m = (x, y, z)$. Then $\Omega^1 m$ is not free because otherwise $\operatorname{proj\,dim}_S m$ should be 1, which is not true. If $\operatorname{sdepth}(\Omega^1 m) = 3$ then follows $\Omega^1 m$ free by the elementary Lemma 3.2.8. Thus $\operatorname{sdepth}(\Omega^1 m) \leq 2 = \operatorname{sdepth}(m)$.

However it remains still the problem in general that if for an exact sequence $0 \to U \to M \to N \to 0$, $\operatorname{sdepth}(M) > \operatorname{sdepth}(N)$ implies $\operatorname{sdepth}(U) \geq \operatorname{sdepth}(N) + 1$. In general this is false (see Example 3.2.5) but we prove this result in a special case.

Lemma 3.2.6. *If* $I \subset S = K[x_1, \ldots, x_n]$ *is a monomial complete intersection, then* $\operatorname{sdepth}(I) \geq \operatorname{sdepth}(S/I) + 1$.

Proof. Let $\{v_1, \ldots, v_m\}$ be the regular sequence of monomials generating I. Since $\operatorname{sdepth}(S/I) = n - m$, by applying Theorem 2.4.1 recursively, and $\operatorname{sdepth}(I) \geq n - m + 1$, by [16], or [17, Proposition 3.4], it follows the desired result. □

In general for any monomial ideal the inequality in above lemma is still an open question. This inequality motivates that $\operatorname{sdepth}(I) \geq \operatorname{sdepth}(J/I) + 1$ for any two monomial ideals $I \subset J \subset S$. But this inequality does not hold as shows the following example:

Example 3.2.7. *Let* $S = K[x, y]$, $I = (xy, y^2)$, $J = I + (x^2)$. *Then we have* $\operatorname{sdepth}(J/I) = 1 = \operatorname{sdepth}(I) = \operatorname{sdepth}(J)$.

Lemma 3.2.8. *If* M *is multigraded* S*-module,* $S = K[x_1, \ldots, x_n]$ *with* $\operatorname{sdepth}(M) = n$ *then* M *is free.*

Proof. If $\operatorname{sdepth}(M) = n$, then we have a Stanley decomposition of the form $M = \oplus_i u_i S$ and $u_i S$ are free S-modules. The direct sum is of linear spaces but it turns out to be of free S-modules. □

Open questions

Depth is a homological invariant, Stanley depth is a combinatorial invariant and dimension is a geometrical invariant. The natural question arises can we compare these three numerical invariant. One relation between depth and Stanley depth is Stanley's conjecture. It is well known the following relation between depth and dimension: if $M \neq 0$ is a finitely generated graded R-module. Then $\operatorname{depth}(M) \leq \dim(R/P)$ for all $P \in \operatorname{Ass}(M)$. In particular, $\operatorname{depth}(M) \leq \dim(M)$ [34]. The following example shows that the first inequality can be strict.

Example 3.2.9. Let $I = (x_2, x_1x_3, x_1x_4, x_1x_5)$ be a monomial ideal of the polynomial ring $S = K[x_1, x_2, x_3, x_4, x_5]$ over the field K. Then $(x_1, x_2) \cap (x_2, x_3, x_4, x_5)$ is the primary decomposition of the ideal I. Let M be the graded S-module S/I. We obtain that $\operatorname{Ass}(M) = \{(x_1, x_2), (x_2, x_3, x_4, x_5)\}$. Since $\dim(M) = \max\{\dim(S/P) : P \in \operatorname{Ass}(M)\}$, we have $\dim(M) = 3$. Since the set of zero divisors of M is $\mathrm{Z}(M) = (x_1, x_2) \cup (x_2, x_3, x_4, x_5)$, we get $x_1 - x_3$ is an M-regular element. We have no homogeneous regular element on $M/(x_1 - x_3)M$, since the maximal ideal $(x_1, x_2, x_3, x_4, x_5)$ belongs to the set $\operatorname{Ass}(M/(x_1 - x_3)M)$, hence $\operatorname{depth}(M) = 1$, which is strictly less than $\dim(S/P)$, where $P \in \operatorname{Ass}(M)$.

Let K be a field and $S = K[x_1, \ldots, x_n]$ be a polynomial ring over K. Soleyman Jahan in [20] proved that if M is a finitely generated multigraded S-module such that $\dim_K M_a \leq 1$ for all $a \in \mathbb{Z}^n$, then $\operatorname{sdepth}(M) \leq \min\{\dim(S/P) : P \in \operatorname{Ass}(M)\}$. In particular, $\operatorname{sdepth}(M) \leq \dim(M)$.

This inequality is still an open question for general finitely generated multigraded

S-modules. We always have

$$\operatorname{depth}(M) \leq \operatorname{sdepth}(M) \leq \dim(M).$$

The following example shows that both inequalities can be strict at the same time.

Example 3.2.10. Let $S = K[x_1, x_2, x_3, x_4]$ be the polynomial ring in three variables over the field K. Consider the ideal $M = (x_1, x_2, x_3, x_4)$. Then since we have the exact sequence

$$0 \to M \to S \to S/M \to 0,$$

we get $\operatorname{depth}(M) = 1$, by Depth Lemma. The Stanley depth of M is 2 by [6, Theorem 2.2]. Note that $\dim(M) = 4$.

This example leads to the following question

Question 3.2.11. *Given a triple $(a, b, c) \in \mathbb{N}^3$ such that $a \leq b \leq c$. Can we find a multigraded S-module M with* $\operatorname{depth}(M) = a$*,* $\operatorname{sdepth}(M) = b$ *and* $\dim(M) = c$*?*

We know some $(a, b, c) \in \mathbb{N}^3$ such that $\operatorname{depth}(M) = a$, $\operatorname{sdepth}(M) = b$ and $\dim(M) = c$. It will be interesting to find a multigraded S-module such that $\operatorname{depth}(M) = 2$, $\operatorname{sdepth}(M) = 4$ and $\dim(M) = 4$.

If M is a finitely generated multigraded S-module and x_k is a monomial which is regular on M, then we have $\operatorname{sdepth}(M/x_kM) \leq \operatorname{sdepth}(M) - 1$ (Proposition 3.1.8).

Question 3.2.12. *If M is finitely generated multigraded S-module and u is a monomial which is regular on M. Is it true that* $\operatorname{sdepth}(M/uM) \leq \operatorname{sdepth}(M) - 1$*?*

If M is almost clean module and u is a monomial which is regular on M, then we have $\operatorname{sdepth}(M/uM) \geq \operatorname{sdepth}(M) - 1$ (see Theorem 3.1.15). If the inequality $\operatorname{sdepth}(M/uM) \leq \operatorname{sdepth}(M) - 1$ holds then for almost clean module M we have the equality.

We see in Theorem 2.5.1 that if $I \subset S$ is a monomial ideal and u a monomial in S such that u is regular on S/I, then S/I is pretty clean if and only if $S/(I, u)$ is pretty clean.

Question 3.2.13. *Let M be a finitely generated multigraded S-module and let u be a monomial which regular on M. Is it true that M is pretty clean if and only if M/uM is pretty clean?*

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