On Jensen's And Related Inequalities

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<u>DECLARATION</u>

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Dedicated to

My parents and respected teachers.

Table of Contents

Table of Contents v

Abstract i

Acknowledgements iii

1 Introduction 1

1.1 Convex Function 1

1.2 Hardy And Polya-Knopp's Inequalities 7

2 Jensen's Inequality For Convex-Concave Anti-Symmetric Functions And Applications 11

2.1 Introduction 11

2.2 Main results 12

2.3 Applications 14

3 On Certain Inequalities Improving The Hermite-Hadamard Inequality 17

3.1 Introduction 17

3.2 Main Results 18

3.3 Applications 21

4 Bounds For Hardy's And Polya-Knopp's Differences **23**

4.1 Introduction 23

4.2 Bounds For Hardy's Differences. 24

4.2.1 Log Convexity Of Boas-Differences. 24

4.2.2 Improvement And Reverse Of Hardy's Inequality 26

4.3 Bounds For Strengthened Hardy and Polya-Knopp's Differences . . . 30

4.3.1 Log Convexity Of Strengthened Hardy-Polya-Knopp Differences. 30

4.3.2 Improvements And Reverses Of Hardy's Inequality 32

4.3.3 Improvements And Reverses of Polya-Knopp Inequality 36

4.4 Bounds For Multidimensional Hardy Type Polya-Knopp Difference . 40

4.4.1 Preliminaries 40

4.4.2 Log-Convexity Of Hardy-Type Differences 42

4.4.3 Improvements And Reverses Of Hardy-Type Inequalities . . . 44

4.4.4 Improvements And Reverses Of Polya-Knopp Inequality . . . 47

5 Some New Refinements Of Strengthened Hardy And Polya-Knopp's Inequalities **50**

5.1 Introduction 50

5.2 The main results 52

5.3 Refinements of strengthened Hardy and Pólya–Knopp's inequalities . 63

6 Applications In Information Theory **67**

6.1 Introduction 67

6.2 log-convexity of $\mathfrak{J}-$divergence 69

6.3 log-convexity of $\mathfrak{L}-$divergence 71

6.4 log-convexity of $\mathfrak{K}-$divergence 72

6.5 log-convexity of $\mathfrak{B}-$divergence 75

6.6 Applications 76

Bibliography **81**

Abstract

Inequalities are one of the most important instruments in many branches of mathematics such as functional analysis, theory of differential and integral equations, interpolation theory, harmonic analysis, probability theory, etc. They are also useful in mechanics, physics and other sciences. A systematic study of inequalities was started in the classical book [31] and continued in [54, 55]. In the eighties and nineties of the last century an impetuous increase of interest in inequalities took place. One result of this fact was a great number of published books on inequalities (see e.g. [4, 5, 37, 39, 38]) and on their applications (see e.g. [2, 11]). Nowadays the theory of inequalities is still being intensively developed. This fact is confirmed by a great number of recent published books (see e.g. [6, 56]) and a huge number of articles on inequalities. Thus, the theory of inequalities may be regarded as an independent area of mathematics. This PhD thesis is devoted to special kind of inequalities, namely Jensen's and some its related inequalities involving Hermite-Hadamard inequality, Hardy and its limit Polya-Knopp inequality.

In the first chapter, called Introduction, some basic notions and results from theory of convex functions and theory of inequalities are being introduced along with classical results of convex functions.

In the second chapter, The weighted Jensen's Inequality for convex-concave antisymmetric functions is proved and some applications are given.

In the third chapter we have discussed the generalized form of Hermite-Hadamard inequality for integrable Convex functions.

In the fourth chapter Some estimates of Hardy, strengthened Hardy-Knopp and

multidimensional Hardy-Polya-Knopp type differences for $p < 0$ and $0 < p < 1$ are calculated.

In the fifth chapter we prove a new general one-dimensional inequality for convex functions and Hardy-Littlewood averages. Furthermore, we apply this result to unify and refine the so-called Boas's inequality and the strengthened inequalities of the Hardy-Knopp-type, deriving their new refinements as special cases of the obtained general relation. In particular, we get new refinements of strengthened versions of the well-known Hardy and Pólya-Knopp's inequalities, while in the last chapter some measures of divergences between vectors in a convex set of $n-$dimensional real vector space are defined in terms of certain types of entropy functions, and their log-convexity properties with some applications in Information theory are discussed.

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Chapter 1

Introduction

1.1 Convex Function

An important mathematical problem is to investigate how function behave under the action of means. The best known case is that of midpoint convex(or Jensen convex) functions, which deals with the arithmetic mean [47, p.2].

J-convex function: A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex in Jensen sense or J-convex if

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2} \tag{1.1.1}$$

for all $x_1, x_2 \in I$.

Convex function: [47, p.7] A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called convex if

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)\, f(x) + \lambda\, f(y) \tag{1.1.2}$$

for all points x and y in I and all $\lambda \in [0, 1]$. It is called strictly convex if the inequality (1.1.2) holds strictly whenever x and y are distinct points and $\lambda \in (0, 1)$. If $-f$ is convex (respectively, strictly convex) then we say that f is concave (respectively, strictly concave). If f is both convex and concave, then f is said to be affine.

In the context of continuity, midpoint convexity means convexity, that is, the following criteria of equivalence of (1.1.1) and (1.1.2) given by J. L. W. V. Jensen [47, p.10] is valid

Theorem 1.1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function. Then f is convex iff f is mid-convex, that is,*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \quad for\ all\ x, y \in I.$$

By mathematical induction we can extend the inequality (1.1.2) to the convex combinations of finitely many points in I. These extensions are known as discrete Jensen inequality and the integral Jensen inequality respectively [47, p.8].

Theorem 1.1.2. (**The discrete case of Jensen's Inequality**) *A real valued function f defined on an interval I is convex iff for all $x_1, ..., x_n \in I$ and all scalars $p_1, ..., p_n \in [0, 1]$ with $P_n = \sum_{i=1}^{n} p_i$ we have*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i), \tag{1.1.3}$$

The above inequality is strict if f is strictly convex, all the points x_i are disjoint and all scalars p_i are positive.

Between the class of convex and mid-convex functions there is another class of functions called class of wright convex functions [56, p.7].

Wright Convex function: A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be wright convex if for each $x \leq y$, $z \geq 0$, x, $y + z \in I$, the following inequality is valid

$$f(x + z) - f(x) \leq f(y + z) - f(y).$$

There is another sub-class of class of convex functions called class of log-convex functions [58, p.18].

Log(or Multiplicative) Convex function: A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be log-convex if f is positive and $\log f$ is convex on I, that is, if f is positive and satisfy

$$f(\alpha\, x \,+\, \beta\, y) \;\leq\; f^{\alpha}(x)\, f^{\beta}(y)$$

for $x, y \in\, I$, $\alpha, \beta > 0$, $\alpha \,+\, \beta = 1$.

J-Log Convex function: A positive function f is $\log -$convex in Jensen sense on an interval I, that is, for each $s,\, t \in I$

$$f(s)\, f(t) \geq f^2 \left(\frac{s+t}{2}\right).$$

This is equivalent to

$$u^2 f(s) + 2uw\, f\left(\frac{s+t}{2}\right) + w^2\, f(t) \geq 0,$$

for each real u, w and $s, t \in I$ [60].

The class of convex functions can be characterized in a variety of ways. Here we discuss some regularity and geometric properties [58, p.2]. A function f convex and finite on a closed interval $[a, b]$ is bounded from above by $M = \max\{f(a), f(b)\}$, since for any $\lambda \in [0, 1]$, $z = \lambda a + (1-\lambda)b$ in the interval,

$$f(z) \leq \lambda f(a) + (1-\lambda) f(b) \leq \lambda M + (1-\lambda)M = M.$$

It is also bounded from below as we see by writing an arbitrary point in the form $\frac{a+b}{2} + t$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} f\left(\frac{a+b}{2} + t\right) + \frac{1}{2} f\left(\frac{a+b}{2} - t\right)$$

or

$$f\left(\frac{a+b}{2} + t\right) \geq 2f\left(\frac{a+b}{2}\right) - f\left(\frac{a+b}{2} - t\right).$$

Using M is upper bound, $-f\left(\frac{a+b}{2} - t\right) \geq -M$, so

$$f\Big(\frac{a+b}{2} + t\Big) \;\geq\; 2\,f\Big(\frac{a+b}{2}\Big) \;-\; M = m.$$

It is easily seen that a convex function may not be continuous at the boundary points of its domain. On the interior it is not only continuous but it satisfies a stronger condition [58, p.4].

Theorem 1.1.3. *If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex, then f satisfies a Lipschitz condition on any closed interval $[a, b]$ contained in the interior of I, that is there is a constant K so that for any two points $x, y \in [a, b]$,*

$$|f(x) - f(y)| \;\leq K\; |x - y|.$$

Consequently, f is absolutely continuous on $[a, b]$ and continuous on interior of I.

The derivative of a convex function is best studied in terms of the left and right derivatives defined by

$$f'_-(x) = \lim_{y\uparrow x} \frac{f(y) - f(x)}{y - x} \quad , \; f'_+(x) = \lim_{y\downarrow x} \frac{f(y) - f(x)}{y - x}$$

The following result for left and right derivatives of the convex functions can be seen in [58, p.5].

Theorem 1.1.4. *If $f : I \to \mathbb{R}$ is convex (strictly convex), then $f'_-(x)$ and $f'_+(x)$ exist and are increasing (strictly increasing) on interior of I.*

The following result about convex functions states that a convex function is differentiable almost everywhere.

Theorem 1.1.5. [*58, p.7*] *If $f : I \to \mathbb{R}$ is convex on the open interval I, then the set E where f' fails to exists is countable. Moreover, f' is continuous on I/E.*

For a twice differentiable function the second derivative test is very useful to check the convexity of a function [58, p.11].

Theorem 1.1.6. *Suppose f'' exists on (a, b). Then f is convex iff $f''(x) \geq 0$. And if $f''(x) > 0$ on (a, b), then f is strictly convex on the interval.*

The geometric characterization depends upon the idea of a support line.

Support line for convex function: A function $f : I \to \mathbb{R}$ is convex if and only if there is at least one line of support for f at each $x_0 \in I$, that is, there exists an affine function $A(x) = f(x_0) + m(x - x_0)$ such that $A(x) \leq f(x)$ for every $x \in I$. If f is differentiable at x_0 then this line in unique and this is tangent line.

Theorem 1.1.7. [*58*, *p*.12] *A function $f : (a, b) \to \mathbb{R}$ is convex iff there is at least one line of support for f at each $x_0 \in (a, b)$.*

The following result is not direct characterization of a convex function but it is closely related to Theorem 1.1.7.

Theorem 1.1.8. [*58*, *p*.12] *Let $f : (a, b) \to \mathbb{R}$ be convex. Then f is differentiable at x_0 iff the line of support for f at x_0 is unique. And in this case,*

$$A(x) = f(x_0) + f'(x_0)(x - x_0)$$

provides this unique support.

Sub-differential (or Generalized derivative): Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, then sub-differential of f at x, denoted by $\partial f(x)$, is defined as

$$\begin{aligned}
\partial f(x) &= \{y \in \mathbb{R} : y \ is \ the \ slope \ of \ a \ support \ line \ for \ f \ at \ x\}. \\
&= \{y \in \mathbb{R} : f(u) - f(x) - y(u - x) \geq 0, u \in I\}.
\end{aligned}$$

Thus ∂f is a set-valued function that is single valued and agrees with the ordinary two-sided derivative f' wherever the latter exists. The domain of ∂f (dom ∂f) is the set of all x in I where f has support and range of ∂f is the set of support slopes.

Note: It may be noted that (int $I=$)$I^{\circ} \subseteq \operatorname{dom} \partial f \subseteq I$.

There is a connection between a convex function and its sub-differential [47, p.30].

Lemma 1.1.9. *Let f be a convex function on an interval I. Then $\partial f(x) \neq \emptyset$ at all interior points of I. Moreover, every function $\omega : I \to \mathbb{R}$ for which $\omega(x) \in \partial f(x)$ whenever $x \in I^{\circ}$ verifies the double inequality*

$$f'_{-}(x) \leq \omega(x) \leq f'_{+}(x),$$

and thus is nondecreasing on I°

Theorem 1.1.10. [*47*, *p.*31] *Let f be a continuous convex function on an interval I and let $\omega : I \to \mathbb{R}$ be a function such that $\omega(x)$ belong to $\partial f(x)$ for all $x \in I^{\circ}$. Then*

$$f(z) = \sup\{f(x) + (z-x)\,\omega(x) : x \in I^{\circ}\} \quad for \ all \ z \in I$$

The notion of convexity can be extended to higher order giving the notion of $n-$convex function [56, p.14].

Let $f : [a,b] \to \mathbb{R}$, then $k-$th order divided difference of f at distinct points $x_0, ..., x_k \in [a,b]$ is defined recursively as

$$[x_i]f = f(x_i)$$

and

$$[x_0, ..., x_k]f = \frac{[x_1, ..., x_k]f - [x_0, ..., x_{k-1}]f}{x_k - x_0}$$

The value of $[x_0, ..., x_k]f$ is independent of the order of the points $x_0, ..., x_k$. The definition may be extended to include the case in which some or all of the points coincide by assuming that $x_0 \leq x_1 \leq ... \leq x_k$ and letting

$$\underbrace{[x, ..., x]}_{(j+1)\ times} f = \frac{f^{(j)}(x)}{j!},$$

provided that $f^{(j)}(x)$ exists. Then f is said to be $n-$convex, $n \geq 0$ on [a, b] iff for all choices of $(n+1)$ distinct points in [a, b],

$$[x_0, ..., x_n]f \geq 0.$$

If this inequality is reversed, then f is said to be $n-$concave on [a, b]. If the inequality is strict, then f is said to be strictly $n-$convex ($n-$concave) function. Let us note that a simple consequence of (1.1.2) is the well known

Hermite Hadamard inequality for a convex function $f : [a, b] \to \mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\ dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1.4}$$

About (1.1.4) see for example recent papers [1, 8, 26]. Other consequences are given in the next section.

1.2 Hardy And Polya-Knopp's Inequalities

Let $p > 1$, $k \neq 1$, and the function F be defined on $\mathbb{R}_+ = (0, \infty)$ by

$$F(x) = \begin{cases} \int_0^x f(t)\,dt, & k > 1; \\ \int_x^\infty f(t)\,dt, & k < 1, \end{cases}$$

then the highly important Hardy's integral inequality

$$\int_0^\infty x^{-k} F^p(x)\,dx \leq \left(\frac{p}{|k-1|}\right)^p \int_0^\infty x^{p-k} f^p(x)\,dx \tag{1.2.1}$$

holds for all non-negative functions f, such that $x^{1-\frac{k}{p}}\ f \in L^p(\mathbb{R}_+)$. This relation was obtained by G. H. Hardy [20] in 1928, although he announced its version with

$k = p > 1$ already in 1920, [23], and then proved it in 1925, [24]. In [20] Hardy also pointed out that if k and F fulfill the conditions of the above result, but $0 < p < 1$, then the sign of inequality in (1.2.1) is reversed, that is,

$$\int_0^\infty x^{-k} F^p(x)\, dx \;\geq\; \left(\frac{p}{|k-1|}\right)^p \int_0^\infty x^{p-k}\; f^p(x)\; dx \tag{1.2.2}$$

holds. On the other hand, the first unweighed Hardy-type inequality for $p < 0$ was considered by K. Knopp [36] in 1928, but in a discrete setting, for sequences of positive real numbers, while general weighted integral Hardy-type inequalities for exponents $p, q < 0$ and $0 < p, q < 1$ were first studied much later, by P. R. Beesack and H. P. Heinig [3] and H. P. Heinig [21]. The unweighed multidimensional Hardy-type inequality for the case $p < 0$ and $0 < p < 1$ were studied in [35] and in [51] weighted versions for these cases were discussed.

Another important classical integral inequality is the so-called Pólya-Knopp's inequality,

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(t)\; dt\right)\, dx \;< e \int_0^\infty f(x)\; dx\,, \tag{1.2.3}$$

which holds for all positive functions $f \in L^1(\mathbb{R}_+)$. This result was first published by K. Knopp [36] in 1928, but it was certainly known before since Hardy himself (see [24, p. 156]) claimed that it was G. Pólya who pointed it out to him earlier. Note that the discrete version of (1.2.3) is surely due to T. Carleman, [12].

It is important to observe that relations (1.2.1) and (1.2.3) are closely related since (1.2.3) can be obtained from (1.2.1) by rewriting it with the function f replaced with $f^{1/p}$ and letting $p \to \infty$. Therefore, Pólya-Knopp's inequality may be considered as a limiting case of Hardy's inequality. Moreover, the constants $\left(\frac{p}{|k-1|}\right)^p$ and e, respectively appearing on the right-hand sides of (1.2.1) and (1.2.3), are the best possible, that is, neither of them can be replaced with any smaller constant.

In 1970, R. P. Boas [9] proved that (1.2.1) and (1.2.3) are just special cases of a much more general inequality

$$\int_0^\infty \Phi\left(\frac{1}{M}\int_0^\infty f(t\,x)\,dm(t)\right)\frac{dx}{x} \;\le\; \int_0^\infty \Phi(f(x))\,\frac{dx}{x} \tag{1.2.4}$$

for continuous convex functions $\Phi : [0,\infty) \to \mathbb{R}$, measurable non-negative functions $f : \mathbb{R}_+ \to \mathbb{R}$, and non-decreasing and bounded functions $m : [0,\infty) \to \mathbb{R}$, where $M = m(\infty) - m(0) > 0$ and the inner integral on the left-hand side of (1.2.4) is a Lebesgue-Stieltjes integral with respect to m. After its author, the relation (1.2.4) was named Boas's inequality (see also [40, Chapter IV, p. 156] and [56, Chapter 8, Theorem 8.1]). In the case of a concave function Φ, (1.2.4) holds with the reversed sign of inequality.

On the other hand, obviously unaware of the mentioned more general Boas's result for Hardy-Littlewood averages, in 2002, S. Kaijser et al. [34] established the so-called general Hardy-Knopp-type inequality for positive functions $f : \mathbb{R}_+ \to \mathbb{R}$,

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \;\le\; \int_0^\infty \Phi(f(x))\,\frac{dx}{x}\,, \tag{1.2.5}$$

where Φ is a convex function on $\mathbb{R}_+$. By taking $\Phi(x) = x^p$ and $\Phi(x) = e^x$, they obtained an elegant new proof of inequalities (1.2.1) and (1.2.3) and showed that both Hardy and Pólya-Knopp's inequality can be derived by using only a convexity argument. Later on, A. Čižmešija et al. [15] generalized the relation (1.2.5) to the so-called strengthened Hardy–Knopp-type inequality by adding a weight function and truncating the range of integration to $(0, b)$. They also obtained a related dual inequality, that is, an inequality with the outer integrals taken over (b,∞) and with the inner integral on the left-hand side taken over (x,∞). These general inequalities provided an unified treatment of the strengthened Hardy and Pólya-Knopp's inequalities from [13, 14, 67, 68].

Theorem 1.2.1. *Suppose* $0 < b \leq \infty$, *let* $u : (0, b) \to \mathbb{R}$ *be a non-negative function such that the function* $x \mapsto \frac{u(x)}{x}$ *is locally integrable in* $(0, b)$, *and the function* v *is defined by*

$$v(t) = t \int_t^b \frac{u(x)}{x^2} dx, \quad t \in (0, b).$$

If the real-valued function ϕ *is convex on* (a, c), *where* $-\infty \leq a < c \leq \infty$, *then the inequality*

$$\int_0^b u(x)\ \phi\Big(\frac{1}{x}\int_0^x f(t)\ dt\Big)\ \frac{dx}{x} \leq \int_0^b v(x)\ \phi(f(x))\ \frac{dx}{x} \tag{1.2.6}$$

holds for all integrable functions $f : (0, b) \to \mathbb{R}$, *such that* $f(x) \in (a, c)$ *for all* $x \in (0, b)$.

Theorem 1.2.2. *For* $0 \leq b \leq \infty$, *let* $u : (b, \infty) \to \mathbb{R}$ *be a non-negative locally integrable function in* (b, ∞), *and the function* v *is defined by*

$$v(t) = \frac{1}{t} \int_b^t u(x)\ dx, \quad t \in (b, \infty).$$

If the real-valued function ϕ *is convex on* $(a,\ c)$, *where* $-\infty \leq a < c \leq \infty$, *then the inequality*

$$\int_b^\infty u(x)\ \phi\Big(x\int_x^\infty f(t)\ \frac{dt}{t^2}\Big)\ \frac{dx}{x} \leq \int_b^\infty v(x)\ \phi(f(x))\ \frac{dx}{x} \tag{1.2.7}$$

holds for all integrable functions $f : (b, \infty) \to \mathbb{R}$, *such that* $f(x) \in (a,\ c)$ *for all* $x \in (b, \infty)$.

Chapter 2

Jensen's Inequality For Convex-Concave Anti-Symmetric Functions And Applications

2.1 Introduction

The famous Jensen's inequality for convex function is given by (1.1.3) [56]. The natural problem in this context is to deduce Jensen's type inequality weakening some of the above assumptions. The classical case is the case of Jensen-convex (or mid-convex) functions is given by (1.1.1). The following theorem was the main motivation for this chapter (see [46] and [56, p.55-56]).

Theorem 2.1.1. *Let* $f : [a, b] \to \mathbb{R}$ *be a Wright convex on* $\left[a, \frac{a+b}{2}\right]$ *and* $f(x) = -f(a + b - x)$. *If* $x_i \in [a, b]$ *and* $\dfrac{x_i + x_{n-i+1}}{2} \in \left[a, \dfrac{a + b}{2}\right]$ *for* $i = 1, 2...., n$, *then* (1.1.3) *for* $p_i = 1$ *is valid.*

Another way of weakening the assumptions for (1.1.3) is relaxing the assumption of positivity of weights $p_i, i = 1, \ldots, n$. The most important result in this direction is the Jensen-Steffensen inequality (see for example [56, p.57]), which states that (1.1.3) holds also if $x_1 \le x_2 \le \cdots \le x_n$ and $0 \le P_k \le P_n$, $P_n > 0$, where $P_k = \sum_{i=1}^{k} p_i$.

The main purpose of this chapter is to prove the weighted version of Theorem 2.1.1. For some related results see [17, 18]. In Section 2.3, to illustrate the applicability of this result, we give a generalization of the famous Ky-Fan inequality.

2.2 Main results

Theorem 2.2.1. ([27]) *Let* $f : (a, b) \to \mathbb{R}$ *be a convex function on* $\left(a, \frac{a+b}{2}\right]$ *and* $f(x) = -f(a+b-x)$ *for every* $x \in (a, b)$. *If* $x_i \in (a, b), p_i > 0$, $\frac{x_i + x_{n-i+1}}{2} \in \left(a, \frac{a+b}{2}\right]$ *and* $\frac{p_i x_i + p_{n-i+1} x_{n-i+1}}{p_i + p_{n-i+1}} \in \left(a, \frac{a+b}{2}\right]$ *for* $i = 1, 2...., n$, *then* (1.1.3) *holds.*

Proof. Without loss of generality we can suppose that $(a, b) = (-1, 1)$. So, f is an odd function. First we consider the case $n = 2$. If $x_1, x_2 \in (-1, 0]$ then we have the known case of Jensen's inequality for convex functions. Thus, we shall assume that $x_1 \in (-1, 0)$ and $x_2 \in (0, 1)$. The equation of the straight line through points $(x_1, f(x_1)), (0, 0)$ is

$$y = \frac{f(x_1)}{x_1} x.$$

Since f is convex on $(-1, 0]$ and $x_1 < \frac{p_1 x_1 + p_2 x_2}{p_1 + p_2} \leq 0$, it follows:

$$f\left(\frac{p_1 x_1 + p_2 x_2}{p_1 + p_2}\right) \leq \frac{f(x_1)}{x_1} \frac{p_1 x_1 + p_2 x_2}{p_1 + p_2}.$$

It is enough to prove that:

$$\frac{f(x_1)}{x_1} \frac{p_1 x_1 + p_2 x_2}{p_1 + p_2} \leq \frac{p_1 f(x_1) + p_2 f(x_2)}{p_1 + p_2}$$

which is obviously equivalent to the inequality:

$$\frac{f(x_1)}{x_1} \leq \frac{f(x_2)}{x_2} = \frac{f(-x_2)}{-x_2}. \tag{2.2.1}$$

Since the function f is convex on $(-1, 0]$ and $f(0) = 0$, by Galvani's theorem it follows that the function $x \mapsto \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$ is increasing on $(-1, 0)$. Therefore, from $\frac{x_1+x_2}{2} \leq 0$ and $x_2 > 0$ we have $x_1 \leq -x_2 < 0$, so (2.2.1) holds.

Now, for an arbitrary $n \in \mathbb{N}$ we have:

$$\begin{aligned}
\sum_{i=1}^{n} p_i f(x_i) &= \frac{1}{2}\sum_{i=1}^{n}\left[p_i f(x_i) + p_{n-i+1} f(x_{n-i+1})\right] \\
&\geq \frac{1}{2}\sum_{i=1}^{n}(p_i + p_{n-i+1}) f\left(\frac{p_i x_i + p_{n-i+1}x_{n-i+1}}{p_i + p_{n-i+1}}\right) \\
&= P_n \cdot \frac{1}{2P_n}\sum_{i=1}^{n}(p_i + p_{n-i+1}) f\left(\frac{p_i x_i + p_{n-i+1}x_{n-i+1}}{p_i + p_{n-i+1}}\right) \\
&\geq P_n f\left(\frac{1}{2P_n}\sum_{i=1}^{n}(p_i + p_{n-i+1})\frac{p_i x_i + p_{n-i+1}x_{n-i+1}}{p_i + p_{n-i+1}}\right) \\
&= P_n f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right),
\end{aligned}$$

so the proof is complete. □

Remark 2.2.1. In fact we have proved that

$$\begin{aligned}
\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) &\geq \frac{1}{2P_n}\sum_{i=1}^{n}(p_i + p_{n-i+1}) f\left(\frac{p_i x_i + p_{n-i+1}x_{n-i+1}}{p_i + p_{n-i+1}}\right) \\
&\geq f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right).
\end{aligned}$$

Remark 2.2.2. Neither of the conditions $\frac{x_i+x_{n-i+1}}{2} \in \left(a, \frac{a+b}{2}\right]$, $i = 1, \ldots, n$ or $\frac{p_i x_i + p_{n-i+1}}{p_i + p_{n-i+1}} \in \left(a, \frac{a+b}{2}\right]$, $i = 1, \ldots, n$, can be removed from the assumptions of Theorem 2.1.1. To see this consider the function $f(x) = -x^3$ on $(-2, 2)$. That the first condition cannot be removed can be seen by considering $x_1 = -\frac{1}{2}$, $x_2 = 1$, $p_1 = \frac{7}{8}$, $p_2 = \frac{1}{8}$. That the second condition cannot be removed can be seen by considering $x_1 = -1$, $x_2 = \frac{3}{4}$, $p_1 = \frac{1}{8}$, $p_2 = \frac{7}{8}$. In both cases (1.1.3) doesn't hold.

Remark 2.2.3. Using Jensen's and Jensen-Steffensen's inequality it is easy to prove the following inequalities (see [6, 45]):

$$2f\left(\frac{a+b}{2}\right)-\frac{1}{P_n}\sum_{i=1}^{n}p_if(x_i)\le f\left(a+b-\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right)$$
$$\le f(a)+f(b)-\frac{1}{P_n}\sum_{i=1}^{n}p_if(x_i), \tag{2.2.2}$$

where f is a convex function on $(a-\varepsilon, b+\varepsilon)$, $\varepsilon>0$, and $x_i\in(a,b)$, $p_i>0$ for $i=1,\ldots,n$. If f is concave the reverse inequalities hold in (2.2.2).

Now, suppose the conditions in Theorem 2.1.1 are fulfilled, except that the function f satisfies $f(x)+f(a+b-x)=2f\left((a+b)/2\right)$. It is immediate (consider the function $g(x)=f(x)-f\left((a+b)/2\right)$) that the inequality (1.1.3) still holds. Using $f(x)=2f\left((a+b)/2\right)-f(a+b-x)$, the inequality (1.1.3) gives:

$$2f\left(\frac{a+b}{2}\right)-\frac{1}{P_n}\sum_{i=1}^{n}p_if(x_i)\le f\left(a+b-\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right),$$

so the left hand side of the inequality (2.2.2) is valid also in this case. On the other hand if $f\left((a+b)/2\right)=0$ (so $f(a)+f(b)=0$), the previous inequality can be written as

$$f\left(a+b-\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right)\ge f(a)+f(b)-\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i,$$

which is reverse of the right hand side inequality of (2.2.2), so the concavity properties of the function f are prevailing in this case.

2.3 Applications

In the following corollary we give a simple proof of a known generalization of the Levinson inequality (see [57], also see [56, p.71-72]). It is easy to prove, using properties of divided differences or using classical case of the Levinson inequality, that if

$f:(0,2a)\to\mathbb{R}$ is a 3-convex function, then the function $g(x)=f(2a-x)-f(x)$ is convex on $(0,a]$ (see [56, p.71-72]).

Corollary 2.3.1. *Let* $f:(0,2a)\to\mathbb{R}$ *be a* $3-$*convex function,* $p_i>0$*,* $x_i\in(0,2a)$*,* $x_i+x_{n+1-i}\le 2a$ *and*

$$\frac{p_ix_i+p_{n+1-i}x_{n+1-i}}{p_i+p_{n+1-i}}\le a,$$

for $i=1,2.....,n$*. Then*

$$\begin{aligned}\frac{1}{P_n}\sum_{i=1}^n p_if(x_i) &- f\left(\frac{1}{P_n}\sum_{i=1}^n p_ix_i\right)\\ &\le \frac{1}{P_n}\sum_{i=1}^n p_if(2a-x_i)-f\left(\frac{1}{P_n}\sum_{i=1}^n p_i(2a-x_i)\right) \qquad (2.3.1)\end{aligned}$$

Proof. It is a simple consequence of Theorem 2.1.1, and the above mentioned fact that $g(x)=f(2a-x)-f(x)$ is convex on $(0,a]$. □

Remark 2.3.1. In fact the following improvement of inequality (2.3.1) is valid:

$$\begin{aligned}\frac{1}{P_n}\sum_{i=1}^n p_if(2a-x_i) &- \frac{1}{P_n}\sum_{i=1}^n p_if(x_i)\\ &\ge \frac{1}{2P_n}\sum_{i=1}^n(p_i+p_{n+1-i})f\left(2a-\frac{p_ix_i+p_{n+1-i}x_{n+1-i}}{p_i+p_{n+1-i}}\right)\\ &- \frac{1}{2P_n}\sum_{i=1}^n(p_i+p_{n+1-i})f\left(\frac{p_ix_i+p_{n+1-i}x_{n+1-i}}{p_i+p_{n+1-i}}\right)\\ &\ge f\left(2a-\frac{1}{P_n}\sum_{i=1}^n p_ix_i\right)-f\left(\frac{1}{P_n}\sum_{i=1}^n p_ix_i\right). \qquad (2.3.2)\end{aligned}$$

A famous inequality due to Ky-Fan states that

$$G_n/G_n'\le A_n/A_n', \qquad (2.3.3)$$

where G_n, G_n' and A_n, A_n' are the weighted geometric and arithmetic means, respectively, defined by

$$G_n=\left(\prod_{i=1}^n x_i^{p_i}\right)^{\frac{1}{P_n}},\ A_n=\frac{1}{P_n}\sum_{i=1}^n p_ix_i,$$

$$G'_n = \left(\prod_{i=1}^{n} (1-x_i)^{p_i} \right)^{\frac{1}{P_n}}, \; A'_n = \frac{1}{P_n} \sum_{i=1}^{n} p_i (1-x_i),$$

where $x_i \in (0, 1/2]$, $i = 1, \ldots, n$ (see [6, p.295]).

The inequality (2.3.3) has stimulated an interest of many researchers. New proofs, improvements and generalizations of (2.3.3) have been found (see for example [61, 62, 63, 64]). Another refinement of (2.3.3) is given in [48].

In the following corollary we give an improvement of the inequality (2.3.3).

Corollary 2.3.2. *Let $p_i > 0$, $x_i \in (0,1)$, $A_2(x_i, x_{n+1-i}) = \frac{p_i x_i + p_{n+1-i} x_{n+1-i}}{p_i + p_{n+1-i}}$ and $x'_i = 1 - x_i$, $i = 1, \ldots, n$. If $x_i + x_{n+1-i} \le 1$ and $A_2(x_i, x_{n+1-i}) \le 1/2$, $i = 1, \ldots, n$, then*

$$\frac{G'_n}{G_n} \ge \left[\prod_{i=1}^{n} \left(\frac{A_2\left(x'_i, x'_{n+1-i}\right)}{A_2\left(x_i, x_{n+1-i}\right)} \right)^{p_i + p_{n+1-i}} \right]^{\frac{1}{2P_n}} \ge \frac{A'_n}{A_n}. \tag{2.3.4}$$

Proof. Set $f(x) = \log x$ and $2a = 1$ in (2.3.2). It follows:

$$\begin{aligned} \frac{1}{P_n} \sum_{i=1}^{n} p_i \log(1-x_i) &- \frac{1}{P_n} \sum_{i=1}^{n} p_i \log x_i \\ &\ge \frac{1}{2P_n} \sum_{i=1}^{n} (p_i + p_{n+1-i}) \log \frac{p_i(1-x_i) + p_{n+1-i}(1-x_{n+1-i})}{p_i + p_{n+1-i}} \\ &- \frac{1}{2P_n} \sum_{i=1}^{n} (p_i + p_{n+1-i}) \log \frac{p_i x_i + p_{n+1-i} x_{n+1-i}}{p_i + p_{n+1-i}} \\ &\ge \log \left(1 - \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right) - \log \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i, \end{aligned}$$

which by obvious rearranging implies (2.3.4). □

Chapter 3

On Certain Inequalities Improving The Hermite-Hadamard Inequality

3.1 Introduction

The classical Hermite-Hadamard inequality (1.1.4) gives us an estimate, from below and from above, of the mean value of a convex function $f : [a, b] \to \mathbb{R}$. See [47, p. 50-51], for details. Inequality (1.1.4) can be easily improved by applying (1.1.4) on each of the subintervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$; summing up side by side we get

$$\begin{aligned}\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] &\leq \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\end{aligned}$$

Usually, the precision in (1.1.4) is estimated via Ostrowski's and Iyengar's inequalities. See [47], p. 63, 191, for details. Based on previous work done by S. S. Dragomir and A.Mcandrew [19], we have proved several better results, that apply to a slightly larger class of functions. We start by estimating the deviation of the support line of a convex function from the mean value. The main ingredient is the existence of the subdifferential.

3.2 Main Results

Theorem 3.2.1. *Assume that f is convex on (a, b). Then*

$$\frac{1}{b-a}\int_a^b f(y)\ dy\ +\ \varphi(x)\left(x\ -\ \frac{a+b}{2}\right) - f(x) \geq$$
$$\left|\frac{1}{b-a}\int_a^b |f(y)\ -\ f(x)|\, dy - |\varphi(x)|\ \frac{(x-a)^2+(b-x)^2}{2(b-a)}\right|$$

for all $x \in (a, b)$. Here $\varphi : (a, b) \to \mathbb{R}$ is any function such that $\varphi(x) \in [f'_-(x), f'_+(x)]$ for all $x \in (a, b)$.

Proof. In fact,

$$f(y)\ \geq\ f(x)\ +\ (y-x)\varphi(x)$$

for all $x, y \in (a, b)$, which yields

$$\begin{aligned} f(y) - f(x) - (y-x)\varphi(x) &= |\ f(y) - f(x) - (y-x)\varphi(x)\ | \\ &\geq\ |\ |\ f(y) - f(x)\ | - |\ y - x\ |\ |\ \varphi(x)\ |\ |\ . \end{aligned} \tag{3.2.1}$$

By integrating side by side we get

$$\begin{aligned} &\int_a^b f(y)dy - (b-a)f(x) + (b-a)\left(x - \frac{a+b}{2}\right)\varphi(x) \\ &\qquad \geq \int_a^b |\ |\ f(y) - f(x)\ | - |\ y - x\ |\ \ |\varphi(x)||\, dy \\ &\qquad \geq \left|\int_a^b |\ f(y) - f(x)\ |\ dy - |\ \varphi(x)\ | \int_a^b |\ y - x\ |\ dy\right| \\ &\qquad = \left|\int_a^b |\ f(y) - f(x)\ |\ dy - |\ \varphi(x)\ |\ \frac{(x-a)^2+(b-x)^2}{2}\right| \end{aligned}$$

and it remains to simplify both sides by $b - a$. □

Theorem 3.2.1 applies for example to convex functions not necessarily defined on interval [a, b], for example, to $f(x) = (1 - x^2)^{-\alpha}$, $x \in (-1, 1)$, for $\alpha \geq 0$.

Theorem 3.2.2. *Assume that $f : [a,b] \to \mathbb{R}$ is a convex function. Then*

$$\frac{1}{2}\left[f(x) + \frac{f(b)(b-x) + f(a)(x-a)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(y)\,dy$$
$$\geq \frac{1}{2}\left|\frac{1}{b-a}\int_a^b \mid f(x) - f(y) \mid \, dy - \frac{1}{b-a}\int_a^b \mid x-y \mid \mid f'(y) \mid dy\right|$$

for all $x \in (a,b)$.

Proof. Without loss of generality we may assume that f is also continuous. See [47, p. 22], (where it is proved that f admits finite limits at the endpoints). In this case f is absolutely continuous and thus it can be recovered from its derivative. The function f is differentiable except for countably many points, and letting $\mathcal{E}$ this exceptional set we have

$$f(x) \;\geq\; f(y) + (x-y)\,f'(y)$$

for all $x \in [a,b]$ and all $y \in [a,b] \backslash \mathcal{E}$. This yields

$$\begin{aligned} f(x) - f(y) - (x-y)f'(y) &= \mid f(x) - f(y) - (x-y)f'(y) \mid \\ &\geq \mid \mid f(x) - f(y) \mid - \mid x-y \mid \cdot \mid f'(y) \mid \mid, \end{aligned}$$

so that by integrating side by side with respect to y we get

$$(b-a)f(x) - 2\int_a^b f(y)\,dy + f(b)(b-x) + f(a)(x-a)$$
$$\geq \left|\int_a^b \mid f(x) - f(y) \mid dy - \int_a^b \mid x-y \mid \mid f'(y) \mid dy\right|$$

equivalently,

$$f(x) + \frac{f(b)(b-x) + f(a)(x-a)}{b-a} - \frac{2}{b-a}\int_a^b f(y)dy$$
$$\geq \frac{1}{b-a}\left|\int_a^b \mid f(x) - f(y) \mid dy - \int_a^b \mid x-y \mid \mid f'(y) \mid dy\right|$$

and the result follows. □

A variant of Theorem 3.2.2 in the case where f is convex only on (a,b) is as follows:

Theorem 3.2.3. *Assume that* $f:[a,b]\to\mathbb{R}$ *is monotone on* $[a,b]$ *and convex on* (a,b)*. Then*

$$\frac{1}{2}\left[f(x)+\frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right]-\frac{1}{b-a}\int_a^b f(y)dy$$
$$\geq\left|\frac{1}{b-a}\int_a^b \operatorname{sgn}(x-y)f(y)dy+\frac{f(x)(a+b-2x)+(a-x)f(a)+(b-x)f(b)}{2(b-a)}\right|$$

for all $x\in(a,b)$*.*

Proof. Consider for example the case where f is nondecreasing on $[a,b]$. Then

$$\begin{aligned}\int_a^b \mid f(x)-f(y)\mid dy &= \int_a^x \mid f(x)-f(y)\mid dy+\int_x^b \mid f(x)-f(y)\mid dy\\ &= (x-a)f(x)-\int_a^x f(y)dy+\int_x^b f(y)dy-(b-x)f(x)\\ &= (2x-a-b)f(x)-\int_a^x f(y)dy+\int_x^b f(y)dy.\end{aligned}$$

As in the proof of Theorem 3.2.2 we may restrict to the case where f is absolutely continuous, which yields

$$\begin{aligned}\int_a^b |x-y|\,|f'(y)|\,dy &= \int_a^x (x-y)f'(y)dy+\int_x^b (y-x)f'(y)dy\\ &= (a-x)f(a)+(b-x)f(b)+\int_a^x f(y)dy-\int_x^b f(y)dy.\end{aligned}$$

By Theorem 3.2.2, we conclude that

$$\frac{1}{2}\left[f(x)+\frac{f(b)(b-x)+f(a)(x-a)}{b-a}\right]-\frac{1}{b-a}\int_a^b f(x)dx\geq$$
$$\frac{1}{2}\left|\frac{2}{b-a}\left[\int_x^b f(y)dy-\int_a^x f(y)dy\right]+\frac{f(x)(2x-a-b)}{b-a}-\frac{(a-x)f(a)+(b-x)f(b)}{b-a}\right|$$

The case where f is nonincreasing can be treated in a similar way. $\square$

3.3 Applications

For $x = (a+b)/2$, Theorem 3.2.3 gives us

$$\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(y)dy$$
$$\geq \left|\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(\frac{a+b}{2}-y\right)f(y)dy+\frac{f(b)-f(a)}{4}\right|, \quad (3.3.1)$$

which is same as obtained in [19, Theorem 2.3]

Now for exponential function (3.3.1) becomes

$$\frac{1}{2}\left[\exp\frac{a+b}{2}+\frac{\exp a+\exp b}{2}\right]-\frac{\exp b-\exp a}{b-a}$$
$$\geq \left|\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(\frac{a+b}{2}-y\right)\exp y\,dy+\frac{\exp b-\exp a}{4}\right|$$

for all $a, b \in \mathbb{R}$, $a < b$, equivalently,

$$\frac{1}{2}\left[\sqrt{ab}+\frac{a+b}{2}\right]-\frac{b-a}{\ln b-\ln a} \geq \left|\frac{b-a}{4}-\frac{a+b-2\sqrt{ab}}{\ln b-\ln a}\right|$$

for all $0 < a < b$.

This is similar to well known inequality:

$$\frac{2}{3}\cdot\sqrt{ab}+\frac{1}{3}\cdot\frac{a+b}{2} > \frac{b-a}{\ln b-\ln a} \quad (3.3.2)$$

In fact (3.3.2) can be embedded into a sequence of interpolating inequalities involving the geometric, the arithmetic, the logarithmic means [65]

$$\sqrt{ab} < \left(\sqrt{ab}\right)^{2/3}\left(\frac{a+b}{2}\right)^{1/3} < \frac{b-a}{\ln b-\ln a} <$$
$$\frac{2}{3}\cdot\sqrt{ab}+\frac{1}{3}\cdot\frac{a+b}{2} < \frac{1}{2}\left(\frac{a+b}{2}+\sqrt{ab}\right) < \frac{a+b}{2}$$

for all $0 < a < b$.

Remark 3.3.1. The extension of Theorems 3.2.1-3.2.3 above to the context of weighted measures is straightforward and we shall omit the details. However, the problem of estimating the Hermite-Hadamard inequality in the case of several variables has been discussed in [49] again rediscovered by M. Bessenyei in [7].

Chapter 4

Bounds For Hardy's And Polya-Knopp's Differences

4.1 Introduction

Kaijser et al. [34] pointed out that (1.2.1) for $k = p$ and (1.2.3) are just special case of much more general Hardy-Knopp-type inequality for positive function f

$$\int_0^\infty \phi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x} \le \int_0^\infty \phi\left(f(x)\right)\frac{dx}{x}, \tag{4.1.1}$$

where ϕ is a convex function on $(0,\infty)$. There are a lot of extensions of (1.2.1). One of the extensions is given by (1.2.4). A. Čižmešija et. al [15] proved the strengthened Hardy-Knopp-type inequality that generalized inequality (4.1.1) given by Theorem 1.2.1. They also formulated its dual result given by Theorem 1.2.2. As a special case the following extensions of (1.2.1) and (1.2.3) and of their dual inequalities were obtained [15]:

$$\int_0^b x^{-k}\left(\int_0^x f(t)dt\right)^p dx < \left(\frac{p}{k-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)\,dx,$$

whenever, $k > 1$ and $0 < \int_0^b x^{p-k} f^p(x)dx < \infty$; and

$$\int_b^\infty x^{-k}\left(\int_x^\infty f(t)\,dt\right)^p dx < \left(\frac{p}{1-k}\right)^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] x^{p-k} f^p(x)\,dx,$$

whenever, $k < 1$ and $0 < \int_b^\infty x^{p-k} f^p(x)dx < \infty$; and

$$\int_0^b x^{\gamma-1} \exp\left(\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f(t)dt\right) dx \, < \, e^{\frac{\gamma}{\alpha}} \int_0^b \left[1-\left(\frac{x}{b}\right)^\alpha\right] x^{\gamma-1} f(x)\, dx,$$

whenever, $\alpha > 0$ and $0 < \int_0^b x^{\gamma-1} f(x)\, dx < \infty$; and

$$\int_b^\infty x^{\gamma-1} \exp\left(-\frac{\alpha}{x^\alpha}\int_x^\infty t^{\alpha-1}\log f(t)\, dt\right) dx \, < \, e^{\frac{\gamma}{\alpha}} \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{-\alpha}\right] x^{\gamma-1} f(x)\, dx,$$

whenever, $\alpha < 0$ and $0 < \int_b^\infty x^{\gamma-1} f(x)\, dx < \infty$.

$$\int_0^\infty x^{-k}\left(\int_0^x f(t)\, dt\right)^p dx \geq \left(\frac{p}{1-k}\right)^p \int_0^\infty x^{p-k} f^p(x)dx$$

for, $0 < p < 1 \, and \, k < 1$; and

$$\int_0^\infty x^{-k}\left(\int_0^x f(t)\, dt\right)^p dx \geq \left(\frac{p}{k-1}\right)^p \int_0^\infty x^{p-k} f^p(x)dx$$

for, $0 < p < 1 \, and \, k > 1$,where $f \in L^p(0,\infty)$ is non-negative function and $\alpha,\ \gamma,\ b,\ p,\ k \in \mathbb{R}$ such that $b > 0$, $\alpha \neq 0$, $p > 1$, $k \neq 1$. Now we shall give some improvements and reverses of these results.

4.2 Bounds For Hardy's Differences.

Bounds for Hardy differences, that is, improvements and reverses of the well-known Hardy Inequality are obtained .

4.2.1 Log Convexity Of Boas-Differences.

Lemma 4.2.1. *Let us define the function,*

$$\varphi_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0, 1; \\ -\log x, & s = 0; \\ x\log x, & s = 1. \end{cases}$$

Then $\varphi_s''(x) = x^{s-2}$ *that is* $\varphi_s(x)$ *is convex for* $x > 0$

The following lemma is equivalent to definition of convex function (see [56, p.2]).

Lemma 4.2.2. *If ϕ is continuous. Then ϕ is convex iff*

$$\phi(s_1)(s_3 - s_2) + \phi(s_2)(s_1 - s_3) + \phi(s_3)(s_2 - s_1) \geq 0.$$

for all s_1, s_2, s_3 in the open interval I such that $s_1 < s_2 < s_3$.

Now we will give $\log -$convexity of Boas-difference, that is, the difference of expressions given on two sides of (1.2.4).

Theorem 4.2.3. *Let the conditions of inequality (1.2.4) be satisfied, and $F : \mathbb{R} \to \mathbb{R}_+$ be a function defined by*

$$F(s) = \int_0^\infty x^{-1}\varphi_s(f(x))dx \, - \int_0^\infty x^{-1}\,\varphi_s\left(M^{-1}\int_0^\infty f(t\,x)\,dm(t)\right)dx. \qquad (4.2.1)$$

Then $F(s)$ is $\log -$convex, that is,

$$[F(p)]^{r-s} \leq [F(r)]^{p-s}[F(s)]^{r-p}, \quad \text{for } -\infty < r < s < p < \infty. \qquad (4.2.2)$$

Proof. Let us consider the function ϕ defined by

$$\phi(x) = u^2\,\varphi_s(x) + 2uw\,\varphi_r(x) + w^2\,\varphi_p(x)\ \text{, where } r = \frac{s+p}{2} \text{ and } u, w \in \mathbb{R}. \qquad (4.2.3)$$

$$\phi''(x) = u^2\,x^{s-2} + 2uw\,x^{r-2} + w^2\,x^{p-2} = \left(ux^{\frac{s}{2}-1} + wx^{\frac{p}{2}-1}\right)^2 \geq 0, \text{ for } x > 0.$$

We have that ϕ is convex for $x > 0$. Therefore (1.2.4) is equivalent to

$$u^2\,F(s) + 2uw\,F(r) + w^2\,F(p) \geq 0,$$

that is,

$$F^2(r) \leq F(s)\,F(p).$$

So F is $\log -$convex in Jensen sense. Since,

$$\lim_{s\to 0} F(s) = F(0) \text{ and } \lim_{s\to 1} F(s) = F(1).$$

F is continuous for $s \in \mathbb{R}$ and therefore $\log -$convex. □

4.2.2 Improvement And Reverse Of Hardy's Inequality

We state and prove an improvement and reverse of Hardy's inequality and of its dual inequality.

Theorem 4.2.4. *Let φ_s and F be as in Lemma 4.2.1 and Theorem 4.2.3 respectively. Let $g \in L^1(0,\infty)$ be non-negative function, then for $p \in \mathbb{R}\backslash\{0,1\}$*

$$\frac{1}{p(p-1)}\left[\left(\frac{p}{k-1}\right)^p \int_0^\infty x^{p-k}\, g^p(x)\, dx - \int_0^\infty x^{-k}\left(\int_0^x g(\rho)\, d\rho\right)^p dx\right] \leq \left(\frac{p}{k-1}\right)^p [H(s)]^{\frac{r-p}{r-s}}\, [H(r)]^{\frac{p-s}{r-s}} \quad (4.2.4)$$

for $-\infty < s < p < r < \infty$; and

$$\frac{1}{p(p-1)}\left[\left(\frac{p}{k-1}\right)^p \int_0^\infty x^{p-k}\, g^p(x) dx - \int_0^\infty x^{-k}\left(\int_0^x g(\rho)\, d\rho\right)^p dx\right] \geq \left(\frac{p}{k-1}\right)^p [H(s)]^{\frac{r-p}{r-s}} [H(r)]^{\frac{p-s}{r-s}} \quad (4.2.5)$$

for $-\infty < r < s < p < \infty$ and for $-\infty < p < r < s < \infty$. Where,

$$H(r) = \int_0^\infty x^{-1}\varphi_r\left(x^{\frac{p-k+1}{p}}\, g(x)\right) dx - \int_0^\infty x^{-1}\varphi_r\left(\frac{k-1}{p}x^{\frac{1-k}{p}}\int_0^x g(\rho)d\rho\right) dx. \quad (4.2.6)$$

Proof. Let, for $\alpha > 0$, $m(t)$ be defined by

$$m(t) = \begin{cases} \alpha^{-1}t^\alpha, & \text{for } 0 \leq t \leq 1; \\ \alpha^{-1}, & \text{for } t > 1. \end{cases}$$

Then $F(s)$ from (4.2.1) becomes

$$\begin{aligned} F_\alpha(s) &= \int_0^\infty x^{-1}\varphi_s(f(x))dx - \int_0^\infty x^{-1}\varphi_s\left(\alpha\int_0^1 f(t\,x) d\left(\frac{t^\alpha}{\alpha}\right)\right) dx. \\ &= \int_0^\infty x^{-1}\varphi_s(f(x))dx - \int_0^\infty x^{-1}\varphi_s\left(\alpha\int_0^1 f(t\,x)\, t^{\alpha-1}\, dt\right) dx. \end{aligned}$$

Put $\rho = tx$, so that $d\rho = x\,dt$; For, $0 \leq t \leq 1$ we have $0 \leq \rho \leq x$.

Now

$$\begin{aligned} F_\alpha(s) &= \int_0^\infty x^{-1}\,\varphi_s(f(x))dx - \int_0^\infty x^{-1}\,\varphi_s\left(\alpha\int_0^x f(\rho)\left(\frac{\rho}{x}\right)^{\alpha-1}\frac{d\rho}{x}\right)\,dx. \\ &= \int_0^\infty x^{-1}\,\varphi_s(f(x))dx - \int_0^\infty x^{-1}\varphi_s\left(\alpha\,x^{-\alpha}\int_0^x \rho^{\alpha-1}f(\rho)\,d\rho\right)\,dx. \end{aligned}$$

Since $F_\alpha(p)$ is log $-$convex, it satisfies

$$[F_\alpha(p)]^{r-s} \leq [F_\alpha(r)]^{p-s}[F_\alpha(s)]^{r-p} \tag{4.2.7}$$

Now put $\alpha = \frac{k-1}{p}$ $(p \neq 0)$ and $f(\rho) = \rho^{1-\alpha}\,g(\rho)$ in (4.2.7) to obtain

$$\int_0^\infty x^{-1}\varphi_p\left(x^{\frac{p-k+1}{p}}g(x)\right)dx - \int_0^\infty x^{-1}\varphi_p\left(\frac{k-1}{p}x^{\frac{1-k}{p}}\int_0^x g(\rho)d\rho\right)dx \leq [H(s)]^{\frac{r-p}{r-s}}[H(r)]^{\frac{p-s}{r-s}} \tag{4.2.8}$$

for $-\infty < s < p < r < \infty$. From (4.2.8) for $p \in \mathbb{R}\backslash\{0,1\}$ we get (4.2.4).

If in (4.2.7) $s \to r$, $p \to s$ and $r \to p$ we have

$$[F_\alpha(p)]^{s-r} \geq [F_\alpha(r)]^{s-p}\ [F_\alpha(s)]^{p-r}.$$

Put $\alpha = \frac{k-1}{p}$ $(p \neq 0)$ and $f(\rho) = \rho^{\alpha-1}\,g(\rho)$ to obtain

$$\int_0^\infty x^{-1}\,\varphi_p\left(x^{\frac{p-k+1}{p}}g(x)\right)dx - \int_0^\infty x^{-1}\,\varphi_p\left(\frac{k-1}{p}x^{\frac{1-k}{p}}\int_0^x g(\rho)\ d\rho\right)dx \geq [H(s)]^{\frac{r-p}{r-s}}[H(r)]^{\frac{p-s}{r-s}} \tag{4.2.9}$$

for $-\infty < p < r < s < \infty$. And from here for $p \in \mathbb{R}\backslash\{0,1\}$ we get (4.2.5).

Now set in (4.2.7) $s \to p$, $p \to r$ and $r \to s$, to obtain

$$[F_\alpha(p)]^{s-r} \geq [F_\alpha(r)]^{s-p}[F_\alpha(s)]^{p-r}.$$

For $\alpha = \frac{k-1}{p}$ $(p \neq 0)$, $f(\rho) = \rho^{\alpha-1}\,g(\rho)$ and $p \in \mathbb{R}\backslash\{0,1\}$. we get (4.2.5) for $-\infty < p < r < s < \infty$. □

Remark 4.2.1. In fact we have proved the more general results. Namely (4.2.8) is valid for $-\infty < s < p < r < \infty$; the inequality (4.2.9) is valid for $-\infty < r < s < p < \infty$, and for $-\infty < p < r < s < \infty$.

Now we give the dual result to Theorem 4.2.4.

Theorem 4.2.5. *Let φ_s and F be as in Lemma 4.2.1 and Theorem 4.2.3 respectively. Let $g \in L^1(0,\infty)$ be non-negative function, then for $p \in \mathbb{R}\backslash\{0,1\}$*

$$\frac{1}{p(p-1)}\left[\left(\frac{p}{1-k}\right)^p \int_0^\infty x^{p-k}\, g^p(x)dx - \int_0^\infty x^{-k}\left(\int_x^\infty g(\rho)\,d\rho\right)^p dx\right] \le \left(\frac{p}{1-k}\right)^p [\tilde{H}(r)]^{\frac{p-s}{r-s}} [\tilde{H}(s)]^{\frac{r-p}{r-s}} \quad (4.2.10)$$

for $-\infty < s < p < r < \infty$; and

$$\frac{1}{p(p-1)}\left[\left(\frac{p}{1-k}\right)^p \int_0^\infty x^{p-k}\, g^p(x)dx - \int_0^\infty x^{-k}\left(\int_x^\infty g(\rho)\,d\rho\right)^p dx\right] \ge \left(\frac{p}{1-k}\right)^p [\tilde{H}(r)]^{\frac{p-s}{r-s}} [\tilde{H}(s)]^{\frac{r-p}{r-s}} \quad (4.2.11)$$

for $-\infty < r < s < p < \infty$ and for $-\infty < p < r < s < \infty$. Where,

$$\tilde{H}(r) = \int_0^\infty x^{-1}\,\varphi_r\left(x^{\frac{p-k+1}{p}}\, g(x)\right) dx - \int_0^\infty x^{-1}\varphi_r\left(\frac{1-k}{p} x^{\frac{1-k}{p}} \int_x^\infty g(\rho)d\rho\right) dx. \quad (4.2.12)$$

Proof. Let, for $\beta > 0$, $m(t)$ be defined by

$$m(t) = \begin{cases} 0, & \text{for } 0 \le t \le 1; \\ \beta^{-1}(1-t^{-\beta}), & \text{for } t > 1. \end{cases}$$

Then $F(s)$ from (4.2.1) becomes

$$\begin{aligned} F_\beta(s) &= \int_0^\infty x^{-1}\varphi_s(f(x))dx - \int_0^\infty x^{-1}\,\varphi_s\left(\int_1^\infty f(tx)\right) d(1-t^{-\beta})dx \\ &= \int_0^\infty x^{-1}\varphi_s(f(x))dx - \int_0^\infty x^{-1}\,\varphi_s\left(\beta\int_1^\infty f(tx)\,t^{-\beta-1}\,dt\right) dx. \end{aligned}$$

Put $\rho = t\,x$, so that $d\rho = x\,dt$; For, $t > 1$ we have $\rho > x$.

Now

$$\begin{aligned} F_\beta(s) &= \int_0^\infty x^{-1}\,\varphi_s(f(x))dx - \int_0^\infty x^{-1}\,\varphi_s\left(\beta\int_x^\infty f(\rho)\,\left(\frac{x}{\rho}\right)^{1+\beta}\frac{d\rho}{x}\right)dx \\ &= \int_0^\infty x^{-1}\,\varphi_s(f(x))dx - \int_0^\infty x^{-1}\,\varphi_s\left(\beta\,x^\beta\int_x^\infty \rho^{-\beta-1}\,f(\rho)d\rho\right)\,dx. \end{aligned}$$

$F_\beta(p)$ is log $-$convex. $i.e.$, it satisfies

$$[F_\beta(p)]^{r-s} \;\le\; [F_\beta(r)]^{p-s}\,[F_\beta(s)]^{r-p}, \quad for \quad -\infty < r < s < p < \infty. \tag{4.2.13}$$

Now put $\beta = \frac{1-k}{p}$ $(p \neq 0)$ and $f(\rho) = \rho^{1+\beta}\,g(\rho)$ in (4.2.13) to obtain

$$\int_0^\infty x^{-1}\,\varphi_p\left(x^{\frac{p-k+1}{p}}\,g(x)\right)dx - \int_0^\infty x^{-1}\,\varphi_p\left(\frac{1-k}{p}\,x^{\frac{1-k}{p}}\int_x^\infty g(\rho)d\rho\right)dx$$
$$\le [\tilde{H}(r)]^{\frac{p-s}{r-s}}[\tilde{H}(s)]^{\frac{r-p}{r-s}}, \tag{4.2.14}$$

for $-\infty < s < p < r < \infty$. From (4.2.14) for $p \in \mathbb{R}\backslash\{0,1\}$ we get (4.2.10). If in (4.2.13) $s \to r$, $p \to s$ and $r \to p$ we have

$$[F_\beta(p)]^{s-r} \ge [F_\beta(r)]^{s-p}\,[F_\beta(s)]^{p-r}.$$

Put $\beta = \frac{1-k}{p}$ $(p \neq 0)$ and $f(\rho) = \rho^{1+\beta}\,g(\rho)$ to obtain

$$\int_0^\infty x^{-1}\,\varphi_p\left(x^{\frac{p-k+1}{p}}\,g(x)\right)dx - \int_0^\infty x^{-1}\,\varphi_p\left(\frac{1-k}{p}\,x^{\frac{1-k}{p}}\int_x^\infty g(\rho)\;d\rho\right)dx$$
$$\ge [\tilde{H}(r)]^{\frac{s-p}{s-r}}\,[\tilde{H}(s)]^{\frac{p-r}{s-r}}, \tag{4.2.15}$$

for $-\infty < r < s < p < \infty$. And from here for $p \in \mathbb{R}\backslash\{0,1\}$ we get (4.2.11). Now set in (4.2.13) $s \to p$, $p \to r$ and $r \to s$. We have

$$[F_\beta(p)]^{s-r} \ge [F_\beta(r)]^{s-p}\,[F_\beta(s)]^{p-r}.$$

For $\beta = \frac{1-k}{p}$ $(p \neq 0)$, $f(\rho) = \rho^{1+\beta}\,g(\rho)$ and $p \in \mathbb{R}\backslash\{0,1\}$ we get (4.2.11). □

Remark 4.2.2. In fact we have proved the more general results. Namely (4.2.14) is valid for $-\infty < s < p < r < \infty$; the inequality (4.2.15) is valid for $-\infty < r < s < p < \infty$ and for $-\infty < p < r < s < \infty$.

4.3 Bounds For Strengthened Hardy and Polya-Knopp's Differences

Here we have obtained the bounds for strengthened Hardy-Knopp differences, that is, improvements and Reverses of Hardy-Knopp type inequality and its dual inequality.

4.3.1 Log Convexity Of Strengthened Hardy-Polya-Knopp Differences.

Lemma 4.3.1. *Let us define another function,*

$$\psi_s(x) = \begin{cases} \frac{1}{s^2} e^{sx}, & s \neq 0; \\ \frac{1}{2} x^2, & s = 0. \end{cases}$$

Then $\psi_s''(x) = e^{sx}$*, that is,* $\psi_s(x)$ *is convex.*

Theorem 4.3.2. *Let the conditions of Theorem 1.2.1 be satisfied and* φ_s *given by Lemma 4.2.1. Let* $\mathfrak{A} : \mathbb{R} \to \mathbb{R}_+$ *be defined by*

$$\mathfrak{A}(s) = \int_0^b v(x)\, \varphi_s(f(x))\, \frac{dx}{x} - \int_0^b u(x) \varphi_s\left(\frac{1}{x} \int_0^x f(t)\, dt\right) \frac{dx}{x}. \tag{4.3.1}$$

Then $\mathfrak{A}$ *is* $\log -$*convex, that is, the following inequality is valid.*

$$[\mathfrak{A}(p)]^{r-s} \leq [\mathfrak{A}(r)]^{p-s}\, [\mathfrak{A}(s)]^{r-p} \quad for \quad -\infty < s < p < r < \infty. \tag{4.3.2}$$

Proof. Let us consider the function ϕ defined by

$$\phi(x) = u^2\, \varphi_s(x) + 2\, u\, w \varphi_r(x) + w^2\, \varphi_p(x), \quad where \quad r = \frac{s+p}{2};\; u, w \in \mathbb{R}.$$

$$\phi''(x) = u^2\, x^{s-2} + 2\, u\, w x^{r-2} + w^2\, x^{p-2} = \; (u\, x^{\frac{s}{2}-1} + w\, x^{\frac{p}{2}-1})^2 \geq \; 0.$$

ϕ is convex for $x \in \mathbb{R}_+$, therefore (1.2.6) is equivalent to

$$u^2\, \mathfrak{A}(s) + 2\, u\, w\, \mathfrak{A}(r) + w^2\, \mathfrak{A}(p) \geq 0,$$

i.e.,

$$\mathfrak{A}^2(r) \; \leq \mathfrak{A}(s)\, \mathfrak{A}(p).$$

So $\mathfrak{A}$ is $\log -$convex in Jensen sense. Since,

$$\lim_{s \to 0} \mathfrak{A}(s) = \mathfrak{A}(0) \quad \textit{and} \quad \lim_{s \to 1} \mathfrak{A}(s) = \mathfrak{A}(1).$$

$\mathfrak{A}$ is continuous for $s \in \mathbb{R}$ and therefore $\log \mathfrak{A}$ is convex. □

Similar consequence of Theorem 1.2.2 is

Theorem 4.3.3. *Let the conditions of Theorem 1.2.2 be satisfied and φ_s be given by Lemma 4.2.1. Let $\mathfrak{Z} : \mathbb{R} \to \mathbb{R}_+$ be defined by*

$$\mathfrak{Z}(s) = \int_b^\infty v(x)\, \varphi_s(f(x))\, \frac{dx}{x} \; - \; \int_b^\infty u(x)\, \varphi_s\left(x \int_x^\infty \frac{f(t)}{t^2}\; dt\right)\, \frac{dx}{x}. \tag{4.3.3}$$

Then $\mathfrak{Z}$ is $\log -$convex, that is, the following inequality is valid.

$$[\mathfrak{Z}(p)]^{r-s} \; \leq \; [\mathfrak{Z}(r)]^{p-s}\, [\mathfrak{Z}(s)]^{r-p} \quad \textit{for} \quad -\infty < s < p < r < \infty. \tag{4.3.4}$$

Similar variants of Theorems 4.3.2 and 4.3.3 for ψ_s can be given as

Theorem 4.3.4. *Let the conditions of Theorem 1.2.1 be satisfied and ψ_s be given by Lemma 4.3.1. Let $\mathfrak{Y} : \mathbb{R} \to \mathbb{R}_+$ be defined by*

$$\mathfrak{Y}(s) = \int_0^b v(x)\, \psi_s(f(x))\, \frac{dx}{x} \; - \; \int_0^b u(x)\, \psi_s\left(\frac{1}{x} \int_0^x f(t)\; dt\right) \frac{dx}{x}. \tag{4.3.5}$$

Then $\mathfrak{Y}$ is $\log -$convex, that is, the following inequality is valid.

$$[\mathfrak{Y}(p)]^{r-s} \; \leq \; [\mathfrak{Y}(r)]^{p-s} [\mathfrak{Y}(s)]^{r-p} \quad \textit{for} \quad -\infty < s < p < r < \infty. \tag{4.3.6}$$

Theorem 4.3.5. *Let the conditions of Theorem 1.2.2 be satisfied and ψ_s be given by Lemma 4.3.1. Let $\mathfrak{X} : \mathbb{R} \to \mathbb{R}_+$ be defined by*

$$\mathfrak{X}(s) = \int_b^\infty v(x)\, \psi_s(f(x))\, \frac{dx}{x} - \int_b^\infty u(x)\, \psi_s\left(x \int_x^\infty \frac{f(t)}{t^2}\, dt\right) \frac{dx}{x}. \tag{4.3.7}$$

Then $\mathfrak{X}$ is log *−convex, that is, the following inequality is valid.*

$$[\mathfrak{X}(p)]^{r-s} \leq [\mathfrak{X}(r)]^{p-s}[\mathfrak{X}(s)]^{r-p} \quad \textit{for} \quad -\infty < s < p < r < \infty. \tag{4.3.8}$$

4.3.2 Improvements And Reverses Of Hardy's Inequality

We state and prove an improvement and reverse of strengthened classical Hardy's inequality and it dual.

Theorem 4.3.6. *Let $k, b \in \mathbb{R}$ be such that $k \neq 1$ and $b > 0$, let f be a non-trivial and non-negative function, and let $p \in \mathbb{R}\backslash\{0, 1\}$*

(i) *If $\frac{k-1}{p} > 0$, then*

$$\frac{1}{p(p-1)}\left[\left(\frac{p}{k-1}\right)^p \int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x) dx - \int_0^b x^{-k}\left(\int_0^x f(t)dt\right)^p dx\right] \leq \left(\frac{p}{k-1}\right)^p [\mathfrak{a}(s)]^{\frac{p-r}{s-r}} [\mathfrak{a}(r)]^{\frac{s-p}{s-r}}, \tag{4.3.9}$$

for $-\infty < s < p < r < \infty$, and

$$\frac{1}{p(p-1)}\left[\left(\frac{p}{k-1}\right)^p \int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x) dx - \int_0^b x^{-k}\left(\int_0^x f(t)dt\right)^p dx\right] \geq \left(\frac{p}{k-1}\right)^p [\mathfrak{a}(s)]^{\frac{p-r}{s-r}} [\mathfrak{a}(r)]^{\frac{s-p}{s-r}}, \tag{4.3.10}$$

for $-\infty < p < r < s < \infty$ and $-\infty < r < s < p < \infty$. Where,

$$\mathfrak{a}(r) = \int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] \varphi_r\left(x^{\frac{p-k+1}{p}} f(x)\right) \frac{dx}{x} - \int_0^b \varphi_r\left(\frac{k-1}{p} x^{\frac{1-k}{p}} \int_0^x f(t)dt\right) \frac{dx}{x}. \tag{4.3.11}$$

(ii) *If* $\frac{1-k}{p} > 0$, *then*

$$\frac{1}{p(p-1)}\left[\left(\frac{p}{1-k}\right)^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] x^{p-k} f^p(x)dx - \int_b^\infty x^{-k}\left(\int_x^\infty f(t)dt\right)^p dx\right]$$
$$\leq \left(\frac{p}{1-k}\right)^p [\mathfrak{b}(s)]^{\frac{r-p}{r-s}} [\mathfrak{b}(r)]^{\frac{p-s}{r-s}}. \quad (4.3.12)$$

for $-\infty < s < p < r < \infty$; *and*

$$\frac{1}{p(p-1)}\left[\left(\frac{p}{1-k}\right)^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] x^{p-k} f^p(x)dx - \int_b^\infty x^{-k}\left(\int_x^\infty f(t)dt\right)^p dx\right]$$
$$\geq \left(\frac{p}{1-k}\right)^p [\mathfrak{b}(s)]^{\frac{r-p}{r-s}} [\mathfrak{b}(r)]^{\frac{p-s}{r-s}}., \quad (4.3.13)$$

for $-\infty < p < r < s < \infty$ *and* $-\infty < r < s < p < \infty$. *Where,*

$$\mathfrak{b}(r) = \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] \varphi_r\left(x^{\frac{p-k+1}{p}} f(x)\right)\frac{dx}{x} -$$
$$\int_b^\infty \varphi_r\left(\frac{1-k}{p} x^{\frac{-k+1}{p}} \int_x^\infty f(t)dt\right)\frac{dx}{x}. \quad (4.3.14)$$

Proof. The proof follows from Theorems 4.3.2 and 4.3.3 by choosing the weight function $u(x) = 1$ (so that $v(x) = 1 - \frac{x}{b}$ and $v(x) = 1 - \frac{b}{x}$).

Consider the case when $k > 1$ first. Let $\alpha > 0$; by replacing the parameter b by $a(= b^\alpha)$ and choosing for f the function $x \mapsto f(x^{\alpha^{-1}})\, x^{\alpha^{-1}-1}$ (4.3.1) becomes

$$\mathfrak{A}_\alpha(s) = \int_0^a \left(1 - \frac{x}{a}\right) \varphi_s\left(f(x^{\alpha^{-1}})\, x^{\alpha^{-1}-1}\right)\frac{dx}{x} -$$
$$\int_0^a \varphi_s\left(\frac{1}{x}\int_0^x f(t^{\alpha^{-1}})\, t^{\alpha^{-1}-1} dt\right)\frac{dx}{x}, \quad (4.3.15)$$

while (4.3.2) becomes

$$[\mathfrak{A}_\alpha(p)]^{r-s} \leq [\mathfrak{A}_\alpha(r)]^{p-s}\, [\mathfrak{A}_\alpha(s)]^{r-p}, \quad (4.3.16)$$

i.e., $\mathfrak{A}_\alpha(s)$ is log $-$convex. Of course, we can give simpler form for $\mathfrak{A}_\alpha$. By the substitutions $l = t^{\alpha^{-1}}$ and $y = x^{\alpha^{-1}}$ respectively we have

$$\mathfrak{A}_\alpha(s) = \alpha\left[\int_0^{a^{\alpha^{-1}}} \left(1 - \left(\frac{y}{b}\right)^\alpha\right) \varphi_s\left(f(y)\, y^{1-\alpha}\right)\frac{dy}{y} - \int_0^{a^{\alpha^{-1}}} \varphi_s\left(\alpha\, y^{-\alpha}\int_0^y f(l)\, dl\right)\frac{dy}{y}\right],$$

i.e.,

$$\mathfrak{A}_\alpha(s) = \alpha \left[\int_0^b \left(1 - \left(\frac{x}{b}\right)^\alpha\right) \varphi_s \left(f(x)x^{1-\alpha}\right) \frac{dx}{x} - \int_0^b \varphi_s \left(\alpha\, x^{-\alpha} \int_0^x f(l)\, dl\right) \frac{dx}{x} \right].$$

For $\alpha = \frac{k-1}{p}$, we have

$$\mathfrak{A}_{\frac{k-1}{p}}(s) = \frac{k-1}{p} \left[\int_0^b \left(1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right) \varphi_s \left(f(x)x^{\frac{p-k+1}{p}}\right) \frac{dx}{x} - \int_0^b \varphi_s \left(\frac{k-1}{p}\, x^{\frac{1-k}{p}} \int_0^x f(l)\, dl\right) \frac{dx}{x} \right]$$

From here (4.3.16) reduces to

$$\int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] \varphi_p \left(f(x)\; x^{\frac{p-k+1}{p}}\right) \frac{dx}{x} - \int_0^b \varphi_p \left(\frac{k-1}{p} x^{\frac{1-k}{p}} \int_0^x f(l)\, dl\right) \frac{dx}{x} \leq [\mathfrak{a}(s)]^{\frac{p-r}{s-r}} [\mathfrak{a}(r)]^{\frac{s-p}{s-r}}. \quad (4.3.17)$$

For $p \in \mathbb{R}\backslash\{0,1\}$, we get (4.3.9).

If in (4.3.16) $s \to r,\; p \to s,\; r \to p$ and $s \to p,\; p \to r,\; r \to s,$ then for $\alpha = \frac{k-1}{p}$ we have

$$\int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] \varphi_p \left(f(x)\; x^{\frac{p-k+1}{p}}\right) \frac{dx}{x} - \int_0^b \varphi_p \left(\frac{k-1}{p} x^{\frac{1-k}{p}} \int_0^x f(l)\, dl\right) \frac{dx}{x} \geq [\mathfrak{a}(s)]^{\frac{p-r}{s-r}} [\mathfrak{a}(r)]^{\frac{s-p}{s-r}}. \quad (4.3.18)$$

And from here for $p \in \mathbb{R}\backslash\{0,1\}$, we get (4.3.10).

Now, suppose that $k < 1$, we choose, again, the weight function $u(x) = 1$ so that $v(x) = 1 - \frac{b}{x}$. Let $\beta > 0$. Now by replacing the parameter b by a $(= b^\beta)$ and by choosing for f the function $x \mapsto f(x^{\beta^{-1}})\, x^{\beta^{-1}+1}$ (4.3.3) becomes

$$\mathfrak{Z}_\beta(s) = \int_a^\infty \left[1 - \frac{a}{x}\right] \varphi_s \left(f(x^{\beta^{-1}}) x^{\beta^{-1}+1}\right) \frac{dx}{x} - \int_a^\infty \varphi_s \left(x \int_x^\infty f(t^{\beta^{-1}}) t^{\beta^{-1}-1}\, dt\right) \frac{dx}{x}, \quad (4.3.19)$$

while (4.3.4) becomes

$$[\mathfrak{Z}_\beta(p)]^{r-s} \;\leq\; [\mathfrak{Z}_\beta(r)]^{p-s} \;\; [\mathfrak{Z}_\beta(s)]^{r-p}. \quad (4.3.20)$$

i.e, $\mathfrak{Z}_\beta(s)$ is log $-$convex. Of course, we can give simpler form for $\mathfrak{Z}_\beta$. By the substitutions $l = t^{\beta^{-1}}$ and $y = x^{\beta^{-1}}$ respectively we have

$$\mathfrak{Z}_\beta(s) = \beta \int_{a^{\beta^{-1}}}^{\infty} \left(1 - \frac{a}{y^\beta}\right) \varphi_s \left(y^{1+\beta} f(y)\right) \frac{dy}{y} - \beta \int_{a^{\beta^{-1}}}^{\infty} \varphi_s \left(\beta\, y^\beta \int_y^\infty f(l)dl\right) \frac{dy}{y}$$

i.e.,

$$\mathfrak{Z}_\beta(s) = \beta \left[\int_b^\infty \left(1 - \left(\frac{b}{x}\right)^\beta\right) \varphi_s \left(x^{1+\beta} f(x)\right) \frac{dx}{x} - \int_b^\infty \varphi_s \left(\beta\, x^\beta \int_x^\infty f(l)dl\right) \frac{dx}{x}\right]$$

For $\beta = \frac{1-k}{p}$, we have

$$\mathfrak{Z}_{\frac{1-k}{p}}(s) = \frac{1-k}{p} \left[\int_b^\infty \left(1 - \left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right) \varphi_s \left(x^{\frac{p-k+1}{p}} f(x)\right) \frac{dx}{x} - \right.$$
$$\left. \int_b^\infty \varphi_s \left(\frac{1-k}{p} x^{\frac{1-k}{p}} \int_x^\infty f(l)dl\right) \frac{dx}{x}\right]$$

From here (4.3.20) reduces to

$$\int_b^\infty \left[1 - \left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] \varphi_p \left(f(x) x^{\frac{p-k+1}{p}}\right) \frac{dx}{x} - \int_b^\infty \varphi_p \left(\frac{1-k}{p} x^{\frac{1-k}{p}} \int_x^\infty f(l)dl\right) \frac{dx}{x}$$
$$\leq [\mathfrak{b}(s)]^{\frac{p-r}{s-r}} [\mathfrak{b}(r)]^{\frac{s-p}{s-r}}. \quad (4.3.21)$$

For $p \in \mathbb{R}\backslash\{0,1\}$, we get (4.3.12).

If in (4.3.20) $s \to r,\ p \to s,\ r \to p;\ \mathit{and}\ s \to p,\ p \to r,\ r \to s$ then for $\beta = \frac{1-k}{p}$ we have

$$\int_b^\infty \left[1 - \left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] \varphi_p \left(f(x) x^{\frac{p-k+1}{p}}\right) \frac{dx}{x} - \int_b^\infty \varphi_p \left(\frac{1-k}{p} x^{\frac{1-k}{p}} \int_x^\infty f(l)dl\right) \frac{dx}{x}$$
$$\geq [\mathfrak{b}(s)]^{\frac{p-r}{s-r}} [\mathfrak{b}(r)]^{\frac{s-p}{s-r}}. \quad (4.3.22)$$

And from here for $p \in \mathbb{R}\backslash\{0,1\}$, we get (4.3.13). □

Remark 4.3.1. In fact we have proved the more general results. Namely (4.3.17) and (4.3.21) are valid $for\ -\infty < s < p < r < \infty$; the inequalities (4.3.18) and (4.3.22) are valid for $-\infty < r < s < p < \infty$, and $-\infty < p < r < s < \infty$.

4.3.3 Improvements And Reverses of Polya-Knopp Inequality

We state and prove an improvement and reverse of Polya-Knopp, inequality and of its dual.

Theorem 4.3.7. *Let $\alpha, \gamma, b \in \mathbb{R}$ be such that $\alpha \neq 0$, $b > 0$, and let f be a positive function,*

(***i***) *If $\alpha > 0$, then*

$$e^{\frac{\gamma}{\alpha}} \int_0^b \left[1 - \left(\frac{x}{b}\right)^\alpha\right] x^{\gamma-1} f(x)dx - \int_0^b x^{\gamma-1} \exp\left(\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f(t)dt\right)dx$$
$$\leq e^{\frac{\gamma}{\alpha}}[\mathfrak{p}(r)]^{\frac{1-s}{r-s}}[\mathfrak{p}(s)]^{\frac{r-1}{r-s}} \quad (4.3.23)$$

for $-\infty < s < 1 < r < \infty$; and

$$e^{\frac{\gamma}{\alpha}} \int_0^b \left[1 - \left(\frac{x}{b}\right)^\alpha\right] x^{\gamma-1} f(x)dx - \int_0^b x^{\gamma-1} \exp\left(\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f(t)dt\right)dx$$
$$\geq e^{\frac{\gamma}{\alpha}}[\mathfrak{p}(r)]^{\frac{1-s}{r-s}}[\mathfrak{p}(s)]^{\frac{r-1}{r-s}} \quad (4.3.24)$$

for $-\infty < 1 < r < s < \infty$ and $-\infty < r < s < 1 < \infty$. Where,

$$\mathfrak{p}(s) = \int_0^b \left[1 - \left(\frac{x}{b}\right)^\alpha\right] \psi_s\left(\log(x^\gamma f(x))\right)\frac{dx}{x} - \int_0^b \psi_s\left[\alpha x^{-\alpha}\int_0^x l^{\alpha-1}\log(l^\gamma f(l))dl\right]\frac{dx}{x}. \quad (4.3.25)$$

(***ii***) *If $\alpha < 0$, then*

$$e^{\frac{\gamma}{\alpha}} \int_b^\infty \left(1 - \left(\frac{b}{x}\right)^{-\alpha}\right) x^{\gamma-1} f(x)dx - \int_b^\infty x^{\gamma-1}\exp\left(-\frac{\alpha}{x^\alpha}\int_x^\infty t^{\alpha-1}\log f(t)dt\right)dx$$
$$\leq e^{\frac{\gamma}{\alpha}}\left[\mathfrak{q}(r)\right]^{\frac{1-s}{r-s}}\left[\mathfrak{q}(s)\right]^{\frac{r-1}{r-s}}. \quad (4.3.26)$$

for $-\infty < s < 1 < r < \infty$ *; and*

$$e^{\frac{\gamma}{\alpha}} \int_b^\infty \left(1 - \left(\frac{b}{x}\right)^{-\alpha}\right) x^{\gamma-1} f(x) dx - \int_b^\infty x^{\gamma-1} \exp\left(-\frac{\alpha}{x^\alpha} \int_x^\infty t^{\alpha-1} \log f(t) dt\right) dx$$
$$\geq e^{\frac{\gamma}{\alpha}} \left[\mathfrak{q}(r)\right]^{\frac{1-s}{r-s}} \left[\mathfrak{q}(s)\right]^{\frac{r-1}{r-s}}. \quad (4.3.27)$$

for $-\infty < 1 < r < s < \infty$ *and* $-\infty < r < s < 1 < \infty$*. Where,*

$$\mathfrak{q}(s) = \int_b^\infty \left(1 - \left(\frac{b}{x}\right)^{-\alpha}\right) \psi_s \left(\log(x^\gamma f(x))\right) \frac{dx}{x} -$$
$$\int_b^\infty \psi_s \left(-\alpha x^{-\alpha} \int_x^\infty l^{\alpha-1} \log(l^\gamma f(l))\, dl\right) \frac{dx}{x}. \quad (4.3.28)$$

Proof. The proof follows from Theorems 4.3.4 and 4.3.5 by choosing the weight function $u(x) = 1$ (so that $v(x) = 1 - \frac{x}{b}$ and $v(x) = 1 - \frac{b}{x}$).

Let $\alpha > 0$. By replacing the parameter b by a $(= b^\alpha)$ and choosing for the function f, $x \mapsto \log\left(x^{\frac{\gamma}{\alpha}} f(x^{\frac{1}{\alpha}})\right)$. Then (4.3.5) becomes

$$\mathfrak{Y}_\alpha(s) = \int_0^a \left(1 - \frac{x}{a}\right) \psi_s \left(\log(x^{\frac{\gamma}{\alpha}} f(x^{\frac{1}{\alpha}}))\right) \frac{dx}{x} -$$
$$\int_0^a \psi_s \left(\frac{1}{x} \int_0^x \log(t^{\frac{\gamma}{\alpha}} f(t^{\frac{1}{\alpha}})) dt\right) \frac{dx}{x}. \quad (4.3.29)$$

while, (4.3.6) becomes

$$[\mathfrak{Y}_\alpha(p)]^{r-s} \leq [\mathfrak{Y}_\alpha(r)]^{p-s} [\mathfrak{Y}_\alpha(s)]^{r-p}. \quad (4.3.30)$$

i.e., $\mathfrak{Y}_\alpha(s)$ is log −convex. Of course we can give simpler form for $\mathfrak{Y}_\alpha$. By the substitutions $l = t^{\alpha^{-1}}$ and $y = x^{\alpha^{-1}}$ respectively we have

$$\mathfrak{Y}_\alpha(s) = \alpha \int_0^{a^{\alpha^{-1}}} \left(1 - \frac{y^\alpha}{a}\right) \psi_s \left(\log(y^\gamma\, f(y))\right) \frac{dy}{y} -$$
$$\alpha \int_0^{a^{\alpha^{-1}}} \psi_s \left(\alpha\, y^{-\alpha} \int_0^y l^{\alpha-1} \log(l^\gamma f(l)) dl\right) \frac{dy}{y}$$

i.e.,

$$\mathfrak{Y}_\alpha(s) = \alpha\left[\int_0^b \left(1-\left(\frac{x}{b}\right)^\alpha\right)\psi_s\left(\log(x^\gamma f(x))\right)\frac{dx}{x} - \int_0^b \psi_s\left(\alpha x^{-\alpha}\int_0^x l^{\alpha-1}\log(l^\gamma f(l))dl\right)\frac{dx}{x}\right]$$

From here (4.3.30) is equivalent to

$$\int_0^b \left(1-\left(\frac{x}{b}\right)^\alpha\right)\psi_p\left(\log(x^\gamma f(x))\right)\frac{dx}{x} - \int_0^b \psi_p\left(\alpha x^{-\alpha}\int_0^x l^{\alpha-1}\, log(l^\gamma f(l))\, dl\right)\frac{dx}{x} \leq [\mathfrak{p}(r)]^{\frac{p-s}{r-s}}[\mathfrak{p}(s)]^{\frac{r-p}{r-s}}. \quad (4.3.31)$$

And from here for $p = 1$ we get (4.3.23).

If in (4.3.30) $s \to r,\ p \to s,\ r \to p$ and $s \to p,\ p \to r,\ r \to s$, then we have

$$\int_0^b \left(1-\left(\frac{x}{b}\right)^\alpha\right)\psi_p\left(\log(x^\gamma f(x))\right)\frac{dx}{x} - \int_0^b \psi_p\left(\alpha x^{-\alpha}\int_0^x l^{\alpha-1}\, \log(l^\gamma f(l))\, dl\right)\frac{dx}{x} \geq [\mathfrak{p}(r)]^{\frac{p-s}{r-s}}[\mathfrak{p}(s)]^{\frac{r-p}{r-s}}. \quad (4.3.32)$$

And from here for $p = 1$ we get (4.3.24).

For the case when $\alpha < 0$, we make substitution $x \mapsto \log\left(x^{-\frac{\gamma}{\alpha}}f(x^{-\frac{1}{\alpha}})\right)$ for the function f and replace parameter b by a $(= b^{-\alpha})$. Then (4.3.7) becomes

$$\mathfrak{X}_\alpha(s) = \int_a^\infty \left(1-\frac{a}{x}\right)\psi_s\left(\log(x^{-\frac{\gamma}{\alpha}}f(x^{-\frac{1}{\alpha}}))\right)\frac{dx}{x} - \int_a^\infty \psi_s\left(x\int_x^\infty \log(t^{-\frac{\gamma}{\alpha}}f(t^{-\frac{1}{\alpha}}))\frac{dt}{t^2}\right)\frac{dx}{x} \quad (4.3.33)$$

while, (4.3.8) becomes

$$[\mathfrak{X}_\alpha(p)]^{r-s} \leq [\mathfrak{X}_\alpha(r)]^{p-s}[\mathfrak{X}_\alpha(s)]^{r-p}. \quad (4.3.34)$$

i.e., $\mathfrak{X}_\alpha(s)$ is $\log -$convex. Of course we can give the simpler form for $\mathfrak{X}_\alpha$. By the substitutions $l = t^{-\alpha^{-1}}$ and $y = x^{-\alpha^{-1}}$ respectively, we have

$$\mathfrak{X}_\alpha(s) = -\alpha \int_{a^{-\alpha^{-1}}}^{\infty} (1 - a\,y^\alpha)\ \psi_s\left(\log(y^\gamma f(y))\right) \frac{dy}{y} + \alpha \int_{a^{-\alpha^{-1}}}^{\infty} \psi_s\left(-\alpha\, y^{-\alpha} \int_y^\infty l^{\alpha-1} \log(l^\gamma f(l)) dl\right) \frac{dy}{y}$$

i.e.,

$$\mathfrak{X}_\alpha(s) = -\alpha \left[\int_b^\infty \left(1 - \left(\frac{x}{b}\right)^\alpha\right) \psi_s\left(\log(x^\gamma f(x))\right) \frac{dx}{x} - \int_b^\infty \psi_s\left(-\alpha x^{-\alpha} \int_x^\infty l^{\alpha-1} \log(l^\gamma f(l)) dl\right) \frac{dx}{x}\right]$$

From here (4.3.34) is equivalent to

$$\int_b^\infty \left(1 - \left(\frac{x}{b}\right)^\alpha\right) \psi_p\left(\log(x^\gamma f(x))\right) \frac{dx}{x} - \int_b^\infty \psi_p\left(-\alpha x^{-\alpha} \int_x^\infty l^{\alpha-1} \log(l^\gamma f(l))\, dl\right) \frac{dx}{x} \le [\mathfrak{q}(r)]^{\frac{p-s}{r-s}}\ [\mathfrak{q}(s)]^{\frac{r-p}{r-s}}. \quad (4.3.35)$$

From here for $p = 1$ we get (4.3.26).

If in (4.3.34) $s \to r,\ \ p \to s,\ \ r \to p$ and $s \to p,\ \ p \to r,\ \ r \to s$, we have

$$\int_b^\infty \left(1 - \left(\frac{x}{b}\right)^\alpha\right) \psi_p\left(\log(x^\gamma f(x))\right) \frac{dx}{x} - \int_b^\infty \psi_p\left(-\alpha\ x^{-\alpha} \int_x^\infty l^{\alpha-1} \log(l^\gamma f(l))\, dl\right) \frac{dx}{x} \ge [\mathfrak{q}(r)]^{\frac{p-s}{r-s}}\ [\mathfrak{q}(s)]^{\frac{r-p}{r-s}}. \quad (4.3.36)$$

And from here for $p = 1$ we get (4.3.27). □

Remark 4.3.2. In fact we have proved the more general results. Namely (4.3.31) and (4.3.35) are valid for $-\infty < s < p < r < \infty$; the inequalities (4.3.32) and (4.3.36) are valid for $-\infty < r < s < p < \infty$ and $-\infty < p < r < s < \infty$.

4.4 Bounds For Multidimensional Hardy Type Polya-Knopp Difference

Here we have established some bounds for multi dimensional Hardy type differences for $p < 0$ and $0 < p < 1$ and of Polya-Knopp differences.

4.4.1 Preliminaries

Throughout all the functions are assumed to be non-negative and non-trivial measurable. In the sequel the notations $\mathbf{b}$, $\mathbf{x}$, $(\mathbf{0},\, \mathbf{b})$, $(\mathbf{b},\, \infty]$ and $[\mathbf{b},\, \infty)$ have their usual meanings, *i.e.*,

$$\mathbf{b} = (b_1, ..., b_n) \in \mathbb{R}^n; \qquad \mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n.$$

$$(\mathbf{0},\, \mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^n : \, 0 < x_j < b_j, \quad 1 \le j \le n\}.$$

$$(\mathbf{b},\, \infty] = \{\mathbf{x} \in \mathbb{R}^n : \, b_j < x_j \le \infty, \quad 1 \le j \le n\}.$$

$$[\mathbf{b},\, \infty) = \{\mathbf{x} \in \mathbb{R}^n : \, b_j \le x_j < \infty, \quad 1 \le j \le n\}.$$

The following two lemma's are well known [51].

Lemma 4.4.1. *Let* $\mathbf{b} \in (\mathbf{0},\, \infty]$, $-\infty \le a < c \le \infty$ *and function* ϕ *is defined on* $[a,\, c]$. *Suppose that the weight function* u *defined on* $(\mathbf{0},\, \mathbf{b})$ *is such that* $\frac{u(x_1,...,x_n)}{x_1^2....x_n^2}$ *is locally integrable on* $(\mathbf{0},\, \mathbf{b})$ *and the weight function* ν *is defined by*

$$\nu(t_1, ..., t_n) = t_1...t_n \int_{t_1}^{b_1} ... \int_{t_n}^{b_n} \frac{u(x_1, ..., x_n)}{x_1^2....x_n^2}\, dx_1...dx_n < \infty, \quad \mathbf{t} \in (\mathbf{0},\, \mathbf{b}).$$

(*i*) *If* ϕ *is convex, then*

$$\int_0^{b_1}\!\!...\!\int_0^{b_n} u(x_1,...,x_n)\phi\left(\frac{1}{x_1...x_n}\int_0^{x_1}\!\!...\!\int_0^{x_n} f(t_1,...,t_n)dt_1...dt_n\right)\frac{dx_1...dx_n}{x_1...x_n}$$
$$\leq \int_0^{b_1}\!\!...\!\int_0^{b_n} \nu(x_1,...,x_n)\phi\left(f(x_1,...,x_n)\right)\frac{dx_1...dx_n}{x_1...x_n} \quad (4.4.1)$$

holds for every function f *on* $(\mathbf{0},\,\mathbf{b})$ *such that* $a < f(x_1,...,x_n) < c$.

(*ii*) *If* ϕ *is concave, then*

$$\int_0^{b_1} ... \int_0^{b_n} u(x_1,...,x_n)\,\phi\left(\frac{1}{x_1...x_n}\int_0^{x_1} ... \int_0^{x_n} f(t_1,...,t_n)\,dt_1...dt_n\right)\,\frac{dx_1...dx_n}{x_1...x_n}$$
$$\geq \int_0^{b_1} ... \int_0^{b_n} \nu(x_1,...,x_n)\,\phi\left(f(x_1,...,x_n)\right)\,\frac{dx_1...dx_n}{x_1...x_n}$$

holds for every function f *on* $(\mathbf{0},\,\mathbf{b})$ *such that* $a < f(x_1,...,x_n) < c$.

Lemma 4.4.2. *Let* $\mathbf{b} \in [\mathbf{0},\,\infty)$, $-\infty \leq a < c \leq \infty$ *and function* ϕ *is defined on* $[a,\,c]$. *Suppose that the weight function* u *defined on* $[\mathbf{b},\,\infty)$ *is such that* $\frac{u(x_1,...,x_n)}{x_1^2....x_n^2}$ *is locally integrable on* $[\mathbf{b},\,\infty)$ *and the weight function* ν *is defined by*

$$\nu(t_1,...,t_n) = \frac{1}{t_1...t_n}\int_{b_1}^{t_1} ... \int_{b_n}^{t_n} u(x_1,...,x_n)\,dx_1...dx_n < \infty, \quad \mathbf{t} \in (\mathbf{b},\,\infty).$$

(*i*) *If* ϕ *is convex, then*

$$\int_{b_1}^{\infty} ... \int_{b_n}^{\infty} u(x_1,...,x_n)\ \phi\left(x_1...x_n\int_{x_1}^{\infty} ... \int_{x_n}^{\infty} f(t_1,...,t_n)\,\frac{dt_1...dt_n}{t_1^2....t_n^2}\right)\ \frac{dx_1...dx_n}{x_1...x_n}$$
$$\leq \int_{b_1}^{\infty} ... \int_{b_n}^{\infty} \nu(x_1,...,x_n)\ \phi\big(f(x_1,...,x_n)\big)\ \frac{dx_1...dx_n}{x_1...x_n} \quad (4.4.2)$$

holds for every function f *on* $[\mathbf{b},\,\infty)$ *such that* $a < f(x_1,...,x_n) < c$.

(*ii*) *If* ϕ *is concave, then*

$$\int_{b_1}^{\infty} ... \int_{b_n}^{\infty} u(x_1,...,x_n)\ \phi\left(x_1...x_n\int_{x_1}^{\infty} ... \int_{x_n}^{\infty} f(t_1,...,t_n)\,\frac{dt_1...dt_n}{t_1^2....t_n^2}\right)\ \frac{dx_1...dx_n}{x_1...x_n}$$
$$\geq \int_{b_1}^{\infty} ... \int_{b_n}^{\infty} \nu(x_1,...,x_n)\ \phi\big(f(x_1,...,x_n)\big)\ \frac{dx_1...dx_n}{x_1...x_n}$$

holds for every function f *on* $[\mathbf{b},\,\infty)$ *such that* $a < f(x_1,...,x_n) < c$.

The special cases of these Lemma's are given by:

($\mathbf{i}$) If $p<0$, $\mathbf{b}\in(\mathbf{0},\infty]$, $k<1$ and f is defined on $(\mathbf{0},\mathbf{b})$, then

$$\int_0^{b_1}\!\!...\!\int_0^{b_n} x_1^{-k}...x_n^{-k}\left(\int_0^{x_1}\!\!...\!\int_0^{x_n} f(t_1,...,t_n)dt_1...dt_n\right)^p dx_1...dx_n \le \left(\frac{p}{k-1}\right)^{np}$$
$$\times\int_0^{b_1}\!\!...\!\int_0^{b_n}\left[1-\left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right]...\left[1-\left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right]x_1^{p-k}...x_n^{p-k}f^p(x_1,...,x_n)dx_1...dx_n.$$

($\mathbf{ii}$) If $p<0$, $\mathbf{b}\in[\mathbf{0},\infty)$, $k>1$ and f is defined on $(\mathbf{b},\infty)$, then

$$\int_{b_1}^{\infty}\!\!...\!\int_{b_n}^{\infty} x_1^{-k}...x_n^{-k}\left(\int_{x_1}^{\infty}\!\!...\!\int_{x_n}^{\infty} f(t_1,...,t_n)dt_1...dt_n\right)^p dx_1...dx_n \le \left(\frac{p}{1-k}\right)^{np}$$
$$\times\int_{b_1}^{\infty}\!\!...\!\int_{b_n}^{\infty}\left[1-\left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right]...\left[1-\left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right]x_1^{p-k}...x_n^{p-k}f^p(x_1,...,x_n)dx_1...dx_n.$$

($\mathbf{iii}$) If $0<p<1$, $\mathbf{b}\in(\mathbf{0},\infty]$, $k>1$ and f is defined on $(\mathbf{0},\mathbf{b})$, then

$$\int_0^{b_1}\!\!...\!\int_0^{b_n} x_1^{-k}...x_n^{-k}\left(\int_0^{x_1}\!\!...\!\int_0^{x_n} f(t_1,...,t_n)dt_1...dt_n\right)^p dx_1...dx_n \ge \left(\frac{p}{k-1}\right)^{np}$$
$$\times\int_0^{b_1}\!\!...\!\int_0^{b_n}\left[1-\left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right]...\left[1-\left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right]x_1^{p-k}...x_n^{p-k}f^p(x_1,...,x_n)dx_1...dx_n.$$

($\mathbf{iv}$) If $0<p<1$, $\mathbf{b}\in[\mathbf{0},\infty)$, $k<1$ and f is defined on $(\mathbf{b},\infty)$, then

$$\int_{b_1}^{\infty}\!\!...\!\int_{b_n}^{\infty} x_1^{-k}...x_n^{-k}\left(\int_{x_1}^{\infty}\!\!...\!\int_{x_n}^{\infty} f(t_1,...,t_n)dt_1...dt_n\right)^p dx_1...dx_n \ge \left(\frac{p}{1-k}\right)^{np}$$
$$\times\int_{b_1}^{\infty}\!\!...\!\int_{b_n}^{\infty}\left[1-\left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right]...\left[1-\left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right]x_1^{p-k}...x_n^{p-k}f^p(x_1,...,x_n)dx_1...dx_n.$$

4.4.2 Log-Convexity Of Hardy-Type Differences

Theorem 4.4.3. *Let the conditions of Lemma 4.4.1 (**i**) be satisfied and φ_s given by Lemma 4.2.1. Let $\mathfrak{A}_n:\mathbb{R}\to\mathbb{R}_+$ be defined by*

$$\mathfrak{A}_n(s)=\int_0^{b_1}...\int_0^{b_n}\nu(x_1,...,x_n)\,\varphi_s\left(f(x_1,...,x_n)\right)\frac{dx_1...dx_n}{x_1...x_n}-$$
$$\int_0^{b_1}\!\!...\!\int_0^{b_n}u(x_1,...,x_n)\varphi_s\left(\frac{1}{x_1...x_n}\int_0^{x_1}\!\!...\!\int_0^{x_n}f(t_1,...,t_n)\,dt_1...dt_n\right)\frac{dx_1...dx_n}{x_1...x_n}. \quad (4.4.3)$$

Then $\mathfrak{A}_n$ *is* $\log -$*convex, i.e., the following inequality is valid*

$$[\mathfrak{A}_n(p)]^{r-s} \leq [\mathfrak{A}_n(r)]^{p-s}[\mathfrak{A}_n(s)]^{r-p} \quad for \quad -\infty < s < p < r < \infty. \tag{4.4.4}$$

Proof. Similar to the proof of Theorem 4.3.2. □

Theorem 4.4.4. *Let the conditions of Lemma 4.4.2 be satisfied and* φ_s *given by Lemma 4.2.1. Let* $\mathfrak{Z}_n : \mathbb{R} \to \mathbb{R}_+$ *be defined by*

$$\mathfrak{Z}_n(s) = \int_{b_1}^{\infty}\!\!...\!\int_{b_n}^{\infty} \nu(x_1,...,x_n)\varphi_s\left(f(x_1,...,x_n)\right)\frac{dx_1...dx_n}{x_1...x_n} - \int_{b_1}^{\infty}\!\!...\!\int_{b_n}^{\infty} u(x_1,...,x_n)\varphi_s\left(x_1...x_n\int_{x_1}^{\infty}\!\!...\!\int_{x_n}^{\infty} f(t_1,...,t_n)\frac{dt_1...dt_n}{t_1^2....t_n^2}\right)\frac{dx_1...dx_n}{x_1...x_n}. \tag{4.4.5}$$

Then $\mathfrak{Z}_n$ *is* $\log -$*convex, that is, the following inequality is valid*

$$[\mathfrak{Z}_n(p)]^{r-s} \leq [\mathfrak{Z}_n(r)]^{p-s}\,[\mathfrak{Z}_n(s)]^{r-p} \quad for \quad -\infty < s < p < r < \infty. \tag{4.4.6}$$

Theorem 4.4.5. *Let the conditions of Lemma 4.4.1 be satisfied and* ψ_s *be given by Lemma 4.3.1. Let* $\mathfrak{Y}_n : \mathbb{R} \to \mathbb{R}_+$ *be defined by*

$$\mathfrak{Y}_n(s) = \int_0^{b_1}\!\!...\!\int_0^{b_n} \nu(x_1,...,x_n)\psi_s\left(f(x_1,...,x_n)\right)\frac{dx_1...dx_n}{x_1...x_n} - \int_0^{b_1}\!\!...\!\int_0^{b_n} u(x_1,...,x_n)\psi_s\left(\frac{1}{x_1...x_n}\int_0^{x_1}\!\!...\!\int_0^{x_n} f(t_1,...,t_n)dt_1...dt_n\right)\frac{dx_1...dx_n}{x_1...x_n}. \tag{4.4.7}$$

Then $\mathfrak{Y}_n$ *is* $\log -$*convex, that is, the following inequality is valid*

$$[\mathfrak{Y}_n(p)]^{r-s} \leq [\mathfrak{Y}_n(r)]^{p-s}[\mathfrak{Y}_n(s)]^{r-p} \quad for \quad -\infty < s < p < r < \infty. \tag{4.4.8}$$

Theorem 4.4.6. *Let the conditions of Lemma 4.4.2 be satisfied and* ψ_s *given by Lemma 4.3.1. Let* $\mathfrak{X}_n : \mathbb{R} \to \mathbb{R}_+$ *be defined by*

$$\mathfrak{X}_n(s) = \int_{b_1}^{\infty}\!\!...\!\int_{b_n}^{\infty} \nu(x_1,...,x_n)\psi_s\left(f(x_1,...,x_n)\right)\frac{dx_1...dx_n}{x_1...x_n} - \int_{b_1}^{\infty}\!\!...\!\int_{b_n}^{\infty} u(x_1,...,x_n)\psi_s\left(x_1...x_n\int_{x_1}^{\infty}\!\!...\!\int_{x_n}^{\infty} f(t_1,...,t_n)\frac{dt_1...dt_n}{t_1^2....t_n^2}\right)\frac{dx_1...dx_n}{x_1...x_n}. \tag{4.4.9}$$

Then $\mathfrak{X}_n$ *is* $\log -$*convex, that is, the following inequality is valid*

$$[\mathfrak{X}_n(p)]^{r-s} \leq [\mathfrak{X}_n(r)]^{p-s}[\mathfrak{X}_n(s)]^{r-p} \quad for \quad -\infty < s < p < r < \infty. \tag{4.4.10}$$

4.4.3 Improvements And Reverses Of Hardy-Type Inequalities

We state and prove an improvement and reverse of Hardy-type inequalities with dual results for both cases $p < 0$ and $0 < p < 1$.

First, consider the case $p < 0$.

Theorem 4.4.7. *Let* $k \in \mathbb{R}\backslash\{1\},\ p < 0,\ \mathbf{b} \in (\mathbf{0}, \infty]$ *and* f *be defined on* $(\mathbf{0}, \mathbf{b})$. *If* $\frac{k-1}{p} > 0$, *then*

$$\left(\frac{p}{k-1}\right)^{np} \int_0^{b_1}\!\!...\!\int_0^{b_n} \left[1-\left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right] ... \left[1-\left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right] f^p(x_1,...,x_n) x_1^{p-k}...x_n^{p-k} \times dx_1...dx_n - \int_0^{b_1}\!\!...\!\int_0^{b_n} x_1^{-k}...x_n^{-k} \left(\int_0^{x_1}\!\!...\!\int_0^{x_n} f(t_1,...,t_n)dt_1...dt_n\right)^p dx_1...dx_n \leq p(p-1)\left(\frac{p}{k-1}\right)^{np} [\mathfrak{s}_n(s)]^{\frac{p-r}{s-r}} [\mathfrak{s}_n(r)]^{\frac{s-p}{s-r}}, \quad (4.4.11)$$

for $-\infty < s < p < r < 0$; *and*

$$\left(\frac{p}{k-1}\right)^{np} \int_0^{b_1}\!\!...\!\int_0^{b_n} \left[1-\left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right] ... \left[1-\left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right] f^p(x_1,...,x_n) x_1^{p-k}...x_n^{p-k} \times dx_1...dx_n - \int_0^{b_1}\!\!...\!\int_0^{b_n} x_1^{-k}...x_n^{-k} \left(\int_0^{x_1}\!\!...\!\int_0^{x_n} f(t_1,...,t_n)dt_1...dt_n\right)^p dx_1...dx_n \geq p(p-1)\left(\frac{p}{k-1}\right)^{np} [\mathfrak{s}_n(s)]^{\frac{p-r}{s-r}} [\mathfrak{s}_n(r)]^{\frac{s-p}{s-r}} \quad (4.4.12)$$

for $-\infty < p < r < s < 0$; *and* $-\infty < r < s < p < 0$. *Where,*

$$\mathfrak{s}_n(r) = \int_0^{b_1}\!\!...\!\int_0^{b_n} \left[1-\left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right] ... \left[1-\left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right] \varphi_r\left(f(x_1,...,x_n) x_1^{\frac{p-k+1}{p}} ... x_n^{\frac{p-k+1}{p}}\right) \frac{dx_1...dx_n}{x_1...x_n} - \int_0^{b_1}\!\!...\!\int_0^{b_n} \varphi_r\left(\left(\frac{k-1}{p}\right)^n x_1^{\frac{1-k}{p}} ... x_n^{\frac{1-k}{p}} \int_0^{x_1}\!\!...\!\int_0^{x_n} f(t_1,...,t_n)dt_1...dt_n\right) \frac{dx_1...dx_n}{x_1...x_n}. \quad (4.4.13)$$

The dual result to Theorem 4.4.7 is given by:

Theorem 4.4.8. *Let* $k \in \mathbb{R}\backslash\{1\}$, $p < 0$, $\mathbf{b} \in [\mathbf{0}, \infty)$ *and* f *be defined on* $[\mathbf{b}, \infty)$. *If* $\frac{1-k}{p} > 0$, *then*

$$\left(\frac{p}{1-k}\right)^{np} \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \left[1 - \left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right] \cdots \left[1 - \left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right] f^p(x_1, ..., x_n) x_1^{p-k} ... x_n^{p-k}$$
$$\times\, dx_1 ... dx_n - \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} x_1^{-k} ... x_n^{-k} \left(\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(t_1, ..., t_n) dt_1 ... dt_n\right)^p dx_1 ... dx_n$$
$$\leq p(p-1)\left(\frac{p}{1-k}\right)^{np} [\mathfrak{t}_n(s)]^{\frac{p-r}{s-r}} [\mathfrak{t}_n(r)]^{\frac{s-p}{s-r}} \quad (4.4.14)$$

for $-\infty < s < p < r < 0$; *and*

$$\left(\frac{p}{1-k}\right)^{np} \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \left[1 - \left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right] \cdots \left[1 - \left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right] f^p(x_1, ..., x_n) x_1^{p-k} ... x_n^{p-k}$$
$$\times\, dx_1 ... dx_n - \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} x_1^{-k} ... x_n^{-k} \left(\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(t_1, ..., t_n) dt_1 ... dt_n\right)^p dx_1 ... dx_n$$
$$\geq p(p-1)\left(\frac{p}{1-k}\right)^{np} [\mathfrak{t}_n(s)]^{\frac{p-r}{s-r}} [\mathfrak{t}_n(r)]^{\frac{s-p}{s-r}} \quad (4.4.15)$$

for $-\infty < p < r < s < 0$; *and* $-\infty < r < s < p < 0$. *Where,*

$$\mathfrak{t}_n(r) = \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \left[1 - \left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right] \cdots \left[1 - \left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right] \varphi_r\left(f(x_1, ..., x_n) x_1^{\frac{p-k+1}{p}} ... x_n^{\frac{p-k+1}{p}}\right)$$
$$\times \frac{dx_1 ... dx_n}{x_1 ... x_n} - \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \varphi_r\left(\left(\frac{1-k}{p}\right)^n x_1^{\frac{1-k}{p}} ... x_n^{\frac{1-k}{p}} \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(t_1, ..., t_n) dt_1 ... dt_n\right) \frac{dx_1 ... dx_n}{x_1 ... x_n}.$$
$$(4.4.16)$$

Now we continue with $0 < p < 1$.

Theorem 4.4.9. *Let* $k \in \mathbb{R}\backslash\{1\}$, $0 < p < 1$, $\mathbf{b} \in (\mathbf{0}, \infty]$ *and* f *be defined on* $(\mathbf{0}, \mathbf{b})$.

If $\frac{k-1}{p} > 0$, *then*

$$-\left(\frac{p}{k-1}\right)^{np}\int_0^{b_1}\!\!..\!\int_0^{b_n}\left[1-\left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right]...\left[1-\left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right]f^p(x_1,...,x_n)x_1^{p-k}...x_n^{p-k}$$
$$\times\, dx_1...dx_n+\int_0^{b_1}\!\!...\!\int_0^{b_n}x_1^{-k}...x_n^{-k}\left(\int_0^{x_1}\!\!...\!\int_0^{x_n}f(t_1,...,t_n)dt_1...dt_n\right)^p dx_1...dx_n$$
$$\leq p(1-p)\left(\frac{p}{k-1}\right)^{np}[\mathfrak{O}_n(s)]^{\frac{p-r}{s-r}}[\mathfrak{O}_n(r)]^{\frac{s-p}{s-r}} \quad (4.4.17)$$

for $0 < s < p < r < 1$; *and*

$$-\left(\frac{p}{k-1}\right)^{np}\int_0^{b_1}\!\!..\!\int_0^{b_n}\left[1-\left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right]...\left[1-\left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right]f^p(x_1,...,x_n)x_1^{p-k}...x_n^{p-k}$$
$$\times\, dx_1...dx_n+\int_0^{b_1}\!\!...\!\int_0^{b_n}x_1^{-k}...x_n^{-k}\left(\int_0^{x_1}\!\!...\!\int_0^{x_n}f(t_1,...,t_n)dt_1...dt_n\right)^p dx_1...dx_n$$
$$\geq p(1-p)\left(\frac{p}{k-1}\right)^{np}[\mathfrak{O}_n(s)]^{\frac{p-r}{s-r}}[\mathfrak{O}_n(r)]^{\frac{s-p}{s-r}} \quad (4.4.18)$$

for $0 < p < r < s < 1$; *and* $0 < r < s < p < 1$. *Where,*

$$\mathfrak{O}_n(r)=\int_0^{b_1}\!\!...\!\int_0^{b_n}\left[1-\left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right]...\left[1-\left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right]\varphi_r\left(f(x_1,...,x_n)x_1^{\frac{p-k+1}{p}}...x_n^{\frac{p-k+1}{p}}\right)$$
$$\times\frac{dx_1...dx_n}{x_1...x_n}-\int_0^{b_1}\!\!...\!\int_0^{b_n}\varphi_r\left(\left(\frac{k-1}{p}\right)^n x_1^{\frac{1-k}{p}}...x_n^{\frac{1-k}{p}}\int_0^{x_1}\!\!...\!\int_0^{x_n}f(t_1,...,t_n)dt_1...dt_n\right)\frac{dx_1...dx_n}{x_1...x_n}.$$

The dual result to Theorem 4.4.9 is given by:

Theorem 4.4.10. *Let* $k \in \mathbb{R}\backslash\{1\}$, $0 < p < 1$, $\mathbf{b} \in [\mathbf{0}, \infty)$ *and* f *be defined on* $(\mathbf{b}, \infty)$. *If* $\frac{1-k}{p} > 0$, *then*

$$-\left(\frac{p}{1-k}\right)^{np}\int_{b_1}^{\infty}\!\!...\!\int_{b_1}^{\infty}\left[1-\left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right]...\left[1-\left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right]f^p(x_1,...,x_n)x_1^{p-k}...x_n^{p-k}$$
$$\times\, dx_1...dx_n+\int_{b_1}^{\infty}\!\!...\!\int_{b_n}^{\infty}x_1^{-k}...x_n^{-k}\left(\int_{x_1}^{\infty}\!\!...\!\int_{x_n}^{\infty}f(t_1,...,t_n)dt_1...dt_n\right)^p dx_1...dx_n$$
$$\leq p(1-p)\left(\frac{p}{1-k}\right)^{np}[\mathfrak{M}_n(s)]^{\frac{p-r}{s-r}}[\mathfrak{M}_n(r)]^{\frac{s-p}{s-r}} \quad (4.4.19)$$

for $0 < s < p < r < 1$; *and*

$$-\left(\frac{p}{1-k}\right)^{np}\int_{b_1}^{\infty}\!\!...\!\int_{b_1}^{\infty}\left[1-\left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right]...\left[1-\left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right]f^p(x_1,...,x_n)x_1^{p-k}...x_n^{p-k}$$

$$\times\, dx_1...dx_n+\int_{b_1}^{\infty}\!\!...\!\int_{b_n}^{\infty}x_1^{-k}...x_n^{-k}\left(\int_{x_1}^{\infty}\!\!...\!\int_{x_n}^{\infty}f(t_1,...,t_n)dt_1...dt_n\right)^p dx_1...dx_n$$

$$\geq p(1-p)\left(\frac{p}{1-k}\right)^{np}[\mathfrak{M}_n(s)]^{\frac{p-r}{s-r}}[\mathfrak{M}_n(r)]^{\frac{s-p}{s-r}} \quad (4.4.20)$$

for $0 < p < r < s < 1$; *and* $0 < r < s < p < 1$. *Where,*

$$\mathfrak{M}_n(r)=\int_{b_1}^{\infty}\!\!...\!\int_{b_n}^{\infty}\left[1-\left(\frac{x_1}{b_1}\right)^{\frac{k-1}{p}}\right]...\left[1-\left(\frac{x_n}{b_n}\right)^{\frac{k-1}{p}}\right]\varphi_r\left(f(x_1,...,x_n)x_1^{\frac{p-k+1}{p}}...x_n^{\frac{p-k+1}{p}}\right)$$

$$\times\frac{dx_1...dx_n}{x_1...x_n}-\int_{b_1}^{\infty}\!\!...\!\int_{b_n}^{\infty}\varphi_r\left(\left(\frac{1-k}{p}\right)^n x_1^{\frac{1-k}{p}}...x_n^{\frac{1-k}{p}}\int_{x_1}^{\infty}\!\!...\!\int_{x_n}^{\infty}f(t_1,...,t_n)dt_1...dt_n\right)\frac{dx_1...dx_n}{x_1...x_n}.$$

4.4.4 Improvements And Reverses Of Polya-Knopp Inequality

We state and prove an improvement and reverse of Polya-Knopp inequality and its dual.

Theorem 4.4.11. *Let* $f:(\mathbf{0},\mathbf{b})\to\mathbb{R}$ *be a function and* $\alpha,\gamma\in\mathbb{R}$ *be such that* $\alpha\neq 0$.
(***i***) *If* $\alpha > 0$, *then*

$$e^{\frac{n\gamma}{\alpha}}\int_0^{b_1}\!\!...\!\int_0^{b_n}\left[1-\left(\frac{x_1}{b_1}\right)^{\alpha}\right]...\left[1-\left(\frac{x_n}{b_n}\right)^{\alpha}\right](x_1...x_n)^{\gamma-1}f(x_1...x_n)dx_1...dx_n-$$

$$\int_0^{b_1}\!\!...\!\int_0^{b_n}(x_1...x_n)^{\gamma-1}\exp\left[\frac{\alpha^n}{(x_1...x_n)^{\alpha}}\int_0^{x_1}\!\!...\!\int_0^{x_n}(t_1...t_n)^{\alpha-1}\log f(t_1...t_n)dt_1...dt_n\right]dx_1...dx_n$$

$$\leq e^{\frac{n\gamma}{\alpha}}[\mathfrak{P}_{n,\alpha}(r)]^{\frac{1-s}{r-s}}[\mathfrak{P}_{n,\alpha}(s)]^{\frac{r-1}{r-s}}, \quad (4.4.21)$$

for $-\infty < s < 1 < r < \infty$; *and*

$$e^{\frac{n\gamma}{\alpha}} \int_0^{b_1} \!\!...\!\! \int_0^{b_n} \left[1-\left(\frac{x_1}{b_1}\right)^{\alpha}\right] ... \left[1-\left(\frac{x_n}{b_n}\right)^{\alpha}\right] (x_1...x_n)^{\gamma-1} f(x_1...x_n) dx_1...dx_n - \int_0^{b_1} \!\!...\!\! \int_0^{b_n} (x_1...x_n)^{\gamma-1} \exp\left[\frac{\alpha^n}{(x_1...x_n)^{\alpha}} \int_0^{x_1} \!\!...\!\! \int_0^{x_n} (t_1...t_n)^{\alpha-1} \log f(t_1...t_n) dt_1...dt_n\right] dx_1...dx_n \geq e^{\frac{n\gamma}{\alpha}} [\mathfrak{P}_{n,\alpha}(r)]^{\frac{1-s}{r-s}} [\mathfrak{P}_{n,\alpha}(s)]^{\frac{r-1}{r-s}}, \quad (4.4.22)$$

for $-\infty < 1 < r < s < \infty$ *and* $-\infty < r < s < 1 < \infty$. *Where,*

$$\mathfrak{P}_{n,\alpha}(s) = \int_0^{b_1} \!\!...\!\! \int_0^{b_n} \left[1-\left(\frac{x_1}{b_1}\right)^{\alpha}\right] ... \left[1-\left(\frac{x_n}{b_n}\right)^{\alpha}\right] \psi_s \left(\log(x_1^{\gamma}...x_n^{\gamma} f(x_1...x_n))\right) \frac{dx_1...dx_n}{x_1...x_n} - \int_0^{b_1} \!\!...\!\! \int_0^{b_n} \psi_s \left(\frac{\alpha^n}{(x_1...x_n)^{\alpha}} \int_0^{x_1} \!\!...\!\! \int_0^{x_n} (t_1...t_n)^{\alpha-1} \log(t_1^{\gamma}...t_n^{\gamma} f(t_1...t_n)) dt_1...dt_n\right) \frac{dx_1...dx_n}{x_1...x_n}. \quad (4.4.23)$$

(***ii***) *If* $\alpha < 0$, *then*

$$e^{\frac{n\gamma}{\alpha}} \int_{b_1}^{\infty} \!\!...\!\! \int_{b_n}^{\infty} \left[1-\left(\frac{x_1}{b_1}\right)^{\alpha}\right] ... \left[1-\left(\frac{x_n}{b_n}\right)^{\alpha}\right] (x_1...x_n)^{\gamma-1} f(x_1...x_n) dx_1...dx_n - \int_{b_1}^{\infty} \!\!...\!\! \int_{b_n}^{\infty} (x_1...x_n)^{\gamma-1} \exp\left[\frac{(-\alpha)^n}{(x_1...x_n)^{\alpha}} \int_{x_1}^{\infty} \!\!...\!\! \int_{x_n}^{\infty} (t_1...t_n)^{\alpha-1} \log f(t_1...t_n)\, dt_1...dt_n\right] dx_1...dx_n \leq e^{\frac{n\gamma}{\alpha}} [\tilde{\mathfrak{P}}_{n,\alpha}(r)]^{\frac{1-s}{r-s}} [\tilde{\mathfrak{P}}_{n,\alpha}(s)]^{\frac{r-1}{r-s}}, \quad (4.4.24)$$

for $-\infty < s < 1 < r < \infty$; *and*

$$e^{\frac{n\gamma}{\alpha}} \int_{b_1}^{\infty} \!\!...\!\! \int_{b_n}^{\infty} \left[1-\left(\frac{x_1}{b_1}\right)^{\alpha}\right] ... \left[1-\left(\frac{x_n}{b_n}\right)^{\alpha}\right] (x_1...x_n)^{\gamma-1} f(x_1...x_n) dx_1...dx_n - \int_{b_1}^{\infty} \!\!...\!\! \int_{b_n}^{\infty} (x_1...x_n)^{\gamma-1} \exp\left[\frac{(-\alpha)^n}{(x_1...x_n)^{\alpha}} \int_{x_1}^{\infty} \!\!...\!\! \int_{x_n}^{\infty} (t_1...t_n)^{\alpha-1} \log f(t_1...t_n)\, dt_1...dt_n\right] dx_1...dx_n \geq e^{\frac{n\gamma}{\alpha}} [\tilde{\mathfrak{P}}_{n,\alpha}(r)]^{\frac{1-s}{r-s}} [\tilde{\mathfrak{P}}_{n,\alpha}(s)]^{\frac{r-1}{r-s}}, \quad (4.4.25)$$

for $-\infty < 1 < r < s < \infty$ *and* $-\infty < r < s < 1 < \infty$. *Where,*

$$\tilde{\mathfrak{P}}_{n,\alpha}(s) = \int_{b_1}^{\infty}...\int_{b_n}^{\infty} \left[1 - \left(\frac{x_1}{b_1}\right)^{\alpha}\right] ... \left[1 \left(\frac{x_n}{b_n}\right)^{\alpha}\right] \psi_s \left(\log(x_1^{\gamma}...x_n^{\gamma} f(x_1...x_n))\right) \frac{dx_1...dx_n}{x_1...x_n}$$
$$- \int_{b_1}^{\infty}...\int_{b_n}^{\infty} \psi_s \left(\frac{(-\alpha)^n}{(x_1...x_n)^{\alpha}} \int_{x_1}^{\infty}...\int_{x_n}^{\infty} (t_1...t_n)^{\alpha-1} \log(t_1^{\gamma}...t_n^{\gamma} f(t_1...t_n)) dt_1...dt_n\right) \frac{dx_1...dx_n}{x_1...x_n}. \tag{4.4.26}$$

Chapter 5

Some New Refinements Of Strengthened Hardy And Polya-Knopp's Inequalities

5.1 Introduction

L.-E. Persson and J. A. Oguntuase obtained a class of refinements of Hardy's inequality (1.2.1) related to an arbitrary $b \in \mathbb{R}_+$ and the outer integrals on both hand sides of (1.2.1) taken over $(0, b)$ or (b, ∞) [53]. These results extend those of D. T. Shum [59] and C. O. Imoru [32, 33] and cover all admissible parameters $p, k \in \mathbb{R}$, $p \neq 0$, $k \neq 1$. Namely, let f be a non-negative integrable function on $(0, b)$, $F(x) = \int_0^x f(t)\, dt$, and let $\frac{p}{k-1} > 0$. If $p \in (-\infty, 0) \cup [1, \infty)$, then

$$\int_0^b x^{-k} F^p(x)\, dx + \frac{p}{k-1}\, b^{1-k}\, F^p(b) \;\leq\; \left(\frac{p}{k-1}\right)^p \int_0^b x^{p-k} f^p(x)\, dx\,, \qquad (5.1.1)$$

while for $p \in (0, 1]$ inequality (5.1.1) holds in the reversed direction. On the contrary, if f is a non-negative integrable function on (b, ∞), $\tilde{F}(x) = \int_x^\infty f(t)\, dt$, and $\frac{p}{k-1} < 0$, then

$$\int_b^\infty x^{-k}\tilde{F}^p(x)\,dx + \frac{p}{1-k}\,b^{1-k}\,\tilde{F}^p\,(b) \;\le\; \left(\frac{p}{1-k}\right)^p \int_b^\infty x^{p-k}\,f^p(x)\,dx \qquad (5.1.2)$$

holds for $p \in (-\infty, 0) \cup [1, \infty)$, while for $p \in (0, 1]$ the sign of inequality in (5.1.2) is reversed. The constant $\left(\frac{p}{|k-1|}\right)^p$ is the best possible for all cases and both inequalities (5.1.1) and (5.1.2).

Motivated by the above observations, in this chapter we consider a general positive Borel measure λ on $\mathbb{R}_+$, such that

$$L = \lambda(\mathbb{R}_+) = \int_0^\infty d\lambda(t) < \infty\,, \qquad (5.1.3)$$

and prove a new weighted Boas-type inequality for this setting. Further, we point out that our result unifies, generalizes and refines relations (1.2.4) and (1.2.5), as well as the strengthened Hardy-Knopp-type inequalities from [15]. More precisely, applying the obtained general relation with some particular weights and a measure λ, we derive new refinements of the above inequalities. Finally, as their special cases, we get new refinements of the strengthened versions of Hardy and Pólya–Knopp's inequalities, completely different from (5.1.1) and (5.1.2) and even hardly comparable with these inequalities.

The chapter is organized in the following way. After this Introduction, in Section 5.2 we introduce some necessary notation and state, prove and discuss a general refined weighted Boas-type inequality. As its particular cases, in the same section we obtain a new refinement of inequality (1.2.4), as well as refinements of (1.2.5) and of the strengthened weighted Hardy-Knopp-type inequalities. Refinements of the strengthened Hardy and Pólya-Knopp's inequalities are presented in the concluding Section 5.3 of the chapter, along with some final remarks.

Conventions. Throughout this chapter, all measures are assumed to be positive, all functions are assumed to be measurable, and expressions of the form $0 \cdot \infty$, $\frac{0}{0}$, $\frac{a}{\infty}$ $(a \in \mathbb{R})$, and $\frac{\infty}{\infty}$ are taken to be equal to zero. As usual, by dx we denote the Lebesgue measure on $\mathbb{R}$, by a weight function (shortly: a weight) we mean a non-negative measurable function on the actual interval, while an interval in $\mathbb{R}$ is any convex subset of $\mathbb{R}$.

5.2 The main results

Let λ be a finite Borel measure on $\mathbb{R}_+$, that is, having property (5.1.3), and $f : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function. By Af we denote its Hardy-Littlewood average, defined in terms of the Lebesgue integral as

$$Af(x) = \frac{1}{L} \int_0^\infty f(tx)\, d\lambda(t) \quad x \in \mathbb{R}_+\,, \tag{5.2.1}$$

where L is defined by (5.1.3).

Now, we can state and prove the main result of this chapter. It is given in the following theorem.

Theorem 5.2.1. *Let λ be a finite Borel measure on $\mathbb{R}_+$, L be defined by* (5.1.3)*, and let u and v be non-negative measurable functions on $\mathbb{R}_+$, where*

$$v(x) = \int_0^\infty u\left(\frac{x}{t}\right) d\lambda(t) < \infty \quad x \in \mathbb{R}_+\,. \tag{5.2.2}$$

Let Φ be a continuous convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \longrightarrow \mathbb{R}$ be any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in I^\circ$. If $f : \mathbb{R}_+ \to \mathbb{R}$ is a measurable function, such that $f(x) \in I$ for all $x \in \mathbb{R}_+$, and $Af(x)$ is defined by (5.2.1)*, then*

$Af(x) \in I$, for all $x \in \mathbb{R}_+$, and the inequality

$$\begin{aligned}
&\frac{1}{L}\int_0^\infty v(x)\ \Phi(f(x))\ \frac{dx}{x} - \int_0^\infty u(x)\ \Phi(Af(x))\ \frac{dx}{x} \\
&\geq \frac{1}{L}\left|\int_0^\infty \int_0^\infty u(x)\,|\Phi(f(tx)) - \Phi(Af(x))|\ d\lambda(t)\,\frac{dx}{x}\right. \\
&\left. - \int_0^\infty \int_0^\infty u(x)\,|\varphi(Af(x))|\,|f(tx) - Af(x)|\ d\lambda(t)\,\frac{dx}{x}\right| \qquad (5.2.3)
\end{aligned}$$

holds.

Proof. For a fixed $x \in \mathbb{R}_+$, let $h_x : \mathbb{R}_+ \to \mathbb{R}$ be defined by $h_x(t) = f(tx) - Af(x)$. Then (5.1.3) and (5.2.1) imply

$$\int_0^\infty h_x(t)\, d\lambda(t) = \int_0^\infty f(tx)\ d\lambda(t)\ -\ Af(x)\int_0^\infty d\lambda(t) = 0\,. \qquad (5.2.4)$$

Now, suppose $x \in \mathbb{R}_+$ is such that $Af(x) \notin I$. Observing that $f(\mathbb{R}_+) \subseteq I$ and that I is an interval in $\mathbb{R}$, we have $h_x(t) > 0$ for all $t \in \mathbb{R}_+$, or $h_x(t) < 0$ for all $t \in \mathbb{R}_+$, that is, the function h_x is either strictly positive or strictly negative. Since this contradicts (5.2.4), we have proved that $Af(x) \in I$, for all $x \in \mathbb{R}_+$. Note that if $Af(x)$ is an endpoint of I for some $x \in \mathbb{R}_+$ (in cases when I is not an open interval), then h_x (or $-h_x$) will be a non-negative function whose integral over $\mathbb{R}_+$, with respect to the measure λ, is equal to 0. Therefore, $h_x \equiv 0$, that is, $f(tx) = Af(x)$ holds for λ-a.e. $t \in \mathbb{R}_+$. To prove inequality (5.2.3), observe that for all $r \in I^\circ$ and $s \in I$ we have

$$\Phi(s)\ -\ \Phi(r)\ -\ \varphi(r)\,(s-r)\ \geq 0,$$

where $\varphi : I \to \mathbb{R}$ is any function such that $\varphi(x) \in \partial\Phi(x)$ for $x \in I^\circ$, and hence

$$\begin{aligned}
\Phi(s)\ -\ \Phi(r)\ -\ \varphi(r)\,(s-r) &= |\Phi(s)\ -\ \Phi(r)\ -\ \varphi(r)\,(s-r)| \\
&\geq ||\Phi(s)\ -\ \Phi(r)|\ -\ |\varphi(r)|\,|s-r||\,. \qquad (5.2.5)
\end{aligned}$$

Especially, in the case when $Af(x) \in I^\circ$, by substituting $r = Af(x)$ and $s = f(tx)$ in (5.2.5), for all $t \in \mathbb{R}_+$ we get

$$\begin{aligned} &\Phi(f(tx)) - \Phi(Af(x)) - \varphi(Af(x))\,[f(tx) - Af(x)] \\ &\geq ||\Phi(f(tx)) - \Phi(Af(x))| - |\varphi(Af(x))|\,|f(tx) - Af(x)|| \,. \end{aligned} \tag{5.2.6}$$

On the other hand, the above analysis provides (5.2.6) to hold even if $Af(x)$ is an endpoint of I, since in that case both sides of inequality (5.2.6) are equal to 0 for λ-a.e. $t \in \mathbb{R}_+$. Multiplying (5.2.6) by $\frac{u(x)}{x}$, then integrating it over $\mathbb{R}_+^2$ with respect to the measures $d\lambda(t)$ and $\frac{dx}{x}$, and applying Fubini's theorem, we obtain the following sequence of inequalities

$$\begin{aligned} &\int_0^\infty \int_0^\infty u(x)\Phi(f(tx))\, d\lambda(t)\, \frac{dx}{x} - \int_0^\infty \int_0^\infty u(x)\,\Phi(Af(x))\, d\lambda(t)\, \frac{dx}{x} \\ &- \int_0^\infty \int_0^\infty u(x)\varphi(Af(x))[f(tx) - Af(x)]\, d\lambda(t)\, \frac{dx}{x} \\ \geq\ & \int_0^\infty \int_0^\infty u(x)\, ||\Phi(f(tx)) - \Phi(Af(x))| - |\varphi(Af(x))|\,|f(tx) - Af(x)||\, d\lambda(t)\, \frac{dx}{x} \\ \geq\ & \int_0^\infty u(x) \bigg| \int_0^\infty |\Phi(f(tx)) - \Phi(Af(x))|\, d\lambda(t) \\ &- |\varphi(Af(x))| \int_0^\infty |f(tx) - Af(x)|\, d\lambda(t) \bigg|\, \frac{dx}{x} \\ \geq\ & \bigg| \int_0^\infty \int_0^\infty u(x)\,|\Phi(f(tx)) - \Phi(Af(x))|\, d\lambda(t)\, \frac{dx}{x} \\ &- \int_0^\infty \int_0^\infty u(x)\,|\varphi(Af(x))|\,|f(tx) - Af(x)|\, d\lambda(t)\, \frac{dx}{x} \bigg| \,. \end{aligned} \tag{5.2.7}$$

Again, by using Fubini's theorem and the substitution $y = tx$, the first integral on the left-hand side of (5.2.7) becomes

$$\begin{aligned} &\int_0^\infty \int_0^\infty u(x)\,\Phi(f(tx))\, d\lambda(t)\, \frac{dx}{x} = \int_0^\infty \int_0^\infty u(x)\,\Phi(f(tx))\, \frac{dx}{x}\, d\lambda(t) \\ =\ & \int_0^\infty \int_0^\infty u\big(\tfrac{y}{t}\big)\,\Phi(f(y))\, \frac{dy}{y}\, d\lambda(t) = \int_0^\infty \Phi(f(y)) \int_0^\infty u\big(\tfrac{y}{t}\big)\, d\lambda(t)\, \frac{dy}{y} \\ =\ & \int_0^\infty v(y)\Phi(f(y))\, \frac{dy}{y}\,, \end{aligned} \tag{5.2.8}$$

while for the second integral we have

$$\int_0^\infty \int_0^\infty u(x)\,\Phi(Af(x))\;d\lambda(t)\;\frac{dx}{x} = L\int_0^\infty u(x)\Phi(Af(x))\;\frac{dx}{x}\,. \qquad (5.2.9)$$

Finally, considering (5.2.4), we similarly get

$$\begin{aligned} &\int_0^\infty \int_0^\infty u(x)\,\varphi(Af(x))[f(tx) - Af(x)]\;d\lambda(t)\;\frac{dx}{x} \\ &= \int_0^\infty u(x)\varphi(Af(x))\Big(\int_0^\infty h_x(t)\,d\lambda(t)\Big)\;\frac{dx}{x} = 0\,, \end{aligned} \qquad (5.2.10)$$

so (5.2.3) holds by combining (5.2.7), (5.2.8), (5.2.9), and (5.2.10). □

Remark 5.2.1. Observe that (5.2.7) provides a pair of inequalities interpolated between the left-hand side and the right-hand side of (5.2.3), that is, further new refinements of (5.2.3).

Remark 5.2.2. If Φ is a concave function (that is, if $-\Phi$ is convex), then (5.2.5) reads

$$\begin{aligned} \Phi(r) - \Phi(s) - \varphi(r)\,(r-s) &= |\Phi(r) - \Phi(s) - \varphi(r)\,(r-s)| \\ &\geq ||\Phi(s) - \Phi(r)| - |\varphi(r)|\,|s-r||\;, \end{aligned}$$

where φ is a real function on I such that $\varphi(x) \in \partial\Phi(x) = [\Phi'_+(x),\, \Phi'_-(x)]$, for all $x \in I^\circ$. Therefore, in this setting (5.2.3) holds by its left-hand side replaced with

$$\int_0^\infty u(x)\,\Phi(Af(x))\;\frac{dx}{x} - \frac{1}{L}\int_0^\infty v(x)\,\Phi(f(x))\;\frac{dx}{x}\,.$$

Moreover, if Φ is an affine function, then (5.2.3) becomes equality.

Since the right-hand side of (5.2.3) is non-negative, as an immediate consequence of Theorem 5.2.1 and Remark 5.2.2 we get the following result, a weighted Boas's inequality.

Corollary 5.2.2. *Suppose λ is a finite Borel measure on $\mathbb{R}_+$, L is defined by* (5.1.3), *u is a non-negative measurable function on $\mathbb{R}_+$, and the function v is defined on $\mathbb{R}_+$ by* (5.2.2). *If Φ is a continuous convex function on an interval $I \subseteq \mathbb{R}$, then the inequality*

$$\int_0^\infty u(x)\,\Phi(Af(x))\,\frac{dx}{x} \;\leq\; \frac{1}{L}\int_0^\infty v(x)\Phi(f(x))\,\frac{dx}{x} \tag{5.2.11}$$

holds for all measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$, such that $f(x) \in I$ for all $x \in \mathbb{R}_+$, where $Af(x)$ is defined by (5.2.1). *For a concave function Φ, the sign of inequality in relation* (5.2.11) *is reversed.*

In the sequel, we analyze some important particular cases of Theorem 5.2.1 and Corollary 5.2.2 and compare them with some results previously known from the literature. Namely, by setting $u(x) \equiv 1$, we obtain a refined Boas-type inequality with $Af(x)$ defined by the Lebesgue integral.

Corollary 5.2.3. *Let λ be a finite Borel measure on $\mathbb{R}_+$ and L be defined by* (5.1.3). *Then the inequality*

$$\begin{aligned} &\int_0^\infty \Phi(f(x))\,\frac{dx}{x} - \int_0^\infty \Phi(Af(x))\,\frac{dx}{x} \\ \geq\; &\frac{1}{L}\left|\int_0^\infty\int_0^\infty |\Phi(f(tx)) - \Phi(Af(x))|\;d\lambda(t)\,\frac{dx}{x}\right. \\ &\left.-\int_0^\infty\int_0^\infty |\varphi(Af(x))|\,|f(tx) - Af(x)|\;d\lambda(t)\,\frac{dx}{x}\right| \end{aligned} \tag{5.2.12}$$

holds for all continuous convex functions Φ on an interval $I \subseteq \mathbb{R}$, real functions φ on I, such that $\varphi(x) \in \partial\Phi(x)$ for $x \in \operatorname{int} I$, and all measurable real functions f on $\mathbb{R}_+$, such that $f(x) \in I$ for all $x \in \mathbb{R}_+$, and $Af(x)$ defined by (5.2.1). *If the function Φ is concave, then* (5.2.12) *holds with*

$$\int_0^\infty \Phi(Af(x))\,\frac{dx}{x} - \int_0^\infty \Phi(f(x))\,\frac{dx}{x}$$

on its left-hand side.

Evidently, Corollary 5.2.3 implies the following analogue of (1.2.4).

Corollary 5.2.4. *If λ is a finite Borel measure on $\mathbb{R}_+$, L is defined by* (5.1.3), *Φ is a continuous convex function on an interval $I \subseteq \mathbb{R}$, f is a measurable real function on $\mathbb{R}_+$ with values in I, and $Af(x)$ is defined by* (5.2.1), *then*

$$\int_0^\infty \Phi(Af(x))\,\frac{dx}{x} \;\le\; \int_0^\infty \Phi(f(x))\,\frac{dx}{x}\,. \tag{5.2.13}$$

If Φ is concave, then the sign of inequality in (5.2.13) *is reversed.*

Remark 5.2.3. Let $m : [0,\infty) \to \mathbb{R}$ be a non-decreasing bounded function and $M = m(\infty) - m(0) > 0$. It is well-known that m induces a finite Borel measure λ on $\mathbb{R}_+$ (and vice versa), such that the related Lebesgue and Lebesgue-Stieltjes integrals are equivalent. Thus, all the above results from this section can be interpreted as for $Af(x)$ defined by the Lebesgue-Stieltjes integral with respect to m, that is, as

$$Af(x) = \frac{1}{M}\int_0^\infty f(tx)\,dm(t)\,, \qquad x \in \mathbb{R}_+\,.$$

Therefore, our results refine and generalize Boas's inequality (1.2.4). Namely, we obtained a refinement of its weighted version.

To conclude this section, we consider measures λ which yield refinements of the Hardy-Knopp-type inequalities mentioned in the Introduction. Especially, for $d\lambda(t) = \chi_{[0,1]}(t)\,dt$ we obtain a refinement of a weighted version of (1.2.5).

Theorem 5.2.5. *Let u be a non-negative function on $\mathbb{R}_+$, such that the function $t \mapsto \frac{u(t)}{t^2}$ is locally integrable in $\mathbb{R}_+$, and let*

$$w(x) = x\int_x^\infty u(t)\,\frac{dt}{t^2}\,, \quad t \in \mathbb{R}_+.$$

If a real-valued function Φ is convex on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in I^\circ$, then the inequality

$$\begin{aligned} &\int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x} - \int_0^\infty u(x)\Phi(Hf(x))\,\frac{dx}{x} \\ \geq\ &\left|\int_0^\infty u(x)\int_0^x |\Phi(f(t)) - \Phi(Hf(x))|\,dt\,\frac{dx}{x^2}\right. \\ &\left.-\int_0^\infty u(x)\,|\varphi(Hf(x))|\int_0^x |f(t) - Hf(x)|\,dt\,\frac{dx}{x^2}\right| \end{aligned} \tag{5.2.14}$$

holds for all functions f on $\mathbb{R}_+$ with values in I and for $Hf(x)$ defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt \tag{5.2.15}$$

for $x \in \mathbb{R}_+$. If Φ is a concave function, then (5.2.14) *holds with*

$$\int_0^\infty u(x)\Phi(Hf(x))\,\frac{dx}{x} - \int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x}$$

on its left-hand side.

Proof. Follows directly from Theorem 5.2.1 and Remark 5.2.2, rewritten with the measure $d\lambda(t) = \chi_{[0,1]}(t)\,dt$. In this setting, we have $L = 1$,

$$Af(x) = \int_0^1 f(tx)\,dt = Hf(x) \quad\text{and}\quad v(x) = \int_0^1 u\left(\frac{x}{t}\right)\,dt = w(x), \quad x \in \mathbb{R}_+\,,$$

so (5.2.14) holds. □

Remark 5.2.4. Let a convex function Φ and functions u, w, f, and Hf be as in Theorem 5.2.5. Observing that the right-hand side of relation (5.2.14) is non-negative, we get

$$\int_0^\infty u(x)\Phi(Hf(x))\,\frac{dx}{x} \leq \int_0^\infty w(x)\;\Phi(f(x))\,\frac{dx}{x}\,. \tag{5.2.16}$$

Moreover, for a concave function Φ relation (5.2.16) holds with the reversed sign of inequality. This result, the so-called weighted Hardy-Knopp-type inequality, was already obtained in [15, Theorem 1], while its particular case (1.2.5), originally proved in [34], follows by setting $u(x) \equiv 1$. Therefore, (5.2.14) may be regarded as a refined weighted inequality of the Hardy-Knopp type and relation (5.2.3) as its generalization.

On the other hand, a dual result to Theorem 5.2.5 can be derived by considering (5.2.3) with $d\lambda(t) = \chi_{[1,\infty)}(t)\,\frac{dt}{t^2}$.

Theorem 5.2.6. *Suppose $u : \mathbb{R}_+ \to \mathbb{R}$ is a non-negative function, locally integrable in $\mathbb{R}_+$, and w is defined on $\mathbb{R}_+$ by*

$$w(x) = \frac{1}{x}\int_0^x u(t)\,dt\,.$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in I^o$, then the inequality

$$\begin{aligned}
&\int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x} - \int_0^\infty u(x)\Phi(\tilde{H}f(x))\,\frac{dx}{x} \\
\geq\; &\left|\int_0^\infty u(x)\int_x^\infty \left|\Phi(f(t)) - \Phi(\tilde{H}f(x))\right|\,\frac{dt}{t^2}\,dx\right. \\
&\left. - \int_0^\infty u(x)\left|\varphi(\tilde{H}f(x))\right|\int_x^\infty \left|f(t) - \tilde{H}f(x)\right|\,\frac{dt}{t^2}\,dx\right|
\end{aligned} \tag{5.2.17}$$

holds for all functions f on $\mathbb{R}_+$ with values in I and for $\tilde{H}f(x)$ defined by

$$\tilde{H}f(x) = x\int_x^\infty f(t)\,\frac{dt}{t^2} \tag{5.2.18}$$

for $x \in \mathbb{R}_+$. In the case when Φ is concave, (5.2.17) *holds if its left-hand side is replaced with*

$$\int_0^\infty u(x)\Phi(\tilde{H}f(x))\,\frac{dx}{x} - \int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x}\,.$$

Proof. Set $d\lambda(t) = \chi_{[1,\infty)}(t)\,\frac{dt}{t^2}$ in Theorem 5.2.1 and Remark 5.2.2. Then

$$Af(x) = \int_1^\infty f(tx)\,\frac{dt}{t^2} = \tilde{H}f(x), \quad v(x) = \int_1^\infty u\left(\frac{x}{t}\right)\frac{dt}{t^2} = w(x), \quad x \in \mathbb{R}_+\,,$$

and $L = 1$, so (5.2.17) holds. □

Remark 5.2.5. As in Remark 5.2.4, note that for a convex function Φ and functions u, w, f, and $\tilde{H}f$ from the statement of Theorem 5.2.6, we have

$$\int_0^\infty u(x)\Phi(\tilde{H}f(x))\,\frac{dx}{x} \le \int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x}\,, \tag{5.2.19}$$

while for a concave Φ relation (5.2.19) holds with the inequality sign $\geq$. Since as a consequence of Theorem 5.2.1 and Theorem 5.2.6 we derived a dual inequality to (5.2.16), relation (5.2.17) can be considered as a refined dual weighted Hardy-Knopp-type inequality and (5.2.3) as its generalization.

Finally, as special cases of Theorem 5.2.5 and Theorem 5.2.6, we formulate refinements of the strengthened Hardy-Knopp-type inequalities.

Corollary 5.2.7. *Suppose $b \in \mathbb{R}_+$, $u : (0,b) \to \mathbb{R}$ is a non-negative function, such that the function $t \mapsto \frac{u(t)}{t^2}$ is locally integrable in $(0,b)$, and the function w is defined by*

$$w(x) = x\int_x^b u(t)\,\frac{dt}{t^2}\,, \quad x \in (0,b)\,.$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in I^\circ$, then the inequality

$$\begin{aligned} &\int_0^b w(x)\Phi(f(x))\,\frac{dx}{x} - \int_0^b u(x)\Phi(Hf(x))\,\frac{dx}{x} \\ \geq\ &\left|\int_0^b u(x)\int_0^x |\Phi(f(t)) - \Phi(Hf(x))|\,dt\,\frac{dx}{x^2}\right. \\ &\left. - \int_0^b u(x)\,|\varphi(Hf(x))|\int_0^x |f(t) - Hf(x)|\,dt\,\frac{dx}{x^2}\right| \end{aligned} \tag{5.2.20}$$

holds for all functions $f : (0,b) \to \mathbb{R}$ with values in I and Hf defined on $(0,b)$ by (5.2.15). If Φ is a concave function, the order of integrals on the left-hand side of (5.2.20) is reversed.

Proof. Let $\hat{u}$, $\hat{w}$, and $\hat{f}$ be defined on $\mathbb{R}_+$ by $\hat{u}(x) = u(x)\chi_{(0,b)}(x)$,

$$\hat{w}(x) = x\int_x^\infty \hat{u}(t)\,\frac{dt}{t^2} = w(x)\chi_{(0,b)}(x)\,,$$

and $\hat{f}(x) = f(x)\chi_{(0,b)}(x) + c\chi_{[b,\infty)}(x)$, where $c \in I$ is arbitrary. Since these functions naturally extend u, w, and f to act on $\mathbb{R}_+$, they evidently fulfill the conditions of Theorem 5.2.5, considered with $\hat{u}$, $\hat{w}$, and $\hat{f}$ instead of u, w, and f respectively. Therefore, (5.2.14) holds and in this setting it becomes (5.2.20). □

Remark 5.2.6. Since the right-hand side of (5.2.20) is non-negative, Corollary 5.2.7 improves a result from [15, Theorem 1]. Hence, it can be considered as a refined strengthened Hardy-Knopp-type inequality.

Remark 5.2.7. For $u(x) \equiv 1$, we have $w(x) = 1 - \frac{x}{b}$ in Corollary 5.2.7, so (5.2.20) reads

$$\begin{aligned} &\int_0^b \left(1 - \frac{x}{b}\right) \Phi(f(x))\,\frac{dx}{x} - \int_0^b \Phi(Hf(x))\,\frac{dx}{x} \\ \geq\ &\left|\int_0^b \int_0^x |\Phi(f(t)) - \Phi(Hf(x))|\,dt\,\frac{dx}{x^2}\right. \\ &\left. - \int_0^b |\varphi(Hf(x))| \int_0^x |f(t) - Hf(x)|\,dt\,\frac{dx}{x^2}\right|\,. \end{aligned} \tag{5.2.21}$$

This relation provides a basis for results in the following section.

A dual result to inequality (5.2.20) is given in the next corollary.

Corollary 5.2.8. *For $b \in \mathbb{R}_+$, let $u : (b,\infty) \to \mathbb{R}$ be a non-negative locally integrable function in (b,∞) and the function w be given by*

$$w(x) = \frac{1}{x}\int_b^x u(t)\,dt\,, \quad x \in (b,\infty)\,.$$

Let Φ be a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ be such that

$\varphi(x) \in \partial\Phi(x)$ for all $x \in I^\circ$. Then the inequality

$$\begin{aligned}
&\int_b^\infty w(x)\ \Phi(f(x))\,\frac{dx}{x} - \int_b^\infty u(x)\ \Phi(\tilde{H}f(x))\,\frac{dx}{x} \\
\geq\ &\left| \int_b^\infty u(x) \int_x^\infty \left|\Phi(f(t)) - \Phi(\tilde{H}f(x))\right| \frac{dt}{t^2}\,dx \right. \\
&\left. - \int_b^\infty u(x) \left|\varphi(\tilde{H}f(x))\right| \int_x^\infty \left|f(t) - \tilde{H}f(x)\right| \frac{dt}{t^2}\,dx \right|
\end{aligned} \tag{5.2.22}$$

holds for all functions f on (b,∞) with values in I and $\tilde{H}f$ defined by (5.2.18)*. For a concave function Φ, the order of integrals on the left-hand side of* (5.2.22) *is reversed.*

Proof. As in the proof of Corollary 5.2.7, inequality (5.2.22) follows by applying Theorem 5.2.6 to the functions $\hat{u}$, $\hat{w}$, and $\hat{f}$, where $\hat{u}(x) = u(x)\chi_{(b,\infty)}(x)$,

$$\hat{w}(x) = \frac{1}{x}\int_0^x \hat{u}(t)\,dt = w(x)\chi_{(b,\infty)}(x)\,,$$

and $\hat{f}(x) = c\chi_{(0,b]}(x) + f(x)\chi_{(b,\infty)}(x)$, for an arbitrary $c \in I$. □

Remark 5.2.8. Note that (5.2.22) refines [15, Theorem 2] since the right-hand side of (5.2.22) is non-negative. Thus, we obtained a refined strengthened dual Hardy-Knopp-type inequality.

Remark 5.2.9. For $u(x) \equiv 1$, (5.2.22) reads

$$\begin{aligned}
&\int_b^\infty \left(1 - \frac{b}{x}\right) \Phi(f(x))\,\frac{dx}{x} - \int_b^\infty \Phi(\tilde{H}f(x))\,\frac{dx}{x} \\
\geq\ &\left| \int_b^\infty \int_x^\infty \left|\Phi(f(t)) - \Phi(\tilde{H}f(x))\right| \frac{dt}{t^2}\,dx \right. \\
&\left. - \int_b^\infty \left|\varphi(\tilde{H}f(x))\right| \int_x^\infty \left|f(t) - \tilde{H}f(x)\right| \frac{dt}{t^2}\,dx \right|\,.
\end{aligned} \tag{5.2.23}$$

Together with (5.2.21), we shall use this inequality to derive refinements of the classical Hardy and Pólya-Knopp's inequalities.

5.3 Refinements of strengthened Hardy and Pólya–Knopp's inequalities

In the previous section, obtained inequalities were discussed with respect to a measure λ and a weight function u, while a convex function Φ remained unspecified. On the contrary, here we consider two particular convex (or concave) functions, namely $\Phi(x) = x^p$ and $\Phi(x) = e^x$, and derive some new refinements of the well-known Hardy and Pólya-Knopp's inequalities, as well as their strengthened versions. Moreover, we show that they are just special cases of the results mentioned.

We start with new refinements of Hardy's inequality, so let $p \in \mathbb{R}$, $p \neq 0$, and $\Phi(x) = x^p$. Obviously, $\varphi(x) = \Phi'(x) = px^{p-1}$, $x \in \mathbb{R}_+$, and the function Φ is convex for $p \in \mathbb{R} \setminus [0, 1)$, concave for $p \in (0, 1]$, and affine for $p = 1$. On the other hand, for a locally integrable function $f : \mathbb{R}_+ \to \mathbb{R}$, as in the Introduction, we denote

$$F(x) = \int_0^x f(t)\, dt \quad \text{and} \quad \tilde{F}(x) = \int_x^\infty f(t)\, dt\,, \quad x \in \mathbb{R}_+. \tag{5.3.1}$$

A new refined strengthened Hardy's inequality is given in the following corollary.

Corollary 5.3.1. *Let* $0 < b \leq \infty$ *and* $p, k \in \mathbb{R}$ *be such that* $p \neq 0$, $k \neq 1$, *and* $\frac{p}{k-1} > 0$. *Let* f *be a non-negative function on* $(0, b)$. *If* $p \in (-\infty, 0) \cup [1, \infty)$, *then the inequality*

$$\begin{aligned} &\left(\frac{p}{k-1}\right)^p \int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)\, dx - \int_0^b x^{-k} F^p(x)\, dx \\ \geq\ &\left| \left(\frac{p}{k-1}\right)^{p-1} \int_0^b x^{\frac{1-k}{p}-1} \int_0^x t^{\frac{k-1}{p}-1} \left| t^{p-k+1} f^p(t) - \left(\frac{k-1}{p}\right)^p x^{1-k} F^p(x) \right| dt\, dx \right. \\ &\left. -|p| \int_0^b x^{-k} F^{p-1}(x) \int_0^x \left| f(t) - \frac{k-1}{p} \cdot \frac{1}{t} \left(\frac{t}{x}\right)^{\frac{k-1}{p}} F(x) \right| dt\, dx \right| \end{aligned} \tag{5.3.2}$$

holds. In the case when $p \in (0, 1)$, *the order of integrals in inequality* (5.3.2) *is reversed.*

Proof. First, let either $p \geq 1$, $k > 1$, or $p < 0$, $k < 1$, and let $\Phi(x) = x^p$ and $\varphi(x) = px^{p-1}$. According to Corollary 5.2.7 and Remark 5.2.7, then (5.2.21) holds. Rewriting it for $a = b^{\frac{k-1}{p}}$ and $x \mapsto f(x^{\frac{p}{k-1}})\, x^{\frac{p}{k-1}-1}$, instead of b and f respectively, we get

$$\begin{aligned}
&\int_0^a \left(1 - \frac{x}{a}\right) x^{p\left(\frac{p}{k-1}-1\right)} f^p(x^{\frac{p}{k-1}})\, \frac{dx}{x} - \int_0^a \left(\frac{1}{x}\int_0^x f(t^{\frac{p}{k-1}})\, t^{\frac{p}{k-1}-1} dt\right)^p \frac{dx}{x} \\
\geq\ & \left|\int_0^a \int_0^x \left| t^{p\left(\frac{p}{k-1}-1\right)} f^p(t^{\frac{p}{k-1}}) - \left(\frac{1}{x}\int_0^x f(r^{\frac{p}{k-1}})\, r^{\frac{p}{k-1}-1} dr\right)^p \right| dt\, \frac{dx}{x^2} \right. \\
& - \int_0^a \left| p \left(\frac{1}{x}\int_0^x f(r^{\frac{p}{k-1}})\, r^{\frac{p}{k-1}-1} dr\right)^{p-1} \right| \\
& \left. \times \int_0^x \left| f(t^{\frac{p}{k-1}})\, t^{\frac{p}{k-1}-1} - \frac{1}{x}\int_0^x f(r^{\frac{p}{k-1}})\, r^{\frac{p}{k-1}-1} dr \right| dt\, \frac{dx}{x^2} \right|,
\end{aligned}$$

so (5.3.2) follows by a sequence of substitutions such as $s = x^{\frac{p}{k-1}}$. The remaining case, that is, when $p \in (0,1)$ and $k > 1$, is a direct consequence of Corollary 5.2.7 and Remark 5.2.7. □

Now, we state and prove a refined strengthened dual Hardy's inequality.

Corollary 5.3.2. *Suppose* $0 \leq b < \infty$ *and* $p, k \in \mathbb{R}$ *are such that* $p \neq 0$, $k \neq 1$, *and* $\frac{p}{k-1} < 0$. *If* $p \in (-\infty, 0) \cup [1, \infty)$, *then the inequality*

$$\begin{aligned}
&\left(\frac{p}{1-k}\right)^p \int_b^\infty \left[1 - \left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] x^{p-k} f^p(x)\, dx - \int_b^\infty x^{-k}\, \tilde{F}^p(x)\, dx \\
\geq\ & \left| \left(\frac{p}{1-k}\right)^{p-1} \int_b^\infty x^{\frac{1-k}{p}-1} \int_x^\infty t^{\frac{k-1}{p}-1} \left| t^{p-k+1} f^p(t) - \left(\frac{1-k}{p}\right)^p x^{1-k} \tilde{F}^p(x) \right| dt\, dx \right. \\
& \left. -|p| \int_b^\infty x^{-k} \tilde{F}^{p-1}(x) \int_x^\infty \left| f(t) - \frac{1-k}{p} \cdot \frac{1}{t}\left(\frac{x}{t}\right)^{\frac{1-k}{p}} \tilde{F}(x) \right| dt\, dx \right| \qquad (5.3.3)
\end{aligned}$$

holds for all non-negative functions f *on* (b, ∞). *In the case when* $p \in (0,1)$, *the order of integrals in inequality* (5.3.3) *is reversed.*

Proof. As in the proof of Corollary 5.3.1, we use $\Phi(x) = x^p$, that is, $\varphi(x) = px^{p-1}$, and rewrite inequality (5.2.23) for b and f respectively replaced with $a = b^{\frac{1-k}{p}}$ and $x \mapsto f(x^{\frac{p}{1-k}})x^{\frac{p}{1-k}+1}$. Relation (5.3.3) then follows by a sequence of substitutions of the form $s = x^{\frac{p}{1-k}}$. Note that we again distinguish two cases. The first one, which yields (5.3.3), holds when either $p \geq 1$, $k < 1$, or $p < 0$, $k > 1$. In the other one, with $p \in (0,1)$ and $k < 1$, the order of integrals on the left-hand side of inequality (5.3.3) is reversed. □

Remark 5.3.1. Observe that for $b = \infty$ the left-hand side of (5.3.2) reads

$$\left(\frac{p}{k-1}\right)^p \int_0^\infty x^{p-k} f^p(x)\,dx - \int_0^\infty x^{-k} F^p(x)\,dx\,,$$

while for $b = 0$ on the left-hand side of (5.3.3) we have

$$\left(\frac{p}{1-k}\right)^p \int_0^\infty x^{p-k} f^p(x)\,dx - \int_0^\infty x^{-k} \tilde{F}^p(x)\,dx\,.$$

Therefore, we obtained a refinement of the classical Hardy's inequality (1.2.1). Also note that for $p = 1$ relations (5.3.2) and (5.3.2) are trivial since their both sides are equal to 0.

Remark 5.3.2. Observe that Corollary 5.3.1 and Corollary 5.3.2 provide new and original refinements of Hardy's inequality although the idea to strengthen and refine (1.2.1) is not new and results of such type already exist in the literature. As in the Introduction, here we just mention the papers [32, 33, 59], and a recent paper [53]. It is important to emphasize that our results are completely different from those given in these papers and even hardly comparable with (5.1.1) and (5.1.2). A third type of refinements of a similar form can be found in another recent paper [50], where a fairly new concept of superquadratic function was used in a crucial way.

Finally, we consider $\Phi(x) = e^x$ to obtain refinements of the strengthened Pólya-Knopp's inequality and of its dual. For a positive function f on $\mathbb{R}_+$ and $x \in \mathbb{R}_+$, we denote

$$Gf(x) = \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right) \quad \text{and} \quad \tilde{G}f(x) = \exp\left(x\int_x^\infty \log f(t)\,\frac{dt}{t^2}\right).$$

Related results are given in the following two corollaries.

Corollary 5.3.3. *Let* $0 < b \leq \infty$ *and* f *be a positive function on* $(0, b)$*. Then*

$$\begin{aligned} & e\int_0^b \left(1 - \frac{x}{b}\right) f(x)\,dx - \int_0^b Gf(x)\,dx \\ \geq & \left|\int_0^b \int_0^x |e\,tf(t) - x\,Gf(x)|\,dt\,\frac{dx}{x^2} - \right. \\ & \left. \int_0^b Gf(x)\int_0^x \left|\log\frac{e\,tf(t)}{x\,Gf(x)}\right|\,dt\,\frac{dx}{x}\right|. \end{aligned} \tag{5.3.4}$$

Corollary 5.3.4. *If* $0 \leq b < \infty$ *and* f *is a positive function on* (b, ∞)*, then*

$$\begin{aligned} & \frac{1}{e}\int_b^\infty \left(1 - \frac{b}{x}\right) f(x)\,dx - \int_b^\infty \tilde{G}f(x)\,dx \\ \geq & \left|\int_b^\infty \int_x^\infty \left|\frac{1}{e}\,tf(t) - x\,\tilde{G}f(x)\right|\,\frac{dt}{t^2}\,dx \right. \\ & \left. - \int_b^\infty x\,\tilde{G}f(x)\int_x^\infty \left|\log\frac{tf(t)}{ex\,\tilde{G}f(x)}\right|\,\frac{dt}{t^2}\,dx\right|. \end{aligned} \tag{5.3.5}$$

Remark 5.3.3. Note that for $b = \infty$ in (5.3.4) we have a refined Pólya-Knopp's inequality, while for $b = 0$ relation (5.3.5) becomes its refined dual inequality.

Chapter 6

Applications In Information Theory

6.1 Introduction

For a function ϕ defined on an interval I of the real line $\mathbb{R}$ the $\mathfrak{J}-$divergence between vectors x, y in a convex set of $n-$dimensional real vector space is defined by [10, 52]

$$\mathfrak{J}_{n,\phi}(x,y) = \sum_{i=1}^{n} \left(\frac{1}{2}[\phi(x_i) + \phi(y_i)] - \phi\left(\frac{x_i + y_i}{2}\right)\right) \tag{6.1.1}$$

and in particular if ϕ is differentiable then $\mathfrak{B}-$divergence is defined by [10, 52]

$$\mathfrak{B}_{n,\phi}(x,y) = \sum_{i=1}^{n} \left[\phi(x_i) - \phi(y_i) - \phi'(y_i)(x_i - y_i)\right], \tag{6.1.2}$$

where $(x, y) \in I^n \times I^n \subseteq \mathbb{R}^n \times \mathbb{R}^n$. When the interval I does not contain the origin, the two alternate measures called $\mathfrak{K}-$ and $\mathfrak{L}-$divergences with $x \neq y$, are defined as [10]

$$\mathfrak{K}_{n,\phi}(x,y) = \sum_{i=1}^{n} (x_i - y_i)\left[\frac{\phi(x_i)}{x_i} - \frac{\phi(y_i)}{y_i}\right] \tag{6.1.3}$$

and

$$\mathfrak{L}_{n,\phi}(x,y) = \sum_{i=1}^{n} \left[x_i\, \phi\left(\frac{y_i}{x_i}\right) + y_i\, \phi\left(\frac{x_i}{y_i}\right) \right]. \tag{6.1.4}$$

The special cases of these divergences for the function [25]

$$\phi_\alpha : \mathbb{R}_+ \to \mathbb{R},\ \alpha > 0\ ,\ defined\ \ by$$

$$\phi_\alpha(x) = \begin{cases} (\alpha-1)^{-1}(x^\alpha - x), & \alpha \neq 1; \\ x \log x, & \alpha = 1. \end{cases} \tag{6.1.5}$$

are denoted by $\mathfrak{J}_{n,\alpha}$, $\mathfrak{K}_{n,\alpha}$ and $\mathfrak{L}_{n,\alpha}$ and are given by the following:

$$\mathfrak{J}_{n,\alpha}(x,y) = \begin{cases} (\alpha\ -\ 1)^{-1} \sum_{i=1}^{n} \left[\frac{1}{2}(x_i^\alpha + y_i^\alpha)\ -\ (\frac{x_i+y_i}{2})^\alpha \right], & \alpha \neq 1; \\ \frac{1}{2} \sum_{i=1}^{n} \left[x_i \log x_i + y_i \log y_i - (x_i\ +\ y_i)\ \log(\frac{x_i+y_i}{2}) \right], & \alpha = 1. \end{cases} \tag{6.1.6}$$

$$\mathfrak{K}_{n,\alpha}(x,y) = \begin{cases} (\alpha-1)^{-1} \sum_{i=1}^{n} (x_i - y_i)\ \left(x_i^{\alpha-1}\ -\ y_i^{\alpha-1}\right), & \alpha \neq 1; \\ \sum_{i=1}^{n} (x_i - y_i)\ (\log x_i\ -\ \log y_i)\,, & \alpha = 1. \end{cases} \tag{6.1.7}$$

$$\mathfrak{L}_{n,\alpha}(x,y) = \begin{cases} (\alpha-1)^{-1} \left[\sum_{i=1}^{n} x_i^\alpha\, y_i^{1-\alpha} + \sum_{i=1}^{n} y_i^\alpha\, x_i^{1-\alpha} - 2 \right], & \alpha \neq 1; \\ \sum_{i=1}^{n}\ (x_i - y_i)\ (\log x_i - \log y_i)\,, & \alpha = 1. \end{cases} \tag{6.1.8}$$

$$\mathfrak{B}_{n,\alpha}(x,y) = \begin{cases} (\alpha-1)^{-1} \sum_{i=1}^{n} \left[x_i^\alpha - \alpha x_i y_i^{\alpha-1} + (\alpha-1)\, y_i^\alpha \right], & \alpha \neq 1; \\ \sum_{i=1}^{n} \left[x_i(\log x_i - 1)\ +\ y_i(1 - \log y_i) \right], & \alpha = 1. \end{cases} \tag{6.1.9}$$

Suppose that all the above divergences are positive for $\alpha > 0$.

In this chapter we shall suppose that $x, y \in S_n = \{(x_1, x_2, ..., x_n) \in I_o^n : \sum_{i=1}^{n} x_i = 1\}$ where $I_o \equiv (0,1)$.

The following Theorem establishes a comparison between $\mathfrak{K}_{n,\phi}$ and $\mathfrak{J}_{n,\phi}$ [10]

Theorem 6.1.1. *For any* $x, y \in \mathbb{R}^n_+$

$$4\,\mathfrak{J}_{n,\phi}(x,y) \;\geq\; \mathfrak{K}_{n,\phi}(x,y) \tag{6.1.10}$$

iff $\frac{\phi(x)}{x}$ *is convex on* $\mathbb{R}_+$, *and in that case equality occur iff* $x = y$.

6.2 log-convexity of $\mathfrak{J}-$divergence

Lemma 6.2.1. *Let us define a function for* $s > 0$,

$$\eta_s(x) = \begin{cases} \frac{x^s}{s-1}, & s \neq 1; \\ x\log x, & s = 1. \end{cases} \tag{6.2.1}$$

Then $\eta_s''(x) = s\,x^{s-2}$, *that is,* $\eta_s(x)$ *is convex for* $x \geq 0$ *with the convention* $0\log 0 = 0$ *also* $\frac{\eta_s(x)}{x}$ *is increasing for* $x > 0$.

Lemma 6.2.2. *Let* f *be* $\log-$*convex function and if* $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, *then the following inequality is valid*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{\frac{1}{x_2-x_1}} \leq \left(\frac{f(y_2)}{f(y_1)}\right)^{\frac{1}{y_2-y_1}}. \tag{6.2.2}$$

By setting $\phi \mapsto \varphi_s$ in (6.1.1) for $x_i,\, y_i > 0$ we get

$$\tilde{\mathfrak{J}}_{n,p}(x,y) = \begin{cases} \frac{\mathfrak{J}_{n,p}(x,y)}{p}, & p \neq 0,1; \\ \sum_{i=1}^{n}\{\log(\frac{x_i+y_i}{2}) \,-\, \log\sqrt{x_i\,y_i}\}, & p = 0; \\ \mathfrak{J}_{n,p}(x,y), & p = 1. \end{cases} \tag{6.2.3}$$

Similarly by setting $\phi \mapsto \eta_s$ in (6.1.1) for $x_i,\, y_i \geq 0$ and $p > 0$ we get $\mathfrak{J}_{n,p}(x,y)$ defined by (6.1.6).

Theorem 6.2.3. $\tilde{\mathfrak{J}}_{n,p}$ *defined by* (6.2.3) *is* $\log-$*convex, that is,*

$$[\tilde{\mathfrak{J}}_{n,p}(x,y)]^{r-s} \leq [\tilde{\mathfrak{J}}_{n,r}(x,y)]^{p-s}\;[\tilde{\mathfrak{J}}_{n,s}(x,y)]^{r-p} \;\; for \;\; -\infty < r < s < p < \infty. \tag{6.2.4}$$

Proof. Let us consider the function Φ defined by

$$\Phi(x) = u^2\varphi_s(x) + 2uw\varphi_r(x) + w^2\varphi_p(x) \ \ , where \ \ r = \frac{s+p}{2} \ \ and \ u, w \in \mathbb{R}$$

$$\Phi''(x) = u^2x^{s-2} + 2uwx^{r-2} + w^2x^{p-2} = (ux^{\frac{s}{2}-1} + wx^{\frac{p}{2}-1})^2 \geq 0, \ \ for \ x > 0.$$

We have that Φ is convex for $x > 0$. For convex function ϕ, $\mathfrak{I}_{n,\phi}(x, y)$ defined by (6.1.1) is non-negative. Therefore (6.1.1) is equivalent to

$$u^2\tilde{\mathfrak{I}}_{n,s} + 2uw\tilde{\mathfrak{I}}_{n,r} + w^2\tilde{\mathfrak{I}}_{n,p} \geq 0$$

that is,

$$\tilde{\mathfrak{I}}^2_{n,r}(x, y) \ \ \leq \tilde{\mathfrak{I}}_{n,s}(x, y)\tilde{\mathfrak{I}}_{n,p}(x, y).$$

So $\tilde{\mathfrak{I}}_{n,s}(x, y)$ is $\log -$convex in Jensen sense. Since,

$$\lim_{s\to 0} \tilde{\mathfrak{I}}_{n,s}(x, y) = \tilde{\mathfrak{I}}_{n,0}(x, y) \ \ and \ \ \lim_{s\to 1} \tilde{\mathfrak{I}}_{n,s}(x, y) = \tilde{\mathfrak{I}}_{n,1}(x, y)$$

$\tilde{\mathfrak{I}}_{n,s}(x, y)$ is continuous for $s \in \mathbb{R}$ and therefore $\log -$convex. □

Corollary 6.2.4. *For p, r, s, $t \in \mathbb{R}$ such that $r \leq s$, $p \leq t$ with $r \neq p$, $s \neq t$ we have*

$$\left(\frac{\tilde{\mathfrak{I}}_{n,p}(x, y)}{\tilde{\mathfrak{I}}_{n,r}(x, y)}\right)^{\frac{1}{p-r}} \ \ \leq \ \ \left(\frac{\tilde{\mathfrak{I}}_{n,t}(x, y)}{\tilde{\mathfrak{I}}_{n,s}(x, y)}\right)^{\frac{1}{t-s}}.$$

Theorem 6.2.5. *$\mathfrak{I}_{n,p}$ defined by (6.1.6) is $\log -$convex, that is,*

$$[\mathfrak{I}_{n,p}(x, y)]^{r-s} \ \ \leq \ \ [\mathfrak{I}_{n,r}(x, y)]^{p-s} \ [\mathfrak{I}_{n,s}(x, y)]^{r-p} \ \ for \ \ -\infty < r < s < p < \infty \qquad (6.2.5)$$

Proof. Similar to proof of Theorem 6.2.3. □

Corollary 6.2.6. *For p, r, s, $t \in \mathbb{R}_+$ such that $r \leq s$, $p \leq t$ with $r \neq p$, $s \neq t$ we have*

$$\left(\frac{\mathfrak{I}_{n,p}(x, y)}{\mathfrak{I}_{n,r}(x, y)}\right)^{\frac{1}{p-r}} \ \ \leq \ \ \left(\frac{\mathfrak{I}_{n,t}(x, y)}{\mathfrak{I}_{n,s}(x, y)}\right)^{\frac{1}{t-s}}.$$

6.3 log-convexity of $\mathfrak{L}-$divergence

Lemma 6.3.1. *If ϕ is convex, then*

$$\phi(1) = \phi\left(\sum_{i=1}^{n} y_i\right) = \phi\left(\sum_{i=1}^{n} x_i \frac{y_i}{x_i}\right) \leq \sum_{i=1}^{n} x_i \phi\left(\frac{y_i}{x_i}\right)$$

From (6.1.4) *we have*

$$\sum_{i=1}^{n}\left[x_i\,\phi\left(\frac{y_i}{x_i}\right) + y_i\,\phi\left(\frac{x_i}{y_i}\right)\right] \geq 2\phi(1)$$

Define

$$\overline{\mathfrak{L}}_{n,\phi}(x,y) = \sum_{i=1}^{n}\left[x_i\,\phi\left(\frac{y_i}{x_i}\right) + y_i\,\phi\left(\frac{x_i}{y_i}\right)\right] - 2\phi(1) \tag{6.3.1}$$

By setting $\phi \mapsto \varphi_s$ in (6.3.1) for $x_i,\ y_i > 0$, we get

$$\tilde{\overline{\mathfrak{L}}}_{n,p}(x,y) = \begin{cases} \frac{\mathfrak{L}_{n,p(x,y)}}{p}, & p \neq 0, 1; \\ \sum_{i=1}^{n}(\log\, x_i - \log\, y_i)(x_i + y_i), & p = 0; \\ \mathfrak{L}_{n,p(x,y)}, & p = 1. \end{cases} \tag{6.3.2}$$

Similarly by setting $\phi \mapsto \eta_s$ in (6.3.1) for $x_i,\ y_i \geq 0$ and $p > 0$ we get $\mathfrak{L}_{n,p}(x,y)$ defined by (6.1.8)

Theorem 6.3.2. $\tilde{\overline{\mathfrak{L}}}_{n,p}$ *defined by* (6.3.2) *is* $\log -$*convex, that is,*

$$\left[\tilde{\overline{\mathfrak{L}}}_{n,p}(x,y)\right]^{r-s} \leq \left[\tilde{\overline{\mathfrak{L}}}_{n,r}(x,y)\right]^{p-s}\left[\tilde{\overline{\mathfrak{L}}}_{n,s}(x,y)\right]^{r-p} \quad for \quad -\infty < r < s < p < \infty \tag{6.3.3}$$

Proof. Let us consider the function Φ defined by

$$\Phi(x) = u^2\varphi_s(x) + 2uw\varphi_r(x) + w^2\varphi_p(x)\ \ , where\ \ r = \frac{s+p}{2}\ \ and\ u, w \in \mathbb{R}$$

$$\Phi''(x) = u^2x^{s-2} + 2uwx^{r-2} + w^2x^{p-2} = \left(ux^{\frac{s}{2}-1} + wx^{\frac{p}{2}-1}\right)^2 \geq 0,\ \ for\ x > 0.$$

We have that Φ is convex for $x > 0$. For convex function ϕ, $\mathfrak{L}_{n,\phi}$ defined by (6.3.1) is non-negative. Therefore (6.3.1) is equivalent to

$$u^2 \,\tilde{\tilde{\mathfrak{L}}}_{n,s} + 2uw\,\tilde{\tilde{\mathfrak{L}}}_{n,r} + w^2\,\tilde{\tilde{\mathfrak{L}}}_{n,p} \geq 0$$

i.e.,

$$\tilde{\tilde{\mathfrak{L}}}^2_{n,r}(x,y) \;\leq \tilde{\tilde{\mathfrak{L}}}_{n,s}(x,y)\;\tilde{\tilde{\mathfrak{L}}}_{n,p}(x,y)$$

So $\tilde{\tilde{\mathfrak{L}}}_{n,s}(x,y)$ is $\log-$convex in Jensen sense. Since,

$$\lim_{s\to 0}\tilde{\tilde{\mathfrak{L}}}_{n,s}(x,y) = \tilde{\tilde{\mathfrak{L}}}_{n,0}(x,y) \;\; and \;\; \lim_{s\to 1}\tilde{\tilde{\mathfrak{L}}}_{n,s}(x,y) = \tilde{\tilde{\mathfrak{L}}}_{n,1}(x,y)$$

$\tilde{\tilde{\mathfrak{L}}}_{n,s}(x,y)$ is continuous for $s \in \mathbb{R}$ and therefore $\log-$convex.

□

Corollary 6.3.3. *For p, r, s, $t \in \mathbb{R}$ such that $r \leq s$, $p \leq t$ with $r \neq p$, $s \neq t$ we have*

$$\left(\frac{\tilde{\tilde{\mathfrak{L}}}_{n,p}(x,y)}{\tilde{\tilde{\mathfrak{L}}}_{n,r}(x,y)}\right)^{\frac{1}{p-r}} \leq \left(\frac{\tilde{\tilde{\mathfrak{L}}}_{n,t}(x,y)}{\tilde{\tilde{\mathfrak{L}}}_{n,s}(x,y)}\right)^{\frac{1}{t-s}}.$$

Theorem 6.3.4. *$\mathfrak{L}_{n,p}$ defined by (6.1.8) is $\log-$convex, that is,*

$$[\mathfrak{L}_{n,p}(x,y)]^{r-s} \;\leq\; [\mathfrak{L}_{n,r}(x,y)]^{p-s}\,[\mathfrak{L}_{n,s}(x,y)]^{r-p} \;\; for \;\; 0 < r < s < p < \infty \qquad (6.3.4)$$

Proof. Similar to proof of Theorem 6.3.2. □

Corollary 6.3.5. *For p, r, s, $t \in \mathbb{R}_+$ such that $r \leq s$, $p \leq t$ with $r \neq p$, $s \neq t$ we have*

$$\left(\frac{\mathfrak{L}_{n,p}(x,y)}{\mathfrak{L}_{n,r}(x,y)}\right)^{\frac{1}{p-r}} \leq \left(\frac{\mathfrak{L}_{n,t}(x,y)}{\mathfrak{L}_{n,s}(x,y)}\right)^{\frac{1}{t-s}}.$$

6.4 log-convexity of $\mathfrak{K}-$divergence

Theorem 6.4.1. *$\mathfrak{K}_{n,p}$ defined by (6.1.7) is $\log-$convex, that is,*

$$[\mathfrak{K}_{n,p}(x,y)]^{r-s} \;\leq\; [\mathfrak{K}_{n,s}(x,y)]^{r-p}\;[\mathfrak{K}_{n,r}(x,y)]^{p-s} \;\; for \;\; -\infty < r < s < p < \infty \quad (6.4.1)$$

Proof. Consider the function

$$F(x) = u^2\,\eta_s(x)\;+\;2u\,w\,\eta_r(x)\;+\;w^2\,\eta_p(x)$$

where u, w, p, $s \in \mathbb{R}$, $x > 0$ and $r = \frac{s+p}{2}$.

Then $\frac{F(x)}{x}$ is an increasing function, that is

$$(x\;-\;y)\left(\frac{F(x)}{x}\;-\;\frac{F(y)}{y}\right) \geq 0.$$

Therefore

$$\sum_{i=1}^{n}(x_i\;-\;y_i)\;\left(\frac{F(x_i)}{x_i}\;-\;\frac{F(y_i)}{y_i}\right) \geq 0.$$

$$\Rightarrow\quad u^2\,\mathfrak{K}_{n,s}\;+\;2\,u\,w\,\mathfrak{K}_{n,r}\;+\;w^2\,\mathfrak{K}_{n,p} \geq 0,$$

i.e.,

$$\mathfrak{K}^2_{n,r}(x,y)\;\;\leq\;\;\mathfrak{K}_{n,p}(x,y)\;\mathfrak{K}_{n,s}(x,y)$$

So $\mathfrak{K}_{n,s}(x,y)$ is log −convex in Jensen sense. Since,

$$\lim_{s\to 0}\mathfrak{K}_{n,s}(x,y) = \mathfrak{K}_{n,0}(x,y)\;\;\textit{and}\;\;\lim_{s\to 1}\mathfrak{K}_{n,s}(x,y) = \mathfrak{K}_{n,1}(x,y)$$

$\mathfrak{K}_{n,s}(x,y)$ is continuous for $s \in \mathbb{R}$ and therefore log −convex.

□

Corollary 6.4.2. *For p, r, s, $t \in \mathbb{R}$ such that $r \leq s$, $p \leq t$ with $r \neq p$, $s \neq t$ we have*

$$\left(\frac{\mathfrak{K}_{n,p}(x,y)}{\mathfrak{K}_{n,r}(x,y)}\right)^{\frac{1}{p-r}} \leq \left(\frac{\mathfrak{K}_{n,t}(x,y)}{\mathfrak{K}_{n,s}(x,y)}\right)^{\frac{1}{t-s}}.$$

Lemma 6.4.3. *Let us define another function*

$$\zeta_p(x) = \begin{cases} \frac{x^{p+1}}{p(p-1)}, & p \neq 0,1; \\ -x\log x, & p{=}0; \\ x^2\log x, & p{=}1. \end{cases} \tag{6.4.2}$$

Then $\psi_p''(x) = \left(\frac{\zeta_p(x)}{x}\right)'' = x^{p-2}$, that is, ψ_p is convex for $x > 0$.

By setting $\phi \mapsto \zeta_p$ in (6.1.1) and (6.1.3) we get

$$\overline{\mathfrak{K}}_{n,p}(x,y) = \begin{cases} \frac{1}{p(p-1)} \sum_{i=1}^{n} (x_i - y_i)\,(x_i^p - y_i^p)\,, & p \neq 0,1; \\ \sum_{i=1}^{n} (x_i - y_i)\,(\log y_i - \log x_i)\,, & p = 0; \\ \sum_{i=1}^{n} (x_i - y_i)\,(x_i \log x_i - y_i \log y_i)\,, & p = 1. \end{cases} \tag{6.4.3}$$

By setting $p \mapsto \alpha - 1$ we get

$$\overline{\mathfrak{K}}_{n,\alpha-1}(x,y) = \begin{cases} \frac{\mathfrak{K}_{n,\alpha}(x,y)}{\alpha-2}, & \alpha \neq 1,2; \\ -\,\mathfrak{K}_{n,\alpha}(x,y), & \alpha = 1; \\ \sum_{i=1}^{n} (x_i - y_i)\,(x_i \log x_i - y_i \log y_i)\,, & \alpha = 2. \end{cases} \tag{6.4.4}$$

and

$$\breve{\mathfrak{J}}_{n,p}(x,y) = \begin{cases} \frac{1}{p\,(p-1)} \sum_{i=1}^{n} \left(\frac{(x_i^{p+1} + y_i^{p+1})}{2} - \left(\frac{x_i + y_i}{2}\right)^{p+1} \right), & p \neq 0,1; \\ \sum_{i=1}^{n} \frac{(-\,x_i \log x_i - y_i \log y_i)}{2} + \left(\frac{x_i + y_i}{2}\right) \log\left(\frac{x_i + y_i}{2}\right), & p = 0; \\ \sum_{i=1}^{n} \frac{\left(x_i^2 \log x_i - y_i^2 \log y_i\right)}{2} - \left(\frac{x_i + y_i}{2}\right)^2 \log\left(\frac{x_i + y_i}{2}\right), & p = 1. \end{cases} \tag{6.4.5}$$

Theorem 6.4.4. *Consider the function*

$$\Lambda_p(x,y) = 4\,\breve{\mathfrak{J}}_{n,p}(x,y) - \overline{\mathfrak{K}}_{n,p}(x,y),$$

then Λ_p *is* $\log -$*convex, that is,*

$$[\Lambda_p(x,y)]^{r-s} \leq [\Lambda_r(x,y)]^{p-s}\,[\Lambda_s(x,y)]^{r-p} \quad for \quad -\infty < r < s < p < \infty \tag{6.4.6}$$

Proof. Let us consider the function f defined by

$$f(x) = u^2\,\zeta_s(x) + 2uw\,\zeta_r(x) + w^2 \zeta_p(x)\ , where\ \ r = \frac{s+p}{2}\ \ and\ u, w \in \mathbb{R}$$

$$\left(\frac{f(x)}{x}\right)'' = u^2\,x^{s-2} + 2uwx^{r-2} + w^2\,x^{p-2} = \left(ux^{\frac{s}{2}-1} + wx^{\frac{p}{2}-1}\right)^2 \geq 0, \ \ for\ x > 0.$$

We have that $\frac{f(x)}{x}$ is convex for $x > 0$. Therefore by (6.1.10) we have

$$u^2\,\Lambda_s + 2\,u\,w\,\Lambda_r + w^2\,\Lambda_p \geq 0,$$

that is,

$$\Lambda_r^2(x,y) \ \leq \ \Lambda_s(x,y)\ \Lambda_p(x,y).$$

So $\Lambda_s(x,y)$ is $\log-$convex in Jensen sense. Since,

$$\lim_{s\to 0}\Lambda_s(x,y) = \Lambda_0(x,y) \ \ and \ \ \lim_{s\to 1}\Lambda_s(x,y) = \Lambda_1(x,y)$$

$\Lambda_s(x,y)$ is continuous for $s \in \mathbb{R}$ and therefore $\log-$convex.

□

Corollary 6.4.5. *For $p,\ r,\ s,\ t \in \mathbb{R}$ such that $r \leq s$, $p \leq t$ with $r \neq p$, $s \neq t$ we have*

$$\left(\frac{\Lambda_p(x,y)}{\Lambda_r(x,y)}\right)^{\frac{1}{p-r}} \ \leq \ \left(\frac{\Lambda_t(x,y)}{\Lambda_s(x,y)}\right)^{\frac{1}{t-s}}.$$

6.5 log-convexity of $\mathfrak{B}-$divergence

By setting $\phi \mapsto \varphi_p$ in (6.1.2) for $x_i,\ y_i > 0$ we get

$$\tilde{\mathfrak{B}}_{n,p}(x,y) = \begin{cases} \frac{\mathfrak{B}_{n,p}(x,y)}{p}, & p \neq 0,1; \\ \sum_{i=1}^{n}\left[\frac{x_i-y_i}{y_i} \ + \log y_i \ - \ \log x_i\right], & p = 0; \\ \mathfrak{B}_{n,p}(x,y), & p = 1. \end{cases} \tag{6.5.1}$$

Similarly by setting $\phi \mapsto \eta_p$ in (6.1.2) for $x_i,\ y_i \geq 0$ and $p > 0$ we get $\mathfrak{B}_{n,p}(x,y)$ defined by (6.1.9).

Theorem 6.5.1. *$\tilde{\mathfrak{B}}_{n,p}$ defined by (6.5.1) is $\log-$convex, that is,*

$$\left[\tilde{\mathfrak{B}}_{n,p}(x,y)\right]^{r-s} \ \leq \ \left[\tilde{\mathfrak{B}}_{n,r}(x,y)\right]^{p-s}\ \left[\tilde{\mathfrak{B}}_{n,s}(x,y)\right]^{r-p} \ \ for \ \ -\infty < r < s < p < \infty. \tag{6.5.2}$$

Proof. Similar to proof of Theorem 6.4.4 □

Corollary 6.5.2. *For* $p,\ r,\ s,\ t\in\mathbb{R}$ *such that* $r\le s$, $p\le t$ *with* $r\ne p$, $s\ne t$ *we have*

$$\left(\frac{\tilde{\mathfrak{B}}_{n,p}(x,y)}{\tilde{\mathfrak{B}}_{n,r}(x,y)}\right)^{\frac{1}{p-r}} \le \left(\frac{\tilde{\mathfrak{B}}_{n,t}(x,y)}{\tilde{\mathfrak{B}}_{n,s}(x,y)}\right)^{\frac{1}{t-s}}.$$

Theorem 6.5.3. $\mathfrak{B}_{n,p}$ *defined by* (6.1.9) *is* $\log-$*convex, that is,*

$$[\mathfrak{B}_{n,p}(x,y)]^{r-s} \le [\mathfrak{B}_{n,r}(x,y)]^{p-s}\,[\mathfrak{B}_{n,s}(x,y)]^{r-p}\ \ for\ \ 0<r<s<p<\infty \tag{6.5.3}$$

Proof. Similar to proof of Theorem 6.4.4. □

Corollary 6.5.4. *For* $p,\ r,\ s,\ t\in\mathbb{R}_+$ *such that* $r\le s$, $p\le t$ *with* $r\ne p$, $s\ne t$ *we have*

$$\left(\frac{\mathfrak{B}_{n,p}(x,y)}{\mathfrak{B}_{n,r}(x,y)}\right)^{\frac{1}{p-r}} \le \left(\frac{\mathfrak{B}_{n,t}(x,y)}{\mathfrak{B}_{n,s}(x,y)}\right)^{\frac{1}{t-s}}.$$

6.6 Applications

The entropy function plays a key role in information theory and has connection with certain special means [66]. We give here some applications in information theory. Moreover in this section all logarithms are considered with base $b>1$.

Lemma 6.6.1. *Suppose* $\xi_k>0$ *and* $p_k>0$ *for* $1\le k\le n$ *with* $\Sigma_{k=1}^n p_k=1$. *Then*

$$\log\left(\Sigma_{k=1}^n p_k\xi_k\right)-\Sigma_{k=1}^n p_k\log\xi_k \ge \left|\Sigma_{k=1}^n p_k\left|\log(\Sigma_{k=1}^n p_k\xi_k)-\log\xi_k\right|-(\Sigma_{k=1}^n p_k\xi_k)^{-1}\frac{\Sigma_{k=1}^n p_k\left|\Sigma_{k=1}^n p_k\xi_k-\xi_k\right|}{\ln b}\right| \tag{6.6.1}$$

Proof. Consider the well known inequality [42]

$$\log\xi\le\frac{1}{\ln b}(\xi-1).$$

For $\xi\to\frac{\xi}{\eta}$ we have

$$\log\frac{\xi}{\eta}\le\frac{1}{\ln b}\left(\frac{\xi}{\eta}-1\right).$$

Set $\xi \to \xi_k$ and $\eta \to \overline{\xi}$ $(= \sum_{k=1}^{n} \xi_k)$ so that

$$\log \overline{\xi} - \log \xi_k - \frac{1}{\ln b}\frac{\overline{\xi} - \xi_k}{\overline{\xi}} \;\; \geq \;\; \left| \left| \log \overline{\xi} - \log \xi_k \right| - \frac{1}{\ln b}\frac{1}{|\overline{\xi}|}\left|\overline{\xi} - \xi_k\right| \right|,$$

i.e.,

$$\Sigma_{k=1}^{n} p_k \left(\log \overline{\xi} - \log \xi_k - \frac{1}{\ln b}\frac{\overline{\xi} - \xi_k}{\overline{\xi}} \right) \geq \Sigma_{k=1}^{n} p_k \left(\left| \left| \log \overline{\xi} - \log \xi_k \right| - \frac{1}{\ln b}\frac{\left|\overline{\xi} - \xi_k\right|}{\mid \overline{\xi} \mid} \right| \right).$$

From here inequality (6.6.1) follows. □

Let X be a discrete valued random variable with finite range $\{x_1, x_2, ..., x_r\}$. Assume $p_k = P\{X = x_k\} > 0$ for $1 \leq k \leq r$. The $b-$entropy of X is defined by [43]

$$H_b(X) = \Sigma_{k=1}^{r} \; p_k \log \left(\frac{1}{p_k} \right).$$

We have the following:

Theorem 6.6.2. *Let X be a discrete valued random variable with finite range $\{x_1, x_2, ..., x_r\}$. Assume $p_k = P\{X = x_k\} > 0$ for $1 \leq k \leq r$. The $b-$entropy of X is defined by [43]*

$$H_b(X) = \Sigma_{k=1}^{r} \; p_k \log \left(\frac{1}{p_k} \right).$$

$$\log r \; - \; H_b(X) \;\; \geq \;\; \left| \Sigma_{k=1}^{r} p_k \left| \log r + \log p_k \right| - \frac{1}{r \ln b} \Sigma_{k=1}^{r} p_k \left| r - \frac{1}{p_k} \right| \right|$$

Proof. Set $n = r$ and $\xi_k = 1/p_k$ for $1 \leq k \leq r$ in Lemma 6.6.1. □

Let X and Y be a pair of random variables with respective ranges $\{x_1, x_2, ..., x_r\}$ and $\{y_1, y_2, ..., y_s\}$. The conditional entropy of X given Y is defined by [43].

$$H_b(X|Y) = \sum_{i,j} p(x_i, y_j) \log \left(\frac{1}{p(x_i|y_j)} \right),$$

where, $p(x_i, y_j) = P\{X = x_i, Y = y_j\}$ and $p(x_i|y_j) = P\{X = x_i|Y = y_j\} = \frac{p(x_i,y_j)}{p(y_j)}$ for $1 \le i \le r$ and $1 \le j \le s$. Without loss of generality we need to define these quantities only for those (i, j) for which $p(x_i, y_j) > 0$. There will be n $(\le r\, s)$ such pairs. The conditional entropy can be interpreted as the amount of uncertainty remaining about X after Y has been observed.

Theorem 6.6.3. *Let X and Y be as above. For a fixed j, $1 \le j \le s$, define $V_j = \{i : p(x_i, y_j) > 0\}$ and $U = \{(i, j) : i \in V_j\}$ and let $r' = \sum_{j=1}^{s} p(y_j) \mid V_j \mid$. Then we have*

$$\log r' - H_b(X|Y) \ \ge \ \left| \sum_{(i,j)\in U} p(x_i, y_j) \left| \log r' - \log \frac{p(y_j)}{p(x_i, y_j)} \right| - \frac{1}{r \ln b} \sum_{(i,j)\in U} p(x_i, y_j) \left| r' - \frac{p(y_j)}{p(x_i, y_j)} \right| \right| \quad (6.6.2)$$

Proof. We may label those pairs (i, j) for which $p(x_i, y_j) > 0$, that is, the pairs $(i, j) \in U$. As $1 \le k \le n$. We then put $p_k = p(x_i, y_j)$ and $\xi_k = \frac{1}{p(x_i|y_j)} = \frac{p(y_j)}{p(x_i,y_j)}$ in Lemma 6.6.1. This gives

$$\log \left(\Sigma_{k=1}^{n} \frac{p(x_i, y_j)}{p(x_i|y_j)} \right) - \Sigma_{k=1}^{n}\ p(x_i, y_j)\ \log \left(p(x_i|y_j)\right)^{-1} \ \ge \left| \Sigma_{k=1}^{n} p(x_i, y_j) \left| \log \left(\Sigma_{k=1}^{n} \frac{p(x_i, y_j)}{p(x_i|y_j)} \right) - \log(p(x_i|y_j))^{-1} \right| - \left(\Sigma_{k=1}^{n} \frac{p(x_i, y_j)}{p(x_i|y_j)} \right)^{-1} \frac{1}{\ln b} \Sigma_{k=1}^{n}\ p(x_i, y_j) \left| \Sigma_{k=1}^{n}\ \frac{p(x_i, y_j)}{p(x_i|y_j)} - \frac{1}{p(x_i|y_j)} \right| \right|$$

i.e.,

$$\log\left(\sum_{(i,j)\in U} p(y_j)\right) - \sum_{(i,j)\in U} p(x_i,y_j)\log\left(p(x_i|y_j)\right)^{-1} \geq \left|\sum_{(i,j)\in U} p(x_i,y_j)\right|\log\left(\sum_{(i,j)\in U} p(y_j)\right) - \log\frac{p(y_j)}{p(x_i,y_j)}\Bigg| - \frac{1}{\ln b}\sum_{(i,j)\in U} p(x_i,y_j)\left|\sum_{(i,j)\in U} p(y_j) - \frac{p(y_j)}{p(x_i,y_j)}\right|\Bigg|$$

Since

$$\sum_{(i,j)\in U} p(y_j) = \sum_{j=1}^{s} p(y_j)\sum_{i\in V_j} 1 = r'.$$

Therefore from here we get (6.6.2). □

A fundamental result related to the the notion of Shannon entropy is the inequality [42]

$$\sum_{i=1}^{r} p_i \log\frac{1}{p_i} \leq \sum_{i=1}^{r} p_i \log\frac{1}{q_i}. \tag{6.6.3}$$

Which is valid for all positive real numbers p_i and q_i with $\sum_{i=1}^{r} p_i = \sum_{i=1}^{r} q_i = 1$. Equality holds in (6.6.3) iff $q_i = p_i$ for $1 \leq i \leq r$ [41, p. 635-650]. This result, some times called the fundamental lemma of information theory, has extensive applications [43]. The following is called Shannon's discrete inequality [44].

Theorem 6.6.4. *Let* $\{p_i : 1 \leq i \leq r\}$ *be a set of positive real numbers with* $\sum_{i=1}^{r} p_i = 1$*. If* $\{q_i : 1 \leq i \leq r\}$ *is a set of non-negative real numbers with* $\sum_{i=1}^{r} q_i = \alpha > 0$*. Then*

$$\sum_{i=1}^{r} p_i \log\frac{1}{p_i} \leq \sum_{i=1}^{r} p_i \log\frac{1}{q_i} + \log\alpha$$

with equality iff $q_i = \alpha\, p_i$ *for* $1 \leq i \leq r$.

Now we shall give improvement of Theorem 6.6.4.

Theorem 6.6.5. *Let the conditions of Theorem 6.6.4 be satisfied. Then*

$$\log \alpha + \sum_{i=1}^{r} p_i \log \frac{1}{q_i} - \sum_{i=1}^{r} p_i \log \frac{1}{p_i} \ \geq \left| \sum_{i=1}^{r} p_i \left| \log \frac{\alpha p_i}{q_i} \right| - \frac{\sum_{i=1}^{r} p_i \left| \alpha - \frac{q_i}{p_i} \right|}{\alpha \ln b} \right| \quad (6.6.4)$$

Proof. Set $\xi_i = q_i/p_i$ and $q_i = \xi_i p_i$ in the Lemma 6.6.1 so that

$$\log \alpha - \sum_{i=1}^{r} p_i (\log q_i - \log p_i) \geq \left| \sum_{i=1}^{r} p_i \left| \log \alpha - \log q_i + \log p_i \right| - \frac{1}{\alpha \ln b} \sum_{i=1}^{r} p_i \left| \alpha - \frac{q_i}{p_i} \right| \right|$$

From here we get (6.6.4). □

The $b-$mutual information between random variables X and Y is defined by [43].

$$I_b(X;Y) := H_b(X) - H_b(X|Y) = \sum_{i,j} p(x_i, y_j) \log \frac{p(x_i, y_j)}{p(x_i)p(y_j)}$$

The following result is valid.

Theorem 6.6.6. *Let* $V = \{(i,j) : p(x_i, y_j) > 0\}$ *and* $S := \sum_{(i,j)\in V} p(x_i)p(y_j)$. *Then*

$$\log S + I_b(X;Y) \ \geq \left| \sum_{(i,j)\in V} p(x_i, y_j) \left| \log S - \log \frac{p(x_i, y_j)}{p(x_i)p(y_j)} \right| - \right.$$
$$\left. \frac{S^{-1}}{\log b} \sum_{(i,j)\in V} p(x_i, y_j) \left| S - \frac{p(x_i)p(y_j)}{p(x_i, y_j)} \right| \right|$$

Proof. This follows the lines of our earlier proofs.

Setting $p_k = p(x_i, y_j)$ and $\xi_k = p(x_i)p(y_j)/p(x_i, y_j)$ for $1 \leq k \leq n$ in Lemma 6.6.1 after suitable relabeling . □

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