Combinatorial and Arithmetic Study of Binomial Edge Ideal

Since 1864

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DECLARATION

I, **Mr. Sohail Zafar** Registration No. **119-GCU-PHD-SMS-08** student at **Abdus Salam School of Mathematical Sciences GC University** in the subject of **Mathematics**, **Year of Admission** (**2008)**, hereby declare that the matter printed in this thesis titled

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is my own work and that

(i) I am not registered for the similar degree elsewhere contemporaneously.

(ii) No direct major work had already been done by me or anybody else on this topic; I worked on, for the Ph. D. degree.

(iii) The work, I am submitting for the Ph. D. degree has not already been submitted elsewhere and shall not in future be submitted by me for obtaining similar degree from any other institution.

Dated: ------------------------ ----------------------------------

Signature

RESEARCH COMPLETION CERTIFICATE

Certified that the research work contained in this thesis titled

"Combinatorial and Arithmetic Study of Binomial Edge Ideal"

has been carried out and completed by Mr. **Sohail Zafar** Registration No. **119-GCU-PHD-SMS-08** under my supervision.

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This thesis is dedicated to my parents, Zafar and Qudsia, for their love, endless support and encouragement.

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Abstract

Let J_G denote the binomial edge ideal of a connected undirected graph G on n vertices. This is the ideal generated by the binomials $x_iy_j - x_jy_i, 1 \leq i < j \leq n$, in the polynomial ring $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ where $\{i, j\}$ is an edge of G. Our aim in this thesis is to compute certain algebraic invariants like dimension, depth, system of parameters, regular sequence, Hilbert series and multiplicity of J_G of some particular classes of binomial edge ideals of graphs. A large amount of information of an ideal is carried by its minimal free resolution. So we give information on the minimal free resolution on certain binomial edge ideals. We also give a complete description of the structure of the modules of deficiencies of binomial edge ideals of some classes of graphs.

A generalization of the concept of a Cohen-Macaulay ring was introduced by S. Goto [7] under the name approximately Cohen-Macaulay. In this thesis we collect a few graphs G such that the associated ring S/J_G is approximately Cohen-Macaulay. We also characterize all the trees that are approximately Cohen-Macaulay.

As more generalized notion than approximately Cohen-Macaulay we also study sequentially Cohen-Macaulay property for binomial edge ideals. We give a nice construction principle in this topic.

The complete graph $\tilde{G}$ on n vertices has the property that $S/J_{\tilde{G}}$ is a Cohen-Macaulay domain with a 1-linear resolution. As one of the main results we clarify the structure of $S/J_{K_{m,n}}$, where $K_{m,n}$ denotes the complete bipartite graph.

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Introduction

There is a recent development in commutative algebra summarized under the name combinatorial commutative algebra. Its main intention is to describe combinatorial aspects, i.e., graphs, polymatroids etc, in data of commutative algebra i.e. polynomial ring and vice versa. Over the last decade or so, the algebraists were interested to study monomial ideals. They relate two monomial ideals to simplicial complex $\Delta:$ the facet ideal whose generators correspond to the facets of Δ and the Stanley-Reisner ideal whose generators correspond to the non-faces of Δ. Villarreal [22], Froberg [6] and Simis, Vasconcelos, and Villarreal [18] were among the early initiator of this discipline. Stanley [19] used it to proof the Upper Bound Conjecture for simplicial spheres. One subject of this recent development is the study of binomial edge ideal as is introduced in [8] and [11], in which binomial ideal relate to the simple graph. In these days this subject is very active field of research.

A graph G on the vertex set $[n]$ we always understand a connected simple graph. Let K denote a field. Let $R = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over the field K. The monomial edge ideal I_G of G is generated by all monomials $x_i x_j$, $i < j$, where $\{i, j\}$ is an edge of G. This notion was studied by Villarreal [23] where it is also discussed under which circumstances R/I_G is Cohen-Macaulay. To characterize all the graphs G such that monomial edge ideal is a Cohen-Macaulay ring is seems to be hopeless.

In a similar way the binomial edge ideal $J_G \subseteq S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$, generated by all binomials $x_iy_j - x_jy_i$, $i < j$, where $\{i, j\}$ forms an edge of G. This is introduced in [8] and [11]. Some nice facts like reduced Grőbner basis, primary decomposition and minimal primes of binomial edge ideals of a simple graph were also given in [8]. They also shows that the binomial edge ideal J_G is the intersection of prime ideals. That is, S/J_G is a reduced ring. In the paper of V. Ene, J. Herzog, T. Hibi [5], the authors start with the systematic investigation of J_G. They found some examples such that binomial edge ideal is Cohen-Macaulay. They also characterize all the trees that are Cohen-Macaulay. Furthermore, the authors believe that it is hopeless to characterize those G such that the binomial edge ideal is Cohen-Macaulay in general.

In this thesis we contribute in order to shed some more light on the correspondence of algebraic notions of commutative algebra in relation to those from combinatorics in the case of binomial edge ideal.

The thesis is structured as follows: In Chapter 1 we remind few necessary definitions and well-known facts on binomial edge ideals, minimal free resolution, Hilbert series, local cohomology, modules of deficiencies and Cohen-Macaulay like properties. At the end we give some important definitions from the graph theory. In Chapter 2 we study some of the properties of the the binomial edge ideal $J_G \subset S$ associated to a complete bipartite graph. We give a complete list of all the modules of deficiencies of the complete bipartite graphs in Theorems 2.2.1, 2.2.2, 2.2.3, 2.2.4 and 2.2.5. Finally we prove the purity of the minimal free resolution of S/J_G in Theorem 2.3.3 and then give the Hilbert series and explicit values of the Betti numbers in Theorem 2.3.4. In chapter 3 we prove that the binomial edge ideal of any cycle is approximately Cohen-Macaulay in Theorem 3.1.5. To compute the Hilbert series of a cycle, we include some investigations on the canonical module of the binomial edge ideal associated to the complete graph (see Lemma 3.1.6 and Theorem 3.1.10). At the end of this chapter we compute all Betti numbers for the binomial edge ideal of a cycle in Theorem 3.1.16. In chapter 4, we give the construction principle for sequentially Cohen-Macaulay rings (In particular approximately Cohen-Macaulay rings) in Lemma 4.1.1. We characterize all the trees that are approximately Cohen-Macaulay in Theorem 4.2.4. Such trees are described as 3-star like. We also compute the Hilbert series of 3-star like trees (see Corollary 4.3.2). After that we give some examples of the graphs G such that the associated binomial edge ideal is sequentially Cohen-Macaulay ring. We also find some basic algebraic invariants of iso-complete graphs and k-deficient graphs. We give the complete description of their module of deficiencies in Theorems 4.5.4, 4.6.3 and 4.6.4. Finally we characterize them in sense of sequentially Cohen-Macaulay property. At the end we discuss some of the applications and open problems.

The main results of my thesis are published or accepted for publication in [25] and [17]. Few results are submitted for publication (see [26]). These results were also reported on the seminars of Algebra at ASSMS.

Chapter 1

Preliminaries and auxiliary results

In this chapter we will introduce the notations used in this thesis. Moreover we summarize a few auxiliary results that we need. Most of the material in this chapter is from commutative algebra ([2], [4], [10] and [23]), homological algebra [21] and graph theory [24].

1.1 Binomial edge ideals

We denote by G a connected undirected graph on n vertices labeled by $[n] = \{1, 2, \ldots, n\}$. For an arbitrary field K let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ denote the polynomial ring in the $2n$ variables. To the graph G one can associate an ideal $J_G \subset S$ generated by all binomials $x_i y_j - x_j y_i$ for $i < j$ such that $\{i, j\}$ forms an edge of G. This Ideal J_G is called **binomial edge ideal** associated to the graph G. This construction was invented by Herzog et al. in [8] and [11]. At first let us recall some of their definitions.

Definition 1.1.1. Fix the previous notation. Let the complete graph on the vertex set $T \subset [n]$ is denoted by $\tilde{G}_T$. Moreover let $G_{[n]\setminus T}$ represent the graph found by deleting all vertices of G that belong to T.

Let $c = c(T)$ represent the number of connected components of $G_{[n]\setminus T}$. Let

$G_1, \ldots, G_c$ are the connected components of $G_{[n]\setminus T}$. Then define

$$P_T(G) = (\cup_{i \in T}\{x_i, y_i\}, J_{\tilde{G}_1}, \ldots, J_{\tilde{G}_{C(T)}}),$$

where $\tilde{G}_i, i = 1, \ldots, c$, denotes the complete graph on the vertex set of the connected component $G_i, i = 1, \ldots, c$.

$P_T(G)$ is a prime ideal because $\cup_{i \in T}\{x_i, y_i\}$ and $J_{\tilde{G}_i}$ for each i are prime ideals and all are in different variables. The following result is important for the understanding of the binomial edge ideal of G.

Lemma 1.1.2. *With the previous notation we have:*

(a) $P_T(G) \subset S$ *is a prime ideal of height* $n - c + |T|$*, where* $|T|$ *denotes the number of elements of* T*.*

(b) $J_G = \cap_{T \subseteq [n]} P_T(G)$*.*

(c) $J_G \subset P_T(G)$ *is a minimal prime if and only if either* $T = \emptyset$ *or* $T \neq \emptyset$ *and* $c(T \setminus \{i\}) < c(T)$ *each* $i \in T$*.*

Proof. For the proof we refer to [8]. □

Therefore J_G is the intersection of prime ideals. That is, S/J_G is a reduced ring. Moreover, we remark that J_G is an ideal generated by quadrics and therefore homogeneous, so that S/J_G is a graded ring with natural grading induced by the $\mathbb{N}$-grading of S.

Remark 1.1.3. If we define a grading on S by setting $\deg x_i = \deg y_i = e_i$ where e_i is the $i-$th unit vector of $\mathbb{N}^n$ then clearly S/J_G is $\mathbb{N}^n$ graded.

1.2 Some basic facts from commutative algebra

In this section we will recall some important definitions and facts from basic commutative algebra that we use several time in our thesis. For the most basic definitions like dimension, depth, height, regular sequence and system of parameters, we refer the book of Matsumura [10]. For the basic properties of functors Ext and Tor we recommend the book of Vermani [21].

1.2.1 Cohen- Macaulay and Gorenstein rings

Let R be a ring. If $\dim R = \operatorname{depth} R$ then R is said to be **Cohen-Macaulay ring**. A Cohen- Macaulay ring R is called **Gorenstein ring** if every ideal generated by a system of parameters of R is irreducible.

1.2.2 Hilbert series

For a homogeneous ideal $I \subset S$ let $H(S/I, t)$ denote the **Hilbert series**, i.e. $H(S/I, t) = \sum_{i \geq 0} (\dim_K [S/I]_i) t^i$. For some basic properties of the Hilbert series we refer the book of Matsumura [10].

1.2.3 Minimal free resolution

First of all we will define the free resolution of a module M.

Definition 1.2.1. Let R be a ring and M be an R-module. A complex of free modules

$$A_\bullet : \cdots \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0$$

which is exact and $coker(f_1) = M$ is called **free resolution** of a module M. If $A_i = 0$ for all $i > n$ such that $A_i \neq 0$ for all $i \leq n$ then $A_\bullet$ is called a **finite free resolution** of length n.

If R is graded ring then $A_\bullet$ is a graded free resolution with all the A_i are graded free modules. Most of the information of a module can be extracted from its minimal free resolution. So in this thesis we mainly focus on the minimal free resolution.

Definition 1.2.2. A **graded minimal free resolution** $A_\bullet$ of an R-module M is a free resolution in which $f_{i+1}(A_{i+1}) \subseteq S_+ A_i$ for all $i \geq 0$.

Note that the minimal finite free resolutions of module M are always unique up to isomorphism. The next theorem is very important in commutative algebra which tells us about the existence of finite minimal free resolution. This theorem is known as Hilbert Syzygy Theorem.

Theorem 1.2.3. *Let R be a polynomial ring over a field K in n variables then every finitely generated graded R-module has a graded finite free resolution of length $\leq n$.*

Proof. For the proof we recommend the book of Eisenbud [4]. □

Now we will use minimal free resolution to define some algebraic invariants of a module. Let M a graded finitely generated S-module. By Theorem 1.2.3, M has a finite minimal graded free resolution:

$$A_\bullet : 0 \to A_p \to \cdots \to A_1 \to A_0 \to M \to 0$$

where $A_i = \bigoplus_j S(-d_{ij})^{\beta_{ij}}$ for $i \geq 0$ and p is called the **projective dimension** of M. The numbers β_{ij} are uniquely determined by M i.e. $\beta_{i,j}(M) = \dim_K \operatorname{Tor}_i^S(K, M)_{i+j}, i, j \in \mathbb{Z}$, and called the **graded Betti numbers** of M. We can also define **Castelnuovo-Mumford regularity** $\operatorname{reg} M = \max\{j \in \mathbb{Z} | \beta_{i,j}(M) \neq 0\}$ and the **Betti table** looks like the following:

	0	1	$\cdots$	p
0	$\beta_{0,0}$	$\beta_{1,0}$	$\cdots$	$\beta_{p,0}$
1	$\beta_{0,1}$	$\beta_{1,1}$	$\cdots$	$\beta_{p,1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
r	$\beta_{0,r}$	$\beta_{1,r}$	$\cdots$	$\beta_{p,r}$

Note that all the $\beta_{i,j}$ outside of the Betti table are zero. Moreover, the module M has a **pure resolution** if there are constants $d_0 < \cdots < d_p$ such that $d_{0i} = d_0, \ldots, d_{pi} = d_p$ for all i. The following theorem relate the depth and the projective dimension of the module. This is also known as Auslander-Buchsbaum formula.

Theorem 1.2.4. *Let $M \neq 0$ be finitely generated S-module then*

$$\operatorname{proj}\dim M + \operatorname{depth} M = \operatorname{depth} S.$$

Proof. For the proof we refer the book of Burns and Herzog [2]. □

We need the following proposition as a tool.

Proposition 1.2.5. *Let $I \subset S$ denote an ideal. Let $\underline{f} = f_1, \ldots, f_r$ denote an S/I-regular sequence. Then $\underline{f}S \cap I = \underline{f}I$.*

Proof. It is clear that $\operatorname{Tor}_1^S(S/\underline{f}S, S/I) \cong \underline{f}S \cap I/\underline{f}I$. Moreover

$$\operatorname{Tor}_1^S(S/\underline{f}S, S/I) \cong H_1(\underline{f}; S/I),$$

where $H_i(\underline{f}; S/I)$ denotes the Koszul homology of $\underline{f}$ with respect to S/I. But these homology modules vanish for $i > 0$ since $\underline{f}$ is S/I-regular sequence. □

1.2.4 Binomial edge ideal associated to complete graph

If $\tilde{G}$ denotes the complete graph on n vertices, then $S/J_{\tilde{G}}$ is the coordinate ring of the Segre embedding $\mathbb{P}_K^1 \times \mathbb{P}_K^n$. This is a variety of minimal degree. Therefore $S/J_{\tilde{G}}$ is Cohen-Macaulay of dimension $n+1$ and has a linear resolution. Also note that $J_{\tilde{G}}$ is a prime ideal.

Theorem 1.2.6. *Let $\tilde{G}$ denotes the complete graph on n vertices then* *(a) The minimal free resolution of* $S/J_{\tilde{G}}$ *is*

$$0 \to S^{b_{n-1}(n)}(-n) \to \cdots \to S^{b_{n-1-i}(n)}(-n+i) \to \cdots \to S^{b_1(n)}(-2) \to S$$

where $b_i(n) = i\binom{n}{i+1}$.

(b) The Hilbert series of $S/J_{\tilde{G}}$ *is*

$$H(S/J_{\tilde{G}}, t) = \frac{1+(n-1)t}{(1-t)^{n+1}}.$$

Proof. (a) is well known (see e.g. [4, Exercise A2.19]). For (b) recall the result in [2], if R is Cohen-Macaulay ring with dimension d and $\underline{x} = x_1, \ldots, x_d$ be the homogenous system of parameters of degree 1 then

$$H(R,t) = \frac{H(R/\underline{x}R, t)}{(1-t)^d}.$$

Now in our case $S/J_{\tilde{G}}$ is Cohen-Macaulay ring with dimension $n+1$, $\underline{x} = x_1, y_1 - x_2, \ldots, y_{n-1} - x_n, y_n$ is system of parameter of degree 1 of $S/J_{\tilde{G}}$ and

$$S/(\underline{x}S, J_{\tilde{G}}) \cong K[y_1, \ldots, y_{n-1}]/(y_1, \ldots, y_{n-1})^2$$

therefore $H(S/(\underline{x}S, J_{\tilde{G}}), t) = 1 + (n-1)t$ and we have

$$H(S/J_{\tilde{G}}, t) = \frac{1+(n-1)t}{(1-t)^{n+1}}.$$

□

1.3 Local cohomology and modules of deficiencies

Let M denote a finitely generated graded S-module. In the thesis we shall use also the local cohomology modules $H^i(M), i \in \mathbb{Z}$ of M with respect to S_+. Note that they are graded Artinian S-modules. We refer to the textbook of Brodmann and Sharp (see [1]) for the basics on it. In particular the characterization of Castelnuovo-Mumford regularity and Gorenstein rings in terms of local cohomology are important for us.

Theorem 1.3.1. *Let M be an S-module then:*

$$\operatorname{reg}(M) = \max\{e(H^i(M)) + i \,|\, \operatorname{depth}(M) \leq i \leq \dim(M)\},$$

where $e(H^i(M)) = \sup\{j \in \mathbb{Z} | [H^i(M)]_j \neq 0\}$.

Theorem 1.3.2. *The d-dimensional ring R is Gorenstein if and only if R is Cohen-Macaulay and $H^d(R) = E$ where E denotes an injective hull of K.*

For our investigations we need the following definition.

Definition 1.3.3. Let M denote a finitely generated graded S-module and $d = \dim M$. Put

$$\omega^i(M) = \operatorname{Ext}_S^{2n-i}(M, S(-2n))$$

for an integer $i \in \mathbb{Z}$ and call it the **i-th module of deficiency**. Moreover we define $\omega(M) = \omega^d(M)$ the **canonical module** of M. We write also $\omega_{2\times}(M) = \omega(\omega(M))$. These modules have been introduced and studied in [14].

Note that by the graded version of Local Duality (see e.g. [1]) there is the natural graded isomorphism $\omega^i(M) \cong \operatorname{Hom}_K(H^i(M), K)$ for all $i \in \mathbb{Z}$. For an integer $i \in \mathbb{N}$ we set

$$(\operatorname{Ass} M)_i = \{\mathfrak{p} \in \operatorname{Ass} M | \dim S/\mathfrak{p} = i\}.$$

In the following we shall summarize some basic facts on the modules of deficiencies.

Proposition 1.3.4. *Let M denote a finitely generated graded S-module and $d = \dim M$.*

(a) $\dim \omega^i(M) \leq i$ *and* $\dim \omega^d(M) = d$.

(b) $(\operatorname{Ass} \omega^i(M))_i = (\operatorname{Ass} M)_i$ *for all* $0 \leq i \leq d$.

(c) M satisfies the Serre condition S_2 if and only if $\dim \omega^i(M) \leq i - 2$ *for all* $0 \leq i < d$.

(d) There is a natural homomorphism $M \to \omega^d(\omega^d(M))$. It is an isomorphism if and only if M satisfies the Serre condition S_2.

(e) For a homogeneous ideal $I \subset S$ there is a natural isomorphism $\omega^d(\omega^d(S/I)) \cong \operatorname{Hom}_S(\omega^d(S/I), \omega^d(S/I)), d = \dim S/I$, and it admits the structure of a commutative Noetherian ring, the S_2-fication of S/I.

(f) The natural map $S/I \to \operatorname{Hom}_S(\omega^d(S/I), \omega^d(S/I)), d = \dim S/I$, sends the unit element to the identity map. Therefore it is a ring homomorphism.

(g) Let S/I is Cohen-Macaulay with $\dim(S/I) > 0$. Suppose that $\omega(S/I)$ is an ideal of S/I then $(S/I)/\omega(S/I)$ is a Gorenstein ring with $\dim(S/I) - 1$.

Proof. The results are shown in [14] and [16]. The proofs in the graded case follow the same line of arguments. □

1.4 Cohen-Macaulay like properties

1.4.1 Approximately Cohen-Macaulay modules

For our purpose here we need a certain generalization of Cohen-Macaulay modules that was originally introduced by S.Goto (see [7]) in the case of local rings.

Definition 1.4.1. Let R denote a commutative ring of finite dimension d and $I \subset R$ an ideal. Then

1. $\operatorname{Assh}_R(R/I) = \{p \in \operatorname{Ass}_R(R/I) : \dim R/p = \dim R/I\}$.

2. $U_R(I) = \cap_{p \in \operatorname{Assh}_R(R/I)} I(p)$ where $I = \cap_{p \in \operatorname{Ass}_R(R/I)} I(p)$ denotes a minimal primary decomposition of I.

That is $U_R(I)$ describes the equidimensional part of the primary decomposition of I.

As an analogue to Goto's notion of approximately Cohen-Macaulay modules we define a graded version of it.

Definition 1.4.2. Let $R = \bigoplus_{i \geq 0} R_i$ denote a standard $K-$algebra, where $K = R_0$. It is called **approximately Cohen-Macaulay** if $R_+ = \bigoplus_{i>0} R_i$ contains a homogeneous element x such that R/x^nR is a Cohen-Macaulay module of $\dim(R) - 1$ for all $n \geq 1$.

A characterization of approximately Cohen-Macaulay modules can be done by the following theorem.

Theorem 1.4.3. *Let $I \subset S$ denote a homogenous ideal and $d = \dim(S/I)$. Then the following conditions are equivalent.*

(i) *S/I is approximately Cohen-Macaulay module.*

(ii) *$S/U_S(I)$ is a $d-$dimensional Cohen-Macaulay module and* $\operatorname{depth}(S/I) \geq d - 1$.

(iii) *$\omega^d(S/I)$ is Cohen-Macaulay module of dimension d, $\omega^{d-1}(S/I)$ is either zero or a $(d-1)-$dimensional Cohen-Macaulay module and $\omega^i(S/I) = 0$ for all $i \neq d, d-1$.*

Proof. In the case of local rings the above theorem was proved by Goto in the paper of [7]. The proof in the graded case follows by the same arguments. We have to adopt the graded situation from the local one. We omit the details. □

Remark 1.4.4. A necessary condition for S/I to be an approximately Cohen-Macaulay module is that $\dim S/p \geq \dim S/I - 1$ for all $p \in \operatorname{Ass}(S/I)$.

We will construct several examples of approximately Cohen-Macaulay modules in this thesis mainly in chapter 3 and 4.

1.4.2 Sequentially Cohen-Macaulay modules

A decreasing sequence $\{M_i\}_{0\leq i\leq d}$ of a d-dimensional S-module M is called **dimension filtration** of M, if M_i/M_{i-1} is either zero or of dimension i for all $i = 0, \ldots, d$, where $M_{-1} = 0$. It was shown (see [15]) that the dimension filtration exists and is uniquely determined.

Definition 1.4.5. An S-module M is called **sequentially Cohen-Macaulay** if the dimension filtration $\{M_i\}_{0\leq i\leq d}$ has the property that M_i/M_{i-1} is either zero or an i-dimensional Cohen-Macaulay module for all $i = 0, \ldots, d$, (see [15]). Note that in [15] this notion was originally called Cohen-Macaulay filtered.

Remark 1.4.6. Note that a sequentially Cohen-Macaulay S-module M with depth $M \geq \dim M - 1$ are approximately Cohen-Macaulay module which we define before.

In this thesis we will exhibit some examples of sequentially Cohen-Macaulay modules that are not approximately Cohen-Macaulay modules. For our purposes here we need the following characterization of sequentially Cohen-Macaulay modules.

Theorem 1.4.7. *Let M be a finitely generated graded S-module with $d = \dim M$. Then the following conditions are equivalent:*

(i) M is a sequentially Cohen-Macaulay.

(ii) For all $0 \leq i < d$ the module of deficiency $\omega^i(M)$ is either zero or an i-dimensional Cohen-Macaulay module.

(iii) For all $0 \leq i \leq d$ the modules $\omega^i(M)$ are either zero or i-dimensional Cohen-Macaulay modules.

Proof. In the case of a local ring admitting a dualizing complex this result was shown in [15, Theorem 5.5]. Similar arguments work also in the case of a finitely generated graded S-module M. Note that the equivalence of (i) and (iii) was announced (without proof) in [20]. □

1.4.3 Canonically Cohen-Macaulay modules

Definition 1.4.8. A finitely generated S-module M is said to be canonically Cohen-Macaulay whenever $\omega(M)$ is Cohen-Macaulay (see [16]).

Note that if M is Cohen-Macaulay, then it is canonically Cohen-Macaulay but the converse is not true. In this thesis we will construct several examples for which converse does not hold.

1.5 Some facts about Graphs

Definition 1.5.1. A set of vertices and a collection of edges that attach pairs of vertices is called **graph**.

For the basic facts on graph theory we recommend the book of D. West [24]. We need the following definitions which we will use in our thesis and note that in all the definitions our graph is connected and simple.

Definitions 1.5.2. **(i)** The number of edges thats are connected with vertex is called **degree** of that vertex.

(ii) A **line** is a graph in which all the vertices are of degree 2 with the exception of two vertices that are of degree 1.

(iii) A **cycle** is a graph in which all the vertices are of degree 2. In particular, for $n = 3$ it is triangle and $n = 4$ it is square.

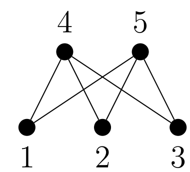
(iv) A **tree** is a graph without cycle.

(v) The **spider graph** is a tree in which the degree of one vertex is at least 3 and all others with degree at most 2.

(vi) The **complete graph** K_n has all possible edges.

(vii) A graph is said to be **bipartite graph** if the vertex set is decomposed into two disjoint sets in such a way there is no two vertices in a same set is adjacent.

(viii) **Complete bipartite graph** is a bipartite graph G in which any two vertices, $v_1 \in V_1$ and $v_2 \in V_2$, v_1v_2 is an edge in G. If $|V_1| = n$ and $|V_2| = m$ then it is usually denoted by $K_{n,m}$. For example $n = 3$ and $m = 2$.



(ix) $K_{1,m}$ is known as **star graph**.

(x) The **join** of two graphs (with disjoint vertex sets) is the union of edges of both graphs and all the possible edges that connects vertices of one graph to the other graph.

Chapter 2

Binomial edge ideal associated to complete bipartite graph

In this chapter we investigate the binomial edge ideal of an important class of graphs, namely the complete bipartite graph $G = K_{m,n}$ (see definitions 1.5.2). As a main result we give a complete list of all the modules of deficiencies of complete bipartite graphs. This is - at least for us - the first time in the literature that there is a complete description of the structure of the modules of deficiencies besides of sequentially Cohen-Macaulay rings or Buchsbaum rings. We also prove the purity of the minimal free resolution and describe explicitly the Betti numbers of complete bipartite graphs. This is the heart of our investigations. It gives in a natural way some non-Cohen-Macaulay rings with pure resolutions. We might relate our investigations as a better understanding of general binomial edge ideals.

2.1 Some algebraic invariants of complete bipartite graphs

Definition 2.1.1. For a sequence of integers $1 \leq i_1 < i_2 < \ldots < i_k \leq n+m$ let $I(i_1, i_2, \ldots, i_k)$ denote the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_{i_1} & x_{i_2} & \cdots & x_{i_k} \\ y_{i_1} & y_{i_2} & \cdots & y_{i_k} \end{pmatrix}.$$

Note that $I(i_1, i_2, \ldots, i_k)$ is the ideal of the complete graph on the vertex set $\{i_1, i_2, \ldots, i_k\}$.

Let J_G be the binomial edge ideal associated to complete bipartite graph on $[n+m]$ vertices and $J_{\tilde{G}}$ be the binomial edge ideal associated to complete graph on $[n+m]$ vertices. We begin with a lemma concerning the dimension of S/J_G.

Lemma 2.1.2. *Let $G = K_{m,n}, m \geq n$, denote the complete bipartite graph. Let $\tilde{G}$ denote the complete graph on $[n+m]$. Let $A_n = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ for $n \geq 1$ and $B_m = (x_{n+1}, \ldots, x_{n+m}, y_{n+1}, \ldots, y_{n+m})$ for $n \geq 2$ and $B_m = S$ for $n = 1$.*

(a) $J_G = J_{\tilde{G}} \cap A_n \cap B_m$ *is the minimal primary decomposition of* J_G.

(b) $\dim S/J_G = \max\{n+m+1, 2m\}$.

(c) $(J_{\tilde{G}} \cap A_n, J_{\tilde{G}} \cap B_m) = J_{\tilde{G}}$.

Proof. We start with the proof of (a). We use the statement proved in Lemma 1.1.2. At first consider the case $m > n = 1$. By view of Lemma 1.1.2 we have to find all $\emptyset \neq T \subseteq [1+m]$ such that $c(T \setminus \{i\}) < c(T)$. Clearly $T_0 = \{1\}$ satisfy the condition because $c(T_0) = m > 1$. Let T denote $T \subset [1+m]$ a subset different of T_0. Then If $1 \in T$ then $c(T) = m+1-|T|$ and $c(T \setminus \{i\}) = m+2-|T|$ for $i \neq 1$ and if $1 \notin T$ then $c(T) = 1$ and $c(T \setminus \{i\}) = 1$ for all $i \in T$. Hence we have the above primary decomposition.

Now consider the case of $m \geq n \geq 2$. As above we have to find all $\emptyset \neq T \subseteq [n+m]$ such that $c(T \setminus \{i\}) < c(T)$ for all $i \in T$. $T_1 = \{1, 2, \ldots, n\}$ satisfy the above condition because $c(T) = m$ and $c(T \setminus \{i\}) = 1$ for all $i \in T$. Similarly $T_2 = \{n+1, n+2, \ldots, n+m\}$ also satisfies the above condition.

Our claim is that no other $T \subseteq [n+m]$ satisfies this condition. If $T_1 \nsubseteq T$ and $T_2 \nsubseteq T$ then $c(T) = 1$ so in this case T does not satisfy the above condition. Now suppose that $T_1 \subsetneq T$ then $c(T) = m - |T \setminus T_1|$ and $c(T \setminus \{i\}) = m + 1 - |T \setminus T_1|$ if $i \in T \setminus T_1$. The same argument works if $T_2 \subsetneq T$. Hence we have $J_G = J_{\tilde{G}} \cap A_n \cap B_m$.

Then the statement on the dimension in (b) is a consequence of the reduced primary decomposition shown in (a). To this end recall that $\dim S/A_n = 2m, \dim S/B_m = 2n$ and $\dim S/J_{\tilde{G}} = n + m + 1$.

For the proof of (c) we use the notation of the Definition 2.1.1. Then it follows that

$$J_{\tilde{G}} \cap A_n = (I(1, \ldots, n, n+i), i = 1, \ldots, m, I(n+1, \ldots, n+m) \cap A_n).$$

Now A_n consists of an $S/I(n+1, \ldots, n+m)$-regular sequence and

$$I(n+1, \ldots, n+m) \cap A_n = A_n I(n+1, \ldots, n+m)$$

by Proposition 1.2.5. Therefore we get

$$J_{\tilde{G}} \cap A_n = (I(1, \ldots, n, n+i), i = 1, \ldots, m, A_n I(n+1, \ldots, n+m))$$

and similarly

$$J_{\tilde{G}} \cap B_m = (I(j, n+1, \ldots, n+m), j = 1, \ldots, n, B_m I(1, \ldots, n)).$$

But this clearly implies that $(J_{\tilde{G}} \cap A_n, J_{\tilde{G}} \cap B_m) = J_{\tilde{G}}$ which proves the statement in (c). $\square$

For the further computations we use the previous Lemma 2.1.2. In particular we use three exact sequences shown in the next statement.

Corollary 2.1.3. *With the previous notation we have the following three exact sequences.*

(1) $0 \to S/J_G \to S/J_{\tilde{G}} \cap A_n \oplus S/J_{\tilde{G}} \cap B_m \to S/J_{\tilde{G}} \to 0.$

(2) $0 \to S/J_{\tilde{G}} \cap A_n \to S/J_{\tilde{G}} \oplus S/A_n \to S/(J_{\tilde{G}}, A_n) \to 0.$

(3) $0 \to S/J_{\tilde{G}} \cap B_m \to S/J_{\tilde{G}} \oplus S/B_m \to S/(J_{\tilde{G}}, B_m) \to 0.$

Proof. The proof is an easy consequence of the primary decomposition as shown in Lemma 2.1.2. We omit the details. □

Note that in case of $n = 1$ we have $B_m = S$ therefore it is enough to consider the exact sequence (2) as (1) and (3) gives no information.

Corollary 2.1.4. *With the previous notation we have that*

$$\operatorname{depth} S/J_G = \begin{cases} m+2, & \text{if} \quad n = 1 \ ; \\ n+2, & \text{if} \quad m \geq n > 1 \end{cases}$$

and $\operatorname{reg} S/J_G \leq 2$.

Proof. The statement is an easy consequence of the short exact sequences shown in Corollary 2.1.3. To this end note that $S/J_{\tilde{G}}, S/(J_{\tilde{G}}, A_n)$ and $S/(J_{\tilde{G}}, B_m)$ are Cohen-Macaulay rings of dimension $n + m + 1, m + 1$ and $n + 1$ respectively. Moreover $\operatorname{reg} S/J_{\tilde{G}} = \operatorname{reg} S/(J_{\tilde{G}}, A_n) = \operatorname{reg} S/(J_{\tilde{G}}, B_m) = 1$. By using the exact sequences it provides the statement on the regularity. For the behavior of the depth respectively the regularity in short exact sequences see [2, Proposition 1.2.9] respectively [4, Corollary 20.19]. □

2.2 The modules of deficiency

The goal of this section is to describe all the local cohomology modules $H^i(S/J_G)$ of the binomial edge ideal associated to complete bipartite graph G. We do this by describing their Matlis duals which by Local Duality are the modules of deficiencies.

We start our investigations with the so-called star graph. That is complete bipartite graph $K_{m,n}$ with $n = 1$. For $m \leq 2$ the ideal J_G is a complete intersection generated by one respectively two quadrics so let us assume that $m > 2$.

Theorem 2.2.1. *Let G denote the star graph $K_{m,1}$. Then the binomial edge ideal $J_G \subset S$ has the following properties:*

(a) $\operatorname{reg} S/J_G = 2$.

(b) $\omega^i(S/J_G) = 0$ *if and only if* $i \notin \{m+2, 2m\}$.

(c) $\omega^{2m}(S/J_G) \cong S/A_1(-2m)$

(d) $\omega^{m+2}(S/J_G)$ *is a* $(m+2)$*-dimensional Cohen-Macaulay module and there is an isomorphism* $\omega^{m+2}(\omega^{m+2}(S/J_G)) \cong (J_{\tilde{G}}, A_1)/J_{\tilde{G}}$.

Proof. We use the short exact sequence of Corollary 2.1.3 (2). It induces a short exact sequence

$$0 \to H^{m+1}(S/(J_{\tilde{G}}, A_1)) \to H^{m+2}(S/J_G) \to H^{m+2}(S/J_{\tilde{G}}) \to 0$$

and an isomorphism $H^{2m}(S/J_G) \cong H^{2m}(S/A_1)$. Moreover the Cohen-Macaulayness of $S/J_{\tilde{G}}, S/A_1$ and $S/(J_{\tilde{G}}, A_1)$ of dimensions $m+2$, $2m$ and $m+1$ respectively imply that $H^i(S/J_G) = 0$ if $i \notin \{m+2, 2m\}$.

The short exact sequence on local cohomology induces the following exact sequence

$$0 \to \omega^{m+2}(S/J_{\tilde{G}}) \to \omega^{m+2}(S/J_G) \to \omega^{m+1}(S/(J_{\tilde{G}}, A_1)) \to 0$$

by Local Duality. Now we apply again local cohomology and take into account that both $\omega^{m+2}(S/J_{\tilde{G}})$ and $\omega^{m+1}(S/(J_{\tilde{G}}, A_1))$ are Cohen-Macaulay modules of dimension $m+2$ and $m+1$ respectively. Then $\operatorname{depth} \omega^{m+2}(S/J_G) \geq m+1$. By applying local cohomology and dualizing again we have

$$0 \to \omega^{m+2}(\omega^{m+2}(S/J_G)) \to S/J_{\tilde{G}} \xrightarrow{f} S/(J_{\tilde{G}}, A_1) \to \omega^{m+1}(\omega^{m+2}(S/J_G)) \to 0.$$

The homomorphism f is induced by the commutative diagram

$$\begin{array}{ccc} S/J_{\tilde{G}} & \to & S/(J_{\tilde{G}}, A_1) \\ \downarrow & & \downarrow \\ \omega_{2\times}(S/J_{\tilde{G}}) & \to & \omega_{2\times}(S/J_{\tilde{G}}, A_1). \end{array}$$

Note that the vertical maps are isomorphisms (see Proposition 1.3.4). Since the upper horizontal map is surjective the lower horizontal map is surjective too. Therefore $\omega^{m+1}(\omega^{m+2}(S/J_G)) = 0$. That is $\operatorname{depth} \omega^{m+2}(S/J_G) = m+2$ and it is a Cohen-Macaulay module. Moreover $\omega^{m+2}(\omega^{m+2}(S/J_G)) \cong (J_{\tilde{G}}, A_1)/J_{\tilde{G}}$. This finally proves the statements in (b), (c) and (d).

It is well known that $\operatorname{reg} S/J_G = \operatorname{reg} S/(J_{\tilde{G}}, A_1) = 1$ and $\operatorname{reg} S/A_1 = 0$. Then an inspection with the short exact sequence of Corollary 2.1.3 shows that $\operatorname{reg} S/J_G = 2$. □

In the next result we will consider the modules of deficiencies of the complete bipartite graph $G = K_{m,n}, n \geq 2$.

Theorem 2.2.2. *Let $m \geq n > 1$ and assume that the pair (m, n) is different from $(n+1, n)$ and $(2n-2, n)$. Then*

(a) $\operatorname{reg} S/J_G = 2$.

(b) $\omega^i(S/J_G) = 0$ *if and only if* $i \notin \{n+2, m+2, 2n, m+n+1, 2m\}$ *and there are*

the following isomorphisms and integers

i	$\omega^i(S/J_G)$	$\operatorname{depth}\omega^i(S/J_G)$	$\dim\omega^i(S/J_G)$
$n+2$	$\omega^{n+1}(S/(J_{\tilde{G}}, B_m))$	$n+1$	$n+1$
$m+2$	$\omega^{m+1}(S/(J_{\tilde{G}}, A_n))$	$m+1$	$m+1$
$2n$	$S/B_m(-2n)$	$2n$	$2n$
$n+m+1$	$\omega^{m+n+1}(S/J_{\tilde{G}})$	$n+m+1$	$n+m+1$
$2m$	$S/A_n(-2m)$	$2m$	$2m$

Proof. Under the assumption of $n+1 < m < 2n-2$ it follows that $2m > m+n+1 > 2n > m+2 > n+2$. Then the short exact sequences (see Corollary 2.1.3) induce the following isomorphisms:

(1) $H^{n+2}(S/J_G) \cong H^{n+2}(S/J_{\tilde{G}} \cap B_m) \cong H^{n+1}(S/(J_{\tilde{G}}, B_m))$,

(2) $H^{m+2}(S/J_G) \cong H^{m+2}(S/J_{\tilde{G}} \cap A_n) \cong H^{m+1}(S/(J_{\tilde{G}}, A_n))$,

(3) $H^{2n}(S/J_G) \cong H^{2n}(S/B_m)$ and

(4) $H^{2m}(S/J_G) \cong H^{2m}(S/A_n)$.

Moreover there is an exact sequence

$$0 \to H^{m+n+1}(S/J_G) \to H^{m+n+1}(S/J_{\tilde{G}} \cap A_n) \oplus H^{m+n+1}(S/J_{\tilde{G}} \cap B_m) \to H^{m+n+1}(S/J_{\tilde{G}}) \to 0$$

and $H^i(S/J_G) = 0$ if $i \notin \{n+2, m+2, 2n, m+n+1, 2m\}$.

Because of the short exact sequences in Corollary 2.1.3 there are isomorphisms

$$H^{m+n+1}(S/J_{\tilde{G}} \cap B_m) \cong H^{m+n+1}(S/J_{\tilde{G}}) \cong H^{m+n+1}(S/J_{\tilde{G}} \cap A_n).$$

So by Local Duality we get

$$0 \to \omega(S/J_{\tilde{G}}) \to \omega(S/J_{\tilde{G}}) \oplus \omega(S/J_{\tilde{G}}) \to \omega^{m+n+1}(S/J_G) \to 0.$$

This implies $\operatorname{depth}\omega(S/J_G) \geq n+m$. Moreover by applying local cohomology and again the Local Duality we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & S/J_G & \to & S/J_{\tilde{G}} \cap A_n \oplus S/J_{\tilde{G}} \cap B_m & \to & S/J_{\tilde{G}} & \to & 0 & \\ & \downarrow & & \downarrow & & \| & & & \\ 0 \to & \Omega(S/J_G) & \to & S/J_{\tilde{G}} \oplus S/J_{\tilde{G}} & \stackrel{f}{\to} & S/J_{\tilde{G}} & \to & \omega^{m+n}(\omega^{m+n+1}(S/J_G)) & \to 0, \end{array}$$

where $\Omega(S/J_G) = \omega^{m+n+1}(\omega^{m+n+1}(S/J_G))$. Now we show that $\omega^{m+n}(\omega^{m+n+1}(S/J_G)) = 0$. This follows since f is easily seen to be surjective. That is, $\omega^{m+n+1}(S/J_G)$ is a $(m+n+1)$-dimensional Cohen-Macaulay module. Moreover f is a split homomorphism and therefore $\Omega(S/J_G) \simeq S/J_{\tilde{G}}$ By duality this implies that $\omega^{m+n+1}(S/J_G) \cong \omega(S/J_{\tilde{G}})$. This completes the proof of the statements in (b). By similar arguments the other cases for (m,n) different of $(n+1,n)$ and $(2n-2,n)$ can be proved. We omit the details. Clearly $\operatorname{reg} S/J_G = 2$ as follows by (b). □

As a next sample of our considerations let us consider the case of the complete bipartite graph $K_{m,n}$ with $(m,n) = (n+1,n)$.

Theorem 2.2.3. *Let $m = n+1$ and $n > 3$. Then:*

(a) $\operatorname{reg} S/J_G = 2$.

(b) $\omega^i(S/J_G) = 0$ *if and only if* $i \notin \{n+2, n+3, 2n, 2n+2\}$ *and there are the following isomorphisms and integers*

i	$\omega^i(S/J_G)$	$\operatorname{depth}\omega^i(S/J_G)$	$\dim\omega^i(S/J_G)$
$n+2$	$\omega^{n+1}(S/(J_{\tilde{G}}, B_{n+1}))$	$n+1$	$n+1$
$n+3$	$\omega^{n+2}(S/(J_{\tilde{G}}, A_n))$	$n+2$	$n+2$
$2n$	$S/B_{n+1}(-2n)$	$2n$	$2n$
$2n+2$	$\omega(S/J_{\tilde{G}}) \oplus S/A_n(-2n-2)$	$2n+2$	$2n+2.$

Proof. By applying the local cohomology functors $H^{\cdot}(-)$ to the exact sequence (2) in Corollary 2.1.3 we get the following:

(1) $H^{n+3}(S/J_{\tilde{G}} \cap A_n) \cong H^{n+2}(S/(J_{\tilde{G}}, A_n))$,

(2) $H^{2n+2}(S/J_{\tilde{G}} \cap A_n) \cong H^{2n+2}(S/J_{\tilde{G}}) \oplus H^{2n+2}(S/A_n)$ and

(3) $H^i(S/J_{\tilde{G}} \cap A_n) = 0$ for $i \neq n+2, 2n+2$.

Similarly, if we apply $H^{\cdot}(-)$ to the exact sequence (3) in Corollary 2.1.3 we get

(4) $H^{n+2}(S/J_{\tilde{G}} \cap B_{n+1}) \cong H^{n+1}(S/(J_{\tilde{G}}, B_{n+1}))$.

(5) $H^{2n}(S/J_{\tilde{G}} \cap B_{n+1}) \cong H^{2n}(S/B_{n+1})$.

(6) $H^{2n+2}(S/J_{\tilde{G}} \cap B_{n+1}) \cong H^{2n+2}(S/J_{\tilde{G}})$.

(7) $H^i(S/J_{\tilde{G}} \cap B_{n+1}) = 0$ for $i \neq n+2, 2n, 2n+2$.

The short exact sequence (1) of Corollary 2.1.3 provides (by applying the local cohomology functor) the vanishing $H^i(S/J_G) = 0$ for all $i \neq n+2, n+3, 2n, 2n+2$. Moreover it induces isomorphisms

$$H^{n+2}(S/J_G) \cong H^{n+1}(S/(J_{\tilde{G}}, B_{n+1})) \text{ and } H^{2n}(S/J_G) \cong H^{2n}(S/B_{n+1})$$

and as $n > 3$ so $2n > n+3$ the isomorphism $H^{n+3}(S/J_G) \cong H^{n+2}(S/(J_{\tilde{G}}, A_n))$. Moreover we get an exact sequence

$$0 \to H^{2n+2}(S/J_G) \to H^{2n+2}(S/J_{\tilde{G}}) \oplus H^{2n+2}(S/A_n) \oplus H^{2n+2}(S/J_{\tilde{G}}) \to H^{2n+2}(S/J_{\tilde{G}}) \to 0.$$

By Local Duality this proves the first three rows in the table of statement (b). By Local Duality we have

$$0 \to \omega(S/J_{\tilde{G}}) \to \omega(S/J_{\tilde{G}}) \oplus \omega(S/A_n) \oplus \omega(S/J_{\tilde{G}}) \to \omega(S/J_G) \to 0.$$

Note that we may write ω instead of ω^{2n+2} because all modules above are canonical modules. First of all the short exact sequence provides that $\operatorname{depth} \omega(S/J_G) \geq 2n+1$. By applying local cohomology and dualizing again we get

$$0 \to \omega_{2\times}(S/J_G) \to S/J_{\tilde{G}} \oplus S/A_n \oplus S/J_{\tilde{G}} \xrightarrow{f} S/J_{\tilde{G}} \to \omega^{2n+1}(\omega(S/J_G)) \to 0.$$

As in the proof of Theorem 2.2.1 we see that f is surjective. Therefore $\omega^{2n+1}(\omega(S/J_G)) = 0$ and $\operatorname{depth} \omega(S/J_G) = 2n+2$. Whence $\omega(S/J_G)$ is a $(2n+2)$-dimensional Cohen-Macaulay module. Then f is a split surjection and $\omega_{2\times}(S/J_G) \cong S/J_{\tilde{G}} \oplus S/A_n$. This implies the isomorphism $\omega(S/J_G) \cong \omega(S/J_{\tilde{G}}) \oplus \omega(S/A_n)$ and this finishes the proof of (b). Clearly $\operatorname{reg} S/J_G = 2$. □

Theorem 2.2.4. *Let $m > 3$ and $n = 2$. Then:*

(a) $\operatorname{reg} S/J_G = 2$.

(b) $\omega^i(S/J_G) = 0$ *if and only if* $i \notin \{4, m+2, m+3, 2m\}$ *and there are the following isomorphisms and integers*

i	$\omega^i(S/J_G)$	$\operatorname{depth} \omega^i(S/J_G)$	$\dim \omega^i(S/J_G)$
4	$\omega^4((J_{\tilde{G}}, B_m)/B_m)$	4	4
$m+2$	$\omega^{m+1}(S/(J_{\tilde{G}}, A_2))$	$m+1$	$m+1$
$m+3$	$\omega(S/J_{\tilde{G}})$	$m+3$	$m+3$
$2m$	$S/A_2(-2m)$	$2m$	$2m$.

Moreover there is an isomorphism $\omega^4(\omega^4(S/J_G) \cong (J_{\tilde{G}}, B_m)/B_m$.

Proof. By applying the local cohomology functors $H^{\cdot}(-)$ to the exact sequence (2) in Corollary 2.1.3 we get the following:

(1) $H^{m+2}(S/J_{\tilde{G}} \cap A_2) \cong H^{m+1}(S/(J_{\tilde{G}}, A_2))$,

(2) $H^{m+3}(S/J_{\tilde{G}} \cap A_2) \cong H^{m+3}(S/J_{\tilde{G}})$,

(3) $H^{2m}(S/J_{\tilde{G}} \cap A_2) \cong H^{2m}(S/A_2)$ and

(4) $H^i(S/J_{\tilde{G}} \cap A_2) = 0$ for $i \neq m+2, m+3, 2m$.

Similarly, if we apply $H^{\cdot}(-)$ to the exact sequence (3) in Corollary 2.1.3 we get the isomorphism $H^{m+3}(S/J_{\tilde{G}} \cap B_m) \cong H^{m+3}(S/J_{\tilde{G}})$ and the exact sequence

$$0 \to H^3(S/(J_{\tilde{G}}, B_m)) \to H^4(S/J_{\tilde{G}} \cap B_m) \to H^4(S/B_m) \to 0.$$

The short exact sequence on local cohomology induces the following exact sequence

$$0 \to \omega^4(S/B_m) \to \omega^4(S/J_{\tilde{G}} \cap B_m) \to \omega^3(S/(J_{\tilde{G}}, B_m)) \to 0$$

by Local Duality. Now we apply again local cohomology and taking into account that both $\omega^4(S/B_m)$ and $\omega^3(S/(J_{\tilde{G}}, B_m))$ are Cohen-Macaulay modules of dimension 4 and 3 respectively. Then $\operatorname{depth} \omega^4(S/J_{\tilde{G}} \cap B_m)) \geq 3$. By applying local cohomology and dualizing again it induces the following exact sequence

$$0 \to \omega^4(\omega^4(S/J_{\tilde{G}} \cap B_m)) \to S/B_m \xrightarrow{f} S/(J_{\tilde{G}}, B_m) \to \omega^3(\omega^4(S/J_{\tilde{G}} \cap B_m)) \to 0.$$

Now the homomorphism f is an epimorphism. $\omega^3(\omega^4(S/J_{\tilde{G}} \cap B_m)) = 0$. That is $\operatorname{depth} \omega^4(S/J_{\tilde{G}} \cap B_m) = 4$ and it is a Cohen-Macaulay module. Moreover $\omega^4(\omega^4(S/J_{\tilde{G}} \cap B_m)) \cong (J_{\tilde{G}}, B_m)/B_m$. With these results in mind the short exact sequence (1) of Corollary 2.1.3 provides (by applying the local cohomology functor) the vanishing $H^i(S/J_G) = 0$ for all $i \neq 4, m+2, m+3, 2m$. Moreover it induces isomorphisms

$$H^4(S/J_G) \cong H^4(S/J_{\tilde{G}} \cap B_m), H^{m+2}(S/J_G) \cong H^{m+1}(S/(J_{\tilde{G}}, A_2))$$

and $H^{2m}(S/J_G) \cong H^{2m}(S/A_2)$. Moreover we get

$$0 \to H^{m+3}(S/J_G) \to H^{m+3}(S/J_{\tilde{G}}) \oplus H^{m+3}(S/J_{\tilde{G}}) \to H^{m+3}(S/J_{\tilde{G}}) \to 0.$$

This implies the isomorphism $\omega^{m+3}(S/J_G) \cong \omega(S/J_{\tilde{G}})$. □

Finally we shall consider the case of the complete bipartite graph with $2n = m+2$. In all of the previous examples we have the phenomenon that $\omega^i(S/J_G)$ is either zero or a Cohen-Macaulay module with $i - 1 \leq \dim \omega^i(S/J_G) \leq i$ for all $i \in \mathbb{Z}$. and the canonical module $\omega(S/J_G) = \omega^d(S/J_G), d = \dim S/J_G$, is a d-dimensional Cohen-Macaulay module. For $2n = m+2$ this is no longer true.

Theorem 2.2.5. *Let $m+2=2n$ and $m>n+1$. Then:*

(a) $\operatorname{reg} S/J_G = 2$.

(b) $\omega^i(S/J_G)=0$ *if and only if* $i \notin \{n+2, m+2=2n, m+n+1, 2m\}$ *and there are the following isomorphisms and integers*

i	$\omega^i(S/J_G)$	$\operatorname{depth}\omega^i(S/J_G)$	$\dim\omega^i(S/J_G)$
$n+2$	$\omega^{n+1}(S/(J_{\tilde{G}}, B_m))$	$n+1$	$n+1$
$m+2$	$\omega^{m+1}(S/(J_{\tilde{G}}, A_n)) \oplus S/B_m(-2n)$	$m+1$	$m+2$
$m+n+1$	$\omega(S/J_{\tilde{G}})$	$m+n+1$	$m+n+1$
$2m$	$S/A_n(-2m)$	$2m$	$2m.$

Proof. Clearly $n+2<2n=m+2<m+n+1<2m$. Then the short exact sequences of Corollary 2.1.3 provide that $H^i(S/J_G)=0$ for all $i \neq n+2, m+2=2n, m+n+1, 2m$. Moreover, it induces the following isomorphisms

(1) $H^{n+2}(S/J_G) \cong H^{n+1}(S/(J_{\tilde{G}}, B_m))$,

(2) $H^{m+2}(S/J_G) \cong H^{m+1}(S/(J_{\tilde{G}}, A_n) \oplus H^{m+2}(S/B_m)$,

(3) $H^{m+n+1}(S/J_G) \cong H^{m+n+1}(S/J_{\tilde{G}})$ and

(4) $H^{2m}(S/J_G) \cong H^{2m}(S/A_n)$.

This easily yields the statements in (a) and (b). □

The difference of the case handled in Theorem 2.2.5 is the fact the $\omega^{m+2}(S/J_G)$ is not Cohen-Macaulay. It is the direct sum of two Cohen-Macaulay modules of dimensions $m+2$ and $m+1$ respectively.

Now we prove some corollaries about the Cohen-Macaulayness and related properties.

Corollary 2.2.6. *Let $J_G \subset S$ denote the binomial edge ideal associated to complete bipartite graph.*

(a) *S/J_G is canonically Cohen-Macaulay and* $\operatorname{depth} \omega^i(S/J_G) \geq i-1$ *for all* $\operatorname{depth} S/J_G \leq i \leq \dim S/J_G$*. Moreover S/J_G is Cohen-Macaulay if and only if $(m,n) \in \{(2,1)(1,1)\}$.*

(b) *S/J_G is sequentially Cohen-Macaulay and not Cohen-Macaulay if and only if $n = 1$ and $m > 2$ or $n = m = 2$.*

Proof. By view of Theorems 2.2.1, 2.2.2, 2.2.3, 2.2.4 and 2.2.5 we get the statements on the Cohen-Macaulayness of $\omega(S/J_G)$ and the estimates of of the depth of $\omega^i(S/J_G)$ for all possible complete bipartite graphs G. By Lemma 2.1.2 and Corollary 2.1.4 the claim on the Cohen-Macaulayness of S/J_G is easily seen. Similar arguments work for the sequentially Cohen-Macaulay property as it is easily seen by the definition. □

2.3 On purity of the free resolution

Let $J_r \subset S, r \leq m+n$, denote the binomial edge ideal corresponding to the complete graph on r vertices. As a technical tool for our further investigations we need the following Lemma.

Lemma 2.3.1. *Let M denote a finitely generated graded S-module. Let $\underline{f} = f_1, \ldots, f_l$ denote an M-regular sequence of forms of degree 1. Then*

$$\operatorname{Tor}_i^S(K, M/\underline{f}M) \cong \oplus_{j=0}^{l} \operatorname{Tor}_{i-j}^S(K,M)^{\binom{l}{j}}(-j).$$

Proof. For the proof of the statement let $l = 1$ and $f = f_1$. Then the short exact sequence $0 \to M(-1) \xrightarrow{f} M \to M/fM \to 0$ provides an isomorphism

$$\operatorname{Tor}_i^S(K, M/fM) \cong \operatorname{Tor}_i^S(K,M) \oplus \operatorname{Tor}_{i-1}^S(K,M)(-1)$$

for all $i \in \mathbb{Z}$. □

For a certain technical reason we need the following Lemma that describes the ideals $J_{\tilde{G}} \cap A_n$ respectively $J_{\tilde{G}} \cap B_m$ as binomial edge ideals.

Lemma 2.3.2. *$J_{\tilde{G}} \cap A_n$ is the binomial edge ideal of the graph G obtained by deleting all edges $\{i, j\}$ of the complete graph on $[n+m]$ vertices such that $n < i < j \leq m+n$. Similarly $J_{\tilde{G}} \cap B_m$ is the binomial edge ideal of the graph where all edges $\{i, j\}$ of the complete graph on $[n+m]$ vertices such that $1 \leq i < j \leq n$ are deleted.*

Proof. Let us consider the ideal $J_{\tilde{G}} \cap A_n$. Look at the primary decomposition of the graph G. We have to find all $\emptyset \neq T \subset [n+m]$ such that $c(T \setminus \{i\}) < c(T)$ for all $i \in T$. If $T = \{1, \ldots, n\}$, then $c(T) = m > 1$ and $c(T \setminus \{i\}) = 1$ for all i. Let $T \subset [n+m]$ denote a subset with $T \neq \{1, \ldots, n\}$. Then clearly the condition $c(T \setminus \{i\}) < c(T)$ for all $i \in T$ can not be satisfied. So the claim follows by Lemma 1.1.2. A similar consideration proves the case of $J_{\tilde{G}} \cap B_m$. □

Now we shall prove that S/J_G has a pure resolution.

Theorem 2.3.3. *Let S/J_G denote the binomial edge ideal associated to complete bipartite graph $K_{m,n}$. Then the Betti diagram has the following form*

	0	1	2	$\cdots$	p
0	1	0	0	$\cdots$	0
1	0	mn	0	$\cdots$	0
2	0	0	$\beta_{2,2}$	$\cdots$	$\beta_{p,2}$

where

$$p = \begin{cases} m, & \text{if } \ n = 1\ ; \\ 2m + n - 2, & \text{if } \ m \geq n > 1. \end{cases}$$

Proof. Because of the regularity and depth of S/J_G, the non-vanishing part of the Betti table is concentrated in the frame of the one given in the statement. Clearly

$\beta_{0,0} = 1$ and $\beta_{i,0} = 0$ for all $i > 0$. Furthermore $\beta_{1,0} = \beta_{2,0} = 0$. Since J_G is minimally generated by mn binomials we get that $\beta_{1,1} = mn$ and $\beta_{1,2} = 0$. Now we have to show that $\beta_{2,1} = 0$ because this implies that $\beta_{i,1} = 0$ for each $i \geq 2$ as a consequence of the minimality of the free resolution. Here we have two cases:

Case(a): Let $m \geq n > 1$. We take the short exact sequence (1) of Corollary 2.1.3. It induces a graded homomorphism of degree zero

$$\operatorname{Tor}_2^S(K, S/J_{\tilde{G}}) = K^{b_2(m+n)}(-3) \to \operatorname{Tor}_1^S(K, S/J_G) = K^{mn}(-2).$$

Therefore it is the zero homomorphism. On the other side it induces a homomorphism

$$\operatorname{Tor}_3^S(K, S/J_{\tilde{G}}) = K^{b_3(m+n)}(-4) \to \operatorname{Tor}_2^S(K, S/J_G),$$

which is the zero homomorphism when restricted to degree 3 since $\operatorname{reg} S/J_G = 2$. Therefore there is a short exact sequence of K-vector spaces

$$0 \to \operatorname{Tor}_2^S(K, S/J_G)_3 \to \operatorname{Tor}_2^S(K, S/J_{\tilde{G}} \cap A_n)_3 \oplus \operatorname{Tor}_2^S(K, S/J_{\tilde{G}} \cap B_m)_3 \to K^{b_2(m+n)} \to 0.$$

That is $\beta_{2,1}(S/J_G) = \beta_{2,1}(S/J_{\tilde{G}} \cap A_n) + \beta_{2,1}(S/J_{\tilde{G}} \cap B_m) - b_2(m+n)$.

In the next step we shall compute $\beta_{2,1}(S/J_{\tilde{G}} \cap A_n)$ and $\beta_{2,1}(S/J_{\tilde{G}} \cap B_m)$. We start with the first of them. To this end we use the short exact sequence (2) of Corollary 2.1.3. At first we note that $\beta_{1,2}(S/J_{\tilde{G}} \cap A_n) = 0$ which is true since $J_{\tilde{G}} \cap A_n$ is minimally generated by quadrics as follows by Lemma 2.3.4. Because of

$$\beta_{3,0}(S/J_{\tilde{G}} \cap A_n) = \beta_{3,0}(S/J_{\tilde{G}}) = \beta_{2,1}(S/A_n) = 0$$

we get the following exact sequence of K-vector spaces.

$$0 \to \operatorname{Tor}_3^S(K, S/A_n)_3 \to \operatorname{Tor}_3^S(K, S/(J_{\tilde{G}}, A_n))_3 \to \operatorname{Tor}_2^S(K, S/J_{\tilde{G}} \cap A_n)_3 \to$$
$$\operatorname{Tor}_2^S(K, S/J_{\tilde{G}})_3 \to \operatorname{Tor}_2^S(K, S/(J_{\tilde{G}}, A_n))_3 \to 0.$$

By counting vector space dimensions this provides that

$$\beta_{2,1}(S/J_{\tilde{G}} \cap A_n) = b_2(m+n) + \beta_{3,0}(S/(J_{\tilde{G}}, A_n)) - \beta_{2,1}(S/(J_{\tilde{G}}, A_n)) - \binom{2n}{3}.$$

Since $(J_{\tilde{G}}, A_n) = (J_m, A_n)$, where $J_m = I(n+1, \ldots, n+m)$ we might use Lemma 2.3.1 for the calculation of these dimensions. Therefore $\operatorname{Tor}_3^S(K, S/(J_{\tilde{G}}, A_n))_3 \cong \operatorname{Tor}_0^S(K, S/J_m)_0^{\binom{2n}{3}}$ and

$$\operatorname{Tor}_2^S(K, S/(J_{\tilde{G}}, A_n))_3 \cong \operatorname{Tor}_2^S(K, S/J_m)_3 \oplus \operatorname{Tor}_1^S(K, S/J_m)_2^{\binom{2n}{1}}.$$

Therefore $\beta_{3,0}(S/(J_{\tilde{G}}, A_n)) = \binom{2n}{3}$ and $\beta_{2,1}(S/(J_{\tilde{G}}, A_n)) = b_2(m) + 2nb_1(m)$. Putting these integers together it follows that

$$\beta_{2,1}(S/J_{\tilde{G}} \cap A_n) = b_2(m+n) - b_2(m) - 2nb_1(m).$$

Interchanging the rôles of m and n we derive a corresponding formula for $\beta_{2,1}(S/J_{\tilde{G}} \cap B_m))$, namely

$$\beta_{2,1}(S/J_{\tilde{G}} \cap B_m) = b_2(m+n) - b_2(n) - 2mb_1(n)$$

Finally we use both expressions in the above formula in order to confirm that $\beta_{2,1}(S/J_G)$ vanishes.

Case(b): Let $m > n = 1$. Because $J_{\tilde{G}} \cap A_1 = J_G$ we might use exact sequence (2) of Corollary 2.1.3. Since $(J_{\tilde{G}}, A_1) = (I(2, \ldots, n+m), A_1)$ the statement in Lemma 2.3.1 imply that $\operatorname{Tor}_3^S(K, S/(J_{\tilde{G}}, A_1))_3 = 0$. Whence there is an exact sequence of K-vector spaces

$$0 \to \operatorname{Tor}_2^S(K, S/J_G)_3 \to \operatorname{Tor}_2^S(K, S/J_{\tilde{G}})_3 \oplus \operatorname{Tor}_2^S(K, S/A_1)_3 \to \operatorname{Tor}_2^S(K, S/(J_{\tilde{G}}, A_1))_3 \to 0.$$

Therefore $\beta_{2,1}(S/J_G) = b_2(m+1) - \beta_{2,1}(S/(J_{\tilde{G}}, A_1))$. Again by the statement of Lemma 2.3.1 it follows that $\beta_{2,1}(S/(J_{\tilde{G}}, A_1)) = b_2(m) + 2b_1(m)$. Finally

$$\beta_{2,1}(S/J_G) = b_2(m+1) - b_2(m) - 2b_1(m) = 0,$$

as required. $\square$

As a final feature of the investigations we will describe the explicit values of the Betti numbers $\beta_{i,2}(S/J_G), 2 \leq i \leq p$, as they are indicated in Theorem 2.3.3.

Theorem 2.3.4. *Let $G = K_{m,n}$ denote the complete bipartite graph with $m \geq n \geq 1$.*

(a) The Hilbert series of S/J_G is

$$H(S/J_G, t) = \tfrac{1}{(1-t)^{m+n+1}}(1 + (m+n-1)t) + \tfrac{1}{(1-t)^{2m}} + \tfrac{1}{(1-t)^{2n}}$$
$$-\tfrac{1}{(1-t)^{m+1}}(1 + (m-1)t) - \tfrac{1}{(1-t)^{n+1}}(1 + (n-1)t).$$

(b) For the multiplicity $e(S/J_G)$ it follows

$$e(S/J_G) = \begin{cases} 1, & \text{if } m > n+1 \text{ or } n = 1 \text{ and } m > 2, \\ 2m, & \text{otherwise.} \end{cases}$$

(c) Let $n = 1$. Then $\beta_{i,2}(S/J_G) = m\binom{m}{i} - \binom{m}{i+1} - \binom{m+1}{i+1}$ for all $2 \leq i \leq p = m$. Let $m \geq n > 1$. Then

$$\beta_{i,2}(S/J_G) = \tbinom{m+n}{i+2} + \tbinom{2n}{i+2} + \tbinom{2m}{i+2} + m\tbinom{m+2n-1}{i+1} + n\tbinom{2m+n-1}{i+1}$$
$$-\tbinom{m+2n}{i+2} - \tbinom{2m+n}{i+2} - (m+n)\tbinom{m+n-1}{i+1}$$

for all $2 \leq i \leq p = 2m + n - 2$.

Proof. In order to prove (a) we use the short exact sequences of Corollary 2.1.3. By the additivity of the Hilbert function we get the following equalities:

$$H(S/J_G, t) = H(S/J_{\tilde{G}} \cap A_n, t) + H(S/J_{\tilde{G}} \cap B_n, t) - H(S/J_{\tilde{G}}, t),$$

$$H(S/J_{\tilde{G}} \cap A_n, t) = H(S/J_{\tilde{G}}, t) + H(S/A_n, t) - H(S/(J_{\tilde{G}} A_n), t), \text{ and}$$

$$H(S/J_{\tilde{G}} \cap B_m, t) = H(S/J_{\tilde{G}}, t) + H(S/B_m, t) - H(S/(J_{\tilde{G}} B_m), t).$$

Substituting the Hilbert series of the complete graphs $S/J_{\tilde{G}}, S/(J_{\tilde{G}}, A_n)$ and $S/(J_{\tilde{G}}, B_m)$ as well as the Hilbert series of the polynomial rings $S/A_n, S/B_m$ we get the desired formula in (a). Then (b) is an easy consequence of (a).

For the proof of (c) we note at first the structure of the finite free resolution of S/J_G

$$0 \to S^{\beta_p}(-p-2) \to \cdots \to S^{\beta_3}(-5) \to S^{\beta_2}(-4) \to S^{\beta_1}(-2) \to S$$

with $\beta_1 = mn$ $\beta_i = \beta_{i,2}(S/J_G), 2 \leq i \leq p$, as shown in Theorem 2.3.3. By the additivity of the Hilbert series this provides the following expression

$$H(S/J_G, t) = \frac{1}{(1-t)^{2m+2n}}(1 - \beta_1 t^2 + \sum_{i=2}^{p}(-1)^i \beta_i t^{i+2})$$

(see also [4, Exercise 19.14]). Now we use the expression of the Hilbert series $H(S/J_G, t)$ as shown in (a) and compare it with the one of the minimal free resolution. By some nasty calculations we derive the formulas for the Betti numbers as given in the statement. □

Remark 2.3.5. Recently S. Saeedi and D. Kiani shows in [13] that $\beta_{2,1} = 2l$ where l is the number of 3-cycles in G. Therefore in an arbitrary bipartite graph $\beta_{i,1} = 0$ for all $i \geq 2$.

Chapter 3

Binomial edge ideal of a cycle

This chapter is devoted to the binomial edge ideal of a cycle. As a main result we prove that the binomial edge ideal of any cycle is approximately Cohen-Macaulay. In order to compute the Hilbert series for the cycle, we include some investigations on the canonical module of the binomial edge ideal associated to complete graph. At the end we discuss the minimal free resolution of cycle by computing its regularity and Betti numbers.

3.1 Some algebraic invariants of cycle

Let $n \geq 3$. We denote the cycle on vertex set $[n]$ by C and its binomial edge ideal by I_C. To study further properties of I_C or S/I_C we shall need few basic properties from [8]. If

$$I_C = \cap_{T\subseteq[n]} P_T(C)$$

then

$$\dim S/P_T(C) = n+1 \;\; if \;\; T = \emptyset \;\text{ and }\; \dim S/P_T(C) \leq n \;\; if \;\; T \neq \emptyset.$$

Hence

$$\dim(S/I_C) = n+1 \;\; and \;\; U_S(I_C) = P_\emptyset(C) = J_{\tilde{G}}.$$

Moreover $P_T(C)$ is minimal prime of S/I_C if either (i) $T=\emptyset$ or (ii) if $T\neq\emptyset$ and $|T|>1$ and no two elements $i,j\in T$ belongs to the same edge of C.

Theorem 3.1.1. *$x_1, y_1-x_2,\dots,y_{n-1}-x_n, y_n$ is the system of parameters for S/I_C.*

Proof. Let $\underline{x}=x_1,y_1-x_2,\dots,y_{n-1}-x_n,y_n$, $M=S/I_C$ and $I=\mathrm{Ann}(M/\underline{x}M)$. Now $M/\underline{x}M=S/(I_C,\underline{x})$. If we replace $y_1=x_2, y_2=x_3,\dots,y_{n-1}=x_n$ in I_C, we get

$$I=(x_1,x_1x_3-x_2^2,x_2x_4-x_3^2,\dots,x_{n-2}x_n-x_{n-1}^2,x_n^2,x_nx_2).$$

Clearly $x_1,x_n\in\mathrm{Rad}(I)$, we need to prove that $x_2,\dots,x_{n-1}\in\mathrm{Rad}(I)$. If $x_1,x_1x_3-x_2^2\in I$ then $x_2^2\in I$. It follows that $x_2\in\mathrm{Rad}(I)$, hence we have a basis of induction. If $x_k\in\mathrm{Rad}(I)$ for $2\le k\le n-2$ then $x_kx_{k+2}-x_{k+1}^2\in I$ as $k+1\le n-1$. Therefore $x_{k+1}^2\in\mathrm{Rad}(I)$ it follows that $x_{k+1}\in\mathrm{Rad}(I)$. This then implies that $\mathrm{Rad}(I)=(x_1,x_2,\dots,x_n)$. Hence I is $m-$primary in the ring $K[x_1,\dots,x_n]$ so $\underline{x}$ is the system of parameters of S/I_C. □

Lemma 3.1.2. *Let I_L be the binomial edge ideal of a line L on vertex set $[n]$, $g=x_1y_n-x_ny_1$ and $J_{\tilde{G}}$ be binomial edge ideal associated to complete graph on vertex set $[n]$ then*

(a) *If $I_L=\cap_{T\subseteq[n]}P_T(L)$ then $g\notin P_T(L)$ for $T\neq\emptyset$.*

(b) $I_L=(I_L:g)\cap J_{\tilde{G}}$.

(c) $I_L:(I_L:g)=J_{\tilde{G}}$.

(d) $I_L:g=I_L:J_{\tilde{G}}$.

Proof. **(a)** Because of $I_L=\cap_{T\subseteq[n]}P_T(L)$, it is known from [8] that $P_T(L)$ is minimal prime of I_L if either (i) $T=\emptyset$ or (ii) if $T\neq\emptyset$ and $1,n\notin T$ and if $|T|>1$ then there are no two elements $i,j\in T$ such that $\{i,j\}$ is an edge of L. If $T=\emptyset$, then $P_T(L) = J_{\tilde{G}}$ is the ideal of complete graph. Now let $T \neq \emptyset$ and $1, n \notin T$. Suppose that $|T| > 1$ then $g \notin P_T(L)$ because x_1, y_1, x_n, y_n does not belongs to $\cup_{i\in T}\{x_i, y_i\}$ and g does not belongs to any $J_{\tilde{G}}$ for any connected component of $[n]\backslash T$.

(b) $I_L : g = \cap_{g\notin P_T(L)} P_T(L)$, Using (a) we have

$$I_L : g = \cap_{T\neq\emptyset} P_T(L)$$

and $I_L = \cap_{T\neq\emptyset} P_T(L) \cap P_\emptyset(L)$ therefore,

$$I_L = (I_L : g) \cap J_{\tilde{G}}.$$

(c) Using (b) we have

$$I_L : (I_L : g) = ((I_L : g) \cap J_{\tilde{G}}) : (I_L : g) = J_{\tilde{G}} : (I_L : g).$$

In order to finish we have to prove that $J_{\tilde{G}} = J_{\tilde{G}} : (I_L : g)$. From the definition $J_{\tilde{G}} \subseteq J_{\tilde{G}} : (I_L : g)$. Now for the other inclusion let $I_L : g = (h_1, h_2, ..., h_r)$, then $J_{\tilde{G}} : (I_L : g) = \cap_{i=1}^{r} J_{\tilde{G}} : h_i$. Now $J_{\tilde{G}} : h_i = J_{\tilde{G}}$ for at least one i, since $J_{\tilde{G}}$ is a prime ideal and $h_i \notin J_{\tilde{G}}$ for at least one i so $J_{\tilde{G}} : (I_L : g) \subseteq J_{\tilde{G}}$ and we are done.

(d) $I_L : g \subseteq I_L : J_{\tilde{G}}$ is trivial. For another inclusion let $f \in I_L : J_{\tilde{G}}$ then $fJ_{\tilde{G}} \subseteq I_L \subseteq P_T(L)$, now $J_{\tilde{G}} \nsubseteq P_T(L)$ for $T \neq \emptyset$, so $f \in P_T(L)$ for $T \neq \emptyset$ which implies $f \in I_L : g$.

□

Definition 3.1.3. [12] Two ideals I and J of height g in S are said to be linked if there is a regular sequence α of height g in their intersection such that $I = \alpha : J$ and $J = \alpha : I$.

It is also known from [12] that I and J are two linked ideals of S then S/I is Cohen-Macaulay if and only if S/J is Cohen-Macaulay.

Lemma 3.1.4. *Let I_L be the binomial edge ideal of a line L on vertex set $[n]$ then $S/I_L : g$ is Cohen-Macaulay of dimension $n+1$.*

Proof. I_L is a complete intersection and using $(b), (c)$ and (d) of Lemma 3.1.2 we have $I_L : g$ and $J_{\tilde{G}}$ are linked ideals. Now by above theorem $S/I_L : g$ is Cohen-Macaulay because $S/J_{\tilde{G}}$ is Cohen-Macaulay. □

Theorem 3.1.5. *S/I_C is approximately Cohen-Macaulay.*

Proof. First we will compute the depth of S/I_C for $n \geq 3$. From above notations $I_C = (I_L, g)$. Consider the exact sequence

$$0 \to S/I_L : g(-2) \to S/I_L \to S/I_C \to 0.$$

Now it follows from the Depth's Lemma that

$$\text{depth}(S/I_C) \geq \min\{\text{depth}(S/I_L : g) - 1, \text{depth}(S/I_L)\} = n.$$

Hence $\text{depth}(S/I_C) \geq n$.

Now $S/U_S(I_C) \cong S/J_{\tilde{G}}$, which is $n+1-$dimensional Cohen-Macaulay ring, so from Theorem 1.4.3 S/I_C is approximately Cohen-Macaulay. □

3.1.1 Hilbert series of the binomial edge ideal of a cycle

Furthermore we will find the Hilbert series of S/I_C. For this we have to introduce a monomial ideal $M = (x_2x_3\cdots x_{n-1}, x_2x_3\cdots x_{n-2}y_{n-1}, \ldots, x_2y_3\cdots y_{n-1}, y_2y_3\cdots y_{n-1})$.

Lemma 3.1.6. *With the notations above we have*

$$\omega(S/J_{\tilde{G}}) \cong (J_{\tilde{G}}, M)/J_{\tilde{G}}.$$

Proof. $J_{\tilde{G}}$ is the ideal of all 2-minors of a generic $2 \times n-$matrix which implies all 2-minors of a generic $2 \times n-$matrix are zero in $S/J_{\tilde{G}}$ hence both rows of this matrix are linearly dependent, therefore $S/J_{\tilde{G}} \cong K[x_1, \ldots, x_n, x_1t, \ldots, x_nt]$. It is known that [2]

$$\omega(S/J_{\tilde{G}}) \cong (x_1, y_1)^{n-2} S/J_{\tilde{G}}$$

therefore

$$\omega(S/J_{\tilde{G}}) \cong (x_1, x_1t)^{n-2} K[x_1, \ldots, x_n, x_1t, \ldots, x_nt].$$

Next we consider the monomial ideal M in $S/J_{\tilde{G}}$

$$MS/J_{\tilde{G}} \cong (x_2x_3 \cdots x_{n-1}, x_2x_3 \cdots x_{n-1}t, \ldots, x_2x_3 \cdots x_{n-1}t^{n-2})K[x_1, \ldots, x_n, x_1t, \ldots, x_nt].$$

Now after multiplying $\omega(S/J_{\tilde{G}})$ by $x_2x_3 \cdots x_{n-1}$ and $MS/J_{\tilde{G}}$ by x_1 respectively, we see that both are isomorphic that is

$$\omega(S/J_{\tilde{G}}) \cong MS/J_{\tilde{G}} \cong (J_{\tilde{G}}, M)/J_{\tilde{G}}.$$

□

Theorem 3.1.7. *$S/(J_{\tilde{G}}, M)$ is Gorenstein of dimension n.*

Proof. From Lemma 1.3.4 (g) and 3.1.6. We have

$$(S/J_{\tilde{G}})/MS/J_{\tilde{G}} \cong S/(J_{\tilde{G}}, M).$$

which is Gorenstein of dimension n. □

Lemma 3.1.8. *With the notations above we have*

$$I_L : g = (I_L, M) \quad \text{and} \quad \text{hence} \quad (I_L : g, J_{\tilde{G}}) = (M, J_{\tilde{G}}).$$

Proof. First we will show that $M \subseteq I_L : g$. Because $I_L \subseteq I_L : g$ it will be enough to prove that

$$(I_L, M)/I_L \subseteq I_L : g/I_L$$

in S/I_L. That is, we have always

$$x_iy_{i+1} \equiv x_{i+1}y_i \bmod I_L$$

for $i = 1, \dots, n-1$. Now let $x_2x_3 \cdots x_{n-1} \in M$. With $g = x_1y_n - x_ny_1$, we get

$$x_2x_3 \cdots x_{n-1}g \equiv x_1 \cdots x_{n-1}y_n - x_2 \cdots x_{n-1}x_ny_1 \bmod I_L.$$

Now put $x_{n-1}y_n \equiv x_ny_{n-1} \bmod I_L$ and $x_{n-2}y_{n-1} \equiv x_{n-1}y_{n-2} \bmod I_L$ and so on. After $(n-1)$ steps it follows that

$$x_2x_3 \cdots x_{n-1}g \equiv 0 \bmod I_L.$$

This proves that $x_2x_3 \cdots x_{n-1} \in I_L : g$. Similarly $y_2y_3 \cdots y_{n-1} \in I_L : g$. Now take any arbitrary element $x_2 \cdots x_iy_{i+1} \cdots y_{n-1} \in M$ then

$$x_2 \cdots x_iy_{i+1} \cdots y_{n-1}g \equiv y_1x_2 \cdots x_iy_{i+1} \cdots y_{n-1}x_n - x_1 \cdots x_iy_{i+1} \cdots y_n \bmod I_L.$$

We have $x_iy_{i+1} \equiv x_{i+1}y_i \bmod I_L$. Therefore

$$x_2 \cdots x_iy_{i+1} \cdots y_{n-1}g \equiv y_1x_2 \cdots x_iy_{i+1} \cdots y_{n-1}x_n - x_1 \cdots x_{i-2}(x_{i-1}y_i)(x_{i+1}y_{i+2})y_{i+3} \cdots y_n \bmod I_L.$$

Now replace $x_{i-1}y_i = x_iy_{i-1}$ *and* $x_{i+1}y_{i+2} = x_{i+2}y_{i+1}$ in S/I_L, we have

$$x_2 \cdots x_iy_{i+1} \cdots y_{n-1}g \equiv y_1x_2 \cdots x_iy_{i+1} \cdots y_{n-1}x_n - x_1 \cdots (x_{i-2}y_{i-1})x_iy_{i+1}(x_{i+2}y_{i+3}) \cdots y_n \bmod I_L.$$

If we continue such replacements, we get that

$$x_2 \cdots x_iy_{i+1} \cdots y_{n-1}g \equiv 0 \bmod I_L.$$

So $M \subseteq I_L : g$ hence

$$(M, J_{\tilde{G}}) \subseteq (I_L : g, J_{\tilde{G}}).$$

S/I_L is a Gorenstein ring and $S/I_L \twoheadrightarrow S/J_{\tilde{G}}$, therefore from Lemma 1.3.4

$$\omega(S/J_{\tilde{G}})(2) \cong \operatorname{Hom}_{S/I_L}(S/J_{\tilde{G}}, S/I_L) \cong (I_L : J_{\tilde{G}})/I_L.$$

Using Lemma 3.1.2 (d) we get

$$\omega(S/J_{\tilde{G}})(2) \cong (I_L : g)/I_L.$$

Now from Lemma 3.1.6 we have that

$$\omega(S/J_{\tilde{G}}) \cong (J_{\tilde{G}}, M)/J_{\tilde{G}}.$$

So there are two expressions for $\omega(S/J_{\tilde{G}})$. As the canonical module is unique up to isomorphism. We want to describe an isomorphism. To this end define a map

$$\phi : \ (J_{\tilde{G}}, M)/J_{\tilde{G}} \to (I_L : g)/I_L$$

which sends $\sum_{i=1}^{n-1} r_i m_i + J_{\tilde{G}}$ to $\sum_{i=1}^{n-1} r_i m_i + I_L$ where $m_1, \ldots, m_{n-1}$ are generators of M. If

$$\sum_{i=1}^{n-1} r_i m_i - \sum_{i=1}^{n-1} r_i' m_i \in J_{\tilde{G}}.$$

Then this implies

$$\sum_{i=1}^{n-1} (r_i - r_i') m_i \in J_{\tilde{G}} \cap M \subseteq J_{\tilde{G}} \cap (I_L, M) = I_L.$$

This follows because of the inclusion

$$I_L \subseteq J_{\tilde{G}} \cap (I_L, M) \subseteq J_{\tilde{G}} \cap (I_L : g) = I_L$$

as follows by Lemma 3.1.2 (b). Hence it's a well define map and clearly a homomorphism.

$$\phi \in \mathrm{Hom}_{S/J_{\tilde{G}}}(\omega(S/J_{\tilde{G}}), \omega(S/J_{\tilde{G}})) \cong S/J_{\tilde{G}} \text{ (see Lemma 1.3.4).}$$

So any homomorphism $\omega(S/J_{\tilde{G}}) \to \omega(S/J_{\tilde{G}})$ is given by multiplication by an element of $S/J_{\tilde{G}}$. Because ϕ is a non-zero homomorphism of degree zero it is in fact an isomorphism. That is

$$Im(\phi) = (I_L, M)/I_L.$$

So finally we get

$$I_L : g = (I_L, M) \;\; and \;\; hence \;\; (I_L : g, J_{\tilde{G}}) = (M, J_{\tilde{G}})$$

□

Lemma 3.1.9. *The Hilbert series of* $S/(I_L : g, J_{\tilde{G}})$ *is*

$$H(S/(I_L : g, J_{\tilde{G}}), t) = \frac{1 + (n-1)t - (n-1)t^{n-2} - t^{n-1}}{(1-t)^{n+1}}$$

Proof. It is known from [2] that

$$H(\omega(S/J_{\tilde{G}}), t) = (-1)^{n+1} H(S/J_{\tilde{G}}, \frac{1}{t}).$$

Therefore,

$$H(\omega(S/J_{\tilde{G}}), t) = \frac{(n-1)t^n + t^{n+1}}{(1-t)^{n+1}}.$$

As we know

$$\omega(S/J_{\tilde{G}}) \cong (M, J_{\tilde{G}})/J_{\tilde{G}}.$$

From the above formula the initial degree of $\omega(S/J_{\tilde{G}})$ is n while the initial degree of $(M, J_{\tilde{G}})/J_{\tilde{G}}$ is $n-2$. Therefore by dividing the non-zero divisor t^2 we get the Hilbert series of $(M, J_{\tilde{G}})/J_{\tilde{G}}$

$$H((M, J_{\tilde{G}})/J_{\tilde{G}}, t) = \frac{(n-1)t^{n-2} + t^{n-1}}{(1-t)^{n+1}}.$$

Consider the exact sequence

$$0 \to (M, J_{\tilde{G}})/J_{\tilde{G}} \to S/J_{\tilde{G}} \to S/(M, J_{\tilde{G}}) \to 0.$$

From Lemma 3.1.8

$$(I_L : g, J_{\tilde{G}}) = (M, J_{\tilde{G}}).$$

Therefore

$$H(S/(I_L : g, J_{\tilde{G}}), t) = H(S/J_{\tilde{G}}, t) - H((M, J_{\tilde{G}})/J_{\tilde{G}}, t).$$

Hence we get the required result. □

Theorem 3.1.10. *Hilbert series of S/I_C is*

$$H(S/I_C, t) = \frac{(1+t)^{n-1} - t^2(1+t)^{n-1} + (n-1)t^n + t^{n+1}}{(1-t)^{n+1}}.$$

In particular, the multiplicity of S/I_C is $e(S/I_C) = n$.

Proof. Hilbert series of binomial edge ideal of I_L is easy to compute because I_L is a complete intersection generated by $n-1$ forms of degree 2. Namely we have

$$H(S/I_L, t) = \frac{(1-t^2)^{n-1}}{(1-t)^{2n}} = \frac{(1+t)^{n-1}}{(1-t)^{n+1}}.$$

Consider the exact sequence

$$0 \to S/I_L \to S/I_L : g \oplus S/J_{\tilde{G}} \to S/(I_L : g, J_{\tilde{G}}) \to 0.$$

Therefore

$$H(S/I_L : g, t) = H(S/(I_L : g, J_{\tilde{G}}), t) + H(S/I_L, t) - H(S/J_{\tilde{G}}, t).$$

So by using Lemma 3.1.9 we get

$$H(S/I_L : g, t) = \frac{(1+t)^{n-1} - (n-1)t^{n-2} - t^{n-1}}{(1-t)^{n+1}}.$$

Consider the another exact sequence and replace $(I_L, g) = I_C$

$$0 \to S/I_L : g(-2) \to S/I_L \to S/I_C \to 0.$$

We have

$$H(S/I_C, t) = H(S/I_L, t) - t^2 H(S/I_L : g, t).$$

and after putting values we get the above series. □

3.1.2 Betti numbers of the binomial edge ideal of a cycle

To compute the Betti numbers of S/I_C we need to understand the modules $I_L : g/I_L$ and $S/I_L : g$. For the start we need the following lemma about the canonical module of $S/I_L : g$.

Lemma 3.1.11. *With the notation before we have,*

(a) $\omega(S/I_L : g) \cong J_{\tilde{G}}/I_L(-2)$.

(b) Minimal number of generators of $\omega(S/I_L : g)$ *is* $\binom{n-1}{2}$.

Proof. From the proof of the Lemma 3.1.8, we have $I_L : g/I_L \cong \omega(S/J_{\tilde{G}})(2)$. Now consider the exact sequence

$$0 \to \omega(S/J_{\tilde{G}})(2) \to S/I_L \to S/I_L : g \to 0.$$

All modules in above exact sequence is Cohen-Macaulay of dimension $n+1$. By applying local cohomology and dualizing it we get the following exact sequence

$$0 \to \omega(S/I_L : g) \to S/I_L(-2) \to S/J_{\tilde{G}}(-2) \to 0.$$

Which implies the isomorphism in (a) and then (a) gives us (b). $\square$

All Tor modules of $I_L : g/I_L$ is given in the following lemma.

Lemma 3.1.12. *We have the following isomorphisms.*

(a) $\operatorname{Tor}_i^S(K, I_L : g/I_L) \cong K^{c_i}(-n+2-i)$ *for* $i = 0, \ldots, n-2$, *where* $c_i = (n-1-i)\binom{n}{i}$.

(b) $\operatorname{Tor}_{n-1}^S(K, I_L : g/I_L) \cong K(-2n+2)$.

Proof. The resolution of $S/J_{\tilde{G}}$ is well known from Theorem 1.2.6.
Since $\operatorname{Ext}_S^{n-1}(S/J_{\tilde{G}}, S(-2n))(2) \cong \omega(S/J_{\tilde{G}})(2) \cong I_L : g/I_L$ therefore we have the above statement. $\square$

Now we will give the Theorem in which we compute all Tor modules of $S/I_L : g$.

Theorem 3.1.13. *With the previous notation we have*

(a) $\operatorname{Tor}_i^S(K, S/I_L : g) \cong K^{\binom{n-1}{i}}(-2i) \oplus K^{c_{i-1}}(-n+3-i)$ *for* $i = 1, \ldots, n-3$,

(b) $\operatorname{Tor}_{n-2}^S(K, S/I_L : g) \cong K^{c_{n-3}}(-2n+5)$,

(c) $\operatorname{Tor}_{n-1}^S(K, S/I_L : g) \cong K^{\binom{n-1}{2}}(-2n+4)$,

(d) $\operatorname{reg}(S/I_L : g) = n-3$.

Proof. Consider the exact sequence

$$0 \to I_L : g/I_L \to S/I_L \to S/I_L : g \to 0.$$

Let $i < n-2$, then the above exact sequence induces a graded homomorphism of degree zero

$$\operatorname{Tor}_i^S(K, I_L : g/I_L) \cong K^{c_i}(-n+2-i) \to \operatorname{Tor}_i^S(K, S/I_L) \cong K^{\binom{n-1}{i}}(-2i).$$

Therefore it is the zero homomorphism so we have the following isomorphism

$$\operatorname{Tor}_i^S(K, S/I_L : g) \cong \operatorname{Tor}_i^S(K, S/I_L) \oplus \operatorname{Tor}_{i-1}^S(K, I_L : g/I_L).$$

Let $i = n-1$, then we have $0 \to K(-2n+2) \to K(-2n+2)$ which is injection so we have the following exact sequence of K-vector spaces.

$$0 \to \operatorname{Tor}_{n-1}^S(K, S/I_L : g) \to \operatorname{Tor}_{n-2}^S(K, I_L : g/I_L) \cong K^{c_{n-2}}(-2n+4) \to$$
$$\operatorname{Tor}_{n-2}^S(K, S/I_L) \cong K^{\binom{n-1}{1}}(-2n+4) \to \operatorname{Tor}_{n-2}^S(K, S/I_L : g) \to \operatorname{Tor}_{n-3}^S(K, I_L : g/I_L) \to 0.$$

By Lemma 3.1.11, we have $\operatorname{Tor}_{n-1}^S(K, S/I_L : g) \cong K^{\binom{n-1}{2}}(-2n+4)$ and because of $c_{n-2} = \binom{n}{2}$ therefore $\operatorname{Tor}_{n-2}^S(K, S/I_L : g) \cong \operatorname{Tor}_{n-3}^S(K, I_L : g/I_L)$. □

Lemma 3.1.14. *The coefficient of the highest power* t^{n-1} *of the numerator of* $H(S/I_C, t)$ *is* $\binom{n-1}{2} - 1$.

Proof. If we expand $(1+t)^{n-1}$ in the numerator of $H(S/I_C, t)$ of Theorem 3.1.10. Last two terms in the numerator cancels and we get $1+(n-1)t+\cdots+(\binom{n-1}{2}-1)t^{n-1}$. $\square$

Now we are ready to say about all Tor modules of S/I_C.

Theorem 3.1.15. *With the previous notation we have*

(a) $\mathrm{Tor}_i^S(K, S/I_C) \cong K^{\binom{n}{i}}(-2i) \oplus K^{c_{i-2}}(-n+2-i)$ *for* $i = 1, \ldots, n-2$,

(b) $\mathrm{Tor}_{n-1}^S(K, S/I_C) \cong K^{c_{n-3}}(-2n+3)$,

(c) $\mathrm{Tor}_n^S(K, S/I_C) \cong K^{\binom{n-1}{2}-1}(-2n+2)$,

(d) $\mathrm{reg}(S/I_C) = n-2$.

Proof. Consider the exact sequence

$$0 \to S/I_L : g(-2) \to S/I_L \to S/I_C \to 0.$$

Let $i < n-1$, then the above exact sequence induces a graded homomorphism of degree zero

$$\mathrm{Tor}_i^S(K, S/I_L : g)(-2) \cong K^{\binom{n-1}{i}}(-2i-2) \oplus K^{c_{i-1}}(-n+1-i) \to \mathrm{Tor}_i^S(K, S/I_L) \cong K^{\binom{n-1}{i}}(-2i).$$

Therefore it is the zero homomorphism so we have the following isomorphism

$$\mathrm{Tor}_i^S(K, S/I_C) \cong \mathrm{Tor}_i^S(K, S/I_L) \oplus \mathrm{Tor}_{i-1}^S(K, S/I_L : g)(-2).$$

Let $i = n$, then we have the following exact sequence of K-vector spaces

$$0 \to \mathrm{Tor}_n^S(K, S/I_C) \to \mathrm{Tor}_{n-1}^S(K, S/I_L : g)(-2) \cong K^{\binom{n-1}{2}}(-2n+2) \to \mathrm{Tor}_{n-1}^S(K, S/I_L) \cong K(-2n+2) \to \mathrm{Tor}_{n-1}^S(K, S/I_C) \to \mathrm{Tor}_{n-2}^S(K, S/I_L : g)(-2) \cong K^{c_{n-3}}(-2n+3) \to 0.$$

It is clear from Lemma 3.1.14 that $\mathrm{Tor}_n^S(K, S/I_C) \cong K^{\binom{n-1}{2}-1}(-2n+2)$ which further implies that $\mathrm{Tor}_{n-1}^S(K, S/I_C) \cong K^{c_{n-3}}(-2n+3)$.

$\square$

As a final result on the binomial edge ideal of a cycle we describe the explicit values of Betti numbers in next theorem.

Theorem 3.1.16. *We have the following non zero Betti numbers for S/I_C on the diagonal of the Betti diagram*

$$\beta_{i,j} = \binom{n}{i}, \text{ if } i = j = 0, \ldots, n-3$$

and the last row of Betti diagram

$$\beta_{i,n-2} = \begin{cases} c_{i-2}, & \text{if } \; i = 2, \ldots, n-3 \; ; \\ \binom{n}{2} + c_{n-4}, & \text{if } \; i = n-2 \; ; \\ c_{n-3}, & \text{if } \; i = n-1 \; ; \\ \binom{n-1}{2} - 1, & \text{if } \; i = n. \end{cases}$$

Proof. It follows from Theorem 3.1.15. □

Chapter 4

A zoo of binomial edge ideals

In this chapter we study some new classes of graphs and its binomial edge ideals with their arithmetic properties. At the start we give a construction principle for sequentially Cohen-Macaulay module. We characterize all the trees which are approximately Cohen-Macaulay and then we give some examples of trees which are sequentially Cohen-Macaulay. We also introduced two new classes of graphs k-deficient and iso-complete graphs and study their sequentially Cohen-Macaulay property. At the end we give some applications and open problems.

4.1 Construction principle

In order to prove the sequentially Cohen-Macaulay property of binomial edge ideals we need the following construction principle. It will be useful also in different circumstances.

Lemma 4.1.1. *Let G be any connected graph on vertices set $[n]$ with at least one vertex of degree 1, choose one of them and label it by n. Let G' be a graph on vertices set $[n+1]$ by attaching one edge $\{n, n+1\}$ to the graph G. Now G is sequentially Cohen-Macaulay if and only if G' is sequentially Cohen-Macaulay.*

Proof. Let J_G and $J_{G'}$ denotes the binomial edge ideal of the corresponding graphs.

Let $\dim(S/J_G) = d$ and $\operatorname{depth}(S/J_G) = t$. Now $J_{G'} = (J_G, f) \subset S'$ where $f = x_n y_{n+1} - x_{n+1} y_n$ and $S' = S[x_{n+1}, y_{n+1}]$. We consider J_G as an ideal in S', therefore $\dim(S'/J_G) = d+2$ and $\operatorname{depth}(S'/J_G) = t+2$.

Now $n \notin T$ for all $T \subseteq [n]$ such that $c(T\backslash\{i\}) < c(T)$. Which implies $x_n, y_n \notin P_T(G)$ for all $P_T(G) \in \operatorname{Ass}(S/J_G)$, hence f is not a zero divisor in S'/J_G and is regular. Therefore $\dim(S'/J_{G'}) = d+1$ and $\operatorname{depth}(S'/J_{G'}) = t+1$.

Consider the exact sequence

$$0 \to S'/J_G(-2) \xrightarrow{f} S'/J_G \to S'/J_{G'} \to 0.$$

Apply $\operatorname{Hom}(S', .)$ to above sequence we get the long exact sequence

$$0 \to \omega^{d+2}(S'/J_G) \xrightarrow{f} \omega^{d+2}(S'/J_G)(2) \to \omega^{d+1}(S'/J_{G'}) \to \omega^{d+1}(S'/J_G) \xrightarrow{f} \omega^{d+1}(S'/J_G)(2)$$
$$\to \cdots \to \omega^{t+2}(S'/J_G) \xrightarrow{f} \omega^{t+2}(S'/J_G)(2) \to \omega^{t+1}(S'/J_{G'}) \to 0.$$

Now we have to show that f is $\omega^i(S'/J_G) - regular$ for all i, $t+1 \leq i \leq d+2$. Suppose contrary that $f \in P$ for some $P \in \operatorname{Ass}_{S'}(\omega^i(S'/J_G))$ with $t+1 \leq i \leq d+2$, which implies $\omega^i(S'/J_G) \neq 0$ then $\omega^i(S'/J_G)$ is i-dimensional Cohen-Macaulay module. It follows that $\operatorname{Ass}_{S'}(\omega^i(S'/J_G))_i = \operatorname{Ass}_{S'}(\omega^i(S'/J_G))$, because a Cohen-Macaulay module is unmixed. By Proposition 1.3.4 (b) it follows that $P \in (\operatorname{Ass}_{S'}(S'/J_G))_i$. But $f \in P$ this contradicts the regularity of f on S'/J_G. Hence f is $\omega^i(S'/J_G) - regular$ for all i, $t+1 \leq i \leq d+2$. Therefore multiplication by f is injective in above exact sequence and we get the following isomorphisms:

$$\omega^i(S'/J_{G'}) \cong \omega^{i+1}(S'/J_G)/f\omega^{i+1}(S'/J_G)(2) \text{ for all } i,\ t+1 \leq i \leq d+1.$$

Hence $\omega^{i+1}(S'/J_G)$ is either zero or Cohen-Macaulay of dimension $i+1$ if and only if $\omega^i(S'/J_{G'})$ is either zero or Cohen-Macaulay of dimension i. Therefore $S'/J_{G'}$ is sequentially Cohen-Macaulay if and only if S/J_G is sequentially Cohen-Macaulay. $\square$

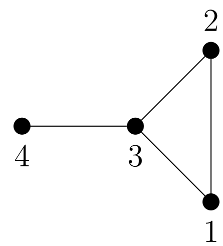


FIGURE (A)

Example 4.1.2. Consider the above graph shown in figure (A). Its binomial edge ideal is Cohen-Macaulay of dimension 5. Now if we add the edges inductively on vertex of degree 1 then it follows from the construction principle that such class of graphs with n vertices is Cohen-Macaulay of dimension $n+1$.

4.2 Characterization of approximately Cohen-Macaulay trees

In this section we will characterize the binomial edge ideal of all the trees that defines an approximately Cohen-Macaulay module. The simplest tree is the line. The binomial edge ideal of a line is a complete intersection. So it defines a Cohen-Macaulay module (see [8]). In our consideration we do not consider the line in detail. To classify all the trees that are approximately Cohen-Macaulay, we have to introduce a new terminology called 3-star like trees.

Definition 4.2.1. A tree G is called 3-star like if there are no vertices of degree ≥ 4 and if either there is at most one vertex of degree 3 or there is at most one edge with the property that both of its vertices are of degree 3. Then the line is of course 3-star like. Other types of 3-star like trees (see the figures 1 and 2).

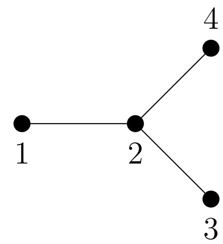


FIGURE 1

Example 4.2.2. Consider the simplest example of the 1st type of 3-star like tree as shown in figure 1. Its binomial edge ideal is approximately Cohen-Macaulay of dimension 6, depth 5 and the Hilbert function is

$$H(S/J_G, t) = \frac{1}{(1-t)^6}(1 + 2t - 2t^3).$$

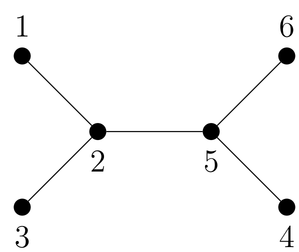


FIGURE 2

Example 4.2.3. The simplest example in case of 2nd type of 3-star like tree is shown above. Its binomial edge ideal is approximately Cohen-Macaulay of dimension 8, depth 7 and the Hilbert Series is

$$H(S/J_G, t) = \frac{1 + 4t + 5t^2 - 3t^4}{(1-t)^8}.$$

Theorem 4.2.4. *Let G be a tree on vertex set $[n]$, then S/J_G is approximately Cohen-Macaulay if and only if G is 3-star like.*

Proof. Suppose the contrary that G is not 3-star like then we have two cases:

Case 1 If G contains at least one vertex of degree $d \geq 4$ say i. Then $T = \{i\}$ and $c(T) = d$ so we have $\dim(S/P_T(G)) = n + d - 1 \geq n + 3$.

Case 2 If G has 2 vertices of degree 3 that are not adjacent say i and j. Then $T = \{i, j\}$ and $c(T) = 5$. Therefore we have $\dim(S/P_T(G)) = n + 3$.

Therefore in both cases $\dim(S/J_G) \geq n + 3$.

On the other hand $J_{\tilde{G}} \in \operatorname{Ass}(S/J_G)$ and $\dim(S/J_{\tilde{G}}) = n+1$ so $\operatorname{depth}(S/J_G) \leq n+1$, therefore S/J_G is not approximately Cohen-Macaulay.

Conversely, in order to prove that any 3-star like tree is approximately Cohen-Macaulay we will use induction on n. For the case of a line there is nothing to prove. For $n = 4$ and 6 it is true see example 4.2.2 and 4.2.3 above. The general case follows by the construction principle in Lemma 4.1.1. □

4.3 Some basic algebraic invariants of 3-star like trees

Now we will discuss some properties of the trees which are approximately Cohen-Macaulay. In Theorem 4.2.4 we have shown that 3-star like trees on vertices set $[n]$ have dimension $n + 2$ and depth $n + 1$. Now the Hilbert series of of 3-star like trees can be easily computed.

Lemma 4.3.1. *With the notations of Lemma 4.1.1, We have*

$$H(S'/J_{G'}, t) = (1 - t^2)H(S'/J_G, t).$$

Proof. Consider the exact sequence

$$0 \to S'/J_G(-2) \xrightarrow{f} S'/J_G \to S'/J_{G'} \to 0.$$

Hence we have a required result. □

Corollary 4.3.2. *Let G be a 3-star like tree on vertex set $[n]$. Then the Hilbert series is*

$$H(S/J_G,t) = \frac{(1+2t-2t^3)(1+t)^{n-4}}{(1-t)^{n+2}} \quad \textit{for } n>3$$
$$\textit{and} \quad \frac{(1+4t+5t^2-3t^4)(1+t)^{n-6}}{(1-t)^{n+2}} \quad \textit{for } n>5 \textit{ respectively}$$

for the first resp. the second type of 3*-starlike trees.*

Proof. Consider the first case of 3-starlike trees. We shall proof this by induction on n. For $n=4$ it is true, see example 4.2.2 resp. 4.2.3. Suppose the claim is true for n. That is,

$$H(S'/J_G,t) = \frac{(1+2t-2t^3)(1+t)^{n-4}}{(1-t)^{n+4}}.$$

Now by Lemma 2.6. we have

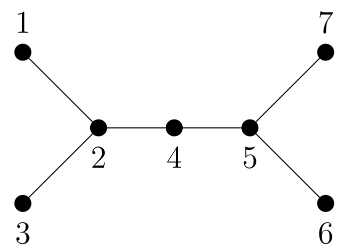
$$H(S'/J_{G'},t) = \frac{(1+2t-2t^3)(1+t)^{n-3}}{(1-t)^{n+3}}$$

as required.

Similar arguments might be used in order to calculate the Hilbert series in the second case of 3-starlike trees. □

4.4 Some examples of trees that are sequentially Cohen-Macaulay

Now we will give some examples of the classes of trees that are sequentially Cohen-Macaulay but not approximately Cohen-Macaulay.



Example 4.4.1. The graph shown above is sequentially Cohen-Macaulay of dimension 10 and depth 8. We check sequentially Cohen-Macaulay property by examining the Cohen-Macaulayness of its non vanishing Ext modules using CoCoA [3] in the view of Theorem 1.4.7. Now we can use Lemma 4.1.1 to conclude that the class of such trees having no vertex of degree 4 and exactly 2 vertices of degree 3 and both have one incident edge with common vertex are sequentially Cohen-Macaulay.

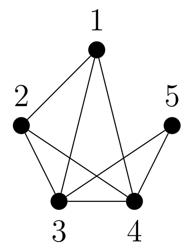
The other example of classes of trees of some interest is spider graphs.

Example 4.4.2. It is well known from the Corollary 2.2.6 that the star graph $K_{1,n}$ is sequentially Cohen-Macaulay, where $K_{m,n}$ denotes the complete bipartite graph. So the binomial edge ideal of spider graphs are sequentially Cohen-Macaulay follows from Lemma 4.1.1. Note that for $n > 3$ it is not approximately Cohen-Macaulay.

4.5 k-deficient graphs

Definition 4.5.1. Let k be the integer such that $1 \leq k \leq n-2$. A graph G is called k-deficient graph if we remove the edges $e_1, e_2, \ldots, e_k$ from the complete graph on $[n]$ where $e_i = \{n, i\}$ and $1 \leq i \leq k$.

For $n = 5$ and $k = 2$ we have the following graph.



For $n \leq 3$, k-deficient graph is a line whose binomial edge ideal is a complete intersection. So it defines a Cohen-Macaulay ring [8]. For our further consideration, we can assume that $n \geq 4$. Let J_G be the binomial edge ideal of k-deficient graph. We begin with the Lemma about some arithmetic invariants of J_G.

Lemma 4.5.2. *Let G denote the k-deficient graph. Let $\tilde{G}$ and $\tilde{G}_k$ be the complete graph on $[n]$ and $[k]$ respectively. Let $L = (J_{\tilde{G}_k}, x_{k+1}, \ldots, x_{n-1}, y_{k+1}, \ldots, y_{n-1})$. Then*

(a) $J_G = J_{\tilde{G}} \cap L$ *is the minimal primary decomposition of* J_G.

(b) $\dim S/J_G = n + 1$ *and* $0 \to S/J_G \to S/L \oplus S/J_{\tilde{G}} \to S/(L, J_{\tilde{G}}) \to 0$ *is an exact sequence of S-modules.*

(c) $\operatorname{depth} S/J_G = k + 3$

(d) The Hilbert series of S/J_G is $H(S/J_G, t) = \frac{1}{(1-t)^{n+1}}(1 + (n-1)t)$
$+\frac{1}{(1-t)^{k+3}}(1 + (k-1)t) - \frac{1}{(1-t)^{k+2}}(1 + kt)$.

Proof. (a) By Lemma 1.1.2 we have to find $T \neq \emptyset \subseteq [n]$ such that $c(T\backslash\{i\}) < c(T)$. If $T_0 = \{k+1, \ldots, n-1\}$ then $c(T_0) = 2$ and $c(T_0\backslash\{i\}) = 1$ for all $i \in T_0$. Let $T \neq T_0$ then if $T_0 \not\subseteq T$ or $n \in T$ then $c(T) = 1$ otherwise $c(T) = c(T\backslash\{i\}) = 2$ for $i \in T \setminus T_0$. If we remove T_0 from $[n]$ then graph $\tilde{G}_k$ and the vertex n is two disconnected components of G, hence we have above primary decomposition.

(b) The dimension and exact sequence is follows from the primary decomposition shown in (a).

(c) Clearly $S/L \cong S'/J_{\tilde{G}_k}$, where $S' = K[x_1, \dots, x_k, x_n, y_1, \dots, y_k, y_n]$ and hence it is Cohen-Macaulay ring of dimension $k+3$. Similarly $S/(L, J_{\tilde{G}})$ is complete graph on $k+1$ vertices therefore it is Cohen-Macaulay ring of dimension $k+2$. Hence depth $S/J_G = k+3$ follows from exact sequence (b) and the depth Lemma.

(d) It is well known that if G be the complete graph on l vertices then $H(S/J_G, t) = \frac{1}{(1-t)^{l+1}}(1+(l-1)t)$ (see Theorem 1.2.6). Therefore we have $H(S/J_{\tilde{G}}, t) = \frac{1}{(1-t)^{n+1}}(1+(n-1)t)$, $H(S/(L, J_{\tilde{G}}), t) = \frac{1}{(1-t)^{k+2}}(1+kt)$ and $H(S/L, t) = \frac{1}{(1-t)^{k+3}}(1+(k-1)t)$ hence from exact sequence (b) we have an above expression of Hilbert series.

□

It is clear from Lemma 4.5.2 that k-deficient graphs are Cohen-Macaulay if and only if $k = n-2$. For further consideration we can assume that $k < n-2$.

Remark 4.5.3. Note that the depth(S/J_G) is independent of n and if we put $k = 1$ then we have a class of graph on vertex set $[n]$ with depth$(S/J_G) = 4$ and it is the smallest possible depth which is known to us for large values of n.

Now we will describe all the modules of deficiencies $\omega^i(S/J_G)$, where G is k-deficient graph.

Theorem 4.5.4. *Let $k < n-2$. Let G is k-deficient graph on $[n]$. Then binomial edge ideal J_G has the following properties.*

(a) $\omega^i(S/J_G) = 0$ *if and only if* $i \notin \{k+3, n+1\}$.

(b) $\omega^{k+3}(S/J_G)$ *is a* $(k+3)$*-dimensional Cohen-Macaulay module and and there is an isomorphism* $\omega^{k+3}(\omega^{k+3}(S/J_G)) \cong (J_{\tilde{G}}, L)/L$.

(c) $\omega^{n+1}(S/J_G) \cong \omega^{n+1}(S/J_{\tilde{G}})$

(d) $\operatorname{reg} S/J_G = 2$.

Proof. After applying local cohomology to the short exact sequence of Lemma 4.5.2 (b) and knowing the Cohen-Macaulayness of the modules $S/J_{\tilde{G}}$, S/L and $S/(L, J_{\tilde{G}})$ we conclude that $H^i(S/J_G) = 0$ for all $i \neq k+3, n+1$. Moreover it induces a short exact sequence

$$0 \to H^{k+2}(S/(L, J_{\tilde{G}})) \to H^{k+3}(S/J_G) \to H^{k+3}(S/L) \to 0$$

and an isomorphism $H^{n+1}(S/J_G) \cong H^{n+1}(S/J_{\tilde{G}})$. The short exact sequence on local cohomology induces the following exact sequence

$$0 \to \omega^{k+3}(S/L) \to \omega^{k+3}(S/J_G) \to \omega^{k+2}(S/(L, J_{\tilde{G}})) \to 0$$

by local duality. Since both $\omega^{k+3}(S/L)$ and $\omega^{k+2}(S/(L, J_{\tilde{G}}))$ are Cohen-Macaulay modules of dimension $k+3$ and $k+2$ resp. Then $\operatorname{depth} \omega^{k+3}(S/J_G) \geq k+2$. By applying local cohomology and dualizing again it induces the following exact sequence (see Proposition 1.3.4(d))

$$0 \to \omega^{k+3}(\omega^{k+3}(S/J_G)) \to S/L \xrightarrow{f} S/(L, J_{\tilde{G}}) \to \omega^{k+2}(\omega^{k+3}(S/J_G)) \to 0.$$

The homomorphism f is induced by the commutative diagram

$$\begin{array}{ccc} S/L & \to & S/(L, J_{\tilde{G}}) \\ \downarrow & & \downarrow \\ \omega_{2\times}(S/L) & \to & \omega_{2\times}(S/(L, J_{\tilde{G}})). \end{array}$$

Note that the vertical maps are isomorphisms (see Proposition 1.3.4). Since the upper horizontal map is surjective the lower horizontal map is surjective too. Therefore $\omega^{k+2}(\omega^{k+3}(S/J_G)) = 0$. That is depth $\omega^{k+3}(S/J_G) = k+3$ and it is a Cohen-Macaulay module. Moreover $\omega^{k+3}(\omega^{k+3}(S/J_G)) \cong (J_{\tilde{G}}, L)/L$. Clearly reg $S/J_G = 2$ follows from (a), (b) and (c). □

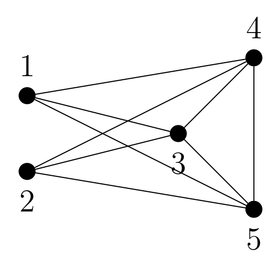
Corollary 4.5.5. *The binomial edge ideal of k-deficient graphs are sequentially Cohen-Macaulay.*

Proof. It follows from Theorem 1.4.7 and 4.5.4 because $\omega^{k+3}(S/J_G)$ is a $(k+3)$-dimensional Cohen-Macaulay module. □

4.6 Iso-complete graphs

Definition 4.6.1. The graph G on vertex set $[n+m]$ is said to be iso-complete graph if it is a join of the graph having n isolated vertices and the complete graph of m vertices.

For example $n = 2$ and $m = 3$ we have,



Let G be the iso-complete graph on $[n+m]$ vertices. For $n = 1$, it is a complete graph so we can assume $n \geq 2$. Following Lemma tell us some of its arithmetic invariants.

Lemma 4.6.2. *Let* $A = (x_{n+1}, \dots, x_{n+m}, y_{n+1}, \dots, y_{n+m})$. *Then* *(a)* $J_G = J_{\tilde{G}} \cap A$ *is the minimal primary decomposition of* J_G.

(b) $\dim S/J_G = \max\{n+m+1, 2n\}$ *and* $0 \to S/J_G \to S/A \oplus S/J_{\tilde{G}} \to S/(A, J_{\tilde{G}}) \to 0$ *is an exact sequence of* S*-modules.*

(c) $\operatorname{depth} S/J_G = n + 2$

(d) The Hilbert series of S/J_G *is* $H(S/J_G, t) = \frac{1}{(1-t)^{m+n+1}}(1 + (m + n - 1)t) + \frac{1}{(1-t)^{2n}} - \frac{1}{(1-t)^{n+1}}(1 + (n - 1)t)$.

Proof. (a) We have to find all $T \neq \emptyset \subseteq [n + m]$ such that $c(T\backslash\{i\}) < c(T)$. If $T = \{n + 1, \ldots, n + m\}$ then $c(T) = n > 1$ and $c(T\backslash\{i\}) = 1$ for all $i \in T$ and for any other T this condition is not satisfied. So we have above primary decomposition.

(b) The dimension and short exact sequence is an easy consequence of (a).

(c) The depth is follows from depth lemma and short exact sequence (b) as $S/J_{\tilde{G}}$, S/A and $S/(A, J_{\tilde{G}})$ are Cohen-Macaulay of dimension $n + m + 1$, $2n$ and $n + 1$ respectively.

(d) The Hilbert function is a consequence of short exact sequence (b) as $H(S/J_{\tilde{G}}, t) = \frac{1}{(1-t)^{n+m+1}}(1 + (n + m - 1)t)$, $H(S/(A, J_{\tilde{G}}), t) = \frac{1}{(1-t)^{k+1}}(1 + (k - 1)t)$ and $H(S/A, t) = \frac{1}{(1-t)^{2n}}$

□

Now we will describe all the modules of deficiencies $\omega^i(S/J_G)$ of the binomial edge ideal associated to iso-complete graph G. Note that for $n = 2$ iso-complete graph is 1-deficient graph and its modules of deficiencies are described in Theorem 4.5.4.

Similarly for $m = 1$ iso-complete graph is a star graph and its modules of deficiencies are given in Theorem 2.2.1. In further discussion we can assume that $n > 2$ and $m > 1$.

Theorem 4.6.3. *Let $n = m + 1$ and $n > 2$. Then:*

(a) $\omega^i(S/J_G) = 0$ *if and only if* $i \notin \{n+2, 2n\}$.

(b) $\omega^{n+2}(S/J_G) \cong \omega^{n+1}(S/(J_{\tilde{G}}, A))$.

(c) $\omega^{2n}(S/J_G) \cong \omega^{2n}(S/J_{\tilde{G}}) \oplus S/A(-2n)$

(d) $\operatorname{reg} S/J_G = 2$.

Proof. Apply local cohomology to the short exact sequence in Lemma 4.6.2 (b) provides the vanishing of $H^i(S/J_G) = 0$ for all $i \neq n+2, 2n$ and induces the homomorphisms $H^{n+2}(S/J_G) \cong H^{n+1}(S/(J_{\tilde{G}}, A))$ and $H^{2n}(S/J_G) \cong H^{2n}(S/J_{\tilde{G}}) \oplus H^{2n}(S/A)$. $\operatorname{reg} S/J_G = 2$ is follows from (a), (b) and (c). □

Theorem 4.6.4. *Let $n \neq m + 1$, $n > 2$ and $m > 1$. Then:*

(a) $\omega^i(S/J_G) = 0$ *if and only if* $i \notin \{n+2, 2n, n+m+1\}$.

(b) $\omega^{n+2}(S/J_G) \cong \omega^{n+1}(S/(J_{\tilde{G}}, A))$.

(c) $\omega^{2n}(S/J_G) \cong S/A(-2n)$

(d) $\omega^{n+m+1}(S/J_G) \cong \omega^{n+m+1}(S/J_{\tilde{G}})$

(e) $\operatorname{reg} S/J_G = 2$.

Proof. Applying local cohomology to the short exact sequence in Lemma 4.6.2 (b) and then use local duality to the induced homomorphisms gives the statement in (a), (b), (c) and (d). $\operatorname{reg} S/J_G = 2$ is a consequence of it. □

Corollary 4.6.5. *Let $J_G \subset S$ denote the binomial edge ideal associated to iso-complete graph.*

(a) S/J_G is a canonically Cohen-Macaulay and $\operatorname{depth} \omega^i(S/J_G) \geq i - 1$ *for all* $\operatorname{depth} S/J_G \leq i \leq \dim S/J_G$.

(b) S/J_G is sequentially Cohen-Macaulay if and only if $n = 2$ or $m = 1$.

Proof. Let $n = 2$ or $m = 1$, then the sequentially Cohen-Macaulay property is follows from Corollary 4.5.5 and 2.2.6 respectively. The converse and statement in (a) easily seen from Theorem 4.6.3 and 4.6.4. □

4.7 Applications and open problems

Recently K. Matsuda and S. Murai in [9] gives the relationship between the Betti numbers of the graph and the Betti numbers of its induced subgraph. The result is as follows:

Theorem 4.7.1. *For a subset $V \subset [n]$ let G_V be the induced subgraph of G on V. Then $\beta_{i,j}(S/J_G) \geq \beta_{i,j}(S/J_{G_V})$ for all i, j.*

They also gives the regularity bounds for the binomial edge ideals. As we prove in Theorem 3.1.15 that the regularity of a cycle graph of k vertices is $k - 2$ therefore we have the following result.

Corollary 4.7.2. *Let the graph G has a cycle of k vertices as its induced subgraph then* $\operatorname{reg}(S/J_G) \geq k - 2$.

The study of binomial edge ideals are recently introduced in combinatorial commutative algebra therefore this field have many open problems for algebraists. We give few of them below.

1. Is that all trees are sequentially Cohen-Macaulay?
2. What is smallest possible depth for the binomial edge ideals with large number of vertices? Is that possible that $\mathrm{depth}(S/J_G) = 3$ for the graph G with the number of vertices more then 10?
3. Describe all graphs G such that $\mathrm{reg}(S/J_G) = 2$ and which of them has pure resolution.
4. Find classes of graphs G for which S/J_G is unmixed, S_2 condition holds for S/J_G or it is connected in codimension 1.
5. Characterization of all graphs G for which $\dim(S/J_G) = \dim(S/J_{\tilde{G}})$ where $\tilde{G}$ is a complete graph.

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