OPINION AGGREGATION IN FUZZY FRAMEWORK AND INCOMPLETE PREFERENCES

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LAHORE PAKISTAN

2013

This work is submitted as a Thesis in the partial fulfilment of the requirement for degree of Doctor of Philosophy in Mathematics, to the Department of Mathematics, Lahore University of Management Sciences, Lahore, Pakistan.

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Abstract

Judgment Aggregation and Preference Aggregation are emerging research areas in many disciplines. Both the theories interrogate the consistency of the collective outcome produced by rational experts. The main idea is that there does not exist any method of aggregation which guarantees consistent collective choices and satisfies certain minimal conditions. This finding, which is now famous to be known as the Discursive Dilemma, is a generalization of classic Condorcet's paradox, discovered by the Marquis de Condorcet in the 18th century.

The early surveillance that enhanced the current development of the field goes back to some work in jurisprudence sparked by Lewis Kornhauser in 1992. The problem was reconstructed and developed further by Christian List, Pettit and Brennan as a more general problem of majority inconsistency. List and Petit proved the first social choice theoretics impossibility results similar to those of Arrow and Sen's impossibility theorem. This work was reinforced and extended in 2006 by several authors, beginning with Pauly and Van Hees and Dietrich. Some strong results on the theory of strategy-proof social choice were put forward by Nehring and Puppe in 2002 which later on helped produce important corollaries for the theory of judgment aggregation.

The work produced by Pigozzi in 2007 and Konieczny and Pino Perez in 2002 suggests that the theory of judgment aggregation and belief merging share similar objective. Moreover, judgment aggregation can be interconnected with probability aggregation as recommended by McConway in 1981 and Mongin 1995. Its linkage with abstract aggregation was highlighted by Wilson in 1975 and Rubinstein and Fishburn in 1986. But modern axiomatic social choice theory was founded by

Arrow. List and Pettit's work in 2004 and Dietrich and List's 2007 paper can be consulted for the understanding of relationship that exists between Arrovian preference aggregation and judgment aggregation.

In the parallel framework of Preference aggregation, experts are encouraged to provide complete and consistent preference relations. On the other hand, demanding a complete preference relation is an idealistic assumption which may not be probable in actuality. Incomplete preferences provided by experts were once discarded which lead to biased collective relations which did not represent choices of experts. To complete such a relation, it is imperative to consider consistency of the resultant completed relation. Literature proposes several methods for completing incomplete fuzzy preference relations and emphasises on their importance in decision making. Zai-Wu et al study a goal programming approach to complete intuitionistic fuzzy preference relations. Alonso et al give an estimation procedure for two tuple fuzzy linguistic preference relations. Two methods for estimating missing pairwise preference values given by Fedrizzi and Giove and Herrera et al are compared by Chiclana. Chiclana deduced that Fedrizzi's method to estimate missing values based on resolution of optimization is a special case of Herrera's estimation method based on known preference values.

Herrera proposed a method to estimate missing values in an incomplete fuzzy preference relation when $(n-1)$ preference values are provided by the expert. A more general condition which includes the case where a complete row or column is given. Estimated preference values that surpassed the unit interval were taken care of with a transformation function. However, consistency of the resultant relation is not assured. Moreover, this can void the originality of preference values provided by experts. Following the trend, this thesis focuses on solutions to inconsistent, indecisive and paradoxical outcomes in judgment aggregation and emphasises on methods to resolve incomplete preference relations provided by experts such that the resultant relations are also consistent.
This thesis is based on four research papers and it builds on the following questions:

- Can the problem of belief aggregation be molded into a framework where complete, consistent and non-paradoxical outcomes are attainable.

- How can Incomplete preference and multiplicative preference relations be completed into complete and additive consistent or Saaty's consistent relations. While ranking consistent relations, can we categorize some ranking methods that are equally efficient or better for these relations.

- Provided additive reciprocal relations, how far are the collective relations from consensus.

To interrogate the above mentioned problems, we have divided this dissertation into seven chapters. Chapter 1 is essentially an introduction aimed at recalling some basic definitions and facts where we fix notations and introduce terminologies to be used in the sequel.

Chapter 2 is concerned with background and literature review of the work in judgment and preference aggregation. The impossibility theorem is listed along with examples of Majority rule and Dictatorship rule to assert how aggregation rules fail to satisfy collective rationality along with other minimal conditions.

In the third chapter, we introduce belief aggregation in fuzzy framework. We propose a distance based approach and study how this structure helps in producing collectively rational outcomes without compromising on systematicity or anonymity. With the help of the illustrated method, the resultant outcomes are consistent and the solutions are free of ties.

Chapter 4 introduces an upper bound condition which ensures complete and consistent collective preference and multiplicative preference relations. The chapter proposes that if preference values provided by experts are "expressible" then the incomplete relation can be completed using consistency properties. The upper bound ensures that the resultant relation is complete with expressible values such that no value transgresses the unit interval and that the completed relation is consistent.

In chapter 5, our focus is on the relations completed in chapter 4. We term such relations as RCI preference and multiplicative preference relations and discuss performance of some ranking methods on complete RCI relations. It is highlighted in the chapter that complete RCI preference relations are additive transitive and complete RCI multiplicative preference relations satisfy Saaty's consistency. For the purpose of comparing ranking methods on these relations, Column wise addition

method is introduced and compared with the performance of Fuzzy borda rule and Shimura's method of ranking. For complete RCI multiplicative preference relations, Fuzzy borda rule for multiplicative preferences is defined and the mentioned procedure is recurred. A ranking method is confirmed to be better than the others if it produces lesser number of ties among alternatives.

Chapter 6 deals with additive reciprocal preference relations which are more general than additive consistent relations. Several preference relations are compiled using ordered weighted averaging operators to formulate different collective relations. In the absence of complete consensus, the metric of distance to consensus is employed to measure how far are the collective relations from consensus.

Chapter 7 concludes the dissertation and gives insight to some possible future work.

Acknowledgements

All praises and thanks are due to Allah, the Merciful. May peace and blessings be upon our leader Muhammad. The success of this study was ensured by the support of a lot of people, many of whom I would like to acknowledge in the following. Foremost, I would like to express my sincere gratitude to my supervisor Dr.Mian Muhammad Awais for the continuous support of my Ph.D research, for his patience, motivation, enthusiasm and encouragement. His guidance from the initial to final level of my research enabled me to develop an understanding of the subject.

Besides my supervisor, I offer my sincere thanks to Dr. Ismat Beg, with whom I started my research work. I am specifically thankful to Dr. Mujahid Abbas who taught me research tactics and instilled in me the ability to consistently work hard. I am indebted to my dear friends Hira Ilyas, Gabriella Pigozzi, Imran Abbas Jadoon, Surajit Borkotoky, Herrera Viedma, Arfa Zehram, Amy Muse Mahmood, Haris Manzoor and my friends from LUMS, Fareeha Khalid, Nabiha Asghar, Javad Ahmed Raheel, Syed Zulkfil, Rabia Ilyas, Ali Basit, Danish Khan, Asma Butt and Nouman Raza for their time, suggestions and moral support throughout these years. Special thanks to Gabriella Pigozzi, Nabiha Asghar and Huma Chaudhry for listening to my concerns and guiding me in life and in research. Nabiha, Gabriella and Huma have been a source of inspiration. In situations when hard work did not seem to be paying off and the process of this study seemed stagnant, they gave me hope and encouraged me to keep moving. Also, I am thankful to Herrera Viedma for providing feedbacks and suggestions for my work. I truly believe that without all of you, I may not have been able to complete this journey. I am glad to have known people who are excellent researchers and even better human beings.

I am in short of words to express my deepest gratitude to my parents for their immense love, support, encouragement, constant patience and prayers throughout my life; without them this effort would have no worth. I am indebted to my husband

Papers from this thesis

On the basis of this work the following research papers have been accepted/submitted.

1. I. Beg and A. Khalid, Belief Aggregation in Fuzzy framework, The Journal of Fuzzy Mathematics, 20(4) (2012), 911-924.

2. A. Khalid and M. M. Awais, Incomplete preference relations: An upper bound condition, Journal of Intelligent and Fuzzy systems, 26 (2014), 1433-1438.

3. A.Khalid and M. M. Awais, Comparing Ranking methods: Complete RCI preference and multiplicative preference relations, Journal of Intelligent and Fuzzy systems (accepted)

4. A.Khalid and M. M. Awais, Comparing distance to consensus of collective relations (submitted)

Contents

Chapter 1

Preliminaries

This chapter presents basic concepts and terminologies used through out this dissertation. A review of some results from the literature is given without proof to keep the chapter with reasonable length. Section 1.1 revisits some important operators in fuzzy set theory. Section 1.2 is focused on classical judgment aggregation and the concept of doctrinal paradox. Section 1.3 concerns on the notion of preference relations in binary two valued logic and introduces fuzzy preference relations along with the concept of consistency in preference relations. Also, ranking methods for preference relations are introduced in this section along with the impression of group consensus.

1.1   Basic Definitions And Operations In Fuzzy Set Theory

Since its inception in 1965, the theory of fuzzy sets has advanced in a variety of ways and in many disciplines. In set theory, to each individual in the universal set, a value of either 0 or 1 is assigned by the characteristic function of the crisp set under consideration. This results in discrimination of members and non members of the crisp set. The function is generalized by Lutfi Zadeh. Larger membership

values denote higher degrees of set membership and vice versa. In this case, each membership function maps elements of a given universal crisp set X, into the real numbers in $[0, 1]$; the most commonly used range of values of membership functions.

Definition 1.1.1. ([72]) A fuzzy set A in X is a function with domain X and values in $[0, 1]$.

Two distinct notations are commonly employed in the literature to denote membership functions. In one of them, the membership function, or the generalized characteristic function of a fuzzy set A in X is denoted by μ_A as

$$\mu_A : X \rightarrow [0, 1].$$

In the other one, the function is denoted as follows

$$A : X \rightarrow [0, 1].$$

Definition 1.1.2. ([72]) Let A and B be two fuzzy sets; The inclusion of A into B and the equality of A and B are defined as:

1. $A \subseteq B$ if and only if $A(x) \leq B(x)$; for all $x \in X$

2. $A = B$ if and only if $A(x) = B(x)$ for all $x \in X$

In this text, we use the second notation. That is, each fuzzy set and the associated membership function are denoted by the same capital letter.

Definition 1.1.3. A decreasing function $\eta : [0, 1] \rightarrow [0, 1]$ is called a fuzzy negation if $\eta(0) = 1$ and $\eta(1) = 0$. A fuzzy negation η is called

1. strict if it is strictly decreasing and continuous;

2. strong if it is an involution, that is, $\eta(\eta(x)) = x$ for all $x \in [0, 1]$.

Definition 1.1.4. ([72]) Complement of a fuzzy set A with respect to the universal set X is defined as

$$A^c(x) = 1 - A(x) \text{ for all } x \in X.$$

Definition 1.1.5. Union and intersection of two fuzzy sets A and B are defined respectively as follows:

$$(A \cup B)(x) = \max[A(x), B(x)]$$

and

$$(A \cap B)(x) = \min[A(x), B(x)]$$

Definition 1.1.6. (Triangular Norm) A triangular norm (t-norm) is a binary operation $\Delta : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying;

1. $1\Delta x = x$. (ordering property)

2. $x\Delta y = y\Delta x$. (commutativity)

3. $x\Delta(y\Delta z) = (x\Delta y)\Delta z$. (associativity)

4. If $w \leq x$ and $y \leq z$ then $w\Delta y \leq x\Delta z$. (monotonicity)

The max and min operations play a key role in the literature of fuzzy sets but these are not the only candidates as fuzzy extensions of the crisp disjunction and conjunction. Hoehle and Trillas propsed the use of t-norms and t-conorms in fuzzy set theory. Yager proposed his operations called Yager t-norms and t-conorms. Zimmermann and Zysno proved that any t-norm and any t-conorm can be used to model fuzzy intersection and union respectively.

Definition 1.1.7. (Fuzzy implication) Fuzzy Implication is a function $\zeta : [0, 1] \times [0, 1] \rightarrow [0, 1]$.

1. $\zeta(x, y) = \zeta(1 - y, 1 - x)$. (contraposition)

2. $\zeta(x, \zeta(y, z)) = \zeta(y, \zeta(x, z))$. (exchange property)

3. $x \leqslant y \Leftrightarrow \zeta(x, y) = 1$ for all $x, y \in [0, 1]$. (boundary condition)

4. $\zeta(1, x) = x$, for all $x \in [0, 1]$. (neutrality of truth)

5. ζ is continuous. (continuity)

Definition 1.1.8. (S-Implication) The fuzzy implication ζ defined by the formula

$$\zeta(a, b) = \nabla(\eta(a), b)$$

for all $a, b \in [0, 1]$ is known as an S-Implication

Definition 1.1.9. (R-Implication) The fuzzy implication ζ defined by the formula

$$\zeta(a, b) = \sup\{x \in [0, 1] : \Delta(a, x) \leq b\}$$

for all $a, b \in [0, 1]$ is known as an R-Implication.

Some important implications borrowed from ([44]) are listed below:

For all $x, y \in [0, 1]$:

1. $\zeta_b(x, y) = \max(1 - x, y)$. (Kleene-Dienes implication)

2. $\zeta_l(x, y) = \min(1 - x + y, 1)$. (Lukasiewicz implication)

3. $\zeta_*(x, y) = 1 - x + xy$. (Reichbach implication)

4. $\zeta_m(x, y) = \max(1 - x, \min(x, y))$. (Zadeh implication)

5. $\zeta_\bigstar(x, y) = \min(\max(1 - x, y), \max(x, 1 - x), \max(y, 1 - y))$.

6. $\zeta_g(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$ (Godel implication)

The class of R-implications is important in our work, we specifically use Lukasiewicz implicator in the first two chapters.

Lemma 1.1.1. *If Δ is left continuous and ζ is the associated R-Implication then the following hold for all $x, y, z \in [0, 1]$*

1. *$\Delta(x, y) \leq z$ if and only if $x \leq \zeta(y, z)$*

2. *$x \leq y$ if and only if $\zeta(x, y) = 1$*

3. *$\Delta(\zeta(x, y), \zeta(y, z)) \leq \zeta(x, z)$*

4. *$\zeta(1, y) = y$*

5. *$\Delta(x, \zeta(x, y)) \leq y$*

This lemma is due to Miyakoshi and Shimbo from 1985

Definition 1.1.10. (Linear Aggregation rule) An aggregation rule is linear if for every profile $(A_1, A_2, ..., A_n)$ and every proposition p in the agenda X, the collective belief on p is the weighted average of the individual opinions on it, i-e

$$A(p) = \omega_1 A_1(p) + \omega_2 A_2(p) + ... + \omega_n A_n(p),$$

where $\omega_i \geq 0$ and $\Sigma_{i=1}^n \omega_i = 1$.

Definition 1.1.11. (Implication Preservation) For all propositions p and q in X and all admissible profiles, if all individuals assign a value 1 to the fuzzy implication $\zeta(p, q)$, then so does the collective belief function representing the beliefs of the group of individuals.

The choice of fuzzy implication in a problem is context dependent. For implication preservation we will choose any implication from the class of R-implications.

Definition 1.1.12. (Elementary Properties of a fuzzy aggregation function)

An aggregation is a function F which assigns to each profile of individual belief sets, a collective belief set.

1. If 0 and 1 are the extremal values, then $F(0, ..., 0) = 0, F(1, ..., 1) = 1$.

   $F(a, a, ..., a) = a$ for all $a \in [0, 1]$. (Idempotence)

2. F is monotonically non-decreasing with respect to each argument if $a'_i > a_i$ implies that $F(a_1, ..., a'_i, ..., a_n) \geq F(a_1, ..., a_i, ..., a_n)$. (Monotonicity)

3. $\min\{a_i\} \leq F(a_1, ..., a_i, ..., a_n) \leq \max\{a_i\}$.(Compensativeness)

Note that property (3) follows from (1) and (2). After the emergence of fuzzy set theory ([72]), the simple task of looking at relations as fuzzy sets on the universe $X \times X$ was accomplished in a celebrated paper by Zadeh ([73]), he introduced the concept of fuzzy relation, defined the notion of equivalence, and gave the concept of fuzzy ordering and similarity measures ([75] [55]). Fuzzy relations have broader utility; compared with crisp relations, one of which is that fuzzy relations have greater expressive power and they are considered as softer models for expressing the strength of links between elements.

1.2 The Doctrinal Paradox

Social choice theory incorporates formal study of mechanisms for collective decision making. In real world problems, groups are not only required to come up with decisions, they are expected to be accompanied with sound justifications. Courts, for instance, have to provide reasons for their assertion of a defendant as liable or innocent. Hiring committees on the other hand, must explain the reason of hiring a particular candidate.

The emerging field of judgment aggregation is an interdisciplinary research area in which groups vote in favor of or against a certain decision called *conclusion*. Conclusion is a sentence which follows from a set of declarative sentences expressing an idea or concept of something which is true or false, known as *premises*. Along with the conclusion, groups are expected to provide reasons for their choice. Reasons, conclusion and the logical connections between them are given in the decision problem. Judgment aggregation studies aggregation from a logical perspective, and considers how multiple sets of logical formulae can be aggregated to a single consistent set. It was observed that majority voting fails to guarantee consistent collective outcomes even when the experts considered for making decisions are rational. This observation, known as *discursive dilemma*, generalizes *Condorcet's paradox of voting* which is highlighted in subsection 1.3. Before defining the formal model of judgment aggregation, a comprehensive review of some notions from binary logic is helpful.

Logical connectives or operators are symbols or words used to connect two or more sentences. For instance,

$$\wedge, \vee, \rightarrow, \leftrightarrow, \neg \text{ and } \models$$

represent conjunction, disjunction, conditional, bi-conditional, negation and logical entailment respectively.

If L is a set of propositions then $P(L)$ denotes the set of subsets of L. $\models$ is an entailment relation on $P(L) \times L$ between $A, B \in L$ and a proposition $p \in L$ as

1. If $p \in A$ then $A \models p$ (Self-entailment).

2. If $A \models p'$ for all $p' \in B$ and $B \models p$, then $A \models p$. (Transitivity)

Let X denote the set of propositions which is closed under negation, $E = \{e_1, e_2, .., e_n\}$ where $n \geq 1$ be the finite set of individual decision makers and $\phi_i, i \in \{1, 2, ..., n\}$

denote the set of propositions accepted by the ith expert. Here, to accept a proposition means to believe it to be true. In judgment aggregation, agents are required to express judgments in the form of yes/no or, equivalently, 1/0 over premises and conclusion.

The collection of the individual judgment sets $\{\phi_1, \phi_2, ..., \phi_n\}$ is known as a *profile*. Now, an aggregation rule F is a function that maps each profile of individual judgment sets to a collective judgment set ϕ which represents the choices of the group of experts. It is assumed that the decision maker is *rational* which means that the judgment set presented is *complete, consistent* and *deductively closed.*

Note that an individual judgment set ϕ_i is *complete* if for any proposition $p \in X$, either $p \in \phi_i$ or its negation $\neg p \in \phi_i$.

ϕ_i is *consistent* if there does not exist a proposition $p \in X$ such that both $p \in \phi_i$ and $\neg p \in \phi_i$.

ϕ_i is *deductively closed* if whenever $\phi_i \models p$ then $p \in \phi_i$. We say that S logically entails a proposition written $S \models \neg p$ is inconsistent. Usually, two rationality conditions are considered for judgment sets: consistency and completeness because together they imply List's deductive closure condition.

Consider for example an agenda $X = \{a, b, a \rightarrow b, \neg b\}$ then the judgment set $A = \Phi$ is consistent but it is not complete. Whereas, $B = \{a, b, a \rightarrow \neg b\}$ is inconsistent but complete and $C = \{a, b, a \rightarrow b\}$ is both consistent and complete. Note that it is deductively closed as well. *Condorcets paradox* or *voting paradox* explains how rational individual behavior begets irrational collective outcomes. More on Condorcet's paradox is elaborated in subsection 1.3. We explain the notion of Doctrinal paradox with the help of the following example.

Example 1.2.1. *Suppose three rational policy makers have to pass their judgment on the following propositions:*

P: *Poverty rate is low.*

Q: *Literacy rate is high.*

R: *Crime rate is low.*

height	P	Q	$P \wedge Q \leftrightarrow R$	R
Policymaker1	*Yes*	*Yes*	*Yes*	*Yes*
Policymaker2	*Yes*	*No*	*Yes*	*No*
Policymaker3	*No*	*Yes*	*Yes*	*No*
Majority	*Yes*	*Yes*	*Yes*	*Yes/No*

Table 1

Each policymaker assigns a binary truth value to the propositions P, Q and R. The paradox is precisely the fact that when majority vote is taken on the premises it gives a contradictory outcome as compared to when majority is calculated on the conclusion. This is called *Doctrinal paradox* which is a generalization of Condorcet paradox.

Note that the agenda in table 1 is $X = \{P, Q, R, \neg P, \neg Q, \neg R\}$ and the judgment sets according to the decision rule $P \wedge Q \leftrightarrow R$ are

$\phi_1 = \{P, Q, R\}$

$\phi_2 = \{P, \neg Q, \neg R\}$

$\phi_3 = \{\neg P, Q, \neg R\}$

It can be seen that the judges are rational, that is, their judgments are complete, consistent and deductively closed. But the conclusion $\phi = \{P, Q, \neg R\}$, which is the collective judgment set, is inconsistent. Since according to the decision rule $P \wedge Q \leftrightarrow R$, the propositions of ϕ cannot be simultaneously true. The first formal model of this problem of inconsistent majority judgments was given by List and Pettit ([50]).

1.3 Preference Relations

1.3.1 Social Choice Theory And Crisp Preference Relations

When a group needs to make a decision, the problem under consideration is to aggregate views of the individuals of that group into a single collective view that adequately reflects the will of the people. The successful completion of this task is the fundamental question of deep philosophical, economic, and political significance that, around the middle of 20th century, has given rise to the field of social choice theory. The key fact of this theory is that individual interactions can result in unexpected, possibly nonsensical, social outcomes.

Social choice theory offers many valuable theoretical insights and research strategies that can help us to comprehend with an improved insight as to why decision making procedures work as they do and how they might be changed for better. It amalgamates the preferences of the many into a social ranking of alternatives ([40]). Voter's paradox or the Condorcet's paradox is the most basic example of how rational individual behavior begets irrational collective decisions. Black in ([11]) pointed out that the paradox was recognized at least as early as 1785 by the Marquis de Condorcet but its implications were not fully understood until the foundations of this theory was laid by Kenneth Arrow and Duncan Black until 1950s.

Consider the set $\{i \in N : xP_iy\}$ to represent the set of all individuals from N who prefer x to y. A *strict preference binary relation* is denoted by P_i. So xP_iy means that x is better than y. In Arrow's original presentation, the preferences or tastes of the voters are represented by *weak preference relations* denoted by R_i. xR_iy means that x is as good as or better than y. Weak preference relation is introduced in literature as $R \subseteq X \times X$ where $\times$ represents the cartesian product of the non empty set X. The comparison of two alternatives x and y, is also represented as the order

pair $(x, y) \in R_i$ to indicate xR_iy. So, the preference relation R_i is thought of as a set containing ordered pairs. Unlike strict preference binary relation, this allows a person to be indifferent between two alternatives. A *complete binary relation* is one in which if there are two alternatives $x, y \in X$ then either xR_iy or yR_ix. A strict preference relation is not complete and it does not allow x to be preferred to itself. Following are some basic properties concerning binary relations.

R_i is reflexive if and only if for all $x \in X, xR_ix$.

R_i is irreflexive if and only if for all $x \in X, xR_ix$ does not hold.

R_i is complete if and only if for all $x, y \in X$, either xR_iy or yR_ix.

R_i is transitive if and only if for all $x, y, z \in X$, if xR_iy and yR_iz then xR_iz.

R_i is symmetric if and only if for all $x, y \in X$, if xR_iy then yR_ix. The formal definition of a preference relation is given as follows.

Definition 1.3.1. A preference relation R is a subset of $X \times X$ which satisfies the two rationality properties: completeness and transitivity.

A preference relation R_i is *cyclic* if and only if its induced strict preference relation P_i is cyclic. That is, there exists a chain of objects $x, ..., z$ such that $xP_i...P_izP_ix$. Otherwise R_i is *acyclic*. We now proceed to discuss the Voter's paradox with the help of the following example in which a committee of voters are considering a finite set of alternatives for election. Each voter is assumed to rank the alternatives according to his preferences in a strict linear order, that is, a complete, irreflexive, transitive relation on the set of alternatives.

Example 1.3.1. *Suppose three voters are choosing by Majority rule or Condorcet rule among three brands of candy to be served at a child's birthday party. Voters are referred to by the numbers 1,2 and 3 for the purpose of notation. The candy brands are Dotch, Candi and Pops. "Candi P_1 Pops" represents that voter 1 prefers Candi*

to Pops. The preference of each voter can be summarized as follows.

Voter 1: Candi P_1 Pops P_1 Dotch

Voter 2: Dotch P_2 Candi P_2 Pops

Voter 3: Pops P_3 Dotch P_3 Candi

Candi wins with the support of voters 1 and 3 in the contest between Candi and Pops brands. Dotch defeats Candi with the support of voters 2 and 3. The situation seems settled until Voter 2 observes that Pops defeats Dotch with the support of Voter 1 and 3.

This voting cycle is an apparent violation of the principle of transitivity, according to which if alternative 1 is preferred over alternative 2 and alternative 2 is preferred over 3 then alternative 1 should be preferred over alternative 3. The voter's paradox is famous by the name *cyclical majority*, coined by Reverend C. L. Dodgson in 1876 to refer to a situation in which no alternative is beatable.

1.3.2 Arrow's Paradox

Arrow's theorem was popularized in 1951 from the book "Social Choice and Individual Values" by Kenneth Arrow. The original paper was titled "A Difficulty in the Concept of Social Welfare" ([1]). This work became the starting point of the modern social choice theory.

Let $\mathbf{R}$ be the set of all weak preference relations on X. This includes all possible orderings of the alternatives. Let $R^n = (R_1, R_2, ..., R_n)$ denote the collection of individual preference relations. Therefore $R^n \in \mathbf{R}^n$. Arrow discussed a social welfare function which is now famous as *social preference function*, a function F that gives back a social preference relation, an element $R \in \mathbf{R}$, for each preference profile $R^n \in \mathbf{R}$. Formally, $F : \mathbf{R}^n \to \mathbf{R}$. The modern version of Arrow's theorem deals with the following four properties of social choice

Universal Domain: For each possible social preference profile R^n, the social preference function prescribes a weak social preference relation. More succinctly, $F :$ $\mathbf{R}^n \to R$.

Pareto Efficiency: If xP_iy for all $i \in N$, then xPy.

Nondictatorship: There is no dictator, a person j such that xP_jy implies xPy.

Independence from Irrelevant alternatives: The social preference relation between x and y is unaffected by changes in the position of an irrelevant alternative, z in the preference profile.

Theorem 1.3.2. *(Arrow's Theorem) If there are three or more alternatives in X, there is no social preference function that satisfies Universal Domain, Pareto Efficiency, Nondictatorship and Independence from Irrelevant alternatives.*

Arrows paradox states that no rank order voting system can convert individual ranks over alternatives into a community-wide ranking while also satisfying certain specified criteria. For instance the Majority rule in example 1.3.1 is acyclic and therefore does not satisfy transitivity.

1.3.3 Preferences And Judgments

Literature on Judgment aggregation has been influenced by earlier work in social choice theory ([47]). In 1976, Robinson defined Judgement as an opinion which is critical and is based on an assessment of a standard of comparison. On the other hand, preference was defined as an opinion which relates to a personal affinity based on experience and knowledge.

The doctrinal paradox or discursive dilemma shows that proposition-wise majority voting conducted by more than three individuals on interconnected propositions may lead to collective judgments on these propositions which are inconsistent.

This paradox illustrates a more general impossibility result which exhibits the non existence of aggregation function that produces consistent collective judgments and satisfy minimal conditions.

The question that whether the new impossibility theorem is a special case of Arrows theorem, or whether there exists absolutely no analogy between the two results was evident to sprout. Although there are differences between judgment aggregation and preference aggregation but the recent results are extensions of persisting results in social choice theory. The discursive dilemma in judgment aggregation resembles Condorcet's paradox of cyclical majority preferences and the various results on judgment aggregation resemble Arrow's theorem in preference aggregation which has lead to the conclusion that judgment aggregation is more generalized. Dietich and List ([26]) suggest that preference aggregation is a special case of judgment aggregation. They prove their claim by constructing an embedding of preference aggregation into judgment aggregation. They illustrate the generality of the judgment aggregation by identifying the analogue of Arrow's theorem in judgment aggregation. They highlight the logical structure underlying Arrow's result.

List et. al ([51]) compared the two theorems and showed that they are not straightforward corollaries of each other. It was further suggested that while the framework of preference aggregation can be mapped into the framework of judgment aggregation, there exists no obvious reverse mapping. They highlighted that the independence condition was one particular minimal condition used in both theorems and that this condition points towards a unifying property underlying both impossibility results.

Since our research is build on the two fields and not on the theory of their inter connection, we do not dig into details of persisting models that study how preference aggregation subsumes into the theory of judgment aggregation.

1.4 Non-Crisp Preference Relations

In certain situations the preference relation is provided with additional information expressing the degree of plausibility of the preferences. This is called a fuzzy preference relation. In a preference relation an expert associates to each pair of alternatives a real number that reflects the preference degree, or the ratio of preference intensity, of the first alternative over, or to that of, the second one. To associate preference values to judgments, two selection models are profoundly used: The fuzzy model, The multiplicative model.

Definition 1.4.1. ([64][33][65][64]) *Fuzzy model:* In this case, preferences are represented by a fuzzy preference relation P. A fuzzy preference on the set of alternatives $X = \{x_1, x_2, ..., x_n\}$ is characterized by a function $\mu_p : X \times X \to [0, 1]$, where $\mu(x_i, x_j) = p_{ij}, i, j \in \{1, 2, ..., n\}, i \neq j$ indicates the preference intensity or the degree of confidence with which alternative x_i is preferred over x_j .

$p_{ij} = \frac{1}{2}$ indicates indifference between the alternatives x_i and x_j.

$p_{ij} = 0$ indicates that alternative x_j is absolutely preferred to x_i and $p_{ij} = 1$ indicates that alternative x_i is absolutely preferred to x_j.

This implies that the scale to use in the fuzzy model is the closed interval $[0, 1]$.

Definition 1.4.2. ([60] [61] [22]) Let $A \subset X \times X$ denote a multiplicative preference relation, the intensity of preference, a_{ij} is measured using a ratio scale, particularly, a $1 - 9$ scale.

$a_{ij} = 1$ indicates indifference between x_i and x_j.

$a_{ij} = 9$ indicates that x_i is absolutely preferred to x_j.

A is multiplicative reciprocal if $a_{ij}.a_{ji} = 1 \forall i, j.$

A reciprocal multiplicative preference relation is consistent if it satisfies Saaty's consistency. That is, if $a_{ij}.a_{jk} = a_{ik} \forall i, j, k.$ where $i, j, k \in \{1, 2, ..., 9\}, i \neq j \neq k.$

Therefore, a crisp preference relation is a special case of a fuzzy preference relation. In order to extend the properties of a crisp preference relation to a fuzzy case, fuzzy relations are defined from a De morgan triple $< \Delta, \nabla, \eta >$ where Δ is a t-norm, ∇ is a t-conorm and η is a fuzzy negation such that $\nabla(x, y) = \eta(\Delta(\eta x, \eta y))$. Some of the properties of a fuzzy preference relation are defined as follows.

P is reflexive if and only if for all $x \in X, P(x, x) = 1$.

P is complete if and only if for all $x, y \in X, \max(P(x, y), P(y, x)) = 1$.

P is transitive if and only if for all $x, y, z \in X, \min(P(x, y), P(y, z)) \leq P(x, z)$.

P is symmetric if and only if for all $x, y \in X, P(x, y) = P(y, x)$.

P is antisymmetric if and only if $P(x, y) + P(y, x) \geq 1$.

It is easy to see that individual preferences can be portrayed as fuzzy relations. We step forward to discuss transitivity properties in fuzzy preference relations and suggest why the notion is important for preference relations in order to give meaning to collective decision making.

1.4.1 Consistency In Preference Relations

Fuzzy preference relation is a representation of the preferences provided by decision makers. Consistency is a prerequisite for rational decision making. If preferences of experts are not consistent, then their aggregation and hence results based on these aggregations will fail to portray choices of the experts in the group that is under consideration.

In a crisp preference relation, the concept of consistency has traditionally been defined in terms of acyclicty. Traditionally, transitivity is used to characterize consistency in fuzzy context. Some of the transitivity properties are as given as follows.

Definition 1.4.3. (Moderate Transitivity) $p_{ik} \geq 0.5, p_{kj} \geq 0.5$ then $p_{ij} \geq \min(p_{ik}, p_{kj})$

Definition 1.4.4. (Strict Transitivity) $p_{ik} \geq 0.5, p_{kj} \geq 0.5$ then $p_{ij} \geq \max(p_{ik}, p_{kj})$

Definition 1.4.5. (Max-min Transitivity) $p_{ik} \geq \min(p_{ij}, p_{jk}) \forall i, j, k$. This has been the traditional requirement to characterize consistency in the case of fuzzy preference relation.

Definition 1.4.6. (Max-max Transitivity) $p_{ik} \geq \max(p_{ij}, p_{jk}) \forall i, j, k$.

Definition 1.4.7. (Triangle Condition) $p_{ij} + p_{jk} \geq p_{ik} \forall i, j, k$.

Definition 1.4.8. $(\Delta - Transitivity) p_{ij} \geq \Delta(p_{ik}, p_{kj}) \forall i, j, k$

Definition 1.4.9. (Additive Transitivity) If $p_{ij} = p_{ik} + p_{kj} - 0.5 \forall i, j, k$.

A consistent fuzzy preference relation should at least satisfy restricted max-max transitivity. Some of the other transitivity properties are max-min transitivity, restricted max-max transitivity and additive transitivity, also discussed by Tanino et. al in ([64] [65]) and ([67]). More generalized transitivity properties based on t-norms were established by Chiclana et.al in ([17]).

1.4.2   Ranking Methods

A ranking method is a function assigning a crisp partial or complete ranking $\preceq$ on X to any fuzzy relation P. A binary relation is partial ranking if it is reflexive and transitive. Several ranking methods are based on scoring functions $S(x, y)$ which assign a real number to each alternative and these methods rank them according to their score. S is a function that assigns to each object y, a numerical score $S(y)$. We say that y is preferred over x if the score of the latter exceeds that of the prior. That is,

$$x \preceq y \text{ if and only if } S(x, y) \leq S(y, x)$$

When there are two alternatives to choose from, the method of simple Majority rule seems to be the most natural and commonly used social choice function. But for

more than two alternatives there is no natural extension of simple majority rule as pointed out two centuries earlier by Marquis De Condorcet. Therefore a variety of rules are used in decision making when there are three or more alternatives involved. These include the following methods: Plurality, Borda, Condorcet, exhaustive voting and double election to name a few. See Black ([8]) for an overview of these methods. Many methods have been envisafed to rank alternatives. With regards to some necessary properties, we can compare the ranking methods. In this section, we give in the form of definition, Shimura's rule of ranking alternatives. Other ranking methods that are used in the study are discussed in chapter 5.

Definition 1.4.10. ([63] [66]) Let x and y be variables defined on universe X. A pairwise function defined as $f_y(x)$ is the membership value of x with respect to y and another membership function $f_x(y)$ is the membership value of y with respect to x. The relativity function given by $f(x|y) = \frac{f_y(x)}{\max[f_y(x), f_x(y)]}$ is a measurement of the membership value of choosing x over y. A general case for n variables is stated as $f(x_i|A') = f(x_i|\{x_1, ..., x_{i-1}, x_{i+1}, ..., x_n\})$ which is the fuzzy measurement of choosing x_i over all alternatives in A' where $A' = \{x_1, ..., x_{i-1}, x_{i+1}, .., x_n\}$. This is used to form an $n \times n$ comparison matrix C, which is the matrix of relativity values with diagonal entries as 1, since relativity function gives $f(x_i|x_i) = 1$. To determine overall ranking, smallest value in each of the rows is found as $C'_i = \min(f(x_i|X)), i = \{1, 2, ..., n\}$ where C'_i is the membership ranking value of the ith variable. Then these variables are ordered from best to worst where $\max(C(X'_i))$ is the most preferred alternative.

As mentioned earlier, consistency is considered as a prerequisite to rational decision making. Consistency issues are also discussed by Zhang et al in ([79]). Once the preference relations representing individual choices are consistent, it is important to rank the alternatives to see the final outcome. More on ranking preference relations

is discussed in chapter 5.

1.4.3 Collective Relations And Consensus

In real world, decision making process usually takes place in an environment where goals, consequences of particular actions and possibilities are not known. Fuzzy set theory as opposed to probability theory allows a flexible framework to work with because it deals with fuzziness of human judgments both quantitatively and qualitatively. In decision making problems it is assumed that m decision makers provide preferences on a set $X = \{1, 2, ..., n\}$ of alternatives where m and n are finite numbers.

Typically, the goal of decision making is to reach consensus. Consensus is generally understood as a unanimous agreement by all experts in the group concerning their choice. The goal of consensus is not the selection of several options but to develop one decision that suits the interests of the entire group under consideration. Despite the simplicity in the definition of the notion of consensus, it is another matter altogether to quantify and attain it.

"There is nothing which is not the subject of debate, and in which men of learning are not of contrary opinions. The most trivial questions escapes not our controversy, and in the most momentous we are not able to give any certain decision. David Hume (1740). " Although consensus is an idealistic situation but it is difficult to attain in practice. We give a review of Ordered weighted averaging (OWA) operators and mention a few commonly used ones that are important for our work. These operators will help us formulate a collective preference relation which represents preferences of the experts involved in the group under consideration. Once a collective preference relation is formulated, we can check if the entire group is at consensus and if not, how far is the collective group from consensus.

Definition 1.4.11. ([70]) A mapping $\mathtt{F} : \mathtt{R}^n \to \mathtt{R}$ with weighting vector $\omega = (\omega_1, \omega_2, ..., \omega_n)$ such that $\omega_i \in [0, 1], 1 \leq i \leq n$ and $\sum_1^n \omega_i = 1$ is an Ordered weighted averaging (OWA) operator of dimension n. Furthermore, $\mathtt{F}(a_1, ..., a_n) = \sum_{j=1}^n \omega_j b_j$ where b_j is the jth largest element of the bag $\langle a_1, a_2, ..., a_n \rangle$.

OWA operator defined is a mean operator that is bounded, monotonic, symmetric and idempotent. Following is a review of some notable OWA aggregations that are needed in this thesis.

Maximum: The weighting vector is $\omega = (1, 0, ..., 0)^T$ in this case and

$$Maximum(a_1, a_2, ..., a_n) = \max\{a_1, a_2, ..., a_n\}.$$

Minimum: The weighting vector is $\omega = (0, 0, ..., 1)$ and

$$Minimum(a_1, a_2, ..., a_n) = \min\{a_1, a_2, ..., a_n\}.$$

Average: The weighting vector is $\omega = (\frac{1}{n}, ..., \frac{1}{n})$ and

$$F(a_1, a_2, ..., a_n) = \frac{a_1 + a_2 + ... + a_n}{n}.$$

Window type OWA operator A window type OWA operator takes average of m arguments about the center. So the weighting vector for this class of operators is as follows.

$$\omega_i = \begin{cases} 0 & i < k \\ 1/m & k \leq i < k + m \\ 0 & i \geq k + m \end{cases}$$

Once we obtain collective relations using OWA operators, we can find distance to consensus with the help of Average certainty and Average fuzziness of those relations.

Definition 1.4.12. ([13]) Two common measures of preference in a relation are Average certainty $\tilde{\zeta}(P)$ and Average fuzziness $\bar{F}(P)$ defined respectively as follows.

$$\tilde{\zeta}(P) = \frac{tr(P)(P^T)}{n(n-1)/2}$$

and

$$\bar{F}(P) = \frac{tr(P^2)}{n(n-1)/2}$$

where $tr()$ and $()^T$ represents the trace and transpose of the fuzzy preference relation.

The measure $\zeta(\tilde{P})$ averages the individual assertiveness of each distinct pair of rankings such that each term maximises the measure when $p_{ij} = 1$ and $\rho_{ij} = 0$ and minimizes the measure when $p_{ij} = p_{ji} = 0.5$. Hence, $\tilde{\zeta}(P)$ is proportional to the overall certainty in P. Consequently, $\bar{F}(P)$ averages the joint preference in P over all distinct pairs in the cartesian space $X \times X$. Each term minimizes the measure when $p_{ij} = p_{ji} = 0.5$ and maximizes the measure when $p_{ij} = 1$ and $p_{ji} = 0$. Therefore, $\bar{F}(P)$ is proportional to the fuzziness or uncertainty about pairwise rankings exhibited by the fuzzy preference relation P.

The two measures are dependent and

$$\bar{F}(P) + \tilde{\zeta}(P) = 1$$

Moreover, the ranges of the two measures are

$$0.5 \leq \tilde{\zeta}(P) \leq 1$$

and

$$0 \leq \bar{F}(P) \leq 0.5$$

These measures are important because they play a role in finding distance to consensus of the collective relations.

Definition 1.4.13. Distance to consensus metric is dependent on average Certainty of a relation and it is defined as $\varpi(P) = 1 - (2(\tilde{\zeta}(P) - 1)^{\frac{1}{2}}$.

We use this metric to see how collective relations exhibit difference in distance to consensus as compared to one another.

Chapter 2

Background/Literature Review

2.1 Judgment Aggregation

In this section we give a brief review of the literature work and define the problem that is tackled in our paper in ([10]). Two escape routes were profoundly used to avoid Doctrinal paradox namely, the *premise based procedure* ([29]) and the *conclusion based procedure* ([58]). Premise based procedure suggests to take the majority vote on premises only and deduce the conclusion. But then we have to decide which propositions should be the premises. Moreover, majority voting on the premises may give two divergent results depending on the choice of premises.

Pigozzi et al ([58]) proposed that in many decision problems the conclusion is more relevant than the reasons for it. According to them, when hiring a candidate for instance, one is more concerned of which new colleague is to join the department than of the reasons for choosing her. They also propose that considering only the individual judgments on the conclusions has also the advantage that unlike the premise based procedure, conclusion based procedure is a strategy-proof procedure. But then, on the other hand, in democratic societies, people have the right to question the process of the decision making process.

Using classical propositional logic, List and Pettit ([50]) formalized judgment aggregation and proved the first social choice theoretic impossibility result similar

to those of Arrow ([2]) and Sen's ([62]) impossibility theorems. Subsequently, several impossibility theorems were proved concluding that there is no non-dictatorial aggregation function that satisfies certain minimal conditions simultaneously.

The minimal conditions are

1. *Universal Domain:* A judgment aggregation function F should accept as input, any logically possible n-tuple profile of judgments, such as each judgment set ϕ_i satisfies conditions of completeness, consistency and deductive closure.

2. *Anonymity:* The collective judgment set ϕ, yielded by F, should be invariant under permutation of the individual judges. That is, for any two profiles $(\phi_1, \phi_2, ..., \phi_n)$ and $(\phi_1', \phi_2', ..., \phi_n')$ in the domain of F, $F(\phi_1, \phi_2, ..., \phi_n) = F(\phi_1', \phi_2', ..., \phi_n')$. This means that no individual's judgment should be given preference in determining collective judgment.

3. *Systematicity:* For any two propositions p and q in X, if every individual makes exactly the same judgment on p as is made on q, then the collective judgment on p should be the same as the collective judgment on q. This means that the proposition should be handled in an even handed way. That is, for any two profiles $(\phi_1, \phi_2, ..., \phi_n)$ and $(\phi_1', \phi_2', ..., \phi_n')$ in the domain of F and any two propositions p and q in X, $[p \in \phi_i \leftrightarrow q \in \phi_i] \rightarrow [p \in F(\phi_1, \phi_2, ..., \phi_n) \leftrightarrow q \in F(\phi_1', \phi_2', ..., \phi_n')]$.

4. *Collective Rationality:* For any profile $(\phi_1, \phi_2, ..., \phi_n)$ in the domain of F, $F(\phi_1, \phi_2, ..., \phi_n)$ is consistent and complete collective judgment on X.

It is noticed that Majority rule satisfies Universal Domain, Anonymity, Systematicity but not collective rationality.

A famous aggregation function which gives consistent collective outcome is *dictatorship* defined as

$$F(\phi_1, \phi_2, ..., \phi_n) = \phi_j$$

for a fixed individual $j \in \{1, 2, ..., n\}$.

Note that dictatorship satisfies Universal Domain, Systematicity and Collective Rationality but it does not satisfy Anonymity. Another example of aggregation function is that of *inverse dictatorship* which is defined as the negation of the individual judgment set of some specific individual. That is, $F(\phi_1, \phi_2, ..., \phi_n) = \neg \phi_j$ for a fixed j. This aggregation rule satisfies Universal Domain and Systematicity but Anonymity is not satisfied and Collective Rationality may or may not be satisfied.

Pigozzi in ([56]) criticized both premise based and conclusion based procedure and suggested an argument based approach. She imported methods from belief merging into judgment aggregation to avoid paradoxical outcomes. Note that belief merging and judgment aggregation share similar objective and their collaboration has been proved fruitful. Belief merging defines a class of operators that produce collective belief from individual and possibly conflicting belief bases. It was pointed out that discursive dilemma disappears with the recognition of the fact that aggregating logically consistent individual judgments does not guarantee a consistent collective outcome and that additional constraints need to be imposed in order to rule out infeasible group judgments.

Using distance based approach in belief merging framework, Pigozzi attempted to highlight all possible interpretations which had the least distance from a profile. That is, the interpretations that best represented the choices of the individuals. It is also mentioned that in the absence of a paradox, Pigozzi's method gives the same result as proposition wise majority voting. The difference between the two procedures is that unlike proposition wise majority voting, the merging operator excludes the inconsistent collective judgments with the help of *integrity constraints* and defines a ranking on all the allowed collective judgments.

This model relaxes the requirement of completeness of belief bases to cater for the possibility of policy makers being indifferent, with respect to a preference, or ignorant, of a certain matter. However, this method leads to indecision in cases where more than one interpretations qualify to have the least distance from the profile. Even in cases where collective rationality is achieved, it is at the expense of systematicity; since certain propositions known as integrity constraints are given preference over other propositions in the agenda.

The crux of the work done in classical judgment aggregation represents the fact that Doctrinal paradox persists to exist and in the covet of attaining a collectively rational outcome, systematicity or anonymity has to be sacrificed. We solve this problem in chapter 3, where the framework of judgment aggregation in classical two valued is generalized to a fuzzy framework and accordingly, policy makers are not bound to completely agree or disagree with every proposition proposed in the agenda. In chapter 3, we explore how belief aggregation in the fuzzy framework can be molded into an optimization problem which helps avoid paradoxical outcomes without the fear of indecision. We further illustrate that depending on the choice of t-norm and fuzzy implication, we can find aggregation functions that produce collectively rational outcome without compromising on systematicity.

2.2 Incomplete Fuzzy Preference Relations

Decision Making is a routine activity and most decision making processes are based on preference relations. Without consistency, aggregation schemes in decision making misrepresent choices of the individuals. In such cases, aggregated preferences are not the best route to follow for collective group of experts. In case of fuzzy

preferences and multiplicative fuzzy preferences, transitivity is a traditional requirement to characterize Saaty's consistency ([60][61]). A consistent fuzzy preference relation should at least satisfy restricted max-max transitivity. Some of the other transitivity properties are max-min transitivity, restricted max-max transitivity and additive transitivity, also discussed in ([65][64]) and ([67]).

Initially it was assumed that the preference relations provided by the decision makers are complete. However, it is not reasonable to expect every decision maker to be certain about the degree of intensity of each alternative over others. An expert may be ambiguous about the problem at hand or may not have sufficient knowledge to discriminate the degree to which some alternatives are better than others. Under such circumstances, naturally, incomplete preference relations are provided by experts.

A partial function $f : X \to Y$ does not map every element in the set X onto an element in the non empty set Y.([36]). The incomplete fuzzy preference relation P and incomplete multiplicative fuzzy preference relation A on X is a fuzzy set on the product set $X \times X$ that is characterized by a partial membership function.

It is certain that discarding incomplete preference relations provided by experts, tend to biased or misrepresentative collective relations. Literature proposes several methods for completing incomplete fuzzy preference relations. Zai-Wu et al ([78]) study a goal programming approach to complete intuitionistic fuzzy preference relations (IFPR) whose equivalent matrices are formulated to avoid the operational difficulty caused by complex operation laws in IFPR. They assume that each decision maker provides weight information to obtain priority vectors. Alonso et al ([6]) give an estimation procedure for two tuple fuzzy linguistic preference relations. They give a transformation function to define additive consistency for such preference relations. Two methods for estimating missing pairwise preference values given

by Fedrizzi and Giove ([32]) and Herrea et al ([36]) are compared by Chiclana et al in ([20]). Chiclana deduced that Fedrizzi's method to estimate missing values based on resolution of optimization is a special case of Herrera's estimation method based on known preference values.

Herrera proposed a method in ([37]) to estimate missing values in an incomplete fuzzy preference relation when $(n-1)$ preference values $\{p_{12}, p_{23}, ..., p_{(n-1),n}\}$ are provided by the expert. A more general condition which includes the case where a complete row or column is given, is provided in ([36]). Estimated preference values that surpassed the unit interval were taken care of with a transformation function defined by Herrera in ([36][37]). These transformation functions result in a complete preference relation with preference values inside the interval $[0, 1]$ but the consistency of the resultant relation is not assured. Moreover, this can void the originality of preference values given by the experts.

We restrict ourselves to the study of incomplete fuzzy preference and multiplicative fuzzy preference relations. It needs to be noticed that although the proposed methods are successful in completing the incomplete fuzzy preference and multiplicative fuzzy preference relations, some of these methods use transformation functions to take care of the surpassed preference values but they are silent about consistency of the resultant completed relation. To bring meaning to the resultant matrices, they need to satisfy some criteria of transitivity because lack of consistency may lead to meaningless solutions. To solve this problem, we propose upper bound conditions in chapter 4 to deal with incomplete fuzzy preference and multiplicative fuzzy preference relations. Additive consistency and Saaty's consistency, along with these upper bound conditions ensure that the missing preference intensities do not surpass their respective codomain. So construction of translation functions, which may void originality of preference values provided by experts, is not required. More importantly,

the fuzzy preference and multiplicative fuzzy preference relations completed using the proposed method are transitive.

2.3   Ranking Of Completed Relations

The upper bound condition in chapter 4 implicity assumes a row or column being provided by an expert to built on. We will refer to such an incomplete preference relation as Row or Column Incomplete preference relations or Row or Column Incomplete multiplicative preference relations. Hereafter, we denote the incomplete relations that satisfy the properties in ([42]) as **RCI preference relations** and **RCI multiplicative preference relations**.

Having dealt with incompleteness of preference relations, we talk about their ranking. It is assumed in the thesis that RCI preference and multiplicative preference relations, provided by the experts, are to be ranked by first completing them using the methods defined in ([42]). This implies that the preference relations to be ranked are additive transitive. Also, they satisfy Saaty's consistency in case of multiplicative preference relations.

It has been noticed that the methods proposed for ranking preference and comparison relations ([12] [66] [63] [5]) in literature may produce different or contradictory outcomes when applied to the same preference relation. We want to investigate which methods are better to rank RCI complete preference and multiplicative preference relations. The answer to this question, naturally holds true for additive transitive and Saaty's multiplicative preference relations.

In the quest of finding the best ranking method for such relations, a performance parameter is required. We describe the performance parameter to be the number of ties produces by a ranking method. Therefore, a method which produces the least

number of ties is the best ranking method for such relations. Finding best ranking methods for RCI complete preference and multiplicative preference relations is the same as finding ranking methods that are equally good to rank these relations. Also, we wish to find the reasons that lead to ties in alternatives in the process of ranking. This question is tackled in chapter 5.

2.4   The Idea Of Consensus

A review of consensus models has been presented by Wade in ([69]) in which complexities of several distance based models are highlighted. Vania ([57] [67]) uses composition of fuzzy relations to aggregate preference relations into a collective one. Concept of fuzzy majority using a linguistic quantifier to aggregate fuzzy preference relation is used by Tanino, Karcprzyk and Chiclana in ([16] [41] [64]).

Several different forms of consensus; Type 1 consensus, Type 2 consensus and Type fuzzy consensus are detailed by Bezdek in ([13]). Type 1 consensus is defined as a consensus in which there is one clear choice. In type 2 consensus there is one clear choice but the remaining (n-1) preferences have definite secondary preference. The third type of consensus is the Type fuzzy consensus in which there is one clear unanimous choice but the remaining (n-1) preferences have infinitely many secondary fuzzy preferences. It is realistic to think about situations where collective groups do not reach a unanimous agreement. Under such circumstances, a distance metric that measures the distance of the collective group from consensus can be measured on a relative scale of extremities of no consensus and complete consensus situations.

A collective relation is more desirable if its distance to consensus on the relative scale is lower. Consensus models depend on collective relations. OWA operators

have actively participated in aggregating preference relations. Fuller ([31]) presents a survey of OWA operators and illustrates their applicability with the help of a real life example. An overview of aggregation operators along with their advantages and disadvantages is given by Gabrisch ([35]). Vitri et al ([68]) use individual centrality approach to achieve consensus.

In chapter 6, we study collective relations obtained using special OWA operators and compare distance to consensus of the resultant relations. There are two assumptions throughout this paper, firstly, the fuzzy preference relations provided by decision makers are complete or complete-able ([42]) and secondly, that they are additive reciprocal ([19]). In short, the input provided by expert are preference relations that must be reciprocal, they may or may not be additive consistent. Since every additive transitive relation is additive reciprocal but the converse is not always true. In section 6.1, we use special ordered weighted averaging operators to formulate collective preference relations $\wp^*, \wp^{\min}, \wp^{\max}, \overline{\wp}$ and $\overline{\overline{\wp}}$. Since the collective relations may not depict complete consensus, we use the distance to consensus metric to compare how far these collective relations are from consensus. Furthermore, an upper and lower bound of distance to consensus on the relevant scale of no consensus (0.0) to complete consensus (1.0), of these collective preferences relations is defined in this section.

Chapter 3

Belief Aggregation In Fuzzy Framework

As mentioned in section 2.1, paradoxical outcomes cannot be avoided in classical propositional logic without compromising on the minimal conditions. The belief aggregation framework that has been opted to avoid paradoxes in literature leads to ties and indecisive outcomes. In this chapter we answer the question raised in section 2.1 and introduce belief aggregation in the fuzzy framework which helps attain consistent and tie-free outcomes without compromising on systematicity or anonymity. Also, in the fuzzy framework, the condition of odd number of decision makers is no longer required.

3.1   Degrees Of Belief

In the framework of classical two-valued logic, individuals are restricted to opt for a Yes or a No, even when they do not completely agree or disagree with a proposition representing a particular idea. The restriction of classical propositional calculus to a two-valued logic has created many paradoxes. For example, the Optimist's conclusion (is the glass half-full or half-empty when the volume is at 500 milliliters). Is the liter-full glass still full if we remove one millimeter of water, 2, 3 or hundred

milliliters? Unfortunately no single milliliter of liquid provides for a transition between full and empty glass. This transition is gradual so that as each milliliter of water is removed, the truth value of the glass being full gradually diminishes from a value 1 at 1000 milliliters to 0 at 0 milliliters.

In most decision making problems propositions representing certain situations are vague. A fuzzy proposition is a statement involving some concept without clearly defined boundaries; statements that tend to express ideas that can be interpreted differently by various individuals.

In this chapter, using a fuzzy logic framework, individuals give degree of truthfulness on crisp or fuzzy propositions. This degree of truthfulness is said to be the belief of the decision maker. Van Hees ([38]) specified several generalizations of the paradox using multivalued logic. He further showed that allowing degrees of belief using multivalued logic does not ensure collective rationality. Duddy and Piggins ([30]) presented a general model in which judgments on propositions were not binary but came in degrees. Triangular norms were used to define totally blocked agendas along with some results. Dietrich and List ([28]) presented a general theory of propositional attitude aggregation and proved two theorems which are of considerable importance. Non-binary beliefs however were represented by probability functions.

We assume that decision makers are rational and they have the freedom to express their opinions on a proposition with which they do not completely agree or disagree. So any number in the interval $[0, 1]$ that best represents their opinions can be opted; 0 representing complete disagreement and 1 representing full agreement with the notion expressed by the proposition.

To answer the problem proposed in section 2.1, we present certain conditions under which allowing degrees of truth values to individuals using fuzzy framework ensures

collective rationality without having to compromise on systematicity or anonymity. We illustrate how the paradox is avoided by molding the problem into an optimization problem where a *unique* optimal fuzzy solution is achieved using a distance based approach in the fuzzy framework. We use the notion of implication preservation and explain how using a specific class of t-norms and fuzzy implications, collective rationality is achieved without violating systematicity or anonymity.

3.2   Preliminaries and reformulation of the problem

Let $N = \{1, 2, ..., n\}$ denote a finite set of individual decision makers where $n \geq 1$. Let ϕ be a finite set of atomic propositions p, q etc. The set of all propositions ϕ_0 is obtained by closing ϕ under the fuzzy connective t-norm (Δ) and fuzzy negation (η). Thus $\phi \subseteq \phi_0$ and $\forall\, p,\, q \in \phi : \eta(p) \in \phi$, $p \Delta q \in \phi$. Let X be a non-empty subset of ϕ_0 and contains all propositions about which the decision makers have to make a decision.

A fuzzy global valuation is a function $v^* : \phi_0 \to [0, 1]$ that satisfies the following conditions:

1. $v^*(\eta(\phi)) = \eta(v^*(\phi)) = 1 - v^*(\phi)$

2. $v^*(\phi \Delta \psi) = \Delta(v^*(\phi), v^*(\psi))$

Let V^* represent the set of all fuzzy global valuations. A valuation is a function $v : X \to [0, 1]$ for which there is some $v^* : \phi_0 \to [0, 1]$ in V^* such that the restriction of v^* to X is v. Let V represent the set of all valuations. A fuzzy aggregation function $A : V^N \to V$ returns for each n-tuple of valuations $(v_1, v_2, ..., v_n)$ an

aggregated valuation $A(v_1, v_2, ..., v_n)$. A decision method D is a function from the set $[0,1]^n$ to the set $[0,1]$.

Definition 3.2.1. (Minimal Agenda Richness) The agenda X contains at least two distinct propositions p, q as well as $p \Delta q$ and $\eta(p \Delta q)$.

Definition 3.2.2. (Anonymity) For any permutation $f : N \to N$, any valuation profile $(v_1, v_2, ..., v_n) \in V^n$ and any proposition $p \in X$,

$$A(v_1, v_2, ..., v_n)(p) = A(v_{f(1)}, v_{f(2)}, ..., v_{f(n)})(p).$$

Definition 3.2.3. (Systematicity) A fuzzy aggregation function $A : V^n \to V$ is systematic if and only if there is a decision method $D : [0,1]^n \to [0,1]$ such that for all $(v_1, v_2, ..., v_n) \in V^n$ and for all $p \in X$,

$$A(v_1, v_2, ..., v_n)(p) = D(v_1(p), v_2(p), ..., v_n(p)).$$

Definition 3.2.4. (Majority voting) Given an agenda X which is closed under negation, an aggregation function F is a majority aggregation function if it assigns to a profile of individual judgment sets $K = \{A_1, A_2, ..., A_n\}$, $(n \geq 3)$, a collective judgment set A which contains propositions from the agenda that are accepted by at least half of the members.

$$F(A_1, A_2, ..., A_n) = \{p : |p \in A_k \text{ where } k \geq \frac{n}{2}|\}.$$

Definition 3.2.5. (Dictatorship) There is some fixed $i \in N$ such that for all $(v_1, v_2, ..., v_n) \in V^n$,

$$A(v_1, v_2, ..., v_n)(p) = v_i(p).$$

As depicted by table 1 of section 1.2, policymaker assigns a binary truth value to the propositions P, Q and R. The paradox is precisely the fact that when majority

vote is taken on the premises it gives a contradictory outcome as compared to when majority is calculated on the conclusion. It is assumed that the decision makers are rational, that is, their judgment sets are complete and consistent. As mentioned in section 3.1, in some decision problems, propositions are "vague" and can have truth values between true and false. Let us reformulate the entire problem in fuzzy logic based framework where individuals can take on values between 0 and 1.

We replace "∧" by t-norm Δ and implication " → " by fuzzy implication $\Rightarrow$. The choice of the fuzzy connectives used in a problem are context dependent. Moreover, if an implication is formed with the help of both t-norms and t-conorms, they are not randomly selected, in fact, they are dual of each other with respect to the fuzzy negation ([54]).

Note that corresponding to each proposition there is a fuzzy set. When for instance a decision maker assigns a truth value to the proposition Q in the example 1.2.1, she basically has a fuzzy set $\acute{A}$ of "countries with high literacy rate" in her mind and she expresses the degree of membership of this specific country in the fuzzy set $\acute{A}$. Degree of truth of proposition $Q : x \in \acute{A}$ is equal to the membership grade of x in the fuzzy set $\acute{A}$. Whenever a decision maker is assigning truth values to a proposition, the above argument will be an underlying assumption but it will not be mentioned.

3.3   Distance Based Approach: The Formal Model

Given a finite set of n individuals (not necessarily odd) and a finite set X of propositions over which individuals have to express their beliefs, a *belief set* of an individual i denoted by A_i is a function $A_i : X \to [0, 1]$. A *profile* $K = (A_1, A_2, ..., A_n)$ is an n-tuple of individual belief sets. An aggregation is a function F that maps to each

profile a collective belief set A. That is,

$$F(A_1, A_2, ..., A_n) = A$$

where $A_i(p)$ is the truth value of a proposition p in the belief set of individual i.

The formal model consists of a propositional language L built from a finite set of propositions P and the fuzzy connectives $(\eta, \Delta, \nabla, \Rightarrow, \Leftrightarrow)$ namely negation, t-norm, t-conorm, fuzzy implication and fuzzy bi-implication respectively ([54]).

Belief set whose elements are the integrity constraints is denoted by IC. The ICs are propositions that should be satisfied by the merged base. These constraints are dependent on the choice of t-norm and implication in our case. ψ maps K and IC into a collective belief set denoted $\psi_{IC}(K)$. This belief set is a result of merging the individual belief sets into one which best represents the beliefs of the individuals.

An interpretation is a function $w : P \rightarrow [0, 1]$. Let W denote the set of all interpretations. A distance between interpretations is a real valued function $d : W \times W \rightarrow R$ such that for all $w, w', w'' \in W$:

1. $d(w, w') \geq 0$.

2. $d(w, w') = 0$ if and only if $w = w'$.

3. $d(w, w') = d(w', w)$.

4. $d(w, w'') \leq d(w, w') + d(w', w'')$.

We choose Euclidean metric as our distance function

$$d^*(w, w') = (\sum_{x \in P} |w(x) - w'(x)|^2)^{\frac{1}{2}}.$$

Euclidean metric helps us find a distance between possibly conflicting interpretations. Our goal is to find an interpretation $w \in W$ which has the least distance from

the profile of belief sets K. This interpretation is required to satisfy the integrity constraint which varies according to our choice of t-norm and implication. The belief merging operator used to find the distance between an interpretation and a profile is defined as

$$D^d(w, A^*) = \Sigma_{i \in N} d(w, A_i).$$

Distance based approach in the fuzzy framework guarantees an outcome without the fear of ending up with indecision. We concede that a wide variety of distance measures exist. Also, the choice of fuzzy connectives to be used in a specific problem are context dependent. For the sake of illustration, we opt for Lukasiewicz t-norm and Lukasiewicz implication in our next example.

We can formulate any aggregation problem as an optimization problem where constraints would vary according to the choice of fuzzy connectives employed. Let w be any arbitrary interpretation. In this case $w(p) = (\theta_1, \theta_2, ..., \theta_{|P|})$ where $|P|$ is the cardinality of P. The optimization problem can now be stated as follows:

Minimize $D^d(w, A^*)$ subject to the fuzzy IC.

Here $w(p) = (\theta_1, \theta_2, ..., \theta_{|P|})$

$\min A_i(j) \leq \theta_j \leq \max A_i(j)$ for $i \in \{1, 2, ..., n\}, j \in \{1, 2, ..., |P|\}$ and $\theta_j \in [0, 1]$. The above optimization problem help us find a unique *optimal* fuzzy aggregation function. An optimal aggregation function produces collective belief set that is closest to the individual belief sets.

Example 3.3.1. *In this example, policy makers express their belief on the same propositions as in table 1.*

	P	Q	R	$P \, \Delta Q \Rightarrow R$
Policymaker1	*0.5*	*0.6*	*0.4*	*1*
Policymaker2	*0.3*	*0.7*	*0.2*	*1*
Policymaker3	*0.8*	*0.4*	*0.3*	*1*
Collective Decision	θ_1	θ_2	θ_3	*1*

Table 2

We choose Lukasiewicz t-norm and implication which is defined as: $\Delta(x, y) = \max(0, x + y - 1)$ and $\zeta(x, y) = \min(1, 1 - x + y)$.

Fuzzy integrity constraint is $(\pi(P)\Delta\pi(Q) \to \pi(R))$ which by the choice of Lukasiewicz t-norm and implication is translated as $\theta_3 \geq \max(0, \theta_1 + \theta_2 - 1)$. Finding collective social choice function in table 2 now becomes an optimization problem. This problem is framed in Matlab. The optimal fuzzy aggregation function gives the solution for table 2 as

$$(\theta_1, \theta_2, \theta_3) = (0.4667979, 0.555718, 0.369934)$$

(for details see appendix 1).

Framing the problem of belief merging into an optimization problem works equally well in cases where voters are allowed to express their beliefs on the implication, that is, when implication is a part of the agenda and hence policy makers can assign truth values to it. For instance, in the next example, we let P, Q and $P \Rightarrow Q$ to be the propositions in the agenda on which committee members are required to express their beliefs.

Example 3.3.2. *Implication chosen in this example is Zadeh's implication defined as* $\zeta(x, y) = \max[1 - x, \min(x, y)]$.

	P	Q	$P \Rightarrow Q$
Policymaker 1	*0.6*	*0.5*	*0.5*
Policymaker 2	*0.8*	*0.9*	*0.8*
Policymaker 3	*0.2*	*0.9*	*0.8*
Collective Decision	θ_1	θ_2	θ_3

Table 3

The fuzzy integrity constraint $\pi(P)\Delta\pi(Q) \to \pi(R)$ is translated as $\theta_3 = \max[1 - \theta_1, \min(\theta_2, \theta_3)]$. The optimal collective outcome obtained is $(\theta_1, \theta_2, \theta_3) = (0.35, 0.65, 0.65)$. (For details see appendix 1). This procedure of tackling the belief aggregation problem as an optimization problem guarantees a unique collective outcome in fuzzy framework but at the expense of systematicity.

3.4 Abiding By The Decision Rule

We now restrict ourselves to the problem where decision rule is not a part of the agenda. Our interest is in the problems where implication is to be preserved. Which means that individuals have to abide by the decision rule and are not allowed to express their beliefs on it. For instance, policy makers in example 3.3.1 did not express their beliefs on the decision rule but it was to be followed by them. Molding the problem into an optimization problem helped us achieve a unique outcome but at the expense of systematicity. We now focus on the class of continuous t-norms and R implications which help us define aggregation rules that satisfy collective rationality without having to compromise with systematicity.

Example 3.4.1. *Let F be an aggregation function such that for every proposition $p \in X$,*

$$F(A_1, A_2, ..., A_n)(p) = \omega_1 A_1(p) + \omega_2 A_2(p) + ... + \omega_n A_n(p).$$

Given that individuals have non-binary beliefs and that fuzzy connectives may be employed according to the nature of the problem, using context dependent R-implications guarantee that the linear aggregation function defined above would produce a collective belief set such that our aggregation rule satisfies universal domain, systematicity, anonymity and collective rationality.

	P	Q	$P\Delta Q \Rightarrow R$	R		
Policymaker 1	p_1	q_1	1	r_1		
Policymaker 2	p_2	q_2	1	r_2		
.	.	.	1	.		
Policymaker i	p_i	q_i	1	r_i		
.	.	.	1	.		
Policymaker n	p_n	q_n	1	r_n		
Collective Decision	θ_1	θ_2	1	θ_3		

Table 4

Here $\theta_1 = \frac{\sum_{i=1}^{n} w_i p_i}{n}$, $\theta_2 = \frac{\sum_{i=1}^{n} w_i q_i}{n}$ and $\theta_3 = \frac{\sum_{i=1}^{n} w_i r_i}{n}$.

Since we do not want to violate anonymity so equal weights are assigned to all the policymakers. The function, as mentioned earlier, produces collective outcome on each proposition denoted by θ_i's, which satisfies collective rationality and implication preservation.

In the following theorem, we answer the question that was raised in chapter 2. We formulated the problem of opinion aggregation in fuzzy framework and deduced that there are aggregation rules which satisfy the minimal conditions and produce collectively rational outcomes. Moreover, these aggregation rules preserve implication when opted from the class of R-implications. The following theorems and remarks build on the assumption that the fuzzy implication chosen in the process of aggregation belong to the class of R-Implication.

Theorem 3.4.2. *A linear aggregation rule satisfies universal domain, collective rationality, independence and implication preservation provided that the implication belongs to the class of R-implications.*

Proof. Assume that F is a linear aggregation rule. Then F satisfies independence and universal domain. If beliefs of all individuals are binary, then the only linear aggregation rule is dictatorial one where a weight of 1 is assigned to a particular individual and the rest are given a weight of 0. Thus, collective rationality and implication preservation are satisfied as well.

For the proof to follow, we assume that the implication opted belongs to the class of R-Implication. However, if beliefs are something between a yes or a no and individuals preserve the implication belonging to the specified class, we need to prove that the collective outcome produced by the linear aggregation rule will also preserve the implication. This is trivial by lemma 1.1.1. Since implication is preserved by the individuals. It means that $A_i(p \Rightarrow q) = 1$ for all $i \in \{1, 2, .., n\}$. So according to lemma 1.1.1(2), $A_i(p) \leq A_i(q)$. This inequality holds true when weighted averages of these real numbers are taken. That is, $F(A_1, A_2, ..., A_n)(p) \leq F(A_1, A_2, ..., A_n)(q)$ which according to lemma 1.1.1(2) imposes that $F(A_1, A_2, ..., A_n)(p \Rightarrow q) = 1$.

Equivalently,

$$F(A_1, A_2, ..., A_n)(p \Rightarrow q) = \omega_1 A_1(p \Rightarrow q) + \omega_2 A_2(p \Rightarrow q) + ... + \omega_n A_n(p \Rightarrow q)$$

where $\sum_{i=1}^{n} \omega_i = 1$ and $A_i(p \Rightarrow q) = 1 \forall i \in \{0, 1, ..., n\}$. This implies that $F(A_1, A_2, ..., A_n)(p \Rightarrow q) = 1$ the collective outcome preserves implication and hence collective rationality and implication preservation are satisfied as well. □

In the following theorem, we prove that linearity of F is confirmed if F is to satisfy the minimal conditions. We further prove that such an F, which satisfies the minimal conditions, is monotonic.

Theorem 3.4.3. *If an aggregation rule F satisfies universal domain, collective rationality, systematicity and implication preservation, then F is linear.*

Proof. Suppose F satisfies the hypothesis. In cases where beliefs are binary, F is clearly a dictatorial rule and thus linear. In case where beliefs are non-binary and implication preservation and collective rationality holds, then it means that $F(A_1, A_2, ..., A_n)(p \Rightarrow q) = 1$ given $A_i(p \Rightarrow q) = 1$ for all $i \in \{1, 2, .., n\}$. It implies that $\omega_1 A_1(p \Rightarrow q) + \omega_2 A_2(p \Rightarrow q) + ... + \omega_n A_n(p \Rightarrow q) = 1$. It further implies that $\omega_1 + ... + \omega_n = 1$. $\square$

Theorem 3.4.4. *If an aggregation function F satisfies universal domain, collective rationality and implication preservation then F is monotonic.*

Proof. Suppose F is an aggregation rule that satisfies the hypothesis. It implies that $F(A_i(p \Rightarrow q)) = 1$ given $A_i(p \Rightarrow q) = 1$. By lemma 1.1.1 it further implies that $F(A_i(p)) \leq F(A_i(q))$ if $A_i(p) \leq A_i(q)$. $\square$

Note that an aggregation rule is strategy proof if it is independent and monotonic ([27]). Also note that since our linear aggregation rule is both independent and monotonic hence it is strategy proof.

Theorem 3.4.5. *Let F be an aggregation function and for all $p \in X$. Let there be a decision method D_p such that for all $(v_1, v_2, ..., v_n) \in V^n$,*

$$A(v_1, v_2, ..., v_n)(p) = D_p(v_1(p), ...v_n(p)).$$

Then the following properties hold:

1. *For every literal p such that $p, \eta(p) \in X$ and for all $x \in [0, 1]^n$, we have:*

$$D_{\eta(p)}(\eta(x_1), ...\eta(x_n)) = \eta(D_p(x_1, x_2, ..., x_n)).$$

2. *For all literals $p \neq q$ and for all $x, y \in [0,1]^n$, where $\eta(p) = 1 - p$ and Δ is the Godel t-norm or the Lukasiewicz t-norm (in cases where $\Delta(x,y) \neq 0$) such that $p \Rightarrow q = 1$ and*

$$\Delta(D_p(x_1, ...x_n), D_q(y_1, ...y_n)) = D_{p \Delta q}(\Delta(x_1, y_1), ..., \Delta(x_n, y_n))$$

Proof. (1). Let $x = (x_1, x_2, ..., x_n) \in [0,1]^n$. Since on the basis of each proposition we can construct a v_i. For any x_i there is at least one v_i and one literal p such that $v_i(p) = x_i$. Then $v_i(\eta(p)) = \eta(x_i)$. Therefore,

$$\begin{aligned}
D_{\eta(p)}(\eta(x_1), ..., \eta(x_n)) &= D_{\eta(p)}(v_1(\eta(p)), ..., v_n(\eta(p))), \\
&= A(v_1, ..., v_n)(\eta(p))
\end{aligned}$$

and by the definition of v we have

$$\begin{aligned}
A(v_1, ..., v_n)(\eta(p)) &= \eta(A(v_1, ..., v_n)(p)) = \eta(D_p(v_1(p), ..., v_n(p)) \\
&= \eta(D_p(x_1, ..., x_n)).
\end{aligned}$$

(2). Consider any $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. By the same argument used in the proof of the first claim, for each propositions p and q there exists a corresponding v_i such that $v_i(p) = x_i$ and $v_i(q) = y_i$. We have

$$\begin{aligned}
\Delta(D_p(x), D_q(y)) &= \Delta(D_p(v_1(p), ..., v_n(p))), D_q(v_1(q), ..., v_n(q))) \\
&= (\Delta(A(v_1, ..., A_n)(p), \Delta(A(v_1, ..., A_n)(q)))) \\
&= A(v_1, ..., v_n)(p \Delta q)) = D_{p \Delta q} \\
&= (\Delta(v_1(p), v_1(q)), ..., \Delta(v_n(p), v_n(q))) \\
&= D_{p \Delta q}(\Delta(x_1, y_1), ..., \Delta(x_n, y_n)).
\end{aligned}$$

$\square$

Example 3.4.6. *In this example, Lukasiewicz t-norm and implication is used for illustration.*

	p	q	$p \Rightarrow q$
Policymaker 1	x_1	y_1	1
Policymaker 2	x_2	y_2	1
Policymaker 3	x_3	y_3	1
Collective Decision	θ_1	θ_2	1

Table 5

Let $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ be the n-tuple representing the truth assignments of n individuals on the propositions $p, q \in X$ such that $p \Rightarrow q = 1$ where $x_i, y_i \in [0, 1]$ and $\Delta(x, y) \neq 0$. Let us define the decision rule D by

$$D_p(x_1, ..., x_n) = \frac{\sum_{i=1}^{n} x_i}{n}.$$

1.

$$
\begin{aligned}
D_{\eta(p)}(\eta(x_1), ..., \eta(x_n)) &= \frac{(1 - x_1) + (1 - x_2) + (1 - x_3)}{3} \\
&= 1 - \frac{(x_1 + x_2 + x_3)}{3} \\
&= \eta(D_p(x_1, x_2, x_3)).
\end{aligned}
$$

2. *Let us consider Lukasiewicz t-norm $\Delta(x, y) = \max(0, x + y - 1)$ such that $\Delta(x, y) \neq 0$.*

$$
\begin{aligned}
\Delta(D_p(x_1, x_2, x_3), D_q(y_1, y_2, y_3)) &= \Delta((\frac{x_1 + x_2 + x_3}{3}), (\frac{y_1 + y_2 + y_3}{3})) \\
&= \max(0, \frac{x_1 + x_2 + x_3}{3} + \frac{y_1 + y_2 + y_3}{3} - 1) \\
&= \frac{x_1 + x_2 + x_3}{3} + \frac{y_1 + y_2 + y_3}{3} - 1.
\end{aligned}
$$

On the other hand

$$D_{p\Delta q}(\max(0, x_1 + y_1 - 1), \max(0, x_2 + y_2 - 1), \max(0, x_3 + y_3 - 1))$$

$$= D_{p\Delta q}(x_1 + y_1 - 1, x_2 + y_2 - 1, x_3 + y_3 - 1)$$

$$= \frac{x_1 + y_1 - 1 + x_2 + y_2 - 1 + x_3 + y_3 - 1}{3}$$

$$= \frac{x_1 + x_2 + x_3}{3} + \frac{y_1 + y_2 + y_3}{3} - 1.$$

Moreover if Godel t-norm $\Delta(x, y) = \min(x, y)$ *is used then*

$$\Delta(D_p(x_1, x_2, x_3), D_q(y_1, y_2, y_3)) = \Delta((\frac{x_1 + x_2 + x_3}{3}), (\frac{y_1 + y_2 + y_3}{3}))$$

$$= \min((\frac{x_1 + x_2 + x_3}{3}), (\frac{y_1 + y_2 + y_3}{3}))$$

$$= \frac{x_1 + x_2 + x_3}{3}$$

using lemma 1.1.1 (2). Also, considering the other side

$$D_{p\Delta q}(\min(x_1, y_1), \min(x_2, y_2), \min(x_3, y_3)) = D_{p\Delta q}(x_1, x_2, x_3)$$

$$= \frac{x_1 + x_2 + x_3}{3},$$

We conclude that if the decision makers of a society are restricted to express their beliefs using crisp values, then the merged outcome based on individual beliefs which themselves are not truly representing the individuals, cannot lead to a collective outcome which best represents the society. Majority voting in classical two valued logic results in Doctrinal Paradox and Distance based merging operators used to attain collective rationality results in a situation of indecision or a tie.

We allowed the decision makers to opt for values from the interval [0, 1] to express their beliefs. We used distance based approach in the fuzzy framework to find an interpretation having the least distance with the profile of individual belief sets. This problem is converted into an optimization problem which helped

us avoid the situation of indecision by producing a unique and optimal collective outcome in the fuzzy framework but at the expense of systematicity. We observed how fuzzy framework not only gives freedom of expression to the decision makers but also provides us with a wider range of fuzzy connectives that can be used according to the nature of the problem at hand. We discovered that the class of R-implications helps us find a collective belief using linear aggregation rules such that the aggregated outcome not only preserves the implication, but also respects collectively rationality without violating systematicity. Linear aggregation functions work equally well in cases where every individual does not have the same power to influence the final decision. In such a case the aggregated belief set will still satisfy collective rationality and implication preservation will still hold provided that the implication used belongs to the class of R-implications.

Chapter 4

Dealing With RCI Preference Relations

4.1 Incomplete Preference Relations: An Upper Bound Condition

Methods given in literature have been successful in estimating missing values in fuzzy preference and multiplicative fuzzy preference relations. However, we highlight that these methods seem unconcerned with the consistency of the resultant completed relations. These methods propose functions to take care of the surpassed estimated values that they produce but these functions may void the originality of the values provided by experts.

We define a method based on additive consistency and an upper bound condition. As mentioned in subsection 2.2, the aim is to attain two purposes: Firstly, this method should produce expressible preference values. Note that an estimated preference value is referred to as **expressible**, if it does not surpass the unit interval $[0, 1]$. Secondly, the resultant complete fuzzy preference and multiplicative preference relation must be consistent and this is precisely the focus of Khalid and Awais in ([42]).

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We refer to an estimated preference value as **crucial** if it can be found using the least and the greatest preference values provided by the expert. Such value is called crucial because it may not always be expressible. It should be noticed that if the crucial values are expressible then without estimating other missing values we can be certain about the expressibility of other missing values.

Example 4.1.1. *Consider the 4×4 incomplete fuzzy preference relation where preference intensities of alternative x_3 over $X = \{x_1, x_2, x_3, x_4\}$ are stated.*

$$
P = \begin{bmatrix}
0.5 & - & - & - \\
- & 0.5 & - & - \\
0.8 & 0.9 & 0.5 & 0.2 \\
- & - & - & 0.5
\end{bmatrix}
\tag{4.1.1}
$$

The crucial value p_{42} in this case is the amalgamation of p_{32} and p_{34} estimated using additive transitivity as $p_{42} = p_{43} + p_{32} - 0.5 = (1 - p_{34}) + p_{32} - 0.5 = 0.8 + 0.9 - 0.5 = 1.2$. The crucial value found is inexpressible.

Transformation functions proposed in the literature would drag the inexpressible values back in the unit interval but the resultant relation will not necessarily be consistent. We now propose a method to find a complete and consistent preference relation containing expressible preference intensities.

Theorem 4.1.2. *If $(n - 1)$ preference values $p_{kj}, j \in \{1, 2, 3, ..., n\}$ are provided by an expert in an $n \times n$ fuzzy preference relation then this incomplete preference relation can be completed with expressible preference degrees only if the greatest value provided δ satisfies the upper bound $\delta \leq 0.5 + \epsilon$ where $\epsilon < 0.5$ is the least preference value provided by the expert.*

Proof. Suppose that an expert provides two preference values p_{ki} and $p_{kj}, i, j \in \{1, 2, 3\}, i \neq j \neq k$ over the set of alternatives $X = \{x_1, x_2, x_3\}$. Let $0 \leq p_{ki} = \epsilon < 0.5$ and p_{kj} be the least and greatest given preference values respectively such that p_{kj} satisfies the upper bound condition.

$$0 \leq p_{kj} \leq 0.5 + \epsilon \tag{4.1.2}$$

We claim that if crucial values are expressible then so are other missing preference values. We identify the crucial value p_{ij} and test its expressibility. Using additive consistency we state that $p_{ij} = p_{ik} + p_{kj} - 0.5 = (1 - p_{ki}) + p_{kj} - 0.5 = p_{kj} - p_{ki} + 0.5$, which is expressible since $p_{kj} - p_{ki} \leq 0.5$ according to equation 4.1. This implies that $p_{ij} \leq 1$. So, p_{ij} and hence $p_{ji} \in [0, 1]$. Therefore the missing entries are expressible. Suppose that $X = \{1, 2, ..., q\}$ and expert provides preference degrees of kth alternative over others $\{p_{k1}, p_{ki}, .., p_{kj}, .., p_{kq}\}$. Where p_{ki} and p_{kj} is the least and greatest respective preference value provided such that they satisfy the upper bound condition $p_{kj} \leq 0.5 + p_{ki}$.

Then the crucial value $p_{ij} = p_{ik} + p_{kj} - 0.5 = (1 - p_{ki}) + p_{kj} - 0.5 \leq (1 - p_{ki}) + (0.5 + p_{ki}) - 0.5 = 1$ is expressible.

We prove the claim that if crucial values are expressible, then other missing preference values are also expressible.

Let $p_{sj}, s \neq j, s, j \in \{1, 2, .., n\}$ be a missing preference value other than the crucial values. Using upper bound condition we prove that this is also expressible.

$$p_{sj} = p_{sk} + p_{kj} - 0.5 \leq p_{sk} + (0.5 + p_{ki}) - 0.5 \tag{4.1.3}$$

We know that, $p_{ki} \leq p_{ks} \leq p_{kj}$ which implies $1 - p_{ki} \geq p_{sk} \geq 1 - p_{kj}$. Using this in equation 4.1.3 provides $p_{sj} \leq p_{sk} + (0.5 + p_{ki}) - 0.5 = p_{sk} + p_{ki} \leq (1 - p_{ki}) + p_{ki} = 1$. Therefore, if crucial values are expressible then so are other missing preference values. $\square$

Relations that can be completed using theorem 4.1.2 are hereafter referred to as **RCI preference relation**.

Corollary 4.1.3. *When missing preference values are estimated using additive consistency along with the upper bound condition then the resultant fuzzy preference relation satisfies additive consistency.*

Proof. The proof directly follows from proof of theorem 4.1.2. □

We next prove that the largest preference value of each row of a complete RCI preference relation obeys the upper bound condition. This is proved in the following corollary. Each row has a least preference value, the expression 0.5+ least preference value exhibits preference values that are more than the intensity of indifference in each row, where + is the addition operation for real numbers. The corollary explains that if a preference relation is additive transitive then the largest preference value of each row in the relation embraces a certain relation with the least preference value of that row.

Corollary 4.1.4. *The largest preference value of any row of a complete additive transitive relation is less than 0.5+least preference value of that row.*

Proof. Suppose on the contrary that for $k \in \{1, 2, ..., n\}, k \neq j \neq i$

$$p_{kj} > p_{ki} + 0.5 \tag{4.1.4}$$

where p_{kj} and $p_{ki} < 0.5$ are the greatest and least preference values of the kth row. Then equation 4.1.4 implies $p_{kj} - p_{ki} - 0.5 > 0$. That is $p_{kj} - (1 - p_{ik}) - 0.5 > 0$. So $p_{ik} + p_{kj} - 0.5 - 1 > 0$. Using additive consistency $p_{ij} - 1 > 0$. Which implies that $p_{ij} > 1$ and therefore it is not expressible. Hence, the greatest preference value in each row of an additive transitive preference relation obeys the upper bound condition. □

Example 4.1.5. *For instance, in the following complete preference relation, notice that the relation $\delta \le 0.5 + \epsilon$ holds true for each row.*

$$P = \begin{bmatrix} 0.5 & 0.7 & 0.8 & 0.8 & 0.3 \\ 0.3 & 0.5 & 0.6 & 0.6 & 0.1 \\ 0.2 & 0.4 & 0.5 & 0.5 & 0 \\ 0.2 & 0.4 & 0.5 & 0.5 & 0 \\ 0.7 & 0.9 & 1 & 1 & 0.5 \end{bmatrix} \tag{4.1.5}$$

Moreover, any preference relation that does not satisfy this relation cannot be additive consistent.

4.2 Upper Bound Condition For Incomplete Multiplicative Preference Relations

Given a reciprocal multiplicative preference relation $A = (a_{ij})$ where $a_{ij} \in [\frac{1}{9}, 9]$, Chiclana et.al in ([18]) proposed a function $f(a_{ij}) = \frac{1}{2}(1 + log_9 a_{ij})$ to evaluate fuzzy reciprocal preference relations corresponding to their respective multiplicative reciprocal preference relations. Since this function is bijective, we can formulate an inverse function

$$a_{ij} = g(p_{ij}) = 9^{2p_{ij}-1} \tag{4.2.1}$$

to find reciprocal multiplicative preference relation corresponding to each reciprocal fuzzy preference relation. We will use this idea to construct an upper bound condition for incomplete multiplicative preference relations.

Example 4.2.1. *Assume that a 4×4 incomplete multiplicative preference relation*

is provided by an expert such that $a_{12} = \frac{1}{8}, a_{13} = 9, a_{14} = 1$.

$$A = \begin{bmatrix} 1 & 1/8 & 9 & 1 \\ 8 & 1 & - & - \\ 1/9 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix} \qquad (4.2.2)$$

Without any condition on the least and greatest multiplicative preference values a_{12} and a_{13} respectively, we use Saaty's consistency to find the crucial preference value $a_{23} = a_{21}.a_{13} = 72$ which is not expressible since it does not belong to $[\frac{1}{9}, 9]$. We use equation 4.2.1 to formulate an upper bonud condition for multiplicative reciprocal preference relations. This condition would confirm the expressibility of missing multiplicative preference values.

Theorem 4.2.2. *If $(n-1)$ preference values $p_{kj}, j \in \{1, 2, 3, ..., n\}$ are provided in an $n \times n$ multiplicative preference relation, then the missing preference intensities are expressible only if the relation $\delta^{mult} \le 9\epsilon^{mult}$ is satisfied by the least ϵ^{mult} and greatest δ^{mult} preference values given by expert.*

Proof. Trivial using equation 4.2.1 and proof of theorem 4.1.2. $\qquad \square$

Corollary 4.2.3. *If an incomplete multiplicative preference relation is completed using upper bound condition of Theorem 4.1.2 and Saaty's consistency then the resultant relation satisfies Saaty's consistency.*

Example 4.2.4. *Suppose that $X = \{x_1, x_2, x_3, x_4\}$ and some knowledge about the preferences is provided such that the situation is modeled by the preference values $a_{12} = 4, a_{13} = \frac{1}{2}, a_{14} = 3$.*

Since $a_{12} \le 9.a_{13}$. Therefore, according to the Theorem 2, the incomplete multiplicative preference relation can be completed with expressible preference values.

$a_{23} = a_{21}.a_{13} = (1/4).(1/2) = 1/8; a_{24} = a_{21}.a_{14} = (1/4).(3) = 3/4; a_{34} = a_{31}.a_{14} = (2).(3) = 6; a_{31} = 1/a_{13} = 2; a_{41} = 1/a_{14} = 1/3; a_{42} = 1/a_{24} = 4/3; a_{43} = 1/a_{34} = 1/6.$

$$A = \begin{bmatrix} 1 & 4 & 1/2 & 3 \\ 1/4 & 1 & 1/8 & 3/4 \\ 2 & 8 & 1 & 6 \\ 1/3 & 4/3 & 1/6 & 1 \end{bmatrix} \qquad (4.2.3)$$

The completed relation satisfies Saaty's consistency. Following is a complete fuzzy preference relation corresponding to the consistent multiplicative preference relation in example 4.2.4. It needs to be noticed that a fuzzy preference relation corresponding to a consistent multiplicative preference relation satisfies additive transitivity.

$$P = \begin{bmatrix} 0.5 & 0.4 & 0.6 & 0.8 \\ 0.6 & 0.5 & 0.7 & 0.9 \\ 0.4 & 80.3 & 0.5 & 0.7 \\ 0.2 & 0.1 & 0.3 & 0.5 \end{bmatrix} \qquad (4.2.4)$$

Remark 4.2.1. If a multiplicative preference relation satisfies Saaty's consistency then the corresponding fuzzy preference relation constructed $f(a_{ij}) = \frac{1}{2}(1 + log_9 a_{ij})$ satisfies additive transitivity property.

Remark 4.2.2. For an additive transitive preference relation with non zero preferences, the corresponding multiplicative preference relation satisfies Saaty's consistency.

Although, the upper bound condition addresses and resolves the issue of incomplete preferences but it needs to be noted that the interval of the values of the incomplete relation is smaller than the interval of predicted values using additive or multiplicative consistency properties. Because of the implication of upper

bound conditions, experts may experience difficulty in expressing preference intensities within the subintervals of $[0, 1]$ or $[1, 1/9]$ in the cases of preference relations and multiplicative preference relations respectively.

In this chapter we tackled the question of incompleteness of preference and multiplicative preference relations. As an answer to the problem highlighted in section 2.2, we proved ways of completing preference relations which guaranteed that the resultant relations are transitive. Once the relations are completed, we can now talk about their rankings. Chapter 5 is focused on comparison of famous ranking methods when applied to complete RCI preference and multiplicative preference relations.

Chapter 5

Comparing Ranking Methods For RCI Preferences

As elaborated earlier in section 2.2, many methods have been employed in completing incomplete preference relations. However, these methods are silent when it comes to consistency of the completed relation. Our work is focused on incomplete preference relations where preference intensities provided by experts satisfy the condition of expressibility and the upper bound condition defined in section 4.1. The upper bound condition implicity assumes a row or column being provided by an expert to build on. As mentioned earlier, we will refer to such an incomplete preference relation as Row or Column Incomplete preference relations or **RCI preference relation.** It is proven in 4.1 that complete RCI relations satisfy additive transitivity property. Having completed the incomplete relations, we move ahead to rank complete RCI relations.

It is assumed that incomplete preference relations are provided by experts which are to be ranked by first completing them using the methods defined in ([42]), equivalently, the relations provided are RCI. This implies that the preference relations that will eventually be ranked would be additive transitive or Saaty's consistent by nature.

Once we have successfully achieved complete and consistent relations, we can

take a step forward to rank the RCI relations. Many methods of ranking preference and comparison relations have been proposed ([12] [66] [63] [5]). These methods may produce contradictory outcomes when applied to the same preference relation. As mentioned in section 2.3,a performance parameter is required to see which method, if any, is better than the rest.

To answer this question, we modify Shimura's rule of ranking comparison matrices ([63]) and make it appropriate to be used for RCI preference and multiplicative preference relations. Also, we propose a ranking method named *Column wise addition method* to rank such relations. We consider the Fuzzy Borda rule for ranking, defined in ([53] [52]), which is an extension of Classical Borda rule ([3]). Also, we propose fuzzy Borda rule for multiplicative preference relations. We compare these methods pairwise to see which method, if any, is more appropriate for such relations. For this purpose, we define the performance parameter to be the number of ties produced by each of these methods. On the other hand, we prove some properties endorsed by additive transitive relations and Saaty's consistent multiplicative relations. These properties help us in the comparison of the ranking methods.

Once we have classified preference relations where ranking methods do not produce contradictory outcomes, we can identify the reason of ties produced by these ranking methods. Properties of additive preference relations and Saaty's consistent multiplicative relations suggest that ties are a consequence of indifferent preference intensities. So to rank such relations without ending up with ties, we can suggest the decision makers to avoid expressing indifference between alternatives.

We begin resolving this problem proposed in section 2.3 by first studying the properties of RCI complete preference and multiplicative preference relations. These properties will give us a better insight of the behavior of ranking methods on these relations.

5.1 Properties Of Complete RCI Preference Relations

In order to see the behavior of ranking methods on RCI complete preference relations or additive transitive relations, we first study the properties confined by these preference relations.

Lemma 5.1.1. *Given that the preferences of an expert are additive transitive, If the expert is indifferent about an alternative x_i over x_j where $i \neq j$, then his corresponding preferences in the ith and j^{th} rows are exactly the same.*

Proof. Let $p_{ij} = 0.5$ for $i, j \in \{1, 2, ..., n\}, i \neq j$. We need to prove that k^{th} element of ith row is the same as k^{th} element of the jth row for $k \in \{1, 2, ..., n\}, k \neq i$.

Sicne $p_{ik} = p_{ij} + p_{jk} - 0.5$ where $p_{ij} = 0.5$, so $p_{ik} = 0.5 + p_{jk} - 0.5$ which implies that $p_{ik} = p_{jk}$ for all $k \in \{1, 2, ..., n\}$.

Therefore, the ith row and j^{th} row are exactly the same. □

Lemma 5.1.2. *If $p_{ij} = 0.5$ for $i, j \in \{1, 2, ..., n\}, i \neq j$ then j^{th} column and i^{th} column have the same preference values.*

Proof. Let $p_{ij} = 0.5$ for $i, j \in \{1, 2, ..., n\}, i \neq j$. We need to prove that k^{th} element of ith column and j^{th} column is the same for $k \in \{1, 2, ..., n\}, k \neq i$.

Since $p_{ki} = p_{kj} + p_{ji} - 0.5$ (where $p_{ji} = 0.5$) which implies that $p_{ki} = p_{kj}$. Hence each element of the ith column and j^{th} column is the same. □

Lemma 5.1.3. *If the least preference value provided by an expert is p_{ij} which lies in the ith row and j^{th} column of an $n \times n$ additive consistent relation then each preference value lying in the ith row and j^{th} column is necessarily less than or equal to 0.5.*

Proof. Let $p_{ij} = \epsilon$ be the least element of an $n \times n$ additive transitive relation and assume on the contrary that there exists a preference value in the ith row which $p_{is_0} > 0.5$ for some $s_0 \in \{1, 2, ..., n\}, s_0 \neq i \neq j$. Then $\epsilon = p_{is_0} + p_{s_0j} - 0.5$ which implies that $\epsilon = (p_{is_0} - 0.5) + p_{s_0j}$.

Note that $p_{is_0} - 0.5 > 0$ by assumption and p_{s_0j} is positive since it lies in $[0, 1]$.

Now if two positive numbers, $p_{is_0} - 0.5$ and p_{s_0j} lying in $[0, 1]$ are being added up to give ϵ then it means that each one of them is individually less than ϵ because otherwise they would add up to a number greater than ϵ, which is a contradiction to $\epsilon = p_{is_0} + p_{s_0j} - 0.5$. Therefore, there does not exist any s_0 in the ith row such that $p_{is_0} > 0.5$.

On similiar lines, one can prove that no entry in the jth column is greater than 0.5. Hence, the row and column entries, in which the least element of an additive transitive relation lie, are always less than or equal to 0.5. □

Lemma 5.1.4. *If p_{ij} is the greatest preference value of an $n \times n$ additive consistent relation then the entries of ith row and j^{th} column p_{is} and p_{sj} respectively are necessarily greater than or equal to 0.5.*

Proof. Let $p_{ij} = \delta$ be the greatest element of an $n \times n$ additive transitive relation and assume that on the contrary $p_{is_0} < 0.5$ for $s_0 \in \{1, 2, ..., n\}, s_0 \neq i \neq j$. Then $p_{ij} = p_{is_0} + p_{s_0j} - 0.5$ which implies that $\delta = (p_{is_0} - 0.5) + p_{s_0j}$.

$(p_{is_0} - 0.5) < 0$ by assumption, therefore, for δ to belong in $[0, 1], p_{s_0j}$ should be greater than δ which is a contradiction to our assumption that δ is the greatest preference value of the relation.

Similarly, Let $p_{ij} = \delta$ be the greatest value of the additive transitive preference relation and suppose that $p_{s_0j} < 0.5$ on the contrary. Then $\delta = p_{is_0} + p_{s_0j} - 0.5$ implies that $\delta = p_{is_0} + (p_{s_0j} - 0.5)$, since $(p_{s_0j} - 0.5) < 0$ therefore, $p_{is_0} > \delta$ which is a contradiction. □

Lemma 5.1.5. *If p_{ij} is the least preference value of an additive transitive relation then the least element of each row (except j^{th} row) will lie in the jth column.*

Proof. Suppose that p_{ij} is the least preference value of an additive transitive relation but on the contrary $p_{kj}, k \in \{1, 2, ..., n\}, k \neq i \neq j$ is not the least element of the kth row. Denote the least preference value of the kth row by $p_{ks}, s \neq j, s \in \{1, 2, ..., n\}$. Then, $p_{ks} < p_{kj}$. So, $p_{ik} + p_{kj} - 0.5 > p_{ik} + p_{ks} - 0.5$. Which implies that $p_{ij} > p_{is}$ which is a contradiction to our assumption that p_{ij} is the least preference value. $\square$

Lemma 5.1.6. *Given that p_{ij} is the least preference value of an additive transitive preference relation. The least element p_{js} of the jth row is greater than the least elements of any other row or equivalently, least element of the jth row is the greatest least element of all rows.*

Proof. We need to prove that the least element p_{js} of the jth row is greater than the least preference value of any other row or equivalently p_{js} is greater than all the least preference values $\{p_{1j}, p_{2j}, ..., p_{j-1,j}, p_{j+1,j}, ..., p_{nj}\}$ of each row, where p_{ij} is the least preference value of the entire relation. According to Lemma 5.1.5 this set of least elements of each row formed above lie in the jth column.

Suppose on the contrary that $p_{js} < p_{kj}$, where p_{kj} is the least value of kth row. Now, according to Lemma 5.1.3, $p_{kj} \leq 0.5$ which makes $p_{js} < p_{kj} \leq 0.5$. Since the relation is reciprocal, p_{sj}, the least value of sth row, which lies in the jth column, is greater than 0.5. This is a contradiction to Lemma 5.1.3. $\square$

Lemma 5.1.7. *In an additive transitive preference relation, sum of preference values in a column which contains the least element, is less than the sum of preferences of any other column.*

Proof. Suppose $p_{ij}, i \neq j, i, j \in \{1, 2, ..., n\}$ is the least element of the entire $n \times n$ preference relation. We need to prove that sum of entries of jth column is less than

sum of entries of kth column where $k \in \{1, 2, .., n\}, k \neq j$ is any other column. If we prove that every preference value in the jth column is less than the corresponding preference values of the kth column, this would prove the statement of our theorem. We know that $p_{ik} > p_{ij}$ since p_{ij} is the least preference value. This could be written as $p_{i1} + p_{1k} - 0.5 > p_{i1} + p_{1j} - 0.5$

which implies that $p_{1k} > p_{1j}$ that is, the first element of the kth row and j^{th} row preserve the order. Similarly,

$p_{it} + p_{tk} - 0.5 > p_{it} + p_{tj} - 0.5$ for all $t \neq i \neq j, t \in \{1, 2, ..., n\}$. So, each preference value from the kth column is greater than its corresponding preference value in the jth column and therefore $\sum_{i=1}^{n} p_{ik} > \sum_{i=1}^{n} p_{ij}$. $\qquad \square$

Lemma 5.1.8. *In an additive transitive relation, sum of preferences of a column which contains the greatest preference value of the relation, is greater than the sum of preferences of any other column.*

Proof. Suppose $p_{ij}, i \neq j, i, j \in \{1, 2, ..., n\}$ is the greatest element of the entire $n \times n$ matrix. We first need to prove that every preference value in the jth column is greater than the corresponding preference values of the kth column. So, $p_{ik} < p_{ij}$ since p_{ij} is the greatest entry. This could be written as

$p_{i1} + p_{1k} - 0.5 < p_{i1} + p_{1j} - 0.5$ which implies that $p_{1k} < p_{1j}$ that is, the first element of the kth column and j^{th} column preserve the order. Similarly, $p_{it} + p_{tk} - 0.5 < p_{it} + p_{tj} - 0.5$ for all $t \neq i, t \neq j, t \in \{1, 2, ..., n\}$ which implies that $p_{tk} < p_{tj}$. So, each preference value from the kth column is less than its corresponding value in the jth column and therefore $\sum_{i=1}^{n} p_{ik} < \sum_{i=1}^{n} p_{ij}$. $\qquad \square$

Lemma 5.1.9. *In an additive transitive relation if any particular preference value p_{ks} in the sth column is less than its corresponding preference value p_{kt} in the tth column then all entries in the sth column are less than the corresponding entries in the tth column.*

Proof. Suppose $p_{ks} < p_{kt}$ which implies that $p_{k1}+p_{1s}-0.5 < p_{k1}+p_{1t}-0.5$ from where we have $p_{1s} < p_{1t}$. Continuing in this way we have $p_{kq} + p_{qs} - 0.5 < p_{kq} + p_{qt} - 0.5$ for all $q \in \{1, 2, .., n\}, q \neq t \neq k$ which implies that $p_{qs} < p_{qt}$. So every preference value of the sth column is less than each corresponding preference value of the tth column. $\square$

Lemma 5.1.10. *In an additive transitive relation if p_{ks} in the kth row in less than its corresponding preference value p_{ts} in the tth column then all entries in the kth row are less than the corresponding entries in the tth row.*

Proof. Suppose $p_{ks} < p_{ts}$ which implies that $p_{k1}+p_{1s}-0.5 < p_{t1}+p_{1s}-0.5$ from where we have $p_{k1} < p_{t1}$. Continuing in this way we have $p_{kq} + p_{qs} - 0.5 < p_{tq} + p_{qs} - 0.5$ for all $q \in \{1, 2, .., n\}, q \neq t \neq s$ which implies that $p_{kq} < p_{tq}$. Hence each preference value of kth row is less than each corresponding preference of the tth row. $\square$

5.2 Properties of Complete RCI Multiplicative Preference Relations

Parallel to subsection 5.1, properties of multiplicative RCI relations are introduced as follows.

Lemma 5.2.1. *Given that the preferences of an expert satisfy Saaty's consistency, If the expert is indifferent about an alternative x_i over x_j where $i \neq j$, then the corresponding preferences in the ith and j^{th} rows are exactly the same.*

Proof. Let $p_{ij} = 1$ for $i, j \in \{1, 2, ..., n\}, i \neq j$. We need to prove that k^{th} element of ith row is the same as k^{th} element of the jth row for $k \in \{1, 2, ..., n\}, k \neq i$.

Sicne $p_{ik} = p_{ij}.p_{jk}$ where $p_{ij} = 1$, so $p_{ik} = p_{jk}$ for all $k \in \{1, 2, ..., n\}$.

Therefore, the ith row and j^{th} row are exactly the same. $\square$

Lemma 5.2.2. *If $p_{ij} = 1$ for $i, j \in \{1, 2, ..., n\}, i \neq j$ then j^{th} column and i^{th} column have the same multiplicative preferences.*

Proof. Let $p_{ij} = 1$ for $i, j \in \{1, 2, ..., n\}, i \neq j$. We need to prove that k^{th} element of ith column and j^{th} column is the same for $k \in \{1, 2, ..., n\}, k \neq i$.

Since $p_{ki} = p_{kj}.p_{ji}$ (where $p_{ji} = 1$). Therefore, $p_{ki} = p_{kj}$. Hence each element of the ith column and j^{th} column is the same. □

Lemma 5.2.3. *If the least multiplicative preference value provided by an expert is p_{ij} which lies in the ith row and j^{th} column of an $n \times n$ Saaty's consistent relation, then each preference value lying in the ith row and j^{th} column is necessarily less than or equal to 1.*

Proof. Let $p_{ij} = \epsilon$ be the least element of an $n \times n$ Saaty's consistent relation. According to the property of reciprocal multiplicative preference relations, $\epsilon < 1$. Because if $\epsilon \geq 1$ then $1/\epsilon$ will be a preference value in the relation that will be smaller than ϵ.

Now, since the relation satisfies Saaty's consistency, therefore $\epsilon = p_{ik}.p_{kj}, \forall i, j, k$ such that $i \neq j \neq k$, where p_{ik} and p_{kj} lie in the interval $[1/9, 9]$. Therefore, trivially p_{ik} and p_{kj} are both less than 1 because this is the only case when they can multiply together to produce a number ϵ less than 1. □

Lemma 5.2.4. *If p_{ij} is the greatest preference value of an $n \times n$ multiplicative preference relation which satisfies Saaty's consistency, then the entries of ith row and j^{th} column p_{is} and p_{sj} respectively are necessarily greater than or equal to 1.*

Proof. Suppose that $p_{ij} = \delta$ is the largest preference value in the relation then $p_{ij} > 1$, because otherwise it cannot be the largest value in a multiplicative reciprocal relation. Now, $\epsilon = p_{ik}.p_{kj}, \forall i, j, k$ such that $i \neq j \neq k$, where p_{ik} and p_{kj} are bound to be less than ϵ. Also p_{ik} and p_{kj} are greater than or equal to 1 because this is the

only case when they can multiply together to produce a number δ greater than 1. Because if the two preferences are less than 1, then the real numbers will multiply to give an even smaller value less than 1. □

Lemma 5.2.5. *If p_{ij} is the least preference value of a Saaty's consistent multiplicative preference relation then the least element of each row (except j^{th} row) will lie in the jth column.*

Proof. Suppose that p_{ij} is the least preference value but on the contrary $p_{kj}, k \in \{1, 2, ..., n\}, k \neq i \neq j$ is not the least element of the kth row. Denote the least preference value of the kth row by $p_{ks}, s \neq j, s \in \{1, 2, ..., n\}$.

Then, $p_{ks} < p_{kj}$. So, $p_{ik}.p_{kj} > p_{ik}.p_{ks}$. Which implies that $p_{ij} > p_{is}$ which is a contradiction to our assumption that p_{ij} is the least preference value. □

Lemma 5.2.6. *Given that p_{ij} is the least preference value of a Saaty's consistent multiplicative preference relation. The least element p_{js} of the jth row is greater than the least elements of any other row or equivalently, least element of the jth row is the greatest least element of all rows.*

Proof. We need to prove that the least element p_{js} of the jth row is greater than the least preference value of any other row or equivalently p_{js} is greater than all the least preference values $\{p_{1j}, p_{2j}, ..., p_{j-1,j}, p_{j+1,j}, ..., p_{nj}\}$ of each row, where p_{ij} is the least preference value of the entire relation. According to Lemma 5.2.5 this set of least elements of each row formed above lie in the jth column.

Suppose on the contrary that $p_{js} < p_{kj}$, where p_{kj} is the least value of kth row. Now, according to Lemma 5.2.3, $p_{kj} \leq 1$ which makes $p_{js} < p_{kj} \leq 1$. Since the relation is multiplicative reciprocal, p_{sj}, the least value of sth row, which lies in the jth column, is greater than 1. This is a contradiction to Lemma 5.2.3. □

Lemma 5.2.7. *In a complete RCI multiplicative preference relation, sum of pref-erence values in a column which contains the least element, is less than the sum of preferences of any other column.*

Proof. Suppose $p_{ij}, i \neq j, i, j \in \{1, 2, ..., n\}$ is the least element of the $n \times n$ mul-tiplicative preference relation. We need to prove that sum of entries of jth column is less than sum of entries of kth column where $k \in \{1, 2, .., n\}, k \neq j$ is any other column. If we prove that every preference value in the jth column is less than the corresponding preference values of the kth column, this would prove the statement of our theorem.

We know that $p_{ik} > p_{ij}$ since p_{ij} is the least preference value. This could be written as $p_{i1}.p_{1k} > p_{i1}.p_{1j}$ which implies that $p_{1k} > p_{1j}$ since p_{i1} cannot consume the value 0. That is, the first element of the kth row and j^{th} row preserve the order. Similarly,

$$p_{it}.p_{tk} > p_{it}.p_{tj} \text{ for all } t \neq i \neq j, t \in \{1, 2, ..., n\}.$$ So, each preference value from the kth column is greater than its corresponding preference value in the jth column and therefore $\sum_{i=1}^{n} p_{ik} > \sum_{i=1}^{n} p_{ij}$. $\qquad \square$

Lemma 5.2.8. *In a complete multiplicative RCI preference relation, sum of pref-erences of a column which contains the greatest preference value of the relation, is greater than the sum of preferences of any other column.*

Proof. Suppose $p_{ij}, i \neq j, i, j \in \{1, 2, ..., n\}$ is the greatest element of the relation. We first need to prove that every preference value in the jth column is greater than the corresponding preference values of the kth column. So, $p_{ik} < p_{ij}$ since p_{ij} is the greatest entry. This could be written as

$p_{i1}.p_{1k} < p_{i1}.p_{1j}$ which implies that $p_{1k} < p_{1j}$. Therefore, the first element of the kth column and j^{th} column preserve the order. Similarly, $p_{it}.p_{tk} < p_{it}.p_{tj}$ for all $t \neq i, t \neq j, t \in \{1, 2, ..., n\}$ which implies that $p_{tk} < p_{tj}$. So, each preference value

from the kth column is less than its corresponding value in the jth column and therefore $\sum_{i=1}^{n} p_{ik} < \sum_{i=1}^{n} p_{ij}$. $\qquad\square$

Lemma 5.2.9. *In a complete multiplicative RCI relation, if any particular preference value p_{ks} in the sth column is less than its corresponding preference value p_{kt} in the tth column then all entries in the sth column are less than the corresponding entries in the tth column.*

Proof. Suppose $p_{ks} < p_{kt}$ which implies that $p_{k1}.p_{1s} < p_{k1}.p_{1t}$ from where we have $p_{1s} < p_{1t}$. Continuing in this way we have $p_{kq}.p_{qs} < p_{kq}.p_{qt}$ for all $q \in \{1, 2, .., n\}, q \neq t \neq k$ which implies that $p_{qs} < p_{qt}$. So every preference value of the sth column is less than each corresponding preference value of the tth column. $\qquad\square$

Lemma 5.2.10. *In a complete multiplicative RCI relation, if p_{ks} in the kth row in less than its corresponding preference value p_{ts} in the tth column then all entries in the kth row are less than the corresponding entries in the tth row.*

5.3   Raking Methods And Their Comparison

The study builds on the assumption that either RCI preference and multiplicative preference relations are provided or complete additive transitive preference relations and complete Saaty's consistent multiplicative preference relations are provided by experts. In case of incomplete preference relations, they can first be completed using the methods defined by Khalid and Awais in ([42]).

We discuss three methods feasible to rank additive transitive or Saaty's consistent multiplicative preference relations.

Relativity functions used in Shimura's method to form a comparison matrix are explained earlier. Accordingly, a relation is ranked as $\max(\min(f(x_i|x_j)))$ for $i \neq j$. The diagonal entries in a comparison matrix do not play any role in $\max(\min(f(x_i)|X))$

since they are always 1 in such matrices. We import this method to rank preference and multiplicative preference relations by modifying it in a way such that the diagonal entries, which are fixed at 0.5 in additive transitive relations, are ignored. We call it modified Shimura's method of ranking.

We name the second method under consideration as the *Column wise addition method of ranking* and define it as follows.

Definition 5.3.1. Given a reciprocal fuzzy preference relation, we say that sum of kth column

$$r(k) = \sum_{i=1, i \neq k}^{n} p_{ik} \tag{5.3.1}$$

denotes the collective preference of the the set of alternatives $\{x_1, ..., .., x_n\}$ over x_k. According to this method, an alternative x_k is preferred over x_m, written as $x_k \succeq x_m$ if $r(k) \leq r(m)$.

This method suggests that $\sum_{i=1, i \neq k}^{n} p_{ik}$ represents the collective preference of the set of alternatives $\{x_1, ..., x_{k-1}, x_{k+1}, .., x_n\}$ over x_k. Hence if

$$\min(\sum_{i=1}^{n} p_{i1}, .., \sum_{i=1}^{n} p_{ij}, ..., \sum_{i=1}^{n} p_{in}) = \sum_{i=1}^{n} p_{ik}.$$

then x_k is the most preferred alternative according to this method.

Another method famous for ranking alternatives is that of Fuzzy Borda method ([5] [53] [52]) which is an extension of the classical Borda rule. The original Borda vote-counting scheme was introduced in 1770 by Jean Charles de Borda. Fuzzy Borda count is a natural extension of Classical Borda rule in which experts numerically express their preference intensities of some alternatives over others. Borda rule is a two stage scheme in the following sense. In the first phase, individual Borda counts are computed, and it would be desirable for these scores to respect the agent's opinions on the alternatives . The second phase aggregates individual scores into a total one for each alternative, and the highest score determines the Borda

winner. Formally, If P is a preference relation of an expert with n alternatives, then according to the Fuzzy Borda rule $\sum_{j=1,p_{ij}>0.5}^{n} p_{ij}$ represents the final value assigned by an expert to an alternative x_i over others. This coincides with the sum of entries greater than 0.5 in the ith row of the preference relation. That is, if

$$\max(\sum_{j=1,p_{1j}>0.5}^{n} p_{1j}, .., \sum_{j=1,p_{ij}>0.5}^{n} p_{ij}, .., \sum_{j=1,p_{nj}>0.5}^{n} p_{ij}) = \sum_{j=1,p_{kj}>0.5}^{n} p_{kj}.$$

then x_k is the most preferred alternative according to this method.

Let us now define fuzzy Borda rule for multiplicative preference relations. In this case, $\sum_{j=1,p_{ij}>1}^{n} p_{ij}$ represents the final value assigned by an expert to an alternative x_i over others. Formally, if

$$\max(\sum_{j=1,p_{1j}>1}^{n} p_{1j}, .., \sum_{j=1,p_{ij}>1}^{n} p_{ij}, .., \sum_{j=1,p_{nj}>1}^{n} p_{ij}) = \sum_{j=1,p_{kj}>1}^{n} p_{kj}.$$

then x_k is the most preferred alternative according to this method.

We discussed three ranking methods, we now intend to identify the best ranking method for Additive transitive preference relations and Saaty's consistent multiplicative preference relations. Setting the performance parameter to be the *number of ties* produced by each of these methods, we scrutinize the performance of these methods. A method is pronounced as the most appropriate or best method to rank such relations, if it produces the least number of ties. We initiate with the modified Shimura's rule to rank additive transitive preference relations. As discussed earlier, we neglect the diagonal entries and find $\min(f(x_i|x_j)), i \neq j$ of each row and write each entry in a separate column. Maximum of the obtained column according to this method reflects the most preferred alternative. Consider the following RCI preference relation where the highlighted intensities are those provided by expert. We complete this relation and rank it according to modified Shimura's rule.

Example 5.3.1.

$$\begin{pmatrix} \mathbf{0.5} & \mathbf{0.3} & \mathbf{0.4} & \mathbf{0.7} & \mathbf{0.6} \\ 0.7 & 0.5 & 0.6 & 0.9 & 0.8 \\ 0.6 & 0.4 & 0.5 & 0.8 & 0.7 \\ 0.3 & 0.1 & 0.2 & 0.5 & 0.4 \\ 0.4 & 0.2 & 0.3 & 0.6 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.6 \\ 0.4 \\ 0.1 \\ 0.2 \end{pmatrix}$$

According to Shimura's method of ranking, we obtain $x_2 \succ x_3 \succ x_1 \succ x_5 \succ x_4$.

Note that this method has not given any ties in this particular example.

Let us now use Fuzzy Borda rule to rank this completed relation.

$$\begin{pmatrix} 0.5 & 0.3 & 0.4 & 0.7 & 0.6 \\ 0.7 & 0.5 & 0.6 & 0.9 & 0.8 \\ 0.6 & 0.4 & 0.5 & 0.8 & 0.7 \\ 0.3 & 0.1 & 0.2 & 0.5 & 0.4 \\ 0.4 & 0.2 & 0.3 & 0.6 & 0.5 \end{pmatrix} = \begin{pmatrix} 1.3 \\ 3.0 \\ 2.1 \\ 0 \\ 0.6 \end{pmatrix}$$

Accordingly, the preference ranking is $x_2 \succ x_3 \succ x_1 \succ x_5 \succ x_4$.

Now rank the preference relation using Column wise addition method.

$$\begin{pmatrix} 0.5 & 0.3 & 0.4 & 0.7 & 0.6 \\ 0.7 & 0.5 & 0.6 & 0.9 & 0.8 \\ 0.6 & 0.4 & 0.5 & 0.8 & 0.7 \\ 0.3 & 0.1 & 0.2 & 0.5 & 0.4 \\ \underline{0.4} & \underline{0.2} & \underline{0.3} & \underline{0.6} & \underline{0.5} \\ 2.5 & 1.5 & 2.0 & 3.5 & 3.0 \end{pmatrix}$$

The alternatives are ranked as $x_2 \succ x_3 \succ x_1 \succ x_5 \succ x_4$ which is the same as the ranking obtained by Fuzzy Borda rule and modified Shimura's rule. This example illustrates that the three methods ranked each alternative in exactly the same way.

Let us now construct an additive preference relation where Shimura's method would give some ties. The purpose is to test how the other two methods rank the tied alternatives which would help us in selecting the best method.

Example 5.3.2. *Consider the following 6×6 RCI Incomplete preference relation, which is first completed and then ranked according to Shimura's method. The preference intensities provided by expert is stated in highlighted.*

$$
\begin{pmatrix}
0.5 & 0.4 & 0.7 & 0.5 & 0.6 & 0.5 \\
0.6 & 0.5 & 0.8 & 0.6 & 0.7 & 0.6 \\
\mathbf{0.3} & \mathbf{0.2} & \mathbf{0.5} & \mathbf{0.3} & \mathbf{0.4} & \mathbf{0.3} \\
0.5 & 0.4 & 0.7 & 0.5 & 0.6 & 0.5 \\
0.4 & 0.3 & 0.6 & 0.4 & 0.5 & 0.4 \\
\underline{0.5} & \underline{0.4} & \underline{0.7} & \underline{0.5} & \underline{0.6} & \underline{0.5} \\
2.8 & 2.2 & 4 & 2.8 & 3.4 & 2.8
\end{pmatrix}
$$

According to Shimura's method, $x_2 \succ x_1 = x_4 = x_6 \succ x_5 \succ x_3$. So, there is a tie between three alternatives x_1, x_4 and x_6. We wish to investigate if any of the other two methods give fewer or more ties while ranking these alternatives. Note that Column wise addition method provides $x_2 \succ x_1 = x_4 = x_6 \succ x_5 \succ x_3$ as reflected by the last row of this relation. Also, Fuzzy Borda rule give the same rank $x_2 \succ x_1 = x_4 = x_6 \succ x_5 \succ x_3$.

We deduce from example 5.3.1 and 5.3.2 that the three methods ranked the alternatives in the same pattern and ended up with the same number of ties, in the same alternatives. Now consider the following multiplicative Saaty's consistent relation.

Example 5.3.3.

$$\begin{pmatrix} 1 & 6 & 4 & 1 \\ 1/6 & 1 & 2/3 & 1/6 \\ 1/4 & 3/2 & 1 & 1/4 \\ 1 & 6 & 4 & 1 \end{pmatrix}$$

It is worth noticing that according to modified Shimura's rule, Column wise addition rule and Fuzzy Borda rule for multiplicative preference relations, the alternatives are ranked as $x_1 = x_4 > x_3 > x_2$.

We now prove the above observations that the three ranking methods are equally good in ranking complete RCI preference and multiplicative preference relations.

Theorem 5.3.4. *Modified Shimura's method, Fuzzy Borda rule and Column wise addition method gives the same preference ranking when applied to complete RCI preference relations.*

Proof. Suppose that $p_{ij}, i \neq j$ is the least preference value in an $n \times n$ additive transitive preference relation. Then according to lemma 5.1.7, sum of entries of the jth column will be least and hence x_j according to the Column wise addition method will be the most preferred alternative.

According to Shimura's rule, using Lemma 5.1.6, j^{th} row will have the greatest least element of all rows and therefore x_j according to this method is the most preferred alternative.

Now considering Fuzzy Borda rule, we have p_{ji} as the greatest element of the entire relation, (since the relation is reciprocal) and so j^{th} row has all elements > 0.5 according to lemma 5.1.4. Therefore addition of all preference intensities in the jth row is the greatest value according to lemma 5.1.4 and 5.1.8. This implies that x_j is the most preferred alternative.

We now investigate if there are any ranking differences among these methods for

the least preferred alternative. Since it has been assumed that p_{ji} is the greatest element of the entire relation, so according to lemma 5.1.8, the column wise addition method would rank x_i as the least preferred alternative since sum of this column is greater than the sum of any other column. Using modified Shimura's rule, since p_{ij} is the least preference value of the relation, therefore it is the lowest value of all least values of each row which makes x_i the least preferred alternative according to Shimura's method. Since p_{ij} is the least preference value therefore the ith row will have all preference values less than 0.5. Therefore, the sum of preferences greater than 0.5 will be 0 according to lemma and so Fuzzy Borda rule will rank x_i as the least preferred alternative.

Let us now choose alternatives other than the least or most preferred ones. Since least preference value of each row will lie in the jth column according to lemma 5.1.5, suppose that $p_{sj}, s \neq j$ is the second lowest value of all the least values of every row. Then according to Shimura's rule, x_s is the second least preferred alternative. We have $p_{sj} < p_{1j}, p_{sj} < p_{2j}, ..., p_{sj} < p_{nj}$ which implies that $p_{js} > p_{j1}, .., p_{js} > p_{j,i-1}, p_{js} > p_{j,i+1}, ., p_{js} > p_{jn}$ where we are ignoring the corresponding entry of the ith column (since x_i has already been declared the most preferred alternative). So, according to lemma 5.1.9, each preference value of sth column is greater than the corresponding values of the other columns(except for the ith column) which implies that sum of preference values of sth column is greater than sum of any other column entries. So,according to the method of Column wise addition, x_s is the second least preferred alternative.

Also, according to lemma 5.1.10, each entry in the sth row is less than the corresponding entries in any other row (except for the ith row) which implies that sum of entries greater than 0.5 in the sth row is less than sum of such entries of any other row and so according to Fuzzy Borda rule, x_s is the second least preferred

alternative. Continuing in this way we find out that the three methods give the same rank to alternatives in X. □

Theorem 5.3.5. *Modified Shimura's method, Fuzzy Borda rule for multiplicative preference relations and Column wise addition method gives the same preference ranking when applied to complete multiplicative RCI preference relations.*

Proof. Suppose that $p_{ij}, i \neq j$ is the least preference value in an $n \times n$ Saaty's consistent preference relation. Then according to lemma 5.2.7, sum of entries of the jth column will be least and hence x_j according to the Column wise addition method will be the most preferred alternative.

According to Shimura's rule, using lemma 5.2.6, j^{th} row will have the greatest least element of all rows and therefore x_j according to this method is the most preferred alternative.

Now considering Fuzzy Borda rule, we have p_{ji} as the greatest element of the entire relation, (since the relation is reciprocal) and so j^{th} row has all elements > 1 according to lemma 5.2.4. Therefore addition of all preference intensities in the jth row is the greatest value according to lemma 5.2.4 and 5.2.8. This implies that x_j is the most preferred alternative.

We now investigate if there are any ranking differences among these methods for the least preferred alternative. Since it has been assumed that p_{ji} is the greatest element of the entire relation, so according to lemma 5.2.8, the column wise addition method would rank x_i as the least preferred alternative since sum of this column is greater than the sum of any other column. Using modified Shimura's rule, since p_{ij} is the least preference value of the relation, therefore it is the lowest value of all least values of each row which makes x_i the least preferred alternative according to Shimura's method. Since p_{ij} is the least preference value therefore the ith row will have all preference values less than 1. Therefore, the sum of preferences greater

than 1 will be 0 according to lemma 5.2.3 and so Fuzzy Borda rule will rank x_i as the least preferred alternative.

Let us now choose alternatives other than the least or most preferred ones. Since least preference value of each row will lie in the jth column according to lemma 5.2.5, suppose that $p_{sj}, s \neq j$ is the second lowest value of all the least values of every row. Then according to Shimura's rule, x_s is the second least preferred alternative. We have $p_{sj} < p_{1j}, p_{sj} < p_{2j}, ..., p_{sj} < p_{nj}$ which implies that $p_{js} > p_{j1}, .., p_{js} > p_{j,i-1}, p_{js} > p_{j,i+1}, ., p_{js} > p_{jn}$ where we are ignoring the corresponding entry of the ith column (since x_i has already been declared the most preferred alternative). So, according to lemma 5.2.9, each preference value of sth column is greater than the corresponding values of the other columns(except for the ith column) which implies that sum of preference values of sth column is greater than sum of any other column entries. So,according to the method of Column wise addition, x_s is the second least preferred alternative.

Also, according to lemma 5.2.10, each entry in the sth row is less than the corresponding entries in any other row (except for the ith row) which implies that sum of entries greater than 1 in the sth row is less than sum of such entries of any other row and so according to Fuzzy Borda rule, x_s is the second least preferred alternative. Continuing in this way we find out that the three methods give the same rank to alternatives in X. $\qquad\square$

Corollary 5.3.6. *Modified Shimura's rule, Fuzzy Borda Rule and Column wise addition method give the same number of ties while ranking an additive transitive preference relation.*

Proof. Suppose that an $n \times n$ additive transitive preference relation needs to be ranked. Without loss of generality, suppose that ties obtained according to modified Shimura's rule are $x_q = x_r = x_s = ... = x_z$ and $x_u = x_v$ where $q, r, s, u, v, ..., z \in$

$\{1, 2, ..., n\}, q \neq r \neq s... \neq z \neq u \neq v.$ This implies that

$$\min\{p_{q1}, ..., p_{q,q-1}, p_{q,q+1}, ..., p_{qn}\} = \min\{p_{r1}, ..., p_{r,r-1}, p_{r,r+1}, ..., p_{rn}\} =$$

$$\min\{p_{s1}, ..., p_{s,s-1}, p_{s,s+1}, .., p_{sn}\} = ... = \min\{p_{z1}, ..., p_{z,z-1}, p_{z,z+1}, .., p_{zn}\}.$$

and

$$\min\{p_{u1}, ..., p_{u,u-1}, p_{u,u+1}, ..., p_{un}\} = \min\{p_{v1}, ..., p_{v,v-1}, p_{v,v+1}, ..., p_{vn}\}.$$

Suppose that the least preference value of the $n \times n$ preference relation is p_{ij} then according to lemma 5.1.5 the least element of each row except the jth row will lie in the jth column. Accordingly,

$$\min\{p_{q1}, p_{q2}, ..., p_{q,q-1}, p_{q,q+1}, ..., p_{qn}\} = p_{qj} \tag{5.3.2}$$

$$\min\{p_{r1}, p_{r2}, ..., p_{r,r-1}, p_{r,r+1}, ..., p_{rn}\} = p_{rj} \tag{5.3.3}$$

$$\min\{p_{s1}, p_{s2}, ..., p_{s,s-1}, p_{s,s+1}, .., p_{sn}\} = p_{sj} \tag{5.3.4}$$

$$\vdots \tag{5.3.5}$$

$$\min\{p_{z1}, p_{z2}, ..., p_{z,z-1}, p_{z,z+1}, .., p_{zn}\} = p_{zj} \tag{5.3.6}$$

and $p_{qj} = p_{rj} = p_{sj} = ... = p_{zj}$. (None of these entries belong to the jth row since according to , if they did, they would be greater than least values of other rows). Now, $p_{qj} = p_{rj} = p_{sj} = ... = p_{zj}$ implies $p_{qk} + p_{kj} - 0.5 = p_{rk} + p_{kj} - 0.5 = p_{sk} + p_{kj} - 0.5 = ... = p_{zk} + p_{zj} - 0.5$. Which implies that $p_{qk} = p_{rk} = p_{sk} = ... = p_{zk}$ for all $k \in \{1, 2, ..., n\}, k \neq q \neq r \neq s... \neq z$.

This means that the entire q^{th}, r^{th}, s^{th} until z^{th} rows are exactly the same. Therefore, according to Fuzzy Borda rule

$$\sum_{k=1, p_{qk}>0.5}^{n} p_{qk} = \sum_{k=1, p_{rk}>0.5}^{n} p_{rk} = \sum_{k=1, p_{sk}>0.5}^{n} p_{sk} = ... = \sum_{k=1, p_{zk}>0.5}^{n} p_{zk}$$

which implies that $x_q = x_r = x_s = ... = x_z$.

Also, since the relation is reciprocal, q^{th}, r^{th}, s^{th} until z^{th} row being the same means that q^{th}, r^{th}, s^{th} until z^{th} columns are exactly the same as well. Therefore, Column wise addition would yet again give a tie between the alternatives and rank them as $x_q = x_r = x_s = ... = x_z$.

Similarly, It can be shown that the other two methods will also give a tie between x_u and x_v. $\square$

Corollary 5.3.7. *Modified Shimura's rule, Fuzzy Borda Rule for multiplicative preference relations and Column wise addition method give the same number of ties while ranking a Saaty's consistent preference relation.*

Therefore, we conclude that the three methods are equally good to rank complete RCI preference and multiplicative preference relations since they produce the same number of ties. We also noticed that these methods produce ties in the same alternatives.

Let us now seek the reason of ties produced when ranking such relations in the following subsection.

5.3.1 Identifying Ties Without Using Ranking Methods

It is settled that ranking methods discussed do not lead to contradictory outcomes and they are equally suitable to rank complete RCI preference and multiplicative preference relations. Without using any ranking method, we proceed to identify by mere observation of the nature of preference values, the alternatives that will end up having a tie.

Theorem 5.3.8. *Number of alternatives that will give a tie in an additive transitive preference relation, if they were ranked, is equal to the number of rows that have*

preference intensities of at least one 0.5, *other than the diagonal value.*

Proof. Please note that the ranking method used for this proof is Shimura's method. Theorem 5.3.4 and Corollary 5.3.6 imply the validity of the proof of other ranking methods for this theorem.

For a 3×3 preference relation, consider the following

$$\begin{pmatrix} 0.5 & p_{12} & 0.5 \\ p_{21} & 0.5 & p_{23} \\ 0.5 & p_{32} & 0.5 \end{pmatrix} = \begin{pmatrix} & p_{12} & \\ & p_{21} = p_{23} & \\ & p_{32} = 1 - p_{23} = 1 - p_{21} = p_{12} & \end{pmatrix}$$

Suppose that the least element of first row is p_{12}. If the least element of the second row is p_{21} then $p_{23} = p_{21}$ since $0.5 = p_{12} + p_{23} - 0.5$ which implies that $p_{12} + p_{23} = 1$ and since the matrix is additive transitive and hence reciprocal, it implies that $1 - p_{12} = p_{23} \Rightarrow p_{21} = p_{23}$. As seen in the above relation, there are two rows with at least one 0.5 and there are two alternatives x_1 and x_3 that have a tie. For an $n \times n$ preference relation, Suppose that there are t rows which have at least one 0.5 preference value other than the diagonal entry. Then using Lemma 5.1.1, there are t rows that are exactly the same which means that minimum preference value of each row and then maximum of minimum preference values of each row will be the same. Hence, according to Shimura's rule there will be ties between t alternatives.

Coversely, suppose that there are $t \leq n$ alternatives that have a tie and without loss of generality suppose that $x_1 = x_2 = ... = x_t$ have a tie. This means that minimum of these rows are the same.

Suppose p_{1j} is the minimum of the first row then according to lemma 5.1.5 and lemma 5.1.6, minimum of the second row is p_{2j} and the minimum of tth row is p_{tj} and $p_{1j} = p_{2j} = ... = p_{t-1,j} = p_{tj}$. So, $p_{1j} = p_{12} + p_{2j} - 0.5$ which implies that

$p_{12} = 0.5$. Similarly $p_{2j} = p_{23} + p_{3j} - 0.5$ which implies that $p_{23} = 0.5$ and so on till $p_{t,j} = p_{t,t-1} + p_{t-1,j} - 0.5$ which implies that $p_{t,t-1} = 0.5$. Which proves that each of these t rows will have at least one preference value, other than the diagonal value, which is equal to 0.5. $\qquad\square$

Theorem 5.3.9. *Number of alternatives that will give a tie in complete RCI multiplicative relations, if they were ranked, is equal to the number of rows that have preference intensities of at least one 1, other than the diagonal value.*

Theorems and corollaries proven in section 5.3 suggest that number of ties is dependent on the alternatives among which indifference is expressed by expert. So now without using any ranking method, these theorems can help in identifying the number of ties in an additive transitive preference relation or a Saaty's consistent preference relation by analysing the nature of the concerned preference intensities in a relation. Note that ties in these relations can be avoided if cases of indifference between distinct alternatives are discouraged. If experts are suggested to express preference intensities such that they are not indifferent between any two alternatives, then as a consequence there will be no tie among those alternatives when they are ranked.

Chapter 6

Distance to consensus

Throughout the sequel, $E = \{e_1, ..., e_m\}$ is the set of decision makers.

In multiplicative aggregation models overall preferences are estimated using geometric means whereas in additive aggregation models overall preferences are estimated by weighted arithmetic means. Fundamental differences between the two models are studied by Choo and Wedley ([22]). This paper focuses on the latter approach. The underlying assumption is that the individual preferences of decision makers are complete or completable, as discussed by Khalid and Awais ([42]) and additive reciprocal. Now, to aggregate experts' preferences, individual preferences are combined into a collective relation such that the resultant relation best represents the experts' choices. Several methods for aggregating preference relations have been discussed and criticized by Popehev and Peneva in ([57]).

Although unanimity is an ideal result but it is not always achieved. In such cases, distance to consensus is a handy tool. Weighted means and ordered weighted averaging operators introduced by Yager ([70]) are well known functions that are widely used in the aggregation process. Although both of these functions are defined through a weighting vector, but there are differences in their behavior. The weighted means allow to weight each information source in relation to their reliability whereas a fundamental aspect of the OWA operator is the re-ordering step, where an aggregate

is not associated with a particular weight but instead a weight is associated with a particular ordered position of aggregate.

We now define collective preference relations obtained using the OWA operators mentioned in subsection 1.4.3. For the discussion to follow, let $\mathtt{F} : [0,1]^m \to [0,1]$ be an ordered weighted averaging operator. Also, let $\wp^{\min}$ be a collective relation obtained using the minimum ordered weighted averaging operator with associated vector $\omega_i^{\min} \in [0,1]$ defined as $\omega^{\min} = (0,0,....,1)$. Then $\mathtt{F}(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)}) = \sum_{t=1}^{m} \omega_t^{\min} a_{ij}^{(t)} = \min(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)}) = \rho_{ij}^{\min}$ which has ijth placement in the collective preference relation $\wp^{\min}$ and $a_{ij}^{(t)}$ is the tth largest preference value of the bag $(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)})$.

Let $\wp^{\max}$ represent the collective relation obtained using the OWA maximum operator with associated m vector $\omega_i^{\max} \in [0,1]$ defined as $\omega^{\max} = (1,0,...,0)$. Then $\mathtt{F}(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)}) = \sum_{t=1}^{m} w_t^{\max} a_{ij}^{(t)} = \max(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)}) = \rho_{ij}^{\max}$ which has ijth placement in $\wp^{\max}$.

Consider $\wp^*$ to represent the collective preference relation using averaging operator with associated m vector $w_i^* \in [0,1]$ defined as $\omega^* = (\frac{1}{m}, ..., \frac{1}{m})$. Then $\mathtt{F}(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)}) = \sum_{t=1}^{m} \omega_t^* a_{ij}^{(t)} = \frac{\rho_{ij}^{(1)} + ... + \rho_{ij}^{(m)}}{m}$ which is ijth preference value in $\wp^*$.

We denote $\overline{\wp}$ by collective preference relation obtained using the ordered weighted averaging operator $\mathtt{F}$ with associated m vector $\overline{\omega_i} \in [0,1]$ defined as $\overline{\omega} = (\frac{1}{m-1}, ..., \frac{1}{m-1}, 0)$. So $\mathtt{F}(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)}) = \sum_{t=1}^{m} \omega_t a_{ij}^{(t)} = \overline{\rho_{ij}}$.

Let $\overline{\overline{\wp}}$ be defined as the collective preference relation obtained using the ordered weighted averaging operator $\mathtt{F}$ with associated m vector $\overline{\overline{\omega}} \in [0,1]$ defined as

$$\overline{\overline{\omega_i}} = \begin{cases} 0 & if \, i < 2, \\ \frac{1}{m-2} & if \, 2 \leq i < m, \\ 0 & if \, i \geq m \end{cases}$$

$$\mathrm{F}(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)}) = \sum_{t=1}^{m} \overline{\overline{\omega_t}} a_{ij}^{(t)} = \overline{\overline{\rho_{ij}}}.$$

Let us illustrate how to find collective relation $P_{\min}$ and P^* with the help of a simple example.

Example 6.0.10. *Let P_1 and P_2 be reciprocal fuzzy preference relations provided by two experts over alternatives $X = \{x_1, x_2, x_3\}$ defined respectively as:*

$$P_1 = \begin{pmatrix} 0 & 0.4 & 0.7 \\ 0.6 & 0 & 0.8 \\ 0.3 & 0.2 & 0 \end{pmatrix}$$

and

$$P_2 = \begin{pmatrix} 0 & 0.5 & 0.7 \\ 0.5 & 0 & 0.4 \\ 0.3 & 0.6 & 0 \end{pmatrix}$$

Then,

$$P^{\min} = \begin{pmatrix} 0 & 0.4 & 0.7 \\ 0.5 & 0 & 0.4 \\ 0.3 & 0.2 & 0 \end{pmatrix}$$

and

$$P^* = \begin{pmatrix} 0 & 0.45 & 0.7 \\ 0.55 & 0 & 0.6 \\ 0.3 & 0.4 & 0 \end{pmatrix}$$

We proceed to discuss the distance to consensus of several collective preference relations in the following section. We intend to compare the collective relations with respect to this metric.

6.1 Comparing distance to consensus of collective preferences

The measure of distance to consensus is dependent on a measure of fuzzy preference relation called Average Certainty ([13] [66]). Consider the antithesis of consensus which is complete ambivalence and denote this relation as M_1. In such a preference relation, all non diagonal entries are 0.5. It can easily be checked that $\varpi(M_1) = 1$. Also consider the converse of M_1 which is the non-fuzzy preference, denoted by M_2, where every pair of alternatives is definitely ranked. In M_2 all non-diagonal elements are equal to 1 or 0; however, there may not be a clear consensus. So for relation M_1 we cannot have consensus and for relation M_2 we may not have consensus.

Consensus measures in fuzzy group decision making are discussed by Cabrerizo et. al in ([14]). For a reciprocal preference relation, three types of consensus Type 1, Type 2 and Type fuzzy consensus is discussed by Bedzek in ([13]) and summarized in ([66])as follows.

1. Type I consensus, M_1', is a consensus in which there is one clear choice and the remaining alternatives all have equal secondary preference.

2. Type II consensus, M_2', there is one clear choice, say alternative i but the remaining n minus 1 alternatives all have definite secondary preference. That is, the ith column is all zeros and $\rho_{kj} = 1$, where $k \neq i$.

3. M_f', Type fuzzy consensus occurs where there is unanimity in deciding the most preferred choice but the remaining alternatives have infinitely many fuzzy secondary preferences.

Mathematically, M_1 and M_2 are logical opposites and so are consensus relations M_1' and M_2'.

$\varpi(P) = 1$ for an M_1 preference relation.

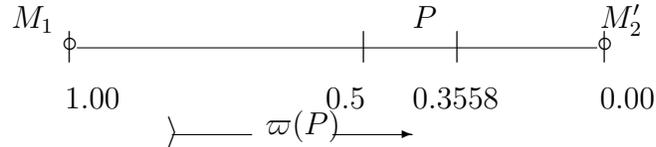$\varpi(P) = 0$ for an M_2 preference relation. The metric $\varpi(P)$ can be thought of

Figure 6.1: Illustration of distance to consensus([13])

as distance between the points $M_1(1.0)$ and $M_2(0.0)$ in n-dimensional space. As number of alternatives increases, it becomes more difficult to develop a consensus choice and rank the remaining pairs of alternatives simultaneously.

If the distance to consensus of a relation is to the left of 0.5 on the line of reference in Figure 1, then it means that this relation is a better solution for the collective group because degree of consensus or distance to complete consensus is lesser and if the intensity of distance measure is to the right of 0.5 then it means that the relation is not a good representation of the group under consideration as the distance to consensus is very complete ambivalence is low.

Example 6.1.1. *Suppose P is a reciprocal relation developed by a small group of people for pairwise preferences in a decision process involving four alternatives, $n = 4$, as shown*

$$\begin{pmatrix} 0 & 0.4 & 0.7 & 0.5 \\ 0.6 & 0 & 0.8 & 0.6 \\ 0.3 & 0.2 & 0 & 0.3 \\ 0.5 & 0.4 & 0.7 & 0 \end{pmatrix}$$

Notice that this group does not reach consensus on their first attempt at ranking the alternatives. Equivalently, this matrix carries none of the properties of a consensus type. However, the group can assess their degree of consensus to evaluate how far they are from consensus prior to subsequent discussions in the decision process. For this relation, $\tilde{\zeta}(P) = 0.5633$ and $\varpi(P) = 0.64419$. So for their first attempt of ranking the four alternatives, the group has a degree of consensus of 0.5633. Note that a value of 0.5 is completely ambivalent (uncertain) and a value of 1.0 is completely certain.

Moreover, the group is $1 - 0.64419 = 0.3558, or 35.5\%$ of the way from complete ambivalence (M_1) toward a Type II consensus as shown graphically in the figure above. It should be noted that the vast majority of group preference situations eventually develop into Type fuzzy consensus; Types I and II are typically only useful as boundary conditions.

Note that a collective preference relation best represents the choices of the individuals if distance to consensus of the collective relation is the least. In this section, we test if distance to consensus of the collective preference relations $\wp^*, \wp^{\min}, \wp^{\max}, \overline{\wp}$ and $\overline{\overline{\wp}}$ are comparable. We define a generic relation for those that can be compared. We refer to the collective preference relation $\overline{\wp}$ as an *optimistic* preference relation in which we ignore the preference value of the expert with the least preference intensity and $\overline{\overline{\wp}}$ as the *moderate* collective preference in which the greatest and the smallest such values are excluded. When we compare distance to consensus of collective preference relations, we are basically using the scale in Figure 1 for relevance.

As mentioned above, if distance to consensus of two collective relations P_1 and P_2 is t_1 and t_2 respectively, where $t_1, t_2 \in [0, 1]$ and $t_1 \leq t_2$ then it implies that P_1 is closer to consensus as compared to P_2. Therefore, it is reasonable to comply with P_1 as a better representation of the collective group. In the following theorems, we compare

the collective relations $\overline{\wp}, \wp^{\min}, \wp^{\max}$ and $\overline{\overline{\wp}}$ to see which of these are comparable and conclude if any of these collective relations produces the least or most distance to consensus as compared to the others.

Theorem 6.1.2. *Distance to consensus of* $\overline{\wp}$ *is less than or equal to distance to consensus of* $\wp^{\min}$.

Proof. Suppose m decision makers have expressed their preferences on alternative x_i over x_j as $(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)})$, where $\rho_{ij}^{(t)}$ is the preference value of expert t. Let $b_{ij}^{(1)}$ and $b_{ij}^{(m)}$ be the largest and smallest preference value of the bag $(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)})$. Then, $b_{ij}^{(m)} \leq b_{ij}^{(1)}$; $b_{ij}^{(m)} \leq b_{ij}^{(2)}$;...;$b_{ij}^{(m)} \leq b_{ij}^{(m)}$. Adding these inequalities gives, $m(b_{ij}^{(m)}) \leq b_{ij}^{(1)} + ... + b_{ij}^{(l)} + ... + b_{ij}^{(m)}$. Subtracting $b_{ij}^{(m)}$ from both sides gives $(m-1)b_{ij}^{(m)} \leq b_{ij}^{(1)} + ... + b_{ij}^{(m-1)}$

Right hand side of the above inequality shows that the least preference value has been excluded from the addition of the rest of the preference values.

So, $b_{ij}^{(m)} \leq \frac{b_{ij}^{(1)}}{m-1} + ... + \frac{b_{ij}^{(m-1)}}{m-1}$.

This is the same as $0.b_{ij}^{(1)} + ... + 0.b_{ij}^{(m-1)} + 1.b_{ij}^{(m)} \leq \frac{b_{ij}^{(1)}}{m-1} + ... + \frac{b_{ij}^{(m-1)}}{m-1} + 0.b_{ij}^{(m)}$. That is,

$\sum_{t=1}^{m} \omega_t^{\min} b_{ij}^{(t)} \leq \sum_{t=1}^{m} \overline{\omega}_t b_{ij}^{(t)}$, where $b_{ij}^{(t)}$ is the tth largest preference value of the bag $(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)})$. This holds true for every $i, j \in \{1, 2, ..., m\}, i \neq j$. That is, $\rho_{ij}^{\min} \leq \overline{\rho_{ij}}$ which means that

$\rho_{12}^{\min} \leq \overline{\rho_{12}}$; $\rho_{13}^{\min} \leq \overline{\rho_{13}}$; ....; $\rho_{n,n-1}^{\min} \leq \overline{\rho_{n,n-1}}$.

Squaring both sides of the above inequalities and adding them gives $(\rho_{12}^{\min})^2 + (\rho_{13}^{\min})^2 + ... + (\rho_{n,n-1}^{\min})^2 \leq (\overline{\rho_{12}})^2 + (\overline{\rho_{13}})^2 + ... + (\overline{\rho_{n,n-1}})^2$. Note that the left and right hand side of this inequality is the trace of the matrix $\wp^{\min}\wp^{\min T}$ and $\overline{\wp\wp}^T$ respectively. That is, $tr(\wp^{\min}\wp^{\min T}) \leq tr(\overline{\wp\wp}^T)$. Dividing both sides with $n(n-1)/2$ gives $\frac{tr(\wp^{\min}\wp^{\min T})}{n(n-1)/2} \leq \frac{tr(\overline{\wp\wp}^T)}{n(n-1)/2}$ which is the Certainty of the preference relations. So $\tilde{\zeta}(\wp^{\min}) \leq \tilde{\zeta}(\overline{\wp})$. Which implies that $\varpi(\wp^{\min}) \geq \varpi(\overline{\wp})$. $\qquad \square$

Theorem 6.1.3. *Distance to consensus of $\overline{\wp}$ is less than or equal to distance to consensus of $\wp^{max}$.*

Proof. Let $b_{ij}^{(1)}$ denote the greatest preference value of the bag $(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)})$. Then, $b_{ij}^{(1)} \geq b_{ij}^{(1)}$; $b_{ij}^{(1)} \geq b_{ij}^{(2)}$;...;$b_{ij}^{(1)} \geq b_{ij}^{(m-1)}$.

This shows that the greatest preference value is compared with all preference values except for the least preference value. Adding these inequalities gives $(m-1)b_{ij}^{(1)} \geq b_{ij}^{(1)} + ... + b_{ij}^{(m-1)}$.

So, $b_{ij}^{(1)} \geq \frac{b_{ij}^{(1)}}{m-1} + ... + \frac{b_{ij}^{(m-1)}}{m-1}$ which is the same as $\sum_{t=1}^{m} \omega_t^{max} b_{ij}^{(t)} \geq \sum_{t=1}^{m} \overline{\omega_t} b_{ij}^{(t)}$, where $b_{ij}^{(t)}$ is the tth largest preference value. This holds true for every $i, j, i \neq j$. Therefore, $\rho_{ij}^{max} \geq \overline{\rho_{ij}}$. That is, $\rho_{12}^{max} \geq \overline{\rho_{12}}$; ...; $\rho_{n,n-1}^{max} \geq \overline{\rho_{n,n-1}}$.

Squaring both sides of the above inequalities and adding them gives $(\rho_{12}^{max})^2 + (\rho_{13}^{max})^2 + ... + (\rho_{n,n-1}^{max})^2 \geq (\overline{\rho_{12}})^2 + (\overline{\rho_{13}})^2 + ... + (\overline{\rho_{n,n-1}})^2$.

Note that the left and right hand side of the above inequality is the trace of the relation $\wp^{max}\wp^{maxT}$ and $\overline{\wp\wp}^T$ respectively.

That is, $tr(\wp^{max}\wp^{maxT}) \geq tr(\overline{\wp\wp}^T)$. Dividing both sides with $n(n-1)/2$ gives $\frac{tr(\wp^{max}\wp^{maxT})}{n(n-1)/2} \geq \frac{tr(\overline{\wp\wp}^T)}{n(n-1)/2}$ which is the Certainty of these preference relations. So we have $\tilde{\zeta}(\wp^{max}) \geq \tilde{\zeta}(\overline{\wp})$. Which implies that $\varpi(\wp^{max}) \geq \varpi(\overline{\wp})$. $\square$

Theorem 6.1.4. *Distance to consensus of $\wp^{min}$ is less than distance to consensus of $\wp^{max}$.*

Proof. Let $b_{ij}^{(1)}$ and $b_{ij}^{(m)}$ denote the greatest and least preference value of the bag $(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)})$ of preference values of alternative x_i over x_j provided by m decision makers. Then, $b_{ij}^{(m)} \leq b_{ij}^{(1)}$. Which can be written as $\sum_{t=1}^{m} \omega_t^{min} b_{ij}^{(t)} \leq \sum_{t=1}^{m} \omega_t^{max} b_{ij}^{(t)}$ where $b_{ij}^{(t)}$ is the tth largest preference value of the bag $(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)})$. Equivalently, $\rho_{ij}^{min} \geq \rho_{ij}^{max}$ which holds true for every $i, j \in \{1, 2, ..., m\}, i \neq j$. So, $\rho_{12}^{min} \geq \rho_{12}^{max}$; ...; $\rho_{n,n-1}^{min} \geq \rho_{n,n-1}^{max}$.

Squaring both sides of the above inequalities and adding them gives $(\rho_{12}^{\min})^2 +$ $(\rho_{13}^{\min})^2 + ... + (\rho_{n,n-1}^{\min})^2 \geq (\rho_{12}^{\max})^2 + (\rho_{13}^{\max})^2 + ... + (\rho_{n,n-1}^{\max})^2.$

Note that the left hand side of this inequality is the trace of the matrix $\wp^{\min}\wp^{\min T}$ and the right hand side of the inequality is trace of the matrix $\wp^{\max}\wp^{\max T}$.

That is, $tr(\wp^{\min}\wp^{\min T}) \geq tr(\wp^{\max}\wp^{\max T})$. So, $\frac{tr(\wp^{\min}P^{\min T})}{n(n-1)/2} \geq \frac{tr(\wp^{\max}\wp^{\max T})}{n(n-1)/2}$ which is the certainity of the two preference relations. So we have $\tilde{\zeta}(\wp^{\min}) \geq \tilde{\zeta}(\wp^{\max})$. Which implies that $1 - (2(\tilde{\zeta}(\wp^{\min})) - 1)^{0.5} \leq 1 - (2(\tilde{\zeta}(\wp^{\max})) - 1)^{0.5}$ which implies that $\varpi(\wp^{\min}) \leq \varpi(\wp^{\max})$. $\qquad\square$

Theorem 6.1.5. *Distance to consensus of $\wp^{\max}$ is less than distance to consensus of $\wp^*$.*

Proof. Let $b_{ij}^{(g)}$ denote the least preference value of the bag of preference values of alternative x_i over x_j. So, $b_{ij}^{(g)} \geq b_{ij}^{(1)}$; $b_{ij}^{(g)} \geq b_{ij}^{(2)}$;...;$b_{ij}^{(g)} \geq b_{ij}^{(m)}$.

Adding both sides gives, $m(b_{ij}^{(g)}) \geq b_{ij}^{(1)} + b_{ij}^{(2)} + ... + b_{ij}^{(m)}$. Which implies that $\rho_{ij}^{(g)} \geq \frac{b_{ij}^{(1)}}{m} + ... + \frac{b_{ij}^{(m)}}{m}$.

That is, $\sum_{t=1}^{m} \omega_t^{\max} b_{ij}^{(t)} \geq \sum_{t=1}^{m} \omega_t^* b_{ij}^{(t)}$, where $b_{ij}^{(t)}$ is the tth largest preference value. This holds true for every $i,j \in \{1, 2, ..., m\}, i \neq j$. Therefore, $\rho_{ij}^{\max} \geq \rho_{ij}^*$. That is, $\rho_{12}^{\max} \geq \rho_{12}^*$;...;$\rho_{n,n-1}^{\max} \geq \rho_{n,n-1}^*$.

Squaring both sides of the above inequalities and adding them gives $(\rho_{12}^{\max})^2 +$ $(\rho_{13}^{\max})^2 + ... + (\rho_{n,n-1}^{\max})^2 \geq (\rho_{12}^*)^2 + (\rho_{13}^*)^2 + ... + (\rho_{n,n-1}^*)^2.$

Note that the left and right hand side of this inequality is the trace of the relation $\wp^{\max}\wp^{\max T}$ and $\wp^* P^{*T}$ respectively. That is, $tr(\wp^{\max}\wp^{\max T}) \geq tr(\wp^* P^{*T})$. Hence $\frac{tr(\wp^{\max}\wp^{\max T})}{n(n-1)/2} \geq \frac{tr(\wp^*\wp^{*T})}{n(n-1)/2}$ and this is the certainty of the two preference relations. So we have $\tilde{\zeta}(\wp^{\max}) \geq \tilde{\zeta}(\wp^*)$. Therefore, $\varpi(\wp^{\max}) \geq \varpi(\wp^*)$. $\qquad\square$

Theorem 6.1.6. *Distance to consensus of a collective fuzzy preference relation $\overline{\wp}$ is less than or equal to distance to consensus of the collective preference relation $\wp^*$.*

Proof. For the sake of convenience, we rearrange the bag $(\rho_{ij}^{(1)}, \rho_{ij}^{(2)}, ..., \rho_{ij}^{(m)})$ as $(b_{ij}^{(1)}, b_{ij}^{(2)}, ..., b_{ij}^{(m)})$ such that the last value $b_{ij}^{(m)}$ is the smallest preference value of the bag and the first value $b_{ij}^{(1)}$ is the greatest value.So,

$b_{ij}^{(m)} \leq b_{ij}^{(1)}; b_{ij}^{(m)} \leq b_{ij}^{(2)}, ..., b_{ij}^{(m)} \leq b_{ij}^{(m-1)}$.

This implies that $(m-1)b_{ij}^{(m)} \leq b_{ij}^{(1)} + b_{ij}^{(2)} + ... + b_{ij}^{(m-1)}$.

Now adding $(m-1)b_{ij}^{(1)} + ... + (m-1)b_{ij}^{(m-1)}$ on both sides gives

$(m-1)b_{ij}^{(1)} + (m-1)b_{ij}^{(2)} + ... + (m-1)b_{ij}^{(m-1)} + (m-1)b_{ij}^{(m)} \leq m(b_{ij}^{(1)} + b_{ij}^{(2)} + ... + b_{ij}^{(m-1)})$.

Which implies that

$(\frac{1}{m})\sum_{t=1}^{m} b_{ij}^{(t)} \leq (\frac{1}{m-1})b_{ij}^{(1)} + (\frac{1}{m-1})b_{ij}^{(2)} + ... + (\frac{1}{m-1})b_{ij}^{(m-1)} + (0)b_{ij}^{(m)}$. Note that left and right hand side of the above inequality is actually $\sum_{t=1}^{m} \omega_t^* b_{ij}^{(t)}$ and $\sum_{t=1}^{m} \omega_t^{min} b_{ij}^{(t)}$ respectively. This implies that, $\rho_{12}^* \leq \overline{\rho_{12}}; ...; \rho_{n,n-1}^* \leq \overline{\rho_{n,n-1}}$. That is $(\rho_{12}^*)^2 + ... + (\rho_{n,n-1}^*)^2 \leq (\overline{\rho_{12}})^2 + ... + (\overline{\rho_{n,n-1}})^2$ which is basically the trace of the two relations $\wp^*$ and $\overline{\wp}$. To find the trace of $\wp^* \wp^{*T}$ and $\overline{\wp}\,\overline{\wp}^T$, we write the following relations.

$$\wp^* = \begin{pmatrix} 0 & (\frac{1}{m})\sum_{t=1}^{m} b_{12}^{(t)} & . & . & . & (\frac{1}{m})\sum_{t=1}^{m} b_{1,n}^{(t)} \\ (\frac{1}{m})\sum_{t=1}^{m} b_{21}^{(t)} & 0 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & .0 & \\ (\frac{1}{m})\sum_{t=1}^{m} b_{n,1}^{(t)} & . & . & . & (\frac{1}{m})\sum_{t=1}^{m} b_{n,n-1}^{(t)} & 0 \end{pmatrix}$$

and

$$\wp^{*T} = \begin{pmatrix} 0 & (\frac{1}{m})\sum_{t=1}^{m} b_{21}^{(t)} & \cdot & \cdot & & \cdot & (\frac{1}{m})\sum_{t=1}^{m} b_{n,1}^{(t)} \\ (\frac{1}{m})\sum_{t=1}^{m} b_{12}^{(t)} & 0 & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & 0 & \\ (\frac{1}{m})\sum_{t=1}^{m} b_{1,n}^{(t)} & \cdot & & \cdot & \cdot & (\frac{1}{m})\sum_{t=1}^{m} b_{n-1,n}^{(t)} & 0 \end{pmatrix}$$

Which gives $tr(\wp\wp^T) = ((\frac{1}{m})\sum_{t=1}^{m} b_{12}^{(t)})^2 + ((\frac{1}{m})\sum_{t=1}^{m} b_{13}^{(t)})^2 + ... + ((\frac{1}{m})\sum_{t=1}^{m} b_{n,n-1}^{(t)})^2$

which is written as $tr(\wp\wp^T) = (\rho_{12}^*)^2 + ... + (\rho_{n,n-1}^*)^2$. On the other hand,

$$\overline{\wp} = \begin{pmatrix} 0 & \frac{b_{12}^{(1)}+b_{12}^{(2)}+....+b_{12}^{(m-1)}}{m-1} & \cdot & \cdot & & \cdot & \frac{b_{1n}^{(1)}+b_{1n}^{(2)}+....+b_{1n}^{(m-1)}}{m-1} \\ \frac{b_{21}^{(1)}+b_{21}^{(2)}+....+b_{21}^{(m-1)}}{m-1} & 0 & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & .0 & \\ (\frac{1}{m})\sum_{t=1}^{m} b_{n,1}^{(t)} & \cdot & & \cdot & \cdot & (\frac{1}{m})\sum_{t=1}^{m} b_{n,n-1}^{(t)} & 0 \end{pmatrix}$$

and

$$\overline{\wp}^T = \begin{pmatrix} 0 & \frac{b_{21}^{(1)}+b_{21}^{(2)}+....+b_{21}^{(m-1)}}{m-1} & \cdot & \cdot & & \cdot & \frac{b_{n1}^{(1)}+b_{n1}^{(2)}+....+b_{n1}^{(m-1)}}{m-1} \\ \frac{b_{12}^{(1)}+b_{12}^{(2)}+....+b_{12}^{(m-1)}}{m-1} & 0 & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & .0.5 & \\ (\frac{1}{m})\sum_{t=1}^{m} b_{1,n}^{(t)} & \cdot & & \cdot & \cdot & (\frac{1}{m})\sum_{t=1}^{m} b_{n-1,n}^{(t)} & 0 \end{pmatrix}$$

So, $tr(\overline{\wp}\overline{\wp}^T) = (\frac{b_{12}^{(1)}+b_{12}^{(2)}+....+b_{12}^{(m-1)}}{m-1})^2 + (\frac{b_{13}^{(1)}+b_{13}^{(2)}+....+b_{13}^{(m-1)}}{m-1})^2 + ... + (\frac{b_{n,n-1}^{(1)}+\rho_{n,n-1}^{(2)}+...+\rho_{n,n-1}^{(m-1)}}{m-1})^2$

which is the same as $tr(\overline{\wp}\overline{\wp}^T) = (\overline{\rho_{12}})^2 + ... + (\overline{\rho_{n,n-1}})^2$.

So, $tr((\wp^*)(\wp^*)^T) \leq tr(\overline{\wp}\overline{\wp}^T)$. Dividing both sides by $n(n-1)/2$ gives us average certainty of the two preference relations. Therefore, $\tilde{\zeta}(\wp^*) \leq \tilde{\zeta}(\overline{\wp})$. Hence,

$\varpi(\wp^*) \geq \varpi(\overline{\wp}).$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 6.1.7. *Distance to consensus of the collective fuzzy preference relation $\overline{\wp}$ is less than the distance to consensus of the collective fuzzy preference relation $\overline{\overline{\wp}}$.*

Proof. Suppose that we have formulated collective preference relations $\overline{\wp}$ and $\overline{\overline{\wp}}$. Let $\rho_{ij}^{(g)}$ represents the greatest value among all preference values assigned to x_i over x_j by the m decision makers and $\rho_{ij}^{(l)}$ is the least of all such preferences. That is, $\max(\rho_{ij}^{(1)}, ... \rho_{ij}^{(m)}) = \rho_{ij}^{(g)}$ and $\min(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)}) = \rho_{ij}^{(l)}$. So, $\rho_{ij}^{(g)} \geq \rho_{ij}^{(1)}; \rho_{ij}^{(g)} \geq \rho_{ij}^{(2)}; ....; \rho_{ij}^{(g)} \geq \rho_{ij}^{(l-1)}; \rho_{ij}^{(g)} \geq \rho_{ij}^{(l+1)}; ...; \rho_{ij}^{(g)} \geq \rho_{ij}^{(g-1)}; \rho_{ij}^{(g)} \geq \rho_{ij}^{(g+1)}; ...; \rho_{ij}^{(g)} \geq \rho_{ij}^{(m)}$.
This shows that $\rho_{ij}^{(g)}$ is not compared to itself and the least value $\rho_{ij}^{(l)}$. Adding them results in the following inequality.

$(m-2)\rho_{ij}^{(g)} \geq \rho_{ij}^{(1)} + .. + \rho_{ij}^{(l-1)} + \rho_{ij}^{(l+1)} + ... + \rho_{ij}^{(g-1)} + \rho_{ij}^{(g+1)} + ... + \rho_{ij}^{(m)}$.
Adding $(m-2)\rho_{ij}^{(1)} + .. + (m-2)\rho_{ij}^{(l-1)} + (m-2)\rho_{ij}^{(l+1)} + ... + (m-2)\rho_{ij}^{(g-1)} + (m-2)\rho_{ij}^{(g+1)} + ... + (m-2)\rho_{ij}^{(m)}$ on both sides gives

$(m-2)\rho_{ij}^{(g)} + (m-2)\rho_{ij}^{(1)} + .. + (m-2)\rho_{ij}^{(l-1)} + (m-2)\rho_{ij}^{(l+1)} + ... + (m-2)\rho_{ij}^{(g-1)} + (m-2)\rho_{ij}^{(g+1)} + ... + (m-2)\rho_{ij}^{(m)} \geq (m-1)\rho_{ij}^{(1)} + (m-1)\rho_{ij}^{(2)} + ... + (m-1)\rho_{ij}^{(l-1)} + (m-1)\rho_{ij}^{(l+1)} + ... + (m-1)\rho_{ij}^{(g-1)} + (m-1)\rho_{ij}^{(g+1)} + ... + (m-1)\rho_{ij}^{(m)}$. This implies that $(m-2)(\rho_{ij}^{(1)} + ... + \rho_{ij}^{(l-1)} + \rho_{ij}^{(l+1)} + ... + \rho_{ij}^{(g-1)} + \rho_{ij}^{(g)} + \rho_{ij}^{(g+1)} + ... + \rho_{ij}^{(m)}) \geq (m-1)(\rho_{ij}^{(1)} + ... + \rho_{ij}^{(l-1)} + \rho_{ij}^{(l+1)} + ... + \rho_{ij}^{(g-1)} + \rho_{ij}^{(g+1)} + ... + \rho_{ij}^{(m)})$. So, $\frac{\rho_{ij}^{(1)} + ... + \rho_{ij}^{(l-1)} + \rho_{ij}^{(l+1)} + ... + \rho_{ij}^{(g)} + ... + \rho_{ij}^{(m)}}{m-1} \geq \frac{\rho_{ij}^{(1)} + ... + \rho_{ij}^{(l-1)} + \rho_{ij}^{(l+1)} + ... + \rho_{ij}^{(g-1)} + \rho_{ij}^{(g+1)} + ... + \rho_{ij}^{(n)}}{m-2}$ which is basically $\sum_{t=1}^{m} \overline{\omega}_t b_{ij}^{(t)} \geq \sum_{t=1}^{m} \overline{\overline{\omega}}_t b_{ij}^{(t)}$ where $b_{ij}^{(t)}$ is the tth largest preference value of the defined bag.

We have $\overline{\rho_{ij}} \geq \overline{\overline{\rho_{ij}}}$ for $i, j \in \{1, 2, ..., n\}$.

Which means, $\overline{\rho_{12}} \geq \overline{\overline{\rho_{12}}}; ...; \overline{\rho_{ij}} \geq \overline{\overline{\rho_{ij}}}; ....; \overline{\rho_{n,n-1}} \geq \overline{\overline{\rho_{n,n-1}}}$. Squaring both sides of each inequality and adding them gives

So, $(\overline{\rho_{12}})^2 + ... + (\overline{\rho_{ij}})^2 + ... + (\overline{\rho_{n,n-1}})^2 \geq (\overline{\overline{\rho_{12}}})^2 + ... + (\overline{\overline{\rho_{ij}}})^2 + ... + (\overline{\overline{\rho_{n,n-1}}})^2$.

We know that this is equal to $tr((\overline{\wp})(\overline{\wp})^T) \geq tr((\overline{\overline{\wp}})(\overline{\overline{\wp}})^T)$. From here we trivially deduce $\tilde{\zeta}(\overline{\wp}) \geq \tilde{\zeta}(\overline{\overline{\wp}})$. Hence, $\varpi(\overline{\wp}) \leq \varpi(\overline{\overline{\wp}})$. $\qquad\qquad\qquad\qquad\square$

It is worth noticing that $\overline{\overline{\wp}}$ is a reciprocal relation. Proof of this is included in the Appendix 2. Distance to consensus of some of the collective relations is compared in this section. It is proved that distance to consensus of $\overline{\wp}$ is less than the distance to consensus of $\overline{\overline{\wp}}$. The comparison made between distance to consensus of these collective relations could be re-written as $\varpi(\overline{\wp}) \leq \varpi(\wp^{\min}) \leq \varpi(\wp^{\max}) \leq \varpi(\wp^*)$ and $\varpi(\overline{\wp}) \leq \varpi(\overline{\overline{\wp}})$. Whereas, $\wp^*$ and $\overline{\overline{\wp}}$ do not exhibit any specific relation. However, if the least $\rho_{ij}^{(l)}$ and greatest $\rho_{ij}^{(m)}$ preferences provided by the decision makers exhibit a specific property $\frac{\rho_{ij}^{(l)}+\rho_{ij}^{(g)}}{2} \leq \frac{\rho_{ij}^{(1)}+\rho_{ij}^{(2)}+...+\rho_{ij}^{(m)}}{m}$ for every i,j, then the relation $\varpi(\wp^*) \leq \varpi(\overline{\overline{\wp}})$ holds, this is written as a remark in Appendix 2. However, it is a strict condition to assume. We now compare distance to consensus of $\wp^{\max}$ with $\overline{\overline{\wp}}$.

Theorem 6.1.8. *Distance to consensus of $\wp^{\max}$ is less than or equal to distance to consensus of $\overline{\overline{\wp}}$.*

Proof. We omit the greatest and least preference values $\rho_{ij}^{(l)}$ and $\rho_{ij}^{(g)}$ respectively from the bag of alternatives $(\rho_{ij}^{(1)}, ..., \rho_{ij}^{(m)})$ and compare the rest. Note that

$$\rho_{ij}^{(1)} \leq \rho_{ij}^{(g)}; ...; \rho_{ij}^{(l-1)} \leq \rho_{ij}^{(g)}; \rho_{ij}^{(1+1)} \leq \rho_{ij}^{(g)}; ....; \rho_{ij}^{(g-1)} \leq \rho_{ij}^{(g)}; \rho_{ij}^{(g+1)} \leq \rho_{ij}^{(g)}; ....; \rho_{ij}^{(m)} \leq \rho_{ij}^{(g)}.$$ Adding the above inequalities gives

$$\rho_{ij}^{(1)} + ... + \rho_{ij}^{(l-1)} + \rho_{ij}^{(l+1)} + ... + \rho_{ij}^{(g-1)} + \rho_{ij}^{(g+1)} + ... + \rho_{ij}^{(m)} \leq (m-2)\rho_{ij}^{(g)}.$$

This implies that

$\frac{\rho_{ij}^{(1)}+...+\rho_{ij}^{(l-1)}+\rho_{ij}^{(l+1)}+...+\rho_{ij}^{(g-1)}+\rho_{ij}^{(g+1)}+...+\rho_{ij}^{(m)}}{m-2} \leq \rho_{ij}^{(g)}$ which is basically $\sum_{t=1}^{m} \overline{\overline{w_t}} b_{ij}^{(t)} \leq \sum_{t=1}^{m} w_t^{\max} b_{ij}^{(t)}$. So,

$$\overline{\overline{\rho_{ij}^{(1)}}} \leq \rho_{(ij)}^{(1)\,\max}; ...; \overline{\overline{\rho_{ij}^{(m)}}} \leq \rho_{(ij)}^{(m)\,\max}.$$

Squaring both sides and adding them gives $(\overline{\overline{\rho_{ij}^{(1)}}})^2 + ... + (\overline{\overline{\rho_{ij}^{(m)}}})^2 \leq ((\rho_{ij}^{(1)\,\max})^2 + ... + (\rho_{ij}^{(m)})^{\max})^2$ which is the trace of the two respective relations. We deduce that $\tilde{\zeta}(\overline{\overline{\wp}}) \leq \tilde{\zeta}(\wp^{\max})$. Hence, $\varpi(\overline{\wp}) \leq \varpi(\overline{\overline{\wp}})$. $\qquad\square$

In the following remark, we rewrite the relation between distance to consensus of the studied collective preference relations.

Remark 6.1.1. The relation between these distances is summarized as $\varpi(\overline{\wp}) \leq \varpi(\wp^{\min}) \leq \varpi(\wp^{\max}) \leq \min(\varpi(\overline{\overline{\wp}}), \varpi(\wp^*))$.

This relation suggests that given the option, $\overline{\wp}$ is a better representation of the collective group, since it has least distance to consensus as compared to other collective relations.

Chapter 7

Conclusion And Future Work

This dissertation addresses the issue of inconsistency faced while aggregating judgments and preferences. We started with the finding that rationality in collective judgment aggregation cannot be attained in two valued logic without having to compromise on systematicity. We propose a solution to this problem by opting fuzzy framework. We highlight that in the field of preference aggregation, collective relations can be problematic if built with the supposition that incomplete preferences are redundant. Emphasizing on the need to include incomplete preferences in the process of decision making, we discuss the methodologies opted by researches to produce complete preference relations. We highlight the problems faced by these methods and propose a method based on upper bound conditions to help resolve the problem of incomplete preferences. We study how this method helps to complete preference and multiplicative preference relations and deduce that the completed relations attained satisfy additive transitivity and Saaty's consistency. Once that is accomplished, we move on to ranking of these relations. For this purpose we employ methods already available in literature and propose a new one to be compared pairwise. The task is to come up with a ranking method that is best for the relations that we have already completed. We lay down the performance parameter to be the number of ties produced by a ranking method. So a method that produces the

least number of ties is the best method to rank such relations. We deduce that the methods behave equally well on the considered relations because of the intrinsic properties of these relations as additive transitive and Saaty's consistent relations. Lastly, we broaden the class of preference relations under consideration from additive transitive relations to a more generalized class of additive reciprocal relations. We form collective relations using ordered weighted averaging operators and with the natural assumption that these collective relations might not exhibit consensus in the group, we compare the distances each of them have from consensus on a scale of 0.0 (no consensus) and 1.0 (complete consensus).

More explicitly, we conclude that if decision makers of a society are restricted to express their beliefs using crisp values, then the merged outcome based on individual beliefs which themselves are not truly representing the individuals, cannot lead to a collective outcome which best represents the society. Majority voting in classical two valued logic results in Doctrinal Paradox. Whereas, distance based merging operators used to attain collective rationality results in a situation of indecision or a tie. We allow decision makers to opt for values from the interval $[0, 1]$ to express their beliefs. We use distance based approach in the fuzzy framework to find an interpretation having the least distance with the profile of individual belief sets. This problem is converted into an optimization problem which helps us in avoiding the situation of indecision by producing a unique and optimal collective outcome in the fuzzy framework but at the expense of systematicity. Fuzzy framework not only gives freedom of expression to the decision makers but also provides us with a wider range of fuzzy connectives that can be used according to the nature of the problem at hand.

Since we do not want to compromise on any of the minimal conditions, we specify the class of R-implications which helps us find a collective belief using linear

aggregation rules such that the aggregated outcome not only preserves the implication but is also collectively rational and systematic. Linear aggregation functions work equally well in cases where every individual does not have the same power to influence the final decision. In such a case the aggregated belief set will still satisfy collective rationality and implication preservation will still hold provided that the implication used belongs to the class of R-implications.

For future work, it will be interesting to see policy makers opting for triangular or trapezoidal fuzzy numbers to express their beliefs. It would be thought provoking to find ways to merge their membership functions and to see how fuzzy optimization techniques can be employed to find an aggregated fuzzy number.

The thesis also focuses on incomplete preference and multiplicative preference relations. Methods proposed in literature to complete such relations are silent on consistency of the resultant relation. The purpose of completing an incomplete relation is to make it useful in the process of decision making. If the resultant relations are not consistent then the purpose of completing incomplete relations at the first place needs attention. While estimating missing values, Herrera ([36][37]) uses transformation functions to bring surpassed values back in the interval $[0, 1]$. In this way the originality of preference values given by the expert may be voided. Moreover, the missing values attained using such transformation functions do not promise consistency of the resultant relation.

For this purpose, we present an upper bound condition for incomplete fuzzy preference and multiplicative fuzzy preference relations which ensures the expressibility of missing preference values. Moreover, this method provides a resultant relation that satisfies additive transitivity and Saaty's consistency and is therefore consistent. Also, It is brought to notice that corresponding to each consistent multiplicative preference relation the fuzzy preference relation found using function proposed

by Chiclana in ([18]) is additive transitive (and vice versa).

Incomplete preference relations that satisfy the property defined by Khalid and Awais are named as RCI preference relations. It is highlighted that completion of such relations by the methods discussed by Khalid and Awais leads to complete preference and multiplicative preference relations which satisfy the property of additive transitivity and Saaty's consistency respectively. We further study the behavior of some ranking methods on these relations. The purpose is to investigate if ranking methods give contradictory ranking outcomes when applied to completed RCI preference and RCI multiplicative preference relations. In order to investigate performance of ranking methods, Shimura's method, Fuzzy Borda rule for fuzzy preferences, Fuzzy Borda rule for multiplicative preference relations and Column wise addition method are selected for this purpose. The performance parameter is defined to be the number of ties produced by each of the methods. Moreover, some interesting properties complied by complete RCI preference relations and multiplicative preference relations are proved. With the help of these properties we conclude that the ranking methods are equally suitable to rank additive transitive preference relations and Saaty's consistent multiplicative preference relations because of the intriguing properties that such relations placate. The properties highlight the role of the least and greatest preference intensities in the formation of the rows and columns of these relations. Moreover, with the help of these properties we identify the cause of ties produced while ranking relations and propose a solution to avoid ties. Chapter 5 suggests that number of ties produced by each of these methods is dependent on indifference of an expert over alternatives. To avoid ties, intensities representing indifference among alternatives need to be discouraged. Moreover, it is stressed that reaching consensus is an idealistic situation while making collective decisions. The fact that distance to consensus of collective preference relations is

not the same when we use different Ordered weighted averaging operators to collect individual additive reciprocal preference relations is highlighted. We formulate an association between distance to consensus of collective relations using particular ordered weighted averaging operators. For this purpose, a lower and upper bound of distances is given such that using ordered weighted averages, distance to consensus of the collective relation remains in this bound.

For future work, more relaxed bounds may be defined to complete preference relations but which do not restrict the experts to express preferences in the dictated subinterval. Also, an upper bound condition for fuzzy preference and multiplicative fuzzy preference relations may be constructed in the case when $(n-1)$ diagonal preference intensities are provided. Moreover, a parallel model in fuzzy linguistic preference relations setting could be formulated. It would be interesting to study if a more generalized class of ranking methods or scoring functions produce the same ranking result when applied to RCI preference and multiplicative preference relations.For future work, the assumption of additive consistency of individual preference relations can be relaxed. Moreover, a larger set of OWA or IOWA operators could be implied to see if a generic relation on distance to consensus can be formulated.

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Appendix 1

1a. Both 1a and 1b are written in Mat Lab.

d=zeros(601,1); d2=zeros(601,1); D=zeros(2000,1); DD=zeros(601,1); d1=zeros(601,1);

d2=zeros(601,1); d3=zeros(601,1);

pm1=[0.5,0.6,0.4]; pm2=[0.3,0.7,0.2]; pm3=[0.8,0.4,0.3]; o1=zeros(601,1); o2=zeros(601,1);

o3=zeros(601,1); o1=[0.3:0.001:0.8]'; o2=[0.1:0.001:0.7]'; o3=[0.1:(0.5-0.1)/600:0.5]';

Bsmallest=10;

for i=1:length(o1)-1; if o3(i)¿(max(0,o1(i)+o2(i)-1))

d1=sqrt((($o1$-pm1(1)).^{2}+$(o2-pm1(2)).^2+(o3(i)-pm1(3)).^2)); d2 = sqrt(((o1-$

$pm2(1)).^2 + (o2 - pm2(2)).^2 + (o3(i) - pm2(3)).^2)); d3 = sqrt(((o1 - pm3(1)).^2 +$

$(o2 - pm3(2)).^2 + (o3(i) - pm3(3)).^2));$

d=(d1+d2+d3);

small=min(d); if(Bsmallest¿small) x=o1(i); y=o2(i); z=o3(i); Bsmallest=small;

end

    end

    end

1b.

d=zeros(1,2); d2=zeros(601,1); D=zeros(2000,1); D=zeros(601,1); d1=zeros(601,1);

d2=zeros(601,1); d3=zeros(601,1);

c1=[0.6,0.5,0.5]; c2=[0.8,0.9,0.8]; c3=[0.2,0.9,0.8]; o1=zeros(601,1); o2=zeros(601,1);

o3=zeros(601,1);

o1=[0.2:0.0001:0.8]'; o2=[0.5:0.0001:0.9]'; o3=[0.5:0.0001:0.8]';

Bsmallest=10; j=1; for i=1:length(o3)-1; if o3(i)==(max((1-o1(i)),min(o1(i),o2(i))))

d1(j)=sqrt(((o1(i)-c1(1)).^{2}+(o2(i)-c1(2)).^{2}+(o3(i)-c1(3)).^{2})); $d2(j) = sqrt(((o1(i)-$

$c2(1)).^2+(o2(i)-c2(2)).^2+(o3(i)-c2(3)).^2)); d3(j) = sqrt(((o1(i)-c3(1)).^2+(o2(i)-$

$c3(2)).^2 + (o3(i) - c3(3)).^2)); v = (d1(j) + d2(j) + d3(j));$

d(j,2)=v; d(j,1)=i; j+1;

end $[\min_v al, index_{4m}in] = min(d(2));$

end

$\min_v alo1(d(index_{4m}in, 1))o2(d(index_{4m}in, 1))o3(d(index_{4m}in, 1))$

Appendix 2

Remark 7.0.2. $\overline{\overline{\varphi}}$ is a reciprocal preference relation.

Proof. Denote $\min(\rho_{ij}^{(1)}, \rho_{ij}^{(2)}, ..., \rho_{ij}^{(m)})$ as $\rho_{ij}^{(l)}$ and $\max(\rho_{ij}^{(1)}, \rho_{ij}^{(2)}, ..., \rho_{ij}^{(m)})$ as $\rho_{ij}^{(g)}$. So,
$\overline{\overline{\rho_{ij}}} = \sum_{t=1}^{m} \overline{\overline{w}} b_{ij}^{(t)} = \frac{(\rho_{ij}^{(1)} + \rho_{ij}^{(2)} + ... + \rho_{ij}^{(l-1)} + \rho_{ij}^{(l+1)} + ... + \rho_{ij}^{(g-1)} + \rho_{ij}^{(g+1)} + ... + \rho_{ij}^{(m)})}{m-2}$. Consider

$$\overline{\overline{\rho_{ji}}} = \frac{(\rho_{ji}^{(1)} + x_{pi}^{(2)} + ... + \rho_{ji}^{(m)} - \min(\rho_{ji}^{(1)}, \rho_{ji}^{(2)}, ..., \rho_{ji}^{(m)}) - \max(\rho_{ji}^{(1)}, \rho_{ji}^{(2)}, ..., \rho_{ji}^{(m)}))}{m-2}.$$
(7.0.1)

Since the m preference relations are additive reciprocal so each $\rho_{ij}^{(t)}$ can be written as $1 - \rho_{ij}^{(t)}$ where $t \in \{1, 2, ..., m\}$. So, $\min(\rho_{ji}^{(1)}, \rho_{ji}^{(2)}, ..., \rho_{ji}^{(m)}) = \min(1 - \rho_{ij}^{(1)}, 1 - \rho_{ij}^{(2)}, .., 1 - \rho_{ij}^{(l)}, 1 - \rho_{ij}^{(g)}, .., 1 - \rho_{ij}^{(m)}) = 1 - \rho_{ij}^{(g)} = \rho_{ji}^{(g)}$.

Also, $\max(\rho_{ji}^{(1)}, \rho_{ji}^{(2)}, ..., \rho_{ji}^{(m)}) = \max(1 - \rho_{ij}^{(1)}, 1 - \rho_{ij}^{(2)}, .., 1 - \rho_{ij}^{(l)}, 1 - \rho_{ij}^{(g)}, .., 1 - \rho_{ij}^{(m)}) = 1 - \rho_{ij}^{(l)} = \rho_{ji}^{(l)}$ and so

$$\overline{\overline{\rho_{ji}}} = \frac{\rho_{ji}^{(1)} + \rho_{ji}^{(2)} + ... + \rho_{ji}^{(m)} - \min(\rho_{ji}^{(1)}, \rho_{ji}^{(2)}, ..., \rho_{ji}^{(m)}) - \max(\rho_{ji}^{(1)}, \rho_{ji}^{(2)}, ..., \rho_{ji}^{(m)})}{m-2} =$$
$$\frac{\rho_{ji}^{(1)} + \rho_{ji}^{(2)} + ... + \rho_{ji}^{(m)} - (\rho_{ji}^{(g)} + \rho_{ji}^{(l)}))}{m-2} = \frac{\rho_{ji}^{(1)} + \rho_{ji}^{(2)} + ... + \rho_{ji}^{(l-1)} + \rho_{ji}^{(l+1)} + ... + \rho_{ji}^{(g-1)} + \rho_{ji}^{(g+1)} + ... + \rho_{ji}^{(m)}}{m-2}.$$

So, $\overline{\overline{\rho_{ij}}} + \overline{\overline{\rho_{ji}}} = \frac{1}{m-2}(\rho_{ij}^{(1)} + \rho_{ij}^{(2)} + ... + \rho_{ij}^{(l-1)} + \rho_{ij}^{(l+1)} + ... + \rho_{ij}^{(g-1)} + \rho_{ij}^{(g+1)} + ... + \rho_{ij}^{(m)} + 1 - \rho_{ij}^{(1)} + 1 - \rho_{ij}^{(2)} + ... + 1 - \rho_{ij}^{(l-1)} + 1 - \rho_{ij}^{(l+1)} + ... + 1 - \rho_{ij}^{(g-1)} + 1 - \rho_{ij}^{(g+1)} + ... + 1 - \rho_{ij}^{(m)}) = \frac{m-2}{m-2} = 1$. Hence, $\overline{\overline{\varphi}}$ is a reciprocal preference relation. $\square$

Remark 7.0.3. If the least and greatest preferences provided by the decision makers exhibit a specific property $\frac{\rho_{ij}^{(l)} + \rho_{ij}^{(g)}}{2} \leq \frac{\rho_{ij}^{(1)} + \rho_{ij}^{(2)} + ... + \rho_{ij}^{(m)}}{m}$ for every i, j then $\varpi(\varphi^*) \leq \varpi(\overline{\overline{\varphi}})$.