APPLICATION OF BIPOLAR FUZZY (SOFT) SET IN ORDERED SEMIGROUPS

by

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Dedication

Dedicated to my parents (especially my father) for their endless love, support and encouragement thorough my life

Abstract

In this research, the ideal theory of ordered semigroups in the framework of bipolar fuzzy sets (briefly BFSs) is expended. The main idea defined in this thesis are that of bipolar fuzzy left (right) ideal (briefly BFL(R)I) in ordered semigroups, bipolar fuzzy interior ideal (briefly BFII) in ordered semigroups, bipolar fuzzy generalized bi-ideal (briefly BFGBI) in ordered semigroups and bipolar fuzzy bi-ideal (briefly BFBI) in ordered semigroups. We also generalize BFL(R)I as (α,β)- BFL(R)I, BFII as (α,β)-BFII, BFGBI as (α,β)-BFGBI and BFBI as (α,β)-BFBI in ordered semigroups. We also generalize BFII as $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-BFII in ordered semigroups.

We examine these newly defined notions in regular, intra-regular and semi-prime ordered semigroups. These classes of ordered semigroups are characterized in framework of the newly defined notions.

As soft sets plays a vital role to deal with uncertainty in the data and has many applications in decisions making problem, we utilize the concept of bipolar fuzzy soft sets (briefly BFSSs) in decision making as well. Decision makings schemes based on fuzzy parameterized bipolar fuzzy soft expert sets and possibility bipolar fuzzy soft expert sets have been examined.

Key Words: *Bipolar Fuzzy Set; Bipolar Fuzzy Ideal; Bipolar Fuzzy Interior Ideal; Bipolar Fuzzy Generalized Bi-Ideal; Bipolar Fuzzy Bi-Ideal; Bipolar Fuzzy Soft Expert Set; Fuzzy Parameterized Bipolar Fuzzy Soft Expert Set; Possibility Bipolar Fuzzy Soft Expert Set; Decision making*

Contents

Research Profile

1. M. Ibrar, A. Khan, B. Davvaz: Characterizations of regular ordered semigroup in terms of (α, β)-bipolar fuzzy generalized bi-ideals, J. Intelligent and fuzzy system, **33**(1), (2017), 365-376.

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SYMBOL	ABBREVIATION
S	Ordered Semigroup
$\mathcal{L}$	Lift ideal of S
$\mathcal{R}$	Right ideal of S
I	Ideal of S
$\mathcal{I}$	Interior ideal of S
$\mathcal{G}$	Generalized bi-ideal of S
B	Bi-ideal of S
U	Initial universe set
E	Set of parameters
$P(U)$	Power set of U
χ_A	Characteristic function of A
$f \circ g$	Bipolar fuzzy product
$C(f;(s,t))$	(s,t)-cut of f
iff	if and only if
BFS	Bipolar fuzzy set
BFLI	Bipolar fuzzy lift ideal
BFRI	Bipolar fuzzy right ideal
BFI	Bipolar fuzzy ideal
BFII	Bipolar fuzzy interior ideal
BFGBI	Bipolar fuzzy generalized bi-ideal
BFBI	Bipolar fuzzy bi-ideal
BFSS	Bipolar fuzzy soft set
BFSES	Bipolar fuzzy soft expert set
FPBFSES	Fuzzy parameterized bipolar fuzzy soft expert set
PBFSES	Possibility bipolar fuzzy soft expert set

Introduction

To separate positive and negative informations, the idea of bipolarity is most important. In bipolarity the positive information represents the possibility while the negative information represents the forbidden or false. In daily life problem, the bipolar information is helpful to make decision. For example, to buy a car, we must take the advantages and disadvantages of the brand. Therefore bipolar information effects the efficiency of decision making. Fuzzy sets are extended to bipolar fuzzy sets (BFS) by increasing the range of fuzzy sets from $[0, 1]$ to $[-1, 1]$. In a BFS, if an element has membership degree 0, it means that it is not related to the required property. If an element has membership degree in the interval $(0, 1]$, it means that it partially satisfies the required property. The elements that satisfy implicit counter property have membership degree in $[-1, 0)$. BFSs are used in an organization, economic, performance, development, evaluate or risk etc. The concepts of BFSs are more interested in mathematics.

Being unclear and probabilistic in nature some mathematical phenomena cannot be measured by classical sets. In 1965, Zadeh [1] was the first to give the idea of fuzzy subset of a set, which could concentrate on these kinds of problem. The idea of fuzzy sets was further explained by Zadeh in [11, 12, 13, 14].

The concept of fuzzy group was proposed by Rosenfeld [2] in 1971 that opened a new aspects for researchers to review numerous notions and conclusions from the area of algebra in the border stream of fuzzification.

Bhakat and Das [4, 5] and Bhakat [6] investigated the notion of (α, β)-fuzzy subgroups. However, the notion of a quasi-coincidence [3] of a fuzzy point with a fuzzy set is of massive importance and plays a major role for producing different types of fuzzy subgroups. With the initiative of Bhakat and Das idea, the research turned to a new direction. The idea of generalized fuzzy bi-ideals in semigroups were presented by Kazanci and Yamak [29]. P. Ming and L. Ming proposed the notion of fuzzy point with a fuzzy subset [30]. Moreover, Davvaz and Mozafar [31] studied $(\in, \in \vee q)$-fuzzy subalgebras and ideals of Lie algebras. The notion of generalized fuzzy filters of R_0-algebra and interval valued $(\in, \in \vee q)$-fuzzy ideals of pseudo-MV algebra was given by Ma et al in [32, 33]. Mordeson et al. [34] applied the fuzzy concept to the various models of fuzzy automata. Many other researchers like Ali [35], Dudek et al. [36], Jun [37], Zhan and

Davvaz [38], Ma et al. [39, 40] and Yuan et al. [41], fuzzified many valuable results from different branches of algebra and proposed at various applied areas of mathematics, physics, computer science, formal languages and coding theory etc. The notion of fuzzy hyperquasigroups were discussed by Zhan et al in [42] and some results were proved in term of this idea. Zhan and Jun [45] characterized the generalization of fuzzy ideals of BCI-algebra. Ma et al [46] proposed the generalization of fuzzy h-ideals of hemirings. The necessary and sufficient conditions for every fuzzy left ideals in ordered semigroup to be a constant function were proved by Khan et at [48]. For a detail study of generalized fuzzy ideal theory of ordered semigroups, we refer the reader to [7, 27, 28, 47, 49, 50, 51, 54, 55, 56, 58, 60, 61, 62, 63].

Zhang [65] was the first to initiate the concept of BFSs. Also Zhang [66] generalized the notions of bipolar fuzziness and interval-based bipolar fuzzy logic. Lee [67] suggested the notion of bipolar valued fuzzy sets as an extension of fuzzy sets. The concept of bipolar fuzzy subalgebra and ideal in BCK/BCI-algebras were presented by Lee [68]. Jun and Park [69] proposed a bipolar fuzzy regularity, regular subalgebra, filter, and closed quasi filter in BCH-algebras. The notion of BFS in Γ- semigroup was presented by Samit [73]. Hee et al. [74] proposed bipolar fuzzy ideals with operators in semigroups. Faruk et al.[75] investigated a group structure on bipolar soft sets. Lee compared the intuitionistic fuzzy sets, interval-valued fuzzy sets and bipolar-valued fuzzy sets in [76]. Jun and Kavikumar [77] introduced the concept of finite state machines in BFS.

The importance of BFS theory is clear from the rapidly increasing high-quality research articles, published in a variety of indexed journals for the past several years.

Most of daily life problems contain vagueness and uncertainty, such as in the area of engineering, medical science, economy, environmental science and social science. Uncertainties and vagueness may be little bit handle by using mathematical algorithm in fuzzy set theory, rough set theory, interval mathematics and probability theory. However, all these theories have their own complexity and flaws as mentioned by Molodtsov which make it inappropriate to be used to deal with uncertainties and vagueness. Molodtsov in 1999 (see [91]) was the first to introduce the concept of soft set theory to overcome these difficulties. He also established its several applications in the field of medical sciences, economics, social sciences, and engineering. Maji et al. (see [92]) introduced the notion of fuzzy soft sets furthered the study of soft sets by initiating the concept of and continued to apply these theories in solving various decision making problems. Currently, research on different generalization of soft sets and its application in different fields are going on. Alkhazaleh and Salleh (see [93]) initiated the notion of soft expert sets. Alkhazaleh and Salleh [94] established the notion of fuzzy parameterization in the field of fuzzy soft sets and mentioned some generalized algorithm for solving problems in decision making. Cagman et al. [95, 96] established the notion of fuzzy parameterized (fuzzy) soft sets and discussed the associated properties. Bashir and Salleh [97] initiated the idea of fuzzy parameterized soft expert sets. Hazaymeh et al. [101] then extend the work of Bashir and Salleh and described the theory of fuzzy parameterized fuzzy soft expert sets. The idea of fuzzy parameterized intuitionistic fuzzy soft expert sets was first initiated by Selvachandranwe and Salleh in [100]. For further study, the reader is refer to see [102, 103, 104, **?**].

Our aim in this thesis is to generalized the concept of bipolar fuzzy ideals in ordered semigroups. The following are the objectives of our research.

Our first aim is to introduce the notion of generalized bipolar fuzzy ideals (BFIs) in S.

Our second objective is to present the application of possibility bipolar fuzzy soft expert sets and fuzzy parameterized bipolar fuzzy soft expert sets in decision making problems.

Thirdly, we give the concept of generalized bipolar fuzzy interior ideals (BFIIs) in S and build the relationship between generalized BFIIs and generalized BFIs in ordered semigroup.

Fourthly, we want to initiate the idea of (α, β)-bipolar fuzzy bi-ideals (BFBIs) in S and characterize ordered semigroups in framework of (α, β)-bipolar fuzzy bi-ideals.

Our fifth objective is to present the notion of generalized bipolar fuzzy generalized bi-ideals (BFGBIs) in S and give some characterizations of different classes of S in context of BFGBIs. Furthermore, we find relationship between generalized BFGBIs and generalized BFBIs in S.

Our six objective is to present the notion of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy left (right, interior) ideals in S and to bring this concept in prime/semiprime BFIs.

Thesis Outlines

This thesis includes five chapters. Throughout this thesis S will denote an ordered semigroups, unless otherwise specified. The literature summary, definitions and basic concepts, which are needed for the subsequent chapters have been given in the first chapter. Also the definitions of BFIs, BFIIs, BFGBIs and BFBIs of ordered semigroup are given and characterize S in framework of these ideals.

In chapter 2, an application of bipolar fuzzy soft expert sets in decision making problems are discussed in detail. The notion of fuzzy parameterized bipolar fuzzy soft expert sets and possibility fuzzy parameterized bipolar fuzzy soft expert sets are introduced. Some basic operations (complement, union and intersection) of these notions are defined in this chapter. Properties and essential laws like De Morgan's laws relevant to these concepts are studied. Lastly, applications of these notions are discussed in decision making problems by using generalized algorithms.

In chapter 3, the concept of (α, β)-BFIs in S are introduced and its related properties are investigated. Characterizations of $(\in, \in \vee \mathrm{q})$-bipolar fuzzy left (resp. right) ideals are provided and discussed relations between its various types. In this chapter, the generalization of an ordered semigroup in the frame work of (α, β)-BFIIs are presented. Different classes namely regular, intra-regular, simple and semi-simple ordered semigroup are characterized in term of (α, β)-BFIIs (resp (α, β)-BFIs). It has been proved that the notion of $(\in, \in \vee \mathrm{q})$-BFIIs and $(\in, \in \vee \mathrm{q})$-BFIs overlap in semi-simple, regular and intra-regular ordered semigroup. Also the upper and lower part of $(\in, \in \vee \mathrm{q})$-BFIIs are discussed.

In chapter 4, we introduce the notion of (α, β)-BFGBI in S. We characterize S in terms of (α, β)-BFGBIs and get some interesting results. We prove that in regular ordered semigroups, the notion of $(\in, \in \vee \mathrm{q})$-BFGBI and $(\in, \in \vee \mathrm{q})$-BFBI coincide. We define upper/lower parts of $(\in, \in \vee \mathrm{q})$-BFGBIs in S. We also give the idea of (α, β)-BFBI of S and obtained few interesting characterization theorems.

In chapter 5, the concept of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFIIs of ordered semigroup are introduced. Different classes of ordered semigroup (regular, intra-regular) are characterized in terms of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFIIs (resp. $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFIs). It has been proved that the concept of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFIIs and $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFIs coincides in regular, intra-regular ordered semigroup. Also the notion of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy prime (semiprime) interior ideals were discussed and get some interesting results.

Chapter 1

Preliminaries

In this chapter, we shall discuss basic definitions and results of fuzzy ordered semigroups and bipolar fuzzy ordered semigroups. We also study some of the background materials that will be of essential for our later research. The main results of this chapter is taken from [78, 15, 80, 67, 81]. In this chapter we also define BFIs, BFIIs, BFGBIs, BFBIs and $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFIIs in S and characterize S in framework of these ideals.

1.1 Basic Definitions and Results

Definition 1.1.1 *[78] A structure $(S, ., \leq)$ is called an ordered semigroup if $(S, .)$ is a semigroup and $(S, \leq)$ is a partial ordered set (poset) and $a \leq b$ implies that $a\xi \leq b\xi$ and $\xi a \leq \xi b$ for all $\xi, a, b \in S$.*

For $\mathcal{A} \subseteq S$, we denote $(\mathcal{A}] = \{a \in S \mid a \leq h \text{ for some } h \in \mathcal{A}\}$. If $\mathcal{A} = \{\xi\}$ then we write $(\xi]$ in place of $(\{\xi\}]$. Let $\mathcal{A}, \mathcal{B} \subseteq S$ then $\mathcal{A}\mathcal{B} = \{\xi\lambda \mid \xi \in \mathcal{A}, \lambda \in \mathcal{B}\}$.

Lemma 1.1.2 *[21, 25] Let $\mathcal{A}, \mathcal{B} \subseteq S$, then*

(1) $\mathcal{A} \subseteq (\mathcal{A}]$.
(2) $(\mathcal{A}](\mathcal{B}] \subseteq (\mathcal{A}\mathcal{B}]$.
(3) $((\mathcal{A}]] = (\mathcal{A}]$.
(4) $((\mathcal{A}](\mathcal{B}]] = (\mathcal{A}\mathcal{B}]$.

Definition 1.1.3 *[79] A* $\emptyset \neq \mathcal{A} \subseteq S$ *is called a right (resp. left) ideal of* S *if*

(i) $\mathcal{AS} \subseteq \mathcal{A}$ (resp. $\mathcal{SA} \subseteq \mathcal{A}$).

(ii) $(\forall \xi_1 \in S)(\forall \xi_2 \in \mathcal{A})(\xi_1 \leq \xi_2 \Longrightarrow \xi_1 \in \mathcal{A})$.

A left as well as right ideal of S is called an ideal of S.

Definition 1.1.4 *[17] A* $\emptyset \neq \mathcal{I} \subseteq S$ *is said to be an interior ideal of* S *if*

(i) $\mathcal{I}^2 \subseteq \mathcal{I}$.

(ii) $\mathcal{SIS} \subseteq \mathcal{I}$.

(ii) $(\forall \xi_1 \in S)(\forall \xi_2 \in \mathcal{I})(\xi_1 \leq \xi_2 \Longrightarrow \xi_1 \in \mathcal{I})$.

Definition 1.1.5 *[19] A* $\emptyset \neq \mathcal{G} \subseteq S$ *is said be a generalized bi-ideal of* S *if*

(i) $\mathcal{GSG} \subseteq \mathcal{G}$.

(ii) $(\forall \xi_1 \in S)(\forall \xi_2 \in \mathcal{G})(\xi_1 \leq \xi_2 \Longrightarrow \xi_1 \in \mathcal{G})$.

Definition 1.1.6 *[18] A* $\emptyset \neq B \subseteq S$ *is said to be a bi-ideal of* S *if*

(i) $B^2 \subseteq B$.

(ii) $BSB \subseteq B$.

(iii) $(\forall \xi_1 \in S)(\forall \xi_2 \in B)(\xi_1 \leq \xi_2 \Longrightarrow \xi_1 \in B)$.

Definition 1.1.7 *[78]An ordered semigroup* S *is regular if*

(i) $\xi \in (\xi S \xi]\ \forall \xi \in S$.

(ii) $\mathcal{A} \subseteq (\mathcal{ASA}]\ \forall \mathcal{A} \subseteq S$.

Definition 1.1.8 *[15] An ordered semigroup* S *is left regular if*

(i) $\xi \in (S\xi^2]\forall \xi \in S$.

(ii) $\mathcal{A} \subseteq (\mathcal{SA}^2]\forall \mathcal{A} \subseteq S$.

Definition 1.1.9 *[15] An ordered semigroup* S *is right regular if*

(i) $\xi \in (\xi^2 S]\forall \xi \in S$.

(ii) $\mathcal{A} \in (\mathcal{A}^2 S]\forall \mathcal{A} \subseteq S$.

Definition 1.1.10 *[78] If S is left as well as right regular then it is called completely regular.*

Definition 1.1.11 *[56] An ordered semigroup S is called intra-regular if*

(1) $\xi \in (S\xi^2 S] \ \forall \xi \in S$.

(2) $\mathcal{A} \subseteq (\mathcal{S}\mathcal{A}^2 S] \ \forall \mathcal{A} \subseteq S$.

Definition 1.1.12 *[26] An ordered semigroup S is said to be left (right) weakly regular if*

(*i*) $\xi \in ((S\xi)^2](\xi \in ((\xi S)^2]) \ \forall \xi \in S$.

(*ii*) $\mathcal{A} \subseteq ((\mathcal{S}\mathcal{A})^2](\mathcal{A} \subseteq ((\mathcal{A}\mathcal{S})^2]) \ \forall \mathcal{A} \subseteq S$.

Definition 1.1.13 *[15] If $T \subseteq S$ then T is called semiprime, if for every $\xi \in S$ such that $\xi^2 \in T$, we have $\xi \in T$.*

Definition 1.1.14 *[15] If $T \subseteq S$ then T is called prime, if for every $\xi_1, \xi_2 \in S$ such that $\xi_1\xi_2 \in T$, we have $\xi_1 \in T$ or $\xi_2 \in T$.*

Proposition 1.1.15 *[80] The following conditions are equivalent for S:*

(*i*) S is regular.

(*ii*) $\mathcal{G} \cap \mathcal{L} \subseteq (\mathcal{G}\mathcal{L}]$ for each left ideal $\mathcal{L}$ and for each generalized bi-ideal $\mathcal{G}$ of S.

(*iii*) $\mathcal{G}(\xi) \cap \mathcal{L}(\xi) \subseteq (\mathcal{G}(\xi)\mathcal{L}(\xi)]$ for every $\xi \in S$.

Proposition 1.1.16 *[80] The following conditions are equivalent for S:*

(*i*) S is regular.

(*ii*) $\mathcal{G} \cap I = (\mathcal{G}I\mathcal{G}]$ for each ideal I and for each generalized bi-ideal $\mathcal{G}$ of S.

(*iii*) $\mathcal{G}(\xi) \cap I(\xi) = (\mathcal{G}(\xi)I(\xi)\mathcal{G}(\xi)]$ for every $\xi \in S$.

Proposition 1.1.17 *[80] The following conditions are equivalent for S:*

(*i*) S is regular.

(*ii*) $\mathcal{R} \cap \mathcal{G} \cap \mathcal{L} \subseteq (\mathcal{R}\mathcal{G}\mathcal{L}]$ for each left ideal $\mathcal{L}$, for each generalized bi-ideal $\mathcal{G}$ and for each right ideal $\mathcal{R}$ of S.

(*iii*) $\mathcal{R}(\xi) \cap \mathcal{G}(\xi) \cap \mathcal{L}(\xi) \subseteq (\mathcal{R}(\xi)\mathcal{G}(\xi)\mathcal{L}(\xi)]$ for every $\xi \in S$.

Proposition 1.1.18 *[79] The following conditions are equivalent for S:*

(1) S is intra-regular.

(2) $\mathcal{R} \cap \mathcal{L} \subseteq (\mathcal{R}\mathcal{L}]$ for every left ideal $\mathcal{L}$ and every right ideal $\mathcal{R}$ of S.

(3) $\mathcal{R}(\xi) \cap \mathcal{L}(\xi) \subseteq (\mathcal{L}(\xi)\mathcal{R}(\xi)]$ for every $\xi \in S$.

Proposition 1.1.19 *[16] For non-empty subsets A and B of S, we have:*

(1) $A \subseteq B$ iff $\chi_A \preceq \chi_B$.

(2) $\chi_A \wedge \chi_B = \chi_{A\cap B}$.

(3) $\chi_A \circ \chi_B = \chi_{(AB]}$.

By a fuzzy subset f of S, we mean a function $f : S \longrightarrow [0, 1]$.

If f and g are two fuzzy subsets of S then we can define $f \wedge g$ and $f \vee g$ as

$$\begin{aligned} f \wedge g(\xi) &= \min\{f(\xi), g(\xi)\} \\ f \vee g(\xi) &= \max\{f(\xi), g(\xi)\} \text{ for all } \xi \in S. \end{aligned}$$

If $A \subseteq S$, then the characteristic function f_A of A is defined as:

$$f_A : S \longrightarrow [0,1] : \xi \longrightarrow f_A(\xi) = \begin{cases} 1 \text{ if } \xi \in A, \\ 0 \text{ if } \xi \notin A. \end{cases}$$

Definition 1.1.20 *[20] A fuzzy subsemigroup f of S is defined as:*

$$(\forall \xi_1, \xi_2 \in S)(f(\xi_1\xi_2) \geq \min\{f(\xi_1), f(\xi_2)\}).$$

Definition 1.1.21 *[16] A fuzzy subset f of S is called a fuzzy left (resp. right) ideal of S if*

(i) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \longrightarrow f(\xi_1) \geq f(\xi_2))$.

(ii) $(\forall \xi_1, \xi_2 \in S)(f(\xi_1\xi_2) \geq f(\xi_2)(\text{resp. } f(\xi_1\xi_2) \geq f(\xi_1)))$.

A fuzzy left as well as fuzzy right ideal of S is called a fuzzy ideal of S.

A level subset $W(f; h)$ of f is defined as

$$W(f; h) = \{\xi \in S \mid f(\xi) \geq h\}$$

for some $h \in (0, 1]$.

Lemma 1.1.22 *[20] Let $\emptyset \neq A \subseteq S$. Then f_A is a fuzzy left (resp. right) ideal of S iff A is left (resp. right) ideal of S.*

Lemma 1.1.23 *[20] A $\emptyset \neq W(f; h) \subseteq S$ is a left (resp. right) ideal of S iff f is a fuzzy left (resp. right) ideal of S.*

Example 1.1.24 *[25] Let $S = \{0, 1, 2, 3, 4, 5\}$ be an ordered semigroup. Define multiplication and order relation as:*

·	0	1	2	3	4	5
0	0	0	0	3	0	0
1	0	1	1	3	1	1
2	0	1	2	3	4	4
3	0	0	3	3	3	3
4	0	1	2	3	4	4
5	0	1	2	3	4	5

$\leq = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (5, 4)\}$.

Right ideal of S are: $\{0, 3\}$, $\{0, 1, 3\}$ and S. Left ideal of S are: $\{0\}$, $\{3\}$, $\{0, 1\}$, $\{0, 3\}$, $\{0, 1, 3\}$, $\{0, 1, 2, 3\}$, $\{0, 1, 3, 4, 5\}$ and S. Define a fuzzy subset f as

S	0	1	2	3	4	5
f	0.8	0.5	0.6	0.4	0.4	0.4

.

Then

$$W(f; h) = \begin{cases} S \text{ if } h \in (0, 0.4] \\ \{0, 1, 3\} \text{ if } h \in (0.4, 0.5] \\ \{0, 3\} \text{ if } h \in (0.5, 0.6] \\ \{0\} \text{ if } h \in (0.6, 0.8] \\ \emptyset \text{ if } h \in (0.8, 1] \end{cases}$$

and $W(f; h)$ are right ideal of S. So by Lemma 1.1.23, f is a fuzzy right ideal of S.

Definition 1.1.25 *[22] A fuzzy subset f of S is said to be a fuzzy interior ideal of S if:*

(i) $(\forall \xi_1, \xi_2 \in S)(f(\xi_1\xi_2) \geq \min\{f(\xi_1), f(\xi_2)\})$.

(ii) $(\forall \xi_1, a, \xi_2 \in S)(f(\xi_1 a \xi_2) \geq f(a))$.

(iii) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Longrightarrow f(\xi_1) \geq f(\xi_2))$.

Lemma 1.1.26 *[22] A $\emptyset \neq W(f;h) \subseteq S$ is an interior ideal of S iff f is a fuzzy interior ideal of S.*

Lemma 1.1.27 *[20] Let $\emptyset \neq \mathcal{I} \subseteq S$. Then $f_{\mathcal{I}}$ is a fuzzy interior ideal of S iff $\mathcal{I}$ is an interior ideal of S.*

Example 1.1.28 *[23] Let $S = \{0, 1, 2, 3, 4\}$ be a set. Define multiplication and order relation on S as follow:*

$\cdot$	0	1	2	3	4
0	0	3	0	3	3
1	0	1	0	3	3
2	0	3	2	3	4
3	0	3	1	3	3
4	0	3	2	3	4

$\leq := \{(0,0), (0,2), (0,3), (0,4), (1,1), (1,3), (1,4), (2,2), (2,4), (3,3), (3,4), (4,4)\}$.

Then clearly S is an ordered semigroup. Interior ideals of S are: $\{0, 1, 3\}, \{0, 2, 3, 4\}$ and S. Define a fuzzy subset f as

S	0	1	2	3	4
f	0.8	0.6	0.5	0.5	0.2

.

Then

$$W(f;h) = \begin{cases} S \text{ if } h \in (0, 0.2] \\ \{0, 1, 3\} \text{ if } h \in (0.2, 0.5] \\ \emptyset \text{ if } h \in (0.5, 1] \end{cases}.$$

Then by Lemma 1.1.26, f is a fuzzy interior ideal of S.

Definition 1.1.29 *[83] A fuzzy subset f of S is said to be a fuzzy generalized bi-ideal of S if:*

(i) $(\forall \xi_1, \xi_2, \xi_3 \in S)(f(\xi_1\xi_2\xi_3) \geq \min\{f(\xi_1), (\xi_3)\})$.

(ii) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Longrightarrow f(\xi_1) \geq f(\xi_2))$.

Lemma 1.1.30 *[83] A $\emptyset \neq \mathcal{G} \subseteq S$ is a generalized bi-ideal of S iff $f_{\mathcal{G}}$ is a fuzzy generalized bi-ideal of S.*

Lemma 1.1.31 *[83] A $\emptyset \neq W(f;h) \subseteq S$ is a generalized bi-ideal of S iff f is a fuzzy generalized bi-ideal of S.*

Example 1.1.32 *[83] Let $S = \{p, q, r, s\}$ be a set. Define multiplication and order relation on S as:*

$\cdot$	p	q	r	s
p	p	p	p	p
q	p	p	p	p
r	p	p	q	p
s	p	p	q	q

$\leq := \{(p,p),(q,q),(r,r),(s,s),(p,q)\}$.

Then clearly S is an ordered semigroup. Generalized bi-ideals of S are: $\{p\}, \{p,q\}, \{p,r\}, \{p,s\}$, $\{p,q,r\}, \{p,q,s\}, \{p,r,s\}$ and S. Some of them are not bi-ideals of S i.e. $\{p,r\}, \{p,s\}$ and $\{p,r,s\}$. All fuzzy subsets f of S which satisfy $f(p) \geq x$ for all $x \in S$ are fuzzy generalized bi-ideal of S. Here we defined a fuzzy subset

S	p	q	r	s
f	0.6	0.1	0.3	0

which is a fuzzy generalized bi-ideal of S but not fuzzy bi-ideal of S because $0.1 = f(q) = f(r \cdot r) \ngeq \min\{f(r), f(r)\} = 0.3$.

Definition 1.1.33 *[16] A fuzzy subset f of S is called a fuzzy bi-ideal of S if:*

(i) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Longrightarrow f(\xi_1) \geq f(\xi_2))$.

(ii) $(\forall \xi_1, \xi_2 \in S)(f(\xi_1 \xi_2) \geq \min\{f(\xi_1), f(\xi_2)\})$.

(iii) $(\forall \xi_1, \xi_2, \xi_3 \in S)(f(\xi_1 \xi_2 \xi_3) \geq \min\{f(\xi_1), (\xi_3)\})$.

Lemma 1.1.34 *[16] A $\emptyset \neq B \subseteq S$ is a bi-ideal of S iff f_B is a fuzzy bi-ideal of S.*

Lemma 1.1.35 *[16] A* $\emptyset \neq W(f;h) \subseteq S$ *is a bi-ideal of* S *iff* f *is a fuzzy bi-ideal of* S*.*

Example 1.1.36 *[23, 24] Let* $S = \{0, 1, 2, 3, 4\}$ *be an ordered semigroup with the following multiplication and order relation:*

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0	3	0
2	0	4	2	2	4
3	0	1	3	3	1
4	0	4	0	2	0

$\leq := \{(0,0), (0,1), (0,2), (0,3), (0,4), (1,1), (2,2), (3,3), (4,4)\}.$

Bi-ideal of S are $\{0\}, \{0,2\}, \{0,2,3\}$ and S. Define a fuzzy subset f on S as

S	0	1	2	3	4
f	0.8	0.5	0.7	0.6	0.5

.

Then

$$W(f;h) = \begin{cases} S \text{ if } h \in (0, 0.5] \\ \{0,2,3\} \text{ if } h \in (0.5, 0.6] \\ \{0,2\} \text{ if } h \in (0.6, 0.7] \\ \{0\} \text{ if } h \in (0.7, 0.8] \\ \emptyset \text{ if } h \in (0.8, 1] \end{cases}$$

By Lemma 1.1.35, f is a fuzzy bi-ideal of S.

Definition 1.1.37 *[67] A bipolar fuzzy set (BFS)* f *defined over universal set* S *is a mapping of the type*

$$f = \{(\xi, f_n(\xi), f_p(\xi)) : \xi \in S\},$$

$$f_n \colon S \longrightarrow [-1, 0]$$

$$f_p \colon S \longrightarrow [0, 1]$$

where $f_p(\xi)$ indicates that how much ξ satisfy some property and $f_n(\xi)$ indicates that how much ξ satisfy some implicit counter-property.

For our convenience we shall use the symbol $f = (S; f_n, f_p)$ for BFS.

If $f = (S; f_n, f_p)$ and $g = (S; g_n, g_p)$ are two BFSs in S. Then

$f \subseteq g$ iff $(\forall \xi \in S)(f_n(\xi) \geq g_n(\xi)$ and $f_p(\xi) \leq g_p(\xi))$,

$f = g$ iff $f \subseteq g$ and $g \subseteq f$,

$f \cap g = (S;\ f_n \vee g_n,\ f_p \wedge g_p)$,

$f \cup g = (S; f_n \wedge g_n, f_p \vee g_p)$.

$f^c = \{\xi, f_n^c(\xi), f_p^c(\xi) : \xi \in S\}$ where $f_n^c(\xi) = -1 - f_n(\xi), \quad f_p^c(\xi) = 1 - f_p(\xi)$.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i|i \in \Lambda\} := \begin{cases} \max\{a_i|i \in \Lambda\} \text{ if } \Lambda \text{ is finite,} \\ \sup\{a_i|i \in \Lambda\} \text{ otherwise,} \end{cases}$$

$$\bigwedge\{a_i|i \in \Lambda\} := \begin{cases} \min\{a_i|i \in \Lambda\} \text{ if } \Lambda \text{ is finite,} \\ \inf\{a_i|i \in \Lambda\} \text{ otherwise.} \end{cases}$$

For a BFS $f = (S; f_n, f_p)$ in S and $(s,t) \in [-1,0) \times (0,1]$, we define

$$N(f;s) := \{\xi \in S | f_n(\xi) \leq s\}$$

$$P(f;t) := \{\xi \in S | f_p(\xi) \geq t\}$$

which are called the negative s-cut and the positive t-cut of $f = (S; f_n, f_p)$, respectively.

The (s,t)-cut of $f = (S; f_n, f_p)$ is denoted by $C(f;(s,t))$ and is defined by

$$C(f;(s,t)) = N(f;s) \cap P(f;t).$$

Bipolar fuzzy subsets $0 = (S; 0_n, 0_p)$ and $1 = (S; 1_n, 1_p)$ are defined as follows:

$0_n(\xi) = 0 = 0_p(\xi)$, $1_n(\xi) = -1$ and $1_p(\xi) = 1$ for all $\xi \in S$. For $\xi \in S$, define $A_\xi = \{(y,z) \in S \times S \mid \xi \leq yz\}$.

Definition 1.1.38 *[81] For BFSs $f = (S; f_n, f_p)$ and $g = (S; g_n, g_p)$ of S, defined the product as $f \circ g = (S; f_n \circ g_n, f_p \circ g_p)$, where*

$$(f_n \circ g_n)(\xi) = \begin{cases} \bigwedge\{f_n(y) \vee g_n(z)\} & \text{if } A_\xi \neq \emptyset \\ 0 & \text{if } A_\xi = \emptyset \end{cases}$$

$$\text{and } (f_p \circ g_p)(\xi) = \begin{cases} \bigvee\{f_p(y) \wedge g_p(z)\} & \text{if } A_\xi \neq \emptyset \\ 0 & \text{if } A_\xi = \emptyset \end{cases}.$$

Definition 1.1.39 *[81] If $\emptyset \neq A \subseteq S$, then the characteristic function of A denoted by $\chi_A = (S; \chi_{nA}, \chi_{pA})$ where χ_{nA} and χ_{pA} are defined as*

$$\chi_{nA}(\xi) = \begin{cases} -1 & \text{if } \xi \in A \\ 0 & \text{if } \xi \notin A \end{cases} \quad \text{and}$$

$$\chi_{pA}(\xi) = \begin{cases} 1 & \text{if } \xi \in A \\ 0 & \text{if } \xi \notin A \end{cases}.$$

1.2 Bipolar fuzzy ideals

In this section, we define bipolar fuzzy left ideals (BFLIs) (resp. bipolar fuzzy right ideals (BFRIs)) of S and characterize S by means of BFLI (resp. BFRI). The results given here are taken from our accepted work (see [86]).

Definition 1.2.1 *[81] A BFS $f = (S; f_n, f_p)$ is said to be BFLI (resp. BFRI) of S if it satisfies:*

1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Longrightarrow f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2))$.

2) $(\forall \xi_1, \xi_2 \in S)(f_n(\xi_1\xi_2) \leq f_n(\xi_2)(\text{resp.} f_n(\xi_1\xi_2) \leq f_n(\xi_1))$ and $f_p(\xi_1\xi_2) \geq f_p(\xi_2)(\text{resp.} f_p(\xi_1\xi_2) \geq f_p(\xi_1)))$.

A BFS is said to be BFI of S if it is both BFLI and BFRI of S.

Theorem 1.2.2 *A BFS $f = (S; f_n, f_p)$ of S is a BFLI of S iff (s,t)-cut $C(f; (s,t))$ is a left ideal of S for all $(s,t) \in [-1, 0) \times (0, 1]$.*

Proof. Let $f = (S; f_n, f_p)$ is a BFLI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in C(f; (s,t))$. Then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since $f = (S; f_n, f_p)$ is a BFLI of S. So $f_n(\xi_1) \leq f_n(\xi_2) \leq s$ and $f_p(\xi_1) \geq f_p(\xi_2) \geq t$. Hence $\xi_1 \in C(f; (s,t))$.

Let $\xi_1 \in S$ and $\xi_2 \in C(f;(s,t))$ then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since $f = (S; f_n, f_p)$ is a BFLI, so $f_n(\xi_1\xi_2) \leq f_n(\xi_2) \leq s$ and $f_p(\xi_1\xi_2) \geq f_p(\xi_2) \geq t$. This implies $\xi_1\xi_2 \in C(f;(s,t))$. Hence $C(f;(s,t))$ is left ideal of S.

Conversely, let $C(f;(s,t))$ is left ideal of S. We show that $f = (\mathrm{S}; f_n, f_p)$ is a BFLI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. Let $f_n(\xi_2) = s$ and $f_p(\xi_2) = t$. This implies $\xi_2 \in C(f;(s,t))$. Since $C(f;(s,t))$ is left ideal of S. This implies $\xi_1 \in C(f;(s,t))$. Hence $f_n(\xi_1) \leq s = f_n(\xi_2)$ and $f_p(\xi_1) \geq t = f_p(\xi_2)$.

Let $\xi_1, \xi_2 \in S$ such that $f_n(\xi_2) = s$ and $f_p(\xi_2) = t$. Then $\xi_2 \in C(f;(s,t))$. So $\xi_1\xi_2 \in C(f;(s,t))$ by left ideal property. Thus $f_n(\xi_1\xi_2) \leq s = f_n(\xi_2)$ and $f_p(\xi_1\xi_2) \geq t = f_p(\xi_2)$. Hence $f = (S; f_n, f_p)$ is a BFLI of S. ■

Theorem 1.2.3 *A BFS $f = (S; f_n, f_p)$ of S is a BFLI of S iff (s,t)-cut $C(f;(s,t))$ is a left ideal of S for all $(s,t) \in [-1,0) \times (0,1]$.*

Theorem 1.2.4 *Let $\mathcal{L} \subseteq S$, then $\mathcal{L}$ is left ideal of S iff $\chi_{\mathcal{L}} = (S; \chi_{n\mathcal{L}}, \chi_{p\mathcal{L}})$ is a BFLI of S.*

Proof. Let $\mathcal{L}$ is left ideal of S and $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. If $\xi_2 \in \mathcal{L}$ then $\xi_1 \in \mathcal{L}$, so

$$\begin{aligned}\chi_{n\mathcal{L}}(\xi_1) &= -1 \leq \chi_{n\mathcal{L}}(\xi_2) \text{ and} \\ \chi_{p\mathcal{L}}(\xi_1) &= 1 \geq \chi_{p\mathcal{L}}(\xi_2).\end{aligned}$$

If $\xi_2 \notin \mathcal{L}$, then

$$\begin{aligned}\chi_{n\mathcal{L}}(\xi_1) &\leq 0 = \chi_{n\mathcal{L}}(\xi_2) \text{ and} \\ \chi_{p\mathcal{L}}(\xi_1) &\geq 0 = \chi_{p\mathcal{L}}(\xi_2).\end{aligned}$$

Let $\xi_1, \xi_2 \in S$. If $\xi_2 \in \mathcal{L}$ then $\xi_1\xi_2 \in \mathcal{L}$, so

$$\begin{aligned}\chi_{n\mathcal{L}}(\xi_1\xi_2) &= -1 \leq \chi_{n\mathcal{L}}(\xi_2) \text{ and} \\ \chi_{p\mathcal{L}}(\xi_1\xi_2) &= 1 \geq \chi_{p\mathcal{L}}(\xi_2).\end{aligned}$$

If $\xi_2 \notin \mathcal{L}$, then

$$\begin{aligned} \chi_{n\mathcal{L}}(\xi_1\xi_2) &\leq 0 = \chi_{n\mathcal{L}}(\xi_2) \text{ and} \\ \chi_{p\mathcal{L}}(\xi_1\xi_2) &\geq 0 = \chi_{p\mathcal{L}}(\xi_2). \end{aligned}$$

Therefore $\chi_{\mathcal{L}} = (S; \chi_{n\mathcal{L}}, \chi_{p\mathcal{L}})$ is a BFLI of S.

Conversely, assume that $\chi_{\mathcal{L}} = (S; \chi_{n\mathcal{L}}, \chi_{p\mathcal{L}})$ is a BFLI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in \mathcal{L}$. Then $\chi_{n\mathcal{L}}(\xi_2) = -1$ and $\chi_{n\mathcal{L}}(\xi_2) = 1$, so $\xi_2 \in C(\chi_{\mathcal{L}}; (-1, 1))$. By Theorem 1.2.3, we have $C(\chi_{\mathcal{L}}; (-1, 1))$ is left ideal of S. Thus $\xi_1 \in C(\chi_{\mathcal{L}}; (-1, 1))$ and so $\chi_{n\mathcal{L}}(\xi_1) \leq -1$ and $\chi_{p\mathcal{L}}(\xi_1) \geq 1$. Hence $\xi_1 \in \mathcal{L}$.

Let $\xi_1 \in S$ and $\xi_2 \in \mathcal{L}$, then $\chi_{n\mathcal{L}}(\xi_2) = -1$ and $\chi_{n\mathcal{L}}(\xi_2) = 1$, so $\xi_2 \in C(\chi_{\mathcal{L}}; (-1, 1))$. Since $C(\chi_{\mathcal{L}}; (-1, 1))$ is left ideal of S, so $\xi_1\xi_2 \in C(\chi_{\mathcal{L}}; (-1, 1))$. Thus $\chi_{n\mathcal{L}}(\xi_1\xi_2) = -1$ and $\chi_{p\mathcal{L}}(\xi_1\xi_2) = 1$. So $\xi_1\xi_2 \in \mathcal{L}$. Hence $\mathcal{L}$ is a left ideal of S. ∎

Theorem 1.2.5 *Let $\mathcal{R} \subseteq S$, then $\mathcal{R}$ is right ideal of S iff $\chi_{\mathcal{R}} = (S; \chi_{n\mathcal{R}}, \chi_{p\mathcal{R}})$ is a BFRI of S.*

Theorem 1.2.6 *A BFS $f = (S; f_n, f_p)$ in S is a BFLI of S iff it satisfies the following conditions, for all $(s, t) \in [-1, 0) \times (0, 1]$.*

$$1)\ (\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2, \frac{\xi_2}{(s,t)} \in f \Longrightarrow \frac{\xi_1}{(s,t)} \in f).$$
$$2)\ (\forall \xi_1, \xi_2 \in S)(\frac{\xi_2}{(s,t)} \in f \Longrightarrow \frac{\xi_1\xi_2}{(s,t)} \in f).$$

Proof. Assume that $f = (S; f_n, f_p)$ is a BFLI of S. Let $\xi_1, \xi_2 \in S$ be such that $\xi_1 \leq \xi_2$ and $\frac{\xi_2}{(s,t)} \in f$ where $(s, t) \in [-1, 0) \times (0, 1]$ then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since $f = (S; f_n, f_p)$ is a BFLI of S. Therefore for $\xi_1 \leq \xi_2$, we have $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2)$. This implies that $f_n(\xi_1) \leq s$ and $f_p(\xi_1) \geq t$, so $\frac{\xi_1}{(s,t)} \in f$.

Let $\xi_1, \xi_2 \in S$ and $\frac{\xi_2}{(s,t)} \in f$ for $(s, t) \in [-1, 0) \times (0, 1]$. Then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since $f = (S; f_n, f_p)$ is BFLI of S so $f_n(\xi_1\xi_2) \leq f_n(\xi_2)$ and $f_p(\xi_1\xi_2) \geq f_p(\xi_2)$. This implies that $f_n(\xi_1\xi_2) \leq s$ and $f_p(\xi_1\xi_2) \geq t$, so $\frac{\xi_1\xi_2}{(s,t)} \in f$.

Conversely, assume the conditions 1 and 2 are true. We show that $f = (S; f_n, f_p)$ is a BFLI of S. Let $\xi_1, \xi_2 \in S$ be such that $\xi_1 \leq \xi_2$. Let $f_n(\xi_2) = s$ and $f_p(\xi_2) = t$. Then $\frac{\xi_2}{(s,t)} \in f$, so by condition 1, $\frac{\xi_1}{(s,t)} \in f$ and so $f_n(\xi_1) \leq s$ and $f_p(\xi_1) \geq t$. Therefore $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2)$. Let $\xi_1, \xi_2 \in S$ be such that $f_n(\xi_2) = s$ and $f_p(\xi_2) = t$. Then $\frac{\xi_2}{(s,t)} \in f$, so by condition 2, $\frac{\xi_1\xi_2}{(s,t)} \in f$, and so $f_n(\xi_1\xi_2) \leq s$ and $f_p(\xi_1\xi_2) \geq t$. Therefore $f_n(\xi_1\xi_2) \leq f_n(\xi_2)$ and $f_p(\xi_1\xi_2) \geq f_p(\xi_2)$. Hence $f = (S; f_n, f_p)$ is an BFLI of S. Same argument can be given for BFRI. ■

1.3 Bipolar fuzzy interior ideals

In this section, we give the definitions and results of BFIIs in ordered semigroups and characterize S by means of BFIIs. The results given here are taken from our published work (see [85]).

Definition 1.3.1 *A BFS $f = (S; f_n, f_p)$ is called BFII of S, if*

(I_1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Rightarrow f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2))$,

(I_2) $(\forall \xi_1, \xi_2 \in S)(f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\})$,

(I_3) $(\forall \xi_1, a, \xi_2 \in S)(f_n(\xi_1 a \xi_2) \leq f_n(a)$ and $f_p(\xi_1 a \xi_2) \geq f_p(a))$.

Example 1.3.2 *Let $S = \{n, k, l, m\}$ be an ordered semigroup. Define multiplication and order relation on S as:*

$\cdot$	n	k	l	m
n	n	n	n	n
k	n	n	n	n
l	n	n	n	k
m	n	n	k	l

$\leq := \{(n,n), (k,k), (l,l), (m,m), (n,k)\}$. *Define a BFS $f = (S; f_n, f_p)$ by*

S	n	k	l	m
f_n	-0.7	-0.4	-0.5	-0.2
f_p	0.8	0.3	0.6	0.1

.

Then by above definition, we have $f = (S; f_n, f_p)$ is a BFII of S.

Theorem 1.3.3 *Let $f = (S; f_n, f_p)$ be a BFS in S. Then $f = (S; f_n, f_p)$ is a BFII of S iff for all $(s,t) \in [-1,0) \times (0,1]$, we have*

(1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2, \frac{\xi_2}{(s,t)} \in f \Longrightarrow \frac{\xi_1}{(s,t)} \in f)$,
(2) $(\forall \xi_1, \xi_2 \in S)(\frac{\xi_1}{(s_1,t_1)} \in f$ and $\frac{\xi_2}{(s_2,t_2)} \in f \Longrightarrow \frac{\xi_1 \xi_2}{(\vee\{s_1,s_2\}, \wedge\{t_1,t_2\})} \in f)$,
(3) $(\forall \xi_1, a, \xi_2 \in S)(\frac{a}{(s,t)} \in f \Longrightarrow \frac{\xi_1 a \xi_2}{(s,t)} \in f)$.

Proof. Suppose $f = (S; f_n, f_p)$ is a BFII of S. Let $\xi_1, \xi_2 \in S$ with $\xi_1 \leq \xi_2$ and $\frac{\xi_2}{(s,t)} \in f$, then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since $f = (S; f_n, f_p)$ is a BFII of S, we have $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2)$. This implies that $f_n(\xi_1) \leq s$ and $f_p(\xi_1) \geq t$ and so $\frac{\xi_1}{(s,t)} \in f$. Let $\xi_1, \xi_2 \in S$ such that $\frac{\xi_1}{(s_1,t_1)} \in f$ and $\frac{\xi_2}{(s_2,t_2)} \in f$. Then $f_n(\xi_1) \leq s_1$, $f_n(\xi_2) \leq s_2$, $f_p(\xi_1) \geq t_1$ and $f_p(\xi_2) \geq t_2$. From definition of BFII, we have

$$f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2) \leq \bigvee\{s_1, s_2\} \text{ and } f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq \bigwedge\{t_1, t_2\}$$

so that $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\}, \wedge\{t_1,t_2\})} \in f$. Let $\xi_1, a, \xi_2 \in S$ such that $\frac{a}{(s,t)} \in f$. Then $f_n(a) \leq s$ and $f_p(a) \geq t$. This implies that $f_n(\xi_1 a \xi_2) \leq f_n(a) \leq s$ and $f_p(\xi_1 a \xi_2) \geq f_p(a) \geq t$, so $\frac{\xi_1 a \xi_2}{(s,t)} \in f$.

Conversely, let $\xi_1, \xi_2 \in S$ with $\xi_1 \leq \xi_2$. Let $f_n(\xi_2) = s$ and $f_p(\xi_2) = t$. Then $\frac{\xi_2}{(s,t)} \in f$, so by condition 1, $\frac{\xi_1}{(s,t)} \in f$ and so $f_n(\xi_1) \leq s$ and $f_p(\xi_1) \geq t$. Therefore $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2)$. Since $\frac{\xi_1}{(f_n(\xi_1), f_p(\xi_1))} \in f$ and $\frac{\xi_2}{(f_n(\xi_2), f_p(\xi_2))} \in f$ for all $\xi_1, \xi_2 \in S$, so by (2), we have $\frac{\xi_1\xi_2}{(\vee\{f_n(\xi_1), f_n(\xi_2)\}, \wedge\{f_p(\xi_1), f_p(\xi_2)\})} \in f$. This implies that, $f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2)$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$. Let $\xi_1, a, \xi_2 \in S$. Since $\frac{a}{(f_n(a), f_p(a))} \in f$ for all $a \in S$, so by (3) we have $\frac{\xi_1 a \xi_2}{(f_n(a), f_p(a))} \in f$. This implies that $f_n(\xi_1 a \xi_2) \leq f_n(a)$ and $f_p(\xi_1 a \xi_2) \geq f_p(a)$. Hence $f = (S; f_n, f_p)$ is BFII of S. ■

Theorem 1.3.4 *Let $f = (S; f_n, f_p)$ be a BFS in S. Then $f = (S; f_n, f_p)$ is a BFII of S iff the non-empty (s,t)-cut $C(f; (s,t))$ of $f = (S; f_n, f_p)$ is an interior ideal of S for all $(s,t) \in [-1,0) \times (0,1]$.*

Proof. Let $f = (S; f_n, f_p)$ is a BFII of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in C(f; (s,t))$, then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since f is a BFII of S so $f_n(\xi_1) \leq f_n(\xi_2) \leq s$ and $f_p(\xi_1) \geq f_p(\xi_2) \geq t$. This implies that $f_n(\xi_1) \leq s$ and $f_p(\xi_1) \geq t$. Hence $\xi_1 \in C(f; (s,t))$.

Let $\xi_1, \xi_2 \in C(f;(s,t))$, then $f_n(\xi_1) \leq s$, $f_n(\xi_2) \leq s$ and $f_p(\xi_1) \geq t$, $f_p(\xi_2) \geq t$. Since f is a BFII of S so $f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2)\} \leq s$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq t$. $\Rightarrow f_n(\xi_1\xi_2) \leq s$ and $f_p(\xi_1\xi_2) \geq t$. Hence $\xi_1\xi_2 \in C(f;(s,t))$.

Let $a \in C(f;(s,t))$ and $\xi_1, \xi_2 \in S$, then $f_n(a) \leq s$ and $f_p(a) \geq t$. Since f is a BFII of S so $f_n(\xi_1 a\xi_2) \leq f_n(a) \leq s$ and $f_p(\xi_1 a\xi_2) \geq f_p(a) \geq t$. This implies that $f_n(\xi_1 a\xi_2) \leq s$ and $f_p(\xi_1 a\xi_2) \geq t$. Hence $\xi_1 a\xi_2 \in C(f;(s,t))$. Thus $C(f;(s,t))$ is an interior ideal of S.

Conversely, assume that $C(f;(s,t))$ is an interior ideal of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. Since $\xi_2 \in C(f;(f_n(\xi_2), f_p(\xi_2)))$ and $C(f;(s,t))$ is an interior ideal of S so $\xi_1 \in C(f;(f_n(\xi_2), f_p(\xi_2)))$. This implies that $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2)$.

As we know that $\xi_1, \xi_2 \in C(f; \bigvee\{(f_n(\xi_1), f_n(\xi_2)\}, \bigwedge\{(f_p(\xi_1), f_p(\xi_2)\})$ and $C(f;(s,t))$ is an interior ideal of S, so $\xi_1\xi_2 \in C(f; \bigvee\{(f_n(\xi_1), f_n(\xi_2)\}, \bigwedge\{(f_p(\xi_1), f_p(\xi_2)\})$. This implies that $f_n(\xi_1\xi_2) \leq \bigvee\{(f_n(\xi_1), f_n(\xi_2)\}$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{(f_p(\xi_1), f_p(\xi_2)\}$.

Let $\xi_1, \xi_2 \in S$ and $a \in C(f;(f_n(a), f_p(a))$. Since $C(f;(s,t))$ is an interior ideal of S, so $\xi_1 a\xi_2 \in C(f; f_n(a), f_p(a))$. This implies that $f_n(\xi_1 a\xi_2) \leq f_n(a)$ and $f_p(\xi_1 a\xi_2) \geq f_p(a)$. Hence $f = (S; f_n, f_p)$ is a BFII of S. ■

Theorem 1.3.5 *Let $\mathcal{I} \subseteq S$, then $\mathcal{I}$ is an interior ideal of S iff $\chi_{\mathcal{I}} = (S, \chi_{n\mathcal{I}}, \chi_{p\mathcal{I}})$ is a BFII of S.*

Proof. Let $\mathcal{I}$ is interior ideal of S and $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. If $\xi_2 \in \mathcal{I}$ then $\xi_1 \in \mathcal{I}$, so

$$\begin{aligned} \chi_{n\mathcal{I}}(\xi_1) &= -1 \leq \chi_{n\mathcal{I}}(\xi_2) \text{ and} \\ \chi_{p\mathcal{I}}(\xi_1) &= 1 \geq \chi_{p\mathcal{I}}(\xi_2). \end{aligned}$$

If $\xi_2 \notin \mathcal{I}$ then

$$\begin{aligned} \chi_{n\mathcal{I}}(\xi_1) &\leq 0 = \chi_{n\mathcal{I}}(\xi_2) \text{ and} \\ \chi_{p\mathcal{I}}(\xi_1) &\geq 0 = \chi_{p\mathcal{I}}(\xi_2). \end{aligned}$$

Let $\xi_1, \xi_2 \in S$. If $\xi_1, \xi_2 \in \mathcal{I}$ then $\xi_1\xi_2 \in \mathcal{I}$. So

$$\begin{aligned} \chi_{n\mathcal{I}}(\xi_1\xi_2) &= -1 \leq (\chi_{n\mathcal{I}}(\xi_1) \vee \chi_{n\mathcal{I}}(\xi_2)) \text{ and} \\ \chi_{p\mathcal{I}}(\xi_1\xi_2) &= 1 \geq (\chi_{p\mathcal{I}}(\xi_1) \wedge \chi_{p\mathcal{I}}(\xi_2)). \end{aligned}$$

If $\xi_1 \notin \mathcal{I}$ or $\xi_2 \notin \mathcal{I}$, then

$$\begin{aligned} \chi_{n\mathcal{I}}(\xi_1\xi_2) &\leq 0 = (\chi_{n\mathcal{I}}(\xi_1) \vee \chi_{n\mathcal{I}}(\xi_2)) \text{ and} \\ \chi_{p\mathcal{I}}(\xi_1\xi_2) &\geq 0 = (\chi_{p\mathcal{I}}(\xi_1) \wedge \chi_{p\mathcal{I}}(\xi_2)). \end{aligned}$$

Let $\xi_1, a, \xi_2 \in S$. If $a \in \mathcal{I}$ then $\xi_1 a \xi_2 \in \mathcal{I}$, so

$$\begin{aligned} \chi_{n\mathcal{I}}(\xi_1 a \xi_2) &= -1 \leq \chi_{n\mathcal{I}}(a) \text{ and} \\ \chi_{p\mathcal{I}}(\xi_1 a \xi_2) &= 1 \geq \chi_{p\mathcal{I}}(a). \end{aligned}$$

If $a \notin \mathcal{I}$, then

$$\begin{aligned} \chi_{n\mathcal{I}}(\xi_1 a \xi_2) &\leq 0 = \chi_{n\mathcal{I}}(a) \text{ and} \\ \chi_{p\mathcal{I}}(\xi_1 a \xi_2) &\geq 0 = \chi_{p\mathcal{I}}(a). \end{aligned}$$

Therefore $\chi_{\mathcal{I}} = (S; \chi_{n\mathcal{I}}, \chi_{p\mathcal{I}})$ is a BFII of S.

Conversely, assume that $\chi_{\mathcal{I}} = (S; \chi_{n\mathcal{I}}, \chi_{p\mathcal{I}})$ is a BFII of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in \mathcal{I}$. Then $\chi_{n\mathcal{I}}(\xi_2) = -1$ and $\chi_{p\mathcal{I}}(\xi_2) = 1$, so $\xi_2 \in C(\chi_{\mathcal{I}}; (-1, 1))$. By Theorem 1.3.4, we have $C(\chi_{\mathcal{I}}; (-1, 1))$ is an interior ideal of S. Thus $\xi_1 \in C(\chi_{\mathcal{I}}; (-1, 1))$ and so $\chi_{n\mathcal{I}}(\xi_1) \leq -1$ and $\chi_{p\mathcal{I}}(\xi_1) \geq 1$. Hence $\xi_1 \in \mathcal{I}$.

Let $\xi_1, \xi_2 \in \mathcal{I}$. Then $\chi_{n\mathcal{I}}(\xi_1) = -1 = \chi_{n\mathcal{I}}(\xi_2)$ and $\chi_{p\mathcal{I}}(\xi_1) = 1 = \chi_{p\mathcal{I}}(\xi_2)$, so $\xi_1, \xi_2 \in C(\chi_{\mathcal{I}}; (-1, 1))$. Since $C(\chi_{\mathcal{I}}; (-1, 1))$ is an interior ideal of S, so $\xi_1\xi_2 \in C(\chi_{\mathcal{I}}; (-1, 1))$. Thus $\chi_{n\mathcal{I}}(\xi_1\xi_2) \leq -1$ and $\chi_{p\mathcal{I}}(\xi_1\xi_2) \geq 1$. Hence $\xi_1\xi_2 \in \mathcal{I}$.

Let $a \in \mathcal{I}$ and $\xi_1, \xi_2 \in S$, then $\chi_{n\mathcal{I}}(a) = -1$ and $\chi_{p\mathcal{I}}(a) = 1$. So $a \in C(\chi_{\mathcal{I}}; (-1, 1))$. Since $C(\chi_{\mathcal{I}}; (-1, 1))$ is an interior ideal of S by Theorem 1.3.4, so $\xi_1 a \xi_2 \in C(\chi_{\mathcal{I}}; (-1, 1))$. This implies $\chi_{n\mathcal{I}}(\xi_1 a \xi_2) \leq -1$ and $\chi_{p\mathcal{I}}(\xi_1 a \xi_2) \geq 1$. Thus $\xi_1 a \xi_2 \in \mathcal{I}$ and hence $\mathcal{I}$ is an interior ideal of S. ■

1.4 Bipolar fuzzy generalized bi-ideals

In this section, we characterize S in context of BFGBI. The results given here are taken from our published work (see [84]).

Definition 1.4.1 *A BFS $f = (S; f_n, f_p)$ is called BFGBI of S, if it satisfies the following axioms.*

(I_1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Rightarrow f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2))$,

(I_2) $(\forall \xi_1, \xi_2, \xi_3 \in S)(f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\})$.

Example 1.4.2 *Consider an ordered semigroup $S = \{n, k, l, m\}$. Define multiplication and order relation on S as:*

$\cdot$	n	k	l	m
n	n	n	n	n
k	n	n	n	n
l	n	n	k	n
m	n	n	k	k

$\leq = \{(n, n), (k, k), (l, l), (m, m), (n, k)\}$. *Every BFGBI may not be a BFBI. For example, the BFS defined by*

S	n	k	l	m
f_n	-0.7	-0.1	-0.5	-0.2
f_p	0.8	0.2	0.4	0.3

is BFGBI of S but is not BFBI of S because $-0.1 = f_n(k) = f_n(l{\cdot}l) \nleq \bigvee\{f_n(l), f_n(l)\} = -0.5$ *and* $0.2 = f_p(k) = f_p(l \cdot l) \ngeq \bigwedge\{f_p(l), f_p(l)\} = 0.4$.

Theorem 1.4.3 *Let $f = (S; f_n, f_p)$ be a BFS in S. Then $f = (S; f_n, f_p)$ is a BFGBI of S iff for all $(s, t) \in [-1, 0) \times (0, 1]$, we have*

(1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2, \frac{\xi_2}{(s,t)} \in f \Longrightarrow \frac{\xi_1}{(s,t)} \in f)$,

(2) $(\forall \xi_1, \xi_2, \xi_3 \in S)(\frac{\xi_1}{(s_1,t_1)} \in f$ and $\frac{\xi_3}{(s_2,t_2)} \in f \Longrightarrow \frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in f)$.

Proof. Suppose that $f = (S; f_n, f_p)$ is a BFGBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\frac{\xi_2}{(s,t)} \in f$, then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since $f = (S; f_n, f_p)$ is BFGBI of S therefore for $\xi_1 \leq \xi_2$, we have $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2)$. This implies that $f_n(\xi_1) \leq s$ and $f_p(\xi_1) \geq t$, and so $\frac{\xi_1}{(s,t)} \in f$. Let $\xi_1, \xi_2, \xi_3 \in S$ and $(s_1,t_1),(s_2,t_2) \in [-1,0) \times (0,1]$ be such that $\frac{\xi_1}{(s_1,t_1)} \in f$ and $\frac{\xi_3}{(s_2,t_2)} \in f$. Then $f_n(\xi_1) \leq s_1$, $f_n(\xi_3) \leq s_2$, $f_p(\xi_1) \geq t_1$ and $f_p(\xi_3) \geq t_2$. It follows from (I_2) that

$$f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3) \leq \bigvee\{s_1, s_2\}$$
$$\text{and } f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq \bigwedge\{t_1, t_2\},$$

so that $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in f$.

Conversely, let $\xi_1, \xi_2 \in S$ be such that $\xi_1 \leq \xi_2$. Let $f_n(\xi_2) = s$ and $f_p(\xi_2) = t$. Then $\frac{\xi_2}{(s,t)} \in f$, this implies that $\frac{\xi_1}{(s,t)} \in f$ and so $f_n(\xi_1) \leq s$ and $f_p(\xi_1) \geq t$. Therefore $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2)$. Let $\xi_1, \xi_2, \xi_3 \in S$. Since $\frac{\xi_1}{(f_n(\xi_1),f_p(\xi_1))} \in f$ and $\frac{\xi_3}{(f_n(\xi_3),f_p(\xi_3))} \in f$ for all $\xi_1, \xi_3 \in S$, so by (2) we have

$$\frac{\xi_1\xi_2\xi_3}{(\vee\{f_n(\xi_1), f_n(\xi_3)\}, \wedge\{f_p(\xi_1), f_p(\xi_3)\})} \in f.$$

This implies that $f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$. ■

Theorem 1.4.4 *Let $f = (S; f_n, f_p)$ be a BFS in S. Then $f = (S; f_n, f_p)$ is a BFGBI of S iff the non-empty (s,t)-cut $C(f;(s,t))$ of $f = (S; f_n, f_p)$ is a generalized bi-ideal of S for all $(s,t) \in [-1,0) \times (0,1]$.*

Proof. Let $f = (S; f_n, f_p)$ is a BFGBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in C(f;(s,t))$, then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since f is a BFGBI of S so $f_n(\xi_1) \leq f_n(\xi_2) \leq s$ and $f_p(\xi_1) \geq f_p(\xi_2) \geq t$. This implies that $f_n(\xi_1) \leq s$ and $f_p(\xi_1) \geq t$. Hence $\xi_1 \in C(f;(s,t))$.

Let $\xi_1, \xi_3 \in C(f;(s,t))$ and $\xi_2 \in S$, then $f_n(\xi_1) \leq s$, $f_n(\xi_3) \leq s$ and $f_p(\xi_1) \geq t$, $f_p(\xi_1) \geq t$. Since f is a BFGBI of S so $f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3)\} \leq s$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq t$. This implies that $f_n(\xi_1\xi_2\xi_3) \leq s$ and $f_p(\xi_1\xi_2\xi_3) \geq t$. Hence $\xi_1\xi_2\xi_3 \in C(f;(s,t))$. Thus $C(f;(s,t))$ is a generalized bi-ideal of S.

Conversely, assume that $C(f;(s,t))$ is a generalized bi-ideal of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. Since $\xi_2 \in C(f;(f_n(\xi_2), f_p(\xi_2)))$, so $\xi_1 \in C(f;(f_n(\xi_2), f_p(\xi_2)))$. This implies that $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2)$.

Since $\xi_1, \xi_3 \in C(f; \bigvee\{(f_n(\xi_1), f_n(\xi_3)\}, \bigwedge\{(f_p(\xi_1), f_p(\xi_3)\})$, so $\xi_1\xi_2\xi_3 \in C(f; \bigvee\{(f_n(\xi_1), f_n(\xi_3)\}$ $, \bigwedge\{(f_p(\xi_1), f_p(\xi_3)\})$. Therefore $f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{(f_n(\xi_1), f_n(\xi_3)\}$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{(f_p(\xi_1), f_p(\xi_3)\}$. Hence $f = (S; f_n, f_p)$ is a BFGBI of S. ■

Theorem 1.4.5 *Let $\mathcal{G} \subseteq S$, then $\mathcal{G}$ is generalized bi-ideal of S iff $\chi_{\mathcal{G}} = (S; \chi_{n\mathcal{G}}, \chi_{p\mathcal{G}})$ is a BFGBI of S.*

Proof. Let $\mathcal{G}$ is generalized bi-ideal of S and $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. If $\xi_2 \in \mathcal{G}$ then $\xi_1 \in \mathcal{G}$, so

$$\begin{aligned} \chi_{n\mathcal{G}}(\xi_1) &= -1 \leq \chi_{n\mathcal{G}}(\xi_2) \text{ and} \\ \chi_{p\mathcal{G}}(\xi_1) &= 1 \geq \chi_{p\mathcal{G}}(\xi_2). \end{aligned}$$

If $\xi_2 \notin \mathcal{G}$ then

$$\begin{aligned} \chi_{n\mathcal{G}}(\xi_1) &\leq 0 = \chi_{n\mathcal{G}}(\xi_2) \text{ and} \\ \chi_{p\mathcal{G}}(\xi_1) &\geq 0 = \chi_{p\mathcal{G}}(\xi_2). \end{aligned}$$

Let $\xi_1, \xi_2, \xi_3 \in S$. If $\xi_1, \xi_3 \in \mathcal{G}$ then $\xi_1\xi_2\xi_3 \in \mathcal{G}$. So

$$\begin{aligned} \chi_{n\mathcal{G}}(\xi_1\xi_2\xi_3) &= -1 \leq (\chi_{n\mathcal{G}}(\xi_1) \vee \chi_{n\mathcal{G}}(\xi_3)) \text{ and} \\ \chi_{p\mathcal{G}}(\xi_1\xi_2\xi_3) &= 1 \geq (\chi_{p\mathcal{G}}(\xi_1) \wedge \chi_{p\mathcal{G}}(\xi_3)). \end{aligned}$$

If $\xi_1 \notin \mathcal{G}$ or $\xi_3 \notin \mathcal{G}$, then

$$\begin{aligned} \chi_{n\mathcal{G}}(\xi_1\xi_2\xi_3) &\leq 0 = (\chi_{n\mathcal{G}}(\xi_1) \vee \chi_{n\mathcal{G}}(\xi_3)) \text{ and} \\ \chi_{p\mathcal{G}}(\xi_1\xi_2\xi_3) &\geq 0 = (\chi_{p\mathcal{G}}(\xi_1) \wedge \chi_{p\mathcal{G}}(\xi_3)). \end{aligned}$$

Therefore $\chi_{\mathcal{G}} = (S, \chi_{n\mathcal{G}}, \chi_{p\mathcal{G}})$ is a BFGBI of S.

Conversely, assume that $\chi_{\mathcal{G}} = (S; \chi_{n\mathcal{G}}, \chi_{p\mathcal{G}})$ is a BFGBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in \mathcal{G}$. Then $\chi_{n\mathcal{G}}(\xi_2) = -1$ and $\chi_{n\mathcal{G}}(\xi_2) = 1$, so $\xi_2 \in C(\chi_{\mathcal{G}}; (-1, 1))$. By Theorem 1.4.4, we have $C(\chi_{\mathcal{G}}; (-1, 1))$ is generalized bi-ideal of S. Thus $\xi_1 \in C(\chi_{\mathcal{G}}; (-1, 1))$ and so

$\chi_{n\mathcal{G}}(\xi_1) \leq -1$ and $\chi_{p\mathcal{G}}(\xi_1) \geq 1$. Hence $\xi_1 \in \mathcal{G}$.

Let $\xi_1, \xi_2, \xi_3 \in S$ such that $\xi_1, \xi_3 \in \mathcal{G}$. Then $\chi_{n\mathcal{G}}(\xi_1) = -1 = \chi_{n\mathcal{G}}(\xi_3)$ and $\chi_{p\mathcal{G}}(\xi_1) = 1 = \chi_{p\mathcal{G}}(\xi_3)$, so $\xi_1, \xi_3 \in C(\chi_{\mathcal{G}}; (-1, 1))$. Since $C(\chi_{\mathcal{G}}; (-1, 1))$ is generalized bi-ideal of S by Theorem 1.4.4, so $\xi_1\xi_2\xi_3 \in C(\chi_{\mathcal{G}}; (-1, 1))$. Therefore $\chi_{n\mathcal{G}}(\xi_1\xi_2\xi_3) \leq -1$ and $\chi_{p\mathcal{G}}(\xi_1\xi_2\xi_3) \geq 1$. So $\xi_1\xi_2\xi_3 \in \mathcal{G}$. Hence $\mathcal{G}$ is generalized bi-ideal of S. ■

1.5 Bipolar fuzzy bi-ideals

In this section, we define BFBIs in S and characterize S in framework of bipolar fuzzy bi-ideals. The results given here are taken from our published work (see [87]).

Definition 1.5.1 *[81] A BFS $f = (S; f_n, f_p)$ is called BFBI of S, if it satisfies:*

(1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Rightarrow f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2))$.

(2) $(\forall \xi_1, \xi_2 \in S)(f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\})$.

(3) $(\forall \xi_1, \xi_2, \xi_3 \in S)(f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\})$.

Example 1.5.2 *Consider an ordered semigroup $S = \{0, 1, 2, 3, 4\}$. Define multiplication and order relation on S as:*

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	2	2	4
3	0	0	2	3	4
4	0	0	2	2	4

$\leq := \{(0, 0), (0, 3), (0, 4), (1, 1)(2, 2), (3, 3), (4, 4)\}$. *Define a BFS $f = (S; f_n, f_p)$ on S as:*

S	0	1	2	3	4
f_n	-0.7	-0.7	-0.6	-0.5	-0.4
f_p	0.6	0.5	0.4	0.3	0.3

.

Then $f = (S; f_n, f_p)$ is a BFBI of S.

Theorem 1.5.3 *A BFS $f = (S; f_n, f_p)$ of S is a BFBI of S iff it satisfies:*

(1)$(\forall \xi_1, \xi_2 \in S)(\forall (s,t) \in [-1,0) \times (0,1])(\xi_1 \leq \xi_2, \frac{\xi_2}{(s,t)} \in f \Rightarrow \frac{\xi_1}{(s,t)} \in f)$.

(2)$(\forall \xi_1, \xi_2 \in S)(\forall (s_1,t_1),(s_2,t_2) \in [-1,0)\times(0,1])(\frac{\xi_1}{(s_1,t_1)} \in f$ and $\frac{\xi_2}{(s_2,t_2)} \in f \Rightarrow \frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in f)$.

(3)$(\forall \xi_1, \xi_2, \xi_3 \in S)(\forall (s_1,t_1),(s_2,t_2) \in [-1,0)\times(0,1])(\frac{\xi_1}{(s_1,t_1)} \in f$ and $\frac{\xi_3}{(s_2,t_2)} \in f \Rightarrow \frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in f)$.

Proof. Let $f = (S; f_n, f_p)$ be a BFBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\frac{\xi_2}{(s,t)} \in f$ where $(s,t) \in [-1,0) \times (0,1]$, then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since $f = (S; f_n, f_p)$ is BFBI of S, therefore for $\xi_1 \leq \xi_2$, we have $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2)$. This implies that $f_n(\xi_1) \leq s$ and $f_p(\xi_1) \geq t$, and so $\frac{\xi_1}{(s,t)} \in f$.

Let $\xi_1, \xi_2 \in S$ such that $\frac{\xi_1}{(s_1,t_1)} \in f$ and $\frac{\xi_2}{(s_2,t_2)} \in f$ for $(s_1,t_1),(s_2,t_2) \in [-1,0) \times (0,1]$. Then $f_n(\xi_1) \leq s_1$, $f_n(\xi_2) \leq s_2$, $f_p(\xi_1) \geq t_1$ and $f_p(\xi_2) \geq t_2$. It follows that

$$f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2) \leq \bigvee\{s_1, s_2\}$$

and

$$f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq \bigwedge\{t_1, t_2\}$$

so that $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in f$.

Let $\xi_1, \xi_2, \xi_3 \in S$ and $(s_1,t_1),(s_2,t_2) \in [-1,0) \times (0,1]$ such that $\frac{\xi_1}{(s_1,t_1)} \in f$ and $\frac{\xi_3}{(s_2,t_2)} \in f$. Then $f_n(\xi_1) \leq s_1$, $f_n(\xi_3) \leq s_2$, $f_p(\xi_1) \geq t_1$ and $f_p(\xi_3) \geq t_2$. Since $f = (S; f_n, f_p)$ is a BFBI of S, so

$$f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3)\} \leq \bigvee\{s_1, s_2\}$$

and

$$f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq \bigwedge\{t_1, t_2\}.$$

So that $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in f$.

Conversely, suppose that conditions (1), (2) and (3) are satisfied. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. Let $f_n(\xi_2) = s$ and $f_p(\xi_2) = t$, so $\frac{\xi_2}{(s,t)} \in f$. Then $\frac{\xi_1}{(s,t)} \in f$ and so $f_n(\xi_1) \leq s$ and $f_p(\xi_1) \geq t$. So $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2)$.

Let $\xi_1, \xi_2 \in S$. Since $\frac{\xi_1}{(f_n(\xi_1), f_p(\xi_1))} \in f$ and $\frac{\xi_2}{(f_n(\xi_2), f_p(\xi_2))} \in f \ \forall \ \xi_1, \xi_2 \in S$, so we have

$$\frac{\xi_1 \xi_2}{(\vee\{f_n(\xi_1), f_n(\xi_2)\}, \wedge\{f_p(\xi_1), f_p(\xi_2)\})} \in f.$$

Therefore $f_n(\xi_1 \xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ and $f_p(\xi_1 \xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$.

Let $\xi_1, \xi_2, \xi_3 \in S$. Since $\frac{\xi_1}{(f_n(\xi_1), f_p(\xi_1))} \in f$ and $\frac{\xi_3}{(f_n(\xi_3), f_p(\xi_3))} \in f$ for all $\xi_1, \xi_3 \in S$, so by (3) we have

$$\frac{\xi_1 \xi_2 \xi_3}{(\vee\{f_n(\xi_1), f_n(\xi_3)\}, \wedge\{f_p(\xi_1), f_p(\xi_3)\})} \in f.$$

This implies that $f_n(\xi_1 \xi_2 \xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$ and $f_p(\xi_1 \xi_2 \xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$. Thus $f = (S; f_n, f_p)$ is a BFBI of S. ■

Proposition 1.5.4 *Let A, B are nonempty subsets of S. Then*

(i) $A \subseteq B$ iff $\chi_A \subseteq \chi_B$.

(ii) $\chi_A \cap \chi_B = \chi_{A \cap B}$.

(iii) $\chi_A \circ \chi_B = \chi_{(AB]}$.

Proof. The proof of (i) and (ii) are straight forward.

(iii) Let $\xi \in (AB]$, then $\chi_{n(AB]}(\xi) = -1$. Since $\xi \leq ab$ for some $a \in A$ and $b \in B$, we have $(a, b) \in A_\xi$ and $A_\xi \neq \emptyset$. Thus we have

$$\begin{aligned}(\chi_{_nA} \circ \chi_{_nB})(\xi) &= \bigwedge_{(y,z) \in A_\xi} \{\chi_{_nA}(y) \vee \chi_{_nB}(z)\} \\ &\leq \{\chi_{_nA}(a) \vee \chi_{_nB}(b)\} = -1.\end{aligned}$$

Therefore $\chi_{n(AB]}(\xi) = -1 = (\chi_{_nA} \circ \chi_{_nB})(\xi)$. Also $\chi_{p(AB]}(\xi) = 1$ and

$$\begin{aligned}(\chi_{pA} \circ \chi_{pB})(\xi) &= \bigvee_{(y,z) \in A_\xi} \{\chi_{pA}(y) \wedge \chi_{pB}(z)\} \\ &\geq \{\chi_{pA}(a) \wedge \chi_{pB}(b)\} = 1.\end{aligned}$$

Hence $\chi_{p(AB]}(\xi) = 1 = (\chi_{pA} \circ \chi_{pB})(\xi)$ and we get the required result. ■

Theorem 1.5.5 *Let $f = (S; f_n, f_p)$ be a BFS in S. Then $f = (S; f_n, f_p)$ is a BFBI of S iff the non-empty (s,t)-cut $C(f; (s,t))$ of $f = (S; f_n, f_p)$ is bi-ideal of S for all $(s,t) \in [-1, 0) \times (0, 1]$.*

Proof. Let $f = (S; f_n, f_p)$ is a BFBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in C(f;(s,t))$, then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since f is a BFBI of S so $f_n(\xi_1) \leq f_n(\xi_2) \leq s$ and $f_p(\xi_1) \geq f_p(\xi_2) \geq t$. This implies that $f_n(\xi_1) \leq s$ and $f_p(\xi_1) \geq t$. Hence $\xi_1 \in C(f;(s,t))$.

Let $\xi_1, \xi_2 \in C(f;(s,t))$, then $f_n(\xi_1) \leq s$, $f_n(\xi_2) \leq s$ and $f_p(\xi_1) \geq t$, $f_p(\xi_2) \geq t$. Since f is a BFBI of S so $f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2)\} \leq s$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq t$. Hence $\xi_1\xi_2 \in C(f;(s,t))$.

Let $\xi_1, \xi_3 \in C(f;(s,t))$ and $\xi_2 \in S$, then $f_n(\xi_1) \leq s$, $f_n(\xi_3) \leq s$ and $f_p(\xi_1) \geq t$, $f_p(\xi_1) \geq t$. Since f is a BFBI of S so $f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3)\} \leq s$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq t$. This implies that $f_n(\xi_1\xi_2\xi_3) \leq s$ and $f_p(\xi_1\xi_2\xi_3) \geq t$. Hence $\xi_1\xi_2\xi_3 \in C(f;(s,t))$ and $C(f;(s,t))$ is bi-ideal of S.

Conversely, assume that $C(f;(s,t))$ is bi-ideal of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. Since $\xi_2 \in C(f;(f_n(\xi_2), f_p(\xi_2)))$ and $C(f;(s,t))$ is bi-ideal of S so $\xi_1 \in C(f;(f_n(\xi_2), f_p(\xi_2)))$. This implies that $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2)$.

As we know that $\xi_1, \xi_2 \in C(f; \bigvee\{(f_n(\xi_1), f_n(\xi_2)\}, \bigwedge\{(f_p(\xi_1), f_p(\xi_2)\})$ and $C(f;(s,t))$ is bi-ideal of S so $\xi_1\xi_2 \in C(f; \bigvee\{(f_n(\xi_1), f_n(\xi_2)\}, \bigwedge\{(f_p(\xi_1), f_p(\xi_2)\})$. This implies that $f_n(\xi_1\xi_2) \leq \bigvee\{(f_n(\xi_1), f_n(\xi_2)\}$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{(f_p(\xi_1), f_p(\xi_2)\}$.

Let $\xi_2 \in S$ and $\xi_1, \xi_3 \in C(f; \bigvee\{(f_n(\xi_1), f_n(\xi_3)\}, \bigwedge\{(f_p(\xi_1), f_p(\xi_3)\})$. Since $C(f;(s,t))$ is bi-ideal of S so $\xi_1\xi_2\xi_3 \in C(f; \bigvee\{(f_n(\xi_1), f_n(\xi_3)\}, \bigwedge\{(f_p(\xi_1), f_p(\xi_3)\})$. Therefore $f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{(f_n(\xi_1), f_n(\xi_3)\}$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{(f_p(\xi_1), f_p(\xi_3)\}$. Hence $f = (S; f_n, f_p)$ is BFBI of S. ■

Theorem 1.5.6 *Let $B \subseteq S$, then B is bi-ideal of S iff $\chi_B = (S; \chi_{nB}, \chi_{pB})$ is a BFBI of S.*

Proof. Let B is bi-ideal of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. If $\xi_2 \in B$ then $\xi_1 \in B$, so

$$\begin{aligned} \chi_{nB}(\xi_1) &= -1 \leq \chi_{nB}(\xi_2) \text{ and} \\ \chi_{pB}(\xi_1) &= 1 \geq \chi_{pB}(\xi_2). \end{aligned}$$

If $\xi_2 \notin B$ then

$$\begin{aligned} \chi_{nB}(\xi_1) &\leq 0 = \chi_{nB}(\xi_2) \text{ and} \\ \chi_{pB}(\xi_1) &\geq 0 = \chi_{pB}(\xi_2). \end{aligned}$$

Let $\xi_1, \xi_2 \in S$. If $\xi_1, \xi_2 \in B$ then $\xi_1\xi_2 \in B$. So

$$\begin{aligned}\chi_{nB}(\xi_1\xi_2) &= -1 \leq (\chi_{nB}(\xi_1) \vee \chi_{nB}(\xi_2)) \text{ and} \\ \chi_{pB}(\xi_1\xi_2) &= 1 \geq (\chi_{pB}(\xi_1) \wedge \chi_{pB}(\xi_2)).\end{aligned}$$

If $\xi_1 \notin B$ or $\xi_2 \notin B$, then

$$\begin{aligned}\chi_{nB}(\xi_1\xi_2) &\leq 0 = (\chi_{nB}(\xi_1) \vee \chi_{nB}(\xi_2)) \text{ and} \\ \chi_{pB}(\xi_1\xi_2) &\geq 0 = (\chi_{pB}(\xi_1) \wedge \chi_{pB}(\xi_2)).\end{aligned}$$

Let $\xi_1, \xi_2, \xi_3 \in S$. If $\xi_1, \xi_3 \in B$ then $\xi_1\xi_2\xi_3 \in B$. So

$$\begin{aligned}\chi_{nB}(\xi_1\xi_2\xi_3) &= -1 \leq (\chi_{nB}(\xi_1) \vee \chi_{nB}(\xi_3)) \text{ and} \\ \chi_{pB}(\xi_1\xi_2\xi_3) &= 1 \geq (\chi_{pB}(\xi_1) \wedge \chi_{pB}(\xi_3)).\end{aligned}$$

If $\xi_1 \notin B$ or $\xi_3 \notin B$, then

$$\begin{aligned}\chi_{nB}(\xi_1\xi_2\xi_3) &\leq 0 = (\chi_{nB}(\xi_1) \vee \chi_{nB}(\xi_3)) \text{ and} \\ \chi_{pB}(\xi_1\xi_2\xi_3) &\geq 0 = (\chi_{pB}(\xi_1) \wedge \chi_{pB}(\xi_3)).\end{aligned}$$

Therefore $\chi_B = (S; \chi_{nB}, \chi_{pB})$ is a BFBI of S.

Conversely, assume that $\chi_B = (S; \chi_{nB}, \chi_{pB})$ is a BFBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in B$. Then $\chi_{nB}(\xi_2) = -1$ and $\chi_{nB}(\xi_2) = 1$, so $\xi_2 \in C(\chi_B; (-1, 1))$. By Theorem 1.5.5, we have $C(\chi_B; (-1, 1))$ is bi-ideal of S. Thus $\xi_1 \in C(\chi_B; (-1, 1))$ and so $\chi_{nB}(\xi_1) \leq -1$ and $\chi_{pB}(\xi_1) \geq 1$. Hence $\xi_1 \in B$.

Let $\xi_1, \xi_2 \in B$. Then $\chi_{nB}(\xi_1) = -1 = \chi_{nB}(\xi_2)$ and $\chi_{pB}(\xi_1) = 1 = \chi_{pB}(\xi_2)$. So $\xi_1, \xi_2 \in C(\chi_B; (-1, 1))$, $\Longrightarrow \xi_1\xi_2 \in C(\chi_B; (-1, 1))$. Thus $\chi_{nB}(\xi_1\xi_2) \leq -1$ and $\chi_{pB}(\xi_1\xi_2) \geq 1$. Hence $\xi_1\xi_2 \in B$.

Let $\xi_1, \xi_2, \xi_3 \in S$ such that $\xi_1, \xi_3 \in B$. Then $\chi_{nB}(\xi_1) = -1 = \chi_{nB}(\xi_3)$ and $\chi_{pB}(\xi_1) = 1 = \chi_{pB}(\xi_3)$, so $\xi_1, \xi_3 \in C(\chi_B; (-1, 1))$. As $C(\chi_B; (-1, 1))$ is bi-ideal of S, so $\xi_1\xi_2\xi_3 \in C(\chi_B; (-1, 1))$. Thus $\chi_{nB}(\xi_1\xi_2\xi_3) \leq -1$ and $\chi_{pB}(\xi_1\xi_2\xi_3) \geq 1$. Hence $\xi_1\xi_2\xi_3 \in B$ and we get the required result. ■

1.6 $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy (interior) ideals

In this section, we define $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy left (resp. right, interior) ideals of S. We also characterize S in context of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy left (resp. right, interior) ideals. The results given are taken from our accepted paper (see [88]).

Definition 1.6.1 *A BFS $f = (S; f_n, f_p)$ of S is called $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy subsemigroup of S where $\alpha_1, \alpha_2 \in [-1, 0]$ and $\beta_1, \beta_2 \in [0, 1]$ if it satisfies:*

1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Longrightarrow \min\{f_n(\xi_1), \alpha_1\} \leq \max\{f_n(\xi_2), \alpha_2\}$ and $\max\{f_p(\xi_1), \beta_1\} \geq \min\{f_p(\xi_2), \beta_2\})$.

2) $(\forall \xi_1, \xi_2 \in S)(\min\{f_n(\xi_1\xi_2), \alpha_1\} \leq \max\{f_n(\xi_1), f_n(\xi_2), \alpha_2\}$ and $\max\{f_p(\xi_1\xi_2), \beta_1\} \geq \min\{(f_p(\xi_1), f_p(\xi_2), \beta_2\})$.

It is important to note that every bipolar fuzzy subsemigroup is $(0, -1; 0, 1)$-bipolar fuzzy subsemigroup of S.

In the following example we show that $f = (S; f_n, f_p)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy subsemigroup of S but $f = (S; f_n, f_p)$ is not be a bipolar fuzzy subsemigroup of S.

Example 1.6.2 *Let $S = \{k, m, n\}$ be an ordered semigroup with following multiplication table and ordered relation.*

$\cdot$	k	m	n
k	k	k	k
m	k	m	k
n	k	k	n

$\leq = \{(k, k), (m, m), (n, n), (m, n)\}$. *Define a BFS $f = (S; f_n, f_p)$ of S as follow:*

S	k	m	n
f_n	-0.8	-0.6	-0.3
f_p	0.3	0.6	0.4

.

Then $f = (S; f_n, f_p)$ is not a bipolar fuzzy subsemigroup of S, since $f_p(m \cdot n) = f_p(k) = 0.3 < 0.4 = \min\{f_p(m), f_p(n)\}$. If $\alpha_1 \in [-0.3, 0]$ and $\alpha_2 \in [-1, 0]$ and $\beta_1 \in [0.6, 1]$ and $\beta_2 \in [0, 1]$. Then $f = (S; f_n, f_p)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy subsemigroup of S.

Definition 1.6.3 *A BFS $f = (S; f_n, f_p)$ of S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFLI (resp. BFRI) of S, where $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$, if it satisfies:*

1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Longrightarrow \min\{f_n(\xi_1), \alpha_1\} \leq \max\{f_n(\xi_2), \alpha_2\}$ and $\max\{f_p(\xi_1), \beta_1\} \geq \min\{f_p(\xi_2), \beta_2\})$.

2) $(\forall \xi_1, \xi_2 \in S)(\min\{f_n(\xi_1\xi_2), \alpha_1\} \leq \max\{f_n(\xi_2), \alpha_2\}$(resp. $\min\{f_n(\xi_1\xi_2), \alpha_1\} \leq \max\{f_n(\xi_1), \alpha_2\}$ and $\max\{f_p(\xi_1\xi_2), \beta_1\} \geq \min\{f_p(\xi_2), \beta_2\}$ (resp $\max\{f_p(\xi_1\xi_2), \beta_1\} \geq \min\{f_p(\xi_1), \beta_2\}$)).

A BFS $f = (S; f_n, f_p)$ of S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFI of S if it is both a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFLI and $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFRI of S.

Theorem 1.6.4 *Let $f = (S; f_n, f_p)$ be a BFS of S. Then f is $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFLI (resp. BFRI) of S for all $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$ iff $C(f; (s, t))(\neq \emptyset)$ is a left (resp. right) ideal of S for all $s \in [-1, 0]$, $t \in [0, 1]$.*

Proof. Straight forward. ■

Theorem 1.6.5 *Let $\mathcal{L} \subseteq S$. Then $\mathcal{L}$ is a left ideal of S iff the characteristic function $\chi_{\mathcal{L}} = (S; \chi_{n\mathcal{L}}, \chi_{p\mathcal{L}})$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFLI of S for all $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$.*

Proof. Let $\mathcal{L}$ is a left ideal of S and $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. If $\xi_2 \in \mathcal{L}$ then $\xi_1 \in \mathcal{L}$, so

$$\begin{aligned} \min\{\chi_{n\mathcal{L}}(\xi_1), \alpha_1\} &= -1 \leq \max\{\chi_{n\mathcal{L}}(\xi_2), \alpha_2\} \text{ and} \\ \max\{\chi_{p\mathcal{L}}(\xi_1), \beta_1\} &= 1 \geq \min\{\chi_{p\mathcal{L}}(\xi_2), \beta_2\}. \end{aligned}$$

If $\xi_2 \notin \mathcal{L}$, then

$$\begin{aligned} \min\{\chi_{n\mathcal{L}}(\xi_1), \alpha_1\} &\leq 0 = \max\{\chi_{n\mathcal{L}}(\xi_2), \alpha_2\} \text{ and} \\ \max\{\chi_{p\mathcal{L}}(\xi_1), \beta_1\} &\geq 0 = \min\{\chi_{p\mathcal{L}}(\xi_2), \beta_2\}. \end{aligned}$$

Let $\xi_1, \xi_2 \in S$. If $\xi_2 \in \mathcal{L}$ then $\xi_1\xi_2 \in \mathcal{L}$, so

$$\begin{aligned} \min\{\chi_{n\mathcal{L}}(\xi_1\xi_2), \alpha_1\} &= -1 \leq \max\{\chi_{n\mathcal{L}}(\xi_1), \alpha_2\} \text{ and} \\ \max\{\chi_{p\mathcal{L}}(\xi_1\xi_2), \beta_1\} &= 1 \geq \min\{\chi_{p\mathcal{L}}(\xi_2), \beta_2\}. \end{aligned}$$

If $\xi_2 \notin \mathcal{L}$, then

$$\begin{aligned} \min\{\chi_{n\mathcal{L}}(\xi_1\xi_2), \alpha_1\} &\leq 0 = \max\{\chi_{n\mathcal{L}}(\xi_2), \alpha_2\} \text{ and} \\ \max\{\chi_{p\mathcal{L}}(\xi_1\xi_2), \beta_1\} &\geq 0 = \min\{\chi_{p\mathcal{L}}(\xi_2), \beta_2\}. \end{aligned}$$

Therefore $\chi_{\mathcal{L}} = (S; \chi_{n\mathcal{L}}, \chi_{p\mathcal{L}})$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFRI of S.

Conversely, assume that $\chi_{\mathcal{L}} = (S; \chi_{n\mathcal{L}}, \chi_{p\mathcal{L}})$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFLI of S for all $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in \mathcal{L}$ then $\chi_{n\mathcal{L}}(\xi_2) = -1$ and $\chi_{p\mathcal{L}}(\xi_2) = 1$, so $\xi_2 \in C(\chi_{\mathcal{L}}; (-1, 1))$. By Theorem 1.6.4, we have $C(\chi_{\mathcal{L}}; (-1, 1))$ is left ideal of S. Thus $\xi_1 \in C(\chi_{\mathcal{L}}; (-1, 1))$ and so $\chi_{n\mathcal{L}}(\xi_1) \leq -1$ and $\chi_{p\mathcal{L}}(\xi_1) \geq 1$. This implies $\xi_1 \in \mathcal{L}$.

Let $\xi_2 \in \mathcal{L}$ and $\xi_1 \in S$, then $\chi_{n\mathcal{L}}(\xi_2) = -1$ and $\chi_{p\mathcal{L}}(\xi_2) = 1$. So $\xi_2 \in C(\chi_{\mathcal{L}}; (-1, 1))$. Since $C(\chi_{\mathcal{L}}; (-1, 1))$ is left ideal of S by Theorem 1.6.4, so $\xi_1\xi_2 \in C(\chi_{\mathcal{L}}; (-1, 1))$ for all $\xi_1 \in S$. This implies that $\chi_{n\mathcal{L}}(\xi_1\xi_2) \leq -1$ and $\chi_{p\mathcal{L}}(\xi_1\xi_2) \geq 1$. So $\xi_1\xi_2 \in \mathcal{L}$ and hence we get the required result. ■

Theorem 1.6.6 *Let $\mathcal{R} \subseteq S$. Then $\mathcal{R}$ is a right ideal of S iff the characteristic function $\chi_{\mathcal{R}} = (S; \chi_{n\mathcal{R}}, \chi_{p\mathcal{R}})$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFRI of S for all $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$.*

Definition 1.6.7 *A BFS $f = (S; f_n, f_p)$ of S is called $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFII of S, where $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$, if it satisfies:*

1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Longrightarrow \min\{f_n(\xi_1), \alpha_1\} \leq \max\{f_n(\xi_2), \alpha_2\}$ and $\max\{f_p(\xi_1), \beta_1\} \geq \min\{f_p(\xi_2), \beta_2\})$.

2) $(\forall \xi_1, \xi_2 \in S)(\min\{f_n(\xi_1\xi_2), \alpha_1\} \leq \max\{f_n(\xi_1), f_n(\xi_2), \alpha_2\}$ and $\max\{f_p(\xi_1\xi_2), \beta_1\} \geq \min\{(f_p(\xi_1), f_p(\xi_2), \beta_2\})$.

3) $(\forall \xi_1, a, \xi_2 \in S)(\min\{f_n(\xi_1 a \xi_2), \alpha_1\} \leq \max\{f_n(a), \alpha_2\}$ and $\max\{f_p(\xi_1 a \xi_2), \beta_1\} \geq \min\{f_p(a), \beta_2\})$.

Theorem 1.6.8 *Let $f = (S; f_n, f_p)$ be a BFS of S. Then f is $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFII of S for all*

$\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$ *iff* $C(f; (s, t))(\neq \emptyset)$ *is an interior ideal of* S *for all* $s \in [-1, 0]$, $t \in [0, 1]$.

Proof. Let $\mathcal{I}$ is a left ideal of S and $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. If $\xi_2 \in \mathcal{I}$ then $\xi_1 \in \mathcal{I}$, so

$$\begin{aligned} \min\{\chi_{n\mathcal{I}}(\xi_1), \alpha_1\} &= -1 \leq \max\{\chi_{n\mathcal{I}}(\xi_2), \alpha_2\} \text{ and} \\ \max\{\chi_{p\mathcal{I}}(\xi_1), \beta_1\} &= 1 \geq \min\{\chi_{p\mathcal{I}}(\xi_2), \beta_2\}. \end{aligned}$$

If $\xi_2 \notin \mathcal{I}$, then

$$\begin{aligned} \min\{\chi_{n\mathcal{I}}(\xi_1), \alpha_1\} &\leq 0 = \max\{\chi_{n\mathcal{I}}(\xi_2), \alpha_2\} \text{ and} \\ \max\{\chi_{p\mathcal{I}}(\xi_1), \beta_1\} &\geq 0 = \min\{\chi_{p\mathcal{I}}(\xi_2), \beta_2\}. \end{aligned}$$

Let $\xi_1, \xi_2 \in S$. If $\xi_1, \xi_2 \in \mathcal{I}$ then $\xi_1\xi_2 \in \mathcal{I}$. So

$$\begin{aligned} \min\{\chi_{n\mathcal{I}}(\xi_1\xi_2), \alpha_1\} &= -1 \leq \max\{\chi_{n\mathcal{I}}(\xi_1), \chi_{n\mathcal{I}}(\xi_2), \alpha_2\} \text{ and} \\ \max\{\chi_{p\mathcal{I}}(\xi_1\xi_2), \beta_1\} &= 1 \geq \min\{\chi_{p\mathcal{I}}(\xi_1), \chi_{p\mathcal{I}}(\xi_2), \beta_2\}. \end{aligned}$$

If $\xi_1 \notin \mathcal{I}$ or $\xi_2 \notin \mathcal{I}$, then

$$\begin{aligned} \min\{\chi_{n\mathcal{I}}(\xi_1\xi_2), \alpha_1\} &\leq 0 = \max\{\chi_{n\mathcal{I}}(\xi_1), \chi_{n\mathcal{I}}(\xi_2), \alpha_2\} \text{ and} \\ \max\{\chi_{p\mathcal{I}}(\xi_1\xi_2), \beta_1\} &\geq 0 = \min\{\chi_{p\mathcal{I}}(\xi_1), \chi_{p\mathcal{I}}(\xi_2), \beta_2\}. \end{aligned}$$

Let $\xi_1, a, \xi_2 \in S$. If $a \in \mathcal{I}$ then $\xi_1 a \xi_2 \in \mathcal{I}$, so

$$\begin{aligned} \min\{\chi_{n\mathcal{I}}(\xi_1 a \xi_2), \alpha_1\} &= -1 \leq \max\{\chi_{n\mathcal{I}}(a), \alpha_2\} \text{ and} \\ \max\{\chi_{p\mathcal{I}}(\xi_1 a \xi_2), \beta_1\} &= 1 \geq \min\{\chi_{p\mathcal{I}}(a), \beta_2\}. \end{aligned}$$

If $a \notin \mathcal{I}$, then

$$\begin{aligned} \min\{\chi_{n\mathcal{I}}(\xi_1 a \xi_2), \alpha_1\} &\leq 0 = \max\{\chi_{n\mathcal{I}}(a), \alpha_2\} \text{ and} \\ \max\{\chi_{p\mathcal{I}}(\xi_1 a \xi_2), \beta_1\} &\geq 0 = \min\{\chi_{p\mathcal{I}}(a), \beta_2\}. \end{aligned}$$

Therefore $\chi_{\mathcal{I}} = (S; \chi_{n\mathcal{I}}, \chi_{p\mathcal{I}})$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFII of S.

Conversely, accept that $\chi_{\mathcal{I}} = (S; \chi_{n\mathcal{I}}, \chi_{p\mathcal{I}})$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFII of S for all $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [-1, 0]$. Let $\xi_1, \xi_2 \in \mathcal{I}$. Then $\chi_{n\mathcal{I}}(\xi_1) = -1 = \chi_{n\mathcal{I}}(\xi_2)$ and $\chi_{p\mathcal{I}}(\xi_1) = 1 = \chi_{p\mathcal{I}}(\xi_2)$, so $\xi_1, \xi_2 \in C(\chi_{\mathcal{I}}; (-1, 1))$. Since $C(\chi_{\mathcal{I}}; (-1, 1))$ is an interior ideal of S by Theorem 1.6.8., so $\xi_1\xi_2 \in C(\chi_{\mathcal{I}}; (-1, 1))$. Thus $\chi_{n\mathcal{I}}(\xi_1\xi_2) \leq -1$ and $\chi_{p\mathcal{I}}(\xi_1\xi_2) \geq 1$. Hence $\xi_1\xi_2 \in \mathcal{I}$.

Let $a \in \mathcal{I}$ and $\xi_1, \xi_2 \in S$, then $\chi_{n\mathcal{I}}(a) = -1$ and $\chi_{p\mathcal{I}}(a) = 1$. So $a \in C(\chi_{\mathcal{I}}; (-1, 1))$. Since $C(\chi_{\mathcal{I}}; (-1, 1))$ is an interior ideal of S by Theorem 1.6.8, so $\xi_1 a \xi_2 \in C(\chi_{\mathcal{I}}; (-1, 1))$. This implies $\chi_{n\mathcal{I}}(\xi_1 a \xi_2) \leq -1$ and $\chi_{p\mathcal{I}}(\xi_1 a \xi_2) \geq 1$. Thus $\xi_1 a \xi_2 \in \mathcal{I}$ and hence we get the required result. ■

Chapter 2

Application of Fuzzy Parameterized Bipolar Fuzzy Soft Expert Sets and Possibility Bipolar Fuzzy Soft Expert Sets in Decision Making Problem

This chapter is composed of our accepted paper [89]. This chapter is divided into five section. In section one, some basic definitions of (fuzzy) soft sets are given. In section two, we discuss the study of fuzzy parameterized bipolar fuzzy soft expert sets (FPBFSESs) and some of its basic operations are defined. In third section, an application of FPBFSES in decision making problems are considered. In section forth, the notion of possibility bipolar fuzzy soft expert sets (PBFSESs) are investigated and its basic operations are discussed. In last section, an application of PBFSES are given in decision making problems by using generalized algorithm.

2.1 Basic Results

In this section, some basic definitions and results of (fuzzy) soft sets are given.

Definition 2.1.1 *[91] Let F is a function given by $F : U \rightarrow P(U)$ where $P(U)$ denotes power set of U then the pair (U, F) is called a soft set over U.*

Definition 2.1.2 *[93] Let us consider U as set of universe and let E be a set of parameters. Let X a set of experts and O be a set of opinions. Let $Z = E \times X \times O$ and $A \subseteq Z$. Consider a function $F : A \rightarrow P(U)$, then the pair (F, A) is called a soft expert set over U.*

For the sake of simplicity, it is supposed that the set of opinions only consist of two values, namely, agree and disagree. However, it is possible to include other options for the set of opinions, including more specific opinions.

Definition 2.1.3 *[98] Consider a set of universe U and let E be a set of parameter. Suppose $P(U)$ represents the set of all BFS on U. Let F is a function given by $F : A \rightarrow P(U)$ then the pair (F, A) is called a bipolar fuzzy soft set (BFSS) over U where $A \subseteq E$.*

Definition 2.1.4 *[98] A BFSS (F, A) is called a null BFSS if $F(e) = \emptyset$ for all $e \in A$.*

Definition 2.1.5 *[98] A BFSS (F, A) is called an absolute BFSS if $F(e) = B^U$ $\forall e \in A$, where B^U is the collection of all bipolar fuzzy subset of U.*

Definition 2.1.6 *[98] Let (G, X) and (H, Y) be two BFSSs over U. Then the union of (G, X) and (H, Y), represented by $(G, X)\hat{\cup}(H, Y)$ is a BFSS classified as $(G, X)\hat{\cup}(H, Y) = (I, Z)$ where $Z = X \cup Y$ and $\forall \tau \in Z$,*

$$I(\alpha) = \begin{cases} G(\tau) \ \text{if} \ \tau \in X - Y, \\ H(\tau) \ \text{if} \ \tau \in Y - X, \\ G(\tau) \cup H(\tau) \ \text{if} \ \tau \in X \cap Y. \end{cases}$$

Definition 2.1.7 *[98] Let (G, X) and (H, Y) be two BFSSs of U. Then the integration of (G, X) and (H, Y), represented by $(G, X)\hat{\cap}(H, Y)$ is a BFSS classified as $(G, X)\hat{\cap}(H, Y) = (I, Z)$ where $Z = X \cup Y$ and $\forall \tau \in Z$,*

$$I(\alpha) = \begin{cases} G(\tau) \ \text{if} \ \tau \in X - Y, \\ H(\tau) \ \text{if} \ \tau \in Y - X, \\ G(\tau) \cap H(\tau) \ \text{if} \ \tau \in X \cap Y. \end{cases}$$

Definition 2.1.8 *[99] Consider a set of universe U and let E be a set of parameter, X be set of experts and O be set of options. Let $Z = E \times X \times O$ and $A \subseteq Z$. Suppose $F : A \rightarrow B^X$ is a function defined by*

$$F(x) = \hat{F}(x)(u_i)\forall u_i \in U,$$

where B^X is the collection of all BFS. Then $F(x)$ is called bipolar fuzzy soft expert set.

2.2 Basic Operations on Fuzzy Parameterized Bipolar Fuzzy Soft Expert Sets

In this section, the notion of FPBFSES are discussed and some interesting results relative to this concept are investigated. We define some basic operations using this theory. Some applicable laws such as De Morgan's laws of this concept are proved.

Definition 2.2.1 *Let $U = \{u_1, u_2, u_3, ..., u_n\}$ be set of universe and $E = \{e_1, e_2, e_3, ..., e_m\}$ be set of parameters, let $A = \{x_1, x_2, x_3, ..., x_i\}$ be a set of experts (agents), and let $O = \{1 = agree, 0 = disagree\}$ be a set of opinions. Let $Z = D \times A \times O$ where D is a fuzzy subset of E then $Z = D \times A \times O$ is defined as*

$$Z = \{(d, x, o) : d \in D, x \in A, o \in O\}.$$

Let G be a mapping given by $G_D : Z \rightarrow B^U$ where B^U represents the set of all BFS on U. Then the pair $(G, X)_D$ is said to be FPBFSES over U.

Example 2.2.2 *Let $U = \{u_1, u_2, u_3\}$ be the set of three cars to be considered, let e_1=beautiful, e_2=cheap and e_3=good condition are the parameters for decision and made a set $E = \{e_1, e_2, e_3\}$, let $A = \{x_1, x_2\}$ be the set of experts and let $D = \{0.9/e_1, 0.6/e_2, 0.2/e_3\}$. Then we have a*

FPBFSES, $(G,X)_D$ *as*

$$(G,X)_D = \left\{ \begin{array}{c} \left(\left(\frac{0.9}{e_1}, x_1, 1\right) = \left\{\frac{u_1}{(-0.2,0.9)}, \frac{u_2}{(-0.4,0.6)}, \frac{u_3}{(-0.9,0.3)}\right\}\right), \\ \left(\left(\frac{0.9}{e_1}, x_2, 1\right) = \left\{\frac{u_1}{(-0.3,0.8)}, \frac{u_2}{(-0.7,0.2)}, \frac{u_3}{(-0.8,0.8)}\right\}\right), \\ \left(\left(\frac{0.6}{e_2}, x_1, 1\right) = \left\{\frac{u_1}{(-0.5,0.7)}, \frac{u_2}{(-0.4,0.4)}, \frac{u_3}{(-0.7,0.1)}\right\}\right), \\ \left(\left(\frac{0.6}{e_2}, x_2, 1\right) = \left\{\frac{u_1}{(-0.5,0.5)}, \frac{u_2}{(-0.1,0.8)}, \frac{u_3}{(-0.2,0.9)}\right\}\right), \\ \left(\left(\frac{0.2}{e_3}, x_1, 1\right) = \left\{\frac{u_1}{(-0.2,1)}, \frac{u_2}{(-0.4,0.6)}, \frac{u_3}{(-0.9,0.3)}\right\}\right), \\ \left(\left(\frac{0.2}{e_3}, x_2, 1\right) = \left\{\frac{u_1}{(-0.6,0.6)}, \frac{u_2}{(-0.7,0.2)}, \frac{u_3}{(-0.1,0.4)}\right\}\right), \\ \left(\left(\frac{0.9}{e_1}, x_1, 0\right) = \left\{\frac{u_1}{(-0.2,0.9)}, \frac{u_2}{(-0.5,0.8)}, \frac{u_3}{(-0.7,0.5)}\right\}\right), \\ \left(\left(\frac{0.9}{e_1}, x_2, 0\right) = \left\{\frac{u_1}{(-0.5,0.9)}, \frac{u_2}{(-0.6,0.3)}, \frac{u_3}{(-0.7,0.5)}\right\}\right), \\ \left(\left(\frac{0.6}{e_2}, x_1, 0\right) = \left\{\frac{u_1}{(-0.4,0.6)}, \frac{u_2}{(-0.5,0.5)}, \frac{u_3}{(-0.7,0.4)}\right\}\right), \\ \left(\left(\frac{0.2}{e_3}, x_2, 0\right) = \left\{\frac{u_1}{(-0.5,0.8)}, \frac{u_2}{(-0.2,0.7)}, \frac{u_3}{(-0.5,0.6)}\right\}\right) \end{array} \right\}.$$

Definition 2.2.3 *A* $(G,X)_D$ *is said to be FPBFSE subset of* $(H,Y)_K$*, denoted by* $(G,X)_D \sqsubseteq (H,Y)_K$ *if*

1)$X \subseteq Y$,

2)For all $\varepsilon \in X$, $G_D(\varepsilon)$ is bipolar fuzzy subset of $H_K(\varepsilon)$.

They are equal if $(G,X)_D \sqsubseteq (H,Y)_K$ and $(H,Y)_K \sqsubseteq (G,X)_D$.

Definition 2.2.4 *An agree-FPBFSES represented by* $(G,X)_{D_1}$ *is defined as*

$$(G,X)_{D_1} = \{G_D(\varepsilon) : \varepsilon \in D \times A \times \{1\}\}.$$

Definition 2.2.5 *A disagree-FPBFSES denoted by* $(G,X)_{D_0}$ *is defined as*

$$(G,X)_{D_0} = \{G_D(\varepsilon) : \varepsilon \in D \times A \times \{0\}\}.$$

Definition 2.2.6 *Let* $(F,X)_D$ *be a FPBFSES over* U*. Then the complement of* $(F,X)_D$*, represented by* $(F,X)^c_D$*, is defined as* $(F,X)^c_D = (F^c, \sim X)_D$ *where* $F^c_D :\sim X \to B^U$ *is a mapping given by:*

$$F^c_D(\varepsilon) = c(F_D(\varepsilon)) \ \forall \varepsilon \sim X,$$

where c is the bipolar fuzzy complement and $\sim X \subset D^c \times A \times O$.

Example 2.2.7 *Let* $(G, X)_D$ *be a FPBFSES as defined in Example 2.2.2. Then by using complement of BFS, we have*

$$(G,X)^c_D = \left\{ \begin{array}{c} \left(\left(\frac{0.1}{e_1}, x_1, 1\right) = \left\{\frac{u_1}{(-0.8,0.1)}, \frac{u_2}{(-0.6,0.4)}, \frac{u_3}{(-0.1,0.7)}\right\}\right), \\ \left(\left(\frac{0.1}{e_1}, x_2, 1\right) = \left\{\frac{u_1}{(-0.7,0.2)}, \frac{u_2}{(-0.3,0.8)}, \frac{u_3}{(-0.2,0.2)}\right\}\right), \\ \left(\left(\frac{0.4}{e_2}, x_1, 1\right) = \left\{\frac{u_1}{(-0.5,0.3)}, \frac{u_2}{(-0.6,0.6)}, \frac{u_3}{(-0.3,0.9)}\right\}\right), \\ \left(\left(\frac{0.4}{e_2}, x_2, 1\right) = \left\{\frac{u_1}{(-0.5,0.5)}, \frac{u_2}{(-0.9,0.2)}, \frac{u_3}{(-0.8,0.1)}\right\}\right), \\ \left(\left(\frac{0.8}{e_3}, x_1, 1\right) = \left\{\frac{u_1}{(-0.8,0)}, \frac{u_2}{(-0.6,0.4)}, \frac{u_3}{(-0.1,0.7)}\right\}\right), \\ \left(\left(\frac{0.8}{e_3}, x_2, 1\right) = \left\{\frac{u_1}{(-0.4,0.4)}, \frac{u_2}{(-0.3,0.8)}, \frac{u_3}{(-0.9,0.6)}\right\}\right), \\ \left(\left(\frac{0.1}{e_1}, x_1, 0\right) = \left\{\frac{u_1}{(-0.8,0.1)}, \frac{u_2}{(-0.5,0.2)}, \frac{u_3}{(-0.3,0.5)}\right\}\right), \\ \left(\left(\frac{0.1}{e_1}, x_2, 0\right) = \left\{\frac{u_1}{(-0.5,0.1)}, \frac{u_2}{(-0.4,0.7)}, \frac{u_3}{(-0.3,0.5)}\right\}\right), \\ \left(\left(\frac{0.4}{e_2}, x_1, 0\right) = \left\{\frac{u_1}{(-0.6,0.4)}, \frac{u_2}{(-0.5,0.5)}, \frac{u_3}{(-0.3,0.6)}\right\}\right), \\ \left(\left(\frac{0.8}{e_3}, x_2, 0\right) = \left\{\frac{u_1}{(-0.5,0.2)}, \frac{u_2}{(-0.8,0.3)}, \frac{u_3}{(-0.5,0.4)}\right\}\right) \end{array} \right\}.$$

Proposition 2.2.8 *Let* $(F, X)_D$ *be a FPBFSES then we have* $((F, X)^c_D)^c = (F, X)_D$.

Proof. Let $(F, X)_D$ be a FPBFSES. Then by definition of complement we have a mapping $(F^c_D)^c :\sim (\sim X) \to B^U$ defined by $(F^c_D)^c(\varepsilon) = c(F^c_D(\sim (\sim \varepsilon)))$ for all $\varepsilon \in \sim (\sim X)$ and $\sim (\sim X) \subseteq ((D^c)^c \times A \times O)$. Since $(D^c)^c = D$, its prove that $\sim (\sim X) = X$ and thus $((F, X)^c_D)^c = (F, X)_D$. ■

Definition 2.2.9 *The union of two FPBFSESs* $(G, X)_D$ *and* $(H, Y)_K$ *over* U*, represented by* $(G, X)_D \Cup (H, Y)_K$*, is the FPBFSES* $(I, Z)_R$ *such that* $Z = \{R \times A \times O\}$ *where* $R = D \cup K$ *and* $\forall \varepsilon \in Z$,

$$I_R(\varepsilon) = G_D(\varepsilon) \hat{\cup} H_K(\varepsilon)$$

where $\hat{\cup}$ *is bipolar fuzzy soft union.*

Example 2.2.10 *Let* $U = \{u_1, u_2, u_3\}$ *be set of universe,* $Z = \{e_1, e_2, e_3\}$ *be set of decision parameter and* $A = \{x_1, x_2\}$ *be set of experts such that* $X = \{e_1, e_2\}$ *and* $Y = \{e_1, e_3\}$. *Suppose* *that* $(G,X)_D$ *and* $(H,Y)_K$ *are two FPBFSESs over a soft universe* U *such that*

$$(G,X)_D = \left\{ \begin{array}{c} \left(\frac{0.8}{e_1}, x_1, 1\right) = \left\{ \left(\frac{u_1}{(-0.4,0.3)}\right), \left(\frac{u_2}{(-0.4,0.6)}\right), \left(\frac{u_3}{(-0.6,0.5)}\right) \right\} \\ \left(\frac{0.8}{e_1}, x_1, 0\right) = \left\{ \left(\frac{u_1}{(-0.6,0.6)}\right), \left(\frac{u_2}{(-0.4,0.7)}\right), \left(\frac{u_3}{(-0.9,0.2)}\right) \right\} \\ \left(\frac{0.6}{e_2}, x_2, 1\right) = \left\{ \left(\frac{u_1}{(-0.5,0.7)}\right), \left(\frac{u_2}{(-0.3,0.8)}\right), \left(\frac{u_3}{(-0.7,0.3)}\right) \right\} \\ \left(\frac{0.6}{e_2}, x_2, 0\right) = \left\{ \left(\frac{u_1}{(-0.7,0.3)}\right), \left(\frac{u_2}{(-0.6,0.2)}\right), \left(\frac{u_3}{(-0.4,0.9)}\right) \right\} \end{array} \right\}$$

and

$$(H,Y)_K = \left\{ \begin{array}{c} \left(\frac{0.5}{e_1}, x_1, 1\right) = \left\{ \left(\frac{u_1}{(-0.2,0.6)}\right), \left(\frac{u_2}{(-0.7,0.3)}\right), \left(\frac{u_3}{(-0.5,0.6)}\right) \right\} \\ \left(\frac{0.5}{e_1}, x_1, 0\right) = \left\{ \left(\frac{u_1}{(-0.67,0.23)}\right), \left(\frac{u_2}{(-0.5,0.6)}\right), \left(\frac{u_3}{(-0.2,0.4)}\right) \right\} \\ \left(\frac{0.7}{e_3}, x_2, 1\right) = \left\{ \left(\frac{u_1}{(-0.11,0.5)}\right), \left(\frac{u_2}{(-0.6,0.7)}\right), \left(\frac{u_3}{(-0.1,0.8)}\right) \right\} \\ \left(\frac{0.7}{e_3}, x_2, 0\right) = \left\{ \left(\frac{u_1}{(-0.71,0.1)}\right), \left(\frac{u_2}{(-0.5,0.5)}\right), \left(\frac{u_3}{(-0.9,0.2)}\right) \right\} \end{array} \right\}.$$

Now by using FPBFSES union, we have $(G,X)_D \Cup (H,Y)_K = (I,Z)_R$ *where*

$$(I,Z)_R = \left\{ \begin{array}{c} \left(\frac{0.8}{e_1}, x_1, 1\right) = \left\{ \left(\frac{u_1}{(-0.4,0.6)}\right), \left(\frac{u_2}{(-0.7,0.6)}\right), \left(\frac{u_3}{(-0.6,0.6)}\right) \right\} \\ \left(\frac{0.8}{e_1}, x_1, 0\right) = \left\{ \left(\frac{u_1}{(-0.67,0.6)}\right), \left(\frac{u_2}{(-0.5,0.7)}\right), \left(\frac{u_3}{(-0.9,0.4)}\right) \right\} \\ \left(\frac{0.6}{e_2}, x_2, 1\right) = \left\{ \left(\frac{u_1}{(-0.5,0.7)}\right), \left(\frac{u_2}{(-0.3,0.8)}\right), \left(\frac{u_3}{(-0.7,0.3)}\right) \right\} \\ \left(\frac{0.6}{e_2}, x_2, 0\right) = \left\{ \left(\frac{u_1}{(-0.7,0.3)}\right), \left(\frac{u_2}{(-0.6,0.2)}\right), \left(\frac{u_3}{(-0.4,0.9)}\right) \right\} \\ \left(\frac{0.7}{e_3}, x_2, 1\right) = \left\{ \left(\frac{u_1}{(-0.11,0.5)}\right), \left(\frac{u_2}{(-0.6,0.7)}\right), \left(\frac{u_3}{(-0.1,0.8)}\right) \right\} \\ \left(\frac{0.7}{e_3}, x_2, 0\right) = \left\{ \left(\frac{u_1}{(-0.71,0.1)}\right), \left(\frac{u_2}{(-0.5,0.5)}\right), \left(\frac{u_3}{(-0.9,0.2)}\right) \right\} \end{array} \right\}.$$

Proposition 2.2.11 *Let* $(G,X)_D$*,* $(H,Y)_K$ *and* $(I,Z)_R$ *are FPBFSESs over* U*. Then we have:*

1)$(G,X)_D \Cup (H,Y)_K = (H,Y)_K \Cup (G,X)_D$,

2)$((G,X)_D \Cup (H,Y)_K) \Cup (I,Z)_R = (G,X)_D \Cup ((H,Y)_K \Cup (I,Z)_R)$.

Proof. 1)Let $(G,X)_D \Cup (H,Y)_K = (I,Z)_R$. Then by Definition 2.2.9, for all $\varepsilon \in Z$, we have $R = D \cup K$ and $I_R(\varepsilon) = G_D(\varepsilon)\hat{\cup}H_K(\varepsilon)$. As we know that the union of fuzzy sets and BFSs are commutative then $R = K \cup D$ and $I_R(\varepsilon) = H_K(\varepsilon)\hat{\cup}G_D(\varepsilon)$. Therefore $(I,Z)_R = (H,Y)_K \Cup (G,X)_D$. Hence $(G,X)_D \Cup (H,Y)_K = (H,Y)_K \Cup (G,X)_D$.

The proof of remaining part is start forward and is therefore omitted. ■

Definition 2.2.12 *The intersection of two FPBFSESs* $(G,X)_D$ *and* $(H,Y)_K$ *over* U*, repre-*

sented by $(G,X)_D \Cap (H,Y)_K$*, is the FPBFSES* $(I,Z)_R$ *such that* $Z = \{R \times A \times O\}$ *where* $R = D \cap K$ *and* $\forall \varepsilon \in Z$*,*

$$I_R(\varepsilon) = G_D(\varepsilon) \hat{\cap} H_K(\varepsilon),$$

where $\hat{\cap}$ *is bipolar fuzzy soft intersection.*

Example 2.2.13 *Consider Example* 2.2.10*. Then* $(G,X)_D \Cap (H,Y)_K = (I,Z)_R$ *where*

$$(I,Z)_R = \left\{ \begin{array}{l} \left(\frac{0.5}{e_1}, x_1, 1\right) = \left\{ \left(\frac{u_1}{(-0.2,0.3)}\right), \left(\frac{u_2}{(-0.4,0.3)}\right), \left(\frac{u_3}{(-0.5,0.5)}\right) \right\} \\ \left(\frac{0.5}{e_1}, x_1, 0\right) = \left\{ \left(\frac{u_1}{(-0.6,0.23)}\right), \left(\frac{u_2}{(-0.5,0.6)}\right), \left(\frac{u_3}{(-0.2,0.2)}\right) \right\} \\ \left(\frac{0.6}{e_2}, x_2, 1\right) = \left\{ \left(\frac{u_1}{(-0.5,0.7)}\right), \left(\frac{u_2}{(-0.3,0.8)}\right), \left(\frac{u_3}{(-0.7,0.3)}\right) \right\} \\ \left(\frac{0.6}{e_2}, x_2, 0\right) = \left\{ \left(\frac{u_1}{(-0.7,0.3)}\right), \left(\frac{u_2}{(-0.6,0.2)}\right), \left(\frac{u_3}{(-0.4,0.9)}\right) \right\} \\ \left(\frac{0.7}{e_3}, x_2, 1\right) = \left\{ \left(\frac{u_1}{(-0.11,0.5)}\right), \left(\frac{u_2}{(-0.6,0.7)}\right), \left(\frac{u_3}{(-0.1,0.8)}\right) \right\} \\ \left(\frac{0.7}{e_3}, x_2, 0\right) = \left\{ \left(\frac{u_1}{(-0.71,0.1)}\right), \left(\frac{u_2}{(-0.5,0.5)}\right), \left(\frac{u_3}{(-0.9,0.2)}\right) \right\} \end{array} \right\}.$$

Proposition 2.2.14 *Let* $(G,X)_D$ *and* $(H,Y)_K$ *are FPBFSESs over* U*. Then the De Morgan's laws holds true:*

$$\text{i) } ((G,X)_D \Cup (H,Y)_K)^c = (G,X)_D^c \Cap (H,Y)_K^c,$$
$$\text{ii) } ((G,X)_D \Cap (H,Y)_K)^c = (G,X)_D^c \Cup (H,Y)_K^c.$$

Proof. i) Let $(G,X)_D$ and $(H,Y)_K$ are FPBFSESs over a universe U. We have

$$\begin{aligned} & (G,X)_D^c \Cap (H,Y)_K^c \\ & = G_D^c(\varepsilon) \hat{\cap} H_K^c(\varepsilon) \\ & = c(G_D(\sim \varepsilon)) \hat{\cap} c(H_K(\sim \varepsilon)) \\ & = c(G_D(\sim \varepsilon) \hat{\cap} H_K(\sim \varepsilon)) \\ & = c(G_D(\varepsilon) \hat{\cup} H_K(\varepsilon)) \\ & = ((G,X)_D \Cup (H,Y)_K)^c. \end{aligned}$$

ii) The proof is simple and is therefore eliminated. ■

Definition 2.2.15 *Let $(G, X)_D$ and $(H, Y)_K$ are two FPBFSESs over a universe U. Then "$(G, X)_D$ AND $(H, Y)_K$", represented by $(G, X)_D \tilde{\wedge} (H, Y)_K = (I, X \times Y)_R$ is a FPBFSES such that $I_R(\alpha, \beta) = (G_D(\alpha) \cap H_K(\beta))$ for all $(\alpha, \beta) \in X \times Y$ and $R = D \times K$.*

Definition 2.2.16 *Let $(G, X)_D$ and $(H, Y)_K$ are two FPBFSESs over a universe U. Then "$(G, X)_D$ OR $(H, Y)_K$", represented by $(G, X)_D \tilde{\vee} (H, Y)_K = (I, X \times Y)_R$ is a FPBFSES such that $I_R(\alpha, \beta) = (G_D(\alpha) \cup H_K(\beta))$ for all $(\alpha, \beta) \in X \times Y$ and $R = D \times K$..*

Proposition 2.2.17 *Let $(G, X)_D$ and $(H, Y)_K$ are FPBFSESs over U. Then the De Morgan's laws holds true:*

$$i)\ ((G, X)_D \tilde{\wedge} (H, Y)_K)^c = (G, X)^c_D \tilde{\vee} (H, Y)^c_K,$$

$$ii)\ ((G, X)_D \tilde{\vee} (H, Y)_K)^c = (G, X)^c_D \tilde{\wedge} (H, Y)^c_K.$$

Proof. i) Suppose $(G, X)_D$ and $(H, Y)_K$ are FPBFSESs over a universe U. Then by Definition 2.2.15 and 2.2.16, we have

$$\begin{aligned} &((G, X)_D \tilde{\wedge} (H, Y)_K)^c \\ &= ((G_D(\alpha) \cap (H_K(\beta))^c \\ &= G^c_D(\alpha) \cup H^c_K(\beta) \\ &= c(G_D(\sim \alpha)) \cup c(H_K(\sim \beta)) \\ &= (G, X)^c_D \tilde{\vee} (H, Y)^c_K. \end{aligned}$$

ii) And

$$\begin{aligned} & ((G, X)_D \tilde{\vee} (H, Y)_K)^c \\ = \ & ((G_D(\alpha) \cup (H_K(\beta))^c \\ = \ & G^c_D(\alpha) \cap H^c_K(\beta) \\ = \ & c(G_D(\sim \alpha)) \cap c(H_K(\sim \beta)) \\ = \ & (G, X)^c_D \tilde{\wedge} (H, Y)^c_K. \end{aligned}$$

■

2.3 Application of Fuzzy Parameterized Bipolar Fuzzy Soft Expert Sets in a Decision Making Problems

To solve hypothetical decision making problems, we establish generalized algorithm which will be applied to FPBFSES model in this section.

Suppose that $U = \{h_1, h_2, h_3\}$ be the set of three houses making the set of universe. Assume that Mr. M wants to buy a house considering four decision parameters $E = \{e_1, e_2, e_3, e_4\}$, where $e_i(1, 2, 3, 4)$ represents the beautiful, wooden, in green surrounding, traffic convenient respectively. Mr. M, his wife and his son have their own opinion making the set of experts $X = \{p, q, r\}$. After critical observation the family constructs the fuzzy sets

$$D = \{\frac{0.9}{e_1}, \frac{0.4}{e_2}, \frac{0.8}{e_3}, \frac{0.5}{e_4}\}$$

and consequently use it to form the following FPBFSES.

$$
(F,Z)_D = \left\{ \begin{array}{c}
(\frac{0.9}{e_1}, p, 1) = \left\{ \left(\frac{h_1}{(-0.3,0.5)} \right), \left(\frac{h_2}{(-0.4,0.5)} \right), \left(\frac{h_3}{(-0.6,0.7)} \right) \right\} \\
(\frac{0.4}{e_2}, p, 1) = \left\{ \left(\frac{h_1}{(-0.3,0.6)} \right), \left(\frac{h_2}{(-0.5,0.7)} \right), \left(\frac{h_3}{(-0.8,0.2)} \right) \right\} \\
(\frac{0.8}{e_3}, p, 1) = \left\{ \left(\frac{h_1}{(-0.6,0.5)} \right), \left(\frac{h_2}{(-0.5,0.3)} \right), \left(\frac{h_3}{(-0.6,0.6)} \right) \right\} \\
(\frac{0.5}{e_4}, p, 1) = \left\{ \left(\frac{h_1}{(-0.4,0.9)} \right), \left(\frac{h_2}{(-0.7,0.7)} \right), \left(\frac{h_3}{(-0.3,0.5)} \right) \right\} \\
(\frac{0.9}{e_1}, q, 1) = \left\{ \left(\frac{h_1}{(-0.8,0.7)} \right), \left(\frac{h_2}{(-0.1,0.2)} \right), \left(\frac{h_3}{(-0.4,1)} \right) \right\} \\
(\frac{0.4}{e_2}, q, 1) = \left\{ \left(\frac{h_1}{(-0.6,0.9)} \right), \left(\frac{h_2}{(-1,0)} \right), \left(\frac{h_3}{(-0.5,0.8)} \right) \right\} \\
(\frac{0.8}{e_3}, q, 1) = \left\{ \left(\frac{h_1}{(-0.6,0.1)} \right), \left(\frac{h_2}{(-0.2,0.6)} \right), \left(\frac{h_3}{(-0.5,0.7)} \right) \right\} \\
(\frac{0.5}{e_4}, q, 1) = \left\{ \left(\frac{h_1}{(-0.5,0.7)} \right), \left(\frac{h_2}{(-0.6,0.8)} \right), \left(\frac{h_3}{(-0.5,0.4)} \right) \right\} \\
(\frac{0.9}{e_1}, r, 1) = \left\{ \left(\frac{h_1}{(-0.8,0.4)} \right), \left(\frac{h_2}{(-0.2,0.8)} \right), \left(\frac{h_3}{(-0.5,0.3)} \right) \right\} \\
(\frac{0.4}{e_2}, r, 1) = \left\{ \left(\frac{h_1}{(-0.78,0.8)} \right), \left(\frac{h_2}{(-0.5,0.7)} \right), \left(\frac{h_3}{(-0.66,0.3)} \right) \right\} \\
(\frac{0.5}{e_4}, r, 1) = \left\{ \left(\frac{h_1}{(-0.1,0.1)} \right), \left(\frac{h_2}{(-0.5,0.7)} \right), \left(\frac{h_3}{(-0.58,0.8)} \right) \right\} \\
(\frac{0.9}{e_1}, p, 0) = \left\{ \left(\frac{h_1}{(-0.4,0.7)} \right), \left(\frac{h_2}{(-0.7,0.6)} \right), \left(\frac{h_3}{(-0.4,0.8)} \right) \right\} \\
(\frac{0.4}{e_2}, p, 0) = \left\{ \left(\frac{h_1}{(-0.3,0.7)} \right), \left(\frac{h_2}{(-0.44,0.11)} \right), \left(\frac{h_3}{(-0.45,0.82)} \right) \right\} \\
(\frac{0.8}{e_3}, p, 0) = \left\{ \left(\frac{h_1}{(-0.7,0.36)} \right), \left(\frac{h_2}{(-0.2,0.8)} \right), \left(\frac{h_3}{(-0.7,0.2)} \right) \right\} \\
(\frac{0.9}{e_1}, q, 0) = \left\{ \left(\frac{h_1}{(-0.9,0.6)} \right), \left(\frac{h_2}{(-0.8,0.2)} \right), \left(\frac{h_3}{(-0.1,0.8)} \right) \right\} \\
(\frac{0.4}{e_2}, q, 0) = \left\{ \left(\frac{h_1}{(-0.8,0.8)} \right), \left(\frac{h_2}{(-0.2,0.9)} \right), \left(\frac{h_3}{(-0.5,0.7)} \right) \right\} \\
(\frac{0.5}{e_4}, q, 0) = \left\{ \left(\frac{h_1}{(-0.3,0.8)} \right), \left(\frac{h_2}{(-0.5,0.8)} \right), \left(\frac{h_3}{(-0.1,0.7)} \right) \right\} \\
(\frac{0.9}{e_1}, r, 0) = \left\{ \left(\frac{h_1}{(-0.4,0.8)} \right), \left(\frac{h_2}{(-0.2,0.8)} \right), \left(\frac{h_3}{(-0.5,0.9)} \right) \right\} \\
(\frac{0.8}{e_3}, r, 0) = \left\{ \left(\frac{h_1}{(-0.7,0.1)} \right), \left(\frac{h_2}{(-0.2,0.8)} \right), \left(\frac{h_3}{(-0.8,0.5)} \right) \right\} \\
(\frac{0.5}{e_4}, r, 0) = \left\{ \left(\frac{h_1}{(-0.4,0.7)} \right), \left(\frac{h_2}{(-0.4,0.9)} \right), \left(\frac{h_3}{(-0.5,0.6)} \right) \right\}
\end{array} \right\}.
$$

Table 1: Value of $c_i = f_i^+(h_i) - f_i^-(h_i)$ for all $h_i \in U$.

U	h_1	h_2	h_3
$(\frac{0.9}{e_1}, p, 1)$	0.8	0.9	1.3
$(\frac{0.4}{e_2}, p, 1)$	0.9	1.2	1
$(\frac{0.8}{e_3}, p, 1)$	1.1	0.8	1.2
$(\frac{0.5}{e_4}, p, 1)$	1.3	1.4	0.8
$(\frac{0.9}{e_1}, q, 1)$	1.3	0.3	1.4
$(\frac{0.4}{e_2}, q, 1)$	1.05	1	1.3
$(\frac{0.8}{e_3}, q, 1)$	0.7	0.8	1.2
$(\frac{0.5}{e_4}, q, 1)$	1.3	1	1.4
$(\frac{0.9}{e_1}, r, 1)$	1.2	1	0.8
$(\frac{0.4}{e_2}, r, 1)$	1.58	1.2	0.96
$(\frac{0.5}{e_4}, r, 1)$	0.2	1.2	1.38
$(\frac{0.9}{e_1}, p, 0)$	1.1	1.1	1.2
$(\frac{0.4}{e_2}, p, 0)$	1	0.55	1.27
$(\frac{0.8}{e_3}, p, 0)$	1.6	1	0.9
$(\frac{0.9}{e_1}, q, 0)$	1.5	1	0.9
$(\frac{0.4}{e_2}, q, 0)$	1.6	1.1	1.2
$(\frac{0.5}{e_4}, q, 0)$	1.1	1.02	0.8
$(\frac{0.9}{e_1}, r, 0)$	1.2	1	1.4
$(\frac{0.8}{e_3}, r, 0)$	0.8	1	1.3
$(\frac{0.5}{e_4}, r, 0)$	1.1	1.3	1.1

Table 2: Numerical grade for agree-FPBFSES.

U	h_1	h_2	h_3
$(\frac{0.9}{e_1}, p, 1)$	0.8	0.9	1.3
$(\frac{0.4}{e_2}, p, 1)$	0.9	1.2	1
$(\frac{0.8}{e_3}, p, 1)$	1.1	0.8	1.2
$(\frac{0.5}{e_4}, p, 1)$	1.3	1.4	0.8
$(\frac{0.9}{e_1}, q, 1)$	1.3	0.3	1.4
$(\frac{0.4}{e_2}, q, 1)$	1.05	1	1.3
$(\frac{0.8}{e_3}, q, 1)$	0.7	0.8	1.2
$(\frac{0.5}{e_4}, q, 1)$	1.3	1	1.4
$(\frac{0.9}{e_1}, r, 1)$	1.2	1	0.8
$(\frac{0.4}{e_2}, r, 1)$	1.58	1.2	.96
$(\frac{0.5}{e_4}, r, 1)$	0.2	1.2	1.38

$$A_j = \sum_{x \in X} \sum_{i=1}^{4} c_{ij}\mu(e_j) : \text{Score } h_1 = 7.222;\ \text{score } h_2 = 6.42;\ \text{score } h_3 = 8.164.$$

Table 3: Numerical grade for disagree-FPBFSES.

U	h_1	h_2	h_3
$(\frac{0.9}{e_1}, p, 0)$	1.1	1.1	1.2
$(\frac{0.4}{e_2}, p, 0)$	1	0.55	1.27
$(\frac{0.8}{e_3}, p, 0)$	1.6	1	0.9
$(\frac{0.9}{e_1}, q, 0)$	1.5	1	0.9
$(\frac{0.4}{e_2}, q, 0)$	1.6	1.1	1.2
$(\frac{0.5}{e_4}, q, 0)$	1.1	1.02	0.8
$(\frac{0.9}{e_1}, r, 0)$	1.2	1	1.4
$(\frac{0.8}{e_3}, r, 0)$	0.8	1	1.3
$(\frac{0.5}{e_4}, r, 0)$	1.1	1.3	1.1

$$D_j = \sum_{x \in X} \sum_{i=1}^{4} c_{ij}\mu(e_j) : \text{Score } h_1 = 7.48;\ \text{score } h_2 = 6.21;\ \text{score } h_3 = 6.848.$$

Table 4: The score $r_j = A_j - B_j$

A_j	B_j	r_j
score $h_1 = 7.222$	score $h_1 = 7.480$	-0.258
score $h_2 = 6.420$	score $h_2 = 6.210$	0.210
score $h_3 = 8.164$	score $h_3 = 6.848$	1.316

Next the generalized algorithm is applied to find the solution of decision making problem. The algorithm may be used is to decide the best house among three different houses. The generalized algorithm is given below:

(1) Input the FPBFSES $(F, A)_D$.

(2) Compute $f_p^+(h_i) - f_p^-(h_i)$ for every $h_i \in U$, where $f_p^+(h_i)$ represent the positive response and $f_p^-(h_i)$ represent the negative response about each $h_i \in U$.

(3) Compute the largest numerical grade for the agree and disagree FPBFSES.

(4) Find the score of every $h_i \in U$ by taking the sum of the products of the numerical grade of each element with member function of fuzzy set D, for the agree-FPBFSES (A_j) and disagree-FPBFSES (B_j).

(5) Find r_j where $r_j = A_j - B_j$ for every $h_i \in U$.

(6) Find s where$s = max_{h_i \in U}\{r_j\}$. Then to choose h_i as the best solution to the problem.

In Table 1, we have $f_p^+(h_i) - f_p^-(h_i)$ for every $h_i \in U$.

The largest numerical grade for the element in agree and disagree FPBFSES are given in Tables 2 and 3 respectively.

In Table 4, we calculate A_j, B_j and r_j. From the computation done from Table 1-4, we obtain $s = max_{h_i \in U}\{r_j\} = r_3$. Thus Mr. M will buy the house h_3.

2.4 Basic Operations on Possibility Bipolar Fuzzy Soft Expert Sets

In this section, the notion of PBFSES are discussed and some interesting results relative to this concept are investigated. We define some basic operations using this theory. Some applicable laws such as De Morgan's laws of this concept are proved.

Definition 2.4.1 *Let $U = \{u_1, u_2, u_3, ..., u_n\}$ be a universe and let $E = \{e_1, e_2, e_3, ..., e_m\}$ be a* *set of parameters. Let $F_p : E \to B^U \times I^U$ be a function defined as follows:*

$$F_p(e) = \left\{ (\frac{u}{F(e)(u)}, p(e)(u)) \right.$$

where $F(e)(u) = (f^+(u), f^-(u))$ and $p(e)(u)$ is a fuzzy subsets of U. Then $F_p(e)$ is called a possibility bipolar fuzzy soft sets over the universe (U, E).

Definition 2.4.2 *Let $Z = E \times X \times O$, where E is the set of parameters, X is the set of experts and O is the set of opinions. Let $F_p : Z \to B^U \times I^U$ be a function defined as*

$$F_p(z) = \left\{ (\frac{u_i}{F(z)(u_i)}, p(z)(u_i)) \forall u_i \in U \right.$$

then F_p is called a PBFSES over soft universe (U, Z), where B^U is the collection of all bipolar fuzzy subsets of U, I^U is the collection of all fuzzy subsets of U, $F(z)(u_i)$ represents the bipolar fuzzy subsets of U and $p(z_i)(u_i)$ represents the fuzzy subsets of U.

Example 2.4.3 *Suppose we have three cars denoted by the set $U = \{c_1, c_2, c_3\}$, let $E = \{e_1 =$Beautiful$, e_2 =$Costly$\}$ be the set of parameters for decision and x_1 and x_2 be two experts. Let $F_p : Z \longrightarrow B^U \times I^U$. Then (F_p, Z) is defined as*

$$(F_p, Z) = \left\{ \begin{array}{c} (e_1, x_1, 1) = \left\{ \left(\frac{c_1}{(-0.6,0.7)}, 0.6 \right), \left(\frac{c_2}{(-0.3,0.4)}, 0.2 \right), \left(\frac{c_3}{(-0.4,0.5)}, 0.3 \right) \right\} \\ (e_2, x_1, 1) = \left\{ \left(\frac{c_1}{(-0.5,0.9)}, 0.8 \right), \left(\frac{c_2}{(-0.4,0.7)}, 0.5 \right), \left(\frac{c_3}{(-0.5,0.6)}, 0.7 \right) \right\} \\ (e_1, x_2, 1) = \left\{ \left(\frac{c_1}{(-0.2,0.4)}, 0.4 \right), \left(\frac{c_2}{(-0.1,0.6)}, 0.9 \right), \left(\frac{c_3}{(-0.7,0.5)}, 0.6 \right) \right\} \\ (e_2, x_2, 1) = \left\{ \left(\frac{c_1}{(-0.6,0.9)}, 0.4 \right), \left(\frac{c_2}{(-0.4,0.6)}, 0.3 \right), \left(\frac{c_3}{(-0.7,0.7)}, 0.5 \right) \right\} \\ (e_1, x_1, 0) = \left\{ \left(\frac{c_1}{(-0.8,0.2)}, 0.3 \right), \left(\frac{c_2}{(-0.5,0.7)}, 0.6 \right), \left(\frac{c_3}{(-0.5,0.6)}, 0.4 \right) \right\} \\ (e_2, x_1, 0) = \left\{ \left(\frac{c_1}{(-0.45,0.05)}, 0.4 \right), \left(\frac{c_2}{(-1,0)}, 0.8 \right), \left(\frac{c_3}{(-0.4,0.7)}, 0.5 \right) \right\} \\ (e_1, x_2, 0) = \left\{ \left(\frac{c_1}{(-0.1,0.8)}, 0.62 \right), \left(\frac{c_2}{(0,1)}, 0.25 \right), \left(\frac{c_3}{(-0.55,0.3)}, 0.99 \right) \right\} \\ (e_2, x_2, 0) = \left\{ \left(\frac{c_1}{(-0.5,0.3)}, 0.5 \right), \left(\frac{c_2}{(-0.5,0.7)}, 0.3 \right), \left(\frac{c_3}{(-0.58,0.8)}, 0.9 \right) \right\} \end{array} \right\}.$$

Definition 2.4.4 *Let (F_p, A) and (G_p, B) are two PBFSES over (U, Z). Then (F_p, A) is called*

PBFSE subset of (G_p, B) *if* A *is subset of* B *and* $\forall\ \tau \in A$, *we have:*

(i) $p(\tau)$ *is fuzzy subset of* $q(\tau)$,

(ii) $F(\tau)$ *is bipolar fuzzy subset of* $G(\tau)$.

Definition 2.4.5 *Let* (F_p, A) *and* (G_p, B) *are two PBFSES. Then* (F_p, A) *and* (G_p, B) *are equal if for all* $\tau \in A$, *we have:*

(i) $p(\tau) = q(\tau)$,

(ii) $F(\tau) = G(\tau)$.

Definition 2.4.6 *A PBFSES* (F_p, A) *is called a null PBFSES, represented by* $(\tilde{\emptyset}, A)$ *and is defined as*

$$(\tilde{\emptyset}, A) = \left\{ (\frac{u}{F(\tau)(u)}, p(\tau)(u)) \right.$$

where $F(\tau)(u) = (0, 0)$ *and* $p(\tau)(u) = 0$ *for all* $\tau \in Z$ *and for all* $u \in U$.

Definition 2.4.7 *A PBFSES* (F_p, A) *is called an absolute PBFSES, represented by* $(F_p, A)_{abs}$ *and is defined as*

$$(F_p, A)_{abs} = \left\{ (\frac{u}{F(\tau)(u)}, p(\tau)(u)) \right.$$

where $F(\tau)(u) = (1, -1)$ *and* $p(\tau)(u) = 1$ *for all* $\tau \in Z$ *and for all* $u \in U$.

Definition 2.4.8 *An agree-PBFSES over* U, *denoted as* $(F_p, A)_1$, *is defined as*

$$(F_p, A)_1 = \left\{ (\frac{u}{F(\tau)(u)}, p(\tau)(u)) \text{ where } \tau \in E \times X \times \{1\} \right. .$$

Definition 2.4.9 *A disagree-PBFSES over* U, *denoted as* $(F_p, A)_0$, *is defined as*

$$(F_p, A)_0 = \left\{ (\frac{u}{F(\tau)(u)}, p(\tau)(u)) \text{ where } \tau \in E \times X \times \{0\} \right. .$$

Definition 2.4.10 *Let* (F_p, A) *be a PBFSES over* (U, Z). *Then* $(F_p, A)^c$ *is defined as*

$$(F_p, A)^c = \left\{ (\frac{u}{\tilde{c}(F(\tau)(u))}, c(p(\tau)(u))) \forall \tau \in A \right.$$

where $\tilde{c}$ is a bipolar fuzzy complement and c is a fuzzy complement.

Example 2.4.11 *Suppose PBFSES (F_p, Z) over (U, Z) as given in Example 2.4.3. Then $(F_p, Z)^c$ is defined as*

$$(F_p, Z)^c = \left\{ \begin{array}{c} (e_1, x_1, 1) = \left\{ \left(\frac{c_1}{(-0.4,0.3)}, 0.4 \right), \left(\frac{c_2}{(-0.7,0.6)}, 0.8 \right), \left(\frac{c_3}{(-0.6,0.5)}, 0.7 \right) \right\} \\ (e_2, x_1, 1) = \left\{ \left(\frac{c_1}{(-0.5,0.1)}, 0.2 \right), \left(\frac{c_2}{(-0.6,0.3)}, 0.5 \right), \left(\frac{c_3}{(-0.5,0.4)}, 0.3 \right) \right\} \\ (e_1, x_2, 1) = \left\{ \left(\frac{c_1}{(-0.8,0.6)}, 0.6 \right), \left(\frac{c_2}{(-0.9,0.4)}, 0.1 \right), \left(\frac{c_3}{(-0.3,0.5)}, 0.4 \right) \right\} \\ (e_2, x_2, 1) = \left\{ \left(\frac{c_1}{(-0.4,0.1)}, 0.6 \right), \left(\frac{c_2}{(-0.6,0.4)}, 0.7 \right), \left(\frac{c_3}{(-0.3,0.3)}, 0.5 \right) \right\} \\ (e_1, x_1, 0) = \left\{ \left(\frac{c_1}{(-0.2,0.8)}, 0.7 \right), \left(\frac{c_2}{(-0.5,0.3)}, 0.4 \right), \left(\frac{c_3}{(-0.5,0.4)}, 0.6 \right) \right\} \\ (e_2, x_1, 0) = \left\{ \left(\frac{c_1}{(-0.55,0.95)}, 0.6 \right), \left(\frac{c_2}{(0,1)}, 0.2 \right), \left(\frac{c_3}{(-0.6,0.3)}, 0.5 \right) \right\} \\ (e_1, x_2, 0) = \left\{ \left(\frac{c_1}{(-0.9,0.2)}, 0.38 \right), \left(\frac{c_2}{(-1,0)}, 0.75 \right), \left(\frac{c_3}{(-0.45,0.7)}, 0.01 \right) \right\} \\ (e_2, x_2, 0) = \left\{ \left(\frac{c_1}{(-0.5,0.7)}, 0.5 \right), \left(\frac{c_2}{(-0.5,0.3)}, 0.7 \right), \left(\frac{c_3}{(-0.42,0.2)}, 0.1 \right) \right\} \end{array} \right\}.$$

Proposition 2.4.12 *Let (F_p, A) be a PBFSES over (U, Z). Then we have*

$$((F_p, A)^c)^c = (F_p, A).$$

Proof. Straight forward. ■

Definition 2.4.13 *Let (F_p, A) and (G_q, B) are two PBFSES over (U, Z). Then the union of (F_p, A) and (G_q, B), represented by $(F_p, A) \Cup (G_q, B) = (H_r, C)$ where $C = A \cup B$ and*

$$r(\tau) = \max(p(\tau), q(\tau)) \forall \tau \in C,$$
$$H(\tau) = F(\tau) \tilde{\cup} G(\tau) \forall \tau \in C,$$

where

$$H(\tau) = \begin{cases} F(\tau) & \tau \in A - B, \\ G(\tau) & \tau \in B - A, \\ F(\tau) \cup G(\tau) & \tau \in A \cap B. \end{cases}$$

Example 2.4.14 *Let $U = \{u_1, u_2, u_3\}$ be set of universe, $Z = \{e_1, e_2, e_3\}$ be set of decision parameter and $X = \{x_1, x_2\}$ be set of experts. Let $A = \{e_1, e_2\}$ and $B = \{e_1, e_3\}$. Assume that*

(F_p, A) *and* (G_q, B) *are two PBFSES such that*

$$(F_p, A) = \left\{ \begin{array}{c} (e_1, x_1, 1) = \left\{ \left(\frac{u_1}{(-0.4,0.3)}, 0.4 \right), \left(\frac{u_2}{(-0.4,0.6)}, 0.6 \right), \left(\frac{u_3}{(-0.6,0.5)}, 0.7 \right) \right\} \\ (e_1, x_1, 0) = \left\{ \left(\frac{u_1}{(-0.6,0.6)}, 0.2 \right), \left(\frac{u_2}{(-0.4,0.7)}, 0.4 \right), \left(\frac{u_3}{(-0.9,0.2)}, 0.3 \right) \right\} \\ (e_2, x_2, 1) = \left\{ \left(\frac{u_1}{(-0.5,0.7)}, 0.8 \right), \left(\frac{u_2}{(-0.3,0.8)}, 0.6 \right), \left(\frac{u_3}{(-0.7,0.3)}, 0.4 \right) \right\} \end{array} \right\}$$

and

$$(G_q, B) = \left\{ \begin{array}{c} (e_1, x_1, 1) = \left\{ \left(\frac{u_1}{(-0.2,0.6)}, 0.5 \right), \left(\frac{u_2}{(-0.7,0.3)}, 0.7 \right), \left(\frac{u_3}{(-0.5,0.6)}, 0.8 \right) \right\} \\ (e_1, x_1, 0) = \left\{ \left(\frac{u_1}{(-0.67,0.23)}, 0.7 \right), \left(\frac{u_2}{(-0.5,0.6)}, 0.4 \right), \left(\frac{u_3}{(-0.2,0.4)}, 0.5 \right) \right\} \\ (e_3, x_2, 1) = \left\{ \left(\frac{u_1}{(-0.11,0.5)}, 0.25 \right), \left(\frac{u_2}{(-0.6,0.7)}, 0.58 \right), \left(\frac{u_3}{(-0.1,0.8)}, 0.8 \right) \right\} \end{array} \right\}.$$

Now by using PBFSES union, we have $(F_p, A) \Cup (G_q, B) = (H_r, C)$ *where*

$$(H_r, C) = \left\{ \begin{array}{c} (e_1, x_1, 1) = \left\{ \left(\frac{u_1}{(-0.4,0.6)}, 0.5 \right), \left(\frac{u_2}{(-0.7,0.6)}, 0.7 \right), \left(\frac{u_3}{(-0.6,0.6)}, 0.8 \right) \right\} \\ (e_1, x_1, 0) = \left\{ \left(\frac{u_1}{(-0.67,0.6)}, 0.7 \right), \left(\frac{u_2}{(-0.5,0.7)}, 0.4 \right), \left(\frac{u_3}{(-0.9,0.4)}, 0.5 \right) \right\} \\ (e_2, x_2, 1) = \left\{ \left(\frac{u_1}{(-0.5,0.7)}, 0.8 \right), \left(\frac{u_2}{(-0.3,0.8)}, 0.6 \right), \left(\frac{u_3}{(-0.7,0.3)}, 0.4 \right) \right\} \\ (e_3, x_2, 1) = \left\{ \left(\frac{u_1}{(-0.11,0.5)}, 0.25 \right), \left(\frac{u_2}{(-0.6,0.7)}, 0.58 \right), \left(\frac{u_3}{(-0.1,0.8)}, 0.8 \right) \right\} \end{array} \right\}.$$

Proposition 2.4.15 *Let* (F_p, A), (G_q, B) *and* (H_r, C) *are three PBFSES over* (U, Z). *Then we have:*

1) $(F_p, A) \Cup (F_p, A) = (F_p, A)$,
2) $(F_p, A) \Cup (\tilde{\emptyset}, A) = (F_p, A)$,
3) $(F_p, A) \Cup (G_q, B) = (G_q, B) \Cup (F_p, A)$,

Proof. Its proof is straight forward and is therefore omitted. ■

Definition 2.4.16 *Let* (F_p, A) *and* (G_q, B) *are two PBFSES over* (U, Z). *Then* $(F_p, A) \Cap (G_q, B) = (H_r, C)$ *where* $C = A \cup B$ *and*

$$r(\tau) = \min(p(\tau), q(\tau)), \forall \tau \in C,$$

$$H(\tau) = F(\tau) \tilde{\cap} G(\tau), \forall \tau \in C,$$

where

$$H(\tau)=\begin{cases} F(\tau) & \tau\in A-B,\\ G(\tau) & \tau\in B-A,\\ F(\tau)\cap G(\tau) & \tau\in A\cap B.\end{cases}$$

Example 2.4.17 *Consider Example* 2.4.14. *Then* $(F_p,A)\Cap(G_q,B)=(H_r,C)$ *where*

$$(H_r,C)=\left\{\begin{array}{c} (e_1,x_1,1)=\left\{\left(\frac{u_1}{(-0.2,0.3)},0.4\right),\left(\frac{u_2}{(-0.4,0.3)},0.6\right),\left(\frac{u_3}{(-0.5,0.5)},0.7\right)\right\}\\ (e_1,x_1,0)=\left\{\left(\frac{u_1}{(-0.6,0.23)},0.2\right),\left(\frac{u_2}{(-0.4,0.6)},0.4\right),\left(\frac{u_3}{(-0.2,0.2)},0.3\right)\right\}\\ (e_2,x_2,1)=\left\{\left(\frac{u_1}{(-0.5,0.7)},0.8\right),\left(\frac{u_2}{(-0.3,0.8)},0.6\right),\left(\frac{u_3}{(-0.7,0.3)},0.4\right)\right\}\\ (e_3,x_2,1)=\left\{\left(\frac{u_1}{(-0.11,0.5)},0.25\right),\left(\frac{u_2}{(-0.6,0.7)},0.58\right),\left(\frac{u_3}{(-0.1,0.8)},0.8\right)\right\}\end{array}\right\}.$$

Proposition 2.4.18 *Let* (F_p,A), (G_q,B) *and* (H_r,C) *are three PBFSES over* (U,Z). *Then the following conditions holds:*

1) $(F_p,A)\Cap(F_p,A)=(F_p,A)$,
2) $(F_p,A)\Cap(F_p,A)_{abs}=(F_p,A)$,
3) $(F_p,A)\Cap(G_q,B)=(G_q,B)\Cap(F_p,A)$,
4) $((F_p,A)\Cap(G_q,B))\Cap(H_r,C)=(F_p,A)\Cap((G_q,B)\Cap(H_r,C))$.

Proof. Its proof is straight forward. ■

Proposition 2.4.19 *Let* (F_p,A) *and* (G_q,B) *are PBFSES over* (U,Z). *Then the De Morgan's laws holds true:*

i) $((F_p,A)\Cup(G_q,B))^c=(F_p,A)^c\Cap(G_q,B)^c$,

ii) $((F_p,A)\Cap(G_q,B))^c=(F_p,A)^c\Cup(G_q,B)^c$.

Proof. i) Let (F_p, A) and (G_q, B) are PBFSES over a soft universe (U, Z). We have

$$
\begin{aligned}
&(F_p, A)^c \Cap (G_q, B)^c \\
&= (F(\tau), p(\tau))^c \tilde{\cap} (G(\tau), q(\tau))^c \\
&= (\tilde{c}F(\tau), cp(\tau)) \tilde{\cap} (\tilde{c}G(\tau), cq(\tau)) \\
&= ((\tilde{c}F(\tau) \tilde{\cap} \tilde{c}G(\tau)), \min(cp(\tau), cq(\tau))) \\
&= (\tilde{c}(F(\tau) \tilde{\cup} G(\tau))), c(\max(p(\tau), q(\tau)))) \\
&= ((F_p, A) \Cup (G_q, B))^c.
\end{aligned}
$$

ii) Its proof is straight forward and is therefore omitted. ■

Definition 2.4.20 *Let (F_p, A) and (G_q, B) are two PBFSES over (U, Z). Then "(F_p, A) AND (G_q, B)", represented by $(F_p, A) \tilde{\wedge} (G_q, B) = (H_r, A \times B)$ is a PBFSES where*

$$(H_r, A \times B) = (H(\alpha, \beta), r(\alpha, \beta))$$

such that $H(\alpha, \beta) = (F(\alpha) \cap G(\beta))$ and $r(\alpha, \beta) = \min(p(\alpha), q(\beta))\ \forall\ (\alpha, \beta) \in A \times B$.

Definition 2.4.21 *Let (F_p, A) and (G_q, B) are two PBFSES over (U, Z). Then "(F_p, A) OR (G_q, B)", represented by $(F_p, A) \tilde{\vee} (G_q, B) = (H_r, A \times B)$ is a PBFSES where*

$$(H_r, A \times B) = (H(\alpha, \beta), r(\alpha, \beta))$$

such that $H(\alpha, \beta) = (F(\alpha) \cup G(\beta))$ and $r(\alpha, \beta) = \max(p(\alpha), q(\beta))\ \forall\ (\alpha, \beta) \in A \times B$.

Proposition 2.4.22 *Let (F_p, A) and (G_q, B) are PBFSES over (U, Z). Then the De Morgan's laws holds true:*

$$\text{i) } ((F_p, A) \tilde{\wedge} (G_q, B))^c = (F_p, A)^c \tilde{\vee} (G_q, B)^c,$$

$$\text{ii) } ((F_p, A) \tilde{\vee} (G_q, B))^c = (F_p, A)^c \tilde{\wedge} (G_q, B)^c.$$

Proof. i) Suppose (F_p, A) and (G_q, B) are PBFSES over (U, Z). Then

$$(F_p, A) = (F(\alpha), p(\alpha))\forall\alpha \in A \subseteq Z,$$
$$(G_q, B) = (G(\beta), q(\beta))\forall\beta \in B \subseteq Z.$$

Then by Definition 2.4.20 and 2.4.21, we have

$$\begin{aligned}
&((F_p, A)\tilde{\wedge}(G_q, B))^c \\
&= ((F(\alpha), p(\alpha))\tilde{\wedge}(G(\beta), q(\beta)))^c \\
&= (F(\alpha) \cap G(\beta), \min(p(\alpha), q(\beta)))^c \\
&= (\tilde{c}(F(\alpha) \cap G(\beta)), c(\min(p(\alpha), q(\beta)))) \\
&= (\tilde{c}F(\alpha) \cup \tilde{c}G(\beta), (\max(cp(\alpha), cq(\beta)))) \\
&= (F(\alpha), p(\alpha))^c\tilde{\vee}(G(\beta), q(\beta))^c \\
&= (F_p, A)^c\tilde{\vee}(G_q, B)^c.
\end{aligned}$$

ii) The proof is simple and is therefore eliminated. ■

2.5 Application of Possibility Bipolar Fuzzy Soft Expert Sets in a Decision Making Problem

In this section we discuss an application of PBFSES in decision making problem by using generalized algorithm.

Suppose that $U = \{m_1 = Nokia, m_2 = Samsung, m_3 = Huawei\}$ be the set of three mobile brands making the set of universe. Suppose that Mr. M wants to buy a mobile phone considering four specification as decision parameters $E = \{e_1, e_2, e_3, e_4\}$, where $e_i(1, 2, 3, 4)$ represents the best camera, high processor, good battery life, fast charging respectively. Mr. M, his wife and his son have their own opinion making the set of experts $X = \{p, q, r\}$. After critical observation the family constructs the following PBFSES.

$$(F_p, Z) = \left\{ \begin{array}{c} (e_1,p,1) = \left\{ \left(\frac{m_1}{(-0.3,0.5)}, 0.4\right), \left(\frac{m_2}{(-0.4,0.5)}, 0.8\right), \left(\frac{m_3}{(-0.6,0.7)}, 0.5\right) \right\} \\ (e_2,p,1) = \left\{ \left(\frac{m_1}{(-0.3,0.6)}, 0.5\right), \left(\frac{m_2}{(-0.5,0.7)}, 0.6\right), \left(\frac{m_3}{(-0.8,0.2)}, 0.1\right) \right\} \\ (e_3,p,1) = \left\{ \left(\frac{m_1}{(-0.6,0.5)}, 0.4\right), \left(\frac{m_2}{(-0.5,0.3)}, 0.5\right), \left(\frac{m_3}{(-0.6,0.6)}, 0.8\right) \right\} \\ (e_4,p,1) = \left\{ \left(\frac{m_1}{(-0.4,0.9)}, 0.7\right), \left(\frac{m_2}{(-0.7,0.7)}, 0.3\right), \left(\frac{m_3}{(-0.3,0.5)}, 0.4\right) \right\} \\ (e_1,q,1) = \left\{ \left(\frac{m_1}{(-0.8,0.7)}, 0.3\right), \left(\frac{m_2}{(-0.1,0.2)}, 0.7\right), \left(\frac{m_3}{(-0.4,1)}, 0.8\right) \right\} \\ (e_2,q,1) = \left\{ \left(\frac{m_1}{(-0.6,0.9)}, 0.6\right), \left(\frac{m_2}{(-1,0)}, 0.7\right), \left(\frac{m_3}{(-0.5,0.8)}, 0.5\right) \right\} \\ (e_3,q,1) = \left\{ \left(\frac{m_1}{(-0.6,0.1)}, 0.6\right), \left(\frac{m_2}{(-0.2,0.6)}, 0.5\right), \left(\frac{m_3}{(-0.5,0.7)}, 0.9\right) \right\} \\ (e_4,q,1) = \left\{ \left(\frac{m_1}{(-0.5,0.7)}, 0.6\right), \left(\frac{m_2}{(-0.6,0.8)}, 0.7\right), \left(\frac{m_3}{(-0.5,0.4)}, 0.5\right) \right\} \\ (e_1,r,1) = \left\{ \left(\frac{m_1}{(-0.8,0.4)}, 0.8\right), \left(\frac{m_2}{(-0.2,0.8)}, 0.2\right), \left(\frac{m_3}{(-0.5,0.3)}, 0.7\right) \right\} \\ (e_2,r,1) = \left\{ \left(\frac{m_1}{(-0.78,0.8)}, 0.2\right), \left(\frac{m_2}{(-0.5,0.7)}, 0.7\right), \left(\frac{m_3}{(-0.66,0.3)}, 0.3\right) \right\} \\ (e_4,r,1) = \left\{ \left(\frac{m_1}{(-0.1,0.1)}, 0.9\right), \left(\frac{m_2}{(-0.5,0.7)}, 0.2\right), \left(\frac{m_3}{(-0.58,0.8)}, 0.8\right) \right\} \\ (e_1,p,0) = \left\{ \left(\frac{m_1}{(-0.4,0.7)}, 0.8\right), \left(\frac{m_2}{(-0.7,0.6)}, 0.7\right), \left(\frac{m_3}{(-0.4,0.8)}, 0.9\right) \right\} \\ (e_2,p,0) = \left\{ \left(\frac{m_1}{(-0.3,0.7)}, 0.3\right), \left(\frac{m_2}{(-0.44,0.11)}, 0.8\right), \left(\frac{m_3}{(-0.45,0.82)}, 0.1\right) \right\} \\ (e_3,p,0) = \left\{ \left(\frac{m_1}{(-0.7,0.36)}, 0.9\right), \left(\frac{m_2}{(-0.2,0.8)}, 0.5\right), \left(\frac{m_3}{(-0.7,0.2)}, 0.8\right) \right\} \\ (e_1,q,0) = \left\{ \left(\frac{m_1}{(-0.9,0.6)}, 0.5\right), \left(\frac{m_2}{(-0.8,0.2)}, 0.2\right), \left(\frac{m_3}{(-0.1,0.8)}, 0.9\right) \right\} \\ (e_2,q,0) = \left\{ \left(\frac{m_1}{(-0.8,0.8)}, 0.6\right), \left(\frac{m_2}{(-0.2,0.9)}, 0.7\right), \left(\frac{m_3}{(-0.5,0.7)}, 0.5\right) \right\} \\ (e_4,q,0) = \left\{ \left(\frac{m_1}{(-0.3,0.8)}, 0.3\right), \left(\frac{m_2}{(-0.5,0.8)}, 0.56\right), \left(\frac{m_3}{(-0.1,0.7)}, 0.7\right) \right\} \\ (e_1,r,0) = \left\{ \left(\frac{m_1}{(-0.4,0.8)}, 0.34\right), \left(\frac{m_2}{(-0.2,0.8)}, 0.6\right), \left(\frac{m_3}{(-0.5,0.9)}, 0.3\right) \right\} \\ (e_3,r,0) = \left\{ \left(\frac{m_1}{(-0.7,0.1)}, 0.1\right), \left(\frac{m_2}{(-0.2,0.8)}, 0.7\right), \left(\frac{m_3}{(-0.8,0.5)}, 0.4\right) \right\} \\ (e_4,r,0) = \left\{ \left(\frac{m_1}{(-0.4,0.7)}, 0.7\right), \left(\frac{m_2}{(-0.4,0.9)}, 0.6\right), \left(\frac{m_3}{(-0.5,0.6)}, 0.4\right) \right\} \end{array} \right\}.$$

	m_1	m_2	m_3
$(e_1, p, 1)$	$0.8, 0.4$	$0.9, 0.8$	$1.3, 0.5$
$(e_2, p, 1)$	$0.9, 0.5$	$1.2, 0.6$	$1, 0.5$
$(e_3, p, 1)$	$1.1, 0.4$	$0.8, 0.5$	$1.2, 0.8$
$(e_4, p, 1)$	$1.3, 0.7$	$1.4, 0.3$	$0.8, 0.4$
$(e_1, q, 1)$	$1.3, 0.3$	$0.3, 0.7$	$1.4, 0.8$
$(e_2, q, 1)$	$1.05, 0.6$	$1, 0.7$	$1.3, 0.5$
$(e_3, q, 1)$	$0.7, 0.6$	$0.8, 0.5$	$1.2, 0.9$
$(e_4, q, 1)$	$1.3, 0.6$	$1, 0.7$	$1.4, 0.5$
$(e_1, r, 1)$	$1.2, 0.8$	$1, 0.2$	$0.8, 0.7$
$(e_2, r, 1)$	$1.58, 0.2$	$1.2, 0.7$	$0.96, 0.3$
$(e_4, r, 1)$	$0.2, 0.9$	$1.2, 0.2$	$1.38, 0.8$
$(e_1, p, 0)$	$1.1, 0.8$	$1.1, 0.7$	$1.2, 0.9$
$(e_2, p, 0)$	$1, 0.3$	$0.55, 0.8$	$1.27, 0.1$
$(e_3, p, 0)$	$1.6, 0.9$	$1, 0.5$	$0.9, 0.8$
$(e_1, q, 0)$	$1.5, 0.5$	$1, 0.2$	$0.9, 0.9$
$(e_2, q, 0)$	$1.6, 0.6$	$1.1, 0.7$	$1.2, 0.5$
$(e_4, q, 0)$	$1.1, 0.3$	$1.02, 0.56$	$0.8, 0.7$
$(e_1, r, 0)$	$1.2, 0.34$	$1, 0.6$	$1.4, 0.3$
$(e_3, r, 0)$	$0.8, 0.1$	$1, 0.7$	$1.3, 0.4$
$(e_4, r, 0)$	$1.1, 0.7$	$1.3, 0.6$	$1.1, 0.4$

Table 2: Numerical grade for agree-PBFSES.

	m_i	Highest numerical grade	Degree of possibility,υ_i
(e_1, p)	m_3	1.3	0.5
(e_2, p)	m_2	1.2	0.6
(e_3, p)	m_3	1.2	0.8
(e_4, p)	m_2	1.4	0.3
(e_1, q)	m_1	1.5	0.3
(e_2, q)	m_1	1.5	0.6
(e_3, q)	m_3	1.2	0.9
(e_4, q)	m_2	1.4	0.7
(e_1, r)	m_1	1.2	0.8
(e_2, r)	m_1	1.58	0.2
(e_4, r)	m_3	1.38	0.8

Score $m_1 = 2.626$; score $m_2 = 2.12$; score $m_3 = 3.794$.

Table 3: Numerical grade for disagree-PBFSES.

	m_i	Highest numerical grade	Degree of possibility,υ_i
(e_1, p)	m_2	1.3	0.7
(e_2, p)	m_3	1.27	0.1
(e_3, p)	m_1	1.6	0.9
(e_1, q)	m_1	1.5	0.5
(e_2, q)	m_1	1.6	0.6
(e_4, q)	m_2	1.3	0.56
(e_1, r)	m_3	1.4	0.3
(e_3, r)	m_3	1.3	0.4
(e_4, r)	m_2	1.3	0.6

Score $m_1 = 3.15$; score $m_2 = 2.418$; score $m_3 = 1.067$.

Table 4: The score $r_j = A_j - B_j$

A_j	B_j	r_j
score $m_1 = 2.626$	score $m_1 = 3.15$	-0.524
score $m_2 = 2.12$	score $m_2 = 2.418$	-0.298
score $m_3 = 3.794$	score $m_3 = 1.067$	2.727

Next the generalized algorithm is applied to PBFSES in ordered to solve decision making problem. The algorithm may be used is to decide the best mobile among three different mobile brands. The generalized algorithm is given below:

(1) Input the PBFSES $(F, A)_D$.

(2) Compute $f_p^+(m_i) - f_p^-(m_i)$ for every $m_i \in U$, where $f_p^+(m_i)$ represent the positive response and $f_p^-(m_i)$ represent the negative response about each $m_i \in U$.

(3) Compute the largest numerical grade for the agree and disagree PBFSES.

(4) Find the score of every $m_i \in U$ by taking the sum of the products of the numerical grade of each element with member function of fuzzy set D, for the agree-PBFSES (A_j) and disagree-PBFSES (B_j).

(5) Find $r_j$where $r_j = A_j - B_j$ for every $m_i \in U$.

(6) Find s where $s = max_{m_i \in U}\{r_j\}$. Then to choose m_i as the best solution to the problem.

In Table 1, we have $f_p^+(m_i) - f_p^-(m_i)$ for every $m_i \in U$.

The largest numerical grade for the element in agree and disagree PBFSES are given in Tables 2 and 3 respectively.

In Table 4, we calculate A_j, B_j and r_j. From the computation done from Table 1-4, we obtain $s = max_{m_i \in U}\{r_j\} = r_3$. So Mr. M will buy Huawei phone.

Chapter 3

Generalized Bipolar Fuzzy Ideals and Generalized Bipolar Fuzzy Interior Ideals in Ordered Semigroups

Ideals play an important role in studying the structure of ordered semigroups. In [49, 50] A. Khan et al., introduced the concept of generalized fuzzy ideals and generalized fuzzy interior ideals in S. In this chapter, we extend their work and brought this concept to BFSs. This chapter is compiled of our published work [84, 85]. This chapter is divided into three sections. In section one, the concept of (α, β)-BFI in S are introduced and its related properties are discussed. We characterize S in the context of $(\in, \in \vee q)$-bipolar fuzzy left (resp. right) ideals and provide relations between its various types. In section two, we also characterize an ordered semigroup in the frame work of (α, β)- BFII. Different classes of ordered semigroups, namely, simple, semisimple, regular and intra-regular ordered semigroups are characterized in terms of (α, β)-BFII (resp. (α, β)-BFI). It has been proved that the notion of $(\in, \in \vee q)$-BFII and $(\in, \in \vee q)$-BFI overlap in semisimple, regular and intra-regular ordered semigroups. In last section, a few characterizations of S in context of the upper/lower parts of $(\in, \in \vee q)$-BFII are investigated.

3.1 (α, β)-Bipolar Fuzzy Ideals

In this section, we define (α, β)-BFLI (resp. BFRI) of S. We characterize S in framework of generalized BFLI (resp. BFRI) and provide relations between its various types. We prove that every characteristic function is $(\in, \in \vee q)$-BFLI (resp. BFRI) of S iff a non-empty subset of S is a left (resp. right) ideal of S. We also show that every BFS is a $(\in, \in \vee q)$-BFLI (resp. BFRI) of S iff the non-empty (s,t)-cut $C(f;(s,t))$ of S is a left (resp. right) ideal of S. The following results are taken from our accepted paper [86].

Definition 3.1.1 *A BFS $f = (S; f_n, f_p)$ is said to be a bipolar fuzzy point (BFP), denoted by $\frac{\xi}{(s,t)}$ if*

$$f_n(y) := \begin{cases} s \text{ if } y \in (\xi], \\ 0 \text{ if } y \notin (\xi], \end{cases} \quad f_p(y) := \begin{cases} t \text{ if } y \in (\xi], \\ 0 \text{ if } y \notin (\xi], \end{cases}$$

for all $(s,t) \in [-1, 0) \times (0, 1]$.

For a BFS $f = (S; f_n, f_p)$ in S and a BFP $\frac{\xi}{(s,t)}$, we say that

1) $\frac{\xi}{(s,t)} \in f$ if $f_n(\xi) \leq s$ and $f_p(\xi) \geq t$,

2) $\frac{\xi}{(s,t)}\ q\ f$ if $f_n(\xi) + s < -1$ and $f_p(\xi) + t > 1$,

3) $\frac{\xi}{(s,t)} \in \vee q\ f$ if $\frac{\xi}{(s,t)} \in f$ or $\frac{\xi}{(s,t)}\ q\ f$,

4) $\frac{\xi}{(s,t)} \in \wedge q\ f$ if $\frac{\xi}{(s,t)} \in f$ and $\frac{\xi}{(s,t)}\ q\ f$.

Let $f = (S, f_n, f_p)$ be a BFS in S such that $f_n(\xi) \geq -0.5$ and $f_p(\xi) \leq 0.5$ for every $\xi \in S$. If $\frac{\xi}{(s,t)} \in \wedge q\ f$, then $f_n(\xi) \leq s$, $f_p(\xi) \geq t$, $f_n(\xi) + s < -1$ and $f_p(\xi) + t > 1$. It follows that $-1 > f_n(\xi) + s \geq 2f_n(\xi)$ and $1 < f_p(\xi) + t \leq 2f_p(\xi)$, so that $f_n(\xi) < -0.5$ and $f_p(\xi) > 0.5$. This mean that $\{\frac{\xi}{(s,t)} \mid \frac{\xi}{(s,t)} \in \wedge q\} = \emptyset$. Therefore the case $\alpha = \in \wedge q$ will be omitted.

Definition 3.1.2 *A BFS $f = (S; f_n, f_p)$ in S is called (α, β)-BFLI (resp. BFRI) of S where $\alpha \neq \in \wedge q$ if for all $(s,t) \in [-1, 0) \times (0, 1]$ it satisfies:*

1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2, \dfrac{\xi_2}{(s,t)} \alpha\ f \Longrightarrow \dfrac{\xi_1}{(s,t)} \beta\ f)$,

2) $(\forall \xi_1, \xi_2 \in S)(\dfrac{\xi_2}{(s,t)} \alpha f \Longrightarrow \dfrac{\xi_1 \xi_2}{(s,t)} \beta f)$(resp. $\dfrac{\xi_1}{(s,t)} \alpha f \Longrightarrow \dfrac{\xi_1 \xi_2}{(s,t)} \beta f)$.

Lemma 3.1.3 *The characteristic function $\chi_{\mathcal{L}} = (S, \chi_{n\mathcal{L}}, \chi_{p\mathcal{L}})$ is an $(\in, \in \vee q)$-BFLI (resp. BFRI) of S iff $\mathcal{L}$ is left (resp. right) ideal of S.*

Proof. Let $\chi_{\mathcal{L}} = (S; \chi_{n\mathcal{L}}, \chi_{p\mathcal{L}})$ is an $(\in, \in \vee$q)-BFLI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in \mathcal{L}$. Then $\chi_{n\mathcal{L}}(\xi_2) = -1$ and $\chi_{p\mathcal{L}}(\xi_2) = 1$, thus $\frac{\xi_2}{(-1,1)} \in \chi_{\mathcal{L}}$. Since $\chi_{\mathcal{L}} = (S; \chi_{n\mathcal{L}}, \chi_{p\mathcal{L}})$ is an $(\in, \in \vee$q)-BFLI of S, so $\frac{\xi_1}{(-1,1)} \in \vee q\chi_{\mathcal{L}}$ i.e $\frac{\xi_1}{(-1,1)} \in \chi_{\mathcal{L}}$ or $\frac{\xi_1}{(-1,1)} q\chi_{\mathcal{L}}$. From both cases, we conclude that $\chi_{n\mathcal{L}}(\xi_1) = -1$ and $\chi_{p\mathcal{L}}(\xi_1) = 1$. Hence $\xi_1 \in \mathcal{L}$.

Let $\xi_1 \in S$ and $\xi_2 \in \mathcal{L}$, then $\chi_{n\mathcal{L}}(\xi_2) = -1$ and $\chi_{p\mathcal{L}}(\xi_2) = 1$. Thus $\frac{\xi_2}{(-1,1)} \in \chi_{\mathcal{L}}$. Since $\chi_{\mathcal{L}} = (S; \chi_{n\mathcal{L}}, \chi_{p\mathcal{L}})$ is an $(\in, \in \vee$q)-BFLI of S, so $\frac{\xi_1\xi_2}{(-1,1)} \in \vee q\chi_{\mathcal{L}}$ i.e $\frac{\xi_1\xi_2}{(-1,1)} \in \chi_{\mathcal{L}}$ or $\frac{\xi_1\xi_2}{(-1,1)} q\chi_{\mathcal{L}}$. From both cases, we conclude that $\chi_{n\mathcal{L}}(\xi_1\xi_2) = -1$ and $\chi_{p\mathcal{L}}(\xi_1\xi_2) = 1$. So $\xi_1\xi_2 \in \mathcal{L}$. Hence $\mathcal{L}$ is left ideal of S.

Conversely, assume that $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\frac{\xi_2}{(s,t)} \in \chi_{\mathcal{L}}$ for $(s,t) \in [-1,0) \times (0,1]$. Then $\chi_{n\mathcal{L}}(\xi_2) \leq s < 0$ and $\chi_{p\mathcal{L}}(\xi_2) \geq t > 0$, which implies that $\chi_{n\mathcal{L}}(\xi_2) = -1$ and $\chi_{p\mathcal{L}}(\xi_2) = 1$. Thus $\xi_2 \in \mathcal{L}$. Since $\mathcal{L}$ is left ideal of S, so $\xi_1 \in \mathcal{L}$. Thus $\chi_{n\mathcal{L}}(\xi_1) = -1 \leq s$ and $\chi_{p\mathcal{L}}(\xi_1) = 1 \geq t$. This implies that $\frac{\xi_1}{(s,t)} \in \chi_{\mathcal{L}}$ and hence $\frac{\xi_1}{(s,t)} \in \vee q\chi_{\mathcal{L}}$.

Let $\xi_1, \xi_2 \in S$ such that $\frac{\xi_2}{(s,t)} \in \chi_{\mathcal{L}}$ for $(s,t) \in [-1,0) \times (0,1]$. Then $\chi_{n\mathcal{L}}(\xi_2) \leq s < 0$ and $\chi_{p\mathcal{L}}(\xi_2) \geq t > 0$. This implies that $\chi_{n\mathcal{L}}(\xi_2) = -1$ and $\chi_{p\mathcal{L}}(\xi_2) = 1$. Thus $\xi_2 \in \mathcal{L}$. Since $\mathcal{L}$ is left ideal of S, so $\xi_1\xi_2 \in \mathcal{L}$. It follows that $\chi_{n\mathcal{L}}(\xi_1\xi_2) = -1 \leq s$ and $\chi_{p\mathcal{L}}(\xi_1\xi_2) = 1 \geq t$. This implies that $\frac{\xi_1\xi_2}{(s,t)} \in \chi_{\mathcal{L}}$ and so $\frac{\xi_1\xi_2}{(st)} \in \vee q\chi_{\mathcal{L}}$. Hence $\chi_{\mathcal{L}} = (S; \chi_{n\mathcal{L}}, \chi_{p\mathcal{L}})$ is an $(\in, \in \vee$q)-BFLI of S. ■

Theorem 3.1.4 *Every $(\in, \in)$-BFLI (resp. BFRI) is an $(\in, \in \vee q)$-BFLI (resp. BFRI) of S.*

Proof. Straight forward. ■

Theorem 3.1.5 *A BFS $f = (S, f_n, f_p)$ in S is an $(\in, \in \vee q)$-BFLI (resp. BFRI) of S iff it satisfies:*

1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Longrightarrow f_n(\xi_1) \leq \vee\{f_n(\xi_2), -0.5\}$ *and* $f_p(\xi_1) \geq \wedge\{f_p(\xi_2), 0.5\})$,

2) $(\forall \xi_1, \xi_2 \in S)(f_n(\xi_1\xi_2) \leq \vee\{f_n(\xi_2), -0.5\}(resp. \vee \{f_n(\xi_1), -0.5\})$ *and*
$f_p(\xi_1\xi_2) \geq \wedge\{f_p(\xi_2), 0.5\}(resp. \wedge \{f_p(\xi_1), 0.5\}))$.

Proof. Assume that $f = (S, f_n, f_p)$ is an $(\in, \in \vee \mathrm{q})$-BFLI of S. Let $\xi_1, \xi_2 \in S$ be such that $\xi_1 \leq \xi_2$. We consider the following four cases.

$$\text{(i) } f_n(\xi_2) > -0.5 \text{ and } f_p(\xi_2) < 0.5,$$

$$\text{(ii) } f_n(\xi_2) \leq -0.5 \text{ and } f_p(\xi_2) \geq 0.5,$$

$$\text{(iii) } f_n(\xi_2) \leq -0.5 \text{ and } f_p(\xi_2) < 0.5,$$

$$\text{(iv) } f_n(\xi_2) > -0.5 \text{ and } f_p(\xi_2) \geq 0.5.$$

For the first case, assume that $f_n(\xi_1) > \vee\{f_n(\xi_2), -0.5\}$ or $f_p(\xi_1) < \wedge\{f_p(\xi_2), 0.5\}$ then $f_n(\xi_1) > f_n(\xi_2)$ or $f_p(\xi_1) < f_p(\xi_2)$. If $f_n(\xi_1) > f_n(\xi_2)$ then $f_n(\xi_1) > s \geq f_n(\xi_2)$ for some $s \in [-1, 0)$. Let $t = f_p(\xi_2)$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}} f$, that is, $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}} f$. This is a contradiction. If $f_p(\xi_1) < f_p(\xi_2)$ then $f_p(\xi_1) < t \leq f_p(\xi_2)$ for some $t \in (0, 1]$. Let $s = f_n(\xi_2)$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$, i.e, $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, a contradiction. Therefore $f_n(\xi_1) \leq \vee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq \wedge\{f_p(\xi_2), 0.5\}$. Let $\xi_1, \xi_2 \in S$ be such that $f_n(\xi_1\xi_2) > \vee\{f_n(\xi_2), -0.5\}$ or $f_p(\xi_1\xi_2) < \wedge\{f_p(\xi_2), 0.5\}$. Then $f_n(\xi_1\xi_2) > f_n(\xi_2)$ or $f_p(\xi_1\xi_2) < f_p(\xi_2)$. If $f_n(\xi_1\xi_2) > f_n(\xi_2)$ then $f_n(\xi_1\xi_2) > s \geq f_n(\xi_2)$ for some $s \in [-1, 0)$. Let $f_p(\xi_2) = t$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$,i.e, $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. If $f_p(\xi_1\xi_2) < f_p(\xi_2)$ then $f_p(\xi_1\xi_2) < t \leq f_p(\xi_2)$ for some $t \in (0, 1]$. Let $f_n(\xi_2) = s$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$, that is, $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, a contradiction. Hence $f_n(\xi_1\xi_2) \leq \vee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq \wedge\{f_p(\xi_2), 0.5\}$.

Now consider the second case, $f_n(\xi_2) \leq -0.5$ and $f_p(\xi_2) \geq 0.5$. This implies that $\frac{\xi_2}{(-0.5,0.5)} \in f$. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. Since $\frac{\xi_2}{(-0.5,0.5)} \in f$. So $\frac{\xi_1}{(-0.5,0.5)} \in \vee \mathrm{q}\ f$. If $\frac{\xi_1}{(-0.5,0.5)} \in f$ then $f_n(\xi_1) \leq -0.5 \leq \vee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq 0.5 \geq \wedge\{f_n(\xi_2), 0.5\}$. If $\frac{\xi_1}{(-0.5,0.5)} \mathrm{q}\ f$ then $f_n(\xi_1) - 0.5 < -1$ and $f_p(\xi_1) + 0.5 > 1$. Thus $f_n(\xi_1) < -0.5$ and $f_p(\xi_1) > 0.5$. Therefore $f_n(\xi_1) \leq -0.5 \leq \vee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq 0.5 \geq \wedge\{f_p(\xi_2), 0.5\}$. Let $\xi_1, \xi_2 \in S$ such that $\frac{\xi_2}{(-0.5,0.5)} \in f$ then $\frac{\xi_1\xi_2}{(-0.5,0.5)} \in \vee \mathrm{q}\ f$. If $\frac{\xi_1\xi_2}{(-0.5,0.5)} \in f$ then $f_n(\xi_1\xi_2) \leq -0.5 \leq \vee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq 0.5 \geq \wedge\{f_p(\xi_2), 0.5\}$. If $\frac{\xi_1\xi_2}{(-0.5,0.5)} \mathrm{q}\ f$ then $f_n(\xi_1\xi_2) - 0.5 < -1$ and $f_p(\xi_1\xi_2) + 0.5 > 1$. So $f_n(\xi_1\xi_2) < -0.5$ and $f_p(\xi_1\xi_2) > 0.5$. Therefore $f_n(\xi_1\xi_2) \leq -0.5 \leq \vee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq 0.5 \geq \wedge\{f_p(\xi_2), 0.5\}$.

Let $\xi_1, \xi_2 \in S$ be such that $\xi_1 \leq \xi_2$ then from case (iii) we have $f_n(\xi_1) \leq -0.5$ and $f_p(\xi_1) \geq f_p(\xi_2)$. Assume that $f_n(\xi_1) > -0.5$ or $f_p(\xi_1) < f_p(\xi_2)$. If $f_n(\xi_1) > -0.5$ then there exists $s \in [-1, 0)$ such that $f_n(\xi_1) > s \geq -0.5 \geq f_n(\xi_2)$. Let $f_p(\xi_2) = t$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$,that is, $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. If $f_p(\xi_1) < f_p(\xi_2)$ then there exists $t \in (0, 1]$ such that $f_p(\xi_1) < t \leq f_p(\xi_2)$. Let $f_n(\xi_2) = s$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$,that is $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. Hence $f_n(\xi_1) \leq -0.5 \leq \vee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq f_p(\xi_2) \geq \wedge\{f_p(\xi_2), 0.5\}$. Let $\xi_1, \xi_2 \in S$ then case (iii) implies that $f_n(\xi_1\xi_2) \leq -0.5$ and $f_p(\xi_1\xi_2) \geq f_p(\xi_2)$. Assume that $f_n(\xi_1\xi_2) > -0.5$ or $f_p(\xi_1\xi_2) < f_p(\xi_2)$. If $f_n(\xi_1\xi_2) > -0.5$ then there exists $s \in [-1, 0)$ such that $f_n(\xi_1\xi_2) > s \geq -0.5 \geq f_n(\xi_2)$. Let $f_p(\xi_2) = t$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$,that is $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, a contradiction. If $f_p(\xi_1\xi_2) < f_p(\xi_2)$ then $f_p(\xi_1\xi_2) < t \leq f_p(\xi_2)$ for some $t \in (0, 1]$. Let $f_n(\xi_2) = s$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}$ f,that is $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. Hence $f_n(\xi_1\xi_2) \leq -0.5 \leq \vee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq f_p(\xi_2) \geq \wedge\{f_p(\xi_2), 0.5\}$.

Let $\xi_1, \xi_2 \in S$ be such that $\xi_1 \leq \xi_2$ then case (iv) implies that $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq 0.5$. Assume that $f_n(\xi_1) > f_n(\xi_2)$ or $f_p(\xi_1) < 0.5$. If $f_n(\xi_1) > f_n(\xi_2)$ then there exists $s \in [-1, 0)$ such that $f_n(\xi_1) > s \geq f_n(\xi_2)$. Let $f_p(\xi_2) = t$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}} f$,that is $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}} f$ which is a contradiction. If $f_p(\xi_1) < 0.5$ then $f_p(\xi_1) < t \leq 0.5 \leq f_p(\xi_2)$ for some $t \in (0, 1]$. Let $f_n(\xi_2) = s$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}} f$,that is $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}} f$ which is a contradiction. Hence $f_n(\xi_1) \leq f_n(\xi_2) \leq \vee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq 0.5 \geq \wedge\{f_p(\xi_2), 0.5\}$.Let $\xi_1, \xi_2 \in S$ then case (iv) implies that $f_n(\xi_1\xi_2) \leq f_n(\xi_2)$ and $f_p(\xi_1\xi_2) \geq 0.5$. Assume that $f_n(\xi_1\xi_2) > f_n(\xi_2)$ or $f_p(\xi_1\xi_2) < 0.5$. If $f_n(\xi_1\xi_2) > f_n(\xi_2)$ then $f_n(\xi_1\xi_2) > s \geq f_n(\xi_2)$ for some $t \in (0, 1]$. Let $f_p(\xi_2) = t$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}} f$,that is $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}} f$ which is a contradiction. If $f_p(\xi_1\xi_2) < 0.5$ then $f_p(\xi_1\xi_2) < t \leq 0.5 \leq f_p(\xi_2)$ for some $t \in (0, 1]$. Let $f_n(\xi_2) = s$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}} f$,that is $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}} f$ which is a contradiction. Hence $f_n(\xi_1\xi_2) \leq f_n(\xi_2) \leq \vee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq 0.5 \geq \wedge\{f_p(\xi_2), 0.5\}$.

Conversely, assume that the conditions 1 and 2 are true. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. Let $(s, t) \in [-1, 0) \times (0, 1]$ such that $\frac{\xi_2}{(s,t)} \in f$. Then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Assume that $\frac{\xi_1}{(s,t)} \overline{\in} f$, then $f_n(\xi_1) > s$ and $f_p(\xi_1) < t$. If $f_n(\xi_1) > s$, then $f_n(\xi_2) \leq -0.5$. Otherwise, we get $f_n(\xi_1) \leq \vee\{f_n(\xi_2), -0.5\} = f_n(\xi_2) \leq s$, a contradiction. If $f_p(\xi_1) < t$, then $f_p(\xi_2) \geq 0.5$. Otherwise we obtain $f_p(\xi_1) > \wedge\{f_p(\xi_2), 0.5\} = f_p(\xi_2) \geq t$, a contrary result. Therefore

$f_n(\xi_1)+s < 2f_n(\xi_1) \leq 2\vee\{f_n(\xi_2),-0.5\} = -1$ and $f_p(\xi_1)+t > 2f_p(\xi_1) \geq 2\wedge\{f_p(\xi_2),0.5\} = 1$. So $\frac{\xi_1}{(s,t)}\mathrm{q}\ f$. Let $\xi_1,\xi_2 \in S$ such that $\frac{\xi_2}{(s,t)} \in f$ for $(s,t) \in [-1,0)\times(0,1]$. Then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Assume that $\frac{\xi_1\xi_2}{(s,t)}\overline{\in} f$, then $f_n(\xi_1\xi_2) > s$ and $f_p(\xi_1\xi_2) < t$. If $f_n(\xi_1\xi_2) > s$ then $f_n(\xi_2) \leq -0.5$. Otherwise, we get $f_n(\xi_1\xi_2) \leq \vee\{f_n(\xi_2),-0.5\} = f_n(\xi_2) \leq s$, a contradiction. If $f_p(\xi_1\xi_2) < t$ then $f_p(\xi_2) \geq 0.5$. Otherwise we obtain $f_p(\xi_1\xi_2) > \wedge\{f_p(\xi_2),0.5\} = f_p(\xi_2) \geq t$, which is a contradiction. It follows that $f_n(\xi_1\xi_2)+s < 2f_n(\xi_1\xi_2) \leq 2\vee\{f_n(\xi_2),-0.5\} = -1$ and $f_p(\xi_1\xi_2)+t > 2f_p(\xi_1\xi_2) \geq 2\wedge\{f_p(\xi_2),0.5\} = 1$. This implies $\frac{\xi_1\xi_2}{(s,t)}\mathrm{q}\ f$. Hence $f = (S, f_n, f_p)$ is an $(\in, \in\vee\mathrm{q})$-BFLI of S. ■

In the following example, it is shown that every generalized BFI may not be BFI.

Example 3.1.6 *Let* $S = \{0,1,2,3,4\}$ *be a set with the following multiplication table and order relation '*$\leq$*'*

Table 1

.	0	1	2	3	4
0	0	3	0	3	3
1	0	1	0	3	3
2	0	3	2	3	4
3	0	3	0	3	3
4	0	3	2	3	4

$\leq := \{(0,0),(0,2),(0,3),(0,4),(1,1),(2,2),(2,4),(3,3),(4,4)\}$ then S is an ordered semi-group. Let $f = (S; f_n, f_p)$ be a BFS in S defined by table 2

Table 2

S	0	1	2	3	4
f_n	-0.75	-0.35	-0.72	-0.58	-0.65
f_p	0.8	0.3	0.7	0.5	0.6

Clearly $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFI of S. Now $\frac{2}{(-0.68,0.68)} \in f$ but $\frac{2\cdot 4}{(-0.68,0.68)} = \frac{4}{(-0.68,0.68)} \overline{\in} f$. Therefore $f = (S; f_n, f_p)$ is not an $(\in, \in)$-BFI of S.

In the following theorem a condition is given for every $(\in, \in \vee q)$-BFLI (resp. BFRI) to be an $(\in, \in)$-BFLI (resp. BFRI) of S.

Theorem 3.1.7 *Let $f = (S; f_n, f_p)$ be an $(\in, \in \vee q)$-BFLI (resp. BFRI) of S such that $(\forall \xi \in S)((f_n(\xi), f_p(\xi)) \in [-0.5, 0) \times (0, 0.5])$. Then $f = (S; f_n, f_p)$ is an $(\in, \in)$-BFLI (resp. BFRI) of S.*

Proof. Suppose $f = (S; f_n, f_p)$ be an $(\in, \in \vee q)$-BFLI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\frac{\xi_2}{(s,t)} \in f$ for $(s, t) \in [-1, 0) \times (0, 1]$ then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. By hypothesis $f_n(\xi_1) \leq \vee\{f_n(\xi_2), -0.5\} \leq s$ and $f_p(\xi_1) \geq \wedge\{f_p(\xi_2), 0.5\} \geq t$. This implies $\frac{\xi_1}{(s,t)} \in f$. Let $\xi_1, \xi_2 \in S$ such that $\frac{\xi_2}{(s,t)} \in f$ then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. By hypothesis $f_n(\xi_1\xi_2) \leq \vee\{f_n(\xi_2), -0.5\} \leq s$ and $f_p(\xi_1\xi_2) \geq \wedge\{f_p(\xi_2), 0.5\} \geq t$. This implies $\frac{\xi_1\xi_2}{(s,t)} \in f$. Hence $f = (S; f_n, f_p)$ is an $(\in, \in)$-BFLI of S. We can also show that $f = (S; f_n, f_p)$ is an $(\in, \in)$-BFRI of S. ■

Theorem 3.1.8 *Let I be an ideal of S. For any $(s, t) \in [-0.5, 0) \times (0, 0.5]$, there exists an $(\in, \in \vee q)$-BFI, $f = (S; f_n, f_p)$ of S such that $N(f; s) = I = P(f; t)$.*

Proof. Let $f = (S; f_n, f_p)$ be a BFS in S defined by

$$f_n(\xi) := \begin{cases} s \text{ if } \xi \in I, \\ 0 \text{ otherwise,} \end{cases} \qquad f_p(\xi_1) := \begin{cases} t \text{ if } \xi \in I, \\ 0 \text{ otherwise,} \end{cases}$$

for all $\xi \in S$ where $(s, t) \in [-0.5, 0) \times (0, 0.5]$. Obviously $N(f; s) = I = P(f; t)$.

Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. Assume that $f_n(\xi_1) > \vee\{f_n(\xi_2), -0.5\}$ for some $\xi_1, \xi_2 \in S$. Let $\vee\{f_n(\xi_2), -0.5\} = s$, then $\xi_2 \in N(f; s) = I$. But $\xi_1 \notin N(f; s) = I$, which is impossible because I is an ideal of S. Thus $f_n(\xi_1) \leq \vee\{f_n(\xi_2), -0.5\}$ for all $\xi_1, \xi_2 \in S$. Assume that $f_p(\xi_1) < \wedge\{f_p(\xi_2), 0.5\}$. Let $\wedge\{f_p(\xi_2), 0.5\} = t$, then $\xi_2 \in P(f; t) = I$. But $\xi_1 \notin P(f; t) = I$, a contrary result because I is an ideal of S. Thus $f_p(\xi_1) \geq \wedge\{f_p(\xi_2), 0.5\}$. Assume that $f_n(\xi_1\xi_2) > \vee\{f_n(\xi_2), -0.5\}$ for some $\xi_1, \xi_2 \in S$. Let $\vee\{f_n(\xi_2), -0.5\} = s$, then $\xi_2 \in N(f; s) = I$. But $\xi_1\xi_2 \notin N(f; s) = I$, a contradiction because I is an ideal of S. Thus $f_n(\xi_1) \leq \vee\{f_n(\xi_2), -0.5\}$ for all $\xi_1, \xi_2 \in S$. Assume that $f_p(\xi_1\xi_2) < \wedge\{f_p(\xi_2), 0.5\}$. Let $\wedge\{f_p(\xi_2), 0.5\} = t$, then $\xi_2 \in$

$P(f;t) = I$. But $\xi_1\xi_2 \notin P(f;t) = I$, a contradiction because I is an ideal of S, thus $f_p(\xi_1\xi_2) \geq \wedge\{f_p(\xi_2), 0.5\}$ for all $\xi_1, \xi_2 \in S$. We conclude that $f = (S; f_n, f_p)$ is an $(\in, \in \vee \text{q})$-BFLI of S. We can easily show that $f = (S; f_n, f_p)$ is an $(\in, \in \vee \text{q})$-BFRI of S. Hence $f = (S; f_n, f_p)$ is an $(\in, \in \vee \text{q})$-BFI of S. ■

Theorem 3.1.9 *Let $f = (S; f_n, f_p)$ is a BFS in S. Then $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFLI (resp. BFRI) of S iff the non-empty (s,t)-cut $C(f;(s,t))$ of S is a left (resp. right) ideal of S for all $(s,t) \in [-0.5, 0) \times (0, 0.5]$.*

Proof. Let $f = (S; f_n, f_p)$ is an $(\in, \in \vee \text{q})$-BFLI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in C(f;(s,t))$. This implies $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee \text{q})$-BFLI of S. Therefore $f_n(\xi_1) \leq \vee\{f_n(\xi_2), -0.5\} \leq s$ and $f_p(\xi_1) \geq \wedge\{f_p(\xi_2), 0.5\} \geq t$. This implies $\xi_1 \in C(f;(s,t))$. Let $\xi_1 \in S$ and $\xi_2 \in C(f;(s,t))$. This implies $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee \text{q})$-BFLI of S. Therefore $f_n(\xi_1\xi_2) \leq \vee\{f_n(\xi_2), -0.5\} \leq s$ and $f_p(\xi_1\xi_2) \geq \wedge\{f_p(\xi_2), 0.5\} \geq t$. This implies that $\xi_1\xi_2 \in C(f;(s,t))$. Hence $C(f;(s,t))$ is a left ideal of S.

Conversely, suppose that $C(f;(s,t))$ is a left ideal of S. Assume that $f = (S; f_n, f_p)$ is not an $(\in, \in \vee \text{q})$-BFLI of S, then for some $a, b \in S$, we have

(i) $a \leq b$ and $f_n(a) > \vee\{f_n(b), -0.5\}$ or $f_p(a) < \wedge\{f_p(b), 0.5\}$.

(ii) $f_n(ab) > \vee\{f_n(b), -0.5\}$ or $f_p(ab) < \wedge\{f_p(b), 0.5\}$.

Let $s = \vee\{f_n(b), -0.5\}$ and $t = \wedge\{f_p(b), 0.5\}$ then $b \in C(f;(s,t))$. If $a \leq b$ and $f_n(a) > \vee\{f_n(b), -0.5\} = s$ then $a \notin N(f;s)$ and so $a \notin C(f;(s,t))$. If $f_n(ab) > \vee\{f_n(b), -0.5\} = s$ then $ab \notin N(f;s)$ and so $ab \notin C(f;(s,t))$ which is impossible because $C(f;(s,t))$ is left ideal of S. Also if $f_p(a) < \wedge\{f_p(b), 0.5\} = t$ then $a \notin P(f;t)$ and so $a \notin C(f;(s,t))$. If $f_p(ab) < \wedge\{f_p(b), 0.5\} = t$ then $ab \notin P(f;t)$. Thus $ab \notin C(f;(s,t))$, a contradiction. Hence $f = (S; f_n, f_p)$ is an $(\in, \in \vee \text{q})$-BFLI of S. Also clearly every $f = (S; f_n, f_p)$ is an $(\in, \in \vee \text{q})$-BFRI of S iff the non-empty (s,t)-cut $C(f;(s,t))$ of S is a right ideal of S. ■

Theorem 3.1.10 *For a BFS $f = (S; f_n, f_p)$ in S, we consider the set*

$$S_\circ = \{\xi \in S | f_n(\xi) \neq 0\} \cap \{\xi \in S | f_p(\xi) \neq 0\}.$$

For $\beta \in \{\in, q\}$ if $f = (S; f_n, f_p)$ is non-zero $(\in, \beta)$-BFLI (resp. BFRI) of S or a non-zero (q, β)-BFLI (resp. BFRI) of S, then the set $S_\circ$ is a left (resp. right) ideal of S.

Proof. We first assume that $f = (S; f_n, f_p)$ is non-zero $(\in, \beta)$-BFLI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in S_\circ$. Then $f_n(\xi_2) \neq 0$ and $f_p(\xi_2) \neq 0$, so $f_n(\xi_2) < 0$ and $f_p(\xi_2) > 0$. Suppose that $f_n(\xi_1) = 0$ or $f_p(\xi_1) = 0$. Clearly $\frac{\xi_2}{(f_n(\xi_2), f_p(\xi_2))} \in f$. Since $f_n(\xi_1) > f_n(\xi_2)$ or $f_p(\xi_1) < f_p(\xi_2)$. This implies $\frac{\xi_1}{(f_n(\xi_2), f_p(\xi_2))} \overline{\in} f$, which is a contradiction. Also $f_n(\xi_1) + f_n(\xi_2) = f_n(\xi_2) \geq -1$ or $f_p(\xi_1) + f_p(\xi_2) = f_p(\xi_2) \leq 1$. This implies that $\frac{\xi_1}{(f_n(\xi_2), f_p(\xi_2))} \overline{q}\, f$, which is a contradiction. Therefore $f_n(\xi_1) \neq 0$ and $f_p(\xi_1) \neq 0$. Hence $\xi_1 \in S_\circ$. Let $\xi_1, \xi_2 \in S$ such that $\xi_2 \in S_\circ$, then $f_n(\xi_2) \neq 0$ and $f_p(\xi_2) \neq 0$, so $f_n(\xi_2) < 0$ and $f_p(\xi_2) > 0$. Suppose that $\xi_1\xi_2 \notin S_\circ$, that is $f_n(\xi_1\xi_2) = 0$ or $f_p(\xi_1\xi_2) = 0$. Clearly $\frac{\xi_2}{(f_n(\xi_2), f_p(\xi_2))} \in f$. Since $f_n(\xi_1\xi_2) > f_n(\xi_2)$ or $f_p(\xi_1\xi_2) < f_p(\xi_2)$. This implies $\frac{\xi_1\xi_2}{(f_n(\xi_2), f_p(\xi_2))} \overline{\in} f$, which is a contradiction. Also $f_n(\xi_1\xi_2) + f_n(\xi_2) = f_n(\xi_2) \geq -1$ or $f_p(\xi_1\xi_2) + f_p(\xi_2) = f_p(\xi_2) \leq 1$. This implies that $\frac{\xi_1\xi_2}{(f_n(\xi_2), f_p(\xi_2))} \overline{q}\, f$, which is not possible. Therefore $f_n(\xi_1\xi_2) \neq 0$ and $f_p(\xi_1\xi_2) \neq 0$. Hence $\xi_1\xi_2 \in S_\circ$.

Now suppose that $f = (S; f_n, f_p)$ is non-zero (q, β)-BFLI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in S_\circ$. Then $f_n(\xi_2) \neq 0$ and $f_p(\xi_2) \neq 0$, so $f_n(\xi_2) < 0$ and $f_p(\xi_2) > 0$. Therefore $f_n(\xi_2) - 1 < -1$ and $f_p(\xi_2) + 1 > 1$, it follows that $\frac{\xi_2}{(-1,1)} \mathrm{q}\, f$. If $f_n(\xi_1) = 0$ or $f_p(\xi_1) = 0$, then $f_n(\xi_1) > -1$ or $f_p(\xi_1) < 1$. This implies $\frac{\xi_1}{(-1,1)} \overline{\in} f$, which is not possible. Also $f_n(\xi_1) - 1 = -1$ or $f_p(\xi_1) + 1 = 1$. This implies $\frac{\xi_1}{(-1,1)} \overline{q}\, f$, which is a contradiction. Therefore $f_n(\xi_1) \neq 0$ and $f_p(\xi_1) \neq 0$. Hence $\xi_1 \in S_\circ$. Let $\xi_1, \xi_2 \in S$ such that $\xi_2 \in S_\circ$. Then $f_n(\xi_2) \neq 0$ and $f_p(\xi_2) \neq 0$, so $f_n(\xi_2) < 0$ and $f_p(\xi_2) > 0$. Therefore $f_n(\xi_2) - 1 < -1$ and $f_p(\xi_2) + 1 > 1$, it follows that $\frac{\xi_2}{(-1,1)} \mathrm{q}\, f$. If $f_n(\xi_1\xi_2) = 0$ or $f_p(\xi_1\xi_2) = 0$, then $f_n(\xi_1\xi_2) > -1$ or $f_p(\xi_1\xi_2) < 1$. This implies $\frac{\xi_1\xi_2}{(-1,1)} \overline{\in} f$, which is a contradiction. Also $f_n(\xi_1\xi_2) - 1 = -1$ or $f_p(\xi_1\xi_2) + 1 = 1$. This implies $\frac{\xi_1\xi_2}{(-1,1)} \overline{q}\, f$, which is a contradiction. Therefore $f_n(\xi_1\xi_2) \neq 0$ and $f_p(\xi_1\xi_2) \neq 0$. Hence $\xi_1\xi_2 \in S_\circ$. We can easily show that $S_\circ$ is right ideal of S. ■

Theorem 3.1.11 *Let $\mathcal{L}$ be a left (resp. right) ideal of S and $f = (S; f_n, f_p)$ a BFS of S such* *that*

$$i)\ (\forall \xi \in S \mid \mathcal{L})(f_n(\xi) = 0 = f_p(\xi)),$$

$$ii)\ (\forall \xi \in \mathcal{L})((f_n(\xi), f_p(\xi)) \in [-1, -0.5] \times [0.5, 1]).$$

Then $f = (S; f_n, f_p)$ *is* $(q, \in \vee q)$*-BFLI (resp. BFRI) of* S.

Proof. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $(s,t) \in [-1, 0) \times (0, 1]$. Let $\frac{\xi_2}{(s,t)}\mathrm{q}\ f$, then $f_n(\xi_2) + s < -1$ and $f_p(\xi_2) + t > 1$. This implies that $\xi_2 \in \mathcal{L}$. Since $\mathcal{L}$ is left ideal of S, so $\xi_1 \in \mathcal{L}$. In order to check $\frac{\xi_1}{(s,t)} \in \vee\mathrm{q}\ f$, we assume the following four cases:

1) $s \geq -0.5$ and $t \leq 0.5$,

2) $s < -0.5$ and $t > 0.5$,

3) $s < -0.5$ and $t \leq 0.5$,

4) $s \geq -0.5$ and $t > 0.5$.

The first case induces $f_n(\xi_1) \leq -0.5 \leq s$ and $f_p(\xi_1) \geq 0.5 \geq t$. This implies $\frac{\xi_1}{(s,t)} \in f$. The second case implies that $f_n(\xi_1) + s < -1$ and $f_p(\xi_1) + t > 1$. Thus $\frac{\xi_1}{(s,t)}\ \mathrm{q}\ f$. Since $\frac{\xi_2}{(s,t)}\ \mathrm{q}\ f$, so case (3) and (4) do not occur. Consequently, $\frac{\xi_1}{(s,t)} \in \vee\mathrm{q}\ f$. Let $\xi_1, \xi_2 \in S$ such that $\frac{\xi_2}{(s,t)}\mathrm{q}\ f$ for $(s,t) \in [-1, 0) \times (0, 1]$, then $f_n(\xi_2) + s < -1$ and $f_p(\xi_2) + t > 1$. This implies that $\xi_2 \in \mathcal{L}$. Since $\mathcal{L}$ is left ideal of S, so $\xi_1\xi_2 \in \mathcal{L}$. In order to check $\frac{\xi_1\xi_2}{(s,t)} \in \vee\mathrm{q}\ f$, we assume the following four cases:

1) $s \geq -0.5$ and $t \leq 0.5$,

2) $s < -0.5$ and $t > 0.5$,

3) $s < -0.5$ and $t \leq 0.5$,

4) $s \geq -0.5$ and $t > 0.5$.

The first case induces $f_n(\xi_1\xi_2) \leq -0.5 \leq s$ and $f_p(\xi_1\xi_2) \geq 0.5 \geq t$. This implies $\frac{\xi_1\xi_2}{(s,t)} \in f$. The second case implies that $f_n(\xi_1\xi_2) + s < -1$ and $f_p(\xi_1\xi_2) + t > 1$. Thus $\frac{\xi_1\xi_2}{(s,t)}\ \mathrm{q}\ f$. Since $\frac{\xi_2}{(s,t)}\ \mathrm{q}\ f$, so case (3) and (4) do not occur. Consequently, $\frac{\xi_1\xi_2}{(s,t)} \in \vee\mathrm{q}\ f$. Similar reasoning applied for right ideals. ■

Theorem 3.1.12 *Every* $(q, \in \vee q)$*-BFLI (resp. BFRI) of* S *is an* $(\in, \in \vee q)$*-BFLI (resp. BFRI) of* S*.*

Proof. We discuss only a case of left ideal. Let $f = (S; f_n, f_p)$ be a $(\mathrm{q}, \in \vee \mathrm{q})$-BFLI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $(s,t) \in [-1,0) \times (0,1]$. Let $\frac{\xi_2}{(s,t)} \in f$, then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Suppose $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\, f$, then $\frac{\xi_1}{(s,t)} \overline{\in}\, f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\, f$. If $\frac{\xi_1}{(s,t)} \overline{\in}\, f$, then $f_n(\xi_1) > s$ or $f_p(\xi_1) < t$. If $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\, f$, then $f_n(\xi_1) + s > -1$ or $f_p(\xi_1) + t < 1$. From the fact that

$$f_n(\xi_1) > s \text{ and } f_n(\xi_1) + s > -1,$$

we have $f_n(\xi_1) > -0.5$ and so $f_n(\xi_1) > \vee\{s, -0.5\}$. Thus

$$\begin{aligned} -1 - f_n(\xi_1) &< -1 - \vee\{s, -0.5\} = \wedge\{-1 - s, -1 + 0.5\} \\ &\leq \wedge\{-1 - f_n(\xi_2), -0.5\} \end{aligned}$$

which implies that there exists $\acute{s} \in [-1, 0)$ such that

$$-1 - f_n(\xi_1) \leq \acute{s} < \wedge\{-1 - f_n(\xi_2), -0.5\}. \tag{3.1}$$

This implies that $f_n(\xi_2) + \acute{s} < -1$. Also from the fact that $f_p(\xi_1) < t$ and $f_p(\xi_1) + t < 1$, we have $f_p(\xi_1) < 0.5$ and so $f_p(\xi_1) < \wedge\{t, 0.5\}$. This implies that

$$\begin{aligned} 1 - f_p(\xi_1) &> 1 - \wedge\{t, 0.5\} = \vee\{1 - t, 1 - 0.5\} \\ &\geq \vee\{1 - f_p(\xi_2), 0.5\}. \end{aligned}$$

Therefore there exists $\acute{t} \in (0, 1]$ such that

$$1 - f_p(\xi_1) \geq \acute{t} > \vee\{1 - f_p(\xi_2), 0.5\}. \tag{3.2}$$

This implies that $f_p(\xi_2) + \acute{t} > 1$. Therefore $\frac{\xi_2}{(\acute{s},\acute{t})}$ q f. Since $f = (S; f_n, f_p)$ is $(\mathrm{q}, \in \vee \mathrm{q})$-BFLI of S. Therefore $\frac{\xi_1}{(\acute{s},\acute{t})} \in \vee\mathrm{q}\ f$. Now the left inequality in (3.1) induces $-1 - f_n(\xi_1) \leq \acute{s}$ and

$f_n(\xi_1) \geq -1 - \acute{s} > -1 + 0.5 = -0.5 > \acute{s}$. Also the left inequality in (3.2) induces $1 - f_p(\xi_1) \geq \acute{t}$ and $f_p(\xi_1) \leq 1 - \acute{t} < 1 - 0.5 = 0.5 < \acute{t}$. Hence $\frac{\xi_1}{(\acute{s},\acute{t})} \overline{\in \vee \mathrm{q}}\ f$, a contradiction. Therefore $\frac{\xi_1}{(s,t)} \in \vee\mathrm{q}$ f.

Let $\xi_1, \xi_2 \in S$ such that $\frac{\xi_2}{(s,t)} \in f$ for $(s,t) \in [-1,0) \times (0,1]$. Then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Suppose $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, then $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$. If $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$, then $f_n(\xi_1\xi_2) > s$ or $f_p(\xi_1\xi_2) < t$. If $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$, then $f_n(\xi_1\xi_2) + s > -1$ or $f_p(\xi_1\xi_2) + t < 1$. From the fact that

$$f_n(\xi_1\xi_2) > s \text{ and } f_n(\xi_1\xi_2) + s > -1,$$

we have $f_n(\xi_1\xi_2) > -0.5$ and so $f_n(\xi_1\xi_2) > \vee\{s, -0.5\}$. Thus

$$\begin{aligned} -1 - f_n(\xi_1\xi_2) &< -1 - \vee\{s, -0.5\} = \wedge\{-1 - s, -1 + 0.5\} \\ &\leq \wedge\{-1 - f_n(\xi_2), -0.5\} \end{aligned}$$

which implies that there exists $\acute{s} \in [-1, 0)$ such that

$$-1 - f_n(\xi_1\xi_2) \leq \acute{s} < \wedge\{-1 - f_n(\xi_2), -0.5\}. \tag{3.3}$$

This implies $f_n(\xi_2) + \acute{s} < -1$. Also from the fact that $f_p(\xi_1\xi_2) < t$ and $f_p(\xi_1\xi_2) + t < 1$, we have $f_p(\xi_1\xi_2) < 0.5$ and so $f_p(\xi_1\xi_2) < \wedge\{t, 0.5\}$. This implies that

$$\begin{aligned} 1 - f_p(\xi_1\xi_2) &> 1 - \wedge\{t, 0.5\} = \vee\{1 - t, 1 - 0.5\} \\ &\geq \vee\{1 - f_p(\xi_2), 0.5\}. \end{aligned}$$

Therefore there exists $\acute{t} \in (0, 1]$ such that

$$1 - f_p(\xi_1\xi_2) \geq \acute{t} > \vee\{1 - f_p(\xi_2), 0.5\}. \tag{3.4}$$

This implies $f_p(\xi_2) + \acute{t} > 1$. Therefore $\frac{\xi_2}{(\acute{s},\acute{t})}$ q f. Since $f = (S; f_n, f_p)$ is $(\mathrm{q}, \in \vee\mathrm{q})$-BFLI of S. Therefore $\frac{\xi_1\xi_2}{(\acute{s},\acute{t})} \in \vee\mathrm{q}\ f$. Now the left inequality in (3.3) induces $-1 - f_n(\xi_1\xi_2) \leq \acute{s}$ and $f_n(\xi_1\xi_2) \geq -1 - \acute{s} > -1 + 0.5 = -0.5 > \acute{s}$. Also the left inequality in (3.4) induces $1 - f_p(\xi_1\xi_2) \geq \acute{t}$ and $f_p(\xi_1\xi_2) \leq 1 - \acute{t} < 1 - 0.5 = 0.5 < \acute{t}$. Hence $\frac{\xi_1\xi_2}{(\acute{s},\acute{t})} \overline{\in \vee \mathrm{q}}\ f$, a contradiction.

Therefore $\frac{\xi_1\xi_2}{(s,t)} \in \vee q\ f$. Hence $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFLI of S. Similar reasoning applied for right ideals. ■

In general every $(\in, \in \vee q)$-BFLI (resp. BFRI) may not be $(q, \in \vee q)$-BFLI (resp. BFRI) as shown in the following example.

Example 3.1.13 *Let $S = \{0, 1, 2, 3, 4\}$ be a set with multiplication table and order relation as in Example* 3.1.6*. Let $f = (S; f_n, f_p)$ be a BFS in S defined by the Table* 3*.*

Table 3

S	0	1	2	3	4
f_n	-0.9	-0.35	-0.72	-0.58	-0.65
f_p	0.8	0.3	0.7	0.5	0.6

Then $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFRI of S. But $f = (S; f_n, f_p)$ is not $(q, \in \vee q)$-BFRI of S because $\frac{0}{(-0.15,0.52)}\ q\ f$ but $\frac{0\cdot 3}{(-0.15,0.52)} = \frac{3}{(-0.15,0.52)} \overline{\in \vee q}\ f$.

We provide a condition for an $(\in, \in \vee q)$-BFLI (resp. BFRI) to be $(q, \in \vee q)$-BFLI (resp. BFRI).

Theorem 3.1.14 *Assume that every bipolar fuzzy point has the value $(s, t) \in [-0.5, 0) \times (0, 0.5]$. Then every $(\in, \in \vee q)$-BFLI (resp. BFRI) of S is a $(q, \in \vee q)$-BFLI (resp. BFRI) of S.*

Proof. Let $f = (S; f_n, f_p)$ be an $(\in, \in \vee q)$-BFLI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\frac{\xi_2}{(s,t)} q\ f$ for $(s, t) \in [-0.5, 0) \times (0, 0.5]$. Then $f_n(\xi_2) + s < -1$ and $f_p(\xi_2) + t > 1$. This implies that $f_n(\xi_2) < -1 - s \leq s$ and $f_p(\xi_2) > 1 - t \geq t$. Thus $\frac{\xi_2}{(s,t)} \in f$. Therefore $\frac{\xi_1}{(s,t)} \in \vee q\ f$. Let $\xi_1, \xi_2 \in S$ such that $\frac{\xi_2}{(s,t)} q\ f$ for $(s, t) \in [-0.5, 0) \times (0, 0.5]$. Then $f_n(\xi_2) + s < -1$ and $f_p(\xi_2) + t > 1$. This implies that $f_n(\xi_2) < -1 - s \leq s$ and $f_p(\xi_2) > 1 - t \geq t$. This implies $\frac{\xi_2}{(s,t)} \in f$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFLI of S. So $\frac{\xi_1\xi_2}{(s,t)} \in \vee q\ f$. Hence $f = (S; f_n, f_p)$ is a $(q, \in \vee q)$-BFLI of S. Same argument is applied for the right ideal. ■

Theorem 3.1.15 *If $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFLI (resp. BFRI) of S, then the set*

$$Q_f^{(s,t)} = \{\xi \in S | \frac{\xi}{(s,t)} q\ f\}$$

is a left (resp. right) ideal of S for all $(s,t) \in [-1,-0.5) \times (0.5,1]$ with $Q_f^{(s,t)} \neq \emptyset$.

Proof. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in Q_f^{(s,t)}$ for all $(s,t) \in [-1,-0.5) \times (0.5,1]$. Then $\frac{\xi_2}{(s,t)}\mathrm{q}\ f$, that is $f_n(\xi_2) + s < -1$ and $f_p(\xi_2) + t > 1$. This implies $f_n(\xi_2) < -1 - s$ and $f_p(\xi_2) > 1 - t$. As $f = (S; f_n, f_p)$ is an $(\in, \in \vee \mathrm{q})$-BFLI of S, so

$$\begin{aligned} f_n(\xi_1) &\leq \vee\{f_n(\xi_2), -0.5\} \\ &= \begin{cases} f_n(\xi_2) & \text{if } f_n(\xi_2) \geq -0.5 \\ -0.5 & \text{if } -0.5 > f_n(\xi_2) \end{cases} \\ &< -1 - s \end{aligned}$$

and

$$\begin{aligned} f_p(\xi_1) &\geq \wedge\{f_p(\xi_2), 0.5\} \\ &= \begin{cases} f_p(\xi_2) & \text{if } f_p(\xi_2) \leq 0.5 \\ 0.5 & \text{if } f_p(\xi_2) > 0.5 \end{cases} \\ &> 1 - t. \end{aligned}$$

Hence $\frac{\xi_1}{(s,t)}\mathrm{q}\ f$, and so $\xi_1 \in Q_f^{(s,t)}$.

Let $\xi_1, \xi_2 \in S$ such that $\xi_2 \in Q_f^{(s,t)}$ for all $(s,t) \in [-1,-0.5) \times (0.5,1]$. Then $\frac{\xi_2}{(s,t)}\mathrm{q}\ f$, that is $f_n(\xi_2) + s < -1$ and $f_p(\xi_2) + t > 1$. This implies $f_n(\xi_2) < -1 - s$ and $f_p(\xi_2) > 1 - t$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee \mathrm{q})$-BFLI of S. So

$$\begin{aligned} f_n(\xi_1\xi_2) &\leq \vee\{f_n(\xi_2), -0.5\} \\ &= \begin{cases} f_n(\xi_2) & \text{if } f_n(\xi_2) \geq -0.5 \\ -0.5 & \text{if } -0.5 > f_n(\xi_2) \end{cases} \\ &< -1 - s \end{aligned}$$

and

$$
\begin{aligned}
f_p(\xi_1\xi_2) &\geq \wedge\{f_p(\xi_2), 0.5\} \\
&= \begin{cases} f_p(\xi_2) & \text{if } f_p(\xi_2) \leq 0.5 \\ 0.5 & \text{if } f_p(\xi_2) > 0.5 \end{cases} \\
&> 1 - t.
\end{aligned}
$$

Hence $\frac{\xi_1\xi_2}{(s,t)}\mathrm{q}\ f$, and so $\xi_1\xi_2 \in Q_f^{(s,t)}$. Therefore $Q_f^{(s,t)}$ is left ideal of S. Same reasoning is given in order to show that $Q_f^{(s,t)}$ is right ideal of S. ■

For any BFS $f = (S; f_n, f_p)$ in S and $(s,t) \in [-1, 0) \times (0, 1]$, we denote $\Sigma = \{\xi_1 \in S \mid \frac{\xi_1}{(s,t)} \in \vee\mathrm{q}\ f\}$. Obviously $\Sigma = C(f; (s,t)) \cup Q_f^{(s,t)}$.

Theorem 3.1.16 *For any BFS $f = (S; f_n, f_p)$ is an $(\in, \in\vee q)$-BFLI (resp. BFRI) of S iff the set $\Sigma = \{\xi_1 \in S \mid \frac{\xi_1}{(s,t)} \in \vee q\ f\}$ is a left (resp. right) ideal of S $\forall (s,t) \in [-1, 0) \times (0, 1]$ with $\Sigma \neq \emptyset$.*

Proof. Suppose $f = (S; f_n, f_p)$ is an $(\in, \in\vee\mathrm{q})$-BFLI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in \Sigma$, then $\frac{\xi_2}{(s,t)} \in \vee\mathrm{q}\ f$, that is $\frac{\xi_2}{(s,t)} \in f$ or $\frac{\xi_2}{(s,t)}\mathrm{q}\ f$ for $(s,t) \in [-1, 0) \times (0, 1]$. If $\frac{\xi_2}{(s,t)} \in f$ then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. We have

$$
f_n(\xi_1) \leq \vee\{f_n(\xi_2), -0.5\} \leq \vee\{s, -0.5\} = \begin{cases} s \text{ if } s \geq -0.5, \\ -0.5 \text{ if } -0.5 > s, \end{cases}
$$

and

$$
f_p(\xi_1) \geq \wedge\{f_p(\xi_2), 0.5\} \geq \wedge\{t, 0.5\} = \begin{cases} t \text{ if } t \leq 0.5, \\ 0.5 \text{ if } t > 0.5. \end{cases}
$$

This implies $\frac{\xi_1}{(s,t)} \in f$ or $f_n(\xi_1) + s \leq -0.5 + s < -1$ and $f_p(\xi_1) + t \geq 0.5 + t > 1$, that is $\frac{\xi_1}{(s,t)}\mathrm{q}$ f. Hence $\frac{\xi_1}{(s,t)} \in \vee\mathrm{q}\ f$ and so $\xi_1 \in \Sigma$. If $\frac{\xi_2}{(s,t)}\mathrm{q}\ f$ then $f_n(\xi_2) + s < -1$ and $f_p(\xi_2) + t > 1$. This implies $f_n(\xi_2) < -1 - s$ and $f_p(\xi_2) > 1 - t$. We have

$$
f_n(\xi_1) \leq \vee\{f_n(\xi_2), -0.5\} \leq \vee\{-1 - s, -0.5\},
$$

and

$$f_p(\xi_1) \geq \wedge\{f_p(\xi_2), 0.5\} \geq \wedge\{1-t, 0.5\}.$$

If $(s,t) \in [-1,-0.5) \times (0.5,1]$ then $f_n(\xi_1) \leq -1-s$ and $f_p(\xi_1) \geq 1-t$. Thus $\xi_1 \in C(f;(-1-s,1-t)) \subseteq \Sigma$. If $(s,t) \in [-0.5,0) \times (0,0.5]$ then $f_n(\xi_1) \leq -0.5 \leq s$ and $f_p(\xi_1) \geq 0.5 \geq t$. Thus $\xi_1 \in C(f;(s,t)) \subseteq \Sigma$. If $(s,t) \in [-0.5,0) \times (0.5,1]$ then $f_n(\xi_1) \leq -0.5 \leq s$ and $f_p(\xi_1) \geq 1-t$. Thus $\xi_1 \in C(f;(s,1-t)) \subseteq \Sigma$. If $(s,t) \in [-1,-0.5) \times (0,0.5]$ then $f_n(\xi_1) \leq -1-s$ and $f_p(\xi_1) \geq 0.5 \geq t$. Thus $\xi_1 \in C(f;(-1-s,t)) \subseteq \Sigma$.

Let $\xi_1, \xi_2 \in S$ such that $\xi_2 \in \Sigma$, then $\frac{\xi_2}{(s,t)} \in \vee\mathrm{q}\ f$, that is, $\frac{\xi_2}{(s,t)} \in f$ or $\frac{\xi_2}{(s,t)}\mathrm{q}\ f$ for $(s,t) \in [-1,0) \times (0,1]$. If $\frac{\xi_2}{(s,t)} \in f$ then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. We have

$$f_n(\xi_1\xi_2) \leq \vee\{f_n(\xi_2), -0.5\} \leq \vee\{s, -0.5\} = \begin{cases} s \text{ if } s \geq -0.5, \\ -0.5 \text{ if } -0.5 > s, \end{cases}$$

and

$$f_p(\xi_1\xi_2) \geq \wedge\{f_p(\xi_2), 0.5\} \geq \wedge\{t, 0.5\} = \begin{cases} t \text{ if } t \leq 0.5, \\ 0.5 \text{ if } t > 0.5. \end{cases}$$

This implies $\frac{\xi_1\xi_2}{(s,t)} \in f$ or $f_n(\xi_1\xi_2) + s \leq -0.5 + s < -1$ and $f_p(\xi_1\xi_2) + t \geq 0.5 + t > 1$, that is, $\frac{\xi_1\xi_2}{(s,t)}\mathrm{q}\ f$. Hence $\frac{\xi_1\xi_2}{(s,t)} \in \vee\mathrm{q}\ f$ and so $\xi_1\xi_2 \in \Sigma$. If $\frac{\xi_2}{(s,t)}\mathrm{q}\ f$ then $f_n(\xi_2)+s < -1$ and $f_p(\xi_2)+t > 1$. This implies $f_n(\xi_2) < -1-s$ and $f_p(\xi_2) > 1-t$. We have

$$f_n(\xi_1\xi_2) \leq \vee\{f_n(\xi_2), -0.5\} \leq \vee\{-1-s, -0.5\},$$

and

$$f_p(\xi_1\xi_2) \geq \wedge\{f_p(\xi_2), 0.5\} \geq \wedge\{1-t, 0.5\}.$$

If $(s,t) \in [-1,-0.5) \times (0.5,1]$ then $f_n(\xi_1\xi_2) \leq -1-s$ and $f_p(\xi_1\xi_2) \geq 1-t$. Thus $\xi_1\xi_2 \in C(f;(-1-s,1-t)) \subseteq \Sigma$. If $(s,t) \in [-0.5,0) \times (0,0.5]$ then $f_n(\xi_1\xi_2) \leq -0.5 \leq s$ and $f_p(\xi_1\xi_2) \geq 0.5 \geq t$. Thus $\xi_1\xi_2 \in C(f;(s,t)) \subseteq \Sigma$. If $(s,t) \in [-0.5,0) \times (0.5,1]$ then $f_n(\xi_1\xi_2) \leq -0.5 \leq s$ and $f_p(\xi_1\xi_2) \geq 1-t$. Thus $\xi_1\xi_2 \in C(f;(s,1-t)) \subseteq \Sigma$. If $(s,t) \in [-1,-0.5) \times (0,0.5]$ then $f_n(\xi_1\xi_2) \leq -1-s$ and $f_p(\xi_1\xi_2) \geq 0.5 \geq t$. Thus $\xi_1\xi_2 \in C(f;(-1-s,t)) \subseteq \Sigma$. Hence Σ is a left ideal of S.

Conversely, suppose that Σ is a left ideal of S. Assume that $f = (S; f_n, f_p)$ is not an $(\in, \in \vee \mathrm{q})$-BFLI of S. Then we can find some $a, b \in S$ such that $a \leq b$ and $f_n(a) > \vee\{f_n(b), 0.5\}$ or $f_p(a) < \wedge\{f_p(b), 0.5\}$. If $f_n(a) > \vee\{f_n(b), 0.5\}$ and $f_p(a) < \wedge\{f_p(b), 0.5\}$ then there exist $s \in [-1, 0)$ and $t \in (0, 1]$ such that $f_n(a) > s \geq \vee\{f_n(b), -0.5\}$ and $f_p(a) < t \leq \wedge\{f_p(b), 0.5\}$. This implies $\frac{b}{(s,t)} \in f$ and so $b \in \Sigma$. Therefore $a \in \Sigma$, since Σ is left ideal of S. This implies $\frac{a}{(s,t)} \in \vee\mathrm{q}\ f$ and so $\frac{a}{(s,t)} \in f$ or $\frac{a}{(s,t)}\mathrm{q}\ f$, that is, $f_n(a) \leq s$ and $f_p(a) \geq t$ or $f_n(a) + s < -1$ and $f_p(a) + t > 1$ which is not possible. If $f_n(a) > \vee\{f_n(b), -0.5\}$ and $f_p(a) \geq \wedge\{f_p(b), 0.5\}$ then there exists $s \in [-1, 0)$ such that $f_n(a) > s \geq \vee\{f_n(b), -0.5\}$ and $f_p(a) \geq t = \wedge\{f_p(b), 0.5\}$ then $b \in C(f; (s, t)) \subseteq \Sigma$. This implies $a \in \Sigma$, that is $\frac{a}{(s,t)} \in \vee\mathrm{q}\ f$, which is impossible. If $f_n(a) \leq \vee\{f_n(b), -0.5\}$ and $f_p(a) < \wedge\{f_p(b), 0.5\}$. Then there exists $t \in (0, 1]$ such that $f_p(a) < t \leq \wedge\{f_p(b), 0.5\}$ and $f_n(a) \leq s = \vee\{f_n(b), -0.5\}$ then $b \in C(f; (s, t)) \subseteq \Sigma$. This implies $a \in \Sigma$, that is $\frac{a}{(s,t)} \in \vee\mathrm{q}\ f$, which is impossible. Therefore $f_n(\xi_1) \leq \vee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq \vee\{f_p(\xi_2), 0.5\}$ for all $\xi_1, \xi_2 \in S$.

Also there exist $a, b \in S$ such that $f_n(ab) > \vee\{f_n(b), 0.5\}$ or $f_p(ab) < \wedge\{f_p(b), 0.5\}$. If $f_n(ab) > \vee\{f_n(b), 0.5\}$ and $f_p(ab) < \wedge\{f_p(b), 0.5\}$ then there exists $s \in [-1, 0)$ such that $f_n(ab) > s \geq \vee\{f_n(b), -0.5\}$ and there exists $t \in (0, 1]$ such that $f_p(ab) < t \leq \wedge\{f_p(b), 0.5\}$. So $\frac{b}{(s,t)} \in f$ and thus $b \in \Sigma$. Therefore $ab \in \Sigma$, since Σ is left ideal of S. So $\frac{ab}{(s,t)} \in \vee\mathrm{q}\ f$, that is, $f_n(ab) \leq s$ and $f_p(ab) \geq t$ or $f_n(ab) + s < -1$ and $f_p(ab) + t > 1$, a contrary result. If $f_n(ab) > \vee\{f_n(b), -0.5\}$ and $f_p(ab) \geq \wedge\{f_p(b), 0.5\}$ then there exists $s \in [-1, 0)$ such that $f_n(ab) > s \geq \vee\{f_n(b), -0.5\}$ and $f_p(ab) \geq t = \wedge\{f_p(b), 0.5\}$ then $b \in C(f; (s, t)) \subseteq \Sigma$. Thus $ab \in \Sigma$, that is, $\frac{ab}{(s,t)} \in \vee\mathrm{q}\ f$, which is impossible. If $f_n(ab) \leq \vee\{f_n(b), -0.5\}$ and $f_p(ab) < \wedge\{f_p(b), 0.5\}$. Then there exists $t \in (0, 1]$ such that $f_p(ab) < t \leq \wedge\{f_p(b), 0.5\}$ and $f_n(ab) \leq s = \vee\{f_n(b), -0.5\}$ then $b \in C(f; (s, t)) \subseteq \Sigma$. So $ab \in \Sigma$, that is, $\frac{ab}{(s,t)} \in \vee\mathrm{q}\ f$, which is impossible. Therefore $f_n(\xi_1\xi_2) \leq \vee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq \vee\{f_p(\xi_2), 0.5\}$ $\forall\ \xi_1, \xi_2 \in S$. Hence $f = (S; f_n, f_p)$ is an $(\in, \in \vee\mathrm{q})$-BFLI of S. ■

3.2 Generalized bipolar fuzzy interior ideals of regular and intra-regular ordered semigroups

In this section, we introduce (α,β)-BFII of S. We characterize different classes of ordered semigroup in context of generalized BFII of S. We prove that the concept of generalized BFI and generalized BFII coincide in regular, intra-regular and semisimple ordered semigroups. We also give a proper example to clarify that every generalized BFII may not be a generalized BFI.

Definition 3.2.1 *A BFS $f=(S;f_n,f_p)$ in S is said to be (α,β)-BFII of S where $\alpha\neq\in\wedge q$ if it satisfy the following axioms. For $(s,t)\in[-1,0)\times(0,1]$*

(1) $(\forall\xi_1,\xi_2\in S)(\xi_1\leq\xi_2, \frac{\xi_2}{(s,t)}\alpha f\Longrightarrow\frac{\xi_1}{(s,t)}\beta f)$,
(2) $(\forall\xi_1,\xi_2\in S)(\frac{\xi_1}{(s_1,t_1)}\alpha f$ and $\frac{\xi_2}{(s_2,t_2)}\alpha f\Longrightarrow\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})}\beta f)$,
(3) $(\forall 2\xi_1,a,\xi_2\in S)(\frac{a}{(s,t)}\alpha f\Longrightarrow\frac{\xi_1 a\xi_2}{(s,t)}\beta f)$.

Lemma 3.2.2 *Let $\emptyset\neq\mathcal{I}\subseteq S$ then $\mathcal{I}$ is an interior ideal of S iff $\chi_{\mathcal{I}}=(S,\chi_{n\mathcal{I}},\chi_{p\mathcal{I}})$ is an $(\in,\in\vee q)$-BFII of S.*

Proof. Let $\chi_{\mathcal{I}}=(S;\chi_{n\mathcal{I}},\chi_{p\mathcal{I}})$ is an $(\in,\in\vee\mathrm{q})$-BFII of S. Let $\xi_1,\xi_2\in S$ such that $\xi_1\leq\xi_2$ and $\xi_2\in\mathcal{I}$. Then $\chi_{n\mathcal{I}}(\xi_2)=-1$ and $\chi_{p\mathcal{I}}(\xi_2)=1$, thus $\frac{\xi_2}{(-1,1)}\in\chi_{\mathcal{I}}$. Since $\chi_{\mathcal{I}}=(S;\chi_{n\mathcal{I}},\chi_{p\mathcal{I}})$ is an $(\in,\in\vee\mathrm{q})$-BFII of S, so $\frac{\xi_1}{(-1,1)}\in\vee\mathrm{q}\chi_{\mathcal{I}}$ i.e $\frac{\xi_1}{(-1,1)}\in\chi_{\mathcal{I}}$ or $\frac{\xi_1}{(-1,1)}\mathrm{q}\chi_{\mathcal{I}}$. From both cases we conclude that $\chi_{n\mathcal{I}}(\xi_1)=-1$ and $\chi_{p\mathcal{I}}(\xi_1)=1$. Hence $\xi_1\in\mathcal{I}$.

Let $\xi_1,\xi_2\in\mathcal{I}$, then $\chi_{n\mathcal{I}}(\xi_1)=\chi_{n\mathcal{I}}(\xi_2)=-1$ and $\chi_{p\mathcal{I}}(\xi_1)=\chi_{p\mathcal{I}}(\xi_2)=1$. Thus $\frac{\xi_1}{(-1,1)}\in\chi_{\mathcal{I}}$ and $\frac{\xi_2}{(-1,1)}\in\chi_{\mathcal{I}}$. Since $\chi_{\mathcal{I}}=(S;\chi_{n\mathcal{I}},\chi_{p\mathcal{I}})$ is an $(\in,\in\vee\mathrm{q})$-BFII of S, so $\frac{\xi_1\xi_2}{(-1,1)}\in\vee\mathrm{q}\chi_{\mathcal{I}}$ i.e $\frac{\xi_1\xi_2}{(-1,1)}\in\chi_{\mathcal{I}}$ or $\frac{\xi_1\xi_2}{(-1,1)}\mathrm{q}\chi_{\mathcal{I}}$. From both cases we conclude that $\chi_{n\mathcal{I}}(\xi_1\xi_2)=-1$ and $\chi_{p\mathcal{I}}(\xi_1\xi_2)=1$. Hence $\xi_1\xi_2\in\mathcal{I}$.

Let $\xi_1,\xi_2\in S$ and $a\in\mathcal{I}$, then $\chi_{n\mathcal{I}}(a)=-1$ and $\chi_{p\mathcal{I}}(a)=1$. Thus $\frac{a}{(-1,1)}\in\chi_{\mathcal{I}}$. Since $\chi_{\mathcal{I}}=(S;\chi_{n\mathcal{I}},\chi_{p\mathcal{I}})$ is an $(\in,\in\vee\mathrm{q})$-BFII of S, so $\frac{\xi_1 a\xi_2}{(-1,1)}\in\vee\mathrm{q}\chi_{\mathcal{I}}$ i.e $\frac{\xi_1 a\xi_2}{(-1,1)}\in\chi_{\mathcal{I}}$ or $\frac{\xi_1 a\xi_2}{(-1,1)}\mathrm{q}\chi_{\mathcal{I}}$. From both cases we conclude that $\chi_{n\mathcal{I}}(\xi_1 a\xi_2)=-1$ and $\chi_{p\mathcal{I}}(\xi_1 a\xi_2)=1$. So $\xi_1 a\xi_2\in\mathcal{I}$. Hence $\mathcal{I}$ is an interior ideal of S.

Conversely, assume that $\mathcal{I}$ is an interior ideal of S. Let $\xi_1,\xi_2\in S$ such that $\xi_1\leq\xi_2$ and $\frac{\xi_2}{(s,t)}\in\chi_{\mathcal{I}}$ for $(s,t)\in[-1,0)\times(0,1]$. Then $\chi_{n\mathcal{I}}(\xi_2)\leq s<0$ and $\chi_{p\mathcal{I}}(\xi_2)\geq t>0$, which implies that $\chi_{n\mathcal{I}}(\xi_2) = -1$ and $\chi_{p\mathcal{I}}(\xi_2) = 1$. Thus $\xi_2 \in \mathcal{I}$. So $\xi_1 \in \mathcal{I}$. It follows that $\chi_{n\mathcal{I}}(\xi_1) = -1 \leq s$ and $\chi_{p\mathcal{I}}(\xi_1) = 1 \geq t$. This implies that $\frac{\xi_1}{(s,t)} \in \chi_{\mathcal{I}}$ and hence $\frac{\xi_1}{(s,t)} \in \vee\mathrm{q}\chi_{\mathcal{I}}$.

Let $\xi_1, \xi_2 \in S$ such that $\frac{\xi_1}{(s_1,t_1)} \in \chi_{\mathcal{I}}$ and $\frac{\xi_2}{(s_2,t_2)} \in \chi_{\mathcal{I}}$ for $(s_1, t_1), (s_2, t_2) \in [-1, 0) \times (0, 1]$. Then $\chi_{n\mathcal{I}}(\xi_1) \leq s_1 < 0, \chi_{n\mathcal{I}}(\xi_2) \leq s_2 < 0$ and $\chi_{p\mathcal{I}}(\xi_1) \geq t_1 > 0, \chi_{p\mathcal{I}}(\xi_2) \geq t_2 > 0$. This implies that $\chi_{n\mathcal{I}}(\xi_1) = \chi_{n\mathcal{I}}(\xi_2) = -1$ and $\chi_{p\mathcal{I}}(\xi_1) = \chi_{p\mathcal{I}}(\xi_2) = 1$. Thus $\xi_1, \xi_2 \in \mathcal{I}$ and so $\xi_1\xi_2 \in \mathcal{I}$. It follows that $\chi_{n\mathcal{I}}(\xi_1\xi_2) = -1 \leq \vee\{s_1, s_2\}$ and $\chi_{p\mathcal{I}}(\xi_1\xi_2) = 1 \geq \wedge\{t_1, t_2\}$. This implies that $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \chi_{\mathcal{I}}$ and so $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \vee\mathrm{q}\chi_{\mathcal{I}}$.

Let $\xi_1, a, \xi_2 \in S$ such that $\frac{a}{(s,t)} \in \chi_{\mathcal{I}}$ for $(s, t) \in [-1, 0) \times (0, 1]$. Then $\chi_{n\mathcal{I}}(a) \leq s < 0$ and $\chi_{p\mathcal{I}}(a) \geq t > 0$. This implies that $\chi_{n\mathcal{I}}(a) = -1$ and $\chi_{p\mathcal{I}}(a) = 1$. Thus $a \in \mathcal{I}$. Since $\mathcal{I}$ is an interior ideal of S, so $\xi_1 a\xi_2 \in \mathcal{I}$. It follows that $\chi_{n\mathcal{I}}(\xi_1 a\xi_2) = -1 \leq s$ and $\chi_{p\mathcal{I}}(\xi_1 a\xi_2) = 1 \geq t$. This implies that $\frac{\xi_1 a\xi_2}{(s,t)} \in \chi_{\mathcal{I}}$ and so $\frac{\xi_1 a\xi_2}{(st)} \in \vee\mathrm{q}\chi_{\mathcal{I}}$. Hence $\chi_{\mathcal{I}} = (S; \chi_{n\mathcal{I}}, \chi_{p\mathcal{I}})$ is an $(\in, \in \vee\mathrm{q})$-BFII of S. ■

Theorem 3.2.3 *A BFS $f = (S; f_n, f_p)$ in S is an $(\in, \in \vee q)$-BFII of S iff it satisfies:*

(1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Longrightarrow f_n(\xi_1) \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq \bigwedge\{f_p(\xi_2), 0.5\})$,

(2) $(\forall \xi_1, \xi_2 \in S)(f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\})$,

(3) $(\forall \xi_1, a, \xi_2 \in S)(f_n(\xi_1 a\xi_2) \leq \bigvee\{f_n(a), -0.5\}$ and $f_p(\xi_1 a\xi_2) \geq \bigwedge\{f_p(a), 0.5\})$.

Proof. Assume that $f = (S; f_n, f_p)$ is an $(\in, \in \vee\mathrm{q})$-BFII of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. We consider the following four cases.

(i) $f_n(\xi_2) > -0.5$ and $f_p(\xi_2) < 0.5$,

(ii) $f_n(\xi_2) \leq -0.5$ and $f_p(\xi_2) \geq 0.5$,

(iii) $f_n(\xi_2) \leq -0.5$ and $f_p(\xi_2) < 0.5$,

(iv) $f_n(\xi_2) > -0.5$ and $f_p(\xi_2) \geq 0.5$.

For the first case, assume that

$$f_n(\xi_1) > \bigvee\{f_n(\xi_2), -0.5\} \text{ or } f_p(\xi_1) < \bigwedge\{f_p(\xi_2), 0.5\}.$$

Then $f_n(\xi_1) > f_n(\xi_2)$ or $f_p(\xi_1) < f_p(\xi_2)$. If $f_n(\xi_1) > f_n(\xi_2)$ then $f_n(\xi_1) > s \geq f_n(\xi_2)$ for some $s \in [-1, 0)$. Let $t = f_p(\xi_2)$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$, that is, $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$. This is a contradiction. If $f_p(\xi_1) < f_p(\xi_2)$ then $f_p(\xi_1) < t \leq f_p(\xi_2)$ for some $t \in (0, 1]$. Let $s = f_n(\xi_2)$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$, that is, $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, a contradiction. Therefore $f_n(\xi_1) \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq \bigwedge\{f_p(\xi_2), 0.5\}$.

Now consider the second case that is, $f_n(\xi_2) \leq -0.5$ and $f_p(\xi_2) \geq 0.5$. This implies that $\frac{\xi_2}{(-0.5,0.5)} \in f$. Let $\xi_1, \xi_2 \in S$ be such that $\xi_1 \leq \xi_2$. Since $\frac{\xi_2}{(-0.5,0.5)} \in f$, this implies $\frac{\xi_1}{(-0.5,0.5)} \in \vee\mathrm{q}\ f$. So $\frac{\xi_1}{(-0.5,0.5)} \in f$ or $\frac{\xi_1}{(-0.5,0.5)} \mathrm{q}\ f$. If $\frac{\xi_1}{(-0.5,0.5)} \in f$ then $f_n(\xi_1) \leq -0.5 \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq 0.5 \geq \bigwedge\{f_p(\xi_2), 0.5\}$. If $\frac{\xi_1}{(-0.5,0.5)} \mathrm{q}\ f$ then $f_n(\xi_1) - 0.5 < -1$ and $f_p(\xi_1) + 0.5 > 1$. This implies that $f_n(\xi_1) < -0.5$ and $f_p(\xi_1) > 0.5$, therefore $f_n(\xi_1) \leq -0.5 \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq 0.5 \geq \bigwedge\{f_p(\xi_2), 0.5\}$.

Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$, then from case (iii) we have $f_n(\xi_1) \leq -0.5$ and $f_p(\xi_1) \geq f_p(\xi_2)$. Assume that $f_n(\xi_1) > -0.5$ or $f_p(\xi_1) < f_p(\xi_2)$. If $f_n(\xi_1) > -0.5$ then there exists $s \in [-1, 0)$ such that $f_n(\xi_1) > s \geq -0.5 \geq f_n(\xi_2)$. Let $f_p(\xi_2) = t$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$,that is, $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, a contrary result. If $f_p(\xi_1) < f_p(\xi_2)$ then $f_p(\xi_1) < t \leq f_p(\xi_2)$ for some $t \in (0, 1]$. Let $f_n(\xi_2) = s$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$,that is, $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, a contradiction. Thus $f_n(\xi_1) \leq -0.5 \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq f_p(\xi_2) \geq \bigwedge\{f_p(\xi_2), 0.5\}$.

Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$, then from case (iv), we have $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq 0.5$. Assume that $f_n(\xi_1) > f_n(\xi_2)$ or $f_p(\xi_1) < 0.5$. If $f_n(\xi_1) > f_n(\xi_2)$ then $f_n(\xi_1) > s \geq f_n(\xi_2)$ for some $s \in [-1, 0)$. Thus $\frac{\xi_2}{(s,0.5)} \in f$ but $\frac{\xi_1}{(s,0.5)} \overline{\in} f$ and $\frac{\xi_1}{(s,0.5)} \overline{\mathrm{q}}\ f$,that is, $\frac{\xi_1}{(s,0.5)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. If $f_p(\xi_1) < 0.5$ then there exists $t \in (0, 1]$ such that $f_p(\xi_1) < t \leq 0.5 \leq f_p(\xi_2)$. Let $s = f_n(\xi_2)$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$,that is, $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. Hence $f_n(\xi_1) \leq f_n(\xi_2) \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq 0.5 \geq \bigwedge\{f_p(\xi_2), 0.5\}$.

Let $\xi_1, \xi_2 \in S$. We consider the four cases.

(i) $\bigvee\{f_n(\xi_1), f_n(\xi_2)\} > -0.5$ and $\bigwedge\{f_p(\xi_1), f_p(\xi_2)\} < 0.5$,

(ii) $\bigvee\{f_n(\xi_1), f_n(\xi_2)\} \leq -0.5$ and $\bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq 0.5$,

(iii) $\bigvee\{f_n(\xi_1), f_n(\xi_2)\} \leq -0.5$ and $\bigwedge\{f_p(\xi_1), f_p(\xi_2)\} < 0.5$,

(iv) $\bigvee\{f_n(\xi_1), f_n(\xi_2)\} > -0.5$ and $\bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq 0.5$.

For the first case, assume that

$$f_n(\xi_1\xi_2) > \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\} \text{ or } f_p(\xi_1\xi_2) < \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\}.$$

Then

$$f_n(\xi_1\xi_2) > \bigvee\{f_n(\xi_1), f_n(\xi_2)\} \text{ or } f_p(\xi_1\xi_2) < \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}.$$

If $f_n(\xi_1\xi_2) > \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ then $f_n(\xi_1\xi_2) > s \geq \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ for some $s \in [-1, 0)$. Let $t = \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$ then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$, i.e, $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, a contradiction. If $f_p(\xi_1\xi_2) < \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$ then $f_p(\xi_1\xi_2) < t \leq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$ for some $t \in (0, 1]$. Let $s = \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$, i.e, $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. Hence $f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\}$.

Now consider the second case that is,

$$\bigvee\{f_n(\xi_1), f_n(\xi_2)\} \leq -0.5 \text{ and } \bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq 0.5.$$

This implies that $\frac{\xi_1}{(-0.5,0.5)} \in f$ and $\frac{\xi_2}{(-0.5,0.5)} \in f$ and so $\frac{\xi_1\xi_2}{(-0.5,0.5)} \in \vee\mathrm{q}\ f$. Thus $\frac{\xi_1\xi_2}{(-0.5,0.5)} \in f$ or $\frac{\xi_1\xi_2}{(-0.5,0.5)}\mathrm{q}\ f$. If $\frac{\xi_1\xi_2}{(-0.5,0.5)} \in f$ then $f_n(\xi_1\xi_2) \leq -0.5 \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq 0.5 \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\}$. If $\frac{\xi_1\xi_2}{(-0.5,0.5)}\mathrm{q}\ f$ then $f_n(\xi_1\xi_2) - 0.5 < -1$ and $f_p(\xi_1\xi_2) + 0.5 > 1$. This implies that $f_n(\xi_1\xi_2) < -0.5$ and $f_p(\xi_1\xi_2) > 0.5$, therefore $f_n(\xi_1\xi_2) \leq -0.5 \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq 0.5 \geq \bigwedge\{f_n(\xi_1), f_p(\xi_2), 0.5\}$.

Case (iii) implies that $f_n(\xi_1\xi_2) \leq -0.5$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$. Assume that $f_n(\xi_1\xi_2) > -0.5$ or $f_p(\xi_1\xi_2) < \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$. If $f_n(\xi_1\xi_2) > -0.5$ then there exists $s \in [-1, 0)$ such that $f_n(\xi_1\xi_2) > s \geq -0.5 \geq \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$. Let $t = \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$ then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$,that is, $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, a contrary result. If $f_p(\xi_1\xi_2) < \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$ then we can find some $t \in (0, 1]$ such that $f_p(\xi_1\xi_2) < t \leq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$. Let $s = \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$,that is, $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. Hence $f_n(\xi_1\xi_2) \leq -0.5 \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\}$.

From case (iv), we have $f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ and $f_p(\xi_1\xi_2) \geq 0.5$. Assume that $f_n(\xi_1\xi_2) > \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ or $f_p(\xi_1\xi_2) < 0.5$. If $f_n(\xi_1\xi_2) > \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ then there exists $s \in [-1, 0)$ such that $f_n(\xi_1\xi_2) > s \geq \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$. It follows that $\frac{\xi_1}{(s,0.5)} \in f$ and $\frac{\xi_2}{(s,0.5)} \in f$ but $\frac{\xi_1\xi_2}{(s,0.5)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,0.5)} \overline{\mathrm{q}}\ f$,that is, $\frac{\xi_1\xi_2}{(s,0.5)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. If $f_p(\xi_1\xi_2) < 0.5$ then there exists $t \in (0, 1]$ such that $f_p(\xi_1\xi_2) < t \leq 0.5 \leq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$. Let $s = \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$,that is, $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. Hence $f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2)\} \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq 0.5 \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\}$.

Let $\xi_1, a, \xi_2 \in S$. We consider the four cases.

$$\text{(i) } f_n(a) > -0.5 \text{ and } f_p(a) < 0.5,$$

$$\text{(ii) } f_n(a) \leq -0.5 \text{ and } f_p(a) \geq 0.5,$$

$$\text{(iii) } f_n(a) \leq -0.5 \text{ and } f_p(a) < 0.5,$$

$$\text{(iv) } f_n(a) > -0.5 \text{ and } f_p(a) \geq 0.5.$$

For the first case, assume that

$$f_n(\xi_1 a\xi_2) > \bigvee\{f_n(a), -0.5\} \text{ or } f_p(\xi_1 a\xi_2) < \bigwedge\{f_p(a), 0.5\}.$$

Then

$$f_n(\xi_1 a\xi_2) > f_n(a) \text{ or } f_p(\xi_1 a\xi_2) < f_p(a).$$

If $f_n(\xi_1 a\xi_2) > f_n(a)$, then we can find some $s \in [-1, 0)$ such that $f_n(\xi_1 a\xi_2) > s \geq f_n(a)$. Let $t = f_p(a)$ then $\frac{a}{(s,t)} \in f$ but $\frac{\xi_1 a\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1 a\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$, i.e, $\frac{\xi_1 a\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, a contrary result. If $f_p(\xi_1 a\xi_2) < f_p(a)$, then $f_p(\xi_1 a\xi_2) < t \leq f_p(a)$ for some $t \in (0, 1]$. Let $s = f_n(a)$ then $\frac{a}{(s,t)} \in f$ but $\frac{\xi_1 a\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1 a\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$, i.e, $\frac{\xi_1 a\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is not possible. Hence $f_n(\xi_1 a\xi_2) \leq \bigvee\{f_n(a), -0.5\}$ and $f_p(\xi_1 a\xi_2) \geq \bigwedge\{f_p(a), 0.5\}$.

Now consider the second case

$$f_n(a) \leq -0.5 \text{ and } f_p(a) \geq 0.5.$$

This implies that $\frac{a}{(-0.5,0.5)} \in f$, so $\frac{\xi_1 a\xi_2}{(-0.5,0.5)} \in \vee \mathrm{q}\ f$. Thus $\frac{\xi_1 a\xi_2}{(-0.5,0.5)} \in f$ or $\frac{\xi_1 a\xi_2}{(-0.5,0.5)} \mathrm{q}\ f$. If $\frac{\xi_1 a\xi_2}{(-0.5,0.5)} \in f$ then $f_n(\xi_1 a\xi_2) \leq -0.5 \leq \bigvee\{f_n(a), -0.5\}$ and $f_p(\xi_1 a\xi_2) \geq 0.5 \geq \bigwedge\{f_p(a), 0.5\}$. If $\frac{\xi_1 a\xi_2}{(-0.5,0.5)} \mathrm{q}\ f$ then $f_n(\xi_1 a\xi_2) - 0.5 < -1$ and $f_p(\xi_1 a\xi_2) + 0.5 > 1$. This implies that $f_n(\xi_1 a\xi_2) < -0.5$ and $f_p(\xi_1 a\xi_2) > 0.5$. Therefore $f_n(\xi_1 a\xi_2) \leq -0.5 \leq \bigvee\{f_n(a), -0.5\}$ and $f_p(\xi_1 a\xi_2) \geq 0.5 \geq \bigwedge\{f_p(a), 0.5\}$.

Case (iii) implies that $f_n(\xi_1 a\xi_2) \leq -0.5$ and $f_p(\xi_1 a\xi_2) \geq f_p(a)$. Assume that $f_n(\xi_1 a\xi_2) > -0.5$ or $f_p(\xi_1 a\xi_2) < f_p(a)$. If $f_n(\xi_1 a\xi_2) > -0.5$, then there exists $s \in [-1, 0)$ such that $f_n(\xi_1 a\xi_2) > s \geq -0.5 \geq f_n(a)$. Let $t = f_p(a)$ then $\frac{a}{(s,t)} \in f$ but $\frac{\xi_1 a\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1 a\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$, i.e, $\frac{\xi_1 a\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, a contrary result. If $f_p(\xi_1 a\xi_2) < f_p(a)$, then we can find some $t \in (0, 1]$ such that $f_p(\xi_1 a\xi_2) < t \leq f_p(a)$. Let $s = f_n(a)$ then $\frac{a}{(s,t)} \in f$ but $\frac{\xi_1 a\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1 a\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$, i.e, $\frac{\xi_1 a\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is impossible. Hence $f_n(\xi_1 a\xi_2) \leq -0.5 \leq \bigvee\{f_n(a), -0.5\}$ and $f_p(\xi_1 a\xi_2) \geq f_p(a) \geq \bigwedge\{f_p(a), 0.5\}$.

From case (iv), we have $f_n(\xi_1 a\xi_2) \leq f_n(a)$ and $f_p(\xi_1 a\xi_2) \geq 0.5$. Assume that

$$f_n(\xi_1 a\xi_2) > f_n(a) \text{ or } f_p(\xi_1 a\xi_2) < 0.5.$$

If $f_n(\xi_1 a\xi_2) > f_n(a)$, then $f_n(\xi_1 a\xi_2) > s \geq f_n(a)$ for some $s \in [-1, 0)$. It follows that $\frac{a}{(s,0.5)} \in f$ but $\frac{\xi_1 a\xi_2}{(s,0.5)} \overline{\in} f$ and $\frac{\xi_1 a\xi_2}{(s,0.5)} \overline{\mathrm{q}}\ f$, i.e, $\frac{\xi_1 a\xi_2}{(s,0.5)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. If $f_p(\xi_1 a\xi_2) < 0.5$, then $f_p(\xi_1 a\xi_2) < t \leq 0.5 \leq f_p(a)$ for some $t \in (0, 1]$. Let $s = f_n(a)$ then $\frac{a}{(s,t)} \in f$ but $\frac{\xi_1 a\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1 a\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$, i.e, $\frac{\xi_1 a\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. Hence $f_n(\xi_1 a\xi_2) \leq f_n(a) \leq \bigvee\{f_n(a), -0.5\}$ and $f_p(\xi_1 a\xi_2) \geq 0.5 \geq \bigwedge\{f_p(a), 0.5\}$.

Conversely, assume that the given three conditions are satisfied. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\frac{\xi_2}{(s,t)} \in f$ for $(s, t) \in [-1, 0) \times (0, 1]$. Then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Assume that $\frac{\xi_1}{(s,t)} \overline{\in} f$, then $f_n(\xi_1) > s$ and $f_p(\xi_1) < t$. If $f_n(\xi_1) > s$ then $f_n(\xi_2) \leq -0.5$. Otherwise, we get $f_n(\xi_1) \leq \bigvee\{f_n(\xi_2), -0.5\} = f_n(\xi_2) \leq s$, a contradiction. If $f_p(\xi_1) < t$ then $f_p(\xi_2) \geq 0.5$. Otherwise we obtain $f_p(\xi_1) > \bigwedge\{f_p(\xi_2), 0.5\} = f_p(\xi_2) \geq t$, which is a contradiction. It follows that $f_n(\xi_1)+s < 2f_n(\xi_1) \leq 2\bigvee\{f_n(\xi_2), -0.5\} = -1$ and $f_p(\xi_1)+t > 2f_p(\xi_1) \geq 2\bigwedge\{f_p(\xi_2), 0.5\} = 1$. This implies that $\frac{\xi_1}{(s,t)} \mathrm{q}\ f$. Let $\frac{\xi_1}{(s_1,t_1)} \in f, \frac{\xi_2}{(s_2,t_2)} \in f$ for $(s_1, t_1), (s_2, t_2) \in [-1, 0) \times (0, 1]$. Then $f_n(\xi_1) \leq s_1$, $f_n(\xi_2) \leq s_2$, $f_p(\xi_1) \geq t_1$ and $f_p(\xi_2) \geq t_2$. Assume that $\frac{\xi_1 \xi_2}{(\vee\{s_1,s_2\}, \wedge\{t_1,t_2\})} \overline{\in} f$, then $f_n(\xi_1 \xi_2) > \bigvee\{s_1, s_2\}$ or $f_p(\xi_1 \xi_2) < \bigwedge\{t_1, t_2\}$. If $f_n(\xi_1 \xi_2) > \bigvee\{s_1, s_2\}$, then $\bigvee\{f_n(\xi_1), f_n(\xi_2)\} \leq$

-0.5. Otherwise, we get $f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\} = \bigvee\{f_n(\xi_1), f_n(\xi_2)\} \leq \bigvee\{s_1, s_2\}$, a contradiction. If $f_p(\xi_1\xi_2) < \bigwedge\{t_1, t_2\}$ then $\bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq 0.5$. Otherwise we obtain $f_p(\xi_1\xi_2) > \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\} = \bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq \bigwedge\{t_1, t_2\}$, a contradiction. Hence $f_n(\xi_1\xi_2) + \bigvee\{s_1, s_2\} < 2f_n(\xi_1\xi_2) \leq 2\bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\} = -1$ and $f_p(\xi_1\xi_2) + \bigwedge\{t_1, t_2\} > 2f_p(\xi_1\xi_2) \geq 2\bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\} = 1$. This implies that $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})}\mathrm{q}\, f$. Let $\xi_1, \xi_2 \in S$ and $\frac{a}{(s,t)} \in f$ for $(s,t) \in [-1, 0) \times (0, 1]$. Then $f_n(a) \leq s$, $f_p(a) \geq t$. Assume that $\frac{\xi_1 a\xi_2}{(s,t)}\overline{\in} f$, then $f_n(\xi_1 a\xi_2) > s$ or $f_p(\xi_1 a\xi_2) < t$. If $f_n(\xi_1 a\xi_2) > s$, then $f_n(a) \leq -0.5$. Otherwise, we get $f_n(\xi_1 a\xi_2) \leq \bigvee\{f_n(a), -0.5\} = f_n(a) \leq s$, a contradiction. If $f_p(\xi_1 a\xi_2) < t$ then $f_p(a) \geq 0.5$. Otherwise we obtain $f_p(\xi_1 a\xi_2) \geq \bigwedge\{f_p(a), 0.5\} = f_p(a) \geq t$, which is a contradiction. It follows that

$$f_n(\xi_1 a\xi_2) + s < 2f_n(\xi_1 a\xi_2) \leq 2\bigvee\{f_n(a), -0.5\} = -1$$

and

$$f_p(\xi_1 a\xi_2) + t > 2f_p(\xi_1 a\xi_2) \geq 2\bigwedge\{f_p(a), 0.5\} = 1.$$

Thus $\frac{\xi_1 a\xi_2}{(s,t)}\mathrm{q}\, f$ and hence $f = (S; f_n, f_p)$ is an $(\in, \in \vee\mathrm{q})$-BFII of S. ■

Theorem 3.2.4 *Let $f = (S; f_n, f_p)$ is a BFS in S. Then $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFII of S iff the non-empty (s,t)-cut $C(f; (s,t))$ of S is an interior ideal of S for all $(s,t) \in [-0.5, 0) \times (0, 0.5]$.*

Proof. Let $f = (S; f_n, f_p)$ is an $(\in, \in \vee\mathrm{q})$-BFII of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in C(f; (s,t))$. This implies $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee\mathrm{q})$-BFII of S. Therefore $f_n(\xi_1) \leq \vee\{f_n(\xi_2), -0.5\} \leq s$ and $f_p(\xi_1) \geq \wedge\{f_p(\xi_2), 0.5\} \geq t$. This implies $\xi_1 \in C(f; (s,t))$. Let $\xi_1, \xi_2 \in C(f; (s,t))$. Then $f_n(\xi_1) \leq s$, $f_n(\xi_2) \leq s$, $f_p(\xi_1) \geq t$ and $f_p(\xi_2) \geq t$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee\mathrm{q})$-BFII of S. Therefore $f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\} \leq s$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\} \geq t$. This implies that $\xi_1\xi_2 \in C(f; (s,t))$. Let $\xi_1, \xi_2, \xi_3 \in S$ such that $\xi_2 \in C(f; (s,t))$. This implies $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee\mathrm{q})$-BFII of S. Therefore $f_n(\xi_1\xi_2\xi_3) \leq \vee\{f_n(\xi_2), -0.5\} \leq s$ and $f_p(\xi_1\xi_2\xi_3) \geq \wedge\{f_p(\xi_2), 0.5\} \geq t$. This implies that $\xi_1\xi_2\xi_3 \in C(f; (s,t))$. Hence $C(f; (s,t))$ is an interior ideal of S.

Conversely, suppose that $C(f; (s,t))$ is an interior ideal of S. Assume that $f = (S; f_n, f_p)$ is not an $(\in, \in \vee\mathrm{q})$-BFII of S, then for some $a, b, c \in S$, we have

(i) $a \leq b$ and $f_n(a) > \bigvee\{f_n(b), -0.5\}$ or $f_p(a) < \bigwedge\{f_p(b), 0.5\}$,

(ii) $f_n(ab) > \bigvee\{f_n(a), f_n(b), -0.5\}$ or $f_p(ab) < \bigwedge\{f_p(a), f_p(b), 0.5\}$,

(iii) $f_n(abc) > \bigvee\{f_n(b), -0.5\}$ or $f_p(abc) < \bigwedge\{f_p(b), 0.5\}$.

Let $s = \bigvee\{f_n(b), -0.5\}$ and $t = \bigwedge\{f_p(b), 0.5\}$ then $b \in C(f;(s,t))$. If $a \leq b$ and $f_n(a) > \vee\{f_n(b), -0.5\} = s$ then $a \notin N(f;s)$ and so $a \notin C(f;(s,t))$. If $f_p(a) < \bigwedge\{f_p(b), 0.5\} = t$ then $a \notin P(f;t)$ and so $a \notin C(f;(s,t))$. In both cases we have a contradiction. Let $s = \bigvee\{f_n(a), f_n(b), -0.5\}$ and $t = \bigwedge\{f_p(a), f_p(b), -0.5\}$ then $a, b \in C(f(s,t))$. If $f_n(ab) > \bigvee\{f_n(a), f_n(b), -0.5\} = s$ then $ab \notin N(f;s)$and so $ab \notin C(f;(s,t))$. If $f_p(ab) < \bigwedge\{f_p(a), f_p(b), 0.5\} = t$ then $ab \notin P(f;t)$ and so $ab \notin C(f;(s,t))$. In both cases we have a contradictory due to the fact that $C(f;(s,t))$ is an interior ideal of S. Let $\bigvee\{f_n(b), -0.5\} = s$ and $\bigwedge\{f_p(b), 0.5\} = t$, then $b \in C(f;(s,t))$. If $f_n(abc) > \bigvee\{f_n(b), -0.5\} = s$ then $abc \notin N(f;s)$ and so $abc \notin C(f;(s,t))$. If $f_p(abc) < \bigwedge\{f_p(b), 0.5\} = t$ then $abc \notin P(f;t)$. Thus $abc \notin C(f;(s,t))$, a contradiction to the fact that $C(f;(s,t))$ is an interior ideal of S. Hence $f = (S; f_n, f_p)$ is an $(\in, \in \vee \mathrm{q})$-BFII of S.

■

Proposition 3.2.5 *Every $(\in, \in \vee q)$-BFI of S is an $(\in, \in \vee q)$-BFII of S.*

Proof. Straight forward. ■

From Example 3.2.5 it is clear that the converse of Proposition 3.2.4 is not true.

Example 3.2.6 *Consider an ordered semigroup $S = \{0, 1, 2, 3\}$ whose multiplication table and order relation are*

Table 1

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

$\leq := \{(0,0), (1,1), (2,2), (3,3), (0,1)\}$. *Let $f = (S; f_n, f_p)$ be a BFS in S defined by*

Table 2

S	0	1	2	3
f_n	-0.8	-0.3	-0.5	-0.1
f_p	0.7	0.4	0.6	0.2

Then clearly $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFII of S, but $f = (S; f_n, f_p)$ is not an $(\in, \in \vee q)$-BFI of S. Because if $\xi_1\xi_2\xi_3 = 0$, then $f_n(\xi_1\xi_2\xi_3) = f_n(0) = -0.8 \leq f_n(\xi_2) \vee -0.5$ and $f_p(\xi_1\xi_2\xi_3) = f_p(0) = 0.7 \geq f_p(\xi_2) \wedge 0.5$. If $\xi_1\xi_2\xi_3 = 1$, then $f_n(\xi_1\xi_2\xi_3) = f_n(1) = -0.3 \leq -0.1 = f_n(\xi_2) \vee -0.5$ and $f_p(\xi_1\xi_2\xi_3) = f_p(1) = 0.4 \geq 0.2 = f_p(\xi_2) \wedge 0.5$. If $\xi_1\xi_2 = 0$, then $f_n(\xi_1\xi_2) = f_n(0) = -0.8 \leq f_n(\xi_1) \vee f_n(\xi_2) \vee -0.5$ and $f_p(\xi_1\xi_2) = f_p(0) = 0.7 \geq f_p(\xi_1) \wedge f_p(\xi_2) \wedge 0.5$. If $\xi_1\xi_2 = 1$, then $f_n(\xi_1\xi_2) = f_n(1) = -0.3 \leq -0.1 = f_n(\xi_1) \vee f_n(\xi_2) \vee -0.5$ and $f_p(\xi_1\xi_2) = f_p(1) = 0.4 \geq 0.2 = f_p(\xi_1) \wedge f_p(\xi_2) \wedge 0.5$. If $\xi_1\xi_2 = 2$, then $f_n(\xi_1\xi_2) = f_n(2) = -0.5 \leq -0.1 = f_n(\xi_1) \vee f_n(\xi_2) \vee -0.5$ and $f_p(\xi_1\xi_2) = f_p(2) = 0.6 \geq 0.2 = f_p(\xi_1) \wedge f_p(\xi_2) \wedge 0.5$. Let $\xi_1, \xi_2 \in S$ with $\xi_1 \leq \xi_2$, then $f_n(\xi_1) \leq f_n(\xi_2) \vee -0.5$ and $f_p(\xi_1) \geq f_p(\xi_2) \wedge 0.5$. But $f_p(2 \cdot 3) = f_p(1) = 0.4 < f_p(2) \wedge 0.5$. Therefore, $f = (S, f_n, f_p)$ is not an $(\in, \in \vee q)$-BFRI of S. Hence $f = (S; f_n, f_p)$ is not an $(\in, \in \vee q)$-BFI of S.

Proposition 3.2.7 *Every $(\in, \in \vee q)$-BFII is an $(\in, \in \vee q)$-BFI in regular ordered semigroup.*

Proof. Let $f = (S; f_n, f_p)$ is an $(\in, \in \vee$q)-BFII and $a, b \in S$. Then $a \leq a\xi a$ for some $\xi \in S$, so

$$\begin{aligned} f_n(ab) &\leq f_n((a\xi a)b) \vee -0.5 = f_n((a\xi)ab) \vee -0.5 \\ &\leq (f_n(a) \vee -0.5) \vee -0.5 = f_n(a) \vee -0.5 \end{aligned}$$

and

$$\begin{aligned} f_p(ab) &\geq f_p((a\xi a)b) \wedge 0.5 = f_p((a\xi)ab) \wedge 0.5 \\ &\geq (f_p(a) \wedge 0.5) \wedge 0.5 = f_p(a) \wedge 0.5. \end{aligned}$$

Similarly, we can show that $f_n(ab) \leq f_n(b) \vee -0.5$ and $f_p(ab) \geq f_p(b) \wedge 0.5$. Thus $f = (S; f_n, f_p)$ is an $(\in, \in \vee$q)-BFI of S. ■

The following corollary is obtained from Proposition 3.2.4 and 3.2.6.

Corollary 3.2.8 *The concept of* $(\in,\in\vee q)$*-BFI and* $(\in,\in\vee q)$*-BFII coincide in regular ordered semigroup.*

Proposition 3.2.9 *Every* $(\in,\in\vee q)$*-BFII is an* $(\in,\in\vee q)$*-BFI in intra-regular ordered semigroup.*

Proof. Let $f=(S;f_n,f_p)$ is an $(\in,\in\vee$q)-BFII and $a,b\in S$. Then $a\leq\xi_1a^2\xi_2$ for some $\xi_1,\xi_2\in S$. So

$$\begin{aligned}f_n(ab)&\leq f_n((\xi_1a^2\xi_2)b)\vee-0.5=f_n((\xi_1a)a(\xi_2b))\vee-0.5\\&\leq(f_n(a)\vee-0.5)\vee-0.5=f_n(a)\vee-0.5\end{aligned}$$

and

$$\begin{aligned}f_p(ab)&\geq f_p((\xi_1a^2\xi_2)b)\wedge0.5=f_p((\xi_1a)a(\xi_2b))\wedge0.5\\&\geq(f_p(a)\wedge0.5)\wedge0.5=f_p(a)\wedge0.5.\end{aligned}$$

Similarly, we can show that $f_n(ab)\leq f_n(b)\vee-0.5$ and $f_p(ab)\geq f_p(b)\wedge0.5$ for every $a,b\in S$. Thus $f=(S;f_n,f_p)$ is an $(\in,\in\vee$q)-BFI of S. ■

The following corollary is obtained from Proposition 3.2.4 and 3.2.8.

Corollary 3.2.10 *The concept of* $(\in,\in\vee q)$*-BFI and* $(\in,\in\vee q)$*-BFII coincide in intra-regular ordered semigroup.*

Proposition 3.2.11 *Every* $(\in,\in\vee q)$*-BFII is an* $(\in,\in\vee q)$*-BFI in semisimple ordered semigroup.*

Proof. Let $f=(S;f_n,f_p)$ is an $(\in,\in\vee$q)-BFII and $a,b\in S$. Then $a\leq\xi_1a\xi_2a\xi_3$ for some $\xi_1,\xi_2,\xi_3\in S$. So

$$\begin{aligned}f_n(ab)&\leq f_n((\xi_1a\xi_2a\xi_3)b)\vee-0.5=f_n((\xi_1a\xi_2)a(\xi_3b))\vee-0.5\\&\leq(f_n(a)\vee-0.5)\vee-0.5=f_n(a)\vee-0.5\end{aligned}$$

and

$$f_p(ab) \geq f_p((\xi_1 a \xi_2 a \xi_3)b) \wedge 0.5 = f_p((\xi_1 a \xi_2)a(\xi_3 b)) \wedge 0.5$$
$$\geq (f_p(a) \wedge 0.5) \wedge 0.5 = f_p(a) \wedge 0.5.$$

Similarly $f_n(ab) \leq f_n(b) \vee -0.5$ and $f_p(ab) \geq f_p(b) \wedge 0.5$ for every $a, b \in S$. Thus $f = (S; f_n, f_p)$ is an $(\in, \in \vee \mathrm{q})$-BFI of S. ■

The following corollary is obtained from Proposition 3.2.4 and 3.2.10.

Corollary 3.2.12 *The concept of* $(\in,\in \vee q)$*-BFI and* $(\in,\in \vee q)$*-BFII coincide in semisimple ordered semigroup.*

Let $f = (S; f_n, f_p)$ is BFS of S, then $I_a \subseteq S$ and its definition is $I_a = \{b \in S \mid f_n(b) \leq \max\{f_n(a), -0.5\}$ and $f_p(b) \geq \min\{f_p(a), 0.5\}\}$.

Proposition 3.2.13 *Let* $f = (S; f_n, f_p)$ *is an* $(\in, \in \vee q)$*-BFRI of* S*. Then for every* $a \in S$, I_a *is right ideal of* S*.*

Proof. Let $f = (S; f_n, f_p)$ is an $(\in, \in \vee \mathrm{q})$-BFRI of S and $a \in S$. Then $I_a \neq \emptyset$, because $a \in I_a$. Let $b \in I_a$ and $\xi \in S$. As $f = (S; f_n, f_p)$ is an $(\in, \in \vee \mathrm{q})$-BFRI of S, we have $f_n(b\xi) \leq \max\{f_n(b), -0.5\}$ and $f_p(b\xi) \geq \min\{f_p(b), 0.5\}$. As $b \in I_a$, we have $f_n(b) \leq \max\{f_n(a), -0.5\}$ and $f_p(b) \geq \min\{f_p(a), 0.5\}$. Thus $f_n(b\xi) \leq \max\{f_n(b), -0.5\} \leq \max\{(f_n(a), -0.5), -0.5\} = \max\{f_n(a), -0.5\}$ and $f_p(b\xi) \geq \min\{f_p(b), 0.5\} \geq \min\{(f_p(a), 0.5), 0.5\} = \min\{f_p(a), 0.5\}$. Hence $b\xi \in I_a$.

Let $\lambda \in I_a$ and $\xi \leq \lambda$. As f is an $(\in, \in \vee \mathrm{q})$-BFRI of S, we have $f_n(\xi) \leq \max\{f_n(\lambda), -0.5\}$ and $f_p(\xi) \geq \min\{f_p(\lambda), 0.5\}$. Thus $f_n(\xi) \leq \max\{f_n(\lambda), -0.5\} \leq \max\{(f_n(a), -0.5), -0.5\} = \max\{f_n(a), -0.5\}$ and $f_p(\xi) \geq \min\{f_p(\lambda), 0.5\} \geq \min\{(f_p(a), 0.5), 0.5\} = \min\{f_p(a), 0.5\}$. Therefore $\xi \in I_a$. Hence I_a is right ideal of S. ■

Proposition 3.2.14 *Let* $f = (S; f_n, f_p)$ *is an* $(\in, \in \vee q)$*-BFLI of* S*. Then* I_a *is a left ideal of* S *for every* $a \in S$*.*

Proof. Let $f = (S; f_n, f_p)$ is an $(\in, \in \vee \mathrm{q})$-BFLI of S and $a \in S$. Then $I_a \neq \emptyset$, because $a \in I_a$. Let $b \in I_a$ and $\xi \in S$. As $f = (S; f_n, f_p)$ is an $(\in, \in \vee \mathrm{q})$-BFLI of S, we have $f_n(\xi b) \leq$

$\max\{f_n(b), -0.5\}$ and $f_p(\xi b) \geq \min\{f_p(b), 0.5\}$. As $b \in I_a$, we have $f_n(b) \leq \max\{f_n(a), -0.5\}$ and $f_p(b) \geq \min\{f_p(a), 0.5\}$. Thus $f_n(\xi b) \leq \max\{f_n(b), -0.5\} \leq \max\{(f_n(a), -0.5), -0.5\} = \max\{f_n(a), -0.5\}$ and $f_p(\xi b) \geq \min\{f_p(b), 0.5\} \geq \min\{(f_p(a), 0.5), 0.5\} = \min\{f_p(a), 0.5\}$. Hence $\xi b \in I_a$.

Let $\lambda \in I_a$ and $\xi \leq \lambda$. As f is an $(\in, \in \vee$q)-BFLI of S, we have $f_n(\xi) \leq \max\{f_n(\lambda), -0.5\}$ and $f_p(\xi) \geq \min\{f_p(\lambda), 0.5\}$. Thus $f_n(\xi) \leq \max\{f_n(\xi_2), -0.5\} \leq \max\{(f_n(a), -0.5), -0.5\} = \max\{f_n(a), -0.5\}$ and $f_p(\xi) \geq \min\{f_p(\lambda), 0.5\} \geq \min\{(f_p(a), 0.5), 0.5\} = \min\{f_p(a), 0.5\}$. Therefore $\xi \in I_a$. Hence I_a is left ideal of S for every $a \in S$. ■

From Proposition 3.2.12 and 3.2.13, we have:

Proposition 3.2.15 *Let $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFI of S. Then I_a is an ideal of S for every $a \in S$.*

3.3 Lower and Upper Part of $(\in, \in \vee$q)-Bipolar Fuzzy Interior Ideals

In this section, we characterize S in framework of lower (upper) of $(\in, \in \vee$q)-BFII.

Definition 3.3.1 *Let $f = (S; f_n, f_p)$ be a BFS in S. We define the upper part $f^+ = (S; f_n^+, f_p^+)$ of $f = (S; f_n, f_p)$ as follows: $f_n^+(\xi) = f_n(\xi) \wedge -0.5$ and $f_p^+(\xi) = f_p(\xi) \vee 0.5$ for all $\xi \in S$. Similarly we define the lower part $f^- = (S; f_n^-, f_p^-)$ of $f = (S; f_n, f_p)$ as follows: $f_n^-(\xi) = f_n(\xi) \vee -0.5$ and $f_p^-(\xi) = f_p(\xi) \wedge 0.5$ for all $\xi \in S$.*

Definition 3.3.2 *The upper part $\chi_A^+ = (S; \chi_{nA}^+, \chi_{pA}^+)$ of the characteristic function $\chi_A = (S; \chi_{nA}, \chi_{pA})$ of $A \neq \emptyset$ is define by $\chi_{nA}^+(\xi) = \begin{cases} -1 \text{ if } \xi \in A, \\ -0.5 \text{ if } \xi \notin A \end{cases}$ and $\chi_{pA}^+(\xi) = \begin{cases} 1 \text{ if } \xi \in A, \\ 0.5 \text{ if } \xi \notin A \end{cases}$. Similarly we can define the lower part $\chi_A^- = (S; \chi_{nA}^-, \chi_{pA}^-)$ of the characteristic function $\chi_A = (S; \chi_{nA}, \chi_{pA})$ of A by $\chi_{nA}^-(\xi) = \begin{cases} -0.5 \text{ if } \xi \in A, \\ 0 \text{ if } \xi \notin A \end{cases}$ and $\chi_{pA}^-(\xi) = \begin{cases} 0.5 \text{ if } \xi \in A, \\ 0 \text{ if } \xi \notin A \end{cases}$.*

Lemma 3.3.3 *[84] Let A, B are nonempty subsets of S. Then*

$$(1)\ (\chi_A \wedge \chi_B)^- = \chi^-_{A\cap B}.$$
$$(2)\ (\chi_A \vee \chi_B)^- = \chi^-_{A\cup B}.$$
$$(3)\ (\chi_A \circ \chi_B)^- = \chi^-_{(AB]}.$$

Lemma 3.3.4 *The lower part $\chi^-_{\mathcal{I}}$ of $\chi_{\mathcal{I}}$ is an $(\in, \in \vee q)$-BFII of S iff $\mathcal{I}$ is an interior ideal of S.*

Proof. The proof follows from Lemma 3.2.2. ■

Lemma 3.3.5 *The lower part χ^-_I of χ_I is an $(\in, \in \vee q)$-BFI of S iff I is an ideal of S.*

Proposition 3.3.6 *Let $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFII of S, then $f^- = (S; f_n^-, f_p^-)$ is BFII of S.*

Proof. Let $f = (S; f_n, f_p)$ is an $(\in, \in \vee$q)-BFII of S. Then for all $\xi_1, \xi_2 \in S$, we get $f_n(\xi_1\xi_2) \leq (f_n(\xi_1) \vee f_n(\xi_2) \vee -0.5)$ and $f_p(\xi_1\xi_2) \geq (f_p(\xi_1) \wedge f_p(\xi_2) \wedge 0.5)$. So $f_n(\xi_1\xi_2) \vee -0.5 \leq (f_n(\xi_1) \vee f_n(\xi_2) \vee -0.5) \vee -0.5 = (f_n(\xi_1) \vee -0.5) \vee (f_n(\xi_2) \vee -0.5)$ and $f_p(\xi_1\xi_2) \wedge 0.5 \leq (f_p(\xi_1) \wedge f_p(\xi_2) \wedge 0.5) \wedge 0.5 = (f_p(\xi_1) \wedge 0.5) \wedge (f_p(\xi_2) \wedge 0.5)$. Hence $f_n^-(\xi_1\xi_2) \leq f_n^-(\xi_1) \vee f_n^-(\xi_2)$ and $f_p^-(\xi_1\xi_2) \geq f_p^-(\xi_1) \wedge f_p^-(\xi_2)$. Since $f_n(\xi_1 a \xi_2) \leq f_n(a) \vee -0.5$ and $f_p(\xi_1 a \xi_2) \geq f_p(a) \wedge 0.5$. So $f_n(\xi_1 a \xi_2) \vee -0.5 \leq (f_n(a) \vee -0.5) \vee -0.5 = f_n(a) \vee -0.5$ and $f_p(\xi_1 a \xi_2) \wedge 0.5 \geq (f_p(a) \wedge 0.5) \wedge 0.5 = f_p(a) \wedge 0.5$. Hence $f_n^-(\xi_1 a \xi_2) \leq f_n^-(a)$ and $f_p^-(\xi_1 a \xi_2) \geq f_p^-(a)$. Let $\xi_1, \xi_2 \in S$ with $\xi_1 \leq \xi_2$. Then $f_n(\xi_1) \leq f_n(\xi_2) \vee -0.5$ and $f_n(\xi_1) \geq f_p(\xi_2) \wedge 0.5$. Thus $f_n(\xi_1) \vee -0.5 \leq (f_n(\xi_2) \vee -0.5) \vee -0.5 = f_n(\xi_2) \vee -0.5$ and $f_n(\xi_1) \wedge 0.5 \geq (f_p(\xi_2) \wedge 0.5) \wedge 0.5 = f_p(\xi_2) \wedge 0.5$. This implies that $f_n^-(\xi_1) \leq f_n^-(\xi_2)$ and $f_p^-(\xi_1) \geq f_p^-(\xi_2)$. Hence $f^- = (S; f_n^-, f_p^-)$ is a BFII of S. ■

Definition 3.3.7 *If every $(\in, \in \vee q)$-BFI in S is constant, that is for every $\xi_1, \xi_2 \in S$, we get $f^-(\xi_1) = f^-(\xi_2)$ then such ordered semigroup is called $(\in, \in \vee q)$-bipolar fuzzy simple.*

Theorem 3.3.8 *An ordered semigroup S is simple iff it is an $(\in, \in \vee q)$-bipolar fuzzy simple.*

Proof. Suppose $f = (S; f_n, f_p)$ is an $(\in, \in \vee \text{q})$-BFI of S and $\xi_1, \xi_2 \in S$. Then I_{ξ_1} is an ideal of S by Proposition 3.2.14. As we are given that S is simple, so $I_{\xi_1} = S$ thus $\xi_2 \in I_{\xi_1}$. Hence $f_n(\xi_2) \leq f_n(\xi_1) \vee -0.5$ and $f_p(\xi_2) \geq f_p(\xi_1) \wedge 0.5$, so $f_n^-(\xi_2) = f_n(\xi_2) \vee -0.5 \leq f_n(\xi_1) \vee -0.5 = f_n^-(\xi_1)$ and $f_p^-(\xi_2) = f_p(\xi_2) \wedge 0.5 \geq f_p(\xi_1) \wedge 0.5 = f_p^-(\xi_1)$. Thus $f^-(\xi_2) \geq f^-(\xi_1)$. Similarly we get $f^-(\xi_2) \leq f^-(\xi_1)$. Hence $f^-(\xi_1) = f^-(\xi_2)$ and S is an $(\in, \in \vee \text{q})$-bipolar fuzzy simple.

Conversely, assume that S has a proper ideal I. Then χ_I^- is an $(\in, \in \vee \text{q})$-BFI of S by Lemma 3.3.5. Let $a \in S$ then χ_I^- is constant function because S is an $(\in, \in \vee \text{q})$-bipolar fuzzy simple, So that is, $\chi_I^-(a) = \chi_I^-(b)$ for every $b \in S$. Thus, for any $\xi_1 \in I$, we have $\chi_{nI}^-(a) = \chi_{nI}^-(\xi_1) = -0.5$ and $\chi_{pI}^-(a) = \chi_{pI}^-(\xi_1) = 0.5$ and so $a \in I$. We get $S = I$, which is a contradiction. Hence, S is simple. ■

Lemma 3.3.9 *[82] For every $\xi \in S$, we have $S = (S\xi S]$ iff S is simple ordered semigroup.*

Theorem 3.3.10 *An ordered semigroup S is simple iff for every $(\in, \in \vee q)$-BFII of S, we have $f^-(\xi_1) = f^-(\xi_2)$ for every $\xi_1, \xi_2 \in S$.*

Proof. Let S is simple and $f = (S; f_n, f_p)$ is an $(\in, \in \vee \text{q})$-BFII of S. Now by Lemma 3.3.9, $S = (S\xi_2 S]$ because $\xi_2 \in S$. Also as $\xi_1 \in S$, so $\xi_1 \in (S\xi_2 S]$. Therefore $\xi_1 \leq a\xi_2 b$ for some $a, b \in S$. As we are given that $f = (S; f_n, f_p)$ is an $(\in, \in \vee \text{q})$-BFII of S, so we have $f_n(\xi_1) \leq f_n(a\xi_2 b) \vee -0.5 \leq f_n(\xi_2) \vee -0.5$ and $f_p(\xi_1) \geq f_p(a\xi_2 b) \wedge 0.5 \geq f_p(\xi_2) \wedge 0.5$. Hence $f_n^-(\xi_1) = f_n(\xi_1) \vee -0.5 \leq f_n(\xi_2) \vee -0.5 = f_n^-(\xi_2)$ and $f_p^-(\xi_1) = f_p(\xi_1) \wedge 0.5 \geq f_p(\xi_2) \wedge 0.5 = f_p^-(\xi_2)$. Thus $f^-(\xi_2) \leq f^-(\xi_1)$. In similar way, we can easily show that $f^-(\xi_2) \geq f^-(\xi_1)$. Therefore $f^-(\xi_1) = f^-(\xi_2)$ for every $\xi_1, \xi_2 \in S$.

Conversely, assume that $f = (S; f_n, f_p)$ is an $(\in, \in \vee \text{q})$-BFII of S, then by Proposition 3.2.10, $f = (S; f_n, f_p)$ is an $(\in, \in \vee \text{q})$-BFI of S. As we are given that, $f^-(\xi_1) = f^-(\xi_2)$ for every $\xi_1, \xi_2 \in S$. So by definition, S is an $(\in, \in \vee \text{q})$-bipolar fuzzy simple and hence S is simple. ■

Theorem 3.3.11 *For every $(\in, \in \vee q)$-BFI $f = (S; f_n, f_p)$ of S, we get $f^-(a) = f^-(a^2)$ for all $a \in S$ iff S be an intra-regular ordered semigroup.*

Proof. Let $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFI of S and let $a \in S$. Then there exist $\xi_1, \xi_2 \in S$ such that $a \leq \xi_1 a^2 \xi_2$. Therefore

$$\begin{aligned} f_n(a) &\leq (f_n(\xi_1 a^2 \xi_2) \vee -0.5) = f_n(\xi_1(a^2\xi_2)) \vee -0.5 \\ &\leq (f_n(a^2\xi_2) \vee -0.5) \vee -0.5 = f_n(a^2) \vee -0.5. \end{aligned}$$

So $f_n^-(a) = f_n(a) \vee -0.5 \leq (f_n(a^2) \vee -0.5) \vee -0.5 = f_n(a^2) \vee -0.5 = f_n^-(a^2)$ and $f_n^-(a^2) = (f_n(a^2) \vee -0.5) \leq (f_n(a) \vee -0.5) \vee -0.5 = f_n(a) \vee -0.5 = f_n^-(a)$. Thus $f_n^-(a) = f_n^-(a^2)$. Also

$$\begin{aligned} f_p(a) &\geq (f_p(\xi_1 a^2 \xi_2) \wedge 0.5) = f_p(\xi_1(a^2\xi_2)) \wedge 0.5 \\ &\geq (f_p(a^2\xi_2) \wedge 0.5) \wedge 0.5 = f_p(a^2) \wedge 0.5. \end{aligned}$$

So $f_p^-(a) = f_n(a) \wedge 0.5 \geq (f_p(a^2) \wedge 0.5) \wedge 0.5 = f_p(a^2) \wedge 0.5 = f_p^-(a^2)$ and $f_p^-(a^2) = (f_p(a^2) \wedge 0.5) \geq (f_p(a) \wedge 0.5) \wedge 0.5 = f_p(a) \wedge 0.5 = f_p^-(a)$. Thus $f_p^-(a) = f_p^-(a^2)$. Hence $f^-(a) = f^-(a^2)$.

Conversely, assume an ideal generated by ξ^2, that is $I(\xi^2) = (\xi^2 \cup S\xi^2 \cup \xi^2 S \cup S\xi^2 S]$. Then $\chi^-_{I(\xi^2)}$ is an $(\in, \in \vee q)$-BFI of S, by Lemma 3.3.5. Also by given information, we have $\chi^-_{I(\xi^2)}(\xi) = \chi^-_{I(\xi^2)}(\xi^2)$. Therefore we have $\xi \in I(\xi^2) = (\xi^2 \cup S\xi^2 \cup \xi^2 S \cup S\xi^2 S]$. This implies that if $a, b \in S$ then we have $\xi \leq \xi^2$ or $\xi \leq a\xi^2$ or $\xi \leq \xi^2 a$ or $\xi \leq a\xi^2 b$. In all cases we conclude that $\xi \in (S\xi^2 S]$. Thus S is an intra-regular ordered semigroup. ■

Theorem 3.3.12 *For an intra-regular ordered semigroup S, we have $f^-(\xi_1\xi_2) = f^-(\xi_2\xi_1)$ for every $\xi_1, \xi_2 \in S$ where $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFI of S.*

Proof. Let $\xi_1, \xi_2 \in S$. Then we have $f^-(\xi_1\xi_2) = f^-((\xi_1\xi_2)^2) = f^-((\xi_1\xi_2)(\xi_1\xi_2)) = f^-(\xi_1(\xi_2\xi_1)\xi_2) \geq f^-(\xi_2\xi_1)$ by Theorem 3.3.11. By the symmetry, we have $f^-(\xi_2\xi_1) \geq f^-(\xi_1\xi_2)$. Thus $f^-(\xi_1\xi_2) = f^-(\xi_2\xi_1)$ for every $\xi_1, \xi_2 \in S$. ■

Theorem 3.3.13 *If S is semisimple ordered semigroup and $f = (S; f_n, f_p)$ and $g = (S; g_n, g_p)$ are $(\in, \in \vee q)$-BFIIs of S. Then $(f \circ g)^- \leq (f \cap g)^-$.*

Proof. If $I_a = \emptyset$, then $(f_n \circ g_n)^-(a) = (f_n \circ g_n)(a) \vee -0.5 = 0 \vee -0.5 = 0 \geq (f_n \vee g_n)(a) \vee -0.5 = (f_n \vee g_n)^-(a)$ and $(f_p \circ g_p)^-(a) = (f_p \circ g_p)(a) \wedge 0.5 = 0 \wedge 0.5 = 0 \leq (f_p \wedge g_p)(a) \wedge 0.5 = (f_p \wedge g_p)^-(a)$. Thus $(f \circ g)^- \leq (f \cap g)^-$.

Let $I_a \neq \emptyset$, then

$$\begin{aligned}(f_n \circ g_n)^-(a) &= (f_n \circ g_n)(a) \vee -0.5 \\ &= \left[\bigwedge_{(y,z)\in A_a} (f_n(y) \vee g_n(z))\right] \vee -0.5 = \bigwedge_{(y,z)\in A_a} (f_n(y) \vee g_n(z) \vee -0.5) \\ &= \bigwedge_{(y,z)\in A_a} ((f_n(y) \vee -0.5) \vee (g_n(z) \vee -0.5) \vee -0.5).\end{aligned}$$

Since S is semisimple and $f = (S; f_n, f_p)$ and $g = (S; g_n, g_p)$ are $(\in, \in \vee\text{q})$-BFIIs of S, so by Proposition 3.2.10, we have f and g are $(\in, \in \vee\text{q})$-BFIs of S. Since $a \leq yz$, we have $f_n(a) \leq f_n(yz) \vee -0.5 \leq f_n(y) \vee -0.5$ and $g_n(a) \leq g_n(yz) \vee -0.5 \leq g_n(z) \vee -0.5$. Thus

$$\begin{aligned}(f_n \circ g_n)^-(a) &= \bigwedge_{(y,z)\in A_a} ((f_n(y) \vee -0.5) \vee (g_n(z) \vee -0.5) \vee -0.5) \\ &\geq (f_n(a) \vee g_n(a) \vee -0.5) = (f_n \vee g_n)(a) \vee -0.5 \\ &= (f_n \vee g_n)^-(a).\end{aligned}$$

In similar way, we can easily show that $(f_p \circ g_p)^-(a) \leq (f_p \wedge g_p)^-(a)$. Thus we conclude that $(f \circ g)^- \leq (f \cap g)^-$. ■

Theorem 3.3.14 *An ordered semigroup S is semisimple iff for every $(\in, \in \vee q)$-BFIIs $f = (S; f_n, f_p)$ and $g = (S; g_n, g_p)$ of S, we have $(f \cap g)^- = (f \circ g)^-$.*

Proof. Let $(f \cap g)^- = (f \circ g)^-$ for every $(\in, \in \vee\text{q})$-BFIIs $f = (S; f_n, f_p)$ and $g = (S; g_n, g_p)$ of S. Let us assume that $\mathcal{I}$ be an interior ideal of S. So $\chi_{\mathcal{I}}^-$ is an $(\in, \in \vee\text{q})$-BFII of S, by Lemma 3.3.8. We have $\chi_{\mathcal{I}}^- = \chi_{\mathcal{I}}^- \cap \chi_{\mathcal{I}}^- = \chi_{\mathcal{I}}^- \circ \chi_{\mathcal{I}}^- = \chi_{(\mathcal{I}^2]}^-$ using Lemma 3.3.3 (3). Thus $\mathcal{I} = (\mathcal{I}^2]$ and S is semisimple.

Conversely, let S is semisimple and $f = (S; f_n, f_p)$ and $g = (S; g_n, g_p)$ are $(\in, \in \vee\text{q})$-BFIIs of S. So for each $a \in S$, then there exist $\xi_1, \xi_2, \xi_3 \in S$ such that $a \leq \xi_1 a \xi_2 a \xi_3 \leq (\xi_1 a \xi_2))(\xi_1 a \xi_2 a \xi_3^2)$.

Thus $(\xi_1 a \xi_2, \xi_1 a \xi_2 a \xi_3^2) \in \mathcal{I}_a$ and $\mathcal{I}_a \neq \emptyset$. So

$$\begin{aligned}(f_n \circ g_n)^-(a) = (f_n \circ g_n)(a) \vee -0.5 &= \bigwedge_{(m,n)\in\mathcal{I}_a} (f_n(m) \vee g_n(n) \vee -0.5) \\ &\leq f_n(\xi_1 a \xi_2) \vee g_n((\xi_1 a \xi_2) a (\xi_3)^2) \vee -0.5 \leq (f_n(a) \vee -0.5) \vee (g_n(a) \vee -0.5) \vee -0.5 \\ &\leq (f_n \vee g_n)(a) \vee -0.5 = (f_n \vee g_n)^-(a).\end{aligned}$$

Also

$$\begin{aligned}(f_p \circ g_p)^-(a) = (f_p \circ g_p)(a) \wedge 0.5 &= \bigvee_{(m,n)\in\mathcal{I}_a} (f_p(m) \wedge g_p(n) \wedge 0.5) \\ &\geq f_p(\xi_1 a \xi_2) \wedge g_p((\xi_1 a \xi_2) a (\xi_3)^2) \wedge 0.5 \geq (f_p(a) \wedge 0.5) \wedge (g_p(a) \wedge 0.5) \wedge 0.5 \\ &\geq (f_p \wedge g_p)(a) \wedge 0.5 = (f_p \wedge g_p)^-(a).\end{aligned}$$

Thus $(f \cap g)^- \leq (f \circ g)^-$. Also by Theorem 3.3.13, we get $(f \cap g)^- \geq (f \circ g)^-$ and so $(f \cap g)^- = (f \circ g)^-$. ∎

Chapter 4

Generalized bipolar fuzzy generalized bi-ideal and generalized bipolar fuzzy bi-ideal in ordered semigroups

This chapter is composed of our published work [84, 87]. Davvaz and Khan [64] characterized regular ordered semigroup by the properties of (α, β)-fuzzy generalized bi-ideals. We extended their work to the notion of BFS. This chapter is divided into three section. In first section, the characterizations of regular ordered semigroups in framework of (α, β)-BFGBI are introduced. It is shown that characteristic function $\chi_{\mathcal{G}} = (S, \chi_{n\mathcal{G}}, \chi_{p\mathcal{G}})$ of $\mathcal{G}$ is an $(\in, \in \vee q)$-BFGBI of S iff $\mathcal{G}$ is a generalized bi-ideal of S. Suitable example is given in order to clear the concept that every $(\in, \in \vee q)$-BFGBI is not a $(\in, \in \vee q)$-BFBI. It is also proved that in regular ordered semigroup, the concepts of $(\in, \in \vee q)$-BFGBI and $(\in, \in \vee q)$-BFBI coincide.

The second section is devoted to the study of upper/lower part of $(\in, \in \vee q)$-BFGBI of S and many characterization theorems of regular ordered semigroups are discussed.

In last section, the notion of (α, β)-BFBIs of S are introduced. We mainly focus on $(\in, \in \vee q)$-BFBIs and obtained some interesting characterization theorems of generalized BFBI.

4.1 (α, β)-Bipolar Fuzzy Generalized Bi-Ideals

In this section, we characterize regular ordered semigroup in terms of (α, β)-BFGBI. We show that every characteristic function $\chi_{\mathcal{G}} = (\mathrm{S}, \chi_{n\mathcal{G}}, \chi_{p\mathcal{G}})$ of $\mathcal{G}$ is an $(\in, \in \vee \mathrm{q})$-BFGBI of S iff $\mathcal{G}$ is a generalized bi-ideal of S. Here an example is given in order to show that every generalized BFGBI may not be generalized BFBI.

Definition 4.1.1 *A BFS $f = (S; f_n, f_p)$ in S is said to be (α, β)-BFGBI of S, where $\alpha \neq \in \wedge q$, if it satisfies the following conditions. For $(s,t) \in [-1, 0) \times (0, 1]$*

(1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2, \frac{\xi_2}{(s,t)} \alpha f \Longrightarrow \frac{\xi_1}{(s,t)} \beta f)$,

(2) $(\forall \xi_1, \xi_2, \xi_3 \in S)(\frac{\xi_1}{(s_1,t_1)} \alpha f$ and $\frac{\xi_3}{(s_2,t_2)} \alpha f \Longrightarrow \frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\}, \wedge\{t_1,t_2\})} \beta f)$.

Theorem 4.1.2 *If $f = (S; f_n, f_p)$ is nonzero (α, β)-BFGBI of S, then the set $S_\circ = \{\xi \in S \mid f_n(\xi) \neq 0\} \cap \{\xi \in S \mid f_p(\xi) \neq 0\}$ is a generalized bi-ideal of S.*

Proof. Let $\xi_1, \xi_2 \in S$, $\xi_1 \leq \xi_2$ and $\xi_2 \in S_\circ$. Then $f_n(\xi_2) \neq 0 \neq f_p(\xi_2)$. Assume that $f_n(\xi_1) = 0 = f_p(\xi_1)$. If $\alpha \in \{\in, \in \vee \mathrm{q}\}$, then $\frac{\xi_2}{(f_n(\xi_2), f_p(\xi_2))} \alpha f$, but $\frac{\xi_1}{(f_n(\xi_2), f_p(\xi_2))} \overline{\beta} f$ for every $\beta \in \{\in, \mathrm{q}, \in \vee \mathrm{q}, \in \wedge \mathrm{q}\}$. This is impossible. Hence $f_n(\xi_1) \neq 0 \neq f_p(\xi_1)$ and $\xi_1 \in S_\circ$. Let $\xi_1, \xi_3 \in S_\circ$. Then $f_n(\xi_1) \neq 0 \neq f_p(\xi_1)$ and $f_n(\xi_3) \neq 0 \neq f_p(\xi_3)$. Assume that $f_n(\xi_1\xi_2\xi_3) = 0 = f_p(\xi_1\xi_2\xi_3)$. If $\alpha \in \{\in, \in \vee \mathrm{q}\}$, then $\frac{\xi_1}{(f_n(\xi_1), f_p(\xi_1))} \alpha f$ and $\frac{\xi_3}{(f_n(\xi_3), f_p(\xi_3))} \alpha f$, but $\frac{\xi_1\xi_2\xi_3}{(f_n(\xi_1) \vee f_n(\xi_3)), (f_p(\xi_1) \wedge f_p(\xi_3))} \overline{\beta} f$ for every $\beta \in \{\in, \mathrm{q}, \in \vee \mathrm{q}, \in \wedge \mathrm{q}\}$. This is impossible. Hence $f_n(\xi_1\xi_2\xi_3) \neq 0 \neq f_p(\xi_1\xi_2\xi_3)$ and $\xi_1\xi_2\xi_3 \in S_\circ$. ■

The central role in (α, β)-BFGBIs is played by $(\in, \in \vee \mathrm{q})$-BFGBIs.

Example 4.1.3 *Let $S = \{0, 1, 2, 3, 4, 5\}$ be an ordered semigroup defined by the following multiplication table:*

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	0	3	4	5
2	2	2	2	2	2	2
3	0	3	0	1	4	5
4	0	4	0	4	4	4
5	0	5	0	5	4	5

$\leq = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5),(5,4)\}$. *Define a BFS as follows:*

S	0	1	2	3	4	5
f_n	-0.9	-0.2	-0.8	-0.3	-0.4	-0.6
f_p	0.8	0.3	0.7	0.4	0.4	0.5

It is easy to prove that $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFGBI of S. But

(i) f is not an $(\in, \in)$-BFGBI of S, since $\frac{3}{(-0.28,0.38)} \in f$ but $\frac{3\cdot 1\cdot 3}{(-0.28\vee -0.28, 0.38\wedge 0.38)} = \frac{1}{(-0.28,0.38)} \overline{\in} f$.

(ii) f is not an $(\mathrm{q}, \in \vee \mathrm{q})$-BFGBI of S, since $\frac{5}{(-0.5,0.6)} \mathrm{q}\, f$ but $\frac{5\cdot 4\cdot 5}{(-0.5\vee -0.5, 0.6\wedge 0.6)} = \frac{4}{(-0.5,0.6)} \overline{\in \vee \mathrm{q}}\, f$.

(iii) f is not an $(\in \vee \mathrm{q}, \in \vee \mathrm{q})$-BFGBI of S, since $\frac{3}{(-0.28,0.38)} \in \vee \mathrm{q}\, f$ but $\frac{3\cdot 1\cdot 3}{(-0.28\vee -0.28, 0.38\wedge 0.38)} = \frac{1}{(-0.28,0.38)} \overline{\in \vee \mathrm{q}}\, f$.

Lemma 4.1.4 *The characteristic function $\chi_{\mathcal{G}} = (S; \chi_{n\mathcal{G}}, \chi_{p\mathcal{G}})$ of $\mathcal{G}$ is an $(\in, \in \vee q)$-BFGBI of S iff $\mathcal{G}$ is a generalized bi-ideal of S.*

Proof. Let $\chi_{\mathcal{G}} = (S; \chi_{n\mathcal{G}}, \chi_{p\mathcal{G}})$ is an $(\in, \in \vee \mathrm{q})$-BFGBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in \mathcal{G}$. Then $\chi_{n\mathcal{G}}(\xi_2) = -1$ and $\chi_{p\mathcal{G}}(\xi_2) = 1$, thus $\frac{\xi_2}{(-1,1)} \in \chi_{\mathcal{G}}$. Since $\chi_{\mathcal{G}} = (S; \chi_{n\mathcal{G}}, \chi_{p\mathcal{G}})$ is an $(\in, \in \vee \mathrm{q})$-BFGBI of S, so $\frac{\xi_1}{(-1,1)} \in \vee \mathrm{q} \chi_{\mathcal{G}}$ i.e $\frac{\xi_1}{(-1,1)} \in \chi_{\mathcal{G}}$ or $\frac{\xi_1}{(-1,1)} \mathrm{q} \chi_{\mathcal{G}}$. From both the cases we conclude that $\chi_{n\mathcal{G}}(\xi_1) = -1$ and $\chi_{p\mathcal{G}}(\xi_1) = 1$. Hence $\xi_1 \in \mathcal{G}$.

Let $\xi_1, \xi_3 \in \mathcal{G}$, then $\chi_{n\mathcal{G}}(\xi_1) = \chi_{n\mathcal{G}}(\xi_3) = -1$ and $\chi_{p\mathcal{G}}(\xi_1) = \chi_{p\mathcal{G}}(\xi_3) = 1$. Thus $\frac{\xi_1}{(-1,1)} \in \chi_{\mathcal{G}}$ and $\frac{\xi_3}{(-1,1)} \in \chi_{\mathcal{G}}$. Since $\chi_{\mathcal{G}} = (S; \chi_{n\mathcal{G}}, \chi_{p\mathcal{G}})$ is an $(\in, \in \vee \mathrm{q})$-BFGBI of S, so $\frac{\xi_1\xi_2\xi_3}{(-1,1)} \in \vee \mathrm{q} \chi_{\mathcal{G}}$ i.e $\frac{\xi_1\xi_2\xi_3}{(-1,1)} \in \chi_{\mathcal{G}}$ or $\frac{\xi_1\xi_2\xi_3}{(-1,1)} \mathrm{q} \chi_{\mathcal{G}}$. From both the cases we conclude that $\chi_{n\mathcal{G}}(\xi_1\xi_2\xi_3) = -1$ and $\chi_{p\mathcal{G}}(\xi_1\xi_2\xi_3) = 1$. Hence $\xi_1\xi_2\xi_3 \in \mathcal{G}$ and $\mathcal{G}$ is generalized bi-ideal of S.

Conversely, assume that $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\frac{\xi_2}{(s,t)} \in \chi_{\mathcal{G}}$ for $(s,t) \in [-1,0) \times (0,1]$. Then $\chi_{n\mathcal{G}}(\xi_2) \leq s < 0$ and $\chi_{p\mathcal{G}}(\xi_2) \geq t > 0$, which implies that $\chi_{n\mathcal{G}}(\xi_2) = -1$ and $\chi_{p\mathcal{G}}(\xi_2) = 1$. Thus $\xi_2 \in \mathcal{G}$. Since $\mathcal{G}$ is a generalized bi-ideal of S, so $\xi_1 \in \mathcal{G}$. It follows that $\chi_{n\mathcal{G}}(\xi_1) = -1 \leq s$ and $\chi_{p\mathcal{G}}(\xi_1) = 1 \geq t$. This implies that $\frac{\xi_1}{(s,t)} \in \chi_{\mathcal{G}}$ and hence $\frac{\xi_1}{(s,t)} \in \vee \mathrm{q} \chi_{\mathcal{G}}$.

Let $\xi_1, \xi_2, \xi_3 \in S$ such that $\frac{\xi_1}{(s_1,t_1)} \in \chi_{\mathcal{G}}$ and $\frac{\xi_3}{(s_2,t_2)} \in \chi_{\mathcal{G}}$ for $(s_1,t_1), (s_2,t_2) \in [-1,0) \times (0,1]$. Then $\chi_{n\mathcal{G}}(\xi_1) \leq s_1 < 0, \chi_{n\mathcal{G}}(\xi_3) \leq s_2 < 0$ and $\chi_{p\mathcal{G}}(\xi_1) \geq t_1 > 0, \chi_{p\mathcal{G}}(\xi_1) \geq t_2 > 0$. This implies that $\chi_{n\mathcal{G}}(\xi_1) = \chi_{n\mathcal{G}}(\xi_3) = -1$ and $\chi_{p\mathcal{G}}(\xi_1) = \chi_{p\mathcal{G}}(\xi_3) = 1$. Thus $\xi_1, \xi_3 \in \mathcal{G}$. Since $\mathcal{G}$

is generalized bi-ideal of S, so $\xi_1\xi_2\xi_3 \in \mathcal{G}$. It follows that $\chi_{n\mathcal{G}}(\xi_1\xi_2\xi_3) = -1 \leq \vee\{s_1, s_2\}$ and $\chi_{p\mathcal{G}}(\xi_1\xi_2\xi_3) = 1 \geq \wedge\{t_1, t_2\}$. This implies that $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \chi_{\mathcal{G}}$ and so $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \vee q\chi_{\mathcal{G}}$. Hence $\chi_{\mathcal{G}} = (S; \chi_{n\mathcal{G}}, \chi_{p\mathcal{G}})$ is an $(\in, \in \vee q)$-BFGBI of S. ■

Theorem 4.1.5 *A BFS $f = (S; f_n, f_p)$ in S is an $(\in, \in \vee q)$-BFGBI of S iff it satisfies the following conditions:*

(1)$(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Longrightarrow f_n(\xi_1) \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq \bigwedge\{f_p(\xi_2), 0.5\})$,

(2)$(\forall \xi_1, \xi_2, \xi_3 \in S)(f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\}$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\})$.

Proof. Suppose that $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFGBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. We consider the following four cases,

(i) $f_n(\xi_2) > -0.5$ and $f_p(\xi_2) < 0.5$,

(ii) $f_n(\xi_2) \leq -0.5$ and $f_p(\xi_2) \geq 0.5$,

(iii) $f_n(\xi_2) \leq -0.5$ and $f_p(\xi_2) < 0.5$,

(iv) $f_n(\xi_2) > -0.5$ and $f_p(\xi_2) \geq 0.5$.

For the first case, assume that

$$f_n(\xi_1) > \bigvee\{f_n(\xi_2), -0.5\} \text{ or } f_p(\xi_1) < \bigwedge\{f_p(\xi_2), 0.5\}.$$

Then $f_n(\xi_1) > f_n(\xi_2)$ or $f_p(\xi_1) < f_p(\xi_2)$. If $f_n(\xi_1) > f_n(\xi_2)$, then there exists $s \in [-1, 0)$ such that $f_n(\xi_1) > s \geq f_n(\xi_2)$. Let $t = f_p(\xi_2)$, then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{q}\, f$, i.e. $\frac{\xi_1}{(s,t)} \overline{\in \vee q}\, f$, which is a contradiction. If $f_p(\xi_1) < f_p(\xi_2)$, then there exists $t \in (0, 1]$ such that $f_p(\xi_1) < t \leq f_p(\xi_2)$. Let $s = f_n(\xi_2)$, so $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{q}\, f$, i.e. $\frac{\xi_1}{(s,t)} \overline{\in \vee q}\, f$, a contradiction. Therefore $f_n(\xi_1) \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq \bigwedge\{f_p(\xi_2), 0.5\}$. Now consider the second case that is, $f_n(\xi_2) \leq -0.5$ and $f_p(\xi_2) \geq 0.5$. This implies that $\frac{\xi_2}{(-0.5,0.5)} \in f$. Since $\frac{\xi_2}{(-0.5,0.5)} \in f$, so $\frac{\xi_1}{(-0.5,0.5)} \in \vee q\, f$. Thus $\frac{\xi_1}{(-0.5,0.5)} \in f$ or $\frac{\xi_1}{(-0.5,0.5)} q\, f$. If $\frac{\xi_1}{(-0.5,0.5)} \in f$, then $f_n(\xi_1) \leq -0.5 \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq 0.5 \geq \bigwedge\{f_p(\xi_2), 0.5\}$. If $\frac{\xi_1}{(-0.5,0.5)} q\, f$, then $f_n(\xi_1) - 0.5 < -1$ and $f_p(\xi_1) + 0.5 > 1$. This implies that $f_n(\xi_1) < -0.5$ and $f_p(\xi_1) > 0.5$. Therefore $f_n(\xi_1) < -0.5 \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) > 0.5 \geq \bigwedge\{f_p(\xi_2), 0.5\}$. From case (iii), we have $f_n(\xi_1) \leq -0.5$ and $f_p(\xi_1) \geq f_p(\xi_2)$. Assume that $f_n(\xi_1) > -0.5$ or $f_p(\xi_1) < f_p(\xi_2)$.

If $f_n(\xi_1) > -0.5$, then there exists $s \in [-1, 0)$ such that $f_n(\xi_1) > s \geq -0.5 \geq f_n(\xi_2)$. Let $f_p(\xi_2) = t$, so $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$, i.e. $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, which is a contradiction. If $f_p(\xi_1) < f_p(\xi_2)$, then there exists $t \in (0, 1]$ such that $f_p(\xi_1) < t \leq f_p(\xi_2)$. Let $f_n(\xi_2) = s$, then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$, i.e. $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, which is a contradiction. Hence, $f_n(\xi_1) \leq -0.5 \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq f_p(\xi_2) \geq \bigwedge\{f_p(\xi_2), 0.5\}$. From case (iv), we have $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq 0.5$. Assume that $f_n(\xi_1) > f_n(\xi_2)$ or $f_p(\xi_1) < 0.5$. If $f_n(\xi_1) > f_n(\xi_2)$, then there exists $s \in [-1, 0)$ such that $f_n(\xi_1) > s \geq f_n(\xi_2)$, so $\frac{\xi_2}{(s,0.5)} \in f$ but $\frac{\xi_1}{(s,0.5)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$, i.e. $\frac{\xi_1}{(s,0.5)} \overline{\in \vee \mathrm{q}}\ f$, which is impossible. If $f_p(\xi_1) < 0.5$, then there exists $t \in (0, 1]$ such that $f_p(\xi_1) < t \leq 0.5 \leq f_p(\xi_2)$. Let $s = f_n(\xi_2)$, then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$,that is, $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, which is a contradiction. Hence, $f_n(\xi_1) \leq f_n(\xi_2) \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq 0.5 \geq \bigwedge\{f_p(\xi_2), 0.5\}$.

Let $\xi_1, \xi_2, \xi_3 \in S$ and consider the four cases.

$$
\begin{aligned}
\text{(i) } \bigvee\{f_n(\xi_1), f_n(\xi_3)\} &> -0.5 \text{ and } \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} < 0.5,\\
\text{(ii) } \bigvee\{f_n(\xi_1), f_n(\xi_3)\} &\leq -0.5 \text{ and } \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq 0.5,\\
\text{(iii) } \bigvee\{f_n(\xi_1), f_n(\xi_3)\} &\leq -0.5 \text{ and } \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} < 0.5,\\
\text{(iv) } \bigvee\{f_n(\xi_1), f_n(\xi_3)\} &> -0.5 \text{ and } \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq 0.5.
\end{aligned}
$$

Let $f_n(\xi_1\xi_2\xi_3) > \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\}$ or $f_p(\xi_1\xi_2\xi_3) < \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\}$ for case (i). Then $f_n(\xi_1\xi_2\xi_3) > \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$ or $f_p(\xi_1\xi_2\xi_3) < \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$. If $f_n(\xi_1\xi_2\xi_3) > \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$, then there exists $s \in [-1, 0)$ such that $f_n(\xi_1\xi_2\xi_3) > s \geq \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$. Let $t = \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$, so $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_3}{(s,t)} \in f$ but $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\mathrm{q}}\ f$, i.e. $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, which is a contradiction. If $f_p(\xi_1\xi_2\xi_3) < \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$ then there exists $t \in (0, 1]$ such that $f_p(\xi_1\xi_2\xi_3) < t \leq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$. Let $s = \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$, then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_3}{(s,t)} \in f$ but $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\mathrm{q}}\ f$,that is, $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, which is impossible. Hence $f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\}$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\}$. Now for case (ii), we have $\bigvee\{f_n(\xi_1), f_n(\xi_3)\} \leq -0.5$ and $\bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq 0.5$. This implies that $\frac{\xi_1}{(-0.5,0.5)} \in f$ and $\frac{\xi_3}{(-0.5,0.5)} \in f$ and so $\frac{\xi_1\xi_2\xi_3}{(-0.5,0.5)} \in \vee \mathrm{q}\ f$. Thus $\frac{\xi_1\xi_2\xi_3}{(-0.5,0.5)} \in f$ or $\frac{\xi_1\xi_2\xi_3}{(-0.5,0.5)} \mathrm{q}\ f$. If $\frac{\xi_1\xi_2\xi_3}{(-0.5,0.5)} \in f$, then $f_n(\xi_1\xi_2\xi_3) \leq -0.5 \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\}$ and $f_p(\xi_1\xi_2\xi_3) \geq 0.5 \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\}$. If $\frac{\xi_1\xi_2\xi_3}{(-0.5,0.5)} \mathrm{q}\ f$, then $f_n(\xi_1\xi_2\xi_3) - 0.5 < -1$ and $f_p(\xi_1\xi_2\xi_3) + 0.5 > 1$. This implies that

$f_n(\xi_1\xi_2\xi_3) < -0.5$ and $f_p(\xi_1\xi_2\xi_3) > 0.5$, therefore $f_n(\xi_1\xi_2\xi_3) \leq -0.5 \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\}$ and $f_p(\xi_1\xi_2\xi_3) \geq 0.5 \geq \bigwedge\{f_n(\xi_1), f_p(\xi_3), 0.5\}$. From case (iii), we have $f_n(\xi_1\xi_2\xi_3) \leq -0.5$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$. Assume that $f_n(\xi_1\xi_2\xi_3) > -0.5$ or $f_p(\xi_1\xi_2\xi_3) < \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$. If $f_n(\xi_1\xi_2\xi_3) > -0.5$, then there exists $s \in [-1, 0)$ such that $f_n(\xi_1\xi_2\xi_3) > s \geq -0.5 \geq \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$. Let $t = \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$, then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_3}{(s,t)} \in f$ but $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\mathrm{q}}\, f$,that is, $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in \vee \mathrm{q}}\, f$ which is a contradiction. If $f_p(\xi_1\xi_2\xi_3) < \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$ then there exists $t \in (0, 1]$ such that $f_p(\xi_1\xi_2\xi_3) < t \leq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$. Let $s = \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$, then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_3}{(s,t)} \in f$ but $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\mathrm{q}}\, f$,that is, $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in \vee \mathrm{q}}\, f$ which is not possible. Hence $f_n(\xi_1\xi_2\xi_3) \leq -0.5 \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\}$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\}$. From case (iv), we have $f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$ and $f_p(\xi_1\xi_2\xi_3) \geq 0.5$. Let $f_n(\xi_1\xi_2\xi_3) > \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$ or $f_p(\xi_1\xi_2\xi_3) < 0.5$. If $f_n(\xi_1\xi_2\xi_3) > \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$, then there exists $s \in [-1, 0)$ such that $f_n(\xi_1\xi_2\xi_3) > s \geq \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$. It follows that $\frac{\xi_1}{(s,0.5)} \in f$ and $\frac{\xi_3}{(s,0.5)} \in f$ but $\frac{\xi_1\xi_2\xi_3}{(s,0.5)} \overline{\in} f$ and $\frac{\xi_1\xi_2\xi_3}{(s,0.5)} \overline{\mathrm{q}}\, f$,that is, $\frac{\xi_1\xi_2\xi_3}{(s,0.5)} \overline{\in \vee \mathrm{q}}\, f$ which is impossible. If $f_p(\xi_1\xi_2\xi_3) < 0.5$, then there exists $t \in (0, 1]$ such that $f_p(\xi_1\xi_2\xi_3) < t \leq 0.5 \leq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$. Let $s = \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$, then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_3}{(s,t)} \in f$ but $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\mathrm{q}}\, f$,that is, $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in \vee \mathrm{q}}\, f$ which is a contradiction. Hence $f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3)\} \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\}$ and $f_p(\xi_1\xi_2\xi_3) \geq 0.5 \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\}$.

Conversely, assume that the conditions (1) and (2) are true. Let $\xi_1, \xi_2 \in S$ be such that $\xi_1 \leq \xi_2$. Let $(s, t) \in [-1, 0) \times (0, 1]$ such that $\frac{\xi_2}{(s,t)} \in f$. Then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Assume that $\frac{\xi_1}{(s,t)} \overline{\in} f$, then $f_n(\xi_1) > s$ and $f_p(\xi_1) < t$. If $f_n(\xi_1) > s$, then $f_n(\xi_2) \leq -0.5$. Otherwise, we get $f_n(\xi_1) \leq \bigvee\{f_n(\xi_2), -0.5\} = f_n(\xi_2) \leq s$, a contradiction. If $f_p(\xi_1) < t$, then $f_p(\xi_2) \geq 0.5$. Otherwise we obtain $f_p(\xi_1) > \bigwedge\{f_p(\xi_2), 0.5\} = f_p(\xi_2) \geq t$, which is a contradiction. It follows that $f_n(\xi_1) + s < 2f_n(\xi_1) \leq 2\bigvee\{f_n(\xi_2), -0.5\} = -1$ and $f_p(\xi_1) + t > 2f_p(\xi_1) \geq 2\bigwedge\{f_p(\xi_2), 0.5\} = 1$. This implies that $\frac{\xi_1}{(s,t)} \mathrm{q}\, f$. Let $\xi_1, \xi_2, \xi_3 \in S$ such that $\frac{\xi_1}{(s_1,t_1)} \in f$ and $\frac{\xi_3}{(s_2,t_2)} \in f$ for (s_1, t_1), $(s_2, t_2) \in [-1, 0) \times (0, 1]$. Then $f_n(\xi_1) \leq s_1$, $f_n(\xi_3) \leq s_2$, $f_p(\xi_1) \geq t_1$ and $f_p(\xi_3) \geq t_2$. Assume that $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\}, \wedge\{t_1,t_2\})} \overline{\in} f$, then $f_n(\xi_1\xi_2\xi_3) > \bigvee\{s_1, s_2\}$ or $f_p(\xi_1\xi_2\xi_3) < \bigwedge\{t_1, t_2\}$. If $f_n(\xi_1\xi_2\xi_3) > \bigvee\{s_1, s_2\}$, then $\bigvee\{f_n(\xi_1), f_n(\xi_3)\} \leq -0.5$. Otherwise, we get $f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\} = \bigvee\{f_n(\xi_1), f_n(\xi_3)\} \leq \bigvee\{s_1, s_2\}$, a contradiction. If $f_p(\xi_1\xi_2\xi_3) < \bigwedge\{t_1, t_2\}$, then $\bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq 0.5$. Otherwise we obtain $f_p(\xi_1\xi_2\xi_3) > \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\}$

$= \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq \bigwedge\{t_1, t_2\}$, which is a contradiction. It follows that

$$\begin{aligned} f_n(\xi_1\xi_2\xi_3) + \bigvee\{s_1, s_2\} &< 2\bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\} = -1 \\ \text{and } f_p(\xi_1\xi_2\xi_3) + \bigwedge\{t_1, t_2\} &> 2\bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\} = 1. \end{aligned}$$

This implies that $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})}$q f. Hence, $f = (S; f_n, f_p)$ is an $(\in, \in \vee$q$)$-BFGBI of S. ■

Theorem 4.1.6 *Let $f = (S; f_n, f_p)$ be a BFS in S. Then $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFGBI of S iff $C(f; (s,t)) \neq \emptyset$ is a generalized bi-ideal of S for all $(s,t) \in [-0.5, 0) \times (0, 0.5]$.*

Proof. Let $f = (S; f_n, f_p)$ is an $(\in, \in \vee$q$)$-BFGBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in C(f; (s,t))$. This implies $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee$q$)$-BFGBI of S. So $f_n(\xi_1) \leq \bigvee\{f_n(\xi_2), -0.5\} \leq s$ and $f_p(\xi_1) \geq \bigwedge\{f_p(\xi_2), 0.5\} \geq t$. This implies $\xi_1 \in C(f; (s,t))$. Let $\xi_1, \xi_2, \xi_3 \in S$ such that $\xi_1, \xi_3 \in C(f; (s,t))$. Then $f_n(\xi_1) \leq s$, $f_n(\xi_3) \leq s$, $f_p(\xi_1) \geq t$ and $f_p(\xi_3) \geq t$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee$q$)$-BFGBI of S. So $f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\} \leq s$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\} \geq t$. This implies that $\xi_1\xi_2\xi_3 \in C(f; (s,t))$. Hence $C(f; (s,t))$ is a generalized bi-ideal of S.

Conversely, suppose that the non-empty (s,t)-cut $C(f; (s,t))$ of $f = (S; f_n, f_p)$ is a generalized bi-ideal of S. Assume that there exist $a, b, c \in S$ such that (i) $a \leq b$ and $f_n(a) > \bigvee\{f_n(b), -0.5\}$ or $f_p(a) < \bigwedge\{f_p(b), 0.5\}$, (ii) $f_n(abc) > \bigvee\{f_n(a), f_n(c), -0.5\}$ or $f_p(abc) < \bigwedge\{f_p(a), f_p(c), 0.5\}$.

Let $s = \bigvee\{f_n(b), -0.5\}$ and $t = \bigwedge\{f_p(b), 0.5\}$ then $b \in C(f; (s,t))$. If $a \leq b$ and $f_n(a) > \vee\{f_n(b), -0.5\} = s$ then $a \notin C(f; (s,t))$. If $f_p(a) < \bigwedge\{f_p(b), 0.5\} = t$, then $a \notin C(f; (s,t))$, which is impossible because $C(f(s,t))$ is generalized bi-ideal of S. Let $\acute{s} = \bigvee\{f_n(a), f_n(c), -0.5\}$ and $\acute{t} = \bigwedge\{f_p(a), f_p(c), 0.5\}$, then $a, c \in C(f(\acute{s}, \acute{t}))$. If $f_n(abc) > \bigvee\{f_n(a), f_n(c), -0.5\} = \acute{s}$ then $abc \notin C(f; (\acute{s}, \acute{t}))$. If $f_p(abc) < \bigwedge\{f_p(a), f_p(c), 0.5\} = \acute{t}$ then $abc \notin C(f; (\acute{s}, \acute{t}))$, which is contradiction to the fact that $C(f; (\acute{s}, \acute{t}))$ is generalized bi-ideal of S. Hence $f = (S; f_n, f_p)$ is an $(\in, \in \vee$q$)$-BFGBI of S. ■

Theorem 4.1.7 *If $f = (S; f_n, f_p)$ is a nonzero $(\in, \in \vee q)$-BFGBI of S, then the set $S_\circ = \{\xi \in S \mid f_n(\xi) \neq 0\} \cap \{\xi \in S \mid f_p(\xi) \neq 0\}$ is a generalized bi-ideal of S.*

Proof. We assume that $f = (S; f_n, f_p)$ is a non-zero $(\in, \in \vee$q$)$-BFGBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in S_\circ$. Then $f_n(\xi_2) \neq 0$ and $f_p(\xi_2) \neq 0$, so $f_n(\xi_2) < 0$ and $f_p(\xi_2) > 0$.

Suppose that $f_n(\xi_1) = 0$ or $f_p(\xi_1) = 0$. Since $\frac{\xi_2}{(f_n(\xi_2), f_p(\xi_2))} \in f$ and $f_n(\xi_1) > f_n(\xi_2)$ or $f_p(\xi_1) < f_p(\xi_2)$. This implies that $\frac{\xi_1}{(f_n(\xi_2), f_p(\xi_2))} \overline{\in} f$, which is a contradiction. Also $f_n(\xi_1) + f_n(\xi_2) = f_n(\xi_2) \geq -1$ or $f_p(\xi_1) + f_p(\xi_2) = f_p(\xi_2) \leq 1$. This implies that $\frac{\xi_1}{(f_n(\xi_2), f_p(\xi_2))} \overline{q}\ f$, which is a contradiction. Therefore $f_n(\xi_1) \neq 0$ and $f_p(\xi_1) \neq 0$. Hence, $\xi_1 \in S_\circ$.

Let $\xi_1, \xi_2, \xi_3 \in S$ such that $\xi_1, \xi_3 \in S_\circ$. Then $f_n(\xi_1) \neq 0$, $f_n(\xi_3) \neq 0$, $f_p(\xi_1) \neq 0$ and $f_p(\xi_3) \neq 0$. So $f_n(\xi_1) < 0$, $f_n(\xi_3) < 0$, $f_p(\xi_1) > 0$ and $f_p(\xi_3) > 0$. Suppose that $\xi_1\xi_2\xi_3 \notin S_\circ$, that is, $f_n(\xi_1\xi_2\xi_3) = 0$ or $f_p(\xi_1\xi_2\xi_3) = 0$. Clearly $\frac{\xi_1}{(f_n(\xi_1), f_p(\xi_1))} \in f$ and $\frac{\xi_3}{(f_n(\xi_3), f_p(\xi_3))} \in f$. Since $f_n(\xi_1\xi_2\xi_3) = 0 > \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$ or $f_p(\xi_1\xi_2\xi_3) = 0 < \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$. This implies $\frac{\xi_1\xi_2\xi_3}{(\vee\{f_n(\xi_1), f_n(\xi_3)\}, \wedge\{f_p(\xi_1), f_p(\xi_3)\})} \overline{\in} f$, which is a contradiction. Also $f_n(\xi_1\xi_2\xi_3) + \bigvee\{f_n(\xi_1), f_n(\xi_3)\} = \bigvee\{f_n(\xi_1), f_n(\xi_3)\} \geq -1$ or $f_p(\xi_1\xi_2\xi_3) + \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} = \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \leq 1$. This implies that $\frac{\xi_1\xi_2\xi_3}{(\vee\{f_n(\xi_1), f_n(\xi_3)\}, \wedge\{f_p(\xi_1), f_p(\xi_3)\})} \overline{q}\ f$, which is a contradiction. Therefore $f_n(\xi_1\xi_2\xi_3) \neq 0$ and $f_p(\xi_1\xi_2\xi_3) \neq 0$. Hence, $\xi_1\xi_2\xi_3 \in S_\circ$. ■

Theorem 4.1.8 *If $f = (S; f_n, f_p)$ is a nonzero $(q, \in \vee q)$-BFGBI of S, then the set $S_\circ = \{\xi \in S \mid f_n(\xi_1) \neq 0\} \cap \{\xi \in S \mid f_p(\xi) \neq 0\}$ is a generalized bi-ideal of S.*

Proof. Suppose that $f = (S; f_n, f_p)$ is a non-zero (q, $\in \vee$q)-BFGBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in S_\circ$. Then $f_n(\xi_2) \neq 0$ and $f_p(\xi_2) \neq 0$, so $f_n(\xi_2) < 0$ and $f_p(\xi_2) > 0$. Therefore $f_n(\xi_2) - 1 < -1$ and $f_p(\xi_2) + 1 > 1$. It follows that $\frac{\xi_2}{(-1,1)} q\ f$. If $f_n(\xi_1) = 0$ or $f_p(\xi_1) = 0$, then $f_n(\xi_1) > -1$ or $f_p(\xi_1) < 1$ and so $\frac{\xi_1}{(-1,1)} \overline{\in} f$, which is a contradiction. Also $f_n(\xi_1) - 1 = -1$ or $f_p(\xi_1) + 1 = 1$. $\Longrightarrow \frac{\xi_1}{(-1,1)} \overline{q}\ f$, which is a contradiction. Therefore $f_n(\xi_1) \neq 0$ and $f_p(\xi_1) \neq 0$. Hence, $\xi_1 \in S_\circ$. Let $\xi_1, \xi_2, \xi_3 \in S$ such that $\xi_1, \xi_3 \in S_\circ$. Then $f_n(\xi_1) \neq 0$, $f_n(\xi_3) \neq 0$, $f_p(\xi_1) \neq 0$ and $f_p(\xi_3) \neq 0$. So $f_n(\xi_1) < 0$, $f_n(\xi_3) < 0$, $f_p(\xi_1) > 0$ and $f_p(\xi_3) > 0$. Therefore $f_n(\xi_1) - 1 < -1$, $f_n(\xi_3) - 1 < -1$, $f_p(\xi_1) + 1 > 1$ and $f_p(\xi_3) + 1 > 1$. It follows that $\frac{\xi_1}{(-1,1)} q$ and $\frac{\xi_3}{(-1,1)} q\ f$. If $f_n(\xi_1\xi_2\xi_3) = 0$ or $f_p(\xi_1\xi_2\xi_3) = 0$, then $f_n(\xi_1\xi_2\xi_3) > -1$ or $f_p(\xi_1\xi_2\xi_3) < 1$. This implies $\frac{\xi_1\xi_2\xi_3}{(-1,1)} \overline{\in} f$, which is not possible. Also $f_n(\xi_1\xi_2\xi_3) - 1 = -1$ or $f_p(\xi_1\xi_2\xi_3) + 1 = 1$. This implies $\frac{\xi_1\xi_2\xi_3}{(-1,1)} \overline{q}\ f$, which is a contradiction. Therefore $f_n(\xi_1\xi_2\xi_3) \neq 0$ and $f_p(\xi_1\xi_2\xi_3) \neq 0$. Hence, $\xi_1\xi_2\xi_3 \in S_\circ$. ■

It is easy to verify that every ($\in, \in \vee$q)-BFBI is an ($\in, \in \vee$q)-BFGBI but the converse is not true in general.

Example 4.1.9 *Let S be an ordered semigroup as given in Example* 1.4.2, *and we define BFS* *as follows:*

S	n	k	l	m
f_n	-0.7	-0.3	-0.5	-0.1
f_p	0.8	0.2	0.4	0

Then clearly $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFGBI of S because for all $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$, we have $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq f_p(\xi_2)$. Also $f_n(\xi_1\xi_2\xi_3) = f_n(0) = -0.7 \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\}$ and $f_p(\xi_1\xi_2\xi_3) = f_p(0) = 0.8 \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\}$ for all $\xi_1, \xi_2, \xi_3 \in S$. But $f_n(l \cdot l) = f_n(k) = -0.3 > -0.5 = \bigvee\{f_n(l), f_n(l), -0.5\}$ and $f_p(l \cdot l) = f_p(k) = 0.2 < 0.4 = \bigwedge\{f_p(l), f_p(l), 0.5\}$. Hence $f = (S; f_n, f_p)$ is not an $(\in, \in \vee q)$-BFBI of S.

From following proposition, we conclude that in regular ordered semigroup, the concept of $(\in, \in \vee$q)-BFGBI and $(\in, \in \vee$q)-BFBI coincide.

Proposition 4.1.10 *If S is regular ordered semigroup, then every $(\in, \in \vee q)$-BFGBI of S is an $(\in, \in \vee q)$-BFBI of S.*

Proof. Let $f = (S; f_n, f_p)$ be a $(\in, \in \vee$q)-BFGBI of S. Let $a, b \in S$, then there exists $\xi \in S$ such that $b \leq b\xi b$ and we have

$$\begin{aligned} f_n(ab) &\leq \bigvee\{f_n(a(b\xi b)), -0.5\} \\ &= \bigvee\{f_n(a(b\xi)b), -0.5\} \\ &\leq \bigvee\{f_n(a), f_n(b), -0.5\}. \end{aligned}$$

Also

$$\begin{aligned} f_p(ab) &\geq \bigwedge\{f_p(a(b\xi b)), 0.5\} \\ &= \bigwedge\{f_p(a(b\xi)b), 0.5\} \\ &\geq \bigwedge\{f_p(a), f_p(b), 0.5\}. \end{aligned}$$

Hence, $f = (S; f_n, f_p)$ is a $(\in, \in \vee$q)-bipolar fuzzy subsemigroup of S. ■

4.2 Upper and Lower Parts of $(\in, \in \vee q)$-Bipolar Fuzzy Generalized Bi-Ideals

In this section we define upper and lower parts of $(\in, \in \vee q)$-BFGBI. We also characterize regular ordered semigroups in framework of upper and lower parts of $(\in, \in \vee q)$-BFGBI.

Definition 4.2.1 *Let $f = (S; f_n, f_p)$ be a BFS in S. We define the upper part $f^+ = (S; f_n^+, f_p^+)$ of $f = (S; f_n, f_p)$ as follows; $f_n^+(\xi) = f_n(\xi) \wedge -0.5$ and $f_p^+(\xi) = f_p(\xi) \vee 0.5$. Similarly we define the lower part $f^- = (S; f_n^-, f_p^-)$ of $f = (S; f_n, f_p)$ as follows; $f_n^-(\xi) = f_n(\xi) \vee -0.5$ and $f_p^-(\xi) = f_p(\xi) \wedge 0.5$.*

Definition 4.2.2 *The upper part $\chi^+ = (S, \chi^+_{nA}, \chi^+_{pA})$ of the characteristic function $\chi_A = (S, \chi_{nA}, \chi_{pA})$ of $A \neq \emptyset$ is define by*

$$\chi^+_{nA}(\xi) = \begin{cases} -1 \ \textit{if} \ \xi \in A \\ -0.5 \ \textit{if} \ \xi \notin A \end{cases},$$

$$\textit{and} \ \chi^+_{pA}(\xi) = \begin{cases} 1 \ \textit{if} \ \xi \in A \\ 0.5 \ \textit{if} \ \xi \notin A \end{cases}.$$

Similarly we can define the lower part $\chi_A^- = (S, \chi^-_{nA}, \chi^-_{pA})$ of the characteristic function $\chi_A = (S, \chi_{nA}, \chi_{pA})$ of A by

$$\chi^-_{nA}(\xi) = \begin{cases} -0.5 \ \textit{if} \ \xi \in A \\ 0 \ \textit{if} \ \xi \notin A \end{cases},$$

$$\textit{and} \ \chi^-_{pA}(\xi) = \begin{cases} 0.5 \ \textit{if} \ \xi \in A \\ 0 \ \textit{if} \ \xi \notin A \end{cases}.$$

Lemma 4.2.3 *If A and B are non-empty subsets of S. Then the following equations hold:*

(1) $(\chi_A \wedge \chi_B)^- = \chi^-_{A \cap B}$,
(2) $(\chi_A \vee \chi_B)^- = \chi^-_{A \cup B}$,
(3) $(\chi_A \circ \chi_B)^- = \chi^-_{(AB]}$.

Proof. The proof of (1) and (2) are obvious.

(3) Let $\xi \in (AB]$. Then $\chi_{n(AB]}(\xi) = -1$ and $\chi_{p(AB]}(\xi) = 1$. Hence $\chi_{n(AB]}(\xi) \vee -0.5 = -1 \vee -0.5 = -0.5$ and $\chi_{p(AB]}(\xi) \wedge 0.5 = 1 \wedge 0.5 = 0.5$, which implies that $\chi^{-}_{n(AB]}(\xi) = -0.5$ and $\chi^{-}_{p(AB]}(\xi) = 0.5$. Since $\xi \in (AB]$, we have $\xi \leq ab$ for some $a \in A$ and $b \in B$. Thus $(a, b) \in A_\xi$ and $A_\xi \neq \emptyset$. So,

$$
\begin{aligned}
&(\chi_{nA} \circ \chi_{nB})^{-}(\xi) \\
&= (\chi_{nA} \circ \chi_{nB})(\xi) \vee -0.5 \\
&= (\bigwedge_{(y,z) \in A_\xi} (\chi_{nA}(y) \vee \chi_{nB}(z))) \vee -0.5 \\
&\leq ((\chi_{nA}(a) \vee \chi_{nB}(b)) \vee -0.5.
\end{aligned}
$$

Since $a \in A$ and $b \in B$, we have $\chi_{nA}(a) = -1$ and $\chi_{nB}(b) = -1$, so

$$
\begin{aligned}
(\chi_{nA} \circ \chi_{nB})^{-}(\xi) &\leq ((\chi_{nA}(a) \vee \chi_{nB}(b)) \vee -0.5 \\
&= (-1 \vee -1) \vee -0.5 = -0.5.
\end{aligned}
$$

Thus $(\chi_{nA} \circ \chi_{nB})^{-}(\xi) = -0.5 = \chi^{-}_{n(AB]}(\xi)$. Also

$$
\begin{aligned}
&(\chi_{pA} \circ \chi_{pB})^{-}(\xi) \\
&= (\chi_{pA} \circ \chi_{pB})(\xi) \wedge 0.5 \\
&= (\bigvee_{(y,z) \in A_\xi} (\chi_{pA}(y) \wedge \chi_{pB}(z))) \wedge 0.5 \\
&\geq ((\chi_{pA}(a) \wedge \chi_{pB}(b)) \wedge 0.5.
\end{aligned}
$$

Since $a \in A$ and $b \in B$, we have $\chi_{pA}(a) = 1$ and $\chi_{pB}(b) = 1$, so

$$
\begin{aligned}
(\chi_{pA} \circ \chi_{pB})^{-}(\xi) &\geq (\chi_{nA}(a) \wedge \chi_{nB}(b)) \wedge 0.5 \\
&= (1 \wedge 1) \wedge 0.5 = 0.5.
\end{aligned}
$$

Thus $(\chi_{pA} \circ \chi_{pB})^{-}(\xi) = 0.5 = \chi^{-}_{p(AB]}(\xi)$. Let $\xi \notin (AB]$, then $\chi_{n(AB]}(\xi) = 0$ and $\chi_{p(AB]}(\xi) = 0$. Hence $\chi^{-}_{n(AB]}(\xi) = 0$ and $\chi^{-}_{p(AB]}(\xi) = 0$.

Let $(y,z) \in A_\xi$. Then,

$$\begin{aligned}(\chi_{nA} \circ \chi_{nB})^{-}(\xi) &= ((\chi_{nA} \circ \chi_{nB})(\xi) \vee -0.5) \\ &= (\bigwedge_{(y,z)\in A_\xi} (\chi_{nA}(y) \vee \chi_{nB}(z))) \vee -0.5\end{aligned}$$

and

$$\begin{aligned}(\chi_{pA} \circ \chi_{pB})^{-}(\xi) &= ((\chi_{pA} \circ \chi_{pB})(\xi) \wedge 0.5) \\ &= (\bigvee_{(y,z)\in A_\xi} (\chi_{pA}(y) \wedge \chi_{pB}(z))) \wedge 0.5\end{aligned}$$

Since $(y,z) \in A_\xi$, we have $\xi \leq yz$. If $y \in A$ and $z \in B$, then $\xi \in (AB]$. This is impossible. If $y \notin A$ and $z \in B$, then

$$(\bigwedge_{(y,z)\in A_\xi} (\chi_{nA}(y) \vee \chi_{nB}(z))) \vee -0.5 = (\bigwedge_{(y,z)\in A_\xi} (0 \vee -1)) \vee -0.5 = 0$$

and

$$(\bigvee_{(y,z)\in A_\xi} (\chi_{pA}(y) \wedge \chi_{pB}(z))) \wedge 0.5 = (\bigvee_{(y,z)\in A_\xi} (0 \wedge 1)) \wedge 0.5 = 0.$$

Hence $(\chi_{nA} \circ \chi_{nB})^{-}(\xi) = 0 = \chi^{-}_{n(AB]}(\xi)$ and $(\chi_{pA} \circ \chi_{pB})^{-}(\xi) = 0 = \chi^{-}_{p(AB]}(\xi)$. Similarly for $y \in A$ and $z \notin B$, we have the same result. Thus $(\chi_A \circ \chi_B)^{-} = \chi^{-}_{(AB]}$. ■

Lemma 4.2.4 *Let $\emptyset \neq \mathcal{G} \subseteq S$ then $\mathcal{G}$ is a generalized bi-ideal of S iff $\chi^{-}_{\mathcal{G}}$ is an $(\in, \in \vee q)$-BFGBI of S.*

Proof. Let $\chi^{-}_{\mathcal{G}}$ is an $(\in, \in \vee$q)-BFGBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. If $\xi_2 \in \mathcal{G}$ then $\chi^{-}_{n\mathcal{G}}(\xi_2) = -0.5$ and $\chi^{-}_{p\mathcal{G}}(\xi_2) = 0.5$. This implies that $\frac{\xi_2}{(-0.5,0.5)} \in \chi^{-}_{\mathcal{G}}$. Since $\chi^{-}_{\mathcal{G}}$ is an $(\in, \in \vee$q)-BFGBI of S, so $\frac{\xi_1}{(-0.5,0.5)} \in \vee$q $\chi^{-}_{\mathcal{G}}$ i.e $\frac{\xi_1}{(-0.5,0.5)} \in \chi^{-}_{\mathcal{G}}$ or $\frac{\xi_1}{(-0.5,0.5)}$ q $\chi^{-}_{\mathcal{G}}$. The case $\frac{\xi_1}{(-0.5,0.5)}$ q $\chi^{-}_{\mathcal{G}}$ is impossible because $\chi^{-}_{n\mathcal{G}}(\xi_1) - 0.5 \geq -1$ and $\chi^{-}_{p\mathcal{G}}(\xi_1) + 0.5 \leq 1$, so $\frac{\xi_1}{(-0.5,0.5)} \in \chi^{-}_{\mathcal{G}}$. This implies that $\chi^{-}_{n\mathcal{G}}(\xi_1) = -0.5$ and $\chi^{-}_{p\mathcal{G}}(\xi_1) = 0.5$. Hence $\xi_1 \in \mathcal{G}$. Let $\xi_1, \xi_3 \in \mathcal{G}$, then $\chi^{-}_{n\mathcal{G}}(\xi_1) = -0.5$, $\chi^{-}_{P\mathcal{G}}(\xi_1) = 0.5$, $\chi^{-}_{n\mathcal{G}}(\xi_3) = -0.5$ and $\chi^{-}_{p\mathcal{G}}(\xi_3) = 0.5$. Therefore $\frac{\xi_1}{(-0.5,0.5)} \in \chi^{-}_{\mathcal{G}}$ and $\frac{\xi_3}{(-0.5,0.5)} \in \chi^{-}_{\mathcal{G}}$. Since $\chi^{-}_{\mathcal{G}}$ is an $(\in, \in \vee$q)-BFGBI of S, so $\frac{\xi_1\xi_2\xi_3}{(-0.5,0.5)} \in \vee$q $\chi^{-}_{\mathcal{G}}$ i.e $\frac{\xi_1\xi_2\xi_3}{(-0.5,0.5)} \in \chi^{-}_{\mathcal{G}}$ or $\frac{\xi_1\xi_2\xi_3}{(-0.5,0.5)}$ q $\chi^{-}_{\mathcal{G}}$. Since $\chi^{-}_{n\mathcal{G}}(\xi_1\xi_2\xi_3) - 0.5 \geq -1$ and $\chi^{-}_{P\mathcal{G}}(\xi_1\xi_2\xi_3) + 0.5 \leq 1$, so $\frac{\xi_1\xi_2\xi_3}{(-0.5,0.5)}$ q $\chi^{-}_{\mathcal{G}}$ is impossible.

Therefore $\frac{\xi_1\xi_2\xi_3}{(-0.5,0.5)} \in \chi^-_{\mathcal{G}}$. This implies that $\chi^-_{n\ \mathcal{G}}(\xi_1\xi_2\xi_3) = -0.5$ and $\chi_{P\mathcal{G}}(\xi_1\xi_2\xi_3) = 0.5$. Thus $\xi_1\xi_2\xi_3 \in \mathcal{G}$ and $\mathcal{G}$ is generalized bi-ideal of S.

Conversely, assume that $\mathcal{G}$ is generalized bi-ideal of S. Then by Theorem 1.4.4 and Lemma 4.1.4, we have $\chi^-_{\mathcal{G}}$ is an $(\in, \in \vee q)$-BFGBI of S. ■

Lemma 4.2.5 *Let $\emptyset \neq I \subseteq S$ then I is an ideal of S iff χ^-_I is an $(\in, \in \vee q)$-BFI of S.*

Proof. The proof follows from Lemma 4.2.4. ■

Proposition 4.2.6 *Let $f = (S; f_n, f_p)$ be an $(\in, \in \vee q)$-BFGBI of S, then $f^- = (S; f^-_n, f^-_p)$ is a BFGBI of S.*

Proof. Let $f = (S; f_n, f_p)$ be an $(\in, \in \vee$q)-BFGBI of S. Let $\xi_1, \xi_2 \in S$, such that $\xi_1 \leq \xi_2$. Then $f_n(\xi_1) \leq f_n(\xi_2) \vee -0.5$ and $f_n(\xi_1) \geq f_p(\xi_2) \wedge 0.5$. Thus $f_n(\xi_1) \vee -0.5 \leq (f_n(\xi_2) \vee -0.5) \vee -0.5 = f_n(\xi_2) \vee -0.5$ and $f_n(\xi_1) \wedge 0.5 \geq (f_p(\xi_2) \wedge 0.5) \wedge 0.5 = f_p(\xi_2) \wedge 0.5$. This implies that $f^-_n(\xi_1) \leq f^-_n(\xi_2)$ and $f^-_p(\xi_1) \geq f^-_p(\xi_2)$. Let $\xi_1, \xi_2, \xi_3 \in S$, then by using 2nd condition of Theorem 4.1.5, $f_n(\xi_1\xi_2\xi_3) \leq f_n(\xi_1) \vee f_n(\xi_3) \vee -0.5$ and $f_n(\xi_1\xi_2\xi_3) \geq f_p(\xi_1) \wedge f_p(\xi_3) \wedge 0.5\}$. Thus

$$\begin{aligned} & f_n(\xi_1\xi_2\xi_3) \vee -0.5 \\ & \leq (f_n(\xi_1) \vee -0.5) \vee (f_n(\xi_3) \vee -0.5) \\ f^-_n(\xi_1\xi_2\xi_3) & \leq f^-_n(\xi_1) \vee f^-_n(\xi_3) \end{aligned}$$

and

$$\begin{aligned} & f_n(\xi_1\xi_2\xi_3) \wedge 0.5 \\ & \geq (f_p(\xi_1) \wedge 0.5) \wedge (f_p(\xi_3) \wedge 0.5) \\ f^-_n(\xi_1\xi_2\xi_3) & \geq f^-_p(\xi_1) \wedge f^-_p(\xi_3). \end{aligned}$$

Therefore, $f^- = (S; f^-_n, f^-_p)$ is a BFGBI of S. ■

In the following theorems, we characterize the regular ordered semigroup in terms of lower parts of $(\in, \in \vee$q)-BFLI (BFRI) and $(\in, \in \vee$q)-BFGBIs.

Theorem 4.2.7 *An ordered semigroup S is regular iff for every $(\in,\in\vee q)$-BFGBI $f=(S;f_n,f_p)$ and every $(\in,\in\vee q)$-BFRI $g=(S;g_n,g_p)$ of S, we have $(f\wedge g)^{-}\preceq(f\circ g)^{-}$.*

Proof. Let $(f\wedge g)^{-}\preceq(f\circ g)^{-}$ for every $(\in,\in\vee$q)-BFGBI $f=(S;f_n,f_p)$ and every $(\in,\in\vee$q)-BFRI $g=(S;g_n,g_p)$ of S. By Proposition 1.1.15, we have only to prove that $\mathcal{G}\cap\mathcal{L}\subseteq(\mathcal{GL}]$ for every generalized bi-ideal $\mathcal{G}$ and every left ideal $\mathcal{L}$ of S. If $a\in\mathcal{G}\cap\mathcal{L}$ then $a\in\mathcal{G}$ and $a\in\mathcal{L}$. By Lemma 4.2.4, $\chi_{\mathcal{G}}^{-}$ is an $(\in,\in\vee$q)-BFGBI of S and by Lemma 4.2.5, $\chi_{\mathcal{L}}^{-}$ is an $(\in,\in\vee$q)-BFRI of S. Then by given hypothesis we get $\chi_{(\mathcal{GL}]}^{-}(a)=\chi_{\mathcal{G}}^{-}(a)\circ\chi_{\mathcal{L}}^{-}(a)\succeq\chi_{\mathcal{G}}^{-}(a)\wedge\chi_{\mathcal{L}}^{-}(a)=\chi_{\mathcal{G}\cap\mathcal{L}}^{-}(a)$, by using Lemma 4.2.3. Thus, $\mathcal{G}\cap\mathcal{L}\subseteq(\mathcal{GL}]$ and S is regular.

Conversely, assume that S is regular and $a\in S$. Then there exists $\xi\in S$ such that $a\leq a\xi a\leq(a\xi a)(\xi a)$. So $(a\xi a,\xi a)\in A_a$ and $A_a\neq\emptyset$. Now

$$\begin{aligned}(f_n\circ g_n)^{-}(a)&=(f_n\circ g_n)(a)\vee-0.5\\&=\Big(\bigwedge_{(y,z)\in A_a}(f_n(y)\vee g_n(z))\Big)\vee-0.5\\&\leq[f_n(a\xi a)\vee g_n(\xi a)]\vee-0.5.\end{aligned}$$

Since $f=(S;f_n,f_p)$ is an $(\in,\in\vee$q)-BFGBI of S, so $f_n(a\xi a)\leq f_n(a)\vee f_n(a)\vee-0.5=f_n(a)\vee-0.5=f_n^{-}(a)$ and $g=(S;g_n,g_p)$ is an $(\in,\in\vee$q)-BFRI of S, so $g_n(\xi a)\leq g_n(a)\vee-0.5=g_n^{-}(a)$. Hence

$$(f_n\circ g_n)^{-}(a)\leq f_n^{-}(a)\vee g_n^{-}(a)=(f_n\vee g_n)^{-}(a).$$

Also

$$\begin{aligned}(f_p\circ g_p)^{-}(a)&=(f_p\circ g_p)(a)\wedge 0.5\\&=\Big(\bigvee_{(y,z)\in A_a}(f_p(y)\wedge g_p(z))\Big)\wedge 0.5\\&\geq[f_p(a\xi a)\wedge g_n(\xi a)]\wedge 0.5.\end{aligned}$$

Since $f=(S;f_n,f_p)$ is an $(\in,\in\vee$q)-BFGBI of S, therefore $f_p(a\xi a)\geq f_p(a)\wedge f_p(a)\wedge 0.5=f_p(a)\wedge 0.5=f_p^{-}(a)$ and $g=(S;g_n,g_p)$ is an $(\in,\in\vee$q)-BFRI of S, so $g_p(\xi a)\geq g_p(a)\wedge 0.5=g_p^{-}(a)$. Hence

$$(f_p\circ g_p)^{-}(a)\geq f_p^{-}(a)\wedge g_p^{-}(a)=(f_p\wedge g_p)^{-}(a).$$

Thus we get $(f \wedge g)^- \preceq (f \circ g)^-$. ■

Theorem 4.2.8 *An ordered semigroup S is regular iff for every $(\in, \in \vee q)$-BFGBI $f = (S; f_n, f_p)$ and every $(\in, \in \vee q)$-BFI $g = (S; g_n, g_p)$ of S, we have $(f \wedge g)^- \preceq (f \circ g \circ f)^-$.*

Proof. Let $(f \wedge g)^- \preceq (f \circ g \circ f)^-$ for every $(\in, \in \vee$q)-BFGBI $f = (S; f_n, f_p)$ and every $(\in, \in \vee$q)-BFI $g = (S; g_n, g_p)$ of S. We show that S is regular. By Proposition 1.1.16, we have only prove that $\mathcal{G} \cap I \subseteq (\mathcal{G} I \mathcal{G}]$ for every generalized bi-ideal $\mathcal{G}$ and every ideal I of S.

Let $a \in \mathcal{G} \cap I$, then $a \in \mathcal{G}$ and $a \in I$. By Lemma 4.2.4, $\chi_{\mathcal{G}}^-$ is an $(\in, \in \vee$q)-BFGBI of S and by Lemma 4.2.5, χ_I^- is an $(\in, \in \vee$q)-BFI of S. Then by given hypothesis we get

$\chi_{(\mathcal{G} I \mathcal{G}]}^-(a) = \chi_{\mathcal{G}}^-(a) \circ \chi_I^-(a) \circ \chi_{\mathcal{G}}^-(a) \succeq \chi_{\mathcal{G}}^-(a) \wedge \chi_I^-(a) = \chi_{\mathcal{G} \cap I}^-(a)$, by using Lemma 4.2.3. Therefore $\mathcal{G} \cap I \subseteq (\mathcal{G} I \mathcal{G}]$ and S is regular.

Conversely, assume that S is regular and $a \in S$. Then there exists $\xi \in S$ such that $a \leq a\xi a \leq (a\xi a)(\xi a) \leq a(\xi a \xi)a$. Hence, $(a(\xi a \xi), a) \in A_a$ and $A_a \neq \emptyset$. Now

$$
\begin{aligned}
&(f_n \circ g_n \circ f_n)^-(a) \\
&= (f_n \circ g_n \circ f_n)(a) \vee -0.5 \\
&= \Big(\bigwedge_{(y,z) \in A_a} (f_n \circ g_n)(y) \vee f_n(z))\Big) \vee -0.5 \\
&= \bigwedge_{(y,z) \in A_a} \bigwedge_{(y_1,z_1) \in A_y} [f_n(y_1) \vee g_n(z_1) \vee f_n(z)] \vee -0.5 \\
&\leq [f_n(a) \vee g_n(\xi a \xi) \vee f_n(a)] \vee -0.5.
\end{aligned}
$$

Since $g = (S; g_n, g_p)$ is an $(\in, \in \vee$q)-BFI of S, so $g_n(\xi a \xi) \leq g_n(\xi a) \vee -0.5 \leq g_n(a) \vee -0.5$. Thus, we have

$$
\begin{aligned}
&[f_n(a) \vee g_n(\xi a \xi) \vee f_n(a)] \vee -0.5 \\
&\leq (f_n(a) \vee -0.5) \vee (g_n(a) \vee -0.5) \\
&= f_n^-(a) \vee g_n^-(a) = (f_n \vee g_n)^-(a).
\end{aligned}
$$

Therefore, $(f_n \circ g_n \circ f_n)^-(a) \leq (f_n \vee g_n)^-(a)$.

In similar way, we can easily show that

$$(f_p \circ g_p \circ f_p)^-(a) \geq (f_p \wedge g_p)^-(a).$$

Hence, $(f \wedge g)^- \preceq (f \circ g \circ f_p)^-$. ■

4.3 (α, β)-Bipolar Fuzzy Bi-Ideals in Ordered Semigroups

In this section we characterize S in framework of generalized BFBI. We prove that every $\chi_B = (S, \chi_{nB}, \chi_{pB})$ of B is an $(\in, \in \vee$q)-BFBI of S iff B is bi-ideal of S. We also provide an example in order to clear the notion that every generalized BFBI may not be BFBI.

Definition 4.3.1 *A BFS $f = (S; f_n, f_p)$ of S is said to be (α, β)-BFBI of S, where $\alpha \neq \in \wedge q$, if the following axioms are satisfied:*

(1) $(\forall \xi_1, \xi_2 \in S)(\forall (s,t) \in [-1,0) \times (0,1])(\xi_1 \leq \xi_2, \frac{\xi_2}{(s,t)} \alpha f \Rightarrow \frac{\xi_1}{(s,t)} \beta f)$
(2) $(\forall \xi_1, \xi_2 \in S)(\forall (s_1,t_1),(s_2,t_2) \in [-1,0) \times (0,1])(\frac{\xi_1}{(s_1,t_1)} \alpha f$ and $\frac{\xi_2}{(s_2,t_2)} \alpha f \Rightarrow \frac{\xi_1 \xi_2}{(\vee\{s_1,s_2\}, \wedge\{t_1,t_2\})} \beta f)$
(3) $(\forall \xi_1, \xi_2, \xi_3 \in S)(\forall (s_1,t_1),(s_2,t_2) \in [-1,0) \times (0,1])(\frac{\xi_1}{(s_1,t_1)} \alpha f$ and $\frac{\xi_3}{(s_2,t_2)} \alpha f \Rightarrow \frac{\xi_1 \xi_2 \xi_3}{(\vee\{s_1,s_2\}, \wedge\{t_1,t_2\})} \beta f)$.

Theorem 4.3.2 *Every $(\in, \in)$-BFBI of S is an $(\in, \in \vee q)$-BFBI of S.*

Proof. Straight forward. ■

Theorem 4.3.3 *A BFS $f = (S, f_n, f_p)$ of S is an $(\in, \in \vee q)$-BFBI of S iff it satisfies the following conditions:*

(1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Longrightarrow f_n(\xi_1) \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq \bigwedge\{f_p(\xi_2), 0.5\})$
(2) $(\forall \xi_1, \xi_2 \in S)(f_n(\xi_1 \xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\}$ and $f_p(\xi_1 \xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\})$
(3) $(\forall \xi_1, \xi_2, \xi_3 \in S)(f_n(\xi_1 \xi_2 \xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\}$ and $f_p(\xi_1 \xi_2 \xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\})$.

Proof. Suppose that $f = (S; f_n, f_p)$ is an $(\in, \in \vee$q)-BFBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$. We consider the following four cases.

(i) $f_n(\xi_2) > -0.5$ and $f_p(\xi_2) < 0.5$,

(ii) $f_n(\xi_2) \leq -0.5$ and $f_p(\xi_2) \geq 0.5$,

(iii) $f_n(\xi_2) \leq -0.5$ and $f_p(\xi_2) < 0.5$,

(iv) $f_n(\xi_2) > -0.5$ and $f_p(\xi_2) \geq 0.5$.

For the first case, assume that

$$f_n(\xi_1) > \bigvee\{f_n(\xi_2), -0.5\} \text{ or } f_p(\xi_1) < \bigwedge\{f_p(\xi_2), 0.5\}$$

then $f_n(\xi_1) > f_n(\xi_2)$ or $f_p(\xi_1) < f_p(\xi_2)$. If $f_n(\xi_1) > f_n(\xi_2)$ then there exists $s \in [-1, 0)$ such that $f_n(\xi_1) > s \geq f_n(\xi_2)$. Let $t = f_p(\xi_2)$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$, that is, $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$. This is not possible. If $f_p(\xi_1) < f_p(\xi_2)$ then there exists $t \in (0, 1]$ such that $f_p(\xi_1) < t \leq f_p(\xi_2)$. Let $s = f_n(\xi_2)$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$, that is, $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, a contrary result. Therefore $f_n(\xi_1) \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq \bigwedge\{f_p(\xi_2), 0.5\}$.

Now consider the second case that is, $f_n(\xi_2) \leq -0.5$ and $f_p(\xi_2) \geq 0.5$. This implies that $\frac{\xi_2}{(-0.5,0.5)} \in f$. Let $\xi_1, \xi_2 \in S$ be such that $\xi_1 \leq \xi_2$. Since $\frac{\xi_2}{(-0.5,0.5)} \in f$, this implies $\frac{\xi_1}{(-0.5,0.5)} \in \vee \mathrm{q}\ f$. So $\frac{\xi_1}{(-0.5,0.5)} \in f$ or $\frac{\xi_1}{(-0.5,0.5)} \mathrm{q}\ f$. If $\frac{\xi_1}{(-0.5,0.5)} \in f$ then $f_n(\xi_1) \leq -0.5 \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq 0.5 \geq \bigwedge\{f_p(\xi_2), 0.5\}$. If $\frac{\xi_1}{(-0.5,0.5)} \mathrm{q}\ f$ then $f_n(\xi_1) - 0.5 < -1$ and $f_p(\xi_1) + 0.5 > 1$. This implies that $f_n(\xi_1) < -0.5$ and $f_p(\xi_1) > 0.5$, therefore $f_n(\xi_1) \leq -0.5 \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq 0.5 \geq \bigwedge\{f_p(\xi_2), 0.5\}$.

Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$, then from case (iii) we have $f_n(\xi_1) \leq -0.5$ and $f_p(\xi_1) \geq f_p(\xi_2)$. Assume that $f_n(\xi_1) > -0.5$ or $f_p(\xi_1) < f_p(\xi_2)$. If $f_n(\xi_1) > -0.5$ then there exists $s \in [-1, 0)$ such that $f_n(\xi_1) > s \geq -0.5 \geq f_n(\xi_2)$. Let $f_p(\xi_2) = t$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$, that is, $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. If $f_p(\xi_1) < f_p(\xi_2)$ then there exists $t \in (0, 1]$ such that $f_p(\xi_1) < t \leq f_p(\xi_2)$. Let $f_n(\xi_2) = s$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$, that is, $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. Hence $f_n(\xi_1) \leq -0.5 \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq f_p(\xi_2) \geq \bigwedge\{f_p(\xi_2), 0.5\}$.

Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$, then from case (iv), we have $f_n(\xi_1) \leq f_n(\xi_2)$ and $f_p(\xi_1) \geq 0.5$. Assume that $f_n(\xi_1) > f_n(\xi_2)$ or $f_p(\xi_1) < 0.5$. If $f_n(\xi_1) > f_n(\xi_2)$ then there exists $s \in [-1, 0)$ such that $f_n(\xi_1) > s \geq f_n(\xi_2)$, then $\frac{\xi_2}{(s,0.5)} \in f$ but $\frac{\xi_1}{(s,0.5)} \overline{\in} f$ and $\frac{\xi_1}{(s,0.5)} \overline{\mathrm{q}}\, f$, that is, $\frac{\xi_1}{(s,0.5)} \overline{\in \vee \mathrm{q}}\, f$ which is impossible. If $f_p(\xi_1) < 0.5$ then there $\exists\, t \in (0, 1]$ such that $f_p(\xi_1) < t \leq 0.5 \leq f_p(\xi_2)$. Let $s = f_n(\xi_2)$ then $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1}{(s,t)} \overline{\in} f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\, f$, that is, $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\, f$ which is a contradiction. Hence $f_n(\xi_1) \leq f_n(\xi_2) \leq \bigvee\{f_n(\xi_2), -0.5\}$ and $f_p(\xi_1) \geq 0.5 \geq \bigwedge\{f_p(\xi_2), 0.5\}$.

Let $\xi_1, \xi_2 \in S$. We consider the four cases.

(i) $\bigvee\{f_n(\xi_1), f_n(\xi_2)\} > -0.5$ and $\bigwedge\{f_p(\xi_1), f_p(\xi_2)\} < 0.5$,

(ii) $\bigvee\{f_n(\xi_1), f_n(\xi_2)\} \leq -0.5$ and $\bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq 0.5$,

(iii) $\bigvee\{f_n(\xi_1), f_n(\xi_2)\} \leq -0.5$ and $\bigwedge\{f_p(\xi_1), f_p(\xi_2)\} < 0.5$,

(iv) $\bigvee\{f_n(\xi_1), f_n(\xi_2)\} > -0.5$ and $\bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq 0.5$.

For the first case, assume that

$$f_n(\xi_1\xi_2) > \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\} \text{ or } f_p(\xi_1\xi_2) < \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\}.$$

Then

$$f_n(\xi_1\xi_2) > \bigvee\{f_n(\xi_1), f_n(\xi_2)\} \text{ or } f_p(\xi_1\xi_2) < \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}.$$

If $f_n(\xi_1\xi_2) > \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ then there exists $s \in [-1, 0)$ such that $f_n(\xi_1\xi_2) > s \geq \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$. Let $t = \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$ then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}\, f$, that is, $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\, f$ which is a contradiction. If $f_p(\xi_1\xi_2) < \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$ then there exists $t \in (0, 1]$ such that $f_p(\xi_1\xi_2) < t \leq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$. Let $s = \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}\, f$, that is, $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\, f$ which is a contradiction. Hence $f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\}$.

Now consider the second case that is,

$$\bigvee\{f_n(\xi_1), f_n(\xi_2)\} \leq -0.5 \text{ and } \bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq 0.5.$$

This implies that $\frac{\xi_1}{(-0.5,0.5)} \in f$ and $\frac{\xi_2}{(-0.5,0.5)} \in f$ and so $\frac{\xi_1\xi_2}{(-0.5,0.5)} \in \vee \mathrm{q}\ f$. Thus $\frac{\xi_1\xi_2}{(-0.5,0.5)} \in f$ or $\frac{\xi_1\xi_2}{(-0.5,0.5)} \mathrm{q}\ f$. If $\frac{\xi_1\xi_2}{(-0.5,0.5)} \in f$ then $f_n(\xi_1\xi_2) \leq -0.5 \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq 0.5 \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\}$. If $\frac{\xi_1\xi_2}{(-0.5,0.5)} \mathrm{q}\ f$ then $f_n(\xi_1\xi_2) - 0.5 < -1$ and $f_p(\xi_1\xi_2) + 0.5 > 1$. This implies that $f_n(\xi_1\xi_2) < -0.5$ and $f_p(\xi_1\xi_2) > 0.5$, therefore $f_n(\xi_1\xi_2) \leq -0.5 \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq 0.5 \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\}$.

Case (iii) implies that $f_n(\xi_1\xi_2) \leq -0.5$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$. Assume that $f_n(\xi_1\xi_2) > -0.5$ or $f_p(\xi_1\xi_2) < \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$. If $f_n(\xi_1\xi_2) > -0.5$ then there exists $s \in [-1, 0)$ such that $f_n(\xi_1\xi_2) > s \geq -0.5 \geq \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$. Let $t = \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$ then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$, that is, $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. If $f_p(\xi_1\xi_2) < \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$ then there exists $t \in (0, 1]$ such that $f_p(\xi_1\xi_2) < t \leq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$. Let $s = \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$, that is, $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}$ f which is a contradiction. Hence $f_n(\xi_1\xi_2) \leq -0.5 \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\}$.

From case (iv), we have $f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ and $f_p(\xi_1\xi_2) \geq 0.5$. Assume that $f_n(\xi_1\xi_2) > \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ or $f_p(\xi_1\xi_2) < 0.5$. If $f_n(\xi_1\xi_2) > \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ then there exists $s \in [-1, 0)$ such that $f_n(\xi_1\xi_2) > s \geq \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$. It follows that $\frac{\xi_1}{(s,0.5)} \in f$ and $\frac{\xi_2}{(s,0.5)} \in f$ but $\frac{\xi_1\xi_2}{(s,0.5)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,0.5)} \overline{\mathrm{q}}\ f$,that is, $\frac{\xi_1\xi_2}{(s,0.5)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. If $f_p(\xi_1\xi_2) < 0.5$ then there exists $t \in (0, 1]$ such that $f_p(\xi_1\xi_2) < t \leq 0.5 \leq \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$. Let $s = \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_2}{(s,t)} \in f$ but $\frac{\xi_1\xi_2}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(s,t)} \overline{\mathrm{q}}\ f$, that is, $\frac{\xi_1\xi_2}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. Hence $f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2)\} \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\}$ and $f_p(\xi_1\xi_2) \geq 0.5 \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\}$.

Let $\xi_1, \xi_2, \xi_3 \in S$. We consider the four cases.

(i) $\bigvee\{f_n(\xi_1), f_n(\xi_3)\} > -0.5$ and $\bigwedge\{f_p(\xi_1), f_p(\xi_3)\} < 0.5$,

(ii) $\bigvee\{f_n(\xi_1), f_n(\xi_3)\} \leq -0.5$ and $\bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq 0.5$,

(iii) $\bigvee\{f_n(\xi_1), f_n(\xi_3)\} \leq -0.5$ and $\bigwedge\{f_p(\xi_1), f_p(\xi_3)\} < 0.5$,

(iv) $\bigvee\{f_n(\xi_1), f_n(\xi_3)\} > -0.5$ and $\bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq 0.5$.

For the first case, assume that

$$f_n(\xi_1\xi_2\xi_3) > \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\} \text{ or } f_p(\xi_1\xi_2\xi_3) < \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\}.$$

Then

$$f_n(\xi_1\xi_2\xi_3) > \bigvee\{f_n(\xi_1), f_n(\xi_3)\} \text{ or } f_p(\xi_1\xi_2\xi_3) < \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}.$$

If $f_n(\xi_1\xi_2\xi_3) > \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$ then there exists $s \in [-1, 0)$ such that $f_n(\xi_1\xi_2\xi_3) > s \geq \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$. Let $t = \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$ then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_3}{(s,t)} \in f$ but $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\mathrm{q}}\, f$, that is, $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in \vee \mathrm{q}}\, f$ which is a contradiction. If $f_p(\xi_1\xi_2\xi_3) < \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$ then there exists $t \in (0, 1]$ such that $f_p(\xi_1\xi_2\xi_3) < t \leq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$. Let $s = \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$ then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_3}{(s,t)} \in f$ but $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\mathrm{q}}\, f$, that is, $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in \vee \mathrm{q}}\, f$ which is a contradiction. Hence $f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\}$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\}$.

Now consider the second case that is,

$$\bigvee\{f_n(\xi_1), f_n(\xi_3)\} \leq -0.5 \text{ and } \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq 0.5.$$

This implies that $\frac{\xi_1}{(-0.5,0.5)} \in f$ and $\frac{\xi_3}{(-0.5,0.5)} \in f$ and so $\frac{\xi_1\xi_2\xi_3}{(-0.5,0.5)} \in \vee \mathrm{q}\, f$. Thus $\frac{\xi_1\xi_2\xi_3}{(-0.5,0.5)} \in f$ or $\frac{\xi_1\xi_2\xi_3}{(-0.5,0.5)} \mathrm{q}\, f$. If $\frac{\xi_1\xi_2\xi_3}{(-0.5,0.5)} \in f$ then $f_n(\xi_1\xi_2\xi_3) \leq -0.5 \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\}$ and $f_p(\xi_1\xi_2\xi_3) \geq 0.5 \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\}$. If $\frac{\xi_1\xi_2\xi_3}{(-0.5,0.5)} \mathrm{q}\, f$ then $f_n(\xi_1\xi_2\xi_3) - 0.5 < -1$ and $f_p(\xi_1\xi_2\xi_3) + 0.5 > 1$. This implies that $f_n(\xi_1\xi_2\xi_3) < -0.5$ and $f_p(\xi_1\xi_2\xi_3) > 0.5$, therefore $f_n(\xi_1\xi_2\xi_3) \leq -0.5 \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\}$ and $f_p(\xi_1\xi_2\xi_3) \geq 0.5 \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\}$.

Case (iii) implies that $f_n(\xi_1\xi_2\xi_3) \leq -0.5$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$. Assume that $f_n(\xi_1\xi_2\xi_3) > -0.5$ or $f_p(\xi_1\xi_2\xi_3) < \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$. If $f_n(\xi_1\xi_2\xi_3) > -0.5$ then there exists $s \in [-1, 0)$ such that $f_n(\xi_1\xi_2\xi_3) > s \geq -0.5 \geq \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$. Let $t = \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$ then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_3}{(s,t)} \in f$ but $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\mathrm{q}}\, f$,that is, $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in \vee \mathrm{q}}\, f$ which is a contradiction. If $f_p(\xi_1\xi_2\xi_3) < \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$ then there exists $t \in (0, 1]$ such that $f_p(\xi_1\xi_2\xi_3) < t \leq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$. Let $s = \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$ then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_3}{(s,t)} \in f$ but $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\mathrm{q}}\, f$,that is, $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in \vee \mathrm{q}}\, f$ which is a contradiction. Hence $f_n(\xi_1\xi_2\xi_3) \leq -0.5 \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\}$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\}$.

From case (iv), we have $f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$ and $f_p(\xi_1\xi_2\xi_3) \geq 0.5$. Assume that

$$f_n(\xi_1\xi_2\xi_3) > \bigvee\{f_n(\xi_1), f_n(\xi_3)\} \text{ or } f_p(\xi_1\xi_2\xi_3) < 0.5.$$

If $f_n(\xi_1\xi_2\xi_3) > \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$ then there exists $s \in [-1, 0)$ such that $f_n(\xi_1\xi_2\xi_3) > s \geq \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$. It follows that $\frac{\xi_1}{(s,0.5)} \in f$ and $\frac{\xi_3}{(s,0.5)} \in f$ but $\frac{\xi_1\xi_2\xi_3}{(s,0.5)} \overline{\in} f$ and $\frac{\xi_1\xi_2\xi_3}{(s,0.5)} \overline{\mathrm{q}}\ f$,that is, $\frac{\xi_1\xi_2\xi_3}{(s,0.5)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. If $f_p(\xi_1\xi_2\xi_3) < 0.5$ then there exists $t \in (0, 1]$ such that $f_p(\xi_1\xi_2\xi_3) < t \leq 0.5 \leq \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$. Let $s = \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$ then $\frac{\xi_1}{(s,t)} \in f$ and $\frac{\xi_3}{(s,t)} \in f$ but $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in} f$ and $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\mathrm{q}}\ f$,that is, $\frac{\xi_1\xi_2\xi_3}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$ which is a contradiction. Hence $f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3)\} \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\}$ and $f_p(\xi_1\xi_2\xi_3) \geq 0.5 \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\}$.

Conversely, assume that the given three conditions are satisfied. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\frac{\xi_2}{(s,t)} \in f$ for $(s, t) \in [-1, 0) \times (0, 1]$. Then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Assume that $\frac{\xi_1}{(s,t)} \overline{\in} f$, then $f_n(\xi_1) > s$ and $f_p(\xi_1) < t$. If $f_n(\xi_1) > s$ then $f_n(\xi_2) \leq -0.5$. Otherwise, we get $f_n(\xi_1) \leq \bigvee\{f_n(\xi_2), -0.5\} = f_n(\xi_2) \leq s$, a contradiction. If $f_p(\xi_1) < t$ then $f_p(\xi_2) \geq 0.5$. Otherwise we obtain $f_p(\xi_1) > \bigwedge\{f_p(\xi_2), 0.5\} = f_p(\xi_2) \geq t$, which is a contradiction. It follows that $f_n(\xi_1)+s < 2f_n(\xi_1) \leq 2\bigvee\{f_n(\xi_2), -0.5\} = -1$ and $f_p(\xi_1)+t > 2f_p(\xi_1) \geq 2\bigwedge\{f_p(\xi_2), 0.5\} = 1$. This implies that $\frac{\xi_1}{(s,t)}\mathrm{q}\ f$. Let $\frac{\xi_1}{(s_1,t_1)} \in f, \frac{\xi_2}{(s_2,t_2)} \in f$ for (s_1, t_1), $(s_2, t_2) \in [-1, 0) \times (0, 1]$. Then $f_n(\xi_1) \leq s_1$, $f_n(\xi_2) \leq s_2$, $f_p(\xi_1) \geq t_1$ and $f_p(\xi_2) \geq t_2$. Assume that $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \overline{\in} f$, then $f_n(\xi_1\xi_2) > \bigvee\{s_1, s_2\}$ or $f_p(\xi_1\xi_2) < \bigwedge\{t_1, t_2\}$. If $f_n(\xi_1\xi_2) > \bigvee\{s_1, s_2\}$, then $\bigvee\{f_n(\xi_1), f_n(\xi_2)\} \leq -0.5$. Otherwise, we get $f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\} = \bigvee\{f_n(\xi_1), f_n(\xi_2)\} \leq \bigvee\{s_1, s_2\}$, a contradiction. If $f_p(\xi_1\xi_2) < \bigwedge\{t_1, t_2\}$ then $\bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq 0.5$. Otherwise we obtain $f_p(\xi_1\xi_2) > \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\} = \bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \geq \bigwedge\{t_1, t_2\}$, which is a contradiction. It follows that $f_n(\xi_1\xi_2)+\bigvee\{s_1, s_2\} < 2f_n(\xi_1\xi_2) \leq 2\bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\} = -1$ and $f_p(\xi_1\xi_2)+ \bigwedge\{t_1, t_2\} > 2f_p(\xi_1\xi_2) \geq 2\bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\} = 1$. This implies that $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})}\mathrm{q}\ f$. Let $\frac{\xi_1}{(s_1,t_1)} \in f, \frac{\xi_3}{(s_2,t_2)} \in f$ for $(s_1, t_1), (s_2, t_2) \in [-1, 0) \times (0, 1]$. Then $f_n(\xi_1) \leq s_1$, $f_n(\xi_3) \leq s_2$, $f_p(\xi_1) \geq t_1$ and $f_p(\xi_3) \geq t_2$. Assume that $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \overline{\in} f$, then $f_n(\xi_1\xi_2\xi_3) > \bigvee\{s_1, s_2\}$ or $f_p(\xi_1\xi_2\xi_3) < \bigwedge\{t_1, t_2\}$. If $f_n(\xi_1\xi_2\xi_3) > \bigvee\{s_1, s_2\}$, then $\bigvee\{f_n(\xi_1), f_n(\xi_3)\} \leq -0.5$. Otherwise, we get $f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\} = \bigvee\{f_n(\xi_1), f_n(\xi_3)\} \leq \bigvee\{s_1, s_2\}$, a contradiction. If $f_p(\xi_1\xi_2\xi_3) < \bigwedge\{t_1, t_2\}$ then $\bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq 0.5$. Otherwise we obtain $f_p(\xi_1\xi_2\xi_3) > \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\} = \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \geq \bigwedge\{t_1, t_2\}$, which is a contradiction.

It follows that

$$f_n(\xi_1\xi_2\xi_3) + \bigvee\{s_1, s_2\} < 2f_n(\xi_1\xi_2\xi_3) \leq 2\bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\} = -1$$

and

$$f_p(\xi_1\xi_2\xi_3) + \bigwedge\{t_1, t_2\} > 2f_p(\xi_1\xi_2\xi_3) \geq 2\bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\} = 1.$$

This implies that $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})}$q f. Hence $f = (S; f_n, f_p)$ is an $(\in, \in \vee$q)-BFBI of S. ■

Proposition 4.3.4 *Let $\emptyset \neq B \subseteq S$ then B is bi-ideal of S iff $\chi_B = (S; \chi_{nB}, \chi_{pB})$ is an $(\in, \in \vee q)$-BFBI of S.*

Proof. Let $\chi_B = (S; \chi_{nB}, \chi_{pB})$ is an $(\in, \in \vee$q)-BFBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in B$. Then $\chi_{nB}(\xi_2) = -1$ and $\chi_{pB}(\xi_2) = 1$, thus $\frac{\xi_2}{(-1,1)} \in \chi_B$. Since $\chi_B = (S; \chi_{nB}, \chi_{pB})$ is an $(\in, \in \vee$q)-BFBI of S, so $\frac{\xi_1}{(-1,1)} \in \vee$q$\chi_B$ i.e $\frac{\xi_1}{(-1,1)} \in \chi_B$ or $\frac{\xi_1}{(-1,1)}$qχ_B. From both cases we conclude that $\chi_{nB}(\xi_1) = -1$ and $\chi_{pB}(\xi_1) = 1$. Hence $\xi_1 \in B$.

Let $\xi_1, \xi_2 \in B$, then $\chi_{nB}(\xi_1) = \chi_{nB}(\xi_2) = -1$ and $\chi_{pB}(\xi_1) = \chi_{pB}(\xi_2) = 1$. Thus $\frac{\xi_1}{(-1,1)} \in \chi_B$ and $\frac{\xi_2}{(-1,1)} \in \chi_B$. Since $\chi_B = (S; \chi_{nB}, \chi_{pB})$ is an $(\in, \in \vee$q)-BFBI of S, so $\frac{\xi_1\xi_2}{(-1,1)} \in \vee$q$\chi_B$ i.e $\frac{\xi_1\xi_2}{(-1,1)} \in \chi_B$ or $\frac{\xi_1\xi_2}{(-1,1)}q\chi_B$. From both cases we conclude that $\chi_{nB}(\xi_1\xi_2) = -1$ and $\chi_{pB}(\xi_1\xi_2) = 1$. Hence $\xi_1\xi_2 \in B$.

Let $\xi_1, \xi_3 \in B$ and $\xi_2 \in S$, then $\chi_{nB}(\xi_1) = \chi_{nB}(\xi_3) = -1$ and $\chi_{pB}(\xi_1) = \chi_{pB}(\xi_3) = 1$. Thus $\frac{\xi_1}{(-1,1)} \in \chi_B$ and $\frac{\xi_3}{(-1,1)} \in \chi_B$. Since $\chi_B = (S; \chi_{nB}, \chi_{pB})$ is an $(\in, \in \vee$q)-BFBI of S, so $\frac{\xi_1\xi_2\xi_3}{(-1,1)} \in \vee$q$\chi_B$ i.e $\frac{\xi_1\xi_2\xi_3}{(-1,1)} \in \chi_B$ or $\frac{\xi_1\xi_2\xi_3}{(-1,1)}q\chi_B$. From both cases we conclude that $\chi_{nB}(\xi_1\xi_2\xi_3) = -1$ and $\chi_{pB}(\xi_1\xi_2\xi_3) = 1$. Hence $\xi_1\xi_2\xi_3 \in B$ and B is bi-ideal of S.

Conversely, assume that $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\frac{\xi_2}{(s,t)} \in \chi_B$ for $(s, t) \in [-1, 0) \times (0, 1]$. Then $\chi_{nB}(\xi_2) \leq s < 0$ and $\chi_{pB}(\xi_2) \geq t > 0$. So $\chi_{nB}(\xi_2) = -1$ and $\chi_{pB}(\xi_2) = 1$. Thus $\xi_2 \in B$ and so $\xi_1 \in B$. It follows that $\chi_{nB}(\xi_1) = -1 \leq s$ and $\chi_{pB}(\xi_1) = 1 \geq t$. This implies that $\frac{\xi_1}{(s,t)} \in \chi_B$ and hence $\frac{\xi_1}{(s,t)} \in \vee$q$\chi_B$.

Let $\xi_1, \xi_2 \in S$ such that $\frac{\xi_1}{(s_1,t_1)} \in \chi_B$ and $\frac{\xi_2}{(s_2,t_2)} \in \chi_B$ for $(s_1, t_1), (s_2, t_2) \in [-1, 0) \times (0, 1]$. Then $\chi_{nB}(\xi_1) \leq s_1 < 0, \chi_{nB}(\xi_2) \leq s_2 < 0$ and $\chi_{pB}(\xi_1) \geq t_1 > 0, \chi_{pB}(\xi_2) \geq t_2 > 0$. This implies that $\chi_{nB}(\xi_1) = \chi_{nB}(\xi_2) = -1$ and $\chi_{pB}(\xi_1) = \chi_{pB}(\xi_2) = 1$. Thus $\xi_1, \xi_2 \in B$, so $\xi_1\xi_2 \in B$. It follows that $\chi_{nB}(\xi_1\xi_2) = -1 \leq \vee\{s_1, s_2\}$ and $\chi_{pB}(\xi_1\xi_2) = 1 \geq \wedge\{t_1, t_2\}$. This implies that

$\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \chi_B$ and so $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \vee q\chi_B$.

Let $\xi_1, \xi_2, \xi_3 \in S$ such that $\frac{\xi_1}{(s_1,t_1)} \in \chi_B$ and $\frac{\xi_3}{(s_2,t_2)} \in \chi_B$ for $(s_1, t_1), (s_2, t_2) \in [-1, 0) \times (0, 1]$. Then $\chi_{nB}(\xi_1) \leq s_1 < 0, \chi_{nB}(\xi_3) \leq s_2 < 0$ and $\chi_{pB}(\xi_1) \geq t_1 > 0, \chi_{pB}(\xi_1) \geq t_2 > 0$. This implies that $\chi_{nB}(\xi_1) = \chi_{nB}(\xi_3) = -1$ and $\chi_{pB}(\xi_1) = \chi_{pB}(\xi_3) = 1$. Thus $\xi_1, \xi_3 \in B$. Since B is bi-ideal of S, so $\xi_1\xi_2\xi_3 \in B$. It follows that $\chi_{nB}(\xi_1\xi_2\xi_3) = -1 \leq \vee\{s_1, s_2\}$ and $\chi_{pB}(\xi_1\xi_2\xi_3) = 1 \geq \wedge\{t_1, t_2\}$. This implies that $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \chi_B$ and so $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \vee q\chi_B$. Hence $\chi_B = (S; \chi_{nB}, \chi_{pB})$ is an $(\in, \in \vee q)$-BFBI of S. ■

Example 4.3.5 *Let $S = \{0, 1, 2, 3, 4\}$ be a set with the following multiplication table and order relation '$\leq$'*

Table 1

.	0	1	2	3	4
0	0	3	0	3	3
1	0	1	0	3	3
2	0	3	2	3	4
3	0	3	0	3	3
4	0	3	2	3	4

$\leq := \{(0, 0), (0, 2), (0, 3), (0, 4), (1, 1), (2, 2), (2, 4), (3, 3), (4, 4)\}$. *Let $f = (S; f_n, f_p)$ be a BFS in S defined by table* 2.

Table 2

S	0	1	2	3	4
f_n	-0.75	-0.35	-0.72	-0.58	-0.65
f_p	0.8	0.3	0.7	0.5	0.6

.

We can easily check that $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFBI of S. Now $\frac{0}{(-0.7,0.7)} \in f$ and $\frac{4}{(-0.6,0.55)} \in f$. But $\frac{0\cdot 4}{(\vee\{-0.7,-0.6\},\ \wedge\{0.7,0.55\})} = \frac{3}{(-0.6,0.55)} \notin f$. Therefore $f = (S; f_n, f_p)$ is not an $(\in, \in)$-BFBI of S.

Theorem 4.3.6 *For a BFS $f = (S; f_n, f_p)$ in S, we consider the set*

$$S_\circ = \{\xi \in S \mid f_n(\xi) \neq 0\} \cap \{\xi \in S \mid f_p(\xi) \neq 0\}.$$

For $\beta \in \{\in, q\}$ if $f = (S; f_n, f_p)$ is non-zero $(\in, \beta)$-BFBI of S or a non-zero (q, β)-BFBI of S, then the set $S_\circ$ is bi-ideal of S.

Proof. We first assume that $f = (S; f_n, f_p)$ is non-zero $(\in, \beta)$-BFBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in S_\circ$. Then $f_n(\xi_2) \neq 0$ and $f_p(\xi_2) \neq 0$, so $f_n(\xi_2) < 0$ and $f_p(\xi_2) > 0$. Suppose that $f_n(\xi_1) = 0$ or $f_p(\xi_1) = 0$. Clearly $\frac{\xi_2}{(f_n(\xi_2), f_p(\xi_2))} \in f$. Since $f_n(\xi_1) > f_n(\xi_2)$ or $f_p(\xi_1) < f_p(\xi_2)$. This implies that $\frac{\xi_1}{(f_n(\xi_2), f_p(\xi_2))} \overline{\in} f$, which is impossible. Also $f_n(\xi_1) + f_n(\xi_2) = f_n(\xi_2) \geq -1$ or $f_p(\xi_1) + f_p(\xi_2) = f_p(\xi_2) \leq 1$. This implies that $\frac{\xi_1}{(f_n(\xi_2), f_p(\xi_2))} \overline{\mathrm{q}}\, f$, which is not possible. Therefore $f_n(\xi_1) \neq 0$ and $f_p(\xi_1) \neq 0$. Hence $\xi_1 \in S_\circ$. Let $\xi_1, \xi_2 \in S_\circ$, then $f_n(\xi_1) \neq 0$, $f_n(\xi_2) \neq 0$, $f_p(\xi_1) \neq 0$ and $f_p(\xi_2) \neq 0$, so $f_n(\xi_1) < 0$, $f_n(\xi_2) < 0$, $f_p(\xi_1) > 0$ and $f_p(\xi_2) > 0$. Suppose that $\xi_1\xi_2 \notin S_\circ$, that is, $f_n(\xi_1\xi_2) = 0$ or $f_p(\xi_1\xi_2) = 0$. Clearly $\frac{\xi_1}{(f_n(\xi_1), f_p(\xi_1))} \in f$ and $\frac{\xi_2}{(f_n(\xi_2), f_p(\xi_2))} \in f$. Since $f_n(\xi_1\xi_2) = 0 > \bigvee\{f_n(\xi_1), f_n(\xi_2)\}$ or $f_p(\xi_1\xi_2) = 0 < \bigwedge\{f_p(\xi_1), f_p(\xi_2)\}$. This implies $\frac{\xi_1\xi_2}{(\vee\{f_n(\xi_1), f_n(\xi_2)\}, \wedge\{f_p(\xi_1), f_p(\xi_2)\})} \overline{\in} f$, which is a contradiction. Also

$$f_n(\xi_1\xi_2) + \bigvee\{f_n(\xi_1), f_n(\xi_2)\} = \bigvee\{f_n(\xi_1), f_n(\xi_2)\} \geq -1$$

or

$$f_p(\xi_1\xi_2) + \bigwedge\{f_p(\xi_1), f_p(\xi_2)\} = \bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \leq 1$$

This implies that $\frac{\xi_1\xi_2}{(\vee\{f_n(\xi_1), f_n(\xi_2)\}, \wedge\{f_p(\xi_1), f_p(\xi_2)\})} \overline{\mathrm{q}}\, f$, which is a contradiction. Therefore $f_n(\xi_1\xi_2) \neq 0$ and $f_p(\xi_1\xi_2) \neq 0$. Hence $\xi_1\xi_2 \in S_\circ$.

Let $\xi_1, \xi_2, \xi_3 \in S$ such that $\xi_1, \xi_3 \in S_\circ$, then $f_n(\xi_1) \neq 0$, $f_n(\xi_3) \neq 0$, $f_p(\xi_1) \neq 0$ and $f_p(\xi_3) \neq 0$, so $f_n(\xi_1) < 0$, $f_n(\xi_3) < 0$, $f_p(\xi_1) > 0$ and $f_p(\xi_3) > 0$. Suppose that $\xi_1\xi_2\xi_3 \notin S_\circ$, that is, $f_n(\xi_1\xi_2\xi_3) = 0$ or $f_p(\xi_1\xi_2\xi_3) = 0$. Clearly $\frac{\xi_1}{(f_n(\xi_1), f_p(\xi_1))} \in f$ and $\frac{\xi_3}{(f_n(\xi_3), f_p(\xi_3))} \in f$. Since $f_n(\xi_1\xi_2\xi_3) = 0 > \bigvee\{f_n(\xi_1), f_n(\xi_3)\}$ or $f_p(\xi_1\xi_2\xi_3) = 0 < \bigwedge\{f_p(\xi_1), f_p(\xi_3)\}$. This implies $\frac{\xi_1\xi_2\xi_3}{(\vee\{f_n(\xi_1), f_n(\xi_3)\}, \wedge\{f_p(\xi_1), f_p(\xi_3)\})} \overline{\in} f$, which is a contradiction. Also $f_n(\xi_1\xi_2\xi_3) + \bigvee\{f_n(\xi_1), f_n(\xi_3)\} = \bigvee\{f_n(\xi_1), f_n(\xi_3)\} \geq -1$ or $f_p(\xi_1\xi_2\xi_3) + \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} = \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \leq 1$. This implies that

$$\frac{\xi_1\xi_2\xi_3}{(\vee\{f_n(\xi_1), f_n(\xi_3)\}, \wedge\{f_p(\xi_1), f_p(\xi_3)\})} \overline{\mathrm{q}}\, f,$$

which is a contradiction. Therefore $f_n(\xi_1\xi_2\xi_3) \neq 0$ and $f_p(\xi_1\xi_2\xi_3) \neq 0$. Hence $\xi_1\xi_2\xi_3 \in S_\circ$ and $S_\circ$ is bi-ideal of S.

Now suppose that $f = (S; f_n, f_p)$ is non-zero (q, β)-BFRI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in S_\circ$. Then $f_n(\xi_2) \neq 0$ and $f_p(\xi_2) \neq 0$, so $f_n(\xi_2) < 0$ and $f_p(\xi_2) > 0$. Therefore $f_n(\xi_2) - 1 < -1$ and $f_p(\xi_2) + 1 > 1$, it follows that $\frac{\xi_2}{(-1,1)}\mathrm{q}\ f$. If $f_n(\xi_1) = 0$ or $f_p(\xi_1) = 0$, then $f_n(\xi_1) > -1$ or $f_p(\xi_1) < 1$ and so $\frac{\xi_1}{(-1,1)}\overline{\in} f$, which is a contradiction. Also $f_n(\xi_1) - 1 = -1$ or $f_p(\xi_1) + 1 = 1. \Longrightarrow \frac{\xi_1}{(-1,1)}\overline{\mathrm{q}}\ f$, which is a contradiction. Therefore $f_n(\xi_1) \neq 0$ and $f_p(\xi_1) \neq 0$. Hence $\xi_1 \in S_\circ$. Let $\xi_1, \xi_2 \in S_\circ$. Then $f_n(\xi_1) \neq 0$, $f_n(\xi_2) \neq 0$, $f_p(\xi_1) \neq 0$ and $f_p(\xi_2) \neq 0$, so $f_n(\xi_1) < 0$, $f_n(\xi_2) < 0$, $f_p(\xi_1) > 0$ and $f_p(\xi_2) > 0$. Therefore $f_n(\xi_1) - 1 < -1$, $f_n(\xi_2) - 1 < -1$, $f_p(\xi_1) + 1 > 1$ and $f_p(\xi_2) + 1 > 1$, it follows that $\frac{\xi_1}{(-1,1)}\mathrm{q}$ and $\frac{\xi_2}{(-1,1)}\mathrm{q}\ f$. If $f_n(\xi_1\xi_2) = 0$ or $f_p(\xi_1\xi_2) = 0$, then $f_n(\xi_1\xi_2) > -1$ or $f_p(\xi_1\xi_2) < 1. \Longrightarrow \frac{\xi_1\xi_2}{(-1,1)}\overline{\in} f$, which is a contradiction. Also $f_n(\xi_1\xi_2) - 1 = -1$ or $f_p(\xi_1\xi_2) + 1 = 1$. This implies $\frac{\xi_1\xi_2}{(-1,1)}\overline{\mathrm{q}}\ f$, which is a contradiction. Therefore $f_n(\xi_1\xi_2) \neq 0$ and $f_p(\xi_1\xi_2) \neq 0$. Hence $\xi_1\xi_2 \in S_\circ$. Let $\xi_1, \xi_2, \xi_3 \in S$ such that $\xi_1, \xi_3 \in S_\circ$. Then $f_n(\xi_1) \neq 0$, $f_n(\xi_3) \neq 0$, $f_p(\xi_1) \neq 0$ and $f_p(\xi_3) \neq 0$, so $f_n(\xi_1) < 0$, $f_n(\xi_3) < 0$, $f_p(\xi_1) > 0$ and $f_p(\xi_3) > 0$. Therefore $f_n(\xi_1) - 1 < -1$, $f_n(\xi_3) - 1 < -1$, $f_p(\xi_1) + 1 > 1$ and $f_p(\xi_3) + 1 > 1$, it follows that $\frac{\xi_1}{(-1,1)}\mathrm{q}$ and $\frac{\xi_3}{(-1,1)}\mathrm{q}\ f$. If $f_n(\xi_1\xi_2\xi_3) = 0$ or $f_p(\xi_1\xi_2\xi_3) = 0$, then $f_n(\xi_1\xi_2\xi_3) > -1$ or $f_p(\xi_1\xi_2\xi_3) < 1. \Longrightarrow \frac{\xi_1\xi_2\xi_3}{(-1,1)}\overline{\in} f$, which is a contradiction. Also $f_n(\xi_1\xi_2\xi_3) - 1 = -1$ or $f_p(\xi_1\xi_2\xi_3) + 1 = 1$. This implies $\frac{\xi_1\xi_2\xi_3}{(-1,1)}\overline{\mathrm{q}}\ f$, which is a contradiction. Therefore $f_n(\xi_1\xi_2\xi_3) \neq 0$ and $f_p(\xi_1\xi_2\xi_3) \neq 0$. Hence $\xi_1\xi_2\xi_3 \in S_\circ$ and $S_\circ$ is bi-ideal of S. ■

Theorem 4.3.7 *Let $f = (S; f_n, f_p)$ be a BFS in S. Then $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFBI of S iff $C(f; (s,t)) \neq \emptyset$ is a bi-ideal of S for all $(s,t) \in [-0.5, 0) \times (0, 0.5]$.*

Proof. Let $f = (S; f_n, f_p)$ is an $(\in, \in \vee \mathrm{q})$-BFBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in C(f; (s,t))$. This implies $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee \mathrm{q})$-BFBI of S. So $f_n(\xi_1) \leq \bigvee\{f_n(\xi_2), -0.5\} \leq s$ and $f_p(\xi_1) \geq \bigwedge\{f_p(\xi_2), 0.5\} \geq t$. This implies $\xi_1 \in C(f; (s,t))$. Let $\xi_1, \xi_2 \in C(f; (s,t))$. Then $f_n(\xi_1) \leq s$, $f_n(\xi_2) \leq s$, $f_p(\xi_1) \geq t$ and $f_p(\xi_2) \geq t$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee \mathrm{q})$-BFBI of S. Therefore $f_n(\xi_1\xi_2) \leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\} \leq s$ and $f_p(\xi_1\xi_2) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\} \geq t$. This implies that $\xi_1\xi_2 \in C(f; (s,t))$. Let $\xi_1, \xi_2, \xi_3 \in S$ such that $\xi_1, \xi_3 \in C(f; (s,t))$. Then $f_n(\xi_1) \leq s$, $f_n(\xi_3) \leq s$, $f_p(\xi_1) \geq t$ and $f_p(\xi_3) \geq t$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee \mathrm{q})$-BFBI of S. So $f_n(\xi_1\xi_2\xi_3) \leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\} \leq s$ and $f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\} \geq t$. This implies that $\xi_1\xi_2\xi_3 \in C(f; (s,t))$.Hence $C(f; (s,t))$ is a bi-ideal of S.

Conversely, suppose that the non-empty (s,t)-cut $C(f;(s,t))$ of $f=(S;f_n,f_p)$ is a bi-ideal of S. Assume that there exist $a,b,c\in S$ such that (i) $a\leq b$ and $f_n(a)>\bigvee\{f_n(b),-0.5\}$ or $f_p(a)<\bigwedge\{f_p(b),0.5\}$,(ii) $f_n(ab)>\bigvee\{f_n(a),f_n(b),-0.5\}$ or $f_p(ab)<\bigwedge\{f_p(a),f_p(b),0.5\}$, (iii) $f_n(abc)>\bigvee\{f_n(a),f_n(c),-0.5\}$ or $f_p(abc)<\bigwedge\{f_p(a),f_p(c),0.5\}$.

Let $s=\bigvee\{f_n(b),-0.5\}$ and $t=\bigwedge\{f_p(b),0.5\}$ then $b\in C(f;(s,t))$. If $a\leq b$ and $f_n(a)>\vee\{f_n(b),-0.5\}=s$ then $a\notin C(f;(s,t))$. If $f_p(a)<\bigwedge\{f_p(b),0.5\}=t$, then $a\notin C(f;(s,t))$, which is impossible because $C(f(s,t))$ is bi-ideal of S. Let $\acute{s}=\bigvee\{f_n(a),f_n(b),-0.5\}$ and $\acute{t}=\bigwedge\{f_p(a),f_p(b),0.5\}$, then $a,b\in C(f(\acute{s},\acute{t}))$. If $f_n(ab)>\bigvee\{f_n(a),f_n(b),-0.5\}=\acute{s}$ then $ab\notin C(f;(\acute{s},\acute{t}))$. If $f_p(ab)<\bigwedge\{f_p(a),f_p(b),0.5\}=\acute{t}$ then $ab\notin C(f;(\acute{s},\acute{t}))$, which is contradiction to the fact that $C(f;(\acute{s},\acute{t}))$ is bi-ideal of S. Let $\acute{s}=\bigvee\{f_n(a),f_n(c),-0.5\}$ and $\acute{t}=\bigwedge\{f_p(a),f_p(c),0.5\}$, then $a,c\in C(f(\acute{s},\acute{t}))$. If $f_n(abc)>\bigvee\{f_n(a),f_n(c),-0.5\}=\acute{s}$ then $abc\notin C(f;(\acute{s},\acute{t}))$. If $f_p(abc)<\bigwedge\{f_p(a),f_p(c),0.5\}=\acute{t}$ then $abc\notin C(f;(\acute{s},\acute{t}))$, which is contradiction to the fact that $C(f;(\acute{s},\acute{t}))$ is bi-ideal of S. Hence $f=(S;f_n,f_p)$ is an $(\in,\in\vee\mathrm{q})$-BFBI of S. ■

Theorem 4.3.8 *Let B be a bi-ideal of S and $f=(S;f_n,f_p)$ a BFS in S such that*

i) $(\forall\xi\in S\setminus B)(f_n(\xi)=0=\ f_p(\xi))$,

ii) $(\forall\xi\in B)((f_n(\xi),f_p(\xi))\in[-1,-0.5]\times[0.5,1])$.

Then $f=(S;f_n,f_p)$ is $(q,\in\vee q)$-BFBI of S.

Proof. Let $\xi_1,\xi_2\in S$ such that $\xi_1\leq\xi_2$ and $(s,t)\in[-1,0)\times(0,1]$. Let $\frac{\xi_2}{(s,t)}\mathrm{q}\ f$ then $f_n(\xi_2)+s<-1$ and $f_p(\xi_2)+t>1$. This implies that $\xi_2\in B$. Since B is bi-ideal of S, so $\xi_1\in B$. In order to check $\frac{\xi_1}{(s,t)}\in\vee\mathrm{q}\ f$, we consider the following four cases:

1) $s\geq-0.5$ and $t\leq0.5$,

2) $s<-0.5$ and $t>0.5$,

3) $s<-0.5$ and $t\leq0.5$,

4) $s\geq-0.5$ and $t>0.5$.

The first case induces $f_n(\xi_1) \leq -0.5 \leq s$ and $f_p(\xi_1) \geq 0.5 \geq t$. This implies $\frac{\xi_1}{(s,t)} \in f$. The second case implies that $f_n(\xi_1) + s < -1$ and $f_p(\xi_1) + t > 1$. Thus $\frac{\xi_1}{(s,t)}$ q f. Since $\frac{\xi_2}{(s,t)}$ q f, so case (3) and (4) do not occur. Consequently, $\frac{\xi_1}{(s,t)} \in \vee$q f. Let $\xi_1, \xi_2 \in S$ such that $\frac{\xi_1}{(s_1,t_1)}$q f and $\frac{\xi_2}{(s_2,t_2)}$q f for $(s_1,t_1),(s_2,t_2) \in [-1,0) \times (0,1]$. Then $f_n(\xi_1) + s_1 < -1$, $f_n(\xi_2) + s_2 < -1$, $f_p(\xi_1) + t_1 > 1$ and $f_p(\xi_2) + t_2 > 1$. This implies that $\xi_1, \xi_2 \in B$. Since B is bi-ideal of S, so $\xi_1\xi_2 \in B$. In order to check $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \vee$q f, we consider the following four cases:

1) $\bigvee\{s_1, s_2\} \geq -0.5$ and $\bigwedge\{t_1, t_2\} \leq 0.5$,

2) $\bigvee\{s_1, s_2\} < -0.5$ and $\bigwedge\{t_1, t_2\} > 0.5$,

3) $\bigvee\{s_1, s_2\} < -0.5$ and $\bigwedge\{t_1, t_2\} \leq 0.5$,

4) $\bigvee\{s_1, s_2\} \geq -0.5$ and $\bigwedge\{t_1, t_2\} > 0.5$.

The first case induces $f_n(\xi_1\xi_2) \leq -0.5 \leq \bigvee\{s_1,s_2\}$ and $f_p(\xi_1\xi_2) \geq 0.5 \geq \bigwedge\{t_1,t_2\}$ and so $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in f$. The second case implies that $f_n(\xi_1\xi_2) + \bigvee\{s_1,s_2\} < -1$ and $f_p(\xi_1\xi_2) + \bigwedge\{t_1,t_2\} > 1$. Thus $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})}$q f. Since $\frac{\xi_1}{(s_1,t_1)}$q f and $\frac{\xi_2}{(s_2,t_2)}$q f, so case (3) and (4) do not occur. Consequently, $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \vee$q f.

Let $\xi_1, \xi_2, \xi_3 \in S$ such that $\frac{\xi_1}{(s_1,t_1)}$q f and $\frac{\xi_3}{(s_2,t_2)}$q f for $(s_1,t_1),(s_2,t_2) \in [-1,0) \times (0,1]$. Then $f_n(\xi_1)+s_1 < -1$, $f_n(\xi_3)+s_2 < -1$, $f_p(\xi_1)+t_1 > 1$ and $f_p(\xi_3)+t_2 > 1$. This implies that $\xi_1, \xi_3 \in B$. Since B is bi-ideals of S, so $\xi_1\xi_2\xi_3 \in B$. In order to check $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \vee$q f, we consider the following four cases:

1) $\bigvee\{s_1, s_2\} \geq -0.5$ and $\bigwedge\{t_1, t_2\} \leq 0.5$,

2) $\bigvee\{s_1, s_2\} < -0.5$ and $\bigwedge\{t_1, t_2\} > 0.5$,

3) $\bigvee\{s_1, s_2\} < -0.5$ and $\bigwedge\{t_1, t_2\} \leq 0.5$,

4) $\bigvee\{s_1, s_2\} \geq -0.5$ and $\bigwedge\{t_1, t_2\} > 0.5$.

The first case induces $f_n(\xi_1\xi_2\xi_3) \leq -0.5 \leq \bigvee\{s_1,s_2\}$ and $f_p(\xi_1\xi_2\xi_3) \geq 0.5 \geq \bigwedge\{t_1,t_2\}$ and so $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in f$. The second case implies that $f_n(\xi_1\xi_2\xi_3)+\bigvee\{s_1,s_2\} < -1$ and $f_p(\xi_1\xi_2\xi_3)+\bigwedge\{t_1,t_2\} > 1$. Thus $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})}$q f. Since $\frac{\xi_1}{(s_1,t_1)}$ q f and $\frac{\xi_3}{(s_2,t_2)}$ q f, so case (3) and (4) do not occur. Consequently, $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \vee$q f. ■

Theorem 4.3.9 *Every $(q, \in \vee q)$-BFBI of S is an $(\in, \in \vee q)$-BFBI of S.*

Proof. Let $f = (S; f_n, f_p)$ be a (q, $\in \vee$q)-BFBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $(s,t) \in [-1, 0) \times (0, 1]$. Let $\frac{\xi_2}{(s,t)} \in f$, then $f_n(\xi_2) \leq s$ and $f_p(\xi_2) \geq t$. Suppose $\frac{\xi_1}{(s,t)} \overline{\in \vee \mathrm{q}}\ f$, then $\frac{\xi_1}{(s,t)} \overline{\in}\ f$ and $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$. If $\frac{\xi_1}{(s,t)} \overline{\in}\ f$, then $f_n(\xi_1) > s$ or $f_p(\xi_1) < t$. If $\frac{\xi_1}{(s,t)} \overline{\mathrm{q}}\ f$, then $f_n(\xi_1) + s > -1$ or $f_p(\xi_1) + t < 1$. From the fact that

$$f_n(\xi_1) > s \text{ and } f_n(\xi_1) + s > -1,$$

we have $f_n(\xi_1) > -0.5$ and so $f_n(\xi_1) > \bigvee\{s, -0.5\}$. Thus

$$\begin{aligned} -1 - f_n(\xi_1) &< -1 - \bigvee\{s, -0.5\} = \bigwedge\{-1 - s, -1 + 0.5\} \\ &\leq \bigwedge\{-1 - f_n(\xi_2), -0.5\} \end{aligned}$$

which implies that there exists $\acute{s} \in [-1, 0)$ such that

$$-1 - f_n(\xi_1) \leq \acute{s} < \bigwedge\{-1 - f_n(\xi_2), -0.5\}. \tag{4.1}$$

This implies $\acute{s} < -1 - f_n(\xi_2)$ and so $f_n(\xi_2) + \acute{s} < -1$. Also from the fact that $f_p(\xi_1) < t$ and $f_p(\xi_1) + t < 1$, we have $f_p(\xi_1) < 0.5$ and so $f_p(\xi_1) < \bigwedge\{t, 0.5\}$. This implies that

$$\begin{aligned} 1 - f_p(\xi_1) &> 1 - \bigwedge\{t, 0.5\} = \bigvee\{1 - t, 1 - 0.5\} \\ &\geq \bigvee\{1 - f_p(\xi_2), 0.5\}. \end{aligned}$$

Therefore there exists $\acute{t} \in (0, 1]$ such that

$$1 - f_p(\xi_1) \geq \acute{t} > \bigvee\{1 - f_p(\xi_2), 0.5\}. \tag{4.2}$$

This implies $\grave{t} > 1 - f_p(\xi_2)$ and so $f_p(\xi_2) + \acute{t} > 1$. Therefore $\frac{\xi_2}{(\acute{s},\acute{t})} \mathrm{q}\ f$. Since $f = (S; f_n, f_p)$ is (q, $\in \vee$q)-BFBI of S so $\frac{\xi_1}{(\acute{s},\acute{t})} \in \vee \mathrm{q}\ f$. Now the left inequality in (4.1) induces $-1 - f_n(\xi_1) \leq \acute{s}$ and $f_n(\xi_1) \geq -1 - \acute{s} > -1 + 0.5 = -0.5 > \acute{s}$. Also the left inequality in (4.2) induces $1 - f_p(\xi_1) \geq \acute{t}$ and $f_p(\xi_1) \leq 1 - \acute{t} < 1 - 0.5 = 0.5 < \acute{t}$. Hence $\frac{\xi_1}{(\acute{s},\acute{t})} \overline{\in \vee \mathrm{q}}\ f$, a contradiction. Therefore $\frac{\xi_1}{(s,t)} \in \vee \mathrm{q}$

f $\forall(s,t) \in [-1,0) \times (0,1]$.

Let $\xi_1, \xi_2 \in S$ such that $\frac{\xi_1}{(s_1,t_1)} \in f$ and $\frac{\xi_2}{(s_2,t_2)} \in f$ for $(s_1,t_1),(s_2,t_2) \in [-1,0) \times (0,1]$. Then $f_n(\xi_1) \leq s_1, f_n(\xi_2) \leq s_2, f_p(\xi_1) \geq t_1$ and $f_p(\xi_2) \geq t_2$. Suppose $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \overline{\in \vee \mathrm{q}}\ f$, then $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \overline{\in} f$ and $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \overline{\mathrm{q}}\ f$. If $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \overline{\in} f$, then $f_n(\xi_1\xi_2) > \bigvee\{s_1,s_2\}$ or $f_p(\xi_1\xi_2) < \bigwedge\{t_1,t_2\}$. If $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \overline{\mathrm{q}}\ f$, then $f_n(\xi_1\xi_2) + \bigvee\{s_1,s_2\} > -1$ or $f_p(\xi_1\xi_2) + \bigwedge\{t_1,t_2\} < 1$. From the fact that

$$f_n(\xi_1\xi_2) > \bigvee\{s_1,s_2\} \text{ and } f_n(\xi_1\xi_2) + \bigvee\{s_1,s_2\} > -1,$$

we have $f_n(\xi_1\xi_2) > -0.5$ and so $f_n(\xi_1\xi_2) > \bigvee\{s_1,s_2,-0.5\}$. Thus

$$\begin{aligned} -1 - f_n(\xi_1\xi_2) < -1 - \bigvee\{s_1,s_2,-0.5\} &= \bigwedge\{-1-s_1,-1-s_2-1+0.5\} \\ &\leq \bigwedge\{-1-f_n(\xi_1),-1-f_n(\xi_2),-0.5\} \end{aligned}$$

which implies that there exists $\acute{s} \in [-1,0)$ such that

$$-1 - f_n(\xi_1\xi_2) \leq \acute{s} < \bigwedge\{-1-f_n(\xi_1),-1-f_n(\xi_2),-0.5\}. \tag{4.3}$$

This implies $\acute{s} < -1 - f_n(\xi_1)$, $\acute{s} < -1 - f_n(\xi_2)$ and so $f_n(\xi_1) + \acute{s} < -1$, $f_n(\xi_2) + \acute{s} < -1$. Also from the fact that $f_p(\xi_1\xi_2) < \bigwedge\{t_1,t_2\}$ and $f_p(\xi_1\xi_2) + \bigwedge\{t_1,t_2\} < 1$, we have $f_p(\xi_1\xi_2) < 0.5$ and so $f_p(\xi_1\xi_2) < \bigwedge\{t_1,t_2,0.5\}$. This implies that

$$\begin{aligned} 1 - f_p(\xi_1\xi_2) > 1 - \bigwedge\{t_1,t_2,0.5\} &= \bigvee\{1-t_1,1-t_2,1-0.5\} \\ &\geq \bigvee\{1-f_p(\xi_1),1-f_p(\xi_2),0.5\}. \end{aligned}$$

Therefore there exists $\acute{t} \in (0,1]$ such that

$$1 - f_p(\xi_1\xi_2) \geq \acute{t} > \bigvee\{1-f_p(\xi_1),1-f_p(\xi_2),0.5\}. \tag{4.4}$$

This implies $\grave{t} > 1 - f_p(\xi_1)$, $\grave{t} > 1 - f_p(\xi_2)$ and so $\grave{t} + f_p(\xi_1) > 1$, $\grave{t} + f_p(\xi_2) > 1$. Therefore $\frac{\xi_1}{(\acute{s},\acute{t})}\mathrm{q}$ f and $\frac{\xi_2}{(\acute{s},\acute{t})}\mathrm{q}\ f$. Since $f = (S; f_n, f_p)$ is $(\mathrm{q}, \in \vee \mathrm{q})$-BFBI of S. Therefore $\frac{\xi_1\xi_2}{(\acute{s},\acute{t})} \in \vee \mathrm{q}\ f$. Now the left inequality in (4.3) induces $-1 - f_n(\xi_1\xi_2) \leq \acute{s}$ and $f_n(\xi_1\xi_2) \geq -1 - \acute{s} > -1 + 0.5 = -0.5 > \acute{s}$.

Also the left inequality in (4.4) induces $1-f_p(\xi_1\xi_2) \geq \acute{t}$ and $f_p(\xi_1\xi_2) \leq 1-\acute{t} < 1-0.5 = 0.5 < \acute{t}$. Hence $\frac{\xi_1\xi_2}{(\acute{s},\acute{t})} \overline{\in \vee \mathrm{q}}\, f$, a contradiction. Therefore $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \vee\mathrm{q}\ f$ for all $(s_1,t_1),(s_2,t_2) \in [-1,0)\times(0,1]$.

Let $\xi_1,\xi_2,\xi_3 \in S$ such that $\frac{\xi_1}{(s_1,t_1)} \in f$ and $\frac{\xi_3}{(s_2,t_2)} \in f$ for $(s_1,t_1),(s_2,t_2) \in [-1,0)\times(0,1]$. Then $f_n(\xi_1) \leq s_1$, $f_n(\xi_2) \leq s_2$, $f_p(\xi_1) \geq t_1$ and $f_p(\xi_2) \geq t_2$. Suppose $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \overline{\in \vee\mathrm{q}}$ f, then $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \overline{\in}\ f$ and $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \overline{\mathrm{q}}\ f$. If $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \overline{\in}\ f$, then $f_n(\xi_1\xi_2\xi_3) > \bigvee\{s_1,s_2\}$ or $f_p(\xi_1\xi_2\xi_3) < \bigwedge\{t_1,t_2\}$. If $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \overline{\mathrm{q}}\ f$, then $f_n(\xi_1\xi_2\xi_3) + \bigvee\{s_1,s_2\} > -1$ or $f_p(\xi_1\xi_2\xi_3) + \bigwedge\{t_1,t_2\} < 1$. From the fact that

$$f_n(\xi_1\xi_2\xi_3) > \bigvee\{s_1,s_2\} \text{ and } f_n(\xi_1\xi_2\xi_3) + \bigvee\{s_1,s_2\} > -1,$$

we have $f_n(\xi_1\xi_2\xi_3) > -0.5$ and so $f_n(\xi_1\xi_2\xi_3) > \bigvee\{s_1,s_2,-0.5\}$. Thus

$$\begin{aligned} -1-f_n(\xi_1\xi_2\xi_3) &< -1-\bigvee\{s_1,s_2,-0.5\} = \bigwedge\{-1-s_1,-1-s_2-1+0.5\} \\ &\leq \bigwedge\{-1-f_n(\xi_1),-1-f_n(\xi_3),-0.5\} \end{aligned}$$

which implies that there exists $\acute{s} \in [-1,0)$ such that

$$-1-f_n(\xi_1\xi_2\xi_3) \leq \acute{s} < \bigwedge\{-1-f_n(\xi_1),-1-f_n(\xi_3),-0.5\}. \tag{4.5}$$

This implies $\acute{s} < -1-f_n(\xi_1)$, $\acute{s} < -1-f_n(\xi_3)$ and so $f_n(\xi_1)+\acute{s} < -1$, $f_n(\xi_3)+\acute{s} < -1$. Also from the fact that $f_p(\xi_1\xi_2\xi_3) < \bigwedge\{t_1,t_2\}$ and $f_p(\xi_1\xi_2\xi_3) + \bigwedge\{t_1,t_2\} < 1$, we have $f_p(\xi_1\xi_2\xi_3) < 0.5$ and so $f_p(\xi_1\xi_2\xi_3) < \bigwedge\{t_1,t_2,0.5\}$. This implies that

$$\begin{aligned} 1-f_p(\xi_1\xi_2\xi_3) &> 1-\bigwedge\{t_1,t_2,0.5\} = \bigvee\{1-t_1,1-t_2,1-0.5\} \\ &\geq \bigvee\{1-f_p(\xi_1),1-f_p(\xi_3),0.5\}. \end{aligned}$$

Therefore there exists $\acute{t} \in (0,1]$ such that

$$1-f_p(\xi_1\xi_2\xi_3) \geq \acute{t} > \bigvee\{1-f_p(\xi_1),1-f_p(\xi_3),0.5\}. \tag{4.6}$$

This implies $\grave{t} > 1-f_p(\xi_1)$, $\grave{t} > 1-f_p(\xi_3)$ and so $\grave{t}+f_p(\xi_1) > 1$, $\grave{t}+f_p(\xi_3) > 1$. Therefore $\frac{\xi_1}{(\acute{s},\acute{t})}\text{q}\ f$ and $\frac{\xi_3}{(\acute{s},\acute{t})}\text{q}\ f$. Since $f = (S; f_n, f_p)$ is $(\text{q}, \in \vee\text{q})$-BFBI of S so $\frac{\xi_1\xi_2\xi_3}{(\acute{s},\acute{t})} \in \vee\text{q}\ f$. Now the left inequality in (4.5) induces $-1 - f_n(\xi_1\xi_2\xi_3) \leq \acute{s}$ and $f_n(\xi_1\xi_2\xi_3) \geq -1-\acute{s} > -1+0.5 = -0.5 > \acute{s}$. Also the left inequality in (4.6) induces $1 - f_p(\xi_1\xi_2\xi_3) \geq \acute{t}$ and $f_p(\xi_1\xi_2\xi_3) \leq 1-\acute{t} < 1-0.5 = 0.5 < \acute{t}$. Hence $\frac{\xi_1\xi_2\xi_3}{(\acute{s},\acute{t})} \overline{\in \vee\text{q}}\ f$, a contradiction. Therefore $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \vee\text{q}\ f$ for all $(s_1,t_1),(s_2,t_2) \in [-1,0) \times (0,1]$. Hence $f = (S; f_n, f_p)$ is an $(\in, \in \vee\text{q})$-BFBI of S. ■

In general every $(\in, \in \vee\text{q})$-BFBI may not be $(\text{q}, \in \vee\text{q})$-BFBI as demonstrated by the upcoming example.

Example 4.3.10 *Let $S = \{0,1,2,3,4\}$ be set with multiplication table and order relation as in Example 4.3.5. Let $f = (S; f_n, f_p)$ be a BFS in S defined by the table 3.*

Table 3

S	0	1	2	3	4
f_n	-0.9	-0.35	-0.72	-0.58	-0.65
f_p	0.8	0.3	0.7	0.5	0.6

Then $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFBI of S. But, $f = (S; f_n, f_p)$ is not $(q, \in \vee q)$-BFBI of S because $\frac{0}{(-0.15,0.7)}q\ f$ and $\frac{4}{(-0.45,0.52)}q\ f$ but $\frac{0\cdot 4}{(\vee\{-0.15,-0.45\},\wedge\{0.7,0.52\})} = \frac{3}{(-0.15,0.52)} \overline{\in \vee q}\ f$.

Theorem 4.3.11 *Let every bipolar fuzzy point has the value $(s,t) \in [-0.5,0) \times (0,0.5]$. Then every $(\in, \in \vee q)$-BFBI of S is an $(q, \in \vee q)$-BFBI of S.*

Proof. Let $f = (S; f_n, f_p)$ be an $(\in, \in \vee\text{q})$-BFBI of S. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\frac{\xi_2}{(s,t)}\text{q}\ f$ for $(s,t) \in [-0.5,0) \times (0,0.5]$. Then $f_n(\xi_2) + s < -1$ and $f_p(\xi_2) + t > 1$. This implies that $f_n(\xi_2) < -1-s \leq s$ and $f_p(\xi_2) > 1-t \geq t$ and so $\frac{\xi_2}{(s,t)} \in f$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee\text{q})$-BFBI of S so $\frac{\xi_1}{(s,t)} \in \vee\text{q}\ f$. Let $\xi_1, \xi_2 \in S$ such that $\frac{\xi_1}{(s_1,t_1)}\text{q}\ f$ and $\frac{\xi_2}{(s_2,t_2)}\text{q}\ f$ for $(s_1,t_1),(s_2,t_2) \in [-0.5,0) \times (0,0.5]$. Then $f_n(\xi_1)+s_1 < -1$, $f_n(\xi_2)+s_2 < -1$, $f_p(\xi_1)+t_1 > 1$ and $f_p(\xi_2)+t_2 > 1$. This implies that $f_n(\xi_1) < -1-s_1 \leq s_1$, $f_n(\xi_2) < -1-s_2 \leq s_2$, $f_p(\xi_1) > 1-t_1 \geq t_1$ and $f_p(\xi_2) > 1-t_2 \geq t_2$. Hence $\frac{\xi_1}{(s_1,t_1)} \in f$ and $\frac{\xi_2}{(s_2,t_2)} \in f$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee\text{q})$-BFBI of S. Therefore $\frac{\xi_1\xi_2}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \vee\text{q}\ f$.

Let $\xi_1, \xi_2, \xi_3 \in S$ such that $\frac{\xi_1}{(s_1,t_1)}\text{q}\ f$ and $\frac{\xi_3}{(s_2,t_2)}\text{q}\ f$ for $(s_1,t_1),(s_2,t_2) \in [-0.5,0) \times (0,0.5]$. Then $f_n(\xi_1)+s_1 < -1$, $f_n(\xi_3)+s_2 < -1$, $f_p(\xi_1)+t_1 > 1$ and $f_p(\xi_3)+t_2 > 1$. This implies that $f_n(\xi_1) < -1 - s_1 \leq s_1$, $f_n(\xi_3) < -1 - s_2 \leq s_2$, $f_p(\xi_1) > 1 - t_1 \geq t_1$ and $f_p(\xi_3) > 1 - t_2 \geq t_2$. Hence $\frac{\xi_1}{(s_1,t_1)} \in f$ and $\frac{\xi_3}{(s_2,t_2)} \in f$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee \mathrm{q})$-BFBI of S so $\frac{\xi_1\xi_2\xi_3}{(\vee\{s_1,s_2\},\wedge\{t_1,t_2\})} \in \vee\mathrm{q}\ f$. Hence $f = (S; f_n, f_p)$ is $(\mathrm{q}, \in \vee\mathrm{q})$-BFBI of S. ■

Definition 4.3.12 *For a BFS $f = (S; f_n, f_p)$ of S and $(s,t) \in [-1,0) \times (0,1]$, we defined*

$$Q_f^{(s,t)} = \{\xi \in S \mid \frac{\xi}{(s,t)} q\ f\}.$$

Theorem 4.3.13 *If $f = (S; f_n, f_p)$ is an $(\in, \in \vee q)$-BFBI of S, then the set*

$$Q_f^{(s,t)} = \{\xi \in S \mid \frac{\xi}{(s,t)} q\ f\}$$

is bi-ideal of S for all $(s,t) \in [-1,-0.5) \times (0.5,1]$ with $Q_f^{(s,t)} \neq \emptyset$.

Proof. Let $\xi_1, \xi_2 \in S$ such that $\xi_1 \leq \xi_2$ and $\xi_2 \in Q_f^{(s,t)}$ for all $(s,t) \in [-1,-0.5) \times (0.5,1]$. Then $\frac{\xi_2}{(s,t)}\mathrm{q}\ f$, that is $f_n(\xi_2) + s < -1$ and $f_p(\xi_2) + t > 1$. This implies $f_n(\xi_2) < -1 - s$ and $f_p(\xi_2) > 1 - t$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee\mathrm{q})$-BFBI of S. It follows that

$$\begin{aligned} f_n(\xi_1) &\leq \bigvee\{f_n(\xi_2), -0.5\} \\ &= \begin{cases} f_n(\xi_2) & \text{if } f_n(\xi_2) \geq -0.5 \\ -0.5 & \text{if } -0.5 > f_n(\xi_2) \end{cases} \\ &< -1 - s \end{aligned}$$

and

$$\begin{aligned} f_p(\xi_1) &\geq \bigwedge\{f_p(\xi_2), 0.5\} \\ &= \begin{cases} f_p(\xi_2) & \text{if } f_p(\xi_2) \leq 0.5 \\ 0.5 & \text{if } f_p(\xi_2) > 0.5 \end{cases} \\ &> 1 - t. \end{aligned}$$

Hence $\frac{\xi_1}{(s,t)}\mathrm{q}\ f$, and so $\xi_1 \in Q_f^{(s,t)}$.

Let $\xi_1, \xi_2 \in Q_f^{(s,t)}$ for all $(s,t) \in [-1,-0.5) \times (0.5,1]$. Then $\frac{\xi_1}{(s,t)}\text{q } f$ and $\frac{\xi_2}{(s,t)}\text{q } f$, that is $f_n(\xi_1) + s < -1,\ f_n(\xi_2) + s < -1, f_p(\xi_1) + t > 1$ and $f_p(\xi_2) + t > 1$. This implies $f_n(\xi_1) < -1 - s, f_n(\xi_2) < -1 - s, f_p(\xi_1) > 1 - t$ and $f_p(\xi_2) > 1 - t$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee\text{q})$-BFBI of S. It follows that

$$\begin{aligned} f_n(\xi_1\xi_2) &\leq \bigvee\{f_n(\xi_1), f_n(\xi_2), -0.5\} \\ &= \begin{cases} \bigvee\{f_n(\xi_1), f_n(\xi_2)\} & \text{if } \bigvee\{f_n(\xi_1), f_n(\xi_2)\} \geq -0.5 \\ -0.5 & \text{if } -0.5 > \bigvee\{f_n(\xi_1), f_n(\xi_2)\} \end{cases} \\ &< -1 - s, \end{aligned}$$

and

$$\begin{aligned} f_p(\xi_1\xi_2) &\geq \bigwedge\{f_p(\xi_1), f_p(\xi_2), 0.5\} \\ &= \begin{cases} \bigwedge\{f_p(\xi_1), f_p(\xi_2)\} & \text{if } \bigwedge\{f_p(\xi_1), f_p(\xi_2)\} \leq 0.5 \\ 0.5 & \text{if } \bigwedge\{f_p(\xi_1), f_p(\xi_2)\} > 0.5 \end{cases} \\ &> 1 - t. \end{aligned}$$

Hence $\frac{\xi_1\xi_2}{(s,t)}\text{q } f$, and so $\xi_1\xi_2 \in Q_f^{(s,t)}$.

Let $\xi_1, \xi_2, \xi_3 \in S$ such that $\xi_1, \xi_3 \in Q_f^{(s,t)}$ for all $(s,t) \in [-1,-0.5) \times (0.5,1]$. Then $\frac{\xi_1}{(s,t)}\text{q } f$ and $\frac{\xi_3}{(s,t)}\text{q } f$, that is $f_n(\xi_1) + s < -1, f_n(\xi_3) + s < -1, f_p(\xi_1) + t > 1$ and $f_p(\xi_3) + t > 1$. This implies $f_n(\xi_1) < -1 - s, f_n(\xi_3) < -1 - s, f_p(\xi_1) > 1 - t$ and $f_p(\xi_3) > 1 - t$. Since $f = (S; f_n, f_p)$ is an $(\in, \in \vee\text{q})$-BFBI of S. It follows that

$$\begin{aligned} f_n(\xi_1\xi_2\xi_3) &\leq \bigvee\{f_n(\xi_1), f_n(\xi_3), -0.5\} \\ &= \begin{cases} \bigvee\{f_n(\xi_1), f_n(\xi_3)\} & \text{if } \bigvee\{f_n(\xi_1), f_n(\xi_3)\} \geq -0.5 \\ -0.5 & \text{if } -0.5 > \bigvee\{f_n(\xi_1), f_n(\xi_3)\} \end{cases} \\ &< -1 - s, \end{aligned}$$

and

$$f_p(\xi_1\xi_2\xi_3) \geq \bigwedge\{f_p(\xi_1), f_p(\xi_3), 0.5\}$$
$$= \begin{cases} \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} & \text{if} \quad \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} \leq 0.5 \\ 0.5 & \text{if} \quad \bigwedge\{f_p(\xi_1), f_p(\xi_3)\} > 0.5 \end{cases}$$
$$> 1 - t.$$

Hence $\frac{\xi_1\xi_2\xi_3}{(s,t)} \mathrm{q}\, f$, and so $\xi_1\xi_2\xi_3 \in Q_f^{(s,t)}$. Therefore $Q_f^{(s,t)}$ is bi-ideal of S. ■

Chapter 5

Characterization of Ordered Semigroups in Terms of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-Bipolar Fuzzy Interior Ideals

This chapter is our accepted paper [88]. Recently much attention has been paid on the characterization of many algebraic structures in terms of new types of BFIs. BFIs are used significantly in ordered semigroup, hemirings and BCK/BCI-algebras. In this chapter the generalization of ordered semigroup in the frame work of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFIIs have been evaluated and well explored. Different classes of S (regular, intra-regular) are characterized in terms of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFIIs (resp $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFIs). It has been proved that the concept of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFIIs and $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFIs coincides in regular, intra-regular ordered semigroup. Also the notion of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy prime (semiprime) ideals are discussed and get some interesting results.

5.1 $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy interior ideals of regular and intra-regular ordered semigroups

In this section, we provide the concept of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFLI (resp. BFRI) and $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFII in S. We prove that every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFI is $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFII but the converse is not true in general. We show that in regular and intra-regular ordered semigroup, the concept of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFI and $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFII coincide. We also show that S is intra-regular iff each $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFII of S, we have $f_n^{(\alpha_2,\alpha_1)}(\xi) = f_n^{(\alpha_2,\alpha_1)}(\xi^2)$ and $f_p^{(\beta_2,\beta_1)}(\xi) = f_p^{(\beta_2,\beta_1)}(\xi^2)$ for all $\xi \in S$ and $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$.

Definition 5.1.1 *A BFS $f = (S; f_n, f_p)$ of S is called $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy subsemigroup of S where $\alpha_1, \alpha_2 \in [-1, 0]$ and $\beta_1, \beta_2 \in [0, 1]$ if it satisfies the following conditions.*

1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Longrightarrow \min\{f_n(\xi_1), \alpha_1\} \leq \max\{f_n(\xi_2), \alpha_2\}$ and $\max\{f_p(\xi_1), \beta_1\} \geq \min\{f_p(\xi_2), \beta_2\})$.

2) $(\forall \xi_1, \xi_2 \in S)(\min\{f_n(\xi_1\xi_2), \alpha_1\} \leq \max\{f_n(\xi_1), f_n(\xi_2), \alpha_2\}$ and $\max\{f_p(\xi_1\xi_2), \beta_1\} \geq \min\{f_p(\xi_1), f_p(\xi_2), \beta_2\})$.

Definition 5.1.2 *A BFS $f = (S; f_n, f_p)$ of S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFLI (resp. BFRI) of S where $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$ if it satisfies the following conditions:*

1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Longrightarrow \min\{f_n(\xi_1), \alpha_1\} \leq \max\{f_n(\xi_2), \alpha_2\}$ and $\max\{f_p(\xi_1), \beta_1\} \geq \min\{f_p(\xi_2), \beta_2\})$.

2) $(\forall \xi_1, \xi_2 \in S)(\min\{f_n(\xi_1\xi_2), \alpha_1\} \leq \max\{f_n(\xi_2), \alpha_2\}$(resp. $\min\{f_n(\xi_1\xi_2), \alpha_1\} \leq \max\{f_n(\xi_1), \alpha_2\})$ and $\max\{f_p(\xi_1\xi_2), \beta_1\} \geq \min\{f_p(\xi_2), \beta_2\}$ (resp $\max\{f_p(\xi_1\xi_2), \beta_1\} \geq \min\{f_p(\xi_1), \beta_2\}))$.

A BFS $f = (S; f_n, f_p)$ of S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFI of S if it is both a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFLI and $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFRI of S.

Definition 5.1.3 *A BFS $f = (S; f_n, f_p)$ of S is called $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFII of S where $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$ if it satisfies the following conditions:*

1) $(\forall \xi_1, \xi_2 \in S)(\xi_1 \leq \xi_2 \Longrightarrow \min\{f_n(\xi_1), \alpha_1\} \leq \max\{f_n(\xi_2), \alpha_2\}$ and $\max\{f_p(\xi_1), \beta_1\} \geq \min\{f_p(\xi_2), \beta_2\})$.

2) $(\forall \xi_1, \xi_2 \in S)(\min\{f_n(\xi_1\xi_2), \alpha_1\} \leq \max\{f_n(\xi_1), f_n(\xi_2), \alpha_2\}$ and $\max\{f_p(\xi_1\xi_2), \beta_1\} \geq \min\{f_p(\xi_1), f_p(\xi_2), \beta_2\})$.

3) $(\forall \xi_1, a, \xi_2 \in S)(\min\{f_n(\xi_1 a\xi_2), \alpha_1\} \leq \max\{f_n(a), \alpha_2\}$ and $\max\{f_p(\xi_1 a\xi_2), \beta_1\} \geq \min\{f_p(a), \beta_2\})$.

Theorem 5.1.4 *Let $f = (S; f_n, f_p)$ be a BFS of S then every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFI is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFII of S.*

Proof. Assume that $f = (S; f_n, f_p)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFI of S then for $\xi_1, \xi_2 \in S$, we have

$$\begin{cases} \min\{f_n(\xi_1\xi_2), \alpha_1\} \leq \max\{f_n(\xi_1), \alpha_2\} \text{ and} \\ \min\{f_n(\xi_1\xi_2), \alpha_1\} \leq \max\{f_n(\xi_2), \alpha_2\}. \end{cases}$$

This implies that $\min\{f_n(\xi_1\xi_2), \alpha_1\} \leq \max\{f_n(\xi_1), f_n(\xi_2), \alpha_2\}$.

And

$$\begin{cases} \max\{f_p(\xi_1\xi_2), \beta_1\} \geq \min\{f_p(\xi_1), \beta_2\} \text{ and} \\ \max\{f_p(\xi_1\xi_2), \beta_1\} \geq \min\{f_p(\xi_2), \beta_2\}. \end{cases}$$

This implies that $\max\{f_p(\xi_1\xi_2), \beta_1\} \geq \min\{f_p(\xi_1), f_p(\xi_2), \beta_2\}$.

Thus $f = (S; f_n, f_p)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- bipolar fuzzy subsemigroup.

Let $\xi_1, a, \xi_2 \in S$ then

$$\begin{aligned} \min\{f_n(\xi_1 a\xi_2), \alpha_1\} &= \min\{\min\{\min\{f_n(\xi_1 a\xi_2), \alpha_1\}, \alpha_1\}, \alpha_1\} \\ &\leq \min\{\min\{\max\{f_n(a\xi_2), \alpha_2\}, \alpha_1\}, \alpha_1\} \\ &= \max\{\min\{f_n(a\xi_2), \alpha_1\}, \min\{\min\{\alpha_2, \alpha_1\}, \alpha_1\}\} \\ &\leq \min\{\max\{\max\{f_n(a), \alpha_2\}, \min\{\alpha_2, \alpha_1\}\}, \alpha_1\} \\ &= \max\{\min\{\max\{f_n(a), \alpha_2\}, \alpha_1\}, \min\{\min\{\alpha_2, \alpha_1\}, \alpha_1\}\} \\ &= \max\{(\min\{f_n(a), \alpha_1\}, \min\{\alpha_2, \alpha_1\}), \min\{\alpha_2, \alpha_1\}\} \\ &= \max\{\min\{f_n(a), \alpha_1\}, \max\{\alpha_2, \alpha_1\}\} \\ &= \min\{\max\{f_n(a), \alpha_2\}, \alpha_1\} \leq \max\{f_p(a), \alpha_2\}. \end{aligned}$$

Also

$$
\begin{aligned}
\max\{f_p(\xi_1 a\xi_2),\beta_1\} &= \max\{\max\{\max\{(f_p(\xi_1 a\xi_2),\beta_1\},\beta_1\},\beta_1\} \\
&\geq \max\{\max\{\min\{(f_p(a\xi_2),\beta_2\},\beta_1\},\beta_1\} \\
&= \max\{\min\{(\max\{f_p(a\xi_2),\beta_1\},\max\{(\beta_2,\beta_1\}\},\beta_1\} \\
&\geq \max\{\min\{\min\{f_p(a),\beta_2\},\max\{\beta_2,\beta_1\}\},\beta_1\} \\
&= \min\{\max\{\min\{f_p(a),\beta_2\},\beta_1\},\max\{\max\{\beta_2,\beta_1\},\beta_1\}\} \\
&= \min\{(\max\{f_p(a),\beta_1\},\max\{\beta_2,\beta_1\}),\max\{\beta_2,\beta_1\}\} \\
&= \min\{\max\{f_p(a),\beta_1\},\max\{\beta_2,\beta_1\}\} \\
&= \max\{\min\{f_p(a),\beta_2\},\beta_1\} \geq \min\{f_p(a),\beta_2\}.
\end{aligned}
$$

Thus $f = (S; f_n, f_p)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFII of S. ■

Example 5.1.5 *Let* $S = \{0, 1, 2, 3\}$ *be an ordered semigroup with multiplication table and order relation given below:*

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

$\leq = \{(0,0),(1,1),(2,2),(3,3),(0,1)\}$.

Define BFS $f = (S; f_n, f_p)$ on S as

S	0	1	2	3
f_n	-0.7	-0.3	-0.5	-0.1
f_p	0.8	0.4	0.6	0.2

Then $f = (S; f_n, f_p)$ is clearly a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFII of S. But $f = (S; f_n, f_p)$ is not a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFI of S because $\max\{f_p(3.2), \beta_1\} = \max\{f_p(1), \beta_1\} = \max\{0.4, \beta_1\} \ngeq \min\{0.6, \beta_2\} = \min\{f_p(2), \beta_2\}$.

Theorem 5.1.6 *Let S be a regular ordered semigroup then every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFII of S is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFI of S for all $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$.*

Proof. Assume that S is regular ordered semigroup and $f = (S; f_n, f_p)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFII of S. Let $\xi_1 \in S$ then there exists $a \in S$ such that $\xi_1 \leq \xi_1 a \xi_1$, so

$$\begin{aligned}
\min\{f_n(\xi_1\xi_2), \alpha_1\} &= \min\{\min\{\min\{f_n(\xi_1\xi_2), \alpha_1\}, \alpha_1\}, \alpha_1\} \\
&\leq \min\{\min\{\max\{f_n((\xi_1 a \xi_1)\xi_2), \alpha_2\}, \alpha_1\}, \alpha_1\} \\
&= \min\{\max\{\min\{(f_n((\xi_1 a)\xi_1(\xi_2)), \alpha_1\}, \min\{\alpha_2, \alpha_1\}\}, \alpha_1\} \\
&\leq \min\{\max\{\max\{f_n(\xi_1), \alpha_2\}, \min\{\alpha_2, \alpha_1\}\}, \alpha_1\} \\
&= \max\{\min\{\max\{f_n(\xi_1), \alpha_2\}, \alpha_1\}, \min\{\min\{\alpha_2, \alpha_1\}, \alpha_1\}\} \\
&= \max\{(\min\{f_n(\xi_1), \alpha_1\}, \min\{\alpha_2, \alpha_1\}), \min\{\alpha_2, \alpha_1\}\} \\
&= \max\{\min\{f_n(\xi_1), \alpha_1\}, \max\{\alpha_2, \alpha_1\}\} \\
&= \min\{\max\{f_n(\xi_1), \alpha_2\}, \alpha_1\} \leq \max\{f_p(\xi_1), \alpha_2\}
\end{aligned}$$

and

$$\begin{aligned}
\max\{f_p(\xi_1\xi_2), \beta_1\} &= \max\{\max\{\max\{f_p(\xi_1\xi_2), \beta_1\}, \beta_1\}, \beta_1\} \\
&\geq \max\{\max\{\min\{f_p((\xi_1 a \xi_1)\xi_2), \beta_2\}, \beta_1\}, \beta_1\} \\
&= \max\{\min\{\max\{f_p((\xi_1 a)\xi_1(\xi_2)), \beta_1\}, \max\{\beta_2, \beta_1\}\}, \beta_1\} \\
&\geq \max\{\min\{\min\{((f_p(\xi_1), \beta_2\}, \max\{\beta_2, \beta_1\}\}, \beta_1\} \\
&= \min\{\max\{\min\{f_p(\xi_1), \beta_2\}, \beta_1\}, \max\{\max\{\beta_2, \beta_1\}, \beta_1\}\} \\
&= \min\{(\max\{(f_p(\xi_1), \beta_1\}, \max\{\beta_2, \beta_1\}), \max\{\beta_2, \beta_1\}\} \\
&= \min\{\max\{(f_p(\xi_1), \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&= \max\{\min f_p(\xi_1), \beta_2), \beta_1\} \geq \min\{f_p(\xi_1), \beta_2\}.
\end{aligned}$$

Thus $f = (S; f_n, f_p)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFRI of S. Similarly we can show that $f = (S; f_n, f_p)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFRI of S. ■

Corollary 5.1.7 *In regular ordered semigroup S, the concept of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFI and $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*

BFII coincide.

Theorem 5.1.8 *Let S be an intra-regular ordered semigroup then every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFII of S is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFI of S for all $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$.*

Proof. Assume that S is an intra-regular ordered semigroup and $f = (S; f_n, f_p)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFII of S. Let $\xi_1 \in S$ then there exist $a, b \in S$ such that $\xi_1 \leq a\xi_1^2 b$, so

$$\begin{aligned}
\min\{f_n(\xi_1\xi_2), \alpha_1\} &= \min\{\min\{\min\{f_n(\xi_1\xi_2), \alpha_1\}, \alpha_1\}, \alpha_1\} \\
&\leq \min\{\min\{\max\{f_n((a\xi_1^2 b)\xi_2), \alpha_2\}, \alpha_1\}, \alpha_1\} \\
&= \min\{\max\{\min\{f_n((a\xi_1)\xi_1(b\xi_2)), \alpha_1\}, \min\{\alpha_2, \alpha_1\}\}, \alpha_1\} \\
&\leq \min\{\max\{\max\{f_n(\xi_1), \alpha_2\}, \min\{\alpha_2, \alpha_1\}\}, \alpha_1\} \\
&= \max\{\min\{\max\{f_n(\xi_1), \alpha_2\}, \alpha_1\}, \min\{\min\{\alpha_2, \alpha_1\}, \alpha_1\}\} \\
&= \max\{(\min\{f_n(\xi_1), \alpha_1\}, \min\{\alpha_2, \alpha_1\}), \min\{\alpha_2, \alpha_1\} \\
&= \max\{\min\{f_n(\xi_1), \alpha_1\}, \max\{\alpha_2, \alpha_1\}\} \\
&= \min\{\max\{f_n(\xi_1), \alpha_2\}, \alpha_1\} \leq \max\{f_p(\xi_1), \alpha_2\}
\end{aligned}$$

and

$$\begin{aligned}
\max\{f_p(\xi_1\xi_2), \beta_1\} &= \max\{\max\{\max\{f_p(\xi_1\xi_2), \beta_1\}, \beta_1\}, \beta_1\} \\
&\geq \max\{\max\{\min\{f_p((a\xi_1^2 b)\xi_2), \beta_2\}, \beta_1\}, \beta_1\} \\
&= \max\{\min\{\max\{f_p((a\xi_1^2 b)\xi_2), \beta_1\}, \max\{\beta_2, \beta_1\}, \beta_1\} \\
&\geq \max\{\min\{\min\{f_p(\xi_1), \beta_2\}, \max\{\beta_2, \beta_1\}\}, \beta_1\} \\
&= \min\{\max\{\min\{f_p(\xi_1), \beta_2\}, \beta_1\}, \max\{\beta_2, \beta_1, \beta_1\}\} \\
&= \min\{(\max\{f_p(\xi_1), \beta_1\}, \max\{\beta_2, \beta_1\}), \max\{\beta_2, \beta_1\}\} \\
&= \min\{\max\{f_p(\xi_1), \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&= \max\{\min\{f_p(\xi_1), \beta_2\}, \beta_1\} \geq \min\{f_p(\xi_1), \beta_2\}.
\end{aligned}$$

Thus $f = (S; f_n, f_p)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFRI of S. Similarly we can show that $f = (S; f_n, f_p)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFRI of S. ■

Corollary 5.1.9 *In intra-regular ordered semigroup S, the concept of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFI and $(\alpha_1, \alpha_2; \beta_1, \beta_2)$- BFII coincide.*

Definition 5.1.10 *Let $f = (S; f_n, f_p)$ be a BFS of S and $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$. Then*

$$\begin{aligned} f_n^{(\alpha_1,\alpha_2)}(\xi) &= \min\{\max\{f_n(\xi), \alpha_1\}, \alpha_2\} \text{ and} \\ f_p^{(\beta_1,\beta_2)}(\xi) &= \max\{\min\{f_p(\xi), \beta_1\}, \beta_2\}. \end{aligned}$$

Set $f^{(\alpha_1,\alpha_2;\beta_1,\beta_2)} = (S; f_n^{(\alpha_1,\alpha_2)}, f_p^{(\beta_1,\beta_2)})$. Then it is a BFS.

Note that

$$\begin{aligned} 1)\ f_n^{(-1,0)}(\xi) &= f_n(\xi) \\ 2)\ f_p^{(1,0)}(\xi) &= f_p(\xi) \\ 3)\ f^{(-1,0;1,0)} &= f = (S; f_n, f_p). \end{aligned}$$

Since χ_I be a characteristic function, we have

$$\chi_{nI}^{(\alpha_2,\alpha_1)}(\xi) = \begin{cases} \min\{\alpha_1, \alpha_2\} \text{ if } \xi \in I \\ \alpha_1 \quad \text{if } \xi \notin I \end{cases}$$

and

$$\chi_{pI}^{(\beta_2,\beta_1)}(\xi) = \begin{cases} \max\{\beta_1, \beta_2\} \text{ if } \xi \in I \\ \beta_1 \quad \text{if } \xi \notin I \end{cases}.$$

Theorem 5.1.11 *The characteristic function $\chi_I^{(\alpha_1,\alpha_2;\beta_1,\beta_2)} = (S; \chi_{nI}^{(\alpha_2,\alpha_1)}, \chi_{pI}^{(\beta_2,\beta_1)})$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFI of S for all $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$ iff I is an ideal of S.*

Proof. The proof follows from Theorem 1.6.5 and 1.6.6. ■

Theorem 5.1.12 *The characteristic function $\chi^{(\alpha_1,\alpha_2;\beta_1,\beta_2)} = (S; \chi_{n\mathcal{I}}^{(\alpha_2,\alpha_1)}, \chi_{p\mathcal{I}}^{(\beta_2,\beta_1)})$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFII of S for all $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$ iff $\mathcal{I}$ is an interior ideal of S.*

Proof. The proof follows from Theorem 1.6.9. ■

Theorem 5.1.13 *The following conditions are equivalent for S.*

1) S is left (right) regular.

2) For each $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFLI (BFRI) $f = (S; f_n, f_p)$ of S, we have $f_p^{(\beta_2,\beta_1)}(\xi) = f_p^{(\beta_2,\beta_1)}(\xi^2)$ and $f_n^{(\alpha_2,\alpha_1)}(\xi) = f_n^{(\alpha_2,\alpha_1)}(\xi^2)$ for all $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$.

Proof. 1) $\Longrightarrow$ 2) Assume that S is left regular. Let $\xi \in S$, then there exists $a \in S$ such that $\xi \leq a\xi^2$. Thus

$$\begin{aligned}
f_p^{(\beta_2,\beta_1)}(\xi) &= \max\{\min\{f_p(\xi), \beta_2\}, \beta_1\} \\
&= \min\{\max\{f_p(\xi), \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&= \min\{\max\{\max\{\max\{f_p(\xi), \beta_1\}, \beta_1\}, \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&\geq \min\{\max\{\max\{\min\{f_p(a\xi^2), \beta_2\}, \beta_1\}, \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&= \min\{\max\{\min\{\max\{f_p(a\xi^2), \beta_1\}, \max\{\beta_2, \beta_1\}\}, \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&\geq \min\{\max\{\min\{\min\{f_p(\xi^2), \beta_2\}, \max\{\beta_2, \beta_1\}\}, \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&= \min\{\min\{\max\{\min\{f_p(\xi^2), \beta_2\}, \beta_1\}, \max\{(\beta_2, \beta_1), \beta_1\}\}, \max\{\beta_2, \beta_1\}\} \\
&= \min\{((\max\{f_p(\xi^2), \beta_1\}, \max\{\beta_2, \beta_1\}), \max\{\beta_2, \beta_1\}), \max\{\beta_2, \beta_1\}\} \\
&= \min\{\max\{f_p(\xi^2), \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&= \max\{\min\{f_p(\xi^2), \beta_2\}, \beta_1\} = f_p^{(\beta_2,\beta_1)}(\xi^2).
\end{aligned}$$

This implies that

$$f_p^{(\beta_2,\beta_1)}(\xi) \geq f_p^{(\beta_2,\beta_1)}(\xi^2).$$

Now

$$\begin{aligned}
f_p^{(\beta_2,\beta_1)}(\xi^2) &= \max\{\min\{f_p(\xi^2), \beta_2\}, \beta_1\} \\
&= \min\{\max\{(f_p(\xi^2), \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&= \min\{\max\{\max\{f_p(\xi^2), \beta_1\}, \beta_1)\}, \max\{\beta_2, \beta_1\}\} \\
&\geq \min\{\max\{\min\{f_p(\xi), \beta_2\}, \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&= \min\{(\max\{f_p(\xi), \beta_1\}, \max\{\beta_2, \beta_1\}), \max\{\beta_2, \beta_1\}\} \\
&= \min\{\max\{f_p(\xi), \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&= \max\{\min\{f_p(\xi), \beta_2\}, \beta_1\} = f_p^{(\beta_2,\beta_1)}(\xi)
\end{aligned}$$

This implies that

$$f_p^{(\beta_2,\beta_1)}(\xi^2) \geq f_p^{(\beta_2,\beta_1)}(\xi).$$

So

$$f_p^{(\beta_2,\beta_1)}(\xi) = f_p^{(\beta_2,\beta_1)}(\xi^2) \text{ for all } \xi \in S.$$

Similarly we can show that

$$f_n^{(\alpha_2,\alpha_1)}(\xi) = f_n^{(\alpha_2,\alpha_1)}(\xi^2) \text{ for all } \xi \in S.$$

2) $\Longrightarrow$ 1) Let $\xi \in S$ and consider the left ideal $\mathcal{L}(\xi^2) = (\xi^2 \cup S\xi^2]$ generated by ξ^2. By Theorem 1.6.5, we have $\chi_{\mathcal{L}(\xi^2)}$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFLI of S and by hypothesis,

$$\chi_{p\mathcal{L}(\xi^2)}^{(\beta_2,\beta_1)}(\xi) = \chi_{p\mathcal{L}(\xi^2)}^{(\beta_2,\beta_1)}(\xi^2)$$

and

$$\chi_{n\mathcal{L}(\xi^2)}^{(\alpha_2,\alpha_1)}(\xi) = \chi_{n\mathcal{L}(\xi^2)}^{(\alpha_2,\alpha_1)}(\xi^2).$$

Since $\xi^2 \in \mathcal{L}(\xi^2)$, we have

$$\begin{aligned} \chi_{p\mathcal{L}(\xi^2)}^{(\beta_2,\beta_1)}(\xi) &= \chi_{p\mathcal{L}(\xi^2)}^{(\beta_2,\beta_1)}(\xi^2) \\ &= \max\{\min\{\chi_{p\mathcal{L}(\xi^2)}(\xi^2), \beta_2\}, \beta_1\} \\ &= \max\{\beta_2, \beta_1\} \end{aligned}$$

and

$$\begin{aligned} \chi_{n\mathcal{L}(\xi^2)}^{(\alpha_2,\alpha_1)}(\xi) &= \chi_{n\mathcal{L}(\xi^2)}^{(\alpha_2,\alpha_1)}(\xi^2) \\ &= \min\{\max\{\chi_{n\mathcal{L}(\xi^2)}(\xi^2), \alpha_2\}, \alpha_1\} \\ &= \min\{\alpha_2, \alpha_1\}. \end{aligned}$$

This implies that $\xi \in \mathcal{L}(\xi^2)$. Hence $\xi \leq \xi^2$ or $\xi \leq a\xi^2$ for some $a \in S$. In both cases we conclude that $\xi \in (S\xi^2]$. Therefore S is left regular. ■

Theorem 5.1.14 *The following conditions are equivalent for S.*

1) S is intra-regular.

2) For each $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFII $f = (S; f_n, f_p)$ of S, we have $f_p^{(\beta_2,\beta_1)}(\xi) = f_p^{(\beta_2,\beta_1)}(\xi^2)$ and $f_n^{(\alpha_2,\alpha_1)}(\xi) = f_n^{(\alpha_2,\alpha_1)}(\xi^2)$ for all $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$.

Proof. 1) $\Longrightarrow$ 2) Assume that S is intra-regular. Let $\xi \in S$, then there exists $a, b \in S$ such that $\xi \leq a\xi^2 b$. Thus

$$\begin{aligned}
f_p^{(\beta_2,\beta_1)}(\xi) &= \max\{\min\{f_p(\xi), \beta_2\}, \beta_1\} \\
&= \min\{\max\{f_p(\xi), \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&= \min\{\max\{\max\{\max\{f_p(\xi), \beta_1\}, \beta_1\}, \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&\geq \min\{\max\{\max\{\min\{f_p(a\xi^2 b), \beta_2\}, \beta_1\}, \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&= \min\{\max\{\min\{\max\{f_p(a\xi^2 b), \beta_1\}, \max\{\beta_2, \beta_1\}\}, \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&\geq \min\{\max\{\min\{\min\{f_p(\xi^2), \beta_2\}, \max\{\beta_2, \beta_1\}\}, \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&= \min\{\max\{\min\{f_p(\xi^2), \beta_2\}, \beta_1\}, \max\{\max\{\beta_2, \beta_1\}, \beta_1)\}, \max\{\beta_2, \beta_1\}\} \\
&= \min\{((\max\{f_p(\xi^2), \beta_1\}, \max\{\beta_2, \beta_1\}), \max\{\beta_2, \beta_1\}), \max\{\beta_2, \beta_1\}\} \\
&= \min\{\max\{f_p(\xi^2), \beta_1\}, \max\{\beta_2, \beta_1\}\} \\
&= \max\{\min\{f_p(\xi^2), \beta_2\}, \beta_1\} = f_p^{(\beta_2,\beta_1)}(\xi^2).
\end{aligned}$$

This implies that

$$f_p^{(\beta_2,\beta_1)}(\xi) \geq f_p^{(\beta_2,\beta_1)}(\xi^2).$$

Now

$$\begin{aligned}
f_p^{(\beta_2,\beta_1)}(\xi^2) &= \max\{\min\{f_p(\xi^2),\beta_2\},\beta_1\} \\
&= \min\{\max\{f_p(\xi^2),\beta_1\},\max\{\beta_2,\beta_1\}\} \\
&= \min\{\max\{\max\{f_p(\xi^2),\beta_1\},\beta_1)\},\max\{\beta_2,\beta_1\}\} \\
&\geq \min\{\max\{\min\{\min\{f_p(\xi),f_p(\xi)\},\beta_2\},\beta_1\},\max\{\beta_2,\beta_1\}\} \\
&= \min\{\max\{\min\{f_p(\xi),\beta_2\},\beta_1\},\max\{\beta_2,\beta_1\}\} \\
&= \min\{(\max\{f_p(\xi),\beta_1\},\max\{\beta_2,\beta_1\}),\max\{\beta_2,\beta_1\}\} \\
&= \min\{\max\{f_p(\xi),\beta_1\},\max\{\beta_2,\beta_1\}\} \\
&= \max\{\min\{f_p(\xi),\beta_2\},\beta_1\} = f_p^{(\beta_2,\beta_1)}(\xi).
\end{aligned}$$

This implies that

$$f_p^{(\beta_2,\beta_1)}(\xi^2) \geq f_p^{(\beta_2,\beta_1)}(\xi).$$

So

$$f_p^{(\beta_2,\beta_1)}(\xi) = f_p^{(\beta_2,\beta_1)}(\xi^2) \text{ for all } \xi \in S.$$

Similarly we can show that

$$f_n^{(\alpha_2,\alpha_1)}(\xi) = f_n^{(\alpha_2,\alpha_1)}(\xi^2) \text{ for all } \xi \in S.$$

2) $\Longrightarrow$ 1) Let $\xi \in S$ and consider an ideal $I(\xi^2) = (\xi^2 \cup S\xi^2 \cup \xi^2 S \cup S\xi^2 S]$ generated by ξ^2. By Theorem 1.6.9 and by result that every $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-BFI of S is a $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-BFII of S, we have $\chi_{I(\xi^2)}$ is a $(\alpha_1,\alpha_2;\beta_1,\beta_2)$-BFII of S and by hypothesis,

$$\chi_{pI(\xi^2)}^{(\beta_2,\beta_1)}(\xi) = \chi_{pI(\xi^2)}^{(\beta_2,\beta_1)}(\xi^2)$$

and

$$\chi_{nI(\xi^2)}^{(\alpha_2,\alpha_1)}(\xi) = \chi_{nI(\xi^2)}^{(\alpha_2,\alpha_1)}(\xi^2).$$

Since $\xi^2 \in I(\xi^2)$, we have

$$\begin{aligned}\chi_{pI(\xi^2)}^{(\beta_2,\beta_1)}(\xi) &= \chi_{pI(\xi^2)}^{(\beta_2,\beta_1)}(\xi^2) \\ &= \max\{\min\{\chi_{pI(\xi^2)}(\xi^2), \beta_2\}, \beta_1\} \\ &= \max\{\beta_2, \beta_1\}\end{aligned}$$

and

$$\begin{aligned}\chi_{nI(\xi^2)}^{(\alpha_2,\alpha_1)}(\xi) &= \chi_{nI(\xi^2)}^{(\alpha_2,\alpha_1)}(\xi^2) \\ &= \min\{\max\{\chi_{nI(\xi^2)}(\xi^2), \alpha_2\}, \alpha_1\} \\ &= \min\{\alpha_2, \alpha_1\}.\end{aligned}$$

This implies that $\xi \in I(\xi^2)$. Hence $\xi \leq \xi^2$ or $\xi \leq a\xi^2$ or $\xi \leq \xi^2 a$ or $\xi \leq a\xi^2 b$ for some $a, b \in S$. In all cases we conclude that $\xi \in (S\xi^2 S]$. Therefore S is intra-regular. ■

Theorem 5.1.15 *Let $f = (S; f_n, f_p)$ be a BFS on a intra-regular ordered semigroup S and $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$. If f is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFII on S then $f_p^{(\beta_2,\beta_1)}(\xi_1\xi_2) = f_p^{(\beta_2,\beta_1)}(\xi_2\xi_1)$ and $f_n^{(\alpha_2,\alpha_1)}(\xi_1\xi_2) = f_n^{(\alpha_2,\alpha_1)}(\xi_2\xi_1)$ for all $\xi_1, \xi_2 \in S$.*

Proof. Let $\xi_1, \xi_2 \in S$. Then by Theorem 5.1.14, we have

$$\begin{aligned}f_p^{(\beta_2,\beta_1)}(\xi_1\xi_2) &= f_p^{(\beta_2,\beta_1)}(\xi_1\xi_2)^2 \\ &= f_p^{(\beta_2,\beta_1)}((\xi_1\xi_2)(\xi_1\xi_2)) \\ &= \max\{\min\{(f_p(\xi_1(\xi_2\xi_1)\xi_2), \beta_2\}, \beta_1\} \\ &= \min\{\max\{f_p(\xi_1(\xi_2\xi_1)\xi_2), \beta_1\}, \max\{\beta_2, \beta_1\}\} \\ &= \min\{\max\{\max\{f_p(\xi_1(\xi_2\xi_1)\xi_2), \beta_1\}, \beta_1\}, \max\{\beta_2, \beta_1\}\} \\ &\geq \min\{\max\{\min\{f_p(\xi_2\xi_1), \beta_2\}, \beta_1\}, \max\{\beta_2, \beta_1\}\} \\ &= \min\{(\max\{f_p(\xi_2\xi_1), \beta_1\}, \max\{\beta_2, \beta_1\}), \max\{\beta_2, \beta_1\}\} \\ &= \min\{\max\{f_p(\xi_2\xi_1), \beta_1\}, \max\{\beta_2, \beta_1\}\} \\ &= \max\{\min\{f_p(\xi_2\xi_1), \beta_2\}, \beta_1\} = f_p^{(\beta_2,\beta_1)}(\xi_2\xi_1).\end{aligned}$$

By symmetry

$$f_p^{(\beta_2,\beta_1)}(\xi_2\xi_1) \geq f_p^{(\beta_2,\beta_1)}(\xi_1\xi_2).$$

Therefore

$$f_p^{(\beta_2,\beta_1)}(\xi_2\xi_1) = f_p^{(\beta_2,\beta_1)}(\xi_1\xi_2).$$

Similarly, we can easily show that

$$f_n^{(\alpha_2,\alpha_1)}(\xi_1\xi_2) = f_n^{(\alpha_2,\alpha_1)}(\xi_2\xi_1).$$

■

5.2 $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-Bipolar Fuzzy Prime (Semiprime) Ideals

In this section, we present the notion of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy prime ideals (BFPIs) of S. We characterize S in framework of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFPI.

Definition 5.2.1 *A $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFI, $f = (S; f_n, f_p)$ of S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFPI of S where $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$, if it satisfies the following condition*

$$\min\{f_n(\xi_1), f_n(\xi_2), \alpha_1\} \leq \max\{f_n(\xi_1\xi_2), \alpha_2\} \text{ and } \max\{f_p(\xi_1), f_p(\xi_2), \beta_1\} \geq \min\{f_p(\xi_1\xi_2), \beta_2\}$$

for all $\xi_1, \xi_2 \in S$.

Definition 5.2.2 *A $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFI, $f = (S; f_n, f_p)$ of S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy semiprime ideal of S where $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$ if it satisfies the following condition*

$$\min\{f_n(\xi), \alpha_1\} \leq \max\{f_n(\xi^2), \alpha_2\} \text{ and } \max\{f_p(\xi), \beta_1\} \geq \min\{f_p(\xi^2), \beta_2\}$$

for all $\xi \in S$.

In the next theorem, we gave a relation between a bipolar fuzzy (α, β)-cut of $f = (S; f_n, f_p)$ and $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFPI of S.

Theorem 5.2.3 *Let $f = (S; f_n, f_p)$ be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFI of S where $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$. Then $C(f; (\alpha, \beta))(\neq \emptyset)$ is a prime ideal of S for all $\alpha \in [-1, 0]$, $\beta \in [0, 1]$ iff $f = (S; f_n, f_p)$ is $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy prime ideal of S.*

Proof. Suppose on contrary that $f = (S; f_n, f_p)$ is not a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy prime ideal of S. Then we can find $a, b \in S$ such that

$$\min\{f_n(a), f_n(b), \alpha_1\} > \max\{f_n(ab), \alpha_2\} \text{ or } \max\{f_p(a), f_p(b), \beta_1\} < \min\{f_p(ab), \beta_2\}.$$

Let $\max\{f_n(ab), \alpha_2\} = \alpha^*$ and $\min\{f_p(ab), \beta_2\} = \beta^*$, then $ab \in C(f; (\alpha^*, \beta^*))$. By assumption, $a \in C(f; (\alpha^*, \beta^*))$ or $b \in C(f; (\alpha^*, \beta^*))$. Also from above inequality we have $f_n(a) > \alpha^*$ or $f_p(a) < \beta^*$. Thus $a \notin C(f; (\alpha^*, \beta^*))$. Also from above argument we have $b \notin C(f; (\alpha^*, \beta^*))$ which is impossible. Hence our supposition is wrong and $f = (S; f_n, f_p)$ be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy prime ideal of S.

Conversely, let $ab \in C(f; (\alpha, \beta))$. Then $f_n(ab) \leq \alpha$ and $f_p(ab) \geq \beta$. We show that $a \in C(f; (\alpha, \beta))$ or $b \in C(f; (\alpha, \beta))$. Assume that $a \notin C(f; (\alpha, \beta))$. By supposition $f = (S; f_n, f_p)$ is a $(f_n(b), \alpha; f_p(b), \beta)$-BFPI of S, so

$$\begin{aligned} \min\{f_n(a), f_n(b), f_n(b)\} &\leq \max\{f_n(ab), \alpha\} = \alpha \text{ and} \\ \max\{f_p(a), f_p(b), f_p(b)\} &\geq \min\{f_p(ab), \beta\} = \beta. \end{aligned}$$

This implies that

$$\begin{aligned} \min\{f_n(a), f_n(b)\} &\leq \alpha \text{ and} \\ \max\{f_p(a), f_p(b)\} &\geq \beta. \end{aligned}$$

Since $a \notin C(f; (\alpha, \beta))$, so $f_n(a) \nleq \alpha$ and $f_p(a) \ngeq \beta$.So $f_n(b) \leq \alpha$ and $f_p(b) \geq \beta$. Hence $b \in C(f; (\alpha, \beta))$ and $C(f; (\alpha, \beta))$ is a prime ideal of S. ■

Theorem 5.2.4 *Let $f = (S; f_n, f_p)$ be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-BFI of S where $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$. Then $C(f; (\alpha, \beta))(\neq \emptyset)$ is a semiprime ideal of S for all $\alpha \in [-1, 0]$, $\beta \in [0, 1]$ iff $f = (S; f_n, f_p)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy semiprime ideal of S.*

Proof. Let $C(f;(\alpha,\beta))(\neq \emptyset)$ is a semiprime ideal of S. Suppose on contrary that $f = (S; f_n, f_p)$ is not a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy semiprime ideal of S. Then we can find $a \in S$ such that

$$\min\{f_n(a), \alpha_1\} > \max\{f_n(a^2), \alpha_2\} \text{ or } \max\{f_p(a), \beta_1\} < \min\{f_p(a^2), \beta_2\}.$$

Let $\max\{f_n(a^2), \alpha_2\} = \alpha^*$ and $\min\{f_p(a^2), \beta_2\} = \beta^*$, then $a^2 \in C(f;(\alpha^*, \beta^*))$. By given hypothesis we have $a \in C(f;(\alpha^*, \beta^*))$. From above inequality we have $f_n(a) > \alpha^*$ or $f_p(a) < \beta^*$. This implies that $a \notin C(f;(\alpha^*, \beta^*))$ which is impossible. Hence our supposition is wrong and $f = (S; f_n, f_p)$ is $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy semiprime ideal of S.

Conversely, let $a^2 \in C(f;(\alpha,\beta))$ for some $a \in S$. Then $f_n(a^2) \leq \alpha$ and $f_p(a^2) \geq \beta$. We show that $a \in C(f;(\alpha,\beta))$. By assumption $f = (S; f_n, f_p)$ is a $(f_n(a), \alpha; f_p(a), \beta)$-bipolar fuzzy semiprime ideal of S, so

$$\begin{aligned} f_n(a) &= \min\{f_n(a), f_n(a)\} \leq \max\{f_n(a^2), \alpha\} \leq \alpha \text{ and} \\ f_p(a) &= \max\{f_p(a), f_p(a)\} \geq \min\{f_p(a^2), \beta\} \geq \beta. \end{aligned}$$

Hence $a \in C(f;(\alpha,\beta))$. Therefore $C(f;(\alpha,\beta))$ is a semiprime ideal of S. ■

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