LINEAR OPERATORS AND RELATED TOPICS IN GEOMETRIC FUNCTION THEORY

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Dedication

My FAMILY

Who
Always have Faith in me and supported me in accomplishing my goals.

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Abbreviations

Set of Complex Number	$\mathbb{C}$
Open unit disk	$\triangle$
The class of normalized analytic functions	∇
Class of univalent functions	S
Class of convex univalent functions	C
Class of close to convex univalent functions	K
Class of starlike univalent functions	S^*
Koebe functions	$Q(z)$
Class of α −convex functions	M_α
Class of analytic functions with real part greater than λ	$P(\lambda)$
Class of bi-univalent functions	$\sum$
Class of m fold bi-univalent functions	$\sum_m$
Schwarz function	$u(z)$
Subordination symbol	$\prec$
Class of uniformly convex functions	U_{CV}
Class of corresponding starlike functions	S_T
Parabolic domain	Ω
Conic domain	Ω_k
Class of k-uniformly convex functions	$k - U_{CV}$
Class of k-starlike functions	$k - S_T$
Class of Janowski functions	ß$[C, D]$
Circular domain	$\Omega[C, D]$
Convolution of y and m	$y * m$

Srivastava-Attiya operator	$J_{\mu,b}$
Hurwitz-Lerch Zeta function	$\phi(\mu,b,z)$
Ruscheweyh derivative operator	D^{λ}
q-Generalized Pochhammer symbol	(v_n^q)
Ruscheweyh q-Differential operator	R_q^{λ}
Salagean q-differential operator	S_q^{λ}
Analytic multivalent functions	$\nabla(p)$
Class of multivalent starlike functions	S_P^*
Class of multivalent convex functions	C_P^*
Class of multivalent starlike functions of order λ	$S_P^*(\lambda)$
Class of multivalent convex functions of order λ	$C_P^*(\lambda)$
Salagean q-differential operator for multivalent functions	$S_{q,p}^{m}$
Class of multivalent k-uniformly starlike functions of type (q,γ,m,p)	$k-US_{q,p}^{\gamma,m}$
Class of k-starlike functions of order α	$SD(k,\alpha)$
Class of k-uniformly convex functions of order α	$KD(k,\alpha)$
Class of analytic bi-univalent functions of type (μ,b,α,λ)	$M_{\Sigma}(\mu,,b,\alpha,\lambda)$
Class of analytic bi-univalent functions of type (μ,b,β,λ)	$M_{\Sigma}(\mu,,b,\beta,\lambda)$
Class of analytic m-Fold bi-univalent functions of type (α,λ,μ)	$S_{\Sigma_m}(\alpha,\lambda,\mu)$
Class of analytic m-Fold bi-univalent functions of type (β,λ,μ)	$S_{\Sigma_m}(\beta,\lambda,\mu)$
Class of analytic bi-univalent functions of type $(\alpha,\beta,\gamma,\lambda)$	$N_{\Sigma}^{q(\alpha,\beta,\gamma,\lambda)}$
Class of analytic bi-univalent functions of type $(\alpha,\beta,\gamma,\lambda)$	$N_{\Sigma}^{q(\alpha,\beta,\gamma,\lambda)}$
Class of analytic bi-univalent functions of type (α,γ,λ)	$N_{\Sigma}^{q(\alpha,\gamma,\lambda)}$
Class of analytic bi-univalent functions of type (α,λ)	$N_{\Sigma}^{q(\alpha,\lambda)}$

Class of k-uniformly starlike functions of type (λ, C, D, β)	$k - US_q(\lambda, A, B, \beta)$
Class of starlike functions of type $(\alpha, \beta, \delta, q)$	$S(\alpha, \beta, \delta, q)$

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ABSTRACT

The current thesis aims at studying linear operators and related topics in Geometric Function Theory. We describe various novel classes of analytic functions by using linear differential operator. Additionally, our focus is on studying subclasses of analytic, univalent and bi-univalent functions. We have studied and investigated some integral properties of aforementioned classes such as inclusion relations, convolution, extreme point and closure property, initial coefficients, upper bounds $|a_n|$, sufficiency criteria, Fekete-Szego inequality and some other problems. We also deal with conic domains, and certain new classes of analytic functions representing conic domains have also been introduced. We have also discussed few applications of these operators which are associated with conic domains.

Introduction

1.1 Historical Background. The branch of complex analysis that cover the geometric properties of analytic functions (AFs) is known as geometric function theory (GFT). GFT is one of the basic branch of a complex analysis. The first person was Cauchy who started to develop the structure of it in during 1814-1831. Since then this theory expainded in various directions. In 1851, Riemann introduced Riemann Mapping Theorem. This theorem gave birth to (GFT). Koebe's [54] in 1907, developed the univalent functions theory which is one of the foundatons of GFT. A complex value function y is univalent if

$$z_1 = z_2 \Rightarrow \; y\left(z_1\right) = y\left(z_2\right), \quad z_1, z_2 \in \triangle.$$

The set of analytic functions on open unit disc $\triangle = \{z \in \mathbb{C} : |z| < 1\}$ is denoted by ∇. A power series expansion $y \in \nabla$ is given by

$$y(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \triangle.$$

A function which is analytic and univalent in $\triangle$ is denoted by $\mathcal{S}$ and also satisfies the normalized condition, namely $y(0)$=0, $y'(0)$=1. The convex functions ($\mathcal{C}$) and starlike functions ($\mathcal{S}^*$) are the subclasses of $\mathcal{S}$. In [7], Alexander proved that $y \in \mathcal{C}$ iff $zy' \in \mathcal{S}^*$. Kaplan [51] defined close to convex functions ($\mathcal{K}$) and verified that the class $\mathcal{K}$ are univalent.

The starlike ($\mathcal{S}^*$) univalent functions defined as:

$$\mathcal{S}^* = \left\{ y : y \in \nabla : \mathfrak{Re}\left(\frac{zy^{'}(z)}{y(z)}\right) > 0, \qquad z \in \triangle \right\}.$$

Caratheodary functions (ß) defined in [21] as:

$$ß = \left\{ \hbar\left(z\right) \in \nabla,\ \hbar\left(0\right) = 1,\ \text{and}\ \hbar\left(z\right) = 1 + \sum_{n=1}^{\infty} a_n z^n,\ \ \text{iff}\ \ Re\left(\hbar\left(z\right)\right) > 0,\ \ z \in \triangle \right\}.$$

Functions $\mathfrak{B}$ play a very important role in the research area of GFT. The classes $\mathcal{S}^*(\beta)$ and $\mathcal{C}(\beta)$ of order β is the subclasses of $\mathcal{S}$ introduced by Robertson in [85], which is defined as:

$$\mathcal{S}^*(\beta) = \left\{ y : y \in \nabla,\ \text{Re}\left(\frac{zy^{'}(z)}{y(z)} \right) > \beta,\ z \in \triangle,\ 0 \leq \beta < 1 \right\}$$

and

$$\mathcal{C}(\beta) = \left\{ y : y \in \nabla,\ \text{Re}\left(\frac{\left(zy^{'}(z)\right)^{'}}{y^{'}(z)} \right) > \beta,\ z \in \triangle,\ 0 \leq \beta < 1 \right\}.$$

One can easily verify that

$$y \in \mathcal{C}(\beta) \iff zy^{'} \in \mathcal{S}^*(\beta).$$

For $\beta = 0$, then $\mathcal{S}^*(\beta)$ and $\mathcal{C}(\beta)$ reduces into $\mathcal{S}^*$ and $\mathcal{C}$ which are the subclasses of $\mathcal{S}$. Above two classes $\mathcal{S}^*(\beta)$ and $\mathcal{C}(\beta)$ were combined by Mocanu in [65] and introduced the single concept of class of M_α, $(0 \leq \alpha \leq 1)$, as:

$$\Re\left[\alpha \left\{ \frac{\left(zy^{'}(z)\right)^{'}}{y^{'}(z)} \right\} + (1-\alpha)\frac{zy^{'}(z)}{y(z)} \right] > 0.$$

The class Υ_m, $m \geq 2$ was given by Lowner [61] in 1917 and after that this concepts was developed systematically by Paatero in [81]. In 1952, Tammi [105] introduced another class χ_m, $m \geq 2$. Then further several authors study these classes see [16, 53, 73, 74, 82]. They worked the development of GFT. Later on Pinchuk [82] introduced another class $\mathfrak{B}_m$, $m \geq 2$. By using the of class $\mathfrak{B}_m$, he defined Υ_m and χ_m as:

$$\begin{aligned} \Upsilon_m &= \left\{ y \in \nabla : \left(\frac{\left(zy^{'}(z)\right)^{'}}{y^{'}(z)} \right) \in \mathfrak{B}_m \right\}, \\ \chi_m &= \left\{ y \in \nabla : \left(\frac{zy^{'}(z)}{y(z)} \right) \in \mathfrak{B}_m \right\}. \end{aligned}$$

If $y, g \in \triangle$, then $y \prec g$, if $\exists$ schwarz function u that is $y(z) = g(u(z))$. Janowski [44] introduced and studied the class $\text{ß}[C, D]$. The class $\text{ß}[C, D]$, $-1 \leq D < C \leq 1$, which satisfy the condition

$$\hbar(z) \prec \frac{1+Cz}{1+Dz}, \quad z \in \triangle.$$

The subclasses $\mathcal{C}$ and $\mathcal{S}^*$ were illustrated through their geometrical properties along with image domains connected with conic regions, for detail see [32, 33]. The most suitable form of Goodman criteria of the subclasses $\mathcal{C}$ and $\mathcal{S}^*$ given by Ronning [86] and Minda [62]. They also derived the parabolic domain. The classes $U_{C,V}$ and S_T be defined by

$$U_{C,V} = \left\{ y \in \nabla : \Re\left[1+\frac{zy''(z)}{y'(z)}\right] > \left|\frac{zy''(z)}{y'(z)}\right|, \quad z \in \triangle \right\}$$

and

$$S_T = \left\{ y \in \nabla : \Re\left[\frac{zy'(z)}{y(z)}\right] > \left|\frac{zy'(z)}{y(z)}-1\right|, \quad z \in \triangle \right\}.$$

This one variable classification move forward to the first conic (parabolic) domain. Here we will define number of subclasses of analytic functions. Further more subclasses of analytic are discussed. New work like inclusion relations, Fekete-Szego inequality, Convolution, initial coefficients, upper bounds $|a_n|$, and sufficiency criteria, will be investigate in this thesis with more clarity and efficiency.

1.2. Preface

Now we will give a short introduction of up coming chapters.

In chapter 1, some preliminary concepts and definitions of GFT will be discuss in this chapter. And these definition and preliminary concepts will be use in the up coming chapters. Also we will briefly discuss the subordination as a main tool. In this chapter we included some lemmas, also we will use them in sucessiive chapters. This chapter include all the previous definitions and it lacks new definitions as well as no new results. References are also quoted with relevant work.

In chapter 2, we defined new operator by using q-calculus theory and by using a newly defined differential operator we defined a new class $k - US_{q,p}^{\gamma,m}$ of multivalent analytic functions. This class generalized the many known classes, see [1, 8, 34, 48, 79, 89, 91]. In this chapter we illustrated structural formula, Fekete–Szego inequality, subordination results and coefficient problem, also we illustrated known corollaries. This chapter is completely published in Journal of Inequalities and Application (2018) 2018:301.

In chapter 3, we defined new classes $M_\Sigma(\mu, b, \alpha, \lambda)$ and $M_\Sigma(\mu, b, \beta, \lambda)$, $0 \leqq \beta < 1, 0 < \alpha \leqq 1, 0 \leqq \lambda \leqq 1, b \in \mathbb{C}$ with $b \neq 0, -1, -2, \ldots\ldots, \mu \in \mathbb{C}$, by using Srivastava-Attiya operator. These classes generalized the known classes $\mathcal{H}_\Sigma^\alpha$ and $\delta_\Sigma^*(\alpha)$, see [17, 92]. We investigated interesting properties such as inclusion results, and initial coefficients bound $|a_2|$ and $|a_3|$, also we illustrated new as well as known corollaries. This chapter is completely published in Bulletin of Mathematical Analysis and Applications Volume 9 Issue 2 (2017), Pages 37-44.

In chapter 4, we defined new classes $\mathcal{S}_{\Sigma_m}(\alpha, \lambda, \mu)$ and $\mathcal{S}_{\Sigma_m}(\beta, \lambda, \mu)$, $m \in N$, $0 < \alpha \leq 1$, $0 \leq \beta < 1$, $0 \leq \mu$, and $0 \leq \lambda \leq 1$. These classes generalized the known classes $\mathcal{S}_{\Sigma,m}^\alpha$, $\mathcal{S}_\Sigma^*(\alpha)$, $\mathcal{H}_\Sigma^m(\alpha)$, $\mathcal{H}_\Sigma(\alpha)$, $\mathcal{N}_\Sigma^{0,m}(\beta, 1)$, $\mathcal{S}_\Sigma^*(\beta)$, $\mathcal{H}_\Sigma^m(\beta)$, $\mathcal{H}_\Sigma(\beta)$, see [10, 17, 39, 92, 93]. In this chapter we proved $|a_{m+1}|$ and $|a_{2m+1}|$ for the functions of these classes and some well known corollaries. This chapter is completely published in Journal of Nonlinear Sciences and Applications, 1 (2018), 425–434.

In chapter 5, by using the Faber polynomial expansions we gave a new subclass $G_\Sigma(\mu, \lambda, \alpha, \beta, C, D)$ for $\alpha \geq 0$, $\lambda \geq 1$, $\mu \geq 0$, $0 \leq \beta \leq 1$ and $-1 \leq D < C \leq 1$. This class generalized the known classes $B_\Sigma(\mu, \lambda, \varphi)$ introduced by Altinkaya and Yalcin [9], and $\mathcal{S}[C, D]$ introduced by Hamidi and Jahangiri [36]. We investigate *n-th* ($n \geq 4$) coefficient of subclass $G_\Sigma(\mu, \lambda, \alpha, \beta, C, D)$ as well as initial coefficients bound $|a_2|$ and $|a_3|$. We will also provide some well-known results as corollaries of our theorems. This chapter is completely published in Journal of Complex Analysis, Article ID 2826514.

In chapter 6, we introduced a subclass of uniformly convex functions by using Ruscheweyh q-differential operator. Also we will investigate many interesting properties such as extreme point, closure theorems, coefficient estimates. This chapter is completely published in Mathematica Slovaca**,** 69 (2019), No. 4, 825-832.

Chapter 1

Preliminary Concepts

In this chapter we will discuss basic concepts of GFT. First we will discuss analytic, and also we will discuss there basic subclasses and will give some basic definition with some of their basic properties and also discuss conic domains with some remarkable properties. Various classical results included as well as explored in this thesis for the sake of completeness. We have quoted some of them and some are proved here as a proof. Some well known preliminary results are listed at the end of this chapter.

2.1: Analytic functions, Univalent and Bi-univalent functions

The role of analytic functions have a great importance in GFT which we can presented as:

2.1.1: Analytic function

Definition 2.1.1: If derivative of function y exist at z_0 and its neighborhood, then y is called analytic function at z_0.

Definition 2.1.2: A function y will belongs to ∇, if y is analytic and normalized, $y(0) = 0$ and $y'(0) = 1$, in open unit disk $\triangle$ and its series expansion given below

$$y(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad z \in \triangle. \tag{2.1.1}$$

2.1.2: Univalent function

Definition 2.1.3: A function y will be univalent in open unit disk $\triangle$. If

$$z_1 \neq z_2 \Rightarrow y(z_1) \neq y(z_2), \qquad z_1, z_2 \in \triangle.$$

Definition 2.1.4: A functions $y \in \mathcal{S}$, if it is analytic, univalent and satisfies the normalized conditions $y(0) = 0$ and $y'(0) = 1$.

$$\mathcal{S} = \{y : y \in \nabla,\ y\left(z\right) \text{ is univalent in } \triangle\}.$$

and

$$Q(z) = \frac{z}{(1-z)^2}, \quad z \in \triangle. \tag{2.1.2}$$

Function (2.1.2) is the example of class $\mathcal{S}$.

Because of extremely nature of Koebe's function, the Koebe's function $Q\left(z\right)$ have vital role in the class $\mathcal{S}$. De Branges proved the Bieberbach conjecture in 1985 for $\mathcal{S}$, the statement of Bieberbach's theorem given in Lemma 2.1.1.

Lemma 2.1.1: [24] Let $y \in \mathcal{S}$ and

$$y\left(z\right) = z + \sum_{n=2}^{\infty} a_n z^n,$$

then $|a_2| \leq 2$.

2.1.3: Bi-univalent Function

Definition 2.1.3: If y and y^{-1} are univalent in $\triangle$, then function $y \in \nabla$ is called a Bi-univalent in $\triangle$. Here we enlist some examples.

$$h_1(z) = \frac{z}{1-z},\ h_2(z) = \log \frac{1}{(1-z)},\ h_3(z) = \frac{1}{2} \log \left(\frac{1+z}{1-z} \right), \qquad z \in \triangle.$$

A functions $y \in \mathcal{S}$ and its inverse y^{-1}, defined by

$$y^{-1}(y(z)) = z \qquad (z \in \triangle)$$

and

$$y(y^{-1}(\varpi)) = \varpi \qquad \left(|\varpi| < r_0(y);\ r_0(y) \geqq \frac{1}{4} \right)$$

where

$$y^{-1}(\varpi) = g(w) = \varpi - a_2^2\varpi + (2a_2^2 - a_3)\varpi^3 - (5a_2^3 - 5a_2a_3 - a_4)\varpi^4 + ... \quad (2.1.3)$$

We will denote set of all bi-univalent functions by Σ. The well known Koebe's functions $Q(z) = \frac{z}{(1-z)^2}$ is not in the class Σ.

2.2. The Class ß and Related Classes

The image domains of several complex valued functions in Geometric Function cover the whole complex plane. In order to complete the study of univalent functions it is necessary that these kind of functions normalized as it has been done in this study. Classes of such a functions to be named as ß, see [31]. Moving further some related classes and subclasses will be discussed along with their basic properties.

Definition 2.2.1: [25, 31] Let $\hbar(z) \in \nabla$, and $Re\left\{\hbar(z)\right\} > 0, \hbar(0) = 1$, are in class ß, that is

$$\left\{\hbar(z) \in ß : \hbar(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \quad \text{iff} \quad Re\left\{\hbar(z)\right\} > 0, \quad z \in \triangle\right\}.$$

The Möbius function $H_o(z)$ is the example of the class ß. In many cases Möbius function $H_o(z)$ used as extremal function for class ß.

Note that:

i) Let $y \in$ß, it is not necessary that y is univalent.

ii) $\hbar(z) = 1 + z^n$, for $n \geq 2$ is not univalent.

iii) For $\hbar \in$ß, then $|a_n| \leq 2$ for $n \geq 1$. The Möbius function $H_o(z)$ is sharp for this inequality.

iii) If ξ_1 and ξ_2 are non-negative and $\xi_1 + \xi_2 = 1$, then set ß is convex and also if $\hbar_1, \hbar_2 \in$ß and $\hbar(z) = \xi_1\hbar_1(z) + \xi_2\hbar_2(z)$, then $\hbar(z) \in$ß, see [25, 31].

In 1911, Herglotz [41] introducing integral form of $\hbar(z)$ from class ß in different technique, which is given as:

$$\hbar(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathfrak{K}(z)d\mu(b), \qquad \text{for all } z \in \triangle, \tag{2.2.2}$$

where

$$\mathfrak{K}(z) = \frac{1+ze^{-ib}}{1-ze^{-ib}}$$

and

$$\int_0^{2\pi} d\mu(b) = 2\pi.$$

The sufficient condition for univalency is given by Noshiro and Warschawski [78, 106] as given below.

Theorem 2.2.1: [78, 106] Let $t \in \mathbb{R}$ and for all $z \in D$, A function $y(z)$ is univalent in $\triangle$. If

$$\Re\left\{e^{it}y'(z)\right\} \geq 0,$$

where D is a convex domain.

Definition 2.2.2: Let $\mathfrak{P}(\lambda)$ is the set of all those whose real part greater then λ in $\triangle$, where $0 \leq \lambda < 1$. If $\hbar \in \mathfrak{P}(\lambda)$, then $\hbar(z)$ can be written as

$$\hbar(z) = (1-\lambda)\,\hbar_1(z) + \lambda, \quad \hbar_1 \in \mathfrak{P}, \quad z \in \triangle.$$

For $\lambda = 0$, we have $\mathfrak{P}(\lambda) = \mathfrak{P}$.

Lemma 2.2.1: [25, 31] Let $\hbar(z)$ be given in (2.2.2) and $\hbar(z) \in \mathfrak{P}(\lambda)$. Then

$$\left|\hbar_n\right| \leq 2(1-\lambda), \text{ for all } n \geq 1.$$

2.3: Subclasses of Univalent Functions

Here we will discuss about the subclasses of univalent functions.

Definition 2.3.1: [25, 31] A domain D which is the subset of $\mathbb{C}$ is a star shaped about any point z_0 if any other point of D joining with z_0 by line segment lies within D. Those functions $y(z)$ in which mapping of domain $\triangle$ is a star-shaped region (with respect to origin) is called starlike function.

Definition 2.3.2: [25, 31] A domain D which is the subset of $\mathbb{C}$ is convex if joining any two points of D by the line segment lies completely in D. Those function $y(z)$ in which image of a domain $\triangle$ is a convex domain is called convex function. Equation (2.3.1) is the example of convex functions.

$$y(z) = \frac{z}{1-z}. \tag{2.3.1}$$

Theorem 2.3.1: [66] A univalent function $y(z)$ which maps $\triangle$ onto a starshaped domain, iff

$$\Re\left(\frac{zy'(z)}{y(z)}\right) > 0, \quad z \in \triangle. \tag{2.3.2}$$

or

$$\left(\frac{zy'(z)}{y(z)}\right) \in \text{ß}.$$

The Koebe's function $Q(z) = \frac{z}{(1-z)^2}$ is the example of starlike functions. Also we note that

$$\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{S}.$$

Theorem 2.3.2: [90] A univalent function $y(z)$ which maps unit disk $\triangle$ onto a convex domain, iff

$$\Re\left(\frac{(zy'(z))'}{y'(z)}\right) > 0, \quad z \in \triangle,$$

or

$$\left(\frac{(zy'(z))'}{y'(z)}\right) \in \text{ß}, \quad z \in \triangle.$$

Relation between starlike and convex functions proved by Alexander [7] in 1915, as

$$y \in \mathcal{C} \Longleftrightarrow zy' \in \mathcal{S}^*.$$

Lemma 2.3.1: [25, 31] Let y be given by (2.1.1) and belongs to $\mathcal{S}^*$, then

$$|a_n| \leq n, \quad n \geq 2, \ z \in \triangle$$

Inequality is sharp for $Q(z) = \frac{z}{(1-z)^2}$.

Lemma 2.3.2: [25, 31] Let $y \in \mathcal{C}$ and be given by (2.1.1), then

$$|a_n| \leq 1, \quad \text{for all} \ \ n \geq 2, \ z \in \triangle \tag{2.3.3}$$

The inequality is sharp for $y(z) = \frac{z}{1-z}$.

Definition 2.3.3: [25, 31] Let $y \in \mathcal{S}^*(\beta)$, $0 \leq \beta < 1$, iff

$$\Re\left(\frac{zy'(z)}{y(z)}\right) > \beta, \quad z \in \triangle.$$

Definition 2.3.4: [25, 31] Let $y \in \mathcal{C}(\beta)$, $0 \leq \beta < 1$, iff

$$\Re\left(\frac{(zy'(z))'}{y'(z)}\right) > \beta, \quad z \in \triangle.$$

For more study see [25, 31].

The idea of $\alpha-$convexity given by Mocanu [65] in 1969 as:

Definition 2.3.5: [65] Let, $\frac{y(z)y'(z)}{z} \neq 0$, and $y \in \nabla$, then $y \in M(\alpha)$, iff

$$\Re\left\{(1-\alpha)\frac{zy'(z)}{y(z)} + \alpha\frac{(zy'(z))'}{y'(z)}\right\} > 0, \ \text{for all} \ \alpha \in \mathbb{R}, \ z \in \triangle.$$

For $\alpha = 0, 1$, we obtain

$$M(0) \equiv \mathcal{S}^* \text{ and } M(1) = \mathcal{C}.$$

For $\alpha \geq 0$, $M(\alpha) \subset \mathcal{S}^*$, and for $\alpha \geq 1$, $M(\alpha) \subset \mathcal{C}$, see [64].

2.4: Subordination

Lindelöf [57] was pioneer of the idea of subordination. Littlewood and Rogosinski were further studied this idea in [58, 87]. The idea of subordination is made up of a Schwarz functions. So we will give the definition of Schwarz function after we will give the definition of subordination.

Definition 2.4.1: If $u(z) \in \nabla$ and satisfy $u(0) = 0$ and $|u(z)| < 1$, then $u(z)$ is called a Schwarz function.

Definition 2.4.2: Let y, $s \in \nabla$, and y are subordinate to s, symbolically $y \prec s$, iff we have an analytic functions u with conditions $u(0) = 0$ and $|u(z)| < 1$, such that

$$y(z) = s(u(z)), \qquad z \in \triangle.$$

Particular if y is univalent in $\triangle$, then $y \prec s$, is equivalent to $y(0) - s(0) = 0$ and $y\,(\triangle) \subset s(\triangle)$.

2.5: Conic Type Regions and Circular Domains

We will briefly discuss conic domains, circular domains in this section. In these research articles [32, 33, 47, 48] the authors talk about the classes $U_{C,V}$ and S_T and give definitions and relations with conic domains.

2.5.1: Uniformly Convex ($U_{C,V}$) and Starlike (S_T) Functions

In the article [32, 33] the class $U_{C,V}$ and S_T was introduced, and after that Rønning [86], Ma and Minda [62], and others studied about it. The conditions of analyticity for the class $U_{C,V}$ and starlike (S_T) functions defined below.

Definition 2.5.1: [32, 33] Let $y \in \nabla$, then $y \in U_{C,V}$, iff

$$\Re\left[1+ (z-\xi)\frac{y''(z)}{y'(z)}\right] > 0, \quad (z, \xi) \in \triangle \times \triangle.$$

2.5.2: Uniformly Starlike Functions

Definition 2.5.2. [32, 33] Let $y \in \nabla$, then $y \in S_T$, iff

$$\Re\left[\frac{(z-\xi)y'(z)}{y(z)- y(\xi)}\right] > 0, \quad (z, \xi) \in \triangle \times \triangle.$$

Putting $\xi = 0$ in definitions 2.5.1 and 2.5.2, we get the $\mathcal{C}$ and $\mathcal{S}^*$. The well-known Alexander's theorem proved a relationship between the $\mathcal{C}$ and $\mathcal{S}^*$as

$$y \in \mathcal{C} \Leftrightarrow zy' \in \mathcal{S}^*$$

The analytical characterizations of the classes $U_{C,V}$ and S_T, given by Ma and Minda [62] in 1994, which is given as:

Definition 2.5.3: [62] Let $y \in \nabla$, then $y \in U_{C,V}$, iff

$$\Re\left[1+\frac{zy''(z)}{y'(z)}\right] > \left|\frac{zy''(z)}{y'(z)}\right|, \qquad z \in \triangle.$$

Definition 2.5.4: [62] Let $y \in \nabla$, then $y \in S_T$, iff

$$\Re\left[\frac{zy'(z)}{y(z)}\right] > \left|\frac{zy'(z)}{y(z)}-1\right|, \qquad z \in \triangle.$$

This classification gave parabolic domain:

$$\Omega = \{\varpi : \Re\left(\varpi(z)\right) > |\varpi(z)-1|\}.$$

Further Kanas and Wisniowska [47, 48] defined the conic domain Ω_k which is the generalization of the Ω parabolic domain and studied it in detail. Let Ω_k denote the following domain

$$\Omega_k = \{\varpi : \Re\left(\varpi(z)\right) > k\left|\varpi(z)-1\right|, \quad k \geq 0\}.$$

Regions

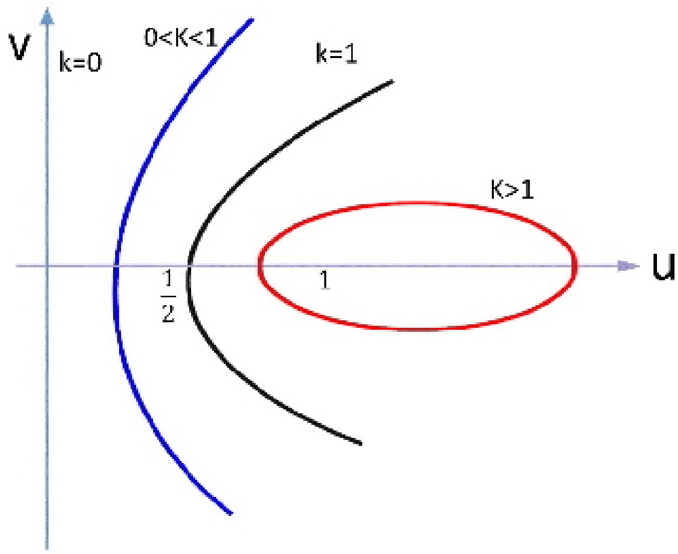

Figure 2.6.1. Conic regions

$_1.jpg$

Here we are going to discuss $\hbar_k(z)$ which we consider as extremal functions for the conic regions be defined as:

Ω_k represents the the right half-plane and

$$\hbar_k(z) = \frac{1+z}{1-z}, \quad \text{for} \quad k = 0.$$

Ω_k represents parabolic regions and in this case

$$\hbar_k(z) = 1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \text{ for } \quad k = 1.$$

Ω_k represents hyperbolic regions

$$\hbar_k(z) = 1+\frac{2}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right\}. \quad \text{for } \ 0 < k < 1.$$

Ω_k will be an elliptic regions

$$\hbar_k(z) = 1+\frac{2}{k^2-1}\sin\left(\frac{\pi}{2K(t)}\int_0^{\frac{\mu(z)}{t}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(xt)^2}}dx\right)+\frac{1}{k^2-1}, \quad \text{for} \quad k > 1.$$

For more study see [47, 48].

Ω_k represent the family of domains for $0 \leq k < \infty$. For detail study of these conic regions, see [68, 69, 91].

Definition 2.5.5: [47] Let $\hbar \in ß(\hbar_k)$, if $\hbar \prec \hbar_k(z)$, $z \in \triangle$.

i) $\hbar(\triangle) \subset \hbar_k(\triangle) = \Omega_k$.

ii) $\Re e \hbar(z) > \frac{k}{k+1}$, such that, $ß(\hbar_k) \subset ß(\frac{k}{k+1})$.

Kanas [47, 48] studied $k-U_{C,V}$ and $k-S_T$ in (1999), (2000).

Definition of these classes given below by making use of the class $ß(\hbar_k)$.

Definition 2.5.6: [47, 48] Let $y \in \nabla$, then $y \in k - U_{C,V}$, iff

$$\Re e\left\{\frac{(zy'(z))'}{y'(z)}\right\} > k\left|\frac{(zy'(z))'}{y'(z)}-1\right|, \qquad k \geq 0, \quad z \in \triangle,$$

above definition can be given as:

$$\left(\frac{(zy'(z))'}{y'(z)}\right) \prec \hbar_k(z), \qquad k \geq 0, \quad z \in \triangle,$$

and $\hbar_k(z)$ defined in (2.15.7).

Definition 2.5.7: [47, 48] Let $y \in \nabla$. Then $y \in k - S_T$, iff

$$\Re e\left[\frac{zy'(z)}{y(z)}\right] > k\left|\frac{zy'(z)}{y(z)} - 1\right|, \qquad k \geq 0, \quad z \in \triangle,$$

above definition can be given as:

$$\left(\frac{zy'(z)}{y(z)}\right) \prec \hbar_k(z), \qquad k \geq 0, \quad z \in \triangle.$$

and $\hbar_k(z)$ defined in (2.15.7).

Theorem 2.5.8. [47] Let $y \in \mathcal{S}$, then $y \in k - U_{C,V}$, for $\mathfrak{r}_0 > |z|$, where

$$\mathfrak{r}_0 = \frac{1}{\left(\sqrt{4k^2+6k+3}\right) 2(k+1)}.$$

Circular Domains

By using Janowski functions first time Janowski [44] studied and defined the circular domain. Definition of circular functions given below:

Definition 2.5.9. Let $\hbar \in \nabla$ and $\hbar(0) = 1$, then $\hbar \in \text{ß}[C, D]$ if

$$\hbar(z) \prec \frac{1+Cz}{1+Dz}, \quad -1 \leq D < C \leq 1.$$

The domain $\Omega[C, D]$ is:

$$\Omega\left[C, D\right] = \left\{\varpi : \left|\varpi - \frac{1-CD}{1-D^2}\right| < \frac{C-D}{1-D^2}\right\}.$$

Geometrically, $\hbar \in \text{ß}[C, D]$ maps $\triangle$ onto the domain $\Omega[C, D]$. The relation which are given below are connected the classes $\text{ß}[C, D]$ and ß as:

$$\hbar(z) \in \text{ß} \quad \text{iff} \quad \frac{(C{+}1)\,\hbar(z) - (C{-}1)}{(D{+}1)\,\hbar(z) - (D{-}1)} \in \text{ß}[C, D].$$

Taking $C = 1,\ D = -1$, then $\text{ß}[C, D] = \text{ß}$. Also $\text{ß}[C, D]$ is a convex set, see [75]. Below Janowski [44] defined Janowski convex functions $\mathcal{C}\,[C, D]$ and Janowski starlike functions $\mathcal{S}^*\,[C, D]$.

Definition 2.5.10: Let $y \in \nabla$ is given by (2.1.1), then $y \in \mathcal{C}\,[C, D]$, iff

$$\left(\frac{(zy'(z))'}{y'(z)}\right) \in \text{ß}[C, D].$$

Definition 2.5.11: Let $y \in \nabla$ is given by (2.1.1), then $y \in \mathcal{S}^*[C, D]$, iff

$$\left(\frac{zy'(z)}{y(z)}\right) \in \text{ß}[C, D].$$

The relation

$$y \in \mathcal{C}[C, D] \qquad \text{iff} \qquad zy' \in \mathcal{S}^*[C, D],$$

is holds, and this relation is called Alexander type relation. For deep study about these classes, see [22, 23, 59, 60, 71, 72, 83].

2.6: Convolution [70]

Here we will discuss interesting properties and definition of convolution for up coming chapters.

Definition 2.6.1: The convolution of y and m given as:

$$(\ y * m)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{4.1.1}$$

and m given as:

$$m(z) = z + \sum_{n=2}^{\infty} b_n z^n\ , \quad z \in \triangle.$$

The identity function under convolution is defined by (2.3.1) and $(y * m)(z) = m(z)$ for all $m \in \nabla$.

$$z(y * m)'(z) = (zy' * m)(z) = (y * zm')(z)$$

and

$$(y * Q)(z) = zy'(z),$$

where $Q(z)$ defined in (2.1.2).

For further study, see [25].

2.7: Srivastava-Attiya operator [95]

By using the convolution idea, Srivastava and Attiya [95] introduced a linear operators in 1984, as:

Definition 2.7.1: Let function $\phi(u, b, \mathrm{z})$ is given by

$$\begin{aligned} \phi(\mu, b, \mathrm{z}) &= \sum_{n=0}^{\infty} \frac{\mathrm{z}^n}{(n+b)^{\mu}}, \\ &= b^{-\mu} + \frac{\mathrm{z}}{(1-b)^{\mu}} + \sum_{n=2}^{\infty} \frac{\mathrm{z}^n}{(n-b)^{\mu}}, \end{aligned}$$

where $\phi(u, b, \mathrm{z})$ is the generalized Hurwitz-Lerch Zeta function and $b \in \mathbb{C}$, $b \neq 0, -1, -2, ..., \mu \in \mathbb{C}$, $\Re e(\mu) > 1$, $\mathrm{z} \in \triangle$.

Srivastava et.al [95] defined the family of linear operators $J_{\mu,b} : \nabla \longrightarrow \nabla$ by using the Hurwitz-Lerch Zeta functions along with the convolution as:

$$J_{\mu,b} y(\mathrm{z}) = G_{\mu,b} * y(\mathrm{z}) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b} \right)^{\mu} a_n \mathrm{z}^n, \tag{2.7.1}$$

and $G_{\mu,b} \in \nabla$ given by

$$G_{\mu,b} = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b} \right)^{\mu} \mathrm{z}^n.$$

By using (2.7.1) and (2.7.2) following identity will hold

$$\mathrm{z}\left[J_{\mu,b}y(\mathrm{z})\right]' = (1+b)J_{\mu-1,b}y(\mathrm{z}) - bJ_{\mu,b}y(\mathrm{z}).$$

Remark 1: For $\mu = 0$, then $J_{\mu,b}$ reduces into identity operator $J_{0,b}$ and for $\mu = -\mu$, then $J_{\mu,b}$ reduces into inverse operator $J_{-\mu,b}$.

Remark 2: Srivastava-Attiya operator defined in (2.7.1) generalizes many known operators for example.

(i) For $\mu = 1$ and $b = 0$, (2.7.1) reduces to the well-known operator defined earlier by Alexander [7].

ii) For $\mu = 1$ and $b = 1$, (2.7.1) reduces into the familiar operator investigated by Libera [56].

(iii) For $\mu = 1$ and $b = \gamma > -1$, $\gamma \in \mathbb{N}$, (2.7.1) reduces to the Bernardi integral operator defined by Bernardi [15].

(iv) For $\mu = \sigma > 0$ and $b = 1$, (2.7.1) reduces to Jung–Kim–Srivastava integral operator [45].

2.8: Ruscheweyh derivative operator (1975)

Definition 2.8.1: Let $y \in \nabla$, Ruscheweyh [88] defined the operator $\mathcal{R}^{\lambda}$ as:

$$\mathcal{R}^{\lambda}y(\mathrm{z}) = y(\mathrm{z}) * \frac{\mathrm{z}}{(1-\mathrm{z})^{\lambda+1}}, \quad \mathrm{z} \in \triangle, \ \lambda \in \mathbb{R}, \lambda > -1.$$

For $\lambda = n \in N_0 = NU\{0\}$, we obtain

$$\mathcal{R}^{n}y(\mathrm{z}) = \frac{\mathrm{z}(\mathrm{z}^{n-1}y(\mathrm{z}))^{(n)}}{n!}. \tag{2.8.1}$$

$\mathcal{R}^{n}y(\mathrm{z})$ is the nth-order Ruscheweyh derivative of $y(\mathrm{z})$.

For $n = 0$, then $\mathcal{R}^{0}y(z) = y(\mathrm{z})$, and $\mathcal{R}^{1}y(\mathrm{z}) = \mathrm{z}y'(\mathrm{z}) = \Gamma_0$, the Alexander differential operator.

The identity (2.8.2) is hold for the operator $\mathcal{R}^{\lambda}y(\mathrm{z})$

$$(\lambda+1)\mathcal{R}^{\lambda+1}y(\mathrm{z})-\lambda\mathcal{R}^{\lambda}y(\mathrm{z}) = \mathrm{z}(\mathcal{R}^{\lambda}y(\mathrm{z}))' \tag{2.8.2}$$

2.9: q-Derivative operator or q-Difference operator

Definition 2.9.1: [43] Jackson defined the q-difference operator or q-derivative operator of analytic functions ∇ as:

$$\partial_q y(\mathrm{z}) = \frac{y(q\mathrm{z}) - y(\mathrm{z})}{(q-1)\,\mathrm{z}}, \qquad (\mathrm{z} \in \triangle,\ q \in (0,1)).$$

and

$$\partial_q \mathrm{z}^n = (n)^q\, \mathrm{z}^{n-1}, \quad \partial_q\left(\sum_{n=1}^{\infty} a_n \mathrm{z}^n\right) = \sum_{n=1}^{\infty} (n)^q\, a_n \mathrm{z}^{n-1}.$$

2.10: q-Generalized Pochhammer symbol

We defined q-generalized Pochhammer symbol as:

$$(v)_n^q = (v)^q\,(v+1)^q\,(v+2)^q \ldots (v+n-1)^q,$$

Let the q-gamma function for $v > 0$ be defined as

$$\Gamma_q(v) = \frac{\Gamma_q\{v+1\}}{(v)} \quad \text{and} \quad \Gamma_q(1) = 1.$$

For $v \in \mathbb{R}$ and $0 < q < 1$, the number $(v)^q$ be defined in [49] as:

$$(v)^q = \frac{1-q^v}{1-q}, \quad (0)^q = 0.$$

The q-number shift factorial for the non-negative integer n, be defined as:

$$(n)^q! = (1)^q\,(2)^q\,(3)^q \ldots (n)^q, \qquad ((0)^q! = 1).$$

For $q \to 1$, then $(n)^q! = n!$, where $q \in (0,1)$.

2.11: Ruscheweyh q-Differential operator

Definition 2.11.1: [49] For $y \in \nabla$, let the Ruscheweyh q-differential operator be defined as:

$$\mathcal{R}_q^{\lambda} y(z) = y(z) * \mathcal{F}_{q,\lambda+1}(z), \qquad (z \in \triangle,\ \lambda > -1) \tag{2.11.1}$$

where

$$\mathcal{F}_{q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q\{n+\lambda\}}{(n-1)^q!\Gamma_q(\lambda+1)} z^n = z + \sum_{n=2}^{\infty} \frac{(\lambda+1)_{n-1}^q}{(n-1)^q!} z^n. \tag{2.11.2}$$

"*" is the symbol of convolution. From (2.11.1) we obtain

$$\mathcal{R}_q^0 y(z) = y(z), \quad \mathcal{R}_q^1 y(z) = z\partial_q y(z)$$

and

$$\mathcal{R}_q^m y(z) = \frac{z\partial_q^n (z^{n-1} y(z))}{(n)^q!}, \qquad (n \in \mathbb{N}).$$

By use of (2.11.1) and (2.11.2), we get

$$\begin{aligned} \mathcal{R}_q^{\lambda} y(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma_q\{n+\lambda\}}{(n-1)^q!\Gamma_q(\lambda+1)} a_n z^n \\ &= z + \sum_{n=2}^{\infty} \frac{(\lambda+1)_{n-1}^q}{(n-1)^q!} a_n z^n \\ &= z + \sum_{n=2}^{\infty} \phi_{n-1} a_n z^n, \end{aligned} \tag{2.11.3}$$

where

$$\phi_{n-1} = \frac{(\lambda+1)_{n-1}^q}{(n-1)^q!}. \tag{2.11.4}$$

we see that when $q \to 1^-$, then

$$\mathcal{F}_{q,\lambda+1}(z) = \frac{z}{(1-z)^{\lambda+1}}.$$

and

$$\mathcal{R}_q^{\lambda} y(z) = y(z) * \frac{z}{(1-z)^{\lambda+1}}. \tag{2.11.5}$$

If $q \to 1^-$, in (2.11.5), we obtain Ruscheweyh differential operator [49]. we verify that

$$z\partial\left(\mathcal{F}_{q,\lambda+1}(z)\right) = \left(1 + \frac{(\lambda)^q}{q^\lambda}\right)\mathcal{F}_{q,\lambda+2}(z) - \frac{(\lambda)^q}{q^\lambda}\mathcal{F}_{q,\lambda+1}(z). \tag{2.11.6}$$

Making use of (2.11.1), (2.11.6), and use the convolution properties, the following identity is hold.

$$z\partial_q\left(\mathcal{R}_q^{\lambda} y(z)\right) = \left(1 + \frac{(\lambda)^q}{q^\lambda}\right)\mathcal{R}_q^{\lambda+1} y(z) - \frac{(\lambda)^q}{q^\lambda}\mathcal{R}_q^{\lambda} y(z). \tag{2.11.7}$$

If $q \to 1^-$, in (2.11.7), we obtain

$$(1+\lambda)\,\mathcal{R}^{\lambda+1} y(z) - \lambda\mathcal{R}^{\lambda} y(z) = z\left(\mathcal{R}_q^{\lambda} y(z)\right)'. \tag{2.11.8}$$

Equation (2.11.8) is the identity of Ruscheweyh differential operator.

2.12: Salagean q-differential operator

Definition 2.12.1: [30] Let $y \in \nabla$, Let Salagean q-differential operator

$$\mathcal{S}_q^0 y(\mathrm{z}) = y(\mathrm{z}), \quad \mathcal{S}_q^1 y(\mathrm{z}) = \mathrm{z}\partial_q y(\mathrm{z}), \cdots, \mathcal{S}_q^m y(\mathrm{z}) = \mathrm{z}\partial_q\left(\mathcal{S}_q^{m-1} y(\mathrm{z})\right), \ \mathrm{z} \in \triangle.$$

A simple calculation implies

$$\mathcal{S}_q^m y(\mathrm{z}) = y(\mathrm{z}) * \mathcal{G}_{q,m}(\mathrm{z}) \tag{2.12.1}$$

and

$$\mathcal{G}_{q,m}(z) = \mathrm{z} + \sum_{n=2}^{\infty} \left\{(n)^q\right\}^m \mathrm{z}^n. \tag{2.12.2}$$

Making use of (2.12.1) and (2.12.2), for y of the form (2.1.1) and series of $\mathcal{S}_q^m y(\mathrm{z})$

$$\mathcal{S}_q^m y(\mathrm{z}) = \mathrm{z}+\sum_{n=2}^{\infty} \{(n)^q\}^m a_n \mathrm{z}^n. \tag{2.12.3}$$

Note that

$$Lim_{q\to 1}\mathcal{G}_{q,m}(\mathrm{z}) = \mathrm{z}+\sum_{n=2}^{\infty} n^m \mathrm{z}^n,$$

and

$$Lim_{q\to 1}\mathcal{S}_q^m y(\mathrm{z}) = \mathrm{z}+\sum_{n=2}^{\infty} n^m a_n \mathrm{z}^n,$$

which is the familiar Salagean derivative [99].

2.13. m-Fold Symmetric functions

Definition 2.13.1. Let $m \in \mathbb{Z}^+$. A domain $\triangle$ is called m-fold symmetric functions if

$$y\left(e^{i\frac{2\pi}{m}}\mathrm{z}\right) = e^{i\frac{2\pi}{m}} y(\mathrm{z}), \qquad y \in \nabla, \quad \mathrm{z} \in \triangle.$$

For all $y \in \mathcal{S}$, we get

$$h(\mathrm{z}) = \sqrt[m]{y(\mathrm{z}^m)}$$

is one one and maps the $\triangle$ into a region with m-fold symmetry.

All m-fold symmetric univalent functions contained in class $\mathcal{S}^m$ and note that $\mathcal{S}^1 = \mathcal{S}$.

The series expansion of $y \in \mathcal{S}^m$ is given as

$$y(\mathrm{z}) = \mathrm{z} + \sum_{n=1}^{\infty} a_{nm+1}\mathrm{z}^{nm+1}. \tag{2.13.1}$$

2.14. Multivalent Functions

Here we will briefly discuss about the class of multivalent functions ($\nabla(p)$) in unit disk $\triangle$. In Complex Analysis the class $\nabla(p)$ have a key role, for detail see [25, 31, 40].

Definition 2.14.1: Let $y \in \nabla(p)$ in open unit disk $\triangle$ have a series expansion of the form

$$y(\mathrm{z}) = \mathrm{z}^p + \sum_{n=p+1}^{\infty} a_n \mathrm{z}^n, \qquad p \in \mathbb{N}. \tag{2.14.1}$$

$\nabla(p)$ denote the set of all analytic multivalent functions.

For $p = 1$, then $\nabla(p) = \nabla$.

Definition 2.14.2: [40] A function $y \in \nabla(p)$ is in $\mathcal{S}_p^*$, iff

$$\Re\left(\frac{zy'(z)}{y(z)}\right) > 0, \qquad (\mathrm{z} \in \triangle). \tag{2.14.2}$$

The class $\mathcal{S}_p^*$ denote multivalent starlike functions in $\triangle$. It is easy to see that $\mathcal{S}^* = \mathcal{S}_1^*$.

Definition 2.14.3: [40] Let $y \in \nabla(p)$ is in $\mathcal{C}_p$, iff

$$\Re\left(1 + \frac{\mathrm{z}y''(\mathrm{z})}{y'(\mathrm{z})}\right) > 0, \qquad (\mathrm{z} \in \triangle). \tag{2.14.3}$$

The class $\mathcal{C}_p$ denote multivalent convex functions in $\triangle$. It is easy to see that $\mathcal{C}_1 = \mathcal{C}$.

The classes $\mathcal{S}_p^*$ and $\mathcal{C}_p$ are satisfy the Alexader relation as

$$y \in \mathcal{C}_p \quad \text{iff} \quad \frac{\mathrm{z}y'}{p} \in \mathcal{S}_p^*.$$

For detail see [80, 98].

Definition 2.14.4: [40] Let $y \in \nabla(p)$, is in $\mathcal{S}_p^*(\lambda)$, iff

$$\Re\left(\frac{\mathrm{z}y'(\mathrm{z})}{y(\mathrm{z})}\right) > \lambda, \qquad (\mathrm{z} \in \triangle). \tag{2.14.4}$$

In $\mathcal{S}_p^*(\lambda)$, $(0 \leq \lambda < p)$, in $\triangle$. For $\lambda = 0$, we have, $\mathcal{S}_p^* = \mathcal{S}_p^*(0)$, and $\mathcal{S}^* = \mathcal{S}_1^*$.

Definition 2.14.5. [40] Let $y \in \nabla(p)$, is in $\mathcal{C}_p(\lambda)$, iff

$$\Re\left(1 + \frac{\mathrm{z}y''(\mathrm{z})}{y'(\mathrm{z})}\right) > \lambda, \qquad (\mathrm{z} \in \triangle). \tag{2.14.5}$$

In $\mathcal{C}_p(\lambda)$, $(0 \leq \lambda < p)$, For $\lambda = 0$, we have, $\mathcal{C}_p(0) = \mathcal{C}_p$ and $\mathcal{C}_1 = \mathcal{C}$.

2.15 Preliminary Results

Here some basic lemmas have discussed, which we will use in up coming chapter.

Lemma 2.15.1 [31, 84] If $\hbar(\mathrm{z}) = 1+\sum_{n=1}^{\infty} p_n \mathrm{z}^n$ have a positive real part and is analytic in $\triangle$, then

$$|p_n| \leq 2 \ (n \in \mathbb{N}.$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}.$$

Lemma 2.15.2 [62] Let $\hbar \in ß$, $u \in \mathbb{C}$ and $\hbar(\mathrm{z}) = 1+p_1\mathrm{z}+p_2\mathrm{z}^2+....$ Then

$$\left| p_2 - u p_1^2 \right| \leq \begin{cases} -4u+2 & \text{if} \quad u \leq 0, \\ 2 & \text{if } 0 \leq u \leq 1, \\ 4u-2 & \text{if} \quad u \geq 1. \end{cases}$$

For detail see **[62].**

Lemma 2.15.3 [26] Let $\hbar(\mathrm{z}) = 1+\sum_{n=1}^{\infty} p_n \mathrm{z}^n$, $\mathrm{z} \in \triangle$. For $u \in \mathbb{C}$

$$\left| p_2 - u p_1^2 \right| \leq 2 \max(1, |2u-1|\,.$$

For sharpness

$$p(\mathrm{z}) = (1+\mathrm{z})\,(1-\mathrm{z})^{-1} \ \text{ or } p(\mathrm{z}) = \left(1+\mathrm{z}^2\right)\left(1-\mathrm{z}^2\right)^{-1}.$$

Lemma 2.15.4 [87] Let $\hbar(\mathrm{z}) \prec \mathcal{H}(\mathrm{z})$, then

$$|p_n| \leq |C_1|\,, \qquad n \geq 1.$$

Where $\mathcal{H}(\mathrm{z})$ is univalent and convex in $\triangle$, and if $\hbar(z) = 1 + \sum_{n=1}^{\infty} p_n \mathrm{z}^n$ and $\mathcal{H}(\mathrm{z}) =$

$1+\sum_{n=1}^{\infty} C_n \mathrm{z}^n$.

Lemma 2.15.5. [52] Let $\hbar(\mathrm{z}) = 1+\sum_{n=1}^{\infty} p_n \mathrm{z}^n$, $\mathrm{z} \in \triangle$. Then for every complex number u,

$$|c_1| \leq 1, \ \ |c_2| \leq 1-|c_1|^2, \ \ \left|c_2-uc_1^2\right| \leq 1+(|u|-1)\left|c_1^2\right|.$$

Lemma 2.15.6. [46] Let $k \in [0,\infty)$, and $\hbar_k \in ß\left(\hbar_k\right)$. If

$$\hbar_k(\mathrm{z}) = 1+T_1(k)\,\mathrm{z}+T_2(k)\,\mathrm{z}^2+..., \qquad \mathrm{z} \in \triangle,$$

then

$$T_1 = T_1(k) = \begin{cases} \frac{2\left(\frac{2}{\pi}\arccos k\right)^2}{1-k^2}, & 1 > k \geq 0, \\ \frac{8}{\pi^2}, & k = 1, \\ \frac{\pi^2}{4k^2(i)^2(1+i)\sqrt{i}} & 1 < k. \end{cases} \tag{2.15.1}$$

$$T_2 := T_2(k) = \mathcal{D}(k)\,T_1(k), \tag{2.15.2}$$

where

$$\mathcal{D}(k) = \begin{cases} \frac{\left(\frac{2}{\pi}\arccos k\right)^2+2}{3}, & 0 \leq k < 1, \\ \frac{8}{\pi^2}, & k = 1, \\ \frac{\left(i^2+6i+1\right)-\pi^2\left(4K^2\right)^2}{24\sqrt{i}K(i)^2(1+i)} & k > 1. \end{cases} \tag{2.15.3}$$

For detail see [47, 48].

Lemma 2.15.7 [76] Let $-1 \leq D < C \leq 1$, $0 \leq k < \infty$, then

$$\hbar_k(\mathrm{z}) = 1+W_1(k)\mathrm{z}+W_2(k)\mathrm{z}^2+..., \qquad \mathrm{z} \in \triangle,$$

and

$$W_1 \ : \ = W_1(k) = \frac{C-D}{2} T_1(k),$$

$$W_2 \ : \ = W_2(k) = \frac{C-D}{4} \left\{ 2\mathcal{D}(k) - (D+1)W_1 \right\} T_1(k),$$

where

$$\hbar_k(\mathrm{z}) = \frac{(C+1)\mathrm{p}_k(\mathrm{z}) - (C-1)}{(D+1)\mathrm{p}_k(\mathrm{z}) - (D-1)}$$

and $T_1(k)$, $\mathcal{D}(k)$ are given by (2.15.1) and (2.15.3).

Lemma 2.15.8 [100] Let $0 \leq k < \infty$, if

$$\hbar_{k,\gamma}(z) = 1 + Q_1 z + Q_2 z^2 + ... \tag{2.15.4}$$

$$Q_1 \ = \ \begin{cases} \frac{2\gamma\left(\frac{2}{\pi}\arccos k\right)^2}{1-k^2}, & 0 \leq k < 1 \\ \frac{8\gamma}{\pi^2}, & k = 1, \\ \frac{\pi^2\gamma}{4(1+t)\sqrt{t}K^2(t)(k^2-1)}, & k > 1, \end{cases} , \tag{2.15.5}$$

$$Q_2 \ = \ \begin{cases} \frac{\left(\frac{2}{\pi}\arccos k\right)^2+2}{3} Q_1, & 0 \leq k < 1 \\ \frac{2}{3} Q_1, & k = 1, \\ \frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24K^2(t)(1+t)\sqrt{t}} Q_1, & k > 1, \end{cases} , \tag{2.15.6}$$

where

$$\hbar_{k,\gamma}(z) = \begin{cases} \frac{1+z}{1-z} & \text{for } k = 0, \\ 1 + B_1 & \text{for } k = 1 \\ 1 + B_2 & \text{for } 0 < k < 1, \\ 1 + B_3, & \text{for } k > 1. \end{cases} \tag{2.15.7}$$

where $B_1 = \frac{2\gamma}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$, $B_2 = \frac{2\gamma}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right\}$,
$B_3 = \frac{\gamma}{k^2-1}\sin\left(\frac{\pi}{2R(i)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(ix)^2}}dx\right) + \frac{\gamma}{1-k^2}$ and $i \in (0,1)$, $k = \cosh\left(\frac{\pi K'(i)}{K(i)}\right)$, $K(i)$ is the first kind of Legendre's complete elliptic integral. For details see [47, 48, 68].

Chapter 2

Application of a q-Salagean type operator on a multivalent functions

3.1: Introduction

The classes $\mathcal{S}_p^*$, $\mathcal{C}_p$, $\mathcal{S}_p^*(\lambda)$and $\mathcal{C}_p^*(\lambda)$ are defined in definition (2.14.2), (2.14.3),(2.14.4) and (2.14.5). The class $k-S_T$ of analytic functions introduced by Kanas and Wisniowska in [48] and is defined as:

$$\Re\left\{1+\frac{1}{\gamma}\left\{\left(\frac{zy'(z)}{y(z)}\right)-1\right\}\right\}>k\left|\frac{1}{\gamma}\left\{\left(\frac{zy'(z)}{y(z)}\right)-1\right\}\right|,\qquad z\in\triangle.$$

Later Hussain at el. [34] introduced the class $k-\mathcal{US}(q,\gamma,m)$ by using a q-Salagean operator, which is given by:

$$\Re\left\{1+\frac{1}{\gamma}\left\{\left(\frac{z\partial_q\mathcal{S}_{q,p}^m y(z)}{\mathcal{S}_{q,p}^m y(z)}\right)-1\right\}\right\}>k\left|\frac{1}{\gamma}\left\{\left(\frac{z\partial_q\mathcal{S}_{q,p}^m y(z)}{\mathcal{S}_{q,p}^m y(z)}\right)-1\right\}\right|,\qquad z\in\triangle.$$

Here we defined new class $k-US_{q,p}^{\gamma,m}$ of AFs by using a newly defined operator. We are declared here that this chapter is completely published in Journal of Inequalities and Applications.

Let

$$\oplus_{p,q}^m(z)=z^p+\sum_{n=p+1}^{\infty}[(n+(p-1))^q]^m z^n. \tag{3.1.1}$$

Making use of (3.1.1), definition 2.9.1 and idea of convolution, we define the operator,

$\mathcal{S}_{q,p}^{m}y(z) : \nabla(p) \to \nabla(p)$, for multivalent functions as:

$$\begin{aligned}\mathcal{S}_{q,p}^{m}y(z) &= \oplus_{p,q}^{m}(z) * y(z),\ m \in \mathbb{N}\cup\{0\},\\ &= z^p + \sum_{n=p+1}^{\infty}[(n+(p-1))^q]^m a_n z^n,\\ &= z^p + \sum_{n=p+1}^{\infty}\psi_n a_n z^n, \qquad (3.1.2)\end{aligned}$$

where

$$\psi_n = [(n+(p-1))^q]^m.$$

Note that $\mathcal{S}_{q,1}^{m}y(z) = \mathcal{S}_{q}^{m}y(z)$, see [30], and $p=1$, $q\to 1$, the operator $\mathcal{S}_{q,p}^{m}y(z) = \mathcal{S}^{m}y(z)$, see [99].

Taking inspiration from [34], and using (3.1.2), we introduced a class $k-US_{q,p}^{\gamma,m}$, of functions $\nabla(p)$ as:

In this chapter we will suppose $k \geq 0$, $m \in \mathbb{N}\cup\{0\}$, $q \in (0,1)$, $\gamma \in \mathbb{C}\backslash\{0\}$ and $p \in \mathbb{N}$.

Definition 3.1.1: Let $y(z) \in \nabla(p)$ is in the class $k-US_{q,p}^{\gamma,m}$, if it satisfies the condition

$$\operatorname{Re}\left\{1+\frac{1}{\gamma}\left\{\frac{1}{(p)^q}\left(\frac{z\partial_q\mathcal{S}_{q,p}^{m}y(z)}{\mathcal{S}_{q,p}^{m}y(z)}\right)-1\right\}\right\} > k\left|\frac{1}{\gamma}\left\{\frac{1}{(p)^q}\left(\frac{z\partial_q\mathcal{S}_{q,p}^{m}y(z)}{\mathcal{S}_{q,p}^{m}y(z)}\right)-1\right\}\right|, \qquad z \in \triangle.$$

(i) For $k-US_{q,1}^{\gamma,m} = k-US_{q}^{\gamma,m}$, introduced by Hussain et.al [34].

(ii) Take $\gamma \in \mathbb{C}\backslash\{0\}$, the class $0-US_{q,1}^{\gamma,o} = \mathcal{S}_q^{*}(\gamma)$ studied by Seoudy and Aouf [89].

(iii) For $\gamma = \frac{1}{1-\alpha}$, with $0 \leq \alpha < 1$, the class $0-US_{q,1}^{\gamma,o} = \mathcal{S}_q^{*}(\alpha)$ introduced in [1].

(iv) For $q \to 1$, and $\gamma = \frac{1}{1-\alpha}$, with $0 \leq \alpha < 1$, the class $k-US_{q,1}^{\gamma,o} = \mathcal{SD}(k,\alpha)$ introduced in [91].

(v) For $q \to 1$, and $\gamma = \frac{2}{1-\alpha}$, with $0 \leq \alpha < 1$, the class $k-US_{q,1}^{\gamma,o} = \mathcal{KD}(k,\alpha)$, introduced in [79].

(vi) For $q \to 1$, and $\gamma = \frac{1}{1-\alpha}$, with $0 \leq \alpha < 1$, the class $1 - US_{q,1}^{\gamma,o} = \mathcal{S}(\alpha)$, studied by Ali et.al [8].

(vii) For $q \to 1$, and $\gamma = \frac{2}{1-\alpha}$, with $0 \leq \alpha < 1$, the class $1 - US_{q,1}^{\gamma,o}$ reduces into the class $C(\alpha)$, in [8].

(viii) For $q \to 1$, the class $k - US_{q,1}^{\gamma,o} = \mathcal{K} - \mathcal{ST}$, see [48].

(ix) For $q \to 1$, and $\gamma = \frac{1}{1-\alpha}$, with $0 \leq \alpha < 1$, the class $0 - US_{q,1}^{\gamma,o} = \mathcal{S}^*(\alpha)$.

Geometric Interpretation

A function $y(z) \in \nabla(p)$, and let $y(z) \in k - US_{q,p}^{\gamma,m}$ iff $\frac{1}{(p)^q}\left(\frac{z\partial_q \mathcal{S}_{q,p}^m y(z)}{\mathcal{S}_{q,p}^m y(z)}\right)$ takes all the values in conic domain $\Omega_{k,\gamma} = h_{k,\gamma}(\triangle)$, such that

$$\Omega_{k,\gamma} = \gamma\Omega_k + (1-\alpha),$$

where

$$\Omega_k = \left\{ u + iv : k\sqrt{(u-1)^2 + v^2} < u \right\}.$$

We know that $\hbar_{k,\gamma}(z)$ is univalent and convex , so definition 3.1.1, we have

$$\frac{1}{(p)^q}\left(\frac{z\partial_q \mathcal{S}_{q,p}^m y(z)}{\mathcal{S}_{q,p}^m y(z)}\right) \prec \hbar_{k,\gamma}(z), \tag{3.1.3}$$

where $\hbar_{k,\gamma}(z)$ is given by (2.15.7). For details see [47, 48, 68]. Moreover, $\hbar_{k,\gamma}(\triangle)$ is convex univalent in $\triangle$, see [47, 48].

3.2: Main Results

Theorem 3.2.1: Let $y(z) \in k - US_{q,p}^{\gamma,m}$. Then

$$\mathcal{S}_{q,p}^{m} y(z) \prec z \exp \int_0^z \frac{(p)^q \left\{ \hbar_{k,\gamma}(u(z)) \right\} - 1}{\zeta} d\xi, \tag{3.2.1}$$

where $u(z) \in \nabla$ in unit disk $\triangle$ and satisfying $u(0) = 0$, $|u(z)| < 1$. and for $|z| = \rho$, we have

$$\exp \left(\int_0^1 \frac{(p)^q \left\{ \hbar_{k,\gamma}(-\rho) \right\} - 1}{\rho} d\rho \right) \leq \left| \frac{\mathcal{S}_{q,p}^{m} y(z)}{z} \right| \leq \exp \left(\int_0^1 \frac{(p)^q \left\{ \hbar_{k,\gamma}(\rho) \right\} - 1}{\rho} d\rho \right), \tag{3.2.2}$$

where $\hbar_{k,\gamma}(z)$ is defined by (2.15.7).

Proof. If $y(z) \in k - US_{q,p}^{\gamma,m}$, and using (3.1.3), we obtain

$$\begin{aligned}
\frac{1}{(p)^q} \left(\frac{z \partial_q \mathcal{S}_{q,p}^{m} y(z)}{\mathcal{S}_{q,p}^{m} y(z)} \right) &\prec \hbar_{k,\gamma}(z) \\
\frac{1}{(p)^q} \left(\frac{z \partial_q \mathcal{S}_{q,p}^{m} y(z)}{\mathcal{S}_{q,p}^{m} y(z)} \right) &= \hbar_{k,\gamma}(u(z)) \\
\frac{\partial_q \mathcal{S}_{q,p}^{m} y(z)}{\mathcal{S}_{q,p}^{m} y(z)} - \frac{1}{z} &= \frac{(p)^q \left\{ \hbar_{k,\gamma}(u(z)) \right\} - 1}{z}.
\end{aligned} \tag{3.2.3}$$

Taking Integral on both side of (3.2.3) and after simple simplification, we get

$$\mathcal{S}_{q,p}^{m} y(z) \prec z \exp \int_0^z \frac{(p)^q \left\{ \hbar_{k,\gamma}(u(z)) \right\} - 1}{\zeta} d\xi. \tag{3.2.4}$$

Which is our required (3.2.1). Nothing that the $h_{k,\gamma}(z)$ maps the disc $|z| < \rho \;\; (0 < \rho \leq 1)$, onto a region and which is convex, symmetric with respect to the real axis, we see

$$\hbar_{k,\gamma}(-\rho\,|z|) \leq \Re\left\{\hbar_{k,\gamma}(u(\rho z)\right\} \leq \hbar_{k,\gamma}(\rho\,|z|) \quad (0<\rho\leq 1, \quad z\in\triangle). \tag{3.2.5}$$

Using (3.2.4) and (3.2.5), we obtain

$$\begin{aligned}\int_0^1 \frac{(p)^q\left\{\hbar_{k,\gamma}(-\rho\,|z|)\right\}-1}{\rho}d\rho \;&\leq\; \Re\int_0^1 \frac{(p)^q\left\{\hbar_{k,\gamma}(u\,(\rho\,(z))\right\}-1}{\rho}d\rho\\ &\leq\; \int_0^1 \frac{(p)^q\left\{\hbar_{k,\gamma}(\rho\,|z|)\right\}-1}{\rho}d\rho,\end{aligned}$$

for $z\in\triangle$. Therefore, (3.2.4), leads us to

$$\int_0^1 \frac{(p)^q\left\{\hbar_{k,\gamma}(-\rho\,|z|)\right\}-1}{\rho}d\rho \leq \log\left|\frac{\mathcal{S}_{q,p}^m y(z)}{z}\right| \leq \int_0^1 \frac{(p)^q\left\{\hbar_{k,\gamma}(\rho\,|z|)\right\}-1}{\rho}d\rho,$$

$$\hbar_{k,\gamma}(-\rho) \leq \hbar_{k,\gamma}(-\rho\,|z|), \quad \hbar_{k,\gamma}(\rho\,|z|) \leq \hbar_{k,\gamma}(\rho),$$

implies that

$$\exp\int_0^1 \frac{(p)^q\left\{\hbar_{k,\gamma}(-\rho)\right\}-1}{\rho}d\rho \leq \left|\frac{\mathcal{S}_{q,p}^m y(z)}{z}\right| \leq \exp\int_0^1 \frac{(p)^q\left\{\hbar_{k,\gamma}(\rho)\right\}-1}{\rho}d\rho.$$

Corollary 3.2.1: [34]. Let $y(z)\in k-US_{q,1}^{\gamma,m}$, then

$$\mathcal{S}_q^m y(z) \prec z\exp\int_0^z \frac{\hbar_{k,\gamma}(u(\xi))-1}{\zeta}d\xi,$$

where $u(z)\in\triangle$ and satisfy the condition $u(0)=0$, $|u(z)|<1$, and for $|z|=\rho$, we have

$$\exp\left(\int_0^1 \frac{\hbar_{k,\gamma}(-\rho)-1}{\rho}d\rho\right) \leq \left|\frac{\mathcal{S}_q^m y(z)}{z}\right| \leq \exp\left(\int_0^1 \frac{\hbar_{k,\gamma}(\rho)-1}{\rho}d\rho\right),$$

where $\hbar_{k,\gamma}(z)$ is defined by (2.15.7).

Theorem 3.2.2: If $y(z) \in k - US_{q,p}^{\gamma,m}$, then

$$|a_{p+1}| \leq \frac{\delta}{\left\{(p+1)_q - (p)_q\right\} \psi_{p+1}},$$

and

$$|a_{n+p-1}| \leq \frac{\delta}{\{(n+p-1)^q - (p)^q\} \psi_{n+p-1}} \prod_{j=1}^{n-2} \left(1 + \frac{\delta}{\{(j+p)^q - (p)^q\}}\right), \text{ for } n \geq 3. \tag{3.2.6}$$

where $\delta = (p)^q |Q_1|$ and Q_1 defined by (2.15.5).

Proof: Let

$$\frac{1}{(p)^q} \left(\frac{z \partial_q \mathcal{S}_{q,p}^m y(z)}{\mathcal{S}_{q,p}^m y(z)} \right) = \hbar(z),$$

$$z \partial_q \mathcal{S}_{q,p}^m y(z) = (p)^q \mathcal{S}_{q,p}^m y(z) \hbar(z), \tag{3.2.7}$$

Let $\hbar(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $\hbar(z) \in \triangle$ with $\hbar(0) = 1$. Then (3.2.7) becomes

$$z^p + \sum_{n=p+1}^{\infty} (n)^q \psi_n a_n z^n = (p)^q \left(\sum_{n=0}^{\infty} c_n z^n\right) \left(z^p + \sum_{n=p+1}^{\infty} \psi_n a_n z^n\right).$$

Now equating the coefficients of z^{n+p-1}, we get

$$\{(n+p-1)^q - (p)^q\} \psi_{n+p-1} a_{n+p-1} = (p)^q \left\{c_1 \psi_{n+p-2} a_{n+p-2} + ... + c_{n-1}\right\}$$

Applying absolute and $|c_n| \leq |Q_1|$, see in ([68]), we get

$$|a_{n+p-1}| \leq \frac{(p)^q |Q_1|}{\{(n+p-1)^q - (p)^q\} \psi_{n+p-1}} \left\{1 + \psi_{p+1} |a_{p+1}| + ... + \psi_{n+p-2} |a_{n+p-2}|\right\}.$$

Let $\delta = (p)^q |Q_1|$, then

$$|a_{n+p-1}| \leq \frac{\delta}{\{(n+p-1)^q - (p)^q\}\psi_{n+p-1}} \{1 + \psi_{p+1}|a_{p+1}| + ... + \psi_{n+p-2}|a_{n+p-2}|\}. \tag{3.2.8}$$

By using principle of mathematical induction on (3.2.8), put $n = 2$ in (3.2.8), we get

$$|a_{p+1}| \leq \frac{\delta}{\{(n+p-1)^q - (p)^q\}\psi_{p+1}}, \tag{3.2.9}$$

hence for $n = 2$, (3.2.6) is hold. Put $n = 3$ in (3.2.8), we obtained

$$|a_{p+2}| \leq \frac{\delta}{\{(p+2)^q - (p)^q\}\psi_{p+2}} \{1 + \psi_{p+1}|a_{p+1}|\}.$$

Using (3.2.9), we have

$$|a_{p+2}| \leq \frac{\delta}{\{(p+2)^q - (p)^q\}\psi_{p+2}} \left\{1 + \frac{\delta}{(p+1)^q - (p)^q}\right\},$$

hence (3.2.6) is true for $n = 3$. Let (3.2.6) is hold for $n \leq k$, that is

$$|a_{k+p-1}| \leq \frac{\delta}{\{(k+p-1)^q - (p)^q\}\psi_{k+p-1}} \prod_{j=1}^{k-2} \left(1 + \frac{\delta}{[(j+p)^q - (p)^q]}\right), \text{ for } n \geq 3.$$

Consider

$$\begin{aligned}
|a_{k+p}| &\leq \frac{\delta}{\{(k+p)^q - (p)^q\}\psi_{k+p}} \{1 + \psi_{p+1}|a_{p+1}| + ... + \psi_{k+p-1}|a_{k+p-1}|\} \\
&\leq \frac{\delta}{\{(k+p)^q - (p)^q\}\psi_{k+p}} \left\{ \begin{array}{c} 1 + \frac{\delta}{(p+1)^q - (p)^q} + \frac{\delta}{(p+2)^q - (p)^q}\left(1 + \frac{\delta}{(p+1)^q - (p)^q}\right) + ... \\ \\ + \frac{\delta}{\{(k+p-1)^q - (p)^q\}} \prod_{j=1}^{k-2}\left(1 + \frac{\delta}{(j+p)^q - (p)^q}\right). \end{array} \right\} \\
&= \frac{\delta}{\{(k+p)^q - (p)^q\}\psi_{k+p}} \prod_{j=1}^{k-1} \left(1 + \frac{\delta}{(j+p)^q - (p)^q}\right).
\end{aligned}$$

Hence result is true $n = k+1$, and (3.2.6) holds for all n, $n \geq 3$.

Corollary 3.2.2. [34] If $y(z) \in k - US_{q,1}^{\gamma,m}$. Then

$$|a_2| \leq \frac{\delta}{\{(2)^q - 1\}\{(2)^q\}^m},$$

and

$$|a_n| \leq \frac{\delta}{\{(n)^q - 1\}\{(n)^q\}^m} \prod_{j=1}^{n-2} \left(1 + \frac{\delta}{(j+1)^q - 1}\right), \text{ for } n \geq 3.$$

Theorem 3.2.3: Let $k \in [0, \infty)$, and let $y(z) \in k - US_{q,p}^{\gamma,m}$ and given by (2.14.1), then for $v \in \mathbb{C}$

$$\left|a_{p+2} - va_{p+1}^2\right| \leq \frac{(p)^q Q_1}{2\left[(2p+1)^q\right]^m \{(p+2)^q - (p)^q\}} \max\left[1, |2\mu - 1|\right], \tag{3.2.10}$$

where

$$\mu = \frac{1}{2}\left\{1 - \frac{Q_2}{Q_1} - Q_1\left(\frac{4(p)^q}{\{(p+1)^q - (p)^q\}} - v\frac{4(p)^q\left[(2p+1)^q\right]^m \{(p+2)^q - (p)^q\}}{\left((2p)^q\right)^{2m}\{(p+1)^q - (p)^q\}}\right)\right\}, \tag{3.2.11}$$

and $(p)_q |Q_1| = \delta$, where Q_1 and Q_2 are given in (2.15.5) and (2.15.6).

Proof. Let $y(z) \in k - US_{q,p}^{\gamma,m}$, then a Schwarz function $u(z)$, along the condition $u(0) = 0$, $|u(z)| < 1$, such that

$$\frac{1}{(p)^q}\left(\frac{z\partial_q \mathcal{S}_{q,p}^m y(z)}{\mathcal{S}_{q,p}^m y(z)}\right) \prec \hbar_{k,\gamma}(z), \quad z \in \triangle,$$

$$\frac{1}{(p)^q}\left(\frac{z\partial_q \mathcal{S}_{q,p}^m y(z)}{\mathcal{S}_{q,p}^m y(z)}\right) = \hbar_{k,\gamma}(u(z)). \tag{3.2.12}$$

Let $\hbar(z) \in ß$ as

$$\hbar(z) = \frac{1 + u(z)}{1 - u(z)},$$

after some simplification

$$u(z)=\frac{c_1}{2}z+\frac{1}{2}(c_2-\frac{c_1^2}{2})z^2+...,$$

and

$$h_{k,\gamma}(u(z))=1+\frac{Q_1c_1}{2}z+\left\{\frac{Q_2c_1^2}{4}+\frac{1}{2}(c_2-\frac{c_1^2}{2})Q_1\right\}z^2+..., \tag{3.2.13}$$

by using (3.2.13) in (3.2.12) and along with (3.1.2), we obtain

$$a_{p+1}=\frac{(p)^q\,Q_1c_1}{\{(2p)^q\}^m\,\{(p+1)^q-(p)^q\}}$$

and

$$a_{p+2}=\frac{(p)^q}{\{(2p+1)^q\}^m\,\{(p+2)^q-(p)^q\}}\left\{\frac{Q_1c_2}{2}+\frac{c_1^2}{4}\left(Q_2-Q_1+\frac{4\,(p)^q\,Q_1^2}{\{(p+1)^q-(p)^q\}}\right)\right\}.$$

For any complex number v and after some calculation we have

$$a_{p+2}-va_{p+1}^2=\frac{(p)^q\,Q_1}{2\,\{(2p+1)^q\}^m\,\{(p+2)^q-(p)^q\}}\left\{c_2-\mu c_1^2\right\}. \tag{3.2.14}$$

Using a Lemma 2.15.3 on (3.2.14) we have

$$\left|a_{p+2}-va_{p+1}^2\right|\leq\frac{(p)^q\,Q_1}{2\,\{(2p+1)^q\}^m\,\{(p+2)^q-(p)^q\}}\max\left[1,|2u-1|\right],$$

where

$$u=\frac{1}{2}\left\{1-\frac{Q_2}{Q_1}-Q_1\left(\frac{4\,(p)^q}{\{(p+1)^q-(p)^q\}}-\mu\frac{4\,(p)^q\,[(2p+1)^q]^m\,\{(p+2)^q-(p)^q\}}{((2p)^q)^{2m}\,\{[(p+1)^q-(p)^q\}}\right)\right\}.$$

Corollary 3.2.3. [34] Let $y(z)\in US_{q,1}^{\gamma,m}$ of the form (2.1.1) and $0\leq k<\infty$, be fixed then $v\in\mathbb{C}$

$$\left|a_3-va_2^2\right|\leq\frac{Q_1}{2\,\{(3)^q\}^m\,\{(3)^q-1\}}\max\left[1,|2\mu-1|\right],$$

where

$$\mu=\frac{1}{2}\left\{1-\frac{Q_2}{Q_1}-Q_1\left(\frac{4}{(2)^q-1}-v\frac{4\left\{(3)^q\right\}^m\left\{(3)^q-1\right\}}{\left((2)^q\right)^{2m}\left\{(2)^q-1\right\}}\right)\right\}.$$

Theorem 3.2.4: Let $y(z)\in\nabla(p)$ with the form (2.14.1) satisfies condition

$$\textstyle\sum_{n=p+1}^{\infty}\left\{\left\{\left|(n)^q-(p)^q\right|\right\}(k+1)+(p)^q\left|\gamma\right|\right\}\left|\psi_n\right|\left|a_n\right|\leq\left|\gamma\right|(p)^q, \tag{3.2.15}$$

then $y(z)\in k-US_{q,p}^{\gamma,m}$.

Proof. Let we note that

$$\begin{aligned}\left|\frac{1}{(p)^q}\left(\frac{z\partial_q\mathcal{S}_{q,p}^m y(z)}{\mathcal{S}_{q,p}^m y(z)}\right)-1\right| &= \left|\frac{z\partial_q\mathcal{S}_{q,p}^m y(z)-(p)^q\,\mathcal{S}_{q,p}^m y(z)}{(p)^q\,\mathcal{S}_{q,p}^m y(z)}\right|\\ &= \left|\frac{\sum_{n=p+1}^{\infty}\psi_n\left\{(n)^q-(p)^q\right\}a_n z^n}{(p)^q z^p+(p)^q\sum_{n=p+1}^{\infty}\psi_n a_n z^n}\right|\\ &\leq \frac{\sum_{n=p+1}^{\infty}\left|\psi_n\left\{(n)^q-(p)^q\right\}\right|\left|a_n\right|}{(p)^q-\sum_{n=p+1}^{\infty}(p)^q\left|\psi_n\right|\left|a_n\right|}. \end{aligned}\tag{3.2.16}$$

From (3.2.15) we get

$$(p)^q-\textstyle\sum_{n=p+1}^{\infty}(p)^q\left|\psi_n\right|\left|a_n\right|>0.$$

For showing $y(z)\in k-US_{q,p}^{\gamma,m}$ it is sufficient that

$$\left|\frac{k}{\gamma}\left\{\frac{1}{(p)^q}\left(\frac{z\partial_q\mathcal{S}_{q,p}^m y(z)}{\mathcal{S}_{q,p}^m y(z)}\right)-1\right\}\right|-\operatorname{Re}\left\{\frac{1}{\gamma}\left\{\frac{1}{(p)^q}\left(\frac{z\partial_q\mathcal{S}_{q,p}^m y(z)}{\mathcal{S}_{q,p}^m y(z)}\right)-1\right\}\right\}\leq 1.$$

From (3.2.16), we get

$$\begin{aligned}
&\left|\frac{k}{\gamma}\left\{\frac{1}{(p)^q}\left(\frac{z\partial_q\mathcal{S}_{q,p}^m y(z)}{\mathcal{S}_{q,p}^m y(z)}\right)-1\right\}\right|-\operatorname{Re}\left\{\frac{1}{\gamma}\left\{\frac{1}{(p)^q}\left(\frac{z\partial_q\mathcal{S}_{q,p}^m y(z)}{\mathcal{S}_{q,p}^m y(z)}\right)-1\right\}\right\} \\
\leq\ & \frac{k}{|\gamma|}\left|\frac{1}{(p)^q}\left(\frac{z\partial_q\mathcal{S}_{q,p}^m y(z)}{\mathcal{S}_{q,p}^m y(z)}\right)-1\right|+\frac{1}{|\gamma|}\left|\frac{1}{(p)^q}\left(\frac{z\partial_q\mathcal{S}_{q,p}^m y(z)}{\mathcal{S}_{q,p}^m y(z)}\right)-1\right| \\
\leq\ & \frac{k+1}{|\gamma|}\left|\frac{1}{(p)^q}\left(\frac{z\partial_q\mathcal{S}_{q,p}^m y(z)}{\mathcal{S}_{q,p}^m y(z)}\right)-1\right| \\
=\ & \frac{k+1}{|\gamma|}\left|\frac{z\partial_q\mathcal{S}_{q,p}^m y(z)-(p)^q\,\mathcal{S}_{q,p}^m y(z)}{(p)^q\,\mathcal{S}_{q,p}^m y(z)}\right| \\
\leq\ & \frac{k+1}{|\gamma|}\left\{\frac{\sum_{n=p+1}^{\infty}|\psi_n\{(n)^q-(p)^q\}|\,|a_n|}{(p)^q-\sum_{n=p+1}^{\infty}(p)^q\,|\psi_n|\,|a_n|}\right\} \\
\leq\ & 1.
\end{aligned}$$

Corollary 3.2.4. [34] Let $y(z) \in \mathcal{A}$ with the form (2.1.1) satisfies

$$\sum_{n=2}^{\infty}\{|(n)^q-1|\,(k+1)+|\gamma|\}\{(n)^q\}^m\,|a_n| \leq |\gamma|\,.$$

Then $y(z) \in k - US_{q,1}^{\gamma,m}$.

For $q \to 1$, $p=1$, $\gamma = 1-\alpha$, $m=0$, with $0 \leq \alpha < 1$, then we have Corollary 3.2.5.

Corollary 3.2.5. [91] A function $y \in \mathcal{A}$ be given in (2.1.1) satisfies

$$\sum_{n=2}^{\infty}\{n(k+1)-(k+\alpha)\}\,|a_n| \leq 1-\alpha.$$

Then $y(z) \in \mathcal{SD}(k,\alpha)$.

For $q \to 1$, $p=1$, $\gamma = 1-\alpha$, $m=0$, with $0 \leq \alpha < 1$ and $k=0$, then we have Corollary 3.2.6.

Corollary 3.2.6. [104] A function $y \in \mathcal{A}$ be given in (2.1.1), satisfies

$$\sum_{n=2}^{\infty}\{n-\alpha\}|a_n| \leq 1-\alpha.$$

Then $y(z) \in \mathcal{SD}(\alpha)$.

Theorem 3.2.5: Let $y(z) \in k-US_{q,p}^{\gamma,m}$. Then $y(\triangle)$ contains an open unit disk of radius

$$r = \frac{\{(p+1)^q-(p)^q\}\psi_{p+1}}{(p+1)\{(p+1)^q-(p)^q\}\psi_{p+1}+\delta},$$

where Q_1 is defined by (2.15.5) and $(p)^q|Q_1| = \delta$.

Proof. Let $u_0 \neq 0 \in \mathbb{C}$ for $z \in \triangle$, $y(z) \neq u_0$, then

$$y_1(z) = \frac{u_0 y(z)}{u_0-y(z)} = z + \left(\frac{1}{u_0}+a_{p+1}\right)z^{p+1}+....$$

We know that $y_1(z) \in \mathcal{S}$, so

$$\left|a_{p+1}+\frac{1}{u_0}\right| \leq p+1.$$

Using (3.2.6), we get

$$\left|\frac{1}{u_0}\right| \leq \frac{\{(p+1)^q-(p)^q\}(p+1)\psi_{p+1}+\delta}{\{(p+1)^q-(p)^q\}\psi_{p+1}}.$$

We obtained

$$|u_0| \geq \frac{\{(p+1)^q-(p)^q\}\psi_{p+1}}{\{(p+1)^q-(p)^q\}(p+1)\psi_{p+1}+\delta}.$$

Corollary 3.2.7. [34] Let $y(z) \in k-US_{q,1}^{\gamma,m}$, Then $y(\triangle)$ contains an open disk of radius

$$r = \frac{\{(2)^q-1\}\{(2)^q\}^m}{2\{(2)^q\}^m\{(2)^q-1\}+Q_1},$$

where Q_1 is given by (2.15.5).

Chapter 3

New subclasses of analytic and bi-univalent functions

4.1: Introduction

A analytic function $y(z) \in \mathcal{S}^*(\beta)$, $(0 \leqq \beta < 1)$, if

$$\Re\left(\frac{zy'(z)}{y(z)} - \beta\right) > 0, \quad z \in \triangle.$$

Moreover, analytic function $y(z) \in \mathcal{C}(\beta)$, $(0 \leqq \beta < 1)$, if

$$\Re\left(1 + \frac{zy''(z)}{y'(z)} - \beta\right) > 0, \quad z \in \triangle.$$

where $y(z)$ given by (2.1.1) which are analytic in $\triangle$.

The convolution of y and s given as:

$$(y * s)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{4.1.1}$$

and s given as:

$$s(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in \triangle.$$

For more detail see definition 2.6.

It is important to note that y and its inverse y^{-1}, defined as

$$y^{-1}(y(z)) = z, \quad z \in \triangle.$$

also

$$y(y^{-1}(\varpi)) = \varpi, \quad \left(|\varpi| < r_0(y);\ r_0(y) \geqq \frac{1}{4}\right).$$

where y is univalent and $y^{-1}(w)$ is given by (2.1.3) .
If y and y^{-1} are univalent in $\triangle$, then function $y \in \nabla$ is called bi-univalent in $\triangle$.
In [92] Srivastava investigated the various subclasses of Σ, also several authors introduced and studied class Σ (see, for example, [11, 20, 94, 96, 97, 107, 108], They obtained only initial coefficients $|a_2|$ and $|a_3|$ in these recent papers.
For $b \in \mathbb{C},\ b \neq 0, -1, -2, \ldots\ldots, \mu \in \mathbb{C},\ z \in \triangle$**,** the linear operator $J_{\mu,b} : \nabla \longrightarrow \nabla$ introduced by Srivastava and Attiya [95] and given by:

$$J_{\mu,b}y(z) = \mathcal{G}_{\mu,b} * y(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^{\mu} a_n z^n, \tag{4.1.2}$$

where $\mathcal{G}_{\mu,b}$ is defined as:

$$\begin{aligned} \mathcal{G}_{\mu,b} &= \frac{[\phi(\mu, b, z) - b^{-\mu}]}{(1+b)^{-\mu}} \\ &= z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^{\mu} z^n. \end{aligned} \tag{4.1.3}$$

For more detail see in section 2.7.
This chapter is completely published in Bulletin of Mathematical Analysis and Applications,Volume 9 Issue 2 (2017), 37-44. By using Srivastava-Attiya operator, we defined new subclasses and also find coefficients estimates $|a_2|$ and $|a_3|$ for functions belonging to class Σ.

Definition 4.1.1 A functions y of the form (2.1.1) and $y \in M_{\Sigma}(\mu, b, \alpha, \lambda)$, if it satisfied condition:

$$\left|\arg\left(\frac{z\,[J_{\mu,b}y(z)]'}{(1-\lambda)z + \lambda J_{\mu,b}y(z)}\right)\right| < \frac{\alpha\pi}{2},\ 0 < \alpha \leqq 1,\ 0 \leqq \lambda \leqq 1,\ z \in \triangle, \tag{4.1.4}$$

and

$$\left|\arg\left(\frac{\varpi\,[J_{\mu,b}g(\varpi)]'}{(1-\lambda)\varpi + \lambda J_{\mu,b}g(\varpi)}\right)\right| < \frac{\alpha\pi}{2},\ \ 0 < \alpha \leqq 1,\ 0 \leqq \lambda \leqq 1,\ \varpi \in \triangle, \tag{4.1.5}$$

where the function g is given by (2.1.3). That is, the extension of y^{-1} to $\triangle$.

Special Cases.

i) For $\mu = 0$, $\lambda = 0$ and $b = 0$ in (4.1.4) and (4.1.5) we have the class $M_\Sigma(0, 0, \alpha, 0) = \mathcal{H}_\Sigma^\alpha$, defined by Srivastava et.al [92].

ii) For $\mu = 0$, $\lambda = 1$ and $b = 0$ in (4.1.4) and (4.1.5) we have the class $M_\Sigma(0, 0, \alpha, 1) = \delta^*_\Sigma(\alpha)$ defined by Brannan and Taha [17].

Definition 4.1.2. Let functions y with the form (2.1.1) and $y \in M_\Sigma(\mu, b, \beta, \lambda)$, if it satisfied condition:

$$\Re e \left(\frac{z\,[J_{\mu,b}y(z)]'}{(1-\lambda)z + \lambda J_{\mu,b}y(z)} \right) > \beta, \quad 0 \leqq \beta < 1, \ 0 \leqq \lambda \leqq 1, \ z \in \triangle, \tag{4.1.6}$$

and

$$\Re e \left(\frac{\varpi\,[J_{\mu,b}g(\varpi)]'}{(1-\lambda)w + \lambda J_{\mu,b}g(\varpi)} \right) > \beta, \quad 0 \leqq \beta < 1, \ 0 \leqq \lambda \leqq 1, \ \varpi \in \triangle, \tag{4.1.7}$$

where the function g is given in (2.1.3).

Special Cases.

i) For $\mu = b = \lambda = 0$, then (4.1.6) and (4.1.7) become $\mathcal{H}_\Sigma(\beta)$ defined in [92].

ii) For $\mu = b = 0$ and $\lambda = 1$, then (4.1.6) and (4.1.7) reduced to the well-known starlike function of order β, see [84].

4.2: Main Results

Theorem 4.2.1: Let $y \in M_\Sigma(\mu, b, \alpha, \lambda)$ $(0 < \alpha \leqq 1, 0 \leqq \lambda \leqq 1)$ and of the form (2.1.1). Then

$$|a_2| \leqq \frac{2\alpha}{\sqrt{\left\{\left\{2\alpha(\lambda^2 - 2\lambda) - (\alpha-1)(2-\lambda)^2\right\}\left(\frac{1+b}{2+b}\right)^{2\mu} + 2\alpha(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}\right\}}}, \qquad (4.2.1)$$

$$|a_3| \leqq \frac{4\alpha^2}{(2-\lambda)^2\left(\frac{1+b}{2+b}\right)^{2\mu}} + \frac{2\alpha}{(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}}. \qquad (4.2.2)$$

Proof. From (4.1.4) and (4.1.5) we will write

$$\frac{z\left[J_{\mu,b}y(z)\right]'}{(1-\lambda)z + \lambda J_{\mu,b}y(z)} = [u(z)]^\alpha \qquad (4.2.3)$$

and

$$\frac{\varpi\left[J_{\mu,b}g(\varpi)\right]'}{(1-\lambda)\varpi + \lambda J_{\mu,b}g(\varpi)} = [l(\varpi)]^\alpha, \qquad (4.2.4)$$

where

$$u(z) = 1 + p_1 z + p_2 z^2 ..., \qquad (4.2.5)$$

and

$$l(\varpi) = 1 + q_1\varpi + q_2^2\varpi.... \qquad (4.2.6)$$

Comparing the coefficients of (4.2.3) and (4.2.4), we have

$$(2-\lambda)\left(\frac{1+b}{2+b}\right)^{\mu} a_2 = \alpha p_1, \qquad (4.2.7)$$

$$(\lambda^2 - 2\lambda)\left(\frac{1+b}{2+b}\right)^{2\mu} a_2^2 + (3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu} a_3 = \frac{1}{2}\left[\alpha(\alpha-1)p_1^2 + 2\alpha p_2\right], \qquad (4.2.8)$$

$$-(2-\lambda)\left(\frac{1+b}{2+b}\right)^{\mu} a_2 = \alpha q_1, \qquad (4.2.9)$$

$$(\lambda^2-2\lambda)\left(\frac{1+b}{2+b}\right)^{2\mu}a_2^2+(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}(2a_2^2-a_3)=\frac{1}{2}\left[\alpha(\alpha-1)q_1^2+2\alpha q_2\right]. \quad (4.2.10)$$

From (4.2.7) and (4.2.9), we have

$$2(2-\lambda)^2\left(\frac{1+b}{2+b}\right)^{2\mu}a_2^2=\alpha^2(p_1^2+q_1^2), \quad (4.2.11)$$

and

$$p_1=-q_1. \quad (4.2.12)$$

From (4.2.8), (4.2.10), (4.2.11) and (4.2.12), we have

$$\left\{\left\{2\alpha(\lambda^2-2\lambda)-(\alpha-1)(2-\lambda)^2\right\}\left(\frac{1+b}{2+b}\right)^{2\mu}+2\alpha(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}\right\}a_2^2$$
$$= \alpha^2(p_2+q_2). \quad (4.2.13)$$

Applying Lemma 2.15.1 on (4.2.13), we have

$$|a_2|\leqq\frac{2\alpha}{\sqrt{\left\{\left\{2\alpha(\lambda^2-2\lambda)-(\alpha-1)(2-\lambda)^2\right\}\left(\frac{1+b}{2+b}\right)^{2\mu}+2\alpha(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}\right\}}}.$$

Which is our required estimates of (4.2.1). Now we find $|a_3|$, for this we subtracting (4.2.10) from (4.2.8) and we have

$$2(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}a_3-2(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}a_2^2=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2). \quad (4.2.14)$$

From (4.2.11), (4.2.12) and (4.2.14), we have

$$a_3=\left[\frac{\alpha^2p_1^2}{(2-\lambda)^2\left(\frac{1+b}{2+b}\right)^{2\mu}}+\frac{\alpha(p_2-q_2)}{2(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}}\right]. \quad (4.2.15)$$

Applying Lemma 2.15.1 once again on (4.2.15) for the coefficients p_2 and q_2, we have

$$|a_3| \leqq \frac{4\alpha^2}{(2-\lambda)^2\left(\frac{1+b}{2+b}\right)^{2\mu}} + \frac{2\alpha}{(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}}.$$

Hence Theorem 4.2.1 is complete.

For $\lambda = 0$, $\mu = 0$ and $b = 0$, we have corollary due to Srivastava et al. [92].

Corollary 4.2.1. Let $y \in \mathcal{H}_{\Sigma}^{\alpha}$ and of the form (2.1.1).Then

$$|a_2| \leqq \alpha\sqrt{\frac{2}{\alpha+2}} \quad \text{and} \quad |a_3| \leqq \frac{\alpha(3\alpha+2)}{3}.$$

Corollary 4.2.2. Let $y \in M_{\Sigma}(1,0,\alpha,\lambda)$, $0 < \alpha \leqq 1, 0 \leqq \lambda \leqq 1$ and of the form (2.1.1) .Then

$$|a_2| \leqq \frac{2\alpha}{\sqrt{\left\{\lambda^2(\alpha+1) - 4(\alpha+\lambda-1)\right\}\frac{1}{4} + \frac{2}{3}\alpha(3-\lambda)}}, \quad |a_3| \leqq \frac{4\alpha^2}{\frac{1}{4}(2-\lambda)^2} + \frac{2\alpha}{\frac{1}{3}(3-\lambda)}.$$

Corollary 4.2.3. Let $y \in M_{\Sigma}(1,1,\alpha,\lambda)$, $0 < \alpha \leqq 1, 0 \leqq \lambda \leqq 1$ and of the form (2.1.1).Then

$$|a_2| \leqq \frac{2\alpha}{\sqrt{\left\{\lambda^2(\alpha+1) - 4(\alpha+\lambda-1)\right\}\frac{4}{9} + \alpha(3-\lambda)}}, \quad |a_3| \leqq \frac{4\alpha^2}{\frac{4}{9}(2-\lambda)^2} + \frac{2\alpha}{\frac{1}{2}(3-\lambda)}.$$

Corollary 4.2.4. Let $y \in M_{\Sigma}(1,\gamma,\alpha,\lambda)$, $0 < \alpha \leqq 1, 0 \leqq \lambda \leqq 1$ and of the form (2.1.1) .Then

$$|a_2| \leqq \frac{2\alpha}{\sqrt{\left\{\lambda^2(\alpha+1) - 4(\alpha+\lambda-1)\right\}\left(\frac{1+\gamma}{2+\gamma}\right)^2 + 2\alpha(3-\lambda)\left(\frac{1+\gamma}{3+\gamma}\right)}},$$

$$|a_3| \leqq \frac{4\alpha^2}{(2-\lambda)^2\left(\frac{1+\gamma}{2+\gamma}\right)^2} + \frac{2\alpha}{(3-\lambda)\left(\frac{1+\gamma}{3+\gamma}\right)^{\mu}}.$$

Theorem 4.2.2. Let $y \in \nabla$ defined by (2.1.1) belonging to $M_{\Sigma}(\mu,b,\beta,\lambda)$ for $0 \leqq \beta <$

$1, 0 \leqq \lambda \leqq 1$. Then

$$|a_2| \leqq \frac{\sqrt{2(1-\beta)}}{\sqrt{\left\{(\lambda^2-2\lambda)\left(\frac{1+b}{2+b}\right)^{2\mu}+(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}\right\}}}, \tag{4.2.16}$$

$$|a_3| \leqq (1-\beta)\left\{\frac{4(1-\beta)}{(2-\lambda)^2\left(\frac{1+b}{2+b}\right)^{2\mu}}+\frac{2}{(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}}\right\}. \tag{4.2.17}$$

Proof. From (4.1.6) and (4.1.7), then

$$\frac{z\left[J_{\mu,b}y(z)\right]'}{(1-\lambda)z+\lambda J_{\mu,b}y(z)} = \beta+(1-\beta)u(z), \tag{4.2.18}$$

$$\frac{\varpi\left[J_{\mu,b}g(\varpi)\right]'}{(1-\lambda)\varpi+\lambda J_{\mu,b}g(\varpi)} = \beta+(1-\beta)l(\varpi), \tag{4.2.19}$$

where

$$u(z) = 1+p_1z+p_2z^2...,$$

and

$$l(\varpi) = 1+q_1\varpi+q_2\varpi^2....$$

Comparing the coefficients of (4.2.18) and (4.2.19), we obtain

$$(2-\lambda)\left(\frac{1+b}{2+b}\right)^{\mu}a_2 = (1-\beta)p_1, \tag{4.2.20}$$

$$(\lambda^2-2\lambda)\left(\frac{1+b}{2+b}\right)^{2\mu}a_2^2+(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}a_3 = (1-\beta)p_2, \tag{4.2.21}$$

$$-(2-\lambda)\left(\frac{1+b}{2+b}\right)^{\mu}a_2 = (1-\beta)q_1, \tag{4.2.22}$$

$$(\lambda^2-2\lambda)\left(\frac{1+b}{2+b}\right)^{2\mu}a_2^2+(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}(2a_2^2-a_3) = (1-\beta)q_2. \tag{4.2.23}$$

From (4.2.20) and (4.2.22), we have

$$2(2-\lambda)^2\left(\frac{1+b}{2+b}\right)^{2\mu}a_2^2 = (1-\beta)^2\left(p_2^2+q_2^2\right), \tag{4.2.24}$$

and

$$p_1 = -q_1. \tag{4.2.25}$$

Adding (4.2.21) and (4.2.23), we have

$$\left[2(\lambda^2 - 2\lambda)\left(\frac{1+b}{2+b}\right)^{2\mu} + 2(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}\right] a_2^2 = (1-\beta)(p_2+q_2). \tag{4.2.26}$$

Applying Lemma 2.15.1 on (4.2.26), we have

$$|a_2| \leqq \frac{\sqrt{2(1-\beta)}}{\sqrt{\left\{(\lambda^2 - 2\lambda)\left(\frac{1+b}{2+b}\right)^{2\mu} + (3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}\right\}}}.$$

Which is our required bound of $|a_2|$ as given in (4.2.16).

Now we find the estimates on $|a_3|$. For this we subtract (4.2.23) from (4.2.21), we have

$$2(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu} a_3 - 2(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu} a_2^2 = (1-\beta)(p_2-q_2). \tag{4.2.27}$$

Substitution the value of a_2^2 from (4.2.24) in (4.2.27), we have

$$a_3 = \left(\frac{(1-\beta)^2\left(p_2^2+q_2^2\right)}{2(2-\lambda)^2\left(\frac{1+b}{2+b}\right)^{2\mu}}\right) + \frac{(1-\beta)(p_2-q_2)}{2(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}}. \tag{4.2.28}$$

Applying Lemma 2.15.1 on (4.2.28) for the coefficient p_2 and q_2, we have

$$|a_3| \leqq (1-\beta)\left\{\frac{4(1-\beta)}{(2-\lambda)^2\left(\frac{1+b}{2+b}\right)^{2\mu}} + \frac{2}{(3-\lambda)\left(\frac{1+b}{3+b}\right)^{\mu}}\right\}.$$

Hence Theorem 4.2.2 is complete.

Corollary 4.2.5: [92]. Let $y(z) \in M_{\Sigma}(0,0,\beta,0)$, $(0 \leqq \beta < 1)$ and of the form (2.1.1). Then

$$|a_2| \leqq \sqrt{\frac{2(1-\beta)}{3}} \quad \text{and} \quad |a_3| \leqq \frac{(1-\beta)(5-3\beta)}{3}.$$

Corollary 4.2.6. Let $y(z) \in M_\Sigma(0, 0, \beta, 1)$, $(0 \leqq \beta < 1)$ and of the form (2.1.1).Then

$$|a_2| \leqq \sqrt{2(1-\beta)} \quad \text{and} \quad |a_3| \leqq (1-\beta)(5-4\beta).$$

Corollary 4.2.7. Let $y(z) \in M_\Sigma(1, 1, \beta, \lambda)$, $(0 \leqq \beta < 1, 0 \leqq \lambda \leqq 1)$ and of the form (2.1.1). Then

$$|a_2| \leqq \frac{\sqrt{2(1-\beta)}}{\sqrt{\frac{4}{9}\left\{(\lambda^2 - 2\lambda) + \frac{1}{2}(3-\lambda)\right\}}}.$$

$$|a_3| \leqq (1-\beta)\left\{\frac{4(1-\beta)}{\frac{4}{9}(2-\lambda)^2} + \frac{2}{\frac{1}{2}(3-\lambda)}\right\}.$$

Chapter 4

Class of m-Fold Symmetric Bi-univalent functions

5.1: Introduction

The definition of m-Fold Symmetric have discussed in section 2.13.1 and also we have discussed about bi-univalent functions in section 2.1.5. Srivastava et al [97], defined the class Σ_m and give a natural extensions of m-Fold symmetric bi-univalent functions. They find the expansion of $g = y^{-1}$ as:

$$g(\varpi) = y^{-1}(\varpi) = \begin{cases} \varpi - a_{m+1}\varpi^{m+1} + ((m+1)a_{m+1}^2 - a_{2m+1})\varpi^{2m+1} \\ - \left\{ \begin{array}{c} \frac{1}{2}(m+1)(3m+2)a_{m+1}^3 \\ -(3m+2)a_{m+1}a_{2m+1} + a_{3m+1}) \end{array} \right\} \varpi^{3m+1} + ..., \end{cases} \tag{5.1.1}$$

where y is given by (2.13.1). The class of all symmetric bi-univalent functions denoted by Σ_m. For $m = 1$, the equation (5.1.1) become (2.1.3) of the class Σ.

In this chapter, we suppose, $z, \varpi \in \triangle$, $y^{-1} = g$, $m \in \mathbb{N}$, $0 < \alpha \leq 1$, $0 \leq \beta < 1$, $0 \leq \mu$, and $0 \leq \lambda \leq 1$, Here we will investigate coefficient estimates $|a_{m+1}|$ and $|a_{2m+1}|$ of newly defined classes of Σ_m . This chapter is completely published in Journal of Nonlinear Sciences and Applications, 11 (2018), 425–434.

Definition 5.1.1: Let $y \in \Sigma_m$, is in $\mathcal{S}_{\Sigma_m}(\alpha, \lambda, \mu)$ if satisfy the condition

$$\left|\arg\left[(1-\lambda)\left(\frac{z^{1-\mu}y'(z)}{(y(z))^{1-\mu}}\right) + \lambda\left(1 + \frac{z^{2-\mu}y''(z)}{[zy'(z)]^{1-\mu}}\right)\right]\right| < \frac{\alpha\pi}{2}, \tag{5.1.2}$$

and

$$\left|\arg\left[(1-\lambda)\left(\frac{\varpi^{1-\mu}g'(\varpi)}{(g(\varpi))^{1-\mu}}\right) + \lambda\left(1 + \frac{\varpi^{2-\mu}g''(\varpi)}{(\varpi g'(\varpi))^{1-\mu}}\right)\right]\right| < \frac{\alpha\pi}{2}. \tag{5.1.3}$$

Special Cases.

(i) For $m = 1$, we have

$$\mathcal{S}_{\Sigma_m}(\alpha, \lambda, \mu) = \mathcal{S}_{\Sigma}(\alpha, \lambda, \mu).$$

(ii) For $\lambda = 0$, we have

$$\mathcal{S}_{\Sigma_m}(\alpha, \lambda, \mu) = \mathcal{R}_{\Sigma_m}(\alpha, \mu).$$

(iii) For $\lambda = 1$, we have

$$\mathcal{S}_{\Sigma_m}(\alpha, \lambda, \mu) = \mathcal{C}_{\Sigma_m}(\alpha, \mu).$$

(iv) For $\lambda = 0$, and $\mu = 0$, we have class $\mathcal{S}^{\alpha}_{\Sigma,m}$ [10].

(v) For $\lambda = 0, m = 1$ and $\mu = 0$, we get $\mathcal{S}^{*}_{\Sigma}(\alpha)$ investigated by Brannan and Taha [17].

(vii) For $\lambda = 0$, $m = 1$, and $\mu = 1$, we have class $\mathcal{H}_{\Sigma}(\alpha)$ introduced by Srivastava et al. [92].

Definition 5.1.2. Let $y \in \Sigma_m$, belongs to $S_{\Sigma_m}(\beta, \lambda, \mu)$ if satisfy the condition

$$\operatorname{Re}\left[(1-\lambda)\left(\frac{z^{1-\mu}y'(z)}{(y(z))^{1-\mu}}\right) + \lambda\left(1 + \frac{z^{2-\mu}y''(z)}{(zy'(z))^{1-\mu}}\right)\right] > \beta, \tag{5.1.4}$$

and

$$\operatorname{Re}\left[(1-\lambda)\left(\frac{\varpi^{1-\mu}g'(\varpi)}{(g(\varpi))^{1-\mu}}\right) + \lambda\left(1 + \frac{\varpi^{2-\mu}g''(\varpi)}{(\varpi g'(\varpi))^{1-\mu}}\right)\right] > \beta. \tag{5.1.5}$$

Special Cases.

(i) For $m = 1$, we have

$$\mathcal{S}_{\Sigma_m}(\beta, \lambda, \mu) = \mathcal{S}_{\Sigma}(\beta, \lambda, \mu).$$

(ii) For $\lambda = 0$, we have

$$\mathcal{S}_{\Sigma_m}(\beta, \lambda, \mu) = \mathcal{R}_{\Sigma_m}(\beta, \mu).$$

(iii) For $\lambda = 1$, we have

$$\mathcal{S}_{\Sigma_m}(\beta, \lambda, \mu) = \mathcal{C}_{\Sigma_m}(\beta, \mu).$$

(iv) For $\lambda = 0$, and $\mu = 0$, we have class which is introduced in [39].

$$\mathcal{S}_{\Sigma_m}(\beta, \lambda, \mu) = \mathcal{N}^0_{\Sigma,m}(\beta, 1).$$

(v) For $\lambda = 0$, $m = 1$ and $\mu = 0$, we have class which is studied in [17].

$$\mathcal{S}_{\Sigma_m}(\beta, \lambda, \mu) = \mathcal{S}^*_{\Sigma}(\beta).$$

(vi) For $\lambda = 0$ and $\mu = 1$, we have class which is introduced by Srivastava et al. [93].

$$\mathcal{S}_{\Sigma_m}(\beta, \lambda, \mu) = \mathcal{H}_{\Sigma,m}(\beta).$$

(vii) For $\lambda = 0$, $m = 1$, and $\mu = 1$, we have class introduced by Srivastava et al. [92].

$$\mathcal{S}_{\Sigma_m}(\beta, \lambda, \mu) = \mathcal{H}_{\Sigma}(\beta).$$

5.2: Main Results

Theorem 5.2.1: Let $y \in \mathcal{S}_{\Sigma_m}(\alpha, \lambda, \mu)$ and of the form (2.13.1). Then

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{\alpha Q_1(\lambda, \mu, m) - (\alpha Q_2(\lambda, \mu, m) + Q_3(\lambda, \mu, m))}},$$

and

$$|a_{2m+1}| \leq \frac{2\alpha}{[(2m+\mu)+\lambda(4m^2-\mu)]} + \frac{2\alpha^2 \left[Q_1(\lambda, \mu, m) + Q_2(\lambda, \mu, m)\right]}{[(2m+\mu)+\lambda(4m^2-\mu)]\left[(m+\mu)+\lambda(m^2-\mu)\right]^2},$$

where $Q_1(\lambda, \mu, m)$, $Q_2(\lambda, \mu, m)$, $Q_3(\lambda, \mu, m)$ given by (5.2.11), (5.2.12), (5.2.14).

Proof. Let $y \in \mathcal{S}_{\Sigma_m}(\alpha, \lambda, \mu)$, then

$$(1-\lambda)\left(\frac{z^{1-\mu}y'(z)}{(y(z))^{1-\mu}}\right) + \lambda\left(1 + \frac{z^{2-\mu}y''(z)}{[zy'(z)]^{1-\mu}}\right) = [u(z)]^{\alpha}, \tag{5.2.1}$$

and

$$(1-\lambda)\left(\frac{\varpi^{1-\mu}g'(\varpi)}{(g(\varpi))^{1-\mu}}\right) + \lambda\left(1 + \frac{\varpi^{2-\mu}g''(\varpi)}{(\varpi g'(\varpi))^{1-\mu}}\right) = [q(\varpi)]^{\alpha}, \tag{5.2.2}$$

where

$$u(z) = 1 + p_m z^m + p_{2m} z^{2m} + ... \tag{5.2.3}$$

and

$$q(\varpi) = 1 + q_m \varpi^m + q_{2m} \varpi^{2m} + ... \tag{5.2.4}$$

Now equating the coefficients in (5.2.1) and (5.2.2), we obtain

$$(m+\mu)+\lambda(m^2-\mu)a_{m+1} = \alpha p_m, \tag{5.2.5}$$

$$\left\{\begin{array}{c}(2m+\mu)+\lambda(4m^2-\mu)a_{2m+1}\\ \\ -Q_2(\lambda,\mu,m)a_{m+1}^2\end{array}\right\} = \alpha p_{2m}+\frac{\alpha(\alpha-1)}{2}p_m^2, \tag{5.2.6}$$

$$-(m+\mu)+\lambda(m^2-\mu)a_{m+1} = \alpha q_m, \tag{5.2.7}$$

$$\left\{\begin{array}{c}Q_1(\lambda,\mu,m)a_{m+1}^2\\ \\ -(2m+\mu)+\lambda(4m^2-\mu)a_{2m+1}\end{array}\right\} = \alpha q_{2m}+\frac{\alpha(\alpha-1)}{2}q_m^2. \tag{5.2.8}$$

From (5.2.5) and (5.2.7), we obtain

$$p_m=-q_m, \tag{5.2.9}$$

and

$$2\left\{(m+\mu)+\lambda(m^2-\mu)\right\}^2 a_{m+1}^2=\alpha^2(p_m^2+q_m^2). \tag{5.2.10}$$

Also form (5.2.6), (5.2.8), and (5.2.10), we have

$$\begin{aligned}\{Q_1(\lambda,\mu,m)-Q_2(\lambda,\mu,m)\}\,a_{m+1}^2 &= \alpha(p_{2m}+q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2)\\ &= \left\{\begin{array}{c}\alpha(p_{2m}+q_{2m})\\ \\ +\frac{(\alpha-1)}{\alpha}\left\{(m+\mu)+\lambda(m^2-\mu)\right\}^2 a_{m+1}^2\end{array}\right\},\end{aligned}$$

where

$$Q_1(\lambda, \mu, m) = \begin{bmatrix} (1-\lambda)(1+2m)(m+\mu) + \lambda m(m+1) \\ \\ \{3m + \mu(m+1) + 1)\} \end{bmatrix}, \tag{5.2.11}$$

$$Q_2(\lambda, \mu, m) = \frac{(1-\mu)}{2!}\left\{(1-\lambda)(\mu+2m) + 2\lambda m(m+1)^2\right\}. \tag{5.2.12}$$

Therefore, we have

$$a_{m+1}^2 = \frac{\alpha^2(p_{2m} + q_{2m})}{\alpha Q_1(\lambda, \mu, m) - [\alpha Q_2(\lambda, \mu, m) + Q_3(\lambda, \mu, m)]}, \tag{5.2.13}$$

where

$$Q_3(\lambda, \mu, m) = (\alpha - 1)\left\{(m+\mu) + \lambda(m^2 - \mu)\right\}^2. \tag{5.2.14}$$

Applying Lemma 2.15.1, on equation (5.2.13) for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{\alpha Q_1(\lambda, \mu, m) - [\alpha Q_2(\lambda, \mu, m) + Q_3(\lambda, \mu, m)]}}.$$

Now we will find the bound on $|a_{2m+1}|$, for this we will subtracting (5.2.8) from (5.2.6), we obtain

$$\begin{bmatrix} 2\{(2m+\mu) + \lambda(4m^2 - \mu)\}\, a_{2m+1} \\ \\ -\{Q_2(\lambda, \mu, m) + Q_1(\lambda, \mu, m)\}\, a_{m+1}^2 \end{bmatrix} = \alpha(p_{2m} - q_{2m}) + \frac{\alpha(\alpha-1)}{2}(p_m^2 - q_m^2). \tag{5.2.15}$$

Then, in view of (5.2.9) and (5.2.10), and applying Lemma 2.15.1, on (5.2.15) for the coefficients p_{2m}, q_{2m}, p_m and q_m we have

$$|a_{2m+1}| \leq \frac{2\alpha}{[(2m+\mu) + \lambda(4m^2 - \mu)]} + \frac{2\alpha^2\,[Q_1(\lambda, \mu, m) + Q_2(\lambda, \mu, m)]}{[(2m+\mu) + \lambda(4m^2 - \mu)]\,[(m+\mu) + \lambda(m^2 - \mu)]^2}.$$

Hence Theorem 5.2.1 is complete.

For $m = 1$, we obtained.

Corollary 5.2.1. Let $y \in \mathcal{S}_\Sigma(\alpha, \lambda, \mu)$ and of the form (2.1.1). Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha Q_{10}(\lambda,\mu) - (\alpha Q_{11}(\lambda,\mu) + Q_{12}(\lambda,\mu))}},$$

and

$$|a_3| \leq \frac{2\alpha}{(2+\mu)+\lambda(4-\mu)} + \frac{2\alpha^2\left[Q_{10}(\lambda,\mu)+Q_{11}(\lambda,\mu)\right]}{\left[(2+\mu)+\lambda(4-\mu)\right]\left[(1+\mu)+\lambda(1-\mu)\right]^2},$$

where

$$Q_{10}(\lambda,\mu) \quad = \quad 3(1-\lambda)(1+\mu) + 4\lambda\left(2+\mu\right), \tag{5.2.16}$$

$$Q_{11}(\lambda,\mu) \quad = \quad \frac{(1-\mu)}{2!}\left\{(1-\lambda)(\mu+2)+8\lambda\right\}, \tag{5.2.17}$$

$$Q_{12}(\lambda,\mu) \quad = \quad (\alpha-1)\left\{(1+\mu)+\lambda(1-\mu)\right\}^2. \tag{5.2.18}$$

For $\lambda = 0$, we obtained.

Corollary 5.2.2. Let $y \in \mathcal{R}_{\Sigma_m}(\alpha, \mu)$ of the form (2.13.1). Then

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{\alpha Q_4(\mu,m) - (\alpha Q_5(\mu,m) + Q_6(\mu,m))}},$$

and

$$|a_{2m+1}| \leq \frac{2\alpha}{(2m+\mu)} + \frac{2\alpha^2\left[Q_4(\mu,m)+Q_5(\mu,m)\right]}{(2m+\mu)(m+\mu)^2},$$

where

$$Q_4(\mu, m) = (1+2m)(m+\mu), \tag{5.2.19}$$

$$Q_5(\mu, m) = \frac{(1-\mu)}{2!}(\mu+2m), \tag{5.2.20}$$

$$Q_6(\mu, m) = (\alpha-1)(m+\mu)^2. \tag{5.2.21}$$

For $\lambda = 1$, we obtained.

Corollary 5.2.3. Let $y \in \mathcal{C}_{\Sigma_m}(\alpha, \mu)$ and of the form (2.13.1). Then

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{\alpha Q_7(\mu, m) - (\alpha Q_8(\mu, m) + Q_9(\mu, m))}},$$

and

$$|a_{2m+1}| \leq \frac{2\alpha}{[(2m+\mu)+(4m^2-\mu)]} + \frac{2\alpha^2\left[Q_7(\mu, m) + Q_8(\mu, m)\right]}{[(2m+\mu)+(4m^2-\mu)]\left[(m+\mu)+(m^2-\mu)\right]^2},$$

where

$$Q_7(\mu, m) = \{3m + \mu(m+1) + 1\}\, m(m+1), \tag{5.2.22}$$

$$Q_8(\mu, m) = \frac{(1-\mu)}{2!}\left\{2m(m+1)^2\right\}, \tag{5.2.23}$$

$$Q_9(\mu, m) = (\alpha-1)\left\{(m+\mu)+(m^2-\mu)\right\}^2. \tag{5.2.24}$$

For $\lambda = 0$, and $\mu = 1$, we obtained.

Corollary 5.2.4. [93] Let $y \in \mathcal{H}_{\Sigma,m}^{\alpha}$ and of the form (2.13.1). Then

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(1+m)(1+m+2\alpha)}},$$

and

$$|a_{2m+1}| \leq \frac{2\alpha^2}{(m+1)} + \frac{2\alpha}{(2m+1)}.$$

For $\lambda = 0, \mu = 1$, and $m = 1$, we obtained.

Corollary 5.2.5. [92] Let $y \in H_{\Sigma}^{\alpha}$ of the form (2.1.1). Then

$$|a_2| \leq \frac{\alpha}{\sqrt{(1+\alpha)}},$$

and

$$|a_3| \leq \frac{2\alpha}{3} + \frac{\alpha^2}{1}.$$

For $\lambda = 0$, and $\mu = 0$, in Theorem, 5.2.1,we get.

Corollary 5.2.6. [10] Let $y \in \mathcal{S}_{\Sigma,m}^{\alpha}$ and of the form (2.13.1). Then

$$|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{\alpha+1}},$$

and

$$|a_{2m+1}| \leq \frac{\alpha}{m} + \frac{2\alpha^2(1+m)}{m^2}.$$

For $\lambda = 0$, $m = 1$ and $\mu = 0$, we obtained.

Corollary 5.2.7. [10] Let $y \in \mathcal{S}_{\Sigma}^{*}(\alpha)$ and of the form (2.1.1). Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha+1}},$$

and

$$|a_3| \leq 4\alpha^2 + \alpha.$$

Theorem 5.2.2. Let $y \in \mathcal{S}_{\Sigma_m}(\beta, \lambda, \mu)$ and of the form (2.13.1), then

$$|a_{m+1}| \leq \frac{2\sqrt{(1-\beta)}}{\sqrt{\{Q_1(\lambda,\mu,m) - Q_2(\lambda,\mu,m)\}}},$$

and

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{\{(2m+\mu)+\lambda(4m^2-\mu)\}} + \frac{2\left\{Q_1(\lambda,\mu,m)+Q_2(\lambda,\mu,m)\right\}(1-\beta)^2}{\{(2m+\mu)+\lambda(4m^2-\mu)\}\left\{(m+\mu)+\lambda(m^2-\mu)\right\}^2},$$

where $Q_1(\lambda, \mu, m)$, $Q_2(\lambda, \mu, m)$, given in (5.2.11) and (5.2.12).

Proof. Let $y \in \mathcal{S}_{\Sigma_m}(\beta, \lambda, \mu)$. Then

$$(1-\lambda)\left(\frac{z^{1-\mu}y'(z)}{(y(z))^{1-\mu}}\right) + \lambda\left(1 + \frac{z^{2-\mu}y''(z)}{(zy'(z))^{1-\mu}}\right) = \beta + (1-\beta)u(z), \tag{5.2.25}$$

and

$$(1-\lambda)\left(\frac{\varpi^{1-\mu}g'(\varpi)}{(g(\varpi))^{1-\mu}}\right) + \lambda\left(1 + \frac{\varpi^{2-\mu}g''(\varpi)}{(\varpi g'(\varpi))^{1-\mu}}\right) = \beta + (1-\beta)q(\varpi), \tag{5.2.26}$$

where

$$u(z) = 1 + p_m z^m + p_{2m} z^{2m} + ...,$$

and

$$q(\varpi) = 1 + q_m \varpi^m + q_{2m} \varpi^{2m} + ...$$

Comparing the coefficients in (5.2.25) and (5.2.26), we get

$$(m+\mu)+\lambda(m^2-\mu)a_{m+1} = (1-\beta)p_m, \quad (5.2.27)$$

$$\left\{(2m+\mu)+\lambda(4m^2-\mu)\right\}a_{2m+1} - Q_2(\lambda,\mu,m)a_{m+1}^2 = (1-\beta)p_{2m}, \quad (5.2.28)$$

$$-(m+\mu)+\lambda(m^2-\mu)a_{m+1} = (1-\beta)q_m, \quad (5.2.29)$$

$$Q_1(\lambda,\mu,m)a_{m+1}^2 - \left\{(2m+\mu)+\lambda(4m^2-\mu)\right\}a_{2m+1} = (1-\beta)q_{2m}. \quad (5.2.30)$$

From (5.2.27) and (5.2.29) we obtain

$$p_m = -q_m, \quad (5.2.31)$$

and

$$2\left\{(m+\mu)+\lambda(m^2-\mu)\right\}^2 a_{m+1}^2 = (1-\beta)^2(p_m^2+q_m^2). \quad (5.2.32)$$

Adding (5.2.28) and (5.2.30), we have

$$\{Q_1(\lambda,\mu,m)-Q_2(\lambda,\mu,m)\}\,a_{m+1}^2 = (1-\beta)(p_{2m}+q_{2m}),$$

therefore we have

$$a_{m+1}^2 = \frac{(1-\beta)(p_{2m}+q_{2m})}{Q_1(\lambda,\mu,m)-Q_2(\lambda,\mu,m)}. \quad (5.2.33)$$

Applying Lemma 2.15.1, on equation (5.2.33) for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}| \leq \frac{2\sqrt{(1-\beta)}}{\sqrt{Q_1(\lambda,\mu,m)-Q_2(\lambda,\mu,m)}}.$$

Now we will find the coefficient estimates on $|a_{2m+1}|$, for this we subtracting (5.2.30) from

(5.2.28) and get

$$\begin{bmatrix} 2\left\{(2m+\mu)+\lambda(4m^2-\mu)\right\}a_{2m+1} \\ \\ -\left\{Q_1(\lambda,\mu,m)+Q_2(\lambda,\mu,m)\right\}a_{m+1}^2 \end{bmatrix} = (1-\beta)(p_{2m}-q_{2m}), \tag{5.2.34}$$

then in view of (5.2.31) and (5.2.32) and applying Lemma 2.15.1, on equation (5.2.34) for the coefficients p_{2m}, q_{2m}, p_m and q_m, we have

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{(2m+\mu)+\lambda(4m^2-\mu)} + \frac{2\left\{Q_1(\lambda,\mu,m)-Q_2(\lambda,\mu,m)\right\}(1-\beta)^2}{\left\{(2m+\mu)+\lambda(4m^2-\mu)\right\}\left\{(m+\mu)+\lambda(m^2-\mu)\right\}^2}.$$

Hence Theorem 5.2.2 is complete.

Corollary 5.2.8. Let $y \in \mathcal{S}_\Sigma(\beta,\lambda,\mu)$ and of the form (2.1.1)).Then

$$|a_2| \leq \frac{2\sqrt{(1-\beta)}}{\sqrt{\left\{Q_{10}(\lambda,\mu)-Q_{11}(\lambda,\mu)\right\}}},$$

and

$$|a_3| \leq \frac{2(1-\beta)}{\left\{(2+\mu)+\lambda(4-\mu)\right\}} + \frac{2(1-\beta)^2\left\{Q_{10}(\lambda,\mu)+Q_{11}(\lambda,\mu)\right\}}{\left\{(1+\mu)+\lambda(1-\mu)\right\}^2\left\{(2+\mu)+\lambda(4-\mu)\right\}}.$$

where $Q_{10}(\lambda,\mu)$, $Q_{11}(\lambda,\mu)$ given (5.2.16), (5.2.17).

For $\lambda = 0$, then we obtained.

Corollary 5.2.9. Let $y \in \mathcal{R}_{\Sigma_m}(\beta,\mu)$ and of the form (2.13.1). Then

$$|a_{m+1}| \leq \frac{2\sqrt{(1-\beta)}}{\sqrt{Q_4(\mu,m)-Q_5(\mu,m)}},$$

and

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{(2m+\mu)} + \frac{2\left\{Q_4(\mu,m)+Q_5(\mu,m)\right\}(1-\beta)^2}{(2m+\mu)(m+\mu)^2},$$

where $Q_4(\mu,m)$, $Q_5(\mu,m)$ given (5.2.19), (5.2.20).

For $\lambda = 1$, we obtained.

Corollary 5.2.10. Let $y \in \mathcal{C}_{\Sigma_m}(\beta, \mu)$ and of the form (2.13.1). Then

$$|a_{m+1}| \leq \frac{2\sqrt{(1-\beta)}}{\sqrt{Q_7(\mu,m) - Q_8(\mu,m)}},$$

and

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{\{(2m+\mu)+(4m^2-\mu)\}} + \frac{2\{Q_7(\mu,m)+Q_8(\mu,m)\}(1-\beta)^2}{\{(2m+\mu)+(4m^2-\mu)\}\{(m+\mu)+(m^2-\mu)\}^2},$$

where $Q_7(\mu, m)$, $Q_8(\mu, m)$ given (5.2.22), (5.2.23).

For $\lambda = 0$, and $\mu = 1$, we obtained.

Corollary 5.2.11. [93] Let $y \in \mathcal{H}^{\beta}_{\Sigma,m}$ and of the form (2.13.1). Then

$$|a_{m+1}| \leq \frac{2\sqrt{(1-\beta)}}{\sqrt{(1+2m)(m+1)}},$$

and

$$|a_{2m+1}| \leq \frac{2(1-\beta)^2}{(m+1)} + \frac{2(1-\beta)}{(2m+1)}.$$

For $\lambda = 0$, $m = 1$, and $\mu = 1$, we obtained.

Corollary 5.2.12. [92] Let $y \in \mathcal{H}^{\beta}_{\Sigma}$ and of the form (2.1.1). Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3}},$$

and

$$|a_3| \leq \frac{2(1-\beta)}{3} + \frac{(1-\beta)^2}{1}.$$

For $\lambda = 0$, and $\mu = 0$, we obtained.

Corollary 5.2.13. [10] Let $y \in \mathcal{S}^{\beta}_{\Sigma_m}$ and of the form (2.1.1). Then

$$|a_{m+1}| \leq \sqrt{\frac{2(1-\beta)}{m}},$$

and

$$|a_{2m+1}| \leq \frac{(1-\beta)}{m} + \frac{2(1+m)(1-\beta)^2}{m^2}.$$

For $\lambda = 0, m = 1$ and $\mu = 0$, we obtained.

Corollary 5.2.14. [10] Let $y \in \mathcal{S}_{\Sigma}^{*}(\beta)$ and of the form (2.1.1). Then

$$|a_2| \leq \sqrt{2(1-\beta)},$$

and

$$|a_3| \leq 4(1-\beta)^2 + (1-\beta).$$

Chapter 5

Coefficients bounds of bi-univalent functions

6.1: Introduction

The definition of q-derivative operator, bi-univalent functions, definition of subordination and Ruscheweyh q-differential operator are discussed in section 2.9.1, 2.1.5, 2.4.2 and 2.11.1. Faber [28] introduced Faber polynomials. Gong [29] and Schiffer [103] demonstrated the significance of the Faber polynomials in GFT. For more study we refer [3, 4, 17, 18, 35, 36, 37, 38, 101, 102].

By using Faber polynomial expansions, Bult [19] defined the class as:

$$\begin{aligned}\Re e\left[(1-\beta)\frac{y(z)}{z}+\beta y'(z)+\gamma z y''(z)\right] &> \alpha,\\ \Re e\left[(1-\beta)\frac{g(\varpi)}{\varpi}+\beta g'(\varpi)+\gamma w g''(\varpi)\right] &> \alpha.\end{aligned}$$

For $\lambda > -1$, $\gamma \geq 1$, $0 \leq \alpha < 1$, z, $\varpi \in \triangle$, and $y^{-1} = g$ is defined by (2.1.3). For $y \in \nabla$, let the Ruscheweyh q-differential operator [49] be given as:

$$\mathcal{R}_q^{\lambda} y(\mathrm{z}) = y(\mathrm{z}) * \mathcal{F}_{q,\lambda+1}(\mathrm{z}), \quad \mathrm{z} \in \triangle,\ \lambda > -1.$$

$$\begin{aligned}\mathcal{R}_q^{\lambda} y(\mathrm{z}) &= \mathrm{z}+\sum_{n=2}^{\infty}\frac{[(\lambda+1)^q]_{n-1}}{[(n-1)^q]!}a_n\mathrm{z}^n = \mathrm{z}+\sum_{n=2}^{\infty}\frac{\Gamma_q\{n+\lambda\}}{[(n-1)^q]!\Gamma_q\{1+\lambda\}}a_n\mathrm{z}^n\\ &= \mathrm{z}+\sum_{n=2}^{\infty}\Psi_{n-1}a_n\mathrm{z}^n. \qquad (6.1.1)\end{aligned}$$

Where

$$\Psi_{n-1} = \frac{\Gamma_q\{n+\lambda\}}{[(n-1)^q]!\Gamma_q\{1+\lambda\}} = \frac{[(\lambda+1)^q]_{n-1}}{[(n-1)^q]!}, \qquad (6.1.2)$$

and

$$\mathcal{F}_{q,\lambda+1}(\mathrm{z}) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q\{n+\lambda\}}{[(n-1)^q]!\Gamma_q\{1+\lambda\}} \mathrm{z}^n = \mathrm{z} + \sum_{n=2}^{\infty} \frac{[(\lambda+1)^q]_{n-1}}{[(n-1)^q]!} \mathrm{z}^n.$$

For detail see section 2.11. By involving Ruscheweyh q-differential operator $\mathcal{R}_q^\lambda f(z)$ we defined new class of analytic bi-univalent functions. By introducing Faber polynomial coefficient techniques we determine general coefficient bound $|a_n|$ for $n \geq 3$, and also estimates on the coefficients $|a_2|$ and $|a_3|$ also we study several related classes and also give some corollaries. This chapter completely published in, Journal of complex analysis June 2017.

Definition 6.1.1 Let $y \in \Sigma$, $0 \leq \gamma$, $-1 < \lambda$, $1 \leq \beta$, and $0 \leq \alpha < 1$, as $y \in \mathcal{N}_\Sigma^{q(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\lambda})}$ iff

$$\frac{(1-\beta)\,\mathcal{R}_q^\lambda y\,(z) + \beta z \partial_q \mathcal{R}_q^\lambda y\,(z) + \gamma z^2 \partial_q^2 \mathcal{R}_q^\lambda y\,(z)}{z} \prec \frac{1+Ez}{1+Fz},\ z \in \triangle,$$

and

$$\frac{(1-\beta)\,\mathcal{R}_q^\lambda g\,(\varpi) + \beta \varpi \partial_q \mathcal{R}_q^\lambda g(\varpi) + \gamma \varpi^2 \partial_q^2 \mathcal{R}_q^\lambda g(\varpi)}{\varpi} \prec \frac{1+E\varpi}{1+F\varpi},\ \varpi \in \triangle,$$

where

$$E = \{1-\alpha(1+q)\}\,,\ F = -q \tag{6.1.3}$$

and $y^{-1} = g$ is given by (2.1.3).

For special value of α, β, γ, we have the following definitions.

Definition 6.1.2 For $\beta = 1$, $-1 < \lambda$, $0 \leq \gamma$, and $0 \leq \alpha < 1$. A function $y \in \Sigma$, belongs to $\mathcal{N}_\Sigma^{q(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\lambda})}$ iff

$$\partial_q \mathcal{R}_q^\lambda y\,(z) + \gamma z \partial_q^2 \mathcal{R}_q^\lambda y\,(z) \prec \frac{1+Ez}{1+Fz},\quad z \in \triangle,$$

and

$$\partial_q \mathcal{R}_q^\lambda g\,(\varpi) + \gamma \varpi \partial_q^2 \mathcal{R}_q^\lambda g\,(\varpi) \prec \frac{1+E\varpi}{1+F\varpi},\quad \varpi \in \triangle,$$

where E, F given by (6.1.3), and $g = y^{-1}$ is given by (2.1.3).

Definition 6.1.3. For $\beta = 1$, $\gamma = 0$, $-1 < \lambda$ and $0 \leq \alpha < 1$. Let $y \in \Sigma$, belongs to

$\mathcal{N}_{\Sigma}^{q(\boldsymbol{\alpha},\boldsymbol{\lambda})}$ iff

$$\partial_q \mathcal{R}_q^{\lambda} y\left(z\right) \prec \frac{1+Ez}{1+Fz},\ z \in \triangle,$$

and

$$\partial_q \mathcal{R}_q^{\lambda} g(\varpi) \prec \frac{1+A\varpi}{1+B\varpi},\ \varpi \in \triangle,$$

where E, F given by (6.1.3), and $g = y^{-1}$ is given by (2.1.3).

Definition 6.1.4 For $\gamma = 0, \beta \geq 1$, $-1 < \lambda$ and $0 \leq \alpha < 1$. A function $y \in \Sigma$, is in $\mathcal{N}_{\Sigma}^{q(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\lambda})}$ iff

$$(1-\beta)\frac{\mathcal{R}_q^{\lambda} y\left(z\right)}{z} + \beta \partial_q \mathcal{R}_q^{\lambda} y\left(z\right) \prec \frac{1+Ez}{1+Fz},\ z \in \triangle,$$

and

$$(1-\beta)\frac{\mathcal{R}_q^{\lambda} g\left(\varpi\right)}{\varpi} + \beta \partial_q \mathcal{R}_q^{\lambda} g(\varpi) \prec \frac{1+E\varpi}{1+F\varpi},\ \varpi \in \triangle,$$

where E, F, $y^{-1}(w) = g(w)$ is given by (6.1.3) and (2.1.3).

It is well known that

$$y\left(z\right) \prec \frac{\left(1-2\alpha\right)z+1}{1-z} \qquad \text{iff } \Re e\left(y\left(z\right)\right) > \alpha.$$

Special Cases

(i) Taking $q \to 1$, then $\mathcal{N}_{\Sigma}^{q(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\mathbf{0})} = \mathcal{N}_{\Sigma}^{(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})}$, see [19].

(ii) Taking $q \to 1$, then $\mathcal{N}_{\Sigma}^{q(\boldsymbol{\alpha},\mathbf{1},\boldsymbol{\gamma},\mathbf{0})} = \mathcal{B}_{\Sigma}^{(\boldsymbol{\alpha},\boldsymbol{\gamma})}$, see [94].

(iii) Taking $q \to 1$, then $\mathcal{N}_{\Sigma}^{q(\boldsymbol{\alpha},\mathbf{1},\mathbf{0},\mathbf{0})} = \mathcal{H}_{\Sigma}^{\boldsymbol{\alpha}}$, see [92].

(iv) Taking $q \to 1$, then $\mathcal{N}_{\Sigma}^{q(\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{0},\mathbf{0})} = \mathcal{N}_{\Sigma}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$, see [27].

6.2 Main Results

A function $y \in \triangle$ with the form (2.1.1), the coefficients of $g = y^{-1}$, by using the Faber polynomial expansion can be expressed as:

$$y^{-1}(\varpi) = g(\varpi) = \varpi + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...)\varpi^n.$$

Where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{[-n]!}{([-2n+1]! \, [n-5]!} a_2^{n-1} + \frac{[-n]!}{[2\{-n+1\}]! \, [n-3]!} a_2^{n-3} a_3 \\ &+ \frac{[-n]!}{[-2n+3]! \, [n-4]!} a_2^{n-4} a_4 \\ &+ \frac{(-n)!}{[2\{-n+2\}]! \, [n-5]!} a_2^{n-5} \left[a_5 + \{-n+2\} a_3^2\right] \\ &+ \frac{[-n]!}{[-2n+5]! \, [n-6]!} a_2^{n-6} \left[a_6 + \{-2n+5\} a_3 a_4\right] \\ &+ \sum_{i \geq 7} a_2^{n-j} \mathcal{V}_j, \end{aligned} \tag{6.2.1}$$

where $\mathcal{V}_i$ is a homogeneous polynomial in the variables $|a_2|, |a_3|, ... |a_n|$, for $7 \leq i \leq n$, see [5]. For details we refer [2, 3]. For $p \in \mathbb{N}$, $n \geq 2$, then expension of K_{n-1}^{p} given as

$$K_{n-1}^{p} = pa_n + \frac{p(p-1)}{2} E_{n-1}^{2} + \frac{p!}{(p-3)!3!} E_{n-1}^{3} + ... + \frac{p!}{(p-n+1)!(n-1)!} E_{n-1}^{n-1},$$

where $E_{n-1}^{p} = E_{n-1}^{p}(a_2, a_3....)$ and by [5],

$$E_{n-1}^{m}(a_2, ..., a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\varsigma_1} ... (a_n)^{\varsigma_{n-1}}}{\varsigma_1!, ..., \varsigma_{n-1}!}, \quad \text{for } n \geq m.$$

While $a_1 = 1$, such that $\varsigma_1, \varsigma_2, ..., \varsigma_n$ are nonnegative integer and satisfying

$$\varsigma_1 + \varsigma_2 + ... + \varsigma_n = m,$$

$$\varsigma_1 + 2\varsigma_2 + ... + (n-1)\varsigma_{n-1} = n-1.$$

Evidently, $E_{n-1}^{n-1}(a_2, ..., a_n) = a_2^{n-1}$, [4], or equivalently,

$$E_n^m(a_{1,}a_2, ..., a_n) = \sum_{n=1}^{\infty} \frac{m!(a_2)^{\varsigma_1}...(a_n)^{\varsigma_{n-1}}}{\varsigma_1!, ..., \varsigma_n!}, \qquad \text{for } n \geq m,$$

while $a_1 = 1$, such that $\varsigma_1, \varsigma_2, ..., \varsigma_n$ are nonnegative integer and satisfying

$$\varsigma_1 + \varsigma_2 + ... + \varsigma_n = m,,$$

$$\varsigma_1 + 2\varsigma_2 + ... + (n)\varsigma_{n-1} = n.$$

It is clear that $E_n^n(a_1, ..., a_n) = E_1^n$, and $E_n^n = a_1^n$, $E_n^1 = a_n$ are the first and last polynomials.

Theorem 6.2.1. Let $0 \leq \gamma$, $1 \leq \beta$, $-1 < \lambda$ and $0 \leq \alpha < 1$. If $y \in \mathcal{N}_\Sigma^{q(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\lambda})}$, For $a_j = 0$, $2 \leq j \leq n-1$. Then

$$|a_n| \leq \frac{[(n-1)^q]!\Gamma_q \{1+\lambda\} (1-\alpha) (1+q)}{\{(1-\beta) + \beta (n)^q + \gamma (n)^q (n-1)^q\} \Gamma_q(n+\lambda)}, \qquad \text{for } n \geq 3. \tag{6.2.2}$$

Proof. Let $y \in \mathcal{N}_\Sigma^{q(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\lambda})}$ be given in (2.1.1), we obtain

$$\frac{(1-\beta) \mathcal{R}_q^\lambda y (z) + \beta \left\{z\partial_q \mathcal{R}_q^\lambda y(z) + \gamma z^2 \partial_q^2 \mathcal{R}_q^\lambda y(z)\right\}}{z}$$

$$= 1 + \sum_{n=2}^{\infty} \{(1-\beta) + \beta (n)^q + \gamma (n)^q (n-1)^q\} \times \Psi_{n-1} a_n z^{n-1}, \tag{2.6.3}$$

and

$$\begin{aligned}&\frac{(1-\beta)\,\mathcal{R}_q^{\lambda}g\,(\varpi)+\beta\left\{\varpi\partial_q\mathcal{R}_q^{\lambda}g(\varpi)+\gamma\varpi^2\partial_q^2\mathcal{R}_q^{\lambda}g(\varpi)\right\}}{\varpi}\\ =\;& 1+\sum_{n=2}^{\infty}\{(1-\beta)+\beta\,(n)^q+\gamma\,(n)^q\,(n-1)^q\}\\ &\times\Psi_{n-1}b_n\varpi^{n-1},\end{aligned}\tag{6.2.4}$$

where, Ψ_{n-1} is given by (2.12.2) and $b_n=\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,...a_n)$.
Since y and $g=y^{-1}\in\mathcal{N}_{\Sigma}^{q(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\lambda})}$ and let a Schwarz function $u(z)=\sum\limits_{n=1}^{\infty}c_nz^n$ and $v(\varpi)=\sum\limits_{n=1}^{\infty}d_n\varpi^n$, where $\varpi,z\in\triangle$. We have

$$\frac{(1-\beta)\,\mathcal{R}_q^{\lambda}y\,(z)+\beta\left\{z\partial_q\mathcal{R}_q^{\lambda}y(z)+\gamma z^2\partial_q^2\mathcal{R}_q^{\lambda}y(z)\right\}}{z}=\frac{1+E(u(z))}{1+F(u(z))},\tag{6.2.5}$$

where

$$\frac{1+E(u(z))}{1+F(u(z))}=1-(E-F)\sum_{n=1}^{\infty}K_n^{-1}(c_{1,}c_2,...,c_n,F)z^n,$$

and

$$\frac{(1-\beta)\,\mathcal{R}_q^{\lambda}g\,(\varpi)+\beta\left\{\varpi\partial_q\mathcal{R}_q^{\lambda}g(\varpi)+\gamma\varpi^2\partial_q^2\mathcal{R}_q^{\lambda}g(\varpi)\right\}}{\varpi}=\frac{1+E(v(\varpi))}{1+F(v(\varpi))},\tag{6.2.6}$$

where

$$\frac{1+E(v(\varpi))}{1+F(v(\varpi))}=1-\sum_{n=1}^{\infty}(E-F)K_n^{-1}(d_{1,}d_2,...,d_n,F)\varpi^n.$$

In general [4, 3] for any $n \geq 2$, $p \in \mathbb{N}$ and an expansion of $K_n^p(k_1, k_2, ..., k_n, F)$ is given as

$$\begin{aligned} K_n^p(k_1, k_2, ..., k_n, F) &= \frac{p!}{\{p-n\}!n!} k_1^n [F]^{n-1} + \frac{p!}{\{p-n+1\}! \{n-2\}!} k_1^{n-2} k_2 [F]^{n-2} \\ &+ \frac{p!}{\{p-n+2\}! \{n-3\}!} \times k_1^{n-3} k_3 (F)^{n-3} \\ &+ \frac{p!}{\{p-n+3\}! \{n-4\}!} k_1^{n-4} \left[k_4 [F]^{n-4} + \frac{p-n+3}{2} k_3^2 (F) \right] \\ &+ \frac{p!}{\{p-n+4\}! \{n-5\}!} k_1^{n-5} \left[k_5 [F]^{n-5} + \{p-n+4\} k_3 k_4 (F) \right] \\ &+ \sum_{j \geq 6} k_1^{n-1} \mathcal{X}_j, \end{aligned}$$

the homogeneous polynomial is represented by $\mathcal{X}_j$.

For the coefficients of $u(z)$ and $v(\varpi)$, using the bounds $|c_n| \leq 1$ and $|d_n| \leq 1$, see in [55]. Equating the coefficients of (6.2.3) and (6.2.5). We obtain

$$\{(1-\beta) + \beta (n)^q + \gamma (n)^q (n-1)^q\} \Psi_{n-1} a_n = -(E-F) K_{n-1}^{-1}(c_1, c_2, ..., c_{n-1}, F). \quad (6.2.7)$$

Similarly corresponding coefficients of (6.2.4) and (6.2.6). We obtain

$$\{(1-\beta) + \beta (n)^q + \gamma (n)^q (n-1)^q\} \Psi_{n-1} b_n = -(E-F) K_{n-1}^{-1}(d_1, d_2, ..., d_{n-1}, F). \quad (6.2.8)$$

If $2 \leq j \leq n-1$, $a_j = 0$, then $b_n = -a_n$

$$\{(1-\beta) + \beta (n)^q + \beta\gamma (n)^q (n-1)^q\} \Psi_{n-1} a_n = -(E-F) c_{n-1}, \quad (6.2.9)$$

$$-\{(1-\beta) + \beta (n)^q + \beta\gamma (n)^q (n-1)^q\} \Psi_{n-1} a_n = -(E-F) d_{n-1}. \quad (6.2.10)$$

Taking the absolute values of (6.2.9) and (6.2.10), we obtain

$$\begin{aligned}|a_n| &\leq \frac{|E-F|}{\{(1-\beta)+\beta\,(n)^q+\gamma\,(n)^q\,(n-1)^q\}\,\Psi_{n-1}}\,|c_{n-1}| \\ &= \frac{|E-F|}{\{(1-\beta)+\beta\,(n)^q+\gamma\,(n)^q\,(n-1)^q\}\,\Psi_{n-1}}\,|d_{n-1}|\,, \\ |a_n| &\leq \frac{|E-F|}{\{(1-\beta)+\beta\,(n)^q+\gamma\,(n)^q\,(n-1)^q\}\,\Psi_{n-1}}, \qquad \text{for} \quad \text{n} \geq 3. \qquad (6.2.11)\end{aligned}$$

On using the values of E, F and Ψ_{n-1} in (6.2.11), one can obtain

$$|a_n| \leq \frac{(1-\alpha)\,(1+q)\,\{(n-1)^q\}!\Gamma_q(1+\lambda)}{\{(1-\beta)+\beta\,(n)^q+\gamma\,(n)^q\,(n-1)^q\}\,\Gamma_q\,\{n+\lambda\}}, \qquad \text{for} \quad \text{n} \geq 3.$$

□

For $\beta = 1$, then we have the corollary.

Corollary 6.2.1. If $y \in \mathcal{N}_{\Sigma}^{q(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\lambda})}$, if $a_j = 0$, $2 \leq j \leq n-1$. Then

$$|a_n| \leq \frac{(1-\alpha)\,(1+q)\,\{(n-1)^q\}!\Gamma_q\,\{1+\lambda\}}{(n)^q\,\{1+\gamma\,(n-1)^q\}\,\Gamma_q\,\{n+\lambda\}}, \qquad \text{for} \quad n \geq 3.$$

where $0 \leq \gamma$, $-1 < \lambda$ and $0 \leq \alpha < 1$.

For $\beta = 1$, $\gamma = 0$, then we obtain.

Corollary 6.2.2: If $y \in \mathcal{N}_{\Sigma}^{q(\boldsymbol{\alpha},\boldsymbol{\lambda})}$, if $a_j = 0$, $2 \leq j \leq n-1$. Then

$$|a_n| \leq \frac{(1-\alpha)\,(1+q)\,\{(n-1)^q\}!\Gamma_q\,\{1+\lambda\}}{(n)^q\,\Gamma_q\,\{n+\lambda\}}, \qquad \text{for } n \geq 3.$$

where $-1 < \lambda$ and $0 \leq \alpha < 1$.

For $\lambda = 0$, $q \to 1$, then we obtain.

Corollary 6.2.3. [19]. If $y \in \mathcal{N}_{\Sigma}^{(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})}$, if $a_j = 0$, $2 \leq j \leq n-1$. Then

$$|a_n| \leq \frac{2\,(1-\alpha)}{1+\beta\,(n-1)+\gamma n\,(n-1)}, \qquad \text{for n} \geq 3.$$

where $1 \leq \beta$, and $0 \leq \gamma$, $0 \leq \alpha < 1$.

For $\beta = 1$, $\lambda = 0$, $q \to 1$, then we obtain.

Corollary 6.2.4: [94]. If $y \in \mathcal{B}_{\Sigma}^{(\boldsymbol{\alpha},\boldsymbol{\gamma})}$, if $a_j = 0$, $n-1 \geq j \geq 2$. Then

$$|a_n| \leq \frac{2(1-\alpha)}{n[1+\gamma(n-1)]}, \qquad \text{for } n \geq 3.$$

where $0 \leq \alpha < 1$, and $0 \leq \gamma$.

For $\beta = 1$, $\gamma = 0$, $\lambda = 0$, $q \to 1$, then we obtain.

Corollary 6.2.5: [92] If $y \in \mathcal{H}_{\Sigma}^{(\boldsymbol{\alpha})}$, if $a_j = 0$, $n-1 \geq j \geq 2$. Then

$$|a_n| \leq \frac{2(1-\alpha)}{n}, \qquad \text{for } n \geq 3.$$

where $0 \leq \alpha < 1$.

For $q \to 1$, $\gamma = 0$, $\lambda = 0$, then we obtain.

Corollary 6.2.6 [35]. If $y \in \mathcal{N}_{\Sigma}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$, and if $a_j = 0$, $2 \leq j \leq n-1$. Then

$$|a_n| \leq \frac{2(1-\alpha)}{1+\beta(n-1)}, \qquad n \geq 3.$$

where $1 \leq \beta$, and $0 \leq \alpha < 1$.

Theorem 6.2.2: If $y \in \mathcal{N}_{\Sigma}^{q(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\lambda})}$. Then

$$|a_2| \leq \min \begin{cases} \frac{(1-\alpha)(1+q)[(1)^q]!\Gamma_q\{1+\lambda\}}{\{(1-\beta)+\beta(2)^q+\gamma(2)^q(1)^q\}\Gamma_q\{2+\lambda\}}, \\ \\ \sqrt{\frac{2|q(1-\alpha)(1+q)|[(2)^q]!\Gamma_q\{1+\lambda\}}{(2)^q\{(1-\beta)+\beta(3)^q+\gamma(3)^q(2)^q\}\Gamma_q\{3+\lambda\}}}. \end{cases} \tag{6.2.12}$$

$$|a_3| \leq \min \begin{cases} \frac{(2)^q}{(1)^q+(1)^q}\left[\frac{(1-\alpha)(1+q)((1)^q)!\Gamma_q\{1+\lambda\}}{\{(1-\beta)+\beta(2)^q+\gamma(2)^q(1)^q\}\Gamma_q\{2+\lambda\})}\right]^2, \\ \\ \frac{(2)^q}{(1)^q+(1)^q}\left[\frac{2q(1-\alpha)(1+q)((2)^q)!\Gamma_q\{1+\lambda\}}{(2)^q\{(1-\beta)+\beta(3)^q+\gamma(3)^q(2)^q\}\Gamma_q\{3+\lambda\}}\right], \end{cases} \tag{6.2.13}$$

$$\left|a_3 - (2)^q a_2^2\right| \leq \frac{q\left((2)^q\right)!\Gamma_q\{1+\lambda\}(1-\alpha)(1+q)}{\{(1-\beta)+\beta(3)^q+\gamma(3)^q(2)^q\}\Gamma_q\{3+\lambda\}}, \tag{6.2.14}$$

where $1 \leq \beta$, and $0 \leq \alpha < 1, -1 < \lambda, 0 \leq \gamma$.

Proof. Taking $n = 2$ in (6.2.9) and $n = 3$ in (6.2.10), we have

$$\{(1-\beta)+\beta(2)^q+\gamma(2)^q(1)^q\}\Psi_1 a_2 = -(E-F)c_1, \tag{6.2.15}$$

$$\{(1-\beta)+\beta(3)^q+\gamma(3)^q(2)^q\}\Psi_2 a_3 = -(E-F)(Fc_1^2-c_2), \tag{6.2.16}$$

$$-\left\{(1-\beta)+\beta(2)^q+\gamma(2)^q(1)_q\right\}\Psi_1 a_2 = -(E-F)d_1, \tag{6.2.17}$$

$$\{(1-\beta)+\beta(3)^q+\gamma(3)^q(2)^q\}\Psi_2\left\{(2)^q a_2^2 - a_3\right\} = -(E-F)(Fd_1^2-d_2). \tag{6.2.18}$$

From (6.2.15) and (6.2.17) we have

$$c_1 = -d_1, \tag{6.2.19}$$

$$|a_2| \leq \frac{|F-E|}{\{(1-\beta)+\beta(2)^q+\gamma(2)^q(1)^q\}\Psi_1}|c_1| = \frac{|F-E|}{\{(1-\beta)+\beta(2)^q+\gamma(2)^q(1)^q\}\Psi_1}|d_1|,$$

$$\leq \frac{|E-F|}{\{(1-\beta)+\beta(2)^q+\gamma(2)^q(1)^q\}\Psi_1}. \tag{6.2.20}$$

Using (6.1.3) and (6.1.2) in (6.2.20), we have

$$|a_2| \leq \frac{\{(1)^q\}!\Gamma_q(1+\lambda)(1+q)(1-\alpha)}{\{(1-\beta)+\beta(2)^q+\gamma(2)^q(1)^q\}\Gamma_q\{2+\lambda\}}.$$

Adding (6.2.16) and (6.2.18) we have

$$(2)^q\{(1-\beta)+\beta(3)^q+\gamma(3)^q(2)^q\}\Psi_2 a_2^2 = -(E-F)\left\{F\left(c_1^2+d_1^2\right)-(c_2+d_2)\right\}. \tag{6.2.21}$$

Taking absolute values of both sides of (6.2.21) and using $|c_2| \leq 1-|c_1|^2$, $|d_2| \leq 1-|d_1|^2$ and $|c_1| \leq 1$, $|d_1| \leq 1$ of Lemma 2.15.5, such that

$$|a_2|^2 \leq \frac{2|E-F||F|}{(2)^q\left\{(1-\beta)+\beta(3)_q+\gamma(3)_q(2)^q\right\}\Psi_2},$$

$$|a_2| \leq \sqrt{\frac{2|E-F||F|}{(2)^q\{(1-\beta)+\beta(3)^q+\gamma(3)^q(2)^q\}\Psi_2}}, \tag{6.2.22}$$

by using (6.1.3) and (6.1.2) in (6.2.22), we have

$$|a_2| \leq \sqrt{\frac{2q[\{(2)^q\}!\Gamma_q\{1+\lambda\}(1-\alpha)(1+q)}{(2)^q\{(1-\beta)+\beta(3)^q+\gamma(3)^q(2)^q\}\Gamma_q\{3+\lambda\}}}.$$

Now in order to find the $|a_3|$, we subtract (6.2.18) from (6.2.16) we have

$$a_3 = \frac{(E-F)\{F(d_1^2-c_1^2)-(c_2-d_2)\}}{\{(1)^q+(1)^q\}\{(1-\beta)+\beta(3)^q+\gamma(3)^q(2)^q\}\Psi_2} + \frac{(2)^q}{(1)^q+(1)^q}a_2^2. \tag{6.2.23}$$

Using the (6.2.19) in (6.2.23), then

$$a_3 = \frac{-(E-F)(c_2-d_2)}{\{(1)^q+(1)^q\}\{(1-\beta)+\beta\,(3)^q+\gamma\,(3)^q\,(2)^q\}\,\Psi_2} + \frac{(2)^q}{(1)^q+(1)^q}a_2^2. \tag{6.2.24}$$

Taking the modulus of (6.2.24), we have

$$|a_3| \leq \frac{|E-F|\,(|c_2|+|d_2|)}{\{(1)^q+(1)^q\}\{(1-\beta)+\beta\,(3)^q+\gamma\,(3)^q\,(2)^q\}\,\Psi_2} + \frac{(2)^q}{(1)^q+(1)^q}\,|a_2|^2\,. \tag{6.2.25}$$

By using $|c_2| \leq 1-|c_1|^2$, $|d_2| \leq 1-|d_1|^2$, of Lemma 2.15.5, and $|c_1| \leq 1$, $|d_1| \leq 1$, in (6.2.25), we have

$$|a_3| \leq \frac{(2)^q}{(1)^q+(1)^q}\,|a_2|^2\,. \tag{6.2.26}$$

Again by using the (6.2.15) in (6.2.26) we have

$$|a_3| \leq \frac{(2)^q}{(1)^q+(1)^q}\left[\frac{(1+q)\,(1-\alpha)}{\{(1-\beta)+\beta\,(2)^q+\gamma\,(2)^q\,(1)^q\}\,\Psi_1}\right]^2. \tag{6.2.27}$$

Using (6.1.3) and (6.1.2) in (6.2.27), we have

$$|a_3| \leq \frac{(2)^q}{(1)^q+(1)^q}\left[\frac{\{(2)^q\}!\Gamma_q\,\{1+\lambda\}\,(1-\alpha)\,(1+q)}{\{(1-\beta)+\beta\,(2)^q+\gamma\,(2)^q\,(1)^q\}\,\Gamma_q\,\{2+\lambda\}}\right]^2. \tag{6.2.28}$$

Again using the (6.2.22) in (6.2.26), we have

$$|a_3| \leq \frac{(2)^q}{(1)^q+(1)^q}\left[\frac{2q\,\{(2)^q\}!\Gamma_q(1+\lambda)\,(1-\alpha)\,(1+q)}{(2)^q\,\{(1-\beta)+\beta\,(3)^q+\gamma\,(3)^q\,(2)^q\}\,\Gamma_q\,\{3+\lambda\}}\right].$$

From (6.2.18), we have

$$\left|a_3-(2)^q\,a_2^2\right| \leq \frac{|(E-F)F|\,|d_1|^2+|E-F|\,|d_2|}{\{(1-\beta)+\beta\,(3)^q+\gamma\,(3)^q\,(2)^q\}\,\Psi_2}. \tag{6.2.29}$$

Using the $|d_2| \leq 1-|d_1|^2$, $|d_1|$ of Lemma 2.15.5 and (6.1.3), (6.1.2) on (6.2.29), we get

$$\left|a_3-(2)^q\,a_2^2\right| \leq \frac{q\,\{(2)^q\}!\Gamma_q(1+\lambda)\,(1-\alpha)\,(1+q)}{\{(1-\beta)+\beta\,(3)^q+\gamma\,(3)^q\,(2)^q\}\,\Gamma_q\,\{3+\lambda\}}.$$

For $\beta = 1$, then we have the corollary.

Corollary 6.2.7. For $-1 < \lambda$, $0 \leq \gamma$, and $0 \leq \alpha < 1$. If $y \in \mathcal{N}_{\Sigma}^{q(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\lambda})}$. Then

$$|a_2| \leq \min \begin{cases} \frac{(1-\alpha)(1+q)\{(1)^q\}!\Gamma_q\{1+\lambda\}}{(2)^q\{1+\gamma(1)^q\}\Gamma_q\{2+\lambda\}}, \\ \\ \sqrt{\frac{2q\{(2)^q\}!\Gamma_q\{1+\lambda\}(1-\alpha)(1+q)}{(2)^q(3)^q\{1+\gamma(2)^q\}\Gamma_q\{3+\lambda\}}}. \end{cases}$$

$$|a_3| \leq \min \begin{cases} \frac{(2)^q}{(1)^q+(1)^q}\left\{\frac{(1-\alpha)(1+q)\{(1)^q\}!\Gamma_q\{1+\lambda\}}{(2)^q\{1+\gamma(1)^q\}\Gamma_q\{2+\lambda\}}\right\}^2, \\ \\ \frac{(2)^q}{(1)^q+(1)^q}\left[\frac{2q\{(2)^q\}!\Gamma_q\{1+\lambda\}(1-\alpha)(1+q)}{(2)^q(3)^q\{1+\gamma(2)^q\}\Gamma_q\{3+\lambda\}}\right], \end{cases}$$

$$\left|a_3 - (2)^q a_2^2\right| \leq \frac{q\{(2)^q\}!\Gamma_q\{1+\lambda\}(1-\alpha)(1+q)}{(3)^q\{1+\gamma(2)^q\}\Gamma_q\{3+\lambda\}}.$$

For $\beta = 1$, $\gamma = 0$, then we have the corollary.

Corollary 6.2.8. For $-1 < \lambda$, $0 \leq \alpha < 1$. If $y \in \mathcal{N}_{\Sigma}^{q(\boldsymbol{\alpha},\boldsymbol{\lambda})}$. Then

$$|a_2| \leq \min \begin{cases} \frac{(1-\alpha)(1+q)\{(1)^q\}!\Gamma_q\{1+\lambda\}}{(2)^q\Gamma_q\{2+\lambda\}}, \\ \\ \sqrt{\frac{2q\{(2)^q\}!\Gamma_q\{1+\lambda\}(1-\alpha)(1+q)}{(2)^q(3)^q\Gamma_q\{3+\lambda\}}}. \end{cases}$$

$$|a_3| \leq \min \begin{cases} \frac{(2)^q}{(1)^q+(1)^q}\left\{\frac{(1-\alpha)(1+q)\{(1)^q\}!\Gamma_q\{1+\lambda\}}{(2)^q\Gamma_q\{2+\lambda\}}\right\}^2, \\ \\ \frac{(2)^q}{(1)^q+(1)^q}\left[\frac{2q\{(2)^q\}!\Gamma_q\{1+\lambda\}(1-\alpha)(1+q)}{(2)^q(3)^q\Gamma_q\{3+\lambda\}}\right], \end{cases}$$

$$\left|a_3 - (2)^q a_2^2\right| \leq \frac{q(1-\alpha)(1+q)\{(2)^q\}!\Gamma_q\{1+\lambda\}}{(3)^q\Gamma_q\{3+\lambda\}}.$$

For $\lambda = 0$, $q \to 1$,then we have the corollary.

Corollary 6.2.9 [19] For $0 \leq \alpha < 1$, $1 \leq \beta$ and $0 \leq \gamma$. If $y \in \mathcal{N}_{\Sigma}^{(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})}$. Then

$$
\begin{aligned}
|a_2| &\leq \begin{cases} \sqrt{\frac{2(1-\alpha)}{1+2\beta+6\gamma}}, & 0 \leq \alpha < 1 - \frac{(1+\beta+2\gamma)^2}{2(1+2\beta+6\gamma)}, \\ \\ \frac{2(1-\alpha)}{1+\beta+2\gamma}, & 1 - \frac{(1+\beta+2\gamma)^2}{2(1+2\beta+6\gamma)} \leq \alpha < 1. \end{cases} \\
|a_3| &\leq \frac{2\left(1-\alpha\right)}{1+2\beta+6\gamma}, \\
\left|a_3 - 2a_2^2\right| &\leq \frac{2\left(1-\alpha\right)}{1+2\beta+6\gamma}.
\end{aligned}
$$

For $\lambda = 0$, $q \to 1$, $\gamma = 0$, then we have the corollary.

Corollary 6.2.10 [35]. For $0 \leq \alpha < 1$ and $1 \leq \beta$. If $y \in \mathcal{N}_{\Sigma}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$. Then

$$
\begin{aligned}
|a_2| &\leq \begin{cases} \sqrt{\frac{2(1-\alpha)}{1+2\beta}}, & 0 \leq \alpha < 1 - \frac{(1+\beta)^2}{2(1+2\beta)}, \\ \\ \frac{2(1-\alpha)}{1+\beta}, & 1 - \frac{(1+\beta)^2}{2(1+2\beta)} \leq \alpha < 1. \end{cases} \\
|a_3| &\leq \frac{2\left(1-\alpha\right)}{1+2\beta}, \\
\left|a_3 - 2a_2^2\right| &\leq \frac{2\left(1-\alpha\right)}{1+2\beta}.
\end{aligned}
$$

For $\lambda = 0$, $q \to 1$, $\beta = 1, \gamma = 0$, then we have the corollary.

Corollary 6.2.11 [19]. For $0 \leq \alpha < 1$, if $y \in \mathcal{H}_{\Sigma}^{(\boldsymbol{\alpha})}$. Then

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\alpha)}{3}}, & 0 \leq \alpha < \frac{1}{3}, \\ \\ (1-\alpha), & \frac{1}{3} \leq \alpha < 1. \end{cases}$$

$$|a_3| \leq \frac{2(1-\alpha)}{3},$$

$$\left|a_3 - 2a_2^2\right| \leq \frac{2(1-\alpha)}{3}.$$

For $\lambda = 0$, $q \to 1$, $\beta = 1$, in Theorem 6.2.2, which give the the corollary.

Corollary 6.2.12 [94]. For $0 \leq \alpha < 1$, $1 \leq \beta$, and $0 \leq \gamma$, if $y \in \mathcal{B}_{\Sigma}^{(\boldsymbol{\alpha},\boldsymbol{\gamma})}$. Then

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\alpha)}{3(1+2\gamma)}}, & 0 \leq \alpha < \frac{1+2\gamma-2\gamma^2}{3(1+2\gamma)}, \\ \\ \frac{2(1-\alpha)}{1+2\gamma}, & \frac{1+2\gamma-2\gamma^2}{3(1+2\gamma)} \leq \alpha < 1. \end{cases}$$

$$|a_3| \leq \frac{2(1-\alpha)}{3(1+2\gamma)}.$$

Chapter 6

A subclass of uniformly convex functions and a corresponding subclass of starlike function with fixed coefficient

7.1 Introduction

q-calculus has provided great encouraged to the researchers because of its various applications in field of mathematics and physics. Some of its very first applications was given by Jackson [42, 43] and also give the introduction of the q-analogue of derivative and integral operator. Later, Aral and Gupta [12, 14] utlizing the q-beta function and gave the definition of q-Baskakov-Durrmeyer operator. Kanas and Raducanu [49] investigated q-Ruscheweyh differential operator, and also studied about its properties. The further study about this differential operator see [6, 63]. The purpose of the this chapter is to use q-analogue of the Ruscheweyh differential operator and then give some interesting applications of this operator.

Goodman [32] introduced and defined the following subclasses of $\mathcal{CV}$, $\mathcal{ST}$ and $\mathcal{UST}$, see also [48, 50] as discussed in section 2.5. In [13], Rønning defined the class $\mathcal{S}_p$ as:

$$y \in \mathcal{S}_p \iff \left| \frac{zy'(z)}{y(z)} - 1 \right| \leq \Re e \left\{ \frac{zy'(z)}{y(z)} \right\}.$$

The class $\mathcal{S}_p$ related to $\mathcal{UCV}$. Further more by introducing the parameter α, $-1 \leq \alpha < 1$, Rønning generalized the class $\mathcal{S}_p$ as:

$$y \in \mathcal{S}_p(\alpha) \iff \left| \frac{zy'(z)}{y(z)} - 1 \right| \leq \Re e \left\{ \frac{zy'(z)}{y(z)} - \alpha \right\}.$$

Many authors have defined convex and starlike classes by using Ruscheweyh differential operator. Here by using Ruscheweyh q-differential operator we defined new class of analytic functions. We have discussed about the q-integer number, q-derivative operator or q-difference operator and Ruscheweyh q-differential operator in section 2.9, 2.10 and 2.11 is generalized as follows:

Kanas and Raducanu [49] defined the Ruscheweyh q-differential operator

$$\mathcal{R}_q^{\delta}y(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q\{n+\delta\}}{(n-1)^q!\,\Gamma_q\{1+\delta\}} a_n z^n = z + \sum_{n=2}^{\infty} \varphi_{n-1} a_n z^n \qquad (z \in \triangle), \quad (7.1.1)$$

where

$$\varphi_{n-1} = \frac{\Gamma_q\{n+\delta\}}{(n-1)^q!\,\Gamma_q\{1+\delta\}}. \quad (7.1.2)$$

For detail see section 2.11.

Throughout in this chapter, we assume that

$$0 < q < 1, \quad \delta > -1, \quad -1 \le \alpha < 1, \quad \text{and} \quad \beta \ge 0.$$

Definition 7.1.1: Let $y(z) \in S(\alpha, \beta, \delta, q)$ and y is given in (2.1.1) and fulfilling the condition

$$\Re e\left\{\frac{z\partial_q(\mathcal{R}_q^{\delta}y(z))}{\mathcal{R}_q^{\delta}y(z)} - \alpha\right\} > \beta\left|\frac{z\partial_q(\mathcal{R}_q^{\delta}y(z))}{\mathcal{R}_q^{\delta}y(z)} - 1\right|. \quad (7.1.3)$$

Where $S(\alpha, \beta, \delta, q)$ denote the subclass of S and also let, $TS(\alpha, \beta, \delta, q) = S(\alpha, \beta, \delta, q) \cap T$.

Let the function of the form

$$y(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \ge 0, \text{ for all } n \ge 2. \quad (7.1.4)$$

be in class T and T is the subclass of $\mathcal{S}$.

Here we investigated necessary, sufficient conditions, extreme point and closure property for the functions $y(z) \in TS(\alpha, \beta, \delta, q)$. This chapter completly published in Journal of Mathematica Slovaca, 69 (2019), No. 4, 825-832.

7.2: Main Results

Theorem 7.2.1: Let $y(z) \in S(\alpha, \beta, \delta, q)$ and y is given in (2.1.1), if

$$\sum_{n=2}^{\infty} \{(n)^q (1+\beta) - (\alpha+\beta)\} \varphi_{n-1} |a_n| \leq 1 - \alpha. \tag{7.2.1}$$

Proof: It is suffices to show that

$$\beta \left| \frac{z\partial_q(\mathcal{R}_q^\delta y(z))}{\mathcal{R}_q^\delta y(z))} - 1 \right| - \text{Re}\left\{ \frac{z\partial_q(\mathcal{R}_q^\delta y(z))}{\mathcal{R}_q^\delta y(z))} - 1 \right\} \leq 1 - \alpha,$$

we have

$$\begin{aligned} & \beta \left| \frac{z\partial_q(\mathcal{R}_q^\delta y(z))}{\mathcal{R}_q^\delta y(z))} - 1 \right| - \text{Re}\left\{ \frac{z\partial_q(\mathcal{R}_q^\delta y(z))}{\mathcal{R}_q^\delta y(z))} - 1 \right\} \\ \leq \ & (1+\beta) \left| \frac{z\partial_q(\mathcal{R}_q^\delta y(z))}{\mathcal{R}_q^\delta y(z))} - 1 \right| \\ \leq \ & \frac{(1+\beta)\sum_{n=2}^{\infty}((n)^q - 1)\varphi_{n-1}|a_n|}{1 - \sum_{n=2}^{\infty}\varphi_{n-1}|a_n|}. \end{aligned}$$

The last inequality is bounded above by $(1-\alpha)$ if

$$\sum_{n=2}^{\infty} \{(n)^q (1+\beta) - (\alpha+\beta)\} \varphi_{n-1} |a_n| \leq 1 - \alpha,$$

Theorem 7.2.2. A function $y(z)$ be defined by (7.1.4) belongs to $TS(\alpha, \beta, \delta, q)$ iff

$$\sum_{n=2}^{\infty} \{(n)^q (1+\beta) - (\alpha+\beta)\} \varphi_{n-1} a_n \leq 1 - \alpha. \tag{7.2.2}$$

Proof. We need only to proof the necessity by using view of theorem (7.2.1). If $y(z) \in$

$TS(\alpha,\beta,\delta,q)$ and $z\in\mathbb{R}$. Then

$$\left\{\frac{1-\sum_{n=2}^{\infty}(n)^{q}\varphi_{n-1}a_{n}z^{n-1}}{1-\sum_{n=2}^{\infty}\varphi_{n-1}a_{n}z^{n-1}}\right\}-\alpha\geq\left\{\beta\left|\frac{\sum_{n=2}^{\infty}\left\{(n)^{q}-1\right\}\varphi_{n-1}z^{n-1}}{1-\sum_{n=2}^{\infty}\varphi_{n-1}a_{n}z^{n-1}}\right|\right\}.$$

When $z\to 1$ along the real axis, we get

$$\sum_{n=2}^{\infty}\left\{(n)^{q}(1+\beta)-(\alpha+\beta)\right\}\varphi_{n-1}a_{n}\leq 1-\alpha.$$

Corollary 7.2.1. A function $y(z)$ be given in (7.1.4) belongs to $TS(\alpha,\beta,\delta,q)$. Then

$$a_{n}\leq\frac{1-\alpha}{\left\{(n)^{q}(1+\beta)-(\alpha+\beta)\right\}\varphi_{n-1}},\quad n\geq 2.$$

Corollary 7.2.2. A function $y(z)$ be given in (7.1.4) belongs to $TS(\alpha,\beta,\delta,q)$. Then

$$a_{2}=\frac{1-\alpha}{\left\{(2)^{q}(1+\beta)-(\alpha+\beta)\right\}\varphi_{1}}. \tag{7.2.3}$$

By fixing the second coefficient in the series representation of elements of $TS(\alpha,\beta,\delta,q)$, we introduce a new subclass $TS_{b}(\alpha,\beta,\delta,q)$ as:

Definition. 7.2.2 Let $0<b\leq 1$ and $y(z)\in TS(\alpha,\beta,\delta,q\ )$. Then $y(z)\in TS_{b}(\alpha,\beta,\delta,q)$ if it can be represented as:

$$y(z)=z-\frac{b(1-\alpha)}{\left\{(2)^{q}(1+\beta)-(\alpha+\beta)\right\}\varphi_{1}}z^{2}-\sum_{n=3}^{\infty}a_{n}z^{n}. \tag{7.2.4}$$

Theorem 7.2.3. Let $y(z)$ given in (7.2.4) belongs to $TS_{b}(\alpha,\beta,\delta,q)$ iff

$$\sum_{n=3}^{\infty}\left\{(n)^{q}(1+\beta)-(\alpha+\beta)\right\}\varphi_{n-1}a_{n}\leq(1-b)(1-\alpha). \tag{7.2.5}$$

Proof. Substituting

$$a_{2}=\frac{b(1-\alpha)}{\left\{(2)^{q}(1+\beta)-(\alpha+\beta)\right\}\varphi_{1}}$$

in (7.2.2), and simple calculation we get required result.

Corollary 7.2.3. Let $y(z)$ be given in (7.2.4) and belongs to $TS_b(\alpha, \beta, \delta, q)$. Then

$$a_n \leq \frac{(1-b)(1-\alpha)}{\{(n)^q (1+\beta) - (\alpha+\beta)\} \varphi_{n-1}}, \quad n \geq 3. \tag{7.2.6}$$

Theorem 7.2.4. Under convex linear combination the class $TS_b(\alpha, \beta, \delta, q)$ is closed.

Proof. Let the functions $y(z)$, $\mathfrak{h}(z) \in TS_b(\alpha, \beta, \delta, q)$. Suppose $y(z)$ is given by (7.2.4) and

$$\mathfrak{h}(z) = z - \frac{b(1-\alpha)}{\{(2)^q (1+\beta) - (\alpha+\beta)\} \varphi_1} z^2 - \sum_{n=3}^{\infty} d_n z^n, \tag{7.2.7}$$

where $d_n \geq 0$. For $0 \leq \lambda \leq 1$, It is sufficient to prove that

$$\mathcal{H}(z) = (1-\lambda)\mathfrak{h}(z) + \lambda y(z), \tag{7.2.8}$$

is also in the class $TS_b(\alpha, \beta, \delta, q)$. From (7.2.4), (7.2.7) and (7.2.8), we obtained

$$\mathcal{H}(z) = z - \frac{b(1-\alpha)}{\{(2)^q (1+\beta) - (\alpha+\beta)\} \varphi_1} z^2 - \sum_{n=3}^{\infty} \{\lambda a_n + (1-\lambda) d_n\} z^n. \tag{7.2.9}$$

As $y(z)$, $\mathfrak{h}(z) \in TS_b(\alpha, \beta, \delta, q)$ and $0 \leq \lambda \leq 1$, so by using Theorem (7.2.3), we obtained

$$\sum_{n=3}^{\infty} \varphi_{n-1} \{(n)^q (1+\beta) - (\alpha+\beta)\} \{\lambda a_n + (1-\lambda) d_n\} \leq (1-b)(1-\alpha). \tag{7.2.10}$$

Again by Theorem (7.2.3) and (7.2.10) we have $\mathfrak{H}(z) \in TS_b(\alpha, \beta, \delta, q)$.

Theorem 7.2.5. For every j $(j = 1, 2, 3, ..., m)$ and let the function

$$y_j(z) = z - \frac{b(1-\alpha)}{\{(2)^q (1+\beta) - (\alpha+\beta)\} \varphi_1} z^2 - \sum_{n=3}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0 \tag{7.2.11}$$

is in $TS_b(\alpha, \beta, \delta, q)$. Then $F(z)$ defined as

$$F(z) = \sum_{j=1}^{m} u_j y_j(z), \tag{7.2.12}$$

is also belong to $TS_b(\alpha, \beta, \delta, q)$, where

$$\sum_{j=1}^{m} u_j = 1. \tag{7.2.13}$$

Proof. By using (7.2.11), (7.2.13) in (7.2.13), we obtained

$$F(z) = z - \frac{b(1-\alpha)}{\{(2)^q (1+\beta) - (\alpha+\beta)\} \varphi_1} z^2 - \sum_{n=3}^{\infty} \left(\sum_{j=1}^{m} u_j a_{n,j} \right) z^n.$$

For $j = 1, 2, 3, ..., m$, since $y_j(z) \in TS_b(\alpha, \beta, \delta, q)$, so by Theorem (7.2.3), we obtained

$$\sum_{n=3}^{\infty} \{(n)^q (1+\beta) - (\alpha+\beta)\} \varphi_{n-1} a_{n,j} \leq (1-b)(1-\alpha). \tag{7.2.14}$$

To prove $F(z) \in TS_b(\alpha, \beta, \delta, q)$ it is sufficient to prove that $F(z)$ satisfies the condition of Theorem (7.2.3). So, consider

$$\begin{aligned} & \sum_{n=3}^{\infty} \{(n)^q (1+\beta) - (\alpha+\beta)\} \varphi_{n-1} \left(\sum_{j=1}^{m} u_j a_{n,j} \right) \\ = & \sum_{j=1}^{m} u_j \left(\sum_{n=3}^{\infty} \{(n)^q (1+\beta) - (\alpha+\beta)\} \varphi_{n-1} a_{n,j} \right). \end{aligned} \tag{7.2.15}$$

Using (7.2.13) , (7.2.14) in (7.2.15), we obtain

$$\sum_{n=3}^{\infty} \{(n)^q (1+\beta) - (\alpha+\beta)\} \varphi_{n-1} \left(\sum_{j=1}^{m} u_j a_{n,j} \right) \leq (1-b)(1-\alpha),$$

which implies $F(z) \in TS_b(\alpha, \beta, \delta, q)$.

Theorem 7.2.6. Let

$$y_2(z) = z - \frac{b(1-\alpha)}{\left\{(2)^q(1+\beta) - (\alpha+\beta)\right\}\varphi_1} z^2 \tag{7.2.16}$$

and

$$y_n(z) = z - \frac{b(1-\alpha)}{\left\{(2)^q(1+\beta) - (\alpha+\beta)\right\}\varphi_1} z^2 - \frac{(1-b)(1-\alpha)}{\left\{(n)^q(1+\beta) - (\alpha+\beta)\right\}\varphi_{n-1}} z^n, \tag{7.2.17}$$

then $y(z) \in TS_b(\alpha, \beta, \delta, q)$ iff $y(z)$ can be written as

$$y(z) = \sum_{n=2}^{\infty} \lambda_n y_n(z), \tag{7.2.18}$$

where

$$\sum_{n=2}^{\infty} \lambda_n = 1$$

and

$$\lambda_n \geq 0.$$

Proof. We suppose that $y(z)$ be defined by (8.2.18). Using (7.2.16), (7.2.17) in (7.2.18), we have

$$f(z) = z - \sum_{n=2}^{\infty} A_n z^n, \tag{7.2.19}$$

where

$$A_2 = \frac{b(1-\alpha)}{\left\{(2)^q(1+\beta) - (\alpha+\beta)\right\}\varphi_1} \tag{7.2.20}$$

and

$$A_n = \frac{(1-b)(1-\alpha)}{\left\{(n)^q(1+\beta) - (\alpha+\beta)\right\}\varphi_{n-1}}, \quad n \geq 3. \tag{7.2.21}$$

To prove $y(z)$ belongs to $TS_b(\alpha, \beta, \delta, q)$, $y(z)$ satisfy the condition of Theorem (7.2.3).

For this, consider

$$\sum_{n=2}^{\infty}\left\{(n)^{q}(1+\beta)-(\alpha+\beta)\right\}\varphi_{n-1}A_n=b(1-\alpha)+\sum_{n=3}^{\infty}\lambda_n(1-b)(1-\alpha).$$

Since $\sum_{n=2}^{\infty}\lambda_n=1$, so above equation can be express as:

$$\sum_{n=2}^{\infty}\left\{(n)^{q}(1+\beta)-(\alpha+\beta)\right\}\varphi_{n-1}A_n=\left[b+(1-\lambda_2)(1-b)\right](1-\alpha)\leq(1-\alpha),$$

which implies $y(z)\in TS_b(\alpha,\beta,\delta,q)$.

Conversely, assume that $y(z)$ given by (4.2.4) belongs to $TS_b(\alpha,\beta,\delta,q)$. So by using (7.2.6), we obtain

$$a_n\leq\frac{(1-b)(1-\alpha)}{\left\{(n)^{q}(1+\beta)-(\alpha+\beta)\right\}\varphi_{n-1}},\quad n\geq 3.$$

By taking

$$\lambda_n=\frac{\left\{(n)^{q}(1+\beta)-(\alpha+\beta)\right\}\varphi_{n-1}a_n}{(1-b)(1-\alpha)},$$

and

$$\lambda_2=1-\sum_{n=3}^{\infty}\lambda_n,$$

hence

$$y(z)=z-\sum_{n=2}^{\infty}A_nz^n.$$

CONCLUSION

The main aim of this research work is to study linear operator and related topics in GFT. In this study, we defined the new subclasses of analytic function in the open unit disk $\triangle$. Also we defined some new subclasses by using Ruschweyah q-differential operator, Salagean q-differential operator and conic domains in the open unit disk $\triangle$. In these newly defined classes we discussed about the analytic and geometric properties. We investigated our main results by using convolution and subordination techniques and at some places we use classical approach. By using these techniques we studied and investigated inclusion relations, coefficient bounds, extreme point, closure theorems, necessary condition for univalent functions, as well as n-th $(n \geq 4)$ coefficient bounds. We gave some results associated with conic domains. We will also mention here that all chapter are completly published.

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