# Properties of Connected Graphs Related to Metric and Partition Dimension

Since 1864


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# Properties of Connected Graphs Related to Metric and Partition Dimension

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# DECLARATION

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**"Properties of Connected Graphs Related to Metric and Partition Dimension"**

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(i) I am not registered for the similar degree elsewhere contemporaneously.

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(iii) The work, I am submitting for the Ph.D. degree has not already been submitted elsewhere and shall not in future be submitted by me for obtaining similar degree from any other institution.

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Certified that the research work contained in this thesis titled

**<u>“Properties of Connected Graphs Related to Metric and Partition Dimension”</u>**

has been carried out and completed by **<u>Mr. Muhammad Imran</u>** Registration No. **23-GCU-PHD-SMS-2005** under my supervision.

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*This dissertation is gratefully dedicated to my parents for their prayers and support.*

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# Introduction

Navigation can be studied in a graph-structured framework in which the navigation agent (which we shall assume to be a point robot) moves from node to node of a "graph space". The robot can locate itself by the presence of distinctly labeled "landmark" nodes in the graph space. For a robot navigating in Euclidean space, visual detection of a distinctive landmark provides information about the direction to the landmark, and allows the robot to determine its position by triangulation. On a graph, however, there is neither the concept of direction nor that of visibility. Instead, we shall assume that a robot navigating on a graph can sense the distances to a set of landmarks.

Evidently, if the robot knows its distances to a sufficiently large set of landmarks, its position on the graph is uniquely determined. This suggests the following problem: given a graph, what are the fewest number of landmarks needed, and where they should be located, so that the distances to the landmarks uniquely determines the robot's position on the graph? This is actually a classical problem about metric spaces. A minimum set of landmarks which uniquely determines the robot's position is called a "metric basis", and the minimum number of landmarks is called the "metric dimension" of a graph.

Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension was introduced by Slater in [36, 37] and studied independently by Harary and Melter in [19]. Slater refereed to the metric dimension of a graph as its "location number" and motivated the study of this invariant by its application to the placement of a minimum number of sonar/loran detecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set.

For a subset $S \subset V(G)$ and a vertex $v$ of a connected graph $G$, the distance $d(v, S)$ between $v$ and $S$ is defined as usually, by $d(v, S) = \min\{d(v, x) : x \in S\}$. If $\Pi = (S_1, S_2, ...., S_k)$ is an ordered $k$- partition of $V(G)$, the representation of $v$ with respect to $\Pi$ is the $k$- tuple $r(v|\Pi) = (d(v, S_1), d(v, S_2), ....., d(v, S_k))$. If the $k$- tuples $r(v|\Pi)$ for $v \in V(G)$ are all distinct, then the partition $\Pi$ is called a resolving partition and the minimum cardinality of a resolving partition of $V(G)$ is called the partition dimension of $G$ and is denoted by $pd(G)$.

These concepts have some applications in chemistry for representing chemical compounds ([9],[25]) or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [31].

This thesis is divided into six chapters. The first two chapters consist of basic concepts and terminology of graphs and distances in graphs. In the third chapter, the metric dimension of plane graphs induced by some classes of convex polytopes has been determined and it was proved that these plane graphs have constant metric dimension and only three vertices appropriately chosen suffice to resolve all the vertices of these graphs.

The fourth chapter deals with the metric dimension of generalized Petersen graphs $P(n, 3)$. In this chapter, we study the metric dimension of the generalized Petersen graphs $P(n,3)$ by giving a partial answer to an open problem raised in [8]. In the fifth chapter, $d$-sets for connected graphs have been defined and it is shown that for a connected graph $G$ of order $n$ and diameter 2 the number of pairs $\{x,y\}$ such that their $d$-sets are equal to $V(G)$ is bounded above by $\lfloor n^2/4 \rfloor$ and it is conjectured that this holds for any connected graph. A lower bound for the metric dimension of $G$, $dim(G)$ is proposed in terms of a family of $d$-sets of $G$ having the property that every subfamily containing at least $r \geq 2$ members has an empty intersection. Three sufficient conditions which guarantee that a family $\mathcal{F} = (G_n)_{n\geq 1}$ of graphs with unbounded order has an unbounded metric dimension are also proposed. Finally, $d$-sets are used to show that $dim(Ne_n) = 3$ when $n$ is odd and 2 otherwise, where $Ne_n$ is the necklace graph of order $2n+2$. In the sixth chapter, we study the metric dimension and partition dimension of some infinite regular graphs generated by tilings of the plane by regular triangles and hexagons. It is shown that these graphs have no finite metric bases but their partition dimension is finite and is evaluated in some cases and it is proved that for every $n \geq 2$ there exists finite induced subgraphs of these graphs having metric dimension equal to $n$ as well as infinite induced subgraphs with metric dimension equal to three. Also some open problems are suggested in the seventh chapter.

# Chapter 1

# Basic concepts

In this chapter, some basic concepts will be elaborated. It includes the concept of distances in graphs and some relevant definitions. As the first part, some definitions, notations and terminology in graph theory that will be used throughout this dissertation, are introduced. In the second section, we give some definitions related with distances in graphs.

### 1.0.1 Preliminaries

A *graph* $G$ consists of a nonempty set $V(G)$ of elements, called *vertices* and a collection $E(G)$ of unordered pairs of vertices called *edges*. A graph is symbolically represented as $G = (V(G), E(G))$, unless otherwise specified, both $V(G)$ and $E(G)$ are finite. The *order* of a graph is the number its vertices, and its *size* is the number of its edges. If $u$ and $v$ are two vertices of a graph and if the unordered pair $\{u, v\}$ is an edge denoted by $e = uv$, we say that $e$ joins $u$ and $v$ or that it is an edge between $u$ and $v$. In this case, the vertices $u$ and $v$ are said to be incident on $e$ and $e$ is incident to both $u$ and $v$.

Two or more edges that join the same pair of distinct vertices are called *parallel edges*. An edge represented by an unordered pair in which the two elements are not distinct is known as a *loop*. A graph with no loops is a *multigraph*. A *simple graph* is a graph with no parallel edges and loops. Unless otherwise stated, the graphs we consider are all simple.

Two distinct vertices $u$ and $v$ in $G$ are *adjacent* if $uv$ is an edge in $G$. In this case, $u$ is called the *neighbor* of $v$, and vice versa. The set of neighbors of a vertex $v$ in a graph $G$ is denoted by $N_G(v)$. The number of neighbors of a vertex is called its *degree*. More generally, the degree of a vertex $v$ in a graph $G$ is the number of edges of $G$ incident with $v$. The degree of a vertex $v$ in graph $G$ is denoted by $d_G(v)$. The minimum, maximum and average degrees of the vertices of a graph $G$ are denoted by $\delta(G)$, $\Delta(G)$ and $d(G)$, respectively. An *isolated vertex* is a vertex of degree zero. A graph is $k$-regular if the degree of each vertex is $k$.

Let $A \subseteq V(G)$. Then $A$ is called a *clique* in $G$ if every two vertices in $A$ are adjacent. On the other hand, $A$ is called *independent set* in $G$ if any two vertices in $A$ are not adjacent in $G$. The clique number of $G$ is denoted by $\omega(G)$, and the independent number of $G$ is denoted by $\alpha(G)$; they are defined as follows:

$$\omega(G) = \max\{|A| : \ A \text{ is a clique in } G\},$$

$$\alpha(G) = \max\{|A| : \ A \text{ is an independent set in } G\}.$$

Two edges $e \neq f$ are adjacent if they have an end vertex in common. If all the vertices of $G$ are pairwise adjacent, then $G$ is *complete*. A complete graph on $n$ vertices is denoted by $K_n$; a $K_3$ is a triangle. A bipartite graph is a simple graph in which the set of vertices can be partitioned into sets $U$ and $V$ such that every edge is between a vertex in $U$ and a vertex in $V$. The complete bipartite graph $K_{m,n}$ is the graph with $m$ vertices in $U$ and $n$ vertices in $V$ in which there is an edge between every vertex in $U$ and every vertex in $V$. The complete bipartite graph $K_{1,n}$ is called a star. Two graphs $G$ and $H$ are *isomorphic* if there are bijections $\theta : V(G) \to V(H)$

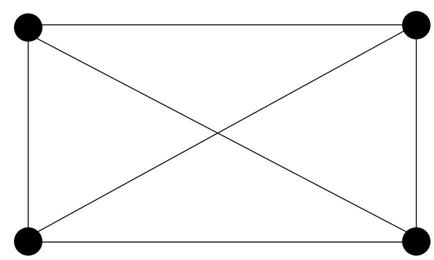

Figure 1.1: The complete graph $K_4$

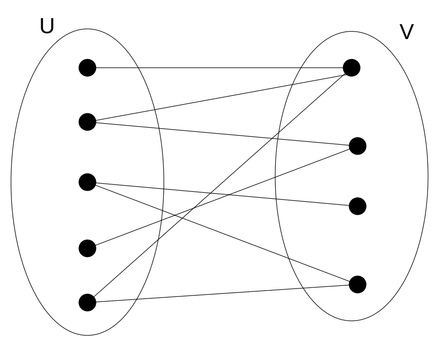


Figure 1.2: A bipartite graph

and $\varphi : E(G) \to E(H)$ such that vertex $v$ and edge $e$ are incident in $G$ if and only if vertex $\theta(v)$ and edge $\varphi(e)$ are incident in $H$. The pair of mappings $(\theta, \varphi)$ is an *isomorphism* from $G$ to $H$; if $G = H$, it is an *automorphism*. Thus, we usually write $G = H$ rather than $G \simeq H$.

The graph $H = (W, F)$ is a subgraph of the graph $G = (V, E)$ if $W$ is a subset of $V$ and $F$ is a subset of $E$. For example, a cyclic graph of order 4 is a subgraph of $K_4$. Any graph $G'$ for which $G$ is a subgraph is called a supergraph of G. Any subgraph $H(V, F)$ of $G(V, E)$ is a spanning subgraph of $G$. A factor of a graph is a spanning subgraph with at least one edge.

### 1.0.2 Connectivity

A subgraph $F$ of a graph $G$ is called an induced subgraph of $G$ if whenever $u$ and $v$ are vertices of $F$ and $uv$ is an edge of $G$, then $uv$ is an edge of $F$ as well. Let $v$ and $w$ be two vertices in a graph. A *walk* between $v$ and $w$ in the graph is a finite alternating sequence $v = v_0, e_1, v_1, e_2, v_2, e_3, ..., e_n, v_n = w$ of vertices and edges of the graph such that each edge $e_i$ in the sequence joins vertices $v_{i-1}$ and $v_i$. The vertices and the edges of the walk need not be distinct. Two walks $v_0, e_1, v_1, e_2, v_2, e_3, ..., e_n, v_n$ and $u_0, f_1, u_1, f_2, u_2, f_3, ...f_m, u_m$ are said to be equal if $n = m$,$v_i = u_i$ for $0 \leq i \leq n$ and $e_i = f_i$ for $1 \leq i \leq n$. Two walks are said to be different if they are not equal. The number of edges in a walk is the length of the walk. A walk in which no edge is repeated is called a *trail*. A walk in which vertices are all distinct is a *path*. Obviously every path is a trail. A closed walk in a graph is a walk between a vertex and itself. A closed walk with no edge repeated is a *circuit*. A cycle is a circuit in which no vertex is repeated. In a simple graph $G$, any cycle consisting of $k$ vertices (that is passing through $k$ vertices) is a $k$-cycle in $G$. It is an odd cycle if $k$ is odd and it is an even cycle if $k$ is even. A pair of vertices in a graph is a connected pair if there is a path between them. A graph $G$ is a *connected graph* if every pair of vertices in a graph $G$ is a connected pair; otherwise , it is a disconnected graph. A connected subgraph $H$

of $G$ is a component of $G$ if $H = H'$ whenever $H'$ is a connected subgraph of $G$ that contains $H$. In other words, a component is a maximal connected subgraph.

A graph is connected if and only if the number of its components is one. If $F$ is a set of edges in a graph $G = (V, E)$ , the graph obtained from $G$ by deleting all the edges belonging to $F$ is denoted by $G - F$. If $F$ consists of the single edge $f$ then $G - F$ is written as $G - f$. A set $F$ of edges in a connected graph $G$ is called a disconnecting set in $G$ if $G - F$ has more than one component. If a disconnecting set $F$ consists of a single edge $f$ then edge $f$ is called a bridge. The following theorem [8] provides a necessary and sufficient condition for an edge of a graph $G$ to be bridge.

**Theorem 1.0.1.** [8] *An edge $e$ of a graph $G$ is a bridge if and only if $e$ lies on no cycle of $G$.*

A graph is said to be $k$-edge connected if every disconnecting set in it has at least $k$ edges. The edge connectivity number $\lambda(G)$ of a graph is the minimum size of a disconnecting set of edges in $G$. A *disconnecting set* $F$ is said to be a cut set if no proper subset of $F$ is a disconnecting set. Analogously, if $W$ is a set of vertices in $G = (V, E)$, then the graph obtained from $G$ by deleting all the vertices belonging to $W$ as well as the edges incident to the vertices in $W$ is denoted by $G - W$. If $W$ consists of a single vertex $w$, the graph $G - W$ is denoted by $G - w$. A set $W$ of vertices in a connected graph $G$ is called a *separating set* (also known as the *vertex cut*) in $G$ if $G - W$ has more than one component. If a separating set consists of a single vertex $w$ then $w$ is known as *cut vertex*. The *connectivity number* $\kappa(G)$ of a graph $G$ is the minimum size of a separating set of vertices in it. Since a complete graph has no separating set, we adopt the convention that the connectivity number of the complete graph of order $n$ is $n - 1$ for all $n$. The following theorem provides us with inequalities concerning the connectivity, edge-connectivity, and minimum degree of a graph.

**Theorem 1.0.2.** [8] *For any graph $G$,*

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

A graph $G$ is said to be $k$-connected if $\kappa(G) \geq k$. Thus $K_n$ is $(n-1)$-connected for all $n$, and a graph that is not complete is $k$-connected if and only if every separating set in it has at least $k$ vertices. The connectivity number of a graph $G$ is zero if and only if $G$ is either the trivial graph $K_1$ or is a disconnected graph. A cyclic graph is 2-connected.

### 1.0.3 Trees and spanning trees

An acyclic graph is a graph with no cycles. A tree is a connected acyclic graph.

**Theorem 1.0.3.** [15] *The following are equivalent in a graph $G$ with $n$ vertices.*

(i) *$G$ is a tree.*

(ii) *There is a unique path between every pair of distinct vertices in $G$.*

(iii) *$G$ is connected and every edge in $G$ is a bridge.*

(iv) *$G$ is connected, and has $n-1$ edges.*

(v) *$G$ is acyclic, and has $n-1$ edges.*

(vi) *$G$ is acyclic, and whenever any two arbitrary nonadjacent vertices in $G$ are joined by an edge, the resulting enlarged graph has a unique cycle.*

(vii) *$G$ is connected, and whenever any two arbitrary nonadjacent vertices in $G$ are joined by an edge, the resulting enlarged graph has a unique cycle.*

An acyclic connected spanning subgraph (if it exists) of $G$ is called a spanning tree in $G$.

**Theorem 1.0.4.** [15] *A graph is connected if and only if it has a spanning tree.*

## 1.1 Distances in Graphs

Facility location problems deal with the task of choosing a site subject to some criterion. For example, in determining where to locate an emergency facility such as hospital or fire station, we would like to minimize the response time between the facility and the location of a possible emergency. Such situations deal with the concept of centrality. In this section, we list the definitions of various kinds of centers and other relevant definitions.

Let $G$ be a connected graph. Then the distance $d(u, v)$ between two vertices $u$ and $v$ is the length of a shortest path between them. The *eccentricity* of a vertex $v \in V(G)$ denoted by $e(v)$ is the distance to a vertex farthest from $v$. Thus $e(v) = max\{d(u, v) : u \in V\}$. The *radius* $r(G)$ is the minimum eccentricity of the vertices of $G$ whereas the *diameter* $d(G)$ is the maximum eccentricity of the vertices of $G$.

Now $v$ is a *central vertex* of $G$ if $e(v) = r(G)$ and the *center* $C(G)$ is the set of all central vertices. Thus the center consists of all vertices having minimum eccentricity. A vertex $v \in V(G)$ is called *peripheral vertex* if $e(v) = d(G)$ and periphery $Per(G)$ is the set of all such vertices. For a vertex $v$ all vertices at distances $e(v)$ from $v$ are called *eccentric vertices.* For a given vertex $u$ in a connected graph $G$, we have discussed seeking a vertex $v$ such that $d(u, v) = e(v)$, that is, $v$ is a vertex that is farthest from $u$. Such a vertex $v$ is called an eccentric vertex of $u$. A vertex $v$ is an *eccentric vertex of the graph G* if $v$ is an eccentric vertex of some vertex of $G$. A connected graph $G$ is an *eccentric graph* if every vertex of $G$ is an eccentric vertex. *The eccentric subgraph* $E_{cc}(G)$ of $G$ is the subgraph of $G$ induced by the set of eccentric vertices of $G$. The

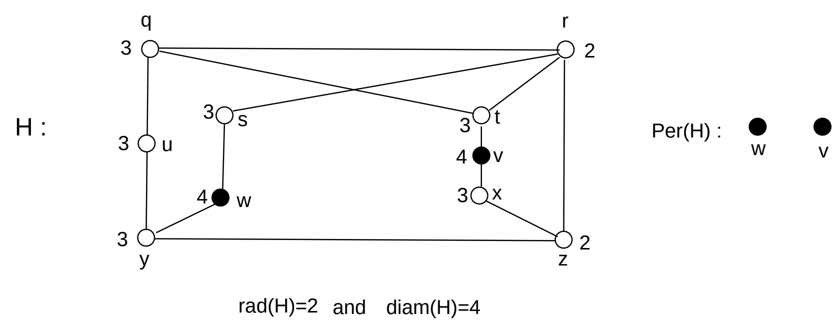


Figure 1.3: The eccentricities of the vertices of a graph

following theorem [8] provides a necessary and sufficient condition for a graph $G$ to be an eccentric subgraph of some graph.

**Theorem 1.1.1.** [8] *A nontrivial graph G is the eccentric subgraph of some graph if and only if every vertex of G has eccentricity* 1 *or no vertex of G has eccentricity* 1*.*

A tree with one central vertex is called a *central tree* and a tree with two central vertices is called *bicentral.* A graph $G$ is a unique eccentric vertex graph if each vertex in $G$ has exactly one eccentric vertex. A graph $G$ is *self-centered* if every vertex is in the center. Thus in a self-centered graph every vertex has the same eccentricity as $r(G) = d(G)$.

**Theorem 1.1.2.** [3] *The center of a tree consists of either a single vertex or a pair of adjacent vertices.*

**Theorem 1.1.3.** [3] *A unique eccentric vertex graph is self-centered if and only if each node of $G$ is eccentric.*

The center of a graph is important in applications involving emergency facilities where response time (distance) to each single location (vertex) in the region (graph) is critical. Suppose instead we consider a service facility such as post office, shopping mall, bank, or power station . When deciding where to locate a post office, we want to minimize the average distance that a person serviced by the post office must travel. This is equivalent to minimizing the total distance traveled by all people within the district. For such situations, the concept of *median* is described.

Let $G$ be a connected graph. The *status* $s(v)$ of a vertex $v$ in $G$ is the sum of the distances from $v$ to each other vertex in $G$. This concept was introduced by Harary [20]. The median $M(G)$ of a graph $G$ is the set of vertices with minimum status. The *minimum status* $ms(G)$ of a graph $G$ is the value of the minimum status; the *total status* $ts(G)$ of a graph $G$ is the sum of all the status values.

# Chapter 2

# Metric dimension and partition dimension of connected graphs

In this chapter, we give the definitions of resolving sets, resolving partitions, metric dimension and partition dimensions which are the key notions of this thesis. We describe the some known results for these invariants and applications of these invariants to different branches of applied sciences.

## 2.1 Resolving Sets and Metric Dimension

If $G$ is a connected graph, the distance $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, ...., w_k\}$ be an ordered set of vertices of $G$ and let $v$ be a vertex of $G$. The representation $r(v|W)$ of $v$ with respect to $W$ is the $k$-tuple $(d(v, w_1), d(v, w_2), ....., d(v, w_k))$. If distinct vertices of $G$ have distinct representations with respect to $W$, then $W$ is called a resolving set for $G$[1]. A resolving set of minimum cardinality is called a basis for $G$ and this cardinality is the metric dimension of $G$, denoted by $dim(G)$.

For a given ordered set of vertices $W = \{w_1, w_2, ...., w_k\}$ of a graph $G$, the $i$-th component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that $W$ is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G)\backslash W$.

For example, consider the graph G of Fig. 2.1. The set $W_1 = \{v_1, v_3\}$ is not a resolv-

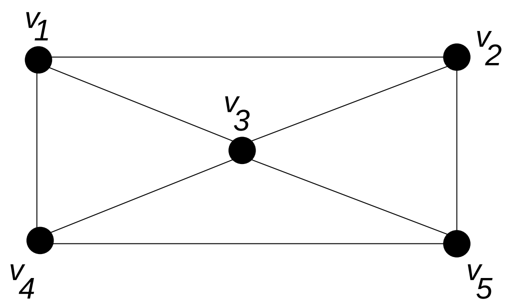


Figure 2.1: A graph with dim(G)=2

ing set of $G$ since $r(v_2|W_1) = (1,1) = r(v_4|W_1)$. On the other hand, $W_2 = \{v_1, v_2, v_3\}$ is a resolving set for $G$ since the representation for the vertices of $G$ with respect to $W_2$ are $r(v_1|W_2) = (0,1,1), r(v_2|W_2) = (1,0,1),\ r(v_3|W_2) = (1,1,0), r(v_4|W_2) = (1,2,1), r(v_5|W_2) = (2,1,1)$. However, $W_2$ is not a minimum resolving set since $W_3 = \{v_1, v_2\}$ is also a resolving set. Since no single vertex constitutes a resolving set for $G$, it follows that $W_3$ is a minimum resolving set implying that $dim(G) = 2$.
A useful property in finding $dim(G)$ is the following:

**Lemma 2.1.1.** *Let $W$ be a resolving set for a connected graph $(G)$ and $u, v \in V(G)$. If $d(u,w) = d(v,w)$ for all vertices $w \in V(G) \setminus \{u, v\}$, then $\{u, v\} \cap W \neq \emptyset$.*

Slater refereed to the metric dimension of a graph as its location number and motivated the study of this invariant by its application to the placement of a minimum number of sonar/loran detecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set ([36],[37]).

## 2.2 Resolving Partitions and Partition Dimension

Another kind of dimension of a connected graph, called partition dimension was introduced in [10, 11] as follows: For a subset $S \subset V(G)$ and a vertex $v$ of a connected graph $G$, the distance $d(v, S)$ between $v$ and $S$ is defined as usually , by $d(v, S) = \min\{d(v, x) : x \in S\}$. If $\Pi = (S_1, S_2, ...., S_k)$ is an ordered $k$- partition of $V(G)$, the representation of $v$ with respect to $\Pi$ is the $k$- tuple $r(v|\Pi) = (d(v, S_1), d(v, S_2), ....., d(v, S_k))$. If the $k$- tuples $r(v|\Pi)$ for $v \in V(G)$ are all distinct, then the partition $\Pi$ is called a resolving partition and the minimum cardinality of a resolving partition of $V(G)$ is called the partition dimension of $G$ and is denoted by $pd(G)$. Let $\Pi = \{S_1, S_2, ...., S_k\}$ be an ordered partition of $V(G)$. If $u \in S_i, v \in S_j$ where $1 \leq i, j \leq k$ and $i \neq j$, then $r(u|\Pi) \neq r(v|\Pi)$ since $d(u, S_i) = 0$ but $d(v, S_i) \neq 0$. Thus, when determining whether a given partition $\Pi$ of $V(G)$ is a resolving partition for $V(G)$, we need only to verify if the vertices of $G$ belonging to the same class of $\Pi$ have distinct representations with respect to $\Pi$. When $d(u, S_i) \neq d(v, S_i)$ we shall say that the class $S_i$ distinguishes vertices $u$ and $v$. Another useful property in determining $pd(G)$ is the following lemma [11].

**Lemma 2.2.1.** *Let $\Pi$ be a resolving partition of $V(G)$ and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all vertices $w \in V(G) \setminus \{u, v\}$, then $u$ and $v$ belong to different classes of $\Pi$.*

These concepts have some applications in chemistry for representing chemical compounds ([9], [25]) or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [31].

## 2.3 Known Results on Metric Dimension and Partition Dimension

If $G$ is a nontrivial connected $k$-dimensional graph of order $n$, then $1 \leq k \leq n-1$. Here we present some results which have been established already. The following result was established in [9].

**Theorem 2.3.1.** *For every pair $n, k$ of integers with $1 \leq k \leq n-1$, there exists a $k$-dimensional graph of order $n$.*

The following two theorems ([9],[19],[36],[37]) characterize connected graphs of order $n$ with dimension $1, n-1$ or $n-2$ and provide the dimensions of some well-known classes of graphs.

For two vertex-disjoint graphs $G$ and $H$, $G \cup H$ is disconnected graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The join $G + H$ consists of $G \cup H$ and all edges joining a vertex of $G$ and a vertex of $H$.

**Theorem 2.3.2.** *Let $G$ be a connected graph of order $n \geq 2$. Then*

*(a) $dim(G) = 1$ if and only if $G = P_n$.*

*(b) $dim(G) = n-1$ if and only if $G = K_n$.*

*(c) for $n \geq 4$, $dim(G) = n-2$ if and only if $G = K_{r,s}$ ,$r, s \geq 1$, $G = K_r + \bar{K}_s, r \geq 1, s \geq 2$ or $G = K_r + (K_1 \cup K_s), r, s \geq 1$.*

We will use the terminology given in [9], [34] and [35]. A vertex of degree at least 3 will be referred to as a major vertex. An end-vertex $u$ of a graph $T$ is said to be a terminal vertex of a major vertex $v$ of $T$ if $d(u, v) < d(u, w)$ for every other major vertex $w$ of $T$. The terminal degree $ter(v)$ of a major vertex $v$ is the number of terminal vertices of $v$. A major vertex $v$ of $T$ is an exterior major vertex of $T$ if it has a positive terminal degree. Let $\sigma(T)$ denote the sum of the terminal degrees of the major vertices of $T$ and let $ex(T)$ denote the number of exterior major vertices of $T$.

**Theorem 2.3.3.** [19] *If $T$ is a tree that is not a path, then*

$$dim(T) = \sigma(T) - ex(T).$$

A fundamental question in graph theory concerns how the value of a parameter is affected by making a small change in the graph. The dimension of a connected graph $G$ is also affected by the addition of a single edge. It has been proved in [9] that the dimension of a tree is also affected by the addition of a single edge. The result suggests that the dimension can increase by at most 1 or decrease at most by 2. We state the theorem.

**Theorem 2.3.4.** [9] *Let $T$ be a tree of order at least 3 and let $e$ be an edge in its complement $\bar{T}$. Then*

$$dim(T) - 2 \leq dim(T + e) \leq dim(T) + 1.$$

*Moreover, for each integer $i$ with $-2 \leq i \leq 1$, there exists a tree $T_i$ and an edge $e_i$ in $\bar{T}_i$ such that*

$$dim(T_i + e_i) = dim(T_i) + i.$$

A *Cartesian product* of two graphs $G$ and $H$, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$, where two vertices $(x, x')$ and $(y, y')$ are adjacent if and only if $x = y$ and $x'y' \in E(H)$ or $x' = y'$ and $xy \in E(G)$. The relationship between the dimension of cartesian product $H \times K_2$ of a connected graph $H$ and $K_2$ and the dimension of $H$ was also established in [9].

**Theorem 2.3.5.** *For every nontrivial connected graph $H$,*

$$dim(H) \leq dim(H \times K_2) \leq dim(H) + 1.$$

The metric dimension of cartesian product of graphs has been studied in [5] and [33]. The following result in [9] gives bounds for the dimension of a graph in terms of its order and diameter.

**Theorem 2.3.6.** *For positive integers $d$ and $n$ with $d < n$, define $f(n, d)$ as the least positive integer $k$ such that $k + d^k \geq n$. Then for a connected graph $G$ of order $n \geq 2$ and diameter $d$,*

$$f(n, d) \leq dim(G) \leq n - d.$$

A sharp lower bound for the dimension of a connected graph $G$ in terms of its maximum degree $\Delta(G)$ was established in [13].

**Theorem 2.3.7.** *Let $G$ be a connected nontrivial connected graph. Then*

$$dim(G) \geq \lceil log_3(\Delta(G) + 1) \rceil$$

*and this bound is sharp.*

In fact, for each pair $k, \triangle$ of integers such that $3^k = \triangle + 1$, there exists a connected graph $G_{k,\triangle}$ such that $dim(G_{k,\triangle}) = k$ and $\triangle(G_{k,\triangle}) = \triangle$.

Now we present some theorems about the partition dimension of graphs. If $G$ is a connected graph of order $n \geq 2$, then certainly $2 \leq pd(G) \leq n$. It was also shown in [11] that every pair $k$, $n$ of integers with $2 \leq k \leq n$ is realizable as the partition dimension and order of some connected graph.

**Theorem 2.3.8.** *For every pair $k, n$ of integers with $2 \leq k \leq n$, there exists a connected graph of order $n$ with partition dimension $k$.*

The partition dimension and metric dimension of a connected graph are related. In [11], the following theorem has been proved.

**Theorem 2.3.9.** *If $G$ is a nontrivial connected graph, then*

$$pd(G) \leq dim(G) + 1.$$

*Moreover, for every pair $a, b$ of positive integers with $\lceil \frac{b}{2} \rceil + 1 \leq a \leq b+1$, there exists a connected graph $G$ such that $pd(G) = a$ and $dim(G) = b$.*

In [11], an open problem was proposed concerning Theorem 2.3.9. Is it the case that

$$pd(G) \geq \lceil \frac{dim(G)}{2} \rceil + 1$$

for every nontrivial connected graph $G$? Tomescu gave a negative answer to this question in [40].

Chartrand and Zhang obtained an improved upper bound for $pd(G)$ in terms of order and diameter of $G$ in [11].

**Theorem 2.3.10.** *If $G$ is a connected graph of order $n \geq 3$ and diameter $d$, then*

$$pd(G) \leq n - d + 1.$$

Chartrand, Salehi and Zhang determined the partition dimension of some well known classes of the graphs in [11] where the connected graphs of order $n$ with $pd(G) = 2, n, n-1$ are characterized.

**Theorem 2.3.11.** *Let $G$ be a nontrivial connected graph of order $n$. Then*

(a) $pd(G) = 2$ *if and only if* $G = P_n$

*(b)* $pd(G) = n$ *if and only if* $G = K_n$ *and*

*(c) for* $n \geq 3$, $pd(G) = n - 1$ *if and only if*

$$G \in \{K_{1,n-1}, K_n - e, K_1 + (K_1 \cup K_{n-1})\}.$$

Tomescu [40] characterized all graphs of order $n \geq 9$ having partition dimension $n-2$ thus completing the characterization of graphs of order $n$ having partition dimension $2, n$ or $n-1$ given by Chartrand, Salehi and Zhang. The list of these graphs includes 23 members.

Chartrand, Salehi, Zhang also studied the partition dimension of a tree in [12]. Although, the partition dimension of some special type of trees, such as paths, stars, double stars, and caterpillars, have been determined, a general formula for the partition dimension of a tree is still unknown. However it was shown that there is no tree of order $n$ with partition dimension $n-2$.

The partition dimension as well as the connected partition dimension of the wheel $W_n$ with $n$ spokes has been determined in [41]. The metric dimension and partition dimension of Cayley digraphs have been studied in [16] and [17].

# Chapter 3

# Families of plane graphs with constant metric dimension

The metric dimension of some classes of plane graphs has been determined in [4], [5], [7], [24], [27] and [39]. In this chapter, we extend this study by considering a class of plane graphs defined in [1] and two classes of plane graphs which are generated by the convex polytopes defined in [2]. We show that these classes of plane graphs have constant metric dimension and only three vertices appropriately chosen suffice to resolve all the vertices of these plane graphs. It is natural to ask for the characterization of families of plane graphs with constant metric dimension.

## 3.1 Notation and preliminary results

A graph $G$ is said to be plane if it is drawn on the Euclidean plane such that edges do not cross each other except at vertices of the graph.

By denoting $G+H$ the join of $G$ and $H$ a *wheel* $W_n$ is defined as $W_n = K_1 + C_n$, for $n \geq 3$, a $fan$ is $f_n = K_1 + P_n$ for $n \geq 1$ and *Jahangir graph* $J_{2n}, (n \geq 2)$ (also known as *gear graph*) is obtained from the *wheel* $W_{2n}$ by alternately deleting $n$ spokes.

Buczkowski *et al.* [4] determined the dimension of *wheel* $W_n$, Caceres *et al.* [7] the dimension of $fan$ $f_n$ and Tomescu and Javaid [39] the dimension of *Jahangir graph* $J_{2n}$.

**Theorem 3.1.1.** *([4], [7], [39]) Let* $W_n$ *be a wheel of order* $n \geq 3$*,* $f_n$ *be fan of order* $n \geq 1$ *and* $J_{2n}$ *be a Jahangir graph. Then*
*(i) For* $n \geq 7$*,* $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$*;*
*(ii) For* $n \geq 7$*,* $dim(f_n) = \lfloor \frac{2n+2}{5} \rfloor$*;*
*(iii) For* $n \geq 4$*,* $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$*.*

The metric dimension of all these plane graphs depends upon the number of vertices in the graph.

On the other hand, we say that a family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if $dim$(G) is finite and does not depend upon the choice of $G$ in $\mathcal{G}$. In [9] it was shown that a graph has metric dimension 1 if and only if it is a *path*, hence paths on $n$ vertices constitute a family of graphs with constant metric dimension. Similarly, *cycles* with $n(\geq 3)$ vertices also constitute such a family of graphs as their metric dimension is 2 and does not depend upon on the number of vertices $n$. In [5] it was proved that

$$dim(P_m \times C_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

Since *prisms* $D_n$ are the trivalent plane graphs obtained by the cross product of path $P_2$ with a cycle $C_n$, this implies that

$$dim(D_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

So, prisms constitutes a family of 3-*regular graphs* with constant metric dimension. Also Javaid *et al.* proved in [24] that the plane graph *antiprism* $A_n$ constitute a family of regular graphs with constant metric dimension as $dim(A_n) = 3$ for every $n \geq 5$. The prism and the antiprism are *Archimedean* convex polytopes defined e.g. in [26]. A *grid* $G_n^m$ is obtained by the cartesian product of two paths $P_n$ by $P_m$. In [27], it was shown that $dim(P_n \times P_m) = 2$, so grids constitute a family of plane graphs with constant metric dimension as their metric dimension is finite and does not depend upon the number of vertices in the graph. Note that the problem of determining whether $dim(G) < k$ is an $NP$-complete problem [18].

In this chapter, we study the metric dimension of some plane graphs which are the convex polytops defined in [1] and [2]. In the second section, we study the metric dimension of graph of convex polytope $\mathbb{D}_n$ consisting of 5-sided faces and $n$-sided faces. In the third section, we investigate the metric dimension of the graph of convex polytope $R_n$ which is obtained by the combination of the graph of a prism and graph of an antiprism. In fourth section, metric dimension of the graph of convex polytope $Q_n$ consisting of 3-sided faces, 4-sided faces, 5-sided faces and $n$-sided faces has been determined.

## 3.2 The graph of convex polytope $\mathbb{D}_n$

Let the graph of antiprism $A_n$ [1] be given. We insert a vertex $a_{n+1}$ inside the $n$-gone $P$ and $b_{n+1}$ inside the $n$-gone $P'$.

We join any vertex $a_i$ of $P$ with the vertex $a_{n+1}$ and any vertex $b_i$ of $P'$ with the vertex $b_{n+1}$ for $i = 1, 2, ..., n$. Thus we obtain the graph $A'_n$. The dual graph to $A'_n$ with vertices $a_1, a_2, \ldots, a_n; b_1, b_2, ..., b_n$; $c_1, c_2, ..., c_n; d_1, d_2, ..., d_n$ (Fig 3.1) is the graph

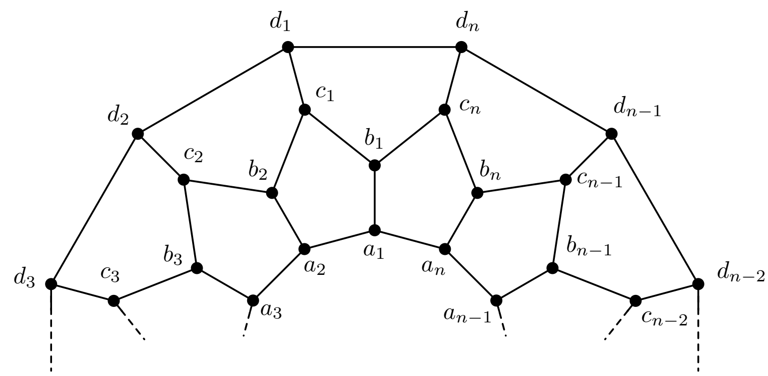


Figure 3.1: The graph of convex polytope $\mathbb{D}_n$

of the convex polytope defined in [1] and will be denoted by $\mathbb{D}_n$. It has 5-sided faces and $n$-sided faces.

For our purpose, we call the cycle induced by $\{a_1, a_2, ..., a_n\}$, the inner cycle, cycle induced by $\{d_1, d_2, ..., d_n\}$, the outer cycle, set of vertices $\{b_1, b_2, ..., b_n\}$ which are connected with the inner cycle, the set of inner vertices, the set of vertices $\{c_1, c_2, ..., c_n\}$ which are connected with the outer cycle, the set of outer vertices.

In the next theorem, we show that the metric dimension of the graph of convex polytope $\mathbb{D}_n$ is 3. Note that the choice of appropriate basis vertices (also referred to as landmarks in [27]) is core of the problem.

**Theorem 3.2.1.** *For $n \geq 4$, let the graph of convex polytopes be $\mathbb{D}_n$; then $dim(\mathbb{D}_n) =$ 3.*

*Proof.* We will prove the above equality by double inequalities. We consider two cases.

**Case(i)** When $n$ is even.

In this case, we can write $n = 2k$, $k \geq 2$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+2}\} \subset V(\mathbb{D}_n)$, we show that $W$ is a resolving set for $\mathbb{D}_n$ in this case. For this we give representations of any vertex of $V(\mathbb{D}_n)$.

Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+2), & 3 \leq i \leq k+1; \\ (2k-i+1, 2k-i+2, i-k-2), & k+3 \leq i \leq 2k. \end{cases}$$

Representations of the set of inner vertices are

$$r(b_i|W) = \begin{cases} (1, 2, k), & i = 1; \\ (i, i-1, k-i+3), & 2 \leq i \leq k+1; \\ (2k-i+2, 2k-i+3, i-k-1), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of the set of outer vertices are

$$r(c_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+3), & 2 \leq i \leq k; \\ (k+1, k+1, 2), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(d_i|W) = \begin{cases} (3, 3, k+2), & i = 1; \\ (i+2, i+1, k-i+4), & 2 \leq i \leq k; \\ (k+2, k+2, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+1), & k+2 \leq i \leq 2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(\mathbb{D}_n) \leq 3$.

On the other hand, we show that $dim(\mathbb{D}_n) \geq 3$. Suppose on contrary that $dim(\mathbb{D}_n) =$

2, then there are following possibilities to be discussed.

(1) Both vertices are in the inner cycle. Without loss of generality we can suppose that one resolving vertex is $a_1$. Suppose that the second resolving vertex is $a_i$ $(2 \leq i \leq k+1)$. Then for $2 \leq i \leq k$, we have $r(a_n|\{a_1, a_i\}) = r(b_1|\{a_1, a_i\})$ and for $i = k+1$ we have $r(a_2|\{a_1, a_{k+1}\}) = r(a_n|\{a_1, a_{k+1}\})$, a contradiction.

(2) Both vertices belong to the set of inner vertices. Without loss of generality we can suppose that one resolving vertex is $b_1$. Suppose that the second resolving vertex is $b_i$ $(2 \leq i \leq k+1)$. Then for $2 \leq i \leq k$, we have $r(a_n|\{b_1, b_i\}) = r(d_n|\{b_1, b_i\})$ and for $i = k+1$ we have $r(b_2|\{b_1, b_{k+1}\}) = r(b_n|\{b_1, b_{k+1}\})$, a contradiction.

(3) Both vertices belong to the set of outer vertices. By the symmetry of the graph, this case is analogous to case (2).

(4) Both vertices are in the outer cycle. By the same argument this case is analogous to case (1).

(5) One vertex is in the inner cycle and other belongs to the set of inner vertices. Without loss of generality we can suppose that one resolving vertex is $a_1$. Suppose that the second resolving vertex is $b_i$ $(1 \leq i \leq k+1)$. Then for $i = 1$, we have $r(a_2|\{a_1, b_1\}) = r(a_n|\{a_1, b_1\})$. If $i = 2$, $r(d_1|\{a_1, b_2\}) = r(b_3|\{a_1, b_2\})$. For $3 \leq i \leq k$, we have $r(a_n|\{a_1, b_i\}) = r(b_1|\{a_1, b_i\})$ and when $i = k+1$, we have $r(a_2|\{a_1, b_{k+1}\}) = r(a_n|\{a_1, b_{k+1}\})$, a contradiction.

(6) One vertex is in the inner cycle and other belongs to the set of outer vertices. Without loss of generality we can suppose that one resolving vertex is $a_1$. Suppose that the second resolving vertex is $c_i$ $(1 \leq i \leq k+1)$. Then for $i = 1$, we have $r(a_3|\{a_1, c_1\}) = r(b_n|\{a_1, c_1\})$. If $i = 2$, $r(a_3|\{a_1, c_2\}) = r(c_1|\{a_1, c_2\})$. For $3 \leq i \leq k-1$, we have $r(b_1|\{a_1, c_i\}) = r(a_n|\{a_1, c_i\})$. When $i = k$, we have $r(c_1|\{a_1, c_k\}) = r(c_n|\{a_1, c_k\})$

and if $i = k+1$, we have $r(b_2|\{a_1, c_{k+1}\}) = r(c_1|\{a_1, c_{k+1}\})$, a contradiction.

(7) One vertex is in the inner cycle and other in the outer cycle. Without loss of generality we can suppose that one resolving vertex is $a_1$. Suppose that the second resolving vertex is $d_i$ $(1 \leq i \leq k+1)$. Then for $i = 1$, we have $r(b_2|\{a_1, d_1\}) = r(c_n|\{a_1, d_1\})$. If $i = 2$, $r(b_2|\{a_1, d_2\}) = r(d_n|\{a_1, d_2\})$. For $3 \leq i \leq k-1$, we have $r(a_2|\{a_1, d_i\}) = r(b_1|\{a_1, d_i\})$ and when $i = k, k+1$ we have $r(b_2|\{a_1, d_i\}) = r(c_1|\{a_1, d_i\})$, a contradiction.

(8) One vertex belongs to the set of inner vertices and other belongs to the set of outer vertices. Without loss of generality we can suppose that one resolving vertex is $b_1$. Suppose that the second resolving vertex is $c_i$ $(1 \leq i \leq k+1)$. Then for $i = 1$, we have $r(a_1|\{b_1, c_1\}) = r(c_n|\{b_1, c_1\})$. If $i = 2$, $r(a_2|\{b_1, c_2\}) = r(d_1|\{b_1, c_2\})$. For $3 \leq i \leq k$, we have $r(a_1|\{b_1, c_i\}) = r(c_1|\{b_1, c_i\})$ and when $i = k+1$, we have $r(a_1|\{b_1, c_{k+1}\}) = r(c_n|\{b_1, c_{k+1}\})$, a contradiction.

(9) One vertex belongs to the set of inner vertices and other in the outer cycle. This case is analogous to case (6).

(10) One vertex belongs to the set of outer vertices and other is in the outer cycle. This case is analogous to case (5).

Hence, from above it follows that there is no resolving set with two vertices for $V(\mathbb{D}_n)$ implying that $dim(\mathbb{D}_n) = 3$ in this case.

**Case(ii)** When $n$ is odd.

In this case, we can write $n = 2k+1$, $k \geq 2$, $k \in \mathbf{Z}^+$. Again we show that $W = \{a_1, a_2, a_{k+2}\} \subset V(\mathbb{D}_n)$ is a resolving set for $\mathbb{D}_n$ in this case. For this we give representations of any vertex of $V(\mathbb{D}_n)$.

Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+2), & 3 \leq i \leq k+1; \\ (2k-i+2, 2k-i+3, i-k-2), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations of the set of inner vertices are

$$r(b_i|W) = \begin{cases} (1, 2, k), & i = 1; \\ (i, i-1, k-i+3), & 2 \leq i \leq k+1; \\ (k+1, k+1, 1), & i = k+2; \\ (2k-i+3, 2k-i+4, i-k-1), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations of the set of outer vertices are

$$r(c_i|W) = \begin{cases} (2, 2, k+2), & i = 1; \\ (i-1, i, k-i+3), & 2 \leq i \leq k+1; \\ (2k-i+3, 2k-i+4, i-k), & k+2 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(d_i|W) = \begin{cases} (3, 3, k+3), & i = 1; \\ (i+2, i+1, k-i+4), & 2 \leq i \leq k+1; \\ (2k-i+4, 2k-i+5, i-k+1), & k+2 \leq i \leq 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(\mathbb{D}_n) \leq 3$.

On the other hand, suppose that $dim(\mathbb{D}_n) = 2$, then there are the same possibilities as in case (i) and contradiction can be deduced analogously. This implies that $dim(\mathbb{D}_n) = 3$ in this case, which completes the proof. □

## 3.3 The graph of convex polytope $R_n$

For $n \geq 5$, by $R_n$ we denote the graph of convex polytope defined in [2] which is obtained as a combination of the graph of a prism and the graph of an antiprism (Fig 3.2). We make the convention that $a_{n+1} = a_1$, $b_{n+1} = b_1$, $c_{n+1} = c_1$, $d_{n+1} = d_1$ to simplify the notation.

For our purpose, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$ the inner cycle, cycle

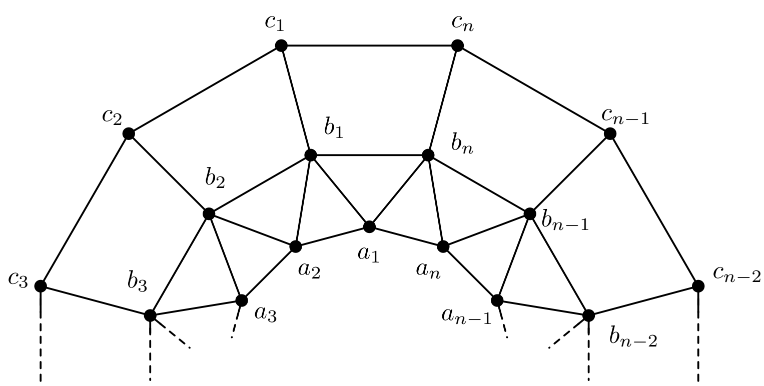


Figure 3.2: The graph of convex polytope $R_n$

induced by $\{b_i : 1 \leq i \leq n\}$ the middle cycle, cycle induced by $\{c_i : 1 \leq i \leq n\}$ the outer cycle.

In the next theorem, we prove that only three vertices suffice to resolve all the vertices of the graph of convex polytope $R_n$. Again, choice of appropriate landmarks is crucial.

**Theorem 3.3.1.** *Let $R_n$ denotes the graph of convex polytope; then $dim(R_n) = 3$ for every $n \geq 5$.*

*Proof.* We will prove the above equality by double inequalities. We consider two cases.

**Case(i)** When $n$ is odd.

In this case, we can write $n = 2k+1$, $k \geq 2$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+2}\} \subset V(R_n)$, we show that $W$ is a resolving set for $R_n$ in this case. For this we give representations of any vertex of $V(R_n)$.

Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+2), & 3 \leq i \leq k+1; \\ (2k-i+2, 2k-i+3, i-k-2), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on middle cycle are

$$r(b_i|W) = \begin{cases} (1, 1, k+1), & i = 1; \\ (i, i-1, k-i+2), & 2 \leq i \leq k+1; \\ (2k-i+2, 2k-i+3, i-k-1), & k+2 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(c_i|W) = \begin{cases} (2, 2, k+2), & i = 1; \\ (i+1, i, k-i+3), & 2 \leq i \leq k+1; \\ (2k-i+3, 2k-i+4, i-k), & k+2 \leq i \leq 2k+1. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(R_n) \leq 3$.

On the other hand, we show that $dim(R_n) \geq 3$. Suppose on contrary that $dim(R_n) = 2$, then there are following possibilities to be discussed.

(1) Both vertices are in the inner cycle. Without loss of generality we can suppose that one resolving vertex is $a_1$. Suppose that the second resolving vertex is $a_i$ $(2 \leq i \leq k+1)$. Then for $2 \leq i \leq k$ we have $r(a_n|\{a_1, a_i\}) = r(b_n|\{a_1, a_i\})$ and

for $i = k+1$ we have $r(a_n|\{a_1, a_{k+1}\}) = r(b_1|\{a_1, a_{k+1}\})$, a contradiction.

(2) Both vertices are in the middle cycle. In this case, we can fix $b_1$ as a resolving vertex. Suppose that the second resolving vertex is $b_i$ $(2 \leq i \leq k+1)$, then $r(a_1|\{b_1, b_i\}) = r(c_1|\{b_1, b_i\})$, a contradiction.

(3) Both vertices are in the outer cycle. Here we can fix $c_1$ as a resolving vertex. Suppose that the second resolving vertex is $c_i$ $(2 \leq i \leq k+1)$. Then for $2 \leq i \leq k$ we have $r(b_1|\{c_1, c_i\}) = r(c_n|\{c_1, c_i\})$ and for $i = k+1$ we have $r(a_1|\{c_1, c_{k+1}\}) = r(b_n|\{c_1, c_{k+1}\})$, a contradiction.

(4) One vertex is in the inner cycle and other in the middle cycle. Without loss of generality we can suppose that one resolving vertex is $a_1$. Suppose that the second resolving vertex is $b_i$ $(1 \leq i \leq k+1)$, then for $i = 1$ we have $r(a_2|\{a_1, b_1\}) = r(b_n|\{a_1, b_1\})$ and for $2 \leq i \leq k+1$, $r(a_2|\{a_1, b_i\}) = r(b_1|\{a_1, b_i\})$, a contradiction.

(5) One vertex is in the inner cycle and other in the outer cycle. We can suppose that one resolving vertex is $a_1$. Suppose that the second resolving vertex is $c_i$ $(1 \leq i \leq k+1)$. Then for $i = 1$, $r(a_2|\{a_1, c_1\}) = r(b_n|\{a_1, c_1\})$ and for $2 \leq i \leq k+1$ we have $r(a_2|\{a_1, c_i\}) = r(b_1|\{a_1, c_i\})$, a contradiction.

(6) One vertex is in the middle cycle and other in the outer cycle. Here we can fix $b_1$ as a resolving vertex. Suppose that the second resolving vertex is $c_i$ $(1 \leq i \leq k+1)$, then for $i = 1$ we have $r(a_1|\{b_1, c_1\}) = r(a_2|\{b_1, c_1\})$ and for $2 \leq i \leq k+1$ we have $r(b_2|\{b_1, c_i\}) = r(c_1|\{b_1, c_i\})$, a contradiction.

Hence, from above it follows that there is no resolving set with two vertices for $V(R_n)$ implying that $dim(R_n) = 3$ in this case.

**Case(ii)** When $n$ is even.

In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(R_n)$, we show that $W$ is a resolving set for $R_n$ in this case. For this we give representations of any vertex of $V(R_n)$.

Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of the vertices on middle cycle are

$$r(b_i|W) = \begin{cases} (1, 1, k), & i = 1; \\ (i, i-1, k-i+1), & 2 \leq i \leq k; \\ (k, k, 1), & i = k+1; \\ (2k-i+1, 2k-i+2, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(c_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \leq i \leq k; \\ (k+1, k+1, 2), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k+1), & k+2 \leq i \leq 2k. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(R_n) \leq 3$ in this case.

On the other hand, suppose that $dim(R_n) = 2$, then there are the same subcases as in case (**i**) and contradiction can be deduced analogously. This implies that $dim(R_n) = 3$ in this case, which completes the proof.

□

## 3.4 The graph of convex polytope $Q_n$

In this section, we shall investigate the metric dimension of the graph of convex polytope $Q_n$ defined in [2] consisting of 3-sided faces, 4-sided faces, 5-sided faces and $n$-sided faces (Fig 3.3). The graph of convex polytope $Q_n$ and the graph of convex polytope $\mathbb{D}_n$ have the same set of vertices; moreover, $E(Q_n) = E(\mathbb{D}_n) \cup \{b_i b_{i+1} : 1 \leq i \leq n\}$.

For our purpose, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$ the inner cycle, cycle

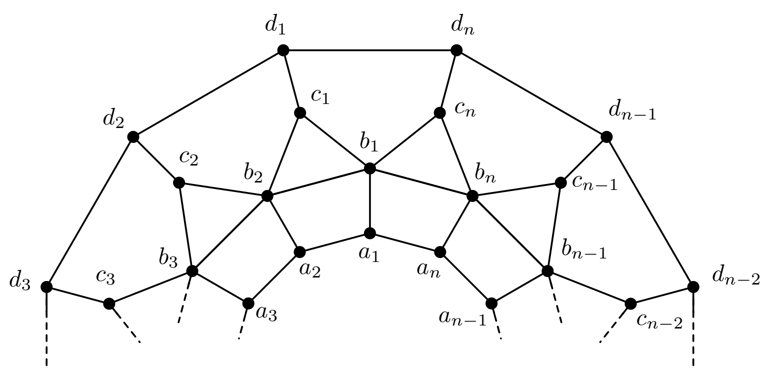


Figure 3.3: The graph of convex polytope $Q_n$

induced by $\{b_i : 1 \leq i \leq n\}$, the middle cycle, set of vertices $\{c_1, c_2, ..., c_n\}$ the set of interior vertices and cycle induced by $\{d_i : 1 \leq i \leq n\}$, the outer cycle.

In the next theorem, we show that with only three vertices, we can resolve all the vertices of the graph of convex polytope $Q_n$. Once gain, choice of appropriate landmarks is very important.

**Theorem 3.4.1.** *Let the graph of convex polytopes be $Q_n$; then $dim(Q_n) = 3$ for every $n \geq 6$.*

*Proof.* We will prove the above equality by double inequalities. We consider two cases.

**Case(i)** When $n$ is even.

In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(Q_n)$, we show that $W$ is a resolving set for $Q_n$ in this case. For this we give representations of any vertex of $V(Q_n)$.

Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of the vertices on middle cycle are

$$r(b_i|W) = \begin{cases} (1, 2, k+1), & i = 1; \\ (i, i-1, k-i+2), & 2 \leq i \leq k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of the set of interior vertices are

$$r(c_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \leq i \leq k+1; \\ (k+1, k+1, 2), & i = k+2; \\ (2k-i+3, 2k-i+4, i-k), & k+3 \leq i \leq 2k. \end{cases}$$

Representations of vertices on outer cycle are

$$r(d_i|W) = \begin{cases} (3,3,k+2), & i = 1; \\ (i+2, i+1, k-i+3), & 2 \le i \le k; \\ (k+2, k+2, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+2), & k+2 \le i \le 2k. \end{cases}$$

We see that there are no two vertices having the same representations implying that $dim(Q_n) \le 3$.

On the other hand, we show that $dim(Q_n) \ge 3$. Suppose on contrary that $dim(Q_n) = 2$, then there are following possibilities to be discussed.

(1) Both vertices are in the inner cycle. Without loss of generality we can suppose that one resolving vertex is $a_1$. Suppose that the second resolving vertex is $a_i$ $(2 \le i \le k+1)$, then for $2 \le i \le k$, $r(a_n|\{a_1, a_i\}) = r(b_1|\{a_1, a_i\})$ and for $i = k+1$ we have $r(a_2|\{a_1, a_{k+1}\}) = r(a_n|\{a_1, a_{k+1}\})$, a contradiction.

(2) Both vertices are in the middle cycle. Without loss of generality we can suppose that one resolving vertex is $b_1$. Suppose that the second resolving vertex is $b_i$ $(2 \le i \le k+1)$, then for $2 \le i \le k$, $r(a_1|\{b_1, b_i\}) = r(c_n|\{b_1, b_i\})$ and for $i = k+1$ we have $r(b_2|\{b_1, b_{k+1}\}) = r(b_n|\{b_1, b_{k+1}\})$, a contradiction.

(3) Both vertices belong to the set of interior vertices. Without loss of generality we can suppose that one resolving vertex is $c_1$. Suppose that the second resolving vertex is $c_i$ $(2 \le i \le k+1)$, then for $2 \le i \le k$, $r(b_1|\{c_1, c_i\}) = r(d_1|\{c_1, c_i\})$ and for $i = k+1$ we have $r(c_2|\{c_1, c_{k+1}\}) = r(c_n|\{c_1, c_{k+1}\})$, a contradiction.

(4) Both vertices are in the outer cycle. Without loss of generality we can suppose that one resolving vertex is $d_1$. Suppose that the second resolving vertex is $d_i$ $(2 \le i \le k+1)$, then for $2 \le i \le k$, $r(c_1|\{d_1, d_i\}) = r(d_n|\{d_1, d_i\})$ and for $i = k+1$ we have $r(d_2|\{d_1, d_{k+1}\}) = r(d_n|\{d_1, d_{k+1}\})$, a contradiction.

(5) One vertex is in the inner cycle and other in the middle cycle. Without loss of generality we can suppose that one resolving vertex is $a_1$. Suppose that the second resolving vertex is $b_i$ $(1 \leq i \leq k+1)$, then for $i = 1$ we have $r(a_2|\{a_1, b_1\}) = r(a_n|\{a_1, b_1\})$ and for $2 \leq i \leq k+1$, $r(a_2|\{a_1, b_i\}) = r(b_1|\{a_1, b_i\})$, a contradiction.

(6) One vertex is in the inner cycle and other in the set of interior vertices. Without loss of generality we can suppose that one resolving vertex is $a_1$. Suppose that the second resolving vertex is $c_i$ $(1 \leq i \leq k+1)$, then for $i = 1$ we have $r(b_2|\{a_1, c_1\}) = r(c_n|\{a_1, c_1\})$. For $2 \leq i \leq k$, $r(a_2|\{a_1, c_i\}) = r(b_1|\{a_1, c_i\})$ and when $i = k+1$ we have $r(b_1|\{a_1, c_{k+1}\}) = r(a_n|\{a_1, c_{k+1}\})$, a contradiction.

(7) One vertex is in the inner cycle and other in the outer cycle. Without loss of generality we can suppose that one resolving vertex is $a_1$. Suppose that the second resolving vertex is $d_i$ $(1 \leq i \leq k+1)$, then for $i = 1$ we have $r(b_2|\{a_1, d_1\}) = r(c_n|\{a_1, d_1\})$ and for $2 \leq i \leq k+1$, $r(a_2|\{a_1, d_i\}) = r(b_1|\{a_1, d_i\})$, a contradiction.

(8) One vertex is in the middle cycle and other in the set of interior vertices. Without loss of generality we can suppose that one resolving vertex is $b_1$. Suppose that the second resolving vertex is $c_i$ $(1 \leq i \leq k+1)$ then for $1 \leq i \leq k$ we have $r(a_1|\{b_1, c_i\}) = r(b_n|\{b_1, c_i\})$ and for $i = k+1$ we have $r(a_1|\{b_1, c_{k+1}\}) = r(c_n|\{b_1, c_{k+1}\})$, a contradiction.

(9) One vertex is in the middle cycle and other in the outer cycle. Without loss of generality we can suppose that one resolving vertex is $b_1$. Suppose that the second resolving vertex is $d_i$ $(1 \leq i \leq k+1)$, then for $1 \leq i \leq k-1$ we have $r(a_1|\{b_1, d_i\}) = r(b_n|\{b_1, d_i\})$ and for $i = k, k+1$ we have, $r(b_n|\{b_1, d_i\}) = r(c_n|\{b_1, d_i\})$, a contradiction.

(10) One vertex is in the set of interior vertices and other in the outer cycle. Without loss of generality we can suppose that one resolving vertex is $c_1$. Suppose that the second resolving vertex is $d_i$ $(1 \leq i \leq k+1)$. Then for $i = 1$, $r(b_1|\{c_1, d_1\}) = r(b_2|\{c_1, d_1\})$, for $i = 2$, $r(b_3|\{c_1, d_2\}) = r(d_n|\{c_1, d_2\})$ and when $3 \leq i \leq k+1$, we have $r(b_3|\{c_1, d_i\}) = r(c_2|\{c_1, d_i\})$, a contradiction.

So from above, we conclude that there is no resolving set with two vertices for $V(Q_n)$ implying that $dim(Q_n) = 3$ in this case.

**Case(ii)** When $n$ is odd.

In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+2}\} \subset V(Q_n)$, we show that $W$ is a resolving set for $Q_n$ in this case. For this we give representations of any vertex of $V(Q_n)$.

Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+2), & 3 \leq i \leq k+1; \\ (2k-i+2, 2k-i+3, i-k-2), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on middle cycle are

$$r(b_i|W) = \begin{cases} (1, 2, k+1), & i = 1; \\ (i, i-1, k-i+3), & 2 \leq i \leq k+1; \\ (k+1, k+1, 1), & i = k+2; \\ (2k-i+3, 2k-i+4, i-k-1), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations of the set of interior vertices are

$$r(c_i|W) = \begin{cases} (2, 2, k+2), & i = 1; \\ (i+1, i, k-i+3), & 2 \leq i \leq k+1; \\ (2k-i+3, 2k-i+4, i-k), & k+2 \leq i \leq 2k+1. \end{cases}$$

Representations of vertices on outer cycle are

$$r(d_i|W) = \begin{cases} (3, 3, k+3), & i = 1; \\ (i+2, i+1, k-i+4), & 2 \leq i \leq k+1; \\ (2k-i+4, 2k-i+5, i-k+1), & k+2 \leq i \leq 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations implying that $dim(Q_n) \leq 3$.

On the other hand, we show that $dim(Q_n) \geq 3$. Suppose on contrary that $dim(Q_n) = 2$, then there are the same possibilities as in case (**i**) and contradiction can be obtained analogously. It follows that $dim(Q_n) = 3$, which completes the proof. □

# Chapter 4

# Metric dimension of generalized Petersen graphs $P(n,3)$

In this chapter, we study the metric dimension of the generalized Petersen graphs $P(n,3)$ by giving a partial answer to an open problem raised in [24]: Is $P(n,m)$ for $n \geq 7$ and $3 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, a family of graphs with constant metric dimension? We prove that the generalized Petersen graphs $P(n,3)$ have metric dimension equal to 3 for $n \equiv 1 (\text{mod } 6)$, $n \geq 25$, and to 4 for $n \equiv 0 (\text{mod } 6)$, $n \geq 24$. For the remaining cases only 4 vertices appropriately chosen suffice to resolve all the vertices of $P(n,3)$, thus implying that $dim(P(n,3)) \leq 4$, except when $n \equiv 2 (\text{mod } 6)$, when $dim(P(n,3)) \leq 5$.

## 4.1 Notation and auxiliary results

In [24] Javaid *et al.* proved that some regular graphs namely generalized Petersen graphs $P(n,2)$, antiprisms $A_n$ and Harary graphs $H_{4,n}$ are families of graphs with constant metric dimension and raised an open problem.

**Open Problem [24]:** Is $P(n,m)$ for $n \geq 7$ and $3 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, a family of graphs with constant metric dimension?

In this chapter, we give a partial answer to this open problem and we show that the generalized Petersen graphs $P(n,3)$ constitute a family of regular graphs having bounded metric dimension and only 4 vertices appropriately chosen suffice to resolve all vertices of the generalized Petersen graphs $P(n,3)$ except $n \equiv 2 (\text{mod } 6)$, when this number equals 5. For $n \equiv 1 (\text{mod } 6)$ a minimal resolving set has cardinality equal to 3.

In what follows all indices $i$ which do not satisfy inequalities $1 \le i \le n$ will be taken modulo $n$.

## 4.2 Upper bounds for metric dimension of generalized Petersen graphs $P(n,3)$

The generalized Petersen graph denoted by $P(n,m)$, where $n \ge 3$ and $1 \le m \le \lfloor \frac{n-1}{2} \rfloor$, is a cubic graph having vertex set

$$V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$$

and edge set

$$E = \{u_i u_{i+1}, u_i v_i, v_i v_{i+m} : 1 \le i \le n\}.$$

Generalized Petersen graphs were first defined by Watkins [44]. For $m = 1$ the generalized Petersen graph $P(n,1)$ is called prism, denoted by $D_n$. In [5] it was shown that

$$dim(P_m \times C_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

Since the prism $D_n$ is actually the cross product of $P_2$ with a cycle $C_n$, this implies that

$$dim(D_n)=\begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

So, prisms constitute a family of 3-regular graphs with bounded metric dimension. In [24] it was proved that $dim(P(n,2)) = 3$ for every $n \geq 5$. Now we will find the metric dimension of the generalized Petersen graphs $P(n,3)$ when $n \equiv 0$ or $1(\text{mod } 6)$ and an upper bound in the remaining cases.

When $m = 3$, $\{u_1, u_2, ...., u_n\}$ induces a cycle in $P(n,3)$ with $u_iu_{i+1}$ $(1 \leq i \leq n)$, as edges. If $n = 3l$ $(l \geq 3)$, then $\{v_1, v_2, \ldots, v_n\}$ induces 3 cycles of length $l$, otherwise it induces a cycle of length $n$ with $v_iv_{i+3}$ $(1 \leq i \leq n)$, as edges. For example, $P(8,3)$ is the *Möbius-Kantor* graph [29].

Since generalized Petersen graphs $P(n,3)$ form an important class of 3-regular graphs with $2n$ vertices and $3n$ edges, it is desirable to find their metric dimensions. For our purpose, we call the cycle induced by $\{u_1, u_2, \ldots, u_n\}$, outer cycle and cycle(s) induced by $\{v_1, v_2, \ldots, v_n\}$ inner cycle(s). Note that the choice of appropriate basis vertices (also referred to as landmarks in [27]) is core of the problem.

**Theorem 4.2.1.** *For the generalized Petersen graph $P(n,3)$ we have*
*(a) $dim(P(n,3)) \leq 3$ for $n \equiv 1(mod\ 6)$ and $n \geq 13$;*
*(b) $dim(P(n,3)) \leq 4$ for $n \equiv 0,3,4,5(mod\ 6)$ and $n \geq 17$.*
*(c) $dim(P(n,3)) \leq 5$ for $n \equiv 2(mod\ 6)$ and $n \geq 8$.*

**Proof:** We denote $W = \{v_1, v_2, v_3, u_4\}$ for $n \equiv 0,3,4,5(\text{mod } 6)$ and $W = \{v_1, v_{3k-1}, v_{6k}\}$ for $n \equiv 1(\text{mod } 6)$, $n = 6k+1$. We show that the set $W$ distinguishes the vertices of $P(n,3)$ for $n \not\equiv 2(\text{mod } 6)$. For this purpose, we give the representation of $V(P(n,3))$ in these cases.

**Case (i)** $n = 6k, k \in \mathbf{Z}^+, k \geq 2$. For every $n \geq 12$, the representations of vertices on the outer cycle are

$r(u_1|W) = (1,2,3,3)$, $r(u_2|W) = (2,1,2,2)$, $r(u_3|W) = (3,2,1,1)$.

$r(u_{2+3i}|W) =$

$$\begin{cases} (3,2,3,1), & i=1; \\ (i+2,i+1,i+2,i+2), & 2 \le i \le k; \\ (2k-i+2,2k-i+1,2k-i+2,2k-i+4), & k+1 \le i \le 2k-1. \end{cases}$$

$r(u_{3+3i}|W) =$

$$\begin{cases} (4,3,2,2), & i=1; \\ (i+3,i+2,i+1,i+3), & 2 \le i \le k-1; \\ (2k-i+1,2k-i+2,2k-i+1,2k-i+3), & k \le i \le 2k-1. \end{cases}$$

$r(u_{4+3i}|W) =$

$$\begin{cases} (i+2,i+3,i+2,i+2), & 1 \le i \le k-1; \\ (2k-i,2k-i+1,2k-i+2,2k-i+2), & k \le i \le 2k-2. \end{cases}$$

Representations of vertices with respect to $W$ on the inner cycles are

$r(v_{1+3i}|W) =$

$$\begin{cases} (i,i+3,i+2,i), & 1 \le i \le k; \\ (2k-i,2k-i+3,2k-i+4,2k-i+2), & k+1 \le i \le 2k-1. \end{cases}$$

$r(v_{2+3i}|W) =$

$$\begin{cases} (i+3,i,i+3,i+1), & 1 \le i \le k; \\ (2k-i+3,2k-i,2k-i+3,2k-i+3), & k+1 \le i \le 2k-1. \end{cases}$$

$r(v_{3+3i}|W) =$

$$\begin{cases} (i+4,i+3,i,i+2), & 1 \le i \le k-1; \\ (2k-i+2,2k-i+3,2k-i,2k-i+2), & k \le i \le 2k-1. \end{cases}$$

**Case (ii)** $n = 6k+1, k \in \mathbf{Z}^+$, $k \geq 2$. For every $n \geq 13$, the representations of vertices with respect to $W$ are the following:

Representations of vertices on the outer cycle are

$$r(u_{3i}|W) = \begin{cases} (i+2, k-i+2, i+2), & 1 \leq i \leq k-1; \\ (k+2, 2, k+1), & i = k; \\ (2k-i+3, i-k+2, 2k-i+1), & k+1 \leq i \leq 2k-1; \\ (3, k+1, 1), & i = 2k. \end{cases}$$

$$r(u_{3i-1}|W) = \begin{cases} (i+1, k-i+1, i+1), & 1 \leq i \leq k; \\ (2k-i+2, i-k+1, 2k-i+2), & k+1 \leq i \leq 2k. \end{cases}$$

$$r(u_{3i-2}|W) = \begin{cases} (i, k-i+2, i+2), & 1 \leq i \leq k; \\ (k+1, 3, k+2), & i = k+1; \\ (2k-i+3, i-k+2, 2k-i+3), & k+2 \leq i \leq 2k; \\ (2, k+2, 2), & i = 2k+1. \end{cases}$$

Representations of vertices with respect to $W$ on the inner cycle are

$$r(v_{3i-1}|W) = \begin{cases} (i+2, k-i, i), & 1 \leq i \leq k-1; \\ (k+1, 0, k), & i = k; \\ (k, 1, k+1), & i = k+1; \\ (2k-i+1, i-k, 2k-i+3), & k+2 \leq i \leq 2k. \end{cases}$$

$$r(v_{3i}|W) = \begin{cases} (i+3, k-i+3, i+3), & 1 \leq i \leq k-2; \\ (i+3, k-i+3, 2k-i), & k-1 \leq i \leq k; \\ (2k-i+4, i-k+3, 2k-i), & k+1 \leq i \leq 2k-2; \\ (5, k+1, 1), & i = 2k-1; \\ (4, k, 0), & i = 2k. \end{cases}$$

$$r(v_{3i+1}|W) = \begin{cases} (0, k+1, 4), & i = 0; \\ (i, k-i+2, i+4), & 1 \leq i \leq k-1; \\ (k, 4, k+3), & i = k; \\ (k+1, 5, k+2), & i = k+1; \\ (2k-i+3, i-k+4, 2k-i+3), & k+2 \leq i \leq 2k-1; \\ (3, k+3, 3), & i = 2k. \end{cases}$$

**Case (iii)** $n = 6k+3, k \in \mathbf{Z}^+$. For $P(9,3)$, the representations of the vertices are $r(u_1|W) = (1,2,3,3)$, $r(u_2|W) = (2,1,2,2)$, $r(u_3|W) = (3,2,1,1)$, $r(u_5|W) = (3,2,3,1)$, $r(u_6|W) = (3,3,2,2)$, $r(u_7|W) = (2,3,3,3)$,

$r(u_8|W) = (3,2,3,4)$, $r(u_9|W) = (2,3,2,4)$, $r(v_4|W) = (1,4,3,1)$,

$r(v_5|W) = (4,1,4,2)$, $r(v_6|W) = (4,4,1,3)$, $r(v_7|W) = (1,4,4,2)$,

$r(v_8|W) = (4,1,4,3)$, $r(v_9|W) = (3,4,1,3)$. For every $n \geq 15$, representations of vertices with respect to $W$ are the following:

Representations of vertices on the outer cycle are

$r(u_1|W) = (1,2,3,3)$, $r(u_2|W) = (2,1,2,2)$, $r(u_3|W) = (3,2,1,1)$.

$r(u_{2+3i}|W) =$

$$\begin{cases} (3,2,3,1), & i = 1; \\ (i+2, i+1, i+2, i+2), & 2 \leq i \leq k; \\ (k+2, k+1, k+2, k+3), & i = k+1; \\ (2k-i+3, 2k-i+2, 2k-i+3, 2k-i+5), & k+2 \leq i \leq 2k. \end{cases}$$

$r(u_{3+3i}|W) =$

$$\begin{cases} (4,3,2,2), & i = 1; \\ (i+3, i+2, i+1, i+3), & 2 \leq i \leq k-1; \\ (k+2, k+2, k+1, k+3), & i = k; \\ (2k-i+2, 2k-i+3, 2k-i+2, 2k-i+4), & k+1 \leq i \leq 2k. \end{cases}$$

$r(u_{4+3i}|W) =$

$$\begin{cases} (i+2, i+3, i+2, i+2), & 1 \leq i \leq k-1; \\ (k+1, k+2, k+2, k+2), & i = k; \\ (2k-i+1, 2k-i+2, 2k-i+3, 2k-i+3), & k+1 \leq i \leq 2k-1. \end{cases}$$

Representations of the vertices with respect to $W$ on the inner cycles are

$r(v_{1+3i}|W) =$

$$\begin{cases} (i, i+3, i+2, i), & 1 \leq i \leq k; \\ (k, k+3, k+3, k+1), & i = k+1; \\ (2k-i+1, 2k-i+4, 2k-i+5, 2k-i+3), & k+2 \leq i \leq 2k. \end{cases}$$

$r(v_{2+3i}|W) =$

$$\begin{cases} (i+3, i, i+3, i+1), & 1 \leq i \leq k; \\ (k+3, k, k+3, k+2), & i = k+1; \\ (2k-i+4, 2k-i+1, 2k-i+4, 2k-i+4), & k+2 \leq i \leq 2k. \end{cases}$$

$r(v_{3+3i}|W) =$

$$\begin{cases} (i+4, i+3, i, i+2), & 1 \leq i \leq k-1; \\ (k+3, k+3, k, k+2), & i = k; \\ (2k-i+3, 2k-i+4, 2k-i+1, 2k-i+3), & k+1 \leq i \leq 2k. \end{cases}$$

**Case (iv)** $n = 6k+4, k \in \mathbf{Z}^+$. For every $n \geq 10$, the representations of the vertices on the outer cycle are

$r(u_1|W) = (1,2,3,3)$, $r(u_2|W) = (2,1,2,2)$, $r(u_3|W) = (3,2,1,1)$.

$r(u_{2+3i}|W) =$

$$\begin{cases} (3,2,3,1), & i = 1; \\ (i+2, i+1, i+2, i+2), & 2 \leq i \leq k; \\ (2k-i+2, 2k-i+3, 2k-i+4, 2k-i+4), & k+1 \leq i \leq 2k. \end{cases}$$

$r(u_{3+3i}|W) =$

$$\begin{cases} (4,3,2,2), & i = 1; \\ (i+3, i+2, i+1, i+3), & 2 \leq i \leq k; \\ (2k-i+3, 2k-i+2, 2k-i+3, 2k-i+5), & k+1 \leq i \leq 2k. \end{cases}$$

$r(u_{4+3i}|W) =$

$$\begin{cases} (i+2, i+3, i+2, i+2), & 1 \leq i \leq k; \\ (2k-i+2, 2k-i+3, 2k-i+2, 2k-i+4), & k+1 \leq i \leq 2k. \end{cases}$$

Representations of vertices on inner cycle are

$r(v_{1+3i}|W) =$

$$\begin{cases} (i, i+3, i+2, i), & 1 \leq i \leq k; \\ (k+1, k+4, k+1, k+1), & i = k+1; \\ (2k-i+4, 2k-i+5, 2k-i+2, 2k-i+4), & k+2 \leq i \leq 2k+1. \end{cases}$$

$r(v_{2+3i}|W) =$

$$\begin{cases} (i+3, i, i+3, i+1), & 1 \leq i \leq k-1; \\ (k+1, k, k+3, k+1), & i = k; \\ (k, k+1, k+4, k+2), & i = k+1; \\ (2k-i+1, 2k-i+4, 2k-i+5, 2k-i+3), & k+2 \leq i \leq 2k. \end{cases}$$

$r(v_{3+3i}|W) =$

$$\begin{cases} (i+4, i+3, i, i+2), & 1 \leq i \leq k-1; \\ (k+4, k+1, k, k+2), & i = k; \\ (k+3, k, k+1, k+3), & i = k+1; \\ (2k-i+4, 2k-i+1, 2k-i+4, 2k-i+4), & k+2 \leq i \leq 2k. \end{cases}$$

**Case (v)** $n = 6k+5, k \in \mathbf{Z}^+$. For every $n \geq 17$, the representations of vertices on the outer cycle are

$r(u_1|W) = (1, 2, 3, 3)$, $r(u_2|W) = (2, 1, 2, 2)$, $r(u_3|W) = (3, 2, 1, 1)$.

$r(u_{2+3i}|W) =$

$$\begin{cases} (3, 2, 3, 1), & i = 1; \\ (i+2, i+1, i+2, i+2), & 2 \leq i \leq k; \\ (k+2, k+2, k+2, k+3), & i = k+1; \\ (2k-i+3, 2k-i+4, 2k-i+3, 2k-i+5), & k+2 \leq i \leq 2k+1. \end{cases}$$

$r(u_{3+3i}|W) =$

$$\begin{cases} (4,3,2,2), & i=1; \\ (i+3,i+2,i+1,i+3), & 2 \le i \le k-1; \\ (k+2,k+2,k+1,k+3), & i=k; \\ (k+1,k+2,k+2,k+3), & i=k+1; \\ (2k-i+2,2k-i+3,2k-i+4,2k-i+4), & k+2 \le i \le 2k. \end{cases}$$

$r(u_{4+3i}|W) =$

$$\begin{cases} (i+2,i+3,i+2,i+2), & 1 \le i \le k-1; \\ (k+2,k+2,k+2,k+2), & i=k; \\ (k+2,k+1,k+2,k+3), & i=k+1; \\ (2k-i+3,2k-i+2,2k-i+3,2k-i+5), & k+2 \le i \le 2k. \end{cases}$$

Representations of vertices with respect to $W$ on inner cycle are

$r(v_{1+3i}|W) =$

$$\begin{cases} (i,i+3,i+2,i), & 1 \le i \le k-1; \\ (k,k+2,k+2,k), & i=k; \\ (k+1,k+1,k+3,k+1), & i=k+1; \\ (k+2,k,k+3,k+2), & i=k+2; \\ (2k-i+5,2k-i+2,2k-i+5,2k-i+5), & k+3 \le i \le 2k+1. \end{cases}$$

$r(v_{2+3i}|W) =$

$$\begin{cases} (i+3, i, i+3, i+1), & 1 \le i \le k-1; \\ (k+3, k, k+2, k+1), & i = k; \\ (k+3, k+1, k+1, k+2), & i = k+1; \\ (k+2, k+2, k, k+2), & i = k+2; \\ (2k-i+4, 2k-i+5, 2k-i+2, 2k-i+4), & k+3 \le i \le 2k+1. \end{cases}$$

$r(v_{3+3i}|W) =$

$$\begin{cases} (i+4, i+3, i, i+2), & 1 \le i \le k-2; \\ (k+2, k+2, k-1, k+1), & i = k-1; \\ (k+1, k+3, k, k+2), & i = k; \\ (k, k+3, k+1, k+2), & i = k+1; \\ (k-1, k+2, k+2, k+1), & i = k+2; \\ (2k-i+1, 2k-i+4, 2k-i+5, 2k-i+3), & k+3 \le i \le 2k. \end{cases}$$

We note that there are no two vertices in the inner cycle(s) with same representations. Also, there are no two vertices in the inner cycle(s) and outer cycle having the same representations and no two vertices on outer cycle having the same representations. This implies that $W = \{v_1, v_2, v_3, u_4\}$ is a resolving set for $V(P(n,3))$ when $n \equiv 0$, 3, 4, 5(mod 6) implying that in these cases $dim(P(n,3)) \le 4$. Also $W = \{v_1, v_{3k-1}, v_{6k}\}$ is a resolving set for $n = 6k+1$, when $dim(P(n,3)) \le 3$.

**Case (vi)** $n = 6k+2, k \in \mathbf{Z}^+$. It is straightforward to verify that $W_1 = \{v_1, v_2, v_3, u_4\}$ and $W_2 = \{v_1, v_2, u_7, u_{10}\}$ are resolving sets for $P(8,3)$ and $P(14,3)$, respectively. For every $n \ge 20$, we show that $W = \{v_1, v_2, v_3, u_4, u_{3k+5}\}$ is a resolving set. For this, we first give representations of vertices with respect to $W' = \{v_1, v_2, v_3, u_4\}$. The representations of the vertices on the outer cycle are

$r(u_1|W') = (1,2,3,3)$, $r(u_2|W') = (2,1,2,2)$, $r(u_3|W') = (3,2,1,1)$.

$r(u_{2+3i}|W') =$

$$\begin{cases} (3,2,3,1), & i=1; \\ (i+2, i+1, i+2, i+2), & 2 \le i \le k; \\ (2k-i+2, 2k-i+3, 2k-i+2, 2k-i+4), & k+1 \le i \le 2k. \end{cases}$$

$r(u_{3+3i}|W') =$

$$\begin{cases} (4,3,2,2), & i=1; \\ (i+3, i+2, i+1, i+3), & 2 \le i \le k-1; \\ (k+1, k+2, k+1, k+3), & i=k; \\ (2k-i+1, 2k-i+2, 2k-i+3, 2k-i+3), & k+1 \le i \le 2k-1. \end{cases}$$

$r(u_{4+3i}|W') =$

$$\begin{cases} (i+2, i+3, i+2, i+2), & 1 \le i \le k-1; \\ (k+2, k+1, k+2, k+2), & i=k; \\ (2k-i+2, 2k-i+1, 2k-i+2, 2k-i+4), & k+1 \le i \le 2k-1. \end{cases}$$

Representations of vertices with respect to $W'$ on inner cycle are

$r(v_{1+3i}|W') =$

$$\begin{cases} (i, i+3, i+2, i), & 1 \le i \le k-1; \\ (k, k+1, k+2, k), & i=k; \\ (k+1, k, k+3, k+1), & i=k+1; \\ (2k-i+4, 2k-i+1, 2k-i+4, 2k-i+4), & k+2 \le i \le 2k. \end{cases}$$

$r(v_{2+3i}|W') =$

$$\begin{cases} (i+3, i, i+3, i+1), & 1 \leq i \leq k-1; \\ (k+3, k, k+1, k+1), & i = k; \\ (k+2, k+1, k, k+2), & i = k+1; \\ (2k-i+3, 2k-i+4, 2k-i+1, 2k-i+3), & k+2 \leq i \leq 2k. \end{cases}$$

$r(v_{3+3i}|W') =$

$$\begin{cases} (i+4, i+3, i, i+2), & 1 \leq i \leq k-2; \\ (k+1, k+2, k-1, k+1), & i = k-1; \\ (k, k+3, k, k+2), & i = k; \\ (k-1, k+2, k+1, k+1), & i = k+1; \\ (2k-i, 2k-i+3, 2k-i+4, 2k-i+2), & k+2 \leq i \leq 2k-1. \end{cases}$$

Consequently, $r(u_{2+3k}|W') = r(u_{4+3k}|W') = (k+2, k+1, k+2, k+2)$;
$r(u_{5+3k}|W') = r(u_{3+3k}|W') = (k+1, k+2, k+1, k+3)$;
$r(u_{3k}|W') = r(v_{5+3k}|W') = (k+2, k+1, k, k+2)$;
$r(v_{8+3k}|W') = r(v_{3k}|W') = (k+1, k+2, k-1, k+1)$.
The vertex $u_{3k+5}$ distinguishes these pairs of vertices with same representations as $d(u_{3k+5}, u_{3k+2}) = 3$, $d(u_{3k+5}, u_{3k+4}) = 1$, $d(u_{3k+5}, u_{3k}) = 5$, $d(u_{3k+5}, v_{3k+5}) = 1$, $d(u_{3k+5}, v_{3k+8}) = 2$ and $d(u_{3k+5}, v_{3k}) = 4$. This suggests that $W = \{v_1, v_2, v_3, u_4, u_{3k+5}\}$ is a resolving set for $V(P(n,3))$ in this case implying that $dim(P(n,3)) \leq 5$. □

## 4.3 Metric dimension of $P(n,3)$ for $n \equiv 0, 1 \pmod 6$

In this section we will prove that $dim(P(n,3)) \geq 3$ for $n \equiv 1 \pmod 6$ and $n \geq 25$ and $dim(P(n,3)) \geq 4$ for $n \equiv 0 \pmod 6$ and $n \geq 24$, yielding exact values of $dim(P(n,3))$ in these cases by Theorem 4.2.1. For this purpose we need some more notations and definitions. Without loss of generality we can suppose that the vertices of the outer cycle are $u_1, u_2, \ldots, u_n$ in the clockwise direction. For two vertices $u_i$ and $u_j$ $(i \neq j)$ we shall define the "clockwise distance" from $u_i$ to $u_j$, denoted by $d^*(u_i, u_j)$ the distance, measured in clockwise direction, from $u_i$ to $u_j$, in the subgraph induced by the outer cycle. For example, $d^*(u_1, u_n) = n - 1$ and $d^*(u_n, u_1) = 1$; in general we have $d^*(u_i, u_j) + d^*(u_j, u_i) = n$. This definition can be extended to any two vertices of $P(n, 3)$ for $i \neq j$ by: $d^*(u_i, v_j) = d^*(v_i, u_j) = d^*(v_i, v_j) = d^*(u_i, u_j)$.

Consider a vertex on the outer cycle, say $u_1$. A vertex $u_i$ is called a good vertex for $u_1$ if $u_i$ and $u_{i+2}$ have equal distances to $u_1$, i. e., $d(u_1, u_i) = d(u_1, u_{i+2})$; otherwise $u_i$ is called a bad vertex for $u_1$. This definition can be extended to vertices of the inner cycle: $u_i$ is a good vertex for $v_1$ if $d(v_1, u_i) = d(v_1, u_{i+2})$ and bad otherwise.

In figure 4.1 we have represented by black dots all good vertices for $u_1$ when $n = 6k + 1 \geq 25$.

It is important to note that the set of good vertices for $v_1$ is deduced from the set of good vertices for $u_1$ by adding 4 new vertices, namely $u_2, u_3, u_{6k-1}$ and $u_{6k-2}$. Similarly, a vertex $u_j$ is said to be good for the pair $\{u_1, u_i\}$ if $d(u_1, u_j) = d(u_1, u_{j+2})$ and $d(u_i, u_j) = d(u_i, u_{j+2})$. If $u_k$ is a good vertex for the pairs $\{u_1, u_i\}$ and $\{u_1, u_j\}$ then $u_k$ is also a good vertex for the triplet $\{u_1, u_i, u_j\}$, i. e., $d(u_1, u_k) = d(u_1, u_{k+2})$, $d(u_i, u_k) = d(u_i, u_{k+2})$ and $d(u_j, u_k) = d(u_j, u_{k+2})$.

Due to the rotational symmetry of $P(n, 3)$ we deduce

**Lemma 4.3.1.** *For any two vertices $u_i$ and $u_j$ on the outer cycle of $P(n, 3)$ we have $d(u_i, u_j) = d(u_{i+r}, u_{j+r})$ for any $1 \leq r \leq n - 1$.*

In order to find good vertices for pairs of vertices belonging to the outer cycle the following lemma will be useful.

**Lemma 4.3.2.** *Let $1 \leq i \leq n - 2$. If $u_j$ is good for $u_1$ and $u_{j-i}$ is also good for $u_1$,*

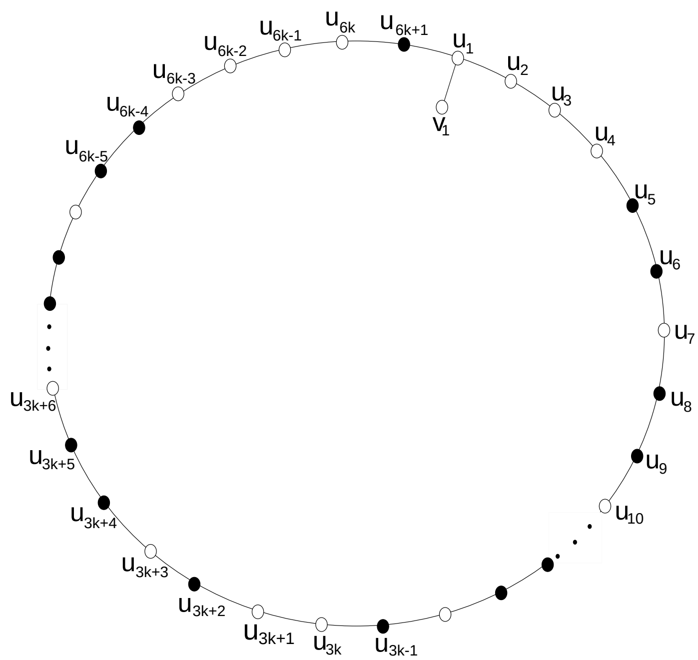


Figure 4.1: Good vertices for $u_1(n = 6k + 1)$

*then* $u_j$ *is good for the pair* $\{u_1, u_{i+1}\}$.

**Proof:** By hypothesis we can write $d(u_1, u_j) = d(u_1, u_{j+2})$ and $d(u_1, u_{j-i}) = d(u_1, u_{j-i+2})$. By Lemma 4.3.1 the last equality is equivalent to $d(u_{i+1}, u_j) = d(u_{i+1}, u_{j+2})$. □

**Theorem 4.3.3.** *$dim(P(n, 3)) = 3$ if $n = 6k + 1$ and $n \geq 25$.*

**Proof:** We shall prove that $dim(P(n, 3)) \geq 3$ in this case, by showing that there is no resolving set of $V(P(n, 3))$ consisting of two vertices, $X$ and $Y$. If both vertices $X$ and $Y$ belong to the outer cycle, we can suppose that $X = u_1$. We distinguish three cases:

1)$d^*(u_1, Y) \equiv 0 \pmod 3$. We choose vertex $u_{6k-4}$. Since vertices $u_{6k-7}$, $u_{6k-10}, \ldots, u_{3k+5}, u_{3k+2}, u_{3k-1}, u_{3k-4}, \ldots, u_8, u_5$ are good vertices for $u_1$, but $u_2, u_{6k}$

and $u_{6k-3}$ are bad vertices for $u_1$ (see fig. 4.1), applying Lemma 4.3.2 we find that $u_{6k-4}$ is a good vertex for any pair $\{u_1, Y\}$ such that $Y \notin \{u_{6k-5}, u_{6k-2}, u_{6k+1}\}$.
But in this case $u_5$ is a good vertex for the pairs $\{u_1, u_{6k-2}\}$ and $\{u_1, u_{6k+1}\}$ if $k \geq 4$ and for $\{u_1, u_{6k-5}\}$ if $k \geq 5$; for $k = 4$ we have that $u_{3k} = u_{12}$ is not a good vertex for $u_1$. It remains to consider the pair $\{u_1, u_{6k-5}\} = \{u_1, u_{19}\}$ when $k = 4$. In this case $u_9$ is a good vertex for this pair. It follows that any pair $\{u_1, Y\}$ cannot be a resolving set having two vertices.

2)$d^*(u_1, Y) \equiv 1 (\text{mod } 3)$. We consider vertex $u_5$. By starting from $u_5$ and going in the counter-clockwise direction at distances 1,4,7,... the only bad vertices encountered are $u_4$, $u_1$ and $u_{6k-1}$. By Lemma 4.3.2 we deduce that $u_5$ is a good vertex for any pair $\{u_1, Y\}$ such that $Y \notin \{u_2, u_5, u_8\}$. In a similar manner we get that $u_{6k-4}$ is a good vertex for the pairs $\{u_1, u_2\}$ and $\{u_1, u_5\}$ if $k \geq 4$ and for $\{u_1, u_8\}$ if $k \geq 5$. For $k = 4$ the vertex $u_{16}$ is a good vertex for $\{u_1, u_8\}$.

3)$d^*(u_1, Y) \equiv 2 (\text{mod } 3)$. In this case in order to minimize the number of bad vertices for the pairs $\{u_1, Y\}$ we choose vertex $u_6$. This vertex is a good vertex for any pair $\{u_1, Y\}$ such that $Y \notin \{u_3, u_6, u_9\}$. Vertex $u_{6k-5}$ is a good vertex for all pairs $\{u_1, u_3\}$, $\{u_1, u_6\}$ and $\{u_1, u_9\}$ for any $k \geq 4$.
If both $X$ and $Y$ belong to the inner cycle we can consider that $X = v_1$ and $Y = v_i$ ($i > 1$); this case can be reduced to the case when $X = u_1$ and $Y = u_i$ since the set of good vertices for $v_p$ includes the set of good vertices for $u_p$ for every $1 \leq p \leq n$.
If $X = u_i$ and $Y = v_i$ then any good vertex for $u_i$ is also a good vertex for $v_i$, hence for the pair $\{X, Y\}$. The remaining case when $X = u_i, Y = v_j$ and $i \neq j$ can also be reduced to the case $X = u_i, Y = u_j$. It follows that there is no resolving set containing two vertices, which concludes the proof. □

**Theorem 4.3.4.** *If $n = 6k$ and $n \geq 24$ then $dim(P(n, 3)) = 4$.*

**Proof:** By Theorem 4.2.1 it is necessary only to show that $dim(P(n,3)) \geq 4$, or that there is no resolving set of $V(P(n,3))$ consisting of three vertices, $X, Y$ and $Z$. By the same reasoning as in the proof of Theorem 4.3.1 it is sufficient to consider only the case when $X, Y, Z$ belong to the outer cycle since the set of good vertices for $v_1$ can be deduced from the set of good vertices for $u_1$ (represented in figure 4.2) by adding vertices $u_2, u_3, u_{6k-2}$ and $u_{6k-3}$. As in the case $n = 6k+1$ we shall see that for any three vertices $X, Y, Z$ such that $d^*(X, Y) < d^*(X, Z)$ it is possible to find a pair of vertices at distance 2 on the outer cycle, $\{u_i, u_{i+2}\}$ having equal distances to $X, Y$ and $Z$, respectively. If $n = 6k$ and $X, Y, Z$ are on the outer cycle, we can suppose that $X = u_1$. By denoting $(x, y) \equiv (a, b)(\text{mod } 3)$ if $x \equiv a(\text{mod } 3)$ and $y \equiv b(\text{mod } 3)$, the following 9 cases can occur: $(d^*(u_1, Y), d^*(u_1, Z))$ is congruent modulo 3 to:
1)(0,0); 2)(1,1); 3)(2,2); 4)(0,1); 5)(0,2); 6)(1,0); 7)(1,2); 8)(2,0); 9)(2,1).

Some of these cases can be reduced to another cases. For example, from case 2 by permutation $X \to Y, Y \to Z, Z \to X$ we obtain case 5 and by permutation $X \to Z, Y \to X, Z \to Y$ we get case 8.
The graph of reducibility between cases is illustrated in figure 4.3.

It follows that it is sufficient to consider only cases 1,2,3,7,9.

*Case* 1. If we choose good vertex $u_6$ and we go in the counter-clockwise direction, reaching vertices $u_3, u_{6k}, u_{6k-6}, \ldots, u_9$ we encounter only two bad vertices, $u_3$ and $u_{6k-3}$ (see figure 4.2). By Lemma 3.2 it follows that $u_6$ is a good vertex for all pairs $\{u_1, Y\}$ where $d^*(u_1, Y) \equiv 0(\text{mod } 3)$ and $Y \notin \{u_4, u_{10}\}$. This implies that $u_6$ is a good vertex for all triplets $\{u_1, Y, Z\}$, unless $Y = u_4$ and $Z \in \{u_7, u_{10}, u_{13}, \ldots, u_{6k-2}\}$; $Y = u_{10}$ and $Z \in \{u_7, u_{13}, u_{16}, \ldots, u_{6k-2}\}$. For these triplets we must find other good

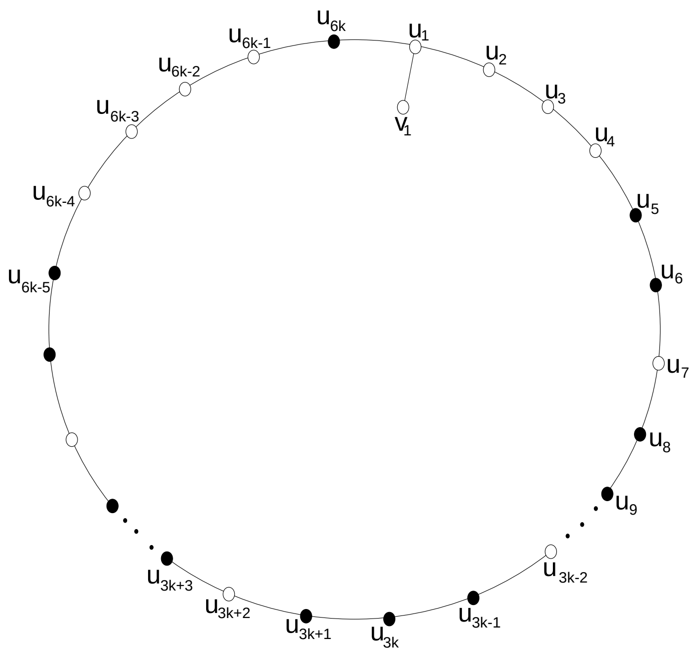


Figure 4.2: Good vertices for $u_1$ ($n = 6k$)

vertices on the outer cycle. Similarly, $u_{3k+6}$ is a good vertex for $u_1$ since $6k - 6 \geq 3k + 6$ and for all pairs $\{u_1, Y\}$ where $d^*(u_1, Y) \equiv 0 (\text{mod } 3)$ and $Y \notin \{u_{3k+4}, u_{3k+10}\}$. Consequently, we have found a good vertex ($u_6$ or $u_{3k+6}$) for all triplets $\{u_1, Y, Z\}$ such that $\{Y, Z\} \neq \{u_4, u_{3k+4}\}, \{u_4, u_{3k+10}\}$,
$\{u_{10}, u_{3k+4}\}, \{u_{10}, u_{3k+10}\}$. Finally, $u_9$ is a good vertex for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 0 (\text{mod } 3)$ and $Y \notin \{u_7, u_{13}\}$. Since $k \geq 4$ we have $3k + 4 > 13$ and $u_9$ is a good vertex for the remaining triplets $\{u_1, Y, Z\}$, where $Y \in \{u_4, u_{10}\}$ and $Z \in \{u_{3k+4}, u_{3k+10}\}$.

*Case* 2. In a similar way we get that $u_{6k-5}$ is a good vertex for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 1 (\text{mod } 3)$ and $Y \notin \{u_{6k-7}, u_{6k-1}\}$, therefore $u_{6k-5}$ is a good vertex for all triplets $\{u_1, Y, Z\}$, unless $Y = u_{6k-7}$ and $Z \in \{u_2, u_5, \ldots, u_{6k-10}, u_{6k-4}, u_{6k-1}\}$;

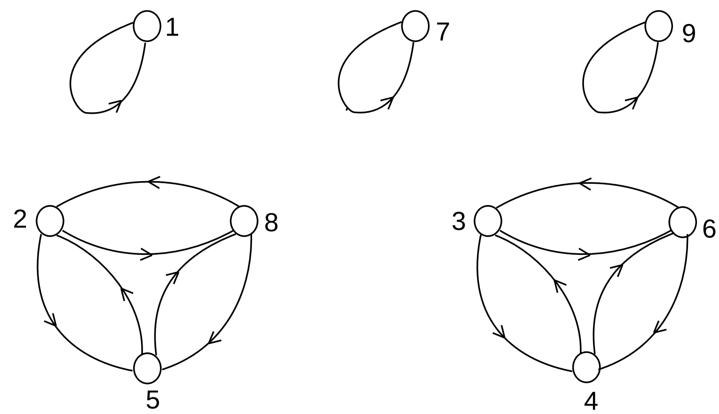


Figure 4.3: Reducibility between cases

$Y = u_{6k-1}$ and $Z \in \{u_2, u_5, \ldots, u_{6k-10}, u_{6k-4}\}$. Since $6k - 8 > 3k - 2$ it follows that $u_{6k-8}$ is a good vertex for $u_1$ and for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 1(\text{mod } 3)$ and $Y \notin \{u_{6k-10}, u_{6k-4}\}$. We have found a good vertex ($u_{6k-5}$ or $u_{6k-8}$) for all triplets $\{u_1, Y, Z\}$ such that $\{Y, Z\} \neq \{u_{6k-7}, u_{6k-10}\}$, $\{u_{6k-7}, u_{6k-4}\}$,
$\{u_{6k-1}, u_{6k-10}\}, \{u_{6k-1}, u_{6k-4}\}$.
Since $k \geq 4$ we find for triplets $\{u_1,\ u_{6k-7},\ u_{6k-10}\}, \{u_1,\ u_{6k-7},\ u_{6k-4}\}$,
$\{u_1,\ u_{6k-1},\ u_{6k-10}\}$ and $\{u_1, u_{6k-1},\ u_{6k-4}\}$ good vertices $u_{3k-4}$, $u_{3k-4}$,
$u_{6k-11}$ and $u_{3k-1}$, respectively (e. g., using Lemma 4.3.2).

*Case* 3. We deduce that $u_5$ is a good vertex for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 2(\text{mod } 3)$ and $Y \notin \{u_3, u_9\}$. It follows that $u_5$ is a good vertex for all triplets $\{u_1, Y, Z\}$, unless $Y = u_3$ and $Z \in \{u_6, u_9, \ldots, u_{6k}\}$; $Y = u_9$ and $Z \in \{u_6, u_{12}, u_{15}, \ldots, u_{6k}\}$. Also $u_{3k-1}$ is a good vertex for $u_1$ and for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 2(\text{mod } 3)$ and $Y \notin \{u_{3k-3}, u_{3k+3}\}$. It follows that there exists a good vertex ($u_5$ or $u_{3k-1}$) for all triplets $\{u_1, Y, Z\}$ such that $\{Y, Z\} \neq \{u_3, u_{3k-3}\}$, $\{u_3, u_{3k+3}\}$, $\{u_9, u_{3k-3}\}$,

$\{u_9, u_{3k+3}\}$ (note that for $k = 4$ the third triplet must be eliminated from the list). Now $u_8$ is a good vertex for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 2(\text{mod } 3)$ and $Y \notin \{u_6, u_{12}\}$. It follows that $u_8$ is a good vertex for all four remaining triplets, except $\{u_1, u_3, u_{12}\}$ for $k = 5$ and $n = 30$. For this last triplet $u_{3k+1} = u_{16}$ is a good vertex.

*Cases* 7 and 9 can be reduced to the case 7 without imposing any inequality between the distances $d^*(X, Y)$ and $d^*(X, Z)$.

Let $A = \{u_2, u_5, u_8, \ldots, u_{6k-1}\}$ and $B = \{u_3, u_6, u_9, \ldots, u_{6k}\}$. It is necessary to prove that for any triplet $\{u_1, Y, Z\}$, where $Y \in A$ and $Z \in B$ there is a good vertex on the outer cycle. From the previous case we have seen that $u_5$ is a bad vertex for pairs $\{u_1, Z\}$, where $Z \in B$ if and only if $Z \in \{u_3, u_9\}$ and a bad vertex for pairs $\{u_1, Y\}$, where $Y \in A$ if and only if $Y \in \{u_2, u_5, u_8, u_{3k+8}, u_{3k+11}, \ldots, u_{6k-1}\}$. It follows that $u_5$ is a good vertex for all triplets $\{u_1, Y, Z\}$, where $Y \in A$ and $Z \in B$, unless: $Y \in A$ and $Z \in \{u_3, u_9\}$; $Y \in \{u_2, u_5, u_8, u_{3k+8}, u_{3k+11}, \ldots, u_{6k-1}\}$ and $Z \in B$ (these sets of pairs $\{Y, Z\}$ will be denoted by $\alpha$ and $\beta$, respectively). For the remaining triplets $\{u_1, Y, Z\}$, where $\{Y, Z\} \in \alpha \cup \beta$ we must find other good vertices on the outer cycle. Consider now vertex $u_{3k+1}$. This vertex is a good vertex for all pairs $\{u_1, Y\}$, where $Y \in A \backslash \{u_{3k-1}, u_{3k+5}\}$. Since $3k - 7 \geq 5$ it follows that $u_{3k+1}$ is also a good vertex for all pairs $\{u_1, Z\}$, where $Z \in \{u_3, u_9\}$. Therefore the set $\alpha$ is reduced to the set $\alpha_1$ of pairs $\{Y, Z\}$ such that $Y \in \{u_{3k-1}, u_{3k+5}\}$ and $Z \in \{u_3, u_9\}$. Now $u_8$ is a good vertex for all pairs $\{u_1, Y\}$, where $Y \in \{u_{14}, u_{17}, \ldots, u_{3k+8}\}$ and all pairs $\{u_1, Z\}$, where $Z \in \{u_3, u_9, u_{15}, u_{18}, \ldots, u_{6k}\}$. Since $3k - 1 \geq 14$ for $k \geq 5$ it follows that $u_8$ is a good vertex for all pairs in $\alpha_1$ if $k \geq 5$ and for $k = 4$ we must consider only the pairs $\{u_{11}, u_3\}$ and $\{u_{11}, u_9\}$. From figure 2 we deduce that $u_{3k+3} = u_{15}$ is a good vertex for $\{u_1, u_{11}, u_3\}$ and $u_{3k+4} = u_{16}$ is a good vertex for $\{u_1, u_{11}, u_9\}$.

It remains to find good vertices for pairs in $\beta$. Since $u_{6k-5}$ is a good vertex for all pairs $\{u_1, Y\}$, where $Y \in A \backslash \{u_{6k-7}, u_{6k-1}\}$ and a good vertex for all pairs $\{u_1, Z\}$, where $Z \in \{u_{3k-3}, u_{3k}, \ldots, u_{6k-9}\}$, $\beta$ is reduced to $\gamma \cup \delta$, where $\gamma$ consists of all pairs $\{Y, Z\}$ with $Y \in \{u_{6k-7}, u_{6k-1}\}$ and $Z \in B$ and $\delta$ of $\{Y, Z\}$ with $Y \in \{u_2, u_5, u_8, u_{3k+8}, u_{3k+11}, \ldots, u_{6k-10}, u_{6k-4}\}$ and $Z \in \{u_3, u_6, \ldots, u_{3k-6}, u_{6k-6}, u_{6k-3}, u_{6k}\}$.

$u_{3k-1}$ is a good vertex for all pairs $\{u_1, Y\}$, where $Y \in \{u_{3k+5}, u_{3k+8}, \ldots,$
$u_{6k-1}\}$ and all pairs $\{u_1, Z\}$, where $Z \in B \backslash \{u_{3k-3}, u_{3k+3}\}$. Therefore $\gamma$ is reduced to $\varepsilon$, consisting of all pairs $\{Y, Z\}$ such that $Y \in \{u_{6k-7}, u_{6k-1}\}$ and $Z \in \{u_{3k-3}, u_{3k+3}\}$ and $\delta$ is reduced to $\mu$, which consists of $\{Y, Z\}$ with $Y \in \{u_2, u_5, u_8\}$ and $Z \in \{u_3, u_6, u_{3k-6}, u_{6k-6}, u_{6k-3}, u_{6k}\}$.

$u_6$ is a good vertex for all pairs $\{u_1, Y\}$, where $Y \in \{u_2, u_{3k+8}, u_{3k+11}, \ldots,$
$u_{6k-1}\}$ and all pairs $\{u_1, Z\}$, where $Z \in \{u_{12}, u_{15}, \ldots, u_{3k+6}\}$.

If $k \geq 5$ then $6k-7 \geq 3k+8$ and $3k-3 \geq 12$; therefore $u_6$ is a good vertex for all triples $\{u_1, Y, Z\}$, where $\{Y, Z\} \in \varepsilon$. If $k = 4$ the pairs in $\varepsilon$ are $\{u_{17}, u_9\}, \{u_{17}, u_{15}\}, \{u_{23}, u_9\}$ and $\{u_{23}, u_{15}\}$; good vertices for these pairs are vertices $u_{24}, u_5, u_{16}$ and $u_9$, respectively, which also are good vertices for $u_1$.

It remains to study the pairs from $\mu$. $u_{3k}$ is a good vertex for all pairs $\{u_1, Y\}$, where $Y \in \{u_2, u_5, \ldots, u_{3k-4}\}$ and all pairs $\{u_1, Z\}$, where $Z \in \{u_{3k+6}, u_{3k+9}, \ldots, u_{6k}\}$. Since $3k - 4 \geq 8$ and $6k - 6 \geq 3k + 6$ it follows that $\mu$ is reduced to the set $\nu$ of pairs $\{Y, Z\}$, where $Y \in \{u_2, u_5, u_8\}$ and $Z \in \{u_3, u_6, u_{3k-6}\}$.

Finally, $u_{3k+1}$ is a good vertex for all pairs $\{u_1, Y\}$, where
$Y \in A \backslash \{u_{3k-1}, u_{3k+5}\}$ and for all pairs $\{u_1, Z\}$, where
$Z \in \{u_3, u_6, \ldots, u_{3k-3}\}$.

Consequently, $u_{3k+1}$ is a good vertex for $u_1$ and all pairs from $\nu$. □

# Chapter 5

# Metric dimension and $d$-sets of connected graphs

A $d$-set relatively to a pair of distinct vertices of a connected graph $G$ is the set of vertices having different distances from these vertices.

In this chapter, it is shown that for a connected graph $G$ of order $n$ and diameter 2 the number of pairs such that their $d$-sets are equal to $V(G)$ is bounded above by $\lfloor n^2/4 \rfloor$ and it is conjectured that this holds for any connected graph.

A lower bound for the metric dimension of $G$, $dim(G)$ is proposed in terms of a family of $d$-sets of $G$ having the property that every subfamily containing at least $r \geq 2$ members has an empty intersection.

Three sufficient conditions which guarantee that a family $\mathcal{F} = (G_n)_{n\geq 1}$ of graphs with unbounded order has an unbounded metric dimension are also proposed.

Finally, $d$-sets are used to show that $dim(Ne_n) = 3$ when $n$ is odd and 2 otherwise, where $Ne_n$ is the necklace graph of order $2n + 2$.

## 5.1 Notation and preliminary results

Let $G$ be a connected graph. The distance $d(u,v)$ between two vertices $u,v \in V(G)$ is the length of a shortest path between them and the diameter of $G$, denoted by $diam(G)$ is $\max_{u,v \in V(G)} d(u,v)$.

For a pair of distinct vertices of $G$, $p = \{x,y\}$ we shall denote by $D(p)$ or $D(x,y)$ the set of vertices $z \in V(G)$ such that $d(z,x) \neq d(z,y)$. Such a set will be called a distinguishing set (or a $d$-set) relatively to the pair $\{x,y\}$.

It is clear that $\{x,y\} \subset D(x,y) \subset V(G)$ for any pair $\{x,y\}$. Some properties of $d$-pairs will be shown in the next section.

The metric dimension of a connected graph $G$ has been defined in [19]. An equivalent definition is the following [9]: Let $W = \{w_1, w_2, ..., w_k\}$ be an ordered set of vertices of $G$ and let $v$ be a vertex of $G$. The representation $r(v|W)$ of $v$ with respect to $W$ is the $k$-tuple $(d(v,w_1), d(v,w_2), ..., d(v,w_k))$. If distinct vertices of $G$ have distinct representations with respect to $W$, then $W$ is called a resolving set for $G$. It is clear that for any pair of distinct vertices $\{x,y\}$ of $G$ there exists a vertex $w_i \in W$ such that $d(x,w_i) \neq d(y,w_i)$, hence $D(x,y) \cap W \neq \emptyset$ for any resolving set $W$.

A resolving set of minimum cardinality is called a basis for $G$ and the number of elements in a basis is the metric dimension of $G$, denoted by $dim(G)$. The problem of determining whether $dim(G) < k$ is an $NP$-complete problem [18].

The property of a given set $W \subset V(G)$ to be resolving set of $G$ can be verified only for vertices from $V(G) \backslash W$, since every vertex $w \in W$ is the only vertex of $G$ whose distance from $W$ is 0.

Let $\mathcal{F}$ be a family of connected graphs $G_n : \mathcal{F} = (G_n)_{n \geq 1}$ depending on $n$ as follows: the order $|V(G)| = \varphi(n)$ and $\lim_{n \to \infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $dim(G) \leq C$ for every $n \geq 1$ then we shall say that $\mathcal{F}$ has bounded metric dimension; otherwise $\mathcal{F}$ has unbounded metric dimension.

If all graphs in $\mathcal{F}$ have the same metric dimension (which does not depend on $n$), $\mathcal{F}$ is called a family with constant metric dimension [24]. A connected graph $G$ has $dim(G) = 1$ if and only if $G$ is a path [9]; cycles $C_n$ have metric dimension 2 for every $n \geq 3$. Also generalized Petersen graphs $P(n, 2)$, antiprisms $A_n$ and Harary graphs $H_{4,n}$ are families of graphs with constant metric dimension [24].

Other families of graphs have unbounded metric dimension: if $W_n$ denotes a wheel with $n$ spokes and $J_{2n}$ the graph deduced from the wheel $W_{2n}$ by alternately deleting $n$ spokes, then $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for every $n \geq 7$ [4] and $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$ [39] for every $n \geq 4$.

An example of a family which has bounded metric dimension is the family of necklaces $(Ne_n)_{n \geq 1}$. The necklace graph, denoted by $Ne_n$ [38] is a cubic Halin graph [30] obtained by joining by a cycle all vertices of degree 1 of a caterpillar (also called comb) having $n$ vertices of degree 3 and $n + 2$ vertices of degree 1, denoted by $u_1, u_2, ..., u_n$ and $v_0, v_1, ..., v_{n+1}$, respectively (see fig. 5.2).

The metric dimension $dim(Ne_n)$ is bounded but not constant and it depends on the parity of $n$. This will be shown in the last section.

The clique number of a graph $G$, denoted by $\omega(G)$, is the maximum number of vertices in a complete subgraph of $G$ and $N(v)$ denotes the set of vertices adjacent with $v$.

## 5.2 Properties of $d$-sets

The following lemma is based on a simple observation: if for a pair of distinct vertices $\{x, y\}$ the distance $d(x, y)$ is even, the middle vertex $v$ of a shortest path between $x$

and $y$ has equal distances to $x$ and to $y$, hence $v \notin D(x, y)$.

**Lemma 5.2.1.** *If $D(x, y) = V(G)$ for a pair $\{x, y\}$ of distinct vertices of a connected graph $G$ then $d(x, y)$ is odd.*

**Lemma 5.2.2.** *If $D(x, y) = \{x, y\}$ for any pair of distinct vertices of a connected graph $G$ then $G$ is a complete graph.*

*Proof.* Let $G$ be a graph of order $n$ satisfying this property and $u, v \in V(G)$, $u \neq v$. If $d(u, v) = k \geq 2$, let $u, x_1, ..., x_k = v$ be a shortest path between them. We have $d(v, u) = k$ and $d(v, x_1) = k - 1$, which implies $v \in D(u, x_1)$ and contradicts the hypothesis. It follows that $d(u, v) = 1$ for any vertex $v \neq u$, or $d(u) = n - 1$. Since $u$ was an arbitrary vertex it follows that $G$ is a complete graph. □

**Theorem 5.2.3.** *If $G$ has $n$ vertices and $diam(G) = 2$ then the number of pairs $\{x, y\}$ such that $D(x, y) = V(G)$ is bounded above by $\lfloor n^2/4 \rfloor$. This bound is reached only for $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.*

*Proof.* Let $G$ be a diameter 2 graph of order $n$ and $a, b$ be a pair such that $D(a, b) = V(G)$.

By Lemma 5.2.1 it follows that $ab \in E(G)$. Also $N(a) \cap N(b) = \emptyset$ since otherwise every vertex $c \in N(a) \cap N(b)$ does not belong to $D(a, b)$ which contradicts the hypothesis. By denoting $A = N(a)$ and $B = N(b)$, $A \cup B$ is a partition of $V(G)$ since otherwise there exists a vertex $c$ such that $d(a, c) = d(b, c) = 2$, hence $D(a, b) \neq V(G)$, a contradiction.

Since every two distinct vertices $u, v \in A$ have a common neighbor which is $a$, we have $D(u, v) \neq V(G)$; a similar situation occurs for $B$. It follows that the number of pairs $\{x, y\}$ such that $D(x, y) = V(G)$ is bounded above by $|A||B| \leq \lfloor n^2/4 \rfloor$ since

$|A| + |B| = n$.

Equality can hold when $A \cup B$ is an equipartition of $V(G)$, i.e. $-1 \leq |A| - |B| \leq 1$ and vertex sets $A$ and $B$ induce subgraphs consisting of isolated vertices. Indeed, if an edge $uv \in E(G)$ has $u, v \in A$ then we have $d(v, u) = d(v, a) = 1$ and $d(u, v) = d(u, a) = 1$, hence both pairs $\{u, a\}$ and $\{v, a\}$ do not satisfy the required property and the number of pairs is strictly less than $\lfloor n^2/4 \rfloor$.

Because $diam(G) = 2$ we deduce that every vertex of $A$ is adjacent to every vertex of $B$, hence only $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ can have $\lfloor n^2/4 \rfloor$ pairs $\{x, y\}$ such that $D(x, y) = V(G)$. But $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ has this property since every pair of adjacent vertices $\{x, y\}$ has $D(x, y) = V(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil})$. □

If a graph $G$ of order $n$ has $diam(G) = n - 1$ then $G$ is a path $P_n$ and every pair of vertices $\{x, y\}$ of $P_n$ such that $d(x, y)$ is odd satisfies $D(x, y) = V(P_n)$; the number of such pairs equals $\lfloor n^2/4 \rfloor$.

By joining by an edge the centers of the stars $K_{1, \lfloor n/2 \rfloor - 1}$ and $K_{1, \lceil n/2 \rceil - 1}$ for any $n \geq 4$ the resulting graph $G$ has diameter 3 and the number of pairs such that $D(x, y) = V(G)$ also equals $\lfloor n^2/4 \rfloor$.

These facts lead to the following conjecture:

**Conjecture**. *For any connected graph $G$ of order $n \geq 2$ the number of pairs $\{x, y\}$ such that $D(x, y) = V(G)$ is bounded above by $\lfloor n^2/4 \rfloor$.*

We can use $d$-sets for obtaining bounds for the metric dimension of a graph $G$.

**Theorem 5.2.4.** *Let $m, r \geq 2$ be two integers. Suppose that a connected graph $G$ has $m$ pairs of vertices $p_1, ..., p_m$ such that $\bigcap_{i \in I} D(p_i) = \emptyset$ for any $I \subset \{1, ..., m\}$ and $|I| \geq r$.*

*Then $dim(G) \geq \lceil m/(r-1) \rceil$ and this inequality is tight.*

*Proof.* Let $W$ be a resolving set of $G$ such that $|W| = dim(G)$. It follows that $W \cap D(p_i) \neq \emptyset$ for any $i = 1, ..., m$. Let $p$ be a natural number such that $p(r-1)+1 \leq m$. We shall prove that

$$|W \cap \bigcup_{i=1}^{pr-p+1} D(p_i)| \geq p+1 \tag{5.2.1}$$

Suppose that $W \cap \bigcup_{i=1}^{pr-p+1} D(p_i) = \{x_1, ..., x_t\}$ and $t \leq p$. Let $A_j = \{i : 1 \leq i \leq pr - p + 1, x_j \in D(p_i)\}$ for any $1 \leq j \leq t$. Since $|W \cap D(p_i)| \geq 1$ for $1 \leq i \leq m$ and $|A_j| \leq r-1$ for any $1 \leq j \leq t$ by the hypothesis, we can write

$$pr - p + 1 = |\bigcup_{j=1}^{t} A_j| \leq \sum_{j=1}^{t} |A_j| \leq t(r-1),$$

a contradiction since $t \leq p$.

Because $W \supset W \cap \bigcup_{i=1}^{pr-p+1} D(p_i)$ from (5.2.1) it follows that $|W| \geq p+1$. Since $p \leq (m-1)/(r-1)$, we can choose $p = \lfloor (m-1)/(r-1) \rfloor$ and we deduce that $dim(G) = |W| \geq \lfloor (m-1)/(r-1) \rfloor + 1 = \lceil m/(r-1) \rceil$.

To see that this bound be reached consider the graph denoted by $ZH_n$ which consists of a zig-zag sequence of hexagons having common edges as in fig. 5.1 for $n = 5$. We get $|V(ZH_n)| = 4n+2$ and $diam(ZH_n) = 2n+1$ for any $n \geq$

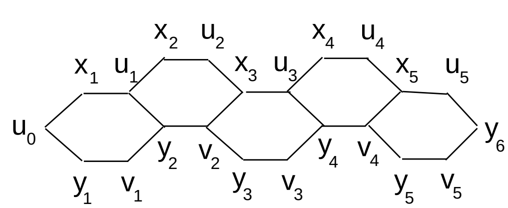


Figure 5.1: $ZH_5$

1. Also $D(x_1, y_1) = D(u_1, v_1) = \{x_1, y_1, u_1, v_1, x_2, u_2\}$; $D(x_2, y_2) = D(u_2, v_2) = \{x_2, y_2, u_2, v_2, y_1, v_1, y_3, v_3\}$; $D(x_3, y_3) = D(u_3, v_3) = \{x_3, y_3, u_3, v_3, x_2, u_2, x_4, u_4\}$ and so on. For any $2 \leq i \leq n-1$ we have $|D(x_i, y_i)| = |D(u_i, v_i)| = 8$. Considering $m = 2n$ pairs $(x_i, y_i)$ and $(u_i, v_i)$ for $1 \leq i \leq n$ the hypothesis of the theorem is satisfied for $r = 7$. It follows that $dim(ZH_n) \geq \lceil \frac{n}{3} \rceil$.

The opposite inequality holds for every $n \geq 7$. Suppose that the hexagons of $ZH_n$ are denoted by $H_1, ..., H_n$ from left to right. For any $n \geq 7$ we can construct a metric basis $W$ of $ZH_n$ of cardinality $\lceil \frac{n}{3} \rceil$ as follows (see Theorem 6.2.2$b$):

We choose any vertex of degree 2 in the hexagons numbered by $2, 5, 8, ..., n-4, n-1$ for $n \equiv 0(mod\ 3)$; $2, 5, 8, ...., n-5, n-2, n$ for $n \equiv 1(mod\ 3)$ and $2, 5, 8, ...., n-6, n-3, n$ for $n \equiv 2(mod\ 3)$. It can be easily verified that any two vertices of $ZH_n$ having equal distances to any vertex of $W$ can be distinguished by other vertices of $W$. □

## 5.3 Graphs with unbounded metric dimension

In this section three sufficient conditions that guarantee that a family $\mathcal{F} = (G_n)_{n \geq 1}$ have unbounded metric dimension are presented.

The observation that the distances from a given vertex to all vertices of a clique can take only 2 distinct values leads to the following property.

**Lemma 5.3.1.** *Suppose that there exists integers $r, d, m \geq 1$ and $C \subset V(G), |C| = m$ where $m \geq (d+1)^r + 1$ such that the distance between any two vertices in $C$ is less than or equal to $d$. Then for any subset $A \subset V(G)$, $|A| = r$ there exist $x, y \in C, x \neq y$ such that $D(x, y) \cap A = \emptyset$.*

*Proof.* Let $A = \{x_1, ..., x_r\}$. We apply induction on $r$. For $r = 1$ we deduce $m \geq d+2$;

all vertices of $C$ being pairwise at a distance at most $d$ apart, the multiset $\{d(x_1, w) : w \in C\}$ contains at most $d+1$ distinct distances. It follows that there exist $x, y \in C$, $x \neq y$ such that $d(x, x_1) = d(y, x_1)$ and the property is verified. Let $r \geq 2$ and suppose that the property is true for $r-1$. If $C$ is such that $|C| = m \geq (d+1)^r + 1$ and $A$ is any set containing $r$ vertices $x_1, ..., x_r$, the multiset $\{d(x_1, w) : w \in C\}$ contains at most $d+1$ distinct distances.

It follows that there exists a subset $C_1 \subset C$ such that the multiset $\{d(x_1, w) : w \in C_1\}$ contains only equal numbers and $|C_1| \geq \lceil ((d+1)^r + 1)/(d+1) \rceil = (d+1)^{r-1} + 1$. Applying the induction hypothesis for $C_1$ and subset $\{x_2, ..., x_r\}$ we find a pair $\{x, y\}$, where $x, y \in C_1 \subset C$, $x \neq y$ such that $D(x, y) \cap \{x_2, ..., x_r\} = \emptyset$. By construction $x_1 \notin D(x, y)$, hence $D(x, y) \cap A = \emptyset$. □

Since every resolving set $W$ of $G$ must have $W \cap D(x, y) \neq \emptyset$ we get $dim(G) \geq r+1$. If $\omega_d(G)$ denotes the maximum number of vertices of a subset $C \subset V(G)$ such that the distance between any two vertices in $C$ is not greater than $d$, we deduce the following consequence.

**Corollary 5.3.2.** *We have $dim(G) \geq \lceil log_{d+1}\omega_d(G) \rceil$ for any connected graph $G$.*

If $d = 1$ then $\omega_1(G) = \omega(G)$, the clique number of $G$ and the following corollary holds.

**Corollary 5.3.3.** *If $G$ is a connected graph with clique number $\omega(G)$, then $dim(G) \geq \lceil log_2\omega(G) \rceil$.*

This inequality is tight for $\omega(G) \leq 5$. For example, take a copy of $K_5$ having vertices $\{x, y, z, t, w\}$. We shall consider three new vertices $a, b, c$ and define a new graph $G$ such that $V(G) = V(K_5) \cup \{a, b, c\}$ and $E(G) = E(K_5) \cup \{ax, ay, by, bz, cz, ct\}$. We have $\omega(G) = 5$ and $dim(G) = 3$, a metric basis being $\{a, b, c\}$. Similar examples of graphs $G$ for which $dim(G) \geq \lceil log_2 \omega(G) \rceil$ can be found for $\omega(G) \leq 4$, but for greater values of $\omega(G)$ the inequality may be strict. We deduce the following consequence for families $\mathcal{F}$ having unbounded clique number.

**Corollary 5.3.4.** *If a family $\mathcal{F} = (G_n)_{n \geq 1}$ has unbounded clique number then it also has unbounded metric dimension.*

Theorem 5.2.4 also has the following consequence concerning families with unbounded metric dimension.

**Corollary 5.3.5.** *Let $r \geq 2$be a fixed natural number. Suppose that family $\mathcal{F} = (G_n)_{n \geq 1}$ has the following property: for each $n$ graph $G_n$ contains $m = \psi(n)$ pairs of vertices $p_1, ..., p_m$ such that $\bigcap_{i \in 1} D(p_i) = \emptyset$ for any $I \subset \{1, ..., m\}$ and $|I| \geq r$.*
*If $\lim_{n \to \infty} \psi(n) = \infty$ then $\lim_{n \to \infty} dim(G_n) = \infty$, which implies that $\mathcal{F}$ has unbounded metric dimension.*

An example of such a family is $\mathcal{F}_1 = (ZH_n)_{n \geq 1}$ considered above. Another example is $\mathcal{F}_2 = (K_{n,n})_{n \geq 1}$ since $dim(K_{n,n}) = 2n - 2$ for every $n \geq 2$ and there exist $n^2 - n$ pairs $\{z, t\}$ contained in the same partite set of $K_{n,n}$ such that $D(x, y) = \{z, t\}$.

**Lemma 5.3.6.** *If for the family $\mathcal{F} = (G_n)_{n \geq 1}$ there exists a constant $C > 0$ such that $dim(G_n) \leq C$ for any $n \geq 1$ then $\mathcal{F}$ has unbounded metric dimension.*

*Proof.* Let $diam(G) = d = d(n) \leq C$ and $W_n = \{w_1, ..., w_k\}$ be a minimum resolving set for $G_n$ containing $k = k(n)$ vertices. The representation $(d(v, w_1), ..., d(v, w_k))$ of a vertex $v \in V(G_n)$ can contain a component equal to 0 only if $v \in W$.
It follows that the number of distinct representations is bounded above by $kd^{k-1} +$

$d^k$. Because the sequence of metric dimensions $(k(n))_{n\geq 1}$ is bounded, there exists a constant $C_1 > 0$ such that $k(n) = dim(G_n) \leq C_1$ for every $n \geq 1$. Since all vertices of $G_n$ have distinct representations we deduce that $kd^{k-1} + d^k \geq |V(G_n)| = \varphi(n) \to \infty$ as $n \to \infty$.

On the other hand $kd^{k-1} + d^k \leq C_1 C^{C_1 - 1} + C^{C_1}$, a contradiction. It follows that $\mathcal{F}$ has unbounded metric dimension. □

## 5.4 Metric dimension of necklace graph

Since necklace graph $Ne_n$ is not a path we have $dim(Ne_n) \geq 2$ for any $n \geq 1$. $Ne_1$ is $K_4$, so $dim(Ne_1) = 3$ and also $dim(Ne_2) = 3$.

**Theorem 5.4.1.** *For every $n \geq 3$ we have*

$$dim(Ne_h) = \begin{cases} 2, & \textit{if } n \textit{ is even };\\ 3, & \textit{if } n \textit{ is odd} \end{cases}$$

*Proof.* **a**) Let $n$ be even. In this case a resolving set of $Ne_n$ is $W = \{v_0, v_{n/2}\}$ since the representations of the vertices with respect to $W$ are the following:

$$r(v_i|W) = \begin{cases} (i, \frac{n}{2} - i), & \text{for } 0 \leq i \leq \frac{n}{2};\\ (n - i + 2, i - \frac{n}{2}), & \text{for } \frac{n}{2} + 1 \leq i \leq n + 1. \end{cases}$$

and

$$r(u_i|W) = \begin{cases} (i, \frac{n}{2} - i + 1), & \text{for } 1 \leq i \leq \frac{n}{2};\\ (n - i + 2, i - \frac{n}{2} + 1), & \text{for } \frac{n}{2} + 1 \leq i \leq n. \end{cases}$$

Since all vertices have distinct representations we obtain $dim(Ne_n) = 2$ in this case.

**b**) When $n$ is odd we show that $W = \{v_0, v_{(n+1)/2}, u_{(n-1)/2}\}$ is a resolving set. The representations of the vertices of $Ne_n$ with respect to $U = \{v_0, v_{(n+1)/2}\}$ are the following:

$$r(v_i|U) = \begin{cases} (i, \frac{n-1}{2} - i + 1), & \text{for } 0 \leq i \leq \frac{n+1}{2}; \\ (n - i + 2, i - \frac{n+}{2}), & \text{for } \frac{n+3}{2} + 1 \leq i \leq n + 1. \end{cases}$$

and

$$r(u_i|U) = \begin{cases} (i, \frac{n-1}{2} - i + 2), & \text{for } 1 \leq i \leq \frac{n+1}{2}; \\ (n - i + 2, i - \frac{n+}{2} + 1), & \text{for } \frac{n+3}{2} \leq i \leq n. \end{cases}$$

$U$ distinguishes all vertices of $Ne_n$ unless $u_i$ and $v_{n+2-i}$ for $1 \leq i \leq \frac{n+1}{2}$. This can be done by $u_{(n-1)/2}$, hence $W$ is a resolving set, thus implying $dim(Ne_n) \leq 3$.

We will show that $dim(Ne_n) \geq 3$, by proving that any resolving set has at least three vertices. Suppose that there exists a resolving set $W$ of $Ne_n$ such that $|W| = 2$. We shall prove that this leads to a contradiction.

By taking into account the action of the automorphism group of $Ne_n$, it is sufficient to consider only the cases when $v_{k-1}, v_k, ...$ or $v_{n+1}$ belongs to $W$, where $k = diam(Ne_n) = \lfloor (n+3)/2 \rfloor$.

**A**. If $v_{n+1} \in W$ we have $d(v_{k-2}, v_{n+1}) = d(u_{k-2}, v_{n+1}) = d(v_{k-1}, u_{n+1}) = d(u_{k-1}, u_{n+1}) = k - 1$ (see fig. 5.2).

Also $D(v_{k-2}, u_{k-1}) = \{v_0\} \cup \{u_k, u_{k+1}, ..., u_n, v_1, v_2 ..., v_{k-3}\} \cup \{v_{k-2}, u_{k-1}\}$; $D(v_{k-1}, u_{k-2}) = \{v_0\} \cup \{u_1, u_2, ..., u_{k-3}, v_k, v_{k+1} ..., v_n\} \cup \{v_{k-1}, u_{k-2}\}$; $D(v_{k-2}, u_{k-1}) \cap D(v_{k-1}, u_{k-2}) = \{v_0\}$ but $d(v_0, v_{k-2}) = d(v_0, u_{k-2}) = k - 2$. It follows that there is no resolving set having two vertices including $v_{n+1}$.

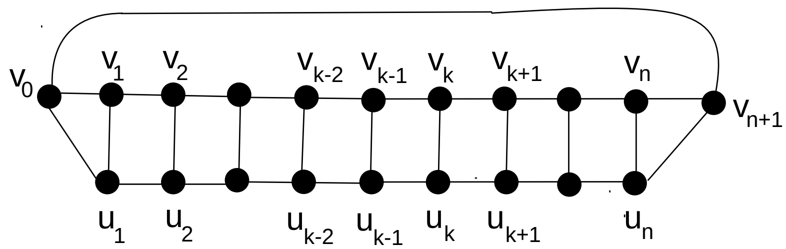


Figure 5.2: $Ne_n$

**B**. If $v_{k-1} \in W$ we deduce $d(v_{k-2}, v_{k-1}) = d(u_{k-1}, v_{k-1}) = d(u_k, v_{k-1}) = 1$. We get (see fig. 5.2)

$D(v_{k-2}, u_{k-1}) = \{v_0\} \cup \{v_1, v_2, ..., v_{k-3}, u_k, u_{k+1}, ..., u_n\} \cup \{v_{k-2}, u_{k-1}\}$; $D(v_k, u_{k-1}) = \{v_{n+1}\} \cup \{u_1, u_2, ..., u_{k-2}, v_{k+1}, ..., v_n\} \cup \{v_k, u_{k-1}\}$; $D(v_{k-2}, u_{k-1}) \cap D(v_k, u_{k-1}) = \{u_{k-1}\}$, but $d(u_{k-1}, v_k) = d(u_{k-1}, v_k) = 2$ and $v_{k-2}, v_k$ have unit distances from $v_{k-1}$. It follows that $v_{k-1} \notin W$, a contradiction.

**C**. $v_k \in W$. In this case $D(v_{k-1}, u_k) = \{v_{n+1}\} \cup \{v_1, , ..., v_{k-2}, u_{k+1}, ..., u_n\} \cup \{v_{k-1}, u_k\}$; $D(v_{k+1}, u_k) = \{v_0, v_{n+1}\} \cup \{v_{k+2}, , ..., v_n, u_2, ..., u_{k-1}\} \cup \{v_{k+1}, u_k\}$ (see fig. 5.2).

We have $D(v_{k+1}, u_k) \cap D(v_{k-1}, u_k) = \{v_{n+1}, u_k\}$. But $v_{n+1}$ has equal distances apart from $v_{k-2}$ and $u_{k-1}$, which have equal distances to $v_k$. Also $u_k$ does not distinguish vertices $v_{k-1}$ and $v_{k+1}$, hence $v_k \notin W$.

**D**. If $v_i \in W$ and $k+1 \leq i \leq n$ we shall consider its antipodal vertex $u_{i-k+1}$ (for which $d(v_i, u_{i-k+1}) = k$) and its neighbor (see fig. 5.3). We get $d(v_i, v_{i-k+1}) = d(v_i, u_{i-k}) = d(v_i, u_{i-k+2}) = k + 1$ and $d(v_i, v_1) = d(v_i, u_1) = n - i + 3$.

Since $v_{n+1}$ and $v_0$ have equal distances to $u_1$ and $v_1$, respectively, it follows that $v_0, v_{n+1} \notin W$.

Also $d(v_j, v_1) = d(v_j, u_1)$ and $d(u_j, v_1) = d(u_j, u_1)$ for every $k + 1 \leq j \leq n$, which

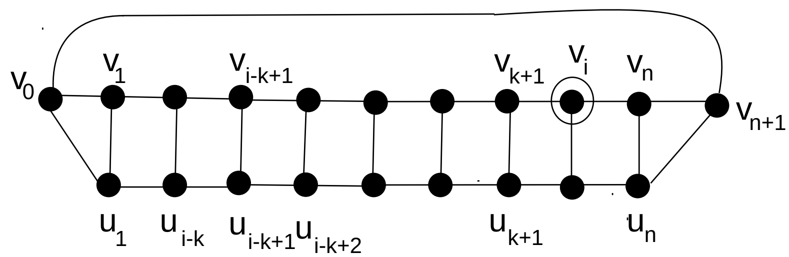


Figure 5.3: $Ne_n$ with two antipodal vertices

implies that

$$(W \backslash \{v_i\}) \cap \{v_0, v_{k+1}, ..., v_{n+1}, u_{k+1}, ..., u_n\} = \emptyset$$

.

For any $x, y \in V(Ne_n)$ and $x \neq y$ denote by $D'(x, y)$ the set $D(x, y) \backslash \{v_0, v_{k+1}, ..., v_{n+1}, u_{k+1}, ..., u_n\}$. We deduce $D'(u_{i-k}, v_{i-k+1}) = \{v_{i-k+2}, ..., v_k, u_1, ..., u_{i-k+1}\} \cup \{u_{i-k}, v_{i-k+1}\}$; $D'(u_{i-k+2}, v_{i-k+1}) = \{v_1, ..., v_{i-k}, u_{i-k+3}, ..., u_k\} \cup \{u_{i-k+2}, v_{i-k+1}\}$ hence $D'(u_{i-k}, v_{i-k+1}) \cap D'(u_{i-k+2}, v_{i-k+1}) = \{v_{i-k+1}\}$. It follows that $W = \{v_i, v_{i-k+1}\}$. But vertices $u_{i-k}$ and $u_{i-k+2}$ have equal distances from both vertices of $W$, a contradiction. Consequently, every resolving set of $V(Ne_n)$ has at least three vertices for $n$ odd. $\square$

# Chapter 6

# On metric and partition dimension of some infinite regular graphs

In this chapter some infinite regular graphs generated by tilings of the plane by regular triangles and hexagons are considered. These graphs have no finite metric bases but their partition dimension is finite and is evaluated in some cases. Also, it is proved that for every $n \geq 2$ there exist finite induced subgraphs of these graphs having metric dimension equal to $n$ as well as infinite induced subgraphs with metric dimension equal to three.

## 6.1 Introduction

It is natural to think that the partition dimension and metric dimension are related; in [4] it was shown that for any nontrivial connected graph $G$ we have $pd(G) \leq dim(G) + 1$.

However, the partition dimension may be much smaller than the metric dimension. Let $(i, j)$ and $(i', j')$ be two points with integral coordinates in $\mathbb{Z}^2$. It is well known that the following definitions yield metrics for $\mathbb{Z}^2$: $d_4((i, j), (i', j')) = |i - i'| + |j - j'|$

(city block distance) and $d_8((i,j),(i',j')) = \max(|i-i'|,|j-j'|)$ (chessboard distance). The indices 4 and 8 are appropriate because they represent the number of points at distance one(the neighbors) from a given point with respect to these two metrics. These two metrics on $\mathbb{Z}^2$ generate two infinite graphs $(\mathbb{Z}^2, \mathcal{E}_4)$ and $(\mathbb{Z}^2, \mathcal{E}_8)$ having the same vertex set $\mathbb{Z}^2$ and the set of edges consisting of all pairs of vertices whose city block and chessboard distances are 1. $(\mathbb{Z}^2, \mathcal{E}_4)$ is a planar 4-regular graph whose regions are unit squares and it is also known as the square lattice graph. $(\mathbb{Z}^2, \mathcal{E}_8)$ is 8-regular and can be obtained from $(\mathbb{Z}^2, \mathcal{E}_4)$ by drawing all diagonals of unit squares. In [31] it was proved that these two graphs have no finite metric bases and for any natural number $n \geq 3$, there exist induced subgraphs of $(\mathbb{Z}^2, \mathcal{E}_4)$ and $(\mathbb{Z}^2, \mathcal{E}_8)$, respectively having metric dimension equal to $n$ and partition dimension equal to three. Also, in [40] it was proved that $pd(\mathbb{Z}^2, \mathcal{E}_4) = 3$ and $pd(\mathbb{Z}^2, \mathcal{E}_8) = 4$.

In what follows we shall consider some infinite regular graphs generated by tilings of the plane by regular hexagons or equilateral triangles.

The graph $G_3$ is a planar 3-regular infinite graph whose regions are regular hexagons of unit side. $V(G_3)$ consists of the vertices of these hexagons, two vertices being adjacent if they are the extremities of a unit side of a hexagon in the tiling. Similarly, $G_6$ is a planar 6-regular infinite graph whose regions are equilateral triangles of unit side. It is also known as the triangular lattice graph. Three kinds of rhombuses of unit side appear in $G_6$. By drawing all diagonals of these rhombuses we get a 12-regular infinite graph denoted by $G_{12}$; if all vertical diagonals are deleted, the resulting 10-regular infinite graph is denoted by $G_{10}$ (see Fig.6.1). The indices $3, 6, 10$ and 12 represent the number of vertices at distance 1 from a given vertex. For sake of simplicity for $G_{10}$ and $G_{12}$ or their subgraphs we shall not draw the corresponding diagonals of the

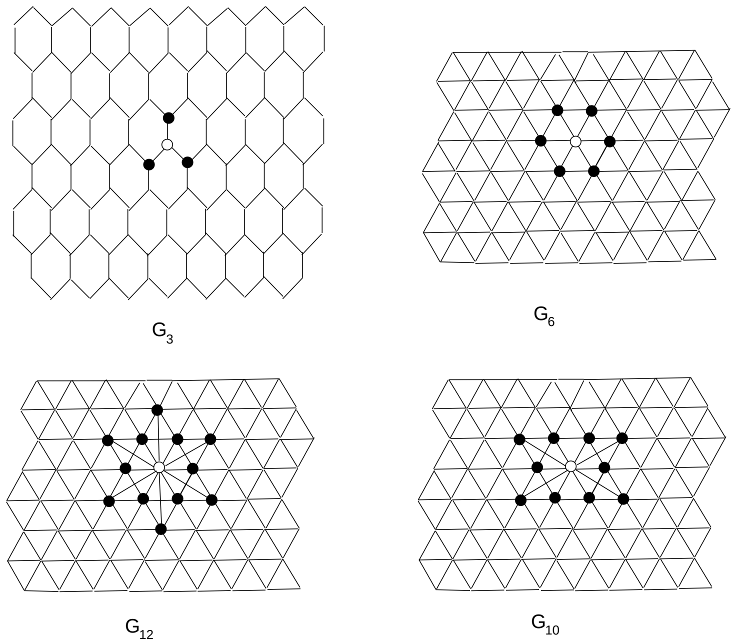


Figure 6.1: Graphs $G_3, G_6, G_{10}$ and $G_{12}$

rhombuses; this will be clear from the context.

## 6.2 Main results

**Lemma 6.2.1.** *The graphs $G_3, G_6, G_{10}$ and $G_{12}$ have no finite metric basis, i.e., $dim(G_3) = dim(G_6) = dim(G_{10}) = dim(G_{12}) = \infty$.*

*Proof.* Figs. $6.2a) - d)$ represent two vertices $x, y$ in $G_3, G_6, G_{10}, G_{12}$ having their Euclidean distances equal to 1 and to $\sqrt{3}$, respectively and subgraphs $G_i(x, y)(i = 3, 6, 10, 12)$ consisting of vertices $z$ such that $d(x, z) = d(z, y)$. Suppose that $G_3$ has a finite metric basis $S$. We can find two vertices $x, y$ and a subset $T \subset G_3(x, y)$

consisting of all vertices $z \in G_3(x,y)$ such that $d(z,x) = d(z,y) \leq k$ for $k$ large enough, such that $S \subset T$. This implies that $d(x,z) = d(y,z)$ for all $z \in S$, a contradiction. The proof is similar for other three graphs. $\square$

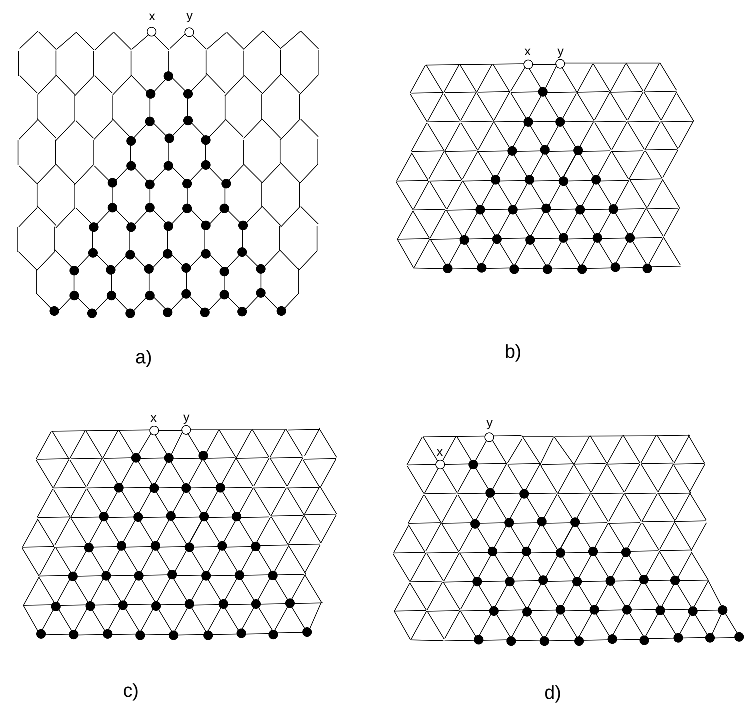


Figure 6.2: Subgraphs of vertices having equal distances to $x$ and $y$.

Fig. 6.3 represents some induced subgraphs of $G_3$ and $G_6$: $F_n$ has $n$ hexagons, $ZZ_n$ is a zig-zag sequence of $n$ hexagons, $H_1$ and $H_2$ are infinite subgraphs of $G_6$ and $L_n$ is an induced subgraph of $G_6$ consisting of $n$ rhombuses with one diagonal.

We have $dim(H_1) = dim(H_2) = 3$ since vertices having the same distance to $B$ are distinguished by $A$ and $C$, respectively.

**Theorem 6.2.2.** *a) For every $n \geq 2$ we have: $dim(F_n) = n$ and $F_n$ has $n!4^n$ metric bases; $dim(L_n) = n$ and $L_n$ has $n!2^n$ metric bases.*

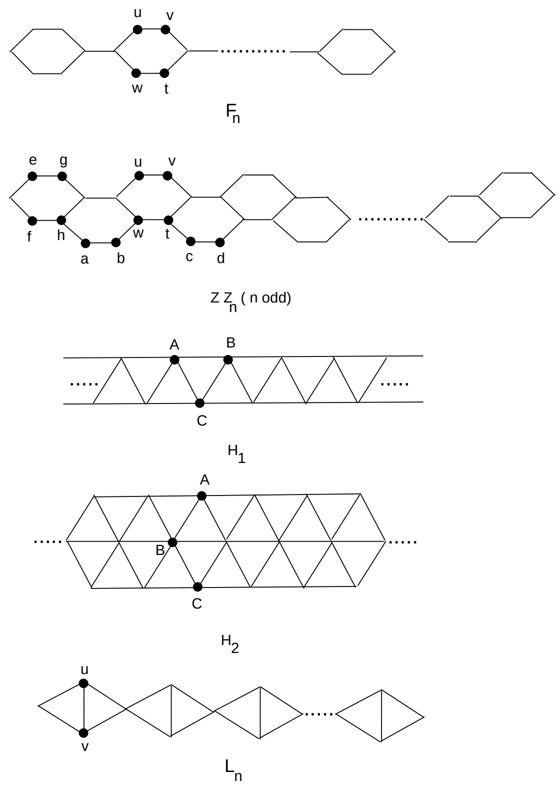


Figure 6.3: Induced subgraphs of $G_3$ and $G_6$.

*b) For every $n \geq 7$ $dim(ZZ_n) = \lceil \frac{n}{3} \rceil$.*

*Proof.* a) $u$ and $w$ have equal distances to all vertices of $F_n$ different from $v$ and $t$ and $v$ and $t$ may be distinguished only by $u$ or $w$ if $v$ and $t$ do not belong to any basis of $F_n$. It follows that at least one of $u, v, w, t$ must belong to any basis of $F_n$.

On the other hand, by choosing exactly one vertex of degree two in each hexagon of $F_n$ ( in $4^n$ ways), the set of these vertices is a metric basis of $F_n$. These vertices can be ordered in $n!$ ways and the result follows. The situation of $L_n$ is similar to $F_n$.

b) Suppose that the hexagons of $ZZ_n$ are denoted by $H_1, \ldots, H_n$ from left to right.

The vertices $e$ and $f$ of $H_1$ (see Fig. 6.3) can be distinguished only by $g, h, a$ or $b$ and vertices $g$ and $h$ only by $e, f, a$ or $b$. We deduce that any resolving set $W$ of $ZZ_n$ must contain at least one vertex from unavoidable set $\{e, f, g, h, a, b\}$ of the pair of hexagons $(H_1, H_2)$. This set will be denoted by $U(H_1, H_2)$. A similar situation occurs for the pair $(H_{n-1}, H_n)$. We shall say that $W$ satisfies the 2 consecutive hexagons (shortly 2CH) property.

Consider now a triplet of consecutive hexagons $(H_2, H_3, H_4)$. The vertices $u$ and $w$ of $H_3$ can be distinguished only by $v, t, a, b, c$ or $d$ and a similar situation also occurs for $v$ and $t$. It follows that any resolving set $W$ of $ZZ_n$ must contain at least one vertex from unavoidable set $\{u, v, w, t, a, b, c, d\}$ of $(H_2, H_3, H_4)$ and a similar property holds for any triplet $(H_i, H_{i+1}, H_{i+2})$, where $2 \le i \le n-3$. This set will be denoted by $U(H_i, H_{i+1}, H_{i+2})$ and we shall say that $W$ satisfies the 3 consecutive hexagons (3CH) property.

Let $B$ be a metric basis of $ZZ_n$. We shall transform $B$ into a set $B'$ such that $|B| \ge |B'|$ , $B'$ contains only vertices of degree 2 and $B'$ also satisfies 2CH and 3CH properties by the following algorithm:

Initially $B' = \emptyset$. We have $U(H_1, H_2) \cap U(H_2, H_3, H_4) = \{a, b\}$. If there exists $x \in \{a, b\}$ such that $x \in B$ then label $x$, assign $B' \leftarrow B' \cup \{x\}$ and consider the next triplet $(H_2, H_3, H_4)$. Otherwise, if there exists $y \in \{e, f, g\}$ such that $y \in B$ then label $y$, define $B' \leftarrow B' \cup \{y\}$ and consider the next triplet. Otherwise, $\{h\} = U(H_1, H_2) \cap B$. In this case label $g$ and do $B' \leftarrow B' \cup \{g\}$. Consequently, only one of the vertices $e, f, g, a, b$ were labeled and included in $B'$.

Consider now the triplet $(H_2, H_3, H_4)$. If $U(H_2, H_3, H_4)$ contains a labeled vertex (or equivalently, $a$ or $b$ has been selected in $B'$), then consider the next triplet

$(H_3, H_4, H_5)$. Otherwise $a$ and $b$ are not labeled. We deduce $U(H_2, H_3, H_4) \cap U(H_3, H_4, H_5) = \{u, v, c, d\}$, $U(H_2, H_3, H_4) \cap U(H_4, H_5, H_6) = \{c, d\}$ and $U(H_2, H_3, H_4) \cap U(H_i, H_{i+1}, H_{i+2}) = \emptyset$ for $i \geq 5$. If there exists $x \in \{c, d\}$ such that $x \in B$ then label it, do $B' \leftarrow B' \cup \{x\}$ and consider the next triplet. Otherwise, if there exists $y \in \{u, v\}$ such that $y \in B$, then label it, assign $B' \leftarrow B' \cup \{y\}$ and consider the next triplet. If $u, v \notin B$ but there exists $z \in \{w, t\}$ such that $z \in B$ then label $u$ (or $v$), $B' \leftarrow B' \cup \{u\}$ and consider the next triplet. Otherwise, there exists $s \in \{a, b\}$ such that $s \in B$. In this case label $s$, define $B' \leftarrow B' \cup \{s\}$ and consider the next triplet $(H_3, H_4, H_5)$.

Each time when the unavoidable set of a triplet has a vertex which has been labeled in the previous steps we shall consider the next triplet (or the last pair). We shall proceed in a similar way for all triplets $(H_i, H_{i+1}, H_{i+2})$ in the order $i = 3, \ldots, n-4$ and also for the triplet $(H_{n-3}, H_{n-2}, H_{n-1})$ and the pair $(H_{n-1}, H_n)$ (like as for $(H_1, H_2)$ and $(H_2, H_3, H_4)$).

Note that $h$ does not belong to $U(H_2, H_3, H_4)$, so its possible replacement by $g$ does not affect the property of $B'$ to satisfy 2CH and 3CH properties. Similarly, $w, t \notin U(H_3, H_4, H_5) \cup U(H_4, H_5, H_6)$, so their possible replacement by vertices of degree 2 $u$ or $v$, respectively, cannot affect the property of $B'$ to satisfy the 3CH property and this holds for all triplets of hexagons of $ZZ_n$. Consequently, $B'$ satisfies 2CH and 3CH properties; moreover, $B'$ has the property that it contains only vertices of degree 2, i. e., it does not contain vertices common to neighboring hexagons $H_i, H_{i+1}$ for $1 \leq i \leq n-1$ and each hexagon has at most one vertex in $B'$.

For $i = 1, \ldots, n$ let $z_i$ denote a binary variable representing the number of vertices of $H_i$ included in $B'$. Since $B'$ satisfies 2CH and 3CH properties we can write:

$$z_1 + z_2 \geq 1$$

$$z_2 + z_3 + z_4 \geq 1$$

$$z_3 + z_4 + z_5 \geq 1$$

$$\cdots\cdots$$

$$z_{n-3} + z_{n-2} + z_{n-1} \geq 1$$

$$z_{n-1} + z_n \geq 1$$

By summing up these inequalities we get:

$$S = z_1 + 2z_2 + 2z_3 + 3(z_4 + \ldots + z_{n-3}) + 2z_{n-2} + 2z_{n-1} + z_n \geq n - 2.$$

We have

$$3|B| \geq 3|B'| = 3\sum_{i=1}^{n} z_i = S + 2z_1 + z_2 + z_3 + z_{n-2} + z_{n-1} + 2z_n \geq n$$

since $2z_1 + z_2 \geq z_1 + z_2 \geq 1; 2z_n + z_{n-1} \geq z_n + z_{n-1} \geq 1$ and $z_3 + z_{n-2} \geq 0$. It follows that $dim(ZZ_n) = |B| \geq |B'| \geq \lceil \frac{n}{3} \rceil$.

By direct evaluation we find that $dim(ZZ_n)$ is equal to: 2 for $n = 1$ or $n = 2$; 3 for $3 \leq n \leq 9$; 4 for $10 \leq n \leq 12$ and so on. For every $n \geq 7$, we can construct a metric basis $B$ of $ZZ_n$ of cardinality $\lceil \frac{n}{3} \rceil$ as follows: we choose any vertex of degree two in the hexagons numbered by: $2, 5, 8, ...., n-4, n-1$ for $n \equiv 0 (mod\, 3)$; $2, 5, 8, ...., n-5, n-2, n$ for $n \equiv 1(mod\ 3)$ and $2, 5, 8, ...., n-6, n-3, n$ for $n \equiv 2(mod\ 3)$. It can be easily verified that any two vertices of $ZZ_n$ having equal distances to any vertex of $B$ can be distinguished by other vertices of $B$, which concludes the proof. □

**Lemma 6.2.3.** *We have* $pd(G_3) = pd(G_6) = 3$.

*Proof.* In [4] it was shown that $pd(G) = 2$ if and only if G is a path and this property also holds for infinite graphs. It follows that $pd(G_3) \geq 3$ and $pd(G_6) \geq 3$.

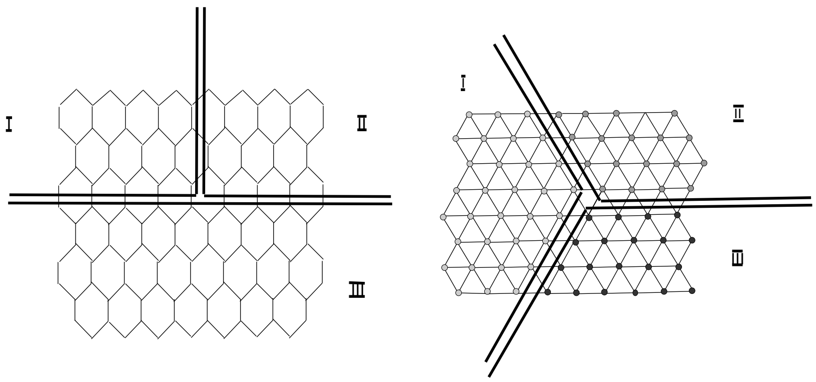


Figure 6.4: Resolving 3-partitions for $V(G_3)$ and $V(G_6)$.

Fig. 6.4 provides resolving 3-partitions of $G_3$ and $G_6$, respectively. It follows that $pd(G_3) = pd(G_6) = 3$. □

The problem of determining partition dimension of $G_{10}$ and $G_{12}$ is much more difficult. We are only able to find some bounds in the next theorems.

Consider now the Euclidean plane containing plane realizations of $G_{10}$ and $G_{12}$ endowed with a rectangular system of axes $xOy$.

**Theorem 6.2.4.** *We have* $4 \leq pd(G_{10}) \leq 5$.

*Proof.* The same argument used in Lemma 6.2.3 implies that $pd(G_{10}) \geq 3$. Suppose that there exists a resolving partition $\Pi = (S_1, S_2, S_3)$ with three classes of $V(G_{10})$. We will show that this leads to a contradiction, which implies $pd(G_{10}) \geq 4$.

**Claim 1.** *In any unit rhombus $PQRS$ which is an induced subgraph of $G_{10}$, having two sides parallel to $Ox$ there are not three vertices belonging to different classes of $\Pi$.*

Suppose that $P \in S_1, Q \in S_2$ and $R \in S_3$. $S$ can belong to $S_1, S_2$ or $S_3$. In any case there exist two vertices of $PQRS$ in $S_i$ having unit distances to $S_j$ and $S_k$

respectively($1 \leq i, j, k \leq 3, i, j$ and $k$ pairwise distinct indices), a contradiction.

**Claim 2**. *In any unit rhombus $PQRS$, induced subgraph of $G_{10}$, no three vertices belong to a class and the fourth to another class of* $\Pi$.

Suppose that $P, Q, R \in S_1$ and $S \in S_2$. Since $P, Q, R$ have equal unit distances to the class $S_2$, we deduce that $d(P, S_3) \neq d(Q, S_3)$, $d(P, S_3) \neq d(R, S_3)$ and $d(Q, S_3) \neq d(R, S_3)$. Since any distance between vertices $P, Q$ and $R$ is 1, one obtains that $d(P, S_3)$ and $d(Q, S_3)$ and $d(R, S_3)$ may differ by at most one, which contradicts these inequalities. A consequence of Claims 1 and 2 follows.

**Claim 3**. *Suppose that $P, Q \in V(G_{10})$ and $d(P, Q) = 1$ such that $P$ and $Q$ belong to the same line having a slope equal to $\sqrt{3}$ or $-\sqrt{3}$ and $P \in S_1, Q \in S_2$. Then all the vertices of $G_{10}$ lying on the horizontal lines $l_P$ and $l_Q$ passing through $P$ and $Q$ belong to the classes $S_1$ and $S_2$ and each unit rhombus, induced subgraph of $G_{10}$ with vertices on $l_P$ and $l_Q$ has two vertices in $S_1$ and two in $S_2$. A similar conclusion holds if $P$ and $Q$ belong to the same horizontal line by considering the lines having slope $\sqrt{3}$ or $-\sqrt{3}$ passing through $P$ and $Q$.*

Without loss of generality, we can assume that there exist two vertices of $G_{10}$, $P(x, y), Q(x - \frac{1}{2}, y - \frac{\sqrt{3}}{2})$ on the same line having slope $\sqrt{3}$ such that $P \in S_1$, $Q \in S_2$ and $d(P, Q) = 1$. Let $P, Q, R, S, P$ be a 4-cycle containing $PQ$ such that $S(x + 1, y)$. By Claims 1 and 2 vertices $R$ and $S$ belong to different classes $S_1, S_2$ of $\Pi$.

Without loss of generality we can suppose that $R \in S_2$ and $S \in S_1$. Since $d(P, S_2) = d(S, S_2) = 1$ it follows that $d(P, S_3) \neq d(S, S_3)$. Because $d(P, S) = 1$ we deduce that $d(P, S_3)$ and $d(S, S_3)$ differ by 1, e.g., $d(P, S_3) = d(S, S_3) + 1$. We obtain that there exists a vertex $T \in S_3$ such that $d(P, S_3) = d(P, T), d(S, S_3) = d(S, T)$ and $d(P, T) = d(S, T) + 1$. By Claim 3 all vertices of $G_{10}$ lying on lines $QR$ and $SP$

belong to $S_1 \cup S_2$.

If $d(S,T) = 1$, since all vertices of $G_{10}$ on $PS$ belong to $S_1 \cup S_2$, we deduce that the slope of $ST$ is $\pm 1/\sqrt{3}$. Without loss of generality we can suppose that this slope is equal to $1/\sqrt{3}$. Let $J$ be on the line $PS$ such that $d(S,J) = d(T,J) = 1$. Because $S \in S_1$ and $T \in S_3$, by Claims 1 and 3 it follows that $J \in S_1$. In this case $S, J \in S_1$, but $d(S,S_2) = d(J,S_2) = 1$ and $d(S,S_3) = d(J,S_3) = 1$, a contradiction.

Suppose that $d(S,T) \geq 2$; then there exists a shortest path between $S$ and $T$ such that the last edge $UT$ of this path incident to $T$ has the slope equal to $1/\sqrt{3}$. Consider the unit rhombus $TVUW$ having diagonal $TU$ and such that $UV$ is parallel to $PS$. Since $U \in S_1 \cup S_2$, we can suppose that $U \in S_1$. By Claim 1 we have $V \in S_1 \cup S_3$.

If $V \in S_1$ then $W \in S_3$, but in this case either $d(S,W) = d(S,T) - 1$ or $d(P,W) = d(S,W) = d(S,T) = d(S,S_3)$, a contradiction.

It follows that $V \in S_3$. Consider the unit rhombus determined by the intersection of the lines $PS, QR$ and $WU, TV$. By Claim 3 two vertices of this rhombus are in $S_1$ and two in $S_3$ and in the same time, two belong to $S_1$ and two belong to $S_2$, a contradiction.

A similar conclusion holds if $U \in S_2$. It follows that $pd(G_{10}) \geq 4$.

Figure 6.5 shows that $pd(G_{10}) \leq 5$, which concludes the proof. $\square$

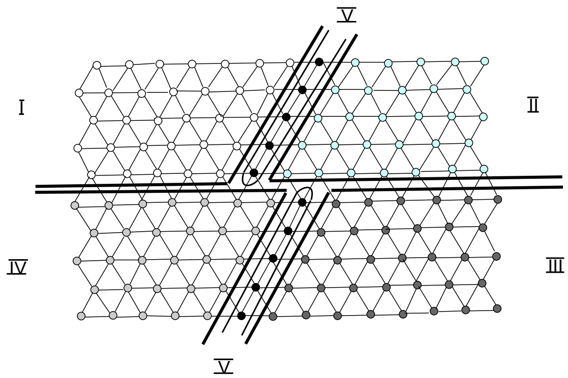


Figure 6.5: A resolving 5-partition of $V(G_{10})$.

**Theorem 6.2.5.** *The following inequalities hold:*

$$4 \leq pd(G_{12}) \leq 6.$$

*Proof.* As for Theorem 6.2.3 we deduce that $pd(G_{12}) \geq 3$. Suppose that there exists a resolving partition $\Pi = (S_1, S_2, S_3)$ of $V(G_{12})$ and we will deduce a contradiction. Since $V(G_{12}) = V(G_{10})$ and all distances equal to 1 in $G_{10}$ are also equal to 1 in $G_{12}$, we deduce that Claims 1, 2 and 3 hold for $G_{12}$. In $G_{12}$ all vertical diagonals of unit rhombus are edges; this leads to a supplementary property of $G_{12}$ expressed as follows:

**Claim 4**. *Let $P, Q \in V(G_{12})$ such that $d(P, Q) = 1$, line $PQ$ is horizontal and $P, Q$ belong to the same class of $\Pi$, say $S_1$. Then all vertices of $G_{12}$ lying on vertical lines passing through $P$ and $Q$ also belong to $S_1$.*

Consider the subgraph of $G_{12}$ having vertices $P, Q, A, B, C, D, E, F, G$, drawn in Fig. 5.6. Suppose that $F \notin S_1$. Without loss of generality we can suppose that $F \in S_2$. By Claim 2 we have $C \notin S_1$. If $C \in S_2$ then by Claims 1 and 2 we deduce $B \in S_2$; thus in $BCGF$ three vertices belong to a class, which contradicts Claim 2. If $C \in S_3$ it follows

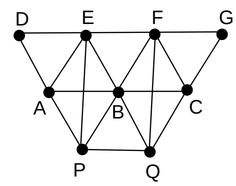


Figure 6.6: A subgraph of $G_{12}$

that $B \in S_3$ and also $G \in S_2$. In this case $B, C \in S_3$ and $d(B, S_1) = d(C, S_1) = 1$ and $d(B, S_2) = d(C, S_2) = 1$, which contradicts the fact that $\Pi$ is a resolving partition of $G_{12}$.

In a similar way, by considering the parallelogram $PQED$ we deduce that $E \in S_1$. By repeating this argument it follows that all vertices of $G_{12}$ lying on lines $EP$ and $FQ$ belong to $S_1$. This concludes the proof of this claim.

Now the proof of the theorem follows the same lines as the proof of Theorem 6.2.3. In the case when $T \in S_3$ is such that $d(P, T) = d(S, T) + 1$, by Claim 4 vertex $T$ cannot belong to the vertical line passing through $S$ and we have the same subcases as for Theorem 2.2.

Fig. 6.7 represents a resolving 6- partition of $V(G_{12})$, hence $pd(G_{12}) \leq 6$. □

We propose the following conjecture:

**Conjecture**. *The following equalities hold:*

$$pd(G_{10}) = 5 \text{ and } pd(G_{12}) = 6.$$

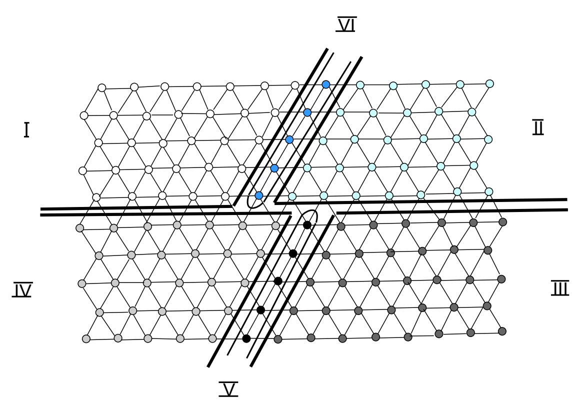


Figure 6.7: A resolving 6-partition of $V(G_{12})$.

# Chapter 7

# Concluding remarks and open problems

We have seen that the metric dimension of the graphs of convex polytopes $\mathbb{D}_n, R_n, Q_n$ is constant and only three vertices appropriately chosen suffice to resolve all the vertices of these classes of convex polytopes. Also the metric dimension of the graphs of prisms and antiprisms (they are Archimedean convex polytopes) is finite and does not depend the number of vertices in the graph.

We proved that the metric dimension of the generalized Petersen graphs $P(n, 3)$ is bounded and determined exact value of the metric dimension when $n \equiv 0$ or $1 (\text{mod } 6)$. In the remaining cases we showed that $dim(P(n, 3)) \leq 4$ when $n \equiv 3, 4, 5 (\text{mod } 6)$, $n \geq 9$ and $dim(P(n, 3)) \leq 5$ when $n \equiv 2 (\text{mod } 6)$ and $n \geq 26$. In the same way as for the case $n \equiv 1 (\text{mod } 6)$ it is not difficult to show that $dim(P(n, 3)) \geq 3$ when $n \equiv 2, 3, 4, 5 (\text{mod } 6)$. In this way, some questions naturally arise from the text.

**1**. Characterize family of plane graphs having constant metric dimension.

**2**. Find the exact value of the metric dimension of $P(n, 3)$ when $n \equiv 2, 3, 4, 5 (\text{mod } 6)$.

**3**. Prove that for any connected graph $G$ of order $n \geq 2$ the number of pairs $\{x, y\}$

such that $D(x, y) = V(G)$ is bounded above by $\lfloor n^2/4 \rfloor$.

**4**. Prove that $pd(G_{10}) = 5$ and $pd(G_{12}) = 6$.

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