ON STUDY OF CERTAIN SUBCLASSES OF ANALYTIC AND MULTIVALENT FUNCTIONS

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By

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Dedicated

To

My Parents

And

Specially

To My Late Mother

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ABSTRACT

In this thesis, we study and define certain families of regular and multivalent functions. In particular, we study, the set $\mathcal{SL}$ of lemniscate of Bernouli, the set of Janowski Bazilevič functions, the set $\mathcal{LD}_{\eta}^{k}(a,c,\rho)$ of regular functions of reciprocal order and the set $\mathcal{ST}_{p}(k,\delta,\eta,q)$ of multivalent functions. Our new defined families generalize the set of Bazilevič functions, the set of convex and starlike functions of reciprocal order and the families of uniformly convex and starlike functions. The family $\mathcal{ST}_{p}(k,\delta,\eta,q)$ of multivalent functions is defined by using newly defined q-analogue of p-valent Ruscheweyh operator. We employ various tools and techniques to study certain geometric properties of these families of functions. We mainly study fourth Hankel determinant of the functions belongs to set $\mathcal{SL}$. Further we investigate coefficient bounds, arc-length problem, growth result, integral preservance and some other interesting properties of these classes of functions.

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0.1 LIST OF NOTATIONS

$\mathbb{C}$	:	Set of complex numbers
$\mathbb{R}$	:	Set of real numbers
$\mathbb{R}^+$	:	Set of positive real numbers
$\mathbb{Z}$	:	Set of integers
$\mathbb{Z}^+$	:	Set of positive integers
$\mathbb{N}$	:	Set of Natural numbers
$\mathfrak{D}$	:	Open unit disc
$\mathfrak{A}$	:	Set of regular functions
$\mathfrak{S}$	:	Set of schlicht functions
$\mathcal{P}$	:	Set of Carathéodory functions
$\mathcal{P}(\zeta)$	:	Set of Carathéodory functions of order ζ
$\mathcal{P}_\eta$	:	Set of Carathéodory functions of complex order η
$\mathcal{C}$	:	Set of convex functions
$\mathcal{S}^*$	:	Set of starlike functions
$\mathcal{K}$	:	Set of close-to-convex functions
$\mathcal{C}^*$	:	Set of quasi-convex functions
$k(z)$	:	Köebe function
$\mathcal{C}(\zeta)$	:	Set of convex functions of order ζ
$\mathcal{S}^*(\zeta)$	:	Set of starlike univalent functions of order ζ
$\mathcal{K}(\zeta)$	:	Class of close-to-convex functions of order ζ
$\Omega[A, B]$	:	Circular domain
$\prec$	:	Subordination symbol
$f * g$	:	Convolution of regular functions f and g
$w\,(z)$	:	Schwarz function
$\mathcal{P}[A, B]$	:	Set of janowski functions
$\mathcal{S}^*\,[A, B]$	:	Set of Janowski starlike functions
$\mathcal{C}\,[A, B]$	:	Set of Janowski convex function

Ω_k	: Conic domain
$\mathcal{UCV}$	: Set of uniformly convex functions
$\mathcal{UST}$	: Set of uniformly starlike functions
$k-\mathcal{UCV}$	: Set of k-uniformly convex functions
$k-\mathcal{ST}$	: Set of k-starlike functions
$\mathcal{B}(\alpha,\beta)$	: Set of Bazilevic function of order α and type β
$\mathfrak{L}(a,c)f(z)$	: Carlson-Shaffer operator
$H_{q,n}(f)$	: qth Hankel determinant
$(x)_n$	: Pochhammer symbol
$\mathfrak{A}_p$	: Set of p-valent analytic functions
$\Phi(a,c;z)$	: Confluent hypergeometric functions
$\mathcal{D}^{\delta}(f(z))$	: Ruscheweyh derivative of $f(z)$
$\partial_q f(z)$	: $q-$ derivative of $f(z)$
$[n,q]!$	: $q-$ number shift factorial of n
$[t,q]_n$	: $q-$ generalized Pochhammer symbol for t
$\Gamma_q(t+1)$	: $q-$ gamma function

Chapter 1

Introduction

1.1 Historical overview

Complex analysis is one of the fundamental branch of mathematics with origins in 19th century and before. It deals with the functions of complex numbers together with their derivatives, manipulation, and other properties. Complex analysis is a wide-ranging subject and its applications can be established not only in mathematics but also in many other fields of science. Among the other fields, Geometric Function Theory (GFT) is one of the most emerging part of complex analysis which deals with geometrical property of regular functions (or analytic functions). The researcher who developed GFT firstly were Cauchy, Riemann and Weierstrass. All of them added original thoughts to it. In 1851, Riemann proved a remarkable Theorem known as Riemann Mapping Theorem [43] in which he allows the researchers to take $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$, the open unit disk with centre at the origin, instead of the arbitrary domain $\mathcal{D}$. This theorem motivates the mathematicians to work in this field. The Theory of schlicht (or univalent) functions is the most prominent area of GFT which was first emerged by Köebe [70] in 1907. The functions which are both regular and schlicht along with normalization condition $f(0) = 0$ and $f'(0) = 1$ form the class $\mathfrak{S}$. This family $\mathfrak{S}$ proved itself as a keystone for the advance and current research in this field. For such function, he also showed the presence of a "covering constant" k that the image of $\mathfrak{D}$ must cover the disk $\{w : |w| < k\}$. Many mathematicians paid consideration to this idea, but in a very short time, the paper of Bieberbach [32] surfaced where the renowned coefficient conjecture was imagined. This familiar conjecture states "If $f(z)$ defined by $f(z) = z + a_2 z^2 + ...$is regular and schlicht in $\mathfrak{D}$, then $|a_n| \leq n$". This legendary thought is considered rich for research and investigation in GFT. Like a challenge for researchers of that time de-Brenges [42], solved it in 1985. Many subfamilies of schlicht functions of the set $\mathfrak{S}$ were introduced with respect to geometric point of view of their image domain, such as the families $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{K}$ of convex, starlike and close-to-convex schlicht functions respectively, following the inclusion $\mathfrak{S} \supset \mathcal{K} \supset \mathcal{S}^* \supset \mathcal{C}$, were of great attention. The set $\mathcal{P}$ of Caratheodory functions played an essential role in GFT. With the help of this family, Study [151] in 1913 and

Nevanlinna [99] in 1921 proved the analytic classification of the families of convex and starlike functions respectively. In 1915, Alexander [7] successfully developed an elegant relation like a bridge between the families of starlike and convex functions. Theory of convex and starlike functions was further nourished by Robertson [130] and he offered with some order, the families of convex and starlike functions.

As mentioned earlier, the image domain $f(\mathfrak{D})$ of regular functions $f(z)$ acquire great importance. In accordance to the geometrical interpretation and other charactristics of image domains, various families have been introduced and studied. Lindelöf [81] in 1973 gave the subordination theory . Keeping the theory of subordination in view Janowski [59] established the idea of circular domain denoted by $\Omega[A, B]$ and then defined the family $\mathcal{P}[A, B]$ of functions that send $\mathfrak{D}$ onto the circular regions, called Janowski functions. These functions were studied by a number of well known scholars and in association with with circular domains various families of regular functions have been introduced, examples can be seen in [87, 105, 104]. Like circular domains, conic domian has an important role in GFT and several famillies are associated to it. The perception of conic domain was given by Goodman[47, 48] in 1991 and introduced $\mathcal{UCV}$ the families of uniformly convex and $\mathcal{UST}$ the family uniformly starlike functions. By construting examples Goodman showd that the Alexander type relation encountered failure between the families of $\mathcal{UCV}$ and $\mathcal{UST}$. He also defined two-variable analytic characterization for these families. In [135], Rönning defined the set $\mathcal{ST}$ as;

$$\mathcal{ST} = \{f(z) : f(z) = zg'(z), \ g(z) \in \mathcal{UCV}\}.$$

Futher he proved in [136] that the inclusions $\mathcal{ST} \subseteq \mathcal{UST}$ and $\mathcal{UST} \subseteq \mathcal{ST}$ does not hold. Later, Rönning (Ma and Minda independently) studied and found most suitable one variable analytic characterizations for these families. Parabolic domain was further generalized by Kanas and Wisniowska [66, 67] by establishing the conic domain Ω_k, $k \geq 0$ in general form which epitomize all the three conical structures, that is, parabola, hyperbola, and ellipse. Conic domain has emerged as most attractive field of GFT for the mathematicians.

The researchers such as Acu, Dziok, Kanas, Noor, Owa, Sokół, Srivastava, Wisniowska, and many others have insert some outstanding research items with respect to quality as well as quantity in the field of GFT. The work like that is extensively encouraging in showing new guidelines in the field. The present and near past research work of the fore mentioned researchers has become a massive source of inspiration and growing demand for today's scholars.

1.2 Geometric function theory in Today's sciences

Geometric Function Theory (GFT) has proved itself to be very significant in the field of mathematics. It is such a commendable and valued subject that its applications almost exists in every field of applied sciences such as theory of partial differential equations, non linear integrable system theory, fluid dynamics, engineering and electronics, modern mathematical physics, etc. Conformal mapping is a proper and useful technique for solving eigen value problems in plane and this way of solving is called conformal transformation details is given in [76]. Research in [30] shows that in construction of finite-gap solutions to non linear integrable systems, the theory of compact Riemann surface is used in great extent. Finding of the uniform potential flow around an assembly of a circular barrier is one of the basic and major problem in fluid mechanics. And physical problem containing a multiply connected domains can be best reflected in it. The Schottky-Klein prime function which is function of complex variable, plays a central role in explaining a range of problems in GFT. There is a significant and inclusive work of Crowdy [40] on application of GFT. Crowdy solved problems involving multiply connected domains by using the Schottky-Klein prime function. More applicability of GFT can be seen in [40, 41, 49, 154, 155].

1.3 Preface

The title "On certain subclasses of analytic and multivalent functions" of this thesis, proved the we are mainly focusing on the study of different subfamilies of regular and multivalent functions. One of the major extension of schlicht functions is the theory of multivalent functions and the tremendous development has been made in this area.

A function $f(z)$ which is regular in a set $\mathcal{D} \subset \mathbb{C}$ is called multivalent or p-valent, see [54], $p \in \mathbb{N} = \{1, 2, \ldots\}$ in $\mathcal{D}$ if for every complex number w, the equation $w - f(z) = 0$ have at most p roots in $\mathcal{D}$ and there will be a complex number w_0 such that the equation $w_0 - f(z) = 0$, has exactly p roots in $\mathcal{D}$. By setting $p - 1 = 0$, we have schlicht functions because in this case $f(z)$ assumes no value more than once in $\mathfrak{D}$. The theory of p-valent functions implied something more than just a generalization of schlicht function theory. The extension of any result from schlicht to p-valent may be trivial or extremely difficult or perhaps false. There are several results that generalize classical results on schlicht functions. The first success in obtaining sharp inequalities for multivalent functions was attained by Hayman in 1955, see [54].

In second chapter we recall some elementary and classical concepts of GFT which gives an essential background for the successive chapters. Initial part is the review some fundamental definitions such as the family of normalized functions which are regular, the family of normalized functions which are schlicht. The set $\mathcal{P}$ containing the functions $p(z)$ with $p(0) = 1$ & $\mathfrak{Re}\, p(z)$ greater than zero, and their generalizations is the main focus to discuss in the third section of this chapter. The next section deals with some common subfamilies of schlicht functions that are defined by geometric and analytic conditions. Some important results associated with these subfamilies are also given. The concepts of subordination and convolution are also discussed in section four as every expert of this field knows its importance and each one is acknowleged as a tool in dealing with different issues of this field. In the fifth section, we give a short overview of circular domain and their applications to define the families of Janowski starlike and Janowski convex functions. In the sixth section, we deals with the conic domains and associated families. These families perform a key role for the description of some of our main results. The concept of domain bounded by right half of the Bernouli lemniscate was first floated by Sokół and Stankiewicz [148]. They defined the region by the following relation $|w^2 - 1| < 1$. The class $\mathcal{SL}$ of starlike functions related with right half of the Bernouli lemniscate is defined as

$$\mathcal{SL} = \left\{ f(z) \in \mathfrak{A} : \left| 1 - \left(zf'(z)/f(z) \right)^2 \right| < 1, \ \ (z \in \mathfrak{D}) \right\}.$$

The geometrical interpretation of the fact $f(z) \in \mathcal{SL}$ is that, for $z \in \mathfrak{D}$, the ratio $zf'(z)/f(z)$ lies in the region bounded by the right half side of the Bernoulli's lemniscate by the inequality $|w^2 - 1| < 1$. It is easily observed that; a function $f(z) \in \mathcal{SL}$ if and only if

$$zf'(z)/f(z) \prec \sqrt{1+z}.$$

Pommerenke [125, 126] introduced the qth Hankel matrices for $q \geq 1$ & $n \geq 0$. In the space of regular functions, to estimate the rate of growth of qth Hankel determinants is one of the major ongoing problems. Several authors studied this problem for different types of regular and schlicht functions, see [26, 60, 129, 157]. The qth Hankel determinant is characterized as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

The second Hankel determinant $H_{2,2}(f)$ and third Hankel determinant $H_{3,1}(f)$ for certain families of regular and schlicht functions is an ongoing research area in geometric function theory. In chapter 3, we study the fourth Hankel determinant of the family $\mathcal{SL}$ for the first time in the literature.

In 1973 the introduction of Janowski function [59] gave rise the idea of the circular domain. Further he investigated certain properties of $\mathcal{S}^*[A,B]$ & $\mathcal{C}[A,B]$, where these notations denotes, respectively, the families of starlike and convex functions related with Janowski functions. In 1923 Loewner's method [89] came forward which was further expalined and improved in 1943 by Kufarev [75]. The idea was to insert the region in a constantly increasing set of domains and then illuminated this class by a differential equation. Using the Loewner-Kufarev differential equation, Bazilevič [29] defined the set of regular functions by imbedding the family of starlike functions and the family $\mathcal{P}$ of Carathéodory functions. The family of Bazilevič functions

$$f(z) = \left\{ (\alpha + i\beta) \int_0^z g^{\alpha}(\xi)\, p(\xi)\, \xi^{i\beta-1} d\xi \right\}^{\frac{1}{\alpha+i\beta}}$$

is a subfamily of schlicht functions in $\mathfrak{D}$, where $g(z) \in \mathcal{S}^*$ with $g(0) = 0,\ p(z)$ is regular with $\mathfrak{Re}p(z) > 0$ in $\mathfrak{D}$, $\alpha > 0$, and $\beta \in \mathbb{R}$. Specific values of $g(z)$, $p(z)$, α and β,different families of schlicht functions such as convex, starlike, close-to-convex. In chapter 4, we introduce the family of Janowski Bazilevič functions. Further, we study Fekete-Szegö inequality, coefficient bounds, arc-length, coefficient difference, integral preserving and some other geometric properties.

In literature of GFT, the applications of differential and integral operators plays an active role. A huge amount of subfamilies of regular and schlicht functions can be traced in the literature which are introduced by using these operators. Carlson and Shaffer [35] used the confluent hypergeometric functions $\Phi(a,c;z)$ and the concept of convolution to define an operator $\mathfrak{L}(a,c)f(z)$. The Carlson-Shaffer operator generalizes the Ruschweyeh derivative operator [139] and Owa-Srivastava fractional differential operator [116]. The operator $\mathfrak{L}(a,c)f(z)$ has been used broadly on the space of regular and schlicht functions in the region $\mathfrak{D}$. The operator $\mathfrak{L}(a,c)f(z)$ sends $\mathfrak{A}$ onto itself and is continuous on $\mathfrak{A}$. This convolution operator provides a convenient representation of differentiation and integration. In the chapter 5, a new subfamily $\mathcal{LD}_{\eta}^{k}(a,c,\rho)$ of regular functions is defined by using Carlson-Shaffer operator. This set of functions generalizes some familiar families such as starlike and convex functions of reciprocal order. Some valuable properties including distortion bounds, coefficient estimates and subordination result of this subfamily of regular functions are derived.

Quantum calculus theory (q-calculus) is simply the classical calculus without using the notion of limits. Recently, the researchers has shown significant interest in the study of q-calculus theory due to its vast applications in various subfields of mathematics and physics,. see detail [18]. Ismail et al. [56] were the first who gave the connection between the q-calculus and Geometric function theory. Uptill now a considerable interest has been developed among the geometric functionalist in q-calculus theory. In chapter 6, by using the function $\Phi_p(q, \delta + p; z)$ and definition of q-derivative along with the idea of convolution, a q-differential operator $\mathcal{L}_q^{\delta+p-1} f(z)$ for p-valent regular functions is introduced. This operator reduces to the differential operator introduced by Goel and Sohi [44] by taking $q = 1$ and further by making $p = 1$, we obtain the familiar Ruscheweyh derivative operator [139]. Furthermore, a subfamily $\mathcal{ST}_p(k, \delta, \eta, q)$ of the set of p-valent functions is defined by using the operator $\mathcal{L}_q^{\delta+p-1}$ and discuss some interesting properties of this set. These functions are closely related with the domains bounded by conic sections.

Chapter 2

Preliminary concepts

This chapter is composed of some classical and elementry concepts of Geometric Function Theory which will provide basis for undersatnding the concepts and will helpfull in proving some results in subsequent chapters. We include some basic results for the sake of completeness and for the other proofs some standard texts are reffered, for example, see [43, 46, 54, 127].

In the initial part, we give some fundamental definitions such as the families of normalized regular and normalized schlicht functions. The family $\mathcal{P}$ containing the function $p(z)$, with $p(0) = 1$ & $\mathfrak{Re}p(z)$ greater than zero, and their generalizations is the main focus to discuss in the third section of this chapter. The next section deals with some common subfamilies of schlicht functions defined in terms of analytic and geometric conditions. Some important results associated with these subfamilies are also given. The concepts of subordination and convolution are also discussed in section four as every expert of this field admits its importance and compatibility in dealing with different issues of this field. In the fifth section, we give a short overview of circular domain and their applications to define the family of Janowski starlike and Janowski convex functions. In the sixth section, we deals with the conic domains and associated families. These families perform a key role for the description of some of our main results.

2.1 Regular and schlicht functions

This section is brief discussion of the family $\mathfrak{A}$ of normalized regular functions and the family $\mathfrak{S}$ of normalized schlicht functions in the region $\mathfrak{A}$. Some important properties of these families are also highlighted here. For detail of this section, see [43, 46, 54, 127].

Regular function

Definition 2.1.1 *A complex-valued function $f(z)$ defined in a domain $\mathcal{D}$ is said to be regular or analytic at a point $z_\circ$, if its derivative exists not only at $z_\circ$ but also in some neighborhood of $z_\circ$. A function $f(z)$ is called regular in a domain $\mathcal{D}$, if it is regular at* *every point in $\mathcal{D}$.*

It is often difficult to handle problems by taking any domain $\mathcal{D}$, so the region $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ is considered instead of $\mathcal{D}$ which is due to the well-known Riemann Mapping Theorem [43, 46] proved by Riemann in 1851. In general, this Theorem is acknowledged as one of the major contribution in the field of Geometric Function Theory.

Theorem 2.1.1 *Let $\mathcal{D} \subset \mathbb{C}$ be a simply-connected domain with atleast two boundary points and $\zeta_0 \in \mathcal{D}$. Then there is a unique regular function $q(z)$ which sends $\mathcal{D}$ conformally onto the region $\mathfrak{D}$ and has the properties $q(\zeta_0) = 0$ & $q'(\zeta_0) > 0$.*

By the set $\mathfrak{A}$, we mean the collection of all those functions which are regular in $\mathfrak{D}$ and normalized by the following power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{2.1.1}$$

Schlicht functions

The famous Riemann Mapping Theorem [43, 46] developed a great interest of researchers in Geometric functions theory which mainly focused on the concept of schlicht functions defined in $\mathfrak{D}$. This concept was first taken into account by Köebe [70] in 1907. A function in which the open unit disk $\mathfrak{D}$ and schlicht domain $f(\mathfrak{D})$ are linked by one-to-one correspondence is called univalent function or schlicht function. Where Schlicht is a German word specifying a region which does not contain any branch point and is not self-overlapping.

Definition 2.1.2 *A function $f(z) \in \mathfrak{A}$ is univalent (Schlicht) in a domain $\mathcal{D}$, if it never takes the same value twice; that is, $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in $\mathcal{D}$ with $z_1 \neq z_2$. In other words, $f(z)$ is one-to-one (injective) mapping from $\mathfrak{D}$ onto image domain $f(\mathfrak{D})$.*

The family of all functions that are regular, schlicht in $\mathcal{D}$ and normalized by the condition $f(0) = f'(0) - 1 = 0$ is denoted by $\mathfrak{S}$. Köebe function $k(z)$ is the leading example of normalized schlicht function, where

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} nz^n. \tag{2.1.2}$$

The Köebe function $k(z)$ maps $\mathfrak{D}$ in a one to one manner onto the the entire complex plane except $(-\infty, -\frac{1}{4}]$. For further details, see [43, 46, 54, 127]. It should be noted that the sum of two schlicht functions is not necessarily to be schlicht. This can be verfied for the functions $z(1-z)^{-1}$ and $z(1+iz)^{-1}$, both of them are schlicht but the derivative of their sum is zero at the point $\frac{1}{2}(1+i)$. On the other hand , the class $\mathfrak{S}$ posses some uncomplicated simple transformations including as rotation, dilation, conjugation and disk automorphism.

In 1916 the bounds of 2nd coefficient for the functions of the set $\mathfrak{S}$ was proved by Bieberbach. Furteher he conjectured that for the functions $f(z)$ in class $\mathfrak{S}$ having the form of (2.1.1), $|a_n| \leq n$, for all $n \geq 2$. To prove this conjecture many attempts were made by different mathematicians, but the credit of proof goes to de-Branges [42] who made a successful attempt in 1985, and is known as de-Branges Theorem.

2.2 Carathéodory functions

This section concerns to the familiar set of Carathéodory functions which gives a central role in GFT. The worth of this is evident from the fact that all the subfamilies of the set $\mathfrak{S}$ of schlicht functions have been linked to the family of Carathéodory functions. This section also deals with the discussion of some of its related families such as $\mathcal{P}(\zeta)$ with $0 \leqq \zeta < 1$ and $\mathcal{P}_\eta$ with $\eta \in \mathbb{C}\backslash\{0\}$. We include here certain basic properties, such as coefficient estimate, distortion bounds and its relation with the family of schlicht functions, for more details, see [43, 46, 54].

The family $\mathcal{P}$ of Carathéodory functions

Definition 2.2.1 *A regular function $q(z)$ belong to the set $\mathcal{P}$ if*

$$q(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad (z \in \mathfrak{D}), \tag{2.2.1}$$

and satisfies the inequality

$$\mathfrak{Re}q(z) > 0, \quad (z \in \mathfrak{D}).$$

It is important to highlight that the function $q(z)$ in $\mathcal{P}$ may not be schlicht. For example $q(z) = 1 + z^n$ is in the family $\mathcal{P}$ for integral value of $n \geq 0$, however for $n \geq 2$, this function $q(z)$ fails to schlicht. The Möbius function given by

$$\mathcal{L}_0(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n, \quad (z \in \mathfrak{D}), \tag{2.2.2}$$

plays the same role in the set $\mathcal{P}$ just as Köebe function $k(z)$ do in the set $\mathfrak{S}$. The family $\mathcal{P}$ can easily be verified for convexity. That is, letting $p_1(z)$ and $p_2(z)$ be in $\mathcal{P}$ and non-negative numbers μ_1 and μ_2 with $\mu_1 + \mu_2 = 1$, then

$$p(z) = \mu_1 p_1(z) + \mu_2 p_2(z), \tag{2.2.3}$$

is also in $\mathcal{P}$. For more details, see [43, 46, 54].

The Herglotz representation theorem

In 1911, Herglotz [55] obtained the integral representation for the functions $q(z) \in \mathcal{P}$. Converse of this result is also true, see [46].

Theorem 2.2.1 *Let the function $d\mu(\tau)$ be non decreasing in $0 \leq \tau \leq 2\pi$ with*

$$\int_0^{2\pi} d\mu(\tau) = 2\pi.$$

Then the function

$$q(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1+e^{-i\tau}z}{1-e^{-i\tau}z} d\mu(\tau) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{L}_0 \left(e^{-i\tau}z\right) d\mu(\tau)$$

is also in the set $\mathcal{P}$.

Theorem 2.2.2 *Let* $q(z) \in \mathcal{P}$ *and having the series form* (2.2.1). *Then*

$$|c_n| \leq 2, \text{ for all } n \in \mathbb{N} = \{1, 2, \cdots\}. \tag{2.2.4}$$

Equality occurs unless $q(z)$ *is a rotation of the function in equation* (2.2.2), *for details see* ([*43*, *46*]).

Theorem 2.2.3 *Let* $q(z) \in \mathcal{P}$. *Then the following inequalities hold*

$$\frac{1-r}{1+r} \leq \mathfrak{Re} q(z) \leq |q(z)| \leq \frac{1+r}{1-r}, \tag{2.2.5}$$

$$\left|\frac{zq'(z)}{q(z)}\right| \leq \frac{2r}{1-r^2}, \tag{2.2.6}$$

$$|zq'(z)| \leq \frac{2r \mathfrak{Re} q(z)}{1-r^2}, \tag{2.2.7}$$

and all are best possible.

For further details, we refer the standard texts books [43, 46].

Lemma 2.2.1 *[54]. Let* $q(z) \in \mathcal{P}$. *Then for* $z = re^{i\theta} \in \mathfrak{D}$

$$\int_0^{2\pi} \left|q(re^{i\theta})\right|^{\varsigma} d\theta < C(\varsigma) \frac{1}{(1-r)^{\varsigma-1}},$$

where $C(\varsigma)$ *is a constant which depends only on* ς, $\varsigma > 1$.

Lemma 2.2.2 *[123]. Let $q(z) \in \mathcal{P}$. Then for $z = re^{i\theta} \in \mathfrak{D}$*

$$\frac{1}{2\pi}\int_{0}^{2\pi} |q(z)|^2 d\theta \leq \frac{1+3r^2}{1-r^2}. \tag{2.2.8}$$

The family $\mathcal{P}(\zeta)$ of Carathéodory functions of order ζ

The terminology of some order for regular functions was first introduced by Robertson [130] in 1936, as the following;

Definition 2.2.2 *A regular function $q(z)$ belongs to the set $\mathcal{P}(\zeta)$, $0 \leq \zeta < 1$, if it has the form* (2.2.1) *and satisfies the condition*

$$\mathfrak{Re}q(z) > \zeta, \quad (z \in \mathfrak{D}).$$

In other way, if a function $q(z) \in \mathcal{P}(\zeta)$, then for $q_1(z) \in \mathcal{P}$ we have

$$q(z) = (1-\zeta)\, q_1(z) + \zeta.$$

Making $\zeta = 0$ in $\mathcal{P}(\zeta)$, we obtain the class $\mathcal{P}$. For details, see [43, 46].

Lemma 2.2.3 *[95]. Let $q(z) \in \mathcal{P}(\zeta)$. Then for, $z = re^{i\theta} \in \mathfrak{D}$ we have*

$$\begin{aligned}
\frac{1+(2\zeta-1)\,r}{1+r} &\leq |q(z)| \leq \frac{1-(2\zeta-1)\,r}{1-r}, \\
\left|\frac{zq'(z)}{q(z)}\right| &\leq \frac{(2-2\zeta)\,r\,(1+r)}{(1-r)\,(1-2\,(\zeta-1)\,r-(2\zeta-1)\,r^2)}, \\
|zq'(z)| &\leq \frac{(2-2\zeta)\,r\mathfrak{Re}q(z)}{(1-r)\,(1-(2\zeta-1)\,r)}.
\end{aligned}$$

The family $\mathcal{P}_\eta$ of Carathéodory functions of complex order η

In 1985, Nasr and Aouf [98] used a complex number $\eta \neq 0$ as order to define the family $\mathcal{P}_\eta$ of Carathéodory functions of complex order η. They defined $\mathcal{P}_\eta$ in the following way.

Definition 2.2.3 *An regular function $q(z) \in \mathcal{P}_\eta$ with non-zero complex number η, if*

$$\mathfrak{Re}\left\{1 + \frac{1}{\eta}\left(q(z) - 1\right)\right\} > 0.$$

In other words $q(z) \in \mathcal{P}_\eta$ if

$$q(z) = \eta q_1(z) + (1 - \eta),$$

where $q_1(z) \in \mathcal{P}$. For $\eta = 1$, we have $\mathcal{P}_1 = \mathcal{P}$.

2.3 Certain subfamilies of schlicht functions

In order to prove the Bieberbach's conjecture the scholars worked hard by using different ideas which laid the invention of several subfamilies of schlicht functions. These families are introduced by natural geometrical conditions. Among them, the families of starlike, convex, close-to-convex and quasi-convex functions are the most prominent subfamilies of schlicht functions. This section is a brief review of some special subfamilies of normalized schlicht functions. We shall study how theses subfamilies are inter related, their basic properties, and numerous other results shall be also investegated. For some detail of these families, see [43, 46].

The family $\mathcal{C}$ of convex schlicht functions

Definition 2.3.1 *A subset $\mathfrak{X} \subset \mathbb{C}$ is convex if for every pair of interior points w_1 and w_2 of $\mathfrak{X}$, the line segment joining w_1 and w_2 is also in the interior of $\mathfrak{X}$, that is, for any*

$w_1, w_2 \in \mathfrak{X}$ *implies*

$$tw_1 + (1-t)\, w_2 \in \mathfrak{X}, \quad for \quad 0 \leq t \leq 1.$$

A function $f(z)$ *is convex if* $f(z)$ *maps the region* $\mathfrak{D}$ *onto a convex domain. The family of all such convex schlicht functions is denoted by* $\mathcal{C}$.

In Theorem 2.3.1, Study [151] in 1913 proved an analytic characterization for the set of convex functions which is given as;

Theorem 2.3.1 *A schlicht function* $f(z) \in \mathcal{C}$ *if and only if*

$$\mathfrak{Re}\frac{zf''(z)}{f'(z)} > -1, \; for \; z \in \mathfrak{D}.$$

The famous example of functions in class $\mathcal{C}$ is $l(z)$, where

$$l(z) = z(1-z)^{-1} = z + \sum_{n=2}^{\infty} z^n, \; z \in \mathfrak{D}. \tag{2.3.1}$$

This function is an extremal functions for the set $\mathcal{C}$ of convex functions.

Theorem 2.3.2 *Let* $f(z) \in \mathcal{C}$ *having the form of* (2.1.1)*. Then for* $z \in \mathfrak{D}$,

$$|a_n| \leq 1, \; for \; all \; n \geq 2.$$

Equality holds for $l(z)$ *given by* (2.3.1).

The family $\mathcal{S}^*$ of starlike schlicht functions

Definition 2.3.2 *A subset* $\mathfrak{X} \subset \mathbb{C}$ *is star-shaped with respect to a fixed point* $z_\circ$ *inside* $\mathfrak{X}$ *if the lin segment joining* $z_\circ$ *to any other points of* $\mathfrak{X}$ *lies wholly in* $\mathfrak{X}$.

Definition 2.3.3 *A function* $f(z)$ *which send* $\mathfrak{D}$ *onto a region which is star-shaped with respect to* $w_\circ$ *is starlike function with respect to* $w_\circ$*, In particular taking* $w_\circ = 0$, *the function $f(z)$ is simply said to be starlike function and family of such functions is denoted by $\mathcal{S}^*$.*

Using the same idea of Theorem 2.3.1, the analytic description for starlike schlicht functions was given by Nevanlinna [99] stated as;

Theorem 2.3.3 *A schlicht function $f(z) \in \mathcal{S}^*$, if and only if*

$$\mathfrak{Re}\frac{zf'(z)}{f(z)} > 0, \ \textit{for } z \in \mathfrak{D}.$$

The köebe function $k(z)$ given in (2.1.2) is an extremal function for the class $\mathcal{S}^*$ of starlike functions. $k(z)$ sends $\mathfrak{D}$ onto the plane $\mathbb{C}$ excluding the strip from $-\infty$ to $-\frac{1}{4}$. The Köebe function $k(z)$ is starlike, but not convex, see [46].
From the discussion above, it is obvious that

$$\mathcal{C} \subset \mathcal{S}^* \subset \mathfrak{S}.$$

The relationship of the families of convex and starlike functions was observed by Alexander [46], these families are related by the following relation

$$f(z) \in \mathcal{C} \iff zf'(z) \in \mathcal{S}^*.$$

Theorem 2.3.4 *Let $f(z)$ be a starlike function and has the form* (2.1.1)*. Then*

$$|a_n| \leq n, \ \textit{for all } n \geq 2.$$

These coefficient bounds are sharp and the köebe function gives the equality.

Convex and starlike schlicht functions of order ζ

Robertson [130] in 1936 introduced the families of starlike schlicht and convex schlicht functions of order ζ $(0 \leq \zeta < 1)$. These classes are denoted with $\mathcal{S}^*(\zeta)$ and $\mathcal{C}(\zeta)$ respectively and are defined by

$$\mathcal{S}^*(\zeta) = \left\{ f(z) \in \mathfrak{S} : \frac{zf'(z)}{f(z)} \in \mathcal{P}(\zeta),\ (z \in \mathfrak{D}) \right\},$$

$$\mathcal{C}(\zeta) = \left\{ f(z) \in \mathfrak{S} : \frac{(zf'(z))'}{f'(z)} \in \mathcal{P}(\zeta),\ (z \in \mathfrak{D}) \right\}.$$

We see that $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{C}(0) \equiv \mathcal{C}$. Strohhacker [150], proved that $\mathcal{C} \subset \mathcal{S}^*(\frac{1}{2})$. It is observed that these two classes also holds the Alexander type relation.

The family $\mathcal{K}$ of close-to-convex functions

Here in this section we discuss another essential subfamilies of schlicht functions namely the family of close-to-convex functions which has a simple geometrical interpretation. in 1952, Kaplan [68] introduced and characterized this family. See also [43, 46].

Definition 2.3.4 *A schlicht function $f(z)$ is close-to-convex if for given a convex function $g(z)$, we have*

$$\mathfrak{Re}\frac{f'(z)}{g'(z)} > 0,\ z \in \mathfrak{D}. \tag{2.3.2}$$

We denote by the symbol $\mathcal{K}$ the family of close-to-convex functions. The functions $f(z)$ of the this family are normalized with $f(0) = 0$ & $f'(0) = 1$. Kaplan in [68], showed that close-to-convex functions are schlicht. The inclusion among the families $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{K}$ is given as

$$\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathfrak{S}.$$

Close-to-convex functions has simple geometric interpretation. The conditions for its description have some similarities to the families of convex and starlike functions. "Let $f(z) \in \mathfrak{A}$ and let the symbol $\mathcal{C}_r$ denotes the image of unit circle $|z| = r$ $(0 < r < 1)$ under $f(z)$. Roughly speaking, $f(z) \in \mathcal{K}$, if and only if none of the curve $\mathcal{C}_r$ makes a "reverse hairpin turn". In more precise words, the requirement is that with the increases of θ, the unit tangent rotates clockwiseor anti clockwise direction in suach amanner that it never come back to any previous position unless its direction is completly reversed. Since we know that

$$\frac{\partial}{\partial\theta}\left[\arg\left\{\frac{\partial}{\partial\theta}f(re^{i\theta})\right\}\right]=1+\mathfrak{Re}\frac{zf''(z)}{f'(z)},\ z=re^{i\theta},$$

the following theorem by Kaplan [68], is is very important for analytic definition for the set of close-to-conex functions.

Theorem 2.3.5 *Let $f(z)\in\mathfrak{A}$ be such that $f'(z)\neq 0$ in $\mathfrak{D}$. Then $f(z)\in\mathcal{K}$ if and only if*

$$\int_{\theta_1}^{\theta_2}\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}d\theta>-\pi,\ z=re^{i\theta}, \tag{2.3.3}$$

for each $r\in(0,1)$ and every pair $\theta_1,\ \theta_2$ with $0\leq\theta_1<\theta_2\leq 2\pi$.

It should be noted that $f(z)=z^n$, for $n>1$ fulfills the inequality $(2.3.3)$, but $f(z)$ is not schlicht and as a result not close-to-convex.

The family $\mathcal{C}^*$ of quasi-convex functions

This section is a briefly discussion of $\mathcal{C}^*$ the family of quasi-convex functions which was investigated by Noor and Thomas [113] in 1980.

Definition 2.3.5 *A function $f(z)\in\mathfrak{A}$ is in set C^*, if and only if for a convex function $g(z)$,the condition*

$$\mathfrak{Re}\frac{(zf'(z))'}{g'(z)}>0,\ z\in\mathfrak{D}.$$

hols. By taking $f(z)=g(z)$, we have $f(z)\in\mathcal{C}$. This shows that $\mathcal{C}\subset\mathcal{C}^*$, that is, every convex schlicht function is quasi-convex function. It can easily be observed that well-known Alexander relation holds among the families $\mathcal{K}$ and $\mathcal{C}^*$. That is,

$$f(z)\in\mathcal{C}^*\iff zf'(z)\in\mathcal{K}.$$

From the above defined classes we summarized the following chain of inclusions

$$(i)\quad \mathcal{C} \;\subset\; \mathcal{C}^* \subset \mathcal{K} \subset \mathfrak{S},$$
$$(ii)\quad \mathcal{C} \;\subset\; \mathcal{S}^* \subset \mathcal{K} \subset \mathfrak{S}.$$

Inclusion given in (i) and (ii) indicate that no relation exists between the classes $\mathcal{S}^*$ and $\mathcal{C}^*$

$$f_0(z) = \frac{1-i}{2}\frac{z}{1-z} - \frac{1+i}{2}\log(1-z).$$

Here $f_0(z) \in \mathcal{C}^*$ but $f_0(z) \notin \mathcal{S}^*$, for details, see [102, 113].

The family of Bazilevič functions

Let $q(z) \in \mathcal{S}^*$ in the region $\mathfrak{D}$ with $g(0) = 0$, $p(z)$ is regular with $\mathfrak{Re}p(z) > 0$ in $\mathfrak{D}$, $\beta > 0$, and $\alpha \in \mathbb{R}$. Then

$$f(z) = \left[(\beta + i\alpha)\int_0^z q^\beta(\zeta)p(\zeta)\zeta^{i\alpha-1}d\zeta\right]^{\frac{1}{\beta+i\alpha}}, \tag{2.3.4}$$

has been shown by Bazilevič [29], see also [124], to be regular and schlicht function in $\mathfrak{D}$. The exponents appearing in the definition of $f(z)$ are considered as principal ones. The collection of all functions defined by (2.3.4) is denoted by the class $\mathcal{B}(\alpha, \beta)$. By taking $\alpha = 0$ in (2.3.4) and then differentiating, we get

$$zf'(z) = p(z)q^\beta(z)f^{1-\beta}(z). \tag{2.3.5}$$

The family of functions $f(z)$ fulfilling (2.3.5) is named as the set of Bazilevič functions of type β and this class was investigated by Thomas [152]. Very few papers appeared on the family $\mathcal{B}(\alpha, \beta)$ in literature, however it can be observed that this set of Bazilevič functions is the largest known subfamily of schlicht functions and contains all the basic researched subfamilies of schlicht functions.

2.4 Subordination and Hadamard product

The aim of the present section is to discuss briefly two major concepts "Convolution and Subordination" of Geometric Function Theory. These techniques have major contribution in the development of this field. Miller and Mocannu in [96] and Ruscheweyh in [138] studied the theory of subordination and convolution in detail respectively and gave its applications as well.

Subordination

The idea of subordination between regular functions was given by Lindelöf [81], however the term subordination and the fundamental results involving subordination was introduced by Littlewood [82, 83] and Rogosinski [131, 132] .For detail see [43, 96].

Definition 2.4.1 *If* $f(z)$ *and* $g(z)$ *are regular in* $\mathfrak{D}$, *then* $f(z)$ *is subordinate to* $g(z)$, *written as* $f(z) \prec g(z)$, *if we can find a Schwarz function* $w(z)$, *which is regular in* $\mathfrak{D}$ *with* $w(0) = 0$ *&* $|w(z)| < 1$, *such that* $f(z) = g(w(z))$, *for* $z \in \mathfrak{A}$. *Moreover, if* $g(z)$ *is schlicht in* $\mathfrak{D}$, *then*

$$f(z) \prec g(z) \quad (z \in \mathfrak{D}) \quad \Leftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathfrak{D}) \subset g(\mathfrak{D}).$$

This is known as subordinate principle or Lindelöf's principle.

Lemma 2.4.4 *[132] Let* $q(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *be subordinate to* $F(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$. *If* $F(z)$ *is schlicht in* $\mathfrak{D}$ *and* $F(\mathfrak{D})$ *is convex, then*

$$|c_n| \leq |d_1|, \text{ for } n \geq 1.$$

Convolution or Hadamard product

One of the noteworthy concept in Geometric Function Theory is convolution or Hadamard product. Using this idea so many difficult problems are solved quite easily. In this section, we need first to discuss convolution concept briefly and then to enlighten some of its important results.

Definition 2.4.2 *Let $f(z)$ and $g(z)$ be two regular functions of the form*

$$f(z) = \sum_{n=1}^{\infty} a_n \ z^n \ \text{and} \ \ g(z) = \sum_{n=1}^{\infty} b_n \ z^n.$$

Then the Hadamard product or convolution of $f(z)$ and $g(z)$ is defined as

$$f(z) * g(z) = \sum_{n=1}^{\infty} a_n \ b_n \ z^n, \ \ |z| < 1.$$

This product is due to Jacques Hadamard [51] who gave for the first time a detail analysis of this concept. The convolution has the algebraic properties of ordinary multiplication. The identity element of convolution is given by the geometric series $l(z)$, given in (2.3.1), since,

$$f(z) * l(z) = f(z),$$

for all $f(z)$ in $\mathfrak{A}$, also see [138].

2.5 The families of Janowski type functions

The concept of circular domain palys an important role in definition of certain subfamilies of regular functions. In 1973 Janowski [59] introduced the circular domain in connection of Janowski functions. He described these functions as follows:

Definition 2.5.1 *A function $q(z)$ is in the family $\mathcal{P}[A, B]$ if it is regular in the region $\mathfrak{D}$ such that $q(0) = 1$ and satisfies*

$$q\left(z\right) \prec \frac{Az+1}{Bz+1}, \ \text{for} \ -1 \leq B < A \leq 1.$$

This definition can be geometrically interpret as; a function $q(z) \in \mathcal{P}[A,B]$ if it sends the region $\mathfrak{D}$ onto the set $\Omega[A,B]$ given by

$$\Omega[A,B] = \left\{\omega : \left|\omega - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}\right\}, \tag{2.5.1}$$

where the domain $\Omega[A,B]$ is an open circular disk, such that $D_1 = \frac{1-A}{1-B}$ and $D_2 = \frac{1+A}{1+B}$ with $0 < D_1 < 1 < D_2$ are the end points of its diameter and its center lies on real axis. The following relation gives a link between the families $\mathcal{P}[A,B]$ and $\mathcal{P}$ of Carathéodory functions

$$q(z) \in \mathcal{P} \Longleftrightarrow \frac{(A+1)\,q(z) - (A-1)}{(B+1)\,q(z) - (B-1)} \in \mathcal{P}[A,B]. \tag{2.5.2}$$

Noor [103] proved that the family $\mathcal{P}[A,B]$ is convex and also observed that $\mathcal{P}[1-2\zeta,-1] = \mathcal{P}(\zeta)$, $(0 \leq \zeta < 1)$ is the set of Carathéodory functions of order ζ.
With the help of the set $\mathcal{P}[A,B]$ we now derive the families $\mathcal{S}^*[A,B]$ and $\mathcal{C}[A,B]$ of Janowski convex and Janowski starlike functions as below;

$$\begin{aligned}
\mathcal{S}^*[A,B] &= \left\{f(z) \in \mathfrak{A} : \frac{zf'(z)}{f(z)} \in \mathcal{P}[A,B],\ (z \in \mathfrak{D})\right\}, \\
\mathcal{C}[A,B] &= \left\{f(z) \in \mathfrak{A} : 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}[A,B],\ (z \in \mathfrak{D})\right\}.
\end{aligned}$$

These families were introduced by Janowski [59]. The extension of Janowski function was discussed by kuroki, Owa and Srivastava [77] by choosing the complex parameters A and B with the following conditions

$$\begin{cases}
(i) \quad A \neq B, \quad |B| < 1, \quad |A| \leq 1 \quad \text{and} \quad \mathfrak{Re}\left(1 - A\overline{B}\right) \geq |A-B| \\[2ex]
(ii) \quad A \neq B, \quad |B| = 1, \quad |A| \leq 1, \quad \text{and} \quad \mathfrak{Re}\left(1 - A\overline{B}\right) > 0.
\end{cases} \tag{2.5.3}$$

Later on, Kuroki and Owa [78] discussed the fact that the condition $|A| \leq 1$ can be omitted from the conditions in part (i) of (2.5.3). Janowski functions are being studied

and extended in different directions by several renowned mathematicians like Arif et al [21], Cho [37], Cho et al [36, 38], Liu and Noor [84], Liu and Patel [85], Liu and Srivastava [86, 108], Noor and Arif [108], Polatoğlu [121], and many more.
We now extend the definition of $\mathcal{P}[A,B]$ by introducing the family $\mathcal{P}[\eta,A,B]$ with the restriction on parameters $-1 \leq B < A \leq 1$ and $\eta \in \mathbb{C}\backslash\{0\}$ as below;

Definition 2.5.2 *A regular function $q(z)$ in $\mathfrak{D}$ with $q(0)=1$ is in the family $\mathcal{P}[\eta,A,B]$ $(\eta \in \mathbb{C}\backslash\{0\})$, if $q(z)$ satisfy the condition given by*

$$q(z) \prec \frac{1+[\eta A+(1-\eta)B]z}{1+Bz}.$$

Or equivalently, a function $q(z) \in \mathcal{P}[\eta,A,B]$ if and only if for $q_1(z) \in \mathcal{P}[A,B]$ we have

$$q(z) = \eta q_1(z) + (1-\eta).$$

By taking $\eta = 1-\zeta$ with $0 \leq \zeta < 1$, then the class $\mathcal{P}[\eta,A,B]$ coincide with $\mathcal{P}[\zeta,A,B]$, defined by Polatoğlu [119, 121] and if we take $\eta = 1$, then $\mathcal{P}[\eta,A,B]$ reduces to the familiar set $\mathcal{P}[A,B]$, defined by Janowski [59]. Also by making $A=1$, $B=-1$ and $\eta=1$ in the set $\mathcal{P}[\eta,A,B]$, we get the most valuable and familiar set $\mathcal{P}$ of functions having real part is positive.

2.6 Conic domains and associated functions

The main purpose of this section is to review or give a short survey on the conic domains. This domain was first introduced by Goodman and several other researchers including Rönning [134, 135, 136], Kanas [61, 62, 63, 65, 66, 67], Noor [109, 110, 111, 112, 107, 114] extensively studied different aspects of conic domains.

Hyperbolic, parabolic and elliptic domains

It is evident that under any complex function $f(z)$ all the circles in $\mathfrak{D}$ which are centered at the origin, are mapped on convex arcs. An interisting question was pointed by Pinchuk, that whether this property remains true if the centers of the circles are not origin. Goodman [47] answered in negative and a new family of functions satisfying this property was defined. We denote the symbol $\mathcal{UCV}$ of the set all such functions. Apart from this, he further defined the family $\mathcal{UST}$ of uniformly starlike functions, under these fuctions the circles lying in $\mathfrak{D}$ centered at points other then origin are mapped onto star-shaped arcs, see [48]. Goodman [47, 48] gave the analytic conditions for functions to be uniformly convex and uniformly starlike as below;

Definition 2.6.1 *Let $f(z)$ be a regualr function. Then $f(z) \in \mathcal{UCV}$, if and only if, for $z, \zeta \in \mathfrak{D}$*

$$\mathfrak{Re}\left\{1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right\} > 0. \tag{2.6.1}$$

and a regular function $f(z) \in \mathcal{UST}$, if and only if, for $z, \zeta \in \mathfrak{D}$

$$\mathfrak{Re}\left\{\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}\right\} > 0. \tag{2.6.2}$$

By making $\zeta = 0$ in (2.6.1) and (2.6.2), the usual families $\mathcal{C}$ and $\mathcal{S}^*$ are obtained. One might expect that $\mathcal{UCV}$ and $\mathcal{UST}$ may be associated to each other by Alexander type relation, but this was contradit in [47] by examples that is, Alexander type relations does not hold between $\mathcal{UCV}$ and $\mathcal{UST}$. Rönning [135], and Ma and Minda [90] illustrated the set $\mathcal{UCV}$ more appropriatly in one variable follow;

Definition 2.6.2 *A function $f(z) \in \mathcal{UCV}$, if and only if for every $z \in \mathfrak{D}$,*

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|. \tag{2.6.3}$$

Rönning [135] introduced the set

$$\mathcal{ST} = \{g(z) : g(z) = zf'(z),\ \ f(z) \in \mathcal{UCV}\}$$

by using Alexander type relation related to $\mathcal{UCV}$. To check whether $\mathcal{UST}$ properly contains $\mathcal{ST}$ or not. Rönning came to know in (see [133]) that none of these is subset of the other, that is,

$$\mathcal{UST} \subsetneq \mathcal{ST} \text{ and } \mathcal{ST} \subsetneq \mathcal{UST}.$$

Also, Rönning [135] has given a one variable classification of the class $\mathcal{ST}$ as follows;

Definition 2.6.3 *A function belongs to $\mathcal{ST}$, if and only if for every $z \in \mathfrak{D}$,*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right|. \tag{2.6.4}$$

The study of these families can be found in research of several authors, see [47, 90, 91, 92, 134, 135].

To illustrate (2.6.3) and (2.6.4) geometrically, let us consider the domain

$$\Lambda_p = \{w : \mathfrak{Re}w > |w-1|\}.$$

It can easily be observed that the set Λ_p is the interior of parabola

$$(\mathrm{Im}w)^2 < 2\mathfrak{Re}w - 1,$$

which is symmetric about real axis and has $\left(\frac{1}{2}, 0\right)$ as its vertex. Therefore we can define the classes $\mathcal{UCV}$ and $\mathcal{ST}$ as follows; a function $f(z) \in \mathcal{ST}$ or $\mathcal{UCV}$, if and only if

$$\frac{zf'(z)}{f(z)} \in \Lambda_p \text{ or } 1 + \frac{zf''(z)}{f'(z)} \in \Lambda_p.$$

Kanas and Wisniowska [66, 67] worked on the generalization of parabolic domain and introduced the conic domain $\Omega_k,\ k \geq 0$. The set Ω_k is given by

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}. \tag{2.6.5}$$

For $k > 1$, the domain Ω_k is elliptic, for $0 < k < 1$, it is hyperbolic and parabolic when $k = 1$. Also the domain Ω_k is a right half plane for $k = 0$.
Now we discuss the families $k - \mathcal{UCV}$ and $k - \mathcal{ST}$, $(0 \leq k < \infty)$, the collection of k-uniformly convex and k-starlike functions in $\mathfrak{D}$. These families can be describes as;

Definition 2.6.4 *Let $f(z) \in \mathfrak{A}$. Then $f(z)$ is said to be in class $k - \mathcal{UCV}$, if and only if, for $z \in \mathfrak{D}$*

$$\mathfrak{Re}\left\{ \frac{(zf'(z))'}{f'(z)} \right\} > k\left| \frac{zf''(z)}{f'(z)} \right|,$$

and a function $f(z)$ is said to be in class $k - \mathcal{ST}$, if and only if, for $z \in \mathfrak{D}$

$$\mathfrak{Re}\left\{ \frac{(zf'(z)}{f(z)} \right\} > k\left| \frac{zf'(z)}{f(z)} - 1 \right|.$$

The sets $k - \mathcal{UCV}$ of k-uniformly convex functions and the corresponding family $k - \mathcal{ST}$ of k-starlike functions were initiated by Kanas and Wisniowska [66, 67] . These classes satisfies usual Alexander type relation.

$$f(z) \in k - \mathcal{UCV} \Leftrightarrow zf'(z) \in k - \mathcal{ST}.$$

Rönning [137] in 1995 introduced the family of $k - \mathcal{UCV}(\zeta)$ and the corresponding family $k - \mathcal{ST}(\zeta)$ of order ζ $(0 \leq \zeta < 1)$ and studies its charactristics of these families. Rönning defined it as;

Definition 2.6.5 *Let $f(z) \in \mathfrak{A}$ and $0 \leq \zeta < 1$. $f(z) \in k - \mathcal{UCV}(\zeta)$, if and only if*

$$\mathfrak{Re}\left\{ \frac{(zf'(z))'}{f'(z)} - \zeta \right\} > k\left| \frac{zf''(z)}{f'(z)} \right|,\ z \in \mathfrak{D}. \tag{2.6.6}$$

and a function $f(z) \in k-\mathcal{ST}(\zeta)$*, if and only if*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}-\zeta\right\} > k\left|\frac{zf'(z)}{f(z)}-1\right|,\ z \in \mathfrak{D}. \tag{2.6.7}$$

From (2.6.6) and (2.6.7), we observed that $\frac{(zf'(z))'}{f'(z)}$ and $\frac{zf'(z)}{f(z)}$ lie in the conic region

$$\Omega_{k,\zeta} = \left\{u+iv : u > k\sqrt{(u-1)^2+v^2}+\zeta\right\},$$

such that $1 \in \Omega_{k,\zeta}$. The conditions (2.6.6) and (2.6.7) can be stated as follow;

$$\frac{(zf'(z))'}{f'(z)} \prec q_{k,\zeta}(z) \text{ and } \frac{zf'(z)}{f(z)} \prec q_{k,\zeta}(z),$$

for different choices of k, the explicit forms of function $q_{k,\zeta}(z)$, we have

$$q_{0,\zeta}(z) = \frac{1+(1-2\zeta)\,z}{1-z}, \text{ and } q_{1,\zeta}(z) = 1+\frac{2\,(1-\zeta)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2.$$

taking $0 < k < 1$, we obtain

$$q_{k,\zeta}(z) = \frac{1-\zeta}{1-k^2}\cos\left\{\frac{2}{\pi}\left(\cos^{-1}k\right)i\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\}+\frac{k^2-\zeta}{k^2+1},$$

and if $k > 1$, then $q_{k,\zeta}(z)$ has the form

$$q_{k,\zeta}(z) = \frac{1-\zeta}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{dx}{\sqrt{1-x^2}\sqrt{1-x^2t^2}}\right)+\frac{k^2-\zeta}{k^2+1},$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0,1)$, $z \in \mathfrak{D}$ and the value of z is chosen so that $k = \cosh\frac{\pi R'(z)}{4R(z)}$, $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$, see [66, 67].

Recently in 2009, Noor et al [114] defined the domain $\Omega_{k,\eta}$, $0<\mathfrak{Re}\eta\leq k+1$, related to Ω_k given by (2.6.5), as follows;

$$\Omega_{k,\eta}=\gamma\Omega_k+(1-\eta)\,.$$

Extremal function for these conic regions, denoted by $p_{k,\eta}(z)$, is regular in $\mathfrak{D}$ and map the open unit disk $\mathfrak{D}$ onto the domain $\Omega_{k,\eta}$ such that

$$p_{k,\eta}(0)=1 \quad \text{and} \quad p'_{k,\eta}(0)>1,$$

where $p_{k,\eta}(z)$ is given by

$$p_{k,\eta}(z)=\begin{cases} \frac{1+(2\eta-1)z}{1-z} & (k=0) \\ 1+\frac{2\eta}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2 & (k=1) \\ 1+\frac{2\eta}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\operatorname{arctanh}\left(\sqrt{z}\right)\right\} & (0<k<1) \\ 1+\frac{\eta}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int\limits_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right)+\frac{1}{k^2-1} & (k>1)\,, \end{cases} \tag{2.6.8}$$

with

$$u(z)=\frac{z-\sqrt{t}}{1-\sqrt{tz}} \quad (0<t<1;\quad z\in\mathfrak{D})\,,\quad k=\cosh\left(\frac{\pi R'(t)}{4R(t)}\right).$$

Here $R(t)$ is the Legendre's complete elliptic integral of the first kind and $R'(t)$ is the complementary integral of $R(t)$ (see [66], [67] and [114]).

Let $\mathcal{P}(p_{k,\eta}(z))$ denotes the set of functions $p(z)$ which are regular in $\mathfrak{D}$ with

$$p(0)=1 \quad \text{and} \quad p(z)\prec p_{k,\gamma}(z) \quad (z\in\mathfrak{D}).$$

Then one can easily prove that $\mathcal{P}\left(p_k\left(z\right)\right) \subset \mathcal{P}\left(\beta_1\right)$, where

$$\beta_1 = \frac{k}{1+k}.$$

For $p\left(z\right) \in \mathcal{P}\left(p_{k,\eta}\left(z\right)\right)$, we have

$$\left|\arg p\left(z\right)\right| \leq \frac{\sigma\pi}{2},$$

where

$$\sigma = \frac{2}{\pi}\arctan\left(\frac{1}{k}\right).$$

So we can write

$$p\left(z\right) = h^{\sigma}\left(z\right) \quad \left(h\left(z\right) \in \mathcal{P}_{\eta}\right).$$

These conic regions have been studied by many researchers, [1, 2, 11, 107, 114, 118, 142].

Chapter 3

Fourth Hankel determinant for a subfamily of schlicht functions

3.1 Introduction

The Hankel determinant $H_{q,n}(f)$ $(q, n \in \mathbb{N} = \{1, 2, \ldots\})$ for a function $f(z) \in \mathfrak{S}$ having the form of (2.1.1) was defined by Pommerenke [125, 126], (see also [19, 21]) as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{3.1.1}$$

For fixed integer q and n, the growth rate of $H_{q,n}(f)$ has been investigated for different subfamilies of schlicht functions. We list here few main contribution of them. The sharp bounds of $|H_{2,2}(f)|$ for the subfamilies $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ of the set $\mathcal{S}$ were investigated by Janteng, Halim and Darus [60], where $\mathcal{R}$ consists of functions having derivatives with positive real part studied by Mac Gregor [93]. They proved the bounds

$$|H_{2,2}(f)| \leq \begin{cases} 1, & \text{for} \quad f \in \mathcal{S}^*, \\ \frac{1}{8}, & \text{for} \quad f \in \mathcal{C}, \\ \frac{4}{9}, & \text{for} \quad f \in \mathcal{R}. \end{cases}$$

For the sets of Bazilevič and close-convex functions, the exact estimate of $|H_{2,2}(f)|$ were obtained by Krishna et al [72] and Silveraj et al [140] respectively. For other known subfamilies of regular and schlicht functions, see the approach of different mathematicians in the research papers [27, 53, 71, 79, 88, 101].

The estimation of $|H_{3,1}(f)|$ is much more difficult as compared to find the bound of $|H_{2,2}(f)|$. The first paper on $H_{3,1}(f)$ appears in 2010 by Babalola [26] in which he obtained the upper bound of $|H_{3,1}(f)|$ for the families of $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$. For the family of $\mathcal{SL}(s,t)$ $(s, t \in \mathbb{Z}$ with $s \neq t)$ which is defined by

$$\mathcal{SL}(s,t) = \left\{ f(z) \in \mathfrak{A} : \left| \left(\frac{(s-t)zf'(z)}{f(sz) - f(tz)} \right) - 1 \right| < 1, \ (z \in \mathfrak{D}) \right\},$$

the bound of $|H_{3,1}(f)|$ was found by Arif et. al [22] which gives the following result

$$|H_{3,1}(f)| \leq \tfrac{43}{576}, \quad \text{for} \quad f(z) \in \mathcal{SL} \cong \mathcal{SL}(1,0),$$

proved by Raza and Malik [129] in 2013. The geometrical interpretation the function $f(z)$ to be a member of class $\mathcal{SL}$ is that, for any $z \in \mathfrak{D}$, the ratio $\dfrac{zf'(z)}{f(z)}$ lies in the region bounded by the right half side of the Bernoulli's lemniscate by holding the inequality $|w^2 - 1| < 1$. It can easily be seen that the condition for function $f(z) \in \mathfrak{A}$, to be in the class $\mathcal{SL}$, if and only if

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}, \tag{3.1.2}$$

where the square root function is considered at principal branch, that is

$$\sqrt{1+z}\,\Big|_{z=0} = 1. \tag{3.1.3}$$

Later on some other well-familiar researcher [12, 28, 73, 74, 143] published their work concerning $|H_{3,1}(f)|$ for different subfamilies of regular and schlicht functions. Recently in 2016, Paweł Zaprawa [157] improved the results of Babalola [26] by proving

$$|H_{3,1}(f)| \leq \begin{cases} 1, & \text{for} \quad f \in \mathcal{S}^*, \\ \tfrac{49}{540}, & \text{for} \quad f \in \mathcal{C}, \\ \tfrac{41}{60}, & \text{for} \quad f \in \mathcal{R}. \end{cases}$$

and claimed that these bounds are still not sharp. Further for the sharpness, he considered the subfamilies of $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ consisting of functions with m-fold symmetry and obtained the sharp bounds. In this chapter, we make a contribution to the subject by deducing fourth Hankel determinant for the class $\mathcal{SL}$. Recalling that Sokól and Stankiewicz [148] introduced the class $\mathcal{SL}$, and further improved by other authors in [146, 147, 145, 52, 9].

3.2 A set of lemmas

The following lemmas are very supportive for proving our main results.

Lemma 3.2.1 *[50] If $p(z) \in \mathcal{P}$ and has the form* (2.2.1) , *then*

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{c_1^2}{2},$$

where the above inquality is proved in [8].

Lemma 3.2.2 *[34] If $p(z) \in \mathcal{P}$ and has the form* (2.2.1), *then*

$$|c_{n+k} - \mu c_n c_k| < 2, \text{ for } 0 \leq \mu \leq 1.$$

This result is due to Ravichandran and Verma [128].

Lemma 3.2.3 *If $p(z) \in \mathcal{P}$ and has the form* (2.2.1), *then*

$$\left| Jc_1^3 - Kc_1c_2 + Lc_3 \right| \leq 2\left(|J| + |K - 2J| + |J - K + L|\right).$$

Proof. It is easy to see that

$$\begin{aligned} &|Jc_1^3 - Kc_1c_2 + Lc_3| = |J\left(c_3 - 2c_1c_2 + c_1^3\right) + (K - 2J)\left(c_3 - c_1c_2\right) + (J - K + L)\, c_3| \\ &\leq |J|\, |c_3 - 2c_1c_2 + c_1^3| + |K - 2J|\, |c_3 - c_1c_2| + |J - K + L|\, |c_3| \\ &\leq 2\left(|J| + |K - 2J| + |J - K + L|\right), \end{aligned}$$

where we have used the Lemma 3.2.2 for $\mu = 1$, $n = 1$, $k = 2$ and a result due to Libra and Zlotkiewicz [80]. ■

Lemma 3.2.4 *[129] If $f(z) \in \mathcal{SL}$ and has the form* (2.1.1), *then*

$$\left| a_3 - a_2^2 \right| \leq \frac{1}{4}.$$

Lemma 3.2.5 *If* $f(z) \in \mathcal{SL}$ *and has the form* (2.1.1), *then*

$$|a_2| \leq \frac{1}{2}, \quad |a_3| \leq \frac{1}{4}, \quad |a_4| \leq \frac{1}{6}, \quad |a_5| \leq \frac{1}{8}.$$

These estimates are sharp.

The first three bounds were obtained by Sokól [147] and the bound for $|a_5|$ was proved in [128].

Lemma 3.2.6 *If* $f(z) \in \mathcal{SL}$ *and has the form* (2.1.1), *then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{16}.$$

This result was found by Sokól [147].

3.3 Main Results

Theorem 3.3.1 *If* $f(z) \in \mathcal{SL}$ *and of the form* (2.1.1), *then*

$$\left|a_3a_5 - a_4^2\right| \leq 0.083924943.$$

Proof. If $f(z) \in \mathcal{SL}$, by using the subordination relation (3.1.2), it follows that

$$\frac{zf'(z)}{f(z)} \prec \Phi(z), \tag{3.3.1}$$

where $\Phi(z) = \sqrt{1+z}$ is considered at principal branch (3.1.3). From (3.3.1), there exists a function w, analytic in the unit disk $\mathcal{U}$, with $|w(z)| \leq 1$ in $\mathcal{U}$, such that

$$\frac{zf'(z)}{f(z)} = \Phi(w(z)),\ z \in \mathcal{U}. \tag{3.3.2}$$

Thus, if we define the function $p(z)$ by

$$p(z) := \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \ldots,\ z \in \mathcal{U}, \tag{3.3.3}$$

it follows that $p(z) \in \mathcal{P}$ and

$$w(z) = \frac{p(z)-1}{p(z)+1},\ z \in \mathcal{U}.$$

From (3.3.2) and the above relation we obtain

$$\frac{zf'(z)}{f(z)} = \sqrt{\frac{2p(z)}{p(z)+1}},\ z \in \mathcal{U}. \tag{3.3.4}$$

Now, according to the power series expansions (2.2.1) and (2.1.1), a simple computation shows that

$$\begin{aligned}\sqrt{\frac{2p(z)}{p(z)+1}} &= 1 + \frac{1}{4}c_1 z + \left(\frac{1}{4}c_2 - \frac{5}{32}c_1^2\right) z^2 + \left(\frac{1}{4}c_3 - \frac{5}{16}c_1 c_2 + \frac{13}{128}c_1^3\right) z^3 \\ &\quad + \left(-\frac{141}{2048}c_1^4 + \frac{39}{128}c_1^2 c_2 - \frac{5}{32}c_2^2 + \frac{1}{4}c_4 - \frac{5}{16}c_1 c_3\right) z^4 \ldots,\end{aligned} \tag{3.3.5}$$

and

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + \left(2a_3 - a_2^2\right) z^2 + \left(3a_4 - 3a_2 a_3 + a_2^3\right) z^3 + \ldots,\ z \in \mathcal{U}. \tag{3.3.6}$$

By comparing (3.3.5) and (3.3.6) , we have

$$
\begin{aligned}
a_2 &= \frac{1}{4}c_1, && (3.3.7)\\
a_3 &= \frac{1}{8}\left(c_2 - \frac{3}{8}c_1^2\right), && (3.3.8)\\
a_4 &= \frac{1}{12}\left(c_3 - \frac{7}{8}c_1c_2 + \frac{13}{64}c_1^3\right), && (3.3.9)\\
a_5 &= \left(-\frac{49}{6144}c_1^4 + \frac{17}{384}c_1^2c_2 - \frac{11}{192}c_1c_3 - \frac{1}{32}c_2^2 + \frac{1}{16}c_4\right), && (3.3.10)\\
a_6 &= -\frac{223}{7680}c_1^3c_2 + \frac{3}{80}c_1^2c_3 + \frac{77}{1920}c_1c_2^2 - \frac{3}{64}c_1c_4 - \frac{5}{96}c_2c_3\\
&\quad + \frac{181}{40960}c_1^5 + \frac{1}{20}c_5, && (3.3.11)\\
a_7 &= \frac{323}{4608}c_1c_2c_3 - \frac{17}{384}c_2c_4 - \frac{19}{480}c_1c_5 - \frac{13}{576}c_3^2 + \frac{19}{1536}c_2^3 + \frac{1}{24}c_6\\
&\quad - \frac{32303}{11796480}c_1^6 - \frac{4717}{184320}c_1^3c_3 + \frac{33}{1024}c_1^2c_4 - \frac{7457}{184320}c_1^2c_2^2 + \frac{30211}{1474560}c_1^4c_2 && (3.3.12)
\end{aligned}
$$

From (3.3.8) , (3.3.9) and (3.3.10) , we obtain

$$
\begin{aligned}
\left|a_3a_5 - a_4^2\right| &= \left|-\frac{89}{147456}c_2c_1^4 + \frac{31}{18432}c_1^2c_2^2 + \frac{23}{4608}c_2c_1c_3 - \frac{1}{256}c_2^3 + \frac{1}{128}c_2c_4 + \frac{103}{1179648}c_1^6\right.\\
&\quad \left. - \frac{5}{36864}c_1^3c_3 - \frac{3}{1024}c_1^2c_4 - \frac{1}{144}c_3^2\right|.
\end{aligned}
$$

Now, re-arranging the above equation, we have

$$
\begin{aligned}
\left|a_3a_5 - a_4^2\right| &\le \left|-\frac{103}{589824}c_1^4\left(c_2 - \frac{c_1^2}{2}\right) + \frac{27}{16384}c_1^2\left(\frac{253}{486}c_1^2 - c_2\right)\left(c_2 - \frac{c_1^2}{2}\right)\right.\\
&\quad - \frac{23}{9216}c_1c_3\left(\frac{5}{92}c_1^2 - c_2\right) + \frac{1}{144}c_3\left(\frac{23}{64}c_1c_2 - c_3\right)\\
&\quad \left. - \frac{1}{256}c_2\left(\frac{37}{64}c_2^2 - c_4\right) - \frac{4}{256}c_4\left(\frac{3}{4}c_1^2 - c_2\right)\right|.
\end{aligned}
$$

Applying triangle inequality and Lemma 3.2.1 and Lemma 3.2.2, we have

$$
\left|a_3a_5 - a_4^2\right| \le \frac{103}{589824}\left|c_1\right|^4\left(2 - \frac{\left|c_1\right|^2}{2}\right) + \frac{27}{4096}\left(2 - \frac{\left|c_1\right|^2}{2}\right) + \frac{23}{2304}\left|c_1\right| + \frac{17}{288}. \quad (3.3.13)
$$

Taking $|c_1| = y \in [0,2]$ in $(3.3.13)$, it gives

$$\left|a_3a_5 - a_4^2\right| \leq \frac{103}{589824}y^4\left(2-\frac{y^2}{2}\right) + \frac{27}{4096}\left(2-\frac{y^2}{2}\right) + \frac{23}{2304}y + \frac{17}{288}. \tag{3.3.14}$$

The above function get its maximum value at $y = 1.573483035$, in $(3.3.14)$, we have

$$\left|a_3a_5 - a_4^2\right| \leq 0.080574496.$$

■

Theorem 3.3.2 *If $f(z) \in \mathcal{SL}$ and of the form* (2.1.1), *then*

$$|a_3a_4 - a_2a_5| \leq \frac{173}{532224}\sqrt{39963} + \frac{1}{24} \simeq 0.1066468.$$

Proof. From $(3.3.7)$, $(3.3.8)$, $(3.3.9)$ and $(3.3.10)$, we have

$$|a_3a_4 - a_2a_5| = \left|-\frac{59}{49152}c_1^5 + \frac{17}{3072}c_1^3c_2 - \frac{1}{96}c_1^2c_3 + \frac{1}{768}c_1c_2^2 + \frac{1}{64}c_1c_4 - \frac{1}{96}c_2c_3\right|.$$

By re-arrangement of the above equation, we get

$$|a_3a_4 - a_2a_5| = \left|\frac{77}{12288}c_1\left(\frac{59}{154}c_1^2 - c_2\right)\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{96}c_2\left(\frac{93}{128}c_1c_2 - c_3\right) - \frac{1}{64}c_1\left(\frac{2}{3}c_1c_3 - c_4\right)\right|.$$

Now applying triangle inequality and Lemmas 3.2.1 and Lemma 3.2.2, we have

$$|a_3a_4 - a_2a_5| \leq \frac{77}{6144}|c_1|\left(2 - \frac{|c_1|^2}{2}\right) + \frac{1}{24} + \frac{1}{32}|c_1|. \tag{3.3.15}$$

Let $|c_1| = y \in [0,2]$, then $(3.3.15)$, becomes

$$|a_3a_4 - a_2a_5| \leq \frac{77}{12288}y\left(2-\frac{y^2}{2}\right) + \frac{1}{24} + \frac{1}{32}y.$$

The above function has its maximum value at $y = \frac{2}{231}\sqrt{39963}$. This implies that

$$|a_3a_4 - a_2a_5| \leq \frac{173}{532224}\sqrt{39963} + \frac{1}{24} \simeq 0.1066468.$$

■

Theorem 3.3.3 *If $f(z) \in \mathcal{SL}$ and of the form* (2.1.1), *then*

$$|a_5 - a_2a_4| \leq \frac{7}{16}.$$

Proof. From (3.3.7), (3.3.9) and (3.3.10), we obtain

$$|a_5 - a_2a_4| = \left| -c_1\left(\frac{25}{2048}c_1^3 - \frac{1}{16}c_1c_2 + \frac{5}{64}c_3\right) - \frac{1}{16}\left(\frac{c_2^2}{2} - c_4\right)\right|.$$

Now by using triangle inequality and Lemmas 3.2.2 and Lemma 3.2.3, we have

$$|a_5 - a_2a_4| \leq \frac{7}{16}.$$

■

Theorem 3.3.4 *If $f(z) \in \mathcal{SL}$ and of the form* (2.1.1), *then*

$$|a_4 - a_2a_3| \leq \frac{1}{6}.$$

This result is sharp for the function $f(z) = z\exp\left(\int_0^z \frac{\sqrt{1+t^3}}{t}dt\right) = z + \frac{1}{6}z^4 - \frac{1}{144}z^7 + \cdots$.

Proof. From (3.3.7), (3.3.8) and (3.3.9), we have

$$|a_4 - a_2a_3| = \left|\frac{11}{384}c_1^3 - \frac{5}{48}c_1c_2 + \frac{1}{12}c_3\right|.$$

Using Lemma 3.2.3, we obtain

$$|a_4 - a_2a_3| \leq \frac{1}{6}.$$

■

Theorem 3.3.5 *If $f(z) \in \mathcal{SL}$ and of the form* (2.1.1)*, then*

$$|a_3a_7 - a_4a_6| \leq \frac{125999}{589824}.$$

Proof. From $(3.3.8)$, $(3.3.9)$, $(3.3.11)$ and $(3.3.12)$, we have

$$\begin{aligned} |a_3a_7 - a_4a_6| &= \left|\frac{19}{12288}c_2^4 + \frac{4493}{83886080}c_1^8 - \frac{1}{240}c_3c_5 - \frac{17}{3072}c_2^2c_4 + \frac{7}{4608}c_2c_3^2\right| \\ &\quad + \frac{1}{192}c_2c_6 - \frac{721}{1474560}c_1^6c_2 - \frac{25}{9216}c_1^2c_2^3 + \frac{9799}{5898240}c_1^4c_2^2 \\ &\quad + \frac{31}{30720}c_1^3c_5 - \frac{127}{61440}c_1^2c_3^2 - \frac{1}{512}c_1^2c_6 + \frac{773}{3932160}c_1^5c_3 - \frac{47}{65536}c_1^4c_4 \\ &\quad + \frac{299}{184320}c_1c_2^2c_3 - \frac{1}{768}c_1c_2c_5 - \frac{331}{737280}c_1^3c_2c_3 + \frac{11}{4096}c_1^2c_2c_4 + \frac{1}{256}c_1c_3c_4\Bigg|. \end{aligned}$$

By re-arranging the above equation, we obtain

$$\begin{aligned} |a_3a_7 - a_4a_6| &= \Bigg|\frac{241}{92160}c_2c_3\left(\frac{299}{964}c_2c_1 - c_3\right) + \frac{149}{24576}c_2^2\left(\frac{299}{2235}c_3c_1 - c_4\right) \\ &\quad - \frac{149}{49152}c_2^2\left(\frac{4537}{47680}c_2^2 - c_4\right) - \frac{1}{768}c_2\left(c_1c_5 - c_6\right) \\ &\quad + \frac{331}{1474560}c_1^3c_3\left(\frac{2319}{2648}c_1^2 - c_2\right) - \frac{31}{30720}c_1^3\left(\frac{331}{1488}c_2c_3 - c_5\right) \\ &\quad - \frac{144139}{188743680}c_1^4\left(\frac{40437}{288278}c_1^2 - c_2\right)\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{240}c_3\left(\frac{15}{16}c_1c_4 - c_5\right) \\ &\quad - \frac{9619}{5242880}c_2^2\left(\frac{169429}{173142}c_1^2 - c_2\right)\left(c_2 - \frac{c_1^2}{2}\right) + \frac{41}{16384}c_4\left(\frac{47}{82}c_1^2 - c_2\right)\left(c_2 - \frac{c_1^2}{2}\right) \\ &\quad + \frac{127}{30720}c_3^2\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{256}c_6\left(c_2 - \frac{c_1^2}{2}\right)\Bigg|. \end{aligned}$$

Applying triangle inequality and Lemmas 3.2.1 and Lemma 3.2.2, the above equation becomes

$$\begin{aligned}|a_3a_7 - a_4a_6| \quad \leq \quad & \frac{144139}{94371840}|c_1|^4\left(2-\frac{|c_1|^2}{2}\right)+\frac{96409}{1966080}\left(2-\frac{|c_1|^2}{2}\right) \\ & +\frac{215}{73728}|c_1|^3+\frac{10649}{92160}. \end{aligned} \tag{3.3.16}$$

Let $|c_1| = y \in [0, 2]$, then (3.3.16) becomes

$$|a_3a_7 - a_4a_6| \leq \frac{144139}{94371840}y^4\left(2-\frac{y^2}{2}\right)+\frac{96409}{1966080}\left(2-\frac{y^2}{2}\right)+\frac{215}{73728}y^3+\frac{10649}{92160}.$$

Clearly the above function is decreasing so by putting $y = 2$, we have

$$|a_3a_7 - a_4a_6| \leq \frac{125999}{589824}.$$

■

Theorem 3.3.6 *If $f(z) \in \mathcal{SL}$ and of the form* (2.1.1)*, then*

$$|a_4a_7 - a_5a_6| \leq 0.2210481986.$$

Proof. From (3.3.9), (3.3.10), (3.3.11) and (3.3.12), it follows that

$$\begin{aligned}|a_4a_7 - a_5a_6| \quad = \quad & \Bigg|-\frac{1}{2304}c_1c_3c_5+\frac{83}{18432}c_1c_2c_3^2-\frac{1}{2304}c_2c_3c_4-\frac{7}{2304}c_1c_2c_6 \\ & +\frac{583}{737280}c_1^3c_2c_4+\frac{669}{655360}c_3c_1^4c_2+\frac{1}{640}c_2^2c_5-\frac{11}{18432}c_3c_2^3 \\ & -\frac{499}{184320}c_1^2c_3c_2^2-\frac{137}{184320}c_1c_2^2c_4-\frac{3}{1280}c_1^2c_3c_4+\frac{31}{46080}c_1^2c_2c_5 \\ & +\frac{3}{1024}c_1c_4^2-\frac{20131}{1811939328}c_1^9-\frac{1}{320}c_4c_5-\frac{137}{1310720}c_1^5c_4+\frac{259}{737280}c_1c_2^4 \\ & -\frac{13}{6912}c_3^3-\frac{5}{18432}c_1^4c_5+\frac{13}{18432}c_1^3c_6+\frac{527}{1105920}c_1^3c_2^3-\frac{10271}{23592960}c_1^5c_2^2 \\ & +\frac{439633}{4529848320}c_1^7c_2-\frac{3}{8192}c_1^3c_3^2-\frac{515}{4718592}c_1^6c_3+\frac{1}{288}c_3c_6\Bigg|. \end{aligned}$$

This implies that

$$\begin{aligned}
|a_4a_7-a_5a_6| &= \left|\frac{18934}{11796480}c_1^2c_3\left(\frac{2575}{18934}c_1^2-c_2\right)\left(c_2-\frac{c_1^2}{2}\right)\right.\\
&+\frac{1}{3840}c_2\left(\frac{2167}{256}c_3c_2-c_5\right)\left(c_2-\frac{c_1^2}{2}\right)+\frac{3}{1024}c_1c_4^2\\
&+\frac{3}{4096}c_1c_3^2\left(c_2-\frac{c_1^2}{2}\right)+\frac{13}{9216}c_1\left(\frac{5}{13}c_1c_5-c_6\right)\left(c_2-\frac{c_1^2}{2}\right)\\
&+\frac{323519}{566231040}c_1^3c_2^2\left(c_2-\frac{c_1^2}{2}\right)+\frac{10739}{56623104}c_1c_2^3\left(c_2-\frac{c_1^2}{2}\right)\\
&+\frac{3431}{2949120}c_1c_4\left(\frac{1233}{6862}c_1^2-c_2\right)\left(c_2-\frac{c_1^2}{2}\right)\\
&+\frac{169489}{1132462080}c_1^5\left(\frac{100655}{677956}c_1^2-c_2\right)\left(c_2-\frac{c_1^2}{2}\right)+\frac{3}{640}c_3c_4\left(c_2-\frac{c_1^2}{2}\right)\\
&-\frac{1}{288}c_3\left(\frac{13}{24}c_3^2-c_6\right)+\frac{793056}{283115520}c_2^3\left(\frac{45761}{793056}c_1c_2-c_3\right)\\
&+\frac{59}{11520}c_2c_3\left(\frac{695}{944}c_1c_3-c_4\right)+\frac{1}{320}c_5\left(\frac{7}{12}c_2^2-c_4\right)\\
&\left.+\frac{5}{3072}c_1c_2\left(\frac{413}{1600}c_2c_4-c_6\right)-\frac{1}{2304}c_3c_1c_5\right|.
\end{aligned}$$

Using triangle inequality and Lemmas 3.2.1 and 3.2.2, we have

$$\begin{aligned}
|a_4a_7-a_5a_6| &\leq \frac{9467}{1474560}|c_1|^2\left(2-\frac{|c_1|^2}{2}\right)+\frac{15109}{368640}\left(2-\frac{|c_1|^2}{2}\right)\\
&+\frac{322063}{35389440}|c_1|\left(2-\frac{|c_1|^2}{2}\right)+\frac{323519}{141557760}|c_1|^3\left(2-\frac{|c_1|^2}{2}\right)\\
&+\frac{169489}{566231040}|c_1|^5\left(2-\frac{|c_1|^2}{2}\right)+\frac{23}{1152}|c_1|+\frac{20677}{184320}. \qquad (3.3.17)
\end{aligned}$$

Let $|c_1|=y\in[0,2]$, then (3.3.17) becomes

$$\begin{aligned}
|a_4a_7-a_5a_6| &\leq \frac{9467}{1474560}y^2\left(2-\frac{y^2}{2}\right)+\frac{15109}{368640}\left(2-\frac{y^2}{2}\right)+\frac{322063}{35389440}y\left(2-\frac{y^2}{2}\right)\\
&+\frac{323519}{141557760}y^3\left(2-\frac{y^2}{2}\right)+\frac{169489}{566231040}y^5\left(2-\frac{y^2}{2}\right)+\frac{23}{1152}y+\frac{20677}{184320}.
\end{aligned}$$

As the above function attain its maximum value at $y = 1.082047787$, so the above equation becomes

$$|a_4a_7 - a_5a_6| \leq 0.2210481986.$$

■

Theorem 3.3.7 *If* $f(z) \in \mathcal{SL}$ *and of the form* (2.1.1), *then*

$$|H_3(1)| \leq \frac{43}{576}.$$

Proof. Since

$$\begin{aligned} |H_3(1)| &= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \\ &= |a_3| \left|a_2a_4 - a_3^2\right| + |a_4| \left|a_4 - a_2a_3\right| + |a_5| \left|a_3 - a_2^2\right|. \end{aligned}$$

Using Lemma 3.2.5, Lemma 3.2.4 and Lemma 3.2.6, we get

$$|H_3(1)| \leq \frac{43}{576}.$$

■

Theorem 3.3.8 *If* $f(z) \in \mathcal{SL}$ *and of the form* (2.1.1), *then*

$$|H_4(1)| \leq 0.06989949554.$$

Proof. Since

$$\begin{aligned} |H_4(1)| \leq\ & \left|a_2a_4 - a_3^2\right| |a_3a_7 - a_4a_6| + |a_2a_3 - a_4| |a_4a_7 - a_5a_6| \\ & + |a_5| \left\{|a_3| \left|a_3a_5 - a_4^2\right| + |a_5| |a_5 - a_2a_4| + |a_6| |a_4 - a_2a_3|\right\} \\ & + |a_4| \left\{|a_4| \left|a_3a_5 - a_4^2\right| + |a_5| |a_2a_5 - a_3a_4| + |a_6| \left|a_2a_4 - a_3^2\right|\right\} \end{aligned}$$

Using Theorem 3.3.1, Theorem 3.3.2, Theorem 3.3.3, Theorem 3.3.4, Theorem 3.3.7, Theorem 3.3.5, Theorem 3.3.6 and Lemma 3.2.5, we have

$$|H_4(1)| \leq 0.06786551485.$$

■

3.4 Bounds of $|H_{4,1}(f)|$ for the Sets $\mathcal{SL}^{(2)}$ and $\mathcal{SL}^{(3)}$

Let $m \in \mathbb{N} = \{1, 2, \ldots\}$. A domain Λ is said to be m-fold symmetric if a rotation of Λ about the origin through an angle $2\pi/m$ carries Λ on itself. It is easy to see that, an analytic function f is m-fold symmetric in $\mathcal{U}$, if

$$f\left(e^{2\pi i/m} z\right) = e^{2\pi i/m} f(z), \ (z \in \mathcal{U}).$$

By $\mathcal{S}^{(m)}$, we mean the set of m-fold univalent functions having the following Taylor series form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \ (z \in \mathcal{U}). \tag{3.4.1}$$

The sub-family $\mathcal{SL}^{(m)}$ of $\mathcal{S}^{(m)}$ is the set of m-fold symmetric starlike functions associated with lemniscate of Bernouli. More intuitively, an analytic function f of the form (3.4.1) belongs to the family $\mathcal{SL}^{(m)}$, if and only if

$$\frac{zf'(z)}{f(z)} = \sqrt{\frac{2p(z)}{p(z)+1}} \text{ with } p \in \mathcal{P}^{(m)},$$

where the set $\mathcal{P}^{(m)}$ is defined by

$$\mathcal{P}^{(m)} = \left\{ p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} c_{mk} z^{mk}, \ (z \in \mathbb{D}) \right\}. \tag{3.4.2}$$

Now we can prove the following theorem

Theorem 3.4.1 *Let $f(z) \in \mathcal{SL}^{(2)}$ be of the form (3.4.1). Then*

$$|H_{4,1}(f)| \leq \frac{13}{3072}.$$

Proof. Since $f(z) \in \mathcal{SL}^{(2)}$, therefore there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$\frac{zf'(z)}{f(z)} = \sqrt{\frac{2p(z)}{p(z)+1}}.$$

For $f \in \mathcal{SL}^{(2)}$, using the series form (3.4.1) and (3.4.2) when $m = 2$, we can write

$$a_3 = \frac{1}{8}c_2, \;\; a_5 = -\frac{1}{32}c_2^2 + \frac{1}{16}c_4, \;\; a_7 = \frac{19}{1536}c_2^3 - \frac{17}{384}c_4c_2 + \frac{1}{24}c_6.$$

It is clear that for $f(z) \in \mathcal{SL}^{(2)}$,

$$H_{4,1}(f) := a_3a_5a_7 - a_3^3a_7 + a_3^2a_5^2 - a_5^3.$$

Therefore

$$H_{4,1}(f) = -\frac{4}{786432}(\frac{1}{4}c_2^2 - c_4)(20\left(\frac{7}{20}c_2^2 - c_4\right)c_2^2 + \left(16c_2c_6 + 48\left(c_2c_6 - c_4^2\right)\right)).$$

Using Lemma 3.2.2 and triangle inequality, we get

$$|H_{4,1}(f)| \leq \frac{8}{786432}(160 + 64 + 192) = \frac{13}{3072}.$$

Hence the proof is complete. ■

Theorem 3.4.2 *If $f(z) \in \mathcal{SL}^{(3)}$ be of the form (3.4.1), then*

$$|H_{4,1}(f)| \leq \frac{8}{3456}$$

Proof. Since $f(z) \in \mathcal{SL}^{(3)}$, therefore there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$\frac{zf'(z)}{f(z)} = \sqrt{\frac{2p(z)}{p(z)+1}}.$$

For $f(z) \in \mathcal{SL}^{(3)}$, using the series form (3.4.1) and (3.4.2) when $m = 3$, we can write

$$a_4 = \frac{1}{12}c_3, \;\; a_7 = -\frac{13}{576}c_3^2 + \frac{1}{24}c_6.$$

It is clear that for $f(z) \in \mathcal{SL}^{(3)}$,

$$H_{4,1}(f) := -a_4^2 a_7 + a_4^4.$$

Therefore

$$\begin{aligned} H_{4,1}(f) &= \frac{17}{82944}c_3^4 - \frac{1}{3456}c_3^2 c_6 \\ &= -\frac{c_3^2}{3456}\left(c_6 - \frac{58752}{82944}c_3^2\right). \end{aligned}$$

Using Lemma 3.2.2 and triangle inequality, we get

$$|H_{4,1}(f)| \leq \frac{8}{3456}.$$

Hence the proof is complete. ∎

Chapter 4

Janowski Bazilevič functions

4.1 Introduction

The set of Bazilevič functions defined in the unit disc $\mathfrak{D}$ is one of the indelible class of GFT. This class was first illustrated in 1955 by Bazilevič [29] . He expressed Bazilevič function by the relation

$$f(z) = \left\{ (\alpha + i\beta) \int\limits_0^z p(t) g^{\alpha}(t) t^{i\beta - 1} dt \right\}^{\frac{1}{\alpha + i\beta}}, \tag{4.1.1}$$

where $p(z) \in \mathcal{P}$, $g(z) \in \mathcal{S}^*$, α, β are real with $\alpha > 0$.Being the largest subfamily of the set of schlicht functions there is little litrature about this family of functions. Pommerenke [124] worked on a characterization of these functions by using subordination chains. By taking $\beta = 0$ in (4.1.1), we get the class of Bazilevič functions of type α given by

$$f(z) = \left\{ \alpha \int\limits_0^z p(\zeta) g^{\alpha}(\zeta) \zeta^{-1} d\zeta \right\}^{\frac{1}{\alpha}}$$

which implies that

$$\mathfrak{Re} \frac{z f'(z)}{f(z)} \left(\frac{f(z)}{g(z)} \right)^{\alpha} > 0, \quad (z \in \mathfrak{D}). \tag{4.1.2}$$

Considering $\alpha = 1$ as a special case of (4.1.2), the well known family of close-to-convex functions is obtained. The family of Bazilevič functions was studied by many researchers for famous Beiberbach conjecture, but still it is unsolved. In this regard Zamorski [156] has also a little contribution on his credit. Thomas [152] in 1968, worked on geometric description of Bazilevič functions for $\beta = 0$. He gave an extention to the results given by Pommerenke and Clunie [39] for bounded close-to-convex functions and by taking $\beta = 0$ he proved that the coefficients of bounded Bazilevič functions satisfy $a_n = \mathrm{O}\left(\frac{1}{n}\right)$. A Fundamental description of the ordinary Bazilevič functions along the lines of Kaplan [68] was given by Shiell Small [144]. Generalization of Bezilevič functions was given by Campbell and Pearce in 1979. The generalized Bezilevič functions by means of differential equation is given as;

$$1+\frac{zf''(z)}{f'(z)}+(\alpha+i\beta-1)\frac{zf'(z)}{f(z)}=\alpha\frac{zg'(z)}{g(z)}+\frac{zp'(z)}{p(z)}+i\beta,\ \text{see [33]}.$$

They showed that each generalized Bazilevič functions is associated with the quadruple (α,β,g,p).

Definition 4.1.1 *Let* $g(z)\in\mathcal{S}^*[A,B]$ *be the function of the form* (2.1.1) *and* $p(z)\in\mathcal{P}[\eta,A,B]$ *with* $p(0)=1$. *Then the function* $f(z)$ *of the form* (2.1.1) *is said to be the generalized Bazilevič function corresponding the quadruple* (α,β,g,p) *if and only if* $f(z)$ *satisfies the following differential equation*

$$1+\frac{zf''(z)}{f'(z)}+(\alpha+i\beta-1)\frac{zf'(z)}{f(z)}=\alpha\frac{zg'(z)}{g(z)}+\frac{zp'(z)}{p(z)}+i\beta,$$

or equivalently

$$\frac{zf'(z)}{f(z)}=\left(\frac{g(z)}{z}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{\alpha+i\beta}p(z).$$

The above differential equation can also be expressed as

$$\frac{z^{1-i\beta}f'(z)}{f^{1-(\alpha+i\beta)}g^{\alpha}(z)}=p(z),\ \ (z\in\mathfrak{D})\,.$$

Since $p(z)\in\mathcal{P}[\eta,A,B]$, therefore

$$1+\frac{1}{\eta}\left\{\frac{z^{1-i\beta}f'(z)}{f^{1-(\alpha+i\beta)}g^{\alpha}(z)}-1\right\}\prec\frac{1+Az}{1+Bz},$$

where $g(z)\in\mathcal{S}^*[A,B]$.

Several research papers appeared recently on classes related with Janowski functions, Bazilevič functions and their generalizations, see [19, 21, 23, 110, 108].

4.2 Preliminary results

Lemma 4.2.1 *Let $p(z) \in \mathcal{P}[\eta, A, B]$ with $-1 \leq B < A \leq 1$, $\eta \in \mathbb{C}\backslash\{0\}$, and has the form* (2.2.1) *. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|p(re^{i\theta})\right|^2 d\theta \leq \frac{1 + \left[|\eta|^2 (A-B)^2 - 1\right] r^2}{1 - r^2}, \ \textit{for } z = re^{i\theta}.$$

Proof. The proof of this Lemma is straight forward but we include it for the sake of completeness. Since $p(z) \in \mathcal{P}[\eta, A, B]$, therefore we have

$$p(z) = \eta p_1(z) + (1-\eta), \quad p_1(z) \in \mathcal{P}[A, B].$$

Let $p_1(z) = 1 + \sum\limits_{n=1}^{\infty} c_n \ z^n$. Then

$$1 + \sum_{n=1}^{\infty} p_n \ z^n = \eta \left(1 + \sum_{n=1}^{\infty} c_n \ z^n\right) + (1-\eta).$$

Comparing the coefficients of z^n, we have

$$p_n = \eta c_n.$$

Since $|c_n| \leq A - B$, therefore $|p_n| \leq |\eta| (A - B)$. Now

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \left|p(re^{i\theta})\right|^2 d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{n=0}^{\infty} p_n z^n\right|^2 d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{n=0}^{\infty} p_n^2 r^{2n} e^{2in\theta}\right| d\theta \\
&= \frac{1}{2\pi}\sum_{n=0}^{\infty} |p_n|^2 r^{2n} \int_0^{2\pi} d\theta, \quad \because \left|e^{i2n\theta}\right| = 1.
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\left|p(re^{i\theta})\right|^2 d\theta &\leq 1+|\eta|^2(A-B)^2\sum_{n=1}^{\infty}r^{2n} \\
&= 1+|\eta|^2(A-B)^2\frac{r^2}{1-r^2} \\
&= \frac{1+\left(|\eta|^2(A-B)^2-1\right)r^2}{1-r^2},
\end{aligned}$$

and thus the proof is completed. ■

Lemma 4.2.2 *[10] Let Ω be the family of regular functions $\omega(z)$,satisfying the condition $\omega(0)=0$ and $|\omega(z)|<1$. If $\omega(z)\in\Omega$ and*

$$\omega(z)=\omega_1 z+\omega_2 z^2+\ldots,\quad (z\in\mathfrak{D}),$$

then

$$\left|\omega_2-t\omega_1^2\right|\leq\begin{cases}-t, & t\leq -1\\ 1, & -1<t<1\\ t, & t\geq 1,\end{cases}$$

and for any complex number t

$$\left|\omega_2-t\omega_1^2\right|\leq\max\{1,|t|\}.$$

Both the above inequalities are best possible.

Lemma 4.2.3 *Let $p(z)=1+\sum_{n=1}^{\infty}p_n z^n\in\mathcal{P}]\eta,A,B]$, $\eta\in\mathbb{C}\backslash\{0\}$, $-1\leq B<A\leq 1$. Then*

$$(i).\quad \left|p_2-\mu p_1^2\right|\leq\begin{cases}|\eta|(A-B)-\mu^2|\eta|^2(A-B)^2, & \mu\leq 0,\\ |\eta|(A-B), & 0<\mu<\frac{2}{|\eta|(A-B)},\\ \mu|\eta|^2(A-B)^2-|\eta|(A-B), & \mu|\eta|^2(A-B)^2-|\eta|(A-B).\end{cases}$$

and for any complex number μ

$$(ii). \quad \left|p_2 - \mu p_1^2\right| \leq |\eta| (A - B) \max\{1, |\mu\eta(A - B) - 1|\}.$$

Both the results are sharp.

Proof. Let $p(z) \in \mathcal{P}[\eta, A, B]$. Then by definition of the set $\mathcal{P}]\eta, A, B]$, we have

$$1 + \frac{1}{\eta}\{p(z) - 1\} \prec \frac{1 + Az}{1 + Bz},$$

or, equivalently

$$p(z) \prec \frac{1 + [\eta A + (1 - \eta)B] z}{1 + Bz} = 1 + \eta(A - B) \sum_{n=1}^{\infty} z^n,$$

and its further gives

$$\begin{aligned} 1 + p_1 z + \ldots &= 1 + \eta(A - B)\omega(z) + \eta(A - B)\omega^2(z) + \ldots \\ &= 1 + \eta(A - B)\left(\omega_1 z + \omega_2 z^2 + \ldots\right) + \eta(A - B)\left(\omega_1 z + \omega_2 z^2 + \ldots\right)^2 + \ldots. \end{aligned}$$

Comparing the coefficients of z and z^2, we obtain

$$\begin{aligned} p_1 &= \eta(A - B)\omega_1 \\ p_2 &= \eta(A - B)\omega_2 + \eta(A - B)\omega_1^2. \end{aligned}$$

By simple computation, we easily obtain

$$\left|p_2 - \mu p_1^2\right| = |\eta| (A - B) \left|\omega_2 - (\mu\eta(A - B) - 1)\omega_1^2\right|.$$

Using Lemma 4.2.2 for $t = [\mu\eta(A - B) - 1]$, we get the required result.

(i). Equality can be attained for the function $p_*(z)$ or one of its rotation, where

$$p_*(z) = \begin{cases} \frac{1+(\eta A+(1-\eta)B)z}{1+Bz}, & \mu < 0 \text{ or } \mu > \frac{1}{|\eta|(A-B)}, \\ \frac{1+(\eta A+(1-\eta)B)z^2}{1+Bz^2}, & 0 < \mu < \frac{1}{|\eta|(A-B)}, \\ \frac{\lambda+1}{2}\frac{1+(\eta A+(1-\eta)B)z}{1+Bz} + \frac{1-\lambda}{2}\frac{1-(\eta A+(1-\eta)B)z}{1-Bz} & 0 \leq \lambda \leq 1,\ \mu = 0. \end{cases}$$

and the equality holds in the case when $\mu = \frac{1}{|\eta|(A-B)}$ if and only if the function is the reciprocal of the function such that equality holds in $\mu = 0$.

(ii). Equality holds for the function

$$p_\circ(z) = \frac{1 + (\eta A + (1-\eta)B)\, z}{1 + Bz} \text{ or } p_\circ(z) = \frac{1 + (\eta A + (1-\eta)B)\, z^2}{1 + Bz^2}.$$

This completes the proof. ■

Lemma 4.2.4 *Let $N(z)$ and $D(z)$ be regular in $\mathfrak{D}$ with $N(0) = 0 = D(0)$ and the image of $\mathfrak{D}$ under $D(z)$ is a many sheeted region which is starlike with respect to the origin, then*

$$\frac{N'(z)}{D'(z)} \in \mathcal{P}[\eta, A, B] \text{ implies } \frac{N(z)}{D(z)} \in \mathcal{P}[\eta, A, B].$$

Proof. Let

$$\frac{N(z)}{D(z)} = p(z).$$

Then $p(z)$ is regular in $\mathfrak{D}$ with $p(0) = 1$. Now by simple computations we have

$$\frac{N'(z)}{D'(z)} = p(z) + \frac{zp'(z)}{p_1(z)},$$

where $p_1(z) = \frac{zD'(z)}{D(z)}$. This implies that $p_1(z) \in \mathcal{P}$. Since $p_1(z) \in \mathcal{P}$, therefore $\frac{1}{p_1(z)} \in \mathcal{P}$. Let $\frac{1}{p_1(z)} = q(z)$. Then

$$\frac{N'(z)}{D'(z)} = p(z) + q(z)zp'(z).$$

Let $p(z) = \eta h_1(z) + (1-\eta)$. Then

$$\frac{N'(z)}{D'(z)} = \eta h_1(z) + (1-\eta) + \eta q(z) z h_1'(z).$$

Therefore

$$\frac{1}{\eta}\left\{\frac{N'(z)}{D'(z)} - (1-\eta)\right\} = h_1(z) + q(z) z h_1'(z) \prec \frac{1+Az}{1+Bz}.$$

Now using a result [96], we have

$$h_1(z) \prec \frac{1+Az}{1+Bz},$$

that is

$$\frac{1}{\eta}\left\{\frac{N(z)}{D(z)} - (1-\eta)\right\} \prec \frac{1+Az}{1+Bz}.$$

Hence

$$\frac{N(z)}{D(z)} \in \mathcal{P}[\eta, A, B],$$

and this completes the proof. ■

Lemma 4.2.5 *[96] Let β_1, γ_1, $A \in \mathbb{C}$ with $\mathfrak{Re}\,(\beta_1+\gamma_1) > 0$ and let $B \in [-1, 0]$ satisfy*

$$\mathfrak{Re}\left[\beta_1(1+AB) + \gamma_1(1+B^2)\right] \geq \left|BA + \overline{\beta_1}B + B\left(\gamma_1 + \overline{\gamma_1}\right)\right|,\ B \in (-1, 0)$$

or

$$\mathfrak{Re}\beta_1(1+A) > 0 \text{ and } \mathfrak{Re}\left[\beta_1(1-A) + 2\gamma_1\right] \geq 0,\ B \in -1.$$

If $p(z) = 1 + p_1 z + p_2(z) + \ldots$ satisfies

$$p(z) = \frac{zp'(z)}{\beta_1 p(z) + \gamma_1} \prec \frac{1+Az}{1+Bz},$$

then

$$p(z) \prec q(z) \prec \frac{1+Az}{1+Bz},$$

where $q(z)$ is best dominant and

$$q(z) = \frac{1}{\beta_1}\left\{\frac{1}{g_1(z)} - \gamma_1\right\},$$
$$g_1(z) = \int_0^1 \left[\frac{1+Btz}{1+Bz}\right]^{\beta_1\left(\frac{A}{B}-1\right)} t^{\beta_1+\gamma_1-1}dt, \quad B \neq 0.$$

4.3 Some properties of the defined new families

Theorem 4.3.1 *Let $g(z)$ be a fuction of the form* (2.1.1) *such that the $g(z) \in \mathcal{S}^*[A,B]$. Then*

$$\left|b_3 - \mu b_2^2\right| \leq \frac{(A-B)}{2}\max\left\{1, \ \left|1+(A-B)(1-\mu)\right|\right\}.$$

Proof. Proof of the result is same as of Lemma 4.2.3. The following function

$$g_*(z) = \begin{cases} z(1+Bz)^{\frac{A-B}{B}} \text{ or } z^2(1+Bz^2)^{\frac{A-B}{B}}, & B \neq 0, \\ ze^{Az} \text{ or } z^2e^{Az^2}, & B = 0. \end{cases}$$

shows that this result is sharp. ■

Theorem 4.3.2 *Let $g(z) \in \mathcal{S}^*[A,B]$. Then*

$$G^{\alpha}(z) = \frac{c+\alpha+i\beta}{z^{c+i\beta}}\int_0^z t^{c+i\beta-1}g^{\alpha}(t)dt, \tag{4.3.1}$$

where $\alpha > 0$, $c > 0$ and β is any real number belong to $\mathcal{S}^*[A,B,\delta]$, where $\delta = \min_{|z|=1}\mathfrak{Re}\,q(z)$ and

$$q(z) = \begin{cases} \frac{1}{\alpha}\frac{\alpha+i\beta+c}{{}_2F_1\left(1;\,\alpha\left(1-\frac{A}{B}\right);\,\alpha+i\beta+c+1;\,\frac{Bz}{1+Bz}\right)} - (c+i\beta), & B \neq 0, \\ \frac{1}{\alpha}\frac{\alpha+i\beta+c}{{}_1F_1(1;\,\alpha+i\beta+c+1;\,-\alpha Az)} - (c+i\beta), & B = 0. \end{cases}$$

Proof. From $(4.3.1)$, we have

$$z^{c+i\beta}G^{\alpha}(z) = (c+\alpha+i\beta)\int_0^z t^{c+i\beta-1}g^{\alpha}(t)dt.$$

This implies that

$$(c+i\beta)\, z^{c+i\beta-1}G^{\alpha}(z)+\alpha z^{c+i\beta}G^{\alpha-1}(z)G'(z)=(c+\alpha+i\beta)\, t^{c+i\beta-1}g^{\alpha}(z).$$

Hence

$$(c+\alpha+i\beta)\,\frac{g^{\alpha}(z)}{G^{\alpha}(z)}=(c+i\beta)+\alpha p(z), \tag{4.3.2}$$

where $p(z)=\frac{g^{\alpha}(z)}{G^{\alpha}(z)}$. By logarithmic differentiation of (4.3.2), we obtain

$$\frac{zg'(z)}{g(z)}=p(z)+\frac{zp'(z)}{\alpha p(z)+(c+i\beta)}.$$

As $g(z)\in\mathcal{S}^{*}[A,B]$, so

$$\frac{zp'(z)}{\alpha p(z)+(c+i\beta)}\prec\frac{1+Az}{1+Bz}.$$

Now using Lemma 4.2.5 , for $\beta_1=\alpha$ and $\gamma_1=c+i\beta$, we obtain

$$p(z)\prec q(z)\prec\frac{1+Az}{1+Bz},$$

where

$$q(z)=\frac{1}{\alpha}\left\{\frac{1}{g_1(z)}-(c+i\beta)\right\},$$

and

$$g_1(z)=\int_0^1\left(\frac{1+Btz}{1+Bz}\right)^{\beta_1\left(\frac{A}{B}-1\right)}t^{\alpha+i\beta+c-1}dt,\quad B\neq 0.$$

Now using the properties of the familiar hypergeometric functions proved in [96], we have

$$g_1(z)=\begin{cases}\frac{1}{\alpha+i\beta+c}{}_2F_1\left(\alpha\left(1-\frac{A}{B}\right);1;\alpha+i\beta+c+1;\frac{Bz}{1+Bz}\right), & B\neq 0,\\ \frac{1}{\alpha+i\beta+c}{}_1F_1\left(1;\alpha+i\beta+c+1;-\alpha Az\right), & B=0.\end{cases}$$

This implies that

$$p(z) \prec q(z) = \begin{cases} \frac{1}{\alpha}\frac{\alpha+i\beta+c}{{}_2F_1\left(1;\,\alpha\left(1-\frac{A}{B}\right);\,\alpha+i\beta+c+1;\,\frac{Bz}{1+Bz}\right)} - (c+i\beta)\,, & B \neq 0, \\ \frac{1}{\alpha}\frac{\alpha+i\beta+c}{{}_2F_1(1;\,\alpha+i\beta+c+1;\,-\alpha Az)} - (c+i\beta)\,, & B = 0, \end{cases}$$

and

$$\mathfrak{Re}\frac{zg'(z)}{g(z)} = \mathfrak{Re}p(z) > \delta = \min_{|z|=1}\mathfrak{Re}q(z).$$

Hence the result follows. ■

Theorem 4.3.3 *Let $g(z) \in \mathcal{S}^*[A,B]$. Then*

$$S(z) = \int_0^z t^{c+i\beta-1}g^{\alpha}(t)dt,$$

is $(\alpha+c)$-valent starlike, where $\alpha > 0$, $c > 0$ and β is real.

Proof. Let $D_1(z) = zS'(z) = z^{c+i\beta}g^{\alpha}(z)$ and $N_1(z) = S(z)$. This implies that

$$\begin{aligned} \mathfrak{Re}\frac{zD_1'(z)}{D_1(z)} &= \mathfrak{Re}\left\{(c+i\beta+\frac{zg'(z)}{g(z)}\right\} \\ &= c+\alpha\frac{zg'(z)}{g(z)}. \end{aligned}$$

Since $g(z) \in \mathcal{S}^*[A,B] \subset \mathcal{S}^*\left(\frac{1-A}{1-B}\right)$, therefore

$$\mathfrak{Re}\frac{zD_1'(z)}{D_1(z)} = c+\alpha\left(\frac{1-A}{1-B}\right) > 0.$$

Also

$$\mathfrak{Re}\frac{D_1'(z)}{N_1'(z)} = \mathfrak{Re}\left\{(c+i\beta+\frac{zg'(z)}{g(z)}\right\} > 0.$$

Now by using Lemma 4.2.4 which is valid even though $\frac{D_1'(0)}{N_1'(0)} \neq 1$, we have

$$\mathfrak{Re}\frac{D_1(z)}{N_1(z)} > 0 \text{ or } \mathfrak{Re}\frac{zS'(z)}{S(z)} > 0.$$

By mean value theorem for harmonic functions

$$\mathfrak{Re}\frac{zS'(z)}{S(z)}\Big|_{z=0} = \frac{1}{2\pi}\int_0^{2\pi} \mathfrak{Re}\frac{re^{i\theta}S'(re^{i\theta})}{S(re^{i\theta})}d\theta.$$

Therefore

$$\begin{aligned}\int_0^{2\pi} \mathfrak{Re}\frac{re^{i\theta}S'(re^{i\theta})}{S(re^{i\theta})}d\theta &= 2\pi\mathfrak{Re}\left\{c+i\beta+\alpha\frac{zg'(z)}{g(z)}\right\}_{z=0} \\ &= 2\pi(c+\alpha).\end{aligned}$$

Now using a result due to [45] , we have $S(z)$ is $(c+\alpha)$-valent starlike. ∎

Theorem 4.3.4 *Let $f(z)$ be a generalized Bezilevič function, presented by quadruple (α,β,g,p) with $g(z)\in\mathcal{S}^*[A,B]$ and $p(z)\in\mathcal{P}[\eta,A,B]$, where each $f(z)$ and $g(z)$ has the form* (2.1.1) *and $p(z)=1+\sum\limits_{n=1}^{\infty}p_nz^n$. Then for $\beta\neq 0$, we have*

$$\left|a_3-\frac{3+\alpha+i\beta}{2(2+\alpha+i\beta)}a_2^2\right| \leq \frac{A-B}{2}\left\{\frac{\alpha+2|\eta|}{|12+(\alpha+i\beta)|}\right\}.$$

Proof. Since $f(z)$ is considered to be generalized Bazilevic function, thus we have

$$1+\frac{zf''(z)}{f'(z)}+(\alpha+i\beta-1)\frac{zf'(z)}{f(z)} = \alpha\frac{zg'(z)}{g(z)}+\frac{zp'(z)}{p(z)}+i\beta.$$

Multiplying $f(z)f'(z)g(z)p(z)$ on both sides, we have

$$(1-i\beta)f(z)f'(z)g(z)p(z)+zp(z)f(z)f''(z)g(z)+(\alpha+i\beta-1)z\left(f'(z)\right)^2g(z)p(z)$$

$$=\alpha zf(z)f'(z)g'(z)p(z)+zf(z)f'(z)g(z)p'(z).$$

Since

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\quad g(z)=\sum_{n=1}^{\infty}b_nz^n \text{ and } p(z)=1+\sum_{n=1}^{\infty}p_nz^n,$$

therefore by the coefficients comparision of z^3, we get

$$\{(1-i\beta)(3a_2+b_2+p_1)+2a_2+(\alpha+i\beta-1)(4a_2+b_2+p_1)\}=\alpha\left(3a_2+2b_2+p_1\right).$$

This implies that

$$(1+\alpha+i\beta)a_2=\alpha b_2+p_1. \tag{4.3.3}$$

like wise comparing the coefficients of z^4 and by the use of the above inequality, we have

$$2(2+\alpha+i\beta)=\alpha(2b_3-b_2^2)+2p_2-p_1^2+a_2^2(3+\alpha+i\beta). \tag{4.3.4}$$

Now from (4.3.3) and (4.3.4) we have

$$\begin{aligned}\left|a_3-\frac{3+\alpha+i\beta}{2(2+\alpha+i\beta)}a_2^2\right| &= \left|\frac{\alpha\left(b_3-\frac{1}{2}b_2^2\right)+(p_2-\frac{1}{2}p_1^2)}{2+\alpha+i\beta}\right| \\ &\leq \frac{\alpha\left|b_3-\frac{1}{2}b_2^2\right|}{|2+\alpha+i\beta|}+\frac{\left|p_2-\frac{1}{2}p_1^2\right|}{|2+\alpha+i\beta|}.\end{aligned}$$

Now using Lemma 4.2.3 and Theorem 4.3.1 for $\mu=\frac{1}{2}$, we obtain

$$\left|b_3-\frac{1}{2}b_2^2\right|\leq\frac{A-B}{2},$$

and

$$\left|p_2-\frac{1}{2}p_1^2\right|\leq|\eta|\,(A-B).$$

Therefore, we get

$$\begin{aligned}\left|a_3-\frac{3+\alpha+i\beta}{2(2+\alpha+i\beta)}a_2^2\right| &\leq \frac{\alpha\left(\frac{A-B}{2}\right)+|\eta|\,(A-B)}{|2+\alpha+i\beta|} \\ &= \frac{\left(\frac{A-B}{2}\right)(\alpha+2\,|\eta|)}{|2+\alpha+i\beta|}.\end{aligned}$$

Equality can be obtain by the function

$$g_\circ(z) = z(1+Bz)^{\frac{A-B}{B}}, B \neq 0$$

and

$$p_\circ(z) = \frac{1+(\eta A+(\eta-1)B)z^2}{1+Bz^2}.$$

This completes the proof. ■

Corollary 4.3.1 *For $A = 1$, $B = -1$, we have the generalized Bazilevič function for which $g(z) \in \mathcal{S}^*$ and $p(z) \in \mathcal{P}(\eta)$, then*

$$\left|a_3 - \frac{3+\alpha+i\beta}{2(2+\alpha+i\beta)}a_2^2\right| \leq \frac{(\alpha+2|\eta|)}{|2+\alpha+i\beta|}$$

Corollary 4.3.2 *For $A = 1$, $B = -1$ and $\eta = 1$, then we have following result proved in [33].*

$$\left|a_3 - \frac{3+\alpha+i\beta}{2(2+\alpha+i\beta)}a_2^2\right| \leq \frac{(\alpha+2)}{|2+\alpha+i\beta|}$$

Corollary 4.3.3 *For $\alpha = 1$, $\beta = 0$, we have $f(z) \in \mathcal{K}$, the class of close-to-convex functions we have*

$$\left|a_3 - \frac{2}{3}a_2^2\right| \leq 1.$$

This result has been proved in [69].

Theorem 4.3.5 *Let $f(z)$ be generalized Bazilevic function associated with quadruple (α, β, g, p), where $g(z) = z + \sum_{n=1}^{\infty} b_n z^n \in \mathcal{S}^*[A, B]$ and $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}[\eta, A, B]$. Then*

$$|a_2| \leq \frac{(A-B)(\alpha+|\eta|)}{|1+\alpha+i\beta|}, \ B \neq 0.$$

(ii) If $f(z)$ is in $(0, \beta, g, p)$, then

$$|a_3| \leq \frac{|\eta|(A-B)}{|2+i\beta|} \max\left\{1, \left|\frac{\eta(A-B)}{2}\left(1-\frac{(3+i\beta)}{(1+i\beta)^2}\right)-1\right|\right\}.$$

the above inequalities are sharp.

Proof. From equation (4.3.3), we get

$$(1+\alpha+i\beta)a_2 = \alpha b_2 + p_1.$$

This implies that

$$|a_2| \le \frac{\alpha|b_2| + |p_1|}{|1+\alpha+i\beta|}$$

Since

$$g(z) = z + \sum_{n=1}^{\infty} b_n z^n \in \mathcal{S}^*[A,B] \text{ and } p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}[\eta, A, B],$$

therefore by using the following second bound for $\mathcal{S}^*[A,B]$ proved in [29] along with the coefficient bound of $\mathcal{P}[\eta, A, B]$, we have

$$|b_2| \le A - B \text{ and } |p_1| \le |\eta|(A-B).$$

This implies that

$$|a_2| \le \frac{(\alpha+|\eta|)(A-B)}{|1+\alpha+i\beta|}.$$

Equality can be obtained by the functions

$$g_\circ(z) = z(1+Bz)^{\frac{A-B}{B}},\ B \neq 0 \ \text{ and } p_\circ(z) = \frac{1+(\eta A + (1-\eta)B)z}{1+Bz}.$$

Thus the result follows. ■

Corollary 4.3.4 *For $A=1$, $B=-1$, we have $g(z) \in \mathcal{S}^*$, $p(z) \in \mathcal{P}(\eta)$, therefore*

$$|a_2| \le \frac{2(\alpha+|\eta|)}{|1+\alpha+i\beta|}.$$

Corollary 4.3.5 *For $A = 1$, $B = -1$ and $\eta = 1$ we have $g(z) \in \mathcal{S}^*$, $p(z) \in \mathcal{P}$, therefore*

$$|a_2| \leq \frac{2(\alpha + 1)}{|1 + \alpha + i\beta|}, \text{ see [33].}$$

Corollary 4.3.6 *For $A = 1$, $B = -1$, $\eta = 1$, $\alpha = 1$ and $\beta = 0$, we have $f(z) \in \mathcal{K}$, therefore*

$$|a_2| \leq 2.$$

Proof. (ii) Since $f(z)$ is in $(0, \beta, g, p)$, therefore from (4.3.3) and (4.3.4), we have

$$(1 + i\beta) a_2 = p_1$$

and

$$2(2 + i\beta)a_3 = 2p_2 - p_1^2 + (3 + i\beta)a_2^2.$$

Therefore

$$\begin{aligned} (2 + i\beta)a_3 &= p_2 - \frac{1}{2}p_1^2 + \frac{(3 + i\beta)p_1^2}{2(1 + i\beta)^2} \\ &= p_2 - \frac{1}{2}\left(1 - \frac{(3 + i\beta)}{(1 + i\beta)^2}\right)p_1^2. \end{aligned}$$

This implies that

$$|a_3| = \frac{1}{|2 + i\beta|}\left|p_2 - \mu p_1^2\right|,$$

where $\mu = \frac{1}{2} - \frac{(3+i\beta)}{2(1+i\beta)^2}$. Now using Lemma 4.2.3 for μ given before, we obtain

$$|a_3| = \frac{|\eta|(A - B)}{|2 + i\beta|} \max\left\{1, \left|\frac{\eta(A - B)}{2}\left(1 - \frac{(3 + i\beta)}{(1 + i\beta)^2}\right) - 1\right|\right\}.$$

Sharpness can be attain by the function

$$p_\circ(z) = \frac{1 + (\eta A + (\eta - 1)B) z^2}{1 + Bz^2}.$$

Hence the proof is complete. ■

Corollary 4.3.7 *For $g(z) \in \mathcal{S}^*$ and $p(z) \in \mathcal{P}(\eta)$, therefore*

$$|a_3| = \frac{2|\eta|}{|2+i\beta|} \max\left\{1, \left|\eta\left(1 - \frac{(3+i\beta)}{(1+i\beta)^2}\right) - 1\right|\right\}.$$

This result is sharp for the function

$$p_\circ(z) = \frac{1+(2\eta-1)z^2}{1-z^2}.$$

Corollary 4.3.8 *For $g(z) \in \mathcal{S}^*$ and $p(z) \in \mathcal{P}$, therefore*

$$|a_3| = \frac{2|\eta|}{|2+i\beta|} \max\left\{1, \left|\frac{(3+i\beta)}{(1+i\beta)^2}\right|\right\}, \text{ see [33].}$$

Theorem 4.3.6 *Let $f(z)$ be generalized Bazilevič function corresponded to the quadruple $(\alpha, 0, g, p)$, where $g(z) \in \mathcal{S}^*[A, B]$ and $p(z) \in \mathcal{P}[\eta, A, B]$. Then for $M(r) = \max\limits_{|z|=r} |f(z)|$ and $0 \le \alpha < 1$*

$$(i).\quad L_r(f(z)) \le C(\eta, A, B) M^{1-\alpha}(r) \left(\frac{1}{1-r}\right)^{\alpha\left(1-\frac{A}{B}\right)},\ B \ne 0$$

and for $\alpha > 1$, $m(r) = \min\limits_{|z|=r} |f(z)|$

$$(ii).\quad L_r(f(z)) \le C(\eta, A, B) m^{\alpha-1}(r) \left(\frac{1}{1-r}\right)^{\alpha\left(1-\frac{A}{B}\right)},\ B \ne 0,$$

where $C(\eta, A, B)$ appearing in (i) and (ii) is a constant and depends upon η, A and B only.

Proof. (i) Since $f(z)$ is generalized Bazilevič function corresponded to the quadruple $(\alpha, 0, g, p)$, therefore

$$zf'(z) = f^{1-\alpha}(z) g^\alpha(z) p(z).$$

We know that

$$L_r(f(z)) = \int_0^{2\pi} |zf'(z)|\, d\theta, \quad z = re^{i\theta},\ 0 < r < 1.$$

This implies that

$$\begin{aligned} L_r(f(z)) &= \int_0^{2\pi} \left|f^{1-\alpha}(z) g^{\alpha}(z) p(z)\right| d\theta \\ &\leq \int_0^{2\pi} \left|f^{1-\alpha}(z)\right| |g^{\alpha}(z)|\, |p(z)|\, d\theta \\ &\leq M^{1-\alpha}(r) \int_0^{2\pi} |g^{\alpha}(z)|\, |p(z)|\, d\theta. \end{aligned}$$

By the use Cauchy-Schwarz inequality, we obtain

$$L_r(f(z)) \leq 2\pi M^{1-\alpha}(r) \left(\frac{1}{2\pi} \int_0^{2\pi} |g(z)|^{2\alpha}\, d\theta \right)^{\frac{1}{2}} \left(\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2\, d\theta \right)^{\frac{1}{2}}.$$

Now using Lemma 4.2.1 along with the result

$$G(r, -A, -B) \leq |g(z)| \leq G(r, A, B),\ \text{see [120]},$$

where

$$G(r, A, B) = \begin{cases} r(1 + Br)^{\frac{A-B}{B}}, & B \neq 0 \\ re^{Ar}, & B = 0, \end{cases}$$

we have

$$L_r(f(z)) \leq 2\pi M^{1-\alpha}(r) \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 + Bre^{i\theta}|^{2\alpha\left(1-\frac{A}{B}\right)}} \right)^{\frac{1}{2}} \left(\frac{1 + \left(|\eta|^2 (A-B)^2 - 1\right) r^2}{1 - r^2} \right)^{\frac{1}{2}} d\theta.$$

Since $1-\left|Bre^{i\theta}\right| \leq \left|1+Bre^{i\theta}\right|$, therefore

$$L_r f(z) \leq \sqrt{\pi} M^{1-\alpha}(r) \frac{|\eta|(A-B)}{(1-r)^{\frac{1}{2}}} \left(\int_0^{2\pi} \frac{1}{(1-|B|r)^{2\alpha\left(1-\frac{A}{B}\right)}} \right)^{\frac{1}{2}}.$$

This implies that

$$L_r f(z) \leq \sqrt{\pi} M^{1-\alpha}(r) \frac{|\eta|(A-B)}{(1-r)^{\frac{1}{2}}} \left(\frac{1}{1-|B|r} \right)^{\alpha\left(1-\frac{A}{B}\right)-\frac{1}{2}}.$$

Since $1-|B|r \geq 1-r$, therefore

$$L_r f(z) \leq C(\eta, A, B) M^{1-\alpha}(r) \left(\frac{1}{1-r} \right)^{\alpha\left(1-\frac{A}{B}\right)}.$$

(ii) When $\alpha > 1$, then$L_r f(z) \leq C(\eta, A, B) m^{\alpha-1}(r) \left(\frac{1}{1-r}\right)^{\alpha\left(1-\frac{A}{B}\right)}$.

Henced proved. ■

Corollary 4.3.9 *Let $g(z) \in \mathcal{S}^*$, $p(z) \in \mathcal{P}(\eta)$, and $0 < \alpha \leq 1$,then*

$$L_r f(z) \leq C(\eta) M^{1-\alpha}(r) \left(\frac{1}{1-r} \right)^{2\alpha}$$

and for $\alpha > 1$

$$L_r f(z) \leq C(\eta) m^{\alpha-1}(r) \left(\frac{1}{1-r} \right)^{2\alpha}.$$

For the last result, we refer [106].

Theorem 4.3.7 *If $g(z) \in \mathcal{S}^*[A, B]$ and $p(z) \in \mathcal{P}[\eta, A, B]$, Let $f(z)$ be the generalized Bazilevič function corresponding to the quadruple $(\alpha, 0, g, p)$,. Then for $0 \leq \alpha < 1$ and $M(r) = \max_{|z|=r} |f(z)|$, we have*

$$|a_n| \leq C_1(\eta, A, B) M^{1-\alpha}(n) n^{\alpha\left(1-\frac{A}{B}\right)-1}, \; n \to \infty$$

and for $\alpha > 1$ *with* $m(r) = \min_{|z|=r} |f(z)|$

$$|a_n| \leq C_1(\eta, A, B) m^{\alpha-1}(n) n^{\alpha\left(1-\frac{A}{B}\right)-1}, \ n \to \infty$$

where $C_1(\eta, A, B)$ *is constant depending upon* η, A *and* B *only.*

Proof. By Cauchy theorem for $z = re^{i\theta}$, $n \geq 2$, we have

$$na_n = \frac{1}{2\pi r^n} \int_0^{2\pi} zf'(z) e^{-in\theta} d\theta.$$

Therefore

$$\begin{aligned} n|a_n| &= \frac{1}{2\pi r^n} \int_0^{2\pi} |zf'(z)| \, d\theta, \\ &= \frac{1}{2\pi r^n} L_r(f(z)). \end{aligned}$$

By using the last Theorem for the case $0 < \alpha \leq 1$, we have

$$n|a_n| = \frac{1}{2\pi r^n} M^{1-\alpha}(r) C(\eta, A, B) \left(\frac{1}{1-r}\right)^{\alpha\left(1-\frac{A}{B}\right)},$$

and now putting $r = 1 - \frac{1}{n}$, we obtain

$$|a_n| = C_1(\eta, A, B) M^{1-\alpha}(n) n^{\alpha\left(1-\frac{A}{B}\right)-1}, \ n \to \infty.$$

For $\alpha > 1$, we have

$$|a_n| = C_1(\eta, A, B) m^{\alpha-1}(n) n^{\alpha\left(1-\frac{A}{B}\right)-1}, \ n \to \infty,$$

Hence completing the proof. ■

Corollary 4.3.10 *For $g(z) \in \mathcal{S}^*$ and $p(z) \in \mathcal{P}(\eta)$, then*

$$|a_n| \leq \begin{cases} C_1(\eta)M^{1-\alpha}(n)n^{2\alpha-1}, & 0 \leq \alpha < 1, \\ C_1(\eta)m^{\alpha-1}(n)n^{2\alpha-1}, & \alpha > 1. \end{cases}$$

For this corollary see [106].

Theorem 4.3.8 *Let $f(z)$ belong to generalized Bazilevic function represented by the quadruple (α, β, g, p), where $p(z) \in \mathcal{P}[\eta, A, B]$ and $g(z) \in \mathcal{S}^*[A, B]$. Then*

$$F(z) = \left[\frac{c+\alpha+i\beta}{z^c}\int_0^z t^{c-1}f^{\alpha+i\beta}(t)dt\right]^{\frac{1}{\alpha+i\beta}} \tag{4.3.5}$$

belongs to the generalized Bazilevič function represented by the quadruple (α, β, G, p), where $G(z) \in \mathcal{S}^*[A, B, \delta]$ and is defined by $(4.3.1)$.

Proof. From $(4.3.5)$, we have

$$F^{\alpha+i\beta}(z) = \frac{c+\alpha+i\beta}{z^c}\int_0^z t^{c-1}(f(t))^{\alpha+i\beta}dt.$$

This implies that

$$z^cF^{\alpha+i\beta}(z) = (c+\alpha+i\beta)\int_0^z t^{c-1}(f(t))^{\alpha+i\beta}dt.$$

Therefore

$$cz^{c-1}F^{\alpha+i\beta}(z) + (\alpha+i\beta)z^cF^{\alpha+i\beta-1}(z)F'(z) = (c+\alpha+i\beta)\,z^{c-1}(f(z))^{\alpha+i\beta}.$$

Hence

$$\frac{z^{1-i\beta}F'(z)}{F^{1-(\alpha+i\beta)}(z)} = \frac{1}{\alpha+i\beta}\left\{(c+\alpha+i\beta)z^{-i\beta}f^{\alpha+i\beta}(z) - cz^{-i\beta}F^{\alpha+i\beta}(z)\right\}.$$

Now from (4.3.1), we have

$$\frac{z^{1-i\beta}F'(z)}{F^{1-(\alpha+i\beta)}G^{\alpha}(z)} = \frac{\frac{1}{\alpha+i\beta}\left\{(c+\alpha+i\beta)z^{-i\beta}f^{\alpha+i\beta}(z) - cz^{-i\beta-C}(c+\alpha+i\beta)\int_0^z t^{c-1}(f(t))^{\alpha+i\beta}dt\right\}}{\frac{(c+\alpha+i\beta)}{z^{c+i\beta}}\int_0^z t^{c+i\beta-1}g^{\alpha}(t)dt}$$

$$= \frac{\frac{1}{\alpha+i\beta}\left\{(z^c f^{\alpha+i\beta}(z) - c\int_0^z t^{c-1}(f(t))^{\alpha+i\beta}dt\right\}}{\int_0^z t^{c+i\beta-1}g^{\alpha}(t)dt} = \frac{N(z)}{D(z)}.$$

Now

$$\frac{N'(z)}{D'(Z)} = \frac{\frac{1}{\alpha+i\beta}\left\{(cz^{c-1}f^{\alpha+i\beta}(z) + (\alpha+i\beta)f^{\alpha+i\beta-1}(z)f'(z) - cz^{c-1}(f(z))^{\alpha+i\beta}\right\}}{z^{c+i\beta-1}g^{\alpha}(z)}$$

$$= \frac{z^{1-i\beta}f'(z)}{f^{1-(\alpha+i\beta)}(z)g^{\alpha}(z)} \in \mathcal{P}[\eta, A, B].$$

From Theorem 4.3.3, we have $D(z) = \int_0^z t^{c+i\beta-1}g^{\alpha}(t)dt$ is $(\alpha + c)$ valent starlike, so by using Lemma 4.2.4, we obtain

$$\frac{z^{1-i\beta}f'(z)}{f^{1-(\alpha+i\beta)}(z)G^{\alpha}(z)} \in \mathcal{P}[\eta, A, B],$$

and thus the proof is complete. ■

Corollary 4.3.11 *Let $A = 1, B = -1$ and $\beta = 0$. Then*

$$G^{\alpha}(z) = \frac{(\alpha+c)}{z^c}\int_0^z t^{c-1}g^{\alpha}(t)dt$$

is starlike of order δ_1,where

$$\delta_1 = \frac{-(1+2c) + \sqrt{(1+2c)^2 + 8\alpha}}{4\alpha}, \text{ see [106].}$$

Hence $G(z)$ is starlike when $g(z)$ is starlike, and as a result

$$F^{\alpha}(z) = \frac{(\alpha + c)}{z^{c}} \int_{0}^{z} t^{c-1} g^{\alpha}(t) dt$$

is Bazilevič functions associated to the quadruple $(\alpha, 0, G, p)$.

Theorem 4.3.9 *Let $p(z) \in \mathcal{P}[\eta, A, B]$, $g(z) \in \mathcal{S}^{*}[A, B]$ and for $\alpha > 0$, the quadruple $(\alpha, 0, g, p) f(z)$ be associated with a generalized Bazilevič function $f(z)$,then*

$$|f(z)|^{\alpha} \leq \frac{(1 - B^2) + (A - B)(|\eta| - B\mathfrak{Re}\eta)}{1 - B} r^{\alpha} \; {}_2F_1\left(\alpha\left(1 - \frac{A}{B}\right) + 1; \alpha; \alpha + 1; -Br\right), \; B \neq 0.$$

Proof. Since $f(z)$ is given to be generalized Bazilevič function in relation with the quadruple $(\alpha, 0, g, p)$, therefore

$$\frac{z f'(z)}{f^{1-\alpha}(z) g^{\alpha}(z)} = p(z),$$

where $p(z) \in \mathcal{P}[\eta, A, B]$ and $g(z) \in \mathcal{S}^{*}[A, B]$. This implies that

$$f^{\alpha}(z) = \alpha \int_{0}^{z} t^{-1} g^{\alpha}(t) p(t) dt,$$

therefore

$$\begin{aligned} |f(z)|^{\alpha} &\leq \alpha \int_{0}^{z} |t^{-1}||g^{\alpha}(t)||p(t)|dt, \\ &= \alpha \int_{0}^{r} t^{-1}|g^{\alpha}(t)||p(t)|dt. \end{aligned}$$

Now by using the following results

$$G(r, -A, -B) \leq |g(z)| \leq G(r, A, B), \text{ see [120]},$$

where

$$G(r,A,B)=\begin{cases} r(1+Br)^{\frac{A-B}{B}}, & B\neq 0 \\ re^{Ar}, & B=0, \end{cases}$$

and

$$|p(z)|\leq \frac{1+|\eta|\,(A-B)r-B\,[(A-B)\mathfrak{Re}\eta+B]\,r^2}{1-B^2r^2},\ \text{see [25]},$$

we have

$$\begin{aligned} |f(z)|^{\alpha} &\leq \alpha\int_0^r t^{-1}\frac{t^{\alpha}}{(1+Bt)^{\alpha\left(1-\frac{A}{B}\right)}}.\frac{1+|\eta|(A-B)t-B\,[(A-B)\mathfrak{Re}\eta+B]\,t^2}{1-B^2t^2}dt \\ &\leq \alpha\frac{(1-B^2)+(A-B)\,(|\eta|-B\mathfrak{Re}\eta)}{1-B}\int_0^r t^{\alpha-1}(1+Bt)^{-\alpha\left(1-\frac{A}{B}\right)-1}dt. \end{aligned}$$

Putting $t=ru$, we have

$$\begin{aligned} |f(z)|^{\alpha} &\leq \alpha\frac{(1-B^2)+(A-B)\,(|\eta|-B\mathfrak{Re}\eta)}{1-B}r^{\alpha}\int_0^r u^{\alpha-1}(1+Bru)^{-\alpha\left(1-\frac{A}{B}\right)-1}dt \\ &= \frac{(1-B^2)+(A-B)\,(|b|-B\mathfrak{Re}\eta)}{1-B}r^{\alpha}\ {}_2F_1\left(\alpha\left(1-\frac{A}{B}\right)+1;\alpha;\alpha+1;-Br\right). \end{aligned}$$

Hence the proof is complete. ■

Corollary 4.3.12 *For $g(z)\in\mathcal{S}^*$ and $p(z)\in\mathcal{P}$, we have*

$$|f(z)|^{\alpha}\leq 2r^{\alpha}\ {}_2F_1\left(2\alpha+1;\alpha;\alpha+1;r\right).$$

Chapter 5

New reciprocal family of regular functions involving linear operator

5.1 Introduction

For several years Geometric function theory faced the challenge of the proof of the Bieberbach (1916) conjecture of coefficient bounds. Many mathematician made attempts in this regard. But de-Branges [42] in 1985 got the credit of this proof. This proof is considered as cornerstone in GFT. Accoring to this conjecture coefficient of a schlicht function $f(z)$, has the form $(2.1.1)$, in the region $\mathfrak{D}$ satisfies $|a_n| \leq n$. Several subfamilies of schlicht functions were also introduced from geometric point of view and explored in effort to resolve this conjecture. These subfamilies comprise the sets of starlike, convex, quasi-convex, close-to-convex and Bazilevič functions. After that Uralegaddi and his coauthors [153] introduced the sets of starlike and convex functions of reciprocal order ρ $(\rho > 1)$ which are given as

$$\mathcal{M}(\rho) = \left\{ f(z) \in \mathcal{A} : \mathfrak{Re}\frac{zf'(z)}{f(z)} < \rho, \ \ (z \in \mathfrak{D}) \right\},$$

$$\mathcal{N}(\rho) = \left\{ f(z) \in \mathcal{A} : \mathfrak{Re}\frac{zf''(z)}{f'(z)} < \rho - 1, \ \ (z \in \mathfrak{D}) \right\}.$$

These families were further considered by the authors in [115, 117]. Using these ideas in above defined classes, Junichi et al. [100] introduced the following classes

Definition 5.1.1 *Let $f(z)$ be a regular function of the form $(2.1.1)$. Then $f(z)$ is in the family $\mathcal{MD}(k, \rho)$ if it satisfies*

$$\mathfrak{Re}\left\{ \frac{zf'(z)}{f(z)} - \rho \right\} < k \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in \mathfrak{D}),$$

for some $\rho\,(\rho > 1)$ and $k\,(k \leq 0)$.

Definition 5.1.2 *A function $f(z) \in \mathfrak{A}$ and has power series represenation $(2.1.1)$, belongs to the set $\mathcal{ND}(k, \rho)$, if and only if*

$$\mathfrak{Re}\left\{ \frac{zf''(z)}{f'(z)} - \rho + 1 \right\} < k \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathfrak{D}),$$

for some $\rho\,(\rho>1)$ *and* $k\,(k\leq 0)$.

We now discuss $\Phi\,(a,c;z)$ which is termed as incomplete beta function and is given as

$$\Phi\,(a,c;z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}\;z^{n}, \tag{5.1.1}$$

where $(x)_n$ denotes the familiar Pochhammer symbol which is given by

$$(x)_n=\frac{\Gamma\,(n+x)}{\Gamma\,(x)}=\begin{cases}1, \quad n=0,\\ x\,(x+1)\,(x+2)\cdots(x+n-1) \quad n\in\mathbb{N}.\end{cases}$$

Carlson and Shaffer [35] defined an operator $\mathfrak{L}(a,c)$ as follows;

$$\mathfrak{L}(a,c)f(z):=\Phi(a,c;z)*f(z)=z+\sum_{n=2}^{\infty}\Lambda_{n-1}(a,c)a_nz^n, \tag{5.1.2}$$

with

$$\Lambda_{n-1}:=\Lambda_{n-1}(a,c)=\frac{(a)_{n-1}}{(c)_{n-1}}, \tag{5.1.3}$$

where $a\in\mathbb{R},\ c\in\mathbb{R}\backslash\mathbb{Z}_0^-$, with $\mathbb{Z}_0^-:=\{0,-1,-2,-3,\cdots\}$. Clearly, under the operator $\mathfrak{L}(a,c)f(z)$, $\mathfrak{A}$ is mapped onto himself and also for $a\neq 0,-1,\cdots$,

$$\mathfrak{L}(a,c)f(z)*\mathfrak{L}(c,a)f(z)=f(z).$$

From (5.1.2) one can easily derived the following identity

$$z\,(\mathfrak{L}(a,c)f(z))'+(a-1)\,\mathfrak{L}(a,c)f(z)=a\mathfrak{L}(a+1,c)f(z). \tag{5.1.4}$$

Also we observe that

$$\mathfrak{L}\,(\delta+1,1)=\mathcal{D}^{\delta}\,(f(z))\,,$$

where $\mathcal{D}^{\delta}(f(z))$ is the familiar Ruscheweyh derivative of $f(z)$ (see [139]). Additionally

$$\mathfrak{L}(2, 2-\mu) f(z) = \Omega_z^{\mu} f(z), \quad (0 \leq \mu < 1;\ z \in \mathfrak{D}),$$

is the Owa-Srivastava fractional differential operator [116].

In the present chapter we define a new subfamily $\mathcal{LD}_{\eta}^{k}(a, c, \rho)$ of regular functions by using Carlson-Shaffer [35] operator. This class generalizes many familiar classes including starlike and convex functions of reciprocal order. Some useful features such as distortion bounds, coefficient estimates and subordination result of this subfamily of regular functions are investigated. We also observed connections of the present results to the relavent earlier works. The contents of this chapter are published in "Iranian Journal of Science Series A, (2016), doi:10.1007/s40995-016-0059-y.", see [24].

5.2 The family $\mathcal{LD}_{\eta}^{k}(a, c, \rho)$

Motivated from different research paper discussed above, we now define a new subfamily $\mathcal{LD}_{\eta}^{k}(a, c, \rho)$ of the set $\mathfrak{A}$ involving the Carlson-Shaffer [35] operator.

Definition 5.2.1 *A regular function $f(z)$ has the form* (2.1.1) *is in the family $\mathcal{LD}_{\eta}^{k}(a, c, \rho)$, if and only if*

$$\mathfrak{Re}\left\{1 + \frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1, c) f(z)}{\mathfrak{L}(a, c) f(z)} - 1\right)\right\} < k\left|\frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1, c) f(z)}{\mathfrak{L}(a, c) f(z)} - 1\right)\right| + \rho,$$

for some $k\ (k \leq 0)$, $\rho\ (\rho > 1)$ and for some $\eta \in \mathbb{C} \setminus \{0\}$.

Special cases

By the choice of different particular values to the parameters involved in the family $\mathcal{LD}_{\eta}^{k}(a, c, \rho)$, we obtain several familiar subfamilies of regular and schlicht functions discussed in earlier literature, we give some of them as follows. That is, we note that

$$\mathcal{LD}_2^0(1,1,\rho) = \mathcal{M}(\rho) \quad \text{and} \quad \mathcal{LD}_1^0(1,1,\rho) = \mathcal{N}(\rho),$$

for these particular families see [115, 117]. Also when we put $\eta = 1, 2$, $c = 1$, and $a = 1$, the family $\mathcal{LD}_\eta^k(a,c,\rho)$ reduces to the sets $\mathcal{MD}(k,\rho)$ and $\mathcal{ND}(k,\rho)$ (see [100]). For $1 < \rho < 4/3$, the families $\mathcal{M}(\rho)$ and $\mathcal{N}(\rho)$ were studied by Uralegaddi et al. [153].

Preliminary lemma

We currently establish the comming result which we needed for the proof of our main result.

Lemma 5.2.1 *Let $t > 0$ be an integer, then we have*

$$\chi \sum_{l=1}^{t} \frac{(\chi)_{l-1}}{(l-1)!} = \frac{(\chi)_t}{(t-1)!}. \tag{5.2.1}$$

Proof. Consider

$$\begin{aligned}
\chi \sum_{l=1}^{t} \frac{(\chi)_{l-1}}{(l-1)!} &= \chi\left(1 + \frac{\chi}{1} + \frac{(\chi)_2}{2!} + \frac{(\chi)_3}{3!} + \frac{(\chi)_4}{4!} + \cdots + \frac{(\chi)_{t-1}}{(t-1)!}\right) \\
&= \chi(1+\chi)\left(1 + \frac{\chi}{2} + \frac{\chi(\chi+2)}{2\times 3} + \cdots + \frac{\chi(\chi+2)\cdots(\chi+t-2)}{2\times\cdots\times(t-1)}\right) \\
&= \chi(1+\chi)\frac{(\chi+2)}{2}\left(1 + \frac{\chi}{3} + \cdots + \frac{\chi(\chi+3)\cdots(\chi+t-2)}{3\times 4\times\cdots\times(t-1)}\right) \\
&= \chi(1+\chi)\frac{(\chi+2)}{2}\frac{(\chi+3)}{3}\left(1 + \frac{\chi}{4} + \cdots + \frac{\chi(\chi+4)\cdots(\chi+t-2)}{4\times\cdots\times(t-1)}\right) \\
&= \chi(1+\chi)\frac{(\chi+2)}{2}\frac{(\chi+3)}{3}\frac{(\chi+4)}{4}\left(1 + \frac{\chi}{5} + \cdots + \frac{\chi\cdots(\chi+t-2)}{5\times 6\times\cdots\times(t-1)}\right) \\
&= \chi(1+\chi)\frac{(\chi+2)}{2}\frac{(\chi+3)}{3}\frac{(\chi+4)}{4}\cdots\left(1 + \frac{\chi}{t-1}\right) \\
&= \frac{(\chi)_t}{(t-1)!},
\end{aligned}$$

and this completes the proof. ■

5.3 Main results

Here we prove some results using of the definition of class of functions $\mathcal{LD}_{\eta}^{k}(a,c,\rho)$.

Theorem 5.3.1 *If* $f(z) \in \mathcal{LD}_{\eta}^{k}(a,c,\rho)$, *then*

$$f(z) \in \mathcal{LD}_{\eta}^{0}\left(a, c, \frac{\rho - k}{1-k}\right).$$

Proof. Since for $k \leq 0$, we can write

$$\begin{aligned}\mathfrak{Re}\left\{1+\frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)}-1\right)\right\} &< \frac{2k}{|\eta|}\left|\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)}-1\right| \\ &\leq k\mathfrak{Re}\left\{1+\frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)}-1\right)\right\}+\rho-k,\end{aligned}$$

which implies that

$$(1-k)\,\mathfrak{Re}\left\{1+\frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)}-1\right)\right\} < \rho - k.$$

After simplification, we obtain

$$\mathfrak{Re}\left\{1+\frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)}-1\right)\right\} < \frac{k-\rho}{k-1},\ \ (\text{for } k \leq 0 \ \&\ \rho > 1), \tag{5.3.1}$$

and this ends the proof. ■

Theorem 5.3.2 *If* $f(z) \in \mathcal{LD}_{\eta}^{k}(a,c,\rho)$ *has given by* (2.1.1), *then*

$$|a_n| \leq \frac{(\sigma)_{n-1}}{\Lambda_{n-1}(n-1)!}, \tag{5.3.2}$$

where

$$\sigma = \frac{a|\eta|(\rho-1)}{1-k}. \tag{5.3.3}$$

Proof. Let us define a function

$$p(z) = \frac{(\rho-k)-(1-k)\left\{1+\frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)}-1\right)\right\}}{\rho-1}. \tag{5.3.4}$$

Then $p(z)$ is regular for all $z \in \mathfrak{D}$ with $p(0)=1$ & $\mathfrak{Re}p(z)>0$. We can write

$$\left\{1+\frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)}-1\right)\right\} = \frac{(\rho-k)-(\rho-1)p(z)}{1-k}. \tag{5.3.5}$$

If we take $p(z) = 1+\sum_{n=1}^{\infty} p_n z^n$, then (5.3.5) can be written as

$$2\mathfrak{L}(a+1,c)f(z) = 2\mathfrak{L}(a,c)f(z) + \eta\left(\mathfrak{L}(a,c)f(z)\right)\left(\frac{1-\rho}{1-k}\sum_{n=1}^{\infty} p_n z^n\right),$$

and this implies that

$$2\left[\sum_{n=2}^{\infty}\left(\frac{n-1}{a}\right)\Lambda_{n-1}(a,c)a_n z^n\right] = \frac{\eta(\rho-1)}{k-1}\left(\sum_{n=1}^{\infty}\Lambda_{n-1}a_n z^n\right)\left(\sum_{n=1}^{\infty}p_n z^n\right).$$

Using Cauchy product

$$\left(\sum_{n=1}^{\infty} x_n\right)\cdot\left(\sum_{n=1}^{\infty} y_n\right) = \sum_{j=1}^{\infty}\sum_{k=1}^{j} x_k y_{k-j},$$

we obtain

$$2\left[\sum_{n=2}^{\infty}\left(\frac{n-1}{a}\right)\Lambda_{n-1}a_n z^n\right] = \frac{\eta(1-\rho)}{1-k}\sum_{n=2}^{\infty}\left(\sum_{j=1}^{n-1}\Lambda_{j-1}a_j p_{n-j}\right)z^n.$$

Comparing the coefficients of z^n on both sides, we have

$$a_n = \frac{a(1-\rho)\eta}{2(n-1)\Lambda_{n-1}(1-k)} \sum_{j=1}^{n-1} \Lambda_{j-1} a_j p_{n-j}.$$

By simple compuatation, we get

$$|a_n| \leq \frac{a(\rho-1)|\eta|}{2(1-k)(n-1)\Lambda_{n-1}} \sum_{j=1}^{n-1} \Lambda_{j-1} |a_j| |p_{n-j}|.$$

Applying the coefficient estimates $|p_n| \leq 2$ ($\forall\ n \geq 1$) for Caratheodory functions [46], we obtain

$$\begin{aligned} |a_n| &\leq \frac{a(\rho-1)|\eta|}{(n-1)(1-k)\Lambda_{n-1}} \sum_{j=1}^{n-1} \Lambda_{j-1} |a_j| \\ &= \frac{\sigma}{(n-1)\Lambda_{n-1}} \sum_{j=1}^{n-1} \Lambda_{j-1} |a_j|, \end{aligned} \tag{5.3.6}$$

where $\sigma = a|\eta|(\rho-1)/(1-k)$. To prove (5.3.2) we use mathematical induction on n. So for $n = 2$ in (5.3.6), we have

$$|a_2| \leq \frac{\sigma}{\Lambda_1} = \frac{(\sigma)_{2-1}}{(2-1)!\Lambda_{2-1}}, \tag{5.3.7}$$

which shows that (5.3.2) is true for $n = 2$. Now taking $n = 3$ in (5.3.6), we get

$$|a_3| \leq \frac{\sigma}{(3-1)\Lambda_{3-1}} \{1 + \Lambda_1 |a_2|\},$$

using (5.3.7), we have

$$|a_3| \leq \frac{\sigma}{2\Lambda_2}(1+\sigma) = \frac{(\sigma)_{3-1}}{(3-1)\Lambda_{3-1}},$$

which implies that (5.3.6) also holds for $n = 3$. Now let us suppose that (5.3.6) is true for all $n \leq t$, that is,

$$|a_t| \leq \frac{(\sigma)_{t-1}}{\Lambda_{t-1}(t-1)!} \quad \text{for} \quad j = 1, 2, \ldots, t. \tag{5.3.8}$$

Using equations (5.3.6) and (5.3.8), we have

$$\begin{aligned} |a_{t+1}| &\leq \frac{\sigma}{t\Lambda_t} \sum_{j=1}^{t} \Lambda_{j-1} |a_j| \\ &\leq \frac{\sigma}{t\Lambda_t} \sum_{j=1}^{t} \Lambda_{j-1} \frac{(\sigma)_{j-1}}{\Lambda_{j-1}(j-1)!} \\ &= \frac{\sigma}{t\Lambda_t} \sum_{j=1}^{t} \frac{(\sigma)_{j-1}}{(j-1)!}. \end{aligned}$$

Applying (5.2.1), we have

$$|a_{t+1}| \leq \frac{1}{t\Lambda_t} \frac{(\sigma)_t}{(t-1)!} = \frac{1}{\Lambda_t} \frac{(\sigma)_t}{t!}.$$

Consequently, by means of induction, we achieved the needed result. ■

Corollary 5.3.1 *If* $f(z) \in \mathcal{LD}_2^k(1,1,\rho) = \mathcal{MD}(k,\rho)$, *then for* $n \geq 2$

$$|a_n| \leq \frac{1}{(n-1)!} \prod_{j=0}^{n-2} \left(j + \frac{2(\rho-1)}{1-k} \right),$$

or equivalently

$$|a_n| \leq \frac{(\delta_{k,\rho})_{n-1}}{(n-1)!},$$

where

$$\delta_{k,\rho} = \frac{2(\rho-1)}{|1-k|}. \tag{5.3.9}$$

Corollary 5.3.2 *If* $f(z) \in \mathcal{LD}_1^k(1,1,\rho) = \mathcal{ND}(k,\rho)$, *then for* $n \geq 2$

$$|a_n| \leq \frac{1}{n!}\prod_{j=0}^{n-2}\left(j + \frac{2(\rho-1)}{1-k}\right),$$

or equivalently

$$|a_n| \leq \frac{(\delta_{k,\rho})_{n-1}}{n!},$$

where the symbol $\delta_{k,\rho}$ *is defined in* (5.3.9).

If $k = 0$, $a = c = 1$, and $\eta = 1, 2$, the following well known results are obtained which were proved by Polatoglu et al. [122].

Corollary 5.3.3 *If* $f(z) \in \mathcal{M}(\rho)$, *then*

$$|a_n| \leq \frac{1}{(n-1)!}\prod_{j=0}^{n-2}(j + 2(1-\rho)), \quad \text{for } n \geq 2.$$

Corollary 5.3.4 *If* $f(z) \in \mathcal{N}(\rho)$, *then*

$$|a_n| \leq \frac{1}{n!}\prod_{j=0}^{n-2}(j + 2(1-\rho)), \quad \text{for } n \geq 2.$$

Theorem 5.3.3 *If a function* $f(z) \in \mathcal{LD}_\eta^k(a,c,\rho)$, *then for* $z \in \mathfrak{D}$

$$\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)} \prec 1 + \eta(\rho_1 - 1) + \frac{\eta(1-\rho_1)}{1-z}, \tag{5.3.10}$$

with

$$\rho_1 = \frac{k-\rho}{k-1}. \tag{5.3.11}$$

Proof. If $f(z) \in \mathcal{LD}_\eta^k(a,c,\rho)$, then by (5.3.1)

$$\mathfrak{Re}\left\{1 + \frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)} - 1\right)\right\} < \rho_1. \tag{5.3.12}$$

Then there exists a Schwarz function $w(z)$ such that

$$\frac{\rho_1 - \left\{1 + \frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)} - 1\right)\right\}}{\rho_1 - 1} = \frac{1+w(z)}{1-w(z)}, \tag{5.3.13}$$

and

$$\mathfrak{Re}\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0, \text{ for } z \in \mathfrak{D}.$$

Therefore, from (5.3.13), we obtain

$$\left\{1 + \frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)} - 1\right)\right\} = \left(\frac{1+w(z)}{1-w(z)}\right)(1-\rho_1) + \rho_1.$$

This gives

$$\begin{aligned}\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)} &= 1 + \frac{\eta(1-\rho_1)}{2}\left(\frac{1+w(z)}{1-w(z)}\right) + \frac{\eta(\rho_1-1)}{2} \\ &= 1 + \eta(\rho_1 - 1) - \frac{\eta(\rho_1-1)w(z)}{1-w(z)}\end{aligned}$$

and hence

$$\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)} \prec 1 + \eta(\rho_1 - 1) - \frac{\eta(\rho_1-1)z}{1-z} \quad (z \in \mathfrak{D}).$$

which was required in (5.3.10). ■

Theorem 5.3.4 *If function $f(z) \in \mathcal{LD}_\eta^k(a,c,\rho)$, then we have*

$$\frac{1-(1+\eta(\rho_1-1))r}{1-r} \leq \mathfrak{Re}\left\{\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)}\right\} \leq \frac{1+(1+\eta(\rho_1-1))r}{1+r}, \tag{5.3.14}$$

for $|z| = r < 1$ and ρ_1 is defined by (5.3.11).

Proof. Using Theorem 5.3.3 and consider the function $\Theta(z)$ defined by

$$\Theta(z) = 1 - \frac{\eta(\rho_1-1)}{1-z} + \eta(\rho_1 - 1) \quad (z \in \mathfrak{D}).$$

Letting $z = re^{i\theta}(0 \leq r < 1)$, we see that

$$\mathfrak{Re}\Theta(z) = 1 + \eta\left(\rho_1 - 1\right) + \frac{\eta\left(1-\rho_1\right)\left(1 - r\cos\theta\right)}{1 + r^2 - 2r\cos\theta}.$$

Let us define

$$\Psi(t) = \frac{1 - rt}{1 + r^2 - 2rt} \quad (t = \cos\theta)\,.$$

Since $\Psi'(t) = \frac{r\left(1-r^2\right)}{\left(1+r^2-2rt\right)^2} \geq 0$, because $r < 1$. Therefore we get

$$1 + \eta\left(\rho_1 - 1\right) + \frac{\eta\left(1-\rho_1\right)}{1-r} \leq \mathfrak{Re}\Theta(z) \leq 1 + \eta\left(\rho_1 - 1\right) + \frac{\eta\left(1-\rho_1\right)}{1+r}.$$

After simplification, we have

$$\frac{1 - \left(1 + \eta\left(\rho_1 - 1\right)\right) r}{1 - r} \leq \mathfrak{Re}\Theta(z) \leq \frac{1 + \left(1 + \eta\left(\rho_1 - 1\right)\right)) r}{1 + r}.$$

Since we note that $\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)} \prec \Theta(z)$, by Theorem 5.3.3 and $\Theta(z)$ is regular in $\mathfrak{D}$, we proved the inequality (5.3.14). ■

Theorem 5.3.5 *If $f(z) \in \mathfrak{A}$ satisfies*

$$\left|\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)} - 1\right| < \frac{(\rho-1)|\eta|}{2(1-k)} \quad (z \in \mathfrak{D})\,, \tag{5.3.15}$$

for a and c as given in (5.1.3), *then $f(z) \in \mathcal{LD}_{\eta}^{k}(a,c,\rho)$ where $k \leq 0$, $\rho > 1$ and $\eta \in \mathbb{C} \setminus \{0\}$.*

Proof. Let us assume that (5.3.15) holds. Then

$$\begin{aligned}
& \left|\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)}-1\right| < \frac{(\rho-1)|\eta|}{2(1-k)} \\
\Rightarrow \quad & \left|\frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)}-1\right)\right| < \frac{\rho-1}{1-k} \\
\Rightarrow \quad & (1-k)\left|\frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)}-1\right)\right| + 1 < \rho \\
\Rightarrow \quad & \left|\frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)}-1\right)\right| + 1 < k\left|\frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)}-1\right)\right| + \rho \\
\Rightarrow \quad & \mathfrak{Re}\left\{\frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)}-1\right)\right\} + 1 < k\left|\frac{2}{\eta}\left(\frac{\mathfrak{L}(a+1,c)f(z)}{\mathfrak{L}(a,c)f(z)}-1\right)\right| + \rho \\
\Rightarrow \quad & f(z) \in \mathcal{LD}_{\eta}^{k}(a,c,\rho),
\end{aligned}$$

and hence the result follows. ■

Corollary 5.3.5 *Let* $f(z) \in \mathfrak{A}$*,having the form of* (2.1.1) *and satisfies*

$$\left|\frac{\sum_{n=2}^{\infty}\frac{n-1}{a}\Lambda_{n-1}a_n z^{n-1}}{1+\sum_{n=2}^{\infty}\Lambda_{n-1}a_n z^{n-1}}\right| < \frac{(\beta-1)|\eta|}{2(1-k)} \quad (z \in \mathfrak{D}), \tag{5.3.16}$$

then $f(z) \in \mathcal{LD}_{\eta}^{k}(a,c,\rho)$ *with* $k\,(k \leq 0)$, $\rho\,(\rho > 1)$, $\eta \in \mathbb{C} \setminus \{0\}$ *and for some* a, c *given in* (5.1.3) ,

Proof. We have

$$\mathfrak{L}(a,c)f(z) = z + \sum_{n=2}^{\infty}\Lambda_{n-1}a_n z^n$$

and by (5.1.4)

$$\mathfrak{L}(a+1,c)f(z) = z + \sum_{n=2}^{\infty}\left(\frac{n-1}{a}+1\right)\Lambda_{n-1}a_n z^n.$$

Therefore, (5.3.15) follows immediately (5.3.16). ■

Theorem 5.3.6 *Let $f(z) \in \mathfrak{A}$, expressible as* (2.1.1) *and satisfies*

$$\sum_{n=2}^{\infty}\left(\frac{n-1}{a}+y\right)|\Lambda_{n-1}|\,|a_n| < y \quad (z \in \mathfrak{D}), \tag{5.3.17}$$

for some a, c given in (5.1.3) *and*

$$y = \frac{(\beta-1)|\eta|}{2(1-k)} > 0,$$

then $f(z) \in \mathcal{LD}_{\eta}^{k}(a,c,\rho)$ with $k\,(k \leq 0)$, $\rho\,(\rho > 1)$ and $\eta \in \mathbb{C} \setminus \{0\}$.

Proof. Suppose that the inequality (5.3.17) is true. Then

$$\begin{aligned}
\sum_{n=2}^{\infty}\left(\frac{n-1}{a}+y\right)|\Lambda_{n-1}||a_n| &< y \\
&\Rightarrow \sum_{n=2}^{\infty}\frac{n-1}{a}|\Lambda_{n-1}||a_n| < y - y\sum_{n=2}^{\infty}|\Lambda_{n-1}||a_n| \\
&\Rightarrow 0 < y - y\sum_{n=2}^{\infty}|\Lambda_{n-1}||a_n| \\
&\Rightarrow 0 < y - y\sum_{n=2}^{\infty}|\Lambda_{n-1}||a_n||z^{n-1}| \\
&\Rightarrow 0 < y\left|1+\sum_{n=2}^{\infty}\Lambda_{n-1}a_n z^{n-1}\right|. \qquad (5.3.18)
\end{aligned}$$

Again consider the inequality (5.3.17), We have

$$\begin{aligned}
\sum_{n=2}^{\infty}\left(\frac{n-1}{a}+y\right)|\Lambda_{n-1}|\,|a_n| &< y \\
&\Rightarrow \sum_{n=2}^{\infty}\left(\frac{n-1}{a}+y\right)|\Lambda_{n-1}|\,|a_n|\,\left|z^{n-1}\right| < y \\
&\Rightarrow \sum_{n=2}^{\infty}\frac{n-1}{a}|\Lambda_{n-1}||a_n|\,\left|z^{n-1}\right| < y - y\sum_{n=2}^{\infty}|\Lambda_{n-1}||a_n||z^{n-1}| \\
&\Rightarrow \left|\sum_{n=2}^{\infty}\frac{n-1}{a}\Lambda_{n-1}a_n z^{n-1}\right| < y\left|1+\sum_{n=2}^{\infty}\Lambda_{n-1}a_n z^{n-1}\right| \\
&\Rightarrow \left|\frac{\sum_{n=2}^{\infty}\frac{n-1}{a}\Lambda_{n-1}a_n z^{n-1}}{1+\sum_{n=2}^{\infty}\Lambda_{n-1}a_n z^{n-1}}\right| < y,
\end{aligned}$$

because of (5.3.18). By (5.3.16) it follows $f\,(z) \in \mathcal{LD}_{\eta}^{k}(a,c,\rho)$. ■

Chapter 6

Some applications of q-analogue of Ruscheweyh differential operator for multivalent functions

6.1 Introduction

Quantum calculus (q-calculus) is simply the study of classical calculus without the notion of limits. The study of q-calculus attracted the researcher due to its applications in various branches of mathematics and physics, see detail [18]. Jackson [57, 58] was the first to give some application of q-calculus and introduced the q-analogue of derivative and integral. Later on Aral and Gupta [15, 16, 17] defined the q-Baskakov Durrmeyer operator by using q-beta function while the author's in [13, 14] discussed the q-generalization of complex operators known as q-Picard and q-Gauss-Weierstrass singular integral operators. Recently, Kanas and Răducanu [64] defined q-analogue of Ruscheweyh differential operator using the concepts of convolution and then studied some of its properties. The application of this differetial operator was further studied by Mohammed and Darus [6] and Mahmood and Sokół [94]. The aim of the current paper is to define a new class of p-valent regular functions associated with conic domains involving q-differetial operator, where multivalent (p-valent) is define as; a function $f(z)$ regular in $\mathcal{D} \subset \mathbb{C}$ is called multivalent (p-valent) function, $p \in \mathbb{N}$ in $\mathcal{D}$ if for every complex number ω, the equation $f(z) = \omega$ does not have more than p roots in $\mathcal{D}$ and there exists a complex number ω_0 such that the equation $f(z) = \omega_0$, has exactly p roots in $\mathcal{D}$, [54].
For a natural number p, let $\mathfrak{A}_p$ stands for the set of all such functions $f(z)$ that are regular and p-valent in the region $\mathfrak{D}$ and obeying the normalization

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n \, z^n \quad (p \in \mathbb{N}). \tag{6.1.1}$$

The convolution of two p-valent regular functions $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n, \;\; (z \in \mathfrak{D}),$$

where each function $f(z)$ and $g(z)$ has the form $(6.1.1)$.

For $0 < q < 1$, the q-derivative of a function $f(z)$ is defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \quad (z \in \mathfrak{D}). \tag{6.1.2}$$

It can easily be seen that for $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and

$$\partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n, q]\, a_n z^{n-1}, \quad (z \in \mathfrak{D}),$$

where

$$[n, q] = \frac{1 - q^n}{1 - q} = 1 + \sum_{l=1}^{n} q^l, \quad [0, q] = 0.$$

For any non-negative integer n the q-number shift factorial is defined by

$$[n, q]! = \begin{cases} 1, & n = 0 \\ [1, q]\,[2, q] \cdots [m, q], & n = m \in \mathbb{N}. \end{cases}$$

Also the q-generalized Pochhammer symbol for $t > 0$ is given by

$$[t, q]_n = \begin{cases} 1, & n = 0 \\ [t, q]\,[t+1, q] \cdots [t+n-1, q], & n \in \mathbb{N}, \end{cases}$$

and for $t > 0$, let q-gamma function is defined as

$$\Gamma_q(t+1) = [t, q]\, \Gamma_q(t) \text{ and } \Gamma_q(1) = 1.$$

We now define the function

$$\Phi_p(q, \delta + p; z) = z^p + \sum_{n=p+1}^{\infty} \frac{[\delta + p, q]_{n-p}}{[n-p, q]!}\, z^n \quad (p \in \mathbb{N}). \tag{6.1.3}$$

It is quite clear that the series defined in (6.1.3) is convergent absolutely in $\mathfrak{D}$. Using the function $\Phi_p(q, \delta + p; z)$ and definition of q-derivative along with the idea of convolutions, we now define the following differential operator $\mathcal{L}_q^{\delta+p-1}:\mathfrak{A}_p\rightarrow\mathfrak{A}_p$ for p-valent regular functions

$$\begin{aligned}\mathcal{L}_q^{\delta+p-1}f(z) &= \Phi_p(q,\delta+p;z)*f(z)\quad(\delta>-p),\\ &= z^p+\sum_{n=p+1}^{\infty}\Psi_n\,a_n\,z^n,\end{aligned}\tag{6.1.4}$$

where

$$\Psi_n=\frac{[\delta+p,q]_{n-p}}{[n-p,q]!}.\tag{6.1.5}$$

Using (6.1.4), one can easily established the following identity

$$[\delta+p,q]\mathcal{L}_q^{\delta+p}f(z)=[\delta,q]\mathcal{L}_q^{\delta+p-1}f(z)+q^{\delta}z\partial_q\mathcal{L}_q^{\delta+p-1}f(z).\tag{6.1.6}$$

If $q\rightarrow 1$, the operator defined in (6.1.4) reduced to the differential operator introduced by Goel and Sohi [44] and further by making $p=1$, we get the well-familiar Ruscheweyh operator [139]. Also for more details on the q-analogue of different differetial operators see the work [4, 5, 97].

We now define a subclass $\mathcal{ST}_p(k,\delta,\eta,q)$ of $\mathfrak{A}_p$ by using $\mathcal{L}_q^{\delta+p-1}$ as follows;

Definition 6.1.1 *Let $k\geq 0$, $\delta>-p$, $\eta\in\mathbb{C}\backslash\{0\}$ and $0<q<1$. A function $f(z)\in\mathfrak{A}_p$ is in the family $\mathcal{ST}_p(k,\delta,\eta,q)$, if it satisfies*

$$\mathfrak{Re}\left\{1+\frac{[2,q]}{\eta}\left(\frac{\mathcal{L}_q^{\delta+p}f(z)}{\mathcal{L}_q^{\delta+p-1}f(z)}-1\right)\right\}>k\left|\frac{[2,q]}{\eta}\left(\frac{\mathcal{L}_q^{\delta+p}f(z)}{\mathcal{L}_q^{\delta+p-1}f(z)}-1\right)\right|.\tag{6.1.7}$$

Geometric interpretation

A multivalent function $f(z)$ is family $\mathcal{ST}_p(k,\delta,\eta,q)$ if and only if $1+\frac{[2,q]}{\eta}\left(\frac{\mathcal{L}_q^{\delta+p}f(z)}{\mathcal{L}_q^{\delta+p-1}f(z)}-1\right)$ takes all the values in the conic region $\Omega_k=p_k(\mathfrak{D})$, where

$$\Omega_k=\left\{u+iv:u^2>k^2\left((u-1)^2+v^2\right)\right\},$$

or equivalently,

$$1+\frac{[2,q]}{\eta}\left(\frac{\mathcal{L}_q^{\delta+p}f(z)}{\mathcal{L}_q^{\delta+p-1}f(z)}-1\right)\prec p_k(z),\quad \Omega_k=p_k\left(\mathfrak{D}\right). \tag{6.1.8}$$

We observed that when $k=0$, the boundary $\partial\Omega_k$ of the set defined above turn into the imaginary axis while its represents a hyperbola for $0<k<1$. In the case $0\leq k<1$, we have

$$p_k(z)=1+\frac{2}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\ \operatorname{arc}\tanh\sqrt{z}\right\},\quad (z\in\mathfrak{D}).$$

For $k=1$ the boundary $\partial\Omega_k$ becomes a parabola and

$$p_1(z)=1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2,\quad (z\in\mathfrak{D}).$$

It is an ellipse when $k>1$ and in this case

$$p_k(z)=1+\frac{1}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{u(z)/\sqrt{t}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\mathrm{d}x\right)+\frac{\eta}{1-k^2},$$

where $u(z)=\frac{z-\sqrt{t}}{1-\sqrt{t}z}$, $z\in\mathfrak{D}$ and $t\in(0,1)$ is picked such that $k=\cosh\left(\frac{\pi R'(t)}{R(t)}\right)$, where the symbol $R(t)$ stands for the familiar Legendre's complete elliptic integral of kind first and $R'(t)$ is complementary integral of $R(t)$, see [66, 67, 114]. Further, $p_k(\mathfrak{D})$ is convex schlicht in $\mathfrak{D}$, see [66, 67]. All the curves of the this class have vertex at $k/(k+1)$. Thus the region Ω_k is elliptic if $k>1$, hyperbolic for $0<k<1$, parabolic when $k=1$ and finally the right half plane for $k=0$. Because $p_k(\mathfrak{D})=\Omega_k$, for several problems of the class $\mathcal{ST}_p\left(k,\delta,\eta,q\right)$, $p_k(z)$ is extremal functions .

Several known classes studied earlier by different author's appears as a special case from this newly define class. We list some of them;

(i). By choosing $p=1$, $\delta=0$, $k=0$ and $\eta=[2,q]b$ with $b\in\mathbb{C}\backslash\{0\}$ in $\mathcal{ST}_p\left(k,\delta,\eta,q\right)$, we get the class $\mathcal{S}_q^*\left(b\right)$, studied by Seoudy and Aouf [141]. Further by taking $b=1-\zeta$ with $0\leq\zeta<1$ in $\mathcal{S}_q^*\left(b\right)$, we get the family $\mathcal{S}_q^*\left(\zeta\right)$ which was investigated by Agrawal and Sahoo [3].

(ii). If we put $p = 1$ and $\eta = [2, q]b$ with $b \in \mathbb{C}\backslash\{0\}$ in $\mathcal{ST}_p(k, \delta, \eta, q)$, then the class $\mathcal{ST}_p(k, \delta, \eta, q)$ reduces to $\mathcal{ST}(k, \delta, \eta, q)$ which was studied recently by Mahmood and Sokół in [94] and further by making $b = 1 - \zeta$ with $0 \le \zeta < 1$, we obtain the class $\mathcal{ST}(k, \delta, \zeta, q)$ studied in [64].

(iii) The class $\mathcal{ST}_p(k, \delta, \eta, q)$ reduced to the class $\mathcal{ST}_p(k, \delta, \eta, 1)$ when $q \to 1$. The class $\mathcal{ST}_p(k, \delta, \eta, 1)$ was studied by Arif, Sokół and Ayaz [20].

(iv). We note that the classes

$$\lim_{q\to 1} \mathcal{ST}_1(k, 0, 2(1-\zeta), q) \equiv \mathcal{ST}(k, \zeta)$$

and

$$\lim_{q\to 1} \mathcal{ST}_1(k, 1, 2(1-\zeta), q) \equiv \mathcal{UCV}(k, \zeta),$$

studied by Bharati et al [31] and Owa [115]. Further by taking $\zeta = 0$ in both the classes $\mathcal{ST}(k, \zeta)$ and $\mathcal{UCV}(k, \zeta)$, we obtain the classes $k - \mathcal{ST}$ and $k - \mathcal{UCV}$ discussed in detail by Kanas and Wiśniowska [66, 67].

We assume during our conversation, except then indicated, that

$$k \ge 0, \ \delta > -p, \ \eta \in \mathbb{C}\backslash\{0\} \text{ and } 0 < q < 1.$$

6.2 A set of lemmas

Lemma 6.2.1 *[63] Let $k \in [0, \infty)$ be fixed and let $p_k(z)$ be defined as above. If*

$$p_k(z) = 1 + Q_1 z + Q_2 z^2 + \cdots,$$

then

$$Q_1 = \begin{cases} \frac{2A^2}{1-k^2} & 0 \leq k < 1, \\ \\ \frac{8}{\pi^2} & k = 1, \\ \\ \frac{\pi^2}{4\sqrt{t}(k^2-1)R^2(t)(1+t)} & k > 1 \end{cases} \tag{6.2.1}$$

and

$$Q_2 = \begin{cases} \frac{\left(A^2+2\right)}{3} Q_1 & 0 \leq k < 1, \\ \\ \frac{2}{3} Q_1 & k = 1, \\ \\ \frac{4R^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}R^2(t)(1+t)} Q_1 & k > 1, \end{cases} \tag{6.2.2}$$

with

$$A = \frac{\cos^{-1} k}{\pi/2},$$

and $t \in (0,1)$ *is choosen so that* $k = \cosh\left(\frac{\pi R'(t)}{R(t)}\right)$, *where* $R(t)$ *stands for the familiar Legendre's complete elliptic integral of kind first.*

Lemma 6.2.2 [92] *If* $h(z) \in \mathcal{P}$ *has the form* (2.2.1), *then the following sharp estimate holds*

$$|c_2 - \nu c_1^2| \leq 2 \max\{1, |2\nu - 1|\} \quad \textit{for all} \quad \nu \in \mathbb{C}.$$

6.3 Main results and their consequences

Theorem 6.3.1 *Let* $f(z) \in \mathfrak{A}_p$ *and has the form* (6.1.1). *If the inequality*

$$\sum_{n=p+1}^{\infty} \left(2(k+1)[n-p,q]q^{\delta+p} + |\eta| \, [\delta+p,q]\right) \frac{\Gamma_q(\delta+n)}{\Gamma_q(\delta+p)[n-p,q]!} |a_n| < \ |\eta| \,, \tag{6.3.1}$$

holds, then the function $f(z) \in \mathcal{ST}_p(k,\delta,\eta,q)$.

Proof. Suppose the inequality (6.3.1) holds. Also let us suppose

$$H(z)=\left\{1+\frac{2}{\eta}\left(\frac{\mathcal{L}_q^{\delta+p}f(z)}{\mathcal{L}_q^{\delta+p-1}f(z)}-1\right)\right\}.$$

Then by simple computation along the identity given in (6.1.6), we have

$$\begin{aligned}
|H(z)-1| &= \frac{2}{[\delta+p,q]|\eta|}\left|\frac{q^{\delta}z\partial_q\left(\mathcal{L}_q^{\delta+p-1}f(z)\right)}{\mathcal{L}_q^{\delta+p-1}f(z)}+[\delta,q]-[\delta+p,q]\right| \\
&= \frac{2q^{\delta+p}}{[\delta+p,q]|\eta|}\left|\frac{\sum\limits_{n=p+1}^{\infty}\frac{[n-p,q]\Gamma_q(\delta+n)}{\Gamma_q(\delta+p)[n-p,q]!}a_n z^{n-p}}{1+\sum\limits_{n=p+1}^{\infty}\frac{\Gamma_q(\delta+n)}{\Gamma_q(\delta+p)[n-p,q]!}a_n z^{n-p}}\right| \\
&\leq \frac{2q^{\delta+p}}{[\delta+p,q]|\eta|}\frac{\sum\limits_{n=p+1}^{\infty}\frac{[n-p,q]\Gamma_q(\delta+n)}{\Gamma_q(\delta+p)[n-p,q]!}|a_n|}{1-\sum\limits_{n=p+1}^{\infty}\frac{\Gamma_q(\delta+n)}{\Gamma_q(\delta+p)[n-p,q]!}|a_n|}.
\end{aligned}$$

Now consider

$$\begin{aligned}
k\left|H(z)-1\right|-\mathfrak{Re}\left\{H(z)-1\right\} &\leq (k+1)\left|H(z)-1\right| \\
&< \frac{2q^{\delta+p}(k+1)}{[\delta+p,q]\,|\eta|}\frac{\sum\limits_{n=p+1}^{\infty}\frac{[n-p,q]\Gamma_q(\delta+n)}{\Gamma_q(\delta+p)[n-p,q]!}\,|a_n|}{1-\sum\limits_{n=p+1}^{\infty}\frac{\Gamma_q(\delta+n)}{\Gamma_q(\delta+p)[n-p,q]!}\,|a_n|}.
\end{aligned}$$

The last expression is bounded by 1 if (6.3.1) holds. This concludes the proof. ■

If we put $p=1$ and $\eta=(1-\zeta)\,[2,q]$ with $0\leq\zeta<1$ in $(6.3.1)$, We acquire the following consequence which was shown by Kanas and Răducanu in [64].

Corollary 6.3.1 *Let* $f(z)\in\mathfrak{A}$ *has the form* $(6.1.1)$. *Then* $f\in\mathcal{ST}\,(k,\delta,\zeta,q)$ *if the following inequality holds*

$$\sum_{n=2}^{\infty}\left([n](k+1)-k-\zeta\right)\frac{\Gamma_q\left(\delta+n\right)}{[n-1,q]!\;\Gamma_q\left(\delta+1\right)}\,|a_n|<1-\zeta.$$

Theorem 6.3.2 *If $f(z) \in \mathcal{ST}_p(k, \delta, \eta, q)$, then*

$$|a_{p+1}| \leq \frac{\eta |Q_1|}{2q^{\delta+p}},$$

and for all $n = 3, 4, \ldots$

$$|a_{n+p-1}| \leq \frac{\sigma}{[n-1, q]\,\Psi_{n+p-1}} \prod_{j=1}^{n-2} \left(1 + \frac{\sigma}{[j, q]}\right), \tag{6.3.2}$$

where Ψ_n is given by (6.1.5) and $\sigma = \frac{\eta[\delta+p,q]|Q_1|}{2q^{\delta+p}}$ with Q_1 is given by (6.2.1).

Proof. Let

$$1 + \frac{2}{\eta}\left(\frac{\mathcal{L}_q^{\delta+p} f(z)}{\mathcal{L}_q^{\delta+p-1} f(z)} - 1\right) = p(z),$$

where $p(z)$ is regular in the region $\mathfrak{D}$.Also it can be re-written as

$$\sum_{n=p+1}^{\infty} \left(\frac{[\delta+n, q]}{[\delta+p, q]} - 1\right) \Psi_n a_n z^n = \frac{\eta}{2} \left(\mathcal{L}_q^{\delta+p-1} f(z)\right) \left(\sum_{n=1}^{\infty} c_n z^n\right).$$

Comparing the coefficients of z^{n+p-1} on both sides

$$\left(\frac{[\delta+n+p-1, q]}{[\delta+p, q]} - 1\right) \Psi_{n+p-1} a_{n+p-1} = \frac{\eta}{2} \left\{c_1 \Psi_{n+p-2} a_{n+p-2} + \ldots + c_{n-1}\right\}.$$

Taking absolute on both sides and then applying the coefficient estimates $|c_n| \leq |Q_1|$ (see [**?**]), we have

$$|a_{n+p-1}| \leq \frac{\sigma}{[n-1, q]\,\Psi_{n+p-1}} \left\{1 + \Psi_{p+1} |a_{p+1}| + \ldots + \Psi_{n+p-2} |a_{n+p-2}|\right\}, \tag{6.3.3}$$

where $\sigma = \frac{\eta[\delta+p,q]|Q_1|}{2q^{\delta+p}}$. We apply the mathematical induction on (6.3.3) so for $n = 2$

$$|a_{p+1}| \leq \frac{\sigma}{\Psi_{p+1}} = \frac{\eta |Q_1|}{2q^{\delta+p}}, \tag{6.3.4}$$

this shows that is true for $n = 2$. Now for $n = 3$

$$|a_{p+2}| \leq \frac{\sigma}{[2,q]\,\Psi_{p+2}}\left\{1 + \Psi_{p+1}\,|a_{p+1}|\right\},$$

using (6.3.4), we obtain

$$|a_{p+2}| \leq \frac{\sigma}{[2,q]\,\Psi_{p+2}}\left(1 + \frac{\sigma}{[1,q]}\right),$$

which is true for $n = 3$. Now assume that (6.3.2) holds for $n \leq t$, that is,

$$|a_{t+p-1}| \leq \frac{\sigma}{[t-1,q]\,\Psi_{t+p-1}}\prod_{j=1}^{t-2}\left(1 + \frac{\sigma}{[j,q]}\right).$$

Consider

$$\begin{aligned}
|a_{n+p}| &\leq \frac{\sigma}{[t,q]\,\Psi_{t+p}}\left\{1 + \Psi_{p+1}\,|a_{p+1}| + \ldots + \Psi_{t+p-1}\,|a_{t+p-1}|\right\} \\
&\leq \frac{\sigma}{[t,q]\,\Psi_{t+p}}\left\{1 + \sigma + \frac{\sigma}{[2,q]}\left(1 + \frac{\sigma}{[1,q]}\right) + \frac{\sigma}{[3,q]}\left(1 + \frac{\sigma}{[1,q]}\right)\left(1 + \frac{\sigma}{[2,q]}\right)\right. \\
&\quad \left. + \cdots + \frac{\sigma}{[t-1,q]}\prod_{j=1}^{t-2}\left(1 + \frac{\sigma}{[j,q]}\right)\right\} \\
&= \frac{\sigma}{[t,q]\,\Psi_{t+p}}\prod_{j=1}^{t-1}\left(1 + \frac{\sigma}{[j,q]}\right).
\end{aligned}$$

Thus, the formula is correct for $n = t + 1$. Hence we conclude that (6.3.2) is true for $n \geq 2$. This concludes the proof. ■

Theorem 6.3.3 *Let $f(z) \in \mathcal{ST}_p(k, \delta, \eta, q)$. Then $f(\mathfrak{D})$ contains an open disk of radius*

$$r = \frac{2q^{\delta+p}}{2(p+1)\,q^{\delta+p} + \eta\,|Q_1|}.$$

Proof. Let $w_0 \in \mathbb{C}\backslash\{0\}$ for $z \in \mathfrak{D}$ & $f(z) \neq w_0$. Then

$$q(z) := \frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_{p+1} + \frac{1}{w_0}\right) z^{p+1} + \ldots.$$

As $q(z)$ is schlicht, therefore

$$\left|a_{p+1} + \frac{1}{w_0}\right| \leq p + 1.$$

Now using Theorem 6.3.2, we have

$$\left|\frac{1}{w_0}\right| \leq p + 1 + \frac{\eta |Q_1|}{2q^{\delta+p}}$$

$$\left|\frac{1}{w_0}\right| \leq \frac{(p+1) 2q^{\delta+p} + \eta |Q_1|}{2q^{\delta+p}}$$

or

$$|w_0| \geq \frac{2q^{\delta+p}}{2(p+1) q^{\delta+p} + \eta |Q_1|},$$

and thus we acquire the needed result. ■

Theorem 6.3.4 *Let $f(z) \in \mathcal{ST}_p(k, \delta, \eta, q)$ with the form* (6.1.1) *and k be fixed such that $0 \leq k < \infty$. Then, for $\mu \in \mathbb{C}$*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{\eta Q_1}{4q^{\delta+p}[\delta + p + 1, q]} \max\{1, |2\nu - 1|\},$$

where

$$\nu = \frac{1}{2} - \frac{Q_2}{2Q_1} - \frac{\eta Q_1 [\delta + p, q]}{4q^{\delta+p}} + \frac{\mu\eta [\delta + p + 1, q] Q_1}{4q^{\delta+p}}, \tag{6.3.5}$$

Q_1 and Q_2 are given by (6.2.1) *and* (6.2.2).

Proof. If $f(z) \in \mathcal{ST}_p(k, \delta, \eta, q)$, then for a Schwarz function $w(z)$ we have

$$1 + \frac{2}{\eta}\left(\frac{\mathcal{L}_q^{\delta+p} f(z)}{\mathcal{L}_q^{\delta+p-1} f(z)} - 1\right) = p_k(w(z)) \quad (z \in \mathfrak{D}). \tag{6.3.6}$$

Let $h(z) \in \mathcal{P}$ be defined by

$$h(z) = \frac{w(z)+1}{-w(z)+1} = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathfrak{D}),$$

and this further gives

$$w(z) = c_1 \frac{z}{2} + \left(2c_2 - c_1^2\right) \frac{z^2}{4} + \cdots,$$

and

$$p_k\left(w(z)\right) = 1 + \frac{Q_1 c_1}{2} z + \left\{ \frac{c_1^2 Q_2}{4} + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) Q_1 \right\} z^2 + \cdots. \tag{6.3.7}$$

Using (6.3.7) in (6.3.6) along with (6.1.4), we obtain

$$a_{p+1} = \frac{\eta Q_1 c_1}{4q^{\delta+p}},$$

and

$$a_{p+2} = \frac{\eta}{2q^{\delta+p}\left[\delta+p+1,q\right]} \left\{ \frac{c_1^2 Q_2}{4} + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) Q_1 + \frac{\eta Q_1^2 c_1^2 \left[\delta+p,q\right]}{8q^{\delta+p}} \right\}.$$

For any complex number μ, we have

$$\begin{aligned} & a_{p+2} - \mu a_{p+1}^2 \\ & = \frac{\eta}{2q^{\delta+p}\left[\delta+p+1,q\right]} \left\{ \frac{c_1^2 Q_2}{4} + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) Q_1 + \frac{\eta Q_1^2 c_1^2 \left[\delta+p,q\right]}{8q^{\delta+p}} \right\} \\ & - \mu \frac{\eta^2 Q_1^2 c_1^2}{16q^{2(\delta+p)}}. \end{aligned} \tag{6.3.8}$$

Then (6.3.8) can be written as

$$a_{p+2} - \mu a_{p+1}^2 = \frac{\eta Q_1}{4q^{\delta+p}\left[\delta+p+1,q\right]} \left\{ c_2 - \nu c_1^2 \right\},$$

where ν is defined by (6.3.5). Now by making simple compuatation along with the use of Lemma 6.2.2, we get the needed result. ■

Theorem 6.3.5 *Let* $k \in [0,\infty)$, $\mathcal{L}_q^{\delta+p-1} f(z) \neq 0$ *in* $\mathfrak{D}$ *and satisfy*

$$\frac{q^{\delta}}{[\delta+p,q]}\left\{\frac{[\delta+p,q]}{q^{\delta}}+\frac{[2,q]}{\eta}\left(\frac{z\partial_q \mathcal{L}_q^{\delta+p-1} f(z)}{\mathcal{L}_q^{\delta+p-1} f(z)}-[p,q]\right)\right\} \prec p_k(z). \tag{6.3.9}$$

Then $f \in \mathcal{ST}_p(k,\delta,\eta,q)$.

Proof. Since $\mathcal{L}_q^{\delta+p-1} f(z) \neq 0$ in $\mathfrak{D}$, therefore we can define the function $p(z)$ by

$$1+\frac{[2,q]}{\eta}\left(\frac{\mathcal{L}_q^{\delta+p} f(z)}{\mathcal{L}_q^{\delta+p-1} f(z)}-1\right)=p(z) \quad (z \in \mathfrak{D}).$$

From the use of identity (6.1.6), we easily obtain

$$\frac{q^{\delta}}{[\delta+p,q]}\left\{\frac{[\delta+p,q]}{q^{\delta}}+\frac{[2,q]}{\eta}\left(\frac{z\partial_q \mathcal{L}_q^{\delta+p-1} f(z)}{\mathcal{L}_q^{\delta+p-1} f(z)}-[p,q]\right)\right\}=p(z).$$

Therefore, using (6.3.9), we have

$$1+\frac{[2,q]}{\eta}\left(\frac{\mathcal{L}_q^{\delta+p} f(z)}{\mathcal{L}_q^{\delta+p-1} f(z)}-1\right)=p(z) \prec p_k(z),$$

and hence we have $f \in \mathcal{ST}_p(k,\delta,\eta,q)$. ■

Chapter 7

Conclusion

This research has largely focused on the development of certain new families of regular and multivalent functions defined in the region $\mathfrak{D}$. We have used various techniques to define these families such as Janowski functions, the Carlson-Shaffer linear operator along with sets of reciprocal order and the q-analogous of multivalent Ruscheweyh differential operator. Particularly we investigated the family of lemniscate of Bernouli, the family of Janowski Bazilevič functions, the family $\mathcal{LD}_{\eta}^{k}(a, c, \rho)$ of regular functions of reciprocal order and lastly the family $\mathcal{ST}_{p}(k, \delta, \eta, q)$ of multivalent functions related with q-analogous of multivalent Ruscheweyh operator. Our newly introduced families generalized the set of Bazilevič functions, the set of convex and starlike functions of reciprocal order and the families of uniformly convex and starlike functions. We have used various tools and techniques to study some geometric properties of these families of functions. We have studied the fourth Hankel determinant of set $\mathcal{SL}$ for the first time in the literature. Further we investigate coefficient bounds, arc legth problem, growth result, integral preservance, Fekete-Szegö inequality and some other interesting properties of these functions. This work will further motivate researchers to find new applications in their related areas and open new directions in this rich area of mathematics.

Chapter 8

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