Maximal and Potential Operators in Weighted Lebesgue Spaces with Non-standard Growth

Since 1864

Name : Muhammad Sarwar

Year of Admission : 2006

Registration No. : 73-GCU-PHD-SMS-06

Abdus Salam School of Mathematical Sciences

GC University Lahore, Pakistan

Maximal and Potential Operators in Weighted Lebesgue Spaces with Non-standard Growth

Submitted to

Abdus Salam School of Mathematical Sciences

GC University Lahore, Pakistan

in the partial fulfillment of the requirements for the award of degree of

Doctor of Philosophy

in

Mathematics

By

Name : Muhammad Sarwar

Year of Admission : 2006

Registration No. : 73-GCU-PHD-SMS-06

Abdus Salam School of Mathematical Sciences

GC University Lahore, Pakistan

DECLARATION

I, **Mr. Muhammad Sarwar** Registration No. **73-GCU-PHD-SMS-06** student at **Abdus Salam School of Mathematical Sciences, GC University Lahore** in the subject of **Mathematics**, **Year of Admission** (**2006)**, hereby declare that the matter printed in this thesis titled

"Maximal and Potential Operators in Weighted Lebesgue Spaces with Non-standard Growth"

is my own work and that

(i) I am not registered for the similar degree elsewhere contemporaneously.

(ii) No direct major work had already been done by me or anybody else on this topic; I worked on, for the Ph. D. degree.

(iii) The work, I am submitting for the Ph. D. degree has not already been submitted elsewhere and shall not in future be submitted by me for obtaining similar degree from any other institution.

Dated: -------------------------   -----------------------------------

Signature

RESEARCH COMPLETION CERTIFICATE

Certified that the research work contained in this thesis titled

"Maximal and Potential Operators in Weighted Lebesgue Spaces with Non-standard Growth"

has been carried out and completed by **Mr. Muhammad Sarwar** Registration No. **73-GCU-PHD-SMS-06** under my supervision.

---------------------------- ------------------------------

Date Supervisor

Dr. Alexander Meskhi

Submitted Through

Prof. Dr. A. D. Raza Choudary --------------------------------

Director General
Abdus Salam School of Mathematical Sciences
GC University Lahore, Pakistan

Controller of Examinations
GC University Lahore,
Pakistan

Dedicated to

My Beloved Parents

&

Brother Muhammad Jalil.

Table of Contents

Abstract

Two–weight criteria of various type for one–sided maximal functions and one–sided potentials are established in variable exponent Lebesgue spaces. Among other results we derive the Hardy–Littlewood, Fefferman–Stein and trace inequalities in these spaces. Weighted estimates for Hardy–type, maximal, potential and singular operators defined by means of a quasi–metric and a doubling measure are derived in $L^{p(x)}$ spaces. In some cases examples of weights guaranteeing the appropriate weighted estimates are given.

Acknowledgements

I am pleased to acknowledge my deepest gratitude, heart-felt regards and obligation to my learned supervisor Prof. Alexander Meskhi whose guidance, creativity and keen interest enabled me to complete this thesis. He has been a great role model as a teacher and as a human being for me.

I would also like to acknowledge Prof. Allah Ditta Raza Chaudhry for his invaluable guidance, constant help and encouragement for the fast four and half years.

My thanks are also for all colleagues from Abdus Salam School of Mathematical Sciences for giving me a chance to improve my academic competence. I am especially thankful to colleagues in Batch IV for the wonderful environment they created in ASSMS.

Of course, at this stage I cannot forget my whole family, especially my parents, my brothers and my sisters for there prayers,continuing help and love. There prayers lightened my sprit to finish this thesis. Without them this work would never have come into existence (literally).

Its my pleasure to acknowledge the financial support by Higher Education Commission and Government of Punjab throughout the process of completing this thesis.

I wish to thank the following: All the foreign professors at ASSMS (for they made ASSMS, a real place of learning) and all the staff members especially; Shokat Ali, Numan Jaffery and Awais Naeem (for so many things).

Lastly, I offer my regards and blessings to all of those who supported me in any respect during the completion of the thesis.

Lahore, Pakistan Muhammad Sarwar

Introduction

The thesis is devoted to weighted estimates with general-type weights for Hardy-type, maximal, potential and singular operators in variable exponent Lebesgue spaces.

In the last two decades a considerable interest of researchers was attracted to the study of mapping properties of differential and integral operators in variable exponent Lebesgue spaces. In the mid-80s, V. V. Zhikov ([93]) started a new line of investigation that was to become related to the study of variable exponent spaces, namely he considered variational integrals with non-standard growth conditions (see also the papers by O. Kováčik in the 80s and 90s). V. V. Zhikov's work was continued by X.-L. Fan from around 1995 and by Yu. A. Alkhutov since 1997. Regularity properties of functionals of the type

$$\int_{\Omega} F(x, |\nabla u|)dx, \quad F(x,z) \approx z^{p(x)},$$

have been intensively investigated by E. Acerbi and G. Mingione and their collaborators. M. Ružička [77] studied the problems in the so called rheological and electrorheological fluids, which lead to the spaces with variable exponent. Differential equations with non-standard growth and corresponding function spaces with variable exponents have been a very active field of investigation in recent years (see the survey papers [80], [37], [81], [32], the monograph [17] and the papers cited therein for the related topics).

The variable exponent Lebesgue spaces first appeared in 1931 in the paper by W.

Orlicz [74], where the author established some properties of $L^{p(x)}$ spaces on the real line. Further development of these spaces was connected with the theory of modular spaces. The first systematic study of modular spaces is due to H. Nakano [70]. The basis of the variable exponent Lebesgue and Sobolev spaces were developed by J. Musielak (see [69], [68]); H. Hudzik; I. I. Sharapudinov; S. Samko; O. Kováčik and J. Rákosník; D. E. Edmunds and J. Rákosník; D. E. Edmunds, J. Lang and A. Nekvinda etc. For the boundary value problems for analytic and harmonic functions in the framework of variable exponent analysis we refer to the papers by V. Kokilashvili and V. Paatashvili; V. Kokilashvili, V. Paatashvili and S. Samko.

The boundedness of Hardy- Littlewood maximal functions in $L^{p(x)}$ spaces first was established by L. Diening under the log-Hölder continuity condition on p. For mapping properties of maximal functions, singular integrals and potentials in $L^{p(x)}$ spaces we emphasize the papers by S. Samko; L. Diening; A. Nekvinda; L. Diening and M. Ružička; D. Cruz-Uribe, A. Fiorenza and Neugebauer; D. Cruz-Uribe, A. Fiorenza, J. M. Martell and C. Perez; D. E. Edmunds, V. Kokilashvili and A. Meskhi; P. Harjulehto, P. P. Hästö and M. Pere; C. Capone, D. Cruz-Uribe, A. Fiorenza; T. Kopaliani; A. Almeida and S. Samko; A. Almeida and H. Rafeiro, etc.

For the weighted inequalities for the classical integral operators in variable exponent function spaces we refer to the papers by V. Kokilashvili and S. Samko; D. E. Edmunds, V. Kokilashvili and A. Meskhi; V.Kokilashvili and A. Meskhi; U. Ashraf, V. Kokilashvili and A. Meskhi; M. Asif, V. Kokilashvili and A. Meskhi; T. Kopaliani; L. Diening and S. Samko; H. Rafeiro and S. Samko; S. Samko and B. Vakulov; S. Samko, E. Shargorodsky and B. Vakulov; V. Kokilashvili, N. Samko and S. Samko; A. Harman and F. I. Mamedov, etc (see also the survey papers [80], [37],[81] and references cited therein).

The One-weight problem under the Muckenhoupt-type condition for the Hardy-Littlewood maximal operator in $L^{p(x)}$ spaces was solved by L. Diening and P. Hästö

[18]. Sawyer-type two-weight criteria for maximal operators were derived in [43].

The thesis is divided into two parts. In the first part various type of two-weight criteria are derived for one-sided operators in variable exponent Lebesgue spaces.

D. E. Edmunds, V. Kokilashvili and A. Meskhi [23] studied the boundedness problems of the unilateral (one-sided) operators in variable exponent Lebesgue spaces on an interval $I \subseteq \mathbb{R}$. In that paper the authors proved that the boundedness of maximal, fractional integral and Calderón-Zygmund type operators with unilateral nature holds in the space $L^{p(x)}$ under the weaker assumptions on $p(x)$ than in the case of bilateral operators. From the results obtained in the latter paper it follows that the unilateral nature of an operator permits the development of better results within the frameworks of variable exponents. This difference between unilateral and bilateral forms of operators was not so essential in the case of constant exponents. It should be emphasized that criteria governing the $L^{p(x)} \to L_v^{q(x)}$ boundedness/compactness for the Riemann-Liouville transform were derived in the paper by U. Ashraf, V. Kokilashvili and A. Meskhi [5] (see also [65], ch.5). One of the novelties of this thesis is to study the two-weight problem for one–sided operators in variable exponent Lebesgue spaces.

The second part of the thesis is devoted to two-weight estimates of integral operators (Hardy–type transforms, maximal functions, potentials, singular integrals) defined on quasi-metric measure spaces in the framework of variable exponent analysis. We derive various type of two-weight sufficient conditions (written in the form of mudulars) ensuring the boundedness of these operators in weighted $L^{p(x)}$ spaces. The derived conditions are simultaneously necessary and sufficient for appropriate two–weight inequalities when exponents of spaces are constants.

It should be stressed that there is a wide range of problems of Mathematical Physics whose solutions are closely connected to the weight problems of integral operators acting between Banach function spaces. We emphasize, for example the very profound impact of trace inequalities on spectral problems of differential operators, and in particular on eigenvalue estimates for Schrödinger operators (see the papers by C. Fefferman; R. Kerman and E. Sawyer; the monograph [27]); and the close connection with the solubility of certain semilinear differential operators with minimal restrictions on the regularity of the coefficients and data. In fact, the existence of positive solutions of certain nonlinear differential equations is equivalent to the validity of a certain two-weighted inequality for a potential-type operator, in which the weights are expressed in terms of coefficients and data (see the papers by K. Hansson; D. R. Adams and M. Pierre; P. Baras and M. Pierre; V. G. Mazya and I. E. Verbitsky; I. E. Verbitsky and R.L. Wheeden). We refer the monographs [63], [28], [38], [82], [73], [59], [20] and references cited therein for the weight theory of integral operators of various type in the classical Lebesgue spaces.

The main results of the thesis are contained in the papers [44, 45, 46, 66, 67]

Chapter 1

Weighted Estimates for One-sided Operators in Variable Exponent Lebesgue Spaces.

1.1 Introduction

This chapter deals with the boundedness of one-sided maximal functions and potentials in weighted Lebesgue spaces with variable exponent. In particular, we derive one-weight inequality for one-sided maximal functions; sufficient conditions (in some cases necessary and sufficient conditions) governing two-weight inequalities for one-sided maximal and potential operators; criteria for the trace inequality for one-sided fractional maximal functions and potentials; Fefferman-Stein type inequality for one-sided fractional maximal functions; generalization of the Hardy-Littlewood theorem for the Riemann-Liouville and Weyl transforms; the one–weight modular inequality for the Riemann-Liouville transform on the cone of decreasing functions from the variable exponent viewpoint. It is worth mentioning that some results of this chapter implies the following fact: the one-weight inequality for one-sided maximal functions automatically holds when both the exponent of the space and the weight are monotonic functions.

Solution of the one-weight problem for one-sided operators in classical Lebesgue spaces was given in [86, 3]. Trace inequalities for one-sided potentials in L^p spaces were characterized in [64, 75, 39]. It should be emphasized that a solution of the two-weight problem in the classical Lebesgue spaces under transparent integral conditions on weights for one-sided maximal functions and potentials in the non-diagonal case are given in the monographs [29](Chapters 2 and 3) and [20](Chapter 2). For Sawyer-type two-weight criteria for one-sided fractional operators we refer to [62], [61], [60].

1.2 Preliminaries

Let Ω be an open set in $\mathbb{R}^n$ and let p be a measurable function on Ω. Suppose that

$$1 \le p_- \le p_+ < \infty, \tag{1.2.1}$$

where p_- and p_+ are the infimum and the supremum respectively of p on Ω. Suppose that ρ is a weight function on Ω, i.e. ρ is an almost everywhere positive locally integrable function on Ω. We say that a measurable function f on Ω belongs to $L^{p(\cdot)}_{\rho}(\Omega)$ (or $L^{p(x)}_{\rho}(\Omega)$) if

$$S_{p,\rho}(f) = \int\limits_{\Omega} \big|f(x)\rho(x)\big|^{p(x)} dx < \infty.$$

It is known that (see, e.g., [57, 49, 48, 78]) $L^{p(\cdot)}_{\rho}(\Omega)$ is a Banach space with the norm

$$\|f\|_{L^{p(\cdot)}_{\rho}(\Omega)} = \inf\big\{\lambda > 0 : S_{p(\cdot),\rho}\big(f/\lambda\big) \le 1\big\}.$$

If $\rho \equiv 1$, then we use the symbol $L^{p(\cdot)}(\Omega)$ (resp. S_p) instead of $L^{p(\cdot)}_{\rho}(\Omega)$ (resp. $S_{p(\cdot),\rho}$). It is clear that $\|f\|_{L^{p(\cdot)}_{\rho}(\Omega)} = \|f\rho\|_{L^{p(\cdot)}(\Omega)}$. It should be also emphasized that when p is constant, then $L^{p(\cdot)}_{\rho}(\Omega)$ coincides with the classical weighted Lebesgue space.

We will use the following notation:

$$p_-(E) := \inf_E p; \quad p_+(E) := \sup_E p, \qquad E \subset \Omega.$$

The following statement is well-known:

Proposition 1.2.1 (**[57, 78]**). *Let E be a measurable subset of Ω. Then the following inequalities hold:*

$$\|f\|^{p_+(E)}_{L^{p(\cdot)}(E)} \le S_{p(\cdot)}(f\chi_E) \le \|f\|^{p_-(E)}_{L^{p(\cdot)}(E)}, \quad \|f\|_{L^{p(\cdot)}(E)} \le 1;$$

$$\|f\|^{p_-(E)}_{L^{p(\cdot)}(E)} \le S_{p(\cdot)}(f\chi_E) \le \|f\|^{p_+(E)}_{L^{p(\cdot)}(E)}, \quad \|f\|_{L^{p(\cdot)}(E)} \ge 1;$$

$$\Big|\int\limits_E f(x)g(x)dx\Big| \le \Big(\frac{1}{p_-(E)} + \frac{1}{(p_+(E))'}\Big) \, \|f\|_{L^{p(\cdot)}(E)} \, \|g\|_{L^{p'(\cdot)}(E)},$$

where $p'(x) = \frac{p(x)}{p(x)-1}$ and $1 < p_- \le p_+ < \infty$.

Let I be an open set in $\mathbb{R}$. In the sequel we shall use the notation:

$$I_+(x,h) := [x, x+h] \cap I, \quad I_-(x,h) := [x-h, x] \cap I;$$

$$I(x,h) := [x-h, x+h] \cap I.$$

Now we introduce the following maximal operators with variable parameter:

$$\big(M_{\alpha(\cdot)}f\big)(x) = \sup_{h>0} \frac{1}{(2h)^{1-\alpha(x)}} \int\limits_{I(x,h)} |f(t)|dt,$$

$$\big(M^-_{\alpha(\cdot)}f\big)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha(x)}} \int\limits_{I_-(x,h)} |f(t)|dt,$$

$$\big(M^+_{\alpha(\cdot)}f\big)(x) = \sup_{h>0} \frac{1}{h^{1-\alpha(x)}} \int\limits_{I_+(x,h)} |f(t)|dt,$$

where $0 < \alpha_- \le \alpha_+ < 1$, I is an open set in $\mathbb{R}$ and $x \in I$.
If $\alpha \equiv 0$, then $M^-_{\alpha(\cdot)}$ and $M^+_{\alpha(\cdot)}$ are the one-sided Hardy-Littlewood maximal operators which are denoted by M^- and M^+ respectively.

To prove the main results we need some statements:

Theorem 1.2.2 ([**12**]). *Let Ω be a bounded open set in $\mathbb{R}^n$. Then the maximal operator*

$$(M_\Omega f)(x) = \sup_{r>0} \frac{1}{r^n} \int\limits_{B(x,r)\bigcap\Omega} |f(y)|dy, \quad x \in \Omega,$$

is bounded in $L^{p(\cdot)}(\Omega)$ if $p \in \mathcal{P}(\Omega)$, that is,

(a) $1 < p_- \leq p(x) \leq p_+ < \infty$;

(b) *p satisfies the log-Hölder continuity (Dini-Lipschitz) condition ($p \in LH(\Omega)$): there exists a positive constant A such that for all $x, y \in \Omega$ with $0 < |x-y| \leq \frac{1}{2}$ the inequality*

$$\big|p(x) - p(y)\big| \leq \frac{A}{\ln \frac{1}{|x-y|}} \tag{1.2.2}$$

holds.

Theorem 1.2.3 ([**9**]). *Let Ω be an open subset of $\mathbb{R}^n$. Suppose that $1 < p_- \leq p_+ < \infty$. Then the maximal operator M_Ω is bounded in $L^{p(\cdot)}(\Omega)$ if*

(i) $p \in \mathcal{P}(\Omega)$;

(ii) $$|p(x) - p(y)| \leq \frac{C}{\ln(e + |x|)} \tag{1.2.3}$$

for all $x, y \in \Omega$, $|y| \geq |x|$.

We shall also make use of the next two results:

Proposition 1.2.4 ([**57, 78**]). *Let $1 \leq p(x) \leq q(x) \leq q_+ < \infty$. Suppose that Ω is an open set in $\mathbb{R}^n$ with $|\Omega| < \infty$, where $|\Omega|$ is the measure of Ω. Then the inequality*

$$\|f\|_{L^{p(\cdot)}(\Omega)} \leq (1 + |\Omega|)\|f\|_{L^{q(\cdot)}(\Omega)}$$

holds.

Proposition 1.2.5 ([**12**]). *Let Ω be an open set in $\mathbb{R}^n$ and let p and q be bounded exponents on Ω. Then*

$$L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$$

if and only if $p(x) \leq q(x)$ *almost everywhere on* Ω *and there is a constant* $0 < K < 1$ *such that*

$$\int_\Omega K^{p(x)q(x)/(|q(x)-p(x)|)} dx < \infty. \tag{1.2.4}$$

Remark 1.2.1. In the previous statement it is used the convention $K^{1/0} := 0$.

To state more results we need the following definitions:

Definition 1.2.1. Let $\mathcal{P}_-(I)$ be the class of all measurable positive functions $p : I \to \mathbb{R}$ satisfying the following condition: there exist a positive constant C_1 such that for a.e $x \in I$ and a.e $y \in I$ with $0 < x - y \leq \frac{1}{2}$ the inequality

$$p(x) \leq p(y) + \frac{C_1}{\ln\left(\frac{1}{x-y}\right)} \tag{1.2.5}$$

holds. Further, we say that p belongs to $\mathcal{P}_+(I)$ if p is a positive function on I and there exists a positive constant C_2 such that for a.e $x \in I$ and a.e $y \in I$ with $0 < y - x \leq \frac{1}{2}$ the inequality

$$p(x) \leq p(y) + \frac{C_2}{\ln\left(\frac{1}{y-x}\right)} \tag{1.2.6}$$

is fulfilled.

Definition 1.2.2. We say that a measurable positive function on I belongs to the class $\mathcal{P}_\infty(I)$ ($p \in \mathcal{P}_\infty(I)$) if (1.2.3) holds for all $x, y \in I$ with $|y| \geq |x|$.

Definition 1.2.3. Let p be a measurable function on an unbounded interval I in $\mathbb{R}$. We say that $p \in \mathcal{G}(I)$ if there is a constant $0 < K < 1$ such that

$$\int_I K^{p(x)p_-/(p(x)-p_-)} dx < \infty.$$

The next result was obtained in [23].

Theorem 1.2.6. *Let I be a bounded interval in $\mathbb{R}$. Suppose that $1 < p_- \leq p_+ < \infty$. Then*

(i) *if $p \in \mathcal{P}_-(I)$, then M^- is bounded in $L^{p(\cdot)}(I)$;*

(ii) *if $p \in \mathcal{P}_+(I)$, then M^+ is bounded in $L^{p(\cdot)}(I)$.*

In the case of an unbounded set we have

Theorem 1.2.7 ([**23**]). *Let I be an arbitrary open set in $\mathbb{R}$. Suppose that $1 < p_- \leq p_+ < \infty$. If $p \in \mathcal{P}_+(I) \cap \mathcal{P}_\infty(I)$, then the operator M^+ is bounded in $L^{p(\cdot)}(I)$. Further, if $p \in \mathcal{P}_-(I) \cap \mathcal{P}_\infty(I)$, then the operator M^- is bounded in $L^{p(\cdot)}(I)$.*

In particular, the previous statement yields

Theorem 1.2.8 ([**23**]). *Let $I = \mathbb{R}_+$ and let $1 < p_- \leq p_+ < \infty$. Suppose that $p \in \mathcal{P}_+(I)$ and there is a positive number a such that $p \in \mathcal{P}_\infty((a, \infty))$. Then M^+ is bounded in $L^{p(\cdot)}(I)$. Further, if $p \in \mathcal{P}_-(I)$ and there is a positive number a such that $p \in \mathcal{P}_\infty((a, \infty))$, then M^- is bounded in $L^{p(\cdot)}(I)$.*

The next statement gives one-weight criteria for the one-sided maximal operators in classical Lebesgue spaces (see [86], [3]).

Theorem 1.2.9 ([**3**]). *Let $I \subseteq \mathbb{R}$ be an interval. Assume that $0 \leq \alpha < 1$ and $1 < p < 1/\alpha$, where p and α are constants ($1/\alpha = \infty$ if $\alpha = 0$). We set $1/q = 1/p - \alpha$.*

(i) *Let $T := M_\alpha^-$. Then the inequality*

$$\left[\int_I |Tf(x)|^q v(x)dx\right]^{1/q} \leq C\left[\int_I |f(x)|^p v^{p/q}(x)dx\right]^{1/p} \tag{1.2.7}$$

holds if and only if

$$\sup_{h>0}\left(\frac{1}{h}\int_{I_+(x,x+h)} v(t)dt\right)^{\frac{1}{q}}\left(\frac{1}{h}\int_{I_-(x-h,x)} v^{-p'/q}(t)dt\right)^{\frac{1}{p'}} < \infty. \tag{1.2.8}$$

(ii) *Let* $T := M_\alpha^+$. *Then* (1.2.7) *holds if and only if*

$$\sup_{h>0}\left(\frac{1}{h}\int\limits_{I_-(x-h,x)} v(t)dt\right)^{\frac{1}{q}}\left(\frac{1}{h}\int\limits_{I_+(x,x+h)} v^{-p'/q}(t)dt\right)^{\frac{1}{p'}} < \infty. \qquad (1.2.9)$$

Definition 1.2.4. Let $I \subseteq \mathbb{R}$ be an interval. Suppose that $1 < p < q < \infty$, where p and q are constants. We say that the weight $v \in A^-_{p,q}(I)$ (resp. $v \in A^+_{p,q}(I)$) if (1.2.8) (resp. (1.2.9)) holds.

If $p = q$, then we denote the class $A^+_{p,q}(I)$ (resp. $A^-_{p,q}(I)$) by $A^+_p(I)$ (resp. $A^-_p(I)$). Notice that $v \in A^+_{p,q}(I)$ (resp. $v \in A^-_{p,q}(I)$) is equivalent to the condition $v \in A^+_{1+q/p'}(I)$ (resp. $v \in A^-_{1+q/p'}(I)$).

Further, we denote by $D(\mathbb{R})$ (resp. $D(\mathbb{R}_+)$) a dyadic lattice in $\mathbb{R}$ (resp. in $\mathbb{R}_+$).

Definition 1.2.5. We say that a measure μ belongs to the class $RD^{(d)}(\mathbb{R}^n)$ (dyadic reverse doubling condition) if there exists a constant $\delta > 1$, such that for all dyadic cubes Q and Q', $Q \subset Q'$, $|Q| = \frac{|Q'|}{2^n}$, the inequality

$$\mu(Q') \geq \delta\mu(Q)$$

holds.

Definition 1.2.6. We say that a measure μ on $\mathbb{R}^n$ satisfies the doubling condition ($\mu \in DC(\mathbb{R}^n)$) if there is a positive number b such that

$$\mu B(x, 2r) \leq b\mu B(x, r)$$

for all $x \in \mathbb{R}^n$ and $r > 0$.

It is known (see [89], p. 11) that if $\mu \in DC(\mathbb{R}^n)$, then $\mu \in RD(\mathbb{R}^n)$, i.e., there are positive constants η_1 and η_2, $0 < \eta_1, \eta_2 < 1$, such that

$$\mu B(x, \eta_1 r) \leq \eta_2\mu B(x, r),$$

for all $x \in \mathbb{R}^n$ and $r > 0$. It is easy to check that if $\mu \in DC(\mathbb{R}^n)$, then $\mu \in RD^{(d)}(\mathbb{R})$.

We shall need some lemmas giving Carleson-Hörmander type inequalities.

Lemma 1.2.10 ([**90**]). *Let $1 < p \le r < \infty$ and let $\rho^{-p'} \in RD^{(d)}(\mathbb{R}^n)$, where ρ is a weight function on $\mathbb{R}^n$. Then there is a positive constant C such that for all non-negative f the inequality*

$$\sum_{Q \in D(\mathbb{R}^n)} \Big(\int_Q \rho^{-p'}(x)dx \Big)^{-\frac{r}{p'}} \Big(\int_Q f(y)dy \Big)^r \le C \Big(\int_{\mathbb{R}^n} (f(x)\rho(x))^p dx \Big)^{\frac{1}{p}}$$

holds.

Lemma 1.2.11 ([**87, 91**]). *Let $u(x) \ge 0$ on $\mathbb{R}^n$; $\{Q_i\}_{i\in A}$ be a countable collection of dyadic cubes in $\mathbb{R}^n$ and $\{a_i\}_{i\in A}$, $\{b_i\}_{i\in A}$ be positive numbers satisfying*

(i) $$\int_{Q_i} u(x)dx \le Ca_i \qquad \textit{for all} \quad i \in A;$$

(ii) $$\sum_{j:\ Q_j \subset Q_i} b_j \le Ca_i \qquad \textit{for all } i \in A.$$

Then

$$\Big(\sum_{i\in I} b_i \Big(\frac{1}{a_i} \int_{Q_i} g(x)u(x)dx \Big)^p \Big)^{\frac{1}{p}} \le C_p \Big(\int_{\mathbb{R}^n} g^p(x)u(x)dx \Big)^{\frac{1}{p}}$$

for all $g \ge 0$ on $\mathbb{R}^n$ and $1 < p < \infty$.

1.3 Hardy-Littlewood One-sided Maximal Functions. One-weight Problem

In this section we discuss the one-weight problem for the one-sided Hardy-Littlewood maximal operators.

We shall apply the following lemma in the proof of the main results of this section:

Lemma 1.3.1 (**[23]**). *Let I be a bounded interval and let p be a measurable function on I such that* (1.2.1) *hold on I. If $p \in \mathcal{P}_+(I)$, then there is a positive constant depending only on p such that for all f, $\|f\|_{L^{p(\cdot)}(I)} \leq 1$, the inequality*

$$\big(M^+ f(x)\big)^{p(x)} \leq C\left(1 + M^+\big(|f|^{p(\cdot)}\big)(x)\right)$$

holds.

The following two theorems are the main results of this section:

Theorem 1.3.2. *Let I be a bounded interval in $\mathbb{R}$ and let p be a measurable function on $\mathbb{R}$ such that $1 < p_- \leq p_+ < \infty$.*

(i) *If $p \in \mathcal{P}_+(I)$ and a weight function w satisfies the condition $w(\cdot)^{p(\cdot)} \in A^+_{p_-}(I)$, then for all $f \in L^{p(\cdot)}_w(I)$ the inequality*

$$\|(Nf)w\|_{L^{p(\cdot)}(I)} \leq C\|wf\|_{L^{p(\cdot)}(I)} \tag{1.3.1}$$

holds, where $N = M^+$.

(ii) *Let $p \in \mathcal{P}_-(I)$ and let $w(\cdot)^{p(\cdot)} \in A^-_{p_-}(I)$. Then inequality* (1.3.1) *holds for all $f \in L^{p(\cdot)}_w(I)$, where $N = M^-$.*

The result similar to Theorem 1.3.2 has been derived in [51], [53] for M_Ω, where $\Omega \subset \mathbb{R}^n$ is a bounded domain.

In the case of unbounded intervals we have the next statement:

Theorem 1.3.3. *Let $I = \mathbb{R}_+$ and let p be a measurable function on $\mathbb{R}_+$ such that $1 < p_- \leq p_+ < \infty$. Suppose that there is a positive number a such that $p(x) \equiv p_c \equiv$ const outside $(0, a)$.*

(i) *If $p \in \mathcal{P}_+(I)$ and $w(\cdot)^{p(\cdot)} \in A^+_{p_-}(I)$, then* (1.3.1) *holds for $N = M^+$.*

(ii) *If $p \in \mathcal{P}_-(I)$ and $w(\cdot)^{p(\cdot)} \in A^-_{p_-}(I)$, then* (1.3.1) *holds for $N = M^-$.*

Theorem 1.3.2 yields the following corollaries:

Corollary 1.3.4. *Let p be an increasing function on an interval $I = (a, b)$ such that $1 < p(a) \le p(b) < \infty$. Suppose that w is an increasing positive function on I. Then the one-weight inequality*

$$\|w^{1/p(\cdot)}(M^+ f)(\cdot)\|_{L^{p(\cdot)}(I)} \le c\|w^{1/p(\cdot)} f(\cdot)\|_{L^{p(\cdot)}(I)}$$

holds.

Corollary 1.3.5. *Let p be a decreasing function on an interval $I = (a, b)$ such that $1 < p(b) \le p(a) < \infty$. Suppose that w is a decreasing positive function on I. Then the one-weight inequality*

$$\|w^{1/p(\cdot)}(M^- f)(\cdot)\|_{L^{p(\cdot)}(I)} \le c\|w^{1/p(\cdot)} f(\cdot)\|_{L^{p(\cdot)}(I)}$$

holds.

Now we prove Theorems 1.3.2 and 1.3.3.

Proof of Theorem 1.3.2. Since the proof of the second part is similar to the first one, we prove only (i). It is enough to show that

$$S_p\left(wM^+(f/w)\right) \le C$$

for f satisfying the condition $\|f\|_{L^{p(\cdot)}(I)} \le 1$.
First we prove that $S_{p^*}\left(\frac{f}{w}\right) < \infty$, where $p^*(x) = \frac{p(x)}{p_-}$.
By using Hölder's inequality we find that

$$S_{p^*}\left(\frac{f}{w}\right) = \int_I [f/w]^{p^*(x)}(x)dx \le \left(\int_I |f(x)|^{p(x)}dx\right)^{\frac{1}{p_-}} \cdot \left(\int_I w(x)^{p(x)(1-(p_-)')}dx\right)^{\frac{1}{(p_-)'}} < \infty$$

because $w^{p(\cdot)}(\cdot) \in A^+_{p_-}(I)$.

Thus Lemma 1.3.1 might be applied for p^*. Consequently,

$$\begin{aligned}
S_p\big(w(M^+f/w)\big) &= \int_I \left[M^+\left(\frac{f}{w}\right)(x)\right]^{p(x)} w^{p(x)}(x)dx \\
&= \int_I \Big(\big[M^+(f/w)(x)\big]^{p^*(x)}\Big)^{p_-} w^{p(x)}(x)dx \\
&\leq C\int_I \left(1+M^+\left(\Big|\frac{f}{w}\Big|^{p^*(\cdot)}\right)(x)\right)^{p_-} (w(x))^{p(x)}dx \\
&\leq C\int_I (w(x))^{p(x)}\,dx + C\int_I \left(M^+\left(\Big|\frac{f}{w}\Big|^{p^*(\cdot)}\right)(x)\right)^{p_-} w^{p(x)}(x)dx \\
&\leq C + C\int_I \big|f/w\big|^{p(x)} w^{p(x)}(x)dx \leq C.
\end{aligned}$$

□

Proof of Theorem 1.3.3. First we prove (i). Without loss of generality we can assume that $M^+f(a) < \infty$. Since M^+ is a sub-linear operator it is enough to prove that $S_{p,w}(M^+f) < \infty$, whenever $S_{p,w}(f) < \infty$. We have

$$\begin{aligned}
&\int_{\mathbb{R}_+} \big(M^+f\big)^{p(x)}(x)w(x)^{p(x)}dx \leq c\Big[\int_0^a \big(M^+f\chi_{[0,a]}\big)^{p(x)}(x)w(x)^{p(x)}dx \\
&\quad + \int_0^a \big(M^+(f\chi_{[a,\infty)})\big)^{p(x)}(x)w(x)^{p(x)}dx + \int_a^\infty \big(M^+(f\chi_{[0,a]})\big)^{p(x)}(x)w(x)^{p(x)}dx \\
&\qquad + \int_a^\infty \big(M^+f\chi_{[a,\infty)}\big)^{p(x)}(x)w(x)^{p(x)}dx\Big] = c[I_1+I_2+I_3+I_4].
\end{aligned}$$

Since $M^+f(x) = M^+(f\chi_{[0,a]})(x)$ for $x \in [0,a]$, using the assumptions $w(\cdot)^{p(\cdot)} \in A^+_{p_-}([0,a])$, $p_+ \in \mathcal{P}_+((0,a))$ and Theorem 1.3.2 we find that $I_1 < \infty$.

Further, the condition $w(\cdot)^{p(\cdot)} \in A^+_{p_-}(I)$ implies that $w(\cdot)^{p(\cdot)} \in A^+_{p_-}((a,\infty))$. Consequently, since $p \equiv p_c \equiv$ const on (a,∞), by Theorem 1.2.9 we have $I_4 < \infty$.

Now observe that $M^+(f\chi_{[0,a]})(x) = 0$ when $x \in (a,\infty)$. Therefore $I_3 = 0$.

It remains to estimate I_2. For this notice that if $x \in (0,a)$, then

$$\begin{aligned}
M^+\big(f\cdot\chi_{[a,\infty)}\big)(x) &= \sup_{h>0}\frac{1}{h}\int_x^{x+h}|f(y)|\chi_{[a,\infty)}(y)dy = \sup_{h>a-x}\frac{1}{h}\int_a^{x+h}|f(y)|\chi_{[a,\infty)}(y)dy \\
&\leq \sup_{h>a-x}\frac{1}{x+h-a}\int_a^{a+(x+h-a)}|f(y)|\chi_{[a,\infty)}(y)dy \leq M^+f(a) < \infty.
\end{aligned}$$

Hence,

$$I_2 \le c \int_0^a w(x)^{p(x)} dx < \infty$$

because $w(\cdot)^{p(\cdot)}$ is locally integrable on $\mathbb{R}_+$.

To prove (ii) we use the notation of the proof of (i) substituting M^+ by M^-. In fact, the proof is similar to that of (i). The only difference is in the estimates of

$$I_2 := \int_0^a \big(M^-(f\chi_{[a,\infty)})\big)^{p(x)}(x) w(x)^{p(x)} dx$$

and

$$I_3 := \int_a^\infty \big(M^-(f \cdot \chi_{[0,a]})(x)\big)^{p(x)} (x) w(x)^{p(x)} dx.$$

Obviously, we have that $I_2 = 0$. Further, we represent I_3 as follows:

$$\begin{aligned} I_3 &= \int_a^\infty \big(M^-(f \cdot \chi_{[0,a]})(x)\big)^{p_c} (x) w(x)^{p_c} dx \\ &= \int_a^{2a} \big(M^-(f \cdot \chi_{[0,a]})(x)\big)^{p_c} (x) w(x)^{p_c} dx + \int_{2a}^\infty \big(M^-(f \cdot \chi_{[0,a]})(x)\big)^{p_c} (x) w(x)^{p_c} dx \\ &=: I_3^{(1)} + I_3^{(2)}. \end{aligned}$$

Observe that for $x \in (a, 2a]$,

$$M^-(f \cdot \chi_{[0,a]})(x) \le \sup_{x-a<h<x} \frac{1}{a-x+h} \int_{a-(a-x+h)}^a |f(y)| dy \le M^- f(a) < \infty.$$

Hence,

$$I_3^{(1)} \le (M^- f)^{p_c}(a) \int_a^{2a} (w(x))^{p_c} dx < \infty.$$

If $x > 2a$, then

$$\big(M^- f\big)(x) \le \frac{1}{a-x} \int_0^a |f(y)| dy.$$

Therefore by using Hölder's inequality with respect to the exponent $p(\cdot)$ (see Proposition 1.2.1) we find that

$$\begin{aligned} I_3^{(2)} &\le \left(\int_{2a}^\infty (w(x))^{p_c} (a-x)^{-p_c} dx\right) \left(\int_0^a |f(x)| dx\right)^{p_c} \\ &\le c \left(\int_{2a}^\infty (w(x))^{p_c} (a-x)^{-p_c} dx\right) \|fw\|^{p_c}_{L^{p(\cdot)}_{([0,a])}} \|w^{-1}\|^{p_c}_{L^{p'(\cdot)}_{([0,a])}} =: cJ_1 \cdot J_2 \cdot J_3. \end{aligned}$$

It is clear that $J_2 < \infty$. Further, since $w(\cdot)^{p(\cdot)} \in A^-_{p_-}\big((a,\infty)\big)$, by Hölder's inequality we have that $w(\cdot)^{p(\cdot)} \in A^-_{p_c}\big((a,\infty)\big)$, because $p_c \geq p_-$. Hence, by applying Theorem 1.2.9 (for $\alpha = 0$) we have that the operator $M^- f := M^-(f\chi_{(a,\infty)})$ is bounded in $L^{p_c}_w((a,\infty))$. Consequently, the Hardy operator

$$H_a f(x) = \frac{1}{x-a}\int_a^x |f(t)|dt, \quad x \in (a,\infty),$$

is bounded in $L^{p_c}_w((a,\infty))$. This implies (see, e.g., [35], [63]) that $J_1 < \infty$.
It remains to see that $J_3 < \infty$. Indeed, Proposition 1.2.4 yields

$$\begin{aligned}
\|w^{-1}\|_{L^{p'(\cdot)}_{([0,a])}} &\leq (1+a)\|w^{-1}\|_{L^{(p_-)'\cdot}([0,a])} \\
&\leq c\|\chi_{\{w^{-1}\geq 1\}}(\cdot)w^{-1}(\cdot)\|_{L^{(p_-)'(\cdot)}([0,a])} + \|\chi_{\{w^{-1}<1\}}(\cdot)w^{-1}(\cdot)\|_{L^{(p_-)'}([0,a])} \\
&\leq c\big\|\chi_{\{w^{-1}\geq 1\}}(\cdot)w^{-\frac{p(\cdot)}{p_-}}(x)\big\|_{L^{(p_-)'}([0,a])} + c \\
&\leq \left(\int_0^a w^{p(x)(1-(p_-)')}(x)dx\right)^{1/(p_-)'} + c.
\end{aligned}$$

Thus $I_3^{(2)} < \infty$. □

1.4 One-sided Fractional Maximal Operators. One-weight Problem

In this section we derive the one-weight inequality for the one-sided fractional maximal operators. Concerning this section the main results are the following statements:

Theorem 1.4.1. *Let I be a bounded interval and let $1 < p_- \leq p_+ < \infty$. Suppose that α is constant satisfying $0 < \alpha < 1/p_+$. Let $q(x) = \frac{p(x)}{1-\alpha p(x)}$.*

(i) *If $p \in \mathcal{P}_+(I)$ and a weight w satisfies the condition $w(\cdot)^{q(\cdot)} \in A^+_{p_-,q_-}(I)$, then the inequality*

$$\|(N_\alpha f)w\|_{L^{q(\cdot)}(I)} \leq C\|wf\|_{L^{p(\cdot)}(I)}, \quad f \in L^{p(\cdot)}_w(I), \tag{1.4.1}$$

holds for $N_\alpha = M_\alpha^+$.

(ii) *Let* $p \in \mathcal{P}_-(I)$ *and let* $w(\cdot)^{q(\cdot)} \in A^-_{p_-,q_-}(I)$. *Then inequality* (1.4.1) *holds for* $N_\alpha = M_\alpha^-$.

Theorem 1.4.2. *Let* $I = \mathbb{R}_+$, $1 < p_- \le p_+ < \infty$ *and let* $p(x) \equiv p_c \equiv const$ *outside some interval* $(0,a)$. *Suppose that* $q(x) = \frac{p(x)}{1-\alpha p(x)}$, *where* α *is constant satisfying* $0 < \alpha < 1/p_+$.

(i) *If* $p \in \mathcal{P}_+(I)$ *and* $w(\cdot)^{q(\cdot)} \in A^+_{p_-,q_-}(I)$, *then* (1.4.1) *holds for* $N_\alpha = M_\alpha^+$.

(ii) *If* $p \in \mathcal{P}_-(I)$ *and* $w(\cdot)^{q(\cdot)} \in A^-_{p_-,q_-}(I)$, *then* (1.4.1) *holds for* $N_\alpha = M_\alpha^-$.

Proof of Theorem 1.4.1. We prove (i). The proof of (ii) is the same. First we show that the inequality

$$M_\alpha^+(f/w)(x) \le \big(M^+\big(f^{p(\cdot)/s(\cdot)}w^{-q(\cdot)/s(\cdot)}\big)(x)\big)^{s(x)/q(x)}\bigg(\int_I f^{p(y)}(y)dy\bigg)^\alpha,$$

holds, where $s(x) = 1 + q(x)/p'(x)$. Indeed, denoting $g(\cdot) := (f(\cdot))^{p(\cdot)/s(\cdot)}(w(\cdot))^{-q(\cdot)/s(\cdot)}$ we see that $f(\cdot)/w(\cdot) = (g(\cdot))^{s(\cdot)/p(\cdot)}w^{q(\cdot)/p(\cdot)-1} = (g(\cdot))^{1-\alpha}g^{s(\cdot)/p(\cdot)+\alpha-1}w^{\alpha q(\cdot)}$.
By using Hölder's inequality with respect to the exponent $(1-\alpha)^{-1}$ and the facts that $s(\cdot)/q(\cdot) = 1-\alpha$, $(s(y)/p(y)+\alpha-1)/\alpha = s(\cdot)$ we have

$$\begin{aligned}
&\frac{1}{h^{1-\alpha}}\int_{I_+(x,x+h)}\frac{f(y)}{w(y)}dy \\
&\qquad \le \bigg(\frac{1}{h}\int_{I_+(x,x+h)} g(y)dy\bigg)^{1-\alpha}\bigg(\int_{I_+(x,x+h)} g^{(s(y)/p(y)+\alpha-1)/\alpha}(y)w^{q(y)}(y)dy\bigg)^\alpha \\
&\qquad \le \big(M^+g(x)\big)^{s(x)/q(x)}\bigg(\int_{I_+(x,x+h)} g^{s(y)}(y)w^{q(y)}(y)\bigg)^\alpha \\
&\qquad \le \big(M^+g(x)\big)^{s(x)/q(x)}\bigg(\int_I f^{p(y)}(y)dy\bigg)^\alpha.
\end{aligned}$$

Now we prove that $S_q\big(wM_\alpha^+(f/w)\big) \le C$, when $S_p(f) \le 1$. By applying the above-derived inequality we find that

$$\begin{aligned}
S_q\big(wM_\alpha^+(f/w)\big) &\le c\int_I \big(M_\alpha^+(f^{p(\cdot)/s(\cdot)}w^{-q(\cdot)/s(\cdot)})\big)^{s(x)}(x)w^{q(x)}(x)dx \\
&= cS_s\big(M^+(f^{p(\cdot)/s(\cdot)}w^{-q(\cdot)/s(\cdot)})w^{q(\cdot)/s(\cdot)}\big).
\end{aligned}$$

Observe now that the condition on the weight w is equivalent to the assumption $w^{q(\cdot)}(\cdot) \in A^+_{s_-}(I)$. On the other hand, $\|f^{p(\cdot)/s(\cdot)}\|_{L^{s(\cdot)}(I)} \le 1$. Therefore taking Theorem 1.3.2 into account we have the desired result. □

Proof of Theorem 1.4.2. (i) Let $f \ge 0$ and let $S_{p,w}(f) < \infty$. We have

$$S_{q,w}(M^+_\alpha f) = \int\limits_I \left(M^+_\alpha f\right)^{q(x)}(x) w(x)^{q(x)} dx$$

$$\le c\Bigg[\int_0^a \left(M^+_\alpha f\chi_{[0,a]}(x)\right)^{q(x)}(x) w(x)^{q(x)} dx + \int_0^a \left(M^+_\alpha (f\cdot \chi_{[a,\infty)})(x)\right)^{q(x)}(x) w(x)^{q(x)} dx$$

$$+\int_a^\infty \left(M^+_\alpha (f\cdot \chi_{[0,a]})(x)\right)^{q(x)}(x) w(x)^{q(x)} dx + \int_a^\infty \left(M^+_\alpha (f\chi_{[a,\infty)})(x)\right)^{q(x)}(x) w(x)^{q(x)} dx\Bigg]$$

$$=: c[I_1 + I_2 + I_3 + I_4].$$

It is easy to see that $I_1 < \infty$ because of Theorem 1.4.1 and the condition $w^{q(\cdot)}(\cdot) \in A^+_{p_-,q_-}([0,a])$. Further, it is obvious that $I_3 < \infty$ because $M^+_\alpha(f\chi_{[0,a]})(x) = 0$ for $x > a$. Further, observe that

$$I_2 \le c\int_0^a w(x)^{q(x)} dx < \infty,$$

where the positive constant depends on α, f, p, a.

It is easy to check that by Hölder's inequality with respect to the power

$$\big((p_c)'/q_c\big)/\big((p_-)'/q_-\big)$$

the condition $w(\cdot)^{q_c} \in A^+_{p_-,q_-}([a,\infty))$ implies $w(\cdot)^{q_c} \in A^+_{p_c,q_c}([a,\infty))$. Hence, by using Theorem 1.2.9 we find that $I_4 < \infty$.

(ii) We keep the notation of the proof of (i) but substitute M^+_α by M^-_α. The only difference between the proofs of (i) and (ii) is in the estimates of I_2 and I_3.

It is obvious that $I_2 = 0$, while for I_3 we have

$$\begin{aligned} I_3 &= \int\limits_a^{2a} \left(M^-_\alpha (f\cdot \chi_{[0,a]})(x)\right)^{q(x)}(x) w(x)^{q(x)} dx + \int\limits_{2a}^{\infty} \left(M^-_\alpha (f\cdot \chi_{[0,a]})(x)\right)^{q_c}(x) w(x)^{q_c} dx \\ &=: I_3^{(1)} + I_3^{(2)}. \end{aligned}$$

If $x > a$, then

$$M_\alpha^- f(x) \le \sup_{x-a<h<x} h^{\alpha-1} \int\limits_{x-h}^{a} |f(y)|dy \le c M_\alpha^- f(a).$$

Consequently,

$$I_3^{(1)} \le c\big(M_\alpha^- f(a)\big)^{q_c} \int\limits_{a}^{2a} \big(w(x)\big)^{q_c} dx < \infty.$$

Now observe that when $x > a$ we have the following pointwise estimates:

$$\begin{aligned} M_\alpha^-(f\chi_{[0,a]})(x) &\le (x-a)^{\alpha-1} \int\limits_0^a |f(y)|dy \\ &\le (x-a)^{\alpha-1} \|fw\|_{L^{p(\cdot)}([0,a])} \|w^{-1}\|_{L^{p'(\cdot)}([0,a])} =: (x-a)^{\alpha-1} J_1 \cdot J_2. \end{aligned}$$

Hence,

$$I_3^{(2)} \le \bigg(\int\limits_{2a}^{\infty} (x-a)^{(\alpha-1)q_c} (w(x))^{q_c} dx \bigg) (J_1 \cdot J_2)^{q_c}.$$

It is obvious that $J_1 < \infty$. Further,

$$J_2 \le \|w^{-1}(\cdot)\chi_{w^{-1}>1}(\cdot)\|_{L^{p'(\cdot)}([0,a])} + \|w^{-1}(\cdot)\chi_{w^{-1}\le 1}(\cdot)\|_{L^{p'(\cdot)}([0,a])} =: J_2^{(1)} + J_2^{(2)}.$$

It is clear that $J_2^{(2)} < \infty$. To estimate $J_2^{(1)}$ observe that by Proposition 1.2.4 we have

$$\begin{aligned} J_2^{(1)} &\le (1+a)\|w^{-1}\chi_{w^{-1}>1}\|_{L^{p_-}([0,a])} \le (1+a)\|w^{-q(\cdot)/q_-}\chi_{w^{-1}>1}\|_{L^{p_-}([0,a])} \\ &\le (1+a)\|w^{-q(\cdot)/q_-}\|_{L^{p_-}([0,a])} < \infty. \end{aligned}$$

Since M_α^- is bounded from $L_w^{p_c}([a,\infty))$ to $L_w^{q_c}([a,\infty))$ we have the Hardy inequality

$$\bigg(\int\limits_a^\infty (x-a)^{(\alpha-1)q_c} w^{q_c}(x) \bigg(\int\limits_a^x |f(t)|dt \bigg)^{q_c} dx \bigg)^{1/q_c} \le c \bigg(\int\limits_a^\infty |f(x)|^{p_c} w^{p_c}(x) dx \bigg)^{1/p_c}.$$

From this inequality it follows that (see, e.g., [35], [63])

$$\int\limits_{2a}^{\infty} (x-a)^{(\alpha-1)q_c} (w(x))^{q_c} dx < \infty.$$

□

1.5 Generalized Fractional Maximal Operators. Two-weight Problem

Let $I=[a,b]$ be a bounded interval and let $I^+:=[b,2b-a)$; $I^-:=[2a-b,a)$.
Let $Q=I_1\times I_2\times\cdots\times I_n$ be a cube in $\mathbb{R}^n$. We denote:

$$Q^+:=I_1^+\times I_2^+\times\cdots\times I_n^+,\quad Q^-:=I_1^-\times I_2^-\times\cdots\times I_n^-.$$

Let α be a measurable function on $\mathbb{R}^n$, $0<\alpha_-\le\alpha(x)\le\alpha_+<n$. Let us define one-sided dyadic fractional maximal functions on $\mathbb{R}^n$:

$$\begin{aligned}\big(M^{+,(d)}_{\alpha(\cdot)}f\big)(x) &= \sup_{\substack{x\in Q\\ Q\in D(\mathbb{R}^n)}}\frac{1}{|Q|^{1-\frac{\alpha(x)}{n}}}\int\limits_{Q^+}|f(y)|dy;\\ \big(M^{-,(d)}_{\alpha(\cdot)}f\big)(x) &= \sup_{\substack{x\in Q\\ Q\in D(\mathbb{R}^n)}}\frac{1}{|Q|^{1-\frac{\alpha(x)}{n}}}\int\limits_{Q^-}|f(y)|dy.\end{aligned}$$

If $\alpha(x)\equiv 0$, then we have one-sided Hardy-Littlewood dyadic maximal functions $M^{+,(d)}$, $M^{-,(d)}$.

In the paper [72] the two-weight weak-type inequality was proved in the classical Lebesgue spaces for the one-sided dyadic Hardy-Littlewood maximal functions defined on $\mathbb{R}^n$.

Theorem 1.5.1. *Let p be constant and let $1<p<q_-\le q_+<\infty$, $0<\alpha_-\le\alpha_+<n$ where q and α are measurable functions on $\mathbb{R}^n$. Suppose that $w^{-p'}\in RD^{(d)}(\mathbb{R}^n)$. Then $M^{+,(d)}_{\alpha(\cdot)}$ is bounded from $L^p_w(\mathbb{R}^n)$ to $L^{q(\cdot)}_v(\mathbb{R}^n)$ if and only if*

$$A:=\sup_{Q,Q\in D(\mathbb{R}^n)}\big\|\chi_Q(\cdot)|Q|^{\frac{\alpha(\cdot)}{n}-1}v(\cdot)\big\|_{L^{q(\cdot)}(\mathbb{R}^n)}\big\|\chi_{Q^+}w^{-1}\big\|_{L^{p'}(\mathbb{R}^n)}<\infty. \tag{1.5.1}$$

Proof. Necessity. Assuming $f=\chi_{Q^+}w^{-p'}$ ($Q\in D(\mathbb{R}^n)$) in the inequality

$$\big\|M^{+,(d)}_{\alpha(\cdot)}f\big\|_{L^{q(\cdot)}_v(\mathbb{R}^n)}\le C\|f\|_{L^p_w(\mathbb{R}^n)} \tag{1.5.2}$$

we have that

$$\Big\|\chi_Q(\cdot)\Big(\frac{1}{|Q|^{1-\frac{\alpha(x)}{n}}}\int\limits_{Q^+} f\Big)\Big\|_{L_v^{q(\cdot)}(\mathbb{R}^n)} = \big\|\chi_Q(\cdot)|Q|^{\frac{\alpha(\cdot)}{n}-1}\big\|_{L_v^{q(\cdot)}(\mathbb{R}^n)}\Big(\int\limits_{Q^+} w^{-p'}(y)dy\Big)$$
$$\leq \big\|M^{+,(d)}_{\alpha(\cdot)}f\big\|_{L_v^{q(\cdot)}(\mathbb{R}^n)} \leq C\Big(\int\limits_{Q^+} w^{-p'}(y)dy\Big)^{\frac{1}{p}}.$$

Thus, to show that (1.5.1) holds it remains to prove that for all dyadic cubes Q, $S_Q = \int\limits_Q w^{-p'}(y)dy < \infty$. Indeed, suppose the contrary that $S_Q = \infty$ for some cube Q. Then $S_Q = \|w^{-1}\|_{L^{p'}(Q)} = \infty$. This implies that there is a non-negative function g such that $g \in L^p(Q)$ and $\int\limits_Q g(y)w^{-1}(y)dy = \infty$. Further, let $Q = \bar{Q}^+$, where $\bar{Q} \in D(\mathbb{R}^n)$. Then taking $f = \chi_Q g w^{-1}$ we have

$$\|f\|_{L^p_w(\mathbb{R}^n)} = \Big(\int\limits_{\bar{Q}^+} g^p(x)dx\Big)^{\frac{1}{p}} < \infty;$$

and

$$\big\|M^{+,(d)}_{\alpha(\cdot)}f\big\|_{L_v^{q(\cdot)}(\mathbb{R}^n)} \geq \big\|\chi_{\bar{Q}}(\cdot)|\bar{Q}|^{\frac{\alpha(\cdot)}{n}-1}\big\|_{L_v^{q(\cdot)}(\mathbb{R}^n)}\Big(\int\limits_{\bar{Q}^+} f(y)dy\Big)$$
$$= \big\|\chi_{\bar{Q}}(\cdot)|\bar{Q}|^{\frac{\alpha(\cdot)}{n}-1}\big\|_{L_v^{q(\cdot)}(\mathbb{R}^n)}\int\limits_{\bar{Q}^+} g(y)w(y)^{-1}dy = \infty.$$

This contradicts (1.5.2).

Sufficiency. For every $x \in \mathbb{R}^n$ we take $Q_x \in D(\mathbb{R}^n)$ ($x \in Q_x$) so that

$$|Q_x|^{\frac{\alpha(x)}{n}-1}\int\limits_{Q_x^+} |f(y)|dy > \frac{1}{2}\,\big(M^{+,(d)}_{\alpha(\cdot)}f\big)(x). \tag{1.5.3}$$

Assume that f be non-negative bounded with compact support. Then it is easy to see that we can take maximal cube Q_x containing x for which (1.5.3) holds. Let $Q \in D(\mathbb{R}^n)$ and let us introduce the set

$$F_Q := \Big\{x \in Q : Q \text{ is maximal for which } |Q|^{\frac{\alpha(x)}{n}-1}\int\limits_{Q^+} f(y)dy > \frac{1}{2}\,M^{+,(d)}_{\alpha(\cdot)}f(x)\Big\}.$$

Dyadic cubes have the following property: if $Q_1, Q_2 \in D(\mathbb{R}^n)$, and $\overset{o}{Q}_1 \bigcap \overset{o}{Q}_2 \neq \emptyset$, then $Q_1 \subset Q_2$ or $Q_2 \subset Q_1$, where $\overset{o}{Q}$ denotes the interior part of a cube Q.

Now observe that $F_{Q_1} \bigcap F_{Q_2} \neq \emptyset$ if $Q_1 \neq Q_2$. Indeed, if $\overset{o}{Q}_1 \bigcap \overset{o}{Q}_2 = \emptyset$, then it is clear. If $\overset{o}{Q}_1 \bigcap \overset{o}{Q}_2 \neq \emptyset$, then $Q_1 \subset Q_2$ or $Q_2 \subset Q_1$. Let us take $x \in F_{Q_1} \bigcap F_{Q_2}$. Then $x \in Q_1$, $x \in Q_2$ and

$$\begin{aligned} \frac{1}{|Q_1|^{1-\frac{\alpha(x)}{n}}} \int\limits_{Q_1^+} f(y)dy &> \frac{1}{2} \left(M^{+,(d)}_{\alpha(\cdot)} f \right)(x); \\ \frac{1}{|Q_2|^{1-\frac{\alpha(x)}{n}}} \int\limits_{Q_2^+} f(y)dy &> \frac{1}{2} \left(M^{+,(d)}_{\alpha(\cdot)} f \right)(x). \end{aligned}$$

If $Q_1 \subset Q_2$, then Q_2 would be the maximal cube for which (1.5.3) holds. Consequently $x \notin F_{Q_1}$ and $x \in F_{Q_2}$. Analogously we have that if $Q_2 \subset Q_1$, then $x \in F_{Q_1}$ and $x \notin F_{Q_2}$. Further, it is clear that $F_Q \subset Q$ and $\bigcup\limits_{Q \in D_m(\mathbb{R}^n)} F_Q = \mathbb{R}^n$, where $D_m(\mathbb{R}^n) = \big\{ Q : Q \in D(\mathbb{R}^n), F_Q \neq \emptyset \big\}$.

Suppose that $\|f\|_{L^p_w(\mathbb{R}^n)} \leq 1$ and that r is a number satisfying the condition $p < r < q_-$. We have

$$\big\| M^{+,(d)}_{\alpha(\cdot)} f \big\|^r_{L^{q(\cdot)}_v(\mathbb{R}^n)} = \big\| v^r \big(M^{+,(d)}_{\alpha(\cdot)} f \big)^r \big\|_{L^{\frac{q(\cdot)}{r}}(\mathbb{R}^n)} = \sup \int\limits_{\mathbb{R}^n} v^r(x) \Big(M^{+,(d)}_{\alpha(\cdot)} f \Big)^r (x) h(x) dx,$$

where the supremum is taken over all functions h, $\|h\|_{L^{\left(\frac{q(\cdot)}{r}\right)'}(\mathbb{R}^n)} \leq 1$. Now for such an h, using Lemma 1.2.10, we have that

$$\begin{aligned} \int\limits_{\mathbb{R}^n} v^r(x) \Big(M^{+,(d)}_{\alpha(\cdot)} f \Big)^r (x) h(x) dx &= \sum_{Q \in D_m(\mathbb{R}^n)} \int\limits_{F_Q} v^r(x) \Big(M^{+,(d)}_{\alpha(x)} f \Big)^r (x) h(x) dx \\ &\leq C \sum_{Q \in D_m(\mathbb{R}^n)} \Big(\int\limits_{F_Q} v^r(x) |Q|^{(\frac{\alpha(x)}{n} - 1) r} h(x) dx \Big) \Big(\int\limits_{Q^+} f(y) dy \Big)^r \\ &\leq C \sum_{Q \in D_m(\mathbb{R}^n)} \big\| v^r(\cdot) |Q|^{(\frac{\alpha(\cdot)}{n} - 1) r} \chi_Q(\cdot) \big\|_{L^{\frac{q(\cdot)}{r}}(\mathbb{R}^n)} \big\| h \big\|_{L^{\left(\frac{q(\cdot)}{r}\right)'}(\mathbb{R}^n)} \Big(\int\limits_{Q^+} f(y) dy \Big)^r \end{aligned}$$

$$= C \sum_{Q\in D_m(\mathbb{R}^n)} \left\| v(\cdot)\ |Q|^{\frac{\alpha(\cdot)}{n}-1}\chi_Q(\cdot)\right\|^r_{L^{q(\cdot)}(\mathbb{R}^n)} \|h\|_{L^{\left(\frac{q(\cdot)}{r}\right)'}(\mathbb{R}^n)} \Big(\int_{Q^+} f(y)dy\Big)^r$$

$$\leq C\ A^r \sum_{Q\in D_m(\mathbb{R}^n)} \Big(\int_{Q^+} w^{-p'}(y)dy\Big)^{-\frac{r}{p}} \Big(\int_{Q^+} f(y)dy\Big)^r \leq CA^r\|f\|^r_{L^p_w(\mathbb{R}^n)}.$$

In the last inequality we used also the fact that $Q^+ \in D(\mathbb{R}^n)$ if and only if $Q \in D(\mathbb{R}^n)$. Let us pass now to an arbitrary f, where $f \in L^p_w(\mathbb{R}^n)$. For such an f we take the sequence $f_m = f\chi_{Q(0,k_m)}\chi_{\{f<j_m\}}$, where

$$Q(0,k_m) := \{(x_1,\cdots,x_n) : |x_i| < k_m,\ i = 1,\cdots,n\}.$$

and $k_m, j_m \to \infty$ as $m \to \infty$. Then it is easy to see that $f_m \to f$ in $L^p_w(\mathbb{R}^n)$ and also pointwise. Moreover, $f_m(x) \leq f(x)$. On the other hand, $\big\{M^{+,(d)}_{\alpha(\cdot)} f_m\big\}$ is a Cauchy sequence in $L^{q(\cdot)}_v(\mathbb{R}^n)$, because

$$\big\|M_{\alpha(\cdot)}f_m - M_{\alpha(\cdot)}f_j\big\|_{L^{q(\cdot)}_v(\mathbb{R}^n)} \leq \big\|M_{\alpha(\cdot)}\big(f_m - f_j\big)\big\|_{L^{q(\cdot)}_v(\mathbb{R}^n)} \leq\ C\big\|f_m - f_j\big\|_{L^p_w(\mathbb{R}^n)}.$$

Since $L^{q(\cdot)}_v(\mathbb{R}^n)$ is a Banach space, there exists $g \in L^{q(\cdot)}_v(\mathbb{R}^n)$ such that

$$\big\|\big(M_\alpha f_m\big) - g\big\|_{L^{q(\cdot)}_v(\mathbb{R}^n)} \to 0.$$

Taking Proposition 1.2.1 into account we can conclude that there is a subsequence $M_{\alpha(\cdot)}f_{m_k}$ which converges to g in $L^{q(\cdot)}_v(\mathbb{R}^n)$ and also almost everywhere. But f_{m_k} converges to f in $L^p_w(\mathbb{R}^n)$ and almost everywhere. Consequently,

$$\|g\|_{L^{q(\cdot)}_v(\mathbb{R}^n)} \leq C\|f\|_{L^p_w(\mathbb{R}^n)}, \tag{1.5.4}$$

where the positive constant C does not depend on f. Now observe that since f_{m_k} is non-decreasing, for fixed $x \in Q,\ Q \in D(\mathbb{R}^n)$, we have that

$$\begin{aligned} |Q|^{\frac{\alpha(x)}{n}-1}\int_{Q^+} f(y)dy &= \lim_{k\to\infty} |Q|^{\frac{\alpha(x)}{n}-1}\int_{Q^+} f_{m_k}(y)dy \\ &\leq \lim_{k\to\infty}\sup_{x\in Q} |Q|^{\frac{\alpha(x)}{n}-1}\int_{Q^+} f_{m_k}(y)dy = \lim_{k\to\infty}\Big(M^{+,(d)}_{\alpha(\cdot)} f_{m_k}\Big)(x) \end{aligned}$$

and the last limit exists because it converges to g almost everywhere. Hence,

$$\left(M^{+,(d)}_{\alpha(\cdot)}f\right)(x) \le \lim_{k\to\infty}\left(M^{+,(d)}_{\alpha(\cdot)}f_{m_k}\right)(x) = g(x),$$

for almost every x. Finally, (1.5.4) yields

$$\|M^{+,(d)}_{\alpha(\cdot)}f\|_{L^{q(\cdot)}_v(\mathbb{R}^n)} \le C\|f\|_{L^p_w(\mathbb{R}^n)}.$$

□

The proof of the next statement is similar to that of Theorem 1.5.1; therefore it is omitted.

Theorem 1.5.2. *Let $1 < p < q_- \le q_+ < \infty$, $0 < \alpha_- \le \alpha_+ < n$, where p is constant and q, α are measurable functions on $\mathbb{R}^n$. Suppose that $w^{-p'} \in RD^{(d)}(\mathbb{R}^n)$. Then $M^{-,(d)}_{\alpha(\cdot)}$ is bounded from $L^p_w(\mathbb{R}^n)$ to $L^{q(\cdot)}_v(\mathbb{R}^n)$ if and only if*

$$\sup_{Q, Q\in D(\mathbb{R}^n)} \left\|\chi_Q(\cdot)|Q|^{\frac{\alpha(\cdot)}{n}-1}v(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|w^{-1}(\cdot)\chi_{Q^-}(\cdot)\right\|_{L^{p'}(\mathbb{R}^n)} < \infty.$$

Let us now consider the case when $p \equiv q \equiv$ const.

Theorem 1.5.3. *Let $1 < p < \infty$, where p is constant. Suppose that $0 < \alpha_- \le \alpha_+ < n$. Then $M^{+,(d)}_{\alpha(\cdot)}$ is bounded from $L^p_w(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$ if and only if*

$$\int_{\mathbb{R}^n} v^p(x)\left(M^{+,(d)}_{\alpha(\cdot)}\left(w^{-p'}\chi_Q\right)(x)\right)^p dx \le C\int_Q w^{-p'}(x)dx < \infty,$$

for all dyadic cubes $Q \subset \mathbb{R}^n$.

Proof. Sufficiency. It is enough to show that the inequality

$$\left\|v\, M^{+,(d)}_{\alpha(\cdot),u}\, f\right\|_{L^p(\mathbb{R}^n)} \le C\left\|u^{\frac{1}{p}}\, f\right\|_{L^p(\mathbb{R}^n)} \tag{1.5.5}$$

holds if for all $Q \in D(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} v^p(x)\left(M^{+,(d)}_{\alpha(\cdot),u}\,\chi_Q\right)^p(x)\, dx \le C\int_Q |f(x)|^p u(x)\, dx,$$

where

$$\left(M^{+,(d)}_{\alpha(\cdot),u}\, f\right)(x) = M^{+,(d)}_{\alpha(\cdot)}\ \big(fu\big)(x).$$

To prove (1.5.5) we argue in the same manner as in the proof of Theorem 1.5.1. Let us construct the set F_Q for $Q \in D(\mathbb{R}^n)$. We have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} v^p(x)\left(M^{+,(d)}_{\alpha(\cdot),u}\right)^p(x)\,dx \\
&\quad \le\ 2^p \sum_{Q\in D_m} \int\limits_{F_Q} v^p(x)\bigg(\frac{1}{|Q|^{1-\frac{\alpha(x)}{n}}}\int\limits_{Q^+} f(y)u(y)dy\bigg)^p dx \\
&\quad =\ C \sum_{Q\in D_m} \bigg(\int\limits_{F_Q} v^p(x)\ |Q|^{\left(\frac{\alpha(x)}{n}-1\right)p}\,dx\bigg)\bigg(\int\limits_{Q^+} f(y)u(y)dy\bigg)^p \\
&\quad =\ C \sum_{Q\in D_m} \bigg(\int\limits_{F_Q} v^p(x)\ |Q|^{\left(\frac{\alpha(x)}{n}-1\right)p}\,dx\bigg)\big(u(Q^+)\big)^p\bigg(\frac{1}{u(Q^+)}\int\limits_{Q^+} f(y)u(y)dy\bigg)^p.
\end{aligned}$$

Taking Lemma 1.2.11 into account it is enough to show that

$$S := \sum_{\substack{j:\ Q_j\subset Q\\ F_{Q_j^-}\neq\emptyset\\ Q_j\in D(\mathbb{R}^n)}} \int\limits_{F_{Q_j^-}} v^p(x)\bigg(\big|Q_j^-\big|^{\frac{\alpha(x)}{n}-1}\int\limits_{Q_j} u(y)dy\bigg)^p dx \le c\int\limits_Q u(y)dy.$$

Indeed, we have

$$\begin{aligned}
S\ &\le \sum_{\substack{j:\ Q_j\subset Q\\ F_{Q_j^-}\neq\emptyset\\ Q_j\in D(\mathbb{R}^n)}} \int\limits_{F_{Q_j^-}} v^p(x)\big(M^{+,(d)}\big(u\,\chi_Q\big)(x)\big)^p dx \\
&= \int\limits_{\bigcup_{Q_j\subset Q} F_{Q_j^-}} v^p(x)\big(M^{+,(d)}\big(u\,\chi_Q\big)(x)\big)^p dx \\
&\le \int\limits_{\mathbb{R}^n} v^p(x)\big(M^{+,(d)}\big(u\,\chi_Q\big)(x)\big)^p dx \le C\int\limits_Q u(y)dy.
\end{aligned}$$

Necessity. Taking the test function $f_Q = \chi_Q w^{-p'}$ in the two-weight inequality

$$\left\| v\ \left(M^{+,(d)}_{\alpha(\cdot)}\ f\right)\right\|_{L^p(\mathbb{R}^n)} \le C\ \left\| f\ w\right\|_{L^p(\mathbb{R}^n)}$$

and observing that $\int\limits_Q w^{-p'}(y)dy < \infty$ for every $Q \in D(\mathbb{R}^n)$ we have the desired result. $\square$

The proof of the next statement is similar to that of the previous theorem. The proof is omitted.

Theorem 1.5.4. *Suppose that $1 < p < \infty$, where p is constant. Then $M^{-,(d)}_{\alpha(\cdot)}$ is bounded from $L^p_w(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$ if and only if there is a positive constant C such that for all $Q \in D(\mathbb{R}^n)$,*

$$\int\limits_{\mathbb{R}^n} v^p(x)\Big(M^{-,(d)}_{\alpha(\cdot)}\Big(w^{-p'}\chi_Q\Big)\Big)^p(x)dx \le C\int\limits_Q w^{-p'}(x)dx < \infty.$$

Let us now discuss the two–weight problem for the one-sided maximal functions $M^+_{\alpha(\cdot)}$, $M^-_{\alpha(\cdot)}$ defined on $\mathbb{R}$.
Recall that by $M^{+,(d)}_{\alpha(\cdot)}$ and $M^{-,(d)}_{\alpha(\cdot)}$ we denote the one-sided dyadic maximal functions. Now we assume that they are defined on $\mathbb{R}$.
Together with these operators we need the following maximal operators:

$$\Big(\bar{M}^+_{\alpha(\cdot)}f\Big)(x) = \sup_{h>0}\ \frac{1}{(h/2)^{1-\alpha(x)}}\int\limits_{x+\frac{h}{2}}^{x+h}|f(y)|dy;$$

$$\Big(\bar{M}^-_{\alpha(\cdot)}f\Big)(x) = \sup_{h>0}\ \frac{1}{(h/2)^{1-\alpha(x)}}\int\limits_{x-h}^{x-\frac{h}{2}}|f(y)|dy;$$

$$\Big(\widetilde{M}^+_{\alpha(\cdot)}f\Big)(x) = \sup_{j\in\mathbb{Z}}\frac{1}{2^{(j-1)(1-\alpha(x))}}\int\limits_{x+2^{j-1}}^{x+2^j}|f(y)|dy.$$

To prove the next statements we need some lemmas.

Lemma 1.5.5. . *Let $f \in L_{loc}(\mathbb{R})$. Then the following pointwise estimates hold:*

$$\big(M^+_{\alpha(\cdot)}f\big)(x) \le \frac{2^{\alpha_+-1}}{1-2^{\alpha_+-1}}\big(\bar{M}^+_{\alpha(\cdot)}f\big)(x);$$

$$\big(M^{-}_{\alpha(\cdot)}f\big)(x) \le \frac{2^{\alpha_+-1}}{1-2^{\alpha_+-1}}\big(\bar{M}^{-}_{\alpha(\cdot)}f\big)(x) \tag{1.5.6}$$

for every $x \in \mathbb{R}$.

Proof. Observe that

$$\frac{1}{h^{1-\alpha(x)}}\int\limits_x^{x+h}|f(t)|dt = \frac{1}{h^{1-\alpha(x)}}\int\limits_x^{x+\frac{h}{2}}|f(t)|dt + \frac{1}{h^{1-\alpha(x)}}\int\limits_{x+\frac{h}{2}}^{x+h}|f(t)|dt$$

$$= 2^{\alpha(x)-1}\frac{1}{(h/2)^{1-\alpha(x)}}\int\limits_x^{x+\frac{h}{2}}|f(t)|dt + 2^{\alpha(x)-1}\frac{1}{(h/2)^{1-\alpha(x)}}\int\limits_{x+\frac{h}{2}}^{x+h}|f(t)|dt$$

$$\le 2^{\alpha(x)-1}\big(M^{+}_{\alpha(\cdot)}f\big)(x) + 2^{\alpha(x)-1}\big(\bar{M}^{+}_{\alpha(\cdot)}f\big)(x).$$

Hence,

$$\big(M^{+}_{\alpha(\cdot)}f\big)(x) \le 2^{\alpha(x)-1}\big(M^{+}_{\alpha(\cdot)}f\big)(x) + 2^{\alpha(x)-1}\big(\bar{M}^{+}_{\alpha(\cdot)}f\big)(x).$$

Consequently,

$$\big(1-2^{\alpha(x)-1}\big)\big(M^{+}_{\alpha(\cdot)}f\big)(x) \le 2^{\alpha(x)-1}\big(\bar{M}^{+}_{\alpha(\cdot)}f\big)(x),$$

which implies

$$\big(M^{+}_{\alpha(\cdot)}f\big)(x) \le \frac{2^{\alpha(x)-1}}{1-2^{\alpha(x)-1}}\big(\bar{M}^{+}_{\alpha(\cdot)}f\big)(x) \le \frac{2^{\alpha_+-1}}{1-2^{\alpha_+-1}}\big(\bar{M}^{+}_{\alpha(\cdot)}f\big)(x).$$

Analogously the inequality (1.5.6) follows. □

Lemma 1.5.6. *The following inequality*

$$\big(\bar{M}^{+}_{\alpha(\cdot)}f\big)(x) \le C\big(\widetilde{M}^{+}_{\alpha(\cdot)}f\big)(x) \tag{1.5.7}$$

holds with a positive constant C *independent of* f *and* x.

Proof. Let us take $h > 0$. Then $h \in [2^{j-1}, 2^j)$ for some $j \in \mathbb{Z}$. Consequently,

$$\frac{1}{(h/2)^{1-\alpha(x)}} \int\limits_{x+h}^{x+\frac{h}{2}} |f(t)|dt \le \frac{1}{(2^{j-2})^{1-\alpha(x)}} \int\limits_{x+2^{j-2}}^{x+2^j} |f(t)|dt$$

$$= \frac{1}{2^{(j-2)1-\alpha(x)}} \int\limits_{x+2^{j-2}}^{x+2^{j-1}} |f(t)|dt + \frac{1}{2^{(j-2)(1-\alpha(x))}} \int\limits_{x+2^{j-1}}^{x+2^j} |f(t)|dt$$

$$= \frac{1}{2^{(j-2)(1-\alpha(x))}} \int\limits_{x+2^{j-2}}^{x+2^{j-1}} |f(t)|dt + \frac{2^{\alpha(x)-1}}{2^{(j-1)(1-\alpha(x))}} \int\limits_{x+2^{j-1}}^{x+2^j} |f(t)|dt$$

$$\le \big(\widetilde{M}^+_{\alpha(\cdot)} f\big)(x) + 2^{\alpha_+-1}\big(\widetilde{M}^+_{\alpha(\cdot)} f\big)(x) = \big(1 + 2^{\alpha_+-1}\big)\big(\widetilde{M}^+_{\alpha(\cdot)} f\big)(x).$$

Hence, (1.5.7) holds for $C = 1 + 2^{\alpha_+-1}$. □

Lemma 1.5.7. *There exists a positive constant C depending only on α such that for all f, $f \in L_{loc}(\mathbb{R})$, and $x \in \mathbb{R}$,*

$$\big(\widetilde{M}^+_{\alpha(\cdot)} f\big)(x) \le C\big(M^{+,(d)}_{\alpha(\cdot)} f\big)(x). \tag{1.5.8}$$

Proof. Let $h = 2^j$ for some integer j. Suppose that I and I' are dyadic intervals such that $I \bigcup I'$ is again dyadic, $|I| = |I'| = 2^{j-1}$ and $[x + \frac{h}{2}, x + h) \subset (I \bigcup I')$. Then $x \in (I \bigcup I')^-$, where $(I \bigcup I')^-$ is dyadic and

$$\int\limits_{x+\frac{h}{2}}^{x+h} |f(t)|dt \le \int\limits_{I \bigcup I'} |f(t)|dt \le 2^{j(1-\alpha(x))}\big(M^{+,(d)}_{\alpha(\cdot)} f\big)(x),$$

whence

$$\big(\widetilde{M}^+_{\alpha(\cdot)} f\big)(x) \le 2^{1-\alpha_-}\big(M^{+,(d)}_{\alpha(\cdot)} f\big)(x).$$

If $I \bigcup I'$ is not dyadic, then we take $I_1 \in D(\mathbb{R})$ with length 2^j containing I'. Consequently, $x \in (I_1)^-$, where I_1^- is dyadic. Observe that $x \in I^-$, where I^- is also dyadic.

Consequently,

$$\int_{x+\frac{h}{2}}^{x+h} |f(t)|dt \le \int_{I\bigcup I_1} |f(t)|dt = \int_I |f(t)|dt + \int_{I_1} |f(t)|dt \le C\ h^{1-\alpha(x)}\big(M^{+,(d)}_{\alpha(\cdot)} f\big)(x),$$

with positive constant C independent of j. Finally, we have (1.5.8). □

Lemma 1.5.8. *There exists a positive constant C depending only on α such that*

$$\big(M^{+,(d)}_{\alpha(\cdot)} f\big)(x) \le C\big(M^{+}_{\alpha(\cdot)} f\big)(x) \tag{1.5.9}$$

for all f, $f \in L_{loc}(\mathbb{R})$, $x \in \mathbb{R}$.

Proof. Let $x \in I$, $I \in D(\mathbb{R})$. Denote $I = [a, b)$. Then $I^+ = [b, 2b-a)$. Let $h = 2b-a-x$. We have

$$\begin{aligned} \frac{1}{|I|^{1-\alpha(x)}} \int_{I^+} |f(t)|dt &\le \frac{2^{1-\alpha(x)}}{|I\bigcup I^+|^{1-\alpha(x)}} \int_x^{x+h} |f(t)|dt \\ &\le 2^{1-\alpha_-} \frac{1}{h^{1-\alpha(x)}} \int_x^{x+h} |f(t)|dt \le 2^{1-\alpha_-} M^{+}_{\alpha(\cdot)} f(x). \end{aligned}$$

Since I is an arbitrary dyadic cube containing x, then (1.5.9) holds for $C = 2^{1-\alpha_-}$. □

Summarizing Lemmas 1.5.5 − 1.5.8, we have the next statement:

Proposition 1.5.9. *There exists positive constants C_1 and C_2 such that for all f, $f \in L_{loc}(\mathbb{R})$ and $x \in \mathbb{R}$ the two-sided inequality*

$$C_1\big(M^{+}_{\alpha(\cdot)} f\big)(x) \le \big(M^{+,(d)}_{\alpha(\cdot)} f\big)(x) \le C_2\big(M^{+}_{\alpha(\cdot)} f\big)(x)$$

holds

Now Theorem1.5.1 (for $n = 1$) and Proposition 1.5.9 yield the following theorem:

Theorem 1.5.10. *Let p, q and α be measurable functions on $I = \mathbb{R}$, $1 < p_- < q_- \le q_+ < \infty$, $0 < \alpha_- \le \alpha_+ < 1$. Suppose also that $p \in \mathcal{G}(I)$. Further, assume that $w^{-(p_-)'} \in RD^{(d)}(I)$. Then $M^+_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}_w(I)$ to $L^{q(\cdot)}_v(I)$ if*

$$B := \sup_{\substack{a\in\mathbb{R}\\ h>0}} \left\|\chi_{(a-h,a)}(\cdot)\ h^{\alpha(\cdot)-1}\right\|_{L^{q(\cdot)}_v(\mathbb{R})} \left\|\chi_{(a,a+h)} w^{-1}\right\|_{L^{(p_-)'}(\mathbb{R})} < \infty.$$

Proof. By using Theorem 1.5.1 we have that the condition $B < \infty$ implies

$$\|M^{+,(d)}_{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\mathbb{R})} \le C\|fw\|_{L^{p_-}(\mathbb{R})}$$

Now Propositions 1.2.5 and 1.5.9 complete the proof. □

Analogously the next statement can be proved:

Theorem 1.5.11. *Let p, q and α be measurable functions on $I := \mathbb{R}$, $1 < p_- < q_- \le q_+ < \infty$, $0 < \alpha_- \le \alpha_+ < 1$. Suppose also that $p \in \mathcal{G}(I)$ and that $w^{-(p_-)'} \in RD^{(d)}(I)$. Then $M^-_{\alpha(\cdot)}$ is bounded from $L^p_w(I)$ to $L^{q(\cdot)}_v(I)$ if*

$$B_1 := \sup_{\substack{a\in I\\ h>0}} \left\|\chi_{(a,a+h)}(\cdot) h^{\alpha(\cdot)-1} v(\cdot)\right\|_{L^{q(\cdot)}(I)} \left\|\chi_{(a-h,a)} w^{-1}\right\|_{L^{(p_-)'}(I)} < \infty.$$

The results of this section imply the following corollaries:

Corollary 1.5.12. *Let $I := \mathbb{R}$ and $1 < p < q_- \le q_+ < \infty$, $0 < \alpha_- \le \alpha_+ < 1$, where p is constant. Assume that $w^{-p'} \in RD^{(d)}(\mathbb{R})$. Then $M^+_{\alpha(\cdot)}$ is bounded from $L^p_w(I)$ to $L^{q(\cdot)}_v(I)$ if and only if*

$$\sup_{\substack{a\in I\\ h>0}} \left\|\chi_{(a-h,a)}(\cdot)\ h^{\alpha(\cdot)-1}\right\|_{L^{q(\cdot)}_v(I)} \left\|\chi_{(a,a+h)} w^{-1}\right\|_{L^{p'}(I)} < \infty.$$

Corollary 1.5.13. *Let $I := \mathbb{R}$ and let $1 < p < q_- \le q_+ < \infty$, where p is constant. Suppose that α is a measurable function on $\mathbb{R}$ satisfying $0 < \alpha_- \le \alpha_+ < 1$. Suppose* *also that* $w^{-(p_-)'} \in RD^{(d)}(I)$*. Then* $M^-_{\alpha(\cdot)}$ *is bounded from from* $L^p_w(I)$ *to* $L^{q(\cdot)}_v(I)$ *if and only if*

$$\sup_{\substack{a\in I\\ h>0}} \left\|\chi_{(a,a+h)}(\cdot)h^{\alpha(\cdot)-1}v(\cdot)\right\|_{L^{q(\cdot)}(I)}\left\|\chi_{(a-h,a)}w^{-1}\right\|_{L^{p'}(I)} < \infty.$$

Corollary 1.5.14. *Let* $I = \mathbb{R}$, $1 < p_- < q_- \le q_+ < \infty$, $0 < \alpha_- \le \alpha_+ < 1$*. Suppose that* $p_- = p(\infty)$ *and* $p \in \mathcal{P}_\infty(I)$*. Assume that* $w^{-(p_-)'} \in RD^{(d)}(\mathbb{R})$*. Then:*

(i) $M^+_{\alpha(\cdot)}$ *is bounded from* $L^p_w(I)$ *to* $L^{q(\cdot)}_v(I)$ *if* $B < \infty$*;*

(ii) $M^-_{\alpha(\cdot)}$ *is bounded from* $L^p_w(I)$ *to* $L^{q(\cdot)}_v(I)$ *if* $B_1 < \infty$*.*

Proof of Corollary 1.5.12. *Sufficiency* is a direct consequence of Theorem 1.5.10. *Necessity* follows immediately by applying the two-weight inequality for the test function $f(x) = \chi_{(a,a+h)}(x)w^{-p'}(x)$ (see also necessity of the proof of Theorem 1.5.1 for the details). □

The proof of Corollary 1.5.13 is similar to that of Corollary 1.5.12.

Proof of Corollary 1.5.14. (i) The result follows from Theorem 1.5.10 because the condition $p \in \mathcal{P}_\infty(I)$ implies that

$$\int_I K^{p(x)p(\infty)/|p(x)-p(\infty)|}dx < \infty.$$

Hence, by using the assumption $p(\infty) = p_-$ we have that $p \in \mathcal{G}(I)$.
The second part of the corollary is obtained in a similar manner; therefore it is omitted. □

The next statement gives the boundedness of $M^+_{\alpha(\cdot)}$ in the diagonal case $p \equiv q \equiv$ *const*.

Theorem 1.5.15. *Let* $I := \mathbb{R}$ *and let* $1 < p < \infty$*, where* p *is constant. Suppose that* $0 < \alpha_- \le \alpha_+ < \infty$*. Then* $M^+_{\alpha(\cdot)}$ *is bounded from* $L^p_w(I)$ *to* $L^p_v(I)$ *if and only if there* *is a positive constant C such that for all bounded intervals $J \subset \mathbb{R}$,*

$$\int_{\mathbb{R}} v^p(x)\Big(M^+_{\alpha(\cdot)}\Big(w^{-p'}\chi_J\Big)(x)\Big)^p dx \le C \int_J w^{-p'}(x)dx < \infty.$$

Proof. Sufficiency follows from Proposition 1.5.9 and Theorem 1.5.3 for $n = 1$.

Necessity. For necessity we take $f = \chi_J\, w^{p'}$ in the two weight inequality

$$\big\|v\; M^+_{\alpha(\cdot)}\; f\big\|_{L^p_v(I)} \le C\; \big\|w\; f\big\|_{L^p_v(I)}$$

and we are done. □

Analogously the following theorem follows:

Theorem 1.5.16. *Let $I := \mathbb{R}$ and let $1 < p < \infty$, where p is constant. Suppose that $0 < \alpha_- \le \alpha_+ < \infty$.. Then $M^-_{\alpha(\cdot)}$ is bounded from $L^p_w(I)$ to $L^p_v(I)$ if and only if*

$$\int_{\mathbb{R}} v^p(x)\Big(M^-_{\alpha(\cdot)}\Big(w^{-p'}\chi_J\Big)(x)\Big)^p dx \le C \int_J w^{-p'}(x)dx < \infty$$

for all bounded intervals $J \subset \mathbb{R}$.

Finally we mention that the results similar to those of this section were derived in [40] for generalized two-sided fractional maximal functions and Riesz potentials.

1.6 Fefferman-Stein Type Inequality

In this section we derive Fefferman-Stein type inequality for the operators $M^-_{\alpha(\cdot)}$, $M^+_{\alpha(\cdot)}$. Notice that this inequality for the classical Riesz potentials for the diagonal case was established by E. Sawyer (see, e.g., [85]).

Theorem 1.6.1. *Let α, p and q be measurable functions on $I = \mathbb{R}$. Suppose that $1 < p_- < q_- \le q_+ < \infty$ and $0 < \alpha_- \le \alpha_+ < 1/p_-$. Suppose that $p \in \mathcal{G}(I)$. Then the following inequalities hold:*

$$\|v(\cdot)(M^+_{\alpha(\cdot)}f)(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})} \le c\|f(\cdot)(\widetilde{N}^-_{\alpha(\cdot)}v)(\cdot)\|_{L^{p(\cdot)}(\mathbb{R})}; \tag{1.6.1}$$

$$\|v(\cdot)(M^{-}_{\alpha(\cdot)}f)(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})} \leq c\|f(\cdot)(\widetilde{N}^{+}_{\alpha(\cdot)}v)(\cdot)\|_{L^{p(\cdot)}(\mathbb{R})}, \tag{1.6.2}$$

where

$$\begin{aligned}
\big(\widetilde{N}^{-}_{\alpha(\cdot)}v\big)(x) &= \sup_{h>0} h^{-1/p_-}\|v(\cdot)h^{\alpha(\cdot)}\chi_{(x-h,x)}(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})},\\
\big(\widetilde{N}^{+}_{\alpha(\cdot)}v\big)(x) &= \sup_{h>0} h^{-1/p_-}\|v(\cdot)h^{\alpha(\cdot)}\chi_{(x,x+h)}(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})}.
\end{aligned}$$

Proof. We prove (1.6.1). The proof of (1.6.2) is the same. First we show that the inequality

$$\|v(\cdot)(M^{+,(d)}_{\alpha(\cdot)}f)(\cdot)\|_{L^{q(\cdot)}(\mathbb{R})} \leq c\|f(\cdot)(\widetilde{N}^{-}_{\alpha(\cdot)}v)(\cdot)\|_{L^{p(\cdot)}(\mathbb{R})}$$

holds. We repeat the arguments of the proof of Theorem 1.5.1 for one-dimensional dyadic intervals J and construct the sets F_J. Taking h, $\|h\|_{L^{(q(\cdot)/r)'}(\mathbb{R})} \leq 1$, where $p_- < r < q_-$, by using Lemma 1.2.10 and Proposition 1.2.5 we have that

$$\begin{aligned}
\int\limits_{\mathbb{R}} v^r(x)\big(M^{+,(d)}_{\alpha()}f(x)\big)^r h(x)dx &= \sum_{J\in D_m(\mathbb{R})}\int_{F_J} v(x)^r\big(M^{+,(d)}_{\alpha(\cdot)}f\big)^r(x)h(x)dx\\
&\leq c\sum_{J\in D_m(\mathbb{R})}\bigg(\int_{F_J} v^r(x)|J|^{(\alpha(x)-1)r}h(x)dx\bigg)\bigg(\int_{J^+} f(t)dt\bigg)^r\\
&\leq c\sum_{J\in D_m(\mathbb{R})}\Big\|v^r(\cdot)|J|^{(\alpha(\cdot)-1)r}h(\cdot)\chi_{F_J}(\cdot)\Big\|_{L^{q(\cdot)/r}(\mathbb{R})}\Big\|h\Big\|_{L^{(q(\cdot)/r)'}(\mathbb{R})}\bigg(\int_{J^+} f(t)dt\bigg)^r\\
&\leq c\sum_{J\in D_m(\mathbb{R})}\Big\|v^r(\cdot)|J|^{(\alpha(\cdot)-1)r}\chi_{F_J}(\cdot)\Big\|_{L^{q(\cdot)/r}(\mathbb{R})}\bigg(\int_{J^+} f(t)dt\bigg)^r\\
&= c\sum_{J\in D_m(\mathbb{R})}\bigg(\int\limits_{J^+} f(x)\Big\|v(\cdot)|J|^{\alpha(\cdot)-1}\chi_{F_J}(\cdot)\Big\|_{L^{q(\cdot)}(\mathbb{R})}dx\bigg)^r\\
&= c\sum_{J\in D_m(\mathbb{R})}|J|^{-r/(p_-)'}\bigg(\int_{J^+} f(x)\Big\|v(\cdot)|J|^{\alpha(\cdot)-1/p_-}\chi_{F_J}(\cdot)\Big\|_{L^{q(\cdot)}(\mathbb{R})}dx\bigg)^r\\
&\leq c\sum_{J\in D_m(\mathbb{R})}|J|^{-r/(p_-)'}\bigg(\int_{J^+} f(x)\big(\widetilde{N}^{-}_{\alpha(\cdot)}v\big)(x)dx\bigg)^r\\
&\leq c\|f(\cdot)\big(\widetilde{N}^{-}_{\alpha(\cdot)}v\big)(\cdot)\|^r_{L^{p_-}(\mathbb{R})} \leq c\|f(\cdot)\widetilde{N}^{-}_{\alpha(\cdot)}v(\cdot)\|^r_{L^{p(\cdot)}(\mathbb{R})}.
\end{aligned}$$

Here we used the inequality

$$\left\| v(\cdot)|J|^{\alpha(\cdot)-1/p_-}\chi_{F_J}(\cdot)\right\|_{L^{q(\cdot)}(\mathbb{R})} \leq C_{\alpha,p}\big(\widetilde{N}^-_{\alpha(\cdot)}v\big)(x), \quad x \in J_+,$$

which follows in the same manner as Lemma 1.5.8 was proved. Now Proposition 1.5.9 completes the proof. □

1.7 The Trace Inequality for One-sided Potentials

Let

$$R_{\alpha(\cdot)}f(x) = \int_{-\infty}^{x} \frac{f(t)}{(x-t)^{1-\alpha(x)}}dt; \qquad x \in \mathbb{R},$$

$$W_{\alpha(\cdot)}f(x) = \int_{x}^{\infty} \frac{f(t)}{(t-x)^{1-\alpha(x)}}dt; \qquad x \in \mathbb{R},$$

where α is a measurable function on $\mathbb{R}$ with $0 < \alpha_- \leq \alpha_+ < 1$. Here we establish criteria which guarantees the boundedness of $R_{\alpha(\cdot)}$ and $W_{\alpha(\cdot)}$ from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}_v(I)$.

It would be useful to have the next result.

Theorem 1.7.1 ([**40**])**.** *Suppose that $1 < p < q_- \leq q_+ < \infty$, where p is constant, and q is a measurable function on $\mathbb{R}$. Let $0 < \alpha_- \leq \alpha_+ < 1$. Then the generalized Riesz potential*

$$T_{\alpha(\cdot)}f(x) = \int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\alpha(x)}}dy, \qquad x \in \mathbb{R},$$

is bounded from $L^p(\mathbb{R})$ to $L^{q(\cdot)}_v(\mathbb{R})$ if and only if

$$\sup_{J\subset\mathbb{R}} \left\|\chi_J(\cdot)\ |J|^{\alpha(\cdot)}\right\|_{L^{q(\cdot)}_v(\mathbb{R})}|J|^{-\frac{1}{p}} < \infty, \tag{1.7.1}$$

where the supremum is taken over all bounded intervals $J \subset \mathbb{R}$.

Now we discuss the main results of this section:

Theorem 1.7.2. *Let $I := \mathbb{R}$ and let measurable functions p, q, and α satisfy the conditions $1 < p_- < q_- \le q_+ < \infty$, $0 < \alpha_- \le \alpha_+ < 1$. Further, suppose that $p \in \mathcal{G}(I)$.*

If

$$\sup_{J\subset\mathbb{R}} \big\|\chi_J(\cdot)\ |J|^{\alpha(\cdot)}\big\|_{L^{q(\cdot)}_v(\mathbb{R})} |J|^{-\frac{1}{p_-}} < \infty,$$

where the supremum is taken over all bounded intervals $J \subset \mathbb{R}$, then $R_{\alpha(\cdot)}$ and $W_{\alpha(\cdot)}$ are bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}_v(I)$.

Proof. The result is a direct consequence of the inequalities

$$\big(R_{\alpha(\cdot)}f\big)(x) \le \big(T_{\alpha(\cdot)}f\big)(x),\ \ \big(W_{\alpha(\cdot)}f\big)(x) \le \big(T_{\alpha(\cdot)}f\big)(x) \quad (f \ge 0),$$

Theorem 1.7.1 and Proposition 1.2.5. □

Theorem 1.7.3. *Let $I := \mathbb{R}$ and let p, q and α satisfy the conditions of Theorem* 1.7.1*. Then the following conditions are equivalent:*

(i) *$R_{\alpha(\cdot)}$ is bounded from $L^p(I)$ to $L^{q(\cdot)}_v(I)$;*

(ii) *$W_{\alpha(\cdot)}$ is bounded from $L^p(I)$ to $L^{q(\cdot)}_v(I)$;*

(iii) *condition* (1.7.1) *holds.*

Proof. The implications (iii) ⇒ (i), (iii) ⇒ (ii) follow from Theorems 1.7.2 and 1.7.1. Let us now show that (i) ⇒ (iii). Let $f(x) = \chi_{(a,a+h)}(x)$, where $a \in \mathbb{R}$ and $h > 0$. Then $\|f\|_{L^p(\mathbb{R})} = h^{\frac{1}{p}}$. On the other hand,

$$\begin{aligned} \big\|R_{\alpha(\cdot)}f\big\|_{L^{q(\cdot)}_v(\mathbb{R})} &\ge \Big\|\chi_{(a,a+h)}(\cdot)\Big(\int_{a-h}^{a} \frac{dt}{(x-t)^{1-\alpha(x)}}\Big)\Big\|_{L^{q(\cdot)}_v(\mathbb{R})} \\ &\ge C\big\|\chi_{(a,a+h)}(\cdot)h^{\alpha(\cdot)}\big\|_{L^{q(\cdot)}_v(\mathbb{R})}. \end{aligned}$$

Hence, (i) implies that

$$\big\|\chi_{(a,a+h)}(\cdot)h^{\alpha(\cdot)}\big\|_{L^{q(\cdot)}_v(\mathbb{R})} h^{-\frac{1}{p}} \le C$$

for all $a \in \mathbb{R}$ and $h > 0$. This implies (iii). Analogously the implication (ii)⇒(iii) can be derived. □

1.8 Hardy-Littlewood Type Inequalities

The results of the previous section enable us to formulate necessary and sufficient conditions governing the Hardy-Littlewood (see [30]) type inequalities for the one-sided potentials. For these inequalities in the classical Lebesgue spaces we refer also to [82]. In particular, we give necessary and sufficient conditions on q, p and α for which $R_{\alpha(\cdot)}$ and $W_{\alpha(\cdot)}$ are bounded from L^p to $L^{q(\cdot)}$, where p is constant.

Theorem 1.8.1. *Let $I = \mathbb{R}$ and let p, q and α satisfy the conditions of* Theorem 1.7.1. *Then the following conditions are equivalent:*

(i) $R_{\alpha(\cdot)}$ *is bounded from $L^p(I)$ to $L^{q(\cdot)}(I)$;*

(ii) $W_{\alpha(\cdot)}$ *is bounded from $L^p(I)$ to $L^{q(\cdot)}(I)$;*

(iii) $\sup\limits_{J \subset \mathbb{R}} \big\| \chi_J(\cdot)\ |J|^{\alpha(\cdot)} \big\|_{L^{q(\cdot)}(J)} |J|^{-\frac{1}{p}} < \infty,$

where the supremum is taken over all bounded intervals J in $\mathbb{R}$.

1.9 Two-weight Inequalities for Monotonic Weights

This section deals with the two-weight estimates of the one-sided maximal functions and one-sided potentials defined on $\mathbb{R}_+ := [0, \infty)$.

Let us consider the following Hardy-type operators:

$$(T_{v,w} f)(x) = v(x) \int_0^x f(y) w(y) dy, \quad x \in \mathbb{R}_+,$$

and

$$(T'_{v,w} f)(x) = v(x) \int_x^\infty f(y) w(y) dy, \quad x \in \mathbb{R}_+.$$

In the sequel we will use the following notation:

$$v_\alpha(x) := \frac{v(x)}{x^{1-\alpha}}, \quad \widetilde{w}(x) := \frac{1}{w(x)}, \quad \overline{w}(x) := \frac{1}{w(x)x}, \quad \overline{w}_\alpha(x) := \frac{1}{x^{1-\alpha}w(x)}.$$

Let us fix a positive number a and let

$$p_0(x) := p_-([0,x]), \quad \widetilde{p_0}(x) := \begin{cases} p_0(x), & \text{if } x \le a; \\ p_c = \text{const}, & \text{if } x > a, \end{cases}$$

$$p_1(x) := p_-([x,a]); \quad \widetilde{p_1}(x) := \begin{cases} p_1(x), & \text{if } x \le a; \\ p_c = \text{const}, & \text{if } x > a, \end{cases}$$

$$I_k := [2^{k-1}, 2^{k+2}]; \quad k \in \mathbb{Z}, \; E_k = [2^k, 2^{k+1}]; \quad k \in \mathbb{Z},$$

where $(0,x)$ and $[0,x]$ are open and close intervals respectively.

The following two results were obtained in [22]:

Theorem 1.9.1. *Let $1 < \widetilde{p_0}(x) \le p(x) \le p_+ < \infty$, and p is a measurable function on $\mathbb{R}_+$. Suppose that there exists a positive number a such that $p(x) = p_c = const$ when $x > a$. If*

$$\sup_{t>0} \int_t^\infty \Big(v(x)\Big)^{p(x)} \left(\int_0^t w(y)^{(\widetilde{p}_0)'(x)} dy \right)^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} dx < \infty,$$

then $T_{v,w}$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$.

Theorem 1.9.2. *Let $1 < \widetilde{p_1}(x) \le p(x) \le p_+ < \infty$,and p is a measurable function on $\mathbb{R}_+$. Suppose that there exists a positive number a such that $p(x) = p_c = const$, when $x > a$. If*

$$\sup_{t>0} \int_0^t (v(x))^{p(x)} \left(\int_t^\infty w(y)^{(\widetilde{p}_1)'(x)} dy \right)^{\frac{p(x)}{(\widetilde{p}_1)'(x)}} dx < \infty,$$

then $T'_{v,w}$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$.

The next two lemmas will be useful for us.

Lemma 1.9.3 ([**5**])**.** Let $1 \leq p_- \leq p(x) \leq q(x) \leq q_+ < \infty$, $p \in LH(\mathbb{R}_+)$ and let $p(x) = p_c = const$, $q(x) = q_c = const$ when $x > a$ for some positive number a. Then there exist a positive constant c such that

$$\sum_i \| f\chi_{I_i} \|_{L^{p(\cdot)}(\mathbb{R}_+)} \| g\chi_{I_i} \|_{L^{q'(\cdot)}(\mathbb{R}_+)} \leq c \| f \|_{L^{p(\cdot)}(\mathbb{R}_+)} \| g \|_{L^{q'(\cdot)}(\mathbb{R}_+)}$$

for all $f$and g with $f \in L^{p(\cdot)}(\mathbb{R}_+)$ and $g \in L^{q'(\cdot)}(\mathbb{R}_+)$.

Lemma 1.9.4 ([**12**])**.** *Let $p \in LH(\mathbb{R}_+)$. Then there exist a positive constant c such that for all open intervals I in $\mathbb{R}_+$ satifying the condition $| I |> 0$ we have*

$$| I |^{p_-(I) - p_+(I)} \leq c.$$

Now we prove some lemmas.

Lemma 1.9.5. *Let $1 < p_- \leq p_0(x) \leq p(x) \leq p_+ < \infty$, where p is a measurable function on $\mathbb{R}_+$, and let $p(x) \equiv p_c \equiv const$ if $x > a$ for some positive constant a. Suppose that v and w are positive increasing functions on $\mathbb{R}_+$ satisfying the condition*

$$B := \sup_{t>0} \int_t^\infty \left(\frac{v(x)}{x} \right)^{p(x)} \left(\int_0^t w(y)^{-(\widetilde{p}_0)'(x)} dy \right)^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} dx < \infty. \tag{1.9.1}$$

Then $v(4x) \leq cw(x)$ for all $x > 0$, where the positive constant c is independent of x.

Proof. First assume that $0 < t < a$. The fact that $\overline{c} = \overline{\lim\limits_{t \to 0}} \frac{v(4t)}{w(t)} < \infty$ follows from the inequalities:

$$\begin{aligned} &\int_t^\infty \left(\frac{v(x)}{x} \right)^{p(x)} \left(\int_0^t w(y)^{-(\widetilde{p}_0)'(x)} dy \right)^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} dx \\ &\quad \geq \int_{4t}^{8t} \left(\frac{v(4t)}{w(t)} \right)^{p(x)} \cdot t^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} \cdot x^{-p(x)} dx \\ &\quad \geq \left(\frac{v(4t)}{w(t)} \right)^{p_-} \int_{4t}^{8t} t^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} \cdot x^{-p(x)} dx \geq c \left(\frac{v(4t)}{w(t)} \right)^{p_-}, \end{aligned}$$

where the positive constant c is independent of a small positive number t.

Further, suppose that δ is a positive number such that $v(4t) \le (\overline{c}+1)w(t)$ when $t < \delta$. If $\delta < a$, then for all $\delta < t < a$, we have that

$$v(4t) \le v(4a) \le \widetilde{c}w(\delta) \le \widetilde{c}w(t),$$

where $\overline{c}$ depends on v, w and δ. Now it is enough to take $c = \max\{(\overline{c}+1), \overline{c}\}$.

Let now $a \le t < \infty$. Then $p(x) \equiv p_c \equiv$ const for $x > t$ and, consequently,

$$B \;\ge\; \sup_{t>0} \left(\int_t^\infty \left(\frac{v(x)}{x} \right)^{p_c} dx \right) \left(\int_0^t w(x)^{-p_c'} dx \right)^{p_c - 1} \ge c \left(\frac{v(4t)}{w(t)} \right)^{p_c}.$$

The lemma is proved. □

The proof of the next lemma is similar to that of the previous one; therefore we omit it.

Lemma 1.9.6. *Let $1 < p_- \le p_1(x) \le p(x) \le p_+ < \infty$, and let $p(x) \equiv p_c \equiv$ const if $x > a$ for some positive constant a. Suppose that v and w are positive decreasing functions on $\mathbb{R}_+$. If*

$$\widetilde{B} := \sup_{t>0} \int_0^t (v(x))^{p(x)} \left(\int_t^\infty (\overline{w}(y))^{(\widetilde{p}_1)'(x)} \, dy \right)^{\frac{p(x)}{(\widetilde{p}_1)'(x)}} dx < \infty, \tag{1.9.2}$$

then $v(x) \le cw(4x)$, where the positive constant c does not depend on $x > 0$.

Theorem 1.9.7. *Let $1 < p_- \le p_+ < \infty$ and let $p \in LH(\mathbb{R}_+)$. Suppose that $p(x) \equiv p_c \equiv$ const if $\in (a, \infty)$ for some positive number a. Let v and w be weights on $\mathbb{R}_+$ such that*

(a) *$T_{v_0,\widetilde{w}}$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$;*

(b) *there exists a positive constant b such that one of the following two conditions hold:*

(i) $\operatorname*{ess\,sup}_{y \in [\frac{x}{4}, 4x]} v(y) \le bw(x)$ *for almost all* $x \in \mathbb{R}_+$;

(ii) $v(x) \le b \operatorname*{ess\,inf}_{y\in[\frac{x}{4},4x]} w(y)$ *for almost all* $x \in \mathbb{R}_+$.

Then M^- *is bounded from* $L^{p(\cdot)}_w(\mathbb{R}_+)$ *to* $L^{p(\cdot)}_v(\mathbb{R}_+)$.

Proof. Suppose that $\|g\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \le 1$. We have

$$\int_0^\infty \big(M^- f(x)\big) v(x) g(x) dx \le \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(M^- f_{1,k}(x)\big) v(x) g(x) dx$$

$$+ \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(M^- f_{2,k}(x)\big) v(x) g(x) dx + \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(M^- f_{3,k}(x)\big) v(x) g(x) dx =: S_1 + S_2 + S_3,$$

where $f_{1,k} = f \cdot \chi_{[0,2^{k-1}]}$, $f_{2,k} = f \cdot \chi_{[2^{k+1},\infty]}$, $f_{3,k} = f \cdot \chi_{[2^{k-1},2^{k+2}]}$.

If $y \in [0, 2^{k-1})$ and $x \in [2^k, 2^{k+1}]$, then $y < x/2$. Hence $x/2 \le x - y$. Consequently, if $h < x/2$, then for $x \in [2^{k-1}, 2^{k+2}]$, we have

$$\frac{1}{h} \int_{x-h}^{x} |\, f_{1,k}(y) \,|\, dy \quad = \quad \frac{1}{h} \int_{x-h}^{x} |\, f \cdot \chi_{[0,2^{k-1}]} \,|\, dy = 0.$$

Further, if $h > \frac{x}{2}$, then

$$\frac{1}{h} \int_{x-h}^{x} |\, f_{1,k}(y) \,|\, dy \quad = \quad \frac{1}{h} \int_{x-h}^{x} |\, f \cdot \chi_{[0,2^{k-1}]} \,|\, dy \le c\, \frac{1}{x} \int_0^x |\, f(y) \,|\, dy.$$

This yields that

$$M^- f_{1,k}(x) \le c\, \frac{1}{x} \int_0^x |\, f(y) \,|\, dy \quad \text{ for } \; x \in [2^k, 2^{k+1}].$$

Hence, due to the boundedness of $T_{v_0,\widetilde{w}}$ in $L^{p(x)}(\mathbb{R}_+)$ we have that

$$\begin{aligned} S_1 \;&\le\; c \int_0^\infty \left(T_{v_0,1}|f|\right)(x)\; g(x) dx \\ &\le\; c\, \|T_{v_0,1}|f|\; \|_{L^{p(\cdot)}(\mathbb{R}_+)} \cdot \|g\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \le c\, \|fw\|_{L^{p(\cdot)}(\mathbb{R}_+)}. \end{aligned}$$

Observe now that $S_2 = 0$ because $f_{2,k} = f \cdot \chi_{[2^{k+2},\infty]}$. Let us estimate S_3. By using condition (i) of (b), boundedness of the operator M^- in $L^{p(\cdot)}(\mathbb{R}_+)$ and Lemma 1.9.3

we have that

$$\begin{aligned} S_3 &\le c\sum_k (\operatorname*{ess\,sup}_{E_k} v)\|M^- f_{3,k}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)} \cdot \|g(\cdot)\chi_{E_k}\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \\ &\le c\sum_k (\operatorname*{ess\,sup}_{E_k} v)\|f(\cdot)\chi_{I_k}\|_{L^{p(\cdot)}(\mathbb{R}_+)} \cdot \|g(\cdot)\chi_{E_k}\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \\ &\le c\,\|f(\cdot)w(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)}. \end{aligned}$$

If condition (ii) of (b) holds, then

$$v(z) \le b \operatorname*{ess\,inf}_{y\in[\frac{z}{4},4z]} w(y) \le b \operatorname*{ess\,inf}_{y\in(2^{k-1},2^{k+2})} w(y) \le bw(x),$$

for $z \in E_k$ and $x \in I_k$. Hence,

$$\operatorname*{ess\,sup}_{E_k} \le bw(x),$$

if $x \in I_k$. Consequently, taking into account this inequality and the estimate of S_3 in the previous case we have the desired result for M^-. □

Theorem 1.9.8. *Let $1 < p_- \le p_+ < \infty$ and let $p \in LH(\mathbb{R}_+)$. Suppose that $p(x) \equiv p_c \equiv const$ if $x > a$, where a is some positive number. Let v and w be weight functions on $\mathbb{R}_+$ such that*

(a) *$T'_{v,\overline{w}}$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$;*

(b) *there exists a positive constant b such that one of the following two conditions holds:*

(i) $\operatorname*{ess\,sup}_{y\in[\frac{x}{4},4x]} v(y) \le bw(x)$ *for almost all* $x \in \mathbb{R}_+$;

(ii) $v(x) \le b \operatorname*{ess\,inf}_{y\in[\frac{x}{4},4x]} w(y)$ *for almost all* $x \in \mathbb{R}_+$.

Then M^+ is bounded from $L^{p(\cdot)}_w(\mathbb{R}_+)$ to $L^{p(\cdot)}_v(\mathbb{R}_+)$.

Proof. Suppose that $\|g\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \leq 1$. We have

$$\int_0^\infty \big(M^+ f(x)\big) v(x) g(x) dx \leq \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(M^+ f_{1,k}(x)\big) v(x) g(x) dx$$

$$+ \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(M^+ f_{2,k}(x)\big) v(x) g(x) dx + \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big(M^+ f_{3,k}(x)\big) v(x) g(x) dx =: S_1 + S_2 + S_3,$$

where $f_{i,k}$, $i = 1, 2, 3$ are defined in the proof of the previous theorem. It is easy to see that $S_1 = 0$. To estimate S_2 observe that

$$M^+ f \cdot \chi_{[2^{k+1},\infty)}(x) \leq c \sup_{j\geq k+2} 2^{-j} \int\limits_{E_j} |f(y)| dy, \quad x \in E_k. \tag{1.9.3}$$

Indeed, notice that if $y \in (2^{k+2}, \infty)$ and $x \in E_k$, then $y - x \geq 2^{k+1}$. Hence,

$$\frac{1}{h} \int_x^{x+h} |\ f_{2,k}(y)\ |\ dy \leq \frac{1}{h} \int\limits_{\{y: y-x<h, y-x>2^{k+1}\}} |f(y)| dy = 0$$

for $h \leq 2^{k+1}$ and $x \in I_k$.

Let now $h > 2^{k+1}$. Then $h \in [2^j, 2^{j+1})$ for some $j \geq k+1$. If $y - x < h$, then it is clear that $y = y - x + x \leq h + x \leq 2^{j+1} + 2^{k+1} \leq 2^{j+1} + 2^j \leq 2^{j+2}$. Consequently, for such an h we have that

$$\begin{aligned}
\frac{1}{h} \int\limits_x^{x+h} |f_{2,k}(y)| dy &= \frac{1}{h} \int\limits_x^{x+h} |f \cdot \chi_{[2^{k+2},\infty)}(y)| dy \leq \frac{1}{h} \int\limits_{\{y: y-x<h, y>2^{k+2}\}} |f(y)| dy \\
&\leq \frac{1}{x} \int\limits_{\{y:\ y\in[2^{k+2}, 2^{j+2}]\}} |f(y)| dy \leq \sum_{i=k+2}^{j+1} 2^{-j} \int\limits_{2^i}^{2^{i+1}} |f(y)| dy
\end{aligned}$$

which proves inequality (1.9.3).

Taking into account estimate (1.9.3) and the boundedness of $T'_{v,\overline{w}}$ in $L^{p(\cdot)}(\mathbb{R}_+)$ we find that

$$S_2 \leq c \sum_k \int\limits_{E_k} v(x) g(x) \bigg(\sup_{j\geq k+1} 2^{-j} \int\limits_{E_j} |f(y)| dy \bigg) dx$$

$$\leq c\sum_k \Big(\int_{I_k} v(x)g(x)dx\Big)\Big(\sum_{j=k+1}^{\infty} 2^{-j}\int_{E_j}|f(y)|dy\Big)$$

$$= c\sum_j 2^{-j}\Big(\int_{E_j}|f(y)|dy\Big)\sum_{k=-\infty}^{j-1}\Big(\int_{E_k} v(x)g(x)dx\Big)$$

$$= c\sum_j 2^{-j}\Big(\int_{E_j}|f(y)|dy\Big)\Big(\int_0^{2^j} v(x)g(x)dx\Big) \leq c\sum_j\int_{E_j}|f(y)|\; y^{-1}\Big(\int_0^y v(x)g(x)dx\Big)dy$$

$$= c\int_{\mathbb{R}_+}|f(y)|\; y^{-1}\Big(\int_0^y v(x)g(x)dx\Big)dy = c\int_{\mathbb{R}_+} v(x)g(x)\Big(\int_x^{\infty}|f(y)|\; y^{-1}dy\Big)dx$$

$$\leq c\;\|g\|_{L^{p'(\cdot)}\mathbb{R}_+}\cdot\|T'_{v(\cdot),1/\cdot}f\|_{L^{p(\cdot)}\mathbb{R}_+} \leq c\|fw\|_{L^{p(\cdot)}\mathbb{R}_+}.$$

To estimate S_3 assume that condition (i) of (b) is satisfied. By Lemma 1.9.3 and the boundedness of the operator M^+ in $L^{p(\cdot)}(\mathbb{R}_+)$ we conclude that

$$\begin{aligned}
S_3 &\leq c\sum_k(\operatorname*{ess\,sup}_{E_k} v)\|M^+ f_{3,k}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)}\cdot\|g(\cdot)\chi_{E_k}\|_{L^{p'(\cdot)}(\mathbb{R}_+)}\\
&\leq c\sum_k(\operatorname*{ess\,sup}_{E_k} v)\|f(\cdot)\chi_{I_k}\|_{L^{p(\cdot)}(\mathbb{R}_+)}\cdot\|g(\cdot)\chi_{E_k}\|_{L^{p'(\cdot)}(\mathbb{R}_+)}\\
&\leq c\sum_k\|f(\cdot)w(\cdot)\chi_{I_k}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)}\cdot\|g(\cdot)\chi_{E_k}\|_{L^{p'(\cdot)}(\mathbb{R}_+)}\\
&\leq c\;\|f(\cdot)w(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)}\cdot\|g(\cdot)\|_{L^{p'(\cdot)}(\mathbb{R}_+)} \leq c\;\|f(\cdot)w(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}_+)}.
\end{aligned}$$

□

Theorem 1.9.9. *Let $1 < p_- \leq p_0(x) \leq p(x) \leq p_+ < \infty$ and let $p \in LH(\mathbb{R}_+)$. Suppose that $p(x) \equiv p_c \equiv const$ if $x > a$, where a is a positive constant. Assume that v and w are positive increasing weights on $(0,\infty)$. If condition (1.9.1) is satisfied, then M^- is bounded from $L^{p(\cdot)}_w(\mathbb{R}^+)$ to $L^{p(\cdot)}_v(\mathbb{R}^+)$.*

Proof. The proof follows by using Lemma 1.9.5 and Theorem 1.9.7. □

Theorem 1.9.10. *Let $1 < p_- \leq p_1(x) \leq p(x) \leq p_+ < \infty$, and let $p \in LH(\mathbb{R}_+)$. Suppose that $p(x) \equiv p_c \equiv const$ if $x > a$, where a is some positive constant. Let v and w be positive decreasing weights on $(0,\infty)$. If condition (1.9.2) is satisfied, then M^+ is bounded from $L^{p(\cdot)}_w(\mathbb{R}^+)$ to $L^{p(\cdot)}_v(\mathbb{R}^+)$.*

Proof. The proof follows immediately from Lemma 1.9.6 and Theorem 1.9.8. □

Now we discuss two-weight estimates for the one-sided potentials defined on $\mathbb{R}_+$:

$$\mathcal{R}_\alpha f(x) = \int\limits_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt;$$

$$\mathcal{W}_\alpha f(x) = \int\limits_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}}dt,$$

where $x > 0$ and $0 < \alpha < 1$.

The following two statements were proved in [23]:

Theorem 1.9.11. *Let $I = \mathbb{R}_+$ and let $p \in \mathcal{P}_+(I)$. Suppose that there exists a positive constant a such that $p \in \mathcal{P}_\infty((a,\infty))$. Suppose that α is a constant on I, $0 < \alpha < \frac{1}{p_I^+}$ and $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Then $\mathcal{W}_\alpha$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

Theorem 1.9.12. *Let $I = \mathbb{R}_+$ and let $p \in \mathcal{P}_-(I)$. Let α be a constant on I, $0 < \alpha < \frac{1}{p_I^+}$ and let $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Suppose that $p \in \mathcal{P}_\infty((a,\infty))$ for some positive number a. Then $\mathcal{R}_\alpha$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

Remark 1.9.1. Theorems 1.9.11 and 1.9.12 are true if we replace the condition $p \in \mathcal{P}_\infty((a,\infty))$ by the condition: p is constant outside an interval $(0,a)$ for some positive number a.

Now we are going to prove the main results regarding the one-sided potentials:

Theorem 1.9.13. *Let* $1 < p_- \le p_+ < \infty$, $\alpha < 1/p_+$, $q(x) = \frac{p(x)}{1-\alpha p(x)}$, $p \in LH(\mathbb{R}_+)$. *Suppose that* $p(x) \equiv p_c \equiv$ *const if* $x > a$, *where* a *is some positive number. Let* v *and* w *be a.e. positive measurable functions on* $\mathbb{R}_+$ *satisfying the conditions:*

(a) $T_{v_\alpha,\widetilde{w}}$ *is bounded in* $L^{p(\cdot)}(\mathbb{R}_+)$,

(b) *there exists a positive constant* b *such that one of the following two conditions hold:*

$$\text{(i)}\quad \operatorname*{ess\,sup}_{y\in[\frac{x}{4},4x]} v(y) \le bw(x) \quad \textit{for almost all } x \in \mathbb{R}_+;$$

$$\text{(ii)}\quad v(x) \le b \operatorname*{ess\,inf}_{y\in[\frac{x}{4},4x]} w(y) \quad \textit{for almost all } x \in \mathbb{R}_+.$$

Then $\mathcal{R}_\alpha$ *is bounded from* $L^{p(\cdot)}_w(\mathbb{R}_+)$ *to* $L^{q(\cdot)}_v(\mathbb{R}_+)$.

Theorem 1.9.14. *Let* $1 < p_- \le p_+ < \infty$, $\alpha < 1/p_+$, $q(x) = \frac{p(x)}{1-\alpha p(x)}$, $p \in LH(\mathbb{R}_+)$. *Suppose that* $p(x) \equiv p_c \equiv$ *const if* $x > a$, *where* a *is some positive number. Let* v *and* w *be a.e. positive measurable functions on* $\mathbb{R}_+$ *satisfying the conditions:*

(a) $T'_{v,\overline{w}_\alpha}$ *is bounded in* $L^{p(\cdot)}(\mathbb{R}_+)$,

(b) *there exists a positive constant* b *such that one of the following two conditions hold:*

$$\text{(i)}\quad \operatorname*{ess\,sup}_{y\in[\frac{x}{4},4x]} v(y) \le bw(x) \quad \textit{for almost all } x \in \mathbb{R}_+;$$

$$\text{(ii)}\quad v(x) \le b \operatorname*{ess\,inf}_{y\in[\frac{x}{4},4x]} w(y) \quad \textit{for almost all } x \in \mathbb{R}_+.$$

Then $\mathcal{W}_\alpha$ *is bounded from* $L^{p(\cdot)}_w(\mathbb{R}_+)$ *to* $L^{q(\cdot)}_v(\mathbb{R}_+)$.

Proof of Theorem 1.9.13. Let $f \ge 0$ and let $\|g\|_{L^{q'(\cdot)}(\mathbb{R}_+)} \le 1$. It is obvious that

$$\int_0^\infty (\mathcal{R}_\alpha f(x))\, v(x)g(x)dx \le \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} (\mathcal{R}_\alpha f_{1,k}(x))\, v(x)g(x)dx$$
$$+ \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} (\mathcal{R}_\alpha f_{2,k}(x))\, v(x)g(x)dx + \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} (\mathcal{R}_\alpha f_{3,k}(x))\, v(x)g(x)dx$$
$$=: S_1 + S_2 + S_3,$$

where $f_{i,k}$, $i = 1, 2, 3$ are defined in the proof of Theorem 1.9.7.

If $y \in [0, 2^{k-1})$ and $x \in [2^k, 2^{k+1}]$, then $y < \frac{x}{2}$. Hence

$$\mathcal{R}_\alpha f_{1,k}(x) \le \frac{c}{x^{1-\alpha}} \int_0^x f(t)dt, \quad x \in [2^{k-1}, 2^{k+2}].$$

By using Hölder's inequality, Theorem 1.9.1, Remark 1.9.1 we find that condition (i) guarantees the estimate

$$S_1 \le c\|fw\|_{L^{p(\cdot)}(\mathbb{R})}.$$

Further, observe that if $x \in [2^k, 2^{k+1})$, then $\mathcal{R}_\alpha f_{2,k}(x) = 0$. Hence $S_2 = 0$.

To estimate S_3 we argue as in the case of the proof of Theorem 1.9.7. □

The proof of Theorem 1.9.14. is similar to that of Theorem 1.9.13; therefore it is omitted.

Now we formulate other results of this section:

Theorem 1.9.15. *Let $1 < p_- \le p_+ < \infty$ and let α be a constant satisfying the condition $\alpha < 1/p_+$. Suppose that $q(x) = \frac{p(x)}{1-\alpha p(x)}$ and $p \in LH(\mathbb{R}_+)$. Assume that $p(x) \equiv p_c \equiv$ const outside some interval $[0, a]$, where a is a positive constant. Let v and w be positive increasing functions on $\mathbb{R}_+$ satisfying the condition*

$$\int\limits_t^\infty (v_\alpha(x))^{q(x)} \left(\int\limits_0^t w^{-(\widetilde{p}_0)'(x)}(y)dy \right)^{\frac{q(x)}{(\widetilde{p}_0)'(x)}} dx < \infty.$$

Then $\mathcal{R}_\alpha$ is bounded from $L^{p(\cdot)}_w(\mathbb{R})$ to $L^{q(\cdot)}_v(\mathbb{R})$.

Theorem 1.9.16. *Let $1 < p_- \le p_+ < \infty$ and let α be a constant satisfying the condition $\alpha < 1/p_+$. Suppose that $q(x) = \frac{p(x)}{1-\alpha p(x)}$ and $p \in LH(\mathbb{R}_+)$. Suppose also that $p(x) \equiv p_c \equiv$ const outside some interval $[0, a]$, where a is a positive constant and that v and w are positive decreasing functions on $\mathbb{R}_+$ satisfying the condition*

$$\sup_{t>0} \int_0^t (v(x))^{p(x)} \left(\int_t^\infty (\overline{w}_\alpha(y))^{(\widetilde{p}_1)'(x)} dy \right)^{\frac{p(x)}{(\widetilde{p}_1)'(x)}} dx < \infty.$$

Then $\mathcal{W}_\alpha$ is bounded from $L^{p(\cdot)}_w(\mathbb{R})$ to $L^{q(\cdot)}_v(\mathbb{R})$.

The proofs of Theorems 1.9.15 and 1.9.16 are based on Theorems 1.9.13, 1.9.14 and the following lemmas:

Lemma 1.9.17. *Let the conditions of* Theorem 1.9.15 *be satisfied. Then there is a positive constant c such that for all $t > 0$ the inequality*

$$v(4t) \leq cw(t)$$

is satisfied.

Lemma 1.9.18. *Let the conditions of* Theorem 1.9.16 *be satisfied. Then there is a positive constant b such that for all $t > 0$ the inequality*

$$v(t) \leq bw(4t)$$

holds.

The proof of Lemma 1.9.17 (resp 1.9.18) is similar to that of Lemma 1.9.5; therefore we omit it.

1.10 Riemann-Liouville Operators on the Cone of Decreasing Functions

In this section necessary and sufficient conditions governing the one-weight inequality for the Riemann-Liouville transform on the cone of decreasing functions for variable exponent are obtained. First we show that the two-sided pointwise estimate

$$c_1 Tf(x) \leq \bar{R}_\alpha f(x)) \leq c_2 Tf(x),$$

holds on the class of functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ which are non-negative and decreasing, where

$$Tf(x) = \frac{1}{x}\int\limits_0^x f(t)dt \quad \text{and} \quad \bar{R}_\alpha f(x) = \frac{1}{x^\alpha}\int\limits_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, \qquad 0 < \alpha < 1.$$

By the symbol $Tf \approx Kf$, where T and K are linear positive operators defined on appropriate classes of functions, we mean that there are positive constants c_1 and c_2 independent of f and x such that

$$c_1 Tf(x) \le Kf(x) \le c_2 Tf(x).$$

Let $p : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a measurable function, satisfying the conditions

$$p_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}_+} p(x) > 0, \quad p_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}_+} p(x) < \infty.$$

Suppose that u is a weight on $(0,\infty)$. Let us define the following local oscillation of p :

$$\varphi_{p(\cdot),u(\delta)} = \operatorname*{ess\,sup}_{x\in(0,\delta)\cap\operatorname{supp} u} p(x) - \operatorname*{ess\,inf}_{x\in(0,\delta)\cap\operatorname{supp} u} p(x).$$

We observe that $\varphi_{p(\cdot),u(\delta)}$ is non-decreasing and positive function such that

$$\lim_{\delta\to\infty} \varphi_{p(\cdot),u(\delta)} = p_u^+ - p_u^-,$$

where p_u^+ and p_u^- denote, respectively the essential supremum and infimum of p on the support of u.

Definition 1.10.1. Let $\mathcal{D}$ be the class of all non–negative decreasing functions on $\mathbb{R}_+$. Suppose that u is a measurable a.e. positive function (weight) on $\mathbb{R}_+$. We denote by $L^{p(x)}(u,\mathbb{R}_+)$ the class of all non–negative functions on $\mathbb{R}_+$ for which

$$S_p(f,u) = \int\limits_{\mathbb{R}_+} |f(x)|^{p(x)} u(x)dx < \infty.$$

By the symbol $L^{p(x)}_{dec}(u,\mathbb{R}_+)$ we mean the class $L^{p(x)}(u,\mathbb{R}_+) \cap \mathcal{D}$.

Now we list the well-known results regarding the one-weight inequality for the operator T. For the following statement we refer to [4].

Theorem 1.10.1. *Let r be constant such that $0 < r < \infty$. Then the inequity*

$$\int_0^\infty u(x)(Tf(x))^r dx \le C \int_0^\infty u(x)(f(x))^r dx, \qquad f \in L^r_{dec}(u, \mathbb{R}_+)$$

holds, if and only if there exists a positive constant C such that for all $s > 0$

$$\int_s^\infty \left(\frac{s}{x}\right)^r u(x)dx \le C \int_0^s u(x)dx. \tag{1.10.1}$$

Condition (1.10.1) *is called the B_r condition and was introduced in* [4].

Theorem 1.10.2 ([**6**]). *Let u be a weight on $(0, \infty)$ and $p : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $0 < p_- \le p_+ < \infty$, and assume that $\varphi_{p(\cdot),u(\delta)} = 0$. The following facts are equivalent:*

(a) *There exists a positive constant C such that for any positive and non-increasing function f,*

$$\int_0^\infty \big(Tf(x)\big)^{p(x)} u(x)dx \le C \int_0^\infty \big(f(x)\big)^{p(x)} u(x)dx.$$

(b) *For any $r, s > 0$,*

$$\int_r^\infty \left(\frac{r}{sx}\right)^{p(x)} u(x) \le C \int_0^r \frac{u(x)}{s^{p(x)}} dx.$$

(c) $p_{|\text{supp u}} \equiv p_0 \quad$ *a.e and $u \in B_{p_0}$.*

Our result in this section is the following statement:

Proposition 1.10.3. *Let u be a weight on $(0, \infty)$ and $p : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $0 < p_- \le p_+ < \infty$. Assume that $\varphi_{p(\cdot),u(\delta)} = 0$. The following facts are equivalent:*

(i) *$\bar{R}_\alpha$ is bounded from $L^{p(x)}_{dec}(u, \mathbb{R}_+)$ to $L^{p(x)}(u, \mathbb{R}_+)$;*

(ii) *condition (b) of* Theorem 1.10.2 *holds;*

(iii) *condition (c) of* Theorem 1.10.2 *holds.*

Proof. In view of Theorem 1.10.2 it is enough to show that the following relation concerning the operators $\bar{R}_\alpha$ and T holds:

$$\bar{R}_\alpha f \approx Tf, \qquad 0 < \alpha < 1, \quad f \in \mathcal{D}.$$

Upper estimate. Represent $\bar{R}_\alpha f$ as follows:

$$\bar{R}_\alpha f(x) = \frac{1}{x^\alpha} \int\limits_0^{x/2} \frac{f(t)}{(x-t)^{1-\alpha}} dt + \frac{1}{x^\alpha} \int\limits_{x/2}^{x} \frac{f(t)}{(x-t)^{1-\alpha}} dt = S_1(x) + S_2(x).$$

Observe that if $t < x/2$, then $x/2 < x - t$. Hence,

$$S_1(x) \le c \frac{1}{x} \int\limits_0^{x/2} f(t) dt \le cTf(x),$$

where the positive constant c does not depend on f and x. Using the fact that f is non-increasing we find that

$$S_2(x) \le cf(x/2) \le cTf(x).$$

The lower estimate follows immediately by using the fact that f is non-negative and the obvious estimate $x - t \le x$ where $0 < t < x$. □

Chapter 2

Integral Operators in $L^{p(x)}$ Spaces Defined on Spaces of Homogeneous Type.

2.1 Introduction

In this chapter we study the two-weight problem for Hardy-type, maximal, potential and singular operators in Lebesgue spaces with non-standard growth defined on quasi-metric measure spaces. In particular, we derive sufficient conditions for the boundedness of these operators in weighted $L^{p(\cdot)}$ spaces which enable us effectively to construct examples of appropriate weights. The conditions are simultaneously necessary and sufficient for corresponding inequalities when the weights are of special type and the exponent p of the space is constant(see, e.g.,[20]). We assume that the exponent p satisfies local log-Hölder continuity condition and if the diameter of X is infinite, then we suppose that p is constant outside some ball. In the framework of variable exponent analysis such a condition first appeared in the paper [12], where the author established the boundedness of the Hardy-Littlewood maximal operator in $L^{p(\cdot)}(\mathbb{R}^n)$. As far as we know, unfortunately, even in the unweighted case, an analog of the log-Hölder condition (at infinity) for $p : X \to [1, \infty)$ which is well-known and natural for the Euclidean spaces is not available. (see [9], [71], [7]). The local log-Hölder continuity condition for the exponent p together with the log-Hölder decay condition guarantees the boundedness of operators of harmonic analysis in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces (see e.g., [10]).

It should be emphasized that in the classical Lebesgue spaces the two-weight problem for fractional integrals is already solved (see [38], [36]) but it is often useful to construct concrete examples of weights from transparent and easily verifiable conditions.

Finally we mention that some examples of weights for appropriate two–weight inequalities are given.

2.2 Preliminaries

Let $X := (X, d, \mu)$ be a topological space with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$ and there exists a non-negative real-valued function (quasi-metric) d on $X \times X$ satisfying the conditions:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) there exists a constant $a_1 > 0$, such that $d(x, y) \leq a_1(d(x, z) + d(z, y))$ for all x, y, $z \in X$;

(iii) there exists a constant $a_0 > 0$, such that $d(x, y) \leq a_0 d(y, x)$ for all x, $y, \in X$.

We assume that the balls $B(x, r) := \{y \in X : d(x, y) < r\}$ are measurable and $0 \leq \mu(B(x, r)) < \infty$ for all $x \in X$ and $r > 0$; for every neighborhood V of $x \in X$, there exists $r > 0$, such that $B(x, r) \subset V$. Throughout the chapter we also suppose that $\mu\{x\} = 0$ and that

$$B(x, R) \setminus B(x, r) \neq \emptyset \tag{2.2.1}$$

for all $x \in X$, positive r and R with $0 < r < R < L$, where

$$L := diam\ (X) = \sup\{d(x, y) : x, y \in X\}.$$

We call the triple (X, d, μ) a quasi-metric measure space. If μ satisfies the doubling condition

$$\mu(B(x, 2r)) \leq c\mu(B(x, r)),$$

where the positive constant c does not depend on $x \in X$ and $r > 0$, then (X, d, μ) is called a space of homogeneous type (SHT). For the definition, examples and some properties of an SHT see, e.g., the monographs [89], [8], [26].

A quasi-metric measure space, where the doubling condition is not assumed is called a non-homogeneous space.

Notice that the condition $L < \infty$ implies that $\mu(X) < \infty$ because we assumed that every ball in X has a finite measure.

Definition 2.2.1. We say that the measure μ satisfies the doubling condition at the point x_0 ($\mu \in DC(x_0)$) if there are positive constants D and D_1 (which might be depended on x_0) such that for all $0 < r < D_1$, the inequality

$$\mu(B(x_0, 2r)) \leq D\mu(B(x_0, r)),$$

holds.

Definition 2.2.2. We say that the measure μ is upper Ahlfors Q- regular if there is a positive constant c_1 such that $\mu B(x, r) \leq c_1 r^Q$ for for all $x \in X$ and $r \in (0, L)$.

Definition 2.2.3. We say that the measure μ is lower Ahlfors $q-$ regular if there is a positive constant c_2 such that $\mu B(x, r) \geq c_2 r^q$ for all $x \in X$ and $r \in (0, L)$.

It is easy to check that if (X, d, μ) is a quasi-metric measure space and $L < \infty$, then μ is lower Ahlfors regular (see also, e.g., [31] for the case when d is a metric).

Despite the fact that the definitions and some properties of the variable exponent Lebesgue spaces defined on $\Omega \subseteq \mathbb{R}^n$ are given in Section 1.2, it would be convenient for the reader to repeat them for quasi-metric measure spaces.

Let p be a non–negative $\mu-$ measurable function on X. Suppose that E is a $\mu-$ measurable set in X. We use the following notation:

$$p_-(E) := \inf_E p; \quad p_+(E) := \sup_E p; \quad p_- := p_-(X); \quad p_+ := p_+(X);$$

$$\overline{B}_{xy} := \overline{B}(x, d(x,y)); \quad g_B := \frac{1}{\mu(B)} \int_B |g(x)| d\mu(x);$$

$$kB(x,r) := B(x,kr); B_{xy} := B(x, d(x,y));$$

$$\overline{B}(x,r) := \{y \in X : \ d(x,y) \leq r\}.$$

Assume that $1 \leq p_- \leq p_+ < \infty$. The variable exponent Lebesgue space $L^{p(\cdot)}(X)$ (sometimes it is denoted by $L^{p(x)}(X)$) is the class of all μ-measurable functions f on X for which $S_p(f) := \int_X |f(x)|^{p(x)} d\mu(x) < \infty$. The norm in $L^{p(\cdot)}(X)$ is defined as follows:

$$\|f\|_{L^{p(\cdot)}(X)} = \inf\{\lambda > 0 : S_p(f/\lambda) \leq 1\}.$$

We need some definitions for the exponent p which will be useful to derive the main result.

Definition 2.2.4. Let (X, d, μ) be a quasi-metric measure space and let $N \geq 1$ be a constant. Suppose that p satisfy the condition $0 < p_- \leq p_+ < \infty$. We say that p belongs to the class $\mathcal{P}(N, x)$, where $x \in X$, if there are positive constants b and c (which might be depend on x) such that

$$\mu(B(x, Nr))^{p_-(B(x,r)) - p_+(B(x,r))} \leq c \tag{2.2.2}$$

holds for all r, $0 < r \leq b$. Further, $p \in \mathcal{P}(N)$ if there are a positive constants b and c such that (2.2.2) holds for all $x \in X$ and all r satisfying the condition $0 < r \leq b$.

Definition 2.2.5. Let (X, d, μ) be an SHT. Suppose that $0 < p_- \leq p_+ < \infty$. We say that $p \in LH(X, x)$ (p satisfies the log-Hölder- type condition at a point $x \in X$) if there are positive constants b and c (which might be depend on x) such that

$$|p(x) - p(y)| \leq \frac{c}{-\ln\big(\mu(B_{xy})\big)} \tag{2.2.3}$$

holds for all y satisfying the condition $d(x, y) \leq b$. Further, $p \in LH(X)$ (p satisfies the log-Hölder type condition on X)if there are positive constants b and c such that (2.2.3) holds for all x, y with $d(x, y) \leq b$.

We shall also need another form of the log-Hölder continuity condition given by the following definition:

Definition 2.2.6. Let (X, d, μ) be a quasi-metric measure space and let $0 < p_- \leq p_+ < \infty$. We say that $p \in \overline{LH}(X, x)$ if there are positive constants b and c (which might be depended on x) such that

$$|p(x) - p(y)| \leq \frac{c}{-\ln d(x, y)} \tag{2.2.4}$$

for all y with $d(x, y) \leq b$. Further, $p \in \overline{LH}(X)$ if (2.2.4) holds for all x, y with $d(x, y) \leq b$.

It is easy to see that if a measure μ is upper Ahlfors Q-regular and $p \in LH(X)$ (resp. $p \in LH(X, x)$), then $p \in \overline{LH}(X)$ (resp. $p \in \overline{LH}(X, x)$. Further, if μ is lower Ahlfors q-regular and $p \in \overline{LH}(X)$ (resp. $p \in \overline{LH}(X, x)$), then $p \in LH(X)$ (resp. $p \in LH(X, x)$).

Remark 2.2.1. It can be checked easily that if (X, d, μ) is an SHT, then $\mu B_{x_0 x} \approx \mu B_{x x_0}$.

Remark 2.2.2. Let (X, d, μ) be an SHT with $L < \infty$. It is known (see, e.g., [31], [41]) that if $p \in \overline{LH}(X)$, then $p \in \mathcal{P}(1)$. Further, if μ is upper Ahlfors Q-regular, then the condition $p \in \mathcal{P}(1)$ implies that $p \in \overline{LH}(X)$.

Proposition 2.2.1. *Let c be a positive constant and let $1 < p_-(X) \le p_+(X) < \infty$ and $p \in LH(X)$ (resp. $p \in \overline{LH}(X)$), then the functions $cp(\cdot)$, $1/p(\cdot)$ and $p'(\cdot)$ belong to $LH(X)$ (resp. $\overline{LH}(X)$). Further if $p \in LH(X,x)$ (resp. $p \in \overline{LH}(X,x)$) then $cp(\cdot)$, $1/p(\cdot)$ and $p'(\cdot)$ belong to $LH(X,x)$ (resp. $p \in \overline{LH}(X,x)$).*

The proof of the latter statement can be checked immediately using the definitions of the classes $LH(X,x)$, $LH(X)$, $\overline{LH}(X,x)$, $\overline{LH}(X)$.

Proposition 2.2.2. *Let (X,d,μ) be an SHT and let $p \in \mathcal{P}(1)$. Then $(\mu B_{xy})^{p(x)} \le c(\mu B_{yx})^{p(y)}$ for all $x, y \in X$ with $\mu(B(x,d(x,y))) \le b$, where b is a small constant and the constant c does not depend on $x, y \in X$.*

Proof. Due to the doubling condition for μ, Remark 2.2.1, the condition $p \in \mathcal{P}(1)$ and the fact $x \in B(y, a_1(a_0+1)d(y,x))$ we have the following estimates:

$$\mu(B_{xy})^{p(x)} \le \mu\big(B(y,a_1(a_0{+}1)d(x,y))\big)^{p(x)} \le c\mu B(y,a_1(a_0{+}1)d(x,y))^{p(y)} \le c(\mu B_{yx})^{p(y)},$$

which proves the statement. □

The proof of the next statement is trivial and follows directly from the definition of the classes $\mathcal{P}(N,x)$ and $\mathcal{P}(N)$. Details are omitted.

Proposition 2.2.3. *Let (X,d,μ) be a quasi-metric measure space and let $x_0 \in X$. Suppose that $N \ge 1$ be a constant. Then the following statements hold:*

(i) *If $p \in \mathcal{P}(N,x_0)$ (resp. $p \in \mathcal{P}(N)$), then there are positive constants r_0, c_1 and c_2 such that for all $0 < r \le r_0$ and all $y \in B(x_0,r)$ (resp. for all x_0, y with $d(x_0,y) < r \le r_0$), we have that*

$$\mu\big(B(x_0,Nr)\big)^{p(x_0)} \le c_1\mu\big(B(x_0,Nr)\big)^{p(y)} \le c_2\mu\big(B(x_0,Nr)\big)^{p(x_0)}.$$

(ii) *Let $p \in \mathcal{P}(N,x_0)$. Then there are positive constants r_0, c_1 and c_2 (in general, depending on x_0) such that for all r ($r \le r_0$) and all $x, y \in B(x_0,r)$ we have*

$$\mu\big(B(x_0,Nr)\big)^{p(x)} \le c_1\mu\big(B(x_0,Nr)\big)^{p(y)} \le c_2\mu\big(B(x_0,Nr)\big)^{p(x)}.$$

(iii) *Let* $p \in \mathcal{P}(N)$. *Then there are positive constants* r_0, c_1 *and* c_2 *such that for all balls* B *with radius* r $(r \le r_0)$ *and all* $x, y \in B$, *we have that*

$$\mu(NB)^{p(x)} \le c_1 \mu(NB)^{p(y)} \le c_2 \mu(NB)^{p(x)}.$$

It is known that (see, e.g., [57], [78]) if f is a measurable function on X and E is a measurable subset of X, then the following inequalities hold:

$$\|f\|^{p_+(E)}_{L^{p(\cdot)}(E)} \le S_p(f\chi_E) \le \|f\|^{p_-(E)}_{L^{p(\cdot)}(E)}, \quad \|f\|_{L^{p(\cdot)}(E)} \le 1;$$
$$\|f\|^{p_-(E)}_{L^{p(\cdot)}(E)} \le S_p(f\chi_E) \le \|f\|^{p_+(E)}_{L^{p(\cdot)}(E)}, \quad \|f\|_{L^{p(\cdot)}(E)} > 1.$$

Further, Hölder's inequality in the variable exponent Lebesgue spaces has the following form:

$$\int_E fg d\mu \le \Big(1/p_-(E) + 1/(p')_-(E)\Big) \|f\|_{L^{p(\cdot)}(E)} \|g\|_{L^{p'(\cdot)}(E)}.$$

Lemma 2.2.4. *Let* (X, d, μ) *be an SHT.*

(i) *If* β *is a measurable function on* X *such that* $\beta_+ < -1$ *and if* r *is a small positive number, then there exists a positive constant* c *independent of* r *and* x *such that*

$$\int_{X \setminus B(x_0, r)} (\mu B_{x_0 y})^{\beta(x)} d\mu(y) \le c \frac{\beta(x)+1}{\beta(x)} \mu(B(x_0, r))^{\beta(x)+1}.$$

(ii) *Suppose that* p *and* α *are measurable functions on* X *satisfying the conditions* $1 < p_- \le p_+ < \infty$ *and* $\alpha_- > 1/p_-$. *Then there exists a positive constant* c *such that for all* $x \in X$ *the inequality*

$$\int_{\overline{B}(x_0, 2d(x_0, x))} \big(\mu B(x, d(x, y))\big)^{(\alpha(x)-1)p'(x)} d\mu(y) \le c\big(\mu B(x_0, d(x_0, x))\big)^{(\alpha(x)-1)p'(x)+1},$$

holds.

Proof. Part (i) was proved in [41] (see also [20], p.372, for constant β). The proof of Part (ii) is given in [20] (Lemma 6.5.2, p. 348) for constant α and p but repeating those arguments we can see that it is also true for variable α and p. Details are omitted. □

Lemma 2.2.5. *Let (X, d, μ) be an SHT. Suppose that $0 < p_- \leq p_+ < \infty$. Then p satisfies the condition $p \in \mathcal{P}(1)$ (resp. $p \in \mathcal{P}(1, x)$) if and only if $p \in LH(X)$ (resp. $p \in LH(X, x)$).*

Proof. We follow [12]. *Necessity.* Let $p \in \mathcal{P}(1)$ and let $x, y \in X$ with $d(x, y) < c_0$ for some positive constant c_0. Observe that $x, y \in B$, where $B := B(x, 2d(x, y))$. By the doubling condition for μ we have that

$$\big(\mu B_{xy}\big)^{-|p(x)-p(y)|} \leq c\big(\mu B\big)^{-|p(x)-p(y)|} \leq c\big(\mu B\big)^{p_-(B)-p_+(B)} \leq C,$$

where C is a positive constant which is greater than 1. Taking now the logarithm in the last inequality we have that $p \in LH(X)$. If $p \in \mathcal{P}(1, x)$, then by the same arguments we find that $p \in LH(X, x)$.

Sufficiency. Let $B := B(x_0, r)$. First observe that If $x, y \in B$, then $\mu B_{xy} \leq c\mu B(x_0, r)$. Consequently, this inequality and the condition $p \in LH(X)$ yield $|p_-(B) - p_+(B)| \leq \frac{C}{-\ln\big(c_0 \mu B(x_0,r)\big)}$. Further, there exists r_0 such that $0 < r_0 < 1/2$ and $c_1 \leq \frac{\ln\big(\mu(B)\big)}{-\ln\big(c_0\mu(B)\big)} \leq c_2$, $0 < r \leq r_0$, where c_1 and c_2 are positive constants. Hence

$$\big(\mu(B)\big)^{p_-(B)-p_+(B)} \leq \Big(\mu(B)\Big)^{\frac{C}{\ln\left(c_0\mu(B)\right)}} = \exp\left(\frac{C\ln\big(\mu(B)\big)}{\ln\big(c_0\mu(B)\big)}\right) \leq C.$$

Let now $p \in LH(X, x)$ and let $B_x := B(x, r)$ where r is a small number. We have that $p_+(B_x) - p(x) \leq \frac{c}{-\ln\left(c_0\mu B(x,r)\right)}$ and $p(x) - p_-(B_x) \leq \frac{c}{-\ln\left(c_0\mu B(x,r)\right)}$ for some positive constant c_0. Consequently,

$$\begin{aligned}\big(\mu(B_x)\big)^{p_-(B_x)-p_+(B_x)} &= \big(\mu(B_x)\big)^{p(x)-p_+(B_x)}\big(\mu(B_x)\big)^{p_-(B_x)-p(x)} \\ &\le c\big(\mu(B_x)\big)^{\frac{-2c}{-\ln(c_0\mu B_x))}} \le C.\end{aligned}$$

□

To present more results we need the following definition:

Definition 2.2.7. A measure μ on X is said to satisfy the reverse doubling condition ($\mu \in RDC(X)$) if there exist constants $A > 1$ and $B > 1$ such that the inequality $\mu\big(B(a, Ar)\big) \ge B\mu\big(B(a, r)\big)$ holds.

Remark 2.2.3. It is known that if all annuli in X are not empty (i.e. condition (2.2.1) holds), then $\mu \in DC(X)$ implies that $\mu \in RDC(X)$ (see, e.g., [89], p. 11, Lemma 20).

Lemma 2.2.6. *Let (X, d, μ) be an SHT. Suppose that there is a point $x_0 \in X$ such that $p \in LH(X, x_0)$. Let A be the constant defined in* Definition 2.2.7. *Then there exist positive constants r_0 and C (which might depend on x_0) such that for all r, $0 < r \le r_0$, the inequality*

$$(\mu B_A)^{p_-(B_A)-p_+(B_A)} \le C$$

holds, where $B_A := B(x_0, Ar) \setminus B(x_0, r)$ and the constant C is independent of r.

Proof. Taking into account condition (2.2.1) and Remark 2.2.3 we have that $\mu \in RDC(X)$. Let $B := B(x_0, r)$. By the doubling and reverse doubling conditions we have that

$$\mu B_A = \mu B(x_0, Ar) - \mu B(x_0, r) \ge (B - 1)\mu B(x_0, r) \ge c\mu(AB).$$

Suppose that $0 < r < c_0$, where c_0 is a sufficiently small constant. Then by using Lemma 2.2.5 we find that

$$(\mu B_A)^{p_-(B_A)-p_+(B_A)} \le c\big(\mu(AB)\big)^{p_-(B_A)-p_+(B_A)} \le c\big(\mu(AB)\big)^{p_-(AB)-p_+(AB)} \le c.$$

$\square$

In the sequel we will use the notation:

$$I_{1,k} := \begin{cases} B(x_0, A^{k-1}L/a_1) & \text{if} \quad L < \infty \\ B(x_0, A^{k-1}/a_1) & \text{if} \quad L = \infty, \end{cases}$$

$$I_{2,k} := \begin{cases} \overline{B}(x_0, A^{k+2}a_1 L) \setminus B(x_0, A^{k-1}L/a_1) & \text{if} \quad L<\infty \\ \overline{B}(x_0, A^{k+2}a_1) \setminus B(x_0, A^{k-1}/a_1) & \text{if} \quad L = \infty, \end{cases}$$

$$I_{3,k} := \begin{cases} X \setminus B(x_0, A^{k+2}La_1) & \text{if} \quad L < \infty \\ X \setminus B(x_0, A^{k+2}a_1) & \text{if} \quad L = \infty, \end{cases}$$

$$E_k := \begin{cases} \overline{B}(x_0, A^{k+1}L) \setminus B(x_0, A^k L) & \text{if} \quad L < \infty \\ \overline{B}(x_0, A^{k+1}) \setminus B(x_0, A^k) & \text{if} \quad L = \infty \end{cases},$$

where the constants A and a_1 are taken respectively from Definition 2.2.7 and the triangle inequality for the quasi-metric d, and L is the diameter of X.

Lemma 2.2.7. *Let (X, d, μ) be an SHT and let $1 < p_-(x) \le p(x) \le q(x) \le q_+(X) < \infty$. Suppose that there is a point $x_0 \in X$ such that $p, q \in LH(X, x_0)$. Assume that if $L = \infty$, then $p(x) \equiv p_c \equiv const$ and $q(x) \equiv q_c \equiv const$ outside some ball $B(x_0, a)$. Then there exists a positive constant C such that*

$$\sum_k \|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)} \|g\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)} \le C\|f\|_{L^{p(\cdot)}(X)} \|g\|_{L^{q'(\cdot)}(X)}$$

for all $f \in L^{p(\cdot)}(X)$ and $g \in L^{q'(\cdot)}(X)$.

Proof. Suppose that $L = \infty$. To prove the lemma first observe that $\mu(E_k) \approx \mu B(x_0, A^k)$ and $\mu(I_{2,k}) \approx \mu B(x_0, A^{k-1})$. This holds because μ satisfies the reverse doubling condition and, consequently,

$$\mu E_k = \mu\Big(\overline{B}(x_0, A^{k+1}) \setminus B(x_0, A^k)\Big) = \mu\overline{B}(x_0, A^{k+1}) - \mu B(x_0, A^k)$$
$$= \mu\overline{B}(x_0, AA^k) - \mu B(x_0, A^k) \geq B\mu B(x_0, A^k) - \mu B(x_0, A^k) = (B-1)\mu B(x_0, A^k)$$

Moreover, the doubling condition yields $\mu E_k \leq \mu B(x_0, AA^k) \leq c\mu B(x_0, A^k)$, where $c > 1$. Hence, $\mu E_k \approx \mu B(x_0, A^k)$.

Further, since we can assume that $a_1 \geq 1$, we find that

$$\mu I_{2,k} = \mu\Big(\overline{B}(x_0, A^{k+2}a_1) \setminus B(x_0, A^{k-1}/a_1)\Big) = \mu\overline{B}(x_0, A^{k+2}a_1) - \mu B(x_0, A^{k-1}/a_1)$$
$$= \mu\overline{B}(x_0, AA^{k+1}a_1) - \mu B(x_0, A^{k-1}/a_1) \geq B\mu B(x_0, A^{k+1}a_1) - \mu B(x_0, A^{k-1}/a_1)$$
$$\geq B^2\mu B(x_0, A^k/a_1) - \mu B(x_0, A^{k-1}/a_1) \geq B^3\mu B(x_0, A^{k-1}/a_1) - \mu B(x_0, A^{k-1}/a_1)$$
$$= (B^3 - 1)\mu B(x_0, A^{k-1}/a_1).$$

Moreover, using the doubling condition for μ we have that

$$\mu I_{2,k} \leq \mu\overline{B}(x_0, A^{k+2}r) \leq c\mu B(x_0, A^{k+1}r) \leq c^2\mu B(x_0, A^k/a_1) \leq c^3\mu B(x_0, A^{k-1}/a_1).$$

This gives the estimates

$$(B^3 - 1)\mu B(x_0, A^{k-1}/a_1) \leq \mu(I_{2,k}) \leq c^3\mu B(x_0, A^{k-1}/a_1).$$

For simplicity assume that $a = 1$. Suppose that m_0 is an integer such that $\frac{A^{m_0-1}}{a_1} > 1$. Let us split the sum as follows:

$$\sum_i \|f\chi_{I_{2,i}}\|_{L^{p(\cdot)}(X)} \cdot \|g\chi_{I_{2,i}}\|_{L^{q'(\cdot)}(X)} = \sum_{i \leq m_0}\Big(\cdots\Big) + \sum_{i > m_0}\Big(\cdots\Big) =: J_1 + J_2.$$

Since $p(x) \equiv p_c = const$, $q(x) = q_c = const$ outside the ball $B(x_0, 1)$, by using Hölder's inequality and the fact that $p_c \leq q_c$, we have

$$J_2 = \sum_{i > m_0} \|f\chi_{I_{2,i}}\|_{L^{p_c}(X)} \cdot \|g\chi_{I_{2,i}}\|_{L^{(q_c)'}(X)} \leq c\|f\|_{L^{p(\cdot)}(X)} \cdot \|g\|_{L^{q'(\cdot)}(X)}.$$

Let us estimate J_1. Suppose that $\|f\|_{L^{p(\cdot)}(X)} \leq 1$ and $\|g\|_{L^{q'(\cdot)}(X)} \leq 1$. Also, by Proposition 2.2.1 we have that $1/q' \in LH(X, x_0)$. Therefore by Lemma 2.2.6 and the fact that $1/q' \in LH(X, x_0)$ we obtain that

$$\mu\big(I_{2,k}\big)^{\frac{1}{q_+(I_{2,k})}} \approx \|\chi_{I_{2,k}}\|_{L^{q(\cdot)}(X)} \approx \mu\big(I_{2,k}\big)^{\frac{1}{q_-(I_{2,k})}}$$

and

$$\mu\big(I_{2,k}\big)^{\frac{1}{q'_+(I_{2,k})}} \approx \|\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)} \approx \mu\big(I_{2,k}\big)^{\frac{1}{q'_-(I_k)}},$$

where $k \leq m_0$. Further, observe that these estimates and Hölder's inequality yield the following chain of inequalities:

$$\begin{aligned}
J_1 &\leq c \sum_{k\leq m_0} \int\limits_{\overline{B}(x_0,A^{m_0+1})} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)} \cdot \|g\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{q(\cdot)}(X)} \cdot \|\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}} \chi_{E_k}(x) d\mu(x) \\
&= c \int\limits_{\overline{B}(x_0,A^{m_0+1})} \sum_{k\leq m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)} \cdot \|g\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{q(\cdot)}(X)} \cdot \|\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}} \chi_{E_k}(x) d\mu(x) \\
&\leq c \Big\| \sum_{k\leq m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{q(\cdot)}(X)}} \chi_{E_k}(x) \Big\|_{L^{q(\cdot)}(\overline{B}(x_0,A^{m_0+1}))} \\
&\times \Big\| \sum_{k\leq m_0} \frac{\|g\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{q'(\cdot)}(X)}} \chi_{E_k}(x) \Big\|_{L^{q'(\cdot)}(\overline{B}(x_0,A^{m_0+1}))} =: cS_1(f) \cdot S_2(g).
\end{aligned}$$

Now we claim that $S_1(f) \leq cI(f)$, where

$$I(f) := \Big\| \sum_{k\leq m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2k}}\|_{L^{p(\cdot)}(X)}} \chi_{E_k(\cdot)} \Big\|_{L^{p(\cdot)}(\overline{B}(x_0,A^{m_0+1}))}$$

and the positive constant c does not depend on f. Indeed, suppose that $I(f) \leq 1$. Then taking into account Lemma 2.2.6 we have that

$$\begin{aligned}
&\sum_{k\leq m_0} \frac{1}{\mu(I_{2,k})} \int_{E_k} \|f\chi_{I_{2,k}}\|^{p(x)}_{L^{p(\cdot)}(X)} d\mu(x) \\
&\leq c \int\limits_{\overline{B}(x_0,A^{m_0+1})} \Big(\sum_{k\leq m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}} \chi_{E_k(x)} \Big)^{p(x)} d\mu(x) \leq c.
\end{aligned}$$

Consequently, since $p(x) \le q(x)$, $E_k \subseteq I_{2,k}$ and $\|f\|_{L^{p(\cdot)}(X)} \le 1$, we find that

$$\sum_{k\le m_0} \frac{1}{\mu(I_{2,k})} \int_{E_k} \|f\chi_{I_{2,k}}\|^{q(x)}_{L^{p(\cdot)}(X)} d\mu(x) \le \sum_{k\le m_0} \frac{1}{\mu(I_{2,k})} \int_{E_k} \|f\chi_{I_{2,k}}\|^{p(x)}_{L^{p(\cdot)}(X)} d\mu(x) \le c.$$

This implies that $S_1(f) \le c$. Thus the desired inequality is proved. Further, let us introduce the following function:

$$\mathbb{P}(y) := \sum_{k\le 2} p_+(\chi_{I_{2,k}})\chi_{E_k(y)}.$$

It is clear that $p(y) \le \mathbb{P}(y)$ because $E_k \subset I_{2,k}$. Hence

$$I(f) \le c \Big\| \sum_{k\le m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2k}}\|_{L^{p(\cdot)}(X)}} \chi_{E_k(\cdot)} \Big\|_{L^{\mathbb{P}(\cdot)}(\overline{B}(x_0, A^{m_0+1}))}$$

for some positive constant c. Then by using this inequality, the definition of the function $\mathbb{P}$, the condition $p \in LH(X)$ and the obvious estimate $\|\chi_{I_{2,k}}\|^{p_+(I_{2,k})}_{L^{p(\cdot)}(X)} \ge c\mu(I_{2,k})$, we find that

$$\begin{aligned}
&\int_{\overline{B}(x_0, A^{m_0+1})} \Big(\sum_{k\le m_0} \frac{\|f\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|_{L^{p(\cdot)}(X)}} \chi_{E_k(x)} \Big)^{\mathbb{P}(x)} d\mu(x) \\
&= \int_{\overline{B}(x_0, A^{m_0+1})} \Big(\sum_{k\le m_0} \frac{\|f\chi_{I_{2,k}}\|^{p_+(I_{2,k})}_{L^{p(\cdot)}(X)}}{\|\chi_{I_{2,k}}\|^{p_+(I_{2,k})}_{L^{p(\cdot)}(X)}} \chi_{E_k(x)} \Big) d\mu(x) \\
&\le c \int_{\overline{B}(x_0, A^{m_0+1})} \Big(\sum_{k\le m_0} \frac{\|f\chi_{I_{2,k}}\|^{p_+(I_{2,k})}_{L^{p(\cdot)}(X)}}{\mu(I_{2,k})} \chi_{E_k(x)} \Big) d\mu(x) \le c \sum_{k\le m_0} \|f\chi_{I_{2,k}}\|^{p_+(I_{2,k})}_{L^{p(\cdot)}(X)} \\
&\le c \sum_{k\le m_0} \int_{I_{2,k}} |f(x)|^{p(x)} d\mu(x) \le c \int_X |f(x)|^{p(x)} d\mu(x) \le c.
\end{aligned}$$

Consequently, $I(f) \le c\|f\|_{L^{p(\cdot)}(X)}$. Hence, $S_1(f) \le c\|f\|_{L^{p(\cdot)}(X)}$. Analogously taking into account the fact that $q' \in DL(X)$ and arguing as above we find that $S_2(g) \le$

$c\|g\|_{L^{q'(\cdot)}(X)}$. Thus summarizing these estimates we conclude that

$$\sum_{i\le m_0} \|f\chi_{I_i}\|_{L^{p(\cdot)}(X)}\|g\chi_{I_i}\|_{L^{q'(\cdot)}(X)} \le c\|f\|_{L^{p(\cdot)}(X)}\|g\|_{L^{q'(\cdot)}(X)}.$$

□

Lemma 2.2.7 for $L^{p(\cdot)}([0,1])$ spaces defined with respect to the Lebesgue measure was derived in [55] (see also [24] for $X = \mathbb{R}^n$, $d(x,y) = |x-y|$ and $d\mu(x) = dx$).

2.3 Hardy-type Transforms

This section is devoted to the sufficient conditions governing two-weight inequalities for Hardy-type operators $T_{v,w}, T'_{v,w}$ defined on quasi-metric measure spaces,where

$$T_{v,w}f(x) = v(x)\int\limits_{B_{x_0x}} f(y)w(y)d\mu(y)$$

and

$$T'_{v,w}f(x) = v(x)\int\limits_{X\setminus \overline{B}_{x_0x}} f(y)w(y)d\mu(y).$$

Let a is a positive constant and let p be a measurable function defined on X. Let us introduce the notation:

$$p_0(x) := p_-(\overline{B}_{x_0x});\;\; \widetilde{p}_0(x) := \begin{cases} p_0(x) & \text{if } \; d(x_0,x) \le a; \\ p_c = \text{const} & \text{if } \; d(x_0,x) > a. \end{cases}$$

$$p_1(x) := p_-\left(\overline{B}(x_0,a)\setminus B_{x_0x}\right);\; \widetilde{p}_1(x) := \begin{cases} p_1(x) & \text{if } \; d(x_0,x) \le a; \\ p_c = \text{const} & \text{if } \; d(x_0,x) > a. \end{cases}$$

Remark 2.3.1. If we deal with a quasi-metric measure space with $L < \infty$, then we will assume that $a = L$. Obviously, $\widetilde{p}_0 \equiv p_0$ and $\widetilde{p}_1 \equiv p_1$ in this case.

Theorem 2.3.1. *Let (X, d, μ) be a quasi-metric measure space . Assume that p and q are measurable functions on X satisfying the condition $1 < p_- \leq \widetilde{p}_0(x) \leq q(x) \leq q_+ < \infty$. In the case when $L = \infty$ suppose that $p \equiv p_c \equiv const$, $q \equiv q_c \equiv const$, outside some ball $\overline{B}(x_0, a)$. If the condition*

$$A_1 := \sup_{0 \leq t \leq L} \int\limits_{t < d(x_0,x) \leq L} \big(v(x)\big)^{q(x)} \Bigg(\int\limits_{d(x_0,x) \leq t} w^{(\widetilde{p}_0)'(x)}(y) d\mu(y) \Bigg)^{\frac{q(x)}{(\widetilde{p}_0)'(x)}} d\mu(x) < \infty,$$

hold, then $T_{v,w}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.

Proof. Here we use the arguments of the proofs of Theorem 1.1.4 in [20] (see p. 7) and of Theorem 2.1 in [22]. First we notice that $p_- \leq p_0(x) \leq p(x)$ for all $x \in X$. Let $f \geq 0$ and let $S_p(f) \leq 1$. First assume that $L < \infty$. We denote

$$I(s) := \int\limits_{d(x_0,y) < s} f(y) w(y) d\mu(y) \quad \text{for } s \in [0, L].$$

Suppose that $I(L) < \infty$. Then $I(L) \in (2^m, 2^{m+1}]$ for some $m \in \mathbb{Z}$. Let us denote $s_j := \sup\{s : I(s) \leq 2^j\}$, $j \leq m$, and $s_{m+1} := L$. Then $\{s_j\}_{j=-\infty}^{m+1}$ is a non-decreasing sequence. It is easy to check that $I(s_j) \leq 2^j$, $I(s) > 2^j$ for $s > s_j$, and $2^j \leq \int\limits_{s_j \leq d(x_0,y) \leq s_{j+1}} f(y) w(y) d\mu(y)$. If $\beta := \lim\limits_{j \to -\infty} s_j$, then $d(x_0, x) < L$ if and only if $d(x_0, x) \in [0, \beta] \cup \bigcup\limits_{j=-\infty}^{m} (s_j, s_{j+1}]$. If $I(L) = \infty$ then we take $m = \infty$. Since $0 \leq I(\beta) \leq I(s_j) \leq 2^j$ for every j, we have that $I(\beta) = 0$. It is obvious that $X = \bigcup\limits_{j \leq m} \{x : s_j < d(x_0, x) \leq s_{j+1}\}$.

Further, we have that

$$S_q(T_{v,w} f) = \int\limits_X (T_{v,w} f(x))^{q(x)} d\mu(x) = \int\limits_X \Bigg(v(x) \int\limits_{B(x_0,\ d(x_0,x))} f(y) w(y) d\mu(y) \Bigg)^{q(x)} d\mu(x)$$

$$= \int\limits_X (v(x))^{q(x)} \Bigg(\int\limits_{B(x_0,\ d(x_0,x))} f(y) w(y) d\mu(y) \Bigg)^{q(x)} d\mu(x)$$

$$\leq \sum_{j=-\infty}^{m} \int\limits_{s_j<d(x_0,x)\leq s_{j+1}} \Big(v(x)\Big)^{q(x)} \Bigg(\int\limits_{d(x_0,y)<s_{j+1}} f(y)w(y)d\mu(y)\Bigg)^{q(x)} d\mu(x).$$

Let us denote

$$B_j(x_0) := \{x \in X : s_{j-1} \leq d(x_0,x) \leq s_j\}.$$

Notice that

$$I(s_{j+1}) \leq 2^{j+1} \leq 4 \int\limits_{B_j(x_0)} w(y)f(y)d\mu(y).$$

Consequently, by this estimate and Hölder's inequality with respect to the exponent $p_0(x)$ we find that

$$\begin{aligned} S_q\big(T_{v,w}f\big) &\leq c\sum_{j=-\infty}^{m} \int\limits_{s_j<d(x_0,x)\leq s_{j+1}} \Big(v(x)\Big)^{q(x)} \Bigg(\int\limits_{B_j(x_0)} f(y)w(y)d\mu(y)\Bigg)^{q(x)} d\mu(x) \\ &\leq c\sum_{j=-\infty}^{m} \int\limits_{s_j<d(x_0,x)\leq s_{j+1}} \big(v(x)\big)^{q(x)} J_k(x)d\mu(x) \end{aligned}$$

where

$$J_k(x) := \Bigg(\int\limits_{B_j(x_0)} f(y)^{p_0(x)}d\mu(y)\Bigg)^{\frac{q(x)}{p_0(x)}} \Bigg(\int\limits_{B_j(x_0)} w(y)^{(p_0)'(x)}d\mu(y)\Bigg)^{\frac{q(x)}{(p_0)'(x)}}.$$

Observe now that $q(x) \geq p_0(x)$. Hence, this fact and the condition $S_p(f) \leq 1$ imply that

$$\begin{aligned} J_k(x) &\leq c\Bigg(\int\limits_{B_j(x_0)\cap\{y:f(y)\leq 1\}} f(y)^{p_0(x)}d\mu(y) + \int\limits_{B_j(x_0)\cap\{y:f(y)>1\}} f(y)^{p(y)}d\mu(y)\Bigg)^{\frac{q(x)}{p_0(x)}} \\ &\qquad \times \Bigg(\int\limits_{B_j(x_0)} w(y)^{(p_0)'(x)}d\mu(y)\Bigg)^{\frac{q(x)}{(p_0)'(x)}} \\ &\leq c\Bigg(\mu\big(B_j(x_0)\big) + \int\limits_{B_j(x_0)\cap\{y:f(y)>1\}} f(y)^{p(y)}d\mu(y)\Bigg)\Bigg(\int\limits_{B_j(x_0)} w(y)^{(p_0)'(x)}d\mu(y)\Bigg)^{\frac{q(x)}{(p_0)'(x)}}. \end{aligned}$$

It follows now that

$$S_q(T_{v,w}f) \le c\Big(\sum_{j=-\infty}^{m} \mu\big(B_j(x_0)\big) \int\limits_{s_j<d(x_0,x)\le s_{j+1}} v(x)^{q(x)}$$

$$\times \Big(\int\limits_{B_j(x_0)} w(y)^{(p_0')(x)} d\mu(y) \Big)^{\frac{q(x)}{(p_0)'(x)}} d\mu(x) + \sum_{j=-\infty}^{m} \Big(\int\limits_{B_j(x_0)\cap\{y: f(y)>1\}} f(y)^{p(y)} d\mu(y) \Big)$$

$$\times \int\limits_{s_j<d(x_0,x)\le s_{j+1}} v(x)^{q(x)} \Big(\int\limits_{B_j(x_0)} w(y)^{(p_0)'(x)} d\mu(y) \Big)^{\frac{q(x)}{(p_0)'(x)}} d\mu(x) \Big) =: c\big(N_1 + N_2\big).$$

Since $L < \infty$ it is obvious that

$$N_1 \le A_1 \sum_{j=-\infty}^{m+1} \mu\big(B_j(x_0)\big) \le CA_1$$

and

$$N_2 \le A_1 \sum_{j=-\infty}^{m+1} \int\limits_{B_j(x_0)} f(y)^{p(y)} d\mu(y) \le C \int\limits_X \big(f(y)\big)^{p(y)} d\mu(y) = A_1 S_p(f) \le A_1.$$

Finally $S_q(T_{v,w}f) \le c\big(CA_1 + A_1\big) < \infty$. Thus $T_{v,w}$ is bounded if $A_1 < \infty$.

Let us now suppose that $L = \infty$. We have

$$\begin{aligned} T_{v,w}f(x) &= \chi_{B(x_0,a)}(x)v(x) \int\limits_{B_{x_0 x}} f(y)w(y)d\mu(y) + \chi_{X\setminus B(x_0,a)}(x)v(x) \int\limits_{B_{x_0 x}} f(y)w(y)d\mu(y) \\ &=: T^{(1)}_{v,w}f(x) + T^{(2)}_{v,w}f(x). \end{aligned}$$

By using already proved result for $L < \infty$ and the fact that diam $\big(B(x_0,a)\big) < \infty$ we find that

$$\|T^{(1)}_{v,w}f\|_{L^{q(\cdot)}\big(B(x_0,a)\big)} \le c\|f\|_{L^{p(\cdot)}\big(B(x_0,a)\big)} \le c,$$

because

$$A_1^{(a)} := \sup_{0\le t\le a} \int\limits_{t<d(x_0,x)\le a} \big(v(x)\big)^{q(x)} \Big(\int\limits_{d(x_0,x)\le t} w^{(p_0)'(x)}(y) d\mu(y) \Big)^{\frac{q(x)}{(p_0)'(x)}} d\mu(x) \le A_1 < \infty.$$

Further, observe that

$$T^{(2)}_{v,w}f(x)=\chi_{X\setminus B(x_0,a)}(x)v(x)\int\limits_{B_{x_0x}} f(y)w(y)d\mu(y)=\chi_{X\setminus B(x_0,a)}(x)v(x)\int\limits_{d(x_0,y)\le a} f(y)w(y)d\mu(y)$$

$$+\chi_{X\setminus B(x_0,a)}(x)v(x)\int\limits_{a\le d(x_0,y)\le d(x_0,x)} f(y)w(y)d\mu(y)=:T^{(2,1)}_{v,w}f(x)+T^{(2,2)}_{v,w}f(x).$$

It is easy to see that (see also Theorem 1.1.3 or 1.1.4 of [20]) the condition

$$\overline{A}_1^{(a)}:=\sup_{t\ge a}\Bigg(\int\limits_{d(x_0,x)\ge t}\big(v(x)\big)^{q_c}d\mu(x)\Bigg)^{\frac{1}{q_c}}\Bigg(\int\limits_{a\le d(x_0,y)\le t} w(y)^{(p_c)'}d\mu(y)\Bigg)^{\frac{1}{(p_c)'}}<\infty$$

guarantees the boundedness of the operator

$$T_{v,w}f(x)=v(x)\int\limits_{a\le d(x_0,y)<d(x_0,x)} f(y)w(y)d\mu(y)$$

from $L^{p_c}\big(X\setminus B(x_0,a)\big)$ to $L^{q_c}\big(X\setminus B(x_0,a)\big)$. Thus $T^{(2,2)}_{v,w}$ is bounded. It remains to prove that $T^{(2,1)}_{v,w}$ is bounded. We have

$$\begin{aligned}\|T^{(2,1)}_{v,w}f\|_{L^{p(\cdot)}(X)} &= \Bigg(\int\limits_{\big(B(x_0,a)\big)^c} v(x)^{q_c}d\mu(x)\Bigg)^{\frac{1}{q_c}}\Bigg(\int\limits_{\overline{B}(x_0,a)} f(y)w(y)d\mu(y)\Bigg)\\ &\le \Bigg(\int\limits_{\big(B(x_0,a)\big)^c} v(x)^{q_c}d\mu(x)\Bigg)^{\frac{1}{q_c}}\|f\|_{L^{p(\cdot)}\big(\overline{B}(x_0,a)\big)}\|w\|_{L^{p'(\cdot)}\big(\overline{B}(x_0,a)\big)}.\end{aligned}$$

Observe now that the condition $A_1<\infty$ guarantees that the integral

$$\int\limits_{\big(B(x_0,a)\big)^c} v(x)^{q_c}d\mu(x)$$

is finite. Moreover, $N:=\|w\|_{L^{p'(\cdot)}\big(\overline{B}(x_0,a)\big)}<\infty$. Indeed, we have that

$$N\le\begin{cases}\Bigg(\int\limits_{\overline{B}(x_0,a)} w(y)^{p'(y)}d\mu(y)\Bigg)^{\frac{1}{\big(p_-(\overline{B}(x_0,a))\big)'}} & \text{if } \|w\|_{L^{p'(\cdot)}(\overline{B}(x_0,a))}\le 1,\\ \Bigg(\int\limits_{\overline{B}(x_0,a)} w(y)^{p'(y)}d\mu(y)\Bigg)^{\frac{1}{\big(p_+(\overline{B}(x_0,a))\big)'}} & \text{if } \|w\|_{L^{p'(\cdot)}(\overline{B}(x_0,a)}>1.\end{cases}$$

Further,

$$\int\limits_{\overline{B}(x_0,a)} w(y)^{p'(y)}d\mu(y) = \int\limits_{\overline{B}(x_0,a)\cap\{w\le 1\}} w(y)^{p'(y)}d\mu(y) + \int\limits_{\overline{B}(x_0,a)\cap\{w>1\}} w(y)^{p'(y)}d\mu(y) =: I_1 + I_2.$$

For I_1, we have that $I_1 \le \mu\big(\overline{B}(x_0,a)\big) < \infty$. Since $L = \infty$ and condition (2.2.1) holds, there exists a point $y_0 \in X$ such that $a < d(x_0,y_0) < 2a$. Consequently, $\overline{B}(x_0,a) \subset \overline{B}(x_0,d(x_0,y_0))$ and $p(y) \ge p_-\big(\overline{B}(x_0,d(x_0,y_0))\big) = p_0(y_0)$, where $y \in \overline{B}(x_0,a)$. Consequently, the condition $A_1 < \infty$ yields $I_2 \le \int\limits_{\overline{B}(x_0,a)} w(y)^{(p_0)'(y_0)}dy < \infty$. Finally we have that $\|T^{(2,1)}_{v,w}f\|_{L^{p(\cdot)}(X)} \le C$. Hence, $T_{v,w}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$. □

The proof of the following statement is similar to that of Theorem 2.3.1; therefore we omit it (see also the proofs of Theorem 1.1.3 in [20] and Theorems 2.6 and 2.7 in [22] for similar arguments).

Theorem 2.3.2. *Let (X,d,μ) be a quasi-metric measure space . Assume that p and q are measurable functions on X satisfying the condition $1 < p_- \le \widetilde{p}_1(x) \le q(x) \le q_+ < \infty$. If $L = \infty$, then we assume that $p \equiv p_c \equiv const$, $q \equiv q_c \equiv const$ outside some ball $B(x_0,a)$. If*

$$B_1 := \sup_{0\le t\le L} \int\limits_{d(x_0,x)\le t} \big(v(x)\big)^{q(x)} \Bigg(\int\limits_{t\le d(x_0,x)\le L} w^{(\widetilde{p}_1)'(x)}(y)d\mu(y)\Bigg)^{\frac{q(x)}{(\widetilde{p}_1)'(x)}} d\mu(x) < \infty,$$

then $T'_{v,w}$ is bounded from $L^{p(\cdot)}(X)$ to$L^{q(\cdot)}(X)$.

Remark 2.3.2. If $p \equiv$ const and $q \equiv$ const, then the condition $A_1 < \infty$ in Theorem 2.3.1 (resp. $B_1 < \infty$ in Theorem 2.3.2) is also necessary for the boundedness of $T_{v,w}$ (resp. $T'_{v,w}$) from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$. See [20], pp.4-5, for the details.

2.4 Potentials

In this section we discuss two-weight problem for the potential operators

$$T_{\alpha(\cdot)}f(x) = \int\limits_X \frac{f(y)}{\mu(B(x,d(x,y)))^{1-\alpha(x)}}d\mu(y), \quad x \in X, \quad 0 < \alpha_- \le \alpha_+ < 1,$$

and

$$I_{\alpha(\cdot)}f(x) = \int\limits_X \frac{f(y)}{d(x,y)^{1-\alpha(x)}}d\mu(y), \quad 0 < \alpha_- \le \alpha_+ < 1$$

on quasi-metric measure spaces, where $0 < \alpha_- \le \alpha_+ < 1$. If $\alpha \equiv$ const, then we denote $T_{\alpha(\cdot)}$ and $I_{\alpha(\cdot)}$ by T_α and I_α respectively.

The boundedness of the Riesz potential operators in $L^{p(\cdot)}(\Omega)$ spaces, where Ω is a domain in $\mathbb{R}^n$ was established in [13], [79], [10], [7].

The following result was obtained in [53]:

Theorem 2.4.1. *Let (X, d, μ) be an SHT. Suppose that $1 < p_- \le p_+ < \infty$ and $p \in \mathcal{P}(1)$. Assume that if $L = \infty$, then $p \equiv$ const outside some ball. Let α be a constant satisfying the condition $0 < \alpha < 1/p_+$. We set $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Then T_α is bounded in $L^{p(\cdot)}(X)$.*

Theorem 2.4.2 ([**42**]). *Let (X, d, μ) be a non-homogeneous space with $L < \infty$ and let N be a constant defined by $N = a_1(1 + 2a_0)$, where the constants a_0 and a_1 are taken from the definition of the quasi-metric d. Suppose that $1 < p_- < p_+ < \infty$, $p, \alpha \in \mathcal{P}(N)$ and that μ is upper Ahlfors 1-regular. We define $q(x) = \frac{p(x)}{1-\alpha(x)p(x)}$, where $0 < \alpha_- \le \alpha_+ < 1/p_+$. Then $I_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$.*

For the statements and their proofs of this section we keep the notation of the previous sections and, in addition, introduce the new notation:

$$v_\alpha^{(1)}(x) := v(x)(\mu B_{x_0 x})^{\alpha-1}, \;\; w_\alpha^{(1)}(x) := w^{-1}(x); \; v_\alpha^{(2)}(x) := v(x);$$

$$w_\alpha^{(2)}(x) := w^{-1}(x)(\mu B_{x_0 x})^{\alpha-1};$$

$$F_x := \begin{cases} \{y \in X : \frac{d(x_0,y)L}{A^2 a_1} \le d(x_0,y) \le A^2 L a_1 d(x_0,x)\}, & \text{if } L < \infty \\ \{y \in X : \frac{d(x_0,y)}{A^2 a_1} \le d(x_0,y) \le A^2 a_1 d(x_0,x)\}, & \text{if } L = \infty, \end{cases},$$

where A and a_1 are constants defined in Definition 2.2.7 and the triangle inequality for d respectively.

The following are the main results in this section:

Theorem 2.4.3. *Let (X, d, μ) be an SHT without atoms. Suppose that $1 < p_- \le p_+ < \infty$ and α is a constant satisfying the condition $0 < \alpha < 1/p_+$. Let $p \in \mathcal{P}(1)$. We set $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Further, if $L = \infty$, then we assume that $p \equiv p_c \equiv const$ outside some ball $B(x_0, a)$. Then the inequality*

$$\|v(T_\alpha f)\|_{L^{q(\cdot)}(X)} \le c\|wf\|_{L^{p(\cdot)}(X)} \tag{2.4.1}$$

holds if the following three conditions are satisfied:

(a) $T_{v_\alpha^{(1)}, w_\alpha^{(1)}}$ *is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$;*

(b) $T_{v_\alpha^{(2)}, w_\alpha^{(2)}}$ *is bounded from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$;*

(c) *there is a positive constant b such that one of the following inequality holds:*

(1) $v_+(F_x) \le bw(x)$ *for* $\mu - a.e. x \in X$; (2) $v(x) \le bw_-(F_x)$ *for* $\mu - a.e. x \in X$.

Proof. For simplicity suppose that $L < \infty$. The proof for the case $L = \infty$ is similar to that of the previous case. Recall that the sets $I_{i,k}$, $i = 1, 2, 3$ and E_k are defined in Section 2.2. Let $f \ge 0$ and let $\|g\|_{L^{q'(\cdot)}(X)} \le 1$.

We have

$$\int_X (T_\alpha f)(x)g(x)v(x)d\mu(x) = \sum_{k=-\infty}^{0} \int_{E_k} (T_\alpha f)(x)g(x)v(x)d\mu(x)$$
$$\leq \sum_{k=-\infty}^{0} \int_{E_k} (T_\alpha f_{1,k})(x)g(x)v(x)d\mu(x) + \sum_{k=-\infty}^{0} \int_{E_k} (T_\alpha f_{2,k})(x)g(x)v(x)d\mu(x)$$
$$+ \sum_{k=-\infty}^{0} \int_{E_k} (T_\alpha f_{3,k})(x)g(x)v(x)d\mu(x) =: S_1 + S_2 + S_3,$$

where $f_{1,k} = f \cdot \chi_{I_{1,k}}$, $f_{2,k} = f \cdot \chi_{I_{2,k}}$, $f_{3,k} = f \cdot \chi_{I_{3,k}}$. Observe that if $x \in E_k$ and $y \in I_{1,k}$, then $d(x_0, y) \leq d(x_0, x)/Aa_1$. Consequently, the triangle inequality for d yields $d(x_0, x) \leq A' a_1 a_0 d(x, y)$, where $A' = A/(A-1)$. Hence, by using Remark 2.2.1 we find that $\mu(B_{x_0 x}) \leq c\mu(B_{xy})$. Applying now condition (a) we have that

$$S_1 \leq c \left\| \left(\mu B_{x_0 x}\right)^{\alpha-1} v(x) \int_{B_{x_0 x}} f(y)d\mu(y) \right\|_{L^{q(x)}(X)} \|g\|_{L^{q'(\cdot)}(X)} \leq c\|f\|_{L^{p(\cdot)}(X)}.$$

Further, observe that if $x \in E_k$ and $y \in I_{3,k}$, then $\mu\big(B_{x_0 y}\big) \leq c\mu\big(B_{xy}\big)$. By condition (b) we find that $S_3 \leq c\|f\|_{L^{p(\cdot)}(X)}$.

Now we estimate S_2. Suppose that $v_+(F_x) \leq bw(x)$. Theorem 2.4.1 and Lemma 2.2.7 yield

$$\begin{aligned}
S_2 &\leq \sum_k \|\big(T_\alpha f_{2,k}\big)(\cdot)\chi_{E_k}(\cdot)v(\cdot)\|_{L^{q(\cdot)}(X)} \|g\chi_{E_k}(\cdot)\|_{L^{q'(\cdot)}(X)} \\
&\leq \sum_k \Big(v_+(E_k)\Big) \|(T_\alpha f_{2,k})(\cdot)\|_{L^{q(\cdot)}(X)} \|g(\cdot)\chi_{E_k}(\cdot)\|_{L^{q'(\cdot)}(X)} \\
&\leq c\sum_k \Big(v_+(E_k)\Big) \|f_{2,k}\|_{L^{p(\cdot)}(X)} \|g(\cdot)\chi_{E_k}(\cdot)\|_{L^{q'(\cdot)}(X)} \\
&\leq c\sum_k \|f_{2,k}(\cdot)w(\cdot)\chi_{I_{2,k}}(\cdot)\|_{L^{p(\cdot)}(X)} \|g(\cdot)\chi_{E_k}(\cdot)\|_{L^{q'(\cdot)}(X)} \\
&\leq c\|f(\cdot)w(\cdot)\|_{L^{p(\cdot)}(X)} \|g(\cdot)\|_{L^{q'(\cdot)}(X)} \leq c\|f(\cdot)w(\cdot)\|_{L^{p(\cdot)}(X)}.
\end{aligned}$$

The estimate of S_2 for the case when $v(x) \leq bw_-(F_x)$ is similar to that of the previous one. Details are omitted. □

Theorems 2.4.3, 2.3.1 and 2.3.2 imply the following statement:

Theorem 2.4.4. *Let (X,d,μ) be an SHT. Suppose that $1<p_-\le p_+<\infty$ and α is a constant satisfying the condition $0<\alpha<1/p_+$. Let $p\in\mathcal{P}(1)$. We set $q(x)=\frac{p(x)}{1-\alpha p(x)}$. If $L=\infty$, then we suppose that $p\equiv p_c\equiv$ const outside some ball $B(x_0,a)$. Then inequality* (2.4.1) *holds if the following three conditions are satisfied:*

(i) $$P_1 := \sup_{0<t\le L} \int\limits_{t<d(x_0,x)\le L} \left(\frac{v(x)}{\big(\mu(B_{x_0x})\big)^{1-\alpha}}\right)^{q(x)} \left(\int\limits_{d(x_0,y)\le t} w^{-(\widetilde{p}_0)'(x)}(y)\,d\mu(y)\right)^{\frac{q(x)}{(\widetilde{p}_0)'(x)}} d\mu(x) < \infty;$$

(ii) $$P_2 := \sup_{0<t\le L} \int\limits_{d(x_0,x)\le t} \big(v(x)\big)^{q(x)} \left(\int\limits_{t<d(x_0,y)\le L} \Big(w(y)\big(\mu B_{x_0y}\big)^{1-\alpha}\Big)^{-(\widetilde{p}_1)'(x)} d\mu(y)\right)^{\frac{q(x)}{(\widetilde{p}_1)'(x)}} d\mu(x) < \infty,$$

(iii) *condition (c) of* Theorem 2.4.3 *holds.*

Remark 2.4.1. If $p=p_c\equiv$ const on X, then the conditions $P_i<\infty$, $i=1,2$, are necessary for (2.4.1). Necessity of the condition $P_1<\infty$ follows by taking the test function $f=w^{-(p_c)'}\chi_{B(x_0,t)}$ in (2.4.1) and observing that $\mu B_{xy}\le c\mu B_{x_0x}$ for those x and y which satisfy the conditions $d(x_0,x)\ge t$ and $d(x_0,y)\le t$ (see also [20], Theorem 6.6.1, p. 418 for the similar arguments), while necessity of the condition $P_2<\infty$ can be derived by choosing the test function $f(x)=w^{-(p_c)'}(x)\chi_{X\setminus B(x_0,t)}(x)\big(\mu B_{x_0x}\big)^{(\alpha-1)((p_c)'-1)}$ and taking into account the estimate $\mu B_{xy}\le\mu B_{x_0y}$ for $d(x_0,x)\le t$ and $d(x_0,y)\ge t$.

The next statement follows in the same manner as the previous one. In this case Theorem 2.4.2 is used instead of Theorem 2.4.1. The proof is omitted.

Theorem 2.4.5. *Let (X,d,μ) be a non-homogeneous space with $L<\infty$. Let N be a constant defined by $N=a_1(1+2a_0)$. Suppose that $1<p_-\le p_+<\infty$, $p,\alpha\in\mathcal{P}(N)$ and that μ is upper Ahlfors 1-regular. We define $q(x)=\frac{p(x)}{1-\alpha(x)p(x)}$, where $0<\alpha_-\le\alpha_+<1/p_+$. Then the inequality*

$$\|v(\cdot)(I_{\alpha(\cdot)}f)(\cdot)\|_{L^{q(\cdot)}(X)} \le c\|w(\cdot)f(\cdot)\|_{L^{p(\cdot)}(X)} \tag{2.4.2}$$

holds if

$$\text{(i)}\quad \sup_{0\le t\le L} \int\limits_{t<d(x_0,x)\le L} \left(\frac{v(x)}{\big(d(x_0,x)\big)^{1-\alpha(x)}}\right)^{q(x)} \left(\int\limits_{\overline{B}(x_0,t)} w^{-(p_0)'(x)}(y)d\mu(y)\right)^{\frac{q(x)}{(p_0)'(x)}} d\mu(x) < \infty;$$

$$\text{(ii)}\quad \sup_{0\le t\le L} \int\limits_{\overline{B}(x_0,t)} \big(v(x)\big)^{q(x)} \left(\int\limits_{t<d(x_0,y)\le L} \big(w(y)d(x_0,y)^{1-\alpha(y)}\big)^{-(p_1)'(x)} d\mu(y)\right)^{\frac{q(x)}{(p_1)'(x)}} d\mu(x) < \infty,$$

(iii) *condition* (c) *of* Theorem 2.4.3 *is satisfied.*

Remark 2.4.2. It is easy to check that if p and α are constants, then conditions (i) and (ii) in Theorem 2.4.5 are also necessary for (2.4.2). This follows easily by choosing appropriate test functions in (2.4.2) (see also Remark 2.4.1)

Theorem 2.4.6. *Let (X, d, μ) be an SHT without atoms. Let $1 < p_- \le p_+ < \infty$ and let α be a constant with the condition $0 < \alpha < 1/p_+$. We set $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Assume that p has a minimum at x_0 and that $p \in LH(X)$. Suppose also that if $L = \infty$, then p is constant outside some ball $B(x_0, a)$. Let v and w be positive increasing functions on $(0, 2L)$. Then the inequality*

$$\|v(d(x_0,\cdot))(T_\alpha f)(\cdot)\|_{L^{q(\cdot)}(X)} \le c\|w(d(x_0,\cdot))f(\cdot)\|_{L^{p(\cdot)}(X)} \tag{2.4.3}$$

holds if

$$I_1 := \sup_{0<t\le L} \int\limits_{t<d(x_0,x)\le L} \left(\frac{v(d(x_0,x))}{\big(\mu(B_{x_0x})\big)^{1-\alpha}}\right)^{q(x)} \left(\int\limits_{d(x_0,y)\le t} w^{-(\widetilde{p}_0)'(x)}(d(x_0,y))d\mu(y)\right)^{\frac{q(x)}{(\widetilde{p}_0)'(x)}} d\mu(x) < \infty$$

for $L = \infty$;

$$J_1 := \sup_{0<t\le L} \int\limits_{t<d(x_0,x)\le L} \left(\frac{v(d(x_0,x))}{\big(\mu(B_{x_0x})\big)^{1-\alpha}}\right)^{q(x)} \left(\int\limits_{d(x_0,y)\le t} w^{-p'(x_0)}(d(x_0,y))d\mu(y)\right)^{\frac{q(x)}{p'(x_0)}} d\mu(x) < \infty$$

for $L < \infty$.

Proof. We prove the theorem for $L = \infty$. The proof for the case when $L < \infty$ is similar. Observe that by Lemma 2.2.5 the condition $p \in LH(X)$ implies $p \in \mathcal{P}(1)$. We will show that the condition $I_1 < \infty$ implies the inequality $\frac{v(A^2 a_1 t)}{w(t)} \leq C$ for all $t > 0$, where A and a_1 are constants defined in Definition 2.2.7 and the triangle inequality for d respectively. Indeed, let us assume that $t \leq b_1$, where b_1 is a small positive constant. Then, taking into account the monotonicity of v and w, and the facts that $\widetilde{p}_0(x) = p_0(x)$ (for small $d(x_0, x)$) and $\mu \in RDC(X)$, we have

$$\begin{aligned} I_1(t) &\geq \int\limits_{A^2 a_1 t \leq d(x_0,x) < A^3 a_1 t} \left(\frac{v(A^2 a_1 t)}{w(t)}\right)^{q(x)} \big(\mu B(x_0,t)\big)^{(\alpha - 1/p_0(x))q(x)} d\mu(x) \\ &\geq \left(\frac{v(A^2 a_1 t)}{w(t)}\right)^{q_-} \int\limits_{A^2 a_1 t \leq d(x_0,x) < A^3 a_1 t} \big(\mu B(x_0,t)\big)^{(\alpha - 1/p_0(x))q(x)} d\mu(x) \geq c \bigg(\frac{v(A^2 a_1 t)}{w(t)}\bigg)^{q_-}. \end{aligned}$$

Hence, $\bar{c} := \overline{\lim\limits_{t \to 0}} \frac{v(A^2 a_1 t)}{w(t)} < \infty$. Further, if $t > b_2$, where b_2 is a large number, then since p and q are constants, for $d(x_0, x) > t$, we have that

$$\begin{aligned} I_1(t) \geq \Bigg(\int\limits_{A^2 a_1 t \leq d(x_0,x) < A^3 a_1 t} & v(d(x_0,x))^{q_c} \big(\mu B(x_0,t)\big)^{(\alpha-1)q_c} d\mu(x) \Bigg) \\ & \times \Bigg(\int\limits_{B(x_0,t)} w^{-(p_c)'}(x) d\mu(x) \Bigg)^{q_c/(p_c)'} d\mu(x) \\ \geq C \bigg(\frac{v(A^2 a_1 t)}{w(t)}\bigg)^{q_c} & \int\limits_{A^2 a_1 t \leq d(x_0,x) < A^3 a_1 t} \big(\mu B(x_0,t)\big)^{(\alpha - 1/p_c)q_c} d\mu(x) \geq c \bigg(\frac{v(A^2 a_1 t)}{w(t)}\bigg)^{q_c}. \end{aligned}$$

In the last inequality we used the fact that μ satisfies the reverse doubling condition. Now we show that the condition $I_1 < \infty$ implies

$$\begin{aligned} \sup_{t>0} I_2(t) := \sup_{t>0} \int\limits_{d(x_0,x) \leq t} (v(d(x_0,x)))^{q(x)} \Bigg(& \int\limits_{d(x_0,y) > t} w^{-(\widetilde{p}_1)'(x)}(d(x_0,y)) \\ & \times \big(\mu(B_{x_0 y})\big)^{(\alpha-1)(\widetilde{p}_1)'(x)} d\mu(y) \Bigg)^{\frac{q(x)}{(\widetilde{p}_1)'(x)}} d\mu(x) < \infty. \end{aligned}$$

Due to monotonicity of functions v and w, the condition $p \in LH(X)$, Proposition 2.2.1, Lemma 2.2.4, Lemma 2.2.5 and the assumption that p has a minimum at x_0, we find that for all $t > 0$,

$$\begin{aligned} I_2(t) &\leq \int\limits_{d(x_0,x)\leq t} \Big(\frac{v(t)}{w(t)}\Big)^{q(x)} \Big(\mu\big(B(x_0,t)\big)\Big)^{(\alpha-1/p(x_0))q(x)} d\mu(x) \\ &\leq c \int\limits_{d(x_0,x)\leq t} \Big(\frac{v(t)}{w(t)}\Big)^{q(x)} \Big(\mu\big(B(x_0,t)\big)\Big)^{\big(\alpha-1/p(x_0)\big)q(x_0)} d\mu(x) \\ &\leq c\Big(\int\limits_{d(x_0,x)\leq t} \Big(\frac{v(A^2a_1t)}{w(t)}\Big)^{q(x)} d\mu(x)\Big)\Big(\mu\big(B(x_0,t)\big)\Big)^{-1} \leq C. \end{aligned}$$

Now Theorem 2.4.4 completes the proof. □

Theorem 2.4.7. *Let (X,d,μ) be an SHT with $L<\infty$. Suppose that p, q and α are measurable functions on X satisfying the conditions: $1 < p_- \leq p(x) \leq q(x) \leq q_+ < \infty$ and $1/p_- < \alpha_- \leq \alpha_+ < 1$. Assume that $\alpha \in LH(X)$ and there is a point $x_0 \in X$ such that $p, q \in LH(X, x_0)$. Suppose also that w is a positive increasing function on $(0, 2L)$. Then the inequality*

$$\|\big(T_{\alpha(\cdot)}f\big)v\|_{L^{q(\cdot)}(X)} \leq c\|w(d(x_0,\cdot))f(\cdot)\|_{L^{p(\cdot)}(X)}$$

holds if the following two conditions are satisfied:

$$\begin{aligned} \widetilde{I}_1 := \sup_{0<t\leq L} \int\limits_{t\leq d(x_0,x)\leq L} &\left(\frac{v(x)}{\big(\mu B_{x_0x}\big)^{1-\alpha(x)}}\right)^{q(x)} \\ &\times \Big(\int\limits_{d(x_0,x)\leq t} w^{-(p_0)'(x)}(d(x_0,y))d\mu(y)\Big)^{\frac{q(x)}{(p_0)'(x)}} d\mu(x) < \infty; \end{aligned}$$

$$\begin{aligned} \widetilde{I}_2 := \sup_{0<t\leq L} \int\limits_{d(x_0,x)\leq t} \big(v(x)\big)^{q(x)} \Bigg(\int\limits_{t\leq d(x_0,x)\leq L} &\Big(w(d(x_0,y)) \\ &\times \big(\mu B_{x_0y}\big)^{1-\alpha(x)}\Big)^{-(p_1)'(x)} d\mu(y)\Bigg)^{\frac{q(x)}{(p_1)'(x)}} d\mu(x) < \infty. \end{aligned}$$

Proof. For simplicity assume that $L = 1$. First observe that by Lemma 2.2.5 we have $p, q \in \mathcal{P}(1, x_0)$ and $\alpha \in \mathcal{P}(1)$. Suppose that $f \geq 0$ and $S_p\big(w(d(x_0, \cdot))f(\cdot)\big) \leq 1$. We will show that $S_q\big(v(T_{\alpha(\cdot)} f)\big) \leq C$.

We have

$$S_q\big(vT_{\alpha(\cdot)}f\big) \leq C_q\Bigg[\int\limits_X \Bigg(v(x) \int\limits_{d(x_0,y)\leq d(x_0,x)/(2a_1)} f(y)\big(\mu B_{xy}\big)^{\alpha(x)-1} d\mu(y)\Bigg)^{q(x)} d\mu(x)$$
$$+ \int\limits_X \Bigg(v(x) \int\limits_{d(x_0,x)/(2a_1)\leq d(x_0,y)\leq 2a_1 d(x_0,x)} f(y)\big(\mu B_{xy}\big)^{\alpha(x)-1} d\mu(y)\Bigg)^{q(x)} d\mu(x)$$
$$+ \int\limits_X \Bigg(v(x) \int\limits_{d(x_0,y)\geq 2a_1 d(x_0,x)} f(y)\big(\mu B_{xy}\big)^{\alpha(x)-1} d\mu(y)\Bigg)^{q(x)} d\mu(x)\Bigg] =: C_q[I_1 + I_2 + I_3].$$

First observe that by virtue of the doubling condition for μ, Remark 2.2.1 and simple calculation we find that $\mu\big(B_{x_0x}\big) \leq c\mu\big(B_{xy}\big)$. Taking into account this estimate and Theorem 2.3.1 we have that

$$I_1 \leq c \int\limits_X \Bigg(\frac{v(x)}{\big(\mu B_{x_0x}\big)^{1-\alpha(x)}} \int\limits_{d(x_0,y)<d(x_0,x)} f(y) d\mu(y)\Bigg)^{q(x)} d\mu(x) \leq C.$$

Further, it is easy to see that if $d(x_0, y) \geq 2a_1 d(x_0, x)$, then the triangle inequality for d and the doubling condition for μ yield that $\mu B_{x_0y} \leq c\mu B_{xy}$. Hence due to Proposition 2.2.2 we see that

$$\big(\mu B_{x_0y}\big)^{\alpha(x)-1} \geq c\big(\mu B_{xy}\big)^{\alpha(y)-1}$$

for such x and y. Therefore, Theorem 2.3.2 implies that $I_3 \leq C$.

It remains to estimate I_2. Let us denote:

$$E^{(1)}(x) := \overline{B}_{x_0x} \setminus B\big(x_0, d(x_0, x)/(2a_1)\big); \;\; E^{(2)}(x) := \overline{B}\big(x_0, 2a_1 d(x_0, x)\big) \setminus B_{x_0x}.$$

Then we have that

$$I_2 \le C\Bigg[\int\limits_X \Big[v(x) \int\limits_{E^{(1)}(x)} f(y)\big(\mu B_{xy}\big)^{\alpha(x)-1} d\mu(y)\Big]^{q(x)} d\mu(x) + \int\limits_X \Big[v(x) \int\limits_{E^{(2)}(x)} f(y)\big(\mu B_{xy}\big)^{\alpha(x)-1} d\mu(y)\Big]^{q(x)} d\mu(x)\Bigg] =: c[I_{21} + I_{22}].$$

Using Hölder's inequality for the classical Lebesgue spaces we find that

$$I_{21} \le \int\limits_X v^{q(x)}(x)\Bigg(\int\limits_{E^{(1)}(x)} w^{p_0(x)}(d(x_0,y))(f(y))^{p_0(x)} d\mu(y)\Bigg)^{q(x)/p_0(x)} \times \Bigg(\int\limits_{E^{(1)}(x)} w^{-(p_0)'(x)}(d(x_0,y))\big(\mu B_{xy}\big)^{(\alpha(x)-1)(p_0)'(x)} d\mu(y)\Bigg)^{q(x)/(p_0)'(x)} d\mu(x).$$

Denote the first inner integral by $J^{(1)}$ and the second one by $J^{(2)}$.

By using the fact that $p_0(x) \le p(y)$, where $y \in E^{(1)}(x)$, we see that

$$J^{(1)} \le \mu(B_{x_0 x}) + \int\limits_{E^{(1)}(x)} (f(y))^{p(y)}\big(w(d(x_0,y))\big)^{p(y)} d\mu(y),$$

while by applying Lemma 2.2.4, for $J^{(2)}$, we have that

$$\begin{aligned} J^{(2)} &\le cw^{-(p_0)'(x)}\Big(\frac{d(x_0,x)}{2a_1}\Big) \int\limits_{E^{(1)}(x)} \Big(\mu B_{xy}\Big)^{(\alpha(x)-1)(p_0)'(x)} d\mu(y) \\ &\le cw^{-(p_0)'(x)}\Big(\frac{d(x_0,x)}{2a_1}\Big)\Big(\mu B_{x_0 x}\Big)^{(\alpha(x)-1)(p_0)'(x)+1}. \end{aligned}$$

Summarizing these estimates for $J^{(1)}$ and $J^{(2)}$ we conclude that

$$I_{21} \le \int\limits_X v^{q(x)}(x)\big(\mu B_{x_0 x}\big)^{q(x)\alpha(x)} w^{-q(x)}\Big(\frac{d(x_0,x)}{2a_1}\Big) d\mu(x) + \int\limits_X v^{q(x)}(x) \times \Bigg(\int\limits_{E^{(1)}(x)} w^{p(y)}(d(x_0,y))(f(y))^{p(y)} d\mu(y)\Bigg)^{q(x)/p_0(x)} \big(\mu B_{x_0 x}\big)^{q(x)(\alpha(x)-1/p_0(x))} \times w^{-q(x)}\Big(\frac{d(x_0,x)}{2a_1}\Big) d\mu(x) =: I_{21}^{(1)} + I_{21}^{(2)}.$$

By applying monotonicity of w, the reverse doubling property for μ with the constants A and B (see Remark 2.2.3), and the condition $\widetilde{I}_1 < \infty$ we have that

$$I_{21}^{(1)} \leq c \sum_{k=-\infty}^{0} \int\limits_{\overline{B}(x_0,A^k)\setminus B(x_0,A^{k-1})} v(x)^{q(x)} \Bigg(\int\limits_{B\left(x_0, \frac{A^{k-1}}{2a_1}\right)} w^{-(p_0)'(x)}(d(x_0,y))d\mu(y) \Bigg)^{\frac{q(x)}{(p_0)'(x)}}$$

$$\times \left(\mu B_{x_0,x}\right)^{\frac{q(x)}{p_0(x)}+(\alpha(x)-1)q(x)} d\mu(x) \leq c \sum_{k=-\infty}^{0} \left(\mu \overline{B}(x_0,A^k)\right)^{q_-/p_+}$$

$$\times \int\limits_{\overline{B}(x_0,A^k)\setminus B(x_0,A^{k-1})} v(x)^{q(x)} \Bigg(\int\limits_{B\left(x_0,A^k\right)} w^{-(p_0)'(x)}(d(x_0,y))d\mu(y) \Bigg)^{\frac{q(x)}{(p_0)'(x)}}$$

$$\times \left(\mu B_{x_0,x}\right)^{q(x)(\alpha(x)-1)} d\mu(x) \leq c \sum_{k=-\infty}^{0} \left(\mu \bar{B}(x_0,A^k) \setminus B(x_0,A^{k-1})\right)^{q_-/p_+}$$

$$\leq c \sum_{k=-\infty}^{0} \int\limits_{\mu\bar{B}(x_0,A^k)\setminus B(x_0,A^{k-1})} \left(\mu B_{x_0,x}\right)^{q_-/p_+-1} d\mu(y) \leq c \int\limits_{X} \left(\mu B_{x_0,x}\right)^{q_-/p_+-1} d\mu(y) < \infty.$$

Due to the facts that $q(x) \geq p_0(x)$, $S_p\big(w\big(d(x_0,\cdot)f(\cdot)\big)\big) \leq 1$, $\widetilde{I}_1 < \infty$ and w is increasing, for $I_{21}^{(2)}$, we find that

$$I_{21}^{(2)} \leq c \sum_{k=-\infty}^{0} \Bigg(\int\limits_{\mu\bar{B}(x_0,A^{k+1}a_1)\setminus B(x_0,A^{k-2})} w^{p(y)}(d(x_0,y))(f(y))^{p(y)} d\mu(y) \Bigg)$$

$$\times \Bigg(\int\limits_{\mu\bar{B}(x_0,A^k)\setminus B(x_0,A^{k-1})} v^{q(x)}(x) \Bigg(\int\limits_{B(x_0,A^{k-1})} w^{-(p_0)'(x)}(d(x_0,y))d\mu(y) \Bigg)^{\frac{q(x)}{(p_0)'(x)}}$$

$$\times \left(\mu B_{x_0,x}\right)^{(\alpha(x)-1)q(x)} d\mu(x) \Bigg) \leq c S_p(f(\cdot)w(d(x_0,\cdot)) \leq c.$$

Analogously, it follows the estimate for I_{22}. In this case we use the condition $\widetilde{I}_2 < \infty$ and the fact that $p_1(x) \leq p(y)$ when $d(x_0,x) \leq d(x_0,y) < 2a_1 d(x_0,x)$. The details are omitted. The theorem is proved. □

Taking into account the proof of Theorem 2.4.7 we can easily derive the following statement the proof of which is omitted:

Theorem 2.4.8. *Let* (X, d, μ) *be an SHT with* $L < \infty$*. Suppose that* p*,* q *and* α *are measurable functions on* X *satisfying the conditions* $1 < p_- \le p(x) \le q(x) \le q_+ < \infty$ *and* $1/p_- < \alpha_- \le \alpha_+ < 1$*. Assume that* $\alpha \in LH(X)$*. Suppose also that there is a point* x_0 *such that* $p, q \in LH(X, x_0)$ *and* p *has a minimum at* x_0*. Let* v *and* w *be positive increasing function on* $(0, 2L)$ *satisfying the condition* $J_1 < \infty$ *(see* Theorem 2.4.6 *). Then inequality* (2.4.3) *is fulfilled.*

Theorem 2.4.9. *Let* (X, d, μ) *be an SHT with* $L < \infty$ *and let* μ *be upper Ahlfors* 1*-regular. Suppose that* $1 < p_- \le p_+ < \infty$ *and that* $p \in \overline{LH}(X)$*. Let* p *have a minimum at* x_0*. Assume that* α *is constant satisfying the condition* $\alpha < 1/p_+$*. We set* $q(x) = \frac{p(x)}{1-\alpha p(x)}$*. If* v *and* w *are positive increasing functions on* $(0, 2L)$ *satisfying the condition*

$$E := \sup_{0 \le t \le L} \int\limits_{t < d(x_0,x) \le L} \left(\frac{v(d(x_0,x))}{\big(d(x_0,x)\big)^{1-\alpha}} \right)^{q(x)} \Bigg(\int\limits_{d(x_0,x) \le t} w^{-(p_0)'(x)}(y) d\mu(y) \Bigg)^{\frac{q(x)}{(p_0)'(x)}} d\mu(x) < \infty,$$

then the inequality

$$\|v\big(d(x_0,\cdot)\big)(I_\alpha f)(\cdot)\|_{L^{q(\cdot)}(X)} \le c\|w\big(d(x_0,\cdot)\big) f(\cdot)\|_{L^{p(\cdot)}(X)}$$

holds.

Proof. The proof is similar to that of Theorem 2.4.6. We only discuss some details. First observe that due to Remark 2.2.2 we have that $p \in \mathcal{P}(N)$, where $N = a_1(1+2a_0)$. It is easy to check that the condition $E < \infty$ implies that $\frac{v(A^2 a_1 t)}{w(t)} \le C$ for all t, where the constant A is defined in Definition 2.2.7 and a_1 is from the triangle inequality for d. Further, Lemmas 2.2.4, 2.2.5, the fact that p has a minimum at x_0 and the inequality

$$\int\limits_{d(x_0,y) > t} \big(d(x_0,y)\big)^{(\alpha-1)(p_1)'(x)} d\mu(y) \le c t^{(\alpha-1)(p_1)'(x)+1},$$

where the constant c does not depend on t and x, yield that

$$\sup_{0\le t\le L} \int\limits_{d(x_0,x)\le t} (v(d(x_0,x)))^{q(x)} \Bigg(\int\limits_{d(x_0,y)>t} \bigg(\frac{w(d(x_0,y))}{\big(d(x_0,y)\big)^{1-\alpha}} \bigg)^{-(p_1)'(x)} d\mu(y) \Bigg)^{\frac{q(x)}{(p_1)'(x)}} d\mu(x) < \infty.$$

Theorem 2.4.5 completes the proof. □

Example 2.4.10. *Let $v(t) = t^{\gamma}$ and $w(t) = t^{\beta}$, where γ and β are constants satisfying the condition $0 \le \beta < 1/(p_-)'$, $\gamma \ge \max\{0, \quad 1-\alpha-\frac{1}{q_+}-\frac{q_-}{q_+}(-\beta+\frac{1}{(p_-)'})\}$. Then (v,w) satisfies the conditions of* Theorem 2.4.6.

2.5 Maximal and Singular Operators

In this section we deal with weighted estimates for the maximal and singular operators defined on X:

$$Mf(x) := \sup_{x\in X, r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f(y)| d\mu(y)$$

and

$$Kf(x) = p.v. \int\limits_{X} k(x,y) f(y) d\mu(y),$$

where $k : X \times X \setminus \{(x,x) : x \in X\} \to \mathbb{R}$ be a measurable function satisfying the conditions:

$$|k(x,y)| \le \frac{c}{\mu B(x, d(x,y))}, \quad x,y \in X, \quad x \ne y;$$

$$|k(x_1,y) - k(x_2,y)| + |k(y,x_1) - k(y,x_2)| \le c\omega\Big(\frac{d(x_2,x_1)}{d(x_2,y)}\Big) \frac{1}{\mu B(x_2, d(x_2,y))}$$

for all x_1, x_2 and y with $d(x_2,y) \ge cd(x_2,x_1)$, where ω is a positive non-decreasing function on $(0,\infty)$ which satisfies the Δ_2 condition: $\omega(2t) \le c\omega(t)$ $(t > 0)$; and the Dini condition: $\int_0^1 \big(\omega(t)/t\big) dt < \infty$.

We also assume that for some constant s, $1 < s < \infty$, and all $f \in L^s(X)$ the limit $Kf(x)$ exists almost everywhere on X and that K is bounded in $L^s(X)$.

It is known (see, e.g., [29]) that if r is constant such that $1 < r < \infty$, (X, d, μ) is an SHT and the weight function $w \in A_r(X)$, i.e.

$$\sup_B \left(\frac{1}{\mu(B)} \int_B w(x) d\mu(x) \right) \left(\frac{1}{\mu(B)} \int_B w^{1-r'}(x) d\mu(x) \right)^{r-1} < \infty,$$

where the supremum is taken over all balls B in X, then the one-weight inequality $\|w^{1/r} K f\|_{L^r(X)} \leq c \|w^{1/r} f\|_{L^r(X)}$ holds.

The boundedness of Calderón-Zygmund operators in $L^{p(\cdot)}(\mathbb{R}^n)$ was establish in [14].

Theorem 2.5.1 (**[50]**). *Let $1 < p_- \leq p_+ < \infty$ and let (X, d, μ) be an SHT. Suppose that $p \in \mathcal{P}(1)$. Then the singular operator K is bounded in $L^{p(\cdot)}(X)$.*

The next statement for metric measure spaces was proved in [31] (see also [41], [42] for quasi-metric measure spaces).

Theorem 2.5.2. *Let (X, d, μ) be an SHT and let $\mu(X) < \infty$. Suppose that $1 < p_- \leq p_+ < \infty$ and $p \in \mathcal{P}(1)$. Then M is bounded in $L^{p(\cdot)}(X)$.*

To prove the next theorem we need the following lemma which can be found in [12] for Euclidian spaces and in [41] for quasi-metric measure spaces.

Lemma 2.5.3. *Let $1 \leq q_- \leq q_+ < \infty$. Suppose that $q \in \mathcal{P}(1)$. Let $\mu(X) < \infty$.Then there is a positive constant c depend on X such that*

$$(Mf(x))^{q(x)} \leq c[M(|f|^{q(\cdot)})(x) + 1]$$

for all $x \in X$.

The next statement was given in the paper by M. Khabazi [34]:

Theorem 2.5.4. *Let (X, d, μ) be an SHT and let $L = \infty$. Suppose that $1 < p_- \leq p_+ < \infty$ and $p \in \mathcal{P}(1)$. Suppose also that $p = p_c = const$ outside a ball $B := B(x_0, R)$ for $x_0 \in X$ and $r > 0$. Then M is bounded in $L^{p(\cdot)}(X)$.*

Before formulating the the next result we introduce the notation:

$$\overline{v}(x) := \frac{v(x)}{\mu(B_{x_0 x})}, \quad \widetilde{w}(x) := \frac{1}{w(x)}, \quad \widetilde{w}_1(x) := \frac{1}{w(x)\mu(B_{x_0 x})}.$$

Theorem 2.5.5. *Let (X, d, μ) be an SHT and let $1 < p_- \leq p_+ < \infty$. Suppose that $p \in LH(X)$. If $L = \infty$, then we assume that p is constant outside a ball $B(x_0, a)$ for some $x_0 \in X$ and $a > 0$. Then the inequality*

$$\|v(Nf)\|_{L^{p(\cdot)}(X)} \leq C\|wf\|_{L^{p(\cdot)}(X)}, \tag{2.5.1}$$

where N is M or K, holds if

(a) $T_{\overline{v},\widetilde{w}}$ *is bounded in* $L^{p(\cdot)}(X)$*;*

(b) $T'_{v,\widetilde{w}_1}$ *is bounded in* $L^{p(\cdot)}(X)$*;*

(c) *there is a positive constant b such that one of the following inequality holds:* (1) $v_+(F_x) \leq bw(x)$ *for* $\mu - a.e. x \in X$; (2) $v(x) \leq bw_-(F_x)$ *for* $\mu - a.e. x \in X$. *where F_x is defined in Section 2.4.*

Proof. First notice that by Lemma 2.2.5 we have that $p \in \mathcal{P}(1)$. Suppose that $L = \infty$ and let $\|g\|_{L^{p'(\cdot)}(X)} \leq 1$. Let us assume that

$$B := B(x, r); \quad h_B := \frac{1}{\mu B} \int_B |h(y)| dy.$$

We have

$$\int_X (Nf)(x)v(x)g(x)d\mu(x) \leq \sum_{j=1}^{3} \left[\sum_{k \in \mathbb{Z}} \int_{E_k} (Nf_{j,k})(x)v(x)g(x)d\mu(x) \right] =: \sum_{j=1}^{3} S_j,$$

where $f_{j,k} := f\chi_{I_{j,k}}$ (recall that the constant A is defined in Definition 2.2.7). We prove the theorem for the case $N = M$. If $x \in E_k$ and $y \in I_{1,k}$, then $\frac{d(x_0,x)}{A'} \leq d(x,y)$, where

$A' := A/(A-1)$. Further, if $r \le \frac{d(x_0,x)}{A'}$, then $B(x,r) \cap \left\{ y : d(x_0,y) \ge \frac{d(x_0,x)}{A'} \right\} = \emptyset$. Consequently, $(f_{1,k})_B = 0$. Let now $r > \frac{d(x_0,x)}{A'}$. Then taking into account Remark 2.2.1 we have

$$(f_{1,k})_B \le \frac{c}{\mu(B_{x_0x})} \int\limits_{B_{x_0x}} |f(y)| d\mu(y)$$

for $x \in E_k$. Hence,

$$Mf_{1,k}(x) \le \frac{c}{\mu(B_{x_0x})} \int\limits_{B_{x_0x}} |f(y)| d\mu(y).$$

Consequently, due to Theorem 2.3.1 and condition (a) we find that

$$S_1 \le c \int\limits_X \big(T_{\overline{v},1}(|f|)(x) g(x) dx \le c \|(T_{\overline{v},1}(|f|)\|_{L^{p(\cdot)}(X)} \|g\|_{L^{p'(\cdot)}(X)} \le c\|fw\|_{L^{p(\cdot)}(X)}.$$

To estimate S_3, first observe that

$$M(f\chi_{I_{3,k}})(x) \le c \sup_{j \ge k+1} \Big(\mu B\big(x, A^j\big)\Big)^{-1} \int\limits_{D_j} |f(y)| d\mu(y), \qquad x \in E_k, \tag{2.5.2}$$

where $D_j := B(x_0, a_1 A^{j+1}) \setminus B(x_0, a_1 A^j)$. To prove (2.5.2) we take r so that $0 < r < A^k$. Then it is easy to see that $B(x,r) \cap I_{3,k} = \emptyset$. Consequently, $(f_{3,k})_B = 0$. Further, let $r \ge A^k$. Then $r \in [A^m, A^{m+1})$ for some $m \ge k$. If $y \in B$, then $d(x_0,y) \le a_1 A^{m+l+1}$ for the integer l defined by $l = \left[\frac{\ln 2}{\ln A}\right] + 1$. On the other hand, there are positive constants b_1 and b_2 such that the inequality

$$\mu B(x_0, A^m) \le b_1 \mu B(x, A^m) \le b_2 \mu B(x_0, A^m),$$

when $x \in E_k$ and $m \ge k$. Consequently, applying the reverse doubling condition, for such r we have

$$(f_{3,k})_B \le \frac{1}{\mu B(x, A^m)} \int\limits_{a_1 A^{k+1} < d(x_0,y) \le a_1 A^{m+l+2}} |f(y)| d\mu(y)$$

$$\le \frac{1}{\mu B(x_0, A^m)} \sum_{j=k+1}^{m+l+1} \int\limits_{D_j} |f(y)| d\mu(y) \le c \sup_{j \ge k+1} \big(\mu B(x, A^i)\big)^{-1} \int\limits_{D_j} |f(y)| d\mu(y) =: \sup_{j \ge k+1} P_j(f),$$

where the positive constant c depends on the constant A.

Further, taking into account condition (b) and the inequality $\sup \leq \sum$, we find that

$$S_3 \leq c \sum_k \Big(\int_{E_k} v(x)g(x)d\mu(x) \Big) \Big(\sum_{j=k+1}^{\infty} P_j(f) \Big)$$

$$= c \sum_j \Big(\mu B(x_0, A^j) \Big)^{-1} \Big(\int_{D_j} |f(y)| d\mu(y) \Big) \sum_{k=-\infty}^{j-1} \Big(\int_{E_k} v(x)g(x)d\mu(x) \Big)$$

$$= c \sum_j \Big(\mu B(x_0, A^j) \Big)^{-1} \Big(\int_{D_j} |f(y)| d\mu(y) \Big) \Big(\int_{B(x_0,A^j)} v(x)g(x)d\mu(x) \Big)$$

$$\leq c \sum_j \Big(\mu B(x_0, A^j) \Big)^{-1} \Big(\int_{D_j} |f(y)| \Big(\mu B(x_0, d(x_0,y)) \Big)^{-1} \Big(\int_{\overline{B}(x_0,d(x_0,y))} v(x)g(x)d\mu(x) \Big) d\mu(y) \Big)$$

$$\leq c \int_X v(x)g(x) \Big(\int_{d(x_0,y) \geq d(x_0,x)} |f(y)| \Big(\mu B(x_0, d(x_0,y))) \Big)^{-1} d\mu(y) \Big) d\mu(x)$$

$$\leq \|g\|_{L^{p'(\cdot)}(X)} \big\| T'_{v(\cdot),d(x_0,\cdot)} f \big\|_{L^{p(\cdot)}(X)} \leq c \|f\|_{L^{p(\cdot)}(X)}.$$

If, for example, (i) of condition (c) is satisfied, then Theorem 2.5.4 and Lemma 2.2.7 yield

$$\begin{aligned} S_2 &\leq \sum_k \Big(v_+(E_k) \Big) \|Mf_{2,k}(\cdot)\|_{L^{p(\cdot)}(X)} \|g(\cdot)\chi_{E_k}(\cdot)\|_{L^{p'(\cdot)}(X)} \\ &\leq \sum_k \Big(v_+(E_k) \Big) \|f\chi_{I_{2,k}}(\cdot)\|_{L^{p(\cdot)}(X)} \|g(\cdot)\chi_{E_k}(\cdot)\|_{L^{p'(\cdot)}(X)} \\ &\leq c \sum_k \|fw\chi_{I_{2,k}}(\cdot)\|_{L^{p(\cdot)}(X)} \|g(\cdot)\chi_{E_k}(\cdot)\|_{L^{p'(\cdot)}(X)} \leq c\|fw(\cdot)\|_{L^{p(\cdot)}(X)}. \end{aligned}$$

When (ii) is satisfied, then by the same arguments we have the desired result.

The proof of the theorem for the operator $N = K$ is similar to that of the case $N = M$. In this case Theorem 2.5.1 is used instead of Theorem 2.5.4 The details are omitted. □

The next two statements are direct consequences of Theorems 2.5.5, 2.3.1, 2.3.2 (see also appropriate statements in Section 2.4). Details are omitted.

Theorem 2.5.6. *Let (X, d, μ) be an SHT and let $1 < p_- \le p_+ < \infty$. Further suppose that $p \in LH(X)$. If $L = \infty$, then we assume that there is a $x_0 \in X$ and a positive constant a such that $p \equiv p_c \equiv const$ outside $B(x_0, a)$. Let N be M or K. Then inequality* (2.5.1) *holds if:*

(a) $$\sup_{0\le t<L} \int\limits_{t\le d(x_0,x)<L} \left(\frac{v(x)}{\mu B_{x_0,x}}\right)^{p(x)} \left(\int\limits_{\overline{B}(x_0,t)} w^{-(\widetilde{p}_0)'(x)}(y)d\mu(y)\right)^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} d\mu(x) < \infty,$$

(b) $$\sup_{0\le t<L} \int\limits_{\overline{B}(x_0,t)} \big(v(x)\big)^{p(x)} \left(\int\limits_{t\le d(x_0,x)<L} \left(\frac{w(y)}{\mu B_{x_0 y}}\right)^{-(\widetilde{p}_1)'(x)} d\mu(y)\right)^{\frac{p(x)}{(\widetilde{p}_1)'(x)}} d\mu(x) < \infty,$$

(c) *condition* (c) *of* Theorem 2.5.5 *is satisfied.*

Theorem 2.5.7. *Let (X, d, μ) be an SHT without atoms. Let $1 < p_- \le p_+ < \infty$. Assume that p has a minimum at x_0 and that $p \in LH(X)$. If $L = \infty$ we also assume that $p \equiv p_c \equiv const$ outside some ball $B(x_0, a)$. Let v and w be positive increasing functions on $(0, 2L)$. Then the inequality*

$$\|v(d(x_0,\cdot)(Nf)(\cdot)\|_{L^{p(\cdot)}(X)} \le c\|w(d(x_0,\cdot))f(\cdot)\|_{L^{p(\cdot)}(X)}, \tag{2.5.3}$$

where N is M or K, holds if the following condition is satisfied:

$$\sup_{0<t<L} \int\limits_{t<d(x_0,x)<L} \left(\frac{v(d(x_0,x))}{\mu(B_{x_0x})}\right)^{p(x)} \left(\int\limits_{\overline{B}(x_0,t)} w^{-(\widetilde{p}_0)'(x)}(d(x_0,y))d\mu(y)\right)^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} d\mu(x) < \infty.$$

Example 2.5.8. *Let (X, d, μ) be a quasi-metric measure space with $L < \infty$. Suppose that $1 < p_- \le p_+ < \infty$ and $p \in LH(X)$. Assume that the measure μ is both upper and lower Ahlfors $1-$ regular. Let there exist $x_0 \in X$ such that p has a minimum at x_0. Then the condition*

$$S := \sup_{0<t\le L} \int\limits_{t<d(x_0,x)<L} \left(\frac{v(d(x_0,x))}{\mu(B_{x_0x})}\right)^{p(x)} \left(\int\limits_{\overline{B}(x_0,t)} w^{-p'(x_0)}(d(x_0,y))d\mu(y)\right)^{\frac{p(x)}{p'(x_0)}} d\mu(x) < \infty$$

is satisfied for the weight functions $v(t) = t^{1/p'(x_0)}$, $w(t) = t^{1/p'(x_0)} \ln \frac{2L}{t}$ *and, consequently, by* Theorem 2.5.7 *inequality* (2.5.3) *holds, where* N *is* M *or* K*.*

Indeed, first observe that v and w are both increasing on $[0, L]$. Further it is easy to check that the condition $p \in LH(X)$, Proposition 2.2.3 and Lemma 2.2.5 implies that

$$\left(\frac{v(d(x_0,x))}{\mu(B_{x_0 x})}\right)^{p(x)} \leq c(d(x_0,x))^{-1}.$$

We have also

$$\left(\int_{\overline{B}(x_0,t)} w^{-p'(x_0)}(d(x_0,y))d\mu(y)\right)^{\frac{p(x)}{p'(x_0)}} = \left(\int_{B(x_0,t)} d(x_0,y)^{-1}\left(\ln\frac{2L}{d(x_0,y)}\right)^{-p'(x_0)} d\mu(y)\right)^{\frac{p(x)}{p'(x_0)}}$$

$$\leq C \ln^{-1}\frac{2L}{t}.$$

Hence,

$$S \leq c \ln\frac{2L}{t} \cdot \ln^{-1}\frac{2L}{t} = c < \infty.$$

Example 2.5.8 for constant p and $X = \mathbb{R}^n$ was presented in [19] (see also [20],Chapter 8 for spaces of homogeneous type)

List of notations

The following notations are introduced in the thesis at the page numbers indicated.

Bibliography

[1] A. Almeida and S. Samko, Fractional and hypersingular operators in variable exponent spaces on metric measure spaces, *Mediterr. J. Math.*, **6**(2009), 215–232.

[2] A. Almeida and H. Rafeiro, Inversion of the Bessel potential operator in weighted variable Lebesgue spaces,*J. Math. Anal. Appl.*, 340:13361346, 2008.

[3] K. Andersen and E. Sawyer, Weighted norm inequalities for the Riemann-Liouville and Weyl fractional operators, *Trans. Amer. Math. Soc.*, **308**(1988), No. 2, 547–558.

[4] M. A. Arino and B. Muckenhoupt, Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions, *Trans. Amer. Math. Soc.*, **320**(1990), No.2, 727–735.

[5] U. Ashraf, V. Kokilashvili and A. Meskhi, Weight characterization of the trace inequality for the generalized Riemann-Liouville transform in $L^{p(x)}$ spaces, *Math. Ineq. Appl.*, **13**(2010), No.1, 63–81.

[6] S. Boza and J. Soria, Weighted Hardy modular inequalities in variable L^p spaces for decreasing functions, *J. Math. Appl.*, **348**(2008), 383-388.

[7] C. Capone, D. Cruz-Uribe SFO and A. Fiorenza, The fractional maximal operator on variable L^p spaces, *Revista Mat. Iberoamericana*, **23**(2007), No. 3, 747–770.

[8] R. R. Coifman and G. Weiss, Analyse harmonique non-commutative sur certains espaces homogénes, *Lecture Notes in Math.*, Vol. **242**, *Springer-Verlag, Berlin*, 1971

[9] D. Cruz-Uribe SFO, A. Fiorenza and C.J. Neugebauer, The maximal function on variable L^p spaces, *Ann. Acad. Sci. Fenn. Math.*, **28**(2003), 223-238, and **29**(2004), 247–249.

[10] D. Cruz-Uribe SFO, A. Fiorenza, J. M. Martell and C. Perez, The boundedness of classical operators on variable L^p spaces, *Ann. Acad. Sci. Fenn. Math.*, **31**(2006), 239–264.

[11] D. Cruz-Uribe, J.M. Martell, and C. Perez, Sharp two-weight inequalities for singular integrals, with applications to the Hilbert transform and the Sarason conjecture, *Adv. Math.*, **216**(2007), pp. 647–676.

[12] L. Diening, Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, *Math. Inequal. Appl.*, **7**(2004), No. 2, 245–253.

[13] L. Diening, Riesz potentials and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$, *Math. Nachr.*, **268**(2004), 31–34.

[14] L. Diening and M. Ružička, Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics, *J. Reine Angew. Math.*, **63**(2003), 197–220.

[15] L. Diening, Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces, *Bull. Sci. Math.*, **129**(2005), No. 8, 657–700.

[16] L. Diening and S. Samko, Hardy inequality in variable exponent Lebesgue spaces, *Frac. Calc. Appl. Anal.*, **10**(2007), 1–18.

[17] L. Diening, P. Harjulehto, P. Hästö and M. Ružička, Lebesgue and Sobolev spaces with variable exponents, *Lecture Notes in Mathematics*, vol.**2017**, *Springer-Verlag, Berlin,* 2011.

[18] L. Diening and P. Hästö, Muckenhoupt weights in variable exponent spaces, Preprint, Available at http://www.helsinki.fi/∼pharjule/varsob/publications.shtml.

[19] D. E. Edmunds and V. Kokilashvili, Two-weight inequalities for singular integrals., *Canadian Math. Bull.*, **38**(1995), 119–125.

[20] D. E. Edmunds, V. Kokilashvili, and A. Meskhi, Bounded and compact integral operators, Mathematics and Its Applications, **543** *Kluwer Academic Publishers, Dordrecht, Boston, London*, 2002.

[21] D. E. Edmunds, V. Kokilashvili, and A. Meskhi, A trace inequality for generalized potentials in Lebesgue spaces with variable exponent, *J. Funct. Spaces Appl.*, **2**(2004), 55–69.

[22] D. E. Edmunds, V. Kokilashvili, and A. Meskhi, On the boundedness and compactness of the weighted Hardy operators in $L^{p(x)}$ spaces, *Georgian Math. J.*, **12**(2005), No. 1 27–44.

[23] D. E. Edmunds, V. Kokilashvili and A. Meskhi, One-sided operators in $L^{p(x)}$ spaces, *Math. Nachr.*, **281**(2008), No. 11, 1525–1548.

[24] D. E. Edmunds, V. Kokilashvili and A. Meskhi, Two-weight estimates in $L^{p(x)}$ spaces with applications to Fourier series, *Houston J. Math.*, **2**(2009), 665–689.

[25] D. E. Edmunds and A. Meskhi, Potential-type operators in $L^{p(x)}$ spaces, *Z. Anal. Anwend.*, **21**(2002), 681–690.

[26] G. B. Folland and E. M. Stein, Hardy spaces on homogeneous groups, *Princeton University Press and University of Tokyo Press, Princeton, New Jersey,* 1982.

[27] M. Frazier, B. Jawerth and G. Weiss, Littlewood-Paley theory and the study of function spaces. *Regional Conference Series in Mathematics, Vol.* **79**, *Amer. Math. Soc. Providence, RI*, 1991.

[28] J. García–Cuerva and J. L. Rubio de Francia, Weighted norm inequalities and related topics, *North-Holland Mathematics Studies,* **116**, *Mathematical Notes,* 104*, North-Holland Publishing Co., Amsterdam,* 1985.

[29] I. Genebashvili, A. Gogatishvili,V.Kokilashvili and M. Krbec, Weight theory for integral transforms on spaces of homogeneous type, *Pitman Monographs and Surveys in Pure and Applied Mathematics*, **92**, *Longman, Harlow*, 1998.

[30] G. H. Hardy, J. E. Littlewood and G. Polya, Inequalities, *Cambridge Univ. Press*, 1934.

[31] P. Harjulehto, P. Hästö, and M. Pere, Variable exponent Lebesgue spaces on metric spaces: the Hardy-Littlewood maximal operator, *Real Anal. Exchange,* **30**(2004-2005), 87–104.

[32] P. Harjulehto, P. Hästö, Út V. Le and M. Nuortio, Overview of differential equations with non-standard growth, *Preprint submitted to Elsevier, February* 4, 2010.

[33] L. Hörmander, L^p estimates for (pluri-) subharmonic functions, *Math. Scand.*, **20**(1967), 65-78.

[34] M. Khabazi, Maximal functions in $L^{p(\cdot)}$ spaces, *Proc. A. Razmadze Math. Inst.*, **135**(2004), 145–146.

[35] V. M. Kokilashvili, On Hardy's inequalities in weighted spaces (Russian), *Soobsch. Akad. Nauk Gruz. SSR*, **96**(1979), 37–40.

[36] V. Kokilashvili, New aspects in weight theory and applications, In" *Function Spaces, Differential Operators and Nonlinear Analysis"* M. Krbec et.al.(eds), *Paseky nad Jizerou, September* 3-9 , 1995, *Prometheus Publishing House, Prague,* 1996, 51-70.

[37] V. Kokilashvili, On a progress in the theory of integral operators in weighted Banach Function Spaces. In *"Function Spaces, Differential Operators and Nonlinear Analysis"*, *Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May* 28– *June* 2, *Math. Inst. Acad. Sci. of Czech Republic, Prague,* 2004.

[38] V. Kokilashvili and M. Krbec, Weighted inequalities in Lorentz and Orlicz spaces, *World Scientific, Singapore, New Jersey, London, Hong Kong,* 1991.

[39] V. Kokilashvili and A. Meskhi, On a trace inequality for one–sided potentials and applications to the solvability of nonlinear integral equations. *Georgian Math. J.*, **8**(2001), No. 3, 521-536.

[40] V. Kokilashvili and A. Meskhi, Weighted criteria for generalized fractional maximal functions and potentials in Lebesgue spaces with variable exponent, *Integr. Trans. Spec. Func.*, **18**(2007), No.9, 609–628.

[41] V. Kokilashvili and A. Meskhi, Boundedness of Maximal and singular integrals in Morrey spaces with variable exponent, *Arm. J. Math.*, **1**(2008), 18-28.

[42] V. Kokilashvili and A. Meskhi, Maximal functions and potentials in variable exponent Morrey spaces with non-doubling measure. *Complex Variables and Elliptic Equations*, **55**(2010), No. 8, 923-936.

[43] V. Kokilashvili and A. Meskhi, Two–weight inequalities for fractional maximal functions and singular integrals in $L^{p(\cdot)}$ spaces, *J. Math. Sci.*, **173**(2011), No. 6, 13-28.

[44] V. Kokilashvili, A. Meskhi and M. Sarwar, One and two weight norm estimates for one-sided operators in $L^{p(x)}$ spaces, *Proc. A. Razmadze Math. Inst.*, **148**(2008), 126-133.

[45] V. Kokilashvili, A. Meskhi and M. Sarwar,One and two weight estimates for one-sided operators in $L^{p(\cdot)}$ spaces. *Eurasian Math. J.*, **1**(2010), No.1, 73-110.

[46] V. Kokilashvili, A. Meskhi and M. Sarwar, Potential operators in variable exponent Lebesgue spaces. Two–weight estimates. *J. Inequal. Appl.*, **2010**(2010), Article ID 329571, 27 pages.

[47] V. Kokilashvili and S. Samko, Singular integrals in weighted Lebesgue spaces with variable exponent, *Georgian Math. J.*, **10**(2003), No. 1, 145-156.

[48] V. Kokilashvili and S. Samko, On Sobolev theorem for Riesz-type potentials in Lebesgue spaces with variable exponent, *Z. Anal. Anwendungen,* **22**(2003), No. 4, 899–910.

[49] V. Kokilashvili and S. Samko, Maximal and fractional operators in weighted $L^{p(x)}$ spaces, *Rev. Mat. Iberoamericana,* **20**(2004), No. 2 493–515.

[50] V. Kokilashvili and S. Samko, Boundedness in Lebesgue spaces with variable exponent of maximal, singular and potential operators, *Izv. Visshikh Uchebn. Zaved. Severo-Kavk. Region,* (2005), pp. 152–157.

[51] V. Kokilashvili and S. Samko, The maximal operator in weighted variable spaces on metric spaces, *Proc. A. Razmadze Math. Inst.*, **144**(2007), 137–144.

[52] V. Kokilashvili and S. Samko, The maximal operator in weighted variable exponent spaces on metric spaces, *Georgian Math. J.*, **15**(2008), 683–712.

[53] V. Kokilashvili and S. Samko, Operators of harmonis analysis in weighted spaces with non–standard growth, *J. Math. Anal. Appl.*, **352**(2009), No. 1, 15-34.

[54] V. Kokilashvili and S. Samko, Boundedness of maximal operators and potential opertators on Carleson curves in Lebesgue spaces with variable exponent, *Acta Mathematica Sinica.* **24**(2008), No. 11, 1775-1800

[55] T. S. Kopaliani, On some structural properties of Banach function spaces and boundedness of certain integral operators, *Czechoslovak Math. J.*, **54**(2004), No. 129, 791–805.

[56] T. S. Kopaliani, Infirmal convolution and Muckenhoupt $A_{p(\cdot)}$ condition in variable L^p spaces, *Arch. Math.*, **89**(2007), 185–192.

[57] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math. J.*, **41**(1991), No. 4, 592–618.

[58] A. Kufner, L. Maligranda and L.–E. Persson, The Hardy inequality–about its history and some related results, *Vydavatelsky Servis Publishing House, Pilsen*, 2007.

[59] A. Kufner and L.-E. Persson, Weighted inequalities of Hardy type, *World Scientific Publishing Co, Singapore, New Jersey, London, Hong Kong*, 2003.

[60] M. Lorente, A Characterization of two-weight norm inequalities for one-sided operators of fractional type, *Canad. J. Math.*, **49**(1997), 839–848.

[61] F. J. Martin-Reyes and A. de la Torre, Two-weight norm inequalities for fractional one-sided maximal operators, *Proc. Amer. Math. Soc.*, **117**(1993), 483–489.

[62] F. J. Martin-Reyes, P. Ortega Salvador and A. de la Torre, Weighted inequalities for one-sided maximal functions, *Trans. Amer. Math. Soc.*, **319**(1990), 514–534.

[63] V. G. Maz'ya , Sobolev spaces, *Springer–Verlag, Berlin–Heidelberg*, 1985.

[64] A. Meskhi, Solution of some weight problems for Riemann-Liouville and Weyl operators, *Georgian Math. J.*, **5**(1998), No. 6, 565–574.

[65] A. Meskhi, Messure of non-compactness for integral operators in weighted Lebesgue spaces, *Nova science publishers, Inc. New york,* 2009.

[66] A. Meskhi, G. Murtaza and M. Sarwar, Two–weight criteria for potentials with product kernels on cones of decreasing functions. *Proc. A. Razmadze Math. Inst.*, **152**(2010), 144-153

[67] A. Meskhi, G. Murtaza and M. Sarwar, Weighted criteria for one-sided potentials with product kernels on cones of decreasing functions, *Math. Ineq. Appl.*,(accepted).

[68] J. Musielak and W. Orlicz, On modular spaces, *Studia Math.*, **18**(1959), 49–65.

[69] J. Musielak, Orlicz spaces and modular spaces, *Lecture Notes in Math.*, **1034**, *Springer-Verlag, Berlin*, 1983.

[70] M. Nakano, Topology of linear topological spaces, *Moruzen Co. Ltd, Tokyo*, 1981.

[71] A. Nekvinda, Hardy-Littlewood maximal operator on $L^{p(\cdot)}(R^n)$, *Math. Ineq. Appl.*, **7**(2004), No. 2, 255–265.

[72] Sh. Ombrosi, Weak weighted inequalities for a dyadic one-sided maximal functions in $\mathbb{R}^n$, *Proc. Amer. Math. Soc.*, **133**(2005), 1769–1775.

[73] B. Opic and A. Kufner, Hardy-type inequalities, *Pitman series,* **219**, *Harlow,* 1990.

[74] W. Orlicz, Uber konjugierte exponentenfolgen, *Studia Math.*, **3**(1931), 200–211.

[75] D. V. Prokhorov, On the boundedness of a class of integral operators, *J. London Math. Soc.*, **61**(2000), No. 2, 617–628.

[76] H. Rafeiro and S. Samko, Hardy type inequality in variable Lebesgue spaces. *Ann. Acad. Sci. Fenn. Math.* **34**(2009), No. 1, 279-289.

[77] M. Ružička, Electrorheological fluids: modeling and mathematical theory, *Lecture Notes in Mathematics*, **1748**, *Springer-Verlag, Berlin*, 2000.

[78] S. Samko, Convolution type operators in $L^{p(x)}$, *Integral Transf. and Spec. Funct.*, **7**(1998), No. 1–2, 123–144.

[79] S. Samko, Convolution and potential type operators in $L^{p(x)}(\mathbb{R}^n)$, *Integr. Transf. and Spec. Funct.*, **7**(1998), No. 3–4, 261–284.

[80] S. Samko, On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators, *Integral Transforms Spec. Funct.*, **16**(2005), No. 56, 461–482.

[81] S. Samko, On some classical operators of variable order in variable exponent analysis, *Operator Theory: Advances and Applications*, Vol. **193**, 281–301, *Birkhauser*, 2009.

[82] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, *Gordon and Breach Sci. Publishers, New York,* 1993.

[83] N.G. Samko, S.G. Samko, and B. Vakulov, Weighted Sobolev theorem in Lebesgue spaces with variable exponent, *J. Math. Anal. Appl.*, **335**(2007), 560-583.

[84] E. Sawyer, Boundedness of classical operators on classical Lorentz spaces, *Studia Math.*, **96**(1990), 145–158.

[85] E. T. Sawyer, Weighted norm inequalities for fractional maximal operators, in: *Seminar on Harmonic Analysis,Montreal, Que.*, 1980, pp. 283–309, *CMS Conf. Proc., 1, Amer. Math. Soc.Providence, R.I.*, 1981.

[86] E. T. Sawyer, Weighted inequalities for one-sided Hardy-Littlewood maximal functions, *Trans. Amer. Math. Soc.*, **297**(1986), 53–61.

[87] E. T. Sawyer, R. L. Wheeden, Carleson conditions for the Poisson integral, *Indiana Univ. Math. J.*, **40**(1991), No. 2, 639–676.

[88] I. I. Sharapudinov, On a topology of the space $L^{p(t)}([0,1])$, *Mat. Zametki,* **26**(1979), 613–632.

[89] J. O. Strömberg and A. Torchinsky, Weighted Hardy spaces, *Lecture Notes in Math.* **1381**, *Springer Verlag, Berlin*, 1989.

[90] K. Tachizawa, On weighted dyadic Carleson's inequalities, *J. Inequal. Appl.*, **6**(2001), NO. 4, 415–433.

[91] I. E. Verbitsky, Weighted norm inequalities for maximal operators and Pisier's theorem on factorization through $L^{p\infty}$, *Integral Equations Operator Theory,* **15**(1992), No.1, 124–153.

[92] A. Volberg, Calderón–Zygmund capacities and operators on nonhomogeneous spaces, *CBMS Regional Conference Series in Mathematics,* Vol. **100**, 2003.

[93] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory. *Math. USSR Izv.* **29**(1987), 33–66.