WEAK FORMS OF SOME NOTIONS IN FUZZY TOPOLOGY

By

MUHAMMAD ATHAR KHARAL

CENTRE FOR ADVANCED STUDIES

IN PURE AND APPLIED MATHEMATICS,

BAHAUDDIN ZAKARIYA UNIVERSITY

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Dr. Bashir Ahmad

Dr. Zahida Akram

CENTRE FOR ADVANCED STUDIES

IN PURE AND APPLIED MATHEMATICS,

BAHAUDDIN ZAKARIYA UNIVERSITY, MULTAN

2013

CERTIFICATE

It is certified that this work has not been submitted for any degree and shall not, in future, be submitted for obtaining Ph. D. degree of any other university.

Candidate

CERTIFICATE

It is certified that the work contained in this thesis is original and carried out by Muhammad Athar Kharal under our supervision.

Dr. Bashir Ahmad

(Supervisor)

Professor of Mathematics,

Centre for Advanced Studies

in Pure and Applied Mathematics,

Bahauddin Zakariya University

Multan, PAKISTAN

Dr. Zahida Akram

(Co-Supervisor)

Associate Professor of Mathematics,

Centre for Advanced Studies

in Pure and Applied Mathematics,

Bahauddin Zakariya University

Multan, PAKISTAN

CERTIFICATE

The thesis titled

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Muhammad Athar Kharal

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Dr. Bashir Ahmad (Supervisor) Professor of Mathematics, Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University Multan, PAKISTAN	**Dr. Zahida Akram** (Co-Supervisor) Associate Professor of Mathematics, Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University Multan, PAKISTAN
Director, Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University Multan, PAKISTAN	**Prof. Dr. Shahid Siddiqui** (External Examiner) Chairman Department of Mathematics University of the Punjab

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BAHAUDDIN ZAKARIYA UNIVERSITY, MULTAN
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Dedicated

To

Bertrand Russell,

who introduced me to the realm of thought.

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Acknowledgements

Abstract

N. Levine [33] introduced the concepts of semi-open sets and semi-continuous mappings in topological spaces. Thereafter many researchers contributed to this area: Andrijevic [7] studied semi-preopen sets in 1986. D.E. Cameron and G. Woods [12] studied the notions of s-continuous and s-open mappings. Cao, Ganester and Reilly studied the links between generalized closed sets and extermal disconnectedness [13]. Dontchev and Ganster [16] studied δ-generalized closed sets. Semi continuous and semi-closed mappings were further studied by Ghosh [21] in 1990.

L. A. Zadeh [59] introduced the seminal notion of fuzzy sets in 1965. C. L. Chang [14] defined and studied the notion of a fuzzy topological space in 1968. Since then much attention has been paid to generalize the basic concepts of Classical Topology in fuzzy setting and thus a modern theory of Fuzzy Topology has been developed. Azad [9] fuzzified the work of Levine, and presented some general properties of fuzzy spaces. Several properties of semi-open fuzzy (resp. semi-closed fuzzy), fuzzy regular open (resp. closed) sets have also been discussed by Azad. Abbas [4] studied fuzzy super irresolute mappings, Ajmal and Azad [5, 6] gave pointwise characterization of fuzzy almost continuity. Caldas, Navalagi and Saraf [10, 11] gave a study of fuzzy weakly semi-open mappings. Jankovic [23] introduced the notion of θ-regular spaces. In 2002 Georgiou and Papadopoulos [20] studied fuzzy θ-convergences.

Ming and Ming [36] defined the notion of fuzzy boundary in fuzzy topo-

logical spaces in 1980, yet there is very little work available on this notion, in present literature. Tang [54] used a limited version of Chang's fuzzy topological space because sufficient material about properties of fuzzy boundary is currently not available. So, we study this concept and establish several of its properties in Chapter 2. We also define the concept of semi fuzzy -boundary and characterized semi-continuous fuzzy functions in terms of semi fuzzy -boundary. Several properties of fuzzy boundary and semi fuzzy -boundary have been obtained, which have been supported by examples. Properties of semi fuzzy -interior, semi fuzzy -closure, fuzzy boundary and semi fuzzy -boundary have been obtained in product related spaces. We give necessary conditions for continuous fuzzy (resp. semi-continuous fuzzy, irresolute fuzzy) functions. Moreover, continuous fuzzy (resp. semi-continuous fuzzy, irresolute fuzzy) functions have been characterized via derived fuzzy(resp. semi-derived fuzzy) sets. The results of this chapter have been published in Advances in Fuzzy Systems Vol. 2008, Article ID 586893, 9 pages doi:10.1155/2008/586893 (MR# 2425456).

In Chapter 3, we studied semi-continuous fuzzy, semi-open fuzzy and almost open fuzzy (Ganguly's sense) mappings. We also define and study properties of almost closed fuzzy mappings. In Chapter 4, we continue the study initiated in Chapter 3 and several properties and characterizations of semi-open fuzzy (semi-closed fuzzy), semi-preopen fuzzy (semi-preclosed fuzzy), semi-precontinuous fuzzy and pre-semi-preopen fuzzy (pre-semi-preclosed fuzzy) mappings have been investigated. Findings of Chapters 3 and 4 have been published in Journal of Fuzzy Mathematics, 16(2)(2008), 341-349 (Zbl# 1146.54302) and vol. 18(1), respectively.

In Capter 5, we further study some properties of semi-open fuzzy sets defined and studied by Zhong [62], semi-preopen fuzzy sets and preopen fuzzy sets. It is also shown that in the class of injective functions, almost open fuzzy (closed) in Nanda's sense and almost quasi-compact fuzzy functions are equivalent. In terms of graph and projections, some interesting characterizations and properties of almost continuous fuzzy functions in Singal's sense are given. Moreover almost continuous fuzzy in Husain's sense, almost weakly continuous fuzzy,nearly almost open (closed) fuzzy functions have been defined and their several characterizations and properties have been obtained. Finally, their equivalences have been established under certain conditions. Results from this chapter have appeared in International Journal of Contemporary Mathematical Sciences 3(34) (2008) 1665-1677 (MR# 2511023).

In 2001, Kresteska [29] pointed out that Lemmas 4.5, 4.7 and Theorems 4.6, 4.8, 4.12 of [52] are incorrect. Since α-continuity does not yield to a straightforward fuzzification of the results from Classical Topology, thus this notion seems promising for Fuzzy Topology. Motivated by such consideration, Chapter 6 studies further, the properties of α-continuous mappings in terms of α-closure of fuzzy sets. Findings of this chapter have been submitted to Korean Annals of Mathematics.

In Chapter 7, our aim is to further contribute to the study of semi-open fuzzy sets by establishing several important fundamental identities and inequalities about their semi-interior and semiclosure. D. E. Cameron and G. Woods [12] introduced the concepts of s-continuous mappings and s-open mappings. They investigated the properties of these mappings and their relationships to properties of semi-open sets. M. Khan and B. Ahmad [25] further worked on the characterizations and properties of s-continuous, s-open and s-closed mappings. In this section, we fuzzify the findings of [12] and [25]. We define s-open and s-closed fuzzy mappings and establish some interesting characterizations of these mappings. It may be noted that the class of s-open (resp. s-closed) fuzzy mappings is a subclass of the class of open (resp. closed) fuzzy mappings. These results have been published by Advances in Fuzzy Systems Volume 2009 (2009), Article ID 303042, 5 pages doi:10.1155/2009/303042.

Chapter 8 comprises a study of simply continuous fuzzy mappings. In Chapter 9, we define and study the notion of α-semicontinuous fuzzy mappings. Results of this capter have been submitted for publication.

Research Publications Based Upon The Material Presented In This Thesis

1. M. Athar, B. Ahmad, *Fuzzy boundary and semi fuzzy -boundary,* Advances in Fuzzy Systems vol. 2008, Article ID 586893, 9 pages doi:10.1155/2008/586893 (MR# 2425456).

2. B. Ahmad, A. Kharal, *On some fuzzy mappings,* Journal of Fuzzy Mathematics 16(2)(2008), 341-349 (Zbl# 1146.54302).

3. A. Kharal, B. Ahmad, *On some fuzzy mappings II,* Journal of Fuzzy Mathematics 18(1)(2010) 245-256.

4. B. Ahmad, M. Athar, *Fuzzy almost continuous functions,* International Journal of Contemporary Mathematical Sciences 3(34) (2008) 1665-1677 (MR# 2511023).

5. A. Kharal, B. Ahmad, *Fuzzy* α-continuous mappings, Korean Annals of Mathematics (to appear).

6. B. Ahmad, M. Athar, *Fuzzy sets, fuzzy s-open and s-closed functions*, Advances in Fuzzy Systems Volume 2009 (2009), Article ID 303042, 5 pages doi:10.1155/2009/303042.

7. A. Kharal, B. Ahmad, *Fuzzy* α-semicontinuous mappings, Journal of Fuzzy Mathematics 20(1)(2012) 71-82.

Contents

Chapter 1

Basic Notions of Fuzzy Topology

After Zadeh's [59] introduction of fuzzy sets, Chang [14] defined and studied the notion of a fuzzy topological space in 1968. Since then much attention has been paid to generalize the basic concepts of Classical Topology in fuzzy setting and thus a modern theory of Fuzzy Topology has been developed.

In recent years, Fuzzy Topology has been found to be very useful in solving many practical problems. Shihong Du *et. al.* [17] fuzzified the very successful 9-intersection Egenhofer model [18, 19] for depicting topological relations in Geographic Information Systems (GIS) query. In [40, 41], El-Naschie showed that the notion of Fuzzy Topology might be relevant to quantum particle physics and quantum gravity in connection with string theory and e^{∞} theory. Tang [54] used a slightly changed version of Chang's fuzzy topological space to model spatial objects for GIS databases and Structured Query Language (SQL) for GIS.

N. Levine [33] introduced the concepts of semi-open sets and semi-continuous mappings in topological spaces. Interestingly, his work found applications in the field of Digital Topology [46]. For example, it was found that digital line is a $T_{\frac{1}{2}}$-space [13], which is a weaker separation axiom based upon semi-open sets. Fuzzy Digital Topology [47] was introduced by A. Rosenfeld, which demonstrated the need for the fuzzification of weaker forms of notions of Classical Topology. Azad [9] carried out this fuzzification in 1981, and presented some general properties of fuzzy spaces. Several properties of semi-open (resp. semi-closed), regular open (resp. closed) fuzzy sets have been discussed. Moreover he defined semi-continuous (resp. semi-open, semi-closed) fuzzy functions and studied the properties of semi-continuous fuzzy function in product related spaces. Finally, he defined and characterized almost continuous fuzzy mappings. In this direction much work followed subsequently e.g. [4, 10, 11, 15, 20, 21, 22, 32, 58, 61].

First, we briefly recall certain definitions and results; for those not described, we refer to [9, 14, 58, 59].

A fuzzy set λ in a set X is a function from X to $[0, 1]$ i.e. $\lambda : X \to [0, 1]$.

Definition 1.1 *[59] Let λ and μ be fuzzy sets in X. Then for all $x \in X$,*

$$\lambda = \mu \iff \lambda(x) = \mu(x),$$

$$\lambda \leq \mu \iff \lambda(x) \leq \mu(x),$$

$$\psi = \lambda \vee \mu \iff \psi(x) = max\{\lambda(x), \mu(x)\},$$

$$\delta = \lambda \wedge \mu \iff \delta(x) = min\{\lambda(x), \mu(x)\},$$

$$\eta = \lambda^c \iff \eta(x) = 1 - \lambda(x).$$

More generally, for a family $\Lambda = \{\lambda_i | i \in I\}$, of fuzzy sets in X, the union $\psi = \vee_i \lambda_i$ and intersection $\delta = \wedge_i \delta_i$ are defined by

$$\psi(x) = sup_i\{\lambda_i(x) | x \in X\},$$

and

$$\delta(x) = inf_i\{\delta_i(x) | x \in X\}.$$

The empty fuzzy set $\widetilde{0}$ is defined as $\widetilde{0}(x) = 0$, for all $x \in X$ and the symbol $\widetilde{1}$ denotes the fuzzy set $\widetilde{1}(x) = 1$, for all $x \in X$.

Definition 1.2 *[14] Let $f : X \to Y$ be a function. Let β be a fuzzy set in Y with membership function $\beta(y)$. Then the inverse of β, written as $f^{-1}(\beta)$, is a fuzzy set in X whose membership function is defined by*

$$f^{-1}(\beta)(x) = \beta(f(x)), for\ all\ x \in X.$$

Conversely, let λ be a fuzzy set in X with membership function $\lambda(x)$. The image of λ, written as $f(\lambda)$, is a fuzzy set in Y whose membership function is given by

$$f(\lambda)(y) = \begin{cases} sup_{x \in f^{-1}(y)}\{\lambda(x)\}, & \textit{if } f^{-1}(y) \textit{ is non-empty,} \\ 0, & \textit{otherwise,} \end{cases}$$

for all $y \in Y$, *where* $f^{-1}(y) = \{x | f(x) = y\}$.

Definition 1.3 *[14] A fuzzy topology is a family* τ *of fuzzy sets in* X, *which satisfies the following conditions:*

(1) $\widetilde{0}, \widetilde{1} \in \tau$.

(2) *If* $\lambda, \mu \in \tau$, *then* $\lambda \wedge \mu \in \tau$.

(3) *If* $\lambda_i \in \tau$ *for each* $i \in I$, *then* $\vee_i \lambda_i \in \tau$.

τ is called a fuzzy topology for X, and the pair (X, τ) is a fuzzy topological space. Every member of τ is called τ-open fuzzy set (or simply open fuzzy set). A fuzzy set is τ-closed if and only if its complement is τ-open.

As in General Topology, the indiscrete fuzzy topology contains only $\widetilde{0}$ and $\widetilde{1}$, while the discrete fuzzy topology contains all fuzzy sets. In the sequel, we write an fts X (or (X, τ)) in place of 'a space X with fuzzy topology τ'.

Definition 1.4 *[14] The closure and interior of a fuzzy set* λ *in an fts* (X, τ) *are denoted and defined as:*

$$Cl\lambda = inf\{\mu \mid \lambda \leq \mu, \mu^c \in \tau\}.$$

$$Int\lambda = sup\{\omega \mid \omega \leq \lambda, \omega \in \tau\}.$$

We mention below some properties of closure and interior of a fuzzy set which have been used in the sequel.

Lemma 1.1 *[56] For fuzzy sets λ and μ in an fts X, we have*

(1) λ *is closed fuzzy(resp. open fuzzy)* $\iff$ $Cl\lambda = \lambda$ *(resp.* $Int\lambda = \lambda$*).*

(2) $\lambda \leq \mu \Rightarrow Cl\lambda \leq Cl\mu$ $(Int\lambda \leq Int\mu)$.

(3) $ClCl\lambda = Cl\lambda$ $(IntInt\lambda = Int\lambda)$.

(4) $Cl\lambda \vee Cl\mu = Cl\,(\lambda \vee \mu)$.

(5) $Cl\lambda \wedge Cl\mu \geq Cl\,(\lambda \wedge \mu)$.

(6) $Int\lambda \vee Int\mu \leq Int\,(\lambda \vee \mu)$.

(7) $Int\lambda \wedge Int\mu = Int\,(\lambda \wedge \mu)$.

(8) $Int\lambda^c = (Cl\lambda)^c$.

(9) $Cl\lambda^c = (Int\lambda)^c$.

Definition 1.5 *[14] A function $f : (X, \tau) \to (Y, \delta)$ is said to be continuous fuzzy if and only if the inverse of each δ-open fuzzy set is τ-open fuzzy.*

Theorem 1.1 *[21, 56] Let $f : X \to Y$ be a function from an fts (X, τ) to another fts (Y, δ). Then following are equivalent*

(1) *f is continuous fuzzy,*

(2) *The inverse of each δ-closed fuzzy set is τ-closed,*

(3) *For each fuzzy set λ in X,* $f(Cl\lambda) \leq Clf(\lambda)$,

(4) *For each fuzzy set ν in Y,* $f^{-1}(Int\nu) \leq Intf^{-1}(\nu)$.

Chapter 2

Fuzzy Boundary and Semi Fuzzy Boundary

Though Ming and Ming [36] defined the notion of fuzzy boundary in fuzzy topological spaces in 1980, yet there is very little work available on this notion, in present literature. One reason, *inter alia,* of Tang's [54] use of a limited version of Chang's fuzzy topological space was the non-availability of sufficient material about properties of fuzzy boundary. So, we study this concept further and establish several of its properties, thus providing sufficient material for researchers to utilize these concepts fruitfully. B. Ahmad and M. Athar [1] defined the concept of semi fuzzy-boundary and characterized semi fuzzy-continuous functions in terms of semi fuzzy-boundary.

In this chapter, we present several properties of fuzzy boundary and semi-fuzzy boundary which have been supported by examples. Properties of semi-fuzzy interior, semi-fuzzy closure, fuzzy boundary and semi-fuzzy boundary have been obtained in product related spaces. We give necessary conditions for continuous (resp. semi-continuous, irresolute) fuzzy functions. Moreover, continuous (resp. semi-continuous, irresolute) fuzzy functions have been characterized via derived (resp. semi-derived) fuzzy sets.

2.1 Fuzzy Boundary

Definition 2.1 *[36] Let λ be a fuzzy set in an fts X. Then the fuzzy boundary of λ is defined as $Bd\lambda = Cl\lambda \wedge Cl\lambda^c$. Obviously, $Bd\lambda$ is a closed fuzzy set.*

Remark 2.1 *In Classical Topology, for an arbitrary set A of a topological space X, we have $A \cup BdA = ClA$, but in Fuzzy Topology we have $\lambda \vee Bd\lambda \leq Cl\lambda$, for an arbitrary fuzzy set λ in X, the converse of which is not true as shown by Ming and Ming [36]. Moreover, we have*

Proposition 2.1 *For fuzzy sets λ and μ in a fts X, the following hold:*

(1) $Bd\lambda = Bd\lambda^c$.

(2) *If λ is closed fuzzy, then $Bd\lambda \leq \lambda$.*

(3) *If λ is open fuzzy , then $Bd\lambda \leq \lambda^c$.*

(4) *Let $\lambda \leq \mu$ and $\mu \in FC(X)$ (resp. $\mu \in FO(X)$). Then $Bd\lambda \leq \mu$ (resp. $Bd\lambda \leq \mu^c$), where $FC(X)$ (resp. $FO(X)$) denotes the class of closed fuzzy (resp. open fuzzy) sets in X.*

(5) $(Bd\lambda)^c = Int\lambda \vee Int\lambda^c$.

Proof. (1) $Bd\lambda = Cl\lambda \wedge Cl\lambda^c = Cl\lambda^c \wedge Cl\lambda = Cl\lambda^c \wedge Cl\,(\lambda^c)^c = Bd\lambda^c$.

(2) $Bd\lambda = Cl\lambda \wedge Cl\lambda^c \leq Cl\lambda = \lambda$, hence $Bd\lambda \leq \lambda$.

(3) λ is open fuzzy implies λ^c is closed fuzzy set. By (2), $Bd\lambda^c \leq \lambda^c$ and by (1) we get $Bd\lambda \leq \lambda^c$.

(4) Since $\lambda \leq \mu$ implies $Cl\lambda \leq Cl\mu$, we have $Bd\lambda = Cl\lambda \wedge Cl\lambda^c \leq Cl\mu \wedge Cl\lambda^c \leq Cl\mu = \mu$, since $\mu \in FC\,(X)$.

(5) $(Bd\lambda)^c = (Cl\lambda \wedge Cl\lambda^c)^c = (Cl\lambda)^c \vee (Cl\lambda^c)^c = Int\lambda^c \vee Int\lambda$. ■

The converse of (2) and (3) of Theorem 2.1 is, in general, not true as is shown by the following:

Example 2.1 *Let $X = \{a, b, c\}$ be a set and τ, the fuzzy topology generated by fuzzy sets $\mu = \{a_{.4}, b_{.8}, c_{.2}\}, \nu = \{a_{.6}, b_{.9}, c_{.1}\}$ and $\omega = \{a_{.5}, b_{.7}, c_{.3}\}$. Then $\tau = \{\widetilde{0}, \mu, \nu, \omega, \{a_{.4}, b_{.8}, c_{.1}\}$,*
$\{a_{.4}, b_{.7}, c_{.2}\}, \{a_{.5}, b_{.7}, c_{.1}\}, \{a_{.4}, b_{.7}, c_{.1}\}, \{a_{.5}, b_{.8}, c_{.3}\}, \{a_{.5}, b_{.8}, c_{.1}\}, \{a_{.6}, b_{.9}, c_{.2}\}$,
$\{a_{.5}, b_{.8}, c_{.2}\}, \{a_{.6}, b_{.9}, c_{.3}\}, \{a_{.5}, b_{.7}, c_{.2}\}, \widetilde{1}\}$. Choose $\lambda = \{a_{.5}, b_{.7}, c_{.9}\}$ and $\psi = \{a_{.5}, b_{.3}, c_{.3}\}$. Calculations give

$$\begin{aligned} Bd\lambda &= \{a_{.5}, b_{.3}, c_{.7}\} \leq \lambda, \text{ but } \lambda \text{ is not closed fuzzy set,} \\ \text{and } Bd\psi &= \{a_{.5}, b_{.3}, c_{.7}\} \leq \psi^c, \text{ but } \psi \text{ is not open fuzzy set.} \end{aligned}$$

Following proposition gives some more properties of fuzzy boundary:

Proposition 2.2 *Let λ be a fuzzy set in an fts X. Then*

(1) $Bd\lambda = Cl\lambda - Int\lambda$.

(2) $BdInt\ \lambda \leq Bd\lambda$.

(3) $BdCl\lambda \leq Bd\lambda$.

(4) $Int\lambda \leq \lambda - Bd\lambda$.

Proof. (1) Since $(Cl\lambda^c)^c = Int\lambda$, therefore we have $Bd\lambda = Cl\lambda \wedge Cl\lambda^c = Cl\lambda - (Cl\lambda^c)^c = Cl\lambda - Int\lambda$. Thus $Bd\lambda = Cl\lambda - Int\lambda$. This proves (1),

(2) $BdInt\lambda = ClInt\lambda - IntInt\lambda = ClInt\lambda - Int\lambda \leq Cl\lambda - Int\lambda = Bd\lambda$,

(3) $BdCl\lambda = ClCl\lambda - IntCl\lambda = Cl\lambda - IntCl\lambda \leq Cl\lambda - Int\lambda = Bd\lambda$,

(4)

$$\begin{aligned}
\lambda - Bd\lambda &= \lambda \wedge (Bd\lambda)^c = \lambda \wedge (Cl\lambda \wedge Cl\lambda^c)^c \\
&= \lambda \wedge (Int\lambda^c \vee Int\lambda) = (\lambda \wedge Int\lambda^c) \vee (\lambda \wedge Int\lambda) \\
&= (\lambda \wedge Int\lambda^c) \vee Int\lambda \geq Int\lambda.
\end{aligned}$$

■

The following example shows that the equality does not hold in Theorem 2.2 (2) – (4):

Example 2.2 *Choose* $\alpha = \{a_{.1}, b_{.4}, c_{.7}\}$, $\beta = \{a_{.6}, b_{.8}, c_{.5}\}$ *and* $\gamma = \{a_{.6}, b_{.3}, c_{.8}\}$*in*

Example 2.1. Then calculations give

$$\begin{aligned} Bd\alpha &= \widetilde{1} \neq \widetilde{0} = BdInt\alpha \\ Bd\beta &= \{a_{.5}, b_{.2}, c_{.7}\} \neq \widetilde{0} = BdCl\beta \\ \gamma - Bd\gamma &= \{a_{.4}, b_{.3}, c_{.2}\} \neq Int\gamma = \widetilde{0}. \end{aligned}$$

Remark 2.2 *In General Topology, the following hold:*

$$\begin{aligned} BdA \cap IntA &= \widetilde{0}. \\ IntA \cup BdA &= ClA. \\ IntA \cup IntA^c \cup BdA &= X. \end{aligned}$$

Whereas, in Fuzzy Topology, we give counter-examples to show that these may not hold in general.

Example 2.3 *In the fts (X, τ) of Example 2.1, we choose fuzzy set $\lambda = \{a_{.4}, b_{.7}, c_{.7}\}$, then calculations give*

$$\begin{aligned} Bd\lambda \wedge Int\lambda &= \{a_{.6}, b_{.3}, c_{.8}\} \wedge \{a_{.4}, b_{.7}, c_{.2}\} \neq \widetilde{0} \\ Int\lambda \vee Bd\lambda &= \{a_{.6}, b_{.7}, c_{.8}\} \neq Cl\lambda = \widetilde{1} \\ Int\lambda \vee Int\lambda^c \vee Bd\lambda &= \{a_{.6}, b_{.7}, c_{.8}\} \neq \tilde{1}. \end{aligned}$$

It is easily seen that $Int\lambda \vee Bd\ \lambda \leq Cl\lambda$.

Theorem 2.1 *Let λ and μ be fuzzy sets in an fts X. Then $Bd(\lambda \vee \mu) \leq Bd\lambda \vee Bd\mu$.*

Proof.

$$
\begin{aligned}
Bd\,(\lambda \vee \mu) &= Cl\,(\lambda \vee \mu) \wedge Cl\,(\lambda \vee \mu)^c \\
&\leq (Cl\lambda \vee Cl\mu) \wedge (Cl\lambda^c \wedge Cl\mu^c) \\
&= [Cl\lambda \wedge (Cl\lambda^c \wedge Cl\mu^c)] \vee [Cl\mu \wedge (Cl\lambda^c \wedge Cl\mu^c)] \\
&= (Bd\lambda \wedge Cl\mu^c) \vee (Bd\mu \wedge Cl\lambda^c) \\
&\leq Bd\lambda \vee Bd\mu.
\end{aligned}
$$

■

In Theorem 2.1, the equality does not hold as is shown by the following:

Example 2.4 *In Example 2.1, choose fuzzy sets* $\alpha = \{a_0, b_0, c_{.2}\}$ *and* $\beta = \{a_{.4}, b_{.8}, c_{.1}\}$. *Then calculations give*

$$Bd\alpha \vee Bd\beta = \{a_{.6}, b_{.2}, c_{.9}\} \neq \{a_{.6}, b_{.2}, c_{.8}\} = Bd\,(\alpha \vee \beta)\,.$$

The following examples show that:

$$Bd\,(\gamma \wedge \delta) \nleq Bd\gamma \wedge Bd\delta \text{ and } Bd\psi \wedge Bd\xi \nleq Bd\,(\psi \wedge \xi)\,.$$

For this, choose $\gamma = \{a_{.4}, b_{.1}, c_{.9}\}$, $\delta = \{a_{.4}, b_1, c_1\}$, $\psi = \{a_{.7}, b_{.4}, c_{.4}\}$ *and* $\xi = \{a_{.4}, b_{.1}, c_{.8}\}$. *Then calculations give*

$$
\begin{aligned}
Bd\,(\gamma \wedge \delta) &= \{a_{.4}, b_{.1}, c_{.9}\} \nleq \{a_{.4}, b_{.1}, c_{.8}\} = Bd\gamma \wedge Bd\delta. \\
Bd\psi \wedge Bd\xi &= \{a_{.4}, b_{.1}, c_{.8}\} \nleq \{a_{.4}, b_{.1}, c_{.7}\} = Bd\,(\psi \wedge \xi)\,.
\end{aligned}
$$

However, we have the following:

Theorem 2.2 *For any fuzzy sets* λ *and* μ *in an fts* X, *we have*

$$Bd\,(\lambda \wedge \mu) \leq (Bd\lambda \wedge Cl\mu) \vee (Bd\mu \wedge Cl\lambda)\,.$$

Proof.

$$\begin{aligned} Bd\,(\lambda \wedge \mu) &= Cl\,(\lambda \wedge \mu) \wedge Cl\,(\lambda \wedge \mu)^c \\ &\leq (Cl\lambda \wedge Cl\mu) \wedge (Cl\lambda^c \vee Cl\mu^c) \\ &= [(Cl\lambda \wedge Cl\mu) \wedge Cl\lambda^c] \vee [(Cl\lambda \wedge Cl\mu) \wedge Cl\mu^c] \\ &= (Bd\lambda \wedge Cl\mu) \vee (Bd\mu \wedge Cl\lambda)\,. \end{aligned}$$

■

Corollary 2.1 *For any fuzzy sets* λ *and* μ *in an fts* X, *we have*

$$Bd\,(\lambda \wedge \mu) \leq Bd\lambda \vee Bd\mu.$$

Example 2.5 *To show that the equality in Theorem 2.2, in general, does not hold, choose* $\lambda = \{a_{.6}, b_{.6}, c_{.7}\}$ *and* $\mu = \{a_{.6}, b_{.2}, c_{.9}\}$ *in the fts* X *defined in Example 2.1. Then calculations give*

$$(Bd\lambda \wedge Cl\mu) \vee (Bd\mu \wedge Cl\lambda) = \mu \nleq \{a_{.6}, b_{.2}, c_{.8}\} = Bd\,(\lambda \wedge \mu)$$

In General Topology, it is known that

$$BdBdBdA = BdBdA,$$

for any subset A of a space X. However, in Fuzzy Topology, we have the following:

Proposition 2.3 *For any fuzzy set λ in an fts X, we have*

(1) $Bd Bd\lambda \leq Bd\lambda.$

(2) $Bd Bd Bd\lambda \leq Bd Bd\lambda.$

Proof. (1) $Bd Bd\lambda = Cl Bd\lambda \wedge Cl\,(Bd\lambda)^c \leq Cl Bd\lambda = Bd\lambda.$

(2)

$$\begin{aligned} Bd Bd Bd\lambda &= Cl Bd Bd\lambda \ \wedge \ Cl\,(Bd Bd\lambda)^c \\ &= Bd Bd\lambda \ \wedge \ Cl\,(Bd Bd\lambda)^c \\ &\leq Bd Bd\lambda. \end{aligned}$$

■

Remark 2.3 *We could not find an example to show that the equality in* (2) *does not hold. However the equality in* (1)*, in general, does not hold as is shown by the following:*

Example 2.6 *Choose $\lambda = \{a_{.2}, b_{.5}, c_{.9}\}$ in fts X of Example 2.1. Then $Bd\lambda = \tilde{1}$ but $Bd Bd\lambda = \tilde{0}$.*

Definition 2.2 *[24] If λ is a fuzzy set of X and μ is a fuzzy set of Y, then $\lambda \times \mu$ is a fuzzy set of $X \times Y$, defined by*

$$(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\}, \ \textit{for each} \ (x, y) \in X \times Y.$$

Definition 2.3 *[9] An fts (X, τ_X) is product related to another fts (Y, τ_Y) if for any fuzzy set ν of X and ξ of Y whenever $\lambda^c \not\geq \nu$ and $\mu^c \not\geq \xi$ imply*

$\lambda^c \times \widetilde{1} \vee \widetilde{1} \times \mu^c \geq \nu \times \xi$, *where* $\lambda \in \tau_X$ *and* $\mu \in \tau_Y$, *there exist* $\lambda_1 \in \tau_X$ *and* $\mu_1 \in \tau_Y$ *such that* $\lambda_1^c \geq \nu$ *and* $\mu_1^c \geq \xi$ *and* $\lambda_1^c \times \widetilde{1} \vee \widetilde{1} \times \mu_1^c = \lambda^c \times \widetilde{1} \vee \widetilde{1} \times \mu^c$.

Theorem 2.3 *[9] Let* X *and* Y *be product related fts's. Then, for a fuzzy set* λ *of* X *and a fuzzy set* μ *of* Y, *we have*

(1) $Cl\,(\lambda \times \mu) = Cl\lambda \times Cl\mu$.

(2) $Int\,(\lambda \times \mu) = Int\lambda \times Int\mu$.

Lemma 2.1 *For fuzzy sets* λ, μ, ν *and* ω *in a set* X, *we have*

$$(\lambda \wedge \mu) \times (\nu \wedge \omega) = (\lambda \times \omega) \wedge (\mu \times \nu)$$

Proof.

$$\begin{aligned}
((\lambda \wedge \mu) \times (\nu \wedge \omega))(x, y) &= \min\left((\lambda \wedge \mu)(x), (\mu \wedge \omega)(y)\right) \\
&= \min\left(\min(\lambda(x), \mu(x)), \min(\nu(y), \omega(y))\right) \\
&= \min\left(\min(\lambda(x), \omega(y)), \min(\mu(x), \nu(y))\right) \\
&= \min\left((\lambda \times \omega)(x, y), (\mu \times \nu)(x, y)\right) \\
&= ((\lambda \times \omega) \wedge (\mu \times \omega))(x, y).
\end{aligned}$$

■

Theorem 2.4 *Let* $X_i, i = 1, 2, ..., n$ *be a family of product related fuzzy topological spaces. If each* λ_i *is a fuzzy set in* X_i, *then*

$$\begin{aligned}
Bd\prod_{i=1}^{n} \lambda_i &= [Bd\lambda_1 \times Cl\lambda_2 \times ... \times Cl\lambda_n] \\
&\vee [Cl\lambda_1 \times Bd\lambda_2 \times Cl\lambda_3 \times ... \times Cl\lambda_n] \vee ... \\
&\vee [Cl\lambda_1 \times Cl\lambda_2 \times ... \times Bd\lambda_n].
\end{aligned}$$

Proof. We use Proposition 2.2(1), Theorem 2.3 and Lemma 2.1 to prove this.

It suffices to prove this for $n = 2$. Consider

$$\begin{aligned}
Bd\,(\lambda_1 \times \lambda_2) &= Cl(\lambda_1 \times \lambda_2) - Int\,(\lambda_1 \times \lambda_2) \\
&= (Cl\lambda_1 \times Cl\lambda_2) - (Int\lambda_1 \times Int\lambda_2) \\
&= (Cl\lambda_1 \times Cl\lambda_2) - ((Int\lambda_1 \wedge Cl\lambda_1) \times (Int\lambda_2 \wedge Cl\lambda_2)) \\
&= (Cl\lambda_1 \times Cl\lambda_2) - ((Int\lambda_1 \times Cl\lambda_2) \wedge (Cl\lambda_1 \times Int\lambda_2)) \quad \text{(by Lemma 2.1)} \\
&= [(Cl\lambda_1 \times Cl\lambda_2) - (Int\lambda_1 \times Cl\lambda_2)] \vee [(Cl\lambda_1 \times Cl\lambda_2) - (Cl\lambda_1 \times Int\lambda_2)] \\
&= [(Cl\lambda_1 - Int\lambda_1) \times Cl\lambda_2] \vee [Cl\lambda_1 \times (Cl\lambda_2 - Int\lambda_2)] \\
&= (Bd\lambda_1 \times Cl\lambda_2) \vee (Cl\lambda_1 \times Bd\lambda_2)\,.
\end{aligned}$$

■

Theorem 2.5 *Let $f : X \to Y$ be a continuous fuzzy function. Then*

$$Bdf^{-1}\,(\mu) \leq f^{-1}\,(Bd\mu)\,,$$

for any fuzzy set μ in Y.

Proof. Let f be continuous fuzzy and μ a fuzzy set in Y. Then

$$\begin{aligned} Bdf^{-1}(\mu) &= Clf^{-1}(\mu) \wedge Cl\left(f^{-1}(\mu)\right)^{c} \\ &\leq Clf^{-1}(Cl\mu) \wedge Clf^{-1}(Cl\mu^{c}) \\ &= f^{-1}(Cl\mu) \wedge f^{-1}(Cl\mu^{c}) \\ &= f^{-1}(Cl\mu \wedge Cl\mu^{c}) \\ &= f^{-1}(Bd\mu). \end{aligned}$$

Therefore $Bdf^{-1}(\mu) \leq f^{-1}(Bd\mu)$. ■

Before closing this section, we give an interesting characterization of continuous fuzzy functions in terms of derived fuzzy set and closure of fuzzy set. It shows that fuzzy continuity, in essence, amounts to preservation of closedness of fuzzy sets. For this we first recall following definitions:

Definition 2.4 *[36] A fuzzy set in X is called a fuzzy point if and only if it takes the value* 0 *for all $x \in X$ except one, say, $e \in X$. If its value at e is α $(0 < \alpha < 1)$ we denote this fuzzy point by e_α. The point $e \in X$ is called support of fuzzy point e_α and is denoted as* $\sup p(e)$.

Definition 2.5 *[36] A fuzzy point e_α is said to be quasi-coincident (also written as q-coincident or Q-coincident) with a fuzzy set λ, denoted by $e_\alpha q\lambda$, if for some $x \in X$, $\alpha > \lambda^c(x)$ or $\alpha + \lambda(x) > 1$.*

Definition 2.6 *[36] A fuzzy set λ in an fts X is called a Q-neighborhood of a fuzzy point e, if there exists a $\mu \in \tau$ such that e q $\mu \leq \lambda$.*

Definition 2.7 *[36] A fuzzy point* e *is called an adherent point of a fuzzy set* λ, *if every Q-neighborhood of* e *is quasi-coincident with* λ.

Definition 2.8 *[36] A fuzzy point* e *is called an accumulation point of a fuzzy set* λ, *if* e *is an adherent point of* λ *and every Q-neighborhood of* e *and* λ *are quasi-coincident at some point different from* $supp(e)$, *whenever* $e \in \lambda$. *The union of all the accumulation points of* λ *is called the derived set of* λ, *denoted as* λ^d. *It is evident that* $\lambda^d \leq Cl\lambda$.

Proposition 2.4 *[36] Let* λ *be a fuzzy set in an fts* X, *then*

(1) $Cl\lambda = \lambda \vee \lambda^d$.

(2) λ *is closed fuzzy iff* $\lambda^d \leq \lambda$.

We use Proposition 2.4 and prove the following:

Theorem 2.6 *Let* $f : X \to Y$ *be a function. Then the following are equivalent:*

(1) $f : X \to Y$ *is continuous fuzzy.*

(2) $f\left(\lambda^d\right) \leq Clf\left(\lambda\right)$, *for any fuzzy set* λ *in* X.

Proof. (1) $\Rightarrow$ (2) Let f be continuous fuzzy mapping and λ a fuzzy set in X. Since $Clf\left(\lambda\right)$ is closed fuzzy in Y, $f^{-1}\left(Clf\left(\lambda\right)\right)$ is closed fuzzy in X. $\lambda \leq f^{-1}\left(Clf\left(\lambda\right)\right)$ gives $Cl\lambda \leq Clf^{-1}\left(Clf\left(\lambda\right)\right) = f^{-1}\left(Clf\left(\lambda\right)\right)$. Therefore $f\left(\lambda^d\right) \leq f\left(Cl\lambda\right) \leq ff^{-1}\left(Clf\left(\lambda\right)\right) \leq Clf\left(\lambda\right)$. Consequently $f\left(\lambda^d\right) \leq Clf\left(\lambda\right)$.

(2) $\Rightarrow$ (1) Suppose $f\left(\lambda^d\right) \leq Clf\left(\lambda\right), \ \lambda \in I^X$. Let μ be any closed fuzzy set in Y, we show that $f^{-1}\left(\mu\right)$ is closed fuzzy in X. By our hypothesis, $f\left(\left[f^{-1}\left(\mu\right)\right]^d\right) \leq Clf\left(f^{-1}\left(\mu\right)\right) \leq Cl\mu = \mu$ or $f\left(\left[f^{-1}\left(\mu\right)\right]^d\right) \leq \mu$ gives $\left[f^{-1}\left(\mu\right)\right]^d \leq f^{-1}\left(f\left[f^{-1}\left(\mu\right)\right]^d\right) \leq f^{-1}\left(\mu\right)$ or $\left[f^{-1}\left(\mu\right)\right]^d \leq f^{-1}\left(\mu\right)$ implies $f^{-1}\left(\mu\right)$ is closed fuzzy in X. Thus f is continuous fuzzy mapping. ■

2.2 Semi Fuzzy Boundary

First, we recall some notions:

Definition 2.9 *[9] Let λ be a fuzzy set in an fts (X, τ). Then λ is called a semi-open fuzzy set of X, if there exists a $\nu \in \tau$ such that $\nu \leq \lambda \leq Cl\nu$.*

Definition 2.10 *[58] Let λ be a fuzzy set in an fts X. Then semi-closure (briefly sCl) and semi-interior (briefly $sInt$) of λ are given as*

$$sCl\lambda = \wedge\{\beta \mid \lambda \leq \beta, \beta \text{ semi-closed fuzzy set}\}.$$
$$sInt\lambda = \vee\{\beta \mid \beta \leq \lambda, \beta \text{ semi-open fuzzy set}\}.$$

Remark 2.4 *In the following theorems, we note that almost all the properties related to semi-interior, semi-closure and semi-boundary of fuzzy sets are analogous to their counterparts in Fuzzy Topology and hence some of the proofs are not given.*

Theorem 2.7 *For fuzzy sets λ and μ in an fts X, we have:*

(1) $sInt(\lambda \vee \mu) \geq sInt\lambda \vee sInt\mu.$

(2) $sInt(\lambda \wedge \mu) = sInt\lambda \wedge sInt\mu.$

(3) $sCl(\lambda \vee \mu) = sCl\lambda \vee sCl\mu.$

(4) $sCl(\lambda \wedge \mu) \leq sCl\lambda \wedge sCl\mu.$

(5) $sClsCl\lambda = sCl\lambda.$

(6) $sIntsInt\lambda = sInt\lambda.$

(7) $(sInt\lambda)^c = sCl(\lambda^c).$

(8) $(sCl\lambda)^c = sInt(\lambda^c).$

Proof. (1) $sInt\lambda$ and $sInt\mu$ are both semi-open fuzzy sets and $\lambda \leq \lambda \vee \mu$, $\mu \leq \lambda \vee \mu$ imply $sInt\lambda \leq sInt(\lambda \vee \mu)$ and $sInt\mu \leq sInt(\lambda \vee \mu)$. Combining, $sInt\lambda \vee sInt\mu \leq sInt(\lambda \vee \mu)$ or

$$sInt(\lambda \vee \mu) \geq sInt\lambda \vee sInt\mu.$$

(2) $\lambda \wedge \mu \leq \lambda$ and $\lambda \wedge \mu \leq \mu$ imply $sInt\,(\lambda \wedge \mu) \leq sInt\lambda,\ sInt\,(\lambda \wedge \mu) \leq sInt\mu$ and therefore $sInt\,(\lambda \wedge \mu) \leq sInt\lambda \wedge sInt\mu$. Conversely $sInt\lambda \leq \lambda$ and $sInt\mu \leq \mu$ implies $sInt\lambda \wedge sInt\mu \leq \lambda \wedge \mu$ and $sInt\lambda \wedge sInt\mu$ is semi-open fuzzy set. But $sInt\,(\lambda \wedge \mu)$ is the largest semi-open fuzzy set contained in $\lambda \wedge \mu$, hence $sInt\lambda \wedge sInt\mu \leq sInt\,(\lambda \wedge \mu)$. This gives the equality.

(3) Follows easily from (2).

(4) Since $\lambda \wedge \mu \leq \lambda,\ \lambda \wedge \mu \leq \mu$

$$\Rightarrow \quad sCl\,(\lambda \wedge \mu) \leq sCl\lambda,\ sCl\,(\lambda \wedge \mu) \leq sCl\mu$$

$$\Rightarrow \quad sCl\,(\lambda \wedge \mu) \leq sCl\lambda \wedge sCl\mu.$$

$(5)-(8)$ Proofs are straightforward. ■

The inequalities (1) and (4) of Theorem 2.7 are irreversible as is shown by the following:

Example 2.7 *Let $X=\{a,b\}$ be a set and $I=\{0,.5,1\}$ the lattice of membership grades for fuzzy sets in X. Let τ be the fuzzy topology on X given as*

$$\tau=\left\{\tilde{0},\{a_{.5},b_{1}\},\{a_{.0},b_{1}\},\{a_{1},b_{.5}\},\{a_{.5},b_{.5}\},\{a_{0},b_{.5}\},\tilde{1}\right\}.$$

We choose fuzzy sets $\alpha=\{a_{.5},b_{0}\},\beta=\{a_{0},b_{.5}\},\gamma=\{a_{.5},b_{1}\}$ and $\delta=\{a_{1},b_{.5}\}$. Then calculations give

$$\begin{aligned} sInt\,(\alpha\vee\beta) &= \{a_{.5},b_{.5}\}\nleq\{a_{0},b_{.5}\}=sInt\alpha\vee sInt\beta. \\ sCl\gamma\wedge sCl\delta &= \{a_{1},b_{.5}\}\nleq\{a_{.5},b_{.5}\}=sCl\,(\alpha\wedge\delta)\,. \end{aligned}$$

Definition 2.11 *[1] Let λ be a fuzzy set in an fts X. Then the semi fuzzy -boundary of λ is defined as $sBd\lambda=sCl\lambda\wedge sCl\lambda^{c}$. Obviously, $sBd\lambda$ is a semi-closed fuzzy set.*

Remark 2.5 *In Fuzzy Topology, we have $\lambda\vee sBd\lambda\leq sCl\lambda$, for an arbitrary fuzzy set λ in X, the equality does not hold as the following example shows:*

Example 2.8 *Let $X=\{a,b\}$ be a set with fuzzy topology $\tau=\{\tilde{0},\{a_{.4},b_{.8}\},$*

$\{a_{.6},b_{.9}\},\{a_{.5},b_{.7}\},\{a_{.5},b_{.2}\},\{a_{.8},b_{.7}\},\{a_{.3},b_{.2}\},\{a_{.4},b_{.7}\},\{a_{.4},b_{.2}\},$

$\{a_{.6}, b_{.7}\}, \{a_{.5}, b_{.8}\}, \{a_{.8}, b_{.8}\}, \{a_{.6}, b_{.8}\}, \{a_{.8}, b_{.9}\}, \tilde{1}\}$. *Choose* $\lambda = \{a_{.4}, b_{.7}\}$, *then calculations give*

$$sCl\lambda = \{a_{.5}, b_{.8}\} \neq \{a_{.5}, b_{.7}\} = \lambda \vee sBd\lambda$$

In the following theorem, (1) – (5) are analogues of Proposition 2.1 and hence we omit their proofs.

Proposition 2.5 *For a fuzzy set* λ *in an fts* X*, the following hold:*

(1) $sBd\lambda = sBd\lambda^c$.

(2) *If* λ *is semi-closed fuzzy , then* $sBd\lambda \leq \lambda$.

(3) *If* λ *is semi-open fuzzy , then* $sBd\lambda \leq \lambda^c$.

(4) *Let* $\lambda \leq \mu$ *and* $\mu \in FSC(X)$ *(resp.* $\mu \in FSO(X)$*). Then* $sBd\lambda \leq \mu$ *(resp.* $sBd\lambda \leq \mu^c$*), where* $FSC(X)$ *(resp.* $FSO(X)$*) denotes the class of semi-closed fuzzy (resp. semi-open fuzzy) sets in* X*.*

(5) $(sBd\lambda)^c = sInt\lambda \vee sInt\lambda^c$.

(6) $sBd\lambda \leq Bd\lambda$.

(7) $sClsBd\lambda \leq Bd\lambda$.

Proof. (6) Since $sCl\lambda \leq Cl\lambda$ and $sCl\lambda^c \leq Cl\lambda^c$, then we have

$$sBd\lambda = sCl\lambda \wedge sCl\lambda^c \leq Cl\lambda \wedge Cl\lambda^c = Bd\lambda.$$

(7)

$$\begin{aligned} sClsBd\lambda &= sCl\left(sCl\lambda \wedge sCl\lambda^c\right) \leq sClsCl\lambda \wedge sClsCl\lambda^c \\ &= sCl\lambda \wedge sCl\lambda^c = sBd\lambda \leq Bd\lambda. \end{aligned}$$

■

The converse of (2) and (3) ,and reverse inequalities of (6) and (7) in Theorem 2.5 are, in general, not true as is shown by the following:

Example 2.9 *Choose* $\lambda = \{a_{.7}, b_{.7}\}$, $\psi = \{a_{.3}, b_{.3}\}$, $\omega = \{a_{.7}, b_{.9}\}$ $\xi = \{a_{.6}, b_1\}$ *in the fts* (X, τ) *defined in Example 2.8. Then calculations give*

$$\begin{aligned} sBd\lambda &= \{a_{.4}, b_{.3}\} \leq \lambda, \text{ but } \lambda \text{ is not semi-closed fuzzy ,} \\ sBd\psi &= \{a_{.4}, b_{.3}\} \leq \psi^c, \text{ but } \psi \text{ is not semi-open fuzzy ,} \\ Bd\omega &= \{a_{.4}, b_{.1}\} \nleq \{a_{.3}, b_{.1}\} = sBd\omega, \\ Bd\xi &= \{a_{.4}, b_{.1}\} \nleq \{a_{.4}, b_0\} = sClsBd\xi. \end{aligned}$$

The following is analogue of Proposition 2.2 and hence we omit its proof.

Proposition 2.6 *Let* λ *be a fuzzy set in an fts* X*. Then we have*

(1) $sBd\lambda = sCl\lambda - sInt\lambda$.

(2) $sBdsInt\ \lambda \leq sBd\lambda$.

(3) $sBdsCl\lambda \leq sBd\lambda$.

(4) $sInt\lambda \leq \lambda - sBd\lambda$.

To show that the inequalities (2), (3) and (4) of Theorem 2.6, are, in general, irreversible, we have:

Example 2.10 *Choose* $\alpha = \{a_{.7}, b_{.5}\}$ *and* $\mu = \{a_{.6}, b_{.3}\}$ *in the fts* X *defined* *in Example 2.8. Then calculations give*

$$\begin{aligned} sBd\alpha &= \{a_{.5}, b_{.8}\} \not\leq \{a_{.5}, b_{.2}\} = sBdsInt\alpha \\ sBd\alpha &= \{a_{.5}, b_{.8}\} \not\leq \{a_{.4}, b_{.2}\} = sBdsCl\alpha \\ \mu - sBd\mu &= \{a_{.5}, b_{.3}\} \not\leq \{a_{.5}, b_{.2}\} = sInt\mu. \end{aligned}$$

Remark 2.6 *In General Topology, the following hold:*

$$\begin{aligned} sBdA \cap sIntA &= \widetilde{0}. \\ sIntA \cup sBdA &= ClA. \\ sIntA \cup sIntA^c \cup sBdA &= X. \end{aligned}$$

Whereas, in Fuzzy Topology, we give counter-examples to show that these may not hold in general.

Example 2.11 *In the fts X of Example 2.8, we choose fuzzy set $\lambda = \{a_{.7}, b_{.5}\}$, then calculations give*

$$\begin{aligned} sBd\lambda \wedge sInt\lambda &= \{a_{.5}, b_{.8}\} \wedge \{a_{.5}, b_{.2}\} \neq \tilde{0}. \\ sInt\lambda \vee sBd\lambda &= \{a_{.5}, b_{.8}\} \neq \{a_{.7}, b_{.8}\} = sCl\lambda. \\ sInt\lambda \vee sInt\lambda^c \vee sBd\lambda &= \{a_{.5}, b_{.8}\} \neq \tilde{1}. \end{aligned}$$

It is easily seen that $sInt\lambda \vee sBd\lambda \leq sCl\lambda$.

Theorem 2.8 *Let λ and μ be fuzzy sets in an fts X. Then $sBd(\lambda \vee \mu) \leq sBd\lambda \vee sBd\mu$.*

The reverse inequality in Theorem 2.8 is, in general, not true as is shown by the following:

Example 2.12 *In Example 2.8, choose fuzzy sets* $\alpha = \{a_{.4}, b_{.7}\}$ *and* $\beta = \{a_{.6}, b_{.2}\}$. *Then calculations give*

$$sBd\alpha \vee sBd\beta = \{a_{.5}, b_{.3}\} \not\leq \{a_{.4}, b_{.3}\} = sBd(\alpha \vee \beta)$$

Following example shows that

$$sBd(\gamma \wedge \delta) \not\leq sBd\gamma \wedge sBd\delta \quad and \quad sBd\lambda \wedge sBd\delta \not\leq sBd(\gamma \wedge \delta).$$

For this choose $\gamma = \{a_{.4}, b_{.8}\}$ *and* $\delta = \{a_{.6}, b_{.3}\}$. *Then calculations give*

$$\begin{aligned} sBd(\gamma \wedge \delta) &= \{a_{.4}, b_{.3}\} \not\leq \{a_{.5}, b_{.2}\} = sBd\gamma \wedge sBd\delta \ and \\ sBd\lambda \wedge sBd\delta &= \{a_{.5}, b_{.2}\} \not\leq \{a_{.4}, b_{.3}\} = sBd(\gamma \wedge \delta) \end{aligned}$$

However, we have the following theorem which is an analogue of Theorem 2.2.

Theorem 2.9 *For any fuzzy sets* λ *and* μ *in an fts* X, *we have*

$$sBd(\lambda \wedge \mu) \leq (sBd\lambda \wedge sCl\mu) \vee (sBd\mu \wedge sCl\lambda).$$

Corollary 2.2 *For any fuzzy sets* λ *and* μ *in an fts* X, *we have*

$$sBd(\lambda \wedge \mu) \leq sBd\lambda \vee sBd\mu.$$

Example 2.13 *To show that the reverse inequality in Theorem 2.9 is, in general, not true, choose fuzzy sets* γ *and* δ *as given in Example 2.8. Then calculations give*

$$(sBd\gamma \wedge sCl\delta) \vee (sBd\gamma \wedge sCl\delta) = \{a_{.5}, b_{.3}\} \not\leq \{a_{.4}, b_{.3}\} = sBd(\gamma \wedge \delta).$$

The analogue of Theorem 2.3 is the following theorem, the proof of which is similar:

Proposition 2.7 *For any fuzzy set λ in an fts X, we have*

(1) $sBdsBd\lambda \leq sBd\lambda$.

(2) $sBdsBdsBd\lambda \leq sBdsBd\lambda$.

Remark 2.7 *As in the case of Theorem 2.3*(2) , *we also do not know whether the equality in Theorem 2.7*(2), *holds or not. However, the reverse inequality of* (1) *is, in general, not true as is shown by the following:*

Example 2.14 *Choose $\lambda = \{a_{.6}, b_{.4}\}$ in fts X of Example 2.8. Then*

$$sBd\lambda = \{a_{.5}, b_{.8}\} \not\leq \{a_{.5}, b_{.2}\} = sBdsBd\lambda$$

Lemma 2.2 *[9] If λ is a fuzzy set of X and μ is a fuzzy set of Y, then*

$$\widetilde{1} - (\lambda \times \mu) = \lambda^c \times \widetilde{1} \vee \widetilde{1} \times \mu^c.$$

Using Lemma 2.2 we have:

Lemma 2.3 *Let λ be a semi-closed fuzzy (resp. semi-open fuzzy [9]) set of an fts X and μ a semi-closed fuzzy (resp. semi-open fuzzy) set of an fts Y. Then $\lambda \times \mu$ is a closed fuzzy (resp. semi-open fuzzy) set of the fuzzy product space $X \times Y$.*

Using Lemma 2.3, we have:

Theorem 2.10 *If λ is a fuzzy set of fts X and μ of fts Y, then*

(1) $sCl\lambda \times sCl\mu \geq sCl\,(\lambda \times \mu)$ *and*

(2) $sInt\lambda \times sInt\mu \leq sInt\,(\lambda \times \mu)$.

Moreover we have

Theorem 2.11 *Let X and Y be product related fts's. Then, for a fuzzy set λ of X and a fuzzy set μ of Y we have*

(1) $sCl\,(\lambda \times \mu) = sCl\lambda \times sCl\mu$.

(2) $sInt\,(\lambda \times \mu) = sInt\lambda \times sInt\mu$.

Proof. (1) For fuzzy sets $\lambda_i's$ of X and $\mu_j's$ of Y, we first note that

(i) $\inf\{\lambda_i, \mu_j\} = \min\{\inf \lambda_i, \inf \mu_j\}$,

(ii) $\inf\left\{\lambda_i \times \widetilde{1}\right\} = (\inf \lambda_i) \times \widetilde{1}$, and

(iii) $\inf\left\{\widetilde{1} \times \mu_j\right\} = \widetilde{1} \times (\inf \mu_j)$

In view of Theorem 2.10, it is sufficient to show that $sCl\,(\lambda \times \mu) \geq sCl\lambda \times sCl\mu$. Let $\lambda_i \in FSO\,(X)$ and $\mu_j \in FSO\,(Y)$. Then

$$\begin{aligned} sCl\,(\lambda \times \mu) &= \inf\left\{(\lambda_i \times \mu_j)^c \mid (\lambda_i \times \mu_j)^c \geq \lambda \times \mu\right\} \\ &= \inf\left\{\lambda_i^c \times \widetilde{1} \vee \widetilde{1} \times \mu_j^c \mid \lambda_i^c \times \widetilde{1} \vee \widetilde{1} \times \mu_j^c \geq \lambda \times \mu\right\} \\ &= \inf\left\{\lambda_i^c \times \widetilde{1} \vee \widetilde{1} \times \mu_j^c \mid \lambda_i^c \geq \lambda \text{ or } \mu_j^c \geq \mu\right\} \\ &= \min\left(\inf\left\{\lambda_i^c \times \widetilde{1} \vee \widetilde{1} \times \mu_j^c \mid \lambda_i^c \geq \lambda\right\}, \inf\left\{\lambda_i^c \times \widetilde{1} \vee \widetilde{1} \times \mu_j^c \mid \mu_j^c \geq \mu\right\}\right) \end{aligned}$$

Since

$$\begin{aligned} \inf\left\{\lambda_i^c \times \widetilde{1} \vee \widetilde{1} \times \mu_j^c \mid \lambda_i^c \geq \lambda\right\} &\geq \inf\left\{\lambda_i^c \times \widetilde{1} \mid \lambda_i^c \geq \lambda\right\} \\ &= \inf\{\lambda_i^c \mid \lambda_i^c \geq \lambda\} \times \widetilde{1} = (sCl\lambda) \times \widetilde{1}, \end{aligned}$$

and

$$\begin{aligned}\inf\left\{\lambda_i^c \times \widetilde{1} \vee \widetilde{1} \times \mu_j^c \mid \mu_j^c \geq \mu\right\} &\geq \inf\left\{\mu_j^c \times \widetilde{1} \mid \mu_j^c \geq \mu\right\} \\ &= \widetilde{1} \times \inf\left\{\mu_j^c \mid \mu_j^c \geq \mu\right\} = \widetilde{1} \times (sCl\mu),\end{aligned}$$

we have

$$sCl(\lambda \times \mu) \geq \min\left(sCl\lambda \times \widetilde{1}, \widetilde{1} \times sCl\mu\right) = sCl\lambda \times sCl\mu.$$

(2) This follows from (1) using the facts that $(sInt\psi)^c = sCl\psi^c$ and $(sCl\psi)^c = sInt\psi^c$. ■

The analogue of Theorem 2.4 is the following, the proof of which is similar:

Theorem 2.12 *Let $X_i, i = 1, 2, ..., n,$ be a family of product related fuzzy topological spaces. If each λ_i is a fuzzy set in X_i, then*

$$\begin{aligned}sBd\prod_{i=1}^{n}\lambda_i &= [sBd\lambda_1 \times sCl\lambda_2 \times ... \times sCl\lambda_n] \\ &\quad \vee[sCl\lambda_1 \times sBd\lambda_2 \times sCl\lambda_3 \times ... \times sCl\lambda_n] \vee ... \\ &\quad \vee[sCl\lambda_1 \times sCl\lambda_2 \times ... \times sBd\lambda_n].\end{aligned}$$

The following theorem gives a necessary condition for semi-continuous fuzzy functions in terms of fuzzy boundary and semi fuzzy -boundary:

Theorem 2.13 *Let $f : X \to Y$ be a semi-continuous fuzzy function. Then we have*

$$sBdf^{-1}(\mu) \leq f^{-1}(Bd\mu),$$

for any fuzzy set μ in Y.

Proof. Let f be semi-continuous fuzzy mapping and μ a fuzzy set in Y. Then $Cl\mu$ is closed fuzzy in Y implies $f^{-1}(Cl\mu)$ is semi-closed fuzzy in X. Therefore

$$\begin{aligned} sBdf^{-1}(\mu) &= sClf^{-1}(\mu) \wedge sCl\left(f^{-1}(\mu)\right)^c \\ &\leq sClf^{-1}(Cl\mu) \wedge sClf^{-1}(Cl\mu^c) \\ &= f^{-1}(Cl\mu) \wedge f^{-1}(Cl\mu^c) \\ &= f^{-1}(Cl\mu \wedge Cl\mu^c) \\ &= f^{-1}(Bd\mu). \end{aligned}$$

Hence $sBdf^{-1}(\mu) \leq f^{-1}(Bd\mu)$. ■

Definition 2.12 *[38] A function $f : X \to Y$ is said to be irresolute fuzzy , if $f^{-1}(\beta)$ is semi-open fuzzy in X, for each semi-open fuzzy set β in Y.*

The following theorem gives a necessary condition of irresolute fuzzy functions in terms of fuzzy boundary and semi fuzzy -boundary, the proof of which is similar to Theorem 2.13.

Theorem 2.14 *Let $f : X \to Y$ be a irresolute fuzzy function. Then we have*

$$sBdf^{-1}(\mu) \leq f^{-1}(sBd\mu),$$

for any fuzzy set μ in Y.

Definition 2.13 *A fuzzy set* λ *in an fts* X *is called a semi fuzzy Q-neighborhood of a fuzzy point* e, *if there exists a semi-open fuzzy set* μ *in* X, *such that* $e\ q\ \mu \leq \lambda$.

Notation 1 *By the phrase 'a fuzzy point* $x_\alpha \in \lambda$*', henceforth in this thesis we shall mean*

$$x_\alpha \in \lambda \iff \lambda(x) \leq \alpha.$$

Theorem 2.15 *A fuzzy point* $e = x_\alpha \in sCl\lambda$, *if each semi-Q-neighborhood of* e *is quasi-coincident with* λ.

Proof. $x_\alpha \in sCl\lambda$ iff for every closed fuzzy set $\psi \geq \lambda$, $x_\alpha \in \psi$. This gives $\psi(x_\alpha) \geq \lambda(x_\alpha)$. Equivalently $x_\alpha \in sCl\lambda$ iff for every semi-open fuzzy set $\beta \leq \lambda^c, \beta(x_\alpha) \leq \lambda^c(x_\alpha)$. That is, for every open fuzzy set β satisfying $\beta(x_\alpha) \geq \lambda^c$, β is not contained in λ^c, or $\beta\ q\ \lambda^{cc} = \lambda$. Thus $x_\alpha \in sCl\lambda$, if every open fuzzy Q-neighborhood β of x_α is quasi-coincident with λ. ■

Definition 2.14 *A fuzzy point* e *is called semi-adherent point of a fuzzy set* λ, *if every semi Q-neighborhood of* e *is quasi-coincident with* λ.

Definition 2.15 *A fuzzy point* e *is called a semi-accumulation point of a fuzzy set* λ, *if* e *is a semi-adherent point of* λ *and every semi-Q-neighborhood of* e *and* λ *are quasi-coincident at some point different from* $supp(e)$, *whenever* $e \in \lambda$. *The union of all the semi-accumulation points of* λ *is called the semi -derived fuzzy set of* λ, *denoted as* λ^{sd}. *It is evident that* $\lambda^{sd} \leq sCl\lambda$.

Proposition 2.8 *Let λ be a fuzzy set in X, then $sCl\lambda = \lambda \vee \lambda^{sd}$.*

Proof. Let $\Omega = \{e \mid e$ is a semi-adherent point of $\lambda\}$. Then from Theorem 2.15, $sCl\lambda = \vee\Omega$. On the other hand, $e \in \Omega$ is either $e \in \lambda$ or $e \notin \lambda$; for the latter case, by Definition 2.15, $e \in \lambda^{sd}$, hence $sCl\lambda = \vee\Omega \leq \lambda \vee \lambda^{sd}$. The reverse inclusion is obvious. ■

Corollary 2.3 *For any fuzzy set λ in an fts X, λ is semi-closed fuzzy iff $\lambda^{sd} \leq \lambda$.*

Definition 2.16 *[9] Let $f : X \rightarrow Y$ be a function from an fts X to another fts Y. Then f is said to be semi-continuous fuzzy function, if $f^{-1}(\lambda)$ is semi-open fuzzy in X, for each open fuzzy set λ in Y.*

We use Corollary 2.3 and characterize semi-continuous fuzzy functions in terms of semi -derived fuzzy set as:

Theorem 2.16 *Let $f : X \rightarrow Y$ be a function. Then the following are equivalent:*

(1) *f is semi-continuous fuzzy .*

(2) *$f(\lambda^{sd}) \leq Clf(\lambda)$, for any fuzzy set λ in X.*

Proof. (1) $\Rightarrow$ (2) Let f be semi-continuous fuzzy and λ a fuzzy set in X. Since $Clf(\lambda)$ is closed fuzzy in Y, $f^{-1}(Clf(\lambda))$ is semi-closed fuzzy in X such that $\lambda \leq f^{-1}(Clf(\lambda))$. $\lambda \leq f^{-1}(sClf(\lambda))$ gives $sCl\lambda \leq sClf^{-1}(Clf(\lambda)) = f^{-1}(Clf(\lambda))$. Therefore $f(\lambda^{sd}) \leq f(sCl\lambda) \leq ff^{-1}(Clf(\lambda)) \leq Clf(\lambda)$. Consequently $f(\lambda^{sd}) \leq Clf(\lambda)$.

(2) $\Rightarrow$ (1) Suppose $f\left(\lambda^{sd}\right) \leq Clf(\lambda)$. Let μ be any closed fuzzy set in Y, we show that $f^{-1}(\mu)$ is semi-closed fuzzy in X. By our hypothesis, $f\left(\left[f^{-1}(\mu)\right]^{sd}\right) \leq Clf\left(f^{-1}(\mu)\right) \leq Cl\mu = \mu$ or $f\left(\left[f^{-1}(\mu)\right]^{sd}\right) \leq \mu$ gives $\left[f^{-1}(\mu)\right]^{sd} \leq f^{-1}\left(f\left[f^{-1}(\mu)\right]^{sd}\right) \leq f^{-1}(\mu)$ or $\left[f^{-1}(\mu)\right]^{sd} \leq f^{-1}(\mu)$ implies $f^{-1}(\mu)$ is semi-closed fuzzy in X. Thus f is semi-continuous fuzzy . ■

Finally, we characterize irresolute fuzzy functions via semi -derived fuzzy set as:

Theorem 2.17 *Let $f : X \rightarrow Y$ be a function. Then the following are equivalent:*

(1) *f is irresolute fuzzy .*

(2) *$f\left(\lambda^{sd}\right) \leq sClf(\lambda)$, for any fuzzy set λ in X.*

Chapter 3

On Some Fuzzy Mappings

First we recall some notion, which are required for this chapter.

Definition 3.1 *[57] A mapping $f : (X, \tau) \rightarrow (Y, \delta)$ is called open fuzzy , if $f(\lambda) \in \delta$ for every $\lambda \in \tau$.*

Definition 3.2 *[9] Let λ be a fuzzy set in an fts (X, τ). Then λ is called a semi-open fuzzy (resp. semi-closed fuzzy) set of X, if there exists a $\nu \in \tau$ (resp. $\nu^c \in \tau$) such that $\nu \leq \lambda \leq Cl\nu$ (resp. $Int\nu \leq \lambda \leq \nu$). The class of semi-open fuzzy (resp. semi-closed fuzzy) sets in X is denoted by $FSO(X)$ (resp. $FSC(X)$).*

Definition 3.3 *[9] A mapping $f : (X, \tau) \rightarrow (Y, \delta)$ is semi-continuous fuzzy , if $f^{-1}(\lambda)$ is a semi-open fuzzy set, for every open fuzzy set λ in Y.*

Then the following is immediate:

Theorem 3.1 *A function $f : X \to Y$ is semi-continuous fuzzy if and only if λ being closed fuzzy in Y implies $f^{-1}(\lambda)$ is semi-closed fuzzy in X.*

3.1 Semi-open fuzzy sets and fuzzy mappings

Definition 3.4 *[9] A mapping $f : X \to Y$ is said to be semi-open fuzzy if for every open fuzzy set λ in X, $f(\lambda)$ is a semi-open fuzzy set in Y.*

It is known [9], that the intersection of two semi-open fuzzy sets needs not be semi-open fuzzy set. However, we have:

Lemma 3.1 *If μ is open fuzzy and λ is semi-open fuzzy , then $\mu \wedge \lambda$ is semi-open fuzzy .*

Proof. Since λ is semi-open fuzzy , there exists a open fuzzy set γ such that $\gamma \leq \lambda \leq Cl\gamma$. It follows from the open fuzzy ness of μ that

$$\mu \wedge Cl\gamma \leq Cl(\mu \wedge \gamma).$$

Hence we have

$$\gamma \wedge \mu \leq \lambda \wedge \mu \leq Cl\gamma \wedge \mu \leq Cl(\gamma \wedge \mu).$$

This implies that $\lambda \wedge \mu$ is semi-open fuzzy because $\gamma \wedge \mu$ is open fuzzy . ■

Converse of Lemma 4.1 is not true in general as shown by following:

Example 3.1 *Let $X = \{a, b, c\}$ be a set, $I = \{0, .3, .5, .7, 1\}$ the lattice for membership grades of fuzzy sets in X and τ, the fuzzy topology on X given as*

$\tau = \{\widetilde{0}, \{a_{.3}, b_{.7}, c_{.5}\},$

$\{a_{.5}, b_{.5}, c_0\}, \{a_{.3}, b_{.5}, c_0\}, \{a_{.5}, b_{.7}, c_{.5}\}, X\}$. *Choose* $\mu = \{a_{.5}, b_{.5}, c_{.5}\}$ *and* $\lambda = \{a_{.5}, b_{.7}, c_{.3}\}$. *Then* $\mu \wedge \lambda = \{a_{.5}, b_{.5}, c_{.3}\} \in FSO(X)$ *but neither* μ *is open fuzzy nor* λ *is semi-open fuzzy in* X.

Lemma 3.2 *Let* $\lambda \in FSO(X)$ *and* $\lambda \leq \beta \leq Cl\lambda$. *Then* $\beta \in FSO(X)$. *In particular* $Cl\lambda$ *is semi-open fuzzy .*

Proof. Since λ is semi-open fuzzy , therefore there exists a open fuzzy set ω such that $\omega \leq \lambda \leq Cl\omega$. Then $\omega \leq \beta$ and $Cl\lambda \leq Cl\omega$. Hence $\omega \leq \beta \leq Cl\omega$ and β is semi-open fuzzy . ■

Example 3.2 *Let* (X, τ) *be the fts as defined in Example 3.1. Choose* $\beta = \{a_1, b_{.7}, c_{.7}\}$ *and* $\lambda = \{a_1, b_{.3}, c_{.7}\}$. *Then* $\lambda \leq \beta \leq Cl\lambda = X$ *and* $\beta \in FSO(X)$ *but* λ *is not semi-open fuzzy in* X.

Definition 3.5 *Let* (X, τ) *be an fts and* Y *a crisp subset of* X. *Then a fuzzy topology* δ *on* Y *is defined as* $\delta = \{\mu \wedge 1_Y : \mu \in \tau\}$ *and the fts* (Y, δ) *is said to be a fuzzy subspace of* X.

Theorem 3.2 *Let* X *be an fts and* Y *a semi-open fuzzy subspace of* X.*Then* $\alpha \in FSO(X)$ *if and only if* $\alpha \in FSO(Y)$ *for each fuzzy set* α *in* X.

Proof. ($\Rightarrow$) If $\alpha \in FSO(X)$, then $\omega \leq \alpha \leq Cl_X(\omega)$, where ω is open fuzzy in X. Now ω is a fuzzy set in Y and thus

$$\omega = \omega \wedge 1_Y \leq \alpha \wedge 1_Y \leq 1_Y \wedge Cl_X(\omega)$$

or

$$\omega \leq \alpha \leq Cl_Y(\omega).$$

Since $\omega = \omega \wedge 1_Y$ is open fuzzy in Y, hence the necessity is proved.

($\Leftarrow$) Let $\alpha \in FSO(Y)$. Then there exists open fuzzy set μ in Y such that $\mu \leq \alpha \leq Cl_Y(\mu)$. Since μ is open fuzzy in Y, there exists a open fuzzy set ν in X such that $\mu = \nu \wedge 1_Y$. Therefore we have

$$\nu \wedge 1_Y \leq \alpha \leq Cl_Y(\nu \wedge 1_Y).$$

Since $Y \in FSO(X)$ and ν is open fuzzy in X, by Lemma 4.1, $\nu \wedge 1_Y$ is semi-open fuzzy in X. Hence by Lemma 4.2, $\alpha \in FSO(X)$. ■

Theorem 3.3 *Let λ be a fuzzy set in an fts (X, τ) and Y a open fuzzy subspace of X such that Y is semi-open fuzzy in X and λ is a fuzzy set in Y. Then $sInt_X\lambda = sInt_Y\lambda$.*

Proof. Since $sInt_X\lambda \in FSO(X)$ and $sInt_X\lambda \leq \lambda \leq 1_Y$. By Theorem 7.1, we get $sInt_X\lambda \in FSO(Y)$. Since $sInt_Y\lambda$ is the largest semi fuzzy open set in Y, therefore $sInt_X\lambda \leq sInt_Y\lambda$.On the other hand

$$sInt_X\lambda \leq \lambda \leq 1_Y \in FSO(X)$$

and again by Theorem 7.1, $sInt_Y\lambda \in FSO(X)$ implies $sInt_Y\lambda \leq sInt_X\lambda$. Hence we have $sInt_X\lambda = sInt_Y\lambda$. ■

Definition 3.6 *[36] A fuzzy point x_α in X is a fuzzy set with membership function defined as:*

$$x_\alpha(x) = \begin{cases} \alpha & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$$

where $0 < \alpha < 1$.

Definition 3.7 *[36] A fuzzy set λ is said to be q-coincident with β, denoted by $\lambda q \beta$, if there exists an $x \in X$ such that $\lambda(x) > \beta^c(x)$, equivalently $\lambda(x) + \beta(x) > 1$.*

In terms of q-coincidence and semi fuzzy -limit points, we characterize a semi-closed fuzzy set by first introducing :

Definition 3.8 *A fuzzy point x_α in an fts X is called semi fuzzy -limit-point of a fuzzy set λ in X, if every semi fuzzy -q-neighborhood of x_α is q-coincident with λ.*

Theorem 3.4 *Let λ be a fuzzy set in an fts X. Then λ is semi-closed fuzzy if and only if λ is q-coincident with all of its semi fuzzy -limit points.*

Proof. Let λ be semi-closed fuzzy set in X and x_α a fuzzy point, which is semi fuzzy -limit point of λ. To show that $x_\alpha q \lambda$, suppose $x_\alpha \not{q} \lambda$. Then $x_\alpha \in \lambda^c$. Clearly λ^c is a fuzzy-semi-open-neighborhood of x_α. But $\lambda \not{q} \lambda^c$ implies that x_α is not a semi fuzzy -limit point, a contradiction, hence $x_\alpha q \lambda$.

Conversely, suppose λ is q-coincident with all of its fuzzy limit points. We have to show that λ is semi-closed fuzzy . Let $x_\alpha q \lambda^c$. Then $x_\alpha \notin \lambda$, implies

x_α is not a semi fuzzy -limit point of λ. Thus there exists a semi-open fuzzy -q-neighborhood μ of x_α such that $x_\alpha q\mu$ and $\mu \not\leq \lambda$. This implies $\mu q\lambda^c$ and hence

$$\lambda^c = \vee\{\mu| \ x_\alpha q\mu \text{ and } \mu \text{ is semi-open fuzzy in } X\}.$$

Thus λ^c is semi-open fuzzy . Hence λ is semi-closed fuzzy . ■

Theorem 3.5 *Let X, Y and Z be fuzzy topological spaces such that Y is fuzzy subspace of Z. Let $f : X \to Y$ be a semi-continuous fuzzy mapping. Then f is semi-continuous fuzzy from X to Z.*

Proof. Let ν be a open fuzzy set in Z. Then $\nu \wedge 1_Y$ is open fuzzy in Y. Since f is semi-continuous fuzzy so $f^{-1}(\nu \wedge 1_Y)$ is semi-open fuzzy in X. But

$$f^{-1}(\nu \wedge 1_Y) = f^{-1}(\nu) \wedge f^{-1}(1_Y) = f^{-1}(\nu).$$

Hence $f^{-1}(\nu)$ is semi-open fuzzy in X whenever ν is open fuzzy in Z. This proves $f : X \to Z$ is semi-continuous fuzzy . ■

Fuzzy continuous image of a semi-open fuzzy set may not be semi-open fuzzy . It is shown by the following:

Example 3.3 *Let $X = \{a, b, c\}$, $Y = \{p, q, r\}$ be two sets and $I = \{0, .3, .5, .7, 1\}$, the lattice of membership grades for fuzzy sets in both X and Y. Fuzzy topologies on X and Y are given as $\tau_X = \left\{\widetilde{0}, \{a_{.5}, b_{.5}, c_1\}, \{a_0, b_{.5}, c_{.5}\}, X\right\}$ and $\tau_Y = \left\{\widetilde{0}, \{a_{.5}, b_1, c_{.5}\}, \{a_{.5}, b_{.5}, c_0\}, Y\right\}$. Define a mapping $f : X \to Y$ as $f(a) = p$, $f(b) = p$, $f(c) = r$ and choose the semi-open fuzzy set $\lambda =$*

$\{a_{.5}, b_{.5}, c_{.5}\}$ *of* X*. Clearly* f *is semi-open fuzzy but calculations give that* $f(\lambda) = \{p_{.5}, q_0, r_{.5}\} \notin FSO(Y)$.

However, using Lemma 4.2 we have the following:

Theorem 3.6 *Let* $f : X \to Y$ *be a fuzzy continuous and semi-open fuzzy mapping. If* $\lambda \in FSO(X)$*, then* $f(\lambda) \in FSO(Y)$*.*

Proof. Since λ is semi-open fuzzy there exists a open fuzzy set ω in X such that $\omega \leq \lambda \leq Cl\omega$. Therefore

$$\begin{aligned} f(\omega) &\leq f(\lambda) \leq f(Cl\omega) \leq Clf(\omega) \\ \text{or } f(\omega) &\leq f(\lambda) \leq Clf(\omega). \end{aligned}$$

Since the fuzzy set $f(\omega)$ is semi-open fuzzy , it follows from Lemma 4.2 that $f(\lambda) \in FSO(Y)$. ■

Corollary 3.1 *Let* $f : X \to Y$ *be fuzzy continuous and open fuzzy . If* $\lambda \in FSO(X)$*, then* $f(\lambda) \in FSO(Y)$*.*

Using Theorem 4.7, we prove the following:

Theorem 3.7 *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two mappings such that* $gof : X \to Z$ *is semi-continuous fuzzy . Then*

(1) *If* g *is a open fuzzy injection, then* f *is semi-continuous fuzzy .*

(2) *If* f *is fuzzy continuous and semi-open fuzzy surjection, then* g *is semi-continuous fuzzy .*

Proof. (1) Let μ be an arbitrary open fuzzy set in Y. Since g is open fuzzy , then $g(\mu)$ is open fuzzy in Z. Also gof is semi-continuous fuzzy and g is a open fuzzy injection, therefore

$$(gof)^{-1}g(\mu) = (f^{-1}og^{-1})g(\mu) = f^{-1}(g^{-1}g(\mu)) = f^{-1}(\mu)$$

is semi-open fuzzy . This proves that f is semi-continuous fuzzy .

(2) Let ν be an arbitrary open fuzzy set in Z. Since f is fuzzy continuous and semi-open fuzzy surjection then by Theorem 4.7

$$\begin{aligned} f((gof)^{-1}(\nu)) &= f(f^{-1}og^{-1})(\nu) \\ &= ff^{-1}(g^{-1}(\nu)) = g^{-1}(\nu) \end{aligned}$$

is semi-open fuzzy in Y. ■

Theorem 3.8 *Let $f : X \to Y$ and $g : Y \to Z$ be mappings. Then*

(1) *If f is semi-continuous fuzzy and g is fuzzy continuous, then gof is semi-continuous fuzzy .*

(2) *If f is open fuzzy and g is semi-open fuzzy , then gof is semi-open fuzzy .*

Proof. (1) Suppose μ is a open fuzzy set in Z. Then $(gof)^{-1}(\mu) = f^{-1}(g^{-1}(\mu))$ is semi-open fuzzy in X and thus gof is semi-continuous fuzzy .

(2) Suppose λ is open fuzzy in X. Then $(gof)(\lambda) = g(f(\lambda))$. f is open fuzzy implies $f(\lambda)$ is open fuzzy in Y. Also g is semi-open fuzzy implies $(gof)(\lambda) = g(f(\lambda))$ is semi-open fuzzy in Z. Consequently gof is a semi-open fuzzy mapping. ■

Definition 3.9 *[38] A mapping $f : (X, \tau) \to (Y, \delta)$ is fuzzy almost open in Ganguly's sense (briefly, f.a.o.G), if $f^{-1}(Cl\lambda) \leq Cl(f^{-1}(\lambda))$, for every open fuzzy set λ in Y.*

We define fuzzy almost closed mapping as :

Definition 3.10 *A mapping $f : X \to Y$ is said to be fuzzy almost closed (briefly f.a.c), if $Int(f^{-1}(\nu)) \leq f^{-1}(Int\nu)$, for every closed fuzzy set ν in Y.*

Now we have

Theorem 3.9 *Let $f : X \to Y$ and $g : Y \to Z$ be f.a.o.G (resp., f.a.c.) mappings. Then $gof : X \to Z$ is f.a.o.G. (resp., f.a.c.), if g is fuzzy continuous.*

Proof. Let λ and μ be open fuzzy sets in Y and Z respectively. Since f and g are f.a.o.G, we have $f^{-1}(Cl\lambda) \leq Cl(f^{-1}\lambda)$ and $g^{-1}(Cl\mu) \leq Clg^{-1}(\mu)$. Let ν be a open fuzzy set in Z. Then

$$\begin{aligned} (gof)^{-1}(Cl\nu) &= (f^{-1}og^{-1})(Cl\nu) \\ &= f^{-1}(g^{-1}(Cl\nu)) \\ &\leq f^{-1}(Clg^{-1}(\nu)) \leq Clf^{-1}(g^{-1}(\nu)) = Cl(gof)^{-1}(\nu) \end{aligned}$$

This proves gof is f.a.o.G. ∎

Theorem 3.10 *Let $f : X \to Y$ and $g : Y \to Z$ be two mappings such that gof is f.a.o.G. Then*

(1) *If g is open fuzzy and fuzzy continuous injection, then f is f.a.o.G.*

(2) *If f is fuzzy continuous surjection then g is f.a.o.G.*

Proof. (1) Let ν be a open fuzzy set in Y. Since g is a open fuzzy and fuzzy continuous injection, we have

$$\begin{aligned} f^{-1}(Cl\nu) &= f^{-1}(g^{-1}og)(Cl\nu) = (gof)^{-1}g(Cl\nu) \\ &\leq (gof)^{-1}Clg(\nu) \leq Cl(gof)^{-1}g(\nu) \\ &= Cl(f^{-1}(\nu)). \end{aligned}$$

This proves that f is f.a.o.G.

(2) Let ν be a open fuzzy set in Z. Since f is fuzzy continuous surjection, we have

$$\begin{aligned} g^{-1}(Cl\nu) &= (fof^{-1})og^{-1}(Cl\nu) = fo(f^{-1}og^{-1})(Cl\nu) \\ &= fo(gof)^{-1}(Cl\nu) \leq f(Cl((gof)^{-1}))(\nu) \\ &\leq Cl(fo(gof)^{-1}(\nu)) = Clff^{-1}g^{-1}(\nu) = Cl(g^{-1}(\nu)). \end{aligned}$$

This proves that g is f.a.o.G. ■

The following is the dual of Theorem 4.12.

Theorem 3.11 *Let $f : X \to Y$ and $g : Y \to Z$ be two mappings such that gof is f.a.c. Then*

(1) *If g is closed fuzzy and fuzzy continuous injection, then f is f.a.c.*

(2) *If f is fuzzy continuous surjection then g is f.a.c.*

Theorem 3.12 *If $f : X \to Y$ is fuzzy continuous and fuzzy almost closed (resp. f.a.o.G), then for every $\nu \in FSC(Y)$ (resp. $\nu \in FSO(Y)$), we have $f^{-1}(\nu) \in FSC(X)$ (resp. $f^{-1}(\nu) \in FSO(X)$).*

Proof. Let ν be an arbitrary semi-closed fuzzy (resp. semi-open fuzzy) set of Y. Then there exists a closed fuzzy (resp. open fuzzy) set ω such that $Int\omega \leq \nu \leq \omega$ (resp. $\omega \leq \nu \leq Cl\omega$). Since f is almost closed (resp. f.a.o.G), we have

$$Intf^{-1}(\omega) \leq f^{-1}(Int\omega) \leq f^{-1}(\nu) \leq f^{-1}(\omega)$$

(resp. $f^{-1}(\omega) \leq f^{-1}(\nu) \leq f^{-1}(Cl\omega) \leq Clf^{-1}(\omega)$) or

$$Intf^{-1}(\omega) \leq f^{-1}(\nu) \leq f^{-1}(\omega).$$

Since f is fuzzy continuous, hence $f^{-1}(\nu) \in FSC(X)$ (resp. $f^{-1}(\nu) \in FSO(X)$). ■

Chapter 4

On Some Fuzzy Mappings-II

4.1 Introduction

In previous chapter we studied semi-continuous fuzzy , semi-open fuzzy and fuzzy almost open (Ganguly's sense) mappings. We also defined and studied properties of fuzzy almost closed mappings. In this chapter several properties and characterizations of semi-open fuzzy (semi-closed fuzzy), semi fuzzy -preopen (semi fuzzy -preclosed), semi fuzzy -precontinuous and fuzzy pre-semi-preopen (fuzzy pre-semi-preclosed) mappings have been investigated further.

4.2 Semi-open and Semi-closed Fuzzy Mappings

Definition 4.1 *[9] Let λ be a fuzzy set in an fts X. Then λ is called a semi-open fuzzy (resp. semi-closed fuzzy) set, if there exists a $\nu \in \tau$ (resp. $\nu^c \in \tau$) such that $\nu \leq \lambda \leq Cl\nu$ (resp. $Int\nu \leq \lambda \leq \nu$). The class of semi-open fuzzy (resp. semi-closed fuzzy) sets in X is denoted by $FSO(X)$ (resp. $FSC(X)$).*

Theorem 4.1 *[9] Let λ be a fuzzy set in a space X, then*

(1) *λ is semi-open fuzzy if and only if $\lambda \leq ClInt\lambda$.*

(2) *λ is semi-closed fuzzy if and only if $IntCl\lambda \leq \lambda$.*

Lemma 4.1 *If λ is semi-open fuzzy and $\lambda \leq \nu$, then $\lambda \leq IntCl\nu$.*

Proof. Since λ is semi-open fuzzy , therefore by Theorem 7.1, $\lambda \leq ClInt\lambda$, and $\lambda \leq \nu$ implies $ClInt\lambda \leq ClInt\nu$. Hence $\lambda \leq ClInt\lambda \leq ClInt\nu$ gives $\lambda \leq ClInt\nu$. ■

The converse of Lemma 4.1 is, in general, not true as shown by the following:

Example 4.1 *Let $X = \{a, b\}$ be a set and $I = \{0, .3, .5, .7, 1\}$ be the lattice of membership grades of fuzzy sets in X. Let τ_X be the fuzzy topology on X generated by fuzzy sets $\{a_{.3}, b_{.7}\}$ and $\{a_{.5}, b_{.5}\}$. Choose $\lambda = \{a_{.7}, b_{.5}\}$ and $\nu = \{a_{.3}, b_1\}$. Then we have $\lambda \leq ClInt\nu = X$ but neither $\lambda \leq \nu$ nor $\lambda \in FSO(X)$.*

Definition 4.2 *[9] A function $f : X \to Y$ is said to be semi-open fuzzy (resp. semi-closed fuzzy), if the image of every open fuzzy (resp. fuzzy closed) set is semi-open fuzzy (resp. semi-closed fuzzy).*

Next, we give a characterization of semi-open fuzzy mappings:

Theorem 4.2 *A mapping $f : X \to Y$ is semi-open fuzzy if and only if for every fuzzy set η in X, $f(Int\eta) \leq ClIntf(\eta)$.*

Proof. Let f be semi-open fuzzy , clearly $f(Int\eta) \leq f(\eta)$. Since f is semi-open fuzzy , therefore $f(Int\eta)$ is semi-open fuzzy . So using Lemma 4.1, we get $f(Int\eta) \leq ClIntf(\eta)$.

Conversely, let γ be any open fuzzy set in X. Then by supposition $f(\gamma) = f(Int\gamma) \leq ClIntf(\gamma)$. By Theorem 7.1, $f(\gamma)$ is therefore a semi-open fuzzy set in Y and consequently f is semi-open fuzzy . ■

Theorem 4.3 *A mapping $f : X \to Y$ is semi-open fuzzy if and only if for every fuzzy set γ in Y, $Intf^{-1}(\gamma) \leq f^{-1}(ClInt\gamma)$*

Proof. ($\Rightarrow$) Let f be semi-open fuzzy and γ a fuzzy set in Y. Then $f(Intf^{-1}(\gamma))$ is semi-open fuzzy in Y because $Int(f^{-1}(\gamma))$ is open in X. This implies $f : Z \to Y$ is semi-open fuzzy mapping, where $Z = \{Int(f^{-1}(\gamma))|\gamma$ is in $Y\}$. By Theorem 7.4, for every fuzzy set α in Z, $f(Int\alpha) \leq ClIntf(\alpha)$ implies $Int\alpha \leq f^{-1}(ClIntf(\alpha))$. Since $\alpha \in Z$, there exists γ_0 in Y such that $\alpha = Intf^{-1}(\gamma_0)$ implies

$$IntIntf^{-1}(\gamma_0) \leq f^{-1}(ClIntf(Intf^{-1}(\gamma_0)))$$

or

$$Intf^{-1}(\gamma_0) \leq f^{-1}(ClIntIntff^{-1}(\gamma_0)) \leq f^{-1}(ClIntInt\gamma_0) = f^{-1}(ClInt\gamma_0)$$

or

$$Intf^{-1}(\gamma_0) \leq f^{-1}(ClInt\gamma_0).$$

(⇐) Let η be open fuzzy in X, let $\gamma = f(\eta)$. It is given that $Intf^{-1}(\gamma) \leq f^{-1}(ClInt\gamma)$ which implies

$$f(Intf^{-1}(\gamma)) \leq ff^{-1}(ClInt\gamma)) \leq ClInt\gamma \tag{4.1}$$

Now $\gamma = f(\eta)$ gives $\eta \leq f^{-1}(\gamma)$ and $\eta = Int\eta \leq Intf^{-1}(\gamma)$ or $f(\eta) \leq f(Intf^{-1}(\gamma))$. Using (4.1), we get $f(\eta) \leq f(Intf^{-1}(\gamma)) \leq ClInt\gamma$ or $f(\eta) \leq ClIntf(\eta)$, that is, $f(Int\eta) \leq ClIntf(\eta)$, hence by Theorem 7.4, f is semi-open fuzzy . ■

Theorem 4.4 *If a function $f : X \to Y$ is semi-closed fuzzy , then for each fuzzy set β in an fts Y and each open fuzzy set μ in an fts X with $\mu \geq f^{-1}(\beta)$, there exists a semi-open fuzzy set ν in Y with $\nu \geq \beta$ such that $f^{-1}(\nu) \leq \mu$.*

Proof. Let μ be an arbitrary open fuzzy set in X with $\mu \geq f^{-1}(\beta)$, where β is a fuzzy set in Y. Clearly $(f(\mu^c))^c = \nu$ (say) is semi-open fuzzy in Y. Since $f^{-1}(\beta) \leq \mu$, then straight forward calculations give that $\beta \leq \nu$. Moreover, we have

$$\begin{aligned} f^{-1}(\nu) &= f^{-1}(Y) - f^{-1}(f(\mu^c)) \\ &= X - f^{-1}(f(\mu^c)) \leq \mu \\ \text{or} \qquad f^{-1}(\nu) &\leq \mu. \end{aligned}$$

■

Theorem 4.5 *Let $f : X \to Y$ be a bijective function from an fts X to an fts Y. If for each fuzzy set β in Y and each open fuzzy set μ in X with $\mu \geq f^{-1}(\beta)$,* *there exists a semi-open fuzzy set ν in Y with $\nu \geq \beta$ such that $f^{-1}(\nu) \leq \mu$, then f is semi-closed.*

Proof. Let ψ be an arbitrary closed fuzzy set in X and $y \in (f(\psi))^c$. Since f is injection, therefore

$$f^{-1}(y) \leq f^{-1}((f(\psi))^c) = \left(f^{-1}f(\psi)\right)^c = \psi^c$$

or $f^{-1}(y) \leq \psi^c$. Since ψ^c is open fuzzy , therefore there exists a semi-open fuzzy set ν_y with $y \in \nu_y$ such that $f^{-1}(\nu_y) \leq \psi^c$. Since f is surjective, we have $y \in \nu_y = f^{-1}f(\nu_y) \leq f(\psi^c) = (f(\psi))^c$ or $y \in \nu_y \leq (f(\psi))^c$. Thus $(f(\psi))^c = \vee\{\nu_y | y \in (f(\psi))^c\}$ is semi-open fuzzy in Y or $f(\psi)$ is semi-closed fuzzy in Y. This proves that f is semi-closed. ■

Combining Theorems 4.4 and 4.5, we get:

Theorem 4.6 *A bijective function $f : X \to Y$ is semi-closed fuzzy if and only if for each fuzzy set β in Y and each open fuzzy set μ in X with $\mu \geq f^{-1}(\beta)$, there exists a semi-open fuzzy set ν in Y with $\nu \geq \beta$ such that $f^{-1}(\nu) \leq \mu$.*

We use Theorem 7.4 and prove:

Theorem 4.7 *Let $f : X \to Y$ be semi-open fuzzy and $g : Y \to Z$ be semi-open fuzzy and fuzzy continuous. Then gof is semi-open fuzzy .*

Proof. Let λ be open fuzzy in X. Then f is semi-open fuzzy , implies $f(\lambda)$ is semi-open fuzzy in Y. Therefore there exists a open fuzzy set μ in Y

such that $\mu \leq f(\lambda) \leq Cl\mu$. Which gives $g(\mu) \leq g(f(\lambda)) \leq g(Cl\mu)$. Since g is fuzzy continuous, therefore we have

$$g(\mu) \leq gof(\lambda) \leq g(Cl\mu) \leq Clg(\mu)$$

or $g(\mu) \leq gof(\lambda) \leq Clg(\mu)$. By supposition, $g(\mu) \in FSO(Y)$. Therefore by Theorem 7.4 gof is semi-open fuzzy . ■

Definition 4.3 *[38] A function $f : X \to Y$ is said to be irresolute fuzzy , if $f^{-1}(\beta)$ is semi-open fuzzy in X, for each semi-open fuzzy set β in Y.*

Theorem 4.8 *Let $f : X \to Y$ and $g : Y \to Z$ be mappings such that $gof : X \to Z$ is semi-closed fuzzy (resp. semi-open fuzzy). Then we have :*

(1) If f is fuzzy continuous and surjective, then g is semi-closed fuzzy (resp. semi-open fuzzy).

(2) If g is irresolute fuzzy and injective, then f is semi-closed fuzzy (resp. semi-open fuzzy).

Proof. (1) Suppose η is an arbitrary closed fuzzy set in Y. f is fuzzy continuous implies $f^{-1}(\eta)$ is closed in X. Since gof is semi-closed fuzzy and f is surjective $(gof)(f^{-1}(\eta)) = g(ff^{-1}(\eta)) = g(\eta)$ is semi-closed fuzzy in Z. This implies that g is semi-closed fuzzy mapping.

(2) Suppose ψ is an arbitrary closed fuzzy set in X. gof is semi-closed fuzzy implies $(gof)(\psi)$ is semi-closed fuzzy in Z. Since g is irresolute fuzzy and injective, we have $g^{-1}((gof)(\psi)) = f(\psi)$, and $f(\psi)$ is semi-closed fuzzy set in Y. This implies that f is semi-closed fuzzy . ■

Lemma 4.2 *Let* λ *and* μ *be fuzzy sets in an fts* X. *Let* $\lambda \in FSO(X)$ *and* $\lambda \leq \mu \leq Cl\lambda$. *Then* $\mu \in FSO(X)$. *In particular,* $Cl\lambda \in FSO(X)$.

Proof. By definition of semi-openness of λ, there exists a open fuzzy set ν in X such that $\nu \leq \lambda \leq Cl\nu$, which implies $\nu \leq \lambda \leq \mu \leq Cl\lambda \leq Cl\nu$ or $\nu \leq \mu \leq Cl\nu$ and hence $\mu \in FSO(X)$. ■

The converse is not true, in general, as shown in the following:

Example 4.2 *Let* (X, τ_X) *be the fts given in Example 4.1. Choose a fuzzy set* $\lambda = \{a_0, b_{.5}\}$ *and a semi-open fuzzy set* $\mu = \{a_{.3}, b_{.5}\}$. *Then* $\lambda \leq \mu \leq Cl\lambda = \{a_{.5}, b_{.5}\}$ *but* $\lambda \notin FSO(X)$.

Theorem 4.9 *Let* $f : X \to Y$ *be fuzzy continuous and semi-open fuzzy and* $\lambda \in FSO(X)$. *Then* $f(\lambda) \in FSO(Y)$.

Proof. Since λ is semi-open fuzzy there exists a open fuzzy set ω in X such that $\omega \leq \lambda \leq Cl\omega$. Since f is fuzzy continuous, therefore $f(\omega) \leq f(\lambda) \leq f(Cl\omega) \leq Clf(\omega)$ or $f(\omega) \leq Clf(\omega)$. Since f is semi-open fuzzy , therefore the fuzzy set $f(\omega)$ is semi-open fuzzy . It follows from Lemma 4.2 that $f(\lambda) \in FSO(Y)$. ■

The converse of Theorem 7.6 is not true, in general, as shown in the following:

Example 4.3 *Let* (X, τ_X) *be the fts in Example 4.1 and* (Y, τ_Y) *another fts with* $Y = \{p, q\}$ *and* τ_Y *generated by fuzzy sets* $\{p_{.7}, q_{.7}\}, \{p_1, q_{.7}\}, \{p_{.3}, q_1\}$

and $\{p_{.5}, q_{.5}\}$. *Let* $f : X \rightarrow Y$ *be defined as* $f(a) = q$, $f(b) = p$. *Then for* $\lambda = \{a_{.7}, b_1\} \in FSO(X)$ *gives* $f(\lambda) = \{a_1, b_{.7}\} \in FSO(X)$. *But* f *is not fuzzy continuous.*

Lemma 4.3 *Let* X_1 *and* X_2 *be two product related fuzzy topological spaces and* $X_1 \times X_2$ *their topological product. If* $\lambda_1 \in FSO(X_1)$ *and* $\lambda_2 \in FSO(X_2)$, *then* $\lambda_1 \times \lambda_2 \in FSO(X_1 \times X_2)$.

Proof. By hypothesis there exist open fuzzy sets μ_1 and μ_2 in X_1 and X_2 respectively, such that $\mu_1 \leq \lambda_1 \leq Cl\mu_1$ and $\mu_2 \leq \lambda_2 \leq Cl\mu_2$. Then $\mu_1 \times \mu_2 \leq \lambda_1 \times \lambda_2 \leq Cl\mu_1 \times Cl\mu_2$. We have $\mu_1 \times \mu_2 \leq \lambda_1 \times \lambda_2 \leq Cl(\mu_1 \times \mu_2)$ ([9] Theorem 3.10) and hence $\lambda_1 \times \lambda_2$ is semi-open fuzzy in $X_1 \times X_2$. ■

Theorem 4.10 *Let* X_1, X_2 *and* Y_1, Y_2 *be product related fuzzy topological spaces. Let* $f_i : X_i \rightarrow Y_i$ *be semi-open fuzzy for* $i = 1, 2$ *and let* $f : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ *be defined as* $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. *Then* f *is semi-open fuzzy .*

Proof. By Lemma 4.3, if $\lambda_1 \in FSO(X_1)$ and $\lambda_2 \in FSO(X_2)$, then $\lambda_1 \times \lambda_2 \in FSO(X_1 \times X_2)$. Let ω_i be a open fuzzy set in $X_i, i = 1, 2$. Then $f(\omega_1 \times \omega_2) = f(\omega_1) \times f(\omega_2)$ is semi-open fuzzy in $Y_1 \times Y_2$. Now if ω is any open fuzzy set in $X_1 \times X_2$, then $\omega = \vee\omega_{i_1} \times \omega_{i_2}$ where ω_{i_1} and ω_{i_2} are open fuzzy sets in X_1 and X_2 respectively. Therefore $f(\omega) = f(\vee\omega_{i_1} \times \omega_{i_2}) = \vee f(\omega_{i_1} \times \omega_{i_2})$ is a semi-open fuzzy set in $Y_1 \times Y_2$. ■

Theorem 4.11 *Let $f_i : X \to X_i \ \ (i = 1, 2)$ and $f : X \to X_1 \times X_2$ be defined as $f(x) = (f_1(x), f_2(x)), \ \ x \in X$. Then f is semi-open fuzzy if and only if both f_1 and f_2 are semi-open fuzzy .*

Proof. Let f be semi-open fuzzy and ω any open fuzzy set in X. Then by Lemma 4.3, $f(\omega) = f_1(\omega) \times f_2(\omega) \in FSO(X_1 \times X_2)$, hence $f_i(\omega) \in FSO(X_i); \ \ i = 1, 2$ and so $f_i \ \ (i = 1, 2)$ is semi-open fuzzy .

Conversely, let f_1 and f_2 be semi-open fuzzy and ω any open fuzzy set in X. Then by Lemma 4.3, $f(\omega) = f_1(\omega) \times f_2(\omega) \in FSO(X_1 \times X_2)$. Consequently f is semi-open fuzzy . ■

Theorem 4.12 *Let $f : X \to Y$ be a bijective mapping. If f is semi-continuous fuzzy , then $sIntf(\eta) \leq f(sInt\eta)$, for any fuzzy set η in X.*

Proof. Let f be semi-continuous fuzzy and η a fuzzy set in X. Semi-continuity of f implies $f^{-1}(sInt\gamma) \leq sIntf^{-1}(\gamma)$, for any fuzzy set γ in Y. Since f is injective, we have $f^{-1}(sIntf(\eta)) \leq sIntf^{-1}f(\eta) = sInt\eta$. The surjectivity of f gives $sIntf(\eta) = ff^{-1}(sIntf(\eta)) \leq f(sInt\eta)$. ■

Converse of Theorem 4.12 is, in general, not true as shown by following:

Example 4.4 *Let $f : X \to Y$ be a mapping given as $f(a) = p$ and $f(b) = q$ from fts's (X, τ_X) to (Y, τ_Y) as defined in Example 4.3. Then $sIntf(\eta) \leq f(sInt\eta)$, for all η in X but f is not semi-continuous fuzzy (in fact not even fuzzy continuous).*

4.3 Semi Preopen and Semi Preclosed Fuzzy Mappings

Definition 4.4 *[49] A fuzzy set λ in an fts X is said to be fuzzy preopen , if $\lambda \leq IntCl\lambda$ (resp. fuzzy preclosed, if $ClInt\lambda \leq \lambda$).*

Definition 4.5 *[44] A fuzzy set λ in an fts X is said to be semi fuzzy -preopen (resp. semi fuzzy -preclosed) if there exists a fuzzy preopen (resp. fuzzy preclosed) set μ such that $\mu \leq \lambda \leq Cl\mu$ (resp. $Int\mu \leq \lambda \leq \mu$).*

The class of semi fuzzy -preopen (resp. semi fuzzy -preclosed) sets of an fts X is denoted by $FSPO(X)$ (resp. $FSPC(X)$).

Definition 4.6 *[44] The semi-preclosure and semi-preinterior of a fuzzy set λ in an fts (X, τ) are denoted and defined as:*

$$spCl\lambda = \wedge\{\mu|\lambda \leq \mu, \mu \text{ is a semi fuzzy -preclosed set in } X\},$$

$$spInt\lambda = \vee\{\omega|\omega \leq \lambda, \lambda \text{ is a semi fuzzy -preopen set in } X\}.$$

Definition 4.7 *[44] A mapping $f : X \to Y$ is said to be semi fuzzy -precontinuous if the inverse image of each open fuzzy set in Y is semi fuzzy -preopen in X.*

Every fuzzy continuous mapping is a semi fuzzy -precontinuous mapping, but the converse is not true in general [55].

Theorem 4.13 *[6] Let $f : X \to Y$ be a open fuzzy and f.a.c.S. function. Then for each open fuzzy set ν in Y, $Clf^{-1}(\nu) = f^{-1}(Cl\nu)$.*

We use this result and prove:

Lemma 4.4 *If $f : X \to Y$ is a open fuzzy , f.a.c.S. and fuzzy preirresolute mapping, then $f^{-1}(\lambda) \in FSPO(X)$ (resp. $f^{-1}(\lambda) \in FSPC(X)$), for each $\lambda \in FSPO(Y)$ (resp. $\lambda \in FSPC(X)$).*

Proof. Suppose λ is a semi fuzzy -preopen set in Y. Then there exists a fuzzy preopen set ν in Y such that $\nu \leq \lambda \leq Cl\nu$. Since f is f.a.c.S., we have $f^{-1}(\nu) \leq f^{-1}(\lambda) \leq f^{-1}(Cl\nu) = Clf^{-1}(\nu)$. Since f is also fuzzy preirresolute and ν is a fuzzy preopen set in Y, $f^{-1}(\nu) \in FPO(X)$. Hence $f^{-1}(\nu) \leq f^{-1}(\lambda) \leq Clf^{-1}(\nu)$ which implies that $f^{-1}(\lambda) \in FSPO(X)$. ■

Theorem 4.14 *[55] Let $f : X \to Y$ be a mapping. Then the following are equivalent:*

(1) *f is semi fuzzy -precontinuous.*

(2) *For each fuzzy set γ in Y, $f^{-1}(Int\gamma) \leq spIntf^{-1}(\gamma)$.*

Remark 2.3 [55] gives that if μ is open fuzzy and λ is semi fuzzy -preopen, then $\mu \wedge \lambda$ is a semi fuzzy -preopen set. It is also known that if λ is a semi fuzzy -preopen set in an fts X such that $\lambda \leq \nu \leq Cl\lambda$, then ν is a semi fuzzy -preopen set. In particular $Cl\lambda$ is semi fuzzy -preopen.

Definition 4.8 *[55] A function $f : X \to Y$ is called semi fuzzy -preopen (resp. semi fuzzy -preclosed), if the image of each open fuzzy (resp. fuzzy closed) set in X is a semi fuzzy -preopen (resp. semi fuzzy -preclosed) set in Y.*

It is known [55] that every open fuzzy mapping is semi fuzzy -preopen but not the converse. Moreover every fuzzy preopen function is semi fuzzy -preopen.

We recall that a mapping $f : X \rightarrow Y$ is called semi fuzzy -preopen (in the sense of Cammaroto and Noiri [8]), if $f(\mu) \in FSPO(Y)$ for each $\mu \in FSO(X)$. Clearly, every semi fuzzy -preopen mapping (in the sense of Cammaroto and Noiri [8]) is a semi fuzzy -preopen mapping in the sense of Definition 4.8.

Lemma 4.5 *The following are equivalent for a fuzzy set λ in an fts X:*

(1) $\lambda \in FSPO(X)$.

(2) $\lambda \leq sIntsCl\lambda$.

We characterize semi fuzzy -preopen mappings as:

Theorem 4.15 *A mapping $f : X \rightarrow Y$ is semi fuzzy -preopen if and only if for every fuzzy set λ in X, $f(Int\lambda) \leq sIntsCl\lambda$.*

Proof. Let f be a semi fuzzy -preopen mapping. Then by hypothesis $f(Int\lambda)$ is a semi fuzzy -preopen set in Y. By Lemma 4.5, $f(Int\lambda) \leq sIntsClf(Int\lambda) \leq sIntsClf(\lambda)$. Conversely, let the given condition hold true and γ any open fuzzy set in X. Then $f(\gamma) = f(Int\gamma) \leq sIntsClf(\gamma)$ which implies that $f(\gamma) \leq sIntsClf(\gamma)$. Thus by Lemma 4.5, $f(\gamma)$ is a semi fuzzy -preopen set in Y. Hence f is a semi fuzzy -preopen mapping. ∎

Next we define:

Definition 4.9 *A function $f : X \to Y$ is called semi fuzzy -preirresolute, if the inverse image of each semi fuzzy -preopen set in Y is a semi fuzzy -preopen set in X.*

Theorem 4.16 *Let X, Y, and Z be fts's and $f : X \to Y$, $g : Y \to Z$ two mappings such that $gof : X \to Z$ is a semi fuzzy -preopen mapping. Then*

(1) *if f is fuzzy continuous and surjective function, then g is semi fuzzy -preopen.*

(2) *if g is semi fuzzy -preirresolute and injective function, then f is semi fuzzy -preopen.*

Proof. (1) Let ν be an arbitrary open fuzzy set in Y. Since gof is semi fuzzy -preopen and f is surjective, then $g(\nu) = gof\,(f^{-1}(\nu))$ is a semi fuzzy -preopen set in Z. This shows that g is a semi fuzzy -preopen mapping.

(2) Let μ be an arbitrary open fuzzy set in X. Then by hypothesis, $gof(\mu)$ is a semi fuzzy -preopen set in Z. Since g is semi fuzzy -irresolute, $f(\mu) = g^{-1}(gof(\mu)) \in FSPO(Y)$ which implies that $f(\mu)$ is a semi fuzzy -preopen set in Y. Hence, f is a semi fuzzy -preopen mapping. ■

Theorem 4.17 *Let $f : X \to Y$ be a mapping from an fts X to another fts Y. Then f is semi fuzzy -preclosed if and only if $spClf\,(\lambda) \leq f(spCl\lambda)$, for each fuzzy set λ in X.*

Proof. Let f be a semi fuzzy -preclosed mapping and λ a closed fuzzy set in X. Then $spClf(\lambda) \in FSPC(Y)$. As $f(\lambda) \leq spClf(\lambda)$, it follows that $spClf(\lambda) \leq f(spCl\lambda)$.

Conversely, assume that $\gamma \in FSPC(X)$. Then $f(\gamma) = f(spCl\gamma) \geq spClf(\gamma)$, thus we obtain that $spClf(\gamma) = f(\gamma)$. Hence, f is a semi fuzzy -preclosed mapping. ■

4.4 Semi Preirresolute Fuzzy Mappings

Recall (Definition 4.9) that a function $f : X \to Y$ is called semi fuzzy -preirresolute, if the inverse image of each semi fuzzy -preopen set in Y is a semi fuzzy -preopen set in X.

Theorem 4.18 *The following statements are equivalent for a function $f : X \to Y$:*

(1) *f is semi fuzzy -preirresolute.*

(2) *For each fuzzy point x of X and each semi fuzzy -pre-neighborhood ν of $f(x)$, there exists a semi fuzzy -preneighborhood μ of x such that $f(\mu) \leq \nu$.*

(3) *For each $x \in X$ and each $\nu \in FSPO(f(x))$, there exists $\mu \in FSPO(x)$ such that $f(\mu) \leq \nu$.*

Proof. (1) $\Rightarrow$ (2) For each $x \in X$, let ν be a semi fuzzy -pre-neighborhood of $f(x)$. Then there exists a semi fuzzy -pre-open set β such that $f(x) \in \beta \leq \nu$. This gives $x \in f^{-1}(\beta) \leq f^{-1}(\nu)$. Since f is semi fuzzy -preopen, therefore $\mu = f^{-1}(\beta)$ is a semi fuzzy -preopen set in X. This gives that $f^{-1}(\nu)$ is a semi fuzzy -preopen neighborhood of x in X and $f(\mu) \leq ff^{-1}(\nu) \leq \nu$.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) Let ν be a semi fuzzy -preopen set in Y such that $f(x) \in \nu$. Then $Clf^{-1}(\nu)$ is semi fuzzy -pre-neighborhood of each $x \in f^{-1}(\nu)$. Thus, each x is a semi fuzzy -preinterior point of $Clf^{-1}(\nu)$ which implies that $f^{-1}(\nu) \leq IntClf^{-1}(\nu) \leq ClIntClf^{-1}(\nu)$. Therefore, $f^{-1}(\nu)$ is a semi fuzzy -preopen set in X and hence f is a semi fuzzy -preirresolute mapping. ■

Definition 4.10 *[50] A mapping $f : X \rightarrow Y$ is called fuzzy preopen, if the image of each open fuzzy set in X is fuzzy preopen in Y.*

Definition 4.11 *A mapping $f : X \rightarrow Y$ is called fuzzy preirresolute, if the inverse image of each fuzzy preopen set in Y is fuzzy preopen in X.*

We now state the following theorems.

Theorem 4.19 *If $f : X \rightarrow Y$ is a fuzzy preopen and fuzzy preirresolute mapping, then f is a semi fuzzy -preirresolute mapping.*

Theorem 4.20 *If $f : X \rightarrow Y$ is semi fuzzy -preirresolute and $g : Y \rightarrow Z$ is semi fuzzy -precontinuous, then gof is a semi fuzzy -precontinuous mapping.*

4.5 Pre-semi-preopen and Pre-semi-preclosed Fuzzy Mappings

First, we define:

Definition 4.12 *A mapping $f : X \to Y$ is called fuzzy pre-semi-preopen (resp. fuzzy pre-semi-preclosed), if the image of each semi fuzzy -preopen (resp. semi fuzzy -preclosed) set in X is a semi fuzzy -preopen (resp. semi fuzzy -preclosed) set in Y.*

Theorem 4.21 *If a mapping $f : X \to Y$ is fuzzy pre-semi-preopen then, $spIntf(\lambda) \leq f(spInt\lambda)$, for every fuzzy set λ in X.*

Proof. Suppose f is a fuzzy pre-semi-preopen mapping and λ a fuzzy set in X. Since $spInt\lambda$ is a semi fuzzy -preopen set, therefore $f(spInt\lambda)$ is a semi fuzzy -preopen set in Y. Since f is a fuzzy pre-semi-preopen mapping, then we obtain $spIntf(\lambda) \leq f(spInt\lambda)$. ■

Theorem 4.22 *Let $f : X \to Y$ and $g : Y \to Z$ be two mappings such that gof is a fuzzy pre-semi-preopen mapping, then*

(1) *if f is a semi fuzzy -preirresolute surjection, then g is a fuzzy pre-semi-preopen (fuzzy pre-semi-preclosed) mapping.*

(2) *if g is a semi fuzzy -preirresolute injection, then f is a fuzzy pre-semi-preopen (fuzzy pre-semi-preclosed) mapping.*

Proof. (1) Let λ be any semi fuzzy -preopen set in Y. Since f is a semi fuzzy -preirresolute mappping therefore $f^{-1}(\lambda)$ is a semi fuzzy -preopen set in X. As gof is a fuzzy pre-semi-preopen mapping and f is surjective, $gof(f^{-1}(\lambda)) = g(\lambda)$ is a semi fuzzy -preopen set in Z. This proves that g is a fuzzy pre-semi-preopen mapping.

(2) Similar to (1). ■

Theorem 4.23 *If a function $f : X \to Y$ is fuzzy pre-semi-pre-closed then for each fuzzy set β in an fts Y and each semi fuzzy -preopen set μ in an fts X with $\mu \geq f^{-1}(\beta)$, there exists a semi fuzzy -preopen set ν in Y with $\nu \geq \beta$ such that $f^{-1}(\nu) \leq \mu$.*

Proof. Let μ be an arbitrary semi fuzzy -preopen set in X with $\mu \geq f^{-1}(\beta)$, where β is a fuzzy set in Y. Clearly $(f(\mu^c))^c = \nu$ (say) is semi fuzzy -preopen in Y. Since $f^{-1}(\beta) \leq \mu$, then straight forward calculations give that $\beta \leq \nu$. Moreover, we have

$$\begin{aligned} f^{-1}(\nu) &= f^{-1}(Y) - f^{-1}(f(X-\mu)) \\ &= X - f^{-1}(f(X-\mu)) \leq \mu \\ \text{or} \quad f^{-1}(\nu) &\leq \mu. \end{aligned}$$

■

Theorem 4.24 *Let $f : X \to Y$ be a bijective function from an fts X to an fts Y. If for each fuzzy set β in Y and each semi fuzzy -preopen set μ in X with $\mu \geq f^{-1}(\beta)$, there exists a semi fuzzy -preopen set ν in Y with $\nu \geq \beta$ such that $f^{-1}(\nu) \leq \mu$, then f is semi-preclosed.*

Proof. Let ψ be an arbitrary semi fuzzy -preclosed set in X and $y \in (f(\psi))^c$. Since f is injective, therefore

$$f^{-1}(y) \leq f^{-1}((f(\psi))^c) = \left(f^{-1}f(\psi)\right)^c \leq \psi^c$$

or $f^{-1}(y) \leq \psi^c$. Since ψ^c is semi-open fuzzy , therefore there exists a semi fuzzy -preopen set ν_y with $y \in \nu_y$ such that $f^{-1}(\nu_y) \leq \psi^c$. Since f is surjective, we have $y \in \nu_y \leq f(\psi)^c$. Thus $f(\psi)^c = \vee\{\nu_y | y \in f(\psi)^c\}$ is semi fuzzy -preopen in Y or $f(\psi)$ is semi fuzzy -preclosed in Y. This proves that f is semi-preclosed. ■

Combining Theorems 4.23 and 4.24, we have:

Theorem 4.25 *A surjective function $f : X \rightarrow Y$ is semi fuzzy -preclosed if and only if for each fuzzy set β in Y and each semi fuzzy -preopen set μ in X with $\mu \geq f^{-1}(\beta)$, there exists a semi fuzzy -preopen set ν in Y with $\nu \geq \beta$ such that $f^{-1}(\nu) \leq \mu$.*

Definition 4.13 *A mapping $f : X \rightarrow Y$ is called fuzzy preclosed preserving, if the image of each fuzzy preclosed set in X is a fuzzy preclosed set in Y.*

Theorem 4.26 *If $f : X \rightarrow Y$ is a fuzzy continuous, fuzzy preclosed preserving injective mapping, then f is a fuzzy pre-semi-preclosed mapping.*

Proof. Let f be a fuzzy continuous, fuzzy preclosed preserving injective mapping. Since λ is a semi fuzzy -preclosed set in X, then there exists a fuzzy preclosed set ψ in X such that $Int\psi \leq \lambda \leq \psi$ and so $f\,(Int\psi) \leq f\,(\lambda) \leq f\,(\psi)$. Since f is a fuzzy continuous injective mapping, $Int(f(\psi)) \leq f(Int\psi)$ and also f is a fuzzy preclosed preserving mapping, $f(\psi)$ is a fuzzy preclosed set in Y. Then, we obtain that $Int(f(\psi)) \leq f(\lambda) \leq f(\psi)$ which implies that $f(\lambda)$ is a semi fuzzy -preclosed set in Y. This proves that f is a fuzzy pre-semi-preclosed mapping. ■

Chapter 5

Almost Continuous Fuzzy Mappings

5.1 Introduction

Now, we further study some properties of semi-open fuzzy sets defined and studied by Zhong [62], semi fuzzy - preopen sets and fuzzy preopen sets. It is also shown that in the class of injective functions, fuzzy almost open (closed) in Nanda's sense and fuzzy almost quasi-compact functions are equivalent. In terms of graph and projections, some interesting characterizations and properties of fuzzy almost continuous functions in Singal's sense are given. Moreover fuzzy almost continuous in Husain's sense, fuzzy almost weakly continuous, fuzzy nearly almost open (closed) functions have been defined and their several characterizations and properties have been obtained. Finally, their equivalences have been established under certain conditions.

5.2 Preliminaries

A fuzzy set λ in a fuzzy topological space (briefly, fts) is said to be fuzzy regularly open [9] (resp. fuzzy regularly closed) (briefly f.r.o (resp. f.r.c)) if $IntCl\lambda = \lambda$ (resp. $ClInt\lambda = \lambda$).

Definition 5.1 *[9] A mapping $f : X \to Y$ from an fts X to another fts Y is called a fuzzy almost continuous mapping in Singal's sense (briefly, f.a.c.S), if $f^{-1}(\lambda)$ is open fuzzy in X, for each fuzzy regularly open set λ in Y.*

Remark 5.1 *Every fuzzy continuous function $f : X \to Y$ at a point $x \in X$, is f.a.c.S. at x, the converse is not true as shown in [9] (Example 7.5).*

Definition 5.2 *[9] A mapping $f : X \to Y$ is said to be open fuzzy if $f(\lambda)$ is open fuzzy set in Y for every open fuzzy set λ in X.*

Theorem 5.1 *[58] Let $f : X \to Y$ be a open fuzzy function. Then for every fuzzy set β in Y, $f^{-1}(Cl\beta) \leq Clf^{-1}(\beta)$.*

Theorem 5.2 *[6] Let $f : X \to Y$ be a f.a.c.S. and η a open fuzzy set in Y. If $x_\alpha \in Clf^{-1}(\eta)$, then $f(x_\alpha) \in Cl\eta$.*

Definition 5.3 *A fuzzy set λ in an fts X is said to be*

(1) *fuzzy α-open [49], if $\lambda \leq IntClInt\lambda$ (resp. α-closed, if $\lambda \geq ClIntCl\lambda$),*

(2) *fuzzy preopen [49], if $\lambda \leq IntCl\lambda$ (resp. preclosed, if $ClInt\lambda \leq \lambda$),*

(3) *semi-open fuzzy (resp. semi-closed fuzzy) [9], if there exists a open fuzzy*

(resp. fuzzy closed) set μ *such that* $\mu \leq \lambda \leq Cl\mu$ *(resp.* $Int\mu \leq \lambda \leq \mu$*).*

It is known [9] that λ *is semi-open fuzzy (resp. semi-closed fuzzy) if and only if* $\lambda \leq ClInt\lambda$ *(resp.* $IntCl\lambda \leq \lambda$*).*

Remark 5.2 (1) *Every fuzzy* α*-open (resp. fuzzy* α*-closed) set is a semi-open fuzzy (resp. semi-closed fuzzy) set.*

(2) *Every fuzzy* α*-open (resp. fuzzy* α*-closed) set is a fuzzy preopen (resp. fuzzy pre-closed) set.*

The classes of all fuzzy α-open, fuzzy preopen, semi-open fuzzy and fuzzy regularly open sets of an fts X are denoted as $\alpha(X)$, $FPO(X)$, $FSO(X)$ and $FRO(X)$ respectively.

Definition 5.4 *[51] Let* λ *be a fuzzy set in an fts* X*. Then its p-closure and p-interior are denoted and defined as:*

$$pCl\lambda = \wedge\{\mu \mid \mu \geq \lambda,\ \mu \text{ is a fuzzy pre-closed set of } X\},$$

$$pInt\lambda = \vee\{\nu \mid \nu \leq \lambda,\ \nu \text{ is a fuzzy pre-open set of } X\}.$$

The definitions of sCl, $sInt$, αCl *and* αInt *are similar.*

Theorem 5.3 *For any fuzzy set* λ *in an fts* X*, the following hold:*

(1) $sCl\lambda = \lambda \vee IntCl\lambda$*.[31]*

(2) $sInt\lambda = \lambda \wedge ClInt\lambda$*.[31]*

(3) $\alpha Cl\lambda \geq \lambda \vee ClIntCl\lambda$*.*

(4) $\alpha Int\lambda \leq \lambda \wedge IntClInt\lambda$.

(5) $pCl\lambda \geq \lambda \vee ClInt\lambda$.*[30]*

(6) $pInt\lambda \leq \lambda \wedge IntCl\lambda$.*[30]*

Proof. (3) Since $\alpha Cl\lambda$ is fuzzy α-closed, we have $ClIntCl\alpha Cl\lambda \leq \alpha Cl\lambda$. Therefore $ClIntCl\lambda \leq \alpha Cl\lambda$, and hence $\lambda \vee ClIntCl\lambda \leq \alpha Cl\lambda$. ■

5.3 Almost Continuous Fuzzy Functions in Singal's Sense

Definition 5.5 *[62] A fuzzy set λ in an fts X is said to be semi fuzzy -preopen, if there exists a fuzzy preopen set μ such that $\mu \leq \lambda \leq Cl\mu$. The class of all the semi fuzzy -preopen sets is denoted as $FSPO\left(X\right)$.*

In General Topology, we have:

Lemma 5.1 *[7]*

(1) *For a subset A and an open set B in a space X, $ClA \cap B \subseteq Cl\left(A \cap B\right)$.*

(2) *For a subset A and a closed set F in a space X, $Int\left(A \cup F\right) \subseteq IntA \cup F$.*

Andrijvic used this lemma to prove Theorem 2.4 [7]. Interestingly this lemma does not hold true in Fuzzy Topology as the following example shows:

Example 5.1 *Let $X = \{a, b\}$ be a set and $\tau = \left\{\widetilde{0}, \{a_{.3}, b_{.7}\}, \{a_{.5}, b_{.5}\}, \{a_{.3}, b_{.5}\}, \{a_{.5}, b_{.7}\}, 1_X\right\}$, the fuzzy topology on X. Choose $\lambda = \{a_{.9}, b_{.3}\}$ and a open fuzzy set $\mu =$*

$\{a_{.3}, b_{.5}\}$. *Then calculations show that*

$$Cl\lambda \wedge \mu = \{a_{.3}, b_{.5}\} \nleq \{a_{.5}, b_{.3}\} = Cl\,(\lambda \wedge \mu)\,.$$

$$Int\,(\lambda \vee \mu) = \{a_{.5}, b_{.5}\} \nleq \{a_{.3}, b_{.5}\} = Int\lambda \vee \mu.$$

Consequently, the exact analogue of Theorem 2.4 of [7] is not true in fuzzy settings. Instead we have:

Theorem 5.4 *For any fuzzy set λ in an fts X, if λ is semi fuzzy -preopen, then $Cl\lambda$ is fuzzy regularly closed.*

Proof. By Theorem 2.3 [44], if λ is semi fuzzy -preopen, then $\lambda \leq ClIntCl\lambda$. This gives $Cl\lambda \leq ClIntCl\lambda$. But $ClIntCl\lambda \leq Cl\lambda$. Therefore, we have $ClIntCl\lambda = Cl\lambda$. This implies $Cl\lambda \in FRC\,(X)\,.$ ■

Theorem 5.5 *[9] Let $f : X \rightarrow Y$ be a function. Then the following are equivalent:*

(1) *f is f.a.c.S.*

(2) *$f^{-1}(\lambda)$ is a closed fuzzy set in X, for every fuzzy regularly closed set λ of Y.*

Using Theorems 5.5 and 5.4, we give characterizations of f.a.c.S. functions:

Theorem 5.6 *The following are equivalent for a function $f : X \rightarrow Y$:*

(1) *f is f.a.c.S.*

(2) *$Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$, for every $\nu \in FSPO(Y)$.*

(3) *$Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$, for every $\nu \in FSO(Y)$.*

(4) *$f^{-1}(\nu) \leq Intf^{-1}(IntCl\nu)$, for every $\nu \in FPO(Y)$.*

Proof. (1) $\Rightarrow$ (2) Let $\nu \in FSPO(Y)$. By Theorem 5.4, $Cl\nu$ is fuzzy regularly closed in Y. Since f is f.a.c.S., then by Theorem 5.5, $f^{-1}(Cl\nu)$ is closed fuzzy in X and we obtain $Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$.

(2) $\Rightarrow$ (3) This is obvious, since each semi-open fuzzy set is semi fuzzy -preopen set.

(3) $\Rightarrow$ (1) Let $\nu \in FRC(Y)$. Then $\nu = ClInt\nu$ and hence $\nu \in FSO(Y)$. Therefore, we have $Clf^{-1}(\nu) \leq f^{-1}(Cl\nu) = f^{-1}(\nu)$. Hence $f^{-1}(\nu)$ is closed fuzzy and by Theorem 5.5 f is f.a.c.S.

(1) $\Rightarrow$ (4) Let $\nu \in FPO(Y)$. Then $\nu \leq IntCl\nu$ and $IntCl\nu$ is fuzzy regularly open. Since f is f.a.c.S. by Theorem 5.5, $f^{-1}(Int\ Cl\nu)$ is open fuzzy in X and hence $f^{-1}(\nu) \leq f^{-1}(IntCl\nu) = Intf^{-1}(IntCl\nu)$.

(4) $\Rightarrow$ (1) Let ν be a fuzzy regularly open set in Y. Then $\nu \in FPO(Y)$ and hence $f^{-1}(\nu) \leq Intf^{-1}(IntCl\nu) = Intf^{-1}(\nu)$. Therefore, $f^{-1}(\nu)$ is open fuzzy in X and f is f.a.c.S. ■

The following is an immediate consequence of Definition 5.4.

Lemma 5.2 *Let x_α be a fuzzy point in an fts X. Then, $x_\alpha \in pCl\lambda$ if and only if $\lambda \wedge \nu \neq \widetilde{0}$, for every fuzzy preopen set ν in X such that $x_\alpha \in \nu$.*

We use Lemma 5.2 and prove:

Theorem 5.7 *If λ is a semi-open fuzzy set in an fts X, then $pCl\lambda = Cl\lambda$.*

Proof. Clearly, $pCl\psi \leq Cl\psi$, for every fuzzy set ψ in X. To prove $Cl\lambda \leq pCl\lambda$ for a semi-open fuzzy set λ, let $x_\alpha \in Cl\lambda$ and $x_\alpha \in \nu$, where ν is a fuzzy preopen set in X. Then $x_\alpha \in \nu \leq IntCl\nu$ and hence $\lambda \wedge IntCl\nu \neq \widetilde{0}$. Since λ is semi-open fuzzy , $\lambda \wedge IntCl\nu \leq ClInt\lambda \wedge IntCl\nu \leq Cl\,(Int\lambda \wedge Cl\nu) \leq Cl\,(\lambda \wedge \nu)$. Therefore we obtain $Cl\,(\lambda \wedge \nu) \neq \widetilde{0}$ and hence $\lambda \wedge \nu \neq \widetilde{0}$. By Lemma 5.2, $x_\alpha \in pCl\lambda$ and hence $Cl\lambda \leq pCl\lambda$. ∎

We use Theorem 5.3 and Theorem 5.4, and prove:

Theorem 5.8 *For a fuzzy set ν in an fts X, the following hold:*

(1) $\alpha Cl\nu = Cl\nu$, *for every* $\nu \in FSPO(X)$.

(2) $sCl\nu = IntCl\nu$, *for every* $\nu \in FPO(X)$.

Proof. (1) Clearly $sCl\nu \leq Cl\nu$. Next, let $\nu \in FSPO(X)$. Then by Theorem 5.4, $\nu \leq ClIntCl\nu$ and by Theorem 5.3 (3), we have $\alpha Cl\nu \geq \nu \vee ClIntCl\nu = Cl\nu$. Hence $\alpha Cl\nu = Cl\nu$.

(2) Let $\nu \in FPO(X)$. Then $\nu \leq IntCl\nu$ and by Theorem 5.3(1), we have $sCl\nu = \nu \vee IntCl\nu = IntCl\nu$. ∎

In view of Theorem 5.8, we have the following theorem, proof of which follows from Theorem 5.6:

Theorem 5.9 *The following are equivalent for a function $f : X \to Y$:*

(1) f *is f.a.c.S.*

(2) $Clf^{-1}(\nu) \leq f^{-1}(\alpha Cl\nu)$, *for every* $\nu \in FSPO(Y)$.

(3) $Clf^{-1}(\nu) \leq f^{-1}(pCl\nu)$, *for every* $\nu \in FSO(Y)$.

(4) $f^{-1}(\nu) \leq Intf^{-1}(sCl\nu)$, *for every* $\nu \in FPO(Y)$.

Definition 5.6 *[39] A function $f : X \to Y$ is said to be fuzzy almost open (resp. fuzzy almost closed) in Nanda's sense, briefly, f.a.o.N (resp. f.a.c.N.), if $f(\mu)$ is open fuzzy (resp. fuzzy closed) in Y, for each fuzzy regularly open (resp. fuzzy regularly closed) set μ in X.*

Theorem 5.10 *[6] Let $f : X \to Y$ be a open fuzzy and f.a.c.S. function. Then for each open fuzzy set ν in Y, $Clf^{-1}(\nu) = f^{-1}(Cl\nu)$.*

We generalize Theorem 5.10 as:

Theorem 5.11 *A function $f : X \to Y$ is f.a.o.N and f.a.c.S. if and only if $Clf^{-1}(\nu) = f^{-1}(Cl\nu)$, for every $\nu \in FSO(Y)$.*

Proof. ($\Rightarrow$) Let $\nu \in FSO(Y)$. Since f is f.a.c.S., by Theorem 5.6(3), $Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$. Since f is fuzzy almost open, we have

$$f^{-1}(Cl\nu) = f^{-1}(ClInt\nu) \leq Clf^{-1}(Int\nu) = Clf^{-1}(\nu)$$

Therefore, we obtain $Clf^{-1}(\nu) = f^{-1}(Cl\nu)$, for every $\nu \in FSO(Y)$.

($\Leftarrow$) It follows from Theorem 5.6(3) that f is fuzzy almost open. Let ψ be any fuzzy regularly closed set in Y. Then $\psi = ClInt\psi$ and hence $\psi \in FSO(Y)$. By the hypothesis, $Clf^{-1}(\psi) = f^{-1}(\psi)$ and hence $f^{-1}(\psi)$ is closed fuzzy in X. Therefore, by Theorem 5.5 f is f.a.c.S. ∎

Next, we define:

Definition 5.7 *A surjective function $f : X \to Y$ is said to be fuzzy almost quasi-compact, if $f^{-1}(\lambda)$ is fuzzy regularly open in X implies λ is open fuzzy in Y.*

Then the following is immediate:

Theorem 5.12 *A bijective function $f : X \to Y$ is fuzzy almost quasi-compact if and only if the image of every fuzzy regularly open (resp. fuzzy regularly closed) inverse set is open fuzzy (resp. fuzzy closed).*

We use Theorem 5.12 and prove following characterizations of fuzzy almost open functions:

Theorem 5.13 *If $f : X \to Y$ is a bijective function, then the following are equivalent:*

(1) f *is fuzzy almost open.*

(2) f *is fuzzy almost closed.*

(3) f *is fuzzy almost quasi-compact.*

(4) f^{-1} *is f.a.c.S.*

Proof. (1) $\Rightarrow$ (2) Let λ be a fuzzy regularly closed set in X. Then λ^c is fuzzy regularly open. Therefore $f(\lambda^c)$ is open fuzzy , or $(f(\lambda))^c$ is open fuzzy . This proves that $f(\lambda)$ is closed fuzzy and consequently f is fuzzy almost closed.

(2) $\Rightarrow$ (3) Let $f^{-1}(\psi)$ be fuzzy regularly closed. Then by Theorem 5.12, $ff^{-1}(\psi)$ is fuzzy closed, that is, ψ is fuzzy closed. This gives that f is fuzzy almost quasi-compact.

(3) $\Rightarrow$ (4) Let μ be a fuzzy regularly open set in X. Then $f^{-1}f(\mu) = \mu$ is fuzzy regularly open. Hence $f(\mu)$ is open fuzzy , that is, $(f^{-1})^{-1}(\mu)$ is open fuzzy implies f^{-1} is fuzzy almost continuous.

(4) $\Rightarrow$ (1) If λ is a fuzzy regularly open set in X. Then by hypothesis, $(f^{-1})^{-1}(\lambda)$ is a open fuzzy set of Y implies f is fuzzy almost open. ■

Theorem 5.14 *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then we have:*
(1) *if a surjective function f is f.a.c.S. and gof is open fuzzy (resp. fuzzy closed), then g is fuzzy almost open (resp. fuzzy almost closed).*
(2) *if f is f.a.c.S. and gof is fuzzy quasi-compact, then g is fuzzy almost quasi-compact.*

Proof. (1) Suppose f is f.a.c.S. and gof is open fuzzy . Let ψ be any fuzzy regularly open set of Y. Since f is fuzzy almost continuous, therefore $f^{-1}(\psi)$ is a open fuzzy subset of X. Since, gof is open fuzzy , therefore $(gof)(f^{-1}(\psi))$ is a open fuzzy subset of Z, that is, $(gof)(f^{-1}(\psi)) = g(\psi)$ is open fuzzy . This proves that g is fuzzy almost open.

(2) Let f be f.a.c.S. and gof fuzzy quasi-compact. Let ψ be a fuzzy set in Z such that $g^{-1}(\psi)$ is fuzzy regularly open set in Y. Then by fuzzy almost-continuity of f, $f^{-1}g^{-1}(\psi)$ is open fuzzy in X. But $f^{-1}(g^{-1}(\psi)) = (gof)^{-1}(\psi)$ is open fuzzy , implies ψ is open fuzzy . This proves that g is fuzzy almost quasi-compact. ■

We generalize Theorem 7.10 [9] and obtain:

Theorem 5.15 *For each $i \in J$, let $f_i : X_i \to Y_i$ be functions where each X_i (resp. Y_i) is product related to X_j (resp. Y_j). Define $f : \Pi X_i \to \Pi Y_i$ by $f((x_i)) = (f_i(x_i))$. If f_i is f.a.c.S., for each $i \in J$, then f is f.a.c.S.*

Proof. Let $j \in J$ be fixed and $x_j \in X_j$. Let ν_j be a fuzzy regularly open set with $f_j(x_j) \in \nu_j$. Take $(x_i) \in \Pi X_i$ whose jth coordinate is x_j. Then the set $\nu = \Pi_{i \neq j} Y_i \times \ \nu_j$ is a open fuzzy set in ΠY_i with $(f_i(x_i)) \in \nu$ and $Cl\nu = \Pi_{i \neq j} Y_i \times \ Cl\nu_j$. Then

$$IntCl\nu = \Pi_{i \neq j} Y_i \times \ IntCl\nu_j = \Pi_{i \neq j} Y_i \times \ \nu_j = \nu,$$

which shows that ν is a fuzzy regularly open set. The fuzzy almost continuity of f gives the existence of a open fuzzy set μ with $(x_i) \in \mu$ such that $f(\mu) \leq \Pi_{i \neq j} Y_i \times \ \nu_j = \nu$. Thus there exists a basic open fuzzy set $\gamma = \Pi Y_i \times \ \gamma_{i_1} \times \gamma_{i_2} \times \ldots \times \gamma_{i_j} \times \ldots \times \gamma_{i_n} \leq \mu$ with $(x_i) \in \gamma$ such that $f(\gamma) \leq \nu$, which implies $f_j(\gamma_j) \leq \nu_j$. This shows that f_j is fuzzy almost continuous and consequently f_i is fuzzy almost continuous, for each $i \in J$. ■

Theorem 5.16 *A mapping $f : X \to Y$ is f.a.c.S. at a fuzzy point $x \in X$, if for every open fuzzy neighborhood ν of $f(x)$, there is a open fuzzy neighborhood μ of x such that $f(\mu) \leq IntCl\nu$.*

Proof. Let x be a fuzzy point in X and let ν be a open fuzzy neighborhood of $f(x)$. Then there exists a open fuzzy set β such that $f(x) \in \beta \leq \nu$ and hence $f(x) \in IntCl\nu = \omega \in FRO(Y)$. Then there exists a open fuzzy set $\mu = f^{-1}(\omega)$ such that $x \in \mu$ and $f(\mu) \leq IntCl\nu$. ■

We use Theorem 5.16 and characterize f.a.c.S. functions in terms of fuzzy projections in product related spaces:

Theorem 5.17 *Let $\{Y_i : i \in J\}$ be a family of fts's, $f : X \to \Pi Y_i$ a function and $p_i : \prod Y_i \to Y_i$, the ith projection function. Then f is f.a.c.S. if and only if $p_i f$ is f.a.c.S., for each $i \in J$.*

Proof. Suppose f is f.a.c.S. Let $x \in X$ and ν_j a fuzzy regularly open set in Y_j such that $(p_j f)(x) = p_j((y_i)) \in \nu_j$, where $f(x) = (y_i) \in \Pi Y_i$. Then

$$\begin{aligned} IntClp_j^{-1}(\nu_j) &= Int(\Pi Y_i \times \ Cl\nu_j) \\ &= \Pi Y_i \times \ IntCl\nu_j = \Pi Y_i \times \ \nu_j = p_j^{-1}(\nu_j) \end{aligned}$$

Therefore $p_j^{-1}(\nu_j)$ is fuzzy regularly open in ΠY_i and $f(x) = (y_i) \in p_j^{-1}(\nu_j)$. Then by Theorem 5.16 f is f.a.c.S. implies the existence of a open fuzzy set μ in X with $x \in \mu$ such that $f(\mu) \leq p_j^{-1}(\nu_j)$. Consequently, $(p_j f)(\mu) = p_j(f(\mu)) \leq p_j(p_j^{-1}(\nu_j)) = \nu_j$, implies $p_j f$ is f.a.c.S. It follows that $p_i f$ is f.a.c.S. for each $i \in J$.

Conversely, suppose $p_i f$ is f.a.c.S. for each $i \in J$. Let $x \in X$ and ν a fuzzy regularly open set with $f(x) = (y_i) \in \nu$. Then there exists a basic open fuzzy set $\Pi_{i \neq i_k} Y_i \times \ \nu_{i_1} \times \ \nu_{i_2} \times \ldots \times \ \nu_{i_n} \leq \nu$ which contains $f(x)$. Thus $f(x) \in \Pi_{i \neq i_k} Y_i \times \ IntCl\nu_{i_1} \times \ IntCl\nu_{i_2} \times \ldots \times \ IntCl\nu_{i_n} \leq IntCl\nu = \nu$. Since each $IntCl\nu_{i_j}$ is fuzzy regularly open in Y_{i_j} and $p_{i_j} f$ is f.a.c.S., then by Theorem 5.16 there exists a open fuzzy set μ_i in X with $x \in \mu_i$ such that $(p_{i_j} f)(\mu_i) \leq IntCl\mu_{i_j}$. Let $\eta = \wedge_{i=1}^{n} \mu_i$. Then $f(\eta) \leq \Pi_{i \neq i_j} Y_i \times IntCl\nu_{i_1} \times \ldots \times IntCl\nu_{i_n} \leq IntCl\nu = \nu$. This shows f is f.a.c.S. ■

In view of Theorem 5.17, the following is immediate:

Theorem 5.18 *Let Y_i, $i \in J$ be the fts's and let $f_i : X \to Y_i$, $i \in J$, be a family of functions. Define $f : X \to \Pi Y_i$ by $f(x) = (f_i(x))$. Then f is f.a.c.S. if and only if each f_i is f.a.c.S.*

5.4 Almost Continuous Fuzzy Functions in Husain's Sense

Definition 5.8 *[6] A function $f : X \to Y$ is said to be a fuzzy almost continuous function in the sense of Husain (briefly, f.a.c.H.) at $x_\alpha \in X$, if for each open fuzzy set ν in Y with $f(x_\alpha) \in \nu$, $Clf^{-1}(\nu)$ is a fuzzy neighborhood of x_α. If f is f.a.c.H. at each point of X, then f is called f.a.c.H.*

Definition 5.9 *A fuzzy set λ is said to be fuzzy dense in another fuzzy set μ, both being fuzzy sets in an fts X, if $Cl\lambda = \mu$.*

Then the following is immediate:

Theorem 5.19 *A function $f : X \to Y$ is f.a.c.H. at $x_\alpha \in X$ if and only if for each open fuzzy set ν in Y with $f(x_\alpha) \in \nu$, there exists a open fuzzy set μ in X with $x_\alpha \in \mu$, such that $f^{-1}(\nu)$ is fuzzy dense in μ.*

Lemma 5.3 *[9] Let $g : X \to X \times Y$ be the graph of a function $f : X \to Y$. If λ is a fuzzy set of X and μ is a fuzzy set in Y, then $g^{-1}(\lambda \times \mu) = \lambda \wedge f^{-1}(\mu)$.*

Using Theorem 5.19 and Lemma 5.3, we have:

Theorem 5.20 *Let $f : X \to Y$ be a function and $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$, be the graph of f. Then g is f.a.c.H. if and only if f is f.a.c.H.*

Proof. $(\Rightarrow)$ Let g be f.a.c.H. and $x_\alpha \in X$. Consider a open fuzzy set ν in Y with $f(x_\alpha) \in \nu$. Then $1_X \times \nu$ is a open fuzzy set in $1_X \times 1_Y$ with $(x_\alpha, f(x_\alpha)) \in 1_X \times \nu$. Since g is f.a.c.H., therefore by Lemma 5.3, $Clg^{-1}(1_X \times \nu) = Cl(1_X \wedge f^{-1}(\nu)) = Clf^{-1}(\nu)$ is a fuzzy neighborhood of x_α. Thus f is f.a.c.H. at the point x_α.

$(\Leftarrow)$ Let f be f.a.c.H. We show that g is a f.a.c.H. at the point $x_\alpha \in X$. Let ω be a open fuzzy set in $1_X \times 1_Y$ such that $g(x_\alpha) = (x_\alpha, f(x_\alpha)) \in \omega$. Then by Theorem 5.19, there exist open fuzzy sets μ in X and ν in Y such that $x_\alpha \in \mu$, $f(x_\alpha) \in \nu$ and $\mu \times \nu \leq \omega$. By Lemma 5.3, $g^{-1}(\mu \times \nu) = \mu \wedge f^{-1}(\nu)$. Since f is f.a.c.H., by Theorem 5.19, there exists a open fuzzy set ψ in X with $x_\alpha \in \psi$ such that $\psi \leq \mu$ and $f^{-1}(\nu)$ is fuzzy dense in ψ. Thus, $g^{-1}(\mu \times \nu) = \mu \wedge f^{-1}(\nu) \geq \psi \wedge f^{-1}(\nu)$ gives $Cl\ g^{-1}(\omega) \geq Clg^{-1}(\mu \times \nu)$ $\geq Cl(\psi \wedge f^{-1}(\nu)) \geq \psi$. Thus $Clg^{-1}(\omega)$ is a fuzzy neighborhood of x_α implies g is f.a.c.H. at the point x_α. ■

The concept of almost weakly continuous functions has been defined and studied by D. S. Jankovic [23]. In fuzzy settings, we define this as:

Definition 5.10 *A function $f : X \rightarrow Y$ is said to be fuzzy almost weakly continuous (briefly, f.a.w.c) $f^{-1}(\nu) \leq IntClf^{-1}(Cl\nu)$, for every open fuzzy set ν in Y.*

The notion of fuzzy almost weakly continuous function is a weaker notion than that of f.a.c.H. as proved in Theorem 5.22.

Theorem 5.21 *For a function $f : X \to Y$, the following are equivalent:*

(1) *f is f.a.w.c.*

(2) *$ClIntf^{-1}(\nu) \leq f^{-1}(Cl\nu)$ for every open fuzzy set ν in Y.*

(3) *For each $x_\alpha \in X$ and each open fuzzy set ν such that $f(x_\alpha) \in \nu$, $Clf^{-1}(Cl\nu)$ is a fuzzy neighborhood of x_α.*

Proof. (1) $\Rightarrow$ (2) Let ν be a open fuzzy set in Y. Then $(Cl\nu)^c$ is open fuzzy in Y and we have

$$\begin{aligned}\left(f^{-1}(Cl\nu)\right)^c &= f^{-1}\left((Cl\nu)^c\right) \\ &\leq IntClf^{-1}\left(Cl\,(Cl\nu)^c\right) \leq \left(ClIntf^{-1}(\nu)\right)^c.\end{aligned}$$

Therefore we obtain $ClIntf^{-1}(\nu) \leq f^{-1}(Cl\nu)$.

(2) $\Rightarrow$ (3) Let $x \in X$ and ν be a open fuzzy set such that $f(x) \in \nu$. Since $(Cl\nu)^c$ is open fuzzy in Y, we have

$$\begin{aligned}\left(IntClf^{-1}(Cl\nu)\right)^c &= ClIntf^{-1}\left((Cl\nu)^c\right) \leq f^{-1}\left(Cl\,(Cl\nu)^c\right) \\ &= f^{-1}\left((IntCl\nu)^c\right) \leq f^{-1}(\nu^c) = \left(f^{-1}(\nu)\right)^c.\end{aligned}$$

Therefore, we obtain $x \in f^{-1}(\nu) \leq IntClf^{-1}(Cl\nu)$ and hence $Clf^{-1}(Cl\nu)$ is a fuzzy neighborhood of x.

(3) $\Rightarrow$ (1) Let ν be a open fuzzy set in Y and $x \in f^{-1}(\nu)$. Then $f(x) \in \nu$ and $Clf^{-1}(Cl\nu)$ is a fuzzy neighborhood of x. Therefore, $x \in IntClf^{-1}(Cl\nu)$ and we obtain $f^{-1}(\nu) \leq IntClf^{-1}(Cl\nu)$. ■

The following is immediate from Theorem 5.21:

Theorem 5.22 *Every f.a.c.H. function $f : X \to Y$ is f.a.w.c. function.*

Finally, we define:

Definition 5.11 *A function $f : X \to Y$ is said to be fuzzy nearly almost open, if there exists a open fuzzy basis $\mathcal{B}$ for the fuzzy topology on Y such that $f^{-1}(Cl\nu) \leq Clf^{-1}(\nu)$, for every $\nu \in \mathcal{B}$.*

Using fuzzy nearly almost open function, we give the partial converse of Theorem 5.22 as:

Theorem 5.23 *If a function $f : X \to Y$ is fuzzy nearly almost open and fuzzy almost weakly continuous, then f is f.a.c.H.*

Proof. Since f is fuzzy nearly almost open, there exists a open fuzzy basis $\mathcal{B}$ for the fuzzy topology on Y such that $f^{-1}(Cl\nu) \leq Clf^{-1}(\nu)$ for every $\nu \in \mathcal{B}$. Let ω be any open fuzzy set of Y. There exists a subfamily $\mathcal{B}_0$ of $\mathcal{B}$ such that $\omega = \vee\{\nu | \nu \in \mathcal{B}_0\}$. Therefore, we obtain

$$\begin{aligned} f^{-1}(\omega) &= f^{-1}(\vee_{\nu\in\mathcal{B}_0}\nu) \\ &= \vee_{\nu\in\mathcal{B}_0} f^{-1}(\nu) \leq \vee_{\nu\in\mathcal{B}_0} IntClf^{-1}(Cl\nu) \\ &\leq \vee_{\nu\in\mathcal{B}_0} IntClf^{-1}(\nu) \leq IntCl\ \left(\vee_{\nu\in\mathcal{B}_0} f^{-1}(\nu)\right) \\ &= IntCl\ f^{-1}(\omega) \end{aligned}$$

This shows that $f^{-1}(\omega) \in FPO(X)$ and hence f is f.a.c.H. ■

Combining Theorems 5.22 and 5.23, we have:

Theorem 5.24 *Let $f : X \to Y$ be a fuzzy nearly almost open function. Then f is f.a.w.c. if and only if f is f.a.c.S.*

5.5 Almost Continuous Fuzzy Functions in Both Senses

In the following theorems, we interconnect the independent notions of f.a.c.S. and f.a.c.H. functions and finally prove their equivalence under certain conditions.

Theorem 5.25 *[6] Let $f : X \to Y$ be a open fuzzy and f.a.c.S. function. Then f is f.a.c.H.*

Theorem 5.26 *[6] Let $f : X \to Y$ be a f.a.c.S. function. Then for each open fuzzy set ν in Y, $Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$.*

Theorem 5.27 *[5] If $f : X \to Y$ is a fuzzy weakly continuous and a open fuzzy function, then f is f.a.c.S.*

We use Theorem 5.27 and prove the partial converse of Theorem 5.25 as well as Theorem 5.26:

Theorem 5.28 *Let $f : X \to Y$ be a open fuzzy and f.a.c.H. function. If for every open fuzzy set ν in Y, $Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$, then f is f.a.c.S.*

Proof. Let f be f.a.c.H. and open fuzzy such that $Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$, for every open fuzzy set ν in Y. We show that f is f.a.c.S. Let $x_\alpha \in X$ such that

$f(x_\alpha) \in \nu$. Since f is f.a.c.H., there exists a open fuzzy set μ in X such that $x_\alpha \in \mu \leq Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$. It follows that $f(\mu) \leq f(f^{-1}(Cl\nu)) \leq Cl\nu$, f is fuzzy weakly continuous. Since f is open fuzzy , then by Theorem 5.27, f is f.a.c.S. ■

Combining Theorems 5.25 and 5.28 we have the following theorem, which improves the main result of [6] (Theorem 5.5):

Theorem 5.29 *Let $f : X \to Y$ be a open fuzzy function satisfying $Clf^{-1}(\nu) \leq f^{-1}(Cl\nu)$, for every open fuzzy ν in Y. Then f is f.a.c.H. if and only if f is f.a.c.S.*

Chapter 6

α-Continuous Fuzzy Mappings

6.1 Introduction

Singal and Rajvanshi [52] studied the class of α-continuous mappings and investigated several of its properties and characterizations. It was noted in [52] (Example 3.3) that though in Classical Topology the collection of α-open sets makes a topology, denoted as τ_α, but the same is not true for fuzzy α-open sets. In 2001, Kresteska [29] pointed out that Lemmas 4.5, 4.7 and Theorems 4.6, 4.8, 4.12 of [52] are incorrect. Since α-continuity does not yield to a straightforward fuzzification of the results from Classical Topology, thus this notion seems promising for Fuzzy Topology. Motivated by such consideration, this chapter studies further, the properties of α-continuous mappings in terms of fuzzy α-closure.

6.2 α-Open Fuzzy Sets

Definition 6.1 *A fuzzy set λ in an fts X is said to be*

(1) *fuzzy α-open [49], if $\lambda \leq IntClInt\lambda$ (resp. α-closed, if $\lambda \geq ClIntCl\lambda$). Clearly every open fuzzy (resp. fuzzy closed) set is fuzzy α-open (resp. fuzzy α-closed). The class of all fuzzy α-open (resp. fuzzy α-closed) sets is denoted by $\alpha O(X)$ (resp. $\alpha C(X)$),*

(2) *fuzzy preopen [49], if $\lambda \leq IntCl\lambda$ (resp. preclosed, if $ClInt\lambda \leq \lambda$),*

(3) *semi-open fuzzy [9], if there exists a open fuzzy set μ such that $\mu \leq \lambda \leq Cl\mu$.*

Remark 6.1 *[52] Arbitrary union of fuzzy α-open sets of an fts X is a fuzzy α-open set.*

Definition 6.2 *[52] The α-closure and α-interior of a fuzzy set λ in an fts (X, τ) are denoted and defined as:*

$$\alpha Cl\lambda = \wedge\{\mu|\lambda \leq \mu, \mu^c \in \alpha C(X)\},$$
$$\alpha Int\lambda = \vee\{\omega|\omega \leq \lambda, \omega \in \alpha O(X)\}.$$

Clearly $\alpha Cl\lambda \leq Cl\lambda$ and $Int\lambda \leq \alpha Int\lambda$.

Definition 6.3 *[52] A mapping $f : X \to Y$ is called*

(1) *fuzzy α-continuous, if the inverse image of each open fuzzy set in Y is a*

fuzzy α-open set in X.

(2) *fuzzy α-open, if the image of each open fuzzy set in X is a fuzzy α-open set in Y.*

We recall the known properties of fuzzy α-open and fuzzy α-closed sets as follows:

Theorem 6.1 *[52] Let λ and μ be fuzzy sets in an fts X. Then*

(1) *λ is fuzzy α-closed if and only if $\lambda = \alpha Cl\lambda$.*

(2) $\lambda \leq \mu \Rightarrow \alpha Cl\lambda \leq \alpha Cl\mu$.

(3) $\alpha Cl\alpha Cl\lambda = \alpha Cl\lambda$.

(4) $\alpha Cl\lambda \vee \alpha Cl\mu = \alpha Cl\left(\lambda \vee \mu\right)$.

(5) $\alpha Cl\left(\lambda \wedge \mu\right) \leq \alpha Cl\lambda \wedge \alpha Cl\mu$.

6.3 Fuzzy α-Continuity

Lemma 6.1 *Let λ and μ be fuzzy sets in an fts X. If either $\lambda \in FSO\left(X\right)$ or $\mu \in FSO\left(X\right)$, then*

$$IntCl\left(\lambda \wedge \mu\right) = IntCl\lambda \wedge IntCl\mu.$$

Proof. For any fuzzy sets λ and μ in X, we generally have

$$IntCl\left(\lambda \wedge \mu\right) \leq IntCl\lambda \wedge IntCl\mu.$$

Assume that $\lambda \in FSO(X)$. Then we have $Cl\lambda = ClInt\lambda$. Therefore,

$$\begin{aligned} IntCl\lambda \wedge IntCl\mu &\leq IntClInt\lambda \wedge Cl\mu \\ &\leq IntCl(Int\lambda \wedge \mu) = IntCl(\lambda \wedge \mu). \end{aligned}$$

■

Proposition 6.1 *Let λ and μ be fuzzy sets in an fts X. Then*

(1) $\lambda \in \alpha O(X)$ if and only if there exists $\nu \in \tau$ such that $\nu \leq \lambda \leq IntCl\nu$.

(2) If $\lambda \in \alpha O(X)$ and $\lambda \leq \mu \leq IntCl\lambda$, then $\mu \in \alpha O(X)$. In particular $IntCl\lambda$ is fuzzy α-open.

Proof. (1) ($\Rightarrow$) follows easily by taking $\nu = Int\lambda$.

($\Leftarrow$)By hypothesis, there exists a open fuzzy set ν such that $\nu \leq \lambda \leq IntCl\nu$ and hence $\nu \leq Int\lambda \leq \lambda \leq IntCl\nu$. Since $\nu \leq Int\lambda, we$ have $IntCl\nu \leq IntClInt\lambda$. Combining this with $\lambda \leq IntCl\nu$, we have $\lambda \leq IntClInt\lambda$.

(2) Since $\lambda \in \alpha O(X)$,

$$\begin{aligned} \mu &\leq IntCl\lambda \leq IntClIntClInt\lambda \\ &= IntClInt\lambda \leq IntClInt\mu. \end{aligned}$$

This shows that $\mu \in \alpha O(X)$. ■

Proposition 6.2 *If λ is a fuzzy set in an fts X, then*

$$Cl\lambda \geq \alpha Cl\lambda \geq ClIntCl\lambda.$$

Proof. Choose $\mu = \alpha Cl\lambda$, then $\lambda \leq \mu$. Since μ is α-closed, then $ClIntCl\lambda \leq ClIntCl\mu \leq \mu = \alpha Cl\lambda$. ■

Theorem 6.2 *[52] Let $f : X \to Y$ be a mapping. Then the following are equivalent:*

(1) f is fuzzy α-continuous.

(2) $f(ClIntCl\lambda) \leq Clf(\lambda)$, for each fuzzy set λ in X.

(3) $ClIntClf^{-1}(\mu) \leq f^{-1}(Cl\mu)$, for each fuzzy set μ in Y.

Theorem 6.3 *A mapping $f : X \to Y$ is fuzzy α-continuous if and only if $f(\alpha Cl\lambda) \leq Clf(\lambda)$, for each fuzzy set λ in X.*

Proof. ($\Rightarrow$) Let $\psi = Clf(\lambda)$ for a fuzzy set λ in X and $\mu = f^{-1}(\psi)$. Then by Theorem 9.10 $f(ClIntCl\mu) \leq Clf(\mu) \leq Cl\psi = \psi$. Hence $ClIntCl\mu \leq f^{-1}(\psi) = \mu$, so that μ is α-closed. Now $\lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}(\psi) = \mu$, so that $\alpha Cl\lambda \leq \alpha Cl\mu = \mu$. Thus we have that $f(\alpha Cl\lambda) \leq f(\mu) \leq \psi = Clf(\lambda)$ as required.

($\Leftarrow$) Follows from Proposition 9.4. ■

The following result can be proved in a similar way.

Theorem 6.4 *A mapping $f : X \to Y$ is fuzzy α-continuous if and only if $\alpha Clf^{-1}(\mu) \leq f^{-1}(Cl\mu)$, for each fuzzy set μ in Y.*

Definition 6.4 *[9] A fuzzy set λ is said to be fuzzy regularly open (resp. fuzzy regularly closed), if $IntCl\lambda = \lambda$ (resp. $ClInt\lambda = \lambda$).*

Definition 6.5 *[9] A mapping $f : X \to Y$ from a fts X to another fts Y is called a fuzzy almost continuous mapping in Singal's sense (briefly, f.a.c.S), if $f^{-1}(\lambda)$ is open fuzzy in X, for each fuzzy regularly open set λ in Y.*

Clearly every f.a.c.S. function is fuzzy α-continuous but the converse is not true, in general, as is shown in the following:

Example 6.1 *Let* $X = \{a, b, c\}$ *and* τ_1, τ_2 *be fuzzy topolgies on* X *generated by the families of fuzzy sets* $\{\{a_{.6}, b_{.6}, c_{.2}\}, \{a_{.1}, b_{.1}, c_{.7}\}, \{a_{.9}, b_{.9}, c_{.4}\}\}$ *and* $\{\{a_{.4}, b_{.9}, c_{.1}\}, \{a_{.7}, b_{.1}, c_{.8}\}, \{a_{.9}, b_{.9}, c_{.4}\}\}$ *respectively. Then the mapping* $f : (X, \tau_1) \to (X, \tau_2)$ *given as:*

$$f(a) = f(b) = b \text{ and } f(c) = a$$

is fuzzy α*-continuous but not f.a.c.S.*

6.4 α-Open and α-Closed Fuzzy Mappings

Definition 6.6 *A mapping* $f : X \to Y$ *is called*

(1) *semi-continuous fuzzy [9], if the inverse image of each open fuzzy set in* Y *is semi-open fuzzy in* X,

(2) *semi-open fuzzy [9], if the image of each open fuzzy set in* X *is semi-open fuzzy in* Y,

(3) *fuzzy precontinuous [49], if the inverse image of each open fuzzy set in* Y *is fuzzy preopen in* X,

(4) *fuzzy preopen [50], if the image of each open fuzzy set in* X *is fuzzy preopen in* Y.

Theorem 6.5 *A fuzzy set* λ *in an fts* X *is fuzzy* α*-open set if and only if* λ *is semi-open fuzzy and fuzzy preopen.*

Proof. Let λ be a fuzzy α-open set in X. By definition we have $\lambda \leq IntClInt\lambda \leq IntCl\lambda$ and $\lambda \leq ClInt\lambda$. Therefore, we obtain $\lambda \in FSO(X) \cap FPO(X)$.

Conversely, let $\lambda \in FSO(X) \cap FPO(X)$. Since $\lambda \in FSO(X)$, $\lambda \leq ClInt\lambda$ and hence it follows from $\lambda \in FPO(X)$ that

$$\lambda \leq IntCl\lambda \leq IntClClInt\lambda = IntClInt\lambda.$$

Therefore, we have $\lambda \in \alpha O(X)$. ■

Corollary 6.1 *A mapping $f : X \to Y$ is fuzzy α-open if and only if it is semi-open fuzzy and fuzzy pre-open.*

Proof. The necessity follows from the definitions.

Conversely, let f be semi-open fuzzy and fuzzy pre-open, and let μ be a open fuzzy set in X. Then $f(\mu) \in FSO(Y) \cap FPO(Y)$. By Theorem 6.5, $f(\mu)$ is a fuzzy α-open set in Y and hence f is fuzzy α-open set. ■

Theorem 6.6 *A mapping $f : X \to Y$ is fuzzy α-continuous if and only if it is semi-continuous fuzzy and fuzzy precontinuous.*

Proof. ($\Rightarrow$) Immediate from the definitions.

($\Leftarrow$) Let f be semi-continuous fuzzy and fuzzy precontinuous mapping and ν, a open fuzzy set in Y. Then $f^{-1}(\nu) \in FSO(X) \cap FPO(X)$. By Theorem 6.5 $f^{-1}(\nu) \in \alpha O(X)$. Hence f is fuzzy α-continuous. ■

Definition 6.7 *[6] A function $f : X \to Y$ is said to be fuzzy almost continuous mapping in the sense of Hussain (briefly, f.a.c.H.) at $x_\alpha \in X$, if for each open fuzzy set ν in Y with $f(x) \in \nu$, $Clf^{-1}(\nu)$ is a fuzzy neighborhood of x_α. If f is f.a.c.H. at each point of X, then f is called f.a.c.H.*

Theorem 6.7 *A mapping $f : X \to Y$ is fuzzy α-continuous if and only if f is f.a.c.H. and semi-continuous fuzzy .*

Proof. This is an immediate consequence of Theorem 6.5. ■

6.5 α-Irresolute Fuzzy Mappings

Definition 6.8 *A mapping $f : X \to Y$ is fuzzy α-irresolute, if the inverse image of every fuzzy α-open set in Y is a fuzzy α-open set in X.*

Theorem 6.8 *Let $f : X \to Y$ be a fuzzy α-irresolute and $g : Y \to Z$, a fuzzy α-continuous mapping. Then $gof : X \to Z$ is fuzzy α-continuous.*

Definition 6.9 *A mapping $f : X \to Y$ is said to be*

(1) *semi fuzzy -irresolute, if the inverse image of each semi-open fuzzy set in Y is semi-open fuzzy in X.*

(2) *fuzzy pre-irresolute, if the inverse image of each fuzzy preopen set in Y is fuzzy preopen in X.*

Corresponding to Theorem 6.8, we have the following results, the proofs of which are straightforward.

Proposition 6.3 *Let $f : X \to Y$ and $g : Y \to Z$ be mappings.*

(1) *If f is semi fuzzy -irresolute and g is semi-continuous fuzzy , then gof is semi-continuous fuzzy .*

(2) *If f is fuzzy pre-irresolute and g is fuzzy precontinuous, then gof is fuzzy precontinuous.*

Our next result relates these classes of 'irresolute fuzzy ' mappings.

Proposition 6.4 *If $f : X \to Y$ is semi fuzzy -irresolute and fuzzy pre-irresolute, then f is fuzzy α-irresolute.*

Proof. Let ν be a fuzzy α-open set in Y. By Theorem 6.5, $\alpha O(Y) = FSO(Y) \cap FPO(Y)$. Since f is semi fuzzy -irresolute and $\nu \in FSO(Y)$, we have $f^{-1}(\nu) \in FSO(X)$. Similarly, f is fuzzy pre-irresolute and $\nu \in FPO(Y)$ implies $f^{-1}(\nu) \in FPO(X)$. Hence $f^{-1}(\nu) \in FSO(X) \cap FPO(X) = \alpha O(X)$. This proves f is fuzzy α-irresolute. ■

Definition 6.10 *[39] A function $f : X \to Y$ is said to be fuzzy almost open (resp. fuzzy almost closed) in Nanda's sense, briefly, f.a.o.N (resp. f.a.c.N.), if $f(\mu)$ is open fuzzy (resp. fuzzy closed) in Y, for each fuzzy regularly open (resp. fuzzy regularly closed) set μ in X.*

We use Proposition 9.2 (1-2) and prove:

Theorem 6.9 *If a mapping $f : X \to Y$ is fuzzy almost open and fuzzy α-continuous, then f is fuzzy α-irresolute.*

Proof. Let μ be a fuzzy α-open set in Y. By Proposition 9.2(1), there exists a open fuzzy set ν in Y such that $\nu \leq \mu \leq IntCl\nu$. Since f is fuzzy α-continuous, $f^{-1}(\nu) \in \alpha O(X) \subset FSO(X)$ and hence $f^{-1}(\nu) \leq ClIntf^{-1}(\nu)$. Put $\psi = \left(f\left((ClIntf^{-1}(\nu))^{c}\right)\right)^{c}$. Since f is fuzzy almost open and $ClIntf^{-1}(\nu)$ is fuzzy regularly closed, then ψ is closed fuzzy in Y . Routine calculations give $\nu \leq \psi$ and $f^{-1}(\psi) \leq ClIntf^{-1}(\nu)$. Thus, $f^{-1}(Cl\nu) \leq ClIntf^{-1}(\nu)$ which implies

$$\begin{aligned} f^{-1}(\nu) &\leq f^{-1}(\mu) \leq f^{-1}(IntCl\nu) \\ &\leq IntClIntf^{-1}(IntCl\nu) \leq IntClf^{-1}(\nu). \end{aligned}$$

It follows from Proposition 9.2(2) that $f^{-1}(\mu) \in \alpha O(X)$. This shows that f is fuzzy α-irresolute. ■

It is known [3] that $sCl\lambda = \lambda \vee IntCl\lambda$ and hence $IntCl\lambda \leq sCl\lambda$.

Theorem 6.10 *A mapping $f : X \to Y$ is semi-open fuzzy if and only if $f^{-1}(sCl\lambda) \leq Clf^{-1}(\lambda)$ for every fuzzy set λ in X.*

Using Theorems 9.16, 9.17 and Proposition 9.2(2) we prove:

Theorem 6.11 *If $f : X \to Y$ is semi-open fuzzy and fuzzy α-continuous mapping, then f is fuzzy α-irresolute.*

Proof. Let μ be a fuzzy α-open set in Y. By Proposition 9.2, there exists a open fuzzy set ν in Y such that $\nu \leq \mu \leq IntCl\nu$. Since f is fuzzy α-continuous,

$f^{-1}(IntCl\nu) \in \alpha O(X)$. It follows from Theorems 9.16 and 9.17 that

$$\begin{aligned} f^{-1}(IntCl\nu) &\leq IntClIntf^{-1}(IntCl\nu) \\ &\leq IntClIntf^{-1}(sCl\nu) \leq IntClf^{-1}(\nu). \end{aligned}$$

Therefore, we obtain $f^{-1}(\nu) \leq f^{-1}(\mu) \leq IntClf^{-1}(\nu)$ and $f^{-1}(\nu) \in \alpha O(X)$. By Proposition 9.2(2), $f^{-1}(\mu) \in \alpha O(X)$. This shows that f is fuzzy α-irresolute. ■

6.6 Almost α-Continuous Fuzzy Mappings

Definition 6.11 *A mapping $f : X \to Y$ is said to be fuzzy almost α-continuous (briefly f.a.α.c.), if $f^{-1}(\nu) \in \alpha O(X)$, for every fuzzy regularly open set ν in Y.*

Definition 6.12 *[60] A fuzzy set λ is said to be fuzzy feebly open, if there exists a open fuzzy set μ such that $\mu \leq \lambda \leq sCl\mu$. Clearly every fuzzy feebly open set is semi-open fuzzy .*

Definition 6.13 *A mapping $f : X \to Y$ is said to be fuzzy almost feebly continuous (resp. fuzzy feebly continuous), if $f^{-1}(\nu)$ is fuzzy feebly open in X, for every fuzzy regularly open (resp. open fuzzy) set ν of Y.*

Lemma 6.2 *[60] If $\lambda \in FPO(X)$, then $sCl\lambda = IntCl\lambda$.*

Lemma 6.3 *For every fuzzy set μ in X, $\mu \in \alpha O(X)$ if and only if μ is fuzzy feebly open in X.*

Proof. By Proposition 9.2(1), $\mu \in \alpha O(X)$ if and only if there exists a open fuzzy set γ such that $\gamma \leq \mu \leq IntCl\gamma$. Therefore Lemma 9.1 gives that μ is fuzzy feebly open. ■

In view of Lemma 6.3, the following is immediate:

Theorem 6.12 *A mapping $f : X \to Y$ is f.a.α.c. (resp. fuzzy α-continuous) if and only if it is fuzzy almost feebly continuous (resp. fuzzy feebly continuous).*

Next, we state the following, proof of which is staraightforward:

Theorem 6.13 *For a mapping $f : X \to Y$, the following are equivalent:*

(1) *f is f.a.α.c.*

(2) *For each $x \in X$ and each open fuzzy set ν in Y with $f(x) \in \nu$, there exists $\mu \in \alpha O(X)$ such that $x \in \mu$ and $f(\mu) \leq IntCl\nu$.*

(3) *$f^{-1}(\psi)$ is fuzzy α-closed in X, for every fuzzy regularly closed set ψ of Y.*

Finally, we define:

Definition 6.14 *A mapping $f : X \to Y$ is said to be semi fuzzy -weakly continuous, if for each $x \in X$ and each open fuzzy set ν in Y with $f(x) \in \nu$, there exists semi-open fuzzy μ in X containing x such that $f(\mu) \leq sCl\nu$.*

Theorem 6.14 *[9] Any union of semi-open fuzzy sets is a semi-open fuzzy set.*

Theorem 6.15 *A mapping $f : X \rightarrow Y$ is semi fuzzy -weakly continuous if and only if $f^{-1}(\nu) \in FSO(X)$, for every fuzzy regularly open set ν of Y.*

Proof. ($\Rightarrow$) Let ν be a fuzzy regularly open set of Y. For each $x \in f^{-1}(\nu)$, there exists $\mu_x \in FSO(X)$ with $x \in \mu_x$ such that $f(\mu_x) \leq sCl(\nu)$. By Lemma 9.1, we have $sCl\nu = IntCl\nu = \nu$ and hence $x \in \mu_x \leq f^{-1}(\nu)$. Therefore, it follows from Theorem 6.14 that $f^{-1}(\nu) \in FSO(X)$.

($\Leftarrow$) Let $x \in X$ and $f(x) \in \nu$, where ν is open fuzzy in Y. Put $\mu = f^{-1}(IntCl\nu)$, then by Lemma 9.1 we have $x \in \mu \in FSO(X)$ and $f(\mu) \leq f^{-1}f(IntCl\nu) \leq IntCl\nu = sCl\nu$. This shows that f is semi fuzzy -weakly continuous. ■

Chapter 7

s-Open and s-Closed Fuzzy Mappings

7.1 Introduction

In this chapter, our aim is to further contribute to the study of semi-open fuzzy sets by establishing several important fundamental identities and inequalities about semi fuzzy -interior and semi fuzzy closure. D. E. Cameron and G. Woods [12] introduced the concepts of s-continuous mappings and s-open mappings. They investigated the properties of these mappings and their relationships to properties of semi-open sets. M. Khan and B. Ahmad [25] further worked on the characterizations and properties of s-continuous, s-open and s-closed mappings. In this section, we fuzzify the findings of [12] and [25]. We define fuzzy s-open and fuzzy s-closed mappings and establish some interesting characterizations of these mappings. It may be noted that the class of fuzzy s-open (resp. s-closed) mappings is a subclass of the class of open fuzzy (resp. closed) mappings.

We first recall following needed results:

Proposition 7.1 *[56] Let X be an fts and λ, μ fuzzy sets in X. Then we have:*

(1) $\lambda \leq \mu \Rightarrow Cl\lambda \leq Cl\mu,\ Int\lambda \leq Int\mu,$

(2) $Int\lambda \leq \lambda \leq Cl\lambda,$

(3) $IntInt\lambda = Int\lambda,$

(4) $ClCl\lambda = Cl\lambda,$

(5) $Cl\lambda^c = (Int\lambda)^c.$

Proposition 7.2 *[14, 56] Let $f : X \to Y$ be a function from an fts (X, τ) to an fts (Y, δ). Then*

(1) *f is fuzzy continuous if and only if the inverse of each δ-closed fuzzy set is τ-fuzzy closed.*

(2) *If f is fuzzy continuous, then $f(Cl\lambda) \leq Clf(\lambda)$, for any fuzzy set λ in X.*

(3) *If f is fuzzy continuous, then $f^{-1}(Int\nu) \leq Intf^{-1}(\nu)$, for any fuzzy set ν in Y.*

7.2 Semi-Open and Semi-Closed Fuzzy sets

Definition 7.1 *[9] Let λ be a fuzzy set in an fts (X, τ). Then λ is called a semi-open fuzzy set of X, if there exists a $\nu \in \tau$ such that $\nu \leq \lambda \leq Cl\nu$. A fuzzy set μ is semi-closed fuzzy iff its complement μ^c is semi-open fuzzy . The* *class of all semi-open fuzzy (resp. semi-closed fuzzy) sets in X is denoted by $FSO(X)$ (resp. $FSC(X)$).*

Definition 7.2 *[58] Let λ be a fuzzy set in an fts X. Then semi-closure (briefly sCl) and semi-interior (briefly $sInt$) of λ are defined as:*

$$\begin{aligned} sCl\lambda &= \bigwedge\{\beta \mid \lambda \leq \beta, \beta \text{ semi-closed fuzzy }\}. \\ sInt\lambda &= \bigvee\{\beta \mid \beta \leq \lambda, \beta \text{ semi-open fuzzy }\}. \end{aligned}$$

and are called the semi fuzzy -closure of λ and semi fuzzy -interior of λ, respectively.

It is immediate that:

(1) $sCl\lambda \geq \lambda$ and $sInt\lambda \leq \lambda$.

(2) $\lambda \leq \mu \Rightarrow sCl\lambda \leq sCl\mu, sInt\lambda \leq sInt\mu$.

Proposition 7.3 *[9] Let λ be a fuzzy set in an fts X. Then λ is*

(1) *semi-closed fuzzy if and only if $IntCl\lambda \leq \lambda$ (resp. $sCl\lambda = \lambda$).*

(2) *semi-open fuzzy if and only if $\lambda \leq ClInt\lambda$ (resp. $sInt\lambda = \lambda$).*

The following gives a characterization of semi-closed fuzzy sets.

Theorem 7.1 *A fuzzy set λ is semi-closed fuzzy if and only if there exists a closed fuzzy set ψ such that $Int\psi \leq \lambda \leq \psi$.*

Proof. ($\Rightarrow$) If λ is semi-closed fuzzy , then λ^c is semi-open fuzzy . Thus there exists a open fuzzy set ω such that $\omega \leq \lambda^c \leq Cl\omega$. Consequently,

$Int\omega^c = (Cl\omega)^c \leq \lambda \leq \omega^c$, and since ω^c is fuzzy closed, the proof of this part is complete.

($\Leftarrow$) Suppose there exists a closed fuzzy set ψ such that $Int\psi \leq \lambda \leq \psi$. We show that λ^c is semi-open. Note that $\psi^c \leq \lambda^c \leq (Int\psi)^c = Cl\psi^c$ implies λ^c is semi-open fuzzy . ■

Proposition 7.4 *If λ is semi-open fuzzy (resp. semi-closed fuzzy), then $Int\lambda, sInt\lambda, sCl\lambda$ and $Cl\lambda$ are semi-open fuzzy (resp. semi-closed fuzzy).*

Proof. Clearly, $Int\lambda$ and $sInt\lambda$ are semi-open fuzzy . Also $\lambda \leq Cl\lambda \leq Cl\lambda$ and $\lambda \leq sCl\lambda \leq Cl\lambda$ imply that $Cl\lambda$ and $sCl\lambda$ are semi-open fuzzy . ■

Proposition 7.5 *A nonvoid nowhere dense fuzzy set λ is semi-closed fuzzy and not semi-open fuzzy .*

Proof. If $\lambda \neq \widetilde{0}$ is nowhere dense, then $IntCl\lambda = \widetilde{0} \leq \lambda$, so that by Proposition 7.3, λ is semi-closed fuzzy . Furthermore, the only open fuzzy subset of λ is $\widetilde{0}$, and $Cl\widetilde{0} = \widetilde{0}$ so that λ is not semi-open fuzzy . ■

The converse of the above theorem is, in general, not true as is shown by following:

Example 7.1 *Let $X = \{a, b, c\}$ be a set and $I = \{0, .3, .5, .7, 1\}$ be the lattice of membership grades for fuzzy sets in X. Let $\mu = \{a_{.7}, b_0, c_1\}$, $\nu = \{a_{.7}, b_{.5}, c_{.3}\}$ and $\omega = \{a_{.5}, b_{.5}, c_{.5}\}$ be fuzzy sets on X, and τ the fuzzy topology generated by μ, ν and ω. Then $\tau = \{\widetilde{0}, \mu, \nu, \omega, \{a_{.5}, b_0, c_{.5}\}, \{a_{.5}, b_{.5}, c_{.3}\}, \{a_{.7}, b_0, c_{.3}\}, \{a_{.7}, b_0, c_{.5}\},$*

$\{a_{.7}, b_{.5}, c_{.5}\}, \{a_{.7}, b_{.5}, c_1\}, X\ \}$. *Calculations give that fuzzy set* ω *is both semi-closed fuzzy and semi-open fuzzy but* $IntCl\omega = \omega$.

Proposition 7.6 *Let* λ, ω *and* ψ *be respectively a fuzzy closed, open fuzzy and a semi-open fuzzy set in an fts* X *such that* $\omega \leq \psi \leq Cl\omega$ *(i.e.* ψ *is semi-open fuzzy). Then* $\omega \leq \lambda$ *implies* $\psi \leq \lambda$.

Proof. $\omega \leq \lambda \Rightarrow Cl\omega \leq Cl\lambda = \lambda$ or $Cl\omega \leq \lambda$. Since $\omega \leq \psi \leq Cl\omega$, therefore $\psi \leq Cl\omega$ gives $\psi \leq Cl\omega \leq \lambda$ or $\psi \leq \lambda$. ■

7.3 Some results on closure, interior, semi-closure and semi-interior

Theorem 7.2 *For fuzzy sets* λ *and* μ *in an fts* X*, we have*

(1) $sInt(\lambda \vee \mu) \geq sInt\lambda \vee sInt\mu$,

(2) $sInt(\lambda \wedge \mu) = sInt\lambda \wedge sInt\mu$,

(3) $sCl(\lambda \vee \mu) = sCl\lambda \vee sCl\mu$,

(4) $sCl(\lambda \wedge \mu) \leq sCl\lambda \wedge sCl\mu$.

Proof. (1) $sInt\lambda$ and $sInt\mu$ are both semi-open fuzzy and $\lambda \leq \lambda \vee \mu$, $\mu \leq \lambda \vee \mu$ imply $sInt\lambda \leq sInt(\lambda \vee \mu)$ and $sInt\mu \leq sInt(\lambda \vee \mu)$. Combining, $sInt\lambda \vee sInt\mu \leq sInt(\lambda \vee \mu)$ or

$$sInt(\lambda \vee \mu) \geq sInt\lambda \vee sInt\mu.$$

(2) $\lambda \wedge \mu \leq \lambda$ and $\lambda \wedge \mu \leq \mu$ imply $sInt(\lambda \wedge \mu) \leq sInt\lambda \wedge sInt\mu$. Conversely $sInt\lambda \leq \lambda$ and $sInt\mu \leq \mu$ imply $sInt\lambda \wedge sInt\mu \leq \lambda \wedge \mu$ and $sInt\lambda \wedge sInt\mu$ is semi-open fuzzy . But $sInt(\lambda \wedge \mu)$ is the largest semi-open fuzzy set contained in $\lambda \wedge \mu$, hence $sInt\lambda \wedge sInt\mu \leq sInt(\lambda \wedge \mu)$. This gives the equality.

(3) Follows easily from (2).

(4) Since $\lambda \wedge \mu \leq \lambda,\ \lambda \wedge \mu \leq \mu$

$$\Rightarrow \quad sCl(\lambda \wedge \mu) \leq sCl\lambda,\ sCl(\lambda \wedge \mu) \leq sCl\mu$$

$$\Rightarrow \quad sCl(\lambda \wedge \mu) \leq sCl\lambda \wedge sCl\mu.$$

■

The inequalities (1) and (4) of Theorem 8.1, are in general irreversible, as is shown by following:

Example 7.2 *Let $X = \{a, b\}$ be a set and $I = \{0, .5, 1\}$ be the lattice of membership grades for fuzzy sets on X. Let $\mu = \{a_{.5}, b_1\}$, $\nu = \{a_{.0}, b_1\}$ and $\omega = \{a_1, b_{.5}\}$ be fuzzy sets on X, and τ the fuzzy topology generated by μ, ν and ω. Then*

$$\tau = \left\{\widetilde{0}, \mu, \nu, \omega, \{a_{.5}, b_{.5}\}, \{a_0, b_{.5}\}, X\right\}.$$

We choose fuzzy sets $\alpha = \{a_{.5}, b_0\}, \beta = \{a_0, b_{.5}\}, \gamma = \{a_{.5}, b_1\}$ and $\delta = \{a_1, b_{.5}\}$. Then calculations give that

$$sInt(\alpha \vee \beta) = \{a_{.5}, b_{.5}\} \not< \{a_0, b_{.5}\} = sInt\alpha \vee sInt\beta.$$

$$sCl\gamma \wedge sCl\delta = \{a_1, b_{.5}\} \not< \{a_{.5}, b_{.5}\} = sCl(\gamma \wedge \delta).$$

Theorem 7.3 *[53] For a fuzzy set λ in an fts X, we have:*

(1) $ClsCl\lambda = Cl\lambda$,

(2) $IntCl\lambda \leq sCl\lambda$.

We use Theorem 7.3 in the following:

Theorem 7.4 *Let λ be a fuzzy set in an fts X. Then we have the following:*

(1) $sClsCl\lambda = sCl\lambda$ $(sIntsInt\lambda = sInt\lambda)$,

(2) $sIntsCl\lambda \geq IntCl\lambda$,

(3) $IntsIntsCl\lambda = IntCl\lambda$,

(4) $sIntsCl\lambda \leq IntCl\lambda \vee \lambda$.

Proof. (1). By the fact that $sCl\lambda$ is semi-closed fuzzy and that λ is semi-closed fuzzy if and only if $sCl\lambda = \lambda$, it follows immediately.

(2). By Theorem 7.3(2), $IntCl\lambda \leq sCl\lambda$, so that $IntCl\lambda \leq sIntsCl\lambda$, since all open fuzzy sets are semi-open fuzzy .

(3). By (2), $IntCl\lambda \leq sIntsCl\lambda$, so that $IntCl\lambda \leq IntsIntsCl\lambda$. Also $sCl\lambda \leq Cl\lambda$, so that $sIntsCl\lambda \leq Cl\lambda$ and $IntsIntsCl\lambda \leq IntCl\lambda$. Consequently, we get (3).

(4). Now $IntCl\lambda \leq IntCl\lambda \vee \lambda \leq Cl\lambda$ so by Theorem 7.1, $IntCl\lambda \vee \lambda$ is semi-closed fuzzy . Also $\lambda \leq IntCl\lambda \vee \lambda$. Since $sCl\lambda$ is the smallest semi-closed fuzzy set with $\lambda \leq sCl\lambda$, therefore $sCl\lambda \leq IntCl\lambda \vee \lambda$. Which implies $sIntsCl\lambda \leq sCl\lambda \leq IntCl\lambda \vee \lambda$ or $sIntsCl\lambda \leq IntCl\lambda \vee \lambda$. ■

The inequalites (2) and (4) of Theorem 8.2 may, in general, not be reversible as is shown in the following:

Example 7.3 *Let* $X = \{a, b, c\}$ *be a set and* $I = \{0, .3, .5, .7, 1\}$ *be the lattice of membership grades for fuzzy sets on* X. *Let* $\mu = \{a_{.5}, b_1, c_{.3}\}$, $\nu = \{a_0, b_{.5}, c_1\}$ *and* $\omega = \{a_{.3}, b_{.7}, c_{.5}\}$ *be fuzzy sets on* X, *and* τ *the fuzzy topology generated by* μ, ν *and* ω. *Then* $\tau = \{\widetilde{0}, \mu, \nu, \omega, \{a_0, b_{.5}, c_{.5}\}, \{a_0, b_{.5}, c_{.3}\}, \{a_{.3}, b_{.7}, c_{.3}\}, \{a_{.5}, b_1, c_{.5}\}, \{a_{.3}, b_{.7}, c_1\}, \{a_{.5}, b_1, c_1\}, X\}$

We choose fuzzy sets $\lambda = \{a_{.7}, b_{.5}, c_{.5}\}$ *and* $\psi = \{a_0, b_0, c_{.5}\}$ *Then calculations give that*

$$sIntsCl\lambda = \{a_{.7}, b_{.5}, c_{.5}\} \nleq \{a_0, b_{.5}, c_{.5}\} = IntCl\lambda$$

$$IntCl\psi \vee \psi = \{a_0, b_0, c_{.5}\} \nleq \{a_0, b_0, c_0\} = sIntsCl\psi$$

Theorem 7.5 *For a fuzzy set* λ *in an fts* X, $sIntsCl\lambda \leq \lambda$ *if and only if* $\lambda = sCl\lambda$ *(i.e.* λ *is semi-closed fuzzy).*

Proof. ($\Rightarrow$) Suppose $sInt(sCl\lambda) \leq \lambda$. Since $sCl\lambda$ is semi-closed fuzzy , so by Theorem 7.1 there exists a closed set ψ such that $Int\psi \leq sCl\lambda \leq \psi$. Thus $Int\psi \leq sIntsCl\lambda \leq \lambda \leq sCl\lambda \leq \psi$ or $Int\psi \leq \lambda \leq \psi$. Hence λ is semi-closed fuzzy and $\lambda = sCl\lambda$ by Proposition 7.3.

($\Leftarrow$) If $\lambda = sCl\lambda$, then $sIntsCl\lambda = sInt\lambda \leq \lambda$. ∎

Theorem 7.6 *For any fuzzy set* λ *in an fts* X, *we have*

(1) $(sInt\lambda)^c = sCl(\lambda^c)$,

(2) $(sCl\lambda)^c = sInt(\lambda^c)$.

Proof. (1) $(sInt\lambda)^c = X - sInt\lambda$

$$\begin{aligned}
&= 1 - \vee\{\mu|\mu \in FSO(X) \text{ and } \mu \leq \lambda\} \\
&= \wedge\{1 - \mu|\mu \in FSO(X) \text{ and } \mu \leq \lambda\} \\
&= \wedge\{\psi|\psi^c \in FSO(X) \text{ and } \psi \geq \lambda^c\};\ \text{where } \psi = 1 - \mu \\
&= sCl(\lambda^c)
\end{aligned}$$

(2) Similar to (1). ■

7.4 s-Open and s-Closed Fuzzy Mappings

First, we define

Definition 7.3 *A function $f : X \to Y$ is said to be fuzzy s-open (resp. fuzzy s-closed) if the image of every semi-open fuzzy (resp. semi-closed fuzzy) set is open fuzzy (resp. fuzzy closed).*

Obviously a fuzzy s-open function is open fuzzy .

Definition 7.4 *[37] A fuzzy point e is called a boundary point of a fuzzy set λ if and only if $e \in Cl\lambda \wedge Cl\lambda^c$. The union of all the boundary points of λ is called a boundary of λ, denoted by $Bd\lambda$. It is clear that*

$$Bd\lambda = Cl\lambda \wedge Cl\lambda^c.$$

Next, we define

Definition 7.5 *Semi boundary (briefly sBd) of a fuzzy set* λ *in an fts* X *is defined as:*

$$sBd\lambda = sCl\lambda \wedge sCl\lambda^c.$$

In the following, we characterize fuzzy s-open mappings in terms of $sInt$, sCl and sBd:

Theorem 7.7 *For a function* $f : X \to Y$ *, a fuzzy set* α *in an fts* X *and a fuzzy set* β *in an fts* Y*, then:*

(1) f *is fuzzy s-open,*

(2) $f(sInt\alpha) \leq Intf(\alpha)$,

(3) $sIntf^{-1}(\beta) \leq f^{-1}(Int\beta)$,

(4) $f^{-1}(Cl\beta) \leq sCl(f^{-1}(\beta))$,

(5) $f^{-1}(Bd(\beta)) \leq sBd(f^{-1}(\beta))$.

Proof. (1) $\Rightarrow$ (2) Obviously $f(sInt\alpha) \leq f(\alpha)$. f is fuzzy s-open gives $f(sInt\alpha)$ is open fuzzy in Y. But $Intf(\alpha)$ is the largest open fuzzy set such that $Intf\,(\alpha) \leq f(\alpha)$. Therefore $f(sInt\alpha) \leq Intf(\alpha)$, for any fuzzy set α in X. This gives (2).

(2) $\Rightarrow$ (3) For any fuzzy set β in Y, $f^{-1}(\beta) = \alpha$ is a fuzzy set in X. Then by (2),

$$f(sIntf^{-1}(\beta)) \leq Int(f(f^{-1}(\beta)) \leq Int(\beta)$$

or $f(sIntf^{-1}(\beta)) \leq Int\beta$ or $sIntf^{-1}(\beta) \leq f^{-1}f(sIntf^{-1}\,(\beta)) \leq f^{-1}(Int\beta)$ or $sIntf^{-1}(\beta) \leq f^{-1}(Int\beta)$. This gives (3).

(3) ⇒ (4) By (3) , we have

$$\begin{aligned} sIntf^{-1}(\beta) &\leq f^{-1}(Int\beta) \\ (f^{-1}(Int\beta))^c &\leq (sIntf^{-1}(\beta))^c \\ &= sCl(f^{-1}(\beta))^c \text{ (by Theorem 8.4(1))} \end{aligned}$$

or $(f^{-1}(Int\beta))^c \leq sCl(f^{-1}(\beta))^c$ or $f^{-1}(Cl\beta^c) \leq sClf^{-1}(\beta^c)$ or $f^{-1}(Cl\psi) \leq sClf^{-1}(\psi)$ where $\psi = \beta^c$, a fuzzy set in Y. This gives (4).

(4) ⇒ (5) For a fuzzy set β in Y, $Bd\beta = Cl\beta \wedge Cl(\beta^c)$ is a closed fuzzy set in Y. Now $f^{-1}(Bd\beta) = f^{-1}(Cl\beta) \wedge f^{-1}(Cl\beta^c)$. Using (4) we have

$$f^{-1}(Bd\beta) \leq sCl(f^{-1}(\beta) \wedge sCl(f^{-1}(\beta^c))$$

or

$$f^{-1}(Bd\beta) \leq sClf^{-1}(\beta) \wedge sCl(f^{-1}(\beta))^c = sBdf^{-1}(\beta).$$

This gives (5). ■

In the following, we characterize fuzzy s-closed mappings as:

Theorem 7.8 *A function $f : X \to Y$ is fuzzy s-closed if and only if $Clf(\lambda) \leq f(sCl\lambda)$, for each fuzzy set λ in an fts X.*

Proof. (⇒) Obviously $f(\lambda) \leq f(sCl\lambda)$. $f(sCl\lambda)$ is fuzzy closed, since f is fuzzy s-closed. But $Clf(\lambda)$ is the smallest closed fuzzy set with $f(\lambda) \leq Clf(\lambda)$. Therefore $Clf(\lambda) \leq f(sCl\lambda)$.

(⇐) Let $\lambda \in FSC(X)$. We show that $f(\lambda)$ is fuzzy closed. By hypothesis, $Clf(\lambda) \leq f(sCl\lambda) = f(\lambda)$ or $Clf(\lambda) \leq f(\lambda)$. This proves that $f(\lambda)$ is fuzzy closed. ■

Theorem 7.9 *If a function $f : X \to Y$ is fuzzy s-closed then for each fuzzy set β in an fts Y and each semi-open fuzzy set μ in an fts X with $\mu \geq f^{-1}(\beta)$, there exists a open fuzzy set ν in Y with $\nu \geq \beta$ such that $f^{-1}(\nu) \leq \mu$.*

Proof. Let μ be an arbitrary semi-open fuzzy set in X with $\mu \geq f^{-1}(\beta)$, where β is a fuzzy set in Y. Clearly $(f(\mu^c))^c = \nu$ (say) is open fuzzy in Y. Since $f^{-1}(\beta) \leq \mu$, then straight forward calculations give that $\beta \leq \nu$. Moreover, we have

$$\begin{aligned} f^{-1}(\nu) &= f^{-1}f(\mu^c) \\ &= \left(f^{-1}f(\mu^c)\right)^c \leq \mu \\ \text{or} \qquad f^{-1}(\nu) &\leq \mu. \end{aligned}$$

■

Theorem 7.10 *Let $f : X \to Y$ be a surjective function from an fts X to an fts Y. If for each fuzzy set β in Y and each semi-open fuzzy set μ in X with $\mu \geq f^{-1}(\beta)$, there exists a open fuzzy set ν in Y with $\nu \geq \beta$ such that $f^{-1}(\nu) \leq \mu$, then f is s-closed.*

Proof. Let ψ be an arbitrary semi-closed fuzzy set in X and $y \in (f(\psi))^c$. Then

$$f^{-1}(y) \leq f^{-1}(f(\psi))^c = \left(f^{-1}f(\psi)\right)^c \leq \psi^c$$

or $f^{-1}(y) \leq \psi^c$. Since ψ^c is semi-open fuzzy , therefore there exists a open fuzzy set ν_y with $y \in \nu_y$ such that $f^{-1}(\nu_y) \leq \psi^c$. Since f is surjective, we have $y \in \nu_y \leq (f(\psi))^c$. Thus$(f(\psi))^c = \vee\{\nu_y | y \in (f(\psi))^c\}$ is open fuzzy in Y or $f(\psi)$ is closed fuzzy in Y. This proves that f is s-closed. ■

Combining Theorems 7.9 and 7.10, we have

Theorem 7.11 *A surjective function $f : X \to Y$ is fuzzy s-closed if and only if for each fuzzy set β in Y and each semi-open fuzzy set μ in X with $\mu \geq f^{-1}(\beta)$, there exists a open fuzzy set ν in Y with $\nu \geq \beta$ such that $f^{-1}(\nu) \leq \mu$.*

Chapter 8

Simply Continuous Fuzzy Mappings

8.1 Simply Open Fuzzy Sets and Simply Continuous Fuzzy Mappings

First, we define

Definition 8.1 *A fuzzy set ω in a space X is said to be fuzzy nowhere dense if, $IntCl\omega = \widetilde{0}$.*

Example 8.1 *In fts (X, τ_X) defined in Example 4.1, the following are the fuzzy nowhere dense sets:*

$$\widetilde{0}, \{a_0, b_{.3}\}, \{a_{.3}, b_0\}, \{a_{.3}, b_{.3}\}, \{a_{.5}, b_0\}, \{a_{.5}, b_{.3}\}, \{a_{.7}, b_0\}, \{a_{.7}, b_{.3}\}$$

Proposition 8.1 *Let λ be a fuzzy nowhere dense set in space X. Then*

1. $Int\lambda = \widetilde{0}$.

2. λ^c is dense in X.

3. $\lambda \leq Cl(Cl\lambda)^c$.

4. Let Y be a subspace of fuzzy topological space X and η be a fuzzy nowhere dense set in X. Then η is fuzzy nowhere dense in Y.

Proof. (1) Obvious.

(2) $Int\lambda \leq \lambda \leq Cl\lambda$ gives $IntInt\lambda \leq IntCl\lambda$ or $(Int\lambda)^c \geq (IntCl\lambda)^c$ using in $Cl\lambda^c = (Int\lambda)^c$ we have $Cl\lambda^c \geq (IntCl\lambda)^c = \widetilde{0}^c = X$ or $X \leq Cl\lambda^c$. That is $Cl\lambda^c = X$ and hence λ^c is dense in X.

(3) Suppose contrary that λ is not in $Cl(Cl\lambda)^c$. But

$$Cl(Cl\lambda)^c = (IntCl\lambda)^c = \widetilde{0}^c = X$$

implies λ is not in X, a contradiction.

(4) Consider

$$IntCl(\eta \wedge Y) \leq Int(Cl\eta \wedge ClY) = IntCl\eta \wedge IntClY = \widetilde{0} \wedge IntClY = \widetilde{0}.$$

■

The inclusion in (3) of Theorem 8.1 cannot be reversed as shown by following:

Example 8.2 *In fts (X, τ_X) of Example 4.1 choose $\lambda = \{a_{.3}, b_{.5}\}$, then $\lambda \leq Cl\,(Cl\lambda)^c = \{a_{.5}, b_{.5}\}$. But λ is not fuzzy nowhere dense.*

Proposition 8.2 *Let λ be a fuzzy dense and ψ, a closed fuzzy set in X such that $\lambda \leq \psi$. Then $\psi = X$ (The only closed fuzzy set containing a dense set is X itself).*

Proof. $\lambda \leq \psi \Rightarrow Cl(\lambda) \leq Cl\psi$ or $X \leq Cl\psi = \psi \Rightarrow \psi = X$. ■

Definition 8.2 *A fuzzy set λ in a space X is called fuzzy simply open if there exist two fuzzy sets μ and ν (either of which may be $\widetilde{0}$), where μ is open fuzzy and ν is fuzzy nowhere dense in X, such that $\mu \vee \nu \leq \lambda \leq Cl(\mu \vee \nu)$.*

Theorem 8.1 *Every semi open fuzzy set is fuzzy simply open.*

Proof. Let λ be a semi open fuzzy set in X. Then there exists a open fuzzy set μ in X such that $\mu \leq \lambda \leq Cl\mu$. Let $\nu = \widetilde{0}$, then obviously $(\mu \vee \nu) \leq \lambda \leq Cl(\mu \vee \nu)$, where μ is open fuzzy in X and ν is nowhere dense. ■

That the converse of this theorem is not true is shown as following:

Example 8.3 *The fts (X, τ_X) as defined in Example 4.1 has 6 open fuzzy , 11 semi open fuzzy and 19 fuzzy simply open sets. The fuzzy set $\{a_{.7}, b_{.3}\}$ is fuzzy simply open but not semi open fuzzy .*

Theorem 8.2 *The union of a finite number of fuzzy simply open sets is fuzzy simply open.*

Proof. Let $\{\lambda i : i \in J\}$ be a finite family of fuzzy simply open sets in X. Then for every λ_i there exists a open fuzzy set μ_i and a nowhere dense fuzzy set ν_i such that

$$(\mu_i \vee \nu_i) \leq \lambda_i \leq Cl(\mu_i \vee \nu_i);\ i \in J$$

implies

$$\vee_i(\mu_i \vee \nu_i) \leq \vee_i \lambda_i \leq \vee_i Cl(\mu_i \vee \nu_i)$$

or

$$(\vee_i\mu_i \vee \vee_i\nu_i) \leq \vee_i\lambda_i \leq Cl(\vee_i\mu_i \vee \vee_i\nu_i)$$

which gives $(\mu \vee \nu) \leq \vee_i\lambda_i \leq Cl(\mu \vee \nu)$ where $\mu = \vee_i\mu_i$ is open fuzzy because μ_i is open fuzzy for all i, and $\nu = \vee_i\nu_i$ is fuzzy nowhere dense because ν_i is nowhere dense for all i. This establishes $\vee_i\lambda_i$ as fuzzy simply open.

Theorem 8.3 *Let λ be a fuzzy simply open set in X. Then there exists a closed fuzzy set ω and a fuzzy nowhere dense set η such that $Int(\omega - \eta) \leq \lambda^c \leq (\omega - \eta)$.*

Proof. Since λ is fuzzy simply open in X, there exists a open fuzzy set μ and a fuzzy nowhere dense set η in X such that $(\mu \vee \eta) \leq \lambda \leq Cl(\mu \vee \eta)$. Taking complements

$$(\mu \vee \eta)^c \geq \lambda^c \geq (Cl(\mu \vee \eta))^c$$

or

$$Int(\mu^c \wedge \eta^c) \leq \lambda^c \leq (\mu^c \wedge \eta^c)$$

Set $\mu^c = \omega$, a closed fuzzy set in X. Then $Int(\omega \wedge \eta^c) \leq \lambda^c \leq (\omega \wedge \eta^c)$ or $Int(\omega - \eta) \leq \lambda^c \leq (\omega - \eta)$. ■

Theorem 8.4 *Let λ be a fuzzy simply open set in X. Then there exists a fuzzy nowhere dense set η such that $\lambda \leq ClInt\lambda \vee Cl\eta$.*

Proof. Since λ is a fuzzy simply open set in X, therefore there exists a open fuzzy set μ and a fuzzy nowhere dense set η such that

$$(\mu \vee \eta) \leq \lambda \leq Cl(\mu \vee \eta) \tag{8.1}$$

which gives

$$Int(\mu \vee \eta) \leq Int\lambda \tag{8.2}$$

Since η is fuzzy nowhere dense and $Int(\mu \vee \eta) \geq Int\mu \vee Int\eta$, by (8.2) we have $Int\mu \vee \widetilde{0} \leq Int(\mu \vee \eta) \leq Int\lambda$. Thus $Int\mu \leq Int\lambda$, or $ClInt\mu \leq ClInt\lambda$ implies

$$ClInt\mu \vee Cl\eta \leq ClInt\lambda \vee Cl\eta \tag{8.3}$$

(8.1) gives $\lambda \leq Cl(\mu \vee \eta) = Cl\mu \vee Cl\eta = ClInt\mu \vee Cl\eta$ or

$$\lambda \leq ClInt\mu \vee Cl\eta \tag{8.4}$$

From (8.3) and (8.4) we have $\lambda \leq ClInt\lambda \vee Cl\eta$. ■

Example 8.4 *Let (X, τ_X) be the fts given in Example 4.1, choosing $\lambda = \{a_{.3}, b_{.7}\}$ and $\eta = \{a_1, b_{.5}\}$ then $\lambda \leq ClInt\lambda \vee Cl\eta = X \vee X$ but η is not fuzzy nowhere dense.*

Theorem 8.5 *Let λ be a fuzzy simply open set in X. If for any fuzzy set μ, $\lambda \leq \mu \leq Cl\lambda$, then μ is also fuzzy simply open.*

Proof. Since λ is fuzzy simply open there exist a open fuzzy set ν and a fuzzy nowhere dense set ω in X such that

$$\nu \vee \omega \leq \lambda \leq Cl(\nu \vee \omega) \tag{8.5}$$

Using (8.5), $\lambda \leq \mu \leq Cl\lambda$, gives

$$\nu \vee \omega \leq \lambda \leq \mu \leq Cl\lambda \tag{8.6}$$

From (8.5)

$$Cl\lambda \leq Cl(\nu \vee \omega) \tag{8.7}$$

From (8.6) and (8.7) $\nu \vee w \leq \lambda \leq \mu \leq Cl\lambda \leq Cl(\nu \vee \omega)$ or $\nu \vee \omega \leq \mu \leq Cl(\nu \vee \omega)$ implies μ is fuzzy simply open. ■

Example 8.5 *In the fts (X, τ_X) as given in Example 4.1, choose $\lambda = \{a_0, b_{.5}\}$ and $\mu = \{a_{.3}, b_{.5}\}$ then $\lambda \leq Cl\lambda = \{a_{.5}, b_{.5}\}$ and $\mu \in F\,\mathrm{Si}\,O\,(X)$ but $\lambda \notin FSO\,(X.)$*

Definition 8.3 *Let X and Y be fts's. Then a mapping $f : X \to Y$ is called fuzzy simply continuous (resp. fuzzy simply irresolute) if and only if the inverse image of each open fuzzy (resp. fuzzy simply open) set in Y is fuzzy simply open in X.*

Example 8.6 *Let $X = \{a, b\}$ and $Y = \{p, q\}$ be two sets with fuzzy topologies τ_X and τ_Y generated by the families $\{\{a_{.3}, b_{.3}\}, \{a_{.3}, b_0\}\}$ and $\{\{p_{.3}, q_{.3}\}, \{p_{.7}, q_{.3}\}, \ \{p_0, q_{.7}\}, \{p_{.5},$ respectively. Membership grades lattice for both the fts's is $I = \{0, 0.3, 0.5, 0.7, 1\}$. Then the function $f : X \to Y$ given as $f\,(a) = q$ and $f\,(b) = p$ is both, fuzzy simply continuous and fuzzy simply irresolute.*

Theorem 8.6 *Let $f : X \to Y$ be fuzzy simply continuous at a fuzzy point $x_\alpha \in X$. Then for every open fuzzy set μ in Y with $f(x) \in \mu$, there exists a fuzzy simply open set λ in X with $x \in \lambda$, such that $f(\lambda) \leq \mu$.*

Proof. Let μ be a open fuzzy set in Y such that $x \in f^{-1}(\mu)$. Then $f^{-1}(\mu)$ is a fuzzy simply open set in X, because f is fuzzy simply continuous. Put

$\lambda = f^{-1}(\mu)$ then $x \in \lambda$ and $f(\lambda) \leq ff^{-1}(\mu) \leq \mu$ or $f(\lambda) \leq \mu$. ■

Theorem 8.7 *Let $f : X \to Y$ and $g : Y \to Z$ be respectively fuzzy simply irresolute and fuzzy simply continuous maps. Then $gof : X \to Z$ is fuzzy simply continuous.*

Proof. Let ω be open fuzzy in Z. Then $g^{-1}(\omega)$ is fuzzy simply open. Since f is fuzzy simply irresolute; $(gof)^{-1}(\omega) = f^{-1}(g^{-1}(\omega))$ is fuzzy simply-open in X. ■

Chapter 9

α-Semicontinuous Fuzzy Mappings

We first recall some required notions as:

Definition 9.1 *A fuzzy set λ in an fts X is said to be*

(1) *fuzzy α-open [49] (resp. α-closed), if $\lambda \leq IntClInt\lambda$ (resp. $\lambda \geq ClIntCl\lambda$),*

(2) *fuzzy preopen [49] (resp. preclosed), if $\lambda \leq IntCl\lambda$ (resp. $ClInt\lambda \leq \lambda$),*

(3) *semi open fuzzy [9], if there exists a open fuzzy set μ such that $\mu \leq \lambda \leq Cl\mu$.*

Note that, λ is semi open fuzzy iff $\lambda \leq ClInt\lambda$. It is clear that every fuzzy α-open (resp. fuzzy α-closed) set is fuzzy preopen (resp. fuzzy preclosed) as well as semi open fuzzy (resp. semi fuzzy closed), whereas fuzzy preopen and semi open fuzzy are independent notions. The class of all fuzzy α-open, fuzzy preopen and semi open fuzzy (resp. fuzzy α-closed, fuzzy preclosed and *semi fuzzy closed) sets are denoted by* $F\alpha O(X), FPO(X), FSO(X)$ *(resp.* $F\alpha C(X), FPC(X), FSC(X)$*).*

Remark 9.1 *[52] Arbitrary union of fuzzy* α*-open sets of an fts* X *is a fuzzy* α*-open set.*

Definition 9.2 *[52] The* α*-closure and* α*-interior of a fuzzy set* λ *in an fts* (X, τ) *are denoted and defined as:*

$$\begin{aligned} \alpha Cl\lambda &= \bigwedge\{\mu \mid \lambda \leq \mu, \mu \in F\alpha C(X)\}, \\ \alpha Int\lambda &= \bigvee\{\omega \mid \omega \leq \lambda, \omega \in F\alpha O(X)\}. \end{aligned}$$

Clearly $\alpha Cl\lambda \leq Cl\lambda$ *and* $Int\lambda \leq \alpha Int\lambda$*.*

We recall some properties of fuzzy α-open and fuzzy α-closed sets as follows:

Theorem 9.1 *[52] Let* λ *and* μ *be fuzzy sets in an fts* X*. Then*

(1) λ *is fuzzy* α*-closed if and only if* $\lambda = \alpha Cl\lambda$*.*

(2) $\lambda \leq \mu \Rightarrow \alpha Cl\lambda \leq \alpha Cl\mu$*.*

(3) $\alpha Cl\alpha Cl\lambda = \alpha Cl\lambda$*.*

The dual of Theorem 9.4 is the following:

Theorem 9.2 *Let* λ *and* μ *be fuzzy sets in an fts* X*. Then*

(1) λ *is fuzzy* α*-open if and only if* $\lambda = \alpha Int\lambda$*.*

(2) $\lambda \leq \mu \Rightarrow \alpha Int\lambda \leq \alpha Int\mu$*.*

(3) $\alpha Int\alpha Int\lambda = \alpha Int\lambda$*.*

Using Theorems 9.4 and 9.5, the following theorem may easily be proved:

Theorem 9.3 *Let* λ *and* μ *be fuzzy sets in an fts* X*. Then*

(1) $\alpha Int\lambda \wedge \alpha Int\mu = \alpha Int\left(\lambda \wedge \mu\right).$

(2) $\alpha Int\left(\lambda \vee \mu\right) \geq \alpha Int\lambda \vee \alpha Int\mu.$

(3) $\alpha Cl\lambda \vee \alpha Cl\mu = \alpha Cl\left(\lambda \vee \mu\right).$

(4) $\alpha Cl\left(\lambda \wedge \mu\right) \leq \alpha Cl\lambda \wedge \alpha Cl\mu.$

The inequalities (2) and (4) of Theorem 9.6 are, in general, irreversible as is clear from the following:

Example 9.1 *Let* $X = \{a, b, c\}$ *and a fuzzy topology on* X *is given as*

$$\tau = \{\widetilde{0}, \{a_{0.3}, b_{0.9}, c_{0.1}\}, \{a_{0.7}, b_{0.1}, c_{0.5}\}, \{a_{0.2}, b_{0.4}, c_{0.7}\}, \{a_{0.3}, b_{0.1}, c_{0.1}\}, \{a_{0.2}, b_{0.4}, c_{0.1}\}, \{a_{0.2}, b_{0}$$

$$\{a_{0.2}, b_{0.1}, c_{0.1}\}, \{a_{0.7}, b_{0.9}, c_{0.5}\}, \{a_{0.3}, b_{0.9}, c_{0.7}\}, \{a_{0.3}, b_{0.9}, c_{0.5}\}, \{a_{0.7}, b_{0.4}, c_{0.7}\}, \{a_{0.7}, b_{0.4}, c$$

$$\{a_{0.7}, b_{0.9}, c_{0.7}\}, \{a_{0.3}, b_{0.4}, c_{0.7}\}, \{a_{0.3}, b_{0.4}, c_{0.1}\}, \{a_{0.3}, b_{0.1}, c_{0.5}\}, \{a_{0.2}, b_{0.4}, c_{0.5}\}, \{a_{0.3}, b_{0.4}, c$$

Choose fuzzy sets $\lambda = \{a_{.7}, b_{1}, c_{.4}\}$ *and* $\mu = \{a_{.4}, b_{1}, c_{.7}\}$*. Then the calculations show that*

$$\alpha Int\left(\lambda \vee \mu\right) = \{a_{.7}, b_{.9}, c_{.7}\} \not\leq \{a_{.4}, b_{.9}, c_{.7}\} = \alpha Int\lambda \vee \alpha Int\mu.$$

Also choose $\gamma = \{a_{0.1}, b_{0}, c_{0}\}, \delta = \{a_{0}, b_{0.4}, c_{0.5}\},$ *then we have*

$$\alpha Cl\gamma \wedge \alpha Cl\delta = \{a_{0.3}, b_{0.1}, c_{0.3}\} \not\leq \{a_{0}, b_{0}, c_{0}\} = \alpha Cl\left(\gamma \wedge \delta\right).$$

9.1 α-Open and α-Closed Fuzzy Sets

Definition 9.3 *A fuzzy set λ in an fts X is said to be*

(1) *fuzzy α-open [49] (resp. α-closed), if $\lambda \leq IntClInt\lambda$ (resp. $\lambda \geq ClIntCl\lambda$),*

(2) *fuzzy preopen [49] (resp. preclosed), if $\lambda \leq IntCl\lambda$ (resp. $ClInt\lambda \leq \lambda$),*

(3) *semi open fuzzy [9], if there exists a open fuzzy set μ such that $\mu \leq \lambda \leq Cl\mu$.*

Note that, λ is semi open fuzzy iff $\lambda \leq ClInt\lambda$. It is clear that every fuzzy α-open (resp. fuzzy α-closed) set is fuzzy preopen (resp. fuzzy preclosed) as well as semi open fuzzy (resp. semi fuzzy closed), whereas fuzzy preopen and semi open fuzzy are independent notions. The class of all fuzzy α-open, fuzzy preopen and semi open fuzzy (resp. fuzzy α-closed, fuzzy preclosed and semi fuzzy closed) sets are denoted by $F\alpha O(X)$, $FPO(X)$, $FSO(X)$ (resp. $F\alpha C(X)$, $FPC(X)$, $FSC(X)$).

Remark 9.2 *[52] Arbitrary union of fuzzy α-open sets of an fts X is a fuzzy α-open set.*

Definition 9.4 *[52] The α-closure and α-interior of a fuzzy set λ in an fts*

(X,τ) are denoted and defined as:

$$\alpha Cl\lambda = \bigwedge\{\mu \mid \lambda \leq \mu, \mu \in F\alpha C(X)\},$$

$$\alpha Int\lambda = \bigvee\{\omega \mid \omega \leq \lambda, \omega \in F\alpha O(X)\}.$$

Clearly $\alpha Cl\lambda \leq Cl\lambda$ and $Int\lambda \leq \alpha Int\lambda$.

We recall some properties of fuzzy α-open and fuzzy α-closed sets as follows:

Theorem 9.4 *[52] Let λ and μ be fuzzy sets in an fts X. Then*

(1) *λ is fuzzy α-closed if and only if $\lambda = \alpha Cl\lambda$.*

(2) *$\lambda \leq \mu \Rightarrow \alpha Cl\lambda \leq \alpha Cl\mu$.*

(3) *$\alpha Cl\alpha Cl\lambda = \alpha Cl\lambda$.*

The dual of Theorem 9.4 is the following:

Theorem 9.5 *Let λ and μ be fuzzy sets in an fts X. Then*

(1) *λ is fuzzy α-open if and only if $\lambda = \alpha Int\lambda$.*

(2) *$\lambda \leq \mu \Rightarrow \alpha Int\lambda \leq \alpha Int\mu$.*

(3) *$\alpha Int\alpha Int\lambda = \alpha Int\lambda$.*

Using Theorems 9.4 and 9.5, the following theorem may easily be proved:

Theorem 9.6 *Let λ and μ be fuzzy sets in an fts X. Then*

(1) *$\alpha Int\lambda \wedge \alpha Int\mu = \alpha Int(\lambda \wedge \mu)$.*

(2) *$\alpha Int(\lambda \vee \mu) \geq \alpha Int\lambda \vee \alpha Int\mu$.*

(3) $\alpha Cl\lambda \vee \alpha Cl\mu = \alpha Cl\left(\lambda \vee \mu\right).$

(4) $\alpha Cl\left(\lambda \wedge \mu\right) \leq \alpha Cl\lambda \wedge \alpha Cl\mu.$

The inequalities (2) and (4) of Theorem 9.6 are, in general, irreversible as is clear from the following:

Example 9.2 *Let $X = \{a, b, c\}$ and a fuzzy topology on X is given as*

$$\tau = \{\widetilde{0}, \{a_{0.3}, b_{0.9}, c_{0.1}\}, \{a_{0.7}, b_{0.1}, c_{0.5}\}, \{a_{0.2}, b_{0.4}, c_{0.7}\}, \{a_{0.3}, b_{0.1}, c_{0.1}\}, \{a_{0.2}, b_{0.4}, c_{0.1}\}, \{a_{0.2}, b_{0}$$

$$\{a_{0.2}, b_{0.1}, c_{0.1}\}, \{a_{0.7}, b_{0.9}, c_{0.5}\}, \{a_{0.3}, b_{0.9}, c_{0.7}\}, \{a_{0.3}, b_{0.9}, c_{0.5}\}, \{a_{0.7}, b_{0.4}, c_{0.7}\}, \{a_{0.7}, b_{0.4},$$

$$\{a_{0.7}, b_{0.9}, c_{0.7}\}, \{a_{0.3}, b_{0.4}, c_{0.7}\}, \{a_{0.3}, b_{0.4}, c_{0.1}\}, \{a_{0.3}, b_{0.1}, c_{0.5}\}, \{a_{0.2}, b_{0.4}, c_{0.5}\}, \{a_{0.3}, b_{0.4},$$

Choose fuzzy sets $\lambda = \{a_{.7}, b_1, c_{.4}\}$ and $\mu = \{a_{.4}, b_1, c_{.7}\}$. Then the calculations show that

$$\alpha Int\left(\lambda \vee \mu\right) = \{a_{.7}, b_{.9}, c_{.7}\} \not\leq \{a_{.4}, b_{.9}, c_{.7}\} = \alpha Int\lambda \vee \alpha Int\mu.$$

Also choose $\gamma = \{a_{0.1}, b_0, c_0\}$, $\delta = \{a_0, b_{0.4}, c_{0.5}\}$, then we have

$$\alpha Cl\gamma \wedge \alpha Cl\delta = \{a_{0.3}, b_{0.1}, c_{0.3}\} \not\leq \{a_0, b_0, c_0\} = \alpha Cl\left(\gamma \wedge \delta\right).$$

9.2 α-Semiopen and α-Semiclosed Fuzzy Sets

First, we define:

Definition 9.5 *A fuzzy set λ in an fts X is said to be fuzzy α-semiopen (resp. fuzzy α-semiclosed), if $\lambda \leq IntClsInt\lambda$ (resp. $\lambda \geq ClIntsCl\lambda$).*

We denote the class of fuzzy α-semiopen sets by $F\alpha SO(X)$. Clearly every fuzzy α-open set is fuzzy α-semiopen. Also the class of fuzzy α-semiopen sets is a subclass of the class of semi open fuzzy sets i.e. $F\alpha SO(X) \subseteq FSO(X)$.

Example 9.3 *Let $X = \{a, b, c\}$ and a fuzzy topology τ on X is given as*

$$\tau = \left\{\widetilde{0}, \{a_{.3}, b_{.1}, c_{.2}\}, \{a_{.7}, b_{.6}, c_{.3}\}, \widetilde{1}\right\}.$$

Then the fuzzy set $\lambda = \{a_{0.3}, b_{0.4}, c_{0.2}\}$ is semi open fuzzy but not fuzzy α-semiopen in an fts (X, τ).

Definition 9.6 *[52] The α-semiclosure and α-semiinterior of a fuzzy set λ in an fts (X, τ) are denoted and defined as:*

$$\alpha sCl\lambda = \bigwedge\{\mu \mid \lambda \leq \mu, \mu \in F\alpha SC(X)\},$$
$$\alpha sInt\lambda = \bigvee\{\omega \mid \omega \leq \lambda, \omega \in F\alpha SO(X)\}.$$

Clearly $\alpha sCl\lambda \leq Cl\lambda$ and $Int\lambda \leq \alpha sInt\lambda$.

Proposition 9.1 *[**?**] For fuzzy sets λ and μ in an fts X, we have*

(1) $sInt(\lambda \vee \mu) \geq sInt\lambda \vee sInt\mu$.

(2) $sInt(\lambda \wedge \mu) = sInt\lambda \wedge sInt\mu$.

(3) $sCl(\lambda \vee \mu) = sCl\lambda \vee sCl\mu$.

(4) $sCl(\lambda \wedge \mu) \leq sCl\lambda \wedge sCl\mu$.

Theorem 9.7 *Let λ and μ be fuzzy sets in an fts X. Then*

(1) *λ is fuzzy α-semiclosed if and only if $\lambda = \alpha sCl\lambda$.*

(2) $\lambda \leq \mu \Rightarrow \alpha sCl\lambda \leq \alpha sCl\mu$.

(3) $\alpha sCl\alpha sCl\lambda = \alpha sCl\lambda$.

Proof. (1) Let $\lambda = \alpha sCl\lambda$, then by definition of α-semiclosure, we have

$$\lambda = \alpha sCl\lambda = \bigwedge \{\rho \mid \rho \text{ is fuzzy } \alpha\text{-semiclosed and } \lambda \leq \rho\}.$$

This shows that $\lambda \in \{\rho \mid \rho \text{ is fuzzy } \alpha\text{-semiclosed and } \lambda \leq \rho\}$. Hence λ is fuzzy α-semiclosed.

Conversely, let λ be fuzzy α-semiclosed set. Clearly,

$$\lambda \in \{\rho \mid \rho \text{ is fuzzy } \alpha\text{-semiclosed and } \lambda \leq \rho\}.$$

Further, $\lambda \leq \rho$ for all such α-semiclosed sets ρ. Hence

$$\lambda = \bigwedge \{\rho \mid \rho \text{ is fuzzy } \alpha\text{-semiclosed and } \lambda \leq \rho\}.$$

(2) Since $\mu \leq \alpha sCl\mu$, and $\lambda \leq \mu$ implies $\lambda \leq \alpha sCl\mu$. Since $\alpha sCl\mu$ is fuzzy α-semiclosed, therefore by (1), $\alpha sCl\lambda \leq \alpha sCl\mu$.

(3) It is obvious. ■

Following is the dual of Theorem 9.7, the proof of which is similar:

Theorem 9.8 *Let λ and μ be fuzzy sets in an fts X. Then*

(1) *λ is fuzzy α-semiopen if and only if $\lambda = \alpha sInt\lambda$.*

(2) $\lambda \leq \mu \Rightarrow \alpha sInt\lambda \leq \alpha sInt\mu$.

(3) $\alpha sInt\alpha sInt\lambda = \alpha sInt\lambda$.

Using Theorem 9.8, we have the following, the proof of which follows from Proposition 9.1 and thus is omitted:

Theorem 9.9 *Let λ and μ be fuzzy sets in an fts X. Then*

(1) $\alpha sInt\lambda \vee \alpha sInt\mu = \alpha sInt\,(\lambda \vee \mu)\,.$

(2) $\alpha sInt\,(\lambda \wedge \mu) \leq \alpha sInt\lambda \wedge \alpha sInt\mu.$

(3) $\alpha sCl\lambda \vee \alpha sCl\mu = \alpha sCl\,(\lambda \vee \mu)\,.$

(4) $\alpha sCl\,(\lambda \wedge \mu) \leq \alpha sCl\lambda \wedge \alpha sCl\mu.$

The inequalities (2) and (4) of Theorem 9.9 are, in general, irreversible as is clear from the following:

Example 9.4 *Let $X = \{a, b, c\}$ and a fuzzy topology τ on X is given as*

$$\tau = \left\{\widetilde{0}, \{a_{.4}, b_0, c_{.8}\}\,, \{a_{.3}, b_{.1}, c_{.2}\}\,, \{a_{.7}, b_{.6}, c_{.3}\}\,, \widetilde{1}\right\}.$$

Choose $\lambda = \{a_{.5}, b_{.7}, c_{.9}\}$ and $\mu = \{a_{.5}, b_{.1}, c_{.7}\}$, then we have:

$$\alpha sInt\lambda \wedge \alpha sInt\mu = \{a_{.4}, b_{.1}, c_{.3}\} \nleq \{a_{.3}, b_{.1}, c_{.5}\} = \alpha sInt\,(\lambda \wedge \mu)\,.$$

Again choose $\nu = \{a_{.4}, b_{.3}, c_{.6}\}$ and $\delta = \{a_{.2}, b_{.5}, c_{.4}\}$. Then calculations show

$$\alpha sCl\nu \wedge \alpha sCl\delta = \{a_{.5}, b_{.3}, c_{.8}\} \nleq \{a_{.5}, b_{.2}, c_{.6}\} = \alpha sCl\,(\nu \wedge \delta)\,.$$

9.3 α-Fuzzy Semicontinuity

First, we define:

Definition 9.7 *A mapping $f : X \to Y$ is called fuzzy α-semicontinuous, if the inverse image of each open fuzzy set in Y is a fuzzy α-semiopen set in X.*

Clearly every fuzzy α-semicontinuous function is semi fuzzy continuous but the converse is not true as is clear from following:

Example 9.5 *Let (X, τ) and (Y, γ) be fts's with $X = \{a, b, c\}, Y = \{w, x, y, z\}$ and*

$$\begin{aligned}
\tau &= \left\{\widetilde{0}, \{a_{.4}, b_0, c_{.8}\}, \{a_{.3}, b_{.1}, c_{.2}\}, \{a_{.7}, b_{.6}, c_{.3}\}, \widetilde{1}\right\}, \\
\gamma &= \left\{\widetilde{0}, \{w_{.3}, x_{.7}, y_{.3}, z_{.2}\}, \{w_{.1}, x_{.5}, y_{.3}, z_{.8}\}, \{w_{.7}, x_{.2}, y_{.7}, z_{.2}\}, \{w_{.6}, x_{.8}, y_{.2}, z_{.6}\}, \widetilde{1}\right\}.
\end{aligned}$$

Then the mapping $f : X \to Y$ given as

$$f(a) = x \quad , \quad f(b) = y \quad , \quad f(c) = w$$

is semi fuzzy continuous but not fuzzy α-semicontinuous.

Proposition 9.2 *Let λ and μ be fuzzy sets in an fts X. Then*

(1) If $\lambda \in F\alpha SO(X)$, then there exists a semi open fuzzy set ν such that $\nu \leq \lambda \leq IntCl\nu$.

(2) If $\nu \in \tau$ such that $\nu \leq \lambda \leq IntCl\nu$, then $\lambda \in F\alpha SO(X)$.

(3) If $\lambda \in F\alpha SO(X)$ and $\lambda \leq \mu \leq IntCl\lambda$, then $\mu \in F\alpha SO(X)$. In particular, $IntCl\lambda$ is fuzzy α-semiopen.

Proof. (1) follows easily by taking $\nu = sInt\lambda$.

(2) By hypothesis, there exists a open fuzzy set ν such that $\nu \leq \lambda \leq IntCl\nu$ and hence $\nu \leq sInt\lambda \leq \lambda \leq IntCl\nu$. Since $\nu \leq sInt\lambda$, we have $IntCl\nu \leq IntClsInt\lambda$. Combining this with $\lambda \leq IntCl\nu$, we have $\lambda \leq IntClsInt\lambda$, that is, $\lambda \in F\alpha SO(X)$.

(3) Since $\lambda \in F\alpha SO(X)$,

$$\begin{aligned} \mu &\leq IntCl\lambda \leq IntClIntClsInt\lambda \\ &= IntClsInt\lambda \leq IntClsInt\mu. \end{aligned}$$

This shows that $\mu \in F\alpha SO(X)$. ■

Corollary 9.1 *$\lambda \in F\alpha SO(X)$ iff there exists $\nu \in \tau$ such that $\nu \leq \lambda \leq IntCl\nu$.*

The dual of Propositon 4.1 is the following, the proof of which is similar.

Proposition 9.3 *Let λ and μ be fuzzy sets in an fts X. Then*

(1) If $\lambda \in F\alpha SC(X)$, then there exists a semi fuzzy closed set ν such that $\nu \leq \lambda \leq sClInt\nu$.

(2) If $\nu \in \tau$ such that $\nu \leq \lambda \leq sClInt\nu$, then $\lambda \in F\alpha SC(X)$.

(3) If $\lambda \in F\alpha SC(X)$ and $\lambda \leq \mu \leq sClInt\lambda$, then $\mu \in F\alpha SC(X)$. In particular, $sClInt\lambda$ is fuzzy α-semiclosed.

The following theorem gives characterizations of fuzzy α-semicontinuous mappings:

Theorem 9.10 *Let $f : X \to Y$ be a mapping. Then the following are equivalent:*

(1) f *is fuzzy* α*-semicontinuous.*

(2) *for each fuzzy singleton* $p \in X$ *and each open fuzzy set* ν *in* Y *and* $f(p) \leq \nu$, *there exists a fuzzy* α*-semiopen set* ω *in* X *such that* $p \leq \omega$ *and* $f(\omega) \leq \nu$.

(3) *the inverse image of each closed fuzzy set in* Y *is fuzzy* α*-semiclosed in* X.

(4) $f(ClIntsCl\lambda) \leq Clf(\lambda)$, *for each fuzzy set* λ *in* X.

(5) $ClIntsClf^{-1}(\mu) \leq f^{-1}(Cl\mu)$, *for each fuzzy set* μ *in* Y.

Proof. (1) $\Rightarrow$ (2) Since ν is open fuzzy in Y and $f(p) \leq \nu$, then $p \leq f^{-1}(\nu)$ and by hypothesis, $f^{-1}(\nu)$ is a fuzzy α-semiopen set. Put $\omega = f^{-1}(\nu)$. Then $p \leq \omega$ and $f(\omega) \leq \nu$.

(2) $\Rightarrow$ (1) Let ν be a open fuzzy set in Y such that $p \leq f^{-1}(\nu)$. By hypothesis, there exists $\omega \in F\alpha SO(X)$ such that $p \leq \omega$ and $f(\omega) \leq \nu$. Then

$$p \leq \omega \leq f^{-1}(\nu) = \bigvee \{\omega \mid \omega \in F\alpha SO(X)\}.$$

Hence $f^{-1}(\nu) \in F\alpha SO(X)$ and therefore f is fuzzy α-semicontinuous.

(1) $\Rightarrow$ (3) Let ρ be a closed fuzzy set in Y. The ρ^c is open fuzzy . Hence $f^{-1}(\rho^c) \in F\alpha SO(X)$, or $(f^{-1}(\rho))^c \in F\alpha SO(X)$. Thus $f^{-1}(\rho)$ is a fuzzy α-semiclosed set in X.

(3) $\Rightarrow$ (4) Let λ be a fuzzy set in X. Then $Clf(\lambda)$ is a closed fuzzy set in Y,

so that $f^{-1}(Clf(\lambda))$ is fuzzy α-semiclosed in X. Thus we have

$$\begin{aligned} f^{-1}(Clf(\lambda)) &\geq ClIntsClf^{-1}(Clf(\lambda)) \\ &\geq ClIntsCl\lambda \\ \text{or } Clf(\lambda) &\geq f(ClIntsCl\lambda). \end{aligned}$$

(4) $\Rightarrow$ (5) Let μ be a fuzzy set in Y. Then $f^{-1}(\mu)$ is a fuzzy set in X. By hypothesis, we have

$$f\left(ClIntsClf^{-1}(\mu)\right) \leq Clf\left(f^{-1}(\mu)\right)$$

or $f(ClIntsClf^{-1}(\mu)) \leq Cl\mu$, that is, $ClIntsClf^{-1}(\mu) \leq f^{-1}(Cl\mu)$.

(5) $\Rightarrow$ (1) Let ν be a open fuzzy set in Y. Let $\omega = \nu^c$. By (5), we have $ClIntsClf^{-1}(\omega) \leq f^{-1}(Cl\omega) = f^{-1}(\omega)$, or $ClIntsClf^{-1}(\nu^c) \leq f^{-1}(\nu^c) = (f^{-1}(\nu))^c$. Therefore by definition $(f^{-1}(\nu))^c$ is fuzzy α-semiclosed. Hence $f^{-1}(\nu)$ is a fuzzy α-semiopen set in X. This gives (1). ■

Proposition 9.4 *If λ is a fuzzy set in an fts X, then*

$$sCl\lambda \geq \alpha sCl\lambda \geq ClIntsCl\lambda.$$

Proof. Choose $\mu = \alpha sCl\lambda$, then $\lambda \leq \mu$. Since μ is α-semiclosed, then $sClIntsCl\lambda \leq ClIntsCl\mu \leq \mu = \alpha sCl\lambda$. ■

We use characterizations (3) and (4) of Theorem 9.10 to prove:

Theorem 9.11 *A mapping $f : X \to Y$ is fuzzy α-semicontinuous if and only if $f(\alpha sCl\lambda) \leq Clf(\lambda)$, for each fuzzy set λ in X.*

Proof. $(\Rightarrow)$ Let $\psi = Clf(\lambda)$, for a fuzzy set λ in X and $\mu = f^{-1}(\psi)$. Then by Theorem 9.10(4) $f(ClIntsCl\mu) \leq Clf(\mu) \leq Clff^{-1}(\psi) \leq Cl\psi = \psi$. Hence $ClIntsCl\mu \leq f^{-1}(\psi) = \mu$ gives μ is α-semiclosed. Now $\lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}(\psi) = \mu$ gives $\alpha sCl\lambda \leq \alpha sCl\mu = \mu$. Thus we have $f(\alpha sCl\lambda) \leq f(\mu) \leq \psi = Clf(\lambda)$ as required.

$(\Leftarrow)$ Follows from Proposition 9.4. ■

We use characterization (5) of Theorem 9.10 to prove:

Theorem 9.12 *A mapping $f : X \to Y$ is fuzzy α-semicontinuous if and only if $\alpha Clf^{-1}(\mu) \leq f^{-1}(sCl\mu)$, for each fuzzy set μ in Y.*

Proof. $(\Rightarrow)$ Let $\psi = f^{-1}(Cl\lambda)$, for a fuzzy set λ in Y and $\mu = f(\psi)$. Then by Theorem 9.10(5)

$$ClIntsClf^{-1}(\mu) \leq f^{-1}(Cl\mu) = f^{-1}Clf(\psi) = f^{-1}Clf^{-1}f(Cl\lambda) \leq f^{-1}(Cl\lambda) = \psi.$$

. Hence $ClIntsCl\mu \leq f(\psi) = \mu$, so that μ is α-semiclosed. Now $\lambda \leq f(f^{-1}(\lambda)) \leq f(\psi) = \mu$, so that $\alpha Cl\lambda \leq \alpha Cl\mu = \mu$. Thus we have that $\alpha Clf^{-1}(\lambda) \leq f^{-1}(\mu) \leq \psi = f^{-1}(sCl\lambda)$ as required.

$(\Leftarrow)$ Follows from Proposition 9.4. ■

9.4 α-Semiopen and α-Semiclosed Fuzzy Mappings

Recall that a mapping $f : X \to Y$ is called semi open fuzzy [9] (resp. fuzzy preopen [50]), if the image of each open fuzzy set in X is semi open fuzzy (resp. fuzzy preopen) in Y. Also f is called semi fuzzy closed if the image of each closed fuzzy set in X is semi-closed fuzzy in Y. Every open fuzzy mapping is semi open fuzzy as well as fuzzy preopen.

Now we define:

Definition 9.8 *A mapping $f : X \to Y$ is called fuzzy α-semiopen (resp. fuzzy α-semiclosed) mapping, if the image of each open fuzzy (resp. fuzzy closed) set in X is a fuzzy α-semiopen (resp. fuzzy α-semiclosed) set in Y.*
Clearly every fuzzy α-semiopen (resp. fuzzy α-semiclosed) mapping is semi open fuzzy (resp. semi fuzzy closed). The converse is not true in general.

Example 9.6 *Let (X, τ) and (Y, γ) be fts's with $X = \{a, b, c\}, Y = \{w, x, y, z\}$ and*

$$\begin{aligned} \tau &= \left\{\widetilde{0}, \{a_{.5}, b_{.4}, c_{.5}\}, \{a_{.8}, b_{.5}, c_{.1}\}, \{a_{.7}, b_{.5}, c_{.8}\}, \widetilde{1}\right\}, \\ \gamma &= \left\{\widetilde{0}, \{w_{.5}, x_{.2}, y_{.3}, z_{.6}\}, \{w_{.1}, x_{.5}, y_{.1}, z_{.8}\}, \{w_{.7}, x_{.3}, y_{.7}, z_{.2}\}, \{w_{.6}, x_{.2}, y_{.2}, z_{.9}\}, \widetilde{1}\right\}. \end{aligned}$$

Consider the following mappings

$$f(a) = z \quad , \quad f(b) = x \quad , \quad f(c) = w,$$

$$g(a) = x \quad , \quad g(b) = z \quad , \quad g(c) = x,$$

$$h(a) = w \quad , \quad h(b) = w \quad , \quad h(c) = y.$$

Then calculations show that $f : X \rightarrow Y$ semi fuzzy pen mapping but not fuzzy α-semiopen. Also $g : X \rightarrow Y$ is open fuzzy but not fuzzy α-semiopen and $h : X \rightarrow Y$ is fuzzy α-semiopen but not open fuzzy mapping.

Theorem 9.13 *A mapping $f : X \rightarrow Y$ is fuzzy α-semiopen, if for each $x \in X$ and each semi open fuzzy set μ in X there exists a fuzzy α-open set ω in Y such that $\omega \leq f(\mu)$.*

Proof. Follows directly from definitions. ■

Theorem 9.14 *If a mapping $f : X \rightarrow Y$ is fuzzy α-semiclosed, then $\alpha sClf(\psi) \leq f(Cl\psi)$, for each fuzzy set ψ in X.*

Proof. If f is fuzzy α-semiclosed, then $f(Cl\psi)$ is a fuzzy α-semiclosed set with $f(\psi) \leq f(Cl\psi)$ and therefore by definition $\alpha sClf(\psi) \leq f(Cl\psi)$. ■

Theorem 9.15 *Let $f : X \rightarrow Y$ be a fuzzy α-semiopen (resp. fuzzy α-semiclosed) mapping. If ω is a fuzzy set in Y and μ is a closed fuzzy (resp. open fuzzy) set in X, such that $f^{-1}(\omega) \leq \mu$, then there exists a fuzzy α-semiclosed (resp. fuzzy α-semiopen) set ν in Y such that $\omega \leq \nu$ and $f^{-1}(\nu) \leq \mu$.*

Proof. Let $\nu = (f(\mu^c))^c$. Since $f^{-1}(\omega) \leq \mu$, we have $f(\mu^c) \leq \omega^c$. This gives $\omega \leq (f(\mu^c))^c = \nu$. Since f is fuzzy α-semiopen, then ν is a fuzzy α-semiclosed set. Then

$$\begin{aligned} f^{-1}(\nu) &= f^{-1}([f(\mu^c)]^c) = \left(f^{-1}(f(\mu^c))\right)^c \\ &\leq (\mu^c)^c = \mu. \end{aligned}$$

Using Theorem 5.3, we prove the following: ■

Corollary 9.2 *If $f : X \to Y$ is a fuzzy α-semiopen mapping, then*

$f^{-1}(ClIntsCl\mu) \leq Clf^{-1}(\mu)$, for each fuzzy set μ in Y.

Proof. (1) $Clf^{-1}(\mu)$ is a closed fuzzy set in X, with $f^{-1}(\mu) \leq Clf^{-1}(\mu)$, for μ in Y. By Theorem 9.15, there exists a fuzzy α-semiclosed set ρ in Y, and $\mu \leq \rho$ such that $f^{-1}(\rho) \leq Clf^{-1}(\mu)$. Thus

$$f^{-1}(ClIntsCl\mu) \leq f^{-1}(ClIntsCl\rho) \leq f^{-1}(\rho) \leq Clf^{-1}(\mu).$$

■

9.5 α-Irresolute Fuzzy Mappings

Recall that a mapping $f : X \to Y$ is said to be irresolute fuzzy [35] (resp. fuzzy pre-irresolute), if the inverse image of each semi open fuzzy (resp. fuzzy preopen) set in Y is semi open fuzzy (resp. fuzzy preopen) in X.

Definition 9.9 *A mapping $f : X \rightarrow Y$ is fuzzy α-irresolute, if the inverse image of every fuzzy α-semiopen set in Y is a fuzzy α-semiopen set in X.*

Definition 9.10 *[39] A function $f : X \rightarrow Y$ is said to be fuzzy almost open (resp. fuzzy almost closed) in Nanda's sense, briefly, f.a.o.N (resp. f.a.c.N.), if $f(\mu)$ is open fuzzy (resp. fuzzy closed) in Y, for each fuzzy regularly open (resp. fuzzy regularly closed) set μ in X.*

Clearly, every open fuzzy mapping is fuzzy almost open.

We use Corollary 9.1 and Proposition 4.1(3) to prove:

Theorem 9.16 *If a mapping $f : X \rightarrow Y$ is fuzzy almost open and fuzzy α-semicontinuous, then f is fuzzy α-irresolute.*

Proof. Let μ be a fuzzy α-semiopen set in Y. By Corollary 9.1, there exists a open fuzzy set ν in Y such that $\nu \leq \mu \leq IntCl\nu$. Since f is fuzzy α-semicontinuous, $f^{-1}(\nu) \in F\alpha SO(X) \subset FSO(X)$ and hence $f^{-1}(\nu) \leq ClIntf^{-1}(\nu)$. Put $\psi = \left(f\left(\left(ClIntf^{-1}(\nu)\right)^c\right)\right)^c$. Since f is fuzzy almost open and $ClIntf^{-1}(\nu)$ is fuzzy regularly closed, then ψ is closed fuzzy in Y . Routine calculations give $\nu \leq \psi$ and $f^{-1}(\psi) \leq ClIntf^{-1}(\nu)$. Thus, $f^{-1}(Cl\nu) \leq ClIntf^{-1}(\nu)$ which implies

$$\begin{aligned} f^{-1}(\nu) &\leq f^{-1}(\mu) \leq f^{-1}(IntCl\nu) \\ &\leq IntClsIntf^{-1}(\nu) \leq IntClf^{-1}(\nu). \end{aligned}$$

It follows from Proposition 9.2(3) that $f^{-1}(\mu) \in F\alpha SO(X)$. This shows that f is fuzzy α-irresolute. ■

It is known [?] that $sCl\lambda = \lambda \vee IntCl\lambda$ and hence $IntCl\lambda \leq sCl\lambda$. Also recall that a mapping $f : X \to Y$ is said to be semi open fuzzy , if the image of every open fuzzy set in X is semi open fuzzy in Y.

Theorem 9.17 *A mapping $f : X \to Y$ is semi open fuzzy if and only if $f^{-1}(sCl\lambda) \leq Clf^{-1}(\lambda)$, for every fuzzy set λ in X.*

Using Theorems 9.16, 9.17 and Proposition 9.2(3), we prove:

Theorem 9.18 *If $f : X \to Y$ is semi open fuzzy and fuzzy α-semicontinuous mapping, then f is fuzzy α-irresolute.*

Proof. Let μ be a fuzzy α-semiopen set in Y. By Proposition 9.2(1), there exists a semi open fuzzy set ν in Y such that $\nu \leq \mu \leq IntCl\nu$. Since f is fuzzy α-semicontinuous, $f^{-1}(IntCl\nu) \in F\alpha SO(X)$ .Thus

$$\begin{aligned} f^{-1}(IntCl\nu) &\leq IntClsIntf^{-1}(IntCl\nu) = IntClf^{-1}(IntCl\nu) \leq IntClf^{-1}(sCl\nu) \quad \text{(by Rema} \\ &\leq IntClf^{-1}(\nu). \quad \text{(by Theorem 6.2)} \end{aligned}$$

Therefore, we obtain $f^{-1}(\nu) \leq f^{-1}(\mu) \leq f^{-1}(IntCl\nu) \leq f^{-1}(sCl\nu) \leq Clf^{-1}(\nu)$ and $f^{-1}(\nu) \in F\alpha O(X)$. By Proposition 9.2(2), $f^{-1}(\mu) \in F\alpha SO(X)$. This shows that f is fuzzy α-irresolute. ■

9.6 Almost α-Continuous Fuzzy Mappings

Definition 9.11 *[60] A fuzzy set λ is said to be fuzzy feebly open, if there exists a open fuzzy set μ such that $\mu \leq \lambda \leq sCl\mu$. Clearly every fuzzy feebly* *open set is semi open fuzzy .*

Next, we define:

Definition 9.12 *A mapping $f : X \to Y$ is said to be fuzzy almost α-continuous (briefly f.a.α.c.), if $f^{-1}(\nu) \in F\alpha O(X)$, for every fuzzy regularly open set ν in Y.*

Definition 9.13 *A mapping $f : X \to Y$ is said to be fuzzy almost feebly continuous (resp. fuzzy feebly continuous), if $f^{-1}(\nu)$ is fuzzy feebly open in X, for every fuzzy regularly open (resp. open fuzzy) set ν of Y.*

Lemma 9.1 *[60] If $\lambda \in FPO(X)$, then $sCl\lambda = IntCl\lambda$.*

Next, we state the following, proof of which is straightforward:

Theorem 9.19 *For a mapping $f : X \to Y$, the following are equivalent:*

(1) *f is f.a.α.c.*

(2) *For each $x \in X$ and each open fuzzy set ν in Y with $f(x) \in \nu$, there exists $\mu \in F\alpha O(X)$ such that $x \in \mu$ and $f(\mu) \leq IntCl\nu$.*

(3) *$f^{-1}(\psi)$ is fuzzy α-closed in X, for every fuzzy regularly closed set ψ of Y.*

Definition 9.14 *[**?**] A mapping $f : X \to Y$ is said to be semi fuzzy -weakly continuous, if for each $x \in X$ and each open fuzzy set ν in Y with $f(x) \in \nu$, there exists semi open fuzzy μ in X with $x \in \mu$ such that $f(\mu) \leq sCl\nu$.*

Theorem 9.20 *A mapping $f : X \to Y$ is semi fuzzy -weakly continuous if and only if $f^{-1}(\nu) \in FSO(X)$, for every fuzzy regularly open set ν of Y.*

Proof. $(\Rightarrow)$ Let ν be a fuzzy regularly open set of Y. For each $x \in f^{-1}(\nu)$, there exists $\mu_x \in FSO(X)$ with $x \in \mu_x$ such that $f(\mu_x) \leq sCl(\nu)$. Since each fuzzy regularly open set is uzzy preopen,therefore, by Lemma 9.1, we have $sCl\nu = IntCl\nu = \nu$ and hence $x \in \mu_x \leq f^{-1}(\nu)$. Therefore, it follows from Theorem 4.3(a) [9] that $f^{-1}(\nu) \in FSO(X)$.

$(\Leftarrow)$ Let $x \in X$ and $f(x) \in \nu$, where ν is open fuzzy in Y. Put $\mu = f^{-1}(IntCl\nu)$, then by Lemma 9.1 we have $x \in \mu \in FSO(X)$ and $f(\mu) \leq ff^{-1}(IntCl\nu) \leq IntCl\nu = sCl\nu$. This shows that f is semi fuzzy -weakly continuous. ■

Chapter 10

Conclusion

Fuzzy Topology was defined by Chang [14] while following the footsteps of L.A. Zadeh's [59] in 1968. Since then much attention has been paid to generalize the basic concepts of Classical Topology in fuzzy setting and thus a modern theory of Fuzzy Topology has been developed.

Fuzzy Topology has been applied in solving many practical problems: for depicting topological relations in Geographic Information Systems (GIS) query [17][18, 19]. In [40, 41], El-Naschie showed that the notion of Fuzzy Topology is relevant to quantum particle physics and quantum gravity in connection with string theory and e^{∞} theory. Tang [54] used a slightly changed version of Chang's fuzzy topological space to model spatial objects for GIS databases and Structured Query Language (SQL) for GIS.

N. Levine [33] introduced the concepts of semi-open sets and semi-continuous mappings in topological spaces. Interestingly, his work found applications in the field of Digital Topology [46]. For example, it was found that digital line is a $T_{\frac{1}{2}}$-space [13], which is a weaker separation axiom based upon semi-open sets. Fuzzy Digital Topology [47] was introduced by A. Rosenfeld, which demonstrated the need for the fuzzification of weaker forms of notions of Classical Topology. Azad [9] carried out this fuzzification in 1981, and presented some general properties of fuzzy spaces. In this thesis we extend the study of fuzzy semi topology. We studied semi-continuous fuzzy, semi-open fuzzy and almost open fuzzy (Ganguly's sense) mappings. We also define and study properties of almost closed fuzzy mappings. We also study the characterizations of semi-open fuzzy (semi-closed fuzzy), semi-preopen fuzzy (semi-preclosed fuzzy), semi-precontinuous fuzzy and pre-semi-preopen fuzzy (pre-semi-preclosed fuzzy) mappings. Further we presented some properties of semi-open fuzzy sets defined and studied by Zhong [62], semi-preopen fuzzy sets and preopen fuzzy sets. It is also shown that in the class of injective functions, almost open fuzzy (closed) in Nanda's sense and almost quasi-compact fuzzy functions are equivalent. In terms of graph and projections, some interesting characterizations and properties of almost continuous fuzzy functions in Singal's sense are given. Moreover almost continuous fuzzy in Husain's sense, almost weakly continuous fuzzy,nearly almost open (closed) fuzzy functions have been defined and their several characterizations and properties have been obtained. Finally, their equivalences have been established under certain conditions.The properties of α-continuous mappings in terms of α-closure of fuzzy sets have been established. Several important fundamental identities and inequalities about semi-interior and semiclosure of fuzzy sets have been found. We define s-open and s-closed fuzzy mappings and establish some interesting characterizations of these mappings. It may be noted that the class of s-open (resp. s-closed) fuzzy mappings is a subclass of the class of open (resp. closed) fuzzy mappings. Finally simply continuous fuzzy mappings and α-semicontinuous fuzzy mappings have been characterized.

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