On Metric Dimension of Some Families of Graphs

Submitted by
Murtaza Ali

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Abstract

For a connected graph $G$ the distance $d(u,v)$ between two vertices $u, v \in V(G)$ is the length of the shortest path between them. A vertex $w$ of a graph $G$ is said to resolve two vertices $u$ and $v$ of $G$ if $d(w,u) \neq d(w,v)$. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of $G$ and let $v$ be a vertex of $G$. The representation of a vertex $v$ with respect to $W$ denoted by $r(v|W)$ is the $k$-tuple $(d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$. If distinct vertices of $G$ have distinct representations with respect to $W$, then $W$ is called a resolving set for $G$. The metric dimension of $G$ denoted by $dim(G)$, is the minimum cardinality of a resolving set of $G$.

Graph structure can be used to study the various concepts of Navigation in space. A work place can be denoted as vertex in the graph, and edges denote the connections between the places. The problem of minimum machines (or Robots) to be placed at certain vertices to trace each and every vertex exactly once is a classical one. This problem can be solved by using networks where places are interconnected in which, the navigating agent moves from one vertex to another in the network. The places or vertices of a network where we place the machines (robots) are called landmarks. The minimum number of machines required to locate each and every vertex of the network is termed the metric dimension and the set of all minimum possible number of landmarks constitute metric basis.

In this thesis, the metric dimension of some well known families of graphs has been investigated. It is shown that the families of graphs obtained from the path graph by the power, middle and total graph operation have a constant metric dimension. We compute the metric dimension of some rotationally symmetric families of graphs and show that only 2 or 3 vertices appropriately chosen suffice to resolve all the vertices of these graphs. In this thesis, we also compute the metric dimension of some families of convex polytopes with pendant edges. It is shown that the metric dimension of these families of graphs is constant and is independent of the order of these graphs.

The metric dimension of the splitting graphs of two families of graph has been computed. We prove that the metric dimension of these graphs is unbounded and depends on the order of the corresponding graph.
Chapter 1

Introduction

1.1 Motivation and problem statement

Navigation can be studied in a graph structured framework in which the navigation agent (which we shall assume to be a point robot) moves from node to node of a graph space. The robot can locate itself by the presence of distinctly labeled landmark nodes in the graph space. For a robot navigating in Euclidean space, visual detection of a distinctive landmark provides information about the direction to the landmark, and allows the robot to determine its position by triangulation. On a graph, however, there is neither the concept of direction nor that of visibility. Instead, we shall assume that a robot navigating on a graph can sense the distances to a set of landmarks, its position on the graph is uniquely determined. This suggests the following problem given a graph, what are the fewest number of landmarks needed, and where they should be located, so that the distances to the landmarks uniquely determines the robot’s position on the graph. This is actually a classical problem about metric spaces. A minimum set of landmarks which uniquely determines the robot’s position is called a metric basis, and the minimum number of landmarks is called the metric dimension of a graph. Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension was introduced by Slater in (Slater. 1975, Slater. 1998) and studied independently by Harary and Melter in (Harary et al. 1976). Slater refereed to the metric dimension of a graph as its location number and motivated the study of this in-variant by its application to the placement of a minimum number of sonar/loran detecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set. For a subset $S \subset V(G)$ and a vertex $v$ of a connected graph $G$, the distance $d(v,S)$ between $v$ and $S$ is defined as usually, by $d(v,S) = \min\{d(v,x) : x \in S\}$. If $\Pi = (S_1,S_2,...,S_n)$ an ordered $k$-partition of $V(G)$, the representation of $v$ with respect to $\Pi$ is the $k$- tuple $r(v|\Pi) = (d(v,S_1),d(v,S_2)...d(v,S_k))$. If the $k$-tuples $r(v|\Pi)$ for $v \in V(G)$ are all distinct, then the partition $\Pi$ is called a resolving partition and the minimum cardinality of a resolving partition of $V(G)$ is called the partition dimension of $G$ and is denoted by $pd(G)$. These concepts have some applications in chemistry for representing
chemical compounds (G. Chartrand et al. 2005 and Johnson. 1993) or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures (Melter et al. 1984). We consider various problems for our future work. Firstly we consider the extensions of some families of graphs whose metric dimension is already calculated, namely by total graphs, middle graphs, line graphs and splitting graphs. We investigate the metric dimension of these graphs. As an extension of Theorem 7 and 8 in (S. Khuller et al. 1996).

**Contribution**

From theorem (2.4) it is understandable that computing the metric dimension of a graph is NP-hard. The area is still open for research and many researchers have worked in this area not only from the point of view of its theoretical development but also due to its wide range of applications in Chemistry, Engineering and many other branches of science. Graph structure can be used to study the various concepts of Navigation in space. A work place can be denoted as node in the graph, and edges denote the connections between the places. The problem of minimum machines (or Robots) to be placed at certain nodes to trace each and every node exactly once is a classical one. This problem can be solved by using networks where places are interconnected in which, the navigating agent moves from one node to another in the network. The places or nodes of a network where we place the machines (robots) are called landmarks. The minimum number of machines required to locate each and every node of the network is termed the metric dimension and the set of all minimum possible number of landmarks constitute metric basis.

The machines, where they are placed at nodes of the network, know their distances to sufficiently large set of landmarks and the positions of these machines on the network are uniquely determined. However there is neither the concept of direction nor that of visibility. Instead we shall assume that a Robot navigating on a graph can sense the distance to a set of landmarks.

Caceres et al. (Caceres et al. 2007) proved an interesting results on cartesian product of some well known families of graphs. They proved that these graphs have constant metric dimension. Javaid et al. (Javaid et al. 2008) determined the metric dimension of some well known families of regular graphs. They have shown that these graphs have constant metric dimension. Khuller et al. (Khuller et al. 1996) proved a nice property about the families of graphs with metric dimension 2. Chartrand et al. (Chartrand et al. 2000) proved that a graph has metric dimension one if and only if it is isomorphic to a path. Therefore, paths on \( n \) vertices constitute a family of graphs with constant metric dimension. Imran et al. (Imran et al. 2010, 2012, in press) extended the work for some new families of graphs which have constant metric dimension. They found the metric dimension of flower graph, convex polytopes, circulant graph and generalized Petersen graphs. They found that these graphs have constant metric dimension. Iswadi et al. (Iswadi et al. 2008) and Yero et al. (Yero et al. 2011) computed the metric dimension of corona product of graphs. Feng et al. (Feng et al. 2012) found metric dimension of line graph \( L(G) \) of digraph. This thesis attempted to further explore this area of study by considering the graphs defined in (Baca 1988, 1992, Imran et al. 2012, Iswadi et al. 2008, Karliraj et al. 2010).
Computing the metric dimension of families of graphs with constant metric dimension is particularly an area of interest to many researchers. Various authors have contributed to this specific problems since the emergence of this notion. In this thesis, we also focus on the computation of metric dimension of families of graphs having constant metric dimension. In particular, we compute the metric dimension of families of graphs having small metric dimension equal to 2 or 3.

We also focus on the computation of metric dimension of graphs with unbounded metric dimension. We compute the metric dimension of two families of graphs and show that their metric dimension is unbounded.

1.1.1 Organization of Thesis

This thesis is divided into eight chapters. The first two chapters consist of basic concepts and terminology of graphs and distances in graphs. In the third chapter, Contains the literature of the research topic of the thesis. A brief Survey related to the topic is also presented in chapter 3.

The author’s contribution to the area of metric dimension starts with Chapter 4. In Chapter 4 the metric dimension of the square, cube, middle and total graphs of the path graph has been determined. The metric dimension of the graph $P_n(1, 2, 3)$ and $H_n$ defined in the previous section, has also been computed in Chapter 4.

Chapter 5 is devoted to the computation of the metric dimension of rotationally symmetric families of graphs. In this chapter, it is shown that the metric dimension of these families of graphs is independent of the order of the graph.

In Chapter 6 we have studied the metric dimension of some new class of convex polytopes with pendant edges and it is shown that these families have a constant metric dimension.

Chapter 7 deals with the metric dimension of the splitting graphs of two families of graphs. The metric dimension of these graphs is shown to be unbounded and depends on the order of the graph.

Chapter 8 deals with final conclusion and some open problems for future research.
Chapter 2

Basic Notions and Methodology

2.1 Introduction

2.2 Basic Definitions

2.2.1 Preliminaries

A graph $G(V, E)$ consists of two types of elements vertices $V(G)$ and edges $E(G)$. Vertex is defined as a node and an edge is the link between two nodes. The number of vertices in a graph $G$ is called the order of the graph, usually denoted by $n$, and the number of edges in a graph $G$ is called the size of the graph commonly denoted by $m$. A graph $G$ of order $n$ and size $m$ is commonly denoted by $G(n, m)$. An edge starting and ending at a vertex $v$ is called a loop. An edge is called a multiple edge if there is another edge with the same end vertices. The multiplicity of an edge is defined as the number of multiple edges joining the same end vertices. The multiplicity of a graph $G$ is the maximum multiplicity of its edges. A graph with no multiple edges or loops is called a simple graph. A graph is a multi-graph if it has multiple edges and no loops, and a pseudograph if it contains both multiple edges and loops. A graph with no edge is called an empty graph. Two graphs $G$ and $H$ are isomorphic if there is a bijective function $f : V(G) \to V(H)$ such that for all $v, w \in V$, and $vw \in E$ if and only if $f(v)f(w) \in F$. An automorphism of a graph $G$ is a mapping $f : V(G) \to V(G)$ such that the edge vertex connectivity is preserved (Kerst and Donald William 2007). A graph $G$ is vertex transitive if for each pair of vertices $v_i, v_j \in V(G)$ there exists an automorphism $f : V(G) \to V(G)$ such that $f(v_i) = v_j$. An edge or vertex is incident to a vertex $u$ if $u$ is one of the endpoints. For a connected graph $G$ of order $n$ and size $m$ the incidence matrix $A = (a_{ij})$ is an $n \times m$ matrix with $a_{ij} = 1$ if the vertex $v_i$ and edge $e_j$ are incident and $0$ otherwise. Two vertices $v_i, v_j \in V(G)$ are said to be adjacent in $G$ if and only if $v_iv_j \in E(G)$. The degree of a vertex $v$ is defined as the number of edges adjacent to vertex $v$. Maximum degree of $G$ is the maximum degree of any vertex of $G$, denoted by $\Delta(G)$. Minimum degree of $G$ is the minimum degree of any vertex of $G$, denoted by $\delta(G)$. For a connected graph $G$ of order $n$ and size $m$ the adjacency matrix $A = (a_{ij})$ is an $n \times n$ matrix with $a_{ij} = 1$ if the vertex $v_i$ and vertex $v_j$ are adjacent and $0$ otherwise.
Pairwise non-adjacent vertices or edges are called independent. More formally, a set of vertices or edges is independent (stable) set if no two of its elements are adjacent. A relation \( R \) on a set \( S \) is irreflexive provided that no element is related to itself; in other words \( xRx \) for no \( x \) in \( S \). A mathematical object is said to be symmetric if it is invariant (looks the same) under a symmetry transformation. A function matrix etc., is symmetric if it remains unchanged in sign when indices are reversed. For example, \( A_{ij} = A_{ji} \) is symmetric. The set \( E \) of edges of a simple graph \( G = (V,E) \) being a set of unordered pairs of elements of \( V \), constitutes an adjacency relation on \( V \). Formally, an adjacency relation is any relation which is irreflexive and symmetric.

The subdivision of an edge \( e = uv \) yields a graph containing one new vertex \( w \) and replacing \( e \) by two new edges \( uw \) and \( vw \). A graph \( H \) is said to be a subdivision, or topological minor of a graph \( G \) if \( H \) is obtained from \( G \) by subdividing some of its edges, that is, by replacing the edges by paths having at most their end vertices in common.

A graph \( H \) is said to be a subgraph of a graph \( G \) if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \) and whose adjacency relation is a subset of that of \( G \) restricted to this subset. A subgraph \( H \) of a graph \( G \) is said to be induced subgraph if for any pair of vertices \( x \) and \( y \) of \( H \), \( xy \in E(H) \) if and only if \( xy \in E(G) \). In other words, \( H \) is an induced subgraph of \( G \) if it has all the edges that are in \( G \) over the same vertex set. A subgraph \( H \) is a spanning subgraph or factor of a graph \( G \) if it has the same vertex set as \( V(G) \). A graph \( G \) is connected if for any two vertices \( v, u \in V(G) \), there is a path whose end points are \( v \) and \( u \). The connectivity \( K(G) \) of a connected graph \( G \) is the minimum number of vertices whose removal disconnects \( G \). When \( K(G) \geq k \), the graph is said to be \( k \)-connected.

A graph \( G \) is called to be separable if it is either disconnected or can be disconnected by deleting one vertex. A graph \( G \) is completely separable if every induced subgraph of \( G \) is separable. A maximal connected subgraph of \( G \) is called a component of \( G \). A vertex which separates two other vertices of the same component is called a cut-vertex. Bridge is an edge which divides a graph in two components. A graph \( G \) is called regular if all vertices of the graph have the same the degree. A 3- regular graph is called cube (Diestel 2005).

An articulation point in a connected graph is a vertex that, if removed would split the graph into two or more connected components. Biconnected graph is a graph with no articulation point. In other words, a graph is biconnected if and only if any vertex is removed, the graph remains connected (Kerst and Donald William 2007).

A path is a non empty graph \( P_n = (V, E) \) with the vertex set \( V(P_n) = \{x_0, x_1, \ldots, x_k\} \) and edge set \( E(P_n) = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\} \).

Distance between two vertices \( v_i \) and \( v_j \) in a connected graph \( G \) is the number of edges in the shortest path joining them. The eccentricity of a vertex \( v \) in a connected graph \( G \) is the maximum distance between \( v \) and any other vertex of \( u \in V(G) \). The maximum eccentricity among all the vertices is called diameter of the graph.
**Metric independence** A collection of pairs of vertices of $G$, no two of which are resolved by the same vertex, is called an independently resolved collection of pairs. Let $V(G) = V_0 \cup V_1 \cup \ldots \cup V_k$ be a partition of the vertex set of $G$. The partite sets $V_0, V_1, V_2, \ldots, V_k$ are called distance partite sets with reference to a vertex $v \in V(G)$ if $V_0 = \{v\}$ and $V_i$ contains those vertices which are at distance $i$ from $v$ for $0 < i < k$ where $k$ is the eccentricity of $v$ in $G$. The *radius* $r$ of a graph is the minimum eccentricity of any vertex.

### 2.2.2 Operations on Graphs

The **complement** of a graph $G$ denoted by $\bar{G}$ is defined as: $V(\bar{G}) \equiv V(G)$ and for any two vertices $x, y \in V(\bar{G})$, $xy \in E(\bar{G})$ if and only if $xy \notin E(G)$. The **union** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ denoted by $G \equiv G_1 \cup G_2$ with $V(G) = V_1 \cup V_2$ and $E(G) = E_1 \cup E_2$. The **join** of two graphs $G_1$ and $G_2$ denoted by $G_1 + G_2$ is defined as $V(G_1 + G_2) = V_1 \cup V_2$ and $E(G_1 + G_2) = E_1 \cup E_2 \cup \{xy|x \in V_1 \text{ and } y \in V(G_2)\}$.

For any connected graph $G$, we write $mG$ for the graph with $m$ components each isomorphic to $G$.

The **Cartesian product** of two graphs $G_1$ and $G_2$ denoted by $G_1 \times G_2$ is a graph with vertex set $V(G_1 \times G_2) = \{(v_1, u_1) | v_1 \in V(G_1) \text{ and } u_1 \in V(G_2)\}$ and an edge $(v_1, u_1)(v_2, u_2)$ exists in $G_1 \times G_2$ if and only if either $v_1 = v_2$ and $u_1u_2 \in E(G_2)$ or $u_1 = u_2$ and $v_1v_2 \in E(G_1)$. The **power** of a graph $G$ denoted by $G^k$ is a graph with $V(G^k) = V(G)$ and $E(G^k) = E(G) \cup \{v_iv_j|d(v_i, v_j) = k, \ v_i, v_j \in V(G)\}$.

Let $G$ be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. To each edge $e_i \in E(G)$ we associate a vertex denoted by $u_i$, thus producing another set of vertices $V'(G) = \{u_1, u_2, \ldots, u_m\}$. For a graph $G$, the **middle** graph $M(G)$ is defined as: $V(M(G)) = V(G) \cup V'(G)$ where two vertices are adjacent if and only if they are either adjacent edges of $G$ or one is a vertex and the other is an edge incident with it (Karlarj et al. 2010).

Let $G$ be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. To each edge $e_i \in E(G)$ we associate a vertex denoted by $u_i$, hence we have a new set of vertices $V'(G) = \{u_1, u_2, \ldots, u_n\}$. For a graph $G$, the **total** graph $T(G)$ is defined in the following way: $V(T(G)) = V(G) \cup V'(G)$. Two vertices $v_i, v_j$ are adjacent in graph $T(G)$ when one of the following hold: (i) $v_i, v_j \in V(G)$ and $v_i \sim v_j$ in $G$ (ii) $u_i, u_j \in V'(G)$ and $u_i \sim u_j$ in $G$ where $\sim$ use for adjacent vertices in $G$ (iii) $v \in V(G)$, $u \in V'(G)$ and $v, u$ are incident in $G$ (Karlarj et al. 2010).

The **splitting** graph of a graph $G$ is defined as: for each vertex $u_i \in V(G)$ $i = 1, 2, \ldots, n$ introduce a new vertex $v_i$. Join $v_i$ to all vertices of $G$ that are adjacent to $u_i$. The graph thus obtained is called the splitting graph of graph $G$ denoted by $S(G)$. If $V(G) = \{u_1, u_2, \ldots, u_n\}$ then $V(S(G)) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$. For simplicity, we write $V(S(G)) = S_1 \cup S_2$ where $S_1 = \{u_1, u_2, \ldots, u_n\}$ and $S_2 = \{v_1, v_2, \ldots, v_n\}$.

Let $G$ be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. The **line graph** of a graph $G$ is denoted by $L(G)$, the vertex set of $L(G)$ is $V(L(G)) = E(G)$.
where two vertices of \( L(G) \) is adjacent iff they are adjacent edges in \( G \).

### 2.2.3 Some Important Families of Graphs

A **tree** is an undirected graph in which any two vertices are connected by a unique path. The **path** \( P_n \) is a tree with exactly two vertices of degree 1, and the remaining \( n - 2 \) vertices of degree 2. A **cycle** graph \( C_n \) is a graph in which every vertex has degree 2 and that the first vertex is adjacent to the last. A **bipartite** graph is a graph whose vertex set can be partitioned into two disjoint sets \( X \) and \( Y \) such that every edge connects a vertex in \( X \) to one in \( Y \) that is, \( X \) and \( Y \) are each independent sets. Equivalently, a bipartite graph is a graph that does not contain any odd length cycles (Kerst and Donald 2007).

The **Kneser graph** \( K(n, k) \) is a graph whose vertices represent the \( k \)-subsets of \( \{1, 2, ..., n\} \) where two vertices are adjacent if and only if they correspond to disjoint subsets. \( K(n, k) \) therefore has \( \binom{n}{k} \) vertices and is a regular graph of degree \( \binom{n - k}{k} \).

A connected graph \( G \) is **distance-regular** if for any vertices \( x \) and \( y \) of \( G \) and any integers \( i, j = 0, 1, ..., d \) (where \( d \) is the graph diameter), the number of vertices at distance \( i \) from \( x \) and distance \( j \) from \( y \) depends only on \( i, j \), and the graph distance between \( x \) and \( y \), independently of the choice of \( x \) and \( y \). Hamming graph \( H(n, k) \) is a distance-regular graph with diameter \( d \). For integers \( d > 1 \) and \( r > 1 \), the Hamming graph \( H(d, r) \) is a graph whose vertex set is the cartesian product of \( d \) copies of size \( r \), where two vertices are adjacent if they differ in precisely one coordinate.

The graph \( P_n(1,2,3) \) is a graph with \( n \) vertices \( \{v_1, v_2, ..., v_n\} \) and \( E(P_n(1,2,3)) = \{v_iv_{i+1}, v_iv_{i+2}, v_{i+1}v_{i+3}: i = 1, 2, ..., n\} \) where \( n + i \) is taken modulo \( n \).

The **prism** graph is obtained from the cartesian product of \( P_2 \) with a cycle \( C_n \), and is denoted by \( D_n \). The **antiprism** graph \( A_n, n \geq 3 \) consists of an outer \( n \)-cycle \( a_1a_2...a_n \) and inner \( 2n \)-cycle \( b_1b_2...b_n \) and a set of \( n \) spokes \( a_ib_i \) and \( a_ib_{i+1}, i = 1, 2, ..., n \) where \( n + i \) taken modulo \( n \) (Gallian 2007).

The graph \( H_n \) is constructed from the graph \( A_n \) as follows: delete all the edges \( \{a_ia_{i+1}: i = 1, 2, ..., n\} \) of \( E(A_n) \) and \( n + i \) is taken modulo \( n \). For the set of vertices \( \{b_1, b_2, ..., b_n\} \), introduce a new set of vertices as \( \{c_1, c_2, ..., c_n\} \). For the graph \( H_n \), the vertex set \( V(H_n) = \{a_1, a_2, ..., a_n\} \cup \{b_1, b_2, ..., b_n\} \cup \{c_1, c_2, ..., c_n\} \) and \( E(H_n) = E(A_n) \setminus \{a_ia_{i+1}: i = 1, 2, ..., n\} \cup \{bc_i| i = 1, 2, ..., n\} \) where \( n + i \) is taken modulo \( n \).

The graph \( A_n^* \) is defined as follows: For each vertex \( a_i, i = 1, 2, ..., n \) of the outer cycle of the antiprism graph, we introduce a new vertex \( c_i, i = 1, 2, ..., n \). For the graph \( A_{2n}^* \) the vertex set \( V(A_{2n}^*) = \{a_1, a_2, ..., a_n\} \cup \{b_1, b_2, ..., b_n\} \cup \{c_1, c_2, ..., c_n\} \) and \( E(A_{2n}^*) = E(A_{2n}) \cup \{a_ic_i| i = 1, 2, ..., n\} \cup \{a_{i-1}c_i| i = 1, 2, ..., n\} \) where \( n \) and \( n + i \) taken modulo \( n \). For all \( i = 1, 2, ..., n \), \( \{b_1, b_2, ..., b_n\} \) are the inner cycle vertices \( \{a_1, a_2, ..., a_n\} \) are the outer cycle vertices and \( \{c_1, c_2, ..., c_n\} \) are the vertices adjacent to the outer cycle vertices of \( A_n \). We define the graph \( A_{2n}^* \) as follows: For each vertex \( a_i, i = 1, 2, ..., n \) of the outer cycle of the antiprism graph, we introduce a new vertex \( c_i \). For graph \( A_{2n}^* \), \( V(A_{2n}^*) = \{a_1, a_2, ..., a_n\} \cup \{b_1, b_2, ..., b_n\} \cup \{c_1, c_2, ..., c_n\} \) and \( E(A_{2n}^*) = E(A_{2n}) \cup \{a_ic_i| i = 1, 2, ..., n\} \) where \( n \) and \( n + i \) taken modulo \( n \).

Where superscript \( n \) of \( A_{2n}^* \) denotes the number of pendant vertices. For all \( i = 1, 2, ..., n \), \( \{b_1, b_2, ..., b_n\} \) are inner cycle vertices and \( \{a_1, a_2, ..., a_n\} \) are outer cycle vertices and \( \{c_1, c_2, ..., c_n\} \) are pendant vertices to outer cycle.
The graph $S_n'$ is constructed from the prism graph $D_n$ by deleting the edges $a_ia_{i+1}$ from $E(D_n)$ for $i = 1, 2, ..., n$ where $n + i$ is taken modulo $n$.

The graph $S_n''$ is obtained from the antiprism graph by deleting the edges $a_ia_{i+1}$ from $E(A_n)$ for $i = 1, 2, ..., n$ and the vertex $a_n$ from $V(A_n)$.

The graph $S_n^*$ is the extension of the graph $S_n'$ as follows: we introduce two new vertices $x, y$ and two new edges $xb_1, yb_n$. Relabel the vertices of $S_n^*$ as $\{u_i = b_i | i = 1, 2, ..., n\}$ and $\{x = v_1, a_1 = v_2, ..., y = v_{n+1}\}$.

A circulant graph is a graph of $n$ vertices $\{V \cup \{D\}\}$ where $n = 1, 2, ..., n$ and the edge set is $E(D_n') = E(D_n) \cup \{a_{i+1}b_i | i = 1, 2, ..., n\}$ where $n + i$ is taken modulo $n$ and $a_0 = b_0 = b_n$. Here $\{a_i\}_{i=1}^n$ are the inner cycle vertices, $\{b_i\}_{i=1}^n$ are outer cycle vertices and $\{a_i\}_{i=1}^n$ are adjacent vertices to outer cycle.

The graph $D_n''$ as an extension of the prism graph defined as follows: For the set of vertices $\{b_1, b_2, ..., b_n\}$ introduce a new set of vertices as $\{a_1, a_2, ..., a_n\}$. For the graph $D_n''$ the vertex set is $V(D_n'') = \{a_1, a_2, ..., a_n\} \cup \{b_1, b_2, ..., b_n\} \cup \{c_1, c_2, ..., c_n\}$ and edge set is $E(D_n'') = E(D_n) \cup \{a_{i+1}b_i | i = 1, 2, ..., n\}$ where $n + i$ is taken modulo $n$ and $a_0 = b_0 = b_n$. Here $\{c_i\}_{i=1}^n$ are the inner cycle vertices, $\{b_i\}_{i=1}^n$ are outer cycle vertices and $\{a_i\}_{i=1}^n$ are auxiliary vertices to outer cycle.

The graph $2C_n + \{a_nb_n\}$ is a family of graphs of order $2n$ obtained from a prism $D_n$ by deleting the spokes $a_ibi$ for $i \in \{1, 2, ..., n - 1\}$.

Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ and $V(P_m) = \{u_0, u_1, ..., u_m\}$. The graph $T_{n,m}$ is a graph of order $n + m$ obtained by joining $v_1$ of $C_n$ with $u_0$ of $P_m$.

Polytope is a geometric object with flat sides, which exists in any general number of dimension. A polygon is a polytope in two dimension, a polyhedron in three dimension. An object is convex set if for every pair of points within the object, every point on the straight line segment that joins them is also within the object. A convex polytope is a special case of a polytope, having the additional property that it is also a convex set of points in the $n$-dimensional space $\mathbb{R}^n$ (Kerst and Donald William 2007).

### 2.2.4 Digraph and some families of digraph

A digraph also known as a directed graph is a graph consisting of a finite set $V$ of vertices and a set $A$ of ordered pairs of distinct vertices called arcs. In a digraph, the number of arcs adjacent to a vertex is the indegree of the vertex, and the number of arcs adjacent from a vertex is the outdegree of that vertex (Balakrishnan 2004). A strongly connected
digraph is a directed graph in which it is possible to reach any node starting from any other node by traversing edges in the directions in which they point. The nodes in a strongly connected digraph therefore must all have indegree at least 1.

The De-Bruijn digraph $B(n,m)$ has the words of length $n \geq 1$ over an alphabet of size $m \geq 1$ as vertices and the adjacency rules are given by: $x_1x_2...x_n \rightarrow y_1y_2...y_n$ if and only if $y_i = x_{i+1}$, for $i = 1,2,...,n-1$. The Kautz digraph $K(n,m)$, $m \geq 2$ is defined as the sub-digraph of $B(n,m)$ induced by the words with different consecutive symbols (Ruiz et al. 2005).

A Cayley digraph, or directed graph, is a graph which is a visual representation of a group. Formally, we say that for any group $G$ and a set of generators $S$ of $G$, we define $Cay(S : G)$ as follows:

1).The elements of $G$ are the vertices of $Cay(S : G)$.
2).For $x$ and $y$ in $G$, there is an arc from $x$ to $y$ if and only if $xs = y$ for some $s \in S$.

There are 4 properties of Cayley digraphs:

1). The digraph is connected that can be get from any vertex $g$ to any vertex $h$ by traveling along consecutive arcs, starting from $g$ and ending to $h$.

**Reasoning:** Every equation $gx = h$ has a solution in a group.

2). At most one arc goes from a vertex $g$ to a vertex $h$.

**Reasoning:** The solution of $gx = h$ is unique.

3). Each vertex $g$ has exactly one arc of each type starting from $g$, and one of each type ending at $g$.

**Reasoning:** For $g \in G$ and each generator $b$ can be computed $gb$, and $(gb^{-1})b = g$.

4). If two different sequences of arc types starting from vertex $g$ lead to the same vertex $h$, then those same sequences of arc types starting from any vertex $u$ will lead to the same vertex $v$.

**Reasoning:** If $qg = h$ and $rg = h$, then $uq = ug^{-1}h = ur$ (Gallian, Joseph A 1986).

### 2.2.5 Summery

The area is still open for research and many researchers have worked in this area not only from the point of view of its theoretical development but also due to its wide range of applications in Chemistry, Engineering and many other branches of science. Graph structure can be used to study the various concepts of Navigation in space.
Chapter 3

Literature Review

3.1 General Idea About Metric Dimension:

3.1.1 Introduction

Since its emergence the metric dimension of graphs has been a hot area of research and has attracted Mathematicians all over the world. The most recent survey on the metric dimension of graphs can be found in (Saenpholphat et al. 2004). In this chapter, we present some of the contributions of various authors to the field of metric dimension and related areas of graphs. These results would be helpful for understanding the basic idea of the metric dimension of graphs. Here all graphs are undirected, simple, finite and connected.

For any two vertices \( v_i, v_j \in V(G) \) the shortest path between \( v_i \) and \( v_j \) is called a geodesic. The distance function in graph \( G \) is represented by \( \phi_d \) is defined as a metric on the vertex set \( V(G) \) of a (labeled) graph \( G \) which satisfies the following inequality for the vertices \( v_i, v_j, v_k \in G \):

**Theorem 3.1.** (Goddard et al. 2011) For all vertices \( v_i, v_j, v_k \in G \), \( \phi_d(v_i, v_j) \leq \phi_d(v_i, v_k) + \phi_d(v_k, v_j) \).

The maximum distance of a vertex \( v \) to any other vertex \( u \) of the graph \( G \) is called the eccentricity of vertex \( v \). For any connected graph \( G \), the diameter of \( G \) is defined as the maximum distance between any two vertices and is represented by \( \text{diam}(G) \). The minimum eccentricity among all vertices of a connected graph \( G \) is called the radius of the graph \( G \) and is represented by \( \text{rad}(G) \). Alternatively, it can be defined that diameter as the maximum eccentricity among all vertices of a graph \( G \). Peripheral vertices of a graph \( G \) are the vertices with maximum eccentricity. Vertices of a graph \( G \) with minimum eccentricity form the center of the graph \( G \). A tree has at most two centers (Goddard et al. 2011). The following inequality holds for any undirected and connected graph \( G \):

**Theorem 3.2.** (Goddard et al. 2011) For a connected graph \( G \), \( \text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G) \).
3.1 General Idea About Metric Dimension:

The upper bound follows from the triangular inequality (Goddard et al. 2011).
A block is defined as the maximal biconnected subgraph of a connected graph $G$. Harary and Norman (Harary and Norman 1953) showed the following result:

**Lemma 3.1.** (Harary and Norman 1953) For a connected graph $G$, the center of $G$ is a connected subgraph of $G$ and this connected subgraph of $G$ lies inside a block of $G$.

A famous result was proved by Jordan and Camille (Jordan and Camille 1869).

**Lemma 3.2.** (Jordan and Camille 1869) Let $G \cong T$, then $\text{diam}(G) = 2\text{rad}(G)$ or $2\text{rad}(G) - 1$. In the first case the center is a single vertex while in the second the center is a pair of adjacent vertices.

In general, there are no structural restrictions on the centre of a graph. Buckley et al. (Buckley et al. 1981) showed that every graph is the centre of some graph. Deleting an edge from a graph $G$ cannot decrease the radius or the diameter of the graph. Indeed removing a bridge disconnects the graph.

Howorka and Edward (Howorka and Edward 1977) defined a graph $G$ to be distance hereditary if for each connected induced subgraph $H$ of $G$ and every pair $x, y$ of vertices in $H, d_H(x, y) = d_G(x, y)$. Also they provided several conditions which characterize distance hereditary graphs:

**Theorem 3.3.** (Howorka and Edward 1977) For a graph $G$ the following are equivalent:

1. $G$ is a completely separable graph.
2. In $G$, every induced path is a shortest path in $G$.
3. Every subpath of a cycle graph of greater than half of cycle length is induced.
4. Every cycle $C$ of length greater or equal to 5 has two chords $e_1, e_2$ of cycle $C$ such that $E(C_n) + \{e_1, e_2\}$ is homeomorphic to $K_4$.

Distances have interesting applications in graph theory. One such application is to uniquely locate the position of a vertex $x$ in a network using distances. A vertex $x$ resolves a pair $y, z$ of vertices in a connected graph $G$ if $d(x, y) \neq d(x, z)$. A set of vertices $W$ is a resolving set of $G$ if every pair of vertices in $G$ is resolved by some vertex of $W$. A resolving set of minimum cardinality is called a metric basis of $G$ and its cardinality is the metric dimension of $G$, denoted by $\text{dim}(G)$. The metric dimension of a graph was firstly introduced independently by Slater (Slater 1975), and Harary et al. (Harary et al. 1976). Slater referred to the metric dimension as the location number. The study of metric dimension was motivated by its application to the placement of a minimum number of sonar detecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the detecting devices. A problem in pharmaceutical chemistry also led to the discovery of the notion of a resolving set of a graph independently (Chartrand et al. 2000). Garey and Johnson (Garey and Johnson 1979) have shown that the problem of computing the metric dimension of a graph is NP-hard. The formula for the metric dimension of trees was computed independently by several authors (Chartrand et al. 2000; Harary et al. 1976; Slater 1975). The metric dimension of a nontrivial path was found to be 1 as any vertex of degree 1 resolves its vertices. Suppose
now $T$ is a tree that contains vertices of degree at least three. A vertex $x$ of degree at least three is an exterior vertex if there is some leaf $y \in T$ such that the $x - y$ path of $T$ contains no vertices of degree exceeding two except for $x$. Let $e_x(T)$ denote the number of exterior vertices of $T$ and $\ell(T)$ the number of leaves of $T$ and $\sigma(T)$ denote the sum of the terminal degrees of the major vertices of $T$. It turns out that a metric basis for a tree can be found by selecting for each exterior vertex all but one of its exterior leaves. Díaz et al. (Díaz et al. 2012) explained that the metric dimension of graph has been apart of research for almost 40 years. The computational complexity of determining the metric dimension of a graph is still very open area for research.

**Theorem 3.4.** (Chartrand et al. 2000) For a connected graph $G$, $\dim(G) = \ell(G) - e_x(G)$ if $G \cong T$.

**Theorem 3.5.** (Díaz et al. 2012) For general graphs metric dimension is NP-hard.

Apart from trees very few exact results for the metric dimension of graphs are known unless the graphs are highly structured (usually vertex transitive). It was claimed (Karliraj et al. 2010) that the metric dimension of the cartesian product of $s$ paths is $s$ but indeed it was only verified that $s$ is an upper bound in this case. Sebő et al. (Sebő et al. 2004) have shown a connection between the solution of a coin evaluating problem and the metric dimension of the $n$-cube represented by $Q_n$. P. Erdös and A. Rényi (Erdös and A. Rényi 1963) showed the following result:

**Theorem 3.6.** (Erdös and A. Rényi 1963) For an $n$-cube graph $Q_n$, $\lim_{n \to \infty} \dim(Q_n) \frac{\log n}{n} = 2$.

### 3.1.2 Connected Resolvability of Graphs

Saenpholphat et al. (Saenpholphat et al. 2003 AJC, Saenpholphat et al. 2003 Czechoslovak Mathematical Journal) introduced the connected resolvability of graphs. A set $W \subseteq V(G)$ is a connected resolving set for $G$ if different vertices of $G$ have different representations with respect to $W$ and the subgraph $\langle W \rangle$ of size $m \geq 1$ induced by $W$ is a connected subgraph of $G$. The minimum order of a connected resolving set in a connected graph $G$ is its connected resolving number denoted by $cr(G)$. A connected resolving set $W$ of $G$ of cardinality $cr(G)$ is called a $cr$-set of $G$.

Saenpholphat et al. found an upper bound for the connected resolving number of a connected graph that is not isomorphic to a path. They also showed that for all integers $k \geq 2$ there exist a connected graph with a unique $cr$-set. Moreover, for every pair $k, r$ of integers with $k \geq 2$ and $0 \leq r \leq k$, there exists a connected graph $G$ with connected resolving number $k$ such that there are exactly $r$ vertices in $G$ that belong to every $cr$-set of $G$.

For a set $S$ of vertices in a connected graph $G$, the Steiner distance $d(S)$ of $S$ is the minimum size of a connected subgraph in $G$ containing all vertices of $S$. Necessarily, each such subgraph is a tree and is called a Steiner tree with respect to $S$ or a Steiner $S$-tree. A basis $W$ of $G$ is said to be a Steiner basis of $G$.

**Proposition 3.1.** (Saenpholphat et al. 2003 AJC) Consider a connected graph $G$ with size greater than one and $G \neq P_n$. If $W$ is a Steiner basis of $G$, then $cr(G) \leq \text{diam}(W) + 1$. 

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Proposition 3.2. (Saenpholphat et al. 2003 AJC) Let $G_1$ and $G_2$ be connected graphs such that $G_2$ is obtained from $G_1$ by adding a vertex with more than one edge incident with $v$, then for any positive integer $N$, $cr(G_2) \leq cr(G_1) - N$.

Theorem 3.7. (Saenpholphat et al. 2003 AJC) For $n \geq 7$,

$$cr(W_n) = \left\lceil \frac{2n + 2}{5} \right\rceil + 1.$$

Theorem 3.8. (Saenpholphat et al. 2003 AJC) For a connected graph $G$ of order $k \geq 2$ there exist a unique cr-set of order $k$.

Theorem 3.9. (Saenpholphat et al. 2003 AJC) For every pair $k, r$ of integers with $k \geq 2$ and $0 \leq r \leq k$, there exist a connected graph $G$ such that $cr(G) = k$ and exactly $r$ vertices of $G$ belong to every cr-set of $G$.

Proposition 3.3. (Saenpholphat et al. 2003 Czechoslovak Mathematical Journal) Let $n \geq 2$. If $G = P_n$ or $G = C_n$ for $n \geq 3$, then $cr(G) = 2$.

Proposition 3.4. (Saenpholphat et al. 2003 Czechoslovak Mathematical Journal) For each integer $k \geq 3$, there is a connected graph $G$ with two cr-sets $S_1$ and $S_2$ of cardinality $k$ such that $S_1$ contains a basis of $G$ and $S_2$ contains no basis of $G$.

Proposition 3.5. (Saenpholphat et al. 2003 Czechoslovak Mathematical Journal) For $k \geq 2$, let $G = K_{n_1,n_2,...,n_k}$ be a complete $k$-partite graph that is not a star. Let $n = n_1 + n_2 + ... + n_k$ and $l$ be the number of one’s in $\{n_i : 1 \leq i \leq k\}$. Then

$$cr(G) = \begin{cases} n - k, & \text{if } l = 0; \\ n + l - (k + 1), & \text{if } l \geq 1; \end{cases}$$

Proposition 3.6. (Saenpholphat et al. 2003 Czechoslovak Mathematical Journal) Let $k, n \geq 2$ be integers:

a). If $G = K_n, P_n$, or $G = K_{n_1,n_2,...,n_k}$ that is not a star, then $cr(G \times K_2) = cr(G)$.

b). If $n \geq 4$, then $cr(C_n \times K_2) = cr(C_n) + 1$.

Proposition 3.7. (Saenpholphat et al. 2003 Czechoslovak Mathematical Journal) If $T$ is a nontrivial tree that is not a path, then $cr(T \times K_2) = cr(T) + 1$.

Theorem 3.10. (Saenpholphat et al. 2003 Czechoslovak Mathematical Journal) Let $G$ be a connected graph of order $n \geq 3$. Then $cr(G) = n - 1$ if and only if $G = K_n$ or $G = K_{1,n-1}$.

Theorem 3.11. (Saenpholphat et al. 2003 Czechoslovak Mathematical Journal) For each pair of integers $k, n$ with $2 \leq k \leq n - 1$, there is a connected graph $G$ of order $n$ with connected resolving number $k$.

Theorem 3.12. (Saenpholphat et al. 2003 Czechoslovak Mathematical Journal) For every pairs $a, b$ of integers with $2 \leq a \leq b$, there exists a connected graph $G$ such that $\dim(G) = a$ and $cr(G) = b$. 

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Theorem 3.13. (Saenpholphat et al. 2003 Czechoslovak Mathematical Journal) Let \(a, b\) and \(n\) be integers with \(n \geq 5\). If \(n - 2 \leq a \leq b = n - 1\) or \(2 \leq a \leq b \leq n - 2\), then there exists a connected graph \(G\) of order \(n\) such that \(\text{dim}(G) = a\) and \(\text{cr}(G) = b\).

Theorem 3.14. (Saenpholphat et al. 2003 Czechoslovak Mathematical Journal) For a nontrivial connected graph \(G\), \(\text{cr}(G) \leq \text{cr}(G \times K_2) \leq \text{cr}(G) + 1\).

3.1.3 One Size and 2-Size Resolving Set

Kwancharone et al. (Kwancharone et al. 2008) have introduced the One size resolving set of a connected graph \(G\). A set \(W \subseteq V(G)\) is a One size resolving set for \(G\) if it satisfies the following:
(i) The size of the subgraph \(<W>\) induced by \(W\) is one and (ii) Different vertices of the graph \(G\) have different representation with respect to \(W\). The minimum order of a One size resolving set in a graph \(G\) is the One size resolving number represented by \(or(G)\) of the graph \(G\). A one resolving set of order \(or(G)\) is known as \(or\)-set of a graph \(G\). They studied the existence of \(or\)-set in graphs and characterized all connected graphs \(G\) of order \(n\) and size \(m \geq 1\) with \(or(G) = |V(G)|\) and \(|V(G)| - 1\).

Theorem 3.15. (Kwancharone et al. 2008) For any connected graph \(G\), \(2 \leq or(G) \leq n\).

Proposition 3.8. (Kwancharone et al. 2008) For \(n \geq 2\), \(or(G) = 2\) if \(G \cong P_n\).

Kwancharone et al. (Kwancharone et al. 2008) verified that for \(n \geq 3\), \(or(G) = 2\) if \(G \cong C_n\). They also found that for a connected graph \(G\) with order \(n \geq 3\), \(or(G) = 2\) iff \(G\) is isomorphic to \(K_3\).

Theorem 3.16. (Kwancharone et al. 2008) Let \(G\) be a connected graph with order greater or equal to 3 and size greater or equal to 2, then \(or(G) = |V(G)| - 1\) if and only if \(G \cong P_3\) or \(G \cong K_3\).

Theorem 3.17. (Kwancharone et al. 2008) Let \(k\) and \(n\) be positive integers with \(k \leq n\). There exist a connected graph \(G\) of order \(n\) with \(or(G) = k\) if and only if \((k,n) = (n - 1,3)\) or \((n,2)\) or \(2 \leq k \leq n - 2\).

Salman et al. (Salman et al. 2012) have extended this idea of One size resolving set of a graph to 2-size resolving set of graphs. A resolving set \(W \subseteq V(G)\) for a connected graph \(G\) of order \(n \geq 3\) is called 2-size resolving set if the size of the subgraph \(<W>\) induced by \(W\) is two. The minimum cardinality of a 2-size resolving set is called the 2-size metric dimension of \(G\) denoted by \(tr(G)\). A 2-size resolving set of cardinality \(tr(G)\) is called a \(tr\)-set. They investigated 2-size resolving sets for some well known classes of graphs.

Theorem 3.18. (Salman et al. 2012) For any connected graph \(G\), \(3 \leq tr(G) \leq n\).

Theorem 3.19. (Salman et al. 2012) Let \(P(n,k)\) be the Petersen graph. Then \(tr(P(n,k)) = 4\).
Salman et al. (Salman et al. 2012) computed 2-size resolving set for some particular graphs. They found that $tr(G) = 3$ if and only $G$ is isomorphic to $P_n$, with $n \geq 3$. Similar result have been established for a cycle graph with $n \geq 4$. They also found $tr(G)$ for a complete graph of order $n \geq 3$. Let $G - e$ be the graph obtained from $G$ by deleting an edge from $G$. Then $tr(G - e) = 3$ if and only if $G \cong K_3$ or $G \cong K_4$. Similarly for a complete graph $G$ of order $n \geq 4$, let $G - 2e$ be the graph obtained from $G$ by deleting two edges from $G$, then $tr(G - 2e) = 3$ if and only if $G \cong K_4$ or $G \cong K_5$.

**Theorem 3.20.** (Salman et al. 2012) Let $G$ be a connected graph of order $|V(G)| \geq 3$. Then $tr(G) = |V(G)|$ if and only if $G$ is isomorphic to $P_5$ or $K_{1,2}$.

**Theorem 3.21.** (Salman et al. 2012) Let $k$ and $n$ be positive integers with $k \leq n$. There exist a connected graph $G$ of order $n$ and $tr(G) = k$ if and only if $n \in \{2,3\}$ and $k = 3$ or $n \geq 5$ and $3 \leq k \leq n - 2$.

### 3.1.4 Some Special Types of Dimensions of Graphs

Chartrand et al. (Gary Chartrand et al. 2000) introduced the partition dimension as follows: For a subset $S \subset V(G)$ of a graph $G$ and for any vertex $v$ of a connected graph $G$ the distance $d(v,S)$ between $v$ and $S$ is defined as $d(v,S) = \min\{d(v,s) : s \in S\}$. If $\mathcal{P} = (S_1,S_2,\ldots,S_k)$ is an ordered $k$-partition of $V(G)$, the representation of $v$ with respect to $\mathcal{P}$ is the $k$-tuple $r(v|\mathcal{P}) = (d(v,S_1), d(v,S_2),\ldots,d(v,S_k))$. If the $k$-tuples $r(v|\mathcal{P})$ for $v \in V(G)$ are all distinct then the partition $\mathcal{P}$ is said to be resolving partition and the minimum number of vertices of a resolving partition of $V(G)$ is said to be partition dimension of a graph $G$ and is represented by $dimp(G)$ or $pd(G)$.

**Theorem 3.22.** (Gary Chartrand et al. 2000) For a connected graph $G$ of size $m \geq 2$, $pd(G) \leq dim(G) + 1$.

**Theorem 3.23.** (Gary Chartrand et al. 2000) Let $a$ and $b$ be positive integers with $\lceil \frac{b}{2} \rceil + 1 \leq a \leq b + 1$ then there exist a connected graph $G$ such that $pd(G) = a$ and $dim(G) = b$.

Gary Chartrand et al. (Gary Chartrand et al. 2000) found a relation between diameter of $G$ and the partition dimension of the graph $G$. They consider a connected graph $G$ of order $n \geq 3$ and diameter $diam(G)$. They showed that $g(|V(G)|, diam(G)) \leq pd(G) \leq (|V(G)| - diam(G) + 1)$, where $g(|V(G)|, diam(G))$ is the least positive integer $k$ for which $(diam(G) + 1)^k \geq |V(G)|$. Also for a connected graph $G$ with $|V(G)| \geq 3$, $pd(G) = |V(G)| - 1$ if and only if $G$ is isomorphic to one of the graphs $K_{1,n-1}$, $K_n - e$, $K_1 + (K_1 \cup K_{n-1})$. Gary Chartrand et al. (Gary Chartrand et al. 2000) computed $pd(G) = 2$ if and only if $G$ is isomorphic to $P_n$ with $|V(G)| \geq 2$.

**Proposition 3.9.** (Gary Chartrand et al. 2000) For a connected graph $G$, $pd(G) = |V(G)|$ if and only if $G$ is isomorphic to $K_n$.

**Corollary 3.1.** (Gary Chartrand et al. 2000) For a connected graph $G$ with $|V(G)| \geq 2$ and $pd(G) = |V(G)| - 1$, then diameter of the graph is two.
Lemma 3.3. (Gary Chartrand et al. 2000) Let \( x, y \in V(G) \) and \( \Pi \) be a resolving partition of \( V(G) \). If \( d(x, z) = d(y, z) \) for all \( z \in V(G) - \{x, y\} \), then \( x \) and \( y \) belong to different elements of \( \Pi \).

Chartrand et al. (Chartrand et al. 2001) introduced the **forcing dimension** of a graph \( G \). For a basis \( W \) of a graph \( G \), a subset \( S \) of \( W \) is called a forcing subset of \( W \) if \( W \) is the unique basis containing \( S \). The **forcing number** \( f_G(W, dim(G)) \) of \( G \) is the minimum number of vertices in a forcing subset for \( W \). The forcing dimension \( f(G, dim(G)) \) of \( G \) is the smallest forcing number among all bases of a graph \( G \). It is shown (Chartrand et al. 2001) that for all integers \( a \) and \( b \), there exist a unique basis for \( k \geq a \) and \( dim(G) = b \) if and only if \( \{a, b\} \neq \{0, 1\} \) where \( f(G) \) represent the forcing dimension of a graph \( G \).

**Proposition 3.10.** (Chartrand et al. 2001) Let \( G \) be a connected graph with size \( m > 1 \). If \( G \) is isomorphic to \( K_n \) or \( C_n \) or \( T \), then \( f(G) = dim(G) \).

**Proposition 3.11.** (Chartrand et al. 2001) For a connected graph \( G \) with \( |V(G)| \geq 2 \), \( dim(G) = |V(G)| - 2 \). If \( G \) is isomorphic to \( K_{rs} \) \((r, s \geq 1)\) or \( G \) is isomorphic to \( K_r + K_s \) \((r \geq 1, s \geq 2)\), then \( f(G) = dim(G) \). If \( G \) is isomorphic to \( K_r + (K_1 \cup K_3) \) \((r, s \geq 1)\), then \( f(G) = dim(G) - 1 \).

Chartrand et al. (Chartrand et al. 2001) showed that the forcing dimension \( f(G) \) is equal to zero if and only if \( G \) has a unique basis, \( f(G) \) is equal to one if and only if \( G \) has at least two different bases but some vertex of \( G \) belongs to exactly one basis, and \( f(G) \) is equal to \( dim(G) \) if and only if no basis of \( G \) is a unique basis containing any of its proper subsets.

**Theorem 3.24.** (Chartrand et al. 2001) For a connected graph \( G \) of metric dimension \( k \), there exist a unique basis for \( k \geq 2 \).

**Theorem 3.25.** (Chartrand et al. 2001) For integer \( a \) and \( b \) with \( 0 \leq a \leq b \) and \( b \geq 1 \) there exist a connected graph \( G \) with size \( m > 1 \) then \( f(G) = a \) and \( dim(G) = b \) if and only if \( \{a, b\} \neq \{0, 1\} \).

Let \( \mathcal{G}_{\beta, \alpha} \) be the class of connected graphs with metric dimension \( \beta \) and diameter \( \alpha \). For a vertex \( x \in G \), the open neighborhood of \( x \) is \( N(x) = \{ y \in V(G) : xy \in E(G) \} \) and the closed neighborhood of \( x \) is \( N[x] = N(x) \cup \{x\} \). Two distinct vertices \( x, y \) are adjacent twins if \( N[x] = N[y] \) and non-adjacent twins if \( N(x) = N(y) \) (Hernando et al. 2010). Observe that if \( x, y \) are adjacent twins then \( xy \in E(G) \), and if \( x, y \) are non-adjacent twins then \( xy \notin E(G) \). If \( x, y \) are adjacent or non-adjacent twins then \( x, y \) are twins. The next lemma follows from the definitions.

**Lemma 3.4.** (Hernando et al. 2010) If \( u, v \) are twins in a connected graph \( G \), then for every vertex \( x \in V(G) - \{u, v\} \), \( d(u, x) = d(v, x) \).

**Lemma 3.5.** (Hernando et al. 2010) The minimum order of a graph in \( \mathcal{G}_{\beta, \alpha} \) is \( \beta + \alpha \).
Theorem 3.27. (Oellermann et al. 2007) For a family of connected graphs it is possible to determine a locating set of the graph \(G\).

Theorem 3.26. (Kratica et al. 2010) Let \(x, y\) and \(z\) be three distinct vertices in a graph. If \(x, y\) are twins and \(y, z\) are twins, then \(x, y\) must be twins.

In the view of the above result, they prove the following very interesting result.

Lemma 3.8. (Hernando et al. 2010) Let \(x, y, z\) be three distinct vertices in a graph. If \(x, y\) are strongly resolved by some vertex of \(G\), then \(y, z\) are twins and \(y, z\) are twins, then \(x, y\) must be twins.

Lemma 3.7. (Hernando et al. 2010) Let \(T\) be a twin-set of a connected graph \(G\) is represented with \(|T| \geq 3\). Then \(\dim(G) = (\dim(G \setminus u) + 1)\) for all \(u \in T\).

3.1.5 Some More Types of Dimension of Graphs

Kratica et al. (Kratica et al. 2008) have defined the strong metric dimension of a graph \(G\). Consider two different vertices \(x\) and \(y\) of a connected graph \(G\), assume that \(N[x, y]\) be the set of all vertices that belong to some geodesic path from \(x\) to \(y\). A vertex \(z\) strongly resolves two vertices \(x\) and \(y\) if \(y \in N[x, z] \lor x \in N[y, z]\). A subset \(W\) of \(V(G)\) is a strong locating (resolving) set of a graph \(G\) if any two different vertices of a connected graph \(G\), are strongly resolved by some vertex of \(W\). The strong metric dimension of a connected graph \(G\) represented by \(sdim(G)\) is defined as the number of vertices in a minimal strong locating set of the graph \(G\). It can be shown that if a vertex \(z\) strongly resolves vertices \(x\) and \(y\) then \(z\) also resolves these vertices. Therefore, all strong locating sets are a locating set as shown in the following theorem.

Theorem 3.26. (Kratica et al. 2008) For a connected graph \(G\), \(\dim(G) \leq sdim(G)\).

Oellermann et al (Oellermann et al. 2007) have shown that the problem of computing the strong metric dimension of a connected graph \(sdim(G)\) is NP-hard. However for particular families of connected graphs it is possible to determine \(sdim(G)\).

Theorem 3.27. (Oellermann et al. 2007) For cycle \((C_n)\) with \(n\) vertices, \(sdim(C_n) = \lceil \frac{n}{2} \rceil\).

Theorem 3.28. (Oellermann et al. 2007) Consider a tree graph \(T\), then \(sdim(T) = \ell(T) - 1\).

Theorem 3.29. (Oellermann et al. 2007) Assume a complete graph \(K_n\) with \(n\) vertices, then \(sdim(K_n) = n - 1\).

Kratica et al. (Kratica et al. 2012) established a new idea about the resolving sets i.e. doubly resolving set and is defined as: Two vertices \(u, v \in G\) and \(n \geq 2\) are said to be doubly resolve vertices \(x\) and \(y\) of a graph \(G\) if \(d(x, u) - d(x, v) \neq d(y, u) - d(y, v)\). A subset \(U = \{v_1, v_2, ..., v_l\} \subseteq V(G)\) is a doubly locating (resolving) set of a connected graph \(G\) if any two distinct vertices of a graph \(G\) are doubly resolved by some two vertices of set \(U\). The minimal doubly locating set problem consists of finding a doubly locating set of a connected graph \(G\) with the minimum number of elements represented by \(\psi(G)\). Note that if \(u, v\) doubly resolve \(x, y\) then \(d(x, u) - d(x, v) \neq 0\) or \(d(x, v) - d(y, v) \neq 0\). Therefore, \(u\) or \(v\) resolve \(x\) and \(y\). Thus a doubly locating set is also a locating set, which is given in the following theorem.
Theorem 3.30. (Kratica et al. 2012) Let $G$ be a connected graph, then $\dim(G) \leq \psi(G)$.

The concept of strong metric dimension and doubly locating set is shown in the following example (Kratica et al. 2012). Consider $C_4$ with vertex set $V(C_4) = \{x_1, x_2, x_3, x_4\}$. The set $S = \{x_1, x_2\}$ is a subset of $V(C_4)$ is a locating set of $C_4$. For this the representations of the vertices of $C_4$ with respect to $S$ are:

$r(x_1, S) = (0, 1)$, $r(x_2, S) = (1, 0)$, $r(x_3, S) = (1, 2)$, $r(x_4, S) = (2, 1)$. It can be shown that a set of single vertex e.g \{x_1\} is not a locating set, as $d(x_2, x_1) = d(x_4, x_1) = 1$. Thus $S$ is a minimal locating set for the graph $C_4$ and $\dim(C_4) = 2$.

Moreover $S$ is a strong locating set of $C_4$. For every set $N[x_1, x_2]$ where $x_1$ is a vertex of the graph $C_4$ and $x_2$ is a vertex of set $S$ have the form: $N[x_1, x_1] = \{x_1\}$, $N[x_2, x_1] = N[x_1, x_2] = \{x_1, x_2\}$, $N[x_3, x_1] = \{x_1, x_3\}$, $N[x_4, x_1] = \{x_1, x_2, x_3, x_4\}$, $N[x_2, x_2] = \{x_2\}$, $N[x_3, x_2] = \{x_1, x_2, x_3, x_4\}$, $N[x_4, x_2] = \{x_2, x_4\}$. It can be checked that for each pair of distinct vertices of the graph $C_4$, there exists a vertex from the set $S$ which strongly resolves them. Consider $x_3 \in N[x_4, x_1]$, the vertex $x_3$ and vertex $x_4$ are strongly resolved by the vertex $x_1$ while vertex $x_2$ and vertex $x_4$ are strongly resolved by vertex $x_2$ as $x_2 \in N[x_4, x_2]$. Therefore, $2 = \dim(C_4) \leq sdim(C_4) \leq |S| = 2$, which shows $sdim(C_4) = 2$.

It can be verified that $S$ is not a doubly locating set because $d(x_3, x_1) - d(x_1, x_1) = 1$ and

\[
d(x_2, x_3) - d(x_1, x_2) = 1.
\]

It can also be shown that none of the subsets consisting of two vertices is a doubly locating set of the graph $C_4$. However $S = \{x_1, x_2, x_3\}$ is a doubly locating set of metric representations with respect to $S$ are:

$r(x_1, S) = (0, 1, 1)$, $r(x_2, S) = (1, 0, 2)$, $r(x_3, S) = (1, 2, 0)$, $r(x_4, S) = (2, 1, 1)$. Therefore, $S$ is a minimal doubly locating set and hence $\psi(C_4) = 3$.

Lemma 3.9. (Kratica et al. 2012) For the Hamming graph $H_{2,q}$ where $q \geq 6$, $\psi(H_{2,q}) \leq \dim(H_{2,q})$.

Theorem 3.31. (Kratica et al. 2012) For $n = 2$ the Hamming graph $H_{n,q}$,

\[
\psi(H_{2,q}) = \begin{cases} 
3, & q = 2, 3; \\
5, & q = 4; \\
\left\lfloor \frac{4q^2 - 2}{3} \right\rfloor, & q \geq 5.
\end{cases}
\]
Theorem 3.32. (Kratica et al. 2012) For the Hamming graph $H_{n,q}$, $sdim(H_{n,q}) = (q-1)q^{n-1}$.

Lemma 3.10. (Kratica et al. 2012) Assume that $H_{n,q}$ be the Hamming graph and let $S$ be a strong locating set then $|S| \geq (q-1)q^{n-1}$.

Lemma 3.11. (Kratica et al. 2012) The set $S = \{(v_1,v_2,...,v_n)\mid 1 \leq v_1 \leq q, 0 \leq v_i \leq q$ for $2 \leq i \leq n\}$ is a strong locating set of $H_{n,q}$.

Corollary 3.2. (Kratica et al. 2012) Let $Q_n$ be the hypercube graph, then $sdim(Q_n) = 2^{n-1}$ for all $n$.

3.1.6 Metric Dimension of some Families of Regular Graphs

Guo et al. (Guo et al. 2011) and Bailey et al. (Bailey et al. 2013) computed the metric dimension of some families of distance regular graphs. Johnson graphs are represented by $J(n,q)$ and used in several branches of Mathematical sciences and Computer Science. The vertices of $J(n,q)$ are the $q$-element subsets of an $n$-element set. Any two vertices are adjacent when they meet in a $(q-1)$-element set.

Theorem 3.33. (Guo et al. 2011) For $e \geq 3$. Then $dim(J(2e+1,e)) \leq 2e$.

Proposition 3.12. (Bailey et al. 2013) For $e \geq 3$. Then $dim(J(n,e)) \leq (e+1)\lceil n/(e+1) \rceil$ for $n = 2e+1$.

Bailey et al. (Bailey et al. 2013) computed the resolving set for Kneser graph $K(n,k)$ when $n \geq 3k$. A resolving set $W$ for the Kneser graph $K(n,k)$ is resolving set for $J(n,k)$. Thus $dim(J(n,k)) \leq dim(K(n,k))$. They found that a set of vertices $W$ is a resolving set for $J(n,k)$ if and only if for any two disjoint non-empty sets $U,V \subset V(J(n,k))$ such that $|U| = |V| \leq k$, there exist a vertex $x \in W$ satisfying $|x \cap U| \neq |x \cap V|$.

Theorem 3.34. (Bailey et al. 2013) For the Johnson graph $J(n,k)$ with $n \geq 2k$,

$$dim(J(n,k)) \leq \left\lceil \frac{k}{k+1} \right\rceil (n+1).$$

Theorem 3.35. (Bailey et al. 2013) For the Kneser graph $K(n,k)$ with $n \geq 2k$,

$$dim(J(n,k)) \leq \left\lceil \frac{n}{2k-1} \right\rceil (2k-1)C_k - 1).$$

Theorem 3.36. (Bailey et al. 2013) Suppose $S$ is a family of k-subsets of $V(J(n,k))$ whose incidence matrix has rank $n$. Then $S$ is a resolving set for the Johnson graph $J(n,k)$.

Let $H$ be the toroidal grid $C_a \times C_b$ with $a,b \geq 10$. Using these Bailey et al. (Bailey et al. 2013) proved that for $ab = n$, the set of all straight paths in $H$ with 4 vertices is a resolving set for $K(n,4)$. Therefore, for such values of $n$, $dim(K(n,4)) \leq 2n$. They also determined the metric dimension for the toroidal grid $C_a \times C_b$ with $a,b \geq 13$. If $n = ab$, then the set of all straight paths in $H$ with 5 vertices is a resolving set for $K(n,5)$. Therefore, for such values of $n$, $dim(K(n,5)) \leq 2n$. 

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Proposition 3.13. \textit{(Bailey et al. 2013)} If \( q \) is a prime power then,
\begin{enumerate}
\item \( \dim(K(q^3,q)) \leq q^2(q+2) \).
\item \( \dim(K((q+1)(q^3+1), q+1)) \leq (q^2+1)(q^3+1) \).
\item \( \dim(K((q^2+1)(q^5+1), q^2+1)) \leq (q^3+1)(q^5+1) \).
\end{enumerate}

Theorem 3.37. \textit{(Bailey et al. 2013)} Let \( H \) be the toroidal grid \( C_a \times C_b \) with \( a,b \geq 16 \). If \( ab = n \), then the set of all straight paths in \( H \) with 6 vertices is a resolving set for \( K(n,6) \). Therefore, for such values of \( n \) \( \dim(K(n,6)) \leq 2n \).

Chartrand et al. (Chartrand et al. 2000) found some results for the metric dimension of \( n \)-regular graphs \( G(n,n) \). Saputro et al. (Saputro et al. 2011) recently found the results for generalized complete \( k \)-partite graphs. Motivated by the result given in lemma (2.18) of this thesis and Lemma (3) (Saputro et al. 2011), researchers focused on graphs which are isomorphic to \( K_{m,m} \backslash E(C_{2m}) \) where \( m \geq 4 \) and \( C_{2m} \) represent the Hamiltonian cycle. Chartrand et al. (Chartrand et al. 2000) calculated this problem for \( m = 4 \) and \( G \) isomorphic to a cycle graph \( C_{2m} \) with \( 2m \) vertices. For \( m \geq 5 \), assume \( S \subset V(G) \) be the set of vertices with \( |S| \geq 2 \). Consider two vertices \( v,w \in S \) and \( Q \) be the shortest path between \( v \) and \( w \) in \( C_{2m} \). Remember that all edges of path \( Q \) are not edges of the edge set of \( G \).

A gap between two vertices \( v \) and \( w \) is the set of vertices in \( Q \setminus \{v,w\} \). The vertices \( v \) and \( w \) are called the termination nodes. Two gaps which have at least one shared end node will be known as adjacent gaps. As a result, gap depends on the cardinality of \( S \), if \( |S| = r \) then \( S \) has \( r \) gaps some of which may be empty. These characterizations were initially introduced by Christopher et al. (Christopher et al. 2003). Moreover, Tomescu and Javaid (Tomescu and Javaid 2007) used this procedure to computed the metric dimension of the gear graph \( J_{2n} \). For a graph \( G \) with basis set \( W \), Saputro et al. (Saputro et al. 2011) observed the following:

1. Each gap of \( W \) has less or equal to four vertices. Otherwise, there is a gap having greater or equal to five vertices \( \{a_1,a_2,a_3,a_4,a_5\} \) of a graph \( G \) where \( a_ja_{j+1} \notin E(G) \) with \( 1 \leq j \leq 4 \). But, for every \( u \in W \), \( d(u,a_2) = d(u,a_4) \) implying that \( r(a_2|W) = r(a_4|W) \), a contradiction.

2. Maximum one gap of \( W \) has four vertices. Else, there exist two distinct gaps \( \{a_1,a_2,a_3,a_4\} \) and \( \{b_1,b_2,b_3,b_4\} \) where \( a_ja_{j+1},b_{j+1}b_{j+1} \notin E(G) \) for \( 1 \leq i \leq 3 \), assume \( u \in W \). If \( a_2 \) and \( b_2 \) are in the similar independent set of vertices, then \( d(u,a_2) = d(u,b_2) \) so \( r(a_2|W) = r(b_2|W) \) or \( d(u,a_2) = d(u,b_3) \), which implies \( r(a_2|W) = r(b_3|W) \), a contradiction.

3. Suppose a gap \( A \) of \( W \) has \( k \) vertices where \( 2 \leq k \leq 4 \), then some adjacent gaps of \( A \) contain at most one vertex. If not, there are \( k+3 \) vertices \( a_1,a_2,...,a_{k+3} \) of a graph \( G \) where \( a_ia_{i+1} \notin E(G) \) with \( 1 \leq i \leq k+2 \) and \( a_{k+1} \) is the single vertex among \( a_1,a_2,...,a_{k+3} \) contained in \( W \). So \( r(a_k|W) = r(a_{k+2}|W) \), a contradiction.

4. Consider any two gaps \( A \) and \( B \) of \( W \) having 3 vertices, such that both of their end nodes are placed in distinct independent sets of a graph \( G \). Then, there are ten vertices \( a_1,a_2,a_3,a_4,a_5,b_1,b_2,b_3,b_4,b_5 \) of the graph \( G \) with \( a_ia_{i+1},b_{i+1}b_{i+1} \notin E(G) \) with \( 1 \leq i \leq 4 \) and \( a_1,a_5,b_1,b_5 \) are the only vertices of \( W \) from the similar independent sets of a graph \( G \). Then \( r(a_3|W) = r(b_3|W) \), a contradiction.
(5). A graph $G$ has either a gap of three vertices or a gap of four vertices. Otherwise, it has two different gaps $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3, b_4\}$ where $a_i a_{i+1}, b_k b_{k+1} \notin E(G)$ for $1 \leq i \leq 2$ and $1 \leq k \leq 3$. If $a_2$ and $b_2$ are in the similar independent set of the graph $G$, then $r(a_2|W) = r(b_2|W)$ otherwise, $r(a_2|W) = r(b_3|W)$, a contradiction.

**Theorem 3.38.** (Saputro et al. 2011) Assume that $G(n,n)$ is an $(n-1)$-regular bipartite graph then $\dim(G) = n-1$ for $n \geq 3$.

**Theorem 3.39.** (Saputro et al. 2011) Consider a connected graph $H$ with $H \cong K_{m,m} \setminus E(C_{2m})$ for $m \geq 5$. Then $\dim(H) = \left\lfloor \frac{4m}{5} \right\rfloor$.

Saputro et al. (Saputro et al. 2011) found results related to $(n-2)$-regular bipartite graph $G$. It can be noted that every $(n-2)$-regular bipartite graph $G(n,n)$ is isomorphic to a graph $K_{n,n} \setminus (E(R_1) \cup E(R_2) \cup \cdots \cup E(R_r))$ in the following results:

**Theorem 3.40.** (Saputro et al. 2011) For $n \geq 4$ and $r \geq 1$, let $R_1, R_2, \ldots, R_r$ be $r$ disjoint even cycles contained in $K_{n,n}$ such that $V(R_1) \cup V(R_2) \cup \cdots \cup V(R_r) = V(K_{n,n})$. For $i \in \{1, 2, \ldots, r\}$ let $G_i \cong K_{n,n} \setminus (E(R_1) \cup E(R_2) \cup \cdots \cup E(R_r))$ and $m_i = \frac{|V(R_i)|}{2}$. For every $i \in \{1, 2, \ldots, r\}$ let $G_i$ be a subgraph of $G$ such that $G_i \cong K_{m_i,m_i} \setminus E(R_i)$. If $k_1$ is the number of cycles $R_i$ where $m_i = 2$ or $m_i \equiv 0 \mod 5$, $k_2$ is the number of cycles $R_i$ where $m_i \equiv 1 \mod 5$ and $k_3$ is the number of cycles $R_i$ where $m_i \equiv 2, 3, 4 \mod 5$ then:

$$
\dim(G) = \begin{cases} 
2, & n = 4; \\
\sum_{i=1}^{r} \dim(G), & n \geq 5 \text{ and } k_1 \in \{r-1, r\} \text{ or } r = 1; \\
\sum_{i=1}^{r} \dim(G) + k_2 + k_3 - 2, & n \geq 5, \ r \geq 2, \ k_1 \leq r-2, \text{ and } k_3 \geq 2; \\
\sum_{i=1}^{r} \dim(G) + k_2 + k_3 - 1, & n \geq 5, \ r \geq 2, \ k_1 \leq r-2, \text{ and } k_3 \in \{0, 1\}. 
\end{cases}
$$

**Lemma 3.12.** (Saputro et al. 2011) For $n \geq 3$, let $G$ be an $(n-1)$-regular bipartite graph. If $W$ is a resolving set of $G$ then $W$ contains at least $n-1$ vertices.

Saputro et al. (Saputro et al. 2011) showed that for an $(n-2)$-regular bipartite graph $G$ and $V(G) = V_1 \cup V_2$ be the partition of $V(G)$ with $|V_1(G)| = |V_2(G)| = n$. Let $G' \subset G$ such that $G'$ is isomorphic to $K_{m,m} \setminus E(C_{2m})$ with $m \in \{2, 3, 4\}$ and let $W$ be a resolving set of $G$ for $n \geq 3$. Assume that $m \in \{2, 3\}$, then $G'$ shares at least two vertices in $W$. For $m = 4$ if $n = 4$ then $G'$ shares at least two vertices in $W$ otherwise $G'$ shares at least three vertices in $W$.

**Lemma 3.13.** (Saputro et al. 2011) For $n \geq 3$, let $G$ be an $(n-1)$-regular bipartite graph with $V_1(G) = \{x_1, \ldots, x_n\}$, $V_2(G) = \{y_1, \ldots, y_n\}$ and $E(G) = \{x_i y_j | i \neq j\}$. Let $W = \{x_1, \ldots, x_{n-1}\}$. Then $W$ is a resolving set of $G$. 

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Lemma 3.14. (Saputro et al. 2011) Let $G$ be an $(n-2)$-regular bipartite graph and $V(G) = V_1 \cup V_2$ be partition of $V(G)$ with $|V_1(G)| = |V_2(G)| = n$ for $n \geq 4$. Assume that $G' \subset G$ such that $G'$ is isomorphic to $K_{m,m} \setminus E(C_{2m})$ with $2 \leq m \leq n$. If $W$ includes a basis of $G'$ then $W$ is a resolving set of $G$.

Saputro et al. (Saputro et al. 2011) introduced the idea of gaps in $(n-2)$-regular bipartite graph. They used this idea to compute the basis for $(n-2)$-regular bipartite graph $G$ with $V(G) = V_1 \cup V_2$ with $|V_1(G)| = |V_2(G)| = n$ for $n \geq 4$. Assume that $G'$ be the subset of $G$ such that $G'$ is isomorphic to $K_{m,m} \setminus E(C_{2m})$ with $m \in \{3,4,...,n\}$. If $m \equiv 1 \pmod{5}$, then a basis of $G'$ has greater or equal to two gaps containing three vertices.

Lemma 3.15. (Saputro et al. 2011) Let $G$ be an $(n-2)$-regular bipartite graph and $V(G) = V_1 \cup V_2$ be partition of $V(G)$ with $|V_1(G)| = |V_2(G)| = n$ for $n \geq 4$. Assume that $G' \subset G$ such that $G'$ is isomorphic to $K_{m,m} \setminus E(C_{2m})$ with $m \in \{3,4,...,n\}$. If $m \equiv 2,3$ or $4 \pmod{5}$ then there exist a basis of $G'$ which contains one gap of three vertices.

3.1.7 Metric Dimension of Digraph

M. Fehr et al. (M. Fehr et al. 2006) computed the metric dimension of Cayley digraphs. A vertex $x$ in a digraph $D$ is said to resolve a pair $u,v$ of vertices of $D$ if the distance from $u$ to $x$ does not equal the distance from $v$ to $x$. A set $S$ of vertices of $D$ is a resolving set for $D$ if every pair of vertices of $D$ is resolved by some vertex of $S$. The smallest cardinality of a resolving set for $D$, denoted by $\text{dim}(D)$, is called the metric dimension for $D$. They found sharp upper and lower bounds for the metric dimension of the Cayley digraphs, and also established the metric independence of the Cayley digraphs. Moreover, the metric dimension of the Cayley digraph of the dihedral group $D_n$ of order $2n$ with a minimum set of generators is found by them. A. Seb et al. (A. Seb et al. 2004) introduced a more restricted invariant than the dimension is called strong metric dimension. For two vertices $u$ and $v$ in a connected graph $G$, the interval $I[u,v]$ between $u$ and $v$ to be the collection of all vertices that belong to some shortest uv path. A vertex $w$ strongly resolves two vertices $u$ and $v$ if $v \in I[u,w]$ or if $u \in I[v,w]$. A set $W$ of vertices in a connected graph $G$ is a strong resolving set for $G$ if every two vertices of $G$ are strongly resolved by some vertex of $W$. The smallest cardinality of a strong resolving set of $G$ is called its strong metric dimension and is denoted by $s\text{dim}(G)$.

Theorem 3.41. (M. Fehr et al. 2006) Let $\Gamma$ be a group of order $n$ and let $\Delta = \{g_1,g_2,...,g_k\}$ be a generating set for $\Gamma$. Let $H = \text{Cay}(\Delta : \Gamma)$. Let $\Delta' = \{(g_1,0),(g_2,0),..., (g_k,0), (e_\Gamma,1)\}$ be a generating set for the group $\Gamma' = \Gamma \oplus Z_m$ where $m \geq 2$ and $e_\Gamma$ is the identity element of $\Gamma$. Then for $H' = \text{Cay}(\Delta' : \Gamma')$, $\text{dim}(H) \leq \text{dim}(H') \leq \text{dim}(H) + m - 1$.

Theorem 3.42. (M. Fehr et al. 2006) Let $m$ and $n$ be positive integers. Let $H'$ be the Cayley digraph for the group $Z_n \oplus Z_m$ with generating set $\{(1,0),(0,1)\}$. Then $\text{dim}(H') = \text{min}(m,n)$. 

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Theorem 3.43. (M. Fehr et al. 2006) Let \( n \) be a positive integer, \( n \geq 3 \). Let \( G \) be the Cayley digraph for the group \( D_n \) with generating set \( \{ R_{360/n}, A \} \), where \( A \) is any reflection in the group \( D_n \). Then \( \text{dim}(G) = n \).

Theorem 3.44. (M. Fehr et al. 2006) For all \( n \geq 2 \), \( m_i(Q_n) = 2 \).

M. Fehr et al. have proved certain combinatorial inequalities to prove theorem 2.45.

Lemma 3.16. (M. Fehr et al. 2006) For all positive integers \( k \), \( \binom{2k}{k} \leq 2^{2k-1} \).

Lemma 3.17. (M. Fehr et al. 2006) For all positive integers \( k \), \( \binom{2k-1}{k-1} \leq 2^{2k-2} \).

Theorem 3.45. (M. Fehr et al. 2006) Let \( n \) be a positive integer, \( n \geq 3 \). Let \( G \) be the Cayley digraph for the group \( D_n \) with generating set \( \{ R_{360/n}, A \} \), where \( A \) is any reflection in the group \( D_n \). Then \( m_i(G) = n \).

Ortrud R. Oellermann et al. (Ortrud R. Oellermann et al. 2006) defined strong metric dimension of graphs and digraph. Let \( G \) be a connected (di)graph. A vertex \( w \) is said to strongly resolve a pair \( u, v \) of vertices of \( G \) if there exists some shortest \( uw \) path containing \( v \) or some shortest \( vw \) path containing \( u \). A set \( W \) of vertices is a strong resolving set for \( G \) if every pair of vertices of \( G \) is strongly resolved by some vertex of \( W \). The smallest cardinality of a strong resolving set for \( G \) is called the strong dimension of \( G \). Moreover, it is shown that computing this invariant is NP-hard.

A vertex cover of a graph is a set \( S \) of vertices of \( G \) such that every edge of \( G \) is incident with at least one vertex of \( S \). The vertex covering number of \( G \), denoted by \( \alpha(G) \), is the smallest cardinality of a vertex cover of \( G \). Two non adjacent vertices \( u \) and \( v \) in a graph \( H \) are false twins if they have the same open neighbourhoods, i.e., if \( N(u) = N(v) \). The relation \( FT \) on \( V(H) \), defined by \( (u,v) \in FT \) if and only if \( u \) and \( v \) are false twins, is an equivalence relation on \( V(H) \), and thus partitions \( V(H) \). Let \( ft(H) \) be the total number of vertices that belong to a non-trivial equivalence class with respect to the equivalence relation \( FT \).

Let \( D \) be a strong digraph, that is a digraph in which every two vertices are mutually reachable, and \( u, v, w \in V(D) \). We say that \( w \) weakly resolves \( u \) and \( v \) if \( d(u,w) \neq d(v,w) \) or \( d(w,u) \neq d(w,v) \). A set \( W \) of vertices weakly resolves \( D \) if every pair of vertices of \( D \) is weakly resolved by some vertex of \( W \). In that case \( W \) is called a weak resolving set of \( D \). A smallest weak resolving set is called a weak basis for \( D \) and its cardinality is called the weak dimension of \( D \) and is denoted by \( wdim(D) \).

Theorem 3.46. (Ortrud R. Oellermann et al. 2006) For any connected graph \( G \), \( sdim(G) = \alpha(G_{SR}) \). Where \( (G_{SR}) \) denoted the strong resolving graph and \( \alpha \) denotes the

Theorem 3.47. (Ortrud R. Oellermann et al. 2006) Let \( H \) be any non-complete connected graph. Then there is a graph \( G^{(H)} \) such that \( H \) is an induced subgraph of \( G^{(H)}_{SR} \) and such that \( \alpha(G^{(H)}_{SR}) = \alpha(H) + ft(H) \).
Theorem 3.48. (Ortrud R. Oellermann et al. 2006) If $D$ is a strong digraph, then $\text{sdim}(D) = \alpha(G_{SR}(D))$.

Theorem 3.49. (Ortrud R. Oellermann et al. 2006) Suppose $n_1$ and $n_2$ are integers such that $n_1 \geq 2$ and $n_2 \geq 3$. Let $D$ be the Cayley digraph $Cay(\Delta : \Gamma)$ for the group $\Gamma = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$ $(2 \leq n_1 \leq n_2)$ with $\Delta = \{(1,0), (0,1)\}$. Then $\text{wdim}(D) = \lceil \frac{n_2}{2} \rceil$.

Lemma 3.18. (Ortrud R. Oellermann et al. 2006) Let $H = Cay(\Delta : \Gamma)$ and suppose $e$ is the identity of $\Gamma$. Let $H' = Cay(\Delta : \Gamma')$ where $\Gamma' = H \oplus \mathbb{Z}_n$ for $n \geq 3$ and $\Delta = (\Delta \times \{0\}) \cup \{(e,1)\}$. Then $\text{wdim}(H) \leq \text{wdim}(H') \leq \text{wdim}(H) + \lceil \frac{n-2}{2} \rceil$.

Theorem 3.50. (Ortrud R. Oellermann et al. 2006) Let $G$ be the graph $Cay(\Delta \cup \Delta^{-1} : \Gamma)$ for the group $\Gamma = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$ $(2 \leq n_1 \leq n_2)$ with generators $\Delta = \{(1,0),(0,1)\}$. Let $n = |V(G)| = n_1n_2$. Then

$$\text{sdim}(G) = \begin{cases} \frac{n}{2}, & \text{if } n_1 \text{ or } n_2 \text{ is even;} \\ \frac{n_1(n_2+1)}{2}, & \text{if } n_1 \text{ is odd}. \end{cases}$$

Feng et al. (Feng et al. 2012) studied the metric dimension of the line graph $L(G)$ of a digraph $G$. In particular, they showed that $\dim(L(G)) = |E(G)| - |V(G)|$ for a strongly connected digraph $G$ except for directed cycles, where $V(G)$ is the vertex set and $E(G)$ is the edge set of $G$. As a corollary, the metric dimension of de Bruijn digraphs and Kautz digraphs is given. Moreover, they have proved that $\lfloor \log_2 \Delta(G) \rfloor \leq \dim(L(G)) \leq |V(G)| - 2$ for a simple connected digraph graph $G$ with at least five vertices where $\Delta(G)$ is the maximum degree of $G$. Finally, they obtained the metric dimension of the line graph of a tree.

Theorem 3.51. (Feng et al. 2012) If $G$ is a strongly connected digraph except for directed cycles, then $\dim(L(G)) = |E(G)| - |V(G)|$.

Corollary 3.3. (Feng et al. 2012) For $d \geq 2$ and $n \geq 1$ be integers. Then

1. $\dim(B(d,n)) = d^{n-1}(d-1)$.
2. $\dim(K(d,n)) = \begin{cases} d, & n = 1; \\ d^{n-2}(d^2-1), & n \geq 1. \end{cases}$

3.1.8 Some Result on Graph Operation

Caceres et al. (Caceres et al. 2007) proved an interesting results about cartesian product of different families of graphs. Caceres et al. (Caceres et al. 2007) computed the metric dimension of cartesian products of graphs. They also introduced doubly locating sets as a valuable tool for finding an upper bound for the metric dimension of graphs in cartesian products of graphs.

Theorem 3.52. (Caceres et al. 2007) For all $n \geq m \geq 1$,

$$\dim(K_n \times K_m) = \begin{cases} \frac{1}{2}(n+m-1), & m \leq n \leq 2m-1; \\ n-1, & n \geq 2m-1. \end{cases}$$
3.1 General Idea About Metric Dimension:

**Theorem 3.53.** (Caceres et al. 2007) Let $G_{n,k}$ be a $k$-connected graph, for all $k \geq 1$ and $n \geq 2$, then $dim(G_{n,k}) \leq 2k$ and $dim(G_{n,k} \times G_{n,k}) \geq n$.

Hernando et al. (Hernando et al. 2005) have found bounds on the metric dimension of some families of graphs. They defined that a vertex $v$ is a boundary vertex of a graph $G$ if there exists a vertex $u$ such that no neighbor of $v$ is further away from $u$ than $v$. The set of all boundary vertices of $G$ is called its boundary $\partial(G)$. A vertex $v$ is called extreme if the subgraph induced by its neighborhood $N(v)$ is a clique. The set of all extreme vertices of $G$ is represented by $Ext(G)$ and $Ext(G) \subseteq \partial(G)$.

**Theorem 3.54.** (Hernando et al. 2005) For any two graphs $G$ and $H$ the following inequalities are satisfied:

1. $dim(G) \leq |\partial(G)|$.
2. $\max\{dim(G), dim(H)\} \leq dim(G \times H) \leq min\{dim(G) + |H|, dim(H) + |G|\} - 1$.
3. $2 \leq dim(G) \leq dim(H) \leq dim(G \times H) \leq min\{dim(G) + |H|, dim(H) + |G|\} - 2$.
4. $dim(G) + dim(H) \leq dim(G + H)$.
5. $dim(G) \leq dim(G \times P_1) \leq dim(G) + 1$.
6. $dim(G \times K_n) \leq dim(G) + n - 2$, if $n \geq 3$.
7. $dim(G \times C_n) \leq \begin{cases} dim(G) + 1, & n = 2k + 1; \\ dim(G) + 2, & n = 2k. \end{cases}$
8. For $n \geq 3$, $dim(P_m \times K_n) = n - 1$.
9. For $m \leq n$,

$$dim(K_m \times K_n) \leq \begin{cases} \frac{n - 1}{2}, & 2m - 2 < n; \\ \frac{2m + 2n - 2}{3}, & 2m - 2 \geq n. \end{cases}$$

Iswadi et al. (Iswadi et al. 2008) and Yero et al. (Yero et al. 2011) computed the metric dimension of corona product of graphs. For two graphs $G$ and $H$ of order $n_1$ and $n_2$ respectively, the graph $G \odot H$ represent the corona product, obtained from $G$ and $H$ by taking one copy of $G$ and $n_1$ copies of $H$ and joining by an edge each vertex from the $i$th copy of $H$ with the $i$th-vertex of connected graph $G$. For integer $k \geq 2$ define the graph $G \odot^k H$ recursively from $G \odot H$ as $G \odot^k H = (G \odot^{k-1} H) \odot H$. They have given several results on the metric dimension of $G \odot^k H$. They also computed that for connected graphs $G$ and $H$ of order $n_1 \geq 2$ and $n_2 \geq 2$, $G \odot^k H$ have unbounded metric dimension.

**Theorem 3.55.** If the diameter of $H$ is less or equal to two then $dim(G \odot^k H) = n_1(n_2 + 1)^{k-1}dim(H)$.

**Theorem 3.56.** If $n_2 \geq 7$ and the diameter of $H$ is greater than five or $H$ is a cycle graph then $dim(G \odot^k H) = n_1(n_2 + 1)^{k-1}dim(K_1 \odot H)$.

The graph $(P_n \times P_m)$ is the cartesian product of $P_n$ and $P_m$ for $n \geq 1$ and $1 \leq m \leq 2$. $(P_n \times P_m) \odot K_1$ is denoted the corona product of $(P_n \times P_m)$ and $K_1$. They have shown that the metric dimension of these families of graphs are dependent on order of the graph.

**Theorem 3.57.** (Iswadi et al. 2008) For $n \geq 3$,

$$dim((K_n \times P_m) \odot K_1) = \begin{cases} n - 1, & m = 1; \\ n, & m = 2. \end{cases}$$

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Corollary 3.4. (Iswadi et al. 2008) For $G \odot K_m$ with $|G| = n$ and $m \geq 2$, $\text{dim} (G \odot K_m) = n(m-1)$.

Lemma 3.19. (Yero et al. 2011) “Let $G = (V, E)$ be a connected graph of order $n \geq 2$ and let $H$ be a graph of order at least 2. Let $H_i = (V_i, E_i)$ be the subgraph of $G \odot H$ corresponding to the $i$th-copy of $H$.

1. If $u, v \in V_i$ then $d_{G \odot H}(u, x) = d_{G \odot H}(v, x)$ for every vertex $x$ of $G \odot H$ not belonging to $V_i$.
2. If $S$ is a resolving set for $G \odot H$ then $V_i \cap S \neq \emptyset$ for every $i \in \{1, \ldots, n\}$.
3. If $S$ is a resolving set for $G \odot H$ of minimum cardinality then $V \cap S = \emptyset$.
4. If $H$ is a connected graph and $S$ is a resolving set for $G \odot H$ then for every $i \in \{1, \ldots, n\}$, $S \cap V_i$ is a resolving set for $H_i$.

Theorem 3.58. (Yero et al. 2011) Let $G$ and $H$ be two connected graphs of order $n_1, n_2 \geq 2$, respectively. Then $\text{dim} (G \odot^k H) \geq n_1(n_2+1)^{k-1}\text{dim}(H)$.

Theorem 3.59. (Yero et al. 2011) Let $G$ be a connected graph of order $n_1 \geq 2$ and let $H$ be a graph of order $n_2 \geq 2$. Let $\alpha$ be the number of connected components of $H$ of order greater than one and let $\beta$ be the number of isolated vertices of $H$. Then

$$\text{dim}(G \odot^k H) \leq \begin{cases} n_1(n_2+1)^{k-1}(n_2 - \alpha - 1), & \alpha \geq 1 \text{ and } \beta \geq 1; \\ n_1(n_2+1)^{k-1}(n_2 - \alpha), & \alpha \geq 1 \text{ and } \beta = 0; \\ n_1(n_2+1)^{k-1}(n_2 - 1), & \alpha \geq 1. \end{cases}$$

Theorem 3.60. (Yero et al. 2011) Let $G$ be a connected graph of order $n_1 \geq 2$ and let $H$ be a graph of order $n_2 \geq 2$. Then $\text{dim} (G \odot^k H) \leq n_1(n_2+1)^{k-1}\text{dim}(K_1 \odot H)$.

Theorem 3.61. (Yero et al. 2011) Let $G$ be a connected graph of order $n_1 \geq 2$ and let $H$ be a graph of order $n_2 \geq 7$. If $\text{diam}(H) \geq 6$ or $H$ is a cycle graph then $\text{dim} (G \odot^k H) = n_1(n_2+1)^{k-1}\text{dim}(K_1 \odot H)$.

Theorem 3.62. (Yero et al. 2011) For any tree $T$ of order $n \geq 3$,

$$\text{dim}(G \odot^k H) = \begin{cases} n_1(T), & k = 1; \\ 2^{k-2}n, & k \geq 2. \end{cases}$$

$n_1(T)$ represents degree one vertices in $T$.

3.1.9 Bounds on Metric Dimension and Graphs with Constant Metric Dimension

Chappell et al. (Chappell et al. 2008) derived sharp bounds on metric dimension for all graph. They established $\text{dim}(G) \geq \log_3(\Delta + 1)$ where $\Delta$ is the maximum degree of the vertex in graph $G$. Let $G$ and $H$ be any two graphs and $G + H$ denote the sum of two graphs. They showed that $\text{dim}(G + H) \leq \text{dim}(G) + \text{dim}(H)$.

Chappell et al. (Chappell et al. 2008) found that for any graph $G$ of order $n$, $\text{dim}(G) \leq$
Chartrand et al. (Chartrand et al. 2000) showed that $G \cong K_n$ if and only if $\dim(G) = |V(G)| - 1$. They also computed bounds for some more classes of graphs which are: $\dim(G) = |V(G)| - 2$ if and only if $G$ is isomorphic to $K_r$, $s \geq 1$ or $G$ is isomorphic to $K_r + K_s$ for $r \geq 1, s \geq 2$ or $G$ is isomorphic to $K_r + (K_1 \cup K_3)$ for $r, s \geq 1$.

For a connected graph $G$ with $|V(G)| \geq 2$ and diameter $\text{diam}(G)$, $f(|V(G)|, \text{diam}(G)) \leq \dim(G) \leq (|V(G)| - \text{diam}(G))$, where $f(|V(G)|, \text{diam}(G))$ denote the least positive integer $k$ for which $k + (\text{diam}(G))^k \geq |V(G)|$.

**Theorem 3.63.** (Chartrand et al. 2000) Let $H$ be a connected graph and $K_2$ be a complete graph, then $\dim(H) \leq \dim(H \times K_2) \leq \dim(H) + 1$.

**Theorem 3.64.** (Chartrand et al. 2000) Consider a connected graph $G$ with $|V(G)| \geq 2$ and diameter $\text{diam}(G)$ then $f(|V(G)|, \text{diam}(G)) \leq \dim(G) \leq |V(G)| - \text{diam}(G)$.

Chartrand et al. (Chartrand et al. 2000) established sharp bounds on the metric dimension of tree and unicyclic graph. They showed that $\dim(G) \leq \dim(G \times K_2) \leq \dim(G) + 1$ for any connected graph $G$. Consider $\sigma(G)$ denote the sum of the terminal degrees of major vertices of graph $G$, and $\text{ex}(G)$ denote the number of exterior major vertices of a graph $G$. Then $\dim(G) \geq \sigma(G) - \text{ex}(G)$.

**Theorem 3.65.** (Chartrand et al. 2000) For a graph $G$, $\dim(G) \geq G - \text{ex}(G)$.

**Theorem 3.66.** (Chartrand et al. 2000) For a tree $T$ with $|T| \geq 3$ and any edge $e$ of $T$ $\dim(T) \leq \dim(T + e) \leq \dim(T) + 1$.

**Theorem 3.67.** (Chartrand et al. 2000) Consider a connected graph $G$ and a connected induced subgraph $H$ of $G$ then for every $\varepsilon > 0$ there exists $\frac{\dim(G)}{\dim(H)} < \varepsilon$ for every $\varepsilon > 0$.

**Theorem 3.68.** (Chartrand et al. 2000) For a connected graph $G$ of order $n$ has metric dimension 1 if and only if $G \cong P_n$.

Cáceres et al. (Cáceres et al. 2007) have proved the following results for cartesian product of two graphs. They have shown that these graphs have constant metric dimension. The metric dimension of the cartesian product of paths and cycle are as follows:

**Theorem 3.69.** (Cáceres et al. 2007) For $m \geq 2, n \geq 2$ the $\dim(P_m \times P_n) = 2$.

**Theorem 3.70.** (Cáceres et al. 2007) For $m \neq 1$,

$$\dim(C_m \times C_n) = \begin{cases} 3, & m \text{ or } n \text{ odd; } \\ 4, & \text{otherwise.} \end{cases}$$

**Theorem 3.71.** (Cáceres et al. 2007) For $m, n \geq 3$,

$$\dim(P_m \times C_n) = \begin{cases} 2, & n \text{ is odd; } \\ 3, & n \text{ is even.} \end{cases}$$
Theorem 3.72. (Hernando et al. 2005) Assume $G$ is a connected graph with $|V(G)| \geq 3$, $\dim(G \times C_n) = 2$ if and only if $G$ is isomorphic to a path and $n$ is odd.

Imran et al. (Imran et al. 2010, 2011, 2012, preprint) have worked on constant metric dimension of convex polytopes and flower graphs. They also found bounds on metric dimension of circulant graphs and generalized Petersen graph which are:

Theorem 3.73. (Imran et al. 2010) Let $S_n$ be the convex polytope graph with $V(S_n) = \{a; b; c; d_i : i = 1, 2, \ldots, n\}$ and $E(S_n) = \{a_i a_{i+1}; b_i b_{i+1}; c_i c_{i+1}; d_i d_{i+1} : i = 1, 2, \ldots, n\}$ for $n \geq 6$, then $\dim(S_n) = 3$.

Theorem 3.74. (Imran et al. 2010) Let $T_n$ be the convex polytope graph with $V(T_n) = \{a; b; c; d_i : i = 1, 2, \ldots, n\}$ and $E(T_n) = \{a_i a_{i+1}; b_i b_{i+1}; c_i c_{i+1}; d_i d_{i+1} : i = 1, 2, \ldots, n\}$ for every $n \geq 6$.

Theorem 3.75. (Imran et al. 2010) Let $U_n$ be the convex polytope graph with $V(U_n) = \{a; b; c; d_i; e_i : i = 1, 2, \ldots, n\}$ and $E(U_n) = \{a_i a_{i+1}; b_i b_{i+1}; c_i c_{i+1}; d_i d_{i+1}; e_i e_{i+1} : i = 1, 2, \ldots, n\}$, $\dim(U_n) = 3$ for all integers $n \geq 6$.

Imran et al. (Imran et al. 2012, preprint) established bounds for different families of graphs. A circulant graph is a graph on $n$ vertices in which a vertex $v_i$ is adjacent to vertex $v_{i+j}$ and $v_{i-j}$ where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n/2$. They computed that the metric dimension of a graph $G$ is equal to $3$ if $G$ is isomorphic $P(n, 3)$ for $n \equiv 1 \ (\mod 6)$ $n \geq 25$ and equal to $4$ for $n \equiv 0 \ (\mod 6)$ $n \geq 24$.

Theorem 3.76. (Imran et al. 2012) For the circulant graphs $C_n(1, 2, 3)$,

$$\dim(C_n(1, 2, 3)) \leq \begin{cases} 4, & n \equiv 2, 3, 4, 5 \pmod{6} \text{ and } n \geq 14; \\ 5, & n \equiv 0 \pmod{6} \text{ and } n \geq 12; \\ 6, & n \equiv 1 \pmod{6} \text{ and } n \geq 13. \end{cases}$$

Lemma 3.20. (Imran et al. 2012) For any two vertices $u_i$ and $u_j$ on the outer cycle of $C_n(1, 2, 3)$, $d(u_i, u_j) = d(u_{i+r}, u_{j+r})$ for any $1 \leq r \leq n - 1$.

Theorem 3.77. (Imran et al. 2012) For $n \equiv 2, 3, 4, 5 \pmod{6}$ and $n \geq 13$, $\dim(C_n(1, 2, 3)) \geq 4$.

Theorem 3.78. (Imran et al. Util. Math, in press) For the generalized Petersen graph $P(n, 3)$,

$$\dim(P(n, 3)) \leq \begin{cases} 3, & n \equiv 1 \pmod{6} \text{ and } n \geq 13; \\ 4, & n \equiv 0, 3, 4, 5 \pmod{6} \text{ and } n \geq 17; \\ 5, & n \equiv 2 \pmod{6} \text{ and } n \geq 8. \end{cases}$$

Consider a vertex $v_i \in G$ is said to be a good vertex for $v_1$, if the distance between $v_i$ and $v_1$ is equal to the distance between $v_i$ and $v_{i+2}$, otherwise $v_i$ is called bad vertex. Applying the above definition Imran et al. (Imran et al. Util. Math, in press) proved that for any two vertices $v_i$ and $v_j$ on the outer cycle of Petersen graph $P(n, 3)$, $d(v_i, v_j) = d(v_{i+r}, v_{j+r})$ for any $1 \leq r \leq n - 1$. 

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3.1 General Idea About Metric Dimension:

**Theorem 3.79.** (Imran et al. Util. Math, in press) \( \dim(P(n,3)) = 3 \) if \( n = 6k + 1 \) and \( n \geq 25 \).

**Theorem 3.80.** (Imran et al. Util. Math, in press) \( \dim(P(n,3)) = 4 \) if \( n = 6k \) and \( n \geq 24 \).

**Theorem 3.81.** (Imran and Bashir in press) Let \( f_{n \times 3} \) be the flower graph then

\[
\dim(f_{n \times 3}) = \begin{cases} 
2, & n = 2k; \\
3, & n = 2k + 1. 
\end{cases}
\]

for every \( n \geq 6 \).

Kousar et al. (Kousar et al. 2010) investigated all graphs whose metric dimension is equal to its diameter. They found all non-isomorphic graphs whose diameter and dimension is two. They characterized these graphs by degree, distance and order.

**Theorem 3.82.** (Kousar et al. 2010) The number of non-isomorphic and connected graphs whose diameter as well as metric dimension is two is exactly \( 37 \).

**Lemma 3.21.** (Kousar et al. 2010) The number of non-isomorphic 2-dimensional connected graphs of diameter two with four or less vertices is exactly four.

**Lemma 3.22.** (Kousar et al. 2010) There are exactly eight non-isomorphic connected graphs of cardinality five such that their metric dimension as well as diameter is two.

**Lemma 3.23.** (Kousar et al. 2010) The number of non-isomorphic and connected graphs of order 6 whose diameter as well as metric dimension is two is exactly 25.

Sudhakara et al. (Sudhakara et al. 2009) characterized graphs having metric dimension two. They defined distance partition of vertex set \( V(G) \) of a connected graph \( G \) with reference to a vertex in it and proved the following results:

**Theorem 3.83.** (Sudhakara et al. 2009) For any vertex \( u \in V \), there exists a shortest path of length \( k \) between \( v_1 \) and \( u \). The path from \( v_1 \) to \( u \) contains exactly one vertex \( w_i \in V \) for \( 1 \leq i \leq k \) and the distance \( d(w_i, u) = k - i \).

**Theorem 3.84.** (Sudhakara et al. 2009) Let \( \{v_1, v_2\} \) be the metric basis of a graph \( G \), and their metric dimension \( \dim(G) = 2 \). Then there exists a unique geodesic path from \( v_1 \) and \( v_2 \).

**Theorem 3.85.** (Sudhakara et al. 2009) Let \( \{v_1, v_2\} \) be the minimum metric basis of \( G \), then \( \deg(v_1) < 3 \) and \( \deg(v_2) < 3 \).

Let \( V(G) = U_0 \cup U_1 \cup \ldots \cup U_k \) be a partition of the vertex set of \( G \). The partite sets \( U_0, U_1, U_2, \ldots, U_k \) are called distance partite sets with reference to a vertex \( u \in V(G) \) if \( U_0 = \{u\} \) and \( U_i \) contains those vertices which are at distance \( i \) from \( u \) for \( 0 < i < k \) where \( k \) is the eccentricity of \( u \) in \( G \). Using this idea Sudhakara et al. (Sudhakara et al. 2009) have shown that \( \{u_i, u_j\} \) is the minimum locating set for a connected graph \( G \). For any vertex \( u \in V(G) \) on the unique shortest path of the locating set vertices, there exists at most one vertex \( w \in V(G) \) adjacent to it in the distance partite set with respect to \( u_i \) to which it belongs. Further, \( u \) has exactly one vertex adjacent to it in the preceding distance partite set.
Theorem 3.86. (Sudhakara et al. 2009) The maximum degree of any vertex \( v \) on the unique shortest path between \( v_1 \) and \( v_2 \) is 5, if \( \{v_1, v_2\} \) be the minimum metric basis of a graph \( G \).

Theorem 3.87. (Sudhakara et al. 2009) Consider distance partite sets \( V_0, V_1, ..., V_k \) with reference to \( v_1 \) where \( \{v_1, v_2\} \) be the minimum metric basis of a connected graph \( G \). Any connected component of the graph \( G \) induced by a distance partite set \( \{V_0, V_1, ..., V_k\} \) is a path and degree of any vertex in the graph \( G \) induced by the set \( \{V_0, V_1, ..., V_k\} \) is at most 2.

Theorem 3.88. (Sudhakara et al. 2009) If the metric dimension of a connected graph \( G \) is two. Then it cannot have \( K_5 - e \) as a subgraph.

Corollary 3.5. (Sudhakara et al. 2009) Let \( G \) be connected graph with \( \dim(G) = 2 \), then \( G \) cannot have \( K_5 \) as a subgraph.

Corollary 3.6. (Sudhakara et al. 2009) For any graph \( G \) with metric dimension two the metric basis of the graph \( G \) cannot have a vertex \( u \) of a subgraph \( K_4 \) of \( G \).

Corollary 3.7. (Sudhakara et al. 2009) The maximum degree of any vertex \( v \in G \) in a graph \( G \) with metric dimension two is 8 and it is realizable.

Theorem 3.89. (Sudhakara et al. 2009) A graph \( G \) cannot have \( K_{3,3} \) as a subgraph. If the metric dimension of a graph \( G \) is two.

Javaid et al. (Javaid et al. 2008) determined that the antiprism graph represented by \( A_n \), generalized Petersen graph represented by \( P(n, 2) \) and Harary graph represented by \( H_{4,n} \) for \( n \equiv 1 \pmod{4} \) are classes of regular graphs having constant metric dimension and raise some questions in a more general setting.

Theorem 3.90. (Javaid et al. 2008) Let \( P(n, 2) \) be the generalized Petersen graph then \( \dim(P(n, 2)) = 3 \) for \( n \geq 5 \).

Theorem 3.91. (Javaid et al. 2008) Let \( A_n \) with \( n \geq 3 \) be an antiprism then \( \dim(A_n) = 3 \).

Theorem 3.92. (Javaid et al. 2008) Let \( H_{4,n} \) be a 4-regular Harary graph with \( n \geq 5 \) then \( \dim(H_{4,n}) = 3 \) when \( n \equiv 0, 2, 3 \pmod{4} \) and \( \dim(H_{4,n}) \leq 4 \) otherwise.
Chapter 4

On Metric Dimension of Path Related Graphs

4.1 Metric Dimension of The Power of Path Graphs

This chapter deals with the computation of metric dimension of $P_k^n$ for $k = 2, 3, M(P_n)$ and $T(P_n)$. It is shown that these graphs are families of graphs with constant metric dimension and only 2 or 3 vertices appropriately chosen suffice to resolve all the vertices of these graphs.

4.1.1 Metric Dimension of The Power of Path Graphs

Theorem 4.1. For $n \geq 4$, $\dim(P_2^n) = 2$.

Proof: Let $V(P_n) = \{v_1, v_2, ..., v_n\}$. By theorem 2.53 (Chartrand et al. 2000), it is clear that $\dim(P_2^n) \geq 2$. We show that $\dim(P_2^n) \leq 2$. Consider the following two cases:

Case(1). For $n = 2k + 1$ where $k \in \mathbb{N}$. Consider the set $W = \{v_1, v_n\} \subset V(P_2^n)$. We show that $W$ is a resolving set for $V(P_2^n)\setminus W$. The representations of the vertices of $V(P_2^n) \setminus W$ with respect to $W$ are:

- $r(v_{2i}|W) = (i, \lceil \frac{n}{2} \rceil - i)$, when $1 \leq i \leq \frac{n-1}{2}$
- $r(v_{2i+1}|W) = (i, \lceil \frac{n-2}{2} \rceil - i)$, when $1 \leq i \leq \frac{n-3}{2}$.

Consider two vertices $v_s$ and $v_t$ where $s \neq t$. Suppose that $r(v_s|W) = r(v_t|W)$, the following four subcases arise:

1. Consider $v_s, v_t \in V(P_2^n) \setminus W$, when $s, t$ are even, then $(\frac{s}{2}, \lceil \frac{n}{2} \rceil - \frac{s}{2}) = (\frac{t}{2}, \lceil \frac{n}{2} \rceil - \frac{t}{2})$. This equality holds iff $s = t$, a contradiction.

2. Let $v_s, v_t \in V(P_2^n) \setminus W$, when $s$ is even and $t$ is odd then $(\frac{s}{2}, \lceil \frac{n}{2} \rceil - \frac{s}{2}) = (\frac{t-1}{2}, \lceil \frac{n-1}{2} \rceil - \frac{t-1}{2})$, a contradiction.

3. Suppose $v_s, v_t \in V(P_2^n) \setminus W$, when $s, t$ are odd then $(\frac{s-1}{2}, \lceil \frac{n-2}{2} \rceil - \frac{s-1}{2}) = (\frac{t-1}{2}, \lceil \frac{n-2}{2} \rceil - \frac{t-1}{2})$. This equality holds iff $s = t$, a contradiction.

4. Consider $v_s, v_t \in V(P_2^n) \setminus W$, when $s$ is odd and $t$ is even then $(\frac{s-1}{2}, \lceil \frac{n-1}{2} \rceil - \frac{s-1}{2}) = (\frac{t}{2}, \lceil \frac{n}{2} \rceil - \frac{t}{2})$, a contradiction.
From above cases it can be concluded that \( r(v_i|W) \neq r(v_j|W) \). Hence \( \text{dim}(P_n^2) \leq 2 \), for \( n = 2k + 1 \).

**Case(2).** For \( n = 2k, k \in \mathbb{N} \). Consider the set \( W = \{v_1, v_2\} \subset V(P_n^2) \). We show that \( W \) is a resolving set for \( V(P_n^2) \). For this we find the representations of the vertices of \( V(P_n^2) \setminus W \) with respect to \( W \). Representations of the vertices are:

\[
r(v_2|W) = (i + 1, i), \quad 1 \leq i \leq n \\
r(v_2|W) = (i, i), \quad 1 \leq i \leq \frac{n}{2}.
\]

For any pair of distinct vertices \( v_s \) and \( v_t \) we have \( r(v_s|W) \neq r(v_t|W) \).

Thus \( \text{dim}(P_n^2) \leq 2 \). From case(1) and case(2), \( \text{dim}(P_n^2) = 2 \).

**Theorem 4.2.** For \( n \geq 4 \),

\[
\text{dim}(P_n^3) = \begin{cases} 
2, & \text{if } n = 4; \\
3, & \text{if } n > 4.
\end{cases}
\]

**Proof:** For \( n = 4 \), \( P_4^3 \cong C_4 \), hence by theorem 2.53 (Chartrand et al. 2000) \( \text{dim}(P_4^3) = 2 \). For \( n > 4 \), consider the set \( W = \{v_1, v_2, v_3\} \subset V(P_n^3) \). We find the representations of \( V(P_n^3) \setminus W \) with respect to \( W \). For \( \text{dim}(P_n^3) \leq 3 \), we have the following six possible cases:

**Case(1).** Consider the representations of all vertices \( v_j \) where \( j \equiv 0 \pmod{3} \). In this case \( j = 3k + 3 \) where \( k \in \mathbb{N} \). Representations of the vertices are:

\[
r(v_{3k+3}|W) = (i + 2, i + 1, i), \quad 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor.
\]

Choose two vertices \( v_s \) and \( v_t \) where \( s \neq t \) such that \( s, t \equiv 0 \pmod{3} \) and consider

\[
r(v_s|W) = r(v_t|W) = (\frac{i + 3}{3}, \frac{i - 3}{3}, \frac{i - 1}{3}) = (\frac{i + 2}{3}, \frac{i}{3}, \frac{i - 2}{3}),
\]

This equality holds iff \( s = t \), a contradiction. Thus all vertices \( v_j \), \( j \equiv 0 \pmod{3} \) have distinct representations.

**Case(2).** Consider the representations of all vertices \( v_j \) where \( j \equiv 1 \pmod{3} \). In this case \( j = 3k + 1 \) where \( k \in \mathbb{N} \). For \( 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor \) the representations of the vertices are:

\[
r(v_{3k+1}|W) = (i, i + 1, i).
\]

Choose two vertices \( v_s \) and \( v_t \) where \( s \neq t \) such that \( s, t \equiv 1 \pmod{3} \). If \( r(v_s|W) = r(v_t|W) \), then

\[
(\frac{s + 1}{3}, \frac{s - 2}{3}, \frac{s + 1}{3}) = (\frac{t + 1}{3}, \frac{t - 2}{3}, \frac{t + 1}{3}),
\]

which holds iff \( s = t \), a contradiction. Thus all the vertices \( v_j \), \( j \equiv 1 \pmod{3} \) have distinct representations.

**Case(3).** Consider the representations of all vertices \( v_j \) where \( j \equiv 2 \pmod{3} \). In this case \( j = 3k + 2 \) where \( k \in \mathbb{N} \). For \( 1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor \), the representations of the vertices are:

\[
r(v_{3k+2}|W) = (i + 1, i, i + 1).
\]

Consider vertices \( v_s \) and \( v_t \) where \( s \neq t \) such that \( s, t \equiv 2 \pmod{3} \). If \( r(v_s|W) = r(v_t|W) \), then

\[
(\frac{s + 1}{3}, \frac{s - 2}{3}, \frac{s + 1}{3}) = (\frac{t + 1}{3}, \frac{t - 2}{3}, \frac{t + 1}{3})
\]

This equality holds iff \( s = t \), a contradiction. Thus all vertices \( v_j \), \( j \equiv 2 \pmod{3} \) have distinct representations.

**Case(4).** Consider the pair of vertices \( v_s \) for \( s \equiv 0 \pmod{3} \) and \( v_t \) for \( t \equiv 1 \pmod{3} \). Consider \( r(v_s|W) = r(v_t|W) \), where

\[
(\frac{s + 3}{3}, \frac{s - 3}{3}, \frac{s - 1}{3}) = (\frac{t + 2}{3}, \frac{t - 3}{3}, \frac{t - 1}{3}),
\]

impossible.

**Case(5).** Consider vertices \( v_s \) for \( s \equiv 0 \pmod{3} \) and \( v_t \) for \( t \equiv 2 \pmod{3} \). Consider \( r(v_s|W) = r(v_t|W) \), implying that

\[
(\frac{s + 3}{3}, \frac{s - 3}{3}, \frac{s - 1}{3}) = (\frac{t + 1}{3}, \frac{t - 2}{3}, \frac{t + 1}{3}),
\]

not possible.

**Case(6).** Choose \( v_s \) for \( s \equiv 1 \pmod{3} \) and \( v_t \) for \( t \equiv 2 \pmod{3} \). If \( r(v_s|W) = r(v_t|W) \),

\[
(\frac{s - 1}{3}, \frac{s + 2}{3}, \frac{s - 3}{3}) = (\frac{t + 1}{3}, \frac{t - 2}{3}, \frac{t + 1}{3}),
\]

not possible.

From all the above cases \( \text{dim}(P_n^3) \leq 3 \).

We show that \( \text{dim}(P_n^3) \geq 3 \), by proving that there is no resolving set \( W \subset V(P_n^3) \) such that
\(|W| = 2\). On contrary, assume a resolving set \(W \subset V(P_n^3)\), \(|W| = 2\). All the possible cases are as follows:

**Case(1)**. Consider the resolving set \(W = \{v_t, v_s\}\), with \(t, s \equiv 0 \mod 3\) and \(t < s\). In this case:
\[
r(v_2|W) = r(v_4|W) = \left(\frac{2}{3}, \frac{2}{3}\right), \quad \text{a contradiction}.
\]

**Case(2)**. Suppose that \(W = \{v_t, v_s\}\), with \(t, s \equiv 1 \mod 3\) and \(t < s\). In this case:
\[
r(v_3|W) = r(v_5|W) = \left(\frac{t-1}{3}, \frac{s-1}{3}\right), \quad \text{a contradiction}.
\]

**Case(3)**. Consider \(W = \{v_t, v_s\}\), with \(t, s \equiv 2 \mod 3\) and \(t < s\). In this case:
\[
r(v_4|W) = r(v_6|W) = \left(\frac{t}{3}, \frac{s}{3}\right), \quad \text{a contradiction}.
\]

**Case(4)**. Consider \(W = \{v_t, v_s\}\), with \(s \equiv 0 \mod 3\), \(s \neq 3\) and \(t \equiv 1 \mod 3\). Then for the pair of vertices \(v_3\) and \(v_5\), \(r(v_3|W) = r(v_5|W) = \left(\frac{t-3}{3}, \frac{s-1}{3}\right), \quad \text{a contradiction}.
\]

Similarly, if \(W = \{v_t, v_s\}\), with \(t \equiv 1 \mod 3\), then for \(v_1\) and \(v_5\), \(r(v_1|W) = r(v_5|W) = \left(2, \frac{t-1}{3}\right), \quad \text{a contradiction}.
\]

**Case(5)**. Consider \(W = \{v_t, v_s\}\), with \(s \equiv 0 \mod 3\) and \(t \equiv 2 \mod 3\). For the pair of vertices \(v_2\) and \(v_4\), \(r(v_2|W) = r(v_4|W) = \left(\frac{s}{3}, \frac{t-2}{3}\right), \quad \text{a contradiction}.
\]

**Case(6)**. Consider \(W = \{v_t, v_s\}\), with \(s \equiv 1 \mod 3\) and \(t \equiv 2 \mod 3\). For the pair of vertices \(v_1\) and \(v_3\), consider \(r(v_1|W) = r(v_3|W) = \left(\frac{t-1}{3}, \frac{s+1}{3}\right), \quad \text{a contradiction}.
\]

From all the above cases, it is concluded that there is no resolving set \(W\) with \(|W| = 2\) for \(V(P_n^3)\) showing that \(dim(P_n^3) \geq 3\).

Thus \(dim(P_n^3) = 3\).

\[4.1\] The Middle Graph of Path Graph

**Theorem 4.3.** For \(n \geq 4\), \(dim(M(P_n)) = 2\).

**Proof:** Let \(M(P_n)\) be the Middle graph of \(P_n\). The order of \(M(P_n)\) is \(2n - 1\). By theorem 2.53 (Chartrand et al. 2000), it is clear that \(dim(M(P_n)) \geq 2\). We prove that \(dim(M(P_n)) \leq 2\). The vertex set of \(M(P_n)\) is \(V(M(P_n)) = V_1 \cup V_2\), where \(V_1 = \{v_1, \ldots, v_n\}\) and \(V_2 = \{e_1, \ldots, e_{n-1}\}\). Where \(\{e_1, e_2, \ldots, e_n\} \subset E(P_n)\). Consider the set \(W = \{e_1, v_n\} \subset V(M(P_n))\). We show that \(W\) is a resolving set for \(V(M(P_n))\). For this we find the representations of the vertices \(V(M(P_n)) \setminus W\) with respect to \(W\). Representations of the vertices are:
\[
r(v_1|W) = (1, n),
\]
\[
r(v_i|W) = (i - 1, n + 1 - i), \text{ where } 2 \leq i \leq n,
\]
\[
r(e_i|W) = (i - 1, n - i), \text{ where } 1 \leq i \leq n.
\]

Consider two vertices \(v_t\) and \(e_t\) where \(s \neq t\), and \(r(v_s|W) = r(v_t|W)\), then \((s - 1, n + 1 - s) = (t - 1, n + 1 - t), \text{ a contradiction, because } s \neq t, \text{ and hence } r(v_s|W) \neq r(v_t|W)\).

Similarly, it can be verified that all vertices in \(M(P_n)\) have distinct representations with respect to \(W\) which yields \(dim(M(P_n)) \leq 2\). We can check that \(r(e_s|W) \neq r(e_t|W)\) and \(r(v_s|W) \neq r(e_t|W)\). Which completes the proof.

\[4.1.3\] The Total Graph of Path Graph

**Theorem 4.4.** For \(n \geq 4\), \(dim(T(P_n)) = 2\).
Proof: \( T(P_n) \) be the total graph of \( P_n \). By theorem 2.53 (Chartrand et al. 2000), \( \text{dim}(T(P_n)) \geq 2 \). We show that \( \text{dim}(T(P_n)) \leq 2 \). Consider the set \( W = \{v_1, e_1\} \subset V(T(P_n)) \). Consider the representations of the vertices of \( V(T(P_n)) \setminus W \) with respect to \( W \). Representations of the vertices are:
\[
\begin{align*}
& r(e_i|W) = (i, i-1), \text{ where } 2 \leq i \leq n-1, \\
& r(v_i|W) = ((i-1), i-1), \text{ where } 2 \leq i \leq n.
\end{align*}
\]
Choose two vertices \( v_s \) and \( e_t \) where \( s \neq t \). Consider \( r(v_s|W) = r(v_t|W) \), then \( (s, s-1) = (t, t-1) \), a contradiction, because \( s \neq t \), so \( r(v_s|W) \neq r(v_t|W) \). Similarly, it ca be verified that all vertices in \( T(P_n) \) have distinct representations with respect to \( W \), yielding \( \text{dim}(T(P_n)) \leq 2 \). Which implies that \( r(e_s|W) \neq r(e_t|W) \) and \( r(v_s|W) \neq r(e_t|W) \). Which completes the proof.

\[\square\]

4.2 Metric dimension of \( P_n(1, 2, 3) \) and \( H_n \)

In this section we computes the metric dimension of \( P_n(1, 2, 3) \) and \( H_n \), defined in chapter 1. It is shown that the metric dimension of these graphs is constant and only 3 vertices that are appropriately chosen will suffice to resolve all the vertices of these graphs.

![Graph P_n(1, 2, 3)](image)

Figure 4.1: Graph \( P_n(1, 2, 3) \)

4.2.1 The \( P_n(1, 2, 3) \) Graph

Theorem 4.5. For \( n \geq 6 \), \( \text{dim}(P_n(1, 2, 3)) = 3 \).

Proof: Consider the set \( W = \{v_1, v_2, v_3\} \subset V(P_n(1, 2, 3)) \), we show that \( W \) is a resolving set for \( V(P_n(1, 2, 3)) \). We find the representations of the vertices of \( V(P_n(1, 2, 3)) \setminus W \) with respect to \( W \). For this the following three possibilities arise:

Case(1). \( n \equiv 0 \pmod{3} \), where \( n \geq 6 \), then the representations of the vertices are:
\[
\begin{align*}
& r(v_3|W) = (i, i, i-1), \text{ where } n \geq 6, \\
& r(v_{3i+1}|W) = (i, i, i), \text{ where } i = 1, 2, \ldots, \left\lfloor \frac{n}{3} \right\rfloor, \\
& r(v_{3i+2}|W) = (i+1, i, i), \text{ where } i = 1, 2, \ldots, \left\lfloor \frac{n}{3} \right\rfloor-1.
\end{align*}
\]
From the above representations for any two vertices \( v_i, v_j \in V(P_n(1, 2, 3)) \) \( i \neq j \) \( r(v_i|W) \neq r(v_j|W) \) implying that \( \text{dim}(P_n(1, 2, 3)) \leq 3 \).

Now we show that \( \text{dim}(P_n(1, 2, 3)) \geq 3 \). Consider a resolving set \( W' \) with \( |W'| = 2 \). The following possible cases arise:

1. Without loss of generality, consider \( W' = \{v_3, v_t\} \), where \( t \equiv 0 \pmod{3} \) and \( 3 < t \leq n \), implying that \( r(v_4|\{v_3, v_t\}) = r(v_5|\{v_3, v_t\}) = (1, \frac{t-3}{3}) \), a contradiction.

2. Without loss of generality, consider \( W' = \{v_4, v_t\} \), where \( t \equiv 1 \pmod{3} \) and \( 4 < t \leq n \),
so \( r(v_1|\{v_4,v_t\}) = r(v_2|\{v_4,v_t\}) = (1,\frac{t-1}{3}) \), a contradiction.

3). Without loss of generality, consider \( W' = \{v_5,v_t\} \), where \( t \equiv 2 \pmod{3} \) and \( 5 < t \leq n \), which gives \( r(v_2|\{v_5,v_t\}) = r(v_3|\{v_5,v_t\}) = (1,\frac{t+2}{3}) \), a contradiction.

4). Without loss of generality, consider \( W' = \{v_3,v_t\} \), where \( t \equiv 1 \pmod{3} \) and \( 3 < t \leq n \). Then \( r(v_1|\{v_3,v_t\}) = r(v_2|\{v_3,v_t\}) = (1,\frac{t-1}{3}) \), a contradiction.

5). Without loss of generality, consider \( W' = \{v_3,v_t\} \), where \( t \equiv 2 \pmod{3} \) and \( 3 < t \leq n \), which shows that \( r(v_2|\{v_3,v_t\}) = r(v_4|\{v_3,v_t\}) = (1,\frac{t-2}{3}) \), a contradiction.

6). Without loss of generality, consider \( W' = \{v_4,v_t\} \), where \( t \equiv 2 \pmod{3} \) and \( 4 < t \leq n \). Then \( r(v_2|\{v_4,v_t\}) = r(v_3|\{v_4,v_t\}) = (1,\frac{t-3}{3}) \), a contradiction.

Hence, from the above cases it follows that there is no resolving set with two vertices for \( V(P_n(1,2,3)) \). Therefore, \( \dim(P_n(1,2,3)) = 3 \).

**Case(2).** \( n \equiv 1 \pmod{3} \), where \( n \geq 6 \), then the representations of the vertices are as follows:

\[ r(v_3|W) = (i,i,i-1), \text{ for } i = 2,...,\frac{n-1}{3} \]

\[ r(v_{3+i}|W) = (i,i,i), \text{ for } i = 1,2,...,\frac{n-1}{3} \]

\[ r(v_{3+i+1}|W) = (i+1,i,i), \text{ for } i = 1,2,...,\frac{n-1}{3} - 1 \]

It follows that for any two vertices \( v_i,v_j \in V(P_n(1,2,3)) \) \( i \neq j \), \( r(v_i|W) \neq r(v_j|W) \), implying that \( \dim(P_n(1,2,3)) \leq 3 \).

For \( \dim(P_n(1,2,3)) \geq 3 \), by proving that there is no resolving set \( W' \) with \( |W'| = 2 \) as in case (1). So \( \dim(P_n(1,2,3)) = 3 \).

**Case(3).** Let \( n \equiv 2 \pmod{3} \), where \( n \geq 6 \), the representations of the vertices are as follows:

\[ r(v_3|W) = (i,i,i-1), \text{ for } i = 2,...,\frac{n-2}{3} \]

\[ r(v_{3+i}|W) = (i,i,i), \text{ for } i = 1,2,...,\frac{n-2}{3} \]

\[ r(v_{3+i+1}|W) = (i+1,i,i), \text{ for } i = 1,2,...,\frac{n-2}{3} \]

Also in this case \( r(v_i|W) \neq r(v_j|W) \) for any \( v_i,v_j \in V(P_n(1,2,3)) \) \( i \neq j \).

Thus, \( \dim(P_n(1,2,3)) \leq 3 \). For \( \dim(P_n(1,2,3)) \geq 3 \), we can show that there is no resolving set \( W' \) with \( |W'| = 2 \) as in case (1). So \( \dim(P_n(1,2,3)) = 3 \).

### 4.2.2 The \( H_n \) Graph

**Theorem 4.6.** For \( n \geq 6 \), \( \dim(H_n) = 3 \).

**Proof:** We discuss the following two cases:

**Case(1).** \( n = 2k, k \geq 3, k \in \mathbb{N} \). Consider the set \( W = \{c_1,a_1,c_{k+1}\} \subset V(H_n) \). We show that \( W \) is a resolving set for \( V(H_n) \). The representations of the vertices of \( V(H_n) \setminus W \) with respect to \( \{c_1,a_1,c_{k+1}\} \) are:

\[ r(c_i|W) = \begin{cases} (i+1,i,3+k-i), & 2 \leq i \leq k; \\ (2k-i+3,2k+3-i,i+1-k), & k+2 \leq i \leq n. \end{cases} \]

\[ r(a_i|W) = \begin{cases} (i+1,i,2+k-i), & 2 \leq i \leq k; \\ (2k-i+2,2k+2-i,i+1-k), & k+1 \leq i \leq n. \end{cases} \]
And

\[ r(b_i|W) = \begin{cases} 
(1, 1, k+1), & i = 1; \\
(i, i-1, k+2-i), & 2 \leq i \leq k+1; \\
(2k+2-i, 2+2k-i, i-k), & k+2 \leq i \leq n. 
\end{cases} \]

For any two vertices \( v_i, v_j \in V(H_n) \) \( i \neq j \), \( r(v_i|W) = r(v_j|W) \) implying that \( dim(H_n) \leq 3 \). We show that \( dim(H_n) \geq 3 \), by proving that there is no resolving set \( W' \) with \( |W'| = 2 \). The following possibilities arise:

(1). Both the vertices of \( W' \) belong to the set \( \{b_1, b_2, ..., b_n\} \subset V(H_n) \). Without loss of generality, suppose that one resolving vertex is \( b_1 \) and the other is \( b_t \), \( 2 \leq t \leq k+1 \). For \( 2 \leq t \leq k \), \( r(c_1|\{b_1, b_t\}) = r(a_n|\{b_1, b_t\}) = (1, t) \). For \( t = k+1 \), \( r(a_k|\{b_1, b_t\}) = r(a_{k+1}|\{b_1, b_t\}) = (k, 1) \), a contradiction.

(2). Both the vertices of the resolving set belong to the set \( \{c_1, c_2, ..., c_n\} \subset V(H_n) \). Without loss of generality, suppose that one resolving vertex is \( c_t \) and the other is \( c_1 \), \( 2 \leq t \leq k+1 \). For \( 2 \leq t \leq k \), \( r(b_n|\{c_1, c_t\}) = r(a_n|\{c_1, c_t\}) = (2, t + 1) \). For \( t = k+1 \), \( r(a_k|\{c_1, c_t\}) = r(a_{k+1}|\{c_1, c_t\}) = (k+1, 2) \), a contradiction.

(3). Both the vertices of \( W' \) belong to \( \{a_1, a_2, ..., a_n\} \subset V(H_n) \). Suppose that one resolving vertex is \( a_t \) and the other is \( a_1 \), \( 2 \leq t \leq k+1 \). For \( 2 \leq t \leq k \), \( r(c_1|\{a_1, a_t\}) = r(b_n|\{a_1, a_t\}) = (t, 2) \). For \( t = k+1 \), \( r(b_{k+1}|\{a_1, a_t\}) = r(b_{k+2}|\{a_1, a_t\}) = (k, 1) \), a contradiction.

(4). One vertex of the resolving set belong to \( \{b_1, b_2, ..., b_n\} \subset V(H_n) \), and the other belong to \( \{c_1, c_2, ..., c_n\} \subset V(H_n) \). Without loss of generality, suppose that one resolving vertex is \( b_1 \) and the other is \( c_t \), \( 1 \leq t \leq k+1 \). For \( 1 \leq t \leq k \), \( r(a_n|\{b_1, c_t\}) = r(b_n|\{b_1, c_t\}) = (1, t + 1) \). For \( t = k+1 \), \( r(a_k|\{b_1, c_t\}) = r(a_{k+1}|\{b_1, c_t\}) = (k+1, 1) \), a contradiction.

(5). One vertex of \( W' \) belong to \( \{b_1, b_2, ..., b_n\} \subset V(H_n) \), and the other belong to \( \{a_1, a_2, ..., a_n\} \subset V(H_n) \). Without loss of generality, suppose that one resolving vertex is \( b_1 \) and the other is \( a_t \), \( 1 \leq t \leq k+1 \). For \( 1 \leq t \leq k \), \( r(a_n|\{b_1, a_t\}) = r(c_1|\{b_1, a_t\}) = (1, t + 1) \). For \( t = k+1 \), \( r(b_k|\{b_1, a_t\}) = r(a_{k+2}|\{b_1, a_t\}) = (k+1, 2) \), a contradiction.

(6). One vertex of \( W' \) belongs to \( \{c_1, c_2, ..., c_n\} \subset V(H_n) \) and the other belongs to \( \{a_1, a_2, ..., a_n\} \subset V(H_n) \). Without loss of generality, suppose that one resolving vertex is \( c_1 \) and the other is \( a_t \), \( 1 \leq t \leq k+1 \). For \( 1 \leq t \leq k-1 \), \( r(a_n|\{c_1, a_t\}) = r(b_n|\{c_1, a_t\}) = (2, t + 1) \). For \( t = k+1 \), \( r(a_k|\{c_1, a_t\}) = r(c_{k+2}|\{c_1, a_t\}) = (k+1, 2) \) similarly, for \( t = k+1 \), \( r(a_k|\{c_1, a_t\}) = r(c_{k+2}|\{c_1, a_t\}) = (k+1, 2) \), a contradiction. Hence it follows that there is no resolving set with two vertices for \( V(H_n) \) implying that \( dim(H_n) = 3 \).

Case(2). For \( n = 2k+1, k \geq 3, \ k \in \mathbb{N} \). Consider the set \( W = \{c_1, a_1, c_{k+1}\} \subset V(H_n) \). We show that \( W \) is a resolving set for \( V(H_n) \). The representations of vertices of \( V(H_n) \setminus W \) with respect to \( W \) are as follows:

\[ r(c_i|W) = \begin{cases} (i+1, i, 3+k-i), & 2 \leq i \leq k; \\
(2k+i+4, 2k+4-i, 1+i-k), & k+2 \leq i \leq n. 
\end{cases} \]
4.2 Metric dimension of $P_n(1,2,3)$ and $H_n$

$$r(a_i|W) = \begin{cases} 
(i+1,i,k+2-i), & 2 \leq i \leq k, \\
(k+2,k+1,2), & i = k+1, \\
(2k+3-i,3+2k-i,i-k+1), & k+2 \leq i \leq n.
\end{cases}$$

$$r(b_j|W) = \begin{cases} 
(1,1,k+1), & i = 1, \\
(i,i-1,2+k-i), & 2 \leq i \leq k+1, \\
(2k+3-i,3+2k-i,i-k), & k+2 \leq i \leq n.
\end{cases}$$

For any two vertices $v_i, v_j \in V(H_n)$ $i \neq j$, $r(v_i|W) \neq r(v_j|W)$. Which implies that $dim(H_n) \leq 3$.

We show that $dim(H_n) \geq 3$, assume that $dim(H_n) = 2$, proceeding on the same lines as in case(1) it can be shown that $dim(H_n) \geq 3$. Finally from case(1) and (2), $dim(H_n) = 3$. \[ \square \]

**Summery**

In this chapter, the metric dimension of some well known families of graphs has been investigated. It is shown that the families of graphs obtained from the path graph by the power, middle, total graph, $P_n(1,2,3)$ and $H_n$ $n3$ operation have a constant metric dimension. We compute the metric dimension of these families of graphs and show that only 2 or 3 vertices appropriately chosen suffice to resolve all the vertices of these graphs. And shown that the metric dimension of these families independent of the order of the graphs.
Chapter 5

Metric Dimension of Rotationally Symmetric Graphs

5.1 Metric Dimension of Möbius Strips Graph

In this chapter the metric dimension of an important family of circulant graph has been discussed. A circulant graph is a graph of $n$ vertices in which the $i$th graph vertex is adjacent to the $(i + j)th$ and $(i - j)th$ graph vertices for each $j$. The circulant graph $C_i n(1,2,...,\lfloor \frac{n}{2} \rfloor)$ gives the complete graph $K_n$ and the graph $C_i n(1)$ gives the cyclic graph $C_n$.

The Möbius strip graph denoted by $M_n$ is a cubic circulant graph with an even number of nodes made from an $n$-cycle by adding edges connecting opposite pairs of nodes in the cycle. It is so named because (with the exception of $M_6 = K_3,3$) $M_n$ has exactly $\frac{n}{2}$ 4–cycles (cycle with size four) which link together by their common edges to form a topological Möbius strip. Möbius strip can also be viewed as a prism with one twisted edge. Möbius strip has many applications in chemistry, chemical stereography, electronics and computer science. Möbius strip $M_n$ as an $n$-cycle by adding edges connecting opposite pairs of vertices in the cycle. Suppose that the vertices of Möbius strip $M_n$ are labelled as $\{v_1, ..., v_n\}$ in the counter clockwise direction.

In the next theorem, we prove that only three vertices suffice to resolve all the vertices for Möbius strip $V(M_n)$ except when $n \equiv 2 \pmod{8}$. In that, case we prove that $3 \leq \text{dim}(M_n) \leq 4$. Note that the choice of an appropriate basis of vertices is the core of the problem.
Theorem 5.1. Let $M_n$ be the Möbius strip, then for every even positive integer $n \geq 8$, $\dim(M_n) = 3$ when $n \not\equiv 2 \pmod{8}$ and $3 \leq \dim(M_n) \leq 4$ otherwise.

Proof: (a) Suppose that $n \not\equiv 2 \pmod{8}$. The following cases arise.

Case(1). $n \equiv 0 \pmod{8}$

In this case, we can write $n = 8k, k \in \mathbb{N}$. Consider the set $W = \{v_1, v_2, v_{4k+1}\} \subset V(M_n)$. We show that $W$ is a resolving set for $M_n$. The representations of the vertices of $V(M_n) \setminus W$ with respect to $W$ are:

$$r(v_{2i+1}|W) = \begin{cases} 
(2i, 2i - 1, 2i + 1), & 1 \leq i \leq k - 1; \\
(2k, 2k - 1, 2k), & i = k; \\
(4k - 2i + 1, 4k - 2i + 2, 4k - 2i), & k + 1 \leq i \leq 2k - 1; \\
(2i - 4k + 1, 2i - 4k, 2i - 4k), & 2k + 1 \leq i \leq 3k - 1; \\
(2k, 2k, 2k), & i = 3k; \\
(8k - 2i, 8k - 2i + 1, 8k - 2i + 1), & 3k + 1 \leq i \leq 4k - 1.
\end{cases}$$

And

$$r(v_{2j}|W) = \begin{cases} 
(2i - 1, 2i - 2, 2i), & 2 \leq i \leq k; \\
(2k, 2k, 2k - 1), & i = k + 1; \\
(4k - 2i + 2, 4k - 2i + 3, 4k - 2i + 1), & k + 2 \leq i \leq 2k; \\
(2i - 4k, 2i - 4k - 1, 2i - 4k - 1), & 2k + 1 \leq i \leq 3k; \\
(8k - 2i + 1, 8k - 2i + 2, 8k - 2i + 2), & 3k + 1 \leq i \leq 4k.
\end{cases}$$

For any two vertices $v_i, v_j \in V(M_n)$ $i \neq j$, $r(v_i|W) \neq r(v_j|W)$, implying that $\dim(M_n) \leq 3$. Now we show that $\dim(M_n) \geq 3$ by proving that there is no resolving set $W$ such that $|W| = 2$. Suppose on contrary that $\dim(M_n) = 2$ i.e there exists a resolving set having exactly two vertices.

Without loss of generality, assume that $W = \{v_1, v_t\}$ where $(2 \leq t \leq 4k + 1)$. Then
for $2 \leq t \leq 4k$, $r(v_n|\{v_1, v_t\}) = r(v_{4k+1}|\{v_1, v_t\}) = (1, t)$ and when $t = 4k + 1$, $r(v_n|\{v_1, v_{4k+1}\}) = r(v_2|\{v_1, v_{4k+1}\}) = (1, 2)$, a contradiction.

Note that there is no resolving set with two vertices for $V(M_n)$, implying that $\dim(M_n) = 3$ in this case.

**Case(2).** $n \equiv 4 \pmod{8}$.

In this case $n = 8k + 4$, $k \in \mathbb{N}$. Consider the set $W = \{v_1, v_2, v_{4k+3}\} \subset V(M_n)$, we show that $W$ is a resolving set for $V(M_n)$. The representations of the vertices of $V(M_n) \setminus W$ with respect to $W$ are:

$$r(v_{2i+1}|W) = \begin{cases} 
(2i, 2i - 1, 2i + 1), & 1 \leq i \leq k; \\
(2k + 1, 2k + 1, 2k), & i = k + 1; \\
(4k - 2i + 3, 4k - 2i + 4, 4k - 2i + 2), & k + 2 \leq i \leq 2k; \\
(2i - 4k - 1, 2i - 4k - 2, 2i - 4k - 2), & 2k + 2 \leq i \leq 3k; \\
(8k - 2i + 4, 8k - 2i + 3, 8k - 2i + 3), & 3k + 1 \leq i \leq 4k + 1.
\end{cases}$$

And

$$r(v_{2i}|W) = \begin{cases} 
(2i - 1, 2i - 2, 2i), & 2 \leq i \leq k; \\
(2k + 1, 2k, 2k + 1), & i = k + 1; \\
(4k - 2i + 4, 4k - 2i + 5, 4k - 2i + 3), & k + 2 \leq i \leq 2k + 1; \\
(2i - 4k - 2, 2i - 4k - 3, 2i - 4k - 3), & 2k + 2 \leq i \leq 3k + 2; \\
(8k - 2i + 5, 8k - 2i + 6, 8k - 2i + 6), & 3k + 3 \leq i \leq 4k + 2.
\end{cases}$$

For any two vertices $v_i, v_j \in V(M_n)$ $i \neq j$, $r(v_i|W) \neq r(v_j|W)$, implying that $\dim(M_n) \leq 3$.

Now we show that $\dim(M_n) \geq 3$ by proving that there is no resolving set $W$ such that $|W| = 2$. Suppose on contrary that $\dim(M_n) = 2$ i.e there exists a resolving set having exactly two vertices.

Assume that one resolving vertex is $v_1$, and the second resolving vertex is $v_t$ ($2 \leq t \leq 4k + 3$). Then for $2 \leq t \leq 4k + 2$ we have $r(v_n|\{v_1, v_t\}) = r(v_{4k+3}|\{v_1, v_t\}) = (1, t)$ and when $t = 4k + 1$, $r(v_n|\{v_1, v_{4k+3}\}) = r(v_2|\{v_1, v_{4k+3}\}) = (1, 2)$, a contradiction.

This signifies that there is no resolving set with two vertices for $V(M_n)$, implying that $\dim(M_n) = 3$ in this case.

**Case(3).** When $n \equiv 6 \pmod{8}$ and $n = 8k + 6$, $k \in \mathbb{N}$. For $n = 6$, $M_6 \cong K_{3,3}$ hence $\dim(M_6) = 4$ because $\dim(K_{n,n}) = 2n - 2$. For every $n \geq 14$, consider the set $W = \{v_1, v_2, v_{4k+3}\} \subset V(M_n)$, we show that $W$ is a resolving set for $M_n$ in this case. For this the representation of the vertices of $V(M_n) \setminus W$ with respect to $W$ are:

$$r(v_{2i+1}|W) = \begin{cases} 
(2i, 2i - 1, 2i + 1), & 1 \leq i \leq k; \\
(2k + 2, 2k + 1, 2k + 1), & i = k + 1; \\
(4k - 2i + 4, 4k - 2i + 5, 4k - 2i + 3), & k + 2 \leq i \leq 2k + 1; \\
(2i - 4k - 2, 2i - 4k - 3, 2i - 4k - 3), & 2k + 2 \leq i \leq 3k + 2; \\
(8k - 2i + 6, 8k - 2i + 7, 8k - 2i + 7), & 3k + 3 \leq i \leq 4k + 2.
\end{cases}$$
show that

\[ V \]

It is proved that there is no resolving set with two vertices for

\[ r \]

In this case,

\[ dim \]

Suppose on contrary that

\[ dim \]

\[ dim \]

\[ dim \]

\[ dim \]

\[ dim \]

\[ dim \]

For any two vertices \( v_i, v_j \in V(M_n) \) \( i \neq j \), \( r(v_i|W) \neq r(v_j|W) \), implying that \( dim(M_n) \leq 3 \).

We show that \( dim(M_n) \geq 3 \) by proving that there is no resolving set \( W \) such that \( |W| = 2 \).

Suppose on contrary that \( dim(M_n) = 2 \) i.e there exists a resolving set having exactly two vertices.

Consider one resolving vertex is \( v_1 \), and the other resolving vertex is \( v_t \) \( (2 \leq t \leq 4k + 4) \).

Then for \( 2 \leq t \leq 4k + 3 \), \( r(v_n|\{v_1, v_t\}) = r(v_{4k+4}|\{v_1, v_t\}) = (1, t) \) and when \( t = 4k + 1 \), \( r(v_n|\{v_1, v_{4k+4}\}) = r(v_2|\{v_1, v_{4k+4}\}) = (1, 2), \) a contradiction.

It is proved that there is no resolving set with two vertices for \( V(M_n) \), implying that \( dim(M_n) = 3 \) in this case.

(b) When \( n \equiv 2 \) (mod 8).

In this case, \( n = 8k + 2, k \in \mathbb{N} \). Consider the set \( W = \{v_1, v_2, v_{4k+2}, v_{6k+2}\} \subset V(M_n) \), we show that \( W \) is a resolving set for \( M_n \). For this first we give the representations of the vertices of \( V(M_n) \) \( \setminus W \) with respect to \( W = \{v_1, v_2, v_{4k+2}\} \) are:

\[ r(v_{2i+1}|W') = \begin{cases} 
(2i - 1, 2i - 2, 2i), & 1 \leq i \leq k; \\
(4k - 2i - 2, 4k - 2i - 3, 4k - 2i - 1), & k + 2 \leq i \leq 2k; \\
(2i - 4k + 4, 2i - 4k + 3, 2i - 4k + 3), & 2k + 1 \leq i \leq 3k; \\
(8k - 2i - 6, 8k - 2i - 5, 8k - 2i - 5), & 3k + 1 \leq i \leq 4k.
\end{cases} \]

And

\[ r(v_{2i}|W') = \begin{cases} 
(2i - 1, 2i - 2, 2i), & 2 \leq i \leq k; \\
(2k + 1, 2k, 2k), & i = k + 1; \\
(4k - 2i - 1, 4k - 2i, 4k - 2i - 2), & k + 2 \leq i \leq 2k; \\
(2i - 4k + 3, 2i - 4k + 2, 2i - 4k + 2), & 2k + 2 \leq i \leq 3k + 1; \\
(8k - 2i - 5, 8k - 2i - 4, 8k - 2i - 4), & 3k + 2 \leq i \leq 4k + 1.
\end{cases} \]

Note that the set \( W' \) can resolve all the vertices of \( M_n \) except the vertices \( v_{2k+2} \) and \( v_{2k+2} \). As \( r(v_{2k+2}|W') = r(v_{4k+2}|W') = (2k + 1, 2k, 2k) \), it suggests that \( W = W' \cup \{v_{2k+2}\} \) is a resolving set for \( M_n \), implying that \( dim(M_n) \leq 4 \).

Now we show that \( dim(M_n) \geq 3 \) by proving that there is no resolving set \( W \) such that \( |W| = 2 \). Suppose on contrary that \( dim(M_n) = 2 \) i.e there exists a resolving set having exactly two vertices.

Without loss of generality, suppose that \( W = \{v_1, v_t\} \) where \( (2 \leq t \leq 4k + 2) \). Then for \( 2 \leq t \leq 4k + 1 \), \( r(v_n|\{v_1, v_t\}) = r(v_{4k+1}|\{v_1, v_t\}) = (1, t) \) and when \( t = 4k + 2 \), \( r(v_n|\{v_1, v_{4k+3}\}) = r(v_2|\{v_1, v_{4k+3}\}) = (1, 2), \) a contradiction.

From above there is no resolving set with two vertices for \( V(M_n) \), implying that \( dim(M_n) \geq 3 \), which completes the proof.
5.2 Prism Related Graphs with Constant Metric Dimension.

This section deals with the study of extension of some families related to the prism graphs. These graphs are denoted by $D_n^*$ and $D_n^0$. It shows that these graphs have constant metric dimension and only 2 or 3 vertices appropriately chosen suffice to resolve all the vertices of these graphs.

![Graphs](image)

Figure 5.2: Graphs $(D_n^*)$ and $(D_n^0)$

5.2.1 The $D_n^*$ Graph

**Theorem 5.2.** For $n \geq 6$, $\dim(D_n^*) = 3$.

**Proof:** Consider two cases.

**Case (1).** Suppose $n = 2k$, $k \geq 3$, $k \in \mathbb{N}$. Consider the set $W = \{c_1, c_2, c_{k+1}\} \subset V(D_n^*)$.

The representations of the vertices of $V(D_n^*) \setminus W$ with respect to $W$ are:

$$r(c_i|W) = \begin{cases} 
(i - 1, i - 2, 1 + k - i), & 3 \leq i \leq k; \\
(2k - i + 1, 2k + 2 - i, i - 1 - k), & k + 2 \leq i \leq n.
\end{cases}$$

$$r(b_i|W) = \begin{cases} 
(1, 2, k + 1), & i = 1; \\
(i, i - 1, k + 2 - i), & 2 \leq i \leq k + 1; \\
(2k + 2 - i, 3 + 2k - i - k), & k + 2 \leq i \leq n.
\end{cases}$$
$V$.

We show that $\dim(D_n^*) \geq 3$ by proving that there is no resolving set $W$ with $|W| = 2$. The following possibilities arise:

(1) Consider the set $W = \{c_1, c_2\}$, with $2 \leq t \leq k + 1$. For $2 \leq t \leq k$,
\[ r(c_1|W) = r(b_1|W) = (1, t). \]
And for $t = k + 1$, $r(c_2|W) = r(c_n|W) = (1, k - 1)$, a contradiction.

(2) Without loss of generality, suppose that $W = \{b_1, b_2\}$, where $2 \leq t \leq k + 1$. For $2 \leq t \leq k$
\[ r(c_1|W) = r(a_1|W) = (1, t), \]
a contradiction.

(3) Let $W = \{a_1, a_2\}$, $2 \leq t \leq k + 1$. For $2 \leq t \leq k$, $r(c_1|W) = r(a_n|W) = (2, t + 1)$. And for $t = k + 1$,
\[ r(b_1|W) = r(b_n|W) = (1, k), \]
a contradiction.

(4) Consider the resolving set $W = \{c_1, b_1\}$, where $1 \leq t \leq k + 1$. For $1 \leq t \leq k$,
\[ r(a_1|W) = r(b_1|W) = (2, t). \]
For $t = k + 1$, $r(b_n|W) = r(b_2|W) = (1, k - 1)$, a contradiction.

(5) Consider the resolving set $W = \{c_1, a_1\}$, where $1 \leq t \leq k + 1$. For $1 \leq t \leq k - 1$,
\[ r(a_1|W) = r(b_{n-1}|W) = (3, t + 1). \]
For $t = k$, $r(b_1|W) = r(c_2|W) = (1, k - 1)$. Similarly, for $t = k + 1$, $r(b_1|W) = r(c_2|W) = (1, k - 1)$, a contradiction.

(6) Consider the resolving set $W = \{b_1, a_1\}$, where $1 \leq t \leq k + 1$. For $1 \leq t \leq k$,
\[ r(a_1|W) = r(c_1|W) = (2, t + 1). \]
For $t = k + 1$, $r(c_k|W) = r(a_{k+2}|W) = (k, 2)$, a contradiction.

Hence it follows that there is no resolving set with two vertices for $V(D_n^*)$ implying that $\dim(D_n^*) = 3$.

Case (2). Suppose $n = 2k + 1, k \geq 3, k \in \mathbb{N}$. Consider the set $W = \{c_1, c_2, c_{k+1}\} \subset V(D_n^*)$.
We show that $W$ is a resolving set for $V(D_n^*)$. The representations of the vertices of $V(D_n^*) \setminus W$ with respect to $W$ are:

\[ r(c_i|W) = \begin{cases} (i - 1, i - 2, k + 1 - i), & 3 \leq i \leq k; \\ (2k + 2 - i, 2k + 2 - i, 1), & i = k + 2; \\ (2k + 2 - i, 2k + 3 - i, i - k - 1), & k + 3 \leq i \leq n. \end{cases} \]

\[ r(b_i|W) = \begin{cases} (1, 2, k + 1), & i = 1; \\ (i, i - 1, k + 2 - i), & 2 \leq i \leq k + 1; \\ (k + 1, k + 1, 2), & i = k + 2; \\ (2k + 3 - i, 2k + 4 - i, i - k), & k + 3 \leq i \leq n. \end{cases} \]

\[ r(a_i|W) = \begin{cases} (2, 3, k + 2), & i = 1; \\ (2, 2, k + 1), & i = 2; \\ (i, i - 1, k + 3 - i), & 3 \leq i \leq k + 1; \\ (k + 2, k + 1, 2), & i = k + 2; \\ (2k + 4 - i, 2k + 5 - i, i - k), & k + 3 \leq i \leq n. \end{cases} \]
Proceeding on same line as in case(1), there are no two vertices having the same representations, implying that \( \dim(D_n^r) \leq 3 \).

We show that \( \dim(D_n^r) \geq 3 \). Assume that \( \dim(D_n^r) = 2 \). As in case(1) contradiction can be established, so \( \dim(D_n^r) \geq 3 \). Thus from case(1) and (2), \( \dim(D_n^r) = 3 \). \( \square \)

### 5.2.2 The \( D_n^r \) Graph

**Theorem 5.3.** For \( n \geq 3 \),

\[
\dim(D_n^r) = \begin{cases} 
2, & n = 2k + 1; \\
3, & n = 2k.
\end{cases}
\]

**Proof:** Consider two cases.

**Case(1).** When \( n = 2k + 1 \), \( k \in \mathbb{N} \). Suppose the set \( W = \{c_1, c_{k+1}\} \subset V(D_n^r) \), we show that \( W \) is resolving set for \( V(D_n^r) \). The representations of the vertices of \( V(D_n^r) \setminus W \) with respect to \( W \) are:

\[
r(c_i|W) = \begin{cases} 
(i-1, k+1-i), & 2 \leq i \leq k; \\
(2k+2-i, i-k-1), & k+2 \leq i \leq n.
\end{cases}
\]

\[
r(b_i|W) = \begin{cases} 
(i, k-i+2), & 1 \leq i \leq k+1; \\
(2k+3-i, i-k), & k+2 \leq i \leq n.
\end{cases}
\]

\[
r(a_i|W) = \begin{cases} 
(i+1, k-i+3), & 1 \leq i \leq k+1; \\
(2k+4-i, i-k+1), & k+2 \leq i \leq n.
\end{cases}
\]

Since these representations are pair wise distinct, it follows that \( \dim(D_n^r) \leq 2 \). By (Chartrand et al. 2000) \( \dim(D_n^r) \geq 2 \). Which implies that \( \dim(D_n^r) = 2 \) in this case.

**Case(2).** When \( n = 2k \), \( k \in \mathbb{N} \). Consider the set \( W = \{c_1, c_2, c_{k+1}\} \subset V(D_n^r) \), we show that \( W \) is a resolving set for \( V(D_n^r) \). The representations of the vertices of \( V(D_n^r) \setminus W \) with respect to \( W \) are:

\[
r(c_i|W) = \begin{cases} 
(i-1, i-2, k+1-i), & 3 \leq i \leq k; \\
(2k+1-i, 2k+2-i, i-k-1), & k+2 \leq i \leq n.
\end{cases}
\]

\[
r(b_i|W) = \begin{cases} 
(1, 2, k+1), & i = 1; \\
(i, i-1, k+2-i), & 2 \leq i \leq k+1; \\
(2k+2-i, 2k+3-i, i-k), & k+2 \leq i \leq n.
\end{cases}
\]

\[
r(a_i|W) = \begin{cases} 
(2, 3, k+2), & i = 1; \\
(i+1, i, k+3-i), & 2 \leq i \leq k+1; \\
(2k+i+3, 2k-i+4, i-k+1), & k+2 \leq i \leq n.
\end{cases}
\]

For any two vertices \( v_i, v_j \in V(D_n^r) \), \( i \neq j \), \( r(v_i|W) \neq r(v_j|W) \), implying that \( \dim(D_n^r) \leq 3 \).

For \( \dim(D_n^r) \geq 3 \) by proving that there is no resolving set \( W \) with \( |W| = 2 \). The following possible cases arise:

(1). Without loss of generality, suppose that \( W = \{c_1, c_t\} \), for \( 2 \leq t \leq k+1 \). For \( 2 \leq t \leq k \),
$r(c_n|W) = r(b_1|W) = (1,t)$. For $t = k + 1$, $r(c_2|W) = r(c_n|W) = (1,k-1)$, a contradiction.

(2). Without loss of generality, suppose that $W = \{b_1,b_t\}$, for $2 \leq t \leq k + 1$. For $2 \leq t \leq k + 1$, $r(c_1|W) = r(a_1|W) = (1,t)$, a contradiction.

(3). Consider the resolving set $W = \{a_1,a_t\}$, where $2 \leq t \leq k + 1$. For $2 \leq t \leq k$, $r(c_1|W) = r(b_1|W) = (2,t+1)$. For $t = k + 1$, $r(c_2|W) = r(a_n|W) = (3,k+1)$, a contradiction.

(4). Consider the resolving set $W = \{c_1,b_t\}$, where $1 \leq t \leq k + 1$. For $1 \leq t \leq k$, $r(a_1|W) = r(b_1|W) = (2,t)$. For $t = k + 1$, $r(b_n|W) = r(b_2|W) = (2,k-1)$, a contradiction.

(5). Consider the resolving set $W = \{c_1,a_t\}$, where $1 \leq t \leq k + 1$. For $1 \leq t \leq k - 1$, $r(a_1|W) = r(b_{n-1}|W) = (3,t+2)$. For $t = k$, $r(b_1|W) = r(c_2|W) = (1,k)$. Similarly, for $t = k + 1$, $r(b_1|\{c_1,a_t\}) = r(c_2|\{c_1,a_t\}) = (1,k+1)$, a contradiction.

(6). Consider the resolving set $W = \{b_1,a_t\}$, where $1 \leq t \leq k + 1$. For $1 \leq t \leq k$, $r(a_1|\{b_1,a_t\}) = r(c_n|\{b_1,a_t\}) = (2,t+2)$. For $t = k + 1$, $r(b_k|\{b_1,a_t\}) = r(b_{k+2}|\{b_1,a_t\}) = (k-1,2)$, a contradiction.

Hence from above it follows that there is no resolving set with two vertices for $V(D^*_n)$ implying that $dim(D^*_n) = 3$. \qed

### 5.3 Antiprism Related Graphs with Constant Metric Dimension

This section contains the metric dimension of some families of graphs which are constructed from antiprism graph. These graphs are denoted by $A^+_n$ and $A^-_n$. These graphs have constant metric dimension and independent of the order of the graph so only 2 or 3 vertices appropriately chosen suffice to resolve all the vertices of these graphs.

#### 5.3.1 The $A^+_n$ Graph

**Theorem 5.4.** Let $n \geq 6$ be an integer then $dim(A^+_n) = 3$.

**Proof:** Case(1). Let $n = 2k$, $k \geq 3$, $k \in N$. Consider the set $W = \{c_1,c_2,c_{k+1}\} \subset V(A^+_n)$, we show that $W$ is a resolving set for $V(A^+_n)$. The representations of the vertices of $V(A^+_n) \setminus W$ with respect to $W$ are:

$$r(c_i|W) = \begin{cases} (i-1,i-2,\frac{n+2}{2}-i), & 3 \leq i \leq k; \\ (n-i+1,n+2-i,i-\frac{n+2}{2}), & k+2 \leq i \leq n. \end{cases}$$

$$r(b_i|W) = \begin{cases} (1,1,\frac{n}{2}), & i = 1; \\ (i,i-1,\frac{n+2}{2}-i), & 2 \leq i \leq k; \\ \left(\frac{n}{2},\frac{n}{2},1\right), & i = k + 1; \\ (n+1-i,n+2-i,i-\frac{n}{2}), & k+2 \leq i \leq n. \end{cases}$$
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Figure 5.3: Graphs \( (A^*_n) \) and \( (A^0_n) \)

\[
r(a_i | W) = \begin{cases} 
(2, 2, \frac{n+2}{2}), & i = 1; \\
(2, 2, \frac{n}{2}), & i = 2; \\
(i, i-1, \frac{n+4}{2} - i), & 3 \leq i \leq k; \\
(\frac{n+1}{2}, \frac{n}{2}, 2), & i = k+1; \\
(n+2-i, n+3-i, i-\frac{n}{2}), & k+2 \leq i \leq n.
\end{cases}
\]

For any two vertices \( v_i, v_j \in V(A^*_n) \) \( i \neq j \), \( r(v_i | W) \neq r(v_j | W) \) implying that \( \dim(A^*_n) \leq 3 \).

To prove the theorem it is sufficient to show that \( \dim(A^*_n) \geq 3 \). Contrarily, assume that there exists a resolving set \( W' \) with \( |W'| = 2 \). The following possible cases arise:

(1). Consider the resolving set \( W = \{c_1, c_i\} \), where \( 2 \leq t \leq k+1 \). For \( 2 \leq t \leq k \), \( r(c_n | W') = r(b_n | W') = (1, t) \). For \( t = k+1 \), \( r(b_1 | W') = r(b_n | W') = (1, k) \), a contradiction.

(2). Assume that \( W' = \{b_1, b_t\} \), where \( 2 \leq t \leq k+1 \). For \( 2 \leq t \leq k \), \( r(b_n | W') = r(a_1 | W') = (1, t) \). For \( t = k+1 \), \( r(a_1 | W') = r(a_2 | W') = (1, k) \), a contradiction.

(3). Consider the resolving set \( W = \{a_1, a_t\} \), where \( 2 \leq t \leq k+1 \). For \( 2 \leq t \leq k \), \( r(c_n | W') = r(a_n | W') = (2, t+1) \). For \( t = k+1 \), \( r(b_1 | W') = r(b_n | W') = (1, k) \), a contradiction.

(4). Assume that \( W' = \{c_1, b_t\} \), where \( 1 \leq t \leq k+1 \). For \( 1 \leq t \leq k \), \( r(b_{k+1} | W') = r(c_{k+1} | W') = (k, k+1-t) \). For \( t = k+1 \), \( r(c_{k+1} | W') = r(a_{k+2} | W') = (k, 1) \), a contradiction.

(5). Assume that \( W' = \{c_1, a_t\} \), where \( 1 \leq t \leq k+1 \). For \( 1 \leq t \leq k \), \( r(a_{k+1} | W') = r(c_{k+1} | W') = (k, k-t+2) \). For \( t = k+1 \), \( r(b_k | W') = r(b_{k+1} | W') = (k, 1) \), a contradiction.

(6). Assume that \( W' = \{b_1, a_t\} \), where \( 1 \leq t \leq k+1 \). Consider \( r(a_n | W') = r(c_n | W') \)
\[ \Rightarrow (2, t + 1), \text{ a contradiction.} \]

Hence from above it follows that there is no resolving set with two vertices for \( V(A_n^n) \).

Therefore, \( \dim(A_n^n) \geq 3 \) which implies that \( \dim(A_n^n) = 3. \)

**Case(2)**. Let \( n = 2k + 1, k \geq 3, k \in \mathbb{N}. \) Consider the set \( W = \{ c_1, c_2, c_{k+1} \} \subset V(A_n^n). \) The representations of the vertices of \( V(A_n^n) \setminus W \) with respect to \( W \) are:

\[
\begin{align*}
\text{for } c_i & : \\
& \begin{cases}
(i - 1, i - 2, \frac{n+1}{2} - i), & 3 \leq i \leq k; \\
(n + 1 - i, n + 2 - i, i - \frac{n+1}{2}), & k + 3 \leq i \leq n.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{for } b_i & : \\
& \begin{cases}
(1, 1, \frac{n-1}{2}), & i = 1; \\
(i - 1, \frac{n+1}{2} - i), & 2 \leq i \leq k; \\
(n + 1 - i, n + 2 - i, i - \frac{n-1}{2}), & k + 2 \leq i \leq n.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{for } a_i & : \\
& \begin{cases}
(2, 2, \frac{n+1}{2}), & i = 1; \\
(2, 2, \frac{n-1}{2}), & i = 2; \\
(i - 1, \frac{n+3}{2} - i), & 3 \leq i \leq k; \\
(n + 2 - i, n + 3 - i, i - \frac{n}{2}), & k + 2 \leq i \leq n.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{for } c_i & : \\
& \begin{cases}
(1, 1, \frac{n}{2}), & i = 1; \\
(i - 1, \frac{n+2}{2} - i), & 2 \leq i \leq k + 1; \\
(n + 1 - i, n + 2 - i, i - \frac{n}{2}), & k + 2 \leq i \leq n.
\end{cases}
\end{align*}
\]

For any two vertices \( v_i, v_j \in V(A_n^n) \) \( i \neq j, \) \( r(v_i|W) \neq r(v_j|W) \) implying that \( \dim(A_n^n) \leq 3. \)

Now we show that \( \dim(A_n^n) \geq 3. \) For this we take contrarily \( \dim(A_n^n) = 2 \) proceeding on the same line as in case(1), a contradiction can be deduced analogously. This implies that \( \dim(A_n^n) \geq 3. \) Finally, from case(1) and case(2), \( \dim(A_n^n) = 3 \) which completes the proof. \( \square \)

### 5.3.2 The \( A_n^n \) Graph

**Theorem 5.5.** Let \( n \geq 6 \) be an integer then \( \dim(A_n^n) = 3. \)

**Proof:** **Case(1).** Let \( n = 2k k \geq 3 k \in \mathbb{N}. \) Consider the set \( W = \{ b_1, b_2, b_{k+1} \} \subset V(A_n^n). \)

The representations of the vertices of \( V(A_n^n) \setminus W \) with respect to \( W \) are:

\[
\begin{align*}
\text{for } b_i & : \\
& \begin{cases}
(i - 1, i - 2, \frac{n+2}{2} - i), & 3 \leq i \leq k; \\
(n - i + 1, n + 2 - i, i - \frac{n+1}{2}), & k + 2 \leq i \leq n.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{for } a_i & : \\
& \begin{cases}
(1, 2, \frac{n+2}{2}), & i = 1; \\
(i - 1, \frac{n+1}{2} - i), & 2 \leq i \leq k + 1; \\
(n + 2 - i, n + 3 - i, i - \frac{n}{2}), & k + 2 \leq i \leq n.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{for } c_i & : \\
& \begin{cases}
(1, 1, \frac{n}{2}), & i = 1; \\
(i - 1, \frac{n+2}{2} - i), & 2 \leq i \leq k; \\
(\frac{n}{2}, \frac{n+1}{2}, 1), & i = k + 1; \\
(n + 1 - i, n + 2 - i, i - \frac{n}{2}), & k + 2 \leq i \leq n.
\end{cases}
\end{align*}
\]

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For any two vertices \( v_i, v_j \in V(A_n^a) \) \( i \neq j \), \( r(v_i|W) \neq r(v_j|W) \) implying that \( \dim(A_n^a) \leq 3 \). Now we show that \( \dim(A_n^a) \geq 3 \). Contrarily assume that there exists a resolving set \( W' \) with \(|W'| = 2\). We have the following possibilities:

1. Consider the resolving set \( W' = \{c_1, c_t\} \), where \( 2 \leq t \leq k + 1 \). For \( 2 \leq t \leq k \), \( r(c_n|W') = r(b_1|W') = (1,t) \). For \( t = k + 1 \), \( r(c_k|W') = r(c_{k+1}|W') = (1,k-1) \), a contradiction.

2. Assume that the resolving set \( W' = \{b_1, b_t\} \), where \( 2 \leq t \leq k + 1 \). For \( 2 \leq t \leq k \), \( r(c_n|W') = r(a_1|W') = (1,t) \). For \( t = k + 1 \), \( r(c_k|W') = r(c_{k+1}|W') = (k,1) \), a contradiction.

3. Consider the resolving set \( W' = \{a_1, a_t\} \), where \( 2 \leq t \leq k + 1 \). For \( 2 \leq t \leq k \), \( r(a_1|W') = r(b_n|W') = (2,t+1) \). For \( t = k + 1 \), \( r(c_k|W') = r(c_{k+1}|W') = (k+1,2) \), a contradiction.

4. Consider the resolving set \( W' = \{c_1, b_t\} \), where \( 1 \leq t \leq k + 1 \). For \( 1 \leq t \leq k \), \( r(a_1|W') = r(b_n|W') = (2,t) \). For \( t = k + 1 \), \( r(c_k|W') = r(c_{k+1}|W') = (k,1) \), a contradiction.

5. Consider the resolving set \( W' = \{c_1, a_t\} \), where \( 1 \leq t \leq k + 1 \). For \( 1 \leq t \leq k - 1 \), \( r(a_1|W') = r(b_{n-1}|W') = (3,t+2) \). For \( k \leq t \leq k + 1 \), \( r(b_2|W') = r(c_2|W') = (1,t-1) \), a contradiction.

6. Consider the resolving set \( W' = \{b_1, a_t\} \), where \( 1 \leq t \leq k + 1 \). For \( 1 \leq t \leq k \), \( r(b_n|W') = r(c_n|W') = (1,t+1) \). For \( t = k + 1 \), \( r(c_k|W') = r(c_{k+1}|W') = (k,2) \), a contradiction.

Hence from above it follows that there is no resolving set with two vertices for \( V(A_n^a) \) implying that \( \dim(A_n^a) = 3 \).

**Case (2).** When \( n = 2k + 1 \), \( k \geq 3 \) \( k \in N \). Consider the set \( W = \{c_1, c_2, c_{k+1}\} \subset V(A_n^a) \), we show that \( W \) is a resolving set for \( V(A_n^a) \). The representations of the vertices of \( V(A_n^a) \setminus W \) with respect to \( W \) are:

\[
\begin{align*}
r(c_i|W) &= \begin{cases} 
(i - 1, i - 2, \frac{n+1}{2} - i), & 3 \leq i \leq k; \\
(\frac{n-1}{2}, \frac{n-1}{2}, 1), & i = k + 2; \\
(n + 1 - i, n + 2 - i, i - \frac{n+1}{2}), & k + 3 \leq i \leq n.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
r(b_i|W) &= \begin{cases} 
(1, 2, \frac{n+1}{2}), & i = 1; \\
(1, 1, \frac{n-1}{2}), & i = 2; \\
(i - 1, i - 2, \frac{n+3}{2} - i), & 3 \leq i \leq k + 1; \\
(\frac{n+1}{2}, \frac{n-1}{2}, 1), & i = k + 2; \\
(n + 2 - i, n + 3 - i, i - \frac{n+1}{2}), & k + 3 \leq i \leq n.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
r(a_i|W) &= \begin{cases} 
(2, 3, \frac{n+3}{2}), & i = 1; \\
(2, 2, \frac{n+1}{2}), & i = 2; \\
(i, i - 1, \frac{n+5}{2} - i), & 3 \leq i \leq k + 1; \\
(\frac{n+3}{2}, \frac{n+1}{2}, 2), & i = k + 2; \\
(n + 3 - i, n + 4 - i, i - \frac{n+1}{2}), & k + 2 \leq i \leq n.
\end{cases}
\end{align*}
\]

For any two vertices \( v_i, v_j \in V(A_n^a) \) \( i \neq j \), \( r(v_i|W) \neq r(v_j|W) \) implying that \( \dim(A_n^a) \leq 3 \). We show that \( \dim(A_n^a) \geq 3 \). Assume that \( \dim(A_n^a) = 2 \). Proceeding on the same way as...
in case(1) $\dim(A^n_n) \geq 3$. Finally, from case(1) and case(2), $\dim(A^n_n) = 3$. For $n \leq 5$ the $\dim(A^n_n) = 3$ is also true, which completes the proof.

**Theorem 5.6.** Let $G \cong 2C_n + \{x_n y_n\}$ then $\dim(G) = 2$ for $n \geq 3$.

**Proof:** By theorem 2.53 (Chartrand et al. 2000), it is clear that, $\dim(G) \geq 2$. We show that the set $W = \{x_1, y_1\}$ is a resolving set for $G$.

**Case (1).** When $n = 2k$ for $k \in \mathbb{N}$. Representations of all vertices from $V(G) \setminus \{x_1, y_1\}$ are as follows:

$$r(y_{i+1}|W) = \begin{cases} (i, i+3), & 1 \leq i \leq k-1; \\ (n-i, n-i+1), & k \leq i \leq n-1. \end{cases}$$

$$r(x_{i+1}|W) = \begin{cases} (i+3, i), & 1 \leq i \leq k-1; \\ (n-i+1, n-i), & k \leq i \leq n-1. \end{cases}$$

It can be verified that all the above representations are distinct. For example suppose that $(s+3, s) = (n-j, n-j+1)$ for some fixed $s$ and $j$. Then $s = n-j-3$ and $s = n-j+1$, a contradiction. 

**Case (2).** When $n = 2k+1$ for $k \in \mathbb{N}$. Representations of the vertices from $V(G) \setminus \{x_1, y_1\}$ are as follows:

$$r(y_{i+1}|W) = \begin{cases} (i, i+3), & 1 \leq i \leq k-1; \\ (k, k+2), & i = k; \\ (n-i, n-i+1), & k+1 \leq i \leq n-1. \end{cases}$$

$$r(x_{i+1}|W) = \begin{cases} (i+3, i), & 1 \leq i \leq k-1; \\ (k+2, k), & i = k; \\ (n-i+1, n-i), & k+1 \leq i \leq n-1. \end{cases}$$

In this also all the above representations are distinct. 

**Theorem 5.7.** For all $n \geq 4$, $\dim(T_{n,m}) = 2$.

**Proof:** By theorem 2.53 (Chartrand et al. 2000), $\dim(T_{n,m}) \geq 2$. We show that there is a resolving set $W$ of cardinality 2.

**Case (1).** When $n = 2k$ for $k \in \mathbb{N}$. Consider the set $W = \{v_k, v_{k+1}\} \subset V(T_{n,m})$. We show that $W$ is a resolving set for the graph $T_{n,m}$. For this the representations of all vertices of $V(G) \setminus W$ are:

$$r(v_i|W) = \begin{cases} (k-i, k-i+1), & 1 \leq i \leq k-1; \\ (i-k, i-k-1), & k+2 \leq i \leq n. \end{cases}$$

Note that all the representations are distinct. Suppose that $(k+s, k+s-1) = (j-k, j-k-1)$ for some fixed $s$ and $j$. Then $j = 2k+s > n$ because $s \geq 1$, a contradiction.

**Case (2).** When $n = 2k+1$, for $k \in \mathbb{N}$. Consider the set $W = \{v_1, v_{n-1}\} \subset V(T_{n,m})$. We show that $W$ is a resolving set for the graph $T_{n,m}$. For this the representations of all vertices
of \( V(G) \setminus W \) are:

\[
\begin{align*}
  r(v_i|W) = \begin{cases} 
    (i - 1, i + 1), & 2 \leq i \leq k - 1; \\
    (i - 1, n - i - 1), & k \leq i \leq k + 1; \\
    (n - i + 1, n - i - 1), & k + 2 \leq i \leq n - 2; \\
    (1, 1), & i = n.
  \end{cases}
\end{align*}
\]

In this case also all the above representations are distinct. \( \square \)

### 5.4 Metric Dimension of Sun Related Graphs

This section of the chapter deals with metric dimension of sun related graphs. It is shown that these graphs have constant metric dimension and only two or three appropriately chosen vertices suffice to resolve all the vertices of these graphs.

#### 5.4.1 The \( S_n' \) Graph

**Theorem 5.8.** For \( n \geq 3 \),

\[
dim(S_n') = \begin{cases} 
  2, & \text{for } 3 \leq n \leq 5; \\
  3, & \text{for } n \geq 6.
\end{cases}
\]

**Proof:** By theorem 2.53 (Chartrand et al. 2000), \( W = \{v_1, v_2\} \) is a resolving set for \( S_n' \) when \( 3 \leq n \leq 5 \). For \( n \geq 6 \) consider the set \( W = \{v_1, v_2, v_k\} \subset V(S_n') \). The representations of \( V(S_n') \setminus W \) vertices are as follows:

\[
\begin{align*}
  r(u_i|W) = \begin{cases} 
    (1, 2, k), & i = 1; \\
    (i - 1, k + 1 - i), & 2 \leq i \leq k; \\
    (k + 1, k, 2), & i = k + 1; \\
    (2k - i + 2, 2k - i + 3, i - k + 1), & k + 2 \leq i \leq n.
  \end{cases}
\end{align*}
\]

And

\[
\begin{align*}
  r(v_i|W) = \begin{cases} 
    (i + 1, i, k + 2 - i), & 3 \leq i \leq k - 1; \\
    (k + 2, k + 1, 3), & i = k + 1; \\
    (2k - i + 3, 2k - i + 4, i - k + 2), & k + 2 \leq i \leq n.
  \end{cases}
\end{align*}
\]

From the above representation for any two vertices \( v_i, v_j \in V(S_n') \) \( i \neq j \), \( r(v_i|W) \neq r(v_j|W) \) implying that \( dim(S_n') \leq 3 \). We now show that \( dim(S_n') \geq 3 \) by proving that there is no resolving set \( W \) with \( |W| = 2 \) for \( V(S_n') \). Contrarily, assume that \( |W| = 2 \) then the following possibilities arise:

1. Consider the resolving set \( W = \{u_1, u_t\} \) for \( 2 \leq t \leq k + 1 \). For \( 2 \leq t \leq k \), \( r(u_n|\{u_1, u_t\}) = r(v_1|\{u_1, u_t\}) = (1, t) \). For \( t = k + 1 \), \( r(u_{t+1}|\{u_1, u_t\}) = r(u_n|\{u_1, u_t\}) = (1, k - 1) \), a contradiction.

2. Without loss of generality, suppose that \( W = \{v_1, v_t\} \) where \( 2 \leq t \leq k + 1 \). For \( 2 \leq t \leq k - 1 \), \( r(v_{t+1}|\{v_1, v_t\}) = r(u_{t+2}|\{v_1, v_t\}) = (t + 2, 3) \). For \( t = k \), \( r(u_{t+2}|\{v_1, v_t\}) =
Thus $dim(V) = (k, 3)$. Similarly for $t = k + 1$, $r(v_{t-1}|\{v_1, v_t\}) = r(u_n|\{v_1, v_t\}) = (3, t)$, a contradiction.

(3). Assume that $W = \{u_1, v_t\}$ where $1 \leq t \leq k + 1$. For $1 \leq t \leq k - 1$, $r(v_n|\{u_1, v_t\}) = r(u_{n-1}|\{u_1, v_t\}) = (2, t + 2)$. For $t = k$, $r(u_t, v_2|\{u_1, v_t\}) = r(v_{t-1}|\{u_1, v_t\}) = (k - 1, 3)$. Similarly for $t = k + 1$, $r(u_t|\{u_1, v_t\}) = r(u_2|\{u_1, v_t\}) = (1, k)$, a contradiction.

Hence from above it follows that there is no resolving set with two vertices for $V(S_n')$. Thus $dim(S_n') = 3$. 

5.4.2 The $S_n''$ Graph

**Theorem 5.9.** For $n \geq 3$,

$$dim(S_n'') = \begin{cases} 2, & \text{for } n = 2k; \\ 3, & \text{for } n = 2k + 1. \end{cases}$$

**Proof:** Case(1). For $n = 2k$, $k \in \mathbb{N}$. Consider the set $W = \{v_1, v_k\} \subset V(S_n'')$. We show that $W$ is resolving set for $V(S_n'')$. The representations of the vertices are as follows:

$$r(u_i|W) = \begin{cases} (1, k), & i = 1; \\ (i - 1, k + 1 - i), & 2 \leq i \leq k; \\ (k, 1), & i = k + 1; \\ (2k - 2 - i, i - k), & k + 2 \leq i \leq 2k. \end{cases}$$

And

$$r(v_i|W) = \begin{cases} (i, k - i + 1), & 2 \leq i \leq k - 1; \\ (2k + 2 - i, i - k + 1), & k + 1 \leq i \leq 2k - 1. \end{cases}$$

Since these representations are pair-wise distinct it follows that $dim(S_n'') \leq 2$. By (Chartrand et al. 2000), it is clear that $dim(S_n'') \geq 2$ showing that $dim(S_n'') = 2$ for even $n$.

Case(2). Let $n = 2k + 1$, $k \in \mathbb{N}$. Consider the set $W = \{v_1, v_2, v_{k+1}\} \subset V(S_n'')$, we show that $W$ is resolving set for $S_n''$. The representations of the vertices of $V(S_n'') \setminus W$ with respect to $W$ are:

$$r(u_i|W) = \begin{cases} (1, 3 - i, k + 2 - i), & 1 \leq i \leq 2; \\ (i - 1, i - 2, k + 2 - i), & 3 \leq i \leq k + 1; \\ (k + 1, k, 1), & i = k + 2; \\ (2k - 3, 2k - i + 4, i - k), & k + 3 \leq i \leq 2k + 1. \end{cases}$$

And

$$r(v_i|W) = \begin{cases} (i, i - 1, k + 2 - i), & 3 \leq i \leq k; \\ (k + 1, k + 1, 2), & i = k + 2; \\ (2k - i + 2, 2k - i + 3, i - k + 1), & k + 3 \leq i \leq 2k. \end{cases}$$

Note, that there are no two vertices having the same representations implying that $dim(S_n'') \leq 3$. We show that $dim(S_n'') \geq 3$ by proving that there is no resolving set having two vertices. Contrarily, suppose that $|W| = 2$, then the following possibilities arise:
Thus, \( \text{dim}(H) \leq t \leq k+1 \). For \( 2 \leq t \leq k-1 \), \( r(u_{n-1}\{u_1,u_t\}) = r(v_{n-1}\{u_1,u_t\}) = (2,t) \). For \( t = k \), \( r(v_t\{u_1,u_t\}) = r(u_{t+1}\{u_1,u_t\}) = (t,1) \), a contradiction. Similarly, for \( t = k+1 \), \( r(v_t\{u_1,u_t\}) = r(u_n\{u_1,u_t\}) = (1,t) \), a contradiction.

(2). Consider the resolving set \( W = \{v_1,v_t\} \) where \( 2 \leq t \leq k+1 \). For \( 2 \leq t \leq k-1 \), \( r(v_{n-1}\{v_1,v_t\}) = r(u_{n-1}\{v_1,v_t\}) = (3,t+2) \). For \( t = k \), \( r(u_{t+2}\{v_1,v_t\}) = r(v_{t+1}\{v_1,v_t\}) = (t+1,1) \). Similarly for \( t = k+1 \), \( r(v_2\{v_1,v_t\}) = r(u_n\{v_1,v_t\}) = (2,k) \), a contradiction.

(3). Consider the resolving set \( W = \{u_1,v_t\} \) where \( 1 \leq t \leq k+1 \). For \( 1 \leq t \leq k \), \( r(v_{t+1}\{u_1,v_t\}) = r(u_{t+2}\{u_1,v_t\}) = (t+1,2) \). For \( t = k+1 \), \( r(u_t\{u_1,v_t\}) = r(u_{t+2}\{u_1,v_t\}) = (k,1) \), a contradiction.

Hence, from above it follows that there is no resolving set with two vertices for \( V(S_n^\ast) \). Thus, \( \text{dim}(S_n^\ast) = 3 \). 

5.4.3 The \( S_n^\ast \) Graph

Theorem 5.10. For \( n \geq 3 \),

\[
\text{dim}(S_n^\ast) = \begin{cases} 
2, & \text{for } n = 2k+1; \\
3, & \text{for } n = 2k, \text{except } n = 4.
\end{cases}
\]

Proof: Case(1). For \( n = 2k+1, k < \mathbb{N} \). Suppose the set \( W = \{v_1,v_{k+1}\} \subseteq V(S_n^\ast) \). We show that \( W \) is resolving set for \( V(S_n^\ast) \). The representations of the vertices of \( V(S_n^\ast) \setminus W \) with respect to \( W \) are:

\[
r(u_i|W) = \begin{cases} 
(i,k+1-i), & 1 \leq i \leq k; \\
(i,1), & i = k+1; \\
(2k+3-i,i-k), & k+2 \leq i \leq 2k+1.
\end{cases}
\]

And

\[
r(v_i|W) = \begin{cases} 
(i,k-i+2), & 2 \leq i \leq k; \\
(2k+4-i,i-k), & k+2 \leq i \leq 2k+2.
\end{cases}
\]

Since these representations are pair wise distinct it follows that \( \text{dim}(S_k^\ast) \leq 2 \). By (Chartrand et al. 2000), it is clear that \( \text{dim}(S_k^\ast) \geq 2 \) showing that \( \text{dim}(S_k^\ast) = 2 \) for odd \( n \).

Case(2). For \( n = 2k, k \in \mathbb{N} \), when \( k = 1 \) then \( \text{dim}(S_k^\ast) = 2 \). For \( k \geq 2 \), suppose \( W = \{v_1,v_2,v_{k+1}\} \subseteq V(S_k^\ast) \), we show that \( W \) is resolving set for \( S_k^\ast \). The representations of the vertices of \( V(S_k^\ast) \setminus W \) with respect to \( W \) are:

\[
r(u_i|W) = \begin{cases} 
(1,1,k), & i = 1; \\
(i,i-1,k+1-i), & 2 \leq i \leq k; \\
(k+1,k,1), & i = k+1; \\
(2k-i+3,2k-i+3,i-k), & k+2 \leq i \leq 2k.
\end{cases}
\]
Thus, $\dim W(2)$. Without loss of generality, suppose that a contradiction. Similarly for $r(2k+4-i, 2k+4-i, i-k)$, $k+3 \leq i \leq 2k+1$; $r(3,3,k+2)$, $i = 2k+2$.

From the above representation for any two vertices $v_i, v_j \in V(S_n^*)$ $i \neq j$, $r(v_i|W) \neq r(v_j|W)$ implying that $\dim(S_n^*) \leq 3$. We now show that $\dim(S_n^*) \geq 3$ by proving that there is no resolving set having two vertices. Contrarily, suppose that $|W| = 2$ then the following possibilities arise:

1. Consider the resolving set $W = \{u_1,u_t\}$ where $2 \leq t \leq k+1$. For $2 \leq t \leq k-1$, $r(v_1|\{u_1,u_t\}) = r(u_n|\{u_1,u_t\}) = (1,t)$. For $t = k$, $r(v_i|\{u_1,u_t\}) = r(u_{t+1}|\{u_1,u_t\}) = (t,1)$, a contradiction. Similarly for $t = k+1$, $r(v_3|\{u_1,u_t\}) = r(v_n|\{u_1,u_t\}) = (1,k-1)$, a contradiction.

2. Without loss of generality, suppose that $W = \{v_1,v_t\}$ where $2 \leq t \leq k+1$. Then for $2 \leq t \leq k$, $r(v_{n+1}|\{v_1,v_t\}) = r(v_n|\{v_1,v_t\}) = (3,t+1)$. For $t = k+1$, $r(v_2|\{v_1,v_t\}) = r(u_n|\{v_1,v_t\}) = (2,k)$, a contradiction.

3. Assume that $W = \{u_1,v_t\}$ where $1 \leq t \leq k+1$. For $1 \leq t \leq k$, $r(v_{n+1}|\{u_1,v_t\}) = r(v_n|\{u_1,v_t\}) = (2,t+1)$. For $t = k+1$, $r(v_2|\{u_1,v_t\}) = r(u_n|\{u_1,v_t\}) = (1,k)$, a contradiction.

Hence from above it follows that there is no resolving set with two vertices for $V(S_n^*)$. Thus, $\dim(S_n^*) = 3$. 

**Summary**

In this chapter the metric dimension of an important family Mobius strip graph denoted by $M_n$ is a cubic circulant graph with an even number of nodes made from an $n$-cycle by adding edges connecting opposite pairs of nodes in the cycle graph has been discussed. In this chapter we also compute the metric dimension of some new rotationally symmetric families of graph which are constructed from the prism graph and antiprism graph. And shown that these graphs have constant metric dimension and independent of the cardinality of graph, so only 2 or 3 vertices appropriately chosen suffice to resolve all the vertices of these graphs.
Chapter 6

Metric Dimension of Some Families of Convex Polytopes with Pendant Edges

6.1 The Convex Polytope Graph $S^n_n$

The $S^n_n$ graph is an extension graph of the convex polytope defined in (Imran et al. 2010). The vertex set of $S^n_n$ graph is $V(S^n_n) = \bigcup_{i=1}^{n} \{a_i, b_i, c_i, d_i, e_i\}$ where $n + i$ is taken modulo $n$ and the $E(S^n_n) = \{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1} : i = 1, 2, ..., n\} \cup \{a_{i+1}b_i, a_i b_i, b_i c_i, c_i d_i, d_i e_i : i = 1, 2, ..., n\}$ where $n + i$ is taken modulo $n$ for $n \geq 6$. The graph $S^n_n$ (superscript $n$ denotes the number of pendant vertices) in (Fig. 5.1) is obtained from a graph of convex polytope defined in (Imran et al. 2010) by adding a pendant edge at each vertex of outer cycle of $S_n$.

For simplification we represent the vertices are as follows: $C_1 = \{a_i : 1 \leq i \leq n\}$, $C_2 = \{b_i : 1 \leq i \leq n\}$, $C_3 = \{c_i : 1 \leq i \leq n\}$, $C_4 = \{d_i : 1 \leq i \leq n\}$ and the set of vertices $\{e_i : 1 \leq i \leq n\}$ is denoted the pendant vertices.

Figure 6.1: The convex polytope graph $S^n_n$
6.1 The Convex Polytope Graph $S_n$

Imran et al. (Imran et al. 2010) have been shown that a graph of convex polytope $S_n$ has constant metric dimension. The following theorem is the extension of $S_n$, we show that the metric dimension of a graph $S_n^m$ is similar as the graph of $S_n$.

**Theorem 6.1.** For every $n \geq 6$, $\dim(S_n^m) = 3$.

**Proof:** Consider the two cases.

**Case(1)** For $n = 2q$, $q \geq 3$, $q \in \mathbb{Z}^+$. Let $W = \{a_1, a_2, a_{q+1}\} \subset V(S_n^m)$, we show that $W$ is a resolving set for $S_n^m$. The representations of the vertices $V(S_n^m) \setminus W$ with respect to $W$ are as follows:

$$r(a_i|W) = \begin{cases} (i-1, i-2, q-i+1), & 3 \leq i \leq q; \\ (2q-i+1, 2q-i+2, i-q-1), & q+2 \leq i \leq 2q. \end{cases}$$

$$r(b_i|W) = \begin{cases} (1, 1, q), & i = 1; \\ (i, i-1, q-i+1), & 2 \leq i \leq q; \\ (q, q, 1), & i = q+1; \\ (2q-i+1, 2q-i+2, i-q), & q+2 \leq i \leq 2q. \end{cases}$$

$$r(c_i|W) = \begin{cases} (2, 2, q+1), & i = 1; \\ (1+i, i, q-i+2), & 2 \leq i \leq q; \\ (q+1, q+1, 2), & i = q+1; \\ (2q-i+3, 2q-i+3, i-q+1), & q+2 \leq i \leq 2q. \end{cases}$$

$$r(d_i|W) = \begin{cases} (3, 3, q+2), & i = 1; \\ (2+i, i+1, q-i+3), & 2 \leq i \leq q; \\ (q+2, q+2, 3), & i = q+1; \\ (2q-i+3, 2q-i+4, i-q+2), & q+2 \leq i \leq 2q. \end{cases}$$

$$r(e_i|W) = \begin{cases} (4, 4, q+3), & i = 1; \\ (3+i, i+2, q-i+4), & 2 \leq i \leq q; \\ (q+3, q+3, 4), & i = q+1; \\ (2q-i+4, 2q-i+5, i-q+3), & q+2 \leq i \leq 2q. \end{cases}$$

For any two vertices $v_i, v_j \in V(S_n^m)$, $i \neq j$, $r(v_i|W) \neq r(v_j|W)$ implying that $\dim(S_n^m) \leq 3$. Now we show that $\dim(S_n^m) \geq 3$ by showing that there is no resolving set $W$ such that $|W| = 2$. Assume on contrary that the metric dimension of a graph $S_n^m$ is two. Then in theorem 2.68, the degree of the vertices belonging to basis set can be at most three. Excluding the vertices of pendant edges all other vertices of a graph $S_n^m$ have degree four or five. The only possibility for $|W| = 2$, when both vertices belonging to the set of pendant vertices. Consider one resolving vertex is $e_1$, and the other is $e_t$ ($2 \leq t \leq q+1$). For $2 \leq t \leq q$, $r(d_{a_1}\{e_1, e_t\}) = r(c_{q+1}\{e_1, e_t\}) = (2, t+1)$ and for $t = q+1$, $r(d_{a_1}\{e_1, e_t\}) = r(c_{q+1}\{e_1, e_t\}) = (2, t-1)$, a contradiction.

Hence, there is no resolving set $W$ with two vertices for $V(S_n^m)$ implying that $\dim(S_n^m) = 3$.

**Case(2)** For $n = 2q+1$, $q \geq 3$, $q \in \mathbb{Z}^+$. Consider the set $W = \{a_1, a_2, a_{q+1}\} \subset V(S_n^m)$ is a resolving set for $S_n^m$. For this the representations of the vertices of $V(S_n^m) \setminus W$ with respect
to $W$ are as follows:

$$r(a_i|W) = \begin{cases} 
(i - 1, i - 2, q - i + 1), & 3 \leq i \leq q; \\
(q, q, 1), & i = q + 2; \\
(2q - i + 3, 2q - i + 4, i - q - 1), & q + 3 \leq i \leq 2q + 1.
\end{cases}$$

$$r(b_i|W) = \begin{cases} 
(1, 1, q), & i = 1; \\
(i, i - 1, q - i + 1), & 2 \leq i \leq q; \\
(q + 1, q, 1), & i = q + 1; \\
(2q - i + 2, 2q - i + 3, i - q), & q + 2 \leq i \leq 2q + 1.
\end{cases}$$

$$r(c_i|W) = \begin{cases} 
(2, 2, q + 1), & i = 1; \\
(i + 1, i, q - i + 2), & 2 \leq i \leq q; \\
(q + 2, q + 1, 2), & i = q + 1; \\
(2q - i + 3, 2q - i + 4, i - q + 1), & q + 2 \leq i \leq 2q + 1.
\end{cases}$$

$$r(d_i|W) = \begin{cases} 
(3, 3, q + 2), & i = 1; \\
(i + 2, 1 + i, q - i + 3), & 2 \leq i \leq q; \\
(q + 3, q + 2, 3), & i = q + 1; \\
(2q - i + 4, 2q - i + 5, i - q + 2), & q + 2 \leq i \leq 2q + 1.
\end{cases}$$

$$r(e_i|W) = \begin{cases} 
(4, 4, q + 3), & i = 1; \\
(i + 3, 2 + i, q - i + 4), & 2 \leq i \leq q; \\
(q + 4, q + 3, 4), & i = q + 1; \\
(2q - i + 5, 2q - i + 6, i - q + 3), & q + 2 \leq i \leq 2q + 1.
\end{cases}$$

For any two vertices $v_i, v_j \in V(S_n^i)$ $i \neq j$, $r(v_i|W) \neq r(v_j|W)$ implying that $\dim(S_n^i) \leq 3$. For other side, consider the $\dim(S_n^i) = 2$, then there are the same possibilities as in case (1) and contradiction can be assumed analogously. Thus $\dim(S_n^i) = 3$ in this case, which completes the proof.

### 6.2 The Convex Polytope $T_n^n$ graph

The $T_n^n$ graph is an extension graph of the convex polytope defined in (Imran et al. 2010). The vertex set of $T_n^n$ graph is $V(T_n^n) = \bigcup_{i=1}^{n} \{a_i, b_i, c_i, d_i, e_i\}$ where $n + i$ is taken modulo $n$ and the edge set is $E(T_n^n) = \{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1}, e_ie_i : i = 1, 2, ..., n\} \cup \{a_{i+1}b_i, b_{i+1}b_i, b_{i+1}c_i, c_{i+1}c_i, c_{i+1}d_i, d_{i+1}d_i, e_i : i = 1, 2, ..., n\}$ where $n + i$ is taken modulo $n$ for $n \geq 6$. The graph $T_n^n$ (superscript $n$ denotes the number of pendant vertices) in (Fig. 5.2) is constructed from a graph of convex polytope defined in (Imran et al. 2010) by adding a pendant edge at each vertex of outer cycle of $T_n$. For simplification we represent the vertices as follows: $C_1 = \{a_i : 1 \leq i \leq n\}$, $C_2 = \{b_i : 1 \leq i \leq n\}$, $C_3 = \{c_i : 1 \leq i \leq n\}$, $C_4 = \{d_i : 1 \leq i \leq n\}$ and the set of vertices $\{e_i : 1 \leq i \leq n\}$ denoted the pendant vertices. Imran et al. (Imran et al. 2010) have been shown that a graph of convex polytope $T_n$ has constant metric dimension . The following theorem is the extension of $T_n$, we show that the metric dimension of a graph $T_n^n$ is similar as the graph of $T_n$. 

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Theorem 6.2. For every $n \geq 6 \dim(T_n^n) = 3$.

Proof: We consider the two cases.

Case(1) For $n = 2q$, $q \geq 3$, $q \in \mathbb{Z}^+$. Consider the set $W = \{a_1, a_2, a_{q+1}\} \subset V(T_n^n)$, we show that $W$ is a resolving set for $T_n^n$. The representations of the vertices of $V(T_n^n) \setminus W$ with respect to $W$ as follows:

\[
\begin{align*}
    r(a_i|W) &= \begin{cases} (i-1, i-2, q-i+1), & 3 \leq i \leq q; \\
                        (2q - i + 2, 2q - i + 3, i - q), & q + 2 \leq i \leq 2q. \end{cases} \\
    r(b_i|W) &= \begin{cases} (1, 1, q), & i = 1; \\
                        (i, i-1, q-i+1), & 2 \leq i \leq q; \\
                        (q, q, 1), & i = q + 1; \\
                        (2q - i + 1, 2q - i + 2, i - q), & q + 2 \leq i \leq 2q. \end{cases} \\
    r(c_i|W) &= \begin{cases} (2, 2, q + 1), & i = 1; \\
                        (1 + i, i, q - i + 2), & 2 \leq i \leq q; \\
                        (q + 1, q + 1, 2), & i = q + 1; \\
                        (2q - i + 2, 2q - i + 3, i - q + 1), & q + 2 \leq i \leq 2q. \end{cases} \\
    r(d_i|W) &= \begin{cases} (3, 3, q + 1), & i = 1; \\
                        (i + 2, i + 1, q - i + 2), & 2 \leq i \leq q - 1; \\
                        (q + 2, q + 1, 3), & i = q; \\
                        (q + 1, q + 2, 3), & i = q + 1; \\
                        (2q - i + 2, 2q - i + 3, i - q + 2), & q + 2 \leq i \leq 2q - 1; \\
                        (3, 3, q + 2), & i = 2q. \end{cases}
\end{align*}
\]
With respect to $W$ when $Hence, there is no resolving set $r$ for $2$ dim.

Now we show that $|W|$ with $\{q\}$, $2$ $i = q$,

For any two vertices $(d, e)$, $\{q\}$, $2$ $i = q + 1$;

For any two vertices $i = q + 1$;

For any two vertices $q + 2$ $i = q$;

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For any two vertices $q + 2$ $i = q$;
Again for any two vertices \( v_i, v_j \in V(T_n^n) \) \( i \neq j \), \( r(v_i|W) \neq r(v_j|W) \) implying that \( \text{dim}(T_n^n) \leq 3 \).

For other side assume that \( \text{dim}(T_n^n) = 2 \), then there are the similar subcases as in case (1) and contradiction can be found analogously. Thus, \( \text{dim}(T_n^n) = 3 \). \( \square \)

### 6.3 The Convex Polytope Graph \( U_n^n \)

The \( U_n^n \) graph is an extension graph of the convex polytope defined in \( \text{Imran et al. 2010} \). The vertex set of \( U_n^n \) graph is \( V(U_n^n) = \bigcup_{i=1}^{n} \{a_i,b_i,c_i,d_i,e_i,f_i\} \) where \( n + i \) is taken modulo \( n \) and the edge set is \( E(U_n^n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, e_i e_{i+1} : i = 1,2,\ldots,n\} \cup \{a_i b_i, b_i c_i, c_i d_i, c_{i+1} d_i, e_i f_i : i = 1,2,\ldots,n\} \) where \( n + i \) is taken modulo \( n \) for \( n \geq 6 \). The graph \( U_n^n \) (superscript \( n \) denotes the number of pendant vertices) in \( \text{(Fig. 5.3)} \) is constructed from the graph of convex polytope defined in \( \text{Imran et al. 2010} \) by adding a pendant edge at each vertex of outer cycle of \( U_n \). For simplification we represent the vertices as follows: \( C_1 = \{a_i : 1 \leq i \leq n\} \), \( C_2 = \{b_i : 1 \leq i \leq n\} \), \( C_3 = \{c_i : 1 \leq i \leq n\} \), \( C_4 = \{e_i : 1 \leq i \leq n\} \) and the set of vertices \( \{f_i : 1 \leq i \leq n\} \) denoted the pendant vertices.

\( \text{Imran et al. (Imran et al. 2010)} \) have been shown that a graph of convex polytope \( T_n \) has constant metric dimension. The following theorem is the extension of \( T_n \), we show that the metric dimension of a graph \( T_n^n \) is similar as the graph of \( T_n \).

![Figure 6.3: The convex polytope graph \( U_n^n \)](image)

**Theorem 6.3.** For every \( n \geq 6 \), \( \text{dim}(U_n^n) = 3 \).

**Proof:** We consider the two cases.

**Case(1)** For \( n = 2q \), \( q \geq 3 \), \( q \in \mathbb{Z}^+ \). Consider the set \( W = \{a_1,a_2,a_{q+1}\} \subset V(U_n^n) \), we show that \( W \) is a resolving set for \( U_n^n \). The representations for any vertex of \( V(U_n^n) \setminus W \)
with respect to $W$ as follows:

$$r(a_i|W) = \begin{cases} 
(i - 1, i - 2, q - i + 1), & 3 \leq i \leq q; \\
(2q - i + 1, 2q - i + 2, i - q - 1), & q + 2 \leq i \leq 2q.
\end{cases}$$

$$r(b_i|W) = \begin{cases} 
(i, i - 1, q - i + 2), & 3 \leq i \leq q; \\
(2q - i + 2, 2q - i + 3, i - q), & q + 2 \leq i \leq 2q.
\end{cases}$$

$$r(c_i|W) = \begin{cases} 
(i + 1, i, q - i + 3), & 3 \leq i \leq q; \\
(2q - i + 3, 2q - i + 4, i - q + 1), & q + 2 \leq i \leq 2q.
\end{cases}$$

$$r(d_i|W) = \begin{cases} 
(3, 3, q + 3), & i = 1; \\
(i + 2, i + 1, q - i + 3), & 2 \leq i \leq q; \\
(q + 3, q + 3), & i = q + 1; \\
(2q - i + 3, 2q - i + 4, i - q + 2), & q + 2 \leq i \leq 2q.
\end{cases}$$

$$r(e_i|W) = \begin{cases} 
(4, 4, q + 4), & i = 1; \\
(i + 3, i + 2, q - i + 4), & 2 \leq i \leq q; \\
(q + 4, q + 4), & i = q + 1; \\
(2q - i + 4, 2q - i + 5, i - q + 3), & q + 2 \leq i \leq 2q.
\end{cases}$$

$$r(f_i|W) = \begin{cases} 
(5, 5, q + 5), & i = 1; \\
(i + 4, i + 3, q - i + 5), & 2 \leq i \leq q; \\
(q + 5, q + 5), & i = q + 1; \\
(2q - i + 5, 2q - i + 6, i - q + 4), & q + 2 \leq i \leq 2q.
\end{cases}$$

For any two vertices $v_i, v_j \in V(U_n^m)$, $i \neq j$, $r(v_i|W) \neq r(v_j|W)$ implying that $\dim(U_n^m) \leq 3$. Now we show that $\dim(U_n^m) \geq 3$. Assume on contrary that $\dim(U_n^m) = 2$, then the following possibilities arise.

(1) Assume that $W \subseteq \{a_i : 1 \leq i \leq n\} \subseteq V(U_n^m)$. Consider that one resolving vertex is $a_1$ and the other resolving vertex is $a_t \ (2 \leq t \leq q + 1)$. Then for $2 \leq t \leq q$, $r(a_1|\{a_1, a_t\}) = r(b_t|\{a_1, a_t\}) = (1, t)$ and for $t = q + 1$, $r(d_2|\{a_1, a_{q+1}\}) = r(a_n|\{a_1, a_{q+1}\}) = (1, q - 1)$, a contradiction.

(2) Consider that $W \subseteq \{c_i, d_i : 1 \leq i \leq n\} \subseteq V(U_n^m)$. Here are three subcases.

1. Consider that $W \subseteq \{c_i : 1 \leq i \leq n\} \subseteq V(U_n^m)$. Let one resolving vertex is $c_1$ and the second resolving vertex is $c_t \ (2 \leq t \leq q + 1)$. For $2 \leq t \leq q$, $r(a_1|\{c_1, c_t\}) = r(b_t|\{c_1, c_t\}) = (2, t + 1)$ and for $t = q + 1$, $r(d_1|\{c_1, c_{q+1}\}) = r(d_{n}|\{c_1, c_{q+1}\}) = (1, q + 3)$, a contradiction.

2. Let $W \subseteq \{d_i : 1 \leq i \leq n\} \subseteq V(U_n^m)$. Consider that one resolving vertex is $d_1$ and the other resolving vertex is $d_t \ (2 \leq t \leq q + 1)$. Then for $2 \leq t \leq q + 1$, $r(b_1|\{d_1, d_t\}) = r(e_n|\{d_1, d_t\}) = (2, t + 1)$, a contradiction.

3. One vertex is in the set $\{c_i : 1 \leq i \leq n\} \subseteq V(U_n^m)$ and other in the set $\{d_i : 1 \leq i \leq n\} \subseteq V(U_n^m)$. Suppose that one resolving vertex is $c_1$ and the other resolving vertex is $d_t \ (2 \leq t \leq q + 1)$. Then for $t = 1$, $r(b_1|\{c_1, d_1\}) = r(d_{n}|\{c_1, d_1\}) = (1, 2)$. If $2 \leq t \leq q$, $r(a_2|\{c_1, d_t\}) = r(b_t|\{c_1, d_t\}) = (1, t + 1)$ and for $t = q + 1$, $r(a_n|\{c_1, d_{q+1}\}) =$
\( r(b_1|\{c_1, d_{q+1}\}) = (1, q + 1) \), a contradiction.

(3) Assume that \( W \subset \{f_i : 1 \leq i \leq n\} \subset V(U_n^m) \). Suppose that one resolving vertex is \( f_1 \).
Assume that the second resolving vertex is \( f_t \) (\( 2 \leq t \leq q + 1 \)).
For \( 2 \leq t \leq q \), \( r(e_n|\{f_1, f_t\}) = r(d_1|\{f_1, f_t\}) = (2, t + 1) \) and for \( t = q + 1 \), \( r(e_2|\{f_1, f_t\}) = r(e_n|\{f_1, f_t\}) = (2, t - 1) \), a contradiction.

(4) Assume that one vertex of the resolving set belong to the set \( \{a_i : 1 \leq i \leq n\} \subset V(U_n^m) \)
and other vertex of the resolving set belong to the set \( \{c_i : 1 \leq i \leq n\} \subset V(U_n^m) \). Then
there are two subcases.

(1) One vertex of the resolving set belong to the set \( \{a_i : 1 \leq i \leq n\} \subset V(U_n^m) \) and other
in the set \( \{c_i : 1 \leq i \leq n\} \subset V(U_n^m) \). Assume that one resolving vertex is \( a_1 \).
Consider that the second resolving vertex is \( c_t \) (\( 1 \leq t \leq q + 1 \)). Then for \( t = 1 \), \( r(a_2|\{a_1, c_1\}) = r(a_n|\{a_1, c_1\}) = (1, 3) \). If \( 2 \leq t \leq q + 1 \), \( r(a_2|\{a_1, b_t\}) = r(b_1|\{a_1, b_t\}) = (1, t) \), a contradiction.

(2) Let one vertex of the resolving set belong to the set \( \{a_i : 1 \leq i \leq n\} \subset V(U_n^m) \) and other
in the set \( \{d_i : 1 \leq i \leq n\} \subset V(U_n^m) \). Suppose that one resolving vertex is \( a_1 \). Suppose that
the second resolving vertex is \( d_t \) (\( 1 \leq t \leq q + 1 \)). Then for \( t = 1 \), we have \( r(d_2|\{a_1, d_1\}) = r(e_n|\{a_1, d_1\}) = (4, 2) \). If \( 2 \leq t \leq q \), \( r(a_2|\{a_1, d_t\}) = r(b_1|\{a_1, d_t\}) = (1, t + 1) \) and
for \( t = q + 1 \), \( r(a_n|\{a_1, d_{q+1}\}) = r(b_1|\{a_1, d_{q+1}\}) = (1, q + 1) \), a contradiction.

(5) Let one vertex of the resolving set belong to the set \( \{a_i : 1 \leq i \leq n\} \subset V(U_n^m) \) and other
vertex of the resolving set belong to the set \( \{f_i : 1 \leq i \leq n\} \subset V(U_n^m) \). Consider that one
resolving vertex is \( a_1 \). Assume that the second resolving vertex is \( f_i \) (\( 1 \leq t \leq q + 1 \)). For \( t = 1 \), \( r(c_2|\{a_1, f_1\}) = r(d_1|\{a_1, f_1\}) = (3, 3) \). For \( 2 \leq t \leq q \), \( r(a_2|\{a_1, f_i\}) = r(b_1|\{a_1, f_i\}) = (1, t + 3) \) and when \( t = q + 1 \), \( r(a_n|\{a_1, f_{q+1}\}) = r(b_1|\{a_1, f_{q+1}\}) = (1, q + 3) \), a contradiction.

(6) Let one vertex of the resolving set belong to the set \( \{c_i : 1 \leq i \leq n\} \subset V(U_n^m) \) and other
vertex of the resolving set belong to the set \( \{f_i : 1 \leq i \leq n\} \subset V(U_n^m) \). Here are the
following two subcases.

(1). One vertex of a resolving set is in the set \( \{c_i : 1 \leq i \leq n\} \subset V(U_n^m) \) and other in the
set of pendant vertices. Assume that one resolving vertex is \( c_1 \). Consider that the second
resolving vertex is \( f_i \) (\( 1 \leq t \leq q + 1 \)). For \( t = 1 \), \( r(d_2|\{c_1, f_1\}) = r(f_2|\{c_1, f_1\}) = (3, 3) \)
and for \( t = 2 \), \( r(a_2|\{c_1, f_2\}) = r(f_{t+1}|\{c_1, f_2\}) = (3, 5) \). If \( 3 \leq t \leq q \), \( r(c_2|\{c_1, f_1\}) = r(e_n|\{c_1, f_1\}) = (2, t + 2) \) and when \( t = q + 1 \), \( r(b_2|\{c_1, f_{q+1}\}) = r(e_n|\{c_1, f_{q+1}\}) = (2, q + 3) \), a contradiction.

(2). One vertex of a resolving set is in the set \( \{d_i : 1 \leq i \leq n\} \subset V(U_n^m) \) and other in the
set of pendant vertices. Assume that one resolving vertex is \( d_1 \). Let the second
resolving vertex is \( f_i \) (\( 1 \leq t \leq q + 1 \)). For \( t = 1 \), \( r(c_1|\{d_1, f_1\}) = r(e_n|\{d_1, f_1\}) = (1, 3) \).
If \( 2 \leq t \leq q \), \( r(e_2|\{d_1, f_t\}) = r(e_n|\{d_1, f_t\}) = (2, t + 1) \) and when \( t = q + 1 \), \( r(e_2|\{d_1, f_{q+1}\}) = r(e_n|\{d_1, f_{q+1}\}) = (2, q) \), a contradiction.

Hence, from above it follows that there is no resolving set \( W \) with 2 vertices for \( V(U_n^m) \)
impllying that \( \text{dim}(U_n^m) = 3 \).

Case(2) For \( n = 2q + 1 \), \( q \geq 3 \), \( q \in \mathbb{Z}^+ \). Let \( W = \{a_1, a_2, a_{q+1}\} \subset V(U_n^m) \), we show that \( W \)
is a resolving set for \( U_n^m \). For this we find the representations of the vertices of \( V(U_n^m) \setminus W \)
with respect to \( W \).
Representations of the vertices as follows:

\[
\begin{align*}
\text{r}(a_i|W) &= \begin{cases} 
(i - 1, i - 2, q - i + 1), & 3 \leq i \leq q; \\
(q, q, 1), & i = q + 2; \\
(2q - i + 2, 2q - i + 3, i - q - 1), & q + 3 \leq i \leq 2q + 1.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{r}(b_i|W) &= \begin{cases} 
(i, i - 1, q - i + 2), & 3 \leq i \leq q; \\
(q + 1, q + 1, 2), & i = q + 2; \\
(2q - i + 3, 2q - i + 4, i - q), & q + 3 \leq i \leq 2q + 1.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{r}(c_i|W) &= \begin{cases} 
(i + 1, i, q - i + 3), & 3 \leq i \leq q; \\
(q + 2, q + 2, 3), & i = q + 2; \\
(2q - i + 4, 2q - i + 5, i - q + 1), & q + 3 \leq i \leq 2q + 1.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{r}(d_i|W) &= \begin{cases} 
(3, 3, q + 2), & i = 1; \\
(i + 2, i + 1, q - i + 3), & 2 \leq i \leq q; \\
(q + 3, q + 2, 3), & i = q + 1; \\
(q + 2, q + 1, 4), & i = q + 2; \\
(q + 1, q, 5), & i = q + 2; \\
(2q - i + 4, 2q - i + 5, i - q + 2), & q + 3 \leq i \leq 2q + 1.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{r}(e_i|W) &= \begin{cases} 
(4, 4, q + 3), & i = 1; \\
(i + 3, i + 2, q - i + 4), & 2 \leq i \leq q; \\
(q + 4, q + 3, 4), & i = q + 1; \\
(q + 3, q + 2, 5), & i = q + 2; \\
(q + 2, q + 1, 6), & i = q + 2; \\
(2q - i + 5, 2q - i + 6, i - q + 3), & q + 3 \leq i \leq 2q + 1.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{r}(f_i|W) &= \begin{cases} 
(5, 5, q + 4), & i = 1; \\
(i + 4, i + 3, q - i + 5), & 2 \leq i \leq q; \\
(q + 5, q + 4, 5), & i = q + 1; \\
(q + 4, q + 3, 6), & i = q + 2; \\
(q + 3, q + 2, 7), & i = q + 2; \\
(2q - i + 6, 2q - i + 7, i - q + 4), & q + 3 \leq i \leq 2q + 1.
\end{cases}
\end{align*}
\]

In this time again for any two vertices \(v_i, v_j \in V(U_n^m)\) \(i \neq j\), \(r(v_i|W) \neq r(v_j|W)\) implying that \(\text{dim}(U_n^m) \leq 3\).

For other side assume that \(\text{dim}(U_n^m) = 2\), then there are the same subcases as in case (1) and contradiction can be found analogously. This implies that \(\text{dim}(U_n^m) = 3\). \(\square\)

**Summary**

In this chapter we compute the metric dimension of some families of convex polytopes with pendent edges. These graphs are obtained from the graphs of convex polytopes defined in (Imran et al. 2010) by adding a pendant edge at each vertex of outer cycle of \(S_n\), \(T_n\), and \(U_n\). We show that the metric dimension of these extended families \(S_n^m, T_n^m\) and \(U_n^m\) is
constant. And shown that these graphs have constant metric dimension and independent of the cardinality of graph, so only 2 or 3 vertices appropriately chosen suffice to resolve all the vertices of these graphs.
Chapter 7
Graph with Unbounded Metric Dimension.

7.1 Metric Dimension of Splitting Graph of $P_n$ and $C_n$

In this chapter, the metric dimension of the splitting graphs of paths and cycle is computed. It is shown that the metric dimension of these graphs is unbounded and depends on the order of the graph.

The splitting graph of a graph $G$ is defined as: for each vertex $u_i \in V(G)$, $i = 1, 2, ..., n$ introduce a new vertex $v_i$. Join $v_i$ to all vertices of $G$ that are adjacent to $u_i$. The graph thus obtained is called the splitting graph of graph $G$ denoted by $S(G)$. If $V(G) = \{u_1, u_2, ..., u_n\}$ then $V(S(G)) = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$. For simplicity we would write $V(S(G)) = S_1 \cup S_2$ where $S_1 = \{u_1, u_2, ..., u_n\}$ and $S_2 = \{v_1, v_2, ..., v_n\}$.

Here we discuss the metric dimension of the splitting graph of path graph and splitting graph of cycle graph. For the splitting path graph we say that $S_1$ is the main path and $S_2$ is the path of vertices adjacent to main path vertices. For any vertex $u_i \in S_1$ the closed neighborhood of $u_i$ is $N[u_i] = \{u_i, u_{i+1}, u_{i-1}, v_{i+1}, v_{i-1}\}$ and for any vertex $v_i \in S_2$ the closed neighborhood of $v_i$ is $N[v_i] = \{v_i, u_{i+1}, u_{i-1}\}$, for all $i$. The relation between $N[u_i]$ and $N[v_i]$ can be written as $N[u_i] \cap N[v_i] = \{u_{i+1}, u_{i-1}\}$. We also note that:

1). For any two consecutive vertices $v_i, v_{i+1} \in S_2$, $d(v_i, v_{i+1}) = 3$, for all $i$.
2). If $u_i \in S_1$ and $v_i \in S_2$ then $d(u_i, v_i) = 2$, for all $i$.
3). Without loss of generality for $u_1 \in S_1$ and $v_1 \in S_2$ we have $d(u_1, u_{i+1}) = d(v_1, u_{i+1}) = d(u_1, v_{i+1}) = i$, for all $i$.

7.1.1 The Splitting Graph of $P_n$

**Proposition 7.1.** Let $S(P_n)$ be splitting graph of path graph with $n \geq 8$, and $W$ is a resolving set of $S(P_n)$, then we have:

1. For any $u_i \in S_1$ and $v_i \in S_2$, $d(u_i, u_{i+1}) = d(v_i, u_{i+1})$ for all $i$. 

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2. Let \( u_i, v_i \in V(S(P_n)) \) then \( d(u_i, x) = d(v_i, x) \) for every vertex \( x \in V(S(P_n)) \setminus \{u_i, v_i\} \), if \( x \neq v_{i+1}, x \neq v_{i-1} \).

3. Let \( W \subset S_1 \) be a resolving set and \( N(u_i), N(u_j) \) are open neighborhoods. If \(|N(u_i) \cap N(u_j)| = 2\), then only \( u_i \) or \( u_j \) must be in \( W \).

4. Let \( W \subset S_2 \) be a resolving set and \( N[v_i], N[v_j] \) are closed neighborhoods. If \(|\hat{N}[v_i] \cap \hat{N}[v_j]| = 1\), then only \( v_i \) or \( v_j \) must be in \( W \).

5. Let \( W \subset S_1 \setminus \{u_{i+2}\} \) be a resolving set of \( S(P_n) \) then \( N[u_i] \cap W = \{u_i\} \) and \( N[v_i] \cap W = \emptyset \) for all \( i \).

6. Let \( W \subset S_2 \setminus \{v_{i+2}\} \) be a resolving set of \( S(P_n) \) then \( N[v_i] \cap W = \{v_i\} \) and \( N[u_i] \cap W = \emptyset \) for all \( i \).

7. \( W \subset S_1 \cup S_2 \) be a resolving set of \( S(P_n) \) if \( N[u_i] \cap W = \{u_i\}, N[u_{i+3}] \cap W = \emptyset \) and \( N[v_i] \cap W = \emptyset \) then \( N[v_{i+3}] \cap W = \{v_{i+3}\} \).

**Proof:**

1. To prove that \( d(u_i, u_{i+1}) = d(v_i, u_{i+1}) \) for all \( i \) we use the idea of closed neighborhood. For any vertex \( u_i \in S_1 \) the closed neighborhood of \( u_i \) is \( N[u_i] = \{u_i, u_{i+1}, u_{i-1}, v_{i+1}, v_{i-1}\} \) and for any vertex \( v_i \in S_2 \) the closed neighborhood of \( v_i \) is \( N[v_i] = \{v_i, u_{i+1}, u_{i-1}\} \), which implies that \( d(u_i, u_{i+1}) = 1, d(u_i, u_{i-1}) = 1, d(u_{i+1}, v_i) = 1, d(u_{i+1}, v_{i-1}) = 1 \) and similarly \( d(v_i, u_{i+1}) = 1, d(u_i, u_{i-1}) = 1 \). The result directly follows from these facts that \( d(u_i, u_{i+1}) = d(v_i, u_{i+1}) \).

2. To prove that \( d(u_i, x) = d(v_i, x) \). On contrary we suppose that \( x = v_{i+1}, x = v_{i-1} \), then we have \( v_{i+1}, v_{i-1} \in N[u_i] \) implying that \( d(u_i, v_{i+1}) = d(u_i, v_{i-1}) = 1 \), and \( v_{i+1}, v_{i-1} \notin N[v_i] \) this implies that \( d(v_i, v_{i+1}) > 1 \) and \( d(u_i, v_{i-1}) > 1 \), a contradiction.

3. Consider \( u_i, u_j \in W \) \( i \neq j \) and \( |N(u_i) \cap N(u_j)| = 2 \). Let \( N(u_i) \cap N(u_j) = \{u_s, v_s\} \), this implies that \( d(u_i, u_s) = 1 = d(u_j, u_s) \) and \( d(u_i, v_s) = 1 = d(u_j, v_s) \). Also by (1) for any
fixed $i$, $1 \leq i \leq n$, $d(v_i, u_j) = d(u_i, v_j)$. Combining these two facts the distance of any two vertices of $W$ from the vertices $u_i, u_j$ are equal.

4. Consider $v_i, v_j \in W$, $i \neq j$ and $|N[v_i] \cap N[v_j]| = 1$. Then we have at least two vertices $u_s, v_s \notin W$ such that $d(v_i, u_s) = d(v_j, u_s)$ and $d(v_i, v_s) = d(v_j, v_s)$. Also by (1) for any fixed $i, 1 \leq i \leq n$, $d(v_i, u_j) = d(u_i, v_j)$. Combining these two facts the distance of any two vertices of $W$ from the vertices $u_i, u_j$ are equal.

5. Let $W \subset S_1$ be a resolving set of $S(P_n)$ then $N[u_i] \cap W = \{u_i\}$ and $N[v_i] \cap W = \emptyset$. Contrarily suppose that $N[u_i] \cap W = \emptyset$, then by (1) all components of $r(u_i|W) = r(v_i|W)$, a contradiction.

6. Let $W \subset S_2$ be a resolving set of $S(P_n)$ then $N[u_i] \cap W = \emptyset$ and $N[v_i] \cap W = \{v_i\}$. Contrarily suppose that $N[v_i] \cap W = \emptyset$, then by (1) all components of $r(u_i|W) = r(v_i|W)$, a contradiction.

7. Let $W \subset S_1 \cup S_2$ be a resolving set of $S(P_n)$ if $N[u_i] \cap W = \{u_i\}$, $N[u_{i+3}] \cap W = \emptyset$ and $N[v_i] \cap W = \emptyset$ then $N[v_{i+3}] \cap W = \{v_{i+3}\}$. On contrary we suppose that $v_{i+3} \notin W$ then by (1), $r(u_{i+3}|W) = r(v_{i+3}|W)$, a contradiction.

\[\text{Theorem 7.1. For } n > 8, \dim(S(P_n)) = \lceil \frac{n}{3} \rceil.\]

**Proof:** Assume $V(S(P_n)) = S_1 \cup S_2$, where $S_1 = \{u_1, \ldots, u_n\}$ and $S_2 = \{v_1, \ldots, v_n\}$. The distances between the vertices of $S(P_n)$ are as follows:

- $d(v_i, v_j) = 3, j = i + 1$.
- $d(v_i, v_j) = |j - i|$, $j \neq i + 1$.
- $d(v_i, u_j) = |j - i|$, $j \neq i$.
- $d(v_i, u_j) = 2, j = i$.
- $d(u_i, u_j) = |j - i|$, $j \neq i$.

We first show that $\dim(S(P_n)) \geq \lceil \frac{n}{4} \rceil$, suppose that a set of vertices $W$ is a basis of $S(P_n)$ with $|W| < \lceil \frac{n}{4} \rceil$. Then the possibilities are: $W \subset S_1$, $W \subset S_2$ or $W \subset S_1 \cup S_2$. For $W \subset S_1$, we suppose that $|W| < \lceil \frac{n}{4} \rceil$ there exists at least three consecutive vertices $v_{i-1}, v_i, v_{i+1}$ which are not in $W$ and for any $v_j \in W$. By proposition 6.1 it is observed that $d(v_i, v_j) = d(u_i, v_j) = |i - j|$. It means that all components of $r(v_i|W) = r(u_i|W)$, a contradiction.

For $W \subset S_2$, we suppose that $|W| < \lceil \frac{n}{4} \rceil$ there exists at least three consecutive vertices $u_{i-1}, u_i, u_{i+1}$ which are not in $W$ and for any $u_j \in W$. By proposition 6.1 it is noted that $d(v_i, v_j) = d(u_i, v_j) = |i - j|$. It means that all components of $r(u_i|W) = r(v_i|W)$, a contradiction.

For $W \subset S_1 \cup S_2$, we suppose that $|W| < \lceil \frac{n}{4} \rceil$ there exists at least two vertices $u_i, v_i$ such that $N[u_i] \cap W = \emptyset$ and $N[v_i] \cap W = \emptyset$ which implying that all components of $r(v_i|W) = r(u_i|W)$, a contradiction. Thus $\dim(S(P_n)) \geq \lceil \frac{n}{3} \rceil$.

We show that, $\dim(S(P_n)) \leq \lceil \frac{n}{4} \rceil$ consider the set $W = \{v_{3i-1} : i = 1, 2, \ldots, \lceil \frac{n}{3} \rceil\}$ be a subset of $V(S(P_n))$. We show that $W$ is a resolving set for $S(P_n)$. For this take the representations of the vertices of $V(S(P_n)) \setminus W$ with respect to $W$. The following cases arise:

**Case(1).** For $u_i, u_j \in S_1 \setminus W$ then by using proposition 6.1, $d(u_i, v_2) \neq d(u_j, v_2)$, for all $u_i, u_j \in S_1 \setminus W$, except $d(u_2, v_2) = d(u_4, v_2)$, but then $d(u_2, v_3) \neq d(u_4, v_3)$ which implying that at least one component of $r(u_i|W)$ is distinct from $r(u_j|W)$.

**Case(2).** For $v_i, v_j \in S_2 \setminus W$. Since $v_2, v_5 \in W$ using proposition 6.1, $d(v_i, v_2) \neq d(v_j, v_2)$,
for all \( v_i, v_j \in S_2 \setminus W \), except \( d(v_1, v_2) = d(v_3, v_2) \), but then \( d(v_1, v_5) \neq d(v_3, v_5) \) implying that at least one component of \( r(v_1|W) \) is distinct from \( r(v_2|W) \).

**Case(3).** For \( u_j \in S_1 \setminus W \) and \( v_i \in S_2 \setminus W \). For \( v_i \) either \( v_i−1, v_i+2, v_i+5 \in W \) or \( v_{i−2}, v_{i+1}, v_{i+4} \in W \). Without loss of generality, let \( v_i−1, v_i+2, v_i+5 \in W \), the by proposition 6.1, \( d(v_i, v_i−1) \neq d(u_j, v_i−1) \), for all \( u_j \in S_1 \setminus W \) and \( v_i \in S_2 \setminus W \) except \( d(v_i, v_i−1) = d(u_i−4, v_i−1) \) and \( d(v_i, v_i−1) = d(u_i+2, v_i−1) \), but then \( d(v_i, v_i+5) \neq d(u_i−4, v_i+5) \) and \( d(v_i, v_i+5) \neq d(u_i+2, v_i+5) \) which implying that at least one component of \( r(u_j|W) \) is distinct from \( r(v_i|W) \).

From all the above cases, \( \dim(S(P_n)) \leq \left\lfloor \frac{n}{3} \right\rfloor \). This implies that \( \dim(S(P_n)) = \left\lfloor \frac{n}{3} \right\rfloor \).

### 7.1.2 The Splitting Graph of \( C_n \)

For a graph \( G \), we call two disjoint subsets of vertices twins if they have the same size and induce subgraphs with the same number of edges (Axenovich, 2013 twins). The vertex set of the splitting graph of cycle is divided into two parts. \( S_1 \) denote the vertex set of the inner cycle and \( S_2 \) denote the vertices adjacent to the \( S_1 \) vertices. For any vertex \( u_i \in S_1 \subset S(C_n) \) the closed neighborhood of \( u_i \) is \( N[u_i] = \{u_i, u_i+1, u_i−1, v_{i+1}, v_{i−1}\} \) and for any vertex \( v_i \in S_2 \subset S(C_n) \) the closed neighborhood of \( v_i \) is \( N[v_i] = \{v_i, v_{i+1}, v_{i−1}\} \). \( N[u_i] \) and \( N[u_{i+3}] \) are twins because \( |N[u_i]| = |N[u_{i+3}]| \) and \( N[v_i] \) and \( N[v_{i+1}] \) are twins because \( |N[v_i]| = |N[v_{i+1}]| \). The relation between these twins are \( N[u_i] \cap N[v_i] = \{u_{i+1}, u_{i−1}\} \), \( N[u_i] \cap N[u_{i+3}] = \emptyset \) and \( N[v_i] \cap N[v_{i+1}] = \emptyset \).

Here we note the following observation:

1. If \( v_1, v_{i+2} \in S_2 \setminus \{v_2, v_n\} \), then \( d(v_1, v_{i+2}) = i + 1 \) for \( i = 1, 2, \ldots, \lfloor n/2 \rfloor \), if \( n \) is even and \( i = 1, 2, \ldots, \lfloor n/2 \rfloor \), if \( n \) is odd.
2. For any \( u_{i+1}, v_{i+1} \in V(S(C_n)) \), then without loss of generality, \( d(u_1, v_{i+1}) = d(v_1, u_{i+1}) = d(v_1, u_{i+1}) = i \) for \( i = 1, 2, \ldots, \lfloor n/2 \rfloor \), if \( n \) is even and \( i = 1, 2, \ldots, \lfloor n/2 \rfloor \), if \( n \) is odd.

**Theorem 7.2.** If \( N[u_i] \) and \( N[v_i] \) are the closed neighborhood, then \( v_1 \) or \( u_i \) is in \( W \). Moreover, if \( v_i \in W \) and \( u_i \notin W \), then \( (W \setminus \{v_i\}) \cup \{u_i\} \) also resolve \( S(C_n) \).

**Proof:** Suppose \( W \subset S_1 \) be a resolving set for \( S(C_n) \) and \( v_i, u_i \notin W \), then we have \( d(u_i, x) = d(v_i, x) \) for all \( x \in S_1 \) and \( x \neq u_i \). It means that all components of \( r(u_i|W) = r(v_i|W) \), a contradiction. Similarly if \( W \subset S_2 \) be a resolving set and \( v_i, u_i \notin W \), then we have \( d(u_i, x) = d(v_i, x) \) for all \( x \in S_1 \) and \( x \neq u_i \). It means that all components of \( r(u_i|W) = r(u_i|W) \), a contradiction. From these it can be deduced that one of \( v_i \) or \( u_i \) must be in \( W \).

**Proposition 7.2.** Let \( S(C_n) \) be splitting graph of cycle graph with \( n \geq 8 \), and \( W \) is a resolving set of \( S(C_n) \), then:

1. For any \( u_i \in S_1 \) and \( v_i \in S_2 \), \( d(u_i, v_i) = |N[u_i] \cap N[v_i]| \).
2. If \( v_i, v_{i+2} \in S_2 \subset S(C_n) \) then \( d(v_i, v_{i+1}) = 3 \) for all \( i \) where \( n + i \) is taken modulo \( n \).
3. Let \( W \subset S_1 \) be a resolving set and \( N(u_i), N(u_j) \) are open neighborhood. If 
\[ |N(u_i) \cap N(u_j)| = 2, \] then only \( u_i \) or \( u_j \) must be in \( W \).
4. Let \( W \subset S_2 \) be a resolving set and \( N[v_i], N[v_j] \) are closed neighborhood. If 
\[ |N[v_i] \cap N[v_j]| = 1, \] then only \( v_i \) or \( v_j \) must be in \( W \).
5. Let \( u_i, v_i \in V(S(C_n)) \setminus \{v_{i+1}, v_{i-1}\} \) then 
\[ d(u_i, x) = d(v_i, x) \] for every vertex 
\[ x \in V(S(C_n)) \setminus \{u_i, v_i\} \] for \( i = 1, 2, ..., \frac{n}{2} \), if \( n \) is even and \( i = 1, 2, ..., \frac{n-1}{2} \), if \( n \) is odd.
6. Let \( W \subset S_1 \) be a resolving set of \( S(C_n) \) then 
\[ N[u_i] \cap W = \{u_i\} \] and 
\[ N[v_i] \cap W = \phi. \]
7. Let \( W \subset S_2 \) be a resolving set of \( S(C_n) \) then 
\[ N[v_i] \cap W = \{v_i\} \] and 
\[ N[u_i] \cap W = \phi. \]
8. Let \( W \subset S_1 \cup S_2 \) be a resolving set of \( S(C_n) \) if \( N[u_i] \cap W = \{u_i\}, N[u_{i+3}] \cap W = \phi \) and 
\[ N[v_i] \cap W = \phi \] then 
\[ N[u_{i+3}] \cap W = \{v_{i+3}\}. \] Moreover if \( W \subset S_1 \cup S_2 \) be a resolving set of \( S(C_n) \) if \( N[u_i] \cap W = \{u_i\}, N[v_{i+3}] \cap W = \phi \) and 
\[ N[u_i] \cap W = \phi \] then 
\[ N[u_{i+3}] \cap W = \{u_{i+3}\}. \]

**Proof:**
1. We know that for any vertex \( u_i \in S_1 \subset S(C_n) \) the closed neighborhood of \( u_i \) is 
\[ N[u_i] = \{u_i, u_{i+1}, u_{i-1}, v_{i+1}, v_{i-1}\} \] and for vertex \( v_i \in S_2 \subset S(C_n) \) the closed neighborhood of \( v_i \) is 
\[ N[v_i] = \{v_i, u_{i+1}, u_{i-1}\}. \] The intersection set of \( N[u_i] \cap N[v_i] = \{u_{i+1}, u_{i-1}\} \), which implies that 
\[ |N[u_i]| \cap N[v_i]| = 2. \]
From definition we know that \( v_i \) is adjacent to those vertices which are adjacent to \( u_i \) inner cycle. The distance between \( u_i \) and \( v_i \) can be written as: 
\[ d(u_i, v_i) = d(u_i, u_{i+1}) + d(u_{i+1}, v_i) \] but from closed neighborhood 
\[ d(u_i, v_i) = 1 \] and 
\[ d(u_{i+1}, v_i) = 1. \] Thus, we have 
\[ d(u_i, v_i) = |N[u_i]| \cap N[v_i]|. \]
2. We know that 
\[ N[v_i] = \{v_i, u_{i+1}, u_{i-1}\}, \] 
\[ N[u_i] = \{u_i, u_{i+1}, u_{i-1}, v_{i+1}, v_{i-1}\}, \] 
\[ N[u_{i+1}] = \{u_{i+1}, u_i, u_{i+2}, v_i, v_{i+2}\} \] and 
\[ N[v_{i+1}] = \{v_{i+1}, u_{i+1}, u_{i+2}\}. \] Distance between \( v_i \) and \( v_{i+1} \) can be written as 
\[ d(v_i, v_{i+1}) = d(v_i, u_{i+1}) + d(u_{i+1}, u_{i+2}) \] but from the given definition 
\[ d(v_i, u_{i+1}) = 1, \] 
\[ d(u_{i+1}, u_{i+2}) = 1 \] and 
\[ d(u_{i+2}, v_{i+1}) = 1. \] Thus, 
\[ d(v_i, v_{i+1}) = 1 + 1 + 1, \] which implies that 
\[ d(v_i, v_{i+1}) = 3. \]
3. Consider \( u_i, u_j \in W \) \( i \neq j \) and 
\[ |N(u_i) \cap N(u_j)| = 2. \] Let \( N(u_i) \cap N(u_j) = \{u_x, v_x\} \), this implies that 
\[ d(u_i, u_x) = 1 = d(u_j, u_x) \] and 
\[ d(u_i, v_x) = 1 = d(u_j, v_x). \] Also by (1) for any fixed \( i, 1 \leq i \leq n, d(v_i, u_j) = d(u_i, v_j). \) Combining these two facts the distance of any two vertices of \( W \) from the vertices \( u_i, u_j \) are equal.
4. Consider \( v_i, v_j \in W \) \( i \neq j \) and 
\[ |N[v_i] \cap N[v_j]| = 1. \] Then we at least two vertices \( u_x, v_x \notin W \) s \( \notin \{i-1, i, i+1, j-1, j+1\} \) such that 
\[ d(v_i, u_x) = d(v_j, u_x) \] and 
\[ d(v_i, v_x) = d(v_j, v_x). \] Also by (1) for any fixed \( i, 1 \leq i \leq n, d(v_i, u_j) = d(u_i, v_j). \) Combining these two facts the distance of any two vertices of \( W \) from the vertices \( u_i, u_j \) are equal.
5. To prove that 
\[ d(u_i, x) = d(v_i, x). \] On contrary we suppose that \( x = v_{i+1}, x = v_{i-1} \), then we have 
\[ v_{i+1}, v_{i-1} \in N[u_i] \] implying that 
\[ d(u_i, v_{i+1}) = d(u_i, v_{i-1}) = 1, \] and 
\[ v_{i+1}, v_{i-1} \notin N[v_i] \] which imply that 
\[ d(v_i, v_{i+1}) > 1 \] and 
\[ d(u_i, v_{i-1}) > 1, \] a contradiction.
6. Let \( W \subset S_1 \) be a resolving set of \( S(P_n) \) then 
\[ N[u_i] \cap W = \{u_i\} \] and 
\[ N[v_i] \cap W = \phi. \] Contrarily suppose that 
\[ N[u_i] \cap W = \phi \] then by (1) all components of 
\[ r(u_i|W) = r(v_i|W), \] a contradiction.
7. Let \( W \subset S_2 \) be a resolving set of \( S(P_n) \) then 
\[ N[u_i] \cap W = \phi \] and 
\[ N[v_i] \cap W = \{v_i\}. \] Contrarily suppose that 
\[ N[v_i] \cap W = \phi \] then by (1) all components of 
\[ r(u_i|W) = r(v_i|W), \]
a contradiction.

8. Let \( W \subseteq S_1 \cup S_2 \) be a resolving set of \( S(P_n) \) if \( N[u_i] \cap W = \{ u_i \}, N[u_{i+3}] \cap W = \emptyset \) and \( N[v_i] \cap W = \emptyset \) then \( N[v_{i+3}] \cap W = \{ v_{i+3} \} \). On contrary we suppose that \( v_{i+3} \notin W \) then by (1) all components of \( r(u_{i+3}) = r(v_{i+3}) \), a contradiction.

\[ \]

Lemma 7.1. Let \( W \) be a resolving set for \( S(C_n) \) and the vertex set of \( V(S(C_n)) = S_1 \cup S_2 \), where \( S_1 = \{ u_1, u_2, \ldots, u_n \} \) and \( S_2 = \{ v_1, v_2, \ldots, v_n \} \). If \( W \subseteq V(S(C_n)) \), be any resolving set for \( V(S(C_n)) \) and \( |W| = \lceil \frac{n}{3} \rceil \). Then the followings must be true.

1. If \( W \subseteq S_1 \) then one of the vertices \( u_{n-1}, u_n, u_{n+1} \in W \) where \( n + i \) is taken modulo \( n \).
2. If \( W \subseteq S_2 \) then at least one of the three consecutive vertices \( v_{i-1}, v_i, v_{i+1} \in W \).

\[ \]

Proof:

1. For \( W \subseteq S_1 \), to show that one of \( u_{n-1}, u_n, u_{n+1} \in W \) where \( n + i \) is taken modulo \( n \). Contrarily suppose that \( u_{n-1}, u_n, u_{n+1} \notin W \) then by proposition 6.2 all components of \( r(v_n) = r(u_n) \), a contradiction. So \( u_n \in W \) or \( v_n \in W \) or \( v_{n-1} \in W \).

2. For \( W \subseteq S_2 \), we show that at least one of the three consecutive vertices \( v_{i-1}, v_i, v_{i+1} \in W \). On contrary, assume that none of these vertices belonging to \( W \). Then by proposition 6.2 there exists a vertex \( u_{i+1} \notin W \) such that all components of \( r(v_{i+1}) = r(u_{i+1}) \), a contradiction. It means that \( W \) is not resolving set. Thus at least one of the \( v_{i-1}, v_i, v_{i+1} \in W \).

\[ \]

Theorem 7.3. For \( n > 7 \), \( dim(S(C_n)) = \lceil \frac{n}{3} \rceil \).

\[ \]

Proof: Assume that \( dim(S(C_n)) \leq \lceil \frac{n}{3} \rceil - 1 \). There is at least one set of three consecutive vertices say \( v_{i-1}, v_i, v_{i+1} \in V(S(C_n)) \), which is not a subset of any resolving set say \( W \). By theorem 6.2 and by proposition 6.2 we have \( r(v_{i-1}) = r(u_{i-1}) \), \( r(v_i) = r(u_i) \), or \( r(v_{i+1}) = r(u_{i+1}) \), a contradiction. Thus \( W \) is not resolving set. Therefore, \( dim(S(C_n)) \geq \lceil \frac{n}{3} \rceil \).

For \( dim(S(C_n)) \leq \lceil \frac{n}{3} \rceil \). Consider the set \( W = \{ v_{3i-1} : i = 1, 2, \ldots, \lceil \frac{n}{3} \rceil \} \). We show that this set is a resolving set for \( V(S(C_n)) \). The representations of the vertices of \( V(S(C_n)) \setminus W \) are discussed in the following cases:

Case(1). Consider two distinct vertices \( u_i, u_j \in S_1 \), then by theorem 6.2 and by proposition 6.2 we have:

\[ r(u_i) = (d(u_i, v_1), d(u_i, v_2), \ldots, d(u_i, v_{n-2})), \]
\[ r(u_j) = (d(u_j, v_1), d(u_j, v_2), \ldots, d(u_j, v_{n-2})). \]

It is observed that in the above tuples at least one component of \( r(u_i) \) elements is distinct from \( r(u_j) \), so \( r(u_i) \neq r(u_j) \). Therefore, \( dim(S(C_n)) \leq \lceil \frac{n}{3} \rceil \).

Case(2). Consider two distinct vertices \( v_i, v_j \in S_2 \), then by theorem 6.2 and by proposition 6.2 we have:

\[ r(v_i) = (d(v_i, v_1), d(v_i, v_2), \ldots, d(v_i, v_{n-2})), \]
\[ r(v_j) = (d(v_j, v_1), d(v_j, v_2), \ldots, d(v_j, v_{n-2})). \]

It is concluded that in the above tuples at least one component of \( r(v_i) \) elements is distinct from \( r(v_j) \), implying that \( r(v_i) \neq r(v_j) \). Therefore, \( dim(S(C_n)) \leq \lceil \frac{n}{3} \rceil \).

Case(3). Consider two distinct vertices \( u_i, v_j \in S_1 \cup S_2 \), then by theorem 6.2 and by proposition 6.2 we have:
\[
\begin{align*}
    r(v_i|W) &= (d(v_i, v_1), d(v_i, v_4), \ldots, d(v_i, v_{n-2})), \\
    r(u_j|W) &= (d(u_j, v_1), d(u_j, v_4), \ldots, d(u_j, v_{n-2})).
\end{align*}
\]

It is showed that in the above tuples at least one component of \( r(u_j|W) \) elements is distinct from \( r(v_j|W) \), implying that \( r(u_j|W) \neq r(v_j|W) \). Therefore, \( \text{dim}(S(P_n)) \leq \lceil \frac{n}{3} \rceil \). Thus, \( \text{dim}(S(C_n)) = \lceil \frac{n}{3} \rceil \).

**Summery**

In this chapter the metric dimension of the splitting graphs of two families of graph has been computed. We prove that the metric dimension of the splitting graph of path graph and splitting graph of cycle graph is unbounded and depends on the order of the corresponding graph. We observed that \( \text{dim}(S(P_n)) = \text{dim}(S(C_n)) = \lceil \frac{n}{3} \rceil \). Note that the choice of an appropriate basis of vertices is the core of the problem.
Chapter 8

Conclusions and Future Work

8.1 Conclusions

This dissertation attempts to compute the metric dimension of some well known families of graphs. The thesis mainly consists of two portions:

1) Computation of the metric dimension of graphs with constant metric dimension. In this portion we find the metric dimension of the path graph by the power, middle and total graph operation and shown that these graphs have constant metric dimension. We compute the metric dimension of some rotationally symmetric families of graphs and show that only 2 or 3 vertices appropriately chosen vertices suffice to resolve all the vertices of these graphs. We also compute the metric dimension of some families of convex polytopes with pendant edges. It has been shown that the metric dimension of these families of graphs is constant and is independent of the order of these graphs.

2) Computation of the metric dimension of graphs with unbounded metric dimension. In this portion the metric dimension of the splitting graphs of two families of graph has been computed. We prove that the metric dimension of these graphs is unbounded and depend on the order of the corresponding graph.

The section is concluded with the following open problems.

Open Problem: Finding the exact value of metric dimension of $P_n^k$ when $n \geq 6$ and $2 \leq k \leq \lceil \frac{n}{2} \rceil$.

Open Problem: Finding the exact value of metric dimension of Möbius strip $M_n$ when $n \equiv 2 \pmod{8}$.

Conjecture: Let $G'$ be the plane graph obtained from rotationally-symmetric graph of convex polytope $G$ by attaching a pendant edge at each vertex of the outer cycle of $G$. If $G$ has constant metric dimension, then $G'$ will always have constant metric dimension.

Open Problem: Characterizing all graphs for which $dim(G) = |G| - 3$.

Open Problem: Characterize graphs $G$ such that $dim(G) = dim(L(G))$.

Open Problem: Find the metric dimension of the $G_n \Box G_m \Box G_s$, for $n \geq 3$, $m \geq 3$ and $s \geq 3$.

Open Problem: Find some condition as in (Sudhakara et al. 2009) for graphs to have
metric dimension 3.

**Open Problem:** Find the metric dimension of the splitting graph of $C_n \Box C_m$, for $n \geq 3$ and $m \geq 3$.

**Open Problem:** Find the metric dimension of the splitting graph of $C_n \Box P_2$, for $n \geq 3$ and $m \geq 3$. 