NUMERICAL SIMULATION OF SOME BOUNDARY LAYER FLOW PROBLEMS

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ABSTRACT

The purpose of this thesis is to present the numerical study of some boundary layer flow and heat transfer problems related to the channel with stretching/shrinking walls, orthogonally moving disks, stretching cylinder, vertical square duct, the stagnation point flow and the pulsatile flow. The study includes the self-similar problems and also the ones for which no similarity solution exists. For the problems of first type, a similarity transformation is used to convert the governing Navier-Stokes equations into a set of nonlinear third or fourth order ODEs. The ones of third order are solved by using the order-reduction technique whereas the later are solved by employing quasi-linearization or pseudo transient approach in which the time plays the role of an iteration parameter until the convergence is reached. In case of second type problems, we have used the three step explicit Runge-Kutta method for the numerical study of the unsteady pulsatile flow of a biofluid in the channel, whereas the numerical simulation in a vertical duct has been carried out by employing the Spectral method and the finite difference method. We have studied the flow and thermal properties of not only the Newtonian fluid but also the biofluid (blood), micropolar and Nano fluids as well. The effects of the governing parameters on different aspects of the problems are discussed through tables and figures.
CHAPTER 1

BRIEF LITERATURE REVIEW

A boundary layer is the layer of fluid very close to a bounding surface where the viscosity is significantly influential. The thickness of the velocity boundary layer is generally defined as the distance from the solid surface at which the viscous flow velocity is 99% of the free-stream velocity. The derivation of the boundary layer equations is considered to be one of the most significant developments in fluid dynamics. The famous Navier–Stokes equations characterizing the viscous flow are greatly simplified within the boundary layer, by employing an order of magnitude analysis. The boundary layer approximations divide the flow into an inviscid portion and the boundary layer, both being governed by easy to solve partial differential equations. It has also been observed that convective heat transfer in a fluid flow is affected only by the velocity values very close to the surface. This observation has led to the concept of thermal boundary layer. The thickness of the thermal boundary layer is defined similarly. That is, the distance from the surface at which the temperature is 99% of the temperature found from an inviscid solution. The Prandtl number describes the ratio of the two thicknesses. The behavior of the boundary layer is of great significance in the design of ships, submarines, offshore platforms, gliders and commercial aircraft.

Self-similarity is an important and powerful tool of fluid dynamics, and has successfully attracted the attention of the research community after the pioneering work of Von Karman [1]. This work was further extended by Cochran [2], Benton [3], Millsaps and Pohlhausen [4], Sparrow and Gregg [5], Attia [6, 7] and Mithal [8]. The numerical solution for the boundary layer flow over a stretching sheet with induced magnetic field was given by Ali et al. [7], using the Keller-box method.
Flows through channels have attracted the research community due to their applications in the fields of microfluidic devices, surface sublimation and filtration (please see Ashraf et al. [8]). Navier Stokes equations, which are the governing equations for the flow problems, have attracted the interest of the researchers. Some of the published papers, such as by Aung [9], Aung et al. [10], Barletta [11], Kumar et al. [12], and Vajravelu and Sastri [13], are concerned with the evaluation of the temperature and velocity profiles in the vertical fully developed flow regimes.

A very good description on the physical properties and classification of the nanofluids may be found in the papers such as Daungthongsuk and Wongwises [14], Wang and Mujumdar [15] and Kumar et al. [16]. Thermal dispersion effects in nanofluids flow in enclosure using a single phase model were analyzed by Khanafer et al. [17], for a differentially heated rectangular cavity. Khaled and Vafai [18] studied the heat transfer enhancement in a horizontal channel, whereas Mokmeli and Saffar-Avval [19] numerically studied the thermal properties of a nanofluid in a straight tube. Numerous researchers have reported results on natural convection in nanofluids considering various flow conditions in different geometries. Few of them are Tiwari and Das [20], Abu Nada [21], Oztop and Abu Nada [22], Nield and Kuznetsov [23], Abu-Nada and Oztop [24], Congedo et al. [25], Beg et al. [26], Rashidi et al. [27], Gorla et al. [28], Gorla and Chamka [29], Beg et al. [30], Kuznetsov and Nield [31], Muthamilselvan et al. [32], Bachok et al. [33], and Ahmad and Pop [34].

During the last decade, several researchers studied the nanofluid flow and heat transfer in cavities using the single phase model. Maïga et al. [35] studied the nanofluid behavior in a uniformly heated tube. Jou and Tzeng [36] considered in their research the differentially heated rectangular two dimensional enclosure with different aspect ratio. Tiwari and Das [37] studied differentially heated two-sided lid-driven
square cavity, Abu-Nada [38] investigated the flow and heat transfer of a nanofluid over a backward facing step.

After the pioneering work of Crane [39], the dynamics of the boundary layer flow over a stretching surface became a popular area of research due to its application in numerous industrial processes such as the production of paper, plastic and glass fiber, crystal growing, cooling of metallic sheets etc., where the desired characteristics of the final products depend upon the stretching process and the rate of cooling. Khan and Pop [40] studied the convective boundary layer flow of a nanofluid past a stretching surface by employing the model of Buongiorno [41] which includes the Brownian diffusion and thermophoresis. Due to the complexity of the Buongiorno’s model, Hamad and Pop [42] employed the usual viscous fluid model to study the boundary layer flow near a stagnation point on a heated permeable stretching surface in a porous medium saturated with a nanofluid, in the presence of heat generation and absorption. Magyari [43] criticized the work of Hamad and Pop [44] in which the effects of Brownian diffusion and thermophoresis were neglected but, in reply, Pop [45] also showed his disagreement with the comments of Magyari [46]. On the other hand, Travis et al. [47] pointed out that, since a nanofluid is a suspension, it cannot be modeled as a Newtonian fluid. Later, this point was taken up by Straughan [48].

In this thesis, we numerically study some flow and heat transfer problems related to the stagnation point flow, channel with stretching/shrinking walls, stretching cylinder, orthogonally moving disks, vertical square duct and the pulsatile flow in a channel. Beside the classical Newtonian fluid, we have considered the micropolar as well as nano fluids with different solid particles, in our studies.
CHAPTER 2

ON THERMAL CHARACTERISTICS OF SLIP FLOW
OUTSIDE A STRETCHING CYLINDER
2.1. ABSTRACT
In this chapter, we present the numerical study of the steady laminar two dimensional MHD nonlinear boundary layer slip flow of an incompressible viscous fluid due to a stretching cylinder, taking the viscous dissipation and the radiation effects into consideration. The study reveals that the thermal reversal near the surface, caused by the viscous dissipation, is remarkably reduced by the slip parameter, the stretching Reynolds number and the thermal radiation as well, whereas an opposite effect is observed for both the Prandtl number and the magnetic parameter. Moreover, the details of the computational procedure given in the paper may also be beneficial for the researchers working on the external flows in different geometries (for example, linearly or nonlinearly stretching/shrinking sheets, rotating and/or expanding disks etc).

2.2. MATHEMATICAL FORMULATION
We consider the steady laminar flow of an incompressible electrically conducting fluid over a stretching cylinder of radius \( a \) in a fluid at rest as shown in Fig. (2.1). The \( z \)-axis is taken along the axis of the cylinder and the \( r \)-axis is considered in the radial direction. The surface of the cylinder is considered to be at constant temperature \( T_w \), the ambient fluid temperature is \( T_\infty \) (with \( T_\infty < T_w \)), and \( B \) is the intensity of the uniform magnetic field in the radial direction.

The governing equations of the flow and heat transfer with boundary layer approximations, taking into account the effects of the thermal radiation and viscous dissipation, may be written as:

Continuity Equation

\[
\frac{\partial (ru)}{\partial r} + \frac{\partial (rw)}{\partial z} = 0, \quad (2.1)
\]

\( z \)-Components of Momentum Equation
\[ u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = \nu \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) - \frac{\sigma B^2}{\rho} w, \quad (2.2) \]

**r- Components of Momentum Equation**

\[ u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right), \quad (2.3) \]

**Heat Equation**

\[ \rho c_p \left( u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial w} \right) = \kappa_0 \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) - \frac{\partial q_r}{\partial r} + \mu \left( \frac{\partial u}{\partial r} \right)^2. \quad (2.4) \]

Here, \( u \) and \( w \) are, respectively, the velocity components along \( r \) and \( z \) directions, \( \nu \) is the dynamic viscosity, \( \sigma \) is the electrical conductivity, \( \rho \) is the density, \( p \) is the pressure, \( c_p \) is the specific heat at constant pressure, \( T \) is the temperature, \( \kappa_0 \) is the thermal conductivity and \( q_r \) is the radiative heat flux of the fluid.

**Boundary Conditions**

\[
\begin{align*}
  r = a : & \ u = 0, w = u_s, T = T_w, \\
  r \to \infty : & \ w = 0, T = T_\infty,
\end{align*}
\quad (2.5)
\]

with \( u_s \) being the slip velocity which is defined as the difference between the velocities of the fluid and the stretching cylinder.

We now employ the Rosseland approximation for thermal radiation in an optically thick layer, we have (by following Devi and Devi [90])

\[ q_r = -\frac{4\sigma^*}{3k^*} \frac{\partial T^4}{\partial y}, \quad (2.6) \]

where \( k^* \) and \( \sigma^* \) are, respectively, the mean absorption coefficient and the Stefan-Boltzmann constant. Now using Taylor series, we express \( T^4 \) as a linear function of temperature as

\[ T^4 = 4T^3_\infty T - 3T^4_\infty. \quad (2.7) \]
Thus Eq. (2.4) in view of Eqs. (2.6) and (2.7) becomes

\[
\rho c_p \left( u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial w} \right) = \kappa_0 \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \frac{16 \sigma^* T^3}{3k^*} \frac{\partial^2 T}{\partial r^2} + \mu \left( \frac{\partial u}{\partial r} \right)^2 .
\] (2.8)

In order to reduce the partial differential Eqs. (2.1-2.3) and (2.8) into ordinary ones, we introduce the following similarity transformations:

\[
\eta = \left( \frac{r}{a} \right)^2, \quad u = -ac f(\eta), \quad w = 2czf'(\eta), \quad \theta(\eta) = \frac{T - T_x}{T_w - T_x},
\] (2.9)

which satisfy Eq. (2.1) identically. This means that the velocity components given in the above equation are compatible with the continuity equation, and therefore represent a possible fluid motion.

Thus, we finally have the following system of equations:

\[
\eta f'' + f^* - Mf' = \text{Re}(f^{\prime 2} - f^{\prime 4}),
\] (2.10)

\[
\frac{p - p_w}{\rho c_v} = -\frac{\text{Re}}{\eta} f^2 - 2f^*,
\] (2.11)

\[
\eta(1 + \frac{4}{3} N_r) \theta'' + (1 + \text{Re Prf} + \frac{2}{3} N_r) \theta' + \text{Pr Ec} \eta f^* = 0 .
\] (2.12)

According to the partial slip condition on the surface of cylinder,

\[
u_s = N \tau \quad \text{with} \quad \tau = \left( \mu \frac{\partial w}{\partial r} \right)_{r=a},
\] (2.13)

where \( N \) being the slip coefficient.

Thus, the boundary conditions given in Eq. (2.5) acquire the form

\[
f(1) = 0, \quad f'(1) = 1 = \lambda f''(1), \quad f'(\infty) = 0, \quad \theta(1) = 1, \quad \theta(\infty) = 0,
\] (2.14)

Where

\[
M = \frac{\sigma B^2 a^2}{4 \nu \rho}, \quad \text{Re} = \frac{a^2 c}{2 \nu}, \quad \text{Ec} = \frac{w_s}{c_p (T_w - T_x)}, \quad N_r = \frac{4 \sigma^* T^3}{k^* \rho k}, \quad \text{Pr} = \frac{\mu c_p}{k} \quad \text{and} \quad \lambda = \frac{2 N \mu}{a}
\]

are, respectively, the magnetic parameter, the stretching Reynolds number, the Eckert
number, the radiation number, the Prandtl number and the normalized slip factor. It is to note that the Eckert number gives the relationship between the kinetic energy and the enthalpy of a flow, and is used to characterize dissipation.

The physical quantities of interest, the skin friction coefficient $C_f$ and Nusstle number $N_u$, are defined as

$$C_f = \frac{\tau_w}{\rho w^2 / 2}, \quad N_u = \frac{aq_w}{k_0(T_w-T_u)},$$

whereas the wall shear stress $\tau_w$ and the wall heat flux $q_w$ are given by

$$\tau_w = \mu \left( \frac{\partial w}{\partial r} \right)_{r=a}, \quad q_w = -k_0 \left( \frac{\partial T}{\partial r} \right)_{r=a}.$$  \hspace{1cm} (2.16)

Using Eqs. (2.9) and (2.16) in Eq. (2.15), we get

$$C_f(Re z/a) = f^*(1), N_u = -2\theta'(1).$$  \hspace{1cm} (2.17)

### 2.3. NUMERICAL SOLUTION

The domain of the present problem is too large for the direct numerical simulation, due to the algebraic decay. To compress the domain we use an exponential transformation $\eta = e^x$ which transforms the domain $[1, \infty)$ to $[0, \infty)$ and yields the following system:

$$f_{xxx} - 2f_{xx} + f_x - Me^{x}f_x = Re \left(f_x^2 - ff_{xx} + ff_x \right),$$ \hspace{1cm} (2.18)

$$\frac{p - p_x}{\rho c^2} = -e^x \left( Re f(x)^2 + 2f_x \right),$$ \hspace{1cm} (2.19)

$$\left(1 + \frac{4}{3}Nr \right)\theta_{xx} + \left( Re Pr f(x)^2 - \frac{2}{3}Nr \right)\theta_x + Pr Ec e^{-2x} \left(f_{xx} - f_x \right)^2 = 0,$$ \hspace{1cm} (2.20)

with boundary conditions

$$f(0) = 0, \lambda f_x(0) = (1 + \lambda)f_x(0) - 1, \theta(0) = 1, \quad f_x(\infty) = 0, \theta(\infty) = 0.$$  

It is important to note that $f^*(1) = f_{xx}(0) - f_x(0)$ and $\theta'(1) = \theta_x(0)$. 
In this chapter, we have employed the domain truncation approach in which the finite length of the domain is so adjusted that profiles thus obtained are compatible with their asymptotic behavior (Pantokratoras [91-92]).

In order to reduce the order of Eq. (2.18) by one (please see Ashraf et al. [23, 86-88]), we substitute $q = f_x$. The order reduction ultimately gives rise to the algebraic system, which is diagonally dominant & therefore possesses better convergence properties when the iterative methods like the successive-over-relaxation (SOR) is applied. Thus, we have to solve the following system of coupled equations:

\[ q = f_x, \quad (2.21) \]

\[ q_{xx} - 2q_x + q - M e^s q = Re \left( q^2 - f q_x + f q \right), \quad (2.22) \]

\[ \frac{p - p_e}{\rho c_v} = -e^s \left( Re f (x)^2 + 2q \right), \quad (2.23) \]

\[ \left( 1 + \frac{4}{3} N_r \right) \theta_{xx} + \left( Re Pr f (x) - \frac{2}{3} N_r \right) \theta_x + Pr Ec e^{-2s} (q_x - q)^2 = 0, \quad (2.24) \]

subject to the boundary conditions

\[ f(0) = 0, \lambda q_x(0) = (1 + \lambda) q(0) - 1, q(R) = 0, \theta(0) = 1, \theta(R) = 0. \quad (2.25) \]

For the numerical solution of the present problem, we first discretize Eqs. (2.22) and (2.24) at a typical grid point $\eta = \eta_i$ by employing central difference approximations for the derivatives. The resultant algebraic system is solved iteratively by employing the SOR method (see Berden and Faires [93]), subject to the appropriate boundary conditions given in Eq. (2.25). On the other hand, Eq. (2.21) may be numerically integrated by using the Simpson’s rule or the 4th order R.K. method, after every iteration of the SOR method. The pseudocode (which looks more or less like a MATLAB code) for a single iteration of the above mentioned computational procedure is given below:
Moreover, \( i = 1 \), \( \eta_n \) correspond to the two boundaries so that \( R = (\eta_n - 1)h \), where \( \eta_n \) is the number of grid points. Further, we define

\[
\Omega = \max \left\{ \left\| f^{(k+1)} - f^{(k)} \right\|_{L_\infty}, \left\| q^{(k+1)} - q^{(k)} \right\|_{L_\infty}, \left\| \theta^{(k+1)} - \theta^{(k)} \right\|_{L_\infty} \right\}.
\]

The tolerance for the iterative process has been set to be \( 10^{-11} \).

Finally, the pressure can be readily found from Eq. (2.23), after solving Eqs. (2.21), (2.22) and (2.24).
2.3.1. ADJUSTING THE RELAXATION PARAMETER FOR CONVERGENCE

The relaxation parameter $\omega$ in the nonlinear case is chosen in such a way that the convergence is reached. From our numerical experience with the above mentioned numerical technique, we know that the choice of the relaxation parameter for convergence of the SOR method becomes especially tiresome when the governing nonlinear equations are highly coupled. To overcome this problem we chose $\omega$ a little higher and consider the quantity $\Omega$ after each iteration. It is worthy to note that a continuous increase in $\Omega$ means a growth of the error during the iterative process, which ultimately leads to the divergence of the iterative method. In our computer program, we implement the SOR method in such a way that the relaxation parameter $\omega$ is decreased by an amount of $\Delta \omega$, as $\Omega$ tends to rise. Our simulations have shown that this technique of reducing $\omega$ during the iterative process as the error grows, ensures the convergence of the method. In our computer code, we have taken $\Delta \omega = 3\%$ of $\omega$.

2.3.2. IMPROVING THE ACCURACY OF THE SOLUTION

We have also used the extrapolation technique for obtaining a solution of better accuracy using the work of Roache and Knupp [94].

The pseudocode for this process is given below:

```plaintext
for $i = 1 : \eta_{\infty}$
    $U_{2i-1}^e \leftarrow \frac{1}{3}(4U_{2i}^f - U_{i}^e)$
    $C_{2i-1} \leftarrow (U_{2i-1}^e - U_{2i-1}^f)$
end for

for $i = 2 : 2 : (2\eta_{\infty} - 2)$
    $U_{i}^e \leftarrow U_{i}^f + \frac{1}{2}(C_{i-1} - C_{i+1})$
end for
```
where \( U^c, U^f \) & \( U^e \) are the coarse grid, finer grid and the extrapolated solutions, respectively. Further, the coarse grid solution is injected into the finer one by employing the following operator:

\[
W_{2i+1} = U^e_i \\
W_{2i} = \frac{1}{2} \left( U^e_i + U^e_{i+1} \right),
\]

where \( W \) is the required initial guess on the fine grid. Moreover, \( \theta \) should be a little higher on the finer grid as compare to its value on the coarse grid. This makes sense as the initial guess for the solution on the finer grid is remarkably closer to the actual solution.

2.4. RESULTS AND DISCUSSION

In this section, we present our numerical results in tabular and graphical forms so that the important features of the solution for a range of values of the governing parameters (the normalized slip factor \( \lambda \), the magnetic parameter \( M \), the stretching Reynolds number \( Re \), the Eckert number \( Ec \), the radiation number \( Nr \) and the Prandtl number \( Pr \) ) affecting the flow and heat transfer characteristics of the problem, may be interpreted.

Table (2.1) shows the convergence of our numerical results for the dimensionless velocity \( \frac{f_x(x)}{e_x} \) as the step-size decreases, which gives us confidence on our computational procedure. It is obvious from Eqs. (2.18), (2.19) and (2.20) that the shear stress is not affected by the Eckert number \( Ec \), the Prandtl number \( Pr \) or the radiation number \( Nr \), due to decoupled equations.

It is clear from Tables (2.2) and (2.3) that the slip tends to reduce the shear stress whereas both the Reynolds number and the magnetic parameter increase it remarkably. It is worthy to note in Table (2.4) that the sign of \( \theta_\eta(1)(=\theta_\eta(0)) \) changes
as the Eckert number increases. Physically, this means a reversal in the direction of heat transfer on the surface. That is, heat has started flowing from the fluid to the surface. Thus, we draw an important result that the viscous dissipation may cause a change in the direction of heat transfer on the cylinder, which is supported by the external magnetic field, as $\theta_s(0)$ increases only in case of thermal reversal. Therefore, the viscous dissipation should be taken into consideration while studying the boundary layer flows around the cylinders.

Moreover, the higher values of both the slip and the Reynolds number have the tendency to reduce the reverse flow of heat, and can even eliminate it (please see Table (2.5)). Table (2.6), on the other hand, shows that the Prandtl number increases thermal reversal whereas the thermal radiation decreases it.

Now, we give the graphical interpretation of our results. The stream surface and hence the streamlines (which are the sections of this surface by the planes parallel to the rz – plane) for the problem are given in Figs. (2.2a) and (2.2b).

Figures (2.3)-(2.6) give the influence of the normalized slip factor $\lambda$ on the velocity, temperature and pressure distributions. It is clear that not only the radial and axial velocity profiles but also the thermal reversal at the surface of the cylinder is reduced by $\lambda$ while increasing the thermal boundary layer. The velocity boundary layer, on the other hand, is also remarkably reduced by the slip factor, which may be more prominently observed in the pressure profiles (please see Fig. (2.6)).

The effect of the external magnetic field on the momentum and thermal aspects of the problem is shown in Figs. (2.7)-(2.10). Obviously, the effect of the magnetic parameter $M$ (also known as the Hartman number) on the profiles is similar to that of the slip, with the exception that the former supports the thermal reversal. It is because the imposition of a magnetic field to an electrically conducting fluid creates a drag
like force called the Lorentz force. This force has the tendency to slow down the fluid flow while increasing its temperature. This is demonstrated by the lowering of the velocity profiles and raising of the temperature distributions, with the increase in $M$. Therefore, the heat transfer rate at the cylinder, which is directly proportional to the temperature difference between the fluid and the cylinder, also increases.

It is important to mention that the stretching Reynolds number indicates the relative significance of the inertia effect compared to the viscous one. For $M = 1, Pr = 2, Nr = 1, Ec = 4$, Figs. (2.11)-(2.14) show that velocity and temperature profiles decrease as the Reynolds number $Re$ increases, while thinning the velocity and thermal boundary layers.

It is clear from Figs. (2.15) and (2.16) that both the Eckert number $Ec$ and the radiation number $Nr$ increase the thermal boundary layer but the former causes the thermal reversal whereas the latter opposes it. Finally, it is obvious from Fig. (2.17) that increasing the Prandtl number $Pr$ supports the thermal reversal and decreases the thermal boundary layer. Physically, when $Pr$ increases, the thermal diffusivity decreases which leads to the decreasing in the energy ability that reduces the thermal boundary layer.

2.5. CONCLUSIONS

We have studied the steady laminar MHD nonlinear slip flow of an incompressible viscous fluid due to a stretching cylinder, with the viscous dissipation and radiation effects. A similarity transformation is used to reduce the governing boundary layer partial differential equations into a set of ordinary ones. The transformed equations are then numerically solved by employing an efficient algorithm based on the finite difference method. Following conclusions may be drawn from the present study:
An increase in the Eckert number may cause a change in the direction of heat transfer near the surface of the cylinder.

The thermal reversal is opposed by the radiation number and the slip parameter, whereas an opposite trend is noted for both the Prandtl number and the magnetic parameter.

The Eckert number, the magnetic parameter, the slip parameter and the radiation number increase the thermal boundary layer whereas both the Reynolds number and the Prandtl number decrease it.

The slip parameter tends to reduce the shear stress whereas both the Reynolds number and the magnetic parameter increase it remarkably.

All the parameters of the problem affecting the velocity boundary layer reduce it.
Table 2.1 Dimensionless velocity \( \frac{f_s(x)}{e^x} \) on three grid sizes and extrapolated values for \( \lambda = 1, M = 0.2, Re = 1, Nr = 0.1, Ec = 2 \) and \( Pr = 7 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>1(^{\text{st}}) grid ( (h = 0.05) )</th>
<th>2(^{\text{nd}}) grid ( (h = 0.025) )</th>
<th>3(^{\text{rd}}) grid ( (h = 0.0125) )</th>
<th>Extrapolated values</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.49307709</td>
<td>0.49307981</td>
<td>0.49308049</td>
<td>0.49308072</td>
</tr>
<tr>
<td>0.5</td>
<td>0.28144541</td>
<td>0.28145019</td>
<td>0.28145138</td>
<td>0.28145178</td>
</tr>
<tr>
<td>1</td>
<td>0.14550929</td>
<td>0.14551555</td>
<td>0.14551712</td>
<td>0.14551764</td>
</tr>
<tr>
<td>1.5</td>
<td>0.06735824</td>
<td>0.06736350</td>
<td>0.06736482</td>
<td>0.06736526</td>
</tr>
<tr>
<td>2</td>
<td>0.02741418</td>
<td>0.02741739</td>
<td>0.02741819</td>
<td>0.02741846</td>
</tr>
<tr>
<td>2.5</td>
<td>0.00954086</td>
<td>0.00954231</td>
<td>0.00954268</td>
<td>0.00954280</td>
</tr>
<tr>
<td>3</td>
<td>0.00272891</td>
<td>0.00272933</td>
<td>0.00272944</td>
<td>0.00272947</td>
</tr>
</tbody>
</table>

Table 2.2 Shear stress on the surface of the cylinder for \( Re = 1, Ec = 2, Nr = 0.1 \) and \( Pr = 7 \)

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( f_{\eta\eta}(1) = f_{\eta\eta}(0) - f_x(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>(-1.011718)</td>
</tr>
<tr>
<td>0.5</td>
<td>(-0.663890)</td>
</tr>
<tr>
<td>1</td>
<td>(-0.473886)</td>
</tr>
<tr>
<td>5</td>
<td>(-0.153202)</td>
</tr>
<tr>
<td>10</td>
<td>(-0.084802)</td>
</tr>
</tbody>
</table>
Table 2.3 Shear stress on the surface of the cylinder for $M = 0.1, Ec = 2, Nr = 0.1$ and $Pr = 7$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$f_{\eta\eta}(1) = f_{xx}(0) - f_x(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Re = 0.1</td>
</tr>
<tr>
<td>0.1</td>
<td>-0.598239</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.471442</td>
</tr>
<tr>
<td>1</td>
<td>-0.375305</td>
</tr>
<tr>
<td>5</td>
<td>-0.147107</td>
</tr>
<tr>
<td>10</td>
<td>-0.084417</td>
</tr>
</tbody>
</table>

Table 2.4 Heat transfer rate on the surface of the cylinder for $Re = 1, \lambda = 0.1, Nr = 0.1$ and $Pr = 7$

<table>
<thead>
<tr>
<th>$M$</th>
<th>$\theta_x(1) = \theta_x(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Ec = 0$</td>
</tr>
<tr>
<td>0</td>
<td>-1.801537</td>
</tr>
<tr>
<td>0.5</td>
<td>-1.716997</td>
</tr>
<tr>
<td>1</td>
<td>-1.649890</td>
</tr>
<tr>
<td>3</td>
<td>-1.449543</td>
</tr>
<tr>
<td>5</td>
<td>-1.303085</td>
</tr>
</tbody>
</table>
**Table 2.5** Heat transfer rate on the surface of the cylinder for $M = 0.1, Nr = 0.1,$ $Ec = 2$ and $Pr = 7$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\theta_y(1) = \theta_x(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Re = 0.1</td>
</tr>
<tr>
<td>0.1</td>
<td>2.373209</td>
</tr>
<tr>
<td>0.5</td>
<td>1.405836</td>
</tr>
<tr>
<td>1</td>
<td>0.804787</td>
</tr>
<tr>
<td>5</td>
<td>-0.078619</td>
</tr>
<tr>
<td>10</td>
<td>-0.151147</td>
</tr>
</tbody>
</table>

**Table 2.6** Heat transfer rate on the surface for $M = 0.1, \lambda = 0.1, Re = 0.1$ and $Ec = 2$

<table>
<thead>
<tr>
<th>$Nr$</th>
<th>$\theta_y(1) = \theta_x(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pr = 1</td>
</tr>
<tr>
<td>0.0</td>
<td>0.373903</td>
</tr>
<tr>
<td>0.5</td>
<td>0.292525</td>
</tr>
<tr>
<td>1.0</td>
<td>0.221041</td>
</tr>
<tr>
<td>1.5</td>
<td>0.173224</td>
</tr>
<tr>
<td>2.0</td>
<td>0.141300</td>
</tr>
</tbody>
</table>
Figure 2.1 Physical model of the problem.

Figure 2.2a Stream surface for the problem.
Figure 2.2b Streamlines for the present problem

Figure 2.3 Radial velocity profiles for $M = 0.2, \Pr = 7, Nr = 0.1, \Re = 1, \ Ec = 2$ and various $\lambda$
Figure 2.4 Axial velocity profiles for $M = 0.2, Pr = 7, Nr = 0.1, Re = 1, Ec = 2$ and various $\lambda$

Figure 2.5 Temperature profiles for $M = 0.2, Pr = 7, Nr = 0.1, Re = 1, Ec = 2$ and various $\lambda$
Figure 2.6 Pressure profiles for $M = 0.2, Pr = 7, Nr = 0.1, Re = 1, Ec = 2$ and various $\lambda$

Figure 2.7 Radial velocity profiles for $\lambda = 0.1, Pr = 7, Nr = 0.1, Re = 1, Ec = 2$ and various $M$
Figure 2.8 Axial velocity profiles for $\lambda = 0.1, \text{Pr} = 7, Nr = 0.1, \text{Re} = 1, Ec = 2$ and various $M$

Figure 2.9 Temperature profiles for $\lambda = 0.1, \text{Pr} = 7, Nr = 0.1, \text{Re} = 1, Ec = 2$ and various $M$
Figure 2.10 Pressure profiles for $\lambda = 0.1, \Pr = 7, Nr = 0.1, \text{Re} = 1, Ec = 2$ and various $M$

Figure 2.11 Radial velocity profiles for $\lambda = 1, \Pr = 7, Nr = 0.1, M = 0.1, Ec = 2$ and various $Re$
Figure 2.12 Axial velocity profiles for $\lambda = 1, \text{Pr} = 7, Nr = 0.1, M = 0.1, Ec = 2$ and various $Re$

Figure 2.13 Temperature profiles for $\lambda = 1, \text{Pr} = 7, Nr = 0.1, M = 0.1, Ec = 2$ and various $Re$
Figure 2.14 Pressure profiles for $\lambda = 1, \Pr = 7, Nr = 0.1, M = 0.1, Ec = 2$ and various $Re$

Figure 2.15 Temperature profiles for $\lambda = 1, \Pr = 7, Nr = 0.1, M = 0.1, Re = 0.1$ and various $Ec$
Figure 2.16 Temperature profiles for $\lambda = 1, \text{Pr} = 7, Ec = 2, M = 0.1, \text{Re} = 0.1$ and various $Nr$

Figure 2.17 Temperature profiles for $\lambda = 1, Nr = 0.1, Ec = 2, M = 0.1, \text{Re} = 0.1$ and various $Pr$
CHAPTER 3

VISCIOUS DISSIPATION EFFECTS IN STAGNATION POINT FLOW TOWARDS A HEATED STATIONARY SURFACE
3.1. INTRODUCTION

The chapter is devoted to the numerical investigation of the problem of two dimensional stagnation point flow. The flow under consideration is steady, incompressible and laminar, whereas the fluid is assumed to be micropolar in nature. A system of nonlinear ODEs is obtained by introducing suitable transformation in the Navier-Stokes equations. The solutions for a set of values of the governing parameters namely, the micropolar parameters, the Eckert number, and the Prandtl number are numerically computed, analyzed and discussed. The study reveals that the reverse flow of heat near the surface may occur due to viscous dissipation which may further be enhanced by the increasing values of the Prandtl number. It may therefore be recommended that the viscous dissipation may not be simply ignored in the stagnation point flows.

3.2. MATHEMATICAL FORMULATION

We assume the viscous dissipation effects in the stagnation point flow, whereas the micropolar model has been employed for the fluid. Further assumptions are the same as given above. If the horizontal and vertical velocity components are denoted by \( u \) and \( v \) (respectively), the system of governing equations may be written as (Lok et al. [96]):

Continuity Equation:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

Momentum Equation:

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \left( \frac{\mu + k}{\rho} \right) \frac{\partial^2 u}{\partial y^2} + \frac{k}{\rho} \frac{\partial \phi}{\partial y},\]

Angular Momentum Equation:
\[
\rho f \left( u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} \right) = \gamma \frac{\partial^2 \phi}{\partial y^2} - k \left( 2 \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial u}{\partial y} \right). \tag{3.3}
\]

It is to mention that:

| \( \phi \) (microrotation component normal to the xy-plane) | \( \rho \) (density) |
| \( k \) (vortax viscosity) | \( \gamma \) (spien gradient viscosity) |
| \( j \) (microineartia density) | \( p \) (pressure) |

All the physical quantities \( \rho, \mu, k, \gamma \) and \( j \) have been taken as fixed, in the present study.

The equation for temperature distribution is written as

\[
\rho c_p \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \kappa_0 \frac{\partial^2 T}{\partial y^2} + \mu \left( \frac{\partial u}{\partial y} \right)^2, \tag{3.4}
\]

whereas:

| \( T \) (temperature) | \( c_p \) (specific heat capacity under constant pressure settings) |
| \( \kappa_0 \) (thermal conductivity) |

The set of appropriate boundary conditions is:

\[
u(x,0) = 0, v(x,0) = 0, u(x, \infty) = U = ax, \phi(x,0) = 0, \phi(x, \infty) = 0, T(x,0) = T_w, T(x, \infty) = T_\infty. \tag{3.5}
\]

Following similarity transformation (compatible with the continuity equation (3.1)) is admissible for the present problem:

\[
\eta = \sqrt[2]{\frac{\alpha}{V}}, \quad p(x, \infty) = p_0 - \frac{\rho a^2}{2} (x^2 + y^2), \quad u(x, y) = axf'(\eta), \quad v(x, y) = -\sqrt{a} \nu f(\eta), \quad \phi(x, y) = -a \sqrt{\frac{a}{V}} xg, \quad \Theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}. \tag{3.6}
\]
After some simplifications, we have the following system

\((1 + C_1) f'' - C_1 g' + 1 = f'^2 - ff'', \quad (3.7)\)

\(C_2 g'' + C_1 C_4 (f'' - 2g) = f'g - fg' \), \quad (3.8)\)

\(\theta'' + \text{Pr} (f\theta' + Ecq'^2) = 0. \quad (3.9)\)

Here, \(C_1 = \frac{\kappa}{\mu}, C_2 = \frac{\mu}{\rho j a}, C_4 = \frac{\gamma}{j \mu}, Ec = \frac{U^2}{c_p(T_w - T_\infty)}\) and \(\text{Pr} = \frac{\mu c_p}{\kappa_0}\) are the vortex viscosisty parameter, the microinertia density parameter, the spien gradient viscoesity parameter, the Eckeart number and the Prandlte number, respectively.

Boundary conditions (3.5) assume the form:

\(f(0) = 0, f'(0) = 0, f'(\infty) = 1, g(0) = 0, g(\infty) = 0, \theta(0) = 1, \theta(\infty) = 0. \quad (3.10)\)

3.3. NUMERICAL SOLUTION

For the numerical solution, we use the computational procedure given in chapter 2.

The order of the governing equations is reduced by using the substitution \(q = f'\), which results in the following system of equations:

\[ q = f' = \frac{df}{d\eta}, \quad (3.11)\]

\[ (1 + C_1) q'' - C_1 g' + 1 = q^2 - fq', \quad (3.12)\]

\[ C_2 g'' + C_1 C_4 (q'' - 2g) - qg + fg' = 0, \quad (3.13)\]

\[ \theta'' + \text{Pr} f\theta' + \text{Pr} Ecq'^2 = 0. \quad (3.14)\]

Again the boundary conditions assume the form:

\(f(0) = 0, q(0) = 0, q(\infty) = 1, g(0) = 0, g(\infty) = 0, \theta(0) = 1, \theta(\infty) = 0. \quad (3.15)\)

The system of differential equations (3.11)-(3.14) is then solved by employing the computational procedure mentioned in the chapter 2.
3.4. RESULTS AND DISCUSSION

We will present the results in tabular as well as in graphical form to study the effect of relevant parameters on the problem.

In Table (3.1), we have given the numerical values of $\eta$ (along with the extrapolated values) at different grid locations, corresponding to three different grid sizes. This clearly indicates the convergence of our numerical results.

For a range of values of the micropolar parameters (that is, the vortax viscoesity parameter $C_1$, the microineartia density parameter $C_2$, and the spien gradient viscoesity parameter $C_3$), the Eckeart number $Ec$ and Prandtl number $Pr$, we present the following quantities:

- shear and couple stresses
- heat transfer rate
- thicknesses of both the velocity and thermal boundary layers
- velocity, microrotation and temperature fields.

The values of all the parameters are taken arbitrarily (please see Ali et al. [97], Guram and Anwar [98] and Takhar et al. [99]).

With fixed $C_1 = 2, C_2 = 0.4$ and $C_3 = 0.5$, Table (3.3) shows that an increase in $Ec$ may cause thermal reversal in the present problem as well. Moreover, the Prandtl number always increases the heat transfer rate $\theta'(0)$ at the surface of the heated sheet.

The effect of $Ec$ on $\theta'(0)$ is presented in Table (3.4), for different sets of values of the micropolar parameters, with fixed $Pr = 0.5$. It is clear that an increase in the values of the micropolar parameters not only decreases the heat transfer rate on the surface but it may also prevent the thermal reversal caused by the viscous dissipation.
With fixed $Ec = 3$, $Pr = 0.5$, Table (3.5) shows that an increase in the micropolar parameters results in the reduction in shear stress $f''(0)$ whereas an opposite effect can be observed for couple stress $g'(0)$. On the other hand, we observe that the sign of $\theta(0)$ changes from positive to negative as values of the micropolar parameters increase, which means that the micropolar structure of the fluids opposes the thermal reversal due to viscous dissipation.

Now we give the interpretation of our graphical results. The stream surface and the streamlines for the problem are given in Figs. (3.1a) and (3.1b). The effect of the micropolar parameters $C_1, C_2 & C_3$ for fixed values of $Ec$ and $Pr$ is shown in Figs. (3.2)-(3.5). An increase in these parameters decreases both the normal and streamwise velocity profiles while increasing the velocity boundary layer thickness, as shown in Figs. (3.2) and (3.3). On the other hand, Fig. (3.4) indicates that the microrotation profiles rise with the increasing values of the micropolar parameters. An increase in the micropolar parameters decreases the temperature profiles near the surface whereas an opposite effect can be seen away from the surface, as shown in Fig. (3.5). Thermal boundary layer thickness also increases with an increase in the micropolar parameters. The effect of $Ec$ on temperature profiles for different cases of the values of the micropolar parameters with fixed $Pr$, is shown in Figs. (3.6)-(3.10). It is clear that the viscous dissipation always causes the thermal reversal near the surface while increasing the thermal boundary layer.

For different values of $Ec$, the influence of $Pr$ on the temperature profiles is presented in Fig. (3.11)-(3.14), with fixed $C_1, C_2 & C_3$. In the absence of viscous dissipation, an increase in $Pr$ does not alter the direction of heat transfer as demonstrated by the Fig. (3.11). On the other hand Figs. (3.12)-(3.14) show that increasing $Pr$ not only
decreases the thermal boundary layer but also facilitates the thermal reversal near the surface while lowering the temperature profiles away from the surface.

3.5. CONCLUSIONS

Following conclusions are the essential outcome of the study:

- Effect of the micropolar fluid structure is to reduce the shear stresses as well as the heat transfer rate at the surface.
- A change in the direction of heat transfer may occur due to viscous dissipation.
- The thermal reversal near the surface is encouraged by the increasing values of the Prandtl number, whereas an opposite trend is obvious for the micropolar parameters.
- Micropolar parameters may add to thicknesses of both the velocity and thermal boundary layer.
- A significant shrinking of the thermal boundary layer has been noted by increasing the Prandtl number.
Table 3.1 Dimensionless normal velocity $f(\eta)$ on three grid levels and extrapolated values for $C_1 = 2$, $C_2 = 0.4$, $C_3 = 0.5$, $Ec = 2$ and $Pr = 0.5$

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$h = 0.09$</th>
<th>$h = 0.045$</th>
<th>$h = 0.0225$</th>
<th>Extrapolated value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1.8</td>
<td>0.8731975</td>
<td>0.8733133</td>
<td>0.8733424</td>
<td>0.8733520</td>
</tr>
<tr>
<td>3.6</td>
<td>2.5494309</td>
<td>2.5495316</td>
<td>2.5495569</td>
<td>2.5495653</td>
</tr>
<tr>
<td>5.4</td>
<td>4.3411765</td>
<td>4.3412629</td>
<td>4.3412847</td>
<td>4.3412919</td>
</tr>
<tr>
<td>7.2</td>
<td>6.1409138</td>
<td>6.1409988</td>
<td>6.1410202</td>
<td>6.1410273</td>
</tr>
<tr>
<td>9</td>
<td>7.9409104</td>
<td>7.9409952</td>
<td>7.9410166</td>
<td>7.9410237</td>
</tr>
</tbody>
</table>

Table 3.2 Values of the micropolar parameters used in the present study

<table>
<thead>
<tr>
<th>Cases</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.4</td>
<td>0.5</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0.5</td>
<td>0.6</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>0.6</td>
<td>0.7</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>0.7</td>
<td>0.8</td>
</tr>
</tbody>
</table>
Table 3.3 Heat transfer rate on the surface for $C_1 = 2$, $C_2 = 0.4$, $C_3 = 0.5$ and various values of $Pr$ and $Ec$

<table>
<thead>
<tr>
<th>$Pr$</th>
<th>$\theta(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Ec = 0$</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.3894575</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.5047505</td>
</tr>
<tr>
<td>1.5</td>
<td>-0.5851450</td>
</tr>
<tr>
<td>2.0</td>
<td>-0.6488796</td>
</tr>
</tbody>
</table>

Table 3.4 Heat transfer rate on the surface for $Pr = 0.5$, with different $Ec$ and various cases of the micropolar parameters

<table>
<thead>
<tr>
<th>Cases</th>
<th>$\theta(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Ec = 0$</td>
</tr>
<tr>
<td>01</td>
<td>-0.4333713</td>
</tr>
<tr>
<td>02</td>
<td>-0.3895185</td>
</tr>
<tr>
<td>03</td>
<td>-0.3705683</td>
</tr>
<tr>
<td>04</td>
<td>-0.3581869</td>
</tr>
<tr>
<td>05</td>
<td>-0.3488841</td>
</tr>
</tbody>
</table>
Table 3.5 Shear & couple stresses and heat transfer rate on the surface for $Pr = 0.5, Ec = 3$ and various cases of the micropolar parameters

<table>
<thead>
<tr>
<th>Cases</th>
<th>$f'(0)$</th>
<th>$\theta'(0)$</th>
<th>$g'(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>1.2326242</td>
<td>0.4669885</td>
<td>0</td>
</tr>
<tr>
<td>02</td>
<td>0.6539339</td>
<td>0.1162702</td>
<td>0.6101800</td>
</tr>
<tr>
<td>03</td>
<td>0.4717224</td>
<td>0.0285995</td>
<td>0.7538036</td>
</tr>
<tr>
<td>04</td>
<td>0.3817171</td>
<td>-0.0164319</td>
<td>0.8043091</td>
</tr>
<tr>
<td>05</td>
<td>0.3273375</td>
<td>-0.0454856</td>
<td>0.8268598</td>
</tr>
</tbody>
</table>
Figure 3.1a Stream surface for the problem

Figure 3.1b Streamlines for the problem
Figure 3.2 Normal velocity profiles for $Pr = 1, Ec = 3$ and various cases of the micropolar parameters

Figure 3.3 Streamwise velocity profiles for $Pr = 1, Ec = 3$ and various cases of the micropolar parameters
Figure 3.4 Microrotation profiles for $\text{Pr} = 1, \text{Ec} = 3$ and various cases of the micropolar parameters

Figure 3.5 Temperature profiles for $\text{Pr} = 1, \text{Ec} = 3$ and various cases of the micropolar parameters
Figure 3.6 Temperature profiles for $Pr = 1, C_1 = 0, C_2 = 0, C_3 = 0$ and various $Ec$

Figure 3.7 Temperature profiles for $Pr = 1, C_1 = 2, C_2 = 0.4, C_3 = 0.5$ and various $Ec$
Figure 3.8 Temperature profiles for $Pr = 1, C_1 = 4, C_2 = 0.5, C_3 = 0.6$ and various $Ec$

Figure 3.9 Temperature profiles for $Pr = 1, C_1 = 6, C_2 = 0.6, C_3 = 0.7$ and various $Ec$
Figure 3.10 Temperature profiles for $Pr=1, C_1=8, C_2=0.7, C_3=0.8$ and various $Ec$.

Figure 3.11 Temperature profiles for $Ec=0, C_1=6, C_2=0.6, C_3=0.7$ and various $Pr$. 
Figure 3.12 Temperature profiles for $Ec = 2, C_1=6, C_2=0.6, C_3=0.7$ and various Pr

Figure 3.13 Temperature profiles for $Ec = 4, C_1=6, C_2=0.6, C_3=0.7$ and various Pr
Figure 3.14 Temperature profiles for $Ec = 6, C_1 = 6, C_2 = 0.6, C_3 = 0.7$ and various $Pr$
CHAPTER 4

MHD MICROPOLAR FLOW IN A CHANNEL WITH

SHRINKING WALLS
4.1. INTRODUCTION

We numerically study the steady hydromagnetic (MHD) properties of a fluid in a channel with shrinking walls. The fluid under consideration is micropolar, electrically conducting, viscous and incompressible. We have employed two distinct numerical techniques are employed to solve the transformed self similar nonlinear ordinary differential equations. One makes use of Quasilinearization, whereas the pseudo transient method has been employed as second approach. The two approaches may be easily extended to the other geometries (for example, sheets, disks, cylinders etc) with possible wall conditions like slip, stretching, rotation, suction and injection.

4.2. MATHEMATICAL FORMULATION

We assume a transverse magnetic field acting on the steady hydromagnetic (MHD) fluid in a channel with shrinking walls. The fluid under consideration is micropolar, electrically conducting, viscous and incompressible. The two walls of the channel are located at \( y = -a \) and \( y = a \), where \( 2a \) is the channel width.

In the present case, the velocity and microrotation fields assume the form:

\[
\mathbf{v} = (u(x,y), v(x,y), 0), \quad \mathbf{\phi} = (0, 0, \phi(x,y)),
\]

(4.1)

where \( \phi \) is the component of the microrotation normal to the \( xy \) – plane.

In view of (4.1), governing equations of the flow and heat transfer for the problem are:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 ,
\]

(4.2)

\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{(\mu + \kappa)}{\rho} \nabla^2 u + \frac{\kappa}{\rho} \frac{\partial \phi}{\partial y} - \frac{\sigma B_0^2}{\rho} u ,
\]

(4.3)
\[
\frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -1 \frac{\partial p}{\partial y} + \left( \frac{\mu + \kappa}{\rho} \right) \nabla^2 v - \kappa \frac{\partial \phi}{\partial x},
\]
\( \ldots \) (4.4)

\[
\rho \left( u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} \right) = \gamma \nabla^2 \phi + \kappa \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) - 2k\phi ,
\]
\( \ldots \) (4.5)

\[
\rho c_p \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \kappa_0 \frac{\partial^2 T}{\partial y^2} ,
\]
\( \ldots \) (4.6)

It is to mention that the physical quantities appearing in Eqs.(4.2)-(4.6) have already been mentioned in chapter 3.

**Boundary Conditions**

\[
u(x, \pm a) = -bx_v(x, \pm a) = 0, \phi(x, \pm a) = 0, T(x, -a) = T_1, T(x, a) = T_2,
\]
\( \ldots \) (4.7)

where \( b > 0 \) is the shrinking rate of the channel walls, and \( T_1 \) and \( T_2 \) are the fixed temperatures of the lower and upper channel walls respectively.

We use following transformation

\[
\eta = \frac{y}{a}, u = bxf'(\eta), v = -abf(\eta), \phi = -\frac{b}{a} xg(\eta), \theta(\eta) = \frac{T - T_2}{T_1 - T_2}
\]
\( \ldots \) (4.8)

After eliminating the pressure term from (4.3) and (4.4), we introduce the above similarity transformation to the resulting equation to get,

\[
(1 + C_1) f'' - C_1 g'' = R(f'g'' - f''g') + Mf'',
\]
\( \ldots \) (4.9)

whereas (4.5) and (4.6), in view of (4.8), are

\[
C_4 g'' + C_5 C_2 (f'' - 2g) = f''g - fg',
\]
\( \ldots \) (4.10)

\[
\theta'' + Pr f' \theta' = 0
\]
\( \ldots \) (4.11)
Here \( C_1 = \frac{\kappa}{\mu} \) is the vortext viscoasity parameter, \( C_2 = \frac{\mu}{\rho j b} \) is the microinnertia
density parameter, \( C_3 = -\frac{\gamma}{\rho ja^2 b} \) is the spein gradeoant viscoesity parameter,
\( R = \frac{\rho a^2 b}{\mu} \) is the shreinkig Reaynold number, \( Pr = \frac{\mu c_p}{\kappa_0} \) is the Pranrdle number and
\( M = \frac{a^2 \sigma B_0^2}{\mu} \) is the magneatic parameter.

Finally, the boundary conditions given in (4.7) are, therefore, reduced to
\[
\begin{align*}
    f(\pm 1) &= 0, \quad f'(\pm 1) = -1, \quad g(\pm 1) = 0, \quad \theta(-1) = 1, \quad \theta(1) = 0.
\end{align*}
\]  
(4.12)

### 4.3. Numerical Solution

A usual approach is to write the nonlinear ODEs in the form of a first order initial
value problem as follows:

Setting: \( f' = p, \ f^{\prime\prime} = q, \ f^{\prime\prime\prime} = r, \ g' = s, \ \theta' = t \) in (4.9)-(4.11), we have
\[
\begin{align*}
    f' &= p, \ p' = q, \ q' = r, \\
    r' &= \frac{1}{(1 + C_1)} (Rpq - Rfr + Mq) + \frac{C_1}{(1 + C_1) C_3} (pg - fs - C_1 C_2 q + 2 C_1 C_2 g), \\
    g' &= s, \ s' = \frac{1}{C_3} (pg - fs - C_1 C_2 q + 2 C_1 C_2 g), \\
    \theta' &= t, \ t' = -Pr Rtf,
\end{align*}
\]  
(4.13)

With the following required boundary conditions
\[
\begin{align*}
    f(-1) &= 0, \ p(-1) = -1, \ g(-1) = 0, \ \theta(-1) = 1, \ q(-1) = \alpha_1, \ r(-1) = \alpha_2, \\
    s(-1) = \alpha_3, \ t(-1) = \alpha_4.
\end{align*}
\]  
(4.14)

Here, \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) that is \( f^{\prime\prime}(-1), f^{\prime\prime\prime}(-1), g'(-1), \theta'(-1) \) are the unknown initial
conditions. Therefore, a shooting methodology is incorporated to solve the above
system, which may be a combination of Runge Kutta method (to solve the 1st order
ODEs) & a four dimensional zero finding algorithm (to find the missing conditions).
It is worthy to note that the missing initial conditions are computed so that the solution satisfies the boundary conditions \( f(0) = 0, p(0) = -1, g(1) = 0, \vartheta(0) = 0 \) of the original boundary value problem.

A serious shortcoming of shooting is the blowing up of the solution, before the initial value problem is completely integrated, and this happens quite often even with very accurate guesses for the initial conditions. This phenomenon is due to the instability of the differential equations and also because of the inherent strong dependence of the solution on the initial conditions of the problem. On the other hand, a finite difference method (FDM) does not suffer from this short coming, and does have a chance as it tends to keep a firm hold on the entire solution at once.

Due to these features, it is desirable to incorporate the FDM in the computational procedure. Therefore, in our earlier works ([23] and [86-88]), we followed an approach different from the usual shooting method and involving FDM, which we describe as follows:

An inspection of (4.9) reveals that it can be readily integrated & becomes,

\[
(1 + C_1)f'' - C_1 g' - R f'' - R f' = \beta,
\]

where \( \beta \) is the constant of integration to be determined.

We consider this third order equation as the following system of coupled 1\textsuperscript{st} order & 2\textsuperscript{nd} order ODEs:

\[
f' = p,
\]

\[
(1 + C_1)p'' - C_1 g' - R (p'^2 - fp') = \beta
\]

We solve these equations along with Eqs. (4.10) and (4.11), subject to the boundary conditions

\[
f(\pm 1) = 0, \quad p(\pm 1) = -1, \quad g(\pm 1) = 0, \quad \vartheta(\pm 1) = 1, \quad \vartheta(0) = 0.
\]
We prefer the SOR method because of its efficiency and simple computer implementation. The details of the solution procedures may be found in the chapter 2.

It is important to mention that the boundary condition \( f_n = 0 \) is used in finding the constant of integration \( \beta \) which is the only unknown as compared to the four missing initial conditions in the shooting approach. Moreover it is chosen in such a way that the absolute difference between the approximate (obtained using the above program) & actual values of \( f_n \) (which is zero in the present problem) is very small. We kept the difference less than \( 10^{-6} \) while adjusting \( \beta \).

### 4.3.1. QUASI-LINEARIZATION METHOD

In quasi-linearization, we construct the sequences of vectors \( \{ f^{(k)} \} \), \( \{ g^{(k)} \} \), \( \{ \theta^{(k)} \} \) which converge to the numerical solutions of (4.9), (4.10) and (4.11) respectively. To construct \( \{ f^{(k)} \} \) we linearize (4.9), by retaining only the first order terms, as follows:

We set: \( G(f, f', f'', f''', f^{(iv)}) \equiv (1 + C_1) f^{(iv)} + Rff'' - Rff'' - Mf'' - C_1 g'' \),

\[
G(f^{(k)}, f'(k), f''(k), f'''(k), f^{(iv)}) + (f'^{(k+1)} - f^{(k)}) \frac{\partial G}{\partial f'(k)} + (f''^{(k+1)} - f''^{(k)}) \frac{\partial G}{\partial f''(k)}
\]

\[
+ (f'''^{(k+1)} - f'''^{(k)}) \frac{\partial G}{\partial f'''(k)} + (f^{(iv)}^{(k+1)} - f^{(iv)}^{(k)}) \frac{\partial G}{\partial f^{(iv)}(k)} = 0,
\]

\[
(1 + C_1) f^{(k+1)iv} + Rf^{(k)} f'(k)'^{iv} - Rf^{(k)} f''(k)''^{iv} - Rf^{(k+1)} f''(k)''^{iv} + Rf^{(k)} = f^{(k+1)}
\]

\[
= R(- f^{(k)}, f'(k) + f''(k) f''(k)) + C_1 g^{(k)}
\]

To solve the ODEs given by Eq. (4.9), we replace the derivatives with their central difference approximations, giving rise to the sequence \( \{ f^{(k)} \} \), generated by the following linear system:
\[ Af^{(k+1)} = B \text{ with } A = A(f^{(k)}) \text{ and } B = B(f^{(k)}, g^{(k)}) \] (4.20)

where \( n \) is the number of grid points. The matrices \( A_{n \times n} \) and \( B_{n \times 1} \) are initialized as follows:

\[ A_{1,1} = 1, B_1 = 0, A_{2,1} = (-4(1 + C_i) + hRf_i - M\alpha^2), A_{2,2} = \left(7(1 + C_i) - hRf_2 + h^2R\alpha + \frac{hR}{2}f_4 + 2M\alpha^2\right), \]

\[ A_{2,3} = (-4(1 + C_i) - hRf_3 - M\alpha^2), A_{2,4} = \left(1 + C_i\right) + \frac{hR}{2}f_2, \]

\[ B_2 = \begin{pmatrix}
    h^2C_i(g_3 - 2g_2 + g_1) + \frac{hR}{2}f_2(-f_2 + 2f_i - 2f_3 + f_4) \\
    -\frac{hR}{2}(f_i - f_1)(f_i - 2f_2 + f_3) + 2h\alpha(1 + C_i)
\end{pmatrix}, \]

\[ A_{j,j-2} = \left(1 + C_i\right) - \frac{hR}{2}f_i, A_{j,j-1} = \left(-4(1 + C_i) + hRf_{j-1} - M\alpha^2\right), \]

\[ A_{j,j} = \left(6(1 + C_i) + \frac{hR}{2}(f_{i+2} - f_{i-2}) + 2M\alpha^2\right), A_{j,j+1} = \left(-4(1 + C_i) - hRf_{j+1} - M\alpha^2\right), \]

\[ A_{j,j+2} = \left(1 + C_i\right) + \frac{hR}{2}f_i, B_i = \begin{pmatrix}
    h^2C_i(g_{i+1} - 2g_i + g_{i-1}) + \frac{hR}{2}f_i(-f_{i+2} + 2f_i + 2f_{i+1} + f_{i+2}) \\
    -\frac{hR}{2}(f_{i+1} - f_{i-1})(f_{i+1} - 2f_i + f_{i-1})
\end{pmatrix}, \]

\[ 2 < i < (n-1) \]
\[ A_{n+1,n+3} = \left( 1 + C_1 \right) - \frac{hR}{2} f_{n+1} \], \quad A_{n+1,n+2} = \left( -4(1 + C_1) + hRf_{n+2} - Mh^2 \right), \]

\[ A_{n+1,n+1} = \left( 6(1 + C_1) + \frac{hR}{2}(f_{i+2} - f_{i-2}) + 2Mh^2 \right), \quad A_{n+1,n} = \left( -4(1 + C_1) - hR f_{i+1} - Mh^2 \right), \]

\[
B_{n+1} = \begin{cases}
\frac{h^2 C_1}{2}(g_{i+2} - 2g_i + g_{i-2}) + \frac{hR}{2} f_i (-f_{i-2} + 2f_{i-1} - 2f_{i+1} + f_{i+2}) \\
- \frac{hR}{2}(f_{i+1} - f_{i-1})(f_{i+1} - 2f_i + f_{i-1})
\end{cases},
\]

\[ A_{n,n} = 1, \quad B_n = 0. \] (4.21)

(The superscripts have been dropped for simplicity & clarity)

On the other hand, Eqs. (4.10) and (4.11) are linear in \( g \) & \( \theta \) respectively. Therefore, in order to generate the sequences \( \{g^{(k)}\} \) & \( \{\theta^{(k)}\} \), these equations may be written as

\[ C_3 g^{(k+1)''} + C_1 C_2 (f^{(k+1)''} - 2g^{(k+1)}) = f^{(k+1)}' g^{(k+1)} - f^{(k+1)} g^{(k+1)',} \quad (4.22) \]

\[ \theta^{(k+1)''} + \text{Pr} R f^{(k+1)} \theta^{(k+1)'} = 0. \quad (4.23) \]

Importantly \( f^{(k+1)} \) is considered to be known in the above equations. We outline the computational procedure as follows:

- Provide the initial guess \( f^{(0)}, g^{(0)} \) & \( \theta^{(0)} \), satisfying the boundary conditions given in (4.12)
- Solve the linear system given by (4.20) to find \( f^{(1)} \)
- Use \( f^{(1)} \) to solve the linear system arising from the FD discretization of (4.22) and (4.23), to get \( g^{(1)} \) and \( \theta^{(1)} \).
Take $f^{(1)}, g^{(1)} \& \theta^{(1)}$ as the new initial guesses & repeat the procedure to generate the sequences $\left\{ f^{(k)} \right\}, \left\{ g^{(k)} \right\} \& \left\{ \theta^{(k)} \right\}$ which, respectively, converge to $f, g \& \theta$ (the numerical solutions of (4.9), (4.10) and (4.11)).

The three sequences are generated until

$$\max \left\{ \| f^{(k+1)} - f^{(k)} \|_{L_{\infty}}, \| g^{(k+1)} - g^{(k)} \|_{L_{\infty}}, \| \theta^{(k+1)} - \theta^{(k)} \|_{L_{\infty}} \right\} < 10^{-6}$$

In this case, we also optimize the relaxation parameter $\omega$ by following Nakamura [101].

The pseudo code (which looks more or less like MATLAB code) for finding the numerical solution on the grid of step-size $h$, is as under:

```plaintext
\begin{align*}
f^{(0)}(1) &= f^{(0)}(n) = 0; g^{(0)}(1) = g^{(0)}(n) = 0; \theta^{(0)}(1) = 1, \theta^{(0)}(n) = 0; \text{ %Boundary Conditions} \\
h &= \frac{2}{(n-1)}, k = 0, \quad \text{ % n = No. of grid points} \\
while(k < \text{max_iter}) \\
f^{(k+1)} &= \text{solve_for_f}(n, R, M, C_1, f^{(k)}, g^{(k)}) \\
g^{(k+1)} &= \text{solve_for_g}(n, R, C_1, C_2, C_3, f^{(k+1)}, g^{(k)}) \\
\theta^{(k+1)} &= \text{solve_for_theta}(n, R, Pr, f^{(k+1)}, \theta^{(k)}) \\
if(\max \left\{ \| f^{(k+1)} - f^{(k)} \|_{x}, \| g^{(k+1)} - g^{(k)} \|_{x}, \| \theta^{(k+1)} - \theta^{(k)} \|_{x} \right\} < \text{TOL}) \text{ break; end} \\
k &= k + 1 \\
endwhile
\end{align*}
```

It is important to note that `solve_for_f`, `solve_for_g` & `solve_for_theta` are the functions to calculate $f^{(k+1)}$, $g^{(k+1)}$ and $\theta^{(k+1)}$, respectively, using the methodology described above.

**4.3.2. PSEUDO TRANSIENT METHOD**

Now, in the second approach, we include a pseudo transient term to each of the governing equations (4.2) to (4.6), again make use of the transformation given in (4.8) along with the dimensionless time $\tau = bt$, and arrive at the following system:
\[(1 + C_1) f^{iv} - C_1 g'' - Mf'' = R \left( \frac{\partial f''}{\partial \tau} + f f'' - ff'' \right), \quad (4.28)\]

\[\frac{\partial g}{\partial \tau} + f' g - fg' = C_3 g'' + C_1 C_2 (f'' - 2g), \quad (4.29)\]

\[\frac{\partial \theta}{\partial \tau} = \frac{1}{R \Pr} \theta'' + f \theta', \quad (4.30)\]

subject to the boundary conditions:

\[f(\pm 1, \tau) = 0, \quad f'(\pm 1, \tau) = -1, \quad g(\pm 1, \tau) = 0, \quad \theta(-1, \tau) = 1, \quad \theta(1, \tau) = 0, \quad \forall \tau \quad (4.31)\]

and the initial conditions:

\[f(\eta, 0) = g(\eta, 0) = \theta(\eta, 0) = 0 \quad \forall \quad -1 < \eta < 1. \quad (4.32)\]

where the primes denote the derivatives w. r. t. \(\eta\).

For the numerical solution we proceed by setting \(f'' = q\) in Eqs. (4.28)-(4.30) and get the following system to be solved:

\[f'' = q, \quad (4.33)\]

\[(1 + C_1) q'' - C_1 g'' - Mq = R \left( \frac{\partial q}{\partial \tau} + f'q - f q' \right), \quad (4.34)\]

\[\frac{\partial g}{\partial \tau} + f' g - fg' = C_3 g'' + C_1 C_2 (q - 2g), \quad (4.35)\]

\[\frac{\partial \theta}{\partial \tau} = \frac{1}{R \Pr} \theta'' + f \theta', \quad (4.36)\]

subject to the conditions given in Eqs. (4.31) and (4.32).

Now, for Eq. (33), we note that

\[q_i = f^{i''}_i = \left( \frac{f_{i-1} - 2f_i + f_{i+1}}{h^2} \right) - \frac{h^2}{12} f_i^{iv} + O(h^4) \quad (4.37)\]

But

\[f_i^{iv} = q_i'' = \left( \frac{q_{i-1} - 2q_i + q_{i+1}}{h^2} \right) + O(h^2) \quad (4.38)\]
Combining Eqs. (4.37) and (4.38), we have the following difference equation for (4.33)

\[
f^j_{i-1} - 2f^j_i + f^j_{i+1} = \frac{h^2}{12} \left( q^j_{i-1} + 10q^j_i + q^j_{i+1} \right) + O(h^4),
\]

(4.39)

whereas the simple central differencing would yield

\[
f^j_{i-1} - 2f^j_i + f^j_{i+1} = h^2q^j_i + O(h^2)
\]

(4.40)

Clearly Eqs. (4.39) and (4.40) are both compact difference schemes but we prefer the former one due to its higher order.

Now, for solving Eqs. (4.34)-(4.36), we treat the linear terms implicitly & nonlinear terms explicitly, employ the forward & central difference approximations for the dimensionless time & spatial derivatives respectively and finally reach the following algebraic system:

\[
f^j_{i-1} - 2f^j_i + f^j_{i+1} = \frac{h^2}{12} \left( q^j_{i-1} + 10h^2q^j_i + q^j_{i+1} \right),
\]

(4.41)

\[
\left\{ \frac{1 + 2dt(1 + C_1)}{R h^2} + \frac{M dt}{R} \right\} q^{j+1}_i - \frac{dt(1 + C_1)}{R h^2} \left( q^{j+1}_{i-1} + q^{j+1}_{i+1} \right) = \frac{dtC_i}{R h^2} \left( g^{j+1}_{i-1} - 2g^{j+1}_i + g^{j+1}_{i+1} \right) - \frac{dt}{2h} \left( f^j_{i+1} - f^j_{i-1} \right) g^j_i + \frac{dt}{2h} \left( q^j_{i-1} - q^j_{i+1} \right) f^j_i + q^j_i,
\]

(4.42)

\[
\left\{ 1 + 2C_i C_2 dt + \frac{2C_i dt}{h^2} \right\} \frac{h}{q^{j+1}_i} - \frac{C_i dt}{h^2} \left( g^{j+1}_{i-1} + g^{j+1}_{i+1} \right) = \frac{dt}{2h} \left( f^j_{i+1} - f^j_{i-1} \right) g^j_i + \frac{dt}{2h} \left( g^j_{i+1} - g^j_{i-1} \right) f^j_i + C_i C_2 q^{j+1+1} dt + q^j_i,
\]

(4.43)

\[
\left\{ \frac{2dt}{R Pr h^2} \right\} \Omega^{j+1} - \frac{dt}{R Pr h^2} \left( \Omega^{j+1}_{i-1} + \Omega^{j+1}_{i+1} \right) = \frac{dt}{2h} \left( \Omega^{j+1}_{i+1} - \Omega^{j+1}_{i-1} \right) f^j_i,
\]

(4.44)

subject to the conditions:

\[
\begin{align*}
&f_i^0 = g_i^0 = \Omega_i^0 = 0, \quad \forall 1 < i < n \\
&f_i^j = f_n^j = f_{n+1}^j = f_{n+2}^j = -h, g_i^j = g_n^j = 0, \quad \forall j
\end{align*}
\]

(4.45)
where the subscript \( i \) denotes the point location & the superscript \( j \) stands for the dimensionless time level \( \tau \). We note that \( q_i^j \) and \( q_n^j \) (boundary condition for \( q \) needed to solve Eq. (4.42) are not available and may be found at every time step, using the relation:

\[
q_i^j = \frac{2}{h^2} \left( f_i^j - f_i^{j+1} + h \right) \\
q_n^j = \frac{2}{h^2} \left( f_n^{j-1} - f_n^j - h \right)
\]

Thus, there are four linear algebraic systems \(((4.41)\) to \((4.44)\)\) to be solved at every time level, which being diagonally dominant, allow us to use the SOR method, with the solution at previous time level as an initial guess. It is to note that the approach is similar to the stream-vorticity formulation for solving the Navier-Stokes equations.

The solution algorithm in the form of a flow chart is given in Fig. (4.1).

Moreover, a good initial guess for the solution on the finer grid can also be obtained by injecting the previous coarse grid solution, by using the operator mentioned in chapter 2.

### 4.4. RESULTS AND DISCUSSION

As usual, the physical quantities of our interest are proportional to \( f^*(-1), g^*(-1) \) and \( \theta^*(-1) \). Due to symmetry of the problem, the results are given only at the lower wall.

The parameters of the study are the Reynoald number \( R \), the meagnetic parameter \( M \), the Preandtl number \( Pr \) and the microplar parameters \( C_1, C_2 \) and \( C_3 \).

Table (4.2) not only gives the convergence of our numerical results as the step-size decreases but also shows that the results obtained by employing the two numerical methods (Quasi-linearization and pseudo transient method) are in excellent comparison. The extrapolated values obtained by employing the two methods, match
up to six decimal places. Table (4.3) shows that the magnetic parameter not only increases both the shear and couple stresses but also enhances the heat transfer rate at the channel walls.

It is also clear from the table that the effect of the external magnetic field on the shear stress is more pronounced as compare to its effect on couple stress and the heat transfer rate. It is due to the reason that the magnetic parameter $M$ does not appear in the angular momentum and heat equations, and affects the two quantities only due the coupled equations (4.9)-(4.11). Table (4.4) predicts that the shear and couple stresses as well as the heat transfer rate at the walls may decrease as the shrinking Reynolds number increases. The increased shrinking rate of the channel walls forces the fluid to move rapidly away from the walls, thus decreasing both the shear and couple stresses. Moreover, the heat generated is kept within the flow region, resulting in decreasing the temperature difference and hence the heat transfer rate.

The influence of the micropolar parameters $C_1, C_2$ and $C_3$ on the heat transfer rate, shear and couple stresses are given in Table (4.5). The first case corresponds to the Newtonian fluid whereas the remaining ones are taken arbitrarily to investigate their effect as chosen in the literature (please see [23] and [86-88]). It is also clear from the table that the role of the micro fluid particles in decreasing the heat transfer rate is not as pronounced as compared to their effect on the shear and couple stresses, as the micropolar parameters do not appear in the heat equation (4.11) and therefore do not directly influence the heat transfer characteristics of the problem. An increase in the Prandtl number results in decreasing the heat transfer rate at the walls as shown in Table (4.6).

The stream surface and the streamlines for the problem are shown in Figs. (4.2a) and (4.2b). It is clear from the Figs. (4.3)-(4.5) that the magnetic field tends to lower the
normal velocity as well as the microrotation profiles while eliminating the parabolic nature of the stream wise velocity profiles and thus decreasing their maximum magnitude. Moreover, it tends to change the temperature rapidly near the walls whereas an opposite trend is noted in the middle of the channel as shown in Fig. (4.6).

Influence of the shrinking Reynolds number $R$ on the velocity, microrotation and temperature fields is given in Figs. (4.7)-(4.10). It is obvious that the effect of $R$ on all the profiles is opposite to that of $M$. Therefore, it may be concluded that the external magnetic field tends to balance the effect of shrinking of the channel walls.

Qualitatively, the influence of the micropolar structure on the flow and heat transfer characteristics of the problem is almost the same as that of $R$ (please see Figs. (4.11)-(4.14)), with the exception that $R$ tends to very slightly lower the microrotation profiles near the walls (which turns into a reduction in couple stresses as observed in Table (4.4)), whereas the micropolar parameters raise them significantly. Finally, it is obvious from the Fig. (4.15) that the Prandtl number affects the temperature profile in the same way as the Reynolds number $R$ does.

4.5. CONCLUSIONS

We have numerically studied the problem of viscous incompressible electrically conducting micropolar fluid in a channel with shrinking walls. It has been observed that the external magnetic field not only increases the shear and couple stresses but also enhances the heat transfer rate at the channel walls. Moreover, the magnetic field tends to balance the influence of the shrinking channel walls. The micropolar structure of the fluid increases the couple stress whereas an opposite trend is noted for the shear stress and the heat transfer rate. The Prandtl number may also decrease the heat transfer rate at the walls.
**Table 4.1** Five cases of values of micropolar parameters $C_1, C_2$ and $C_3$

<table>
<thead>
<tr>
<th>Cases</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (Newtonian)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0.5</td>
<td>0.7</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>0.8</td>
<td>1.0</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>1.2</td>
<td>1.5</td>
</tr>
</tbody>
</table>
Table 4.2 \( f(\eta), g(\eta) \) and \( \theta(\eta) \) on three grid sizes and extrapolated values for

\[ C_1 = 3, C_2 = 0.8, C_3 = 1, \; Pr = 0.1, M = 10 \text{ and } R = 20 \]

\[ f(\eta) \]

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>1st grid ( h = 0.02 )</th>
<th>2nd grid ( h = 0.01 )</th>
<th>3rd grid ( h = 0.005 )</th>
<th>Extrapolated values ( h = 0.005 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.8</td>
<td>-0.145624</td>
<td>-0.145635</td>
<td>-0.145635</td>
<td>-0.145637</td>
</tr>
<tr>
<td>-0.6</td>
<td>-0.194488</td>
<td>-0.194505</td>
<td>-0.194507</td>
<td>-0.194510</td>
</tr>
<tr>
<td>-0.4</td>
<td>-0.169717</td>
<td>-0.169736</td>
<td>-0.169738</td>
<td>-0.169739</td>
</tr>
<tr>
<td>-0.2</td>
<td>-0.096668</td>
<td>-0.096678</td>
<td>-0.096680</td>
<td>-0.096681</td>
</tr>
</tbody>
</table>

\[ g(\eta) \]

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>1st grid ( h = 0.02 )</th>
<th>2nd grid ( h = 0.01 )</th>
<th>3rd grid ( h = 0.005 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.8</td>
<td>0.252740</td>
<td>0.252781</td>
<td>0.252785</td>
</tr>
<tr>
<td>-0.6</td>
<td>0.318390</td>
<td>0.318410</td>
<td>0.318450</td>
</tr>
<tr>
<td>-0.4</td>
<td>0.265999</td>
<td>0.266016</td>
<td>0.266052</td>
</tr>
<tr>
<td>-0.2</td>
<td>0.147448</td>
<td>0.147458</td>
<td>0.147476</td>
</tr>
</tbody>
</table>

\[ \theta(\eta) \]

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>1st grid ( h = 0.02 )</th>
<th>2nd grid ( h = 0.01 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.8</td>
<td>0.911942</td>
<td>0.911940</td>
</tr>
<tr>
<td>-0.6</td>
<td>0.818886</td>
<td>0.818883</td>
</tr>
<tr>
<td>-0.4</td>
<td>0.718544</td>
<td>0.718542</td>
</tr>
<tr>
<td>-0.2</td>
<td>0.611336</td>
<td>0.611335</td>
</tr>
</tbody>
</table>
Table 4.3  The effect of magnetic field on shear stress, couple stresses and heat transfer rate with \( C_1 = 3, C_2 = 0.8, C_3 = 1, Pr = 0.1 \) and \( R = 20 \)

<table>
<thead>
<tr>
<th>M</th>
<th>( f''(-1) )</th>
<th>( g'(-1) )</th>
<th>( \theta'(-1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.336806</td>
<td>1.829962</td>
<td>-0.078361</td>
</tr>
<tr>
<td>50</td>
<td>4.276544</td>
<td>1.963079</td>
<td>-0.136539</td>
</tr>
<tr>
<td>100</td>
<td>5.589657</td>
<td>2.029472</td>
<td>-0.175580</td>
</tr>
<tr>
<td>150</td>
<td>6.613072</td>
<td>2.070440</td>
<td>-0.203320</td>
</tr>
<tr>
<td>200</td>
<td>7.468431</td>
<td>2.098830</td>
<td>-0.224287</td>
</tr>
</tbody>
</table>

Table 4.4  The effect of Reynolds number on sheer and couple stresses and heat transfer rate with \( C_1 = 3, C_2 = 0.8, C_3 = 1, Pr = 0.1 \) and \( M = 50 \)

<table>
<thead>
<tr>
<th>R</th>
<th>( f''(-1) )</th>
<th>( g'(-1) )</th>
<th>( \theta'(-1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.636654</td>
<td>1.979758</td>
<td>-0.50000000</td>
</tr>
<tr>
<td>20</td>
<td>4.276544</td>
<td>1.963079</td>
<td>-0.447433</td>
</tr>
<tr>
<td>50</td>
<td>3.726069</td>
<td>1.935625</td>
<td>-0.367804</td>
</tr>
<tr>
<td>100</td>
<td>2.824160</td>
<td>1.884691</td>
<td>-0.240081</td>
</tr>
<tr>
<td>200</td>
<td>1.456117</td>
<td>1.787907</td>
<td>-0.065795</td>
</tr>
</tbody>
</table>
**Table 4.5** The effect of micropolar parameters on shear stress, couple stress and heat transfer rate with $R = 20$, $M = 200$ and $Pr = 1$

<table>
<thead>
<tr>
<th>Cases</th>
<th>$f'(-1)$</th>
<th>$g'(-1)$</th>
<th>$\theta'(-1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (Newtonian)</td>
<td>13.094595</td>
<td>0</td>
<td>-0.319873</td>
</tr>
<tr>
<td>2</td>
<td>9.902448</td>
<td>0.642689</td>
<td>-0.273448</td>
</tr>
<tr>
<td>3</td>
<td>8.402035</td>
<td>1.318745</td>
<td>-0.244798</td>
</tr>
<tr>
<td>4</td>
<td>7.468431</td>
<td>2.098830</td>
<td>-0.224287</td>
</tr>
<tr>
<td>5</td>
<td>6.816418</td>
<td>2.673407</td>
<td>-0.208653</td>
</tr>
</tbody>
</table>

Table 4.6 The effect of Prandtl number on heat transfer rates with $C_1 = 3, C_2 = 0.8, C_3 = 1, R = 20$ and $M = 200$

<table>
<thead>
<tr>
<th>$Pr$</th>
<th>$\theta'(-1)$</th>
<th>$\theta'(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.464810</td>
<td>-0.464810</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.341412</td>
<td>-0.341412</td>
</tr>
<tr>
<td>1</td>
<td>-0.224287</td>
<td>-0.224287</td>
</tr>
<tr>
<td>5</td>
<td>-0.003572</td>
<td>-0.003572</td>
</tr>
<tr>
<td>10</td>
<td>-0.000010</td>
<td>-0.000010</td>
</tr>
</tbody>
</table>
Figure 4.1 Flow chart for the Pseudo transient method
Figure 4.2a Stream surface for the problem

Figure 4.2b Streamlines for the problem
Figure 4.3 Normal velocity profiles for $C_1 = 3, C_2 = 0.8, C_3 = 1, Pr = 1, R = 20$ and various $M$

Figure 4.4 Streamwise velocity profiles for $C_1 = 3, C_2 = 0.8, C_3 = 1, Pr = 1, R = 20$ and various $M$
Figure 4.5 Microrotation profiles for $C_1 = 3, C_2 = 0.8, C_3 = 1$, $Pr = 1$, $R = 20$ and various $M$

Figure 4.6 Temperature profiles for $C_1 = 3, C_2 = 0.8, C_3 = 1$, $Pr = 1$, $R = 20$ and various $M$
Figure 4.7 Normal velocity profiles when $C_1 = 3, C_2 = 0.8, C_3 = 1$, $Pr = 1$, $M = 50$ and various $R$

Figure 4.8 Streamwise velocity profiles when $C_1 = 3, C_2 = 0.8, C_3 = 1$, $Pr = 1$, $M = 50$ and various $R$
Figure 4.9 Microrotation profiles when $C_1 = 3, C_2 = 0.8, C_3 = 1, Pr = 1, M = 50$ and various $R$

Figure 4.10 Temperature Profiles when $C_1 = 3, C_2 = 0.8, C_3 = 1, Pr = 1, M = 50$ and various $R$
Figure 4.11 Normal velocity profiles for five cases of $C_1, C_2$ and $C_3$ when $R = 20$, $M = 200$ and $Pr = 10$

Figure 4.12 Streamwise velocity profiles for five cases of $C_1, C_2$ and $C_3$ when $R = 20$, $M = 200$ and $Pr = 10$
Figure 4.13 Microrotation profiles for five cases of $C_1, C_2$ and $C_3$ when $R = 20,$

$M = 200$ and $Pr = 10$

Figure 4.14 Temperature profiles for five cases of $C_1, C_2$ and $C_3$ when $R = 20,$

$M = 200$ and $Pr = 10$
Figure 4.15 Temperature profiles for $C_1 = 3, C_2 = 0.8, C_3 = 1$, $R = 20$, $M = 200$ and various Pr
CHAPTER 5

NUMERICAL SIMULATION OF NANOFLUID FLOW

DUE TO TWO ORTHOGONALLY MOVING DISKS
5.1. INTRODUCTION
We consider the flow of a nanofluid (homogeneous Titanium dioxide and water mixture) between two orthogonally moving porous coaxial disks with suction. A similarity transformation converts the governing partial differential equations into a system of nonlinear coupled ODEs in the dimensionless form, which are then solved numerically. We consider the effect of the governing parameters namely, the wall expansion ratio, the permeability Reynoalds number, the Prandtle number, the Schmidt number, and the Eckeart number on different aspects of the problem.

5.2. MATHEMATICAL FORMULATION
We consider the nanofluid between two orthogonally moving disks (of same permeability) with suction.

The flow is supposed to be:

- laminar
- unsteady
- viscous
- incompressible
- electrically conducting

We further assume that the distance between the disks is \( 2a(t) \), whereas the disks are capable of moving uniformly at the rate \( a'(t) \).

The cylindrical coordinate system is noted to be the most appropriate one for the present problem.
With \( u \) and \( w \) being the velocity components in the \( r \) and \( z \) directions respectively, the governing equations for the problem (taking the viscous dissipation effect into account) are:

**Continuity Equation**

\[
\frac{\partial u}{\partial r} + u \frac{\partial u}{\partial r} + \frac{\partial u}{\partial z} = 0,
\]

\( \text{(5.1)} \)

**\( r \)-Component of Momentum Equation**

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho_{nf}} \frac{\partial p}{\partial r} + \nu_{nf} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} \right),
\]

\( \text{(5.2)} \)

**\( z \)-Component of Momentum Equation**

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho_{nf}} \frac{\partial p}{\partial z} + \nu_{nf} \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right),
\]

\( \text{(5.3)} \)

**Heat Equation**

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} = \alpha_{nf} \frac{\partial^2 T}{\partial z^2} + \frac{\mu_{nf}}{(\rho c_p)_{nf}} \left( \frac{\partial u}{\partial z} \right)^2,
\]

\( \text{(5.4)} \)

**Concentration Equation**

\[
\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial r} + w \frac{\partial C}{\partial z} = D \nabla^2 C,
\]

\( \text{(5.5)} \)

whereas:

\( u \) (velocity components in the \( r \) directions)

\( w \) (velocity components in the \( z \) directions)

\( C \) (concentration)

\( p \) (pressure)

\( T \) (temperature)

\( \alpha_{nf} \) (thermal diffusivity)

\( \rho_{nf} \) (density)

\( \nu_{nf} \) (kinematics viscosity)
In literature, it is found that:

\[ \nu_{nf} = \frac{\mu_{nf}}{\rho_{nf}}, \mu_{nf} = \frac{\mu_f}{(1-\varphi)^{2.5}}, \rho_{nf} = (1-\varphi)\rho_f + \varphi\rho_s, \alpha_{nf} = \frac{k_{nf}}{(\rho c_p)_{nf}} \]

\[ (\rho c_p)_{nf} = (1-\varphi)(\rho c_p)_f + \varphi(\rho c_p)_s \]

\[ \frac{k_{nf}}{k_f} = \frac{(k_s + 2k_f) - 2\varphi(k_f - k_s)}{(k_s + 2k_f) + \varphi(k_f - k_s)} \] (5.6)

Where,

- \( \rho_s \) (density of the solid)
- \( \rho_f \) (density of the base-fluid)
- \( (\rho c_p)_{nf} \) (heat capacitance of the nanofluid)
- \( k_{nf} \) (effective thermal conductivity of the nanofluid)

Boundary Conditions

\[ \begin{align*}
  z = -a(t); \quad u = 0, w = -Aa'(t), T = T_1, C = C_1, \\
  z = a(t); \quad u = 0, w = Aa'(t), T = T_2, C = C_2,
\end{align*} \] (5.7)

where \( A \) quantifies the wall permibility and the prime represents the time-derivative.

The pressure term is eliminated from the Eqs. (5.1)-(5.5), and the introduction of the following transformation

\[ \eta = \frac{z}{a}, \quad u = \frac{r}{a^2} F_\eta(\eta, t), \quad w = \frac{2\nu_f}{a} F(\eta, t), \quad \theta = \frac{T - T_2}{T_1 - T_2}, \quad \chi(\eta) = \frac{C - C_2}{C_1 - C_2}, \]

(5.8)

leads to:

\[ \frac{\nu_{nf}}{\nu_f} F_{\eta\eta\eta\eta} + \alpha (3F_{\eta\eta} + \eta F_{\eta\eta\eta}) - 2FF_{\eta\eta\eta\eta} - \frac{a^2}{\nu_f} F_{\eta\eta\eta\eta\eta} = 0, \]

(5.9)

\[ \theta_{\eta\eta} - \frac{\nu_f}{\alpha_{nf}} (2F - \eta\alpha) \theta_\eta + (1-\varphi)^{2.5} F_{\eta\eta}^2 Ec Pr \left( \frac{k_f}{k_{nf}} \right) - \frac{a^2}{\alpha_{nf}} \theta = 0, \] (5.10)
\[ D\chi'' = a^2 \chi_i + \nu_f \chi' (2F - \eta \alpha) \]  

(5.11)

Above transformation converts the boundary conditions to:

\[ \eta = -1; \ F = -R, \ F_\alpha = 0, \ \theta = 1, \ \chi = 1, \ \text{and} \ \eta = 1; \ F = R, \ F_\alpha = 0, \ \theta = 0, \ \chi = 0. \]  

(5.12)

It is to note that:

\[ T_1, \ T_2 (\text{fixed temperatures at the lower and upper disks}) \]

\[ C_1, \ C_2 (\text{fixed concentrations at the two disks}) \]

\[ \alpha = \frac{a a'(t)}{\nu_f} (\text{wall expansion ratio}) \]

\[ R = \frac{Aa a'}{2\nu_f} (\text{permeability Reynold's number}) \]

\[ \Pr = \left( \frac{\mu c_p}{k_f} \right)_f (\text{Prandtl number}) \]

\[ Sc = \frac{\nu_f}{D} (\text{Schmidt number}) \]

\[ Ec = \frac{U^2}{(T_1 - T_2) \left( c_p \right)_f} (\text{Eckart number}) \]

\[ U = \frac{ru_f}{a^2} (\text{reference velocity}) \]

The identical satisfaction of the continuity equation ensures that the proposed velocity field represents the possible motion of the fluid.

Let us define \( f = \frac{F}{R} \), and assume the case when \( \alpha \) is a constant, \( f = f(\eta) \) and \( \theta = \partial(\eta) \), which leads to \( \theta_i = 0, \ f_{\eta \eta} = 0 \) and \( \chi_i = 0 \). Thus we have the following equations

\[ \frac{\nu_{sf}}{\nu_f} f_{\eta \eta \eta \eta} + \alpha (3 f_{\eta \eta} + \eta f_{\eta \eta \eta}) - 2 R f f_{\eta \eta \eta} = 0, \]

(5.13)

\[ \theta_{\eta \eta} - \frac{\nu_f}{\alpha_{sf}} (2Rf - \eta \alpha) \theta_{\eta} + R^2 \left( 1 - \phi \right)^{-2.5} f_{\eta \eta}^2 E_c \Pr \left( \frac{k_f}{k_{sf}} \right) = 0, \]

(5.14)
\( \chi'' = \text{Sc}(2Rf - \eta \alpha) \chi' \)

(5.15)

\( \eta = -1; f = -1, f_0 = 0, \theta = 1, \chi = 1, \) and \( \eta = 1; f = 1, f_0 = 0, \theta = 0, \chi = 0. \) (5.16)

5.3. NUMERICAL SOLUTION

For the numerical solution of Eqs. (5.13)-(5.15), subject to the boundary conditions given in Eq. (5.16), we have used quasi-linearization to construct \( \{ \theta^{(k)} \} \) and \( \{ \chi^{(k)} \} \), as follows:

We set:

\[
G(f, f_\eta, f_{\eta\eta}, f_{\eta\eta\eta}, f_{\eta\eta\eta\eta}) = \frac{\nu_f}{\nu_f} f_{\eta\eta\eta\eta} + \alpha(3f_{\eta\eta} + \eta f_{\eta\eta\eta}) - 2Rf_{\eta\eta\eta},
\]

\[
G(f^{(k)}, f_\eta^{(k)}, f_{\eta\eta}^{(k)}, f_{\eta\eta\eta}^{(k)}, f_{\eta\eta\eta\eta}^{(k)}) + \left( f^{(k+1)} - f^{(k)} \right) \frac{\partial G}{\partial f^{(k)}} + \left( f^{(k+1)} - f^{(k)} \right) \frac{\partial G}{\partial f^{(k)}}
\]

\[
+ \left( f_{\eta\eta}^{(k+1)} - f_{\eta\eta}^{(k)} \right) \frac{\partial G}{\partial f_{\eta\eta}^{(k)}} + \left( f_{\eta\eta\eta}^{(k+1)} - f_{\eta\eta\eta}^{(k)} \right) \frac{\partial G}{\partial f_{\eta\eta\eta}^{(k)}} + \left( f_{\eta\eta\eta\eta}^{(k+1)} - f_{\eta\eta\eta\eta}^{(k)} \right) \frac{\partial G}{\partial f_{\eta\eta\eta\eta}^{(k)}} = 0,
\]

\[
\frac{\nu_f}{\nu_f} f_{\eta\eta\eta\eta}^{(k+1)} + \left( \eta \alpha - 2Rf^{(k)} \right) f_{\eta\eta}^{(k+1)} + 3\alpha f_{\eta\eta}^{(k+1)} - 2Rf_{\eta\eta}^{(k)} f^{(k+1)} = -2Rf_{\eta\eta}^{(k)} f^{(k)}
\]

(5.17)

Eqs. (5.14)-(5.15) may be written as

\[
\theta_{\eta\eta}^{(k+1)} - \frac{\nu_f}{\alpha_f} \left( 2Rf^{(k+1)} - \eta \alpha \right) \theta_{\eta\eta}^{(k+1)} + R^2 \left( (1 - \phi)^{-2.5} f_{\eta\eta}^{(k+1)} \right)^2 Ec Pr \left( \frac{k_f}{k_{af}} \right) = 0
\]

(5.18)

\[
\chi_{\eta\eta}^{(k+1)} - \text{Sc} \left( 2Rf^{(k+1)} - \eta \alpha \right) \chi_{\eta\eta}^{(k+1)} = 0,
\]

(5.19)

Eqs. (5.17)-(5.19) are then solved by using the procedures given in chapter 4.

5.4. RESULTS AND DISCUSSION
The physical quantities of our interest are the shear stress, and the heat and mass transfer rates at the disks. Due to symmetry of the problem, the results are given only at the lower disk. The parameters for the present study are the Reynolds number $R$, the nanoparticle friction parameter $\phi$, the wall expansion ratio $\alpha$, the Eckert number $Ec$ and the Schmidt number $Sc$. We note that if $\alpha < 0$ if the disks are approaching, and $\alpha > 0$ when the disks are moving away from each other. Further, $Re < 0$ for suction. We will study the effect of the governing parameters on the following quantities:

- shear stress $f''(-1)$
- heat transfer rate $\theta'(-1)$
- mass transfer rate at the disks $\chi'(-1)$
- velocity $(f(\eta), f'(\eta))$, concentration $\chi(\eta)$, and temperature fields $\theta(\eta)$.

For the water-based nanofluid with $TiO_2$ solid particles, we take the following values: $\rho_f = 997.1, \rho_s = 4250, k_s = 8.9538, k_f = 0.613, (c_p)_s = 686.2, (c_p)_f = 4179$ and $Pr = 6.2$. It is noted that the case $\phi = 0$ corresponds to the pure water.

In Table (5.1), we have given the numerical values of $f(\eta)$ (along with the extrapolated values) at different grid locations, corresponding to three different grid sizes. This clearly indicates the convergence of our numerical results.

It is obvious from the Table (5.2) that the nanoparticles significantly affect the heat transfer rate only. The Reynolds number, on the other hand, has the effect of remarkable reduction on the shear stress and the heat transfer rate, only when the disks are approaching. The trend is, however, reversed for the receding disks (Table (5.3)). Table (5.4) predicts that the shear stress, and the heat and mass transfer rates at
the disks may increase with the magnitude of the wall expansion ratio $\alpha$ only when the disks are approaching. Table (5.5) shows that the viscous dissipation always increases the heat transfer rate at the disks for both the cases of $\alpha$. Moreover, the Eckert number (which characterises the viscous dissipation) is more influential in case of approaching disks. Table (5.6) predicts a remarkable decrease in mass transfer rate with increasing Schmidt number, irrespective of the movement of the disks.

The stream surface and the streamlines for the present problem are shown in Figs. (5.2a) and (5.2b) respectively. It has been observed that only the temperature profiles are significantly affected by the nanoparticle volume fraction parameter $\phi$, for both the cases of $\alpha$. That is why, in Figures (5.3) and (5.4), we have given the influence of the parameter $\phi$ on the temperature distribution only. Moreover, the temperature distribution near the disks (and hence the heat transfer rate) is not affected remarkably by the parameter $\phi$ for $\alpha > 0$, whereas the thermal distribution is raised across the entire domain when the disks are approaching ($\alpha < 0$).

For the fixed $\phi = 0.1, Sc = 0.1$ and $Ec = 0.1$, the influence of the Reynolds number $R$ on the velocity, temperature and concentration distributions is given in the (Figures (5.5)-(5.8)) for $\alpha = 5$, whereas the effect on the profiles for $\alpha = -5$ is given in the Figures (5.9)-(5.12). For the receding disks, both the axial velocity and the maximum value of the radial velocity decrease with the Reynolds number. On the other hand, the temperature distribution increases across the entire domain whereas the concentration profiles are raised only in the lower half of the plane $z = 0$. For $\alpha < 0$, it has been noted that the velocity profiles are affected by $R$ in an opposite manner as in case of receding disks, but the influence is the same for the concentration profiles. The Reynolds number $R$ tends to eliminate the flattened nature of the temperature
profile by lowering it near the disks and significantly raising in the middle of the
domain. Moreover, the behavior of the concentration profiles remains the same
whether the disks are approaching or receding.

The influence of the wall expansion ratio \( \alpha \) on the velocity, temperature and
concentration profiles is given in the Figs. (5.13)-(5.16) for the fixed
\( \phi = 0.1, Ec = 0.1, Sc = 0.5 \) and \( R = -1 \). As \( \alpha \) varies from negative to positive, the
magnitude of the axial velocity increases on the entire domain, whereas the radial
velocity increases only in the middle of the region between the disks. On the other
hand, the temperature profiles are remarkably lowered near the disks, and raised for a
small region in the range \(-0.4 < \eta < -0.1\). The concentration profiles are raised only in
the lower half of the plane \( z = 0 \).

For the fixed \( R = -1, \phi = 0.1 \) and \( Ec = 0.1 \), Figs. (5.17)-(5.18) give the influence of the
Schmidt number on the concentration profiles for \( \alpha = \pm 5 \). It is noted that, for the
approaching disks, the concentration profiles are lowered only in the lower half of the
plane \( z = 0 \), but the trend is reversed for the receding disks.

Finally, the effect of the Eckert number on the thermal characteristics of the problem
is presented in the Figs. (5.19)-(5.20), for both the cases of \( \alpha \). For the approaching
disks, the temperature profiles are raised across the whole domain whereas, in the
other case, the profiles are not much affected near the disks. As a result the heat
transfer rate at the disks is relatively low for different \( Ec \) in case of \( \alpha > 0 \).

5.5. CONCLUSIONS

We have numerically studied how the governing parameters, namely the Reaynolds
number \( R \), the nanoparticle friction parameter \( \phi \), the wall expeansion ratio \( \alpha \), the
Eckeart number \( Ec \) and the Schmidt number \( Sc \) influence the flow, heat and heat
transfer characteristics of the unsteady, laminar, incompressible flow of a nanofluid
(containing $TiO_2$ solid particles) between two orthogonally moving porous coaxial disks with suction.

When the disks are moving away ($\alpha > 0$):

- shear stress at the disks increases with $R$ whereas an opposite effect is observed for $\alpha$
- heat transfer rate increases with $R$, $Ec$ and $\phi$ but an opposite trend is noted for $\alpha$
- $\alpha, R$ and $Sc$ decrease the mass transfer rate

For the approaching disks ($\alpha < 0$):

- shear stress at the disks increases with $\alpha$ whereas an opposite effect is observed for $R$
- heat transfer rate decreases with $R$ but an opposite trend is noted for $\alpha, \phi$ and $Ec$
- mass transfer decreases with both $R$ and $Sc$, but increases with $\alpha$

Moreover, it has also been noted that the shear stress as well as the mass transfer is not significantly influenced by $\phi$ whether the disks are approaching or moving away.

Table 5.1 Dimensionless velocity $f(\eta)$ on three grid sizes and extrapolated values for
$R = -5, \phi = 0.1, \alpha = -1, Ec = 0.1$ and $Sc = 1$.

\begin{align*}
    f(\eta) \\
    \eta & \quad \text{1st grid} & \quad \text{2nd grid} & \quad \text{3rd grid} & \quad \text{Extrapolated} \\
    & (h = 0.02) & (h = 0.01) & (h = 0.005) & \text{values} \\
    0.20 & 0.2858747 & 0.2858997 & 0.2859059 & 0.2859080 \\
    0.40 & 0.5524110 & 0.5524554 & 0.5524664 & 0.5524701 \\
    0.60 & 0.7782718 & 0.7783239 & 0.7783369 & 0.7783412 \\
    0.80 & 0.9380806 & 0.9381214 & 0.9381315 & 0.9381349
\end{align*}

Table 5.2 Effect of the nanoparticle volume fraction parameter $\phi$ on $f'(1)$, $\theta'(1)$ and $\chi'(1)$ for $R = -5$, $Ec = 0.1$ and $Sc = 0.1$.

\begin{align*}
    \phi & \quad \alpha = 5 & \quad \alpha = -5 \\
    f'(1) & \quad \theta'(1) & \quad \chi'(1) & \quad f'(1) & \quad \theta'(1) & \quad \chi'(1) \\
    0 & 1.3191 & 0.2949 & -0.2650 & 3.6440 & 5.8936 & -0.3960 \\
    0.05 & 1.3164 & 0.3402 & -0.2649 & 3.6514 & 6.7055 & -0.3960 \\
    0.10 & 1.3169 & 0.3975 & -0.2649 & 3.6501 & 7.6396 & -0.3960 \\
    0.15 & 1.3200 & 0.4704 & -0.2650 & 3.6414 & 8.7284 & -0.3960 \\
    0.20 & 1.3261 & 0.5646 & -0.2651 & 3.6259 & 10.0136 & -0.3959
\end{align*}

Table 5.3 Effect of the permeability Reynolds number $R$ on $f'(1)$, $\theta'(1)$ and $\chi'(1)$ for $\phi = 0.1$, $Ec = 0.1$ and $Sc = 0.1$.
Table 5.4 Effect of the wall expansion ratio $\alpha$ on $f''(-1)$ and $\theta'(-1)$ for

$\phi = 0.1, R = -5, Ec = 0.1$ and $Sc = 0.1$

<table>
<thead>
<tr>
<th>$R$</th>
<th>$f''(-1)$</th>
<th>$\theta'(-1)$</th>
<th>$\chi'(-1)$</th>
<th>$f''(-1)$</th>
<th>$\theta'(-1)$</th>
<th>$\chi'(-1)$</th>
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<td>2.7695</td>
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Table 5.5 Effect of the Eckert number $Ec$ on $\theta'(-1)$ for $R = -5, \phi = 0.1$ and $Sc = 0.1$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$f''(-1)$</th>
<th>$\theta'(-1)$</th>
<th>$\chi'(-1)$</th>
</tr>
</thead>
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<tr>
<td>-5</td>
<td>3.6501</td>
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<td>2.9411</td>
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<td>-0.2875</td>
</tr>
<tr>
<td>5</td>
<td>1.3169</td>
<td>0.3975</td>
<td>-0.2649</td>
</tr>
</tbody>
</table>
\[ \theta'(-1) \]

<table>
<thead>
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<th>( Ec )</th>
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<th>( \alpha = -5 )</th>
</tr>
</thead>
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<tr>
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</table>

**Table 5.6** Effect of the Schmidt number \( Sc \) on \( \chi'(-1) \) for \( R = -5 \) and \( \phi = 0.1 \)

\[ \chi'(-1) \]

<table>
<thead>
<tr>
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<th>( \alpha = -5 )</th>
</tr>
</thead>
<tbody>
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<td>-0.5000</td>
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<tr>
<td>1.0</td>
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<td>-0.0317</td>
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</table>
Figure 5.1 Physical model of the problem

Figure 5.2a: Stream surface for the problem
Figure 5.2b Streamlines for the problem

Figure 5.3 Temperature profiles for $R = -1, Ec = 0.1, Sc = 0.1, \alpha = 5$ and various $\phi$
Figure 5.4 Temperature profiles for $R = -1, Ec = 0.1, Sc = 0.1, \alpha = -5$ and various $\phi$

Figure 5.5 Axial velocity profiles for $\phi = 0.1, Ec = 0.1, Sc = 0.1, \alpha = 5$ and various $R$
Figure 5.6 Radial velocity profiles for $\phi = 0.1, Ec = 0.1, Sc = 0.1, \alpha = 5$ and various $R$

Figure 5.7 Temperature profiles for $\phi = 0.1, Ec = 0.1, Sc = 0.1, \alpha = 5$ and various $R$
Figure 5.8 Concentration profiles for $\phi = 0.1, Ec = 0.1, Sc = 0.1, \alpha = 5$ and various $R$

Figure 5.9 Axial velocity profiles for $\phi = 0.1, Ec = 0.1, Sc = 0.1, \alpha = -5$ and various $R$
Figure 5.10 Radial velocity profiles for $\phi = 0.1, Ec = 0.1, Sc = 0.1, \alpha = -5$ and various $R$

Figure 5.11 Temperature profiles for $\phi = 0.1, Ec = 0.1, Sc = 0.1, \alpha = -5$ and various $R$
Figure 5.12 Concentration profiles for $\phi = 0.1, Ec = 0.1, Sc = 0.1, \alpha = -5$ and various $R$

Figure 5.13 Axial velocity profiles for $\phi = 0.1, Ec = 0.1, Sc = 0.5, R = -1$ and various $\alpha$
Figure 5.14 Radial velocity profiles for $\phi = 0.1, Ec = 0.1, Sc = 0.5, R = -1$ and various $\alpha$

Figure 5.15 Temperature profiles for $\phi = 0.1, Ec = 0.1, Sc = 0.5, R = -1$ and various $\alpha$
Figure 5.16 Concentration profiles for $\phi = 0.1, Ec = 0.1, Sc = 0.5, R = -1$ and various $\alpha$

Figure 5.17 Concentration profiles for $\phi = 0.1, Ec = 0.1, \alpha = -5, R = -1$ and various $Sc$
Figure 5.18 Concentration profiles for $\phi = 0.1, Ec = 0.1, \alpha = 5, R = -1$ and various $Sc$

Figure 5.19 Temperature profiles for $\phi = 0.1, Sc = 0.1, \alpha = 5, R = -1$ and various $Ec$
Figure 5.20 Temperature profiles for $\phi = 0.1$, $Sc = 0.1$, $\alpha = -5$, $R = -1$ and various $Ec$.
CHAPTER 6

UNSTEADY PULSATILE MAGNETORHEOLOGICAL BIOFLUID THROUGH A POROUS CHANNEL
6.1. INTRODUCTION

The purpose of this chapter is to numerically study the interaction of an external magnetic field with the flow of a biofluid through a Darcy-Forchheimer porous channel, due to an oscillatory pressure gradient, in the presence of wall transpiration as well as chemical reaction considerations. We have noticed that if the Reynolds number of the wall transpiration flow is increased, the velocity of the main flow direction is raised. Similar effect has also been observed for the rheological parameter and the Darcy parameter, whereas an opposite trend has been noted for both the Forchheimer quadratic drag parameter and the magnetic parameter. Further, an increase in the Reynolds number results in straightening the concentration profile, thus making it an almost linear function of the dimensionless spatial variable.

6.2. MATHEMATICAL FORMULATION

We consider the two dimensional, incompressible, laminar, electrically-conducting, rheological-biofluid through a porous medium inside a parallel plate channel with wall transpiration, under the action of a transverse magnetic field. The flow is considered to be pulsatile and is driven by a pressure gradient with steady and oscillatory components, whereas the wall transpiration is modeled by considering the injection at lower plate and the suction at the upper one. The geometry of the problem suggests that the Cartesian coordinate system is the most suitable, with the channel plates being located at \(y = \pm a\). By employing the Nakamura and Sawada model, the governing momentum equation of the problem is:

\[
\frac{\partial u}{\partial \tau} + v_0 \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + v_y \left(1 + \frac{1}{\beta} \right) \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2}{\rho} \frac{u}{k_p} - B u, \tag{6.1}
\]

where \(\tau\) is the dimensional time, \(u\) is the horizontal velocity component, \(v_0\) is the wall transpiration velocity, \(\rho\) is the density, \(p\) is the pressure, \(u_B\) is the kinematic
viscosity, $\sigma$ is the electrical conductivity, $B_0$ is the magnetic field strength, $k_p$ is the permeability of the porous medium, and $b$ is the inertial drag coefficient. Further, $\beta$ is the rheological parameter which gives the measure of rheological properties of the non Newtonian fluid. Rheological properties are a combination of viscous, elastic, and plastic properties, which change most often non-linearly. It is worthy to mention that the above equation is a combination of the Navier-Stokes equation and the Forchheimer law, last two terms in the equation are due to this law.

It is worthy to note that the biofluid is taken as blood, which has been treated as the Casson fluid in the present problem. It is worthy to mention that Eq. (6.1) differs from typical Navier-Stokes equation due to the factor $\left(1+\frac{1}{\beta}\right)$, which is present because of the Casson model for the blood (Boyd et al. [104]). Obviously, as $\beta$ approaches zero, Eq. (6.1) reduces to the governing equation for the ordinary Newtonian fluid.

The governing equation for heat transfer is:

$$\frac{\partial T}{\partial \tau} + \nu_0 \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2},$$  \hspace{1cm} (6.2)

where $\alpha$ is the thermal diffusivity, and $T$ is the temperature of the biofluid. In Eq. (6.2), the first and second derivative terms stand for thermal convection and diffusion respectively. Since the two channel walls are located at different temperature, therefore the diffusion has been taken into account. Initially, no wall transpiration effects (suction/ injection) are assumed at the channel walls. Therefore, linear variation of temperature across the channel is considered. After this, it is the imposition of suction/injection which is responsible for the change in temperature profile.
We further consider the mass transfer of a chemically reactive species in the channel. Blood is assumed to contain some chemically reactive species, and the mixture thus formed is homogeneous. Further, $C$ is the concentration of this species in the mixture, and has the units $mol/m^3$ (Please see Sharma and Borgohain [105]). Moreover, 1st order homogeneous and irreversible reaction is assumed to be taking place in the fluid because of chemical reactive nature of the species. The species is assumed to have constant concentrations $C_1$ and $C_2$ (respectively) at the lower and upper channel walls. The governing equation for the mass transfer (due to the difference in concentrations at the two walls) is:

$$\frac{\partial C}{\partial \tau} + v_0 \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2} - k(C - C_m),$$

(6.3)

where $C$ is the species concentration, $D$ is the mass diffusivity (analogous to thermal diffusivity). Moreover, $k$ is the reaction rate constant, and is the measure how fast the chemical reaction is taking place in the biofluid inside the channel. Further, the negative term on the right hand side of the equation shows that the chemical reaction taking place in the fluid, is destroying the species. Finally, $C_m = \frac{(C_1 + C_2)}{2}$ is the characteristic concentration, and has been used as reference concentration in the mathematical modeling of the mass transfer phenomenon.

The boundary conditions for the present problem (for $t > 0$) are:

$$\begin{align*}
u = 0, T = T_1, C = C_1 & \text{ at } y = -a, \\
u = 0, T = T_2, C = C_2 & \text{ at } y = a
\end{align*}$$

(6.4)

where $T_1, T_2$ and $C_1, C_2$ are the fixed temperatures and concentrations at the lower and upper walls of the channel, respectively.

We define the following dimensionless parameters,
\[ U = \frac{u}{v_0}, \xi = \frac{x}{a}, \eta = \frac{y}{a}, t = \frac{v_0}{T_2}, P = \frac{p}{\rho v_0^2}, \theta = \frac{T - T_m}{T_2 - T_m}, \phi = \frac{C - C_m}{C_2 - C_m}, \]  

which reduce the governing equations to:

\[ \frac{\partial U}{\partial t} + \frac{\partial U}{\partial \eta} = -\frac{\partial P}{\partial \xi} + \frac{1}{R} \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 U}{\partial \eta^2} - N_m U - \frac{1}{\lambda} U - N_f U^2 \]  

\[ \frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial \eta} = \frac{1}{R \Pr} \frac{\partial^2 \theta}{\partial \eta^2} \]  

\[ \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial \eta} = \frac{1}{Sc} \frac{\partial^2 \phi}{\partial \eta^2} - \frac{\gamma}{R} \phi \]  

where \( R \left( = \frac{av_0}{v_b} \right) \) is the Reynolds number, \( \lambda \left( = \frac{k_p v_0}{a v_b} \right) \) is the Darcy parameter, \( N_f (= ab) \) is the Forchheimer quadratic drag parameter, \( N_m \left( = \frac{\sigma B_0 a}{\rho v_0} \right) \) is the magnetic parameter, \( Sc \left( = \frac{av_0}{D} \right) \) is the Schmidt number, \( \gamma \left( = \frac{k a^2}{v_b} \right) \) is the chemical reaction parameter and \( \Pr \left( = \frac{v_b}{\alpha} \right) \) is the Prandtl number.

The pressure gradient is defined as:

\[ -\frac{\partial P}{\partial \xi} = P_s + P_0 \cos(\omega t), \]  

where \( P_0 \) is the static and \( P_s \) is the oscillatory components (with \( \omega \) being its frequency) of the pressure gradient.

The boundary conditions finally get the form:

\[ \eta = -1: U = 0, \theta = \varphi = -1, \eta = 1: U = 0, \theta = \varphi = 1 \quad \forall \ t > 0 \]  

(6.10)
Initially, we assume that the fluid is at rest, and the temperature and concentration distributions vary linearly with $\eta$ (that is, across the channel), which results in the following initial conditions:

$$U = 0, \theta = \phi = \eta \quad \forall \quad -1 \leq \eta \leq 1 \quad \text{at} \quad t = 0 \quad (6.11)$$

### 6.3. NUMERICAL SOLUTION

We solve Eqs. (6.6)-(6.8) numerically subject to the initial and boundary conditions given in Eqs. (6.10) and (6.11), by employing the three step explicit Runge-Kutta method for time-integration whereas the spatial derivatives are approximated by the corresponding central differences. We elaborate the computational procedure for Eq. (6.6) only (the remaining equations have been treated in a similar manner):

If $U^{(n)}$ represents the numerical solution of Eq. (6.6) at $t = t_n$ (where $t_n = (n-1)\Delta t$), we obtain the numerical solution $U^{(n+1)}$ at the next time level by using the following explicit scheme:

$$
\begin{align*}
\frac{U_1^{(n)} - U_1^{(n)}}{dt/2} & = H\left(U_1^{(n)}\right), \\
\frac{U_2^{(n)} - U_2^{(n)}}{dt/2} & = H\left(U_2^{(n)}\right), \\
\frac{U_3^{(n)} - U_3^{(n)}}{dt} & = H\left(U_3^{(n)}\right), \\
K^{(n)} & = H\left(U_3^{(n)}\right), \\
U^{(n+1)} & = \frac{1}{3}\left(-U^{(n)} + U_1^{(n)} + 2U_2^{(n)} + U_3^{(n)}\right) + \frac{\Delta t}{6}K^{(n)},
\end{align*}
$$

(6.12)

where,

$$H\left(U^{(n)}\right) = \left(\frac{\partial P}{\partial \xi}\right)^{(n)} + \left[\frac{1}{R}\left(1 + \frac{1}{\beta}\right)\left(U_{1o}^{(n)} - 2U_j^{(n)} + U_{1o}^{(n)}\right) + \left(U_{1o}^{(n)} - U_{1o}^{(n)}\right)\frac{\Delta t}{6}\right] \left(M + \frac{1}{\lambda} + N_f\right)U^{(n)}.$$
with $h = \frac{2}{(N-1)}$ being the uniform step size along the $\eta$ - direction. It is to note that $i = 1, N$ correspond to the two boundaries located at $\eta = -1, 1$ respectively. In this way, we keep on marching along the time until we reach the desired time level.

6.4. RESULTS AND DISCUSSION

We numerically calculate the velocity, temperature and concentration distributions across the channel for various values of the dimensionless parameters. The biofluid is taken to be blood and following values of the parameters (with corresponding SI units) are considered:

\[
\begin{align*}
\rho &= 1050 \text{ kgm}^{-3}, 2\alpha = 0.6 \times 10^{-2} \text{ m}, V_0 = 0.05 \times 10^{-2} \text{ ms}^{-1}, D = 10^{-5} \text{ m}^2 \text{s}^{-1}, \mu = 3.2 \times 10^{-3} \text{ Nms}^{-2}, \\
c_p &= 14.286 \text{ Jkg}^{-1} \text{ K}^{-1}, k = 2.2 \times 10^{-3} \text{ Jm}^{-1} \text{s}^{-1} \text{ K}^{-1}, \sigma = 0.8 \text{ Sm}^{-1}, Pr = 21, R = 0.5, \beta = 4, \omega = 8, \\
P_0 &= 7, P_s = 10, \lambda = 5, N_f = 0.002, Sc = 0.15 \text{ and } N_m = 0.3.
\end{align*}
\]

These values correspond to very weak inertial effects whereas the value of $Sc$, is a reasonable approximation for pharmaceutical species.

Table 6.1 shows the convergence of our numerical results for $U(\eta)$ with the above mentioned values of the governing parameters at $t = 0.3$, as the step-size $h$ decreases. This gives us confidence on our computational procedure. The schematic diagram for the problem is shown in the Figure 6.1.

Figure 6.2 shows the time variation of velocity distribution across the channel. The number of cycles is controlled by the parameter $\omega$. It is observed that the peak value of the velocity in the first cycle is significantly lower than the ones corresponding to the next cycles. It is due to the reason that initially the fluid is at rest, and because of inertia, the pressure gradient $\frac{\partial P}{\partial \xi}$ is not as effective as in the forthcoming cycles when the fluid is set in motion. Figure 6.3 predicts high velocity gradient near $t = 0$ on the
application of pressure gradient to the stationary fluid. It may also be observed that cycles are elongated across the channel width and, after the first one, almost periodic. Figures 6.4 and 6.5 show the temperature distribution across the channel as the time passes. Initially, when the fluid is stagnant in the absence of any deriving force, the temperature is assumed to be varying linearly across the channel. But once the fluid comes out of its stationary state, the temperature distribution is concentrated in a small portion towards the upper channel wall, and in the rest of the channel, the thermal profile is almost flat.

Now, we consider the effect of the various parameters on the velocity and temperature distribution at the arbitrary dimensionless time (say at $t = 0.3$). Figure 6.6 gives the effect of the transpiration Reynolds number on the velocity profiles. Clearly, the effect of $Re$ is to accelerate the fluid across the channel. Moreover, for uniform properties of the fluid, an increase in $Re$ would mean an increase in the suction velocity at the upper plate (located at $\eta = 1$). The profiles, for this reason, are tilted toward the upper channel wall. It is clear from the Fig. 6.7 that the Reynolds number affects the temperature profile in the small regions near the walls. Therefore, the thermal distribution remains an almost linear function of $\eta$, in the middle of the channel.

Figure 6.8 gives the effect of the rheological parameter $\beta$ on the fluid velocity at $t = 0.3$. An increase in $\beta$ reflects a stronger Newtonian behavior meaning thereby a decrease in the viscosity, which results into an increase in the velocity as shown in the Fig. 6.8. The effect of the Darcy parameter $\lambda$ is given in the Fig. 6.9. An increase in $\lambda$ would mean a reduction in the Darcian drag force (quantified by the term $-\frac{U}{\lambda}$ in Eq. (6.6) ). This would turn into decreasing the resistance to the flow and thus
accelerating the biofluid. Thus, the net effect of $\lambda$ is to accelerate the fluid across the channel as given by the Fig. 6.9.

An increase in the Forchheimer quadratic drag parameter $N_f$ would enhance the resistance to the flow (characterized by the term $-N_f U^2$ in Eq. (6.6)) and therefore will reduce the fluid velocity as shown in the Fig. 6.10. Increasing the magnetic parameter $N_m$ decreases the velocity distribution across the channel, as shown in the Fig. 6.11. The external magnetic field exerts a drag like force, called the Lorentz force which tends to slow down the fluid motion.

As expected, the effect of the steady component $P_s$ of the pressure gradient is to remarkably accelerate the flow (please see Fig. 6.12). On the other hand, $P_0$ contributes to the oscillatory component of the deriving force, and its influence on the velocity distribution depends on the sign of the term $\cos(\omega t)$, at any instant of time.

For example, Fig. 6.13 shows a decrease in the fluid velocity with $P_0$ at $t = 0.3$, because the term is negative at this instant of time. However, the fluid was accelerated with $P_0$ at $t = 0.1$ due to the positive sign of the same term.

Figure 6.14 shows how the thermal distribution across the channel is affected by the Prandtl number $Pr$. It is noted that an increase in $Pr$ influences the temperature profile near the channel walls only.

Finally, the influence of the Reynolds number $Re$, the Schmidt number $Sc$ and the chemical reaction parameter $\gamma$ on the concentration profiles is given in the Figs. 6.15-6.17. It is obvious from the Fig. 6.15 that an increase in the Reynolds number tends to straighten the concentration profile, thus making it an almost linear function of $\eta$.

The influence of the chemical reaction parameter on the concentration distribution is
qualitatively opposite to that of the Reynolds number, as it tends to eliminate the
linear nature of the profiles (Fig. 6.16). From the Fig. 6.17, we note that the Schmidt
number does not influence the concentration profiles near the lower channel wall
whereas the lowering of the profiles is observed in the rest of the channel.

6.5. CONCLUSIONS

In this chapter, we have numerically studied the unsteady pulsatile magneto-
rheological flow of a non Newtonian biofluid through a porous channel, taking the
chemical reaction effects into consideration. Three step explicit Runge-Kutta method
has been used for the time integration whereas the central difference approximations
are employed for the spatial derivatives. It has been observed that, as the Reynolds
number is increased, the velocity profile across the channel is raised. Similar effect
has also been observed for the rheological parameter and the Darcy parameter,
whereas an opposite trend has been noted for both the Forchheimer quadratic drag
parameter and the magnetic parameter. The effect of the steady component of the
pressure gradient is to remarkably accelerate the flow whereas that of oscillatory
component is time dependent. An increase in the Prandtl number influences the
temperature profile near the channel walls only. It has also been noted that the
Reynolds number tends to straighten the concentration profile, thus making it an
almost linear function of the dimensionless spatial variable. The influence of the
chemical reaction parameter on the concentration distribution is qualitatively opposite
to that of the Reynolds number, as it tends to eliminate the linear nature of the
profiles. Moreover, the Schmidt number does not significantly influence the
concentration profiles near the lower channel wall whereas the lowering of the profiles is observed in the rest of the channel.

Table 6.1  Dimensionless velocity $U(\eta)$ on three grid sizes and extrapolated values

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>1$^{\text{st}}$ grid ($h = 0.02$)</th>
<th>2$^{\text{nd}}$ grid ($h = 0.01$)</th>
<th>3$^{\text{rd}}$ grid ($h = 0.005$)</th>
<th>Extrapolated values</th>
</tr>
</thead>
<tbody>
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<td>-1</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0.480709</td>
<td>0.480695</td>
<td>0.480690</td>
</tr>
<tr>
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<td>0.897773</td>
<td>0.897766</td>
</tr>
<tr>
<td>-0.4</td>
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<td>1.228991</td>
<td>1.228961</td>
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</tr>
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<td>1.456121</td>
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</tr>
<tr>
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<td>1.564306</td>
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<tr>
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</tr>
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</tr>
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</tr>
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<td>0</td>
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</tbody>
</table>
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$P_0 = 7, \lambda = 5, N_f = 0.002, \text{Sc} = 0.15$ and $N_m = 0.3$.

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$P_0 = 7, \lambda = 5, N_f = 0.002, \text{Sc} = 0.15$ and $N_m = 0.3.$
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\[ R = 0.5, \beta = 4, \omega = 8, P_s = 10, P_0 = 7, \lambda = 5, N_f = 0.002, \text{Sc} = 0.15 \text{ and } N_m = 0.3. \]

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$R = 0.5, \omega = 8, P_c = 10, P_0 = 7, \beta = 4, \lambda = 5, Sc = 0.15$ and $N_f = 0.002$. 

Figure 6.12 Effect of the steady component of pressure gradient on the velocity profiles for $R = 0.5$, $\omega = 8$, $N_m = 0.3$, $P_0 = 7$, $\beta = 4$, $\lambda = 5$, $Sc = 0.15$, and $N_f = 0.002$.

Figure 6.13 Effect of the steady component of pressure gradient on the velocity profiles for $R = 0.5$, $\omega = 8$, $N_m = 0.3$, $P_0 = 10$, $\beta = 4$, $\lambda = 5$, $Sc = 0.15$, and $N_f = 0.002$. 
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\[ R = 0.5, \omega = 8, N_m = 0.3, P_0 = 7, P_s = 10, \beta = 4, \lambda = 5, Sc = 0.15, \text{ and } N_f = 0.002 \]

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Figure 6.17 Effect of the Schmidt number on the concentration profiles for $\omega=8, N_m=0.3, P_0=7, P_s=10, \beta=4, \lambda=5, \gamma=1, R=0.5$ and $N_r=0.002$. 

[Graph showing concentration profiles with different chemical reaction parameters and Schmidt numbers.]
CHAPTER 7
FLOW REVERSAL IN A WATER-BASED NANOFUID
INSIDE A VERTICAL SQUARE DUCT
7.1. INTRODUCTION

This chapter is devoted to the numerical study of the mixed convection in an incompressible, electrically conducting water-based nanofluid containing silver particles, inside a vertical square duct, using the spectral method and the finite difference method. The flow is considered to be laminar and hydrodynamically as well as thermally developed, whereas the thermal boundary condition of constant heat flux per unit axial length with constant peripheral temperature at any cross section is assumed. For different values of the relaxation parameter, the efficiency of the SOR method used to solve the linear system arising from the finite difference discretization of the governing equations, is compared with the spectral method at high Raleigh number. We have noted that the flow reversal due to high Raleigh number may be controlled by applying an external magnetic field of suitable strength.

7.2. MATHEMATICAL FORMULATION
We consider the hydrodynamically as well as thermally developed incompressible laminar flow of a water-based nanofluid containing silver particles, in a vertical square duct of side $L$, subject to constant heat flux per unit axial length with constant peripheral temperature at any cross section. The wall thickness of the duct is assumed to be negligible so that the assumption of infinite wall conductivity in the outward direction may be more realistic, which also means the same temperature on the outside duct surface and on the solid-fluid interface. The fluid is assumed to be electrically conducting with external magnetic field acting on it. Moreover, compared with the imposed one, the induced magnetic field is assumed to be negligible.

The velocity field for the problem is

$$\vec{V} = (0, 0, w(x, y)),$$

and the governing equations of motion and heat transfer are

**Continuity Equation**

$$\frac{\partial w}{\partial z} = 0, \quad (7.1)$$

**$x$, $y$- Components of Momentum Equation**

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \quad (7.2)$$

**$z$- Components of Momentum Equation**

$$-\frac{\partial p}{\partial z} + \mu_{nf} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + g(T - T_w)\beta_{nf} - \sigma_{nf} B_0^2 w = 0, \quad (7.3)$$

**Heat Equation**

$$w \frac{\partial T}{\partial z} = \alpha_{nf} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right), \quad (7.4)$$

**Boundary Conditions**
\[ \begin{align*}
&w(x,0) = w(x,L) = w(0,y) = w(L,y) = 0, \\
&T(x,0) = T(x,L) = T(0,y) = T(L,y) = T_w(z) \quad \text{(7.5)},
\end{align*} \]

whereas,

\[
\mu_{nf} = \frac{\mu_f}{(1-\phi)^2}, \quad \rho_{nf} = (1-\phi)\rho_f + \phi\rho_s, \quad \alpha_{nf} = \frac{k_{nf}}{(\rho c_p)_{nf}},
\]

\[
\frac{(\rho c_p)_{nf}}{f} = (1-\phi)\frac{(\rho c_p)_{f}}{f} + \phi\frac{(\rho c_p)_{s}}{f}, \quad k_{nf} = \frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)}k_f,
\]

\[
\frac{(\rho \beta)_{nf}}{f} = (1-\phi)(\rho \beta)_{f} + \phi(\rho \beta)_{s}, \quad \frac{\sigma_{nf}}{\sigma_f} = 1 + \frac{3\left(\frac{\sigma_s}{\sigma_f} - 1\right)}{\left(\frac{\sigma_s}{\sigma_f} + 2\right) - \left(\frac{\sigma_s}{\sigma_f} - 1\right)\phi}.
\]

Physical quantities appearing in the above mentioned expression have already been mentioned in chapter 5.

Equation (7.2) implies \( p = p(z) \), and for hydrodynamically developed flow,

\[
\frac{dp}{dz} = \text{Constant}.
\]

We consider the energy balance over a cross section of the duct and arrive at (Kays et al. [106]):

\[
\dot{Q}'dz = w_m(\rho c_p)_{nf} L^2 dT_b \quad \text{or} \quad \frac{dT_b}{dz} = \frac{\dot{Q}'}{w_m(\rho c_p)_{nf} L^2}, \quad (7.6)
\]

where \( \dot{Q}' \) is the heat transfer rate per unit axial length, and \( T_b \) is the bulk temperature of the fluid defined as:

\[
T_b = \frac{\int_{\sqrt{L}}^L \int_{-\sqrt{L}}^{\sqrt{L}} wT \, dx \, dy}{\int_{\sqrt{L}}^L \int_{-\sqrt{L}}^{\sqrt{L}} \, w \, dx \, dy}.
\]
We introduce the following dimensionless variables

\[ X = \frac{x}{L}, Y = \frac{y}{L}, W = -\frac{\mu_f w}{L^2 \left( \frac{dp}{dz} \right)} \psi = \frac{k_f (T - T_w)}{Q'} W_m \quad (7.7) \]

where

\[ W_m = \int_0^1 \int_0^1 W \, dx \, dy = -\frac{\mu_f w_m}{L^2 \left( \frac{dp}{dz} \right)} \quad \text{with} \quad w_m = \frac{1}{L^2} \int_0^L \int_0^L w \, dx \, dy = \int_0^1 \int_0^1 w \, dx \, dy, \]

which implies

\[ \frac{dp}{dz} = -\frac{\mu_f w_m}{L^2 W_m}. \quad (7.8) \]

For thermally fully-developed flow subject to a constant heat transfer (per unit axial length) boundary condition, it is well known (Incropera and Dewitt [107]) that

\[ \frac{\partial \psi}{\partial z} = 0 = \frac{\partial T}{\partial z} - \frac{dT_w}{dz} \quad \text{and} \quad \frac{dT_w}{dz} = \frac{dT_h}{dz}. \quad (7.9) \]

Combining Eqs. (7.6) and (7.9)

\[ \frac{\partial T}{\partial z} = \frac{dT_w}{dz} = \frac{dT_h}{dz} = \frac{\dot{Q}'}{w_m (\rho c_p)_m L^2}, \]

which implies

\[ \frac{\partial^2 T}{\partial z^2} = 0. \]

Therefore, the governing Eqs. (7.3) and (7.4) are reduced to

\[ 1 + (1 - \phi)^{-1.5} \left( \frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} \right) + \left( 1 - \phi + \phi \left( \frac{\rho c_p}_s \right) \left( 1 - \phi + \phi \left( \rho \beta \right) \right) \right) Ra \psi \]

\[ -M^2 \left[ \frac{3 \left( \frac{\sigma_s}{\sigma_f} - 1 \right) \phi}{\left( \frac{\sigma_s}{\sigma_f} + 2 \right) - \left( \frac{\sigma_s}{\sigma_f} - 1 \right) \phi} \right] W = 0, \quad (7.10) \]
\[ \left( \frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2} \right) = \frac{(k_s + 2k_f) + \phi(k_f - k_s)}{(k_s + 2k_f) - 2\phi(k_f - k_s)} W, \] 

(7.11)

where \( Ra = \frac{\rho_f g BL^4}{\alpha_f u_f} \frac{dT_b}{dz} \) is the Raleigh number (Dong and Ebadian [108]), \( \alpha_f \)

\[ \left( -\frac{k_f}{(\rho c_p)_f} \right) \] is the thermal diffusivity of the base-fluid, and \( M = B_0 L \sqrt{\frac{\sigma_f}{\mu_f}} \) is the magnetic parameter. On the other hand, the boundary conditions are

\[ \psi(X, 0) = \psi(X, 1) = \psi(0, Y) = \psi(1, Y) = 0, \]

\[ W(X, 0) = W(X, 1) = W(0, Y) = W(1, Y) = 0 \]

(7.12)

### 7.3. NUMERICAL SOLUTION

We use both the finite difference method (FDM) and the spectral method to solve the coupled Eqs. (7.10) and (7.11), subject to the boundary conditions given in Eq. (7.12).

For the FDM, we first stretch the grid towards the boundaries by employing the transformation:

\[ X = \frac{1}{2}(1 - \cos \pi \zeta), Y = \frac{1}{2}(1 - \cos \pi \eta), \]

(7.13)

due to the expected higher gradients near the duct walls.

The governing Eqs. (7.10) and (7.11), in view of (7.13), get the form:

\[ 1 + (1 - \phi)^{-2.5} \left( \left\{ \frac{\partial^2 \zeta}{\partial X^2} \right\} \frac{\partial W}{\partial \zeta} + \left( \frac{\partial \zeta}{\partial X} \right)^2 \frac{\partial^2 W}{\partial Y^2} \right) + \left( \frac{\partial \eta}{\partial Y} \right)^2 \frac{\partial^2 W}{\partial \eta^2} + \left( \frac{\partial^2 W}{\partial \eta^2} \right) \]

\[ + \left\{ 1 - \phi + \phi \left( \frac{\rho c_p}{\rho c_p} \right) (\rho c_p) \right\} Ra \psi - M^2 \left\{ 1 + \frac{3(\sigma_s - \sigma_f)}{(\sigma_s + 2\sigma_f) - (\sigma_s - \sigma_f) \phi} \right\} W = 0, \]

(7.14)
\[
\left\{ \frac{\partial^2 \zeta}{\partial x^2} \frac{\partial \psi}{\partial \zeta} + \left( \frac{\partial \zeta}{\partial x} \right)^2 \frac{\partial^2 \psi}{\partial \zeta^2} + \frac{\partial^2 \eta}{\partial \zeta^2} \frac{\partial \psi}{\partial \eta} + \left( \frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2 \psi}{\partial \eta^2} \right\} = \frac{\left( k_i + 2k_f \right) + \phi(k_f - k_i)}{\left( k_i + 2k_f - 2\phi(k_f - k_i) \right)} W_i, 
\]

(7.15)

We then use the central difference approximations for the derivatives, at a typical grid point \((\zeta_i, \eta_j)\), in the above equations, which yields the following system of algebraic equations:

\[
\left\{ \left\{ \frac{2}{\pi} \cos \pi \zeta_i \right\}^2 + \left\{ \frac{h}{k} \right\}^2 \left\{ \frac{2}{\pi} \cos \pi \eta_j \right\}^2 + 2(1 - \phi)^{2.5} h^2 M^2 \left\{ 1 + \frac{3(\sigma_s - \sigma_f)\phi}{(\sigma_s + 2\sigma_f) - (\sigma_s - \sigma_f)\phi} \right\} \right\} W_{i,j} 
= \left\{ \left\{ \frac{2}{\pi} \cos \pi \zeta_i \right\}^2 + \frac{4\cos \pi \zeta_i}{\pi \sin^3 \pi \zeta_i} \right\} W_{i-1,j} + \left\{ \left\{ \frac{2}{\pi} \cos \pi \eta_j \right\}^2 - \frac{4\cos \pi \eta_j}{\pi \sin^3 \pi \eta_j} \right\} W_{i+1,j} 
+ \left\{ \left\{ \frac{2}{\pi} \cos \pi \eta_j \right\}^2 - \frac{4\cos \pi \eta_j}{\pi \sin^3 \pi \eta_j} \right\} W_{i,j-1} + \left\{ \left\{ \frac{2}{\pi} \cos \pi \eta_j \right\}^2 - \frac{4\cos \pi \eta_j}{\pi \sin^3 \pi \eta_j} \right\} W_{i,j+1} 
+ 2h^2 \left\{ 1 + \phi + \phi \left( \frac{\rho \beta_i}{\rho \beta_f} \right) \right\} (1 - \phi)^{2.5} \right\} \psi_{i,j} 
\]

(7.16)

\[
\left\{ \left\{ \frac{2}{\pi} \cos \pi \zeta_i \right\}^2 + \left\{ \frac{h}{k} \right\}^2 \left\{ \frac{2}{\pi} \cos \pi \eta_j \right\}^2 \right\} \psi_{i,j} 
= \left\{ \left\{ \frac{2}{\pi} \cos \pi \zeta_i \right\}^2 + \frac{4\cos \pi \zeta_i}{\pi \sin^3 \pi \zeta_i} \right\} \psi_{i-1,j} + \left\{ \left\{ \frac{2}{\pi} \cos \pi \eta_j \right\}^2 - \frac{4\cos \pi \eta_j}{\pi \sin^3 \pi \eta_j} \right\} \psi_{i+1,j} 
+ \left\{ \left\{ \frac{2}{\pi} \cos \pi \eta_j \right\}^2 + \frac{4\cos \pi \eta_j}{\pi \sin^3 \pi \eta_j} \right\} \psi_{i,j-1} + \left\{ \left\{ \frac{2}{\pi} \cos \pi \eta_j \right\}^2 - \frac{4\cos \pi \eta_j}{\pi \sin^3 \pi \eta_j} \right\} \psi_{i,j+1} 
- 2h^2 \left( \frac{k_i + 2k_f}{k_i + 2k_f - 2\phi(k_f - k_i)} \right) W_{i,j}, 
\]

(7.17)

where
\[
\zeta_i = (i - 1)h, \eta_j = (j - 1)k, 
\]

(7.18)

are the equi-spaced grid points in \(\zeta \eta\)-plane, and
\[ h = \frac{1}{(n_i - 1)}, k = \frac{1}{(n_j - 1)} \]

are the step sizes along the \( \zeta \) and \( \eta \) – directions, respectively. The computational and physical domains of the problem are shown in Figs. (7.1a) and (7.1b). We note that, for the equi-spaced grid points in \( \zeta \eta \)-plane (shown in the Fig. (7.1a), the corresponding points in XY-plane are stretched towards the walls of the duct, as shown in the Fig. (7.1b). The above mentioned coupled linear algebraic systems are then solved iteratively by employing the Successive over relaxation method (SOR).

The numerical solution thus obtained is finally represented again in the XY-plane.

In order to solve Eqs. (7.10) and (7.11) by using the spectral method, we (considering the boundary conditions) set:

\[
W(X, Y) = \sum_{p=1}^{N} \sum_{q=1}^{N} a_{p,q} \sin(p \pi X) \sin(q \pi Y) \\
\psi(X, Y) = \sum_{p=1}^{N} \sum_{q=1}^{N} b_{p,q} \sin(p \pi X) \sin(q \pi Y)
\]

(7.19)

Now substituting Eq. (7.18) in Eqs. (7.10) and (7.11), and using

\[
\int_{0}^{1} \sin(m \pi t) \sin(n \pi t) dt = \begin{cases} \frac{1}{2} & m = n, \\ 0 & m \neq n \end{cases} \quad \text{(where } t \text{ is either } X \text{ or } Y) 
\]

the unknown coefficients are found to be:

\[
a_{p,q} = \frac{1}{pq \pi^2} \left(1 - \cos(p \pi)(1 - \cos(q \pi)) \right) \left[ \frac{\Delta_1 \Delta_2 \Delta_3 \text{Ra}}{4 \pi^2 (p^2 + q^2)} + 0.25(1 - \phi)^{-2.5} \pi^2 (p^2 + q^2) + \frac{1}{4} M^2 \right] \\
b_{p,q} = -\frac{\Delta_3 a_{p,q}}{\pi^2 (p^2 + q^2)}
\]

(7.20)

where

\[
\Delta_1 = 1 - \phi + (\rho c_p)_{\rho} \Delta_2 = 1 - \phi + (\rho \beta) \Delta_3 = \frac{(k_s + 2k_f) + \phi(k_f - k_s)}{(k_s + 2k_f) - 2\phi(k_f - k_s)}
\]
Thus, the solution at the grid point \((X_i, Y_j)\) is given by:

\[
W(X_i, Y_j) = \sum_{p=1}^{N} \sum_{q=1}^{N} a_{(p,q)} \sin(p \pi X_i) \sin(q \pi Y_j)
\]

\[
\psi(X_i, Y_j) = \sum_{p=1}^{N} \sum_{q=1}^{N} b_{(p,q)} \sin(p \pi X_i) \sin(q \pi Y_j)
\]

(7.21)

It is clear from the above equation that the coefficients \(a_{p,q}\) and \(b_{p,q}\) do not depend upon the orientation of the grid points and are dependent only upon the structure of the governing equations. Thus, on the stretched grid, only the definitions of \(X_i\) and \(Y_j\) in Eq. (7.21) will undergo a change.

Hence, on the uniformly spaced grid (which corresponds to the XY-plane in Fig. (7.1a)), we have:

\[
W_{i,j} = \sum_{p=1}^{N} \sum_{q=1}^{N} a_{(p,q)} \sin(p \pi (i-1)h) \sin(q \pi (j-1)k)
\]

\[
\psi_{i,j} = \sum_{p=1}^{N} \sum_{q=1}^{N} b_{(p,q)} \sin(p \pi (i-1)h) \sin(q \pi (j-1)k)
\]

(7.22)

On the other hand, the solution on the un-equispaced stretched grid is given by:

\[
W_{i,j} = \sum_{p=1}^{N} \sum_{q=1}^{N} a_{(p,q)} \sin \left( p \pi \left( \frac{1 - \cos(p \pi (i-1)h)}{2} \right) \right) \sin \left( q \pi \left( \frac{1 - \cos(q \pi (j-1)k)}{2} \right) \right)
\]

\[
\psi_{i,j} = \sum_{p=1}^{N} \sum_{q=1}^{N} b_{(p,q)} \sin \left( p \pi \left( \frac{1 - \cos(p \pi (i-1)h)}{2} \right) \right) \sin \left( q \pi \left( \frac{1 - \cos(q \pi (j-1)k)}{2} \right) \right)
\]

(7.23)

which corresponds to the \(\zeta\eta\) coordinate system, shown in the Fig. 1b.

7.4 RESULTS AND DISCUSSION

Physical quantities of our interest are the Darcy (or Moody) friction factor \(f\) and the Nusselt number \(Nu\), given by:

\[
f = - \frac{dp}{dz} \frac{L}{\rho_{nf} w_m^2/2} \quad \text{and} \quad \frac{\dot{Q}'}{k_{nf}} = \frac{\dot{Q}'}{k_{nf}} = \frac{T_w - T_h}{k_{nf}}.
\]
Usually the friction factor is combined in a product with the Reynolds number: \( \text{Re} = \frac{\rho_f w_m L}{\mu_f} \) which, in view of Eqs. (7.7) and (7.13), can be written as:

\[
f \text{Re} = \left( \frac{\rho_f}{\rho_{nf}} \right) \frac{2}{W_m} \]

(7.24)

\[
N_u = \frac{W_m^2 \left( \frac{k_f}{k_{nf}} \right)}{\int_0^1 \int_0^1 W(X,Y) \psi(X,Y) dX dY} = \frac{4W_m^2 \left( \frac{k_f}{k_{nf}} \right)}{\pi^2 \int_0^1 \int_0^1 W(\zeta,\eta) \psi(\zeta,\eta) \sin(\pi\eta) \sin(\pi\zeta) d\eta d\zeta}
\]

(7.25)

where

\[
W_m = \int_0^1 \int_0^1 W(X,Y) dX dY = \frac{\pi^2}{4} \int_0^1 \int_0^1 W(\zeta,\eta) \sin(\pi\eta) \sin(\pi\zeta) d\eta d\zeta
\]

We have considered water-based nanofluids containing silver nanoparticles (Thermophysical properties of water and the solid particles are given in Table (7.1)). For the nanofluid, we shall study the effects of the nanoparticle volume fraction parameter \( \phi \), the Raleigh number \( Ra \) and the magnetic parameter \( M \) on the physical quantities (\( f \text{Re} \) and the Nusselt number \( Nu \)), and also on the dimensionless temperature \( \psi \) and velocity fields \( W \).

Table (7.2) gives the comparison of the Spectral method and the finite difference method (FDM) on both the uniform and stretched grids, as well as the grid independence study for both the methods. It is clear that the numerical results for \( f \text{Re} \) and \( Nu \) converge as the number of grid points are increased. Moreover, compared with the FDM, more accurate results are obtained using the Spectral method on the coarser grids. It is also noted that the stretched grid remarkably improves the accuracy of the solution, particularly for the FDM.
Tables (7.3a)-(7.3d) give the influence of the grid stretching on the efficiency of the SOR method (used to solve the linear algebraic equations arised from the FD discretization of the governing PDEs), for different values of the relaxation parameter $\omega$ on two different grids with $51 \times 51$ and $101 \times 101$ points respectively. It is obvious that the grid stretching significantly contributes to the efficiency of the iterative method on both the grids. Thus the use of stretched grid may be recommended for the SOR method due to the enhanced efficiency as well as the accuracy of the method.

Orientation of the regular and stretched grids are shown in Figs. (7.1a) and (7.1b), whereas Figs. (7.2a) and (7.2b) show the effect of the relaxation parameter $\omega$ on the CPU time (in seconds) required to solve the Eqs. (7.10) and (7.11) which are discretized by using the compact finite difference scheme and then solved by employing the SOR method, on $51 \times 51$ and $101 \times 101$ grids respectively. The nanoparticle volume fraction $\phi$ is fixed at 0.1 whereas four different values of the Raleigh number $Ra$ are considered ($Ra = 5000, 10000, 20000, 30000$) with the fixed $M = 5$. During the simulation process, we noted that the convergence was quite slow below $\omega = 1$ whereas the iterative process simply failed to converge at large $Ra$ for $\omega > 1.7$. That is why, $\omega$ is chosen to lie in the range $(1 < \omega < 1.7)$ in the Figs.(7.2a) and (7.2b).

It is clear that the efficiency of the SOR method is remarkably influenced by the choice of the relaxation parameter $\omega$. It is also obvious from the Figs. (7.2a) and (7.2b) that, on the $51 \times 51$ grid, the lowest CPU time for SOR method is nearly 0.3 seconds, whereas the CPU time required to solve the problem with the above mentioned values of the parameters ($\phi$ and $Ra$) using the spectral method was noted to be just about 0.0468 seconds. On the other hand, the two CPU times (for the SOR method and the spectral method) are, respectively, 4 seconds (approx.) and 0.2808...
seconds on the $101 \times 101$ grid, which clearly depicts the supremacy of the Spectral method over the compact finite difference scheme.

Figures (7.3a)-(7.4b) give the dimensionless velocity and temperature distributions across the duct for $\phi = 0.10, M = 5$ and $Ra = 20000$. It is obvious from Figs. (7.3a) and (7.3b) that the velocity in the centre of the duct is different both quantitatively and qualitatively. Due to the no slip condition, the velocity gradients are high near the walls. Moreover, the triangle like region near each corner of the duct is formed due to the influence of the two rigid boundaries. Figs. (7.4a) and (7.4b), on the other hand, predict an equi-temperature region in the middle of the duct at high Raleigh number.

In order to observe the effect of the physical parameters on the velocity and temperature distribution, we consider the sections of the two surfaces (velocity and temperature) by the plane $y = 0.5$, thus, giving the temperature or velocity profiles in the middle of the duct.

The effect of the Raleigh number on the velocity and temperature distributions across the middle of the duct is shown in Figs. (7.5) and (7.6), which clearly predict that the increase in $Ra$ remarkably reduces the fluid velocity to such an extent that the reverse flow may occur in the centre of the duct. Moreover, $Ra$ decreases the temperature distribution while flattening the temperature profiles in the center of the duct, thus creating an equi-temperature region (Fig. (7.6)).

On the other hand, nanoparticle volume fraction $\phi$ tends to narrow down the region of the flow reversal in the duct while flattening the temperature profiles as shown in the Figs. (7.7) and (7.8). Thus, from the flow reversal point of view, the two physical parameters ($\phi$ and $Ra$) tend to act against each other.

The effect of the external magnetic field on the velocity and temperature profiles is given in Figs. (7.9) and (7.10). The magnetic field exerts a drag like force called the
Lorentz force which not only lowers the velocity profiles near the duct walls but also tends to reduce the flow reversal while rendering a parabolic shape to the thermal distribution. Thus, it is concluded that the flow reversal due to high Raleigh number may be controlled by applying an external magnetic field of suitable strength.

Influence of the nanoparticle volume fraction $\phi$ on $Re$ and $Nu$ with different $Ra$ for $M = 10$ is given in Figs. (7.11) and (7.12), respectively. It is clear that $f Re$ is more sensitive to $\phi$ at the smaller values of the later, and the sensitivity increases with $Ra$ whereas $Nu$ varies almost linearly with $\phi$. For the fixed $Ra = 30000$, Figs. (7.13) and (7.14) show how $f Re$ and $Nu$ are affected by the nanoparticle volume fraction $\phi$ for different values of the magnetic parameter $M$. For $\phi$ in the range $0 < \phi < 0.20$, an increase in the magnetic parameter increases $f Re$, whereas an opposite trend is obvious for $Nu$.

7.5. CONCLUSIONS

In this study, we numerically investigate the mixed convection in the hydrodynamically as well as thermally developed flow of a silver-water nanofluid in a vertical square duct, using the spectral method and the finite difference method (FDM). Thermal boundary condition of constant heat flux per unit axial length with constant peripheral temperature of the heated wall is assumed. We notice the remarkable efficiency of the spectral method compared with the FDM, for solving the governing partial differential equations of the problem. We also observe that the Raleigh number remarkably reduces the fluid velocity and, in the middle of the duct, flow reversal may even occur at high Raleigh number while flattening the temperature field (thus developing an equi-temperature region). Moreover, it is concluded that the flow reversal due to high Raleigh number may be controlled by applying an external
magnetic field of suitable strength. Moreover, the Nusselt number is found to be almost a linear function of the nanoparticle volume fraction parameter, for different values of the Raleigh number and the magnetic parameter. On the other hand, \( f \Re \) is found to be more sensitive to nanoparticle volume fraction parameter for the smaller values of the later.

<table>
<thead>
<tr>
<th>Table 7.1</th>
<th>Thermo-physical properties of water and nanoparticles</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Water</td>
</tr>
<tr>
<td>( \rho ) ( (kgm^{-3}) )</td>
<td>997.1</td>
</tr>
<tr>
<td>( C_p ) ( (Jkg^{-1}K^{-1}) )</td>
<td>4179</td>
</tr>
<tr>
<td>( k ) ( (Wm^{-1}K^{-1}) )</td>
<td>0.613</td>
</tr>
<tr>
<td>( \beta ) ( (K^{-1}) )</td>
<td>( 21 \times 10^5 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 7.2</th>
<th>Grid independence study for ( \phi=0.1, Ra=3000 ) and ( M=5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grid</td>
<td>Spectral Method</td>
</tr>
<tr>
<td></td>
<td>( f \ Re )</td>
</tr>
<tr>
<td>Stretched / Uniform Grid</td>
<td>31\times31</td>
</tr>
<tr>
<td>Stretched / Uniform Grid</td>
<td>51\times51</td>
</tr>
<tr>
<td>Stretched / Uniform Grid</td>
<td>61\times61</td>
</tr>
<tr>
<td>Stretched / Uniform Grid</td>
<td>81\times81</td>
</tr>
<tr>
<td>Stretched / Uniform Grid</td>
<td>101\times101</td>
</tr>
</tbody>
</table>
Table 7.3a Influence of the Raleigh number \((Ra)\) on the number of iterations required by the SOR method on the \(51 \times 51\) grid, for \(\phi=0.1, M=5\) and \(\omega=1.0, 1.2\)

<table>
<thead>
<tr>
<th>Ra</th>
<th>(\omega = 1.00)</th>
<th>(\omega = 1.20)</th>
<th>(%)</th>
<th>(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform Grid</td>
<td>Stretched Grid</td>
<td>Uniform Grid</td>
<td>Stretched Grid</td>
<td>Uniform Grid</td>
</tr>
<tr>
<td>5000</td>
<td>1657</td>
<td>856</td>
<td>48.34</td>
<td>1109</td>
</tr>
<tr>
<td>10000</td>
<td>1596</td>
<td>892</td>
<td>44.11</td>
<td>1198</td>
</tr>
<tr>
<td>15000</td>
<td>1646</td>
<td>825</td>
<td>49.88</td>
<td>1207</td>
</tr>
<tr>
<td>20000</td>
<td>1719</td>
<td>787</td>
<td>54.22</td>
<td>1242</td>
</tr>
<tr>
<td>30000</td>
<td>1768</td>
<td>751</td>
<td>57.52</td>
<td>1259</td>
</tr>
</tbody>
</table>

Table 7.3b Influence of the Raleigh number \((Ra)\) on the number of iterations required by the SOR method on the \(51 \times 51\) grid, for \(\phi=0.1, M=5\) and \(\omega=1.4, 1.6\)

<table>
<thead>
<tr>
<th>Ra</th>
<th>(\omega = 1.40)</th>
<th>(\omega = 1.60)</th>
<th>(%)</th>
<th>(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform Grid</td>
<td>Stretched Grid</td>
<td>Uniform Grid</td>
<td>Stretched Grid</td>
<td>Uniform Grid</td>
</tr>
<tr>
<td>5000</td>
<td>827</td>
<td>422</td>
<td>48.97</td>
<td>540</td>
</tr>
<tr>
<td>10000</td>
<td>851</td>
<td>427</td>
<td>49.82</td>
<td>586</td>
</tr>
<tr>
<td>15000</td>
<td>847</td>
<td>420</td>
<td>50.41</td>
<td>645</td>
</tr>
</tbody>
</table>
Table 7.3c Influence of the Raleigh number \( (Ra) \) on the number of iterations required by the SOR method on the \( 101 \times 101 \) grid, for \( \phi=0.1, M=5 \) and \( \omega=1.0, 1.2 \)

<table>
<thead>
<tr>
<th>Ra</th>
<th>Uniform Grid</th>
<th>Stretched Grid</th>
<th>%age Diff.</th>
<th>Uniform Grid</th>
<th>Stretched Grid</th>
<th>%age Diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5000</td>
<td>5423</td>
<td>3275</td>
<td>39.61</td>
<td>3628</td>
<td>2215</td>
<td>38.95</td>
</tr>
<tr>
<td>10000</td>
<td>5512</td>
<td>3133</td>
<td>43.16</td>
<td>3687</td>
<td>2119</td>
<td>42.53</td>
</tr>
<tr>
<td>15000</td>
<td>5200</td>
<td>2918</td>
<td>43.88</td>
<td>3921</td>
<td>1973</td>
<td>49.68</td>
</tr>
<tr>
<td>20000</td>
<td>5683</td>
<td>2806</td>
<td>50.62</td>
<td>3801</td>
<td>1898</td>
<td>50.07</td>
</tr>
<tr>
<td>30000</td>
<td>5620</td>
<td>2535</td>
<td>54.89</td>
<td>4070</td>
<td>1830</td>
<td>55.04</td>
</tr>
</tbody>
</table>

Table 7.3d Influence of the Raleigh number \( (Ra) \) on the number of iterations required by the SOR method on the \( 101 \times 101 \) grid, for \( \phi=0.1, M=5 \) and \( \omega=1.4, 1.6 \)

<table>
<thead>
<tr>
<th>Ra</th>
<th>Uniform Grid</th>
<th>Stretched Grid</th>
<th>%age Diff.</th>
<th>Uniform Grid</th>
<th>Stretched Grid</th>
<th>%age Diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5000</td>
<td>2818</td>
<td>1446</td>
<td>48.69</td>
<td>1655</td>
<td>955</td>
<td>42.30</td>
</tr>
<tr>
<td>10000</td>
<td>2722</td>
<td>1385</td>
<td>49.12</td>
<td>1786</td>
<td>900</td>
<td>49.61</td>
</tr>
<tr>
<td>15000</td>
<td>2807</td>
<td>1392</td>
<td>50.41</td>
<td>1804</td>
<td>894</td>
<td>50.54</td>
</tr>
<tr>
<td></td>
<td>20000</td>
<td>2931</td>
<td>1332</td>
<td>54.55</td>
<td>1857</td>
<td>900</td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
<td>------</td>
<td>------</td>
<td>-------</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td>30000</td>
<td>2825</td>
<td>1272</td>
<td>54.97</td>
<td>1994</td>
<td>858</td>
<td>56.97</td>
</tr>
</tbody>
</table>

**Figure 7.1a** Uniform grid for the problem

**Figure 7.1b** Orientation of the stretched grid for the problem
Figure 7.2(a) CPU time for the SOR method against the relaxation parameter $\omega$, for

$\phi=0.1, n=51, M=5$ and various $Ra$

Figure 7.2b CPU time for the SOR method against the relaxation parameter $\omega$, for

$\phi=0.1, n=101, M=5$ and various $Ra$
Figure 7.3a  Dimensionless velocity $W$ for $\phi = 0.1, Ra = 20000$ and $M = 5$

Figure 7.3b  Contours of the dimensionless velocity $W$ for $\phi = 0.1, Ra = 20000$ and $M = 5$
Figure 7.4a Dimensionless temperature $\psi$ for $\phi = 0.1, Ra = 20000$ and $M = 5$

Figure 7.4b Contours of the dimensionless temperature $\psi$ for $\phi = 0.1, Ra = 20000$ and $M = 5$
Figure 7.5 Velocity profiles in the middle of the duct with $\phi=0.1, M=5$ and various values of $Ra$

Figure 7.6 Temperature profiles in the middle of the duct with $\phi=0.1, M=5$ and various values of $Ra$
Figure 7.7 Velocity profiles in the middle of the duct with $Ra = 30000$, $M = 5$ and various values of $\phi$.

Figure 7.8 Temperature profiles in the middle of the duct with $Ra = 30000$, $M = 5$ and various $\phi$. 
Figure 7.9 Velocity profiles in the middle of the duct with $Ra=30000$, $\phi=0.1$ and various $M$

Figure 7.10 Temperature profiles in the middle of the duct with $Ra=30000$, $\phi=0.1$ and various $M$
Figure 7.11 Influence of the nanoparticle volume fraction $\phi$ on $f \ Re$ for $M = 10$ with different $Ra$

Figure 7.12 Influence of the nanoparticle volume fraction $\phi$ on $Nu$ for $M = 10$ with different $Ra$
Figure 7.13 Influence of the nanoparticle fraction $\phi$ on $f \Re$ with $Ra = 30000$ and different $M$

Figure 7.14 Influence of the nanoparticle fraction $\phi$ on $Nu$ with $Ra = 30000$ and different $M$
CHAPTER 8
GENERAL CONCLUSIONS AND FUTURE DIRECTIONS
8.1. GENERAL CONCLUSIONS

Before we give the results concluded in various chapters of the thesis, we state that the governing parameters of the problems discussed in this thesis have been mentioned in the relevant chapters. These parameters are all dimensionless groups of material and flow properties, and/or geometric dimensions of the domain. The traditional way of studying the different physical aspects of the fluid dynamics problems is to specify the values of these dimensionless groups rather than specifying the particular fluid properties and the domain dimensions. Obviously, the results thus obtained are applicable to the flow problems with particular values of material properties and the dimensions of the domain, falling in the ranges considered in the studies carried out in this thesis. Therefore, the results described below are true only for the range of parameters considered in the relevant chapters of the thesis.

Chapter 1 of the thesis has been devoted to the basic definition and concepts. In chapter 2, we numerically investigated the steady laminar two dimensional MHD nonlinear boundary layer slip flow of an incompressible viscous fluid due to a stretching cylinder, taking the viscous dissipation and the radiation effects into consideration. We used a similarity transformation for converting the governing Navier-Stokes equations into a set of nonlinear ordinary ones of second or third order. The order reduction approach was employed in which the third order equation was first written as a system of coupled first and second order equations, whereas the central difference approximations were used for the derivatives appearing in the governing equations. We employed the exponential transformation to compress the computational domain of the problem, as the domain was too large for the direct numerical simulation due to the algebraic decay. For the numerical solution, we have used the domain truncation technique in which the semi-infinite domain was replaced by a finite domain whose length was chosen in such a way that an increase in the
length did not have any remarkable influence on the numerical solution. Further, the length of the domain was noted to depend upon the parameters of the problem and, therefore, it was different for the different sets of the parameters. We observed that the velocity and temperature profiles obtained in this way were compatible with their asymptotic behavior. The governing parameters affecting the flow and heat transfer characteristics of the problem, were the normalized slip factor, the magnetic parameter, the stretching Reynolds number, the Eckert number, the radiation number and the Prandtl number. We noted that the slip factor had the tendency to reduce the shear stress whereas both the Reynolds number and the magnetic parameter increased it remarkably. We further noted that the viscous dissipation could cause a change in the direction of heat transfer on the cylinder, which was supported by the external magnetic field and the Prandtl number whereas the thermal radiation opposed it. Viscous dissipation should, therefore, be taken into consideration while studying the boundary layer flows around the cylinders. Moreover, the higher values of both the slip and the Reynolds number had the ability to reduce the thermal reversal, and could even eliminate it. Both the radial and axial velocity profiles were lowered by the slip factor while increasing the thermal boundary layer. The velocity boundary layer, on the other hand, was also remarkably reduced by the slip factor, which could be more prominently observed in the pressure profiles. Effect of the magnetic parameter on the profiles were similar to that of the slip, with the exception that the former supported the thermal reversal. The velocity and temperature profiles decreased as the Reynolds number increased, while thinning the velocity and thermal boundary layers. Both the Eckert number and the radiation number increased the thermal boundary layer but the former caused the thermal reversal whereas the later opposed it. Finally, the details of the computational procedure given in the chapter
may also be beneficial for the researchers working on the external flows in different geometries (for example, linearly or nonlinearly stretching/shrinking sheets, rotating and/or expanding disks etc.).

We presented the numerical investigation of the problem of steady laminar boundary layer stagnation point flow of a micropolar fluid towards a heated surface in the presence of viscous dissipation, in chapter 3. The transformed self-similar governing ordinary differential equations were solved by employing the computational procedure mentioned in chapter 2. The governing parameters of the problem were noted to be the micropolar parameters, the Eckert number, and the Prandtl number. For a range of values of the micropolar parameters (that is, the vortex viscosity parameter, the microinertia density parameter, and the spin gradient viscosity parameter), the Eckert number and Prandtl number, we presented the shear and couple stresses, the heat transfer rate, the velocity and thermal boundary layer thicknesses, and the velocity and the microrotation fields. We noted that an increase in the Eckert number caused the thermal reversal in the present problem as well. It was obvious that an increase in the values of the micropolar parameters not only decreased the heat transfer rate on the sheet but it could also prevent the thermal reversal. The micropolar structure of the fluid was further responsible for the reduction in the shear stress whereas an opposite effect was also observed for the couple stress. Moreover, the micropolar parameters decreased both the normal and streamwise velocity profiles while increasing not only the velocity boundary layer thickness but also the microrotation distribution across the domain. On the other hand, the micropolar parameters lowered the temperature profiles near the surface whereas an opposite effect was observed away from the surface. Thermal boundary layer thickness also increased with an increase in the values of the micropolar parameters.
In chapter 4, we numerically studied the steady flow and heat transfer characteristics of a viscous incompressible electrically conducting micropolar fluid in a channel with shrinking walls. We were mainly interested in numerically solving the transformed self-similar 4th order nonlinear ordinary differential equations. The numerical approaches we chose were different from the classical shooting methodology, in which the nonlinear ODEs were written in the form of a first order initial value problem, and were then solved numerically by employing a combination of Runge-Kutta method (to solve the 1st order ODEs) and a four dimensional zero finding algorithm (to find the missing conditions). We pointed out that a serious shortcoming of the shooting was the blowing up of the solution, before the initial value problem was completely integrated, and this happened quite often even with very accurate guesses for the initial conditions. That is why, we needed some alternative approach which did not require finding any unknown. In this chapter, we discussed two alternative approaches: one was based on Quasi-linearization while, in the second one, we included pseudo transient terms in the governing equations and looked for the steady state solution. In quasi-linearization, we constructed the sequences of linear ODEs whose numerical solution vectors converge to the numerical solutions of the governing equations. To construct the sequences, we linearized the equations by retaining only the first order terms. For solving the ODEs, we replaced the derivatives with their central difference approximations, giving rise to the sequences of linear systems. The sequences were generated until the relative difference between two consecutive vectors of the sequence was less than a desired tolerance. It was noted that the coefficient matrix for the fourth order ODE was pentadiagonal & not diagonally dominant, and hence the SOR method might fail or work very poorly. Therefore, some direct method like LU factorization or
Guassian elimination with full pivoting (to ensure stability) was noted to be more suitable for this purpose. In the second approach, we included a pseudo transient term to each of the governing equations, which were solved by treating the linear terms implicitly & nonlinear terms explicitly. Further, the forward and the central difference approximations for the dimensionless time and spatial derivatives were employed, respectively. The resulting linear algebraic systems were solved at every time level which, being diagonally dominant, allowed us to use the SOR method with the solution at previous time level as an initial guess. Moreover, for both the methods (Quasi-linearization and pseudo transient), we improved the order of accuracy of the solution by solving the problem again on the grids with smaller step sizes, and then using the Richardson extrapolation which was carried out at not only the common grid points but also at the skipped points. Moreover, a good initial guess for the solution on the finer grid could also be obtained by injecting the previous coarse grid solution, by using the operator mentioned in chapter 2. From the numerical simulations, we observed that the external magnetic field not only increased the shear and couple stresses but also enhanced the heat transfer rate at the channel walls. Moreover, the magnetic field had the tendency to balance the influence of the shrinking channel walls. The micropolar structure of the fluid increased the couple stress whereas an opposite trend was noted for the shear stress and the heat transfer rate, whereas the Prandtl number could decrease the heat transfer rate at the walls.

In chapter 5, we presented the study of flow, heat and mass transfer in an unsteady viscous incompressible water-based nanofluid (containing Titanium dioxidenanoparticles) between two orthogonally moving porous coaxial disks with suction. The governing nonlinear coupled ordinary differential equations, in the dimensionless form, were numerically solved by employing the algorithms described
in chapter 4. Due to symmetry of the problem, the numerical values for the shear stress, and the heat and mass transfer rates were given only at the lower disk. The parameters for the study were the Reynolds number, the nanoparticle volume fraction parameter, the wall expansion ratio, the Eckert number and the Schmidt number. It was noted that the wall expansion ratio was negative or positive, according to the case when the disks were approaching each other or moving away, whereas the Reynolds number was negative for suction. It was noted that the addition of the nanoparticles to the water remarkably increased the heat transfer rate at the disks, without significantly altering the shear stress and the mass transfer rate. It was also noted that the mass transfer rate was least influenced by the nanoparticle volume fraction parameter, as the parameter did not explicitly appear in the mass concentration equation. Further, only the temperature profiles across the two disks were significantly affected by the nano structure in the fluid. It has been observed that the effect of the suction was to reduce the mass transfer rate at the disks whether the disks were approaching or moving away from each other. On the other hand, the Reynolds number significantly reduced the shear stress and the heat transfer rate in case of approaching disks whereas an opposite trend was noted when the disks were moving away. Shear stress as well as the heat and mass transfer rates at the disks increased with the magnitude of the wall expansion ratio only when the disks were approaching. Moreover, the Eckert number (which characterised the viscous dissipation) was more influential in case of approaching disks. The position of viscous layer was noted to lie in the middle of the region between the two disks, due to the same amount of suctions being considered at the two disks. However, it might shift towards either of the disks for the asymmetrical case. Moreover, the axial velocity assumed its dimensionless value -1 at the lower disk and acquired its maximum value 1 at the upper one, with a point of inflection lying at
the middle (due to the symmetry of the problem) where it changed its concavity. For
the receding disks, both the axial velocity and the maximum value of the radial
velocity decreased with the Reynolds number. On the other hand, the temperature
distribution increased across the entire domain whereas the concentration profiles
were raised only in the lower half of the domain. The Reynolds number had the
tendency to eliminate the linear nature of the temperature profile by significantly
raising it, in the middle of the domain.

Chapter 6 was devoted to the numerical study of the interaction of an external
magnetic field with the flow of an incompressible, laminar, electrically-conducting,
biofluid through a porous medium inside a parallel plate channel with wall
transpiration. The flow was considered to be pulsatile and was driven by a pressure
gradient with steady and oscillatory components, whereas the wall transpiration was
modeled by considering the injection at lower the plate and the suction at the upper
one. The governing momentum equation was a combination of the Navier-Stokes
equation and the Forchheimer law. It is worthy to note that the biofluid was taken as
blood, which was treated as the casson fluid in the present problem. Initially, no wall
transpiration effects (suction/ injection) were assumed at the channel walls. Moreover,
we considered the mass transfer of a chemically reactive species in the channel. For
the numerical solution of the governing partial differential equations, we used the
three step explicit Runge-Kutta method for the time integration whereas the central
difference approximations were employed for the spatial derivatives. We noticed that
the velocity of the main flow is raised with the Reynolds number. Similar effect has
also been observed for the rheological parameter and the Darcy parameter, whereas an
opposite trend was noticed for both the Forchheimer quadratic drag parameter and the
magnetic parameter. Further, an increase in the Reynolds number resulted in
straightening the concentration profile, thus making it an almost linear function of the
dimensionless spatial variable.

In chapter 7, we considered the hydrodynamically as well as thermally
developed incompressible laminar flow of a water-based nanofluid containing silver
particles, in a vertical square duct, subject to constant heat flux per unit axial length
with constant peripheral temperature at any cross section. The wall thickness of the
duct was assumed to be negligible so that the assumption of infinite wall conductivity
in the outward direction might be more realistic, which also meant the same
temperature on the outside duct surface and on the solid-fluid interface. The fluid was
assumed to be electrically conducting with external magnetic field acting on it.
Moreover, compared with the imposed one, the induced magnetic field was assumed
to be negligible. We used both the finite difference method (FDM) and the spectral
method to solve the coupled governing equations, subject to the appropriate boundary
conditions. For the FDM, the grid was first stretched towards the boundaries by
employing a suitable transformation, due to the expected higher gradients near the
duct walls. The coupled linear algebraic systems arising due to the FD discretization
of the transformed governing equations were then solved iteratively by employing the
Successive over relaxation method (SOR). For the spectral method, the same non-
uniform stretched grid was used. We have compared the efficiency of both the
methods by comparing the CPU times taken by the two methods to solve the problem
with the same set of governing parameters on the similar grids. During the simulation
process, we noted that the convergence was quite slow when the relaxation parameter
was less than unity whereas the iterative process simply failed to converge at large
Raleigh number (up to $5 \times 10^4$) when the parameter was greater than 1.7. It was
obvious that the grid stretching significantly contributed to the efficiency of the
iterative method on different grids. Thus the use of stretched grid may be recommended for the SOR method due to the enhanced efficiency as well as the accuracy of the method. It was noted that, on the $51 \times 51$ grid, the lowest CPU time for the SOR method is nearly 0.3 seconds, whereas the CPU time required to solve the problem with the same values of the parameters using the spectral method was noted to be just about 0.0468 seconds. On the other hand, the two CPU times (for the SOR method and the spectral method) are, respectively, 4 seconds (approx.) and 0.2808 seconds on the $101 \times 101$ grid, which clearly indicated the supremacy of the Spectral method over the compact finite difference scheme. We further noted that the flow reversal due to high Raleigh number might be controlled by applying an external magnetic field of suitable strength.
8.2. POSSIBLE FUTURE WORK

In the chapter 2, we presented the numerical solution of the steady laminar two dimensional MHD nonlinear boundary layer slip flow of an incompressible viscous fluid due to a stretching cylinder. We considered the viscous dissipation as well as the radiation effects in the problem. We further assumed the presence of an external magnetic field which was being uniformly applied on the fluid, irrespective of its position w.r.t. the origin of the coordinate system being used. In order to make the problem physically more realistic, we intend to consider the spatially varying magnetic field. This will remarkably alter the mathematical modeling of the problem, but the results thus obtained will have more significant physical meanings. We have seen quite a few references on this type of problems in different geometries.

The numerical investigation of the problem of stagnation point flow and heat transfer of a micropolar fluid towards a heated surface, carried out in chapter 3 may be extended to the case of curved stretching sheet, coiled in a circle of fixed radius. A curvilinear coordinate system may be employed for the mathematical modeling of the problem. Further the governing parameters will now include the dimensionless radius of curvature which will also be taken into consideration during the numerical simulations. Physical characteristics of the momentum and thermal aspects of the problem may be explored by analyzing the fluid velocity, angular velocity, temperature profile, the skin-friction coefficient, couple wall stress and the local Nusselt number.

In chapter 4, we discussed two distinct computational approaches for the numerical solution of the self-similar nonlinear ordinary differential equations arising in the problem of micropolar fluid flow in a channel due to its wall stretching/shrinking. Differential transform method (DTM) is one of the semi numerical techniques which
has gained popularity in the recent years. We intend to compare the performance (in terms of both accuracy and efficiency) of the two techniques discussed in chapter 4, with the DTM.

We are also working on the numerical study of mixed convection in an enclosure with heat flux from some parts of its vertical walls, by using the pseudo transient method discussed in chapter 4, for the numerical solution of the complete Navier-Stokes equations in the steam-vorticity formulation. We have already developed the computer code for the problem in the MATLAB environment. As a sample output, the stream function, vorticity and temperature distribution for the problem (for certain values of the governing parameters) are given below.

![Figure 8.1](image.png)

**Figure 8.1** Temperature distribution when the lids are at rest (179)
Figure 8.2 Temperature distribution when the top lid is moving.

Figure 8.3 Temperature distribution when both the lids are moving in the same direction.
Figure 8.4 Temperature distribution when both the lids are moving in the opposite directions.

Figure 8.5 Streamlines when only the top lid is moving.
Figure 8.6 Streamlines when both the lids are moving in the same direction

Figure 8.7 Streamlines when both the lids are moving in the opposite directions
Figure 8.8 Vorticity distribution when only the top lid is moving

Figure 8.9 Vorticity distribution when both the lids are moving in the same direction
The study of momentum and thermal characteristics of a water-based nanofluid between two orthogonally moving porous disks, undertaken in the chapter 5 is intended to extend to the non-symmetric case when one disk is porous while the other is non porous. We may further consider the mass transfer of a chemically reactive species between the two disks. The nanofluid may be assumed to contain some chemically reactive species, and the mixture thus formed is homogeneous. Moreover, 1st order homogeneous and irreversible reaction may also assumed to be taking place in the fluid because of chemical reactive nature of the species.

In chapter 6, we studied the interaction of an external magnetic field with the flow through a Darcy-Forchheimer porous channel, due to an oscillatory pressure gradient, in the presence of wall transpiration considerations. We intend to extend this work by employing the Eringen’s micropolar fluid model to simulate the blood (considered as a micropolar fluid) flow in the present geometry. The boundary condition of constant wall temperature may be replaced by the condition of constant
wall heat flux. Further, a comparison may be made between the Darcy model and the Darcy-Forchheimer model of the porous medium inside the channel. We also wish to employ the spectral method for the numerical simulation of the modified problems, due to its supremacy (established in chapter 7) over the finite difference method.

Numerical study of the mixed convection in a water-based nanofluid inside a vertical square duct, presented in chapter 7 may lead to the exploration of the micropolar fluid behavior in the present geometry under the conditions of constant heat flux per unit axial length with one or more adiabatic walls. For the proposed study, we will have to incorporate two more differential equations (responsible for the two components of the microrotation vector) in the mathematical model of the problem. The study may further be extended to the circular duct with sinusoidal temperature distribution over the duct periphery.