Numerical Solution of Differential Equations by Haar Wavelets

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Dedication

Dedicated to my father (Late), beloved mother, brothers and sisters
List of publications

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Abstract

This work, addresses numerical methods based on Haar wavelets for solution of ordinary, partial and time fractional partial differential equations. In nonlinear cases, Haar wavelets is combined with Quasilinearization technique. The highest order derivative is then approximated by Haar wavelet series. The solution and lower order derivative is obtained through integration. Collocation approach is then applied for calculation of wavelets coefficients. The scheme is tested with different problems in case of ordinary differential equations. For partial differential equations the temporal part is discretized with finite differences while spatial part is approximated by one and two dimensional Haar wavelets for (1+1) and (1+2)-dimensional equations respectively. For validation, the proposed method is applied to solve various test problems. The obtained results are matched with exact and those presented before. It has been observed that the proposed method is suitable for solving (1+1)-dimensional partial differential equation. The suggested method is then combined with two dimensional Haar wavelets to solve (1+2)-dimensional problems. To check the efficiency of the scheme, variety of linear and nonlinear problems have been solved and reported good accuracy. In both cases convergence of the scheme has been discussed. Next the scheme is extended for solution of time fractional partial differential equations of constant and variable order. Computations show that proposed method has good results.
# Contents

1 Introduction ............................................. 1  
   1.1 Motivation ........................................... 1  
   1.2 Literature survey .................................... 2  
   1.3 Wavelets and its importance .......................... 4  
   1.4 Haar wavelets and their integrals ..................... 5  
   1.5 Function approximation via Haar wavelet series ......... 6  
   1.6 Thesis outline ....................................... 7  

2 Solution of nonlinear fluid flow problems ................. 8  
   2.1 Quasilinearization .................................... 8  
   2.2 Description of the method ............................ 9  
   2.3 Numerical examples ................................... 10  
   2.4 Conclusion ........................................... 18  

3 Solution of (1+1)-dimensional PDEs ........................ 19  
   3.1 Method description ................................... 21  
   3.2 Error analysis ....................................... 23  
   3.3 Test problems ........................................ 25  
   3.4 Conclusion ........................................... 48  

4 Solution of fourth order PDEs arising in Euler-Bernoulli Beam models ................. 49  
   4.1 Description of the method ............................ 50
4.2 Test problems ......................................................... 53
4.3 Conclusion .......................................................... 60

5 Solution of (1+2)-dimensional PDEs and coupled PDEs 61
  5.1 Method description ............................................... 63
  5.2 Convergence analysis ............................................ 65
  5.3 Test problems .................................................... 68
  5.4 Conclusion ........................................................ 78

6 Solution of time fractional diffusion wave equations 85
  6.1 Introduction ....................................................... 85
  6.2 Description of the method ..................................... 87
  6.3 Test problems .................................................... 89
  6.4 Conclusion ........................................................ 95

7 Solution of variable order time fractional dispersion and diffusion models 96
  7.1 Introduction ....................................................... 96
  7.2 Method description ............................................. 98
  7.3 Test problems .................................................... 101
  7.4 Conclusion ........................................................ 107

8 Conclusion ............................................................ 108
## List of Figures

### Chapter 2

2.1 Solution profile corresponding to example 2.3.1 ........................................ 12  
2.2 Solution profile corresponding to example 2.3.2 ........................................ 15

### Chapter 3

3.1 Solution profiles of example 3.3.1 (case 1), (a) : $R_e = 1, \tau = 0.001$, (b) : $R_e = 10, \tau = 0.01$, (c) : $R_e = 100, \tau = 0.01$, (d) : $R_e = 200, \tau = 0.01, \delta x = 0.0125$. ........................................ 27  
3.2 3D plot of example 3.3.1 (case 1), (a) : $R_e = 1, \tau = 0.001$, (b) : $R_e = 10, \tau = 0.01$. ........................................ 31  
3.3 Solution profile of example 3.3.1 (case 2), (a) : $R_e = 1, \tau = 0.001, R_e = 10$, (b) : $\tau = 0.01, R_e = 100, (c) : \tau = 0.01, R_e = 200, (d) : \tau = 0.01, \delta x = 0.0125$. ........................................ 35  
3.4 3D plot of example 3.3.1 (case 2), (a) : $R_e = 1, \tau = 0.001$, (b) : $R_e = 10, \tau = 0.01$. ........................................ 36  
3.5 Solution profile of example 3.3.2 at $t=0.0125$. ........................................ 37  
3.6 Propagation of single solitary wave. ......................................................... 41  
3.7 Interaction of double solitary Waves. ......................................................... 42  
3.8 Interaction of three solitary Waves. ......................................................... 45  
3.9 Undular bore (case 1). ......................................................... 45  
3.10 Undular bore (case 2). ......................................................... 46  
3.11 Soliton collision. ......................................................... 47

### Chapter 4

ix
4.1 Graphical solution of problem 4.2.1, (a): Exact an approximate at \( t = 4, \tau = 0.001 \) (b): Absolute error in (a), (c): Exact, (d): Approximate at \( t = 4, \tau = 0.01, J = 4 \).

4.2 Graphical solution of problem 4.2.2, (a): Exact an approximate at \( t = 4, \tau = 0.001 \) (b): Absolute error in (a), (c): Exact 3D plot, (d): Approximate 3D plot at \( t = 4, \tau = 0.01, J = 5 \).

4.3 Graphical solution of problem 4.2.3 with exact (a): Exact an approximate at \( t = 4, \tau = 0.001 \) (b): Absolute error in (a), (c): Exact 3D plot, (d): Approximate 3D plot at \( t = 4, \tau = 0.01, J = 5 \).

Chapter 5

5.1 Exact, approximate solutions, Error and contour plot of example 5.3.1 at \( t = 0.2, J = 4 \).

5.2 Graphical solution and absolute error of example 5.3.2 at \( t = 1, \tau = 0.01, \theta = 1/2 \) for case (i).

5.3 Graphical solution and absolute error of example 5.3.2 at \( t = 1, \tau = 0.01, \theta = 1/2 \) for case (ii).

5.4 Exact, approximate solutions, error and contour plot of example 5.3.3 at \( t = 0.5, \delta t = 0.0025 \).

5.5 Graphical solution and absolute error of example 5.3.5 at \( t = 1, \tau = 0.01, \theta = 1/2 \).

5.6 Graphical solution and absolute error of example 5.3.5 at \( t = 0.2, \tau = 0.001, \theta = 1/2 \).

5.7 Exact, approximate, error and contour plot of \( \psi(x, y, t) \) when \( t = 0.5, \tau = 0.001, R_e = 80 \).

5.8 Exact, approximate, error and contour plot of \( u(x, y, t) \) when \( t = 0.5, \tau = 0.001, R_e = 80 \).

Chapter 6

6.1 Graphical behavior of problem 6.3.1 \( t=1, \xi = 1.5 \).

6.2 Graphical behavior of problem 6.3.2 at \( t=0.3, \xi = 1.1 \).
6.3  Graphical behavior of problem 6.3.3, at t=0.5, \( \xi = 1.9 \).

Chapter 7

7.1  Solution profile of example 7.3.1 at \( t = 1, \ \theta = 1 \), with \( \tau = \frac{1}{10} \).

7.2  Solution profile of example 7.3.2 at \( t = 1, \ \theta = 1 \), with \( \tau = \frac{1}{100} \).

7.3  Solution profile of example 7.3.3 at \( t = 1 \) with \( \tau = 0.001 \).

7.4  Solution profile of example 7.3.4.
List of Tables

2.1 Comparison corresponding to example 2.3.1 ........................................ 12
2.2 Comparison of three methods corresponding to example 2.3.2 ................. 14
2.3 Comparison corresponding to example 2.3.3 ........................................ 17

3.1 Comparison of present method with exact and available results in literature of example 3.3.1 (case 1) when $t = 0.01$, $\tau = 0.0001$, $Re = 1$. ................................. 26
3.2 Comparison of present method with exact and available results in literature of example 3.3.1 (case 1) when $t = 0.1$, $\tau = 0.001$, $Re = 1$. ................................. 27
3.3 Comparison of present method, exact and method available in literature of example 3.3.1 (case 1) when $\tau = 0.01$, $Re = 10$. ................................. 28
3.4 Comparison of present method with exact and available results in literature of example 3.3.1 (case 1) when $\tau = 0.01$, $Re = 100$. ................................. 29
3.5 Comparison of present method with exact and available results in literature of example 3.3.1 (case 1) when $\tau = 0.01$, $Re = 200$. ................................. 30
3.6 Comparison between present results, exact and those available in literature for example 3.3.1 (case 2) when $t = 0.1$, $\tau = 0.001$, $Re = 1.0$. ................................. 31
3.7 Comparison between present results, exact and those available in literature for example 3.3.1 (case 2) when $\tau = 0.01$, $Re = 10$. ................................. 32
3.8 Comparison between present results, exact and those available in literature for example 3.3.1 (case 2) when $\tau = 0.01$, $Re = 100$. ................................. 33
3.9 Comparison between present results, exact and those available in literature for example 3.3.1 (case 2) when $\tau = 0.01$, $Re = 200$. ................................. 34
3.10 Comparison of present results with those available in literature of example 3.3.2 when \( t = 0.0125, \tau = 0.00025, J = 5 \). .......................... 38

3.11 Error norms and invariants of single solitary wave. .......................... 41

3.12 Invariants of two solitary wave. .................................................. 43

3.13 Interaction of three solitary waves. .................................................. 44

4.1 Absolute error at different points in problem 4.2.1. .......................... 55

4.2 Absolute error at \( x = 0.5 \) of problem 4.2.2 .................................. 56

4.3 Maximum error norms of problem 4.2.2 at different times with \( \tau = 0.001 \) .......................... 57

4.4 Maximum error norms of problem 4.2.3 at different times with \( \tau = 0.001 \) .......................... 58

5.1 Exact, approximate solutions and Error norms corresponding to example 5.3.1 when \( t = 0.2, \tau = 0.001 \) and \( J = 5 \). .......................... 79

5.2 Error norms of example 5.3.2 at \( t = 1, \tau = 0.01, \theta = 1/2 \) case (i). .......................... 80

5.3 Error norms of example 5.3.2 at \( t = 1, \tau = 0.01, \theta = 1/2 \) case(ii). .......................... 80

5.4 Exact vs approximate solutions and error norms corresponding to example 5.3.3. .......................... 81

5.5 Error norms of example 5.3.5 at \( t = 0.2, 1.0, \tau = 0.01, \theta = 1, J = 3 \). .......................... 82

5.6 Error norms of example 5.3.5 at \( t = 0.2, 0.5, \tau = 0.01, \theta = 1/2, J = 3 \). .......................... 82

5.7 Comparison of exact and approximate solutions of example 5.3.6 at different times when \( \tau = 0.0001 \). .......................... 83

5.8 Comparison of error norms of example 5.3.6 at different times and Reynolds numbers. 84

6.1 Error norms of problem 6.3.1 at different times and \( \xi \). .......................... 90

6.2 maximum error norms of problem 6.3.2 at different times and \( \xi \) and \( \tau = 0.001 \). .......................... 91

6.3 maximum error norms of problem 6.3.3, at different times and \( \xi \). .......................... 93

6.4 Absolute error at different points of 6.3.3 when \( t = 0.1, 0.5, \tau = 0.01, \xi = 1.9, J = 4 \) and \( \theta = \frac{1}{2} \). .......................... 94

7.1 Comparison of absolute error of example 7.3.1 with previous work for \( J = 6, t = 1, \tau = 0.001, \theta = 1 \). .......................... 102
7.2 Comparison of maximum absolute error of example 7.3.2 with previous work for

\[ J = 4, \ t = 1, \ \theta = 1. \] 104

7.3 Absolute error at different points of example 7.3.4 when \( t = 0.01, \ \tau = 0.001, \ J = 4 \) and \( \theta = \frac{1}{2} \). 106
Chapter 1

Introduction

1.1 Motivation

An equation which connects function and its derivatives termed as differential equation. Further differential equation is categorised in two different classes, ordinary differential equations (ODEs) and partial differential equations (PDEs). Application of ODEs seizes diverse area of science and engineering. They describe the relation of various dynamical quantities with each other. Maximum real world problems can be modeled via PDEs which initiated in 18th century when ODEs were unable to explain some problems in geometry and mechanics [1]. This subject was then developed by eminent mathematician in different directions. Waves and string phenomenons were studied by Joseph-Louis Lagrange and Leonared Euler. Daniel Bernoulli and Euler initially investigated potential theory, then remarkably stretched by Adrien-Marie Legendre and Pierre-Simon Laplace. Fluid mechanics, Electromagnetic theory and Quantum mechanics phenomenons based on well known PDEs namely: Navier-Stokes, Maxwell’s, and Schrodinger equations respectively. For more information regarding applications of PDEs we refer to [1]. Due to widespread application of differential equations there exits lot of analytical methods to solve these equations. Generically closed form solution for all kind of differential equations are quite complicated, therefore numerical methods are needed to solve these equations.
1.2 Literature survey

In earlier work several semi-analytical methods exist for solution of differential equations. Some well-known methods are Perturbation Method [2], Homotopy Analysis method [3], Homotopy Perturbation Method [4], Adomian Decomposition Method [5], Optimal Homotopy Asymptotic Method (OHAM) [6] etc. Similarly, in numerical methods finite difference methods (FDM) are easy and popular for solution of differential equations. In 1768, Euler applied FDM for solution of differential equations but has not given more emphases up to the start of 20th century [7]. Quoting from [8] Runge published a paper in 1895, on the extension of Euler approximation and claimed good accuracy. The innovative work for solution of PDEs was performed by Courant et al. [9] in 1928. In first half of 20th century these methods were extended to time dependent PDEs in which John Von Neumann, Crank and Nicolson contributed well [10,11]. The use of FDM are not only beneficial in the study of integer order calculus, it is equally important in study of fractional calculus (FC). The idea of FC was originated in a scientific discussion of L’Hopital and Leibniz in 1696, and was extended by Liouville and Riemann [12]. Application of FDM are suitable for solution space fractional partial differential equations (SFPDEs) as well as time fractional partial differential equations (TFPDEs). Researchers have applied different approaches for numerical solution of fractional PDEs in last few decades. Diethelm et al. [13] used predictor-corrector method for TFPDEs. Chen et al. [14] proposed an implicit method for time fractional Fokker-Planck equation. The authors in [15] studied time fractional diffusion equations via FDM. For more details on the study of SFPDEs, TFPDEs and time-SFPDES one can see [16–21]. Although FDM are very efficient and easy to apply in regular domain but are not more accurate in case of irregular domains.

To overcome this difficulty finite element methods (FEM) are very flexible and easy to apply in complex geometry. Actually in FEM the domain is sub divided in to small pieces known as elements, then variational or weak formulation approach can be used for approximation of differential equation. Rayleigh studied variational approach for boundary value problem in 1894 and 1896, Ritz applied the same approach and developed Rayleigh-Ritz method [11]. Courant [22] used this
method with triangular element for equilibrium and vibrational problems, which was considered pioneer work on FEM in 1943. Later on FEM was developed by Turner et al. [23], where they solved some engineering problems.

Meshless methods have been introduced in 1970. Hardy [24] solved some problems in geodesy in 1971 by using multiquadric radial functions. Later on the Hardy work was extended by Kansa [25, 26]. Gingold and Monaghn [27] proposed a meshless method known as smooth particle hydrodynamics method to solve some astrophysical problems. Meshless methods have given more emphases during 1990s. Different generalization of meshless methods have been developed such as meshless local Petrove-Galerkin method [28], element free Galerkin method [29] etc. Haq and Marjan contributed well using different meshless methods (see [30–35]). In meshless methods specially in application of radial basis functions the selection of shape parameter is a critical problem.

To remove this ambiguity, wavelet based numerical methods gain great importance to solve differential equations since 1990. These wavelets contain different families like Daubechies, Symlet, Coiflet, etc. Wavelets have some interesting properties such as to detect singularities and express the function in different resolution level for the purpose of accuracy. In this direction Lazaar et al. [36] investigated wavelets technique for numerical solution of PDEs. Comincioli et al. [37] studied evolutions equations numerically via wavelets. Tangborn [38] adopted wavelet transform for approximation of Kalman filter system. In [39] Luo et al. introduced an algorithm based on Gaussian wavelets for vibrational modelling. Recently Oruc [40] applied Hermite wavelets for solution of two dimensional PDEs. For more information about wavelets one can see [41–45]. Amongst different wavelets families Haar wavelets are the simplest. Haar wavelets consist of square box function. Primarily the idea of these functions was presented by Alfred Haar in 1910 but they have been applied to calculus problems since 1997. Haar functions are quite simple mathematically but are discontinuous at the partitioning points of the interval and hence not differentiable at these points. Due to this reason the direct implementation of Haar functions for differential equations is not possible. To avoid this difficulty Cattani [46, 47] used interpolating splines to regularize Haar wavelets. Alternate approach was used by Chen and Hasio [48]. They endorsed to approximate the
highest order derivative with Haar wavelet series instead of solution. The lower order derivatives and solution can be obtained through repeated integrations. Later on this technique was applied to solve variety of differential equations. Lepik [49, 50, 52–55] introduced a numerical method for the solution of ODEs, PDEs, and Integral equations using one and two dimensional Haar wavelets. The same author [56] applied non-uniform Haar wavelets for solution of differential equations where abrupt changes occurs. Jiwari [57] used Haar wavelets coupled with quasilinearization for solution of Burgers’ equation. Mittal et al. [58] studied system of viscous Burgers’ equations with the help of Haar wavelets. Oruc [59] applied finite difference hybrid scheme combined with Haar wavelets for the solution of modified Burgers’ equation. Kumar [60] solved system of Burgers’ equations by finite difference Haar wavelet technique. Somayeh et al. [61, 62] investigated semi-analytical approach for solving Hunter-Saxton and foam drainage equations. The same authors [63] implemented Haar wavelets based scheme for the solution of two dimensional system of PDEs. Mittal and Pandit [64] solved unsteady squeezing nanofluid problems via Haar wavelets.

1.3 Wavelets and its importance

Wavelet word is basically derived from French word “ondelette” which means ”small wave”. Simply wavelet is mathematical function localized both in time and frequency. Therefore, wavelets have numerous application in different area of science and engineering. We encounter some of them below.

- In signal processing the less important part of a whole signal can be removed with a transformation using wavelets which is known as compression application. This simply tells that a signal can be decompose in to low and high frequency part from which the useful one can be used.

- Fourier transform is a powerful tool for data analysis but it does not predict abrupt changes efficiently because it represents the data in the form of sinusoidal wave which is not localized in time. This can be done with wavelets because they are localized in time.
• In music one can obtain the information about frequency using Fourier transform, but cannot say anything about the tune. This can be done with the aid of wavelets.

• Wavelets can be applied in graphics for radiosity and for the representation of surfaces and curves.

• The most common application of wavelets in the field of numerical analysis is the solution of differential, integral and integro-differential equations, which is quite new area of research nowadays.

1.4 Haar wavelets and their integrals

Let us assume \( x \in [A, B] \). Next the interval is subdivided into \( 2^M \) intervals of equal length \( \delta x = \frac{B - A}{2^M} \), where \( M = 2^J \) and \( J \) denote the maximal level of resolution. Further two parameters \( j = 0, \cdots, J \) and \( k = 0, \cdots, 2^j - 1 \) are introduced. These parameters show the integer decomposition of wavelet number \( i = 2^j + k + 1 \). The Haar wavelet family which consist of series of square waves is defined for \( i \geq 1 \) as follows [50] :

\[
\begin{align*}
H_1(x) &= \begin{cases} 
1, & x \in [A, B] \\
0, & \text{otherwise}
\end{cases} \\
H_i(x) &= \begin{cases} 
1, & x \in [\zeta_1(i), \zeta_2(i)) \\
-1, & x \in [\zeta_2(i), \zeta_3(i)) \\
0, & \text{otherwise}
\end{cases}
\end{align*}
\]

where

\[
\zeta_1(i) = A + 2k\nu \delta x, \quad \zeta_2(i) = A + (2k + 1)\nu \delta x, \quad \zeta_3(i) = A + 2(k + 1)\nu \delta x, \quad \nu = \frac{M}{2^j}.
\]

The parameter \( i \) is used to define the width of \( i^{th} \) wavelet given as follows:

\[
\zeta_3(i) - \zeta_1(i) = 2\nu \delta x = \frac{B - A}{2^j}.
\]
Chapter 1 Introduction

From Eq. (1.3) it is clear that wavelet width is inversely proportional the value of $j$. This shows that wavelet width will become narrow as $j$ increases that is why $j$ is known as dilatation parameter. Similarly, the translation parameter $k$ is helpful to locate the position of wavelets. When $k$ starting from 0 and reaches up to $2^j - 1$, then $\zeta_1(i)$ will move from initial point $A$ to final point $\frac{A+(2^j-1)B}{2^j}$.

Haar wavelets are orthogonal and satisfy the relation

$$\int_A^B H_i(x) H_s(x) dx = \begin{cases} \frac{B-A}{2^j}, & i = s \\ 0, & i \neq s. \end{cases}$$

(1.4)

To compute the numerical solution of $n^{th}$ order differential using Haar wavelets one needs the integrals of these wavelets. These integrals will appear in the repeated form which is define as

$$R_{i,\beta}(x) = \int_A^x \cdots \int_A^x h_i(z) dz^{\beta} = \frac{1}{(\beta-1)!} \int_A^x (x-z)^{\beta-1} h_i(z) dz,$$

(1.5)

where

$$\beta = 1, 2, \ldots n, \quad i = 1, 2, \ldots 2^M.$$

The analytical expression for these integral is described as follows:

$$R_{1,\beta}(x) = \frac{(x-A)^\beta}{\beta!}.$$  

(1.6)

$$R_{i,\beta}(x) = \begin{cases} 0, & x < \zeta_i(x) \\ \frac{1}{\beta!} \left[(x - \zeta_1(i))^\beta - 2(x - \zeta_2(i))^\beta + (x - \zeta_3(i))^\beta\right], & x \in [\zeta_1(i), \zeta_2(i)) \\ \frac{1}{\beta!} \left[(x - \zeta_1(i))^\beta - 2(x - \zeta_2(i))^\beta + (x - \zeta_3(i))^\beta\right] + \left[(x - \zeta_2(i))^\beta - 2(x - \zeta_3(i))^\beta + (x - \zeta_4(i))^\beta\right], & x \in \zeta_2(i), \zeta_3(i) \end{cases}$$

(1.7)

1.5 Function approximation via Haar wavelet series

Haar wavelets possess orthogonality property, therefore form basis. Any function $h(x)$ which is square integrable that is $|\int_A^B h^2(x) dx| < \infty$ in $[A,B]$ can be express as

$$h(x) = \sum_{i=1}^{\infty} \alpha_i H_i(x), \quad i = 2^j + k + 1,$$

(1.8)
where \( \alpha_i \) are unknown constants and \( H_i(x) \) are Haar wavelets. In actual computation \( h(x) \) can be written as

\[
h(x) = \sum_{i=1}^{2M} \alpha_i H_i(x). \quad (1.9)
\]

To calculate \( \alpha_i \) in Eq. (1.9), we will use collocation approach. Therefore the collocation points are described as

\[
x_p = \frac{y_{p-1} + y_p}{2}, \quad p = 1, \cdots, 2M, \quad (1.10)
\]

where \( y_p \) are grid points given by

\[
y_p = A + p\delta x, \quad p = 0, \cdots, 2M. \quad (1.11)
\]

Substitution \( x_p \), in Eq. (1.9) the resultant expression is

\[
h(x_p) = \sum_{i=1}^{2M} \alpha_i H_i(x_p). \quad (1.12)
\]

In more compact form Eq. (1.12) can be expressed as

\[
H^t = HC^t, \quad (1.13)
\]

where \( C^t \) represent coefficients matrix and \( H \) is Haar matrix evaluated at collocation points. In Eq. (1.13) there are \( 2M \) linear equations from which wavelet coefficients can be computed. Once these coefficients determined the approximation can be obtained from Eq. (1.9).

### 1.6 Thesis outline

The main theme of this dissertation, is the application of Haar wavelets to solve differential equations numerically. In Chapter 2 we give the method description for nonlinear ODEs. In Chapter 3 we suggest a mixed numerical scheme based on finite differences and Haar wavelets to solve (1+1)-dimensional PDEs. Chapter 4 is the extension of the scheme to higher order PDEs. In chapter 5, we apply two dimensional Haar wavelets coupled with finite differences to solve two dimensional linear and nonlinear PDEs and system of PDEs. In Chapter 6 and 7 the proposed scheme is applied to constant and variable order time fractional PDEs of (1+1) and (1+2)-dimensional problems.
Chapter 2

Solution of nonlinear fluid flow problems

Squeezing flow is the flow where moving flow material is compressed between the two plates. The applications of such flows cover a diverse area of engineering. Some mechanical machinery is designed to work using the principle of squeezing flows. Compression moulding processes of metals, polymers and lubrication systems involve squeezing fluids. In biological applications, valves and diarthrodial joints are examples of squeezing flows. Bubble film boundaries spread out biaxially and contract in width during foams formation which is another example of squeezing flow. Nasogastric tubes and syringes work under the influence of squeezing flows of movable disk. Some phenomena in food consumption also use squeezing flow principle. Similarly, Magnetic Hydro Dynamic (MHD) squeezing flow is equally important and is using as a lubricant to prohibit the accidental deviation of lubricant viscosity with temperature.

2.1 Quasilinearization

Quasilinearization technique is generalized form of Newton-Raphson technique for functional equations and converges quadratically [51]. Consider a non-linear $n^{th}$ order differential equation of the
Chapter 2 Numerical solution of non-linear ODEs

form

$$\psi^{(n)}(x) = u, \quad u = u \left( x, \psi, \psi^{(1)}, \psi^{(2)}, \ldots, \psi^{(n-1)} \right), \quad x \in \Omega$$  \hspace{1cm} (2.1)

with initial and boundary conditions

$$\psi^{(i-1)}(x) = \mu_i, \quad x \in \partial \Omega,$$  \hspace{1cm} (2.2)

where $\psi^i$ represents $i^{th}$ derivative of $\psi$, $\mu_i$ are given constants, $\Omega$ and $\partial \Omega$ are the domain and boundary of the domain respectively. According to [65] the Quasilinearization procedure reduces Eq. (2.1) to

$$\psi^{(n)}_{\gamma+1}(x) = u + \sum_{i=0}^{n-1} \left( \psi^{(i)}_{\gamma+1} - \psi^{(i)}_{\gamma} \right) \frac{\partial^i u}{\partial \psi^{(i)}_{\gamma}}.$$  \hspace{1cm} (2.3)

The above equation is linear $n^{th}$ order differential equation equation in iterated form. In Eq. (2.3) $\gamma$ stands for the number of iteration. It is clear from Eq. (2.3) that the value of $\psi$ at $(\gamma + 1)^{th}$ iteration can be computed if it is known at $\gamma^{th}$ iteration. In the following section a brief discussion of the Haar wavelets collocation method (HWCM) is given which will be used for the solution of nonlinear problems.

2.2 Description of the method

This section is devoted to explain the proposed method for non-linear ODEs. Instead of the solution we approximate the highest order derivative in Eq. (2.3) by Haar wavelets series as [52]:

$$\psi^{(n)}_{\gamma+1}(x) = \sum_{i=1}^{2M} \alpha_i \mathbb{H}_i(x).$$  \hspace{1cm} (2.4)

where $\alpha_i$ are unknowns and $\mathbb{H}_i(x)$ are Haar wavelets. Integrating Eq. (2.4) $(n - q)$ times one can get

$$\psi^{(q)}_{\gamma+1}(x) = \sum_{i=1}^{2M} \alpha_i \mathcal{R}_{i,n-q}(x) + \sum_{j=0}^{n-q-1} \frac{1}{j!} [x - x^*]^j \mu^{(q+j)},$$  \hspace{1cm} (2.5)

where $x^*$ is the initial point of the domain. Using Eqs. (2.4), (2.5) in Eq. (2.3) and then using collocation points $x_p$ (see Eq.(1.10)), we get a system of $2M$ linear equations in unknowns $\alpha_i$.  


Once the values of these unknowns are computed, the solution of the problem can be obtained from Eq. (2.5) with \( q = 0 \). It is to be noted that the HWCM method requires the initial conditions of \( \psi^{(i)}(x^*) \), \( i = 0, 1, 2, \ldots, n - 1 \). If some of them are missing then they can be computed with the help of any other kind of known conditions. For example if the conditions \( \Psi^{(r)}(x_v) = \Gamma_v \) for \( x_v \in \Omega, r < n \) is any non-negative integer, are known instead of \( \psi^{(r)}(x^*) \). We need to find \( \psi^{(r)}(x^*) \).

From Eq. (2.5) one can obtain the system of linear equation corresponding to \( x_v \) as

\[
\Gamma_v = \sum_{i=1}^{2M} \alpha_i R_{n-q}(x_v) + \sum_{j=0}^{n-q-1} \frac{1}{j!} [x_v - x^*]^j .
\]

(2.6)

The solution of this system will give the required conditions.

### 2.3 Numerical examples

Here we apply the proposed method to three different problems namely: Squeezing Flow problem, MHD squeezing flow problem and problem of Third Grade Fluid Flow on Moving Belt. These problems are nonlinear and have wide range of applications in different fields. For nonlinearity quasilinearization technique is applied. The results obtained are compared with those available in the literature. From the computations it is evident that such kind of nonlinear problems can be solved easily with the help of proposed method.

#### Example 2.3.1

Consider the following squeezing flow problem problem [66]

\[
\psi^{(4)}(x) + R_e \psi(x)\psi'''(x) = 0, \quad \psi(0) = 0, \quad \psi''(0) = 0, \quad \psi(1) = 1, \quad \psi'(1) = 0, \quad (2.7)
\]

where \( R_e \) is the Reynolds number. Applying the quasilinearization to Eq. (2.8) we can write

\[
\psi^{(4)}_{\gamma+1}(x) + R_e \psi_{\gamma+1}\psi''_{\gamma}(x) + R_e \psi'''_{\gamma+1}(x)\psi_{\gamma+1}(x) = R_e \psi_{\gamma}(x)\psi'''_{\gamma}(x),
\]

\[
\psi_{\gamma+1}(0) = 0, \quad \psi''_{\gamma+1}(0) = 0, \quad \psi_{\gamma+1}(1) = 1, \quad \psi'_{\gamma+1}(1) = 0.
\]

(2.8)
Chapter 2 Numerical solution of non-linear ODEs

Approximating the highest order derivative in Eq. (2.8) by Haar wavelets series and then integrating four times lead to

\[
\psi_{\gamma+1}^{(4)}(x) = \sum_{i=1}^{2M} \alpha_i H_i(x),
\]

\[
\psi_{\gamma+1}^{j}(x) = \sum_{i=1}^{2M} \alpha_i R_i,4-j(x) + \sum_{i=j}^{3} \frac{x^{i-j}}{(i-j)!} \psi_{\gamma+1}^{(i)}(0), \quad j = 0, 1, 2, 3. \tag{2.9}
\]

As discussed earlier HWCM needs initial conditions for its computations. In Eq. (2.8) only two initial conditions are known, therefore, we require two more initial conditions which have been calculated from Eq. (2.9) and are given as follows:

\[
\psi_{\gamma+1}'(0) = \frac{1}{2} \left\{ 3 + \sum_{i=1}^{2M} \alpha_i R_{i,3}(1) - 3 \sum_{i=1}^{2M} \alpha_i R_{i,4}(1) \right\}
\]

\[
\psi_{\gamma+1}'''(0) = -3 \left\{ 1 + \sum_{i=1}^{2M} \alpha_i R_{i,3}(1) - \sum_{i=1}^{2M} \alpha_i R_{i,4}(1) \right\}. \tag{2.10}
\]

Putting values from Eqs. (2.9) and (2.10) in (2.8) and then collocation points \(x_p\) (see Eq. (1.10)), the following system of linear equations can be obtained

\[
\sum_{i=1}^{2M} \alpha_i \left[ \left\{ \frac{1}{2} R_e (x_p - x_p^3) R_{i,3}(1) - \frac{1}{2} R_e (3x_p - x_p^3) R_{i,4}(1) - R_e R_{i,4}(x_p) \right\} \psi_{\gamma+1}'''(x_p) \right.
\]

\[
+ \left\{ R_e R_{i,1}(x_p) - 3 R_e R_{i,3}(1) + 3 R_e R_{i,4}(1) \right\} y_{\gamma}(x_p) + H(x_p) \right]
\]

\[
= R_e \psi_{\gamma}'''(x_p) y_{\gamma}(x_p) - \frac{3}{2} R_e x_p \psi_{\gamma}'''(x_p) + \frac{1}{2} R_e x_p^3 \psi_{\gamma}'''(x_p) + 3 R_e \psi_{\gamma}(x_p). \tag{2.11}
\]

The system of equations (2.11) have been solved for the unknowns \(\alpha_i, \quad i = 1, 2, \ldots, 2M\). Using values of parameters \(J = 4, \quad R_e = 1, 3\), and unknowns the required solution has been obtained from Eq. (2.9). In order to start the approximation we have used

\[
\psi_0(x_p) = 0, \quad \psi_0'''(x_p) = 0, \quad p = 1, 2, \ldots, 2M.
\]

In Table 2.1 the results with \(R_e = 1, 3\), \(J = 4\) are shown and are compared with results of OHAM [66]. From the table it is clear that the results of proposed technique are in good agreement
Table 2.1: Comparison corresponding to example 2.3.1

with those available in literature. The solution profile are given in Fig. 2.1 which also agree mutually.

\[ R_e = 1 \quad \quad \quad \quad \quad R_e = 3 \]

Figure 2.1: Solution profile corresponding to example 2.3.1
Example 2.3.2

Here we take the differential equation for MHD squeezing flow [67]

\[ \psi^{(4)}(x) + \Re \psi(x) \psi'''(x) - M_e^2 \psi'(x) = 0, \quad \psi(0) = 0, \quad \psi''(0) = 0, \quad \psi(1) = 1, \quad \psi'(1) = 0, \quad (2.12) \]

Applying quasilinearization Eq. (2.12) takes the form

\[ \psi^{(4)}_{\gamma+1}(x) + \Re \psi_{\gamma}(x) \psi'''_{\gamma+1}(x) - M_e^2 \psi'_{\gamma+1}(x) + \Re \psi'''_{\gamma}(r) \psi_{\gamma+1}(x) = \Re \psi_{\gamma}(x) \psi'''_{\gamma}(x), \]

\[ \psi_{\gamma+1}(0) = 0, \quad \psi''_{\gamma+1}(0) = 0, \quad \psi_{\gamma+1}(1) = 1, \quad \psi'_{\gamma+1}(1) = 0. \quad (2.13) \]

Following the procedure as adopted in Example 2.3.1 the system of equations so obtained is

\[ \sum_{i=1}^{2M} \alpha_i \left[ H_i(x_p) - M_e^2 R_{i,2}(x_p) + \Re R_{i,1}(x_p) + 3M_e^2 x_p \{ R_{i,3}(1) - R_{i,4}(1) \} ight. 
\]

\[ -3 \Re \left\{ R_{i,3}(1) - R_{i,4}(1) \right\} \psi_{\gamma}(x_p) + \frac{\Re}{2} \left\{ x_p - x_p^3 \right\} R_{i,3}(1) \psi'''_{\gamma}(x_p) 
\]

\[ + \frac{\Re}{2} \left\{ x_p^3 - 3x_p \right\} R_{i,k}(1) + \Re R_{i,4}(x_p) \psi'''_{\gamma}(x_p) \right] 
\]

\[ = \Re \psi_{\gamma}(x_p) \psi'''_{\gamma}(x_p) - 3M_e^2 x_p + 3 \Re \psi_{\gamma}(x_p) + \frac{\Re}{2} \left\{ x_p^3 - 3x_p \right\} \psi'''_{\gamma}(x_p). \]

Eq. (2.14) is a system of \( 2M \) equations in so many unknowns. The obtained system has been solved for \( \Re = 1, 5, \quad M_e = 1, 5, \quad J = 4 \) and then the required solution is computed from Eq. (2.9).

It is to noted that the following starting approximation has been used in the computations

\[ \psi_0(x_p) = 0, \quad \psi'''_0(x_p) = 0, \quad p = 1, 2, \ldots, \quad 2M. \]

The results obtained with the help of HWCM and are given in Table 2.2. From the table it is evident that the results of HWCM, and that of RK4, available in literature [67], are in good agreement with each other. The solution profile using the proposed technique are shown in Fig. 2.2.
### Chapter 2 Numerical solution of non-linear ODEs

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Table 2.2: Comparison of three methods corresponding to example 2.3.2

14
Example 2.3.3

Consider differential equation of thin film flow of third grade fluid with moving belt [68]

\[ \psi''(x) + 6U \left\{ (\psi'(x))^2 \right\} \psi''(x) = m, \quad \text{with} \quad \psi(0) = 1, \quad \psi'(1) = 0, \quad (2.15) \]

where \( m = \frac{\tilde{\rho} \tilde{\kappa}^2}{\tilde{\sigma} U_0} \), \( U = \frac{(U_1 + U_2)}{\kappa \sigma^2} U_0 \), \( \psi \) is velocity of fluid, \( \tilde{\rho} \) is density, \( \tilde{\kappa} \) is dynamic viscosity, \( \tilde{\sigma} \) is thickness of fluid film, \( U_0 \) is the speed of belt, and \( U_1, U_2 \) are material constants. After the quasilinearization the Eq. (2.15) can be written as:

\[ \psi''(x) + \frac{12mU}{1 + 6U} \left\{ \psi'(x) \right\}^2 \psi''(x) = \frac{m}{1 + 6U} \left\{ \psi'(x) \right\}^2 + \frac{12mU}{1 + 6U} \left\{ \psi'(x) \right\}^2, \quad (2.16) \]

with

\[ \psi_{\gamma+1}(x) = 1, \quad \psi'_{\gamma+1}(x) = 0. \]

Approximating the highest order derivative by Haar wavelets series and integrating twice we have

\[ \psi''_{\gamma+1}(x) = \sum_{i=1}^{2M} \alpha_i \mathcal{H}_i(x), \]

\[ \psi^{(j)}_{\gamma+1}(x) = \sum_{i=1}^{2M} \alpha_i \mathcal{R}_{i,2-j}(x) + \sum_{i=j}^{1} \frac{x^{i-j}}{(i-j)!} \psi^{(i)}_{\gamma+1}(0), \quad j = 0, 1. \quad (2.17) \]
Using the condition at $x = 1$ in Eq. (2.17) one can write

$$
\psi_{\gamma+1}'(0) = -\sum_{i=0}^{2M} \alpha_i R_{i,1}(1).
$$  

(2.18)

Putting values from Eq. (2.17) and Eq. (2.18) in Eq. (2.16) and then the collocation points $x_p$ leads to the following system of linear equations

$$
\sum_{i=1}^{2M} \alpha_i \left[ \mathbb{H}(x_p) + \frac{12mU \psi'_\gamma(x_p)}{\{1 + 6U \psi'_\gamma(x_p)\}^2} R_{i,2}(x_p) - \frac{12mU \psi'_\gamma(x_p)}{\{1 + 6U \psi'_\gamma(x_p)\}^2} R_{i,2}(1) \right] = \frac{m}{1 + 6U \{\psi'_\gamma(x_p)\}^2} + \frac{12mU \{\psi'_\gamma(x_p)\}^2}{\left[1 + 6U \{\psi'_\gamma(x_p)\}^2\right]^2}, \quad p = 1,2,\ldots,2M.
$$  

(2.19)

Eq. (2.19) represents a system of $2M$ equations which has been solved for unknowns using $m = 0.5, U = 0.5$ and the required approximate solution has been obtained from Eq. (2.17). For initial approximation the following values have been taken

$$
\psi_0(x_p) = 1, \quad \psi'_0(x_p) = 0, \quad p = 1,2,\ldots,2M.
$$

The results of Example 3.3.3 with the aid of proposed method and comparison with OHAM [68] are listed in Table 2.3 which illustrate conformity of the method.
Chapter 2  
Numerical solution of non-linear ODEs

\[ Re = 0.5 \quad U = 0.5 \quad J = 5 \]

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Table 2.3: Comparison corresponding to example 2.3.3
2.4 Conclusion

In this chapter, a numerical method based on Haar wavelets has been implemented to compute approximate solutions of two squeezing flow problems and one third garde fluid problem. All these three problems are nonlinear. Quasilinearization method has been used to tackle the nonlinearity. It has been noted that such kind of nonlinear problems can easily be solved with the help of proposed method.
Chapter 3

Solution of (1+1)-dimensional PDEs

In this chapter, we study the numerical solution of the (1+1)-dimensional time dependent PDEs of the following types:

Type 1

\[
\psi_t + c_1 \psi \psi_x + c_2 \psi \psi_{xx} = \frac{c_3}{Re} \psi_{xx} + c_4 (\psi_x)^2, \quad x \in \Phi, \; t > 0,
\]

associate with initial and boundary conditions

\[
\begin{aligned}
\psi(x, 0) &= \chi(x), \quad x \in \Phi \\
\psi(x, t) &= \varsigma_1(t), \quad \psi(x, t) = \varsigma_2(t), \quad x \in \partial \Phi, \; t > 0,
\end{aligned}
\]

where \( \Phi = [a, b], \; \partial \Phi \) denote domain and boundary of the domain respectively, \( c_i (i = 1, 2, 3, 4) \), are real constants and \( Re \) represent Reynolds number.

Type 2

\[
\psi_t + \psi \psi_y - \mu \psi_{yyt} = 0, \quad y \in \Phi, \; t > 0,
\]

with initial and boundary conditions

\[
\begin{aligned}
\psi(y, 0) &= \chi_1(y), \quad y \in \Phi \\
\psi(a, t) &= \varsigma_3(t), \quad \psi(b, t) = \varsigma_4(t), \quad t > 0.
\end{aligned}
\]
If $c_1 = c_3 = 1$, $c_2 = c_4 = 0$, Eq. (3.1) reduces to nonlinear Burgers’ equation. Also for $c_1 = c_3 = 0$, $c_2 = -1$, $c_4 = 1$, it reduces to Boussinesq equation arising in infiltration phenomena. Similarly Eq. (3.3) is well-known Equal Width (EW) equation.

Burgers’ equation was initiated by Bateman [69] in 1915. Later on in 1948 Burgers’ [70, 71] studied this equation for serval structures of turbulent fluid. Burgers’ equation is equivalent to Navier-Stokes equations and chiefly containing convection and diffusion terms. The ground water flows serve a vital importance in different fields such as fluid dynamics, biomathematics, chemical engineering and environmental problems. The infiltration phenomena was observed by Boussinesq in 1903. Mathematical form of this process give rise Boussinesq equation. Burgers’ and Boussinesq equations have been studied by many authors with the help of different numerical methods such as: cubic B-spline finite elements [72], Chebyshev spectral method [73], quartic B-spline differential quadrature method [74], FEM [75], a parameter-uniform implicit difference scheme [76], a novel numerical scheme [77], HAM [78], semi-implicit finite difference method [79], variational iteration technique [80], FDM for Boussinesq equation [81] etc. The EW equation was initially studied by Morrison et al. [82] for modeling of waves in shallow water channel. The EW equation has been solved analytically by different authors using specific boundary conditions. Gardner et al. [83] studied the interaction of solitary waves using cubic B-spline finite element method. The same author [84] also studied undular bore behavior of EW equation by Petrov Galerkin method using quadratic B-spline spatial finite elements. Archilla [85] applied spectral fourier discretization technique for the numerical solution of EW equation. Khalifa and Raslan [86] investigated invariant imbedding method together with finite differences for analysis of Maxwellain conditions and single solitary wave for this equation. Collocation method using quintic B-spline finite elements was proposed by Raslan [87] to study single, interaction of solitary waves and undular bore phenomena. Dogan [88] implemented Galerkin method with linear B-spline (shape functions) for studying single solitary wave and development of undular bore in EW equation. Zaki [89] studied EW equation applying least-squares method combined with space-times linear finite elements. Zaki [90] also analyzed the train of EW solitary waves generated using boundary forcing by Petrov-Galerkin finite element scheme coupled with quadratic B-splines as trial functions. Marjan [33] implemented
radial basis functions for solving EW equation.

The aim of this chapter is to develop a numerical scheme based on finite differences and Haar wavelets for numerical solution of (1+1)-dimensional PDEs addressed in Eqs. (3.1)-(3.2) and (3.3)-(3.4). The results obtained through proposed numerical scheme are matched with earlier work. Computation shows that present method is reliable, efficient and give good results.

3.1 Method description

In this section, the proposed method has been discussed and applied for the solution of concern PDEs. For sake of simplicity we proceed for Burger' equation and similar process can be extended to Boussinesq equation equation as well. When $c_1 = c_3 = 1$, $c_2 = c_4 = 0$, Eq. (3.1) reduces to

$$
\psi_t + \psi \psi_x = \frac{1}{R_e} \psi_{xx}.
$$

(3.5)

Applying finite difference to temporal part and $\theta$-weighted $(0 \leq \theta \leq 1)$ scheme to Eq. (3.5) in the following way

$$
\frac{\psi^{j+1} - \psi^j}{\tau} + \theta (\psi \psi_x)^{j+1} + (1 - \theta) (\psi \psi_x)^j = \frac{1}{R_e} \left[ \theta (\psi_{xx})^{j+1} + (1 - \theta) (\psi_{xx})^j \right],
$$

(3.6)

where $\tau$ is the time step size and $\psi^j = \psi(x, t^j)$. The corresponding boundary conditions defined in Eq. (3.2) takes the form

$$
\psi(a, t^{j+1}) = \varsigma_1(t^{j+1}); \quad \psi(b, t^{j+1}) = \varsigma_2^{j+1}.
$$

(3.7)

To linearize the nonlinear term $(\psi \psi_x)^{j+1}$ in Eq. (3.6) applying quasilinearization technique, we obtain

$$
(\psi \psi_x)^{j+1} \approx \psi^{j+1} (\psi_x)^j + (\psi_x)^{j+1} \psi^j - \psi^j (\psi_x)^j.
$$

(3.8)

Using Eq. (3.8) in Eq. (3.6), we get

$$
\psi^{j+1} + \tau \left[ \theta \psi^{j+1} (\psi_x)^j + \theta (\psi_x)^{j+1} \psi^j - \frac{\theta}{R_e} (\psi_{xx})^{j+1} \right] = \psi^j + \tau (1 - \theta) \left[ \frac{1}{R_e} (\psi_{xx})^j - (\psi \psi_x)^j \right] + \tau \theta (\psi \psi_x)^j.
$$

(3.9)
Chapter 3  Solution of (1+1)-dimensional PDEs

Next, approximating the highest order derivative in Eq. (3.9) by Haar wavelets series as

\[(\psi_{xx})^{j+1} = \sum_{i=1}^{2M} \alpha_i \mathcal{H}_i(x). \tag{3.10}\]

Integrating Eq. (3.10) twice in domain \([a, x]\) gives

\[
\psi_x^{j+1} = \sum_{i=1}^{2M} \alpha_i \mathcal{R}_i,1(x) + [\psi_x](a)]^{j+1}
\]

\[
\psi^{j+1} = \sum_{i=1}^{2M} \alpha_i \mathcal{R}_i,2(x) + (x-a) [\psi_x](a)]^{j+1} + \psi^{j+1}(a). \tag{3.12}\]

Using boundary condition at \(x = b\) (refer to Eq. (3.2)) in Eq. (3.12), the unknown term \([\psi_x](a)]^{j+1}\) is computed as

\[
\psi_x^{j+1}(a) = \zeta_2^{j+1}(b) - \zeta_1^{j+1}(a) - \frac{1}{b-a} \sum_{i=1}^{2M} \alpha_i \mathcal{R}_i,2(b). \tag{3.13}\]

Substituting value from Eq. (3.13) in Eqs. (3.11) and (3.12), one can write

\[
\psi_x^{j+1} = \sum_{i=1}^{2M} \alpha_i \{\mathcal{R}_i,11(x) - \frac{1}{b-a} \mathcal{R}_i,2(b)\} + \zeta_2^{j+1}(b) - \zeta_1^{j+1}(a)
\]

\[
\psi^{j+1} = \sum_{i=1}^{2M} \alpha_i \{\mathcal{R}_i,2(x) - \frac{x-a}{b-a} \mathcal{R}_i,2(b)\} + (x-a) [\zeta_2^{j+1}(b) - \zeta_1^{j+1}(a)]
\]

\[
+ \zeta_1^{j+1}(a). \tag{3.15}\]

Putting values from Eqs. (3.10), (3.14) and (3.15) in Eq. (3.9) and using the collocation points, \(x_p\) (Eq. 1.10), leads to the following system of algebraic equations

\[
\sum_{i=1}^{2M} \alpha_i [Q(i,p) + \tau \theta Q(i,p) (\psi_i)_x = x_p + \tau \theta U(i,p) Y^j x = x_p - \frac{\theta \tau}{R_e} \mathcal{H}(i,p)] = \]

\[
\psi_x^j x = x_p + \tau (1 - \theta) F(p) + \tau \theta (\psi_i)_x^j x = x_p - \Psi(p)(1 + \tau \theta) - \theta \left[\zeta_2^{j+1} - \zeta_1^{j+1}\right] \psi_x^j x = x_p, \tag{3.16}\]

where

\[
Q(i,p) = \mathcal{R}_i,2(x_p) - \left(\frac{x-a}{b-a}\right) \mathcal{R}_i,2(b), \quad U(i,p) = \mathcal{R}_i,1(x) - \frac{1}{b-a} \mathcal{R}_i,2(b),
\]

\[
F(p) = \left(\frac{x-a}{b-a}\right) \psi_x^j x = x_p \left[\zeta_2^{j+1}(b) - \zeta_1^{j+1}(a)\right] + \zeta_1^{j+1}(a),
\]

\[
\Psi(p) = \frac{1}{R_e} (\psi_{xx})^j x = x_p - (\psi \psi_x)^j x = x_p, \quad \mathcal{H}(i,p) = \mathcal{H}_i(x_p).
\]
Solving the system of equations Eq. (3.16), give the wavelet coefficients $\alpha_i$. Once these coefficients figured out, then approximate solution can be obtained from Eq. (3.15). For numerical computation $\theta = \frac{1}{2}$ and the domain $[a, b] = [0, 1]$ have been used.

### 3.2 Error analysis

In this section, the error analysis of the current scheme Eq. (3.15) has been examined via asymptotic expansion. The following lemma [91] is required for the proof of the convergence theorem.

**Lemma 3.2.1**

If $\psi(x) \in L^2(R)$ with $|\psi'(x)| \leq \rho$, for all $x \in (0, 1)$, $\rho > 0$ and $\psi(x) = \sum_{i=0}^{\infty} \alpha_i \mathbb{H}_i(x)$. Then $|\alpha_i| \leq \frac{\rho}{2^{j+1}}$.

**Theorem 3.2.2**

If $\psi(x)$ and $\psi_{2^J}$ be exact and approximate solution of Eq. (3.1), then the error norm $\| E_J \|$ at $J^{th}$ resolution level is

$$\| E_J \| \leq \frac{4\rho}{3} \left( \frac{1}{2^{j+1}} \right)^2.$$  \hspace{1cm} (3.17)

**Proof**

At $J^{th}$ resolution level (refer to Eq. (3.15))

$$| E_J | = | \psi - \psi_{2^J}(x) | = \left| \sum_{i=2^{J+1}}^{\infty} \alpha_i [R_{i,2}(x) - xR_{i,2}(1)] \right|$$

$$= \left| \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^{j-1}} \alpha_{2^{j}+k+1} [R_{2^{j}+k+1,2}(x) - xR_{2^{j}+k+1,2}(1)] \right|$$  \hspace{1cm} (3.18)
From Eq. (3.18), the error norm can be obtained as

\[
\| E_J \|_2^2 = \left| \int_0^1 \left( \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \alpha_{2^j+k+1} [R_{2^j+k+1,2}(x) - xR_{2^j+k+1,2}(1)] \right)^2 \right|
\]

\[
= \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \sum_{\tilde{j}=J+1}^{\infty} \sum_{\tilde{k}=0}^{2^{\tilde{j}}-1} \alpha_{2^j+k+1} \alpha_{2^{\tilde{j}}+\tilde{k}+1} \left| \int_0^1 [R_{2^j+k+1,2}(x) - xR_{2^j+k+1,2}(1)] [R_{2^{\tilde{j}}+\tilde{k}+1,2}(x) - xR_{2^{\tilde{j}}+\tilde{k}+1,2}(1)] \, dx \right|. \quad (3.19)
\]

Using Lemma 3.2.1, the above equation can be written as

\[
\| E_J \|_2^2 \leq \rho^2 \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \sum_{\tilde{j}=J+1}^{\infty} \sum_{\tilde{k}=0}^{2^{\tilde{j}}-1} \frac{1}{2^{j+1}} \frac{1}{2^{\tilde{j}+1}} \left| \int_0^1 [R_{2^j+k+1,2}(x) - xR_{2^j+k+1,2}(1)] [R_{2^{\tilde{j}}+\tilde{k}+1,2}(x) - xR_{2^{\tilde{j}}+\tilde{k}+1,2}(1)] \, dx \right|. \quad (3.20)
\]

As Haar wavelet integral are monotonically increasing and absolute maximum will occur in the interval \( x > \zeta_3 \) (see [91]). Therefore, from (1.6) one can write

\[
R_{i,n}(x) = \frac{1}{n!} \left[ (x - \zeta_1)^n - 2(x - \zeta_2)^n + (x - \zeta_3)^n \right]
\]

\[
= \frac{1}{n!} \sum_{k=2}^{n} \binom{n}{k} (x - \zeta_2)^{n-k} \left[ \left( \frac{1}{2^{j+1}} \right)^k + \left( -\frac{1}{2^{j+1}} \right)^k \right]
\]

\[
\leq \frac{1}{n!} \sum_{k=2}^{n} \binom{n}{k} (1 - \zeta_2)^{n-k} \left[ \left( \frac{1}{2^{j+1}} \right)^k + \left( -\frac{1}{2^{j+1}} \right)^k \right]
\]

\[
= R_{i,n}(1). \quad (3.21)
\]

Above inequality implies that

\[
R_{i,n}(x) \leq R_{i,n}(1). \quad (3.22)
\]

From Eq. (3.21), we can also write

\[
R_{i,2}(1) \leq \frac{1}{(2^{j+1})^2}. \quad (3.23)
\]

Since

\[
|R_{i,2}(x) - xR_{i,2}(1)| \leq 2|R_{i,2}(1)|. \quad (3.24)
\]
Using Eqs. (3.23) and (3.24) in Eq. (3.20), the error norm reduces to
\[ \| E_J \|^2 \leq 4\rho^2 \sum_{j=J+1}^{\infty} \sum_{j=J+1}^{2^j - 1} \sum_{k=0}^{\infty} \left( \frac{1}{2^j+1} \right)^3 \left( \frac{1}{2^{j+1}} \right)^3 (1 - \zeta_1) \]
which gives
\[ \| E_J \| \leq \frac{4\rho}{3} \left( \frac{1}{2^{J+1}} \right)^2. \] (3.26)
Hence proved.

### 3.3 Test problems

Here the suggested techniques is applied to selected problems. The obtained results are matched with earlier work which shows effectiveness of the proposed scheme.

**Example 3.3.1**

**Case 1**

Consider Eq. (3.1) with initial and boundary conditions
\[ \psi(x,0) = \sin(\pi x), \quad x \in [0,1], \quad \psi(0,t) = \psi(1,t) = 0 \quad t > 0. \] (3.27)

Applying the scheme discussed earlier and replacing \( x \) by \( x_p \), we get the following system of equations
\[ \sum_{i=1}^{2M} \alpha_i \left[ Q + \frac{\tau}{2} Q (\psi_x)^3 + \frac{\tau}{2} U \psi^3 - \frac{\tau\theta}{Re} \mathcal{H}_i(x_p) \right] = \psi^3 + \frac{\tau}{2Re} (\psi_{xx})^3, \] (3.28)
where
\[ Q = \mathcal{R}_{i,2}(x_p) - x_p \mathcal{R}_{i,2}(1), \quad U = \mathcal{R}_{i,1}(x_p) - \mathcal{R}_{i,2}(1), \quad p = 1, 2, \ldots, 2M. \]

Solving the system of linear equations Eq. (3.28) gives the required unknowns \( \alpha_i \).
Table 3.1: Comparison of present method with exact and available results in literature of example 3.3.1 (case 1) when $t = 0.01$, $\tau = 0.0001$, $R_e = 1$.

<table>
<thead>
<tr>
<th>x</th>
<th>[92]</th>
<th>[93]</th>
<th>Present Method $J = 5$</th>
<th>Present Method $J = 7$</th>
<th>Exact</th>
</tr>
</thead>
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<td>0.11460</td>
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</tr>
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</tr>
</tbody>
</table>

**Case 2**

In this case we take the following initial and boundary conditions for Eq. (3.1),

$$
\psi(x,0) = 4x(x-1), \quad x \in [0,1], \quad \psi(0,t) = \psi(1,t) = 0, \quad t > 0.
$$

(3.29)

The solution has been obtained for different values of $R_e$ in both cases (Case 1 and 2). The results are compared with the results of different methods in literature and are shown in Tables 3.1-3.5 (case 1) and 3.6-3.9 (case 2). From the tables it is obvious that the present results are better than the other techniques. The solution profiles for $J = 7$ are plotted in Figs. 3.1-3.4 at different times.
Chapter 3  
Solution of (1+1)-dimensional PDEs

Table 3.2: Comparison of present method with exact and available results in literature of example 3.3.1 (case 1) when \( t = 0.1, \tau = 0.001, R_e = 1 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>[92]</th>
<th>[93]</th>
<th>Present ((J = 5))</th>
<th>Present ((J = 7))</th>
<th>Exact</th>
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</thead>
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</table>

![Figure 3.1](image-url)
Table 3.3: Comparison of present method, exact and method available in literature of example 3.3.1 (case 1) when $\tau = 0.01$, $R_e = 10$.

<table>
<thead>
<tr>
<th>x</th>
<th>t</th>
<th>[92]</th>
<th>[93]</th>
<th>Present ($J = 5$)</th>
<th>Present ($J = 7$)</th>
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Table 3.4: Comparison of present method with exact and available results in literature of example 3.3.1 (case 1) when $\tau = 0.01$, $R_e = 100$.

<table>
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<th>[93]</th>
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<th>Present ($J = 7$)</th>
<th>Exact</th>
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Table 3.5: Comparison of present method with exact and available results in literature of example 3.3.1 (case 1) when $\tau = 0.01$, $R_e = 200$.

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Table 3.6: Comparison between present results, exact and those available in literature for example 3.3.1 (case 2) when $t = 0.1$, $\tau = 0.001, R_e = 1.0$.

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Figure 3.2: 3D plot of example 3.3.1 (case 1), (a) $R_e = 1, \tau = 0.001$, (b) $R_e = 10, \tau = 0.01$. 
Table 3.7: Comparison between present results, exact and those available in literature for example 3.3.1 (case 2) when $\tau = 0.01$, $R_e = 10$.

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Table 3.8: Comparison between present results, exact and those available in literature for example 3.3.1 (case 2) when $\tau = 0.01$, $R_e = 100$.

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Table 3.9: Comparison between present results, exact and those available in literature for example 3.3.1 (case 2) when $\tau = 0.01$, $R_c = 200$.

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<td>0.02434</td>
</tr>
<tr>
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</table>
Figure 3.3: Solution profile of example 3.3.1 (case 2), (a) : $R_e = 1$, $\tau = 0.001$, $R_e = 10$, (b) : $\tau = 0.01$, $R_e = 100$, (c) : $\tau = 0.01$, $R_e = 200$, (d) : $\tau = 0.01$, $\delta x = 0.0125$. 
Example 3.3.2

In this example, we consider Boussinesq equation by choosing \( c_1 = c_3 = 0, \ c_2 = -1, \ c_4 = 1 \) in Eq. (3.1)

\[
\psi_t - \psi \psi_{xx} = (\psi_x)^2, \tag{3.30}
\]

combined with initial and boundary conditions

\[
\psi(x, 0) = 1 - x^2, \quad x \in [0, 1], \quad \psi(0, t) = 1, \quad \psi(1, t) = 0, \quad t > 0. \tag{3.31}
\]

Applying the proposed scheme, we obtain

\[
\psi^{j+1} - \tau \theta (\psi \psi_{xx})^{j+1} - \tau \theta \left[ (\psi_x)^2 \right]^{j+1} = \psi^j + (1 - \theta) \tau \left[ (\psi \psi_{xx})^j + (\psi_x)^2 \right]. \tag{3.32}
\]

Linearizing the nonlinear terms by formulas

\[
[\psi \psi_{xx}]^{j+1} = \psi^{j+1} (\psi_{xx})^2 + (\psi_{xx})^{j+1} \psi^j - (\psi \psi_{xx})^{j+1}, \tag{3.33}
\]

\[
\left[ (\psi_x)^2 \right]^{j+1} = 2 (\psi_x)^{j+1} (\psi_x)^j - (\psi_x)^2. \tag{3.34}
\]

Figure 3.4: 3D plot of example 3.3.1 (case 2), (a) : \( Re = 1, \tau = 0.001 \), (b) : \( Re = 10, \tau = 0.01 \).
The approximations of \((\psi_{xx})^{j+1}, (\psi_x)^{j+1}\) and \(\psi^{j+1}\) are given by

\[
(\psi_{xx})^{j+1} = \sum_{i=1}^{2M} \alpha_i H_i(x),
\]

(3.35)

\[
(\psi_x)^{j+1} = \sum_{i=1}^{2M} \alpha_i [R_i(x) - R_{i,2}(1)] - 1,
\]

(3.36)

\[
\psi^{j+1} = \sum_{i=1}^{2M} \alpha_i [R_{i,2}(x) - xR_{i,2}(1)] - x + 1.
\]

(3.37)

Using Eqs. (3.33)-(3.37) in Eq. (3.32), we get the following system of equations when \(x = x_p(1.10)\),

\[
\sum_{i=1}^{2M} \alpha_i \left[ Q - \frac{\tau}{2} Q (\psi_{xx})^j - \frac{\tau}{2} H_i(x_p) \psi^j - \tau U (\psi_x)^j \right]
\]

\[
= \psi^j + \left[ x - 1 + \frac{\tau}{2} (1 - x) (\psi_{xx})^j - \tau (\psi_x)^j \right], \quad p = 1, 2, \ldots, 2M,
\]

(3.38)

where \(Q\) and \(U\) are the same as given in Example 3.3.1. Here also Eq. (3.38) generates \(2M\) number of equations in so many unknowns. The system has been solved for \(\alpha_i\) and then approximate solution has been obtained from Eq. (3.37). The results computed results are compared with those of finite differences and are listed in Table 3.10. The results of the present method are in good agreement with those available in literature. Solution profile is plotted in Fig. 3.5.

![Figure 3.5: Solution profile of example 3.3.2 at t=0.0125.](image)

37
Table 3.10: Comparison of present results with those available in literature of example 3.3.2 when $t = 0.0125$, $\tau = 0.00025$, $J = 5$.

<table>
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Example 3.3.3

In this problem, we consider Eq. (3.3) with associated initial and boundary conditions. We apply Haar wavelets in the interval \([0, 1]\) for this reason the arbitrary interval \([a, b]\) should be transformed to \([0, 1]\). Applying the transformation \(x = \frac{y-a}{L}\) with \(L = b-a\), then Eq. (3.3) takes the form

\[
\psi_t + \frac{1}{L} \left[ \psi \psi_x \right] - \frac{\mu}{L^2} \left[ \psi_{xxx} \right] = 0, \quad x \in [0, 1], \quad t > 0.
\]

(3.39)

\(\psi(0, t) = \psi_1(t), \quad \psi(1, t) = \psi_2(t), \quad \psi(x, 0) = \chi_2(x).\)

Using forward difference and \(\theta\)-weighted scheme for time and spatial derivatives respectively, Eq. (3.39) at \(j^{th}\) time level transforms to

\[
\frac{\psi^{j+1} - \psi^j}{\tau} + \frac{1}{L} \left[ \theta (\psi \psi_x)^{j+1} + (1 - \theta) (\psi \psi_x)^j \right] - \frac{\mu}{L^2} \left[ (\psi_{xx})^{j+1} - (\psi_{xx})^j \right] = 0.
\]

(3.40)

Replacing the nonlinear term in Eq. (3.40) by formula

\[
(\psi \psi_x)^{j+1} \approx \psi^{j+1} (\psi_x)^j + \psi^j (\psi_x)^{j+1} - \psi^j (\psi_x)^2.
\]

(3.41)

Substituting (3.41) in (3.40) gives

\[
\psi^{j+1} + \frac{\tau \theta}{L} \left[ \psi^{j+1} (\psi_x)^j + (\psi_x)^{j+1} \psi^j \right] - \frac{\mu}{L^2} \left[ (\psi_{xx})^{j+1} - (\psi_{xx})^j \right] = 0.
\]

(3.42)

The boundary and initial conditions are

\[
\psi(0, t^{j+1}) = \psi_1^{j+1}, \quad \psi(1, t^{j+1}) = \psi_2^{j+1}, \quad \psi(x, 0) = \chi_2(x).
\]

(3.43)

The required system of algebraic can be obtained from Eq. (3.42) as discussed in example 3.3.1 by putting the values of \(\psi_{xx}, \psi_x, \psi\) and \(x = x_p\). We apply the proposed scheme to observe single solitary wave, iteration of two and three solitary waves, undular bore and soliton collision. In order to check efficiency of the technique \(L_2\) and \(L_\infty\) norms are computed which are defined as follows:

\[
L_2 = \left( \delta y \sum_{i=1}^{N} \left| \psi_i^{ext} - \psi_i^{app} \right|^2 \right)^{\frac{1}{2}}
\]

(3.44)

\[
L_\infty = \max_{1 \leq i \leq N} \left| \psi_i^{ext} - \psi_i^{app} \right|
\]
where $\psi_{i}^{\text{ext}}$ and $\psi_{i}^{\text{app}}$ are the exact and numerical solutions respectively, and $\delta y = \frac{b-a}{N}$. The invariants of the problem under consideration are given as

\begin{align*}
\Upsilon_1 &= \int_a^b \psi dy \simeq \delta y \sum_{i=1}^{N} \psi_i^2, \\
\Upsilon_2 &= \int_a^b [\psi^2 - \mu (\psi_y)^2] dy \simeq \delta y \sum_{i=1}^{N} [(\psi_i^2)^2 + \mu (\psi_y^i)^2], \\
\Upsilon_3 &= \int_a^b \psi^3 dy \simeq \delta y \sum_{i=1}^{N} (\psi_i^3)^3. \tag{3.45}
\end{align*}

**Single solitary wave solution**

In case of single solitary wave consider Eq. (3.39) combined with the following boundary and initial condition

\begin{align*}
Y(0,t) &= 0, \quad Y(1,t) = 0, \quad Y(x,0) = 3\omega \text{sech}^2(\rho(xL + a - y_o)). \tag{3.46}
\end{align*}

The exact solution of this problem is

\begin{align*}
Y(y,t) &= 3\omega \text{sech}^2[\rho(y - y_o - \omega t)]. \tag{3.47}
\end{align*}

where $\omega$ is the wave velocity, $\rho = 1/2\sqrt(\mu)$ which measure width of the wave pulse. Initial condition has been extracted from exact solution and values of the parameters used are $\mu = 1$, $\omega = 0.1$, $\rho = \left(\frac{1}{4\rho}\right)^{\frac{1}{2}}$, $J = 10$, $\tau = 0.05$, $t = 80$, $a = 0$ and $b = 30$. The exact values of the invariants in single solitary solution are $\Upsilon_1 = 1.2$, $\Upsilon_2 = 0.2880$, $\Upsilon_3 = 0.05760$. The numerical values of these invariants are shown in Table 3.11 which shows validity of the conservation laws. The error norms $L_{\infty}$ and $L_2$ are also given in Table 3.11 and the results obtained are compared with the results of [33, 83, 88, 89, 94–97] available in literature. From the table it is clear that the results obtained with the help of the present method are better as compared to the other techniques. The graphs of the solution at $t = 0, 20, 40, 60, 80$ are given in Fig. 3.6 from which the propagation of wave can be observed as the time passes.
Table 3.11: Error norms and invariants of single solitary wave.

<table>
<thead>
<tr>
<th>Method</th>
<th>t</th>
<th>$\Upsilon_1$</th>
<th>$\Upsilon_2$</th>
<th>$\Upsilon_3$</th>
<th>$L_\infty$</th>
<th>$L_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present method</td>
<td>80</td>
<td>1.19999</td>
<td>0.28799</td>
<td>0.05759</td>
<td>1.2647e-05</td>
<td>2.2685e-05</td>
</tr>
<tr>
<td>[94]</td>
<td>80</td>
<td>1.19998</td>
<td>0.28798</td>
<td>0.05759</td>
<td>5.300e-05</td>
<td>5.600e-05</td>
</tr>
<tr>
<td>[88]</td>
<td>80</td>
<td>1.23387</td>
<td>0.29915</td>
<td>0.06097</td>
<td>1.642e-02</td>
<td>2.469e-02</td>
</tr>
<tr>
<td>[95]</td>
<td>80</td>
<td>1.19995</td>
<td>0.28798</td>
<td>0.05759</td>
<td>2.100e-02</td>
<td>2.900e-05</td>
</tr>
<tr>
<td>[96]</td>
<td>80</td>
<td>1.20004</td>
<td>0.28799</td>
<td>0.05760</td>
<td>7.300e-05</td>
<td>1.250e-04</td>
</tr>
<tr>
<td>[83]</td>
<td>80</td>
<td>1.1910</td>
<td>0.2855</td>
<td>0.05582</td>
<td>2.646e-03</td>
<td>3.849e-03</td>
</tr>
<tr>
<td>[97]</td>
<td>80</td>
<td>1.20003</td>
<td>0.28801</td>
<td>0.05761</td>
<td>2.029e-05</td>
<td>3.119e-05</td>
</tr>
<tr>
<td>[89]</td>
<td>80</td>
<td>1.1964</td>
<td>0.2858</td>
<td>0.0569</td>
<td>4.373e-03</td>
<td>7.444e-03</td>
</tr>
<tr>
<td>[33][Ode113($\phi_{2,3}$)]</td>
<td>80</td>
<td>1.19399</td>
<td>0.28656</td>
<td>0.05731</td>
<td>1.362e-05</td>
<td>4.813e-05</td>
</tr>
<tr>
<td>[33][Ode113(MQ)]</td>
<td>80</td>
<td>1.19399</td>
<td>0.28656</td>
<td>0.05731</td>
<td>1.363e-05</td>
<td>5.343e-05</td>
</tr>
</tbody>
</table>

Figure 3.6: Propagation of single solitary wave.
Interaction of two solitary waves

For interaction of two solitary waves we take Eq. (3.39) together with initial and boundary conditions as follows:

\[ \psi(x, 0) = 3 \sum_{i=1}^{2} \omega_i \text{sech}^2(\rho_i(xL + a - y_i - \omega_i)), \quad \psi(0, t) = \psi(1, t) = 0. \tag{3.48} \]

The initial condition represents the sum of two solitary waves with amplitude \(3\omega_1\) and \(3\omega_2\). In numerical computations for the solution of the problem \(a = 0, b = 80, t = 30, \omega_1 = 1.5, \omega_2 = 0.75,\]
\(J = 8, y_1 = 10, y_2 = 25, \rho_1 = \rho_2 = 0.5.\) The exact values of the three invariants are \(\Upsilon_1 = 27,\]
\(\Upsilon_2 = 81\) and \(\Upsilon_3 = 218.7.\) For different values of \(t\) the invariants are calculated and are shown in Table 3.12. The results are compared with those of the methods available in literature. It can be seen that the results are in good agreements with the exact values. The profile of the solution at different time levels are given in Fig. 3.7. From the figure it can be noted that the large wave is moving faster than that of the small wave. The large wave catches the small wave and move ahead as time increases.

\[ \begin{align*}
  t &= 1 & t &= 20 & t &= 30 \\
  \begin{array}{c}
    \text{Figure 3.7: Interaction of double solitary Waves.}
  \end{array}
\end{align*} \]
### Table 3.12: Invariants of two solitary wave.

<table>
<thead>
<tr>
<th>Method</th>
<th>t</th>
<th>$\gamma_1$</th>
<th>$\gamma_2$</th>
<th>$\gamma_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present method</td>
<td>1</td>
<td>27.00001</td>
<td>80.98166</td>
<td>218.62715</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>27.00001</td>
<td>80.97718</td>
<td>218.59942</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>27.00001</td>
<td>80.97652</td>
<td>218.59257</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>27.00001</td>
<td>80.98113</td>
<td>218.62301</td>
</tr>
<tr>
<td>[95]</td>
<td>30</td>
<td>27.00003</td>
<td>81.01719</td>
<td>218.70650</td>
</tr>
<tr>
<td>[87]</td>
<td>20</td>
<td>27.02642</td>
<td>80.99261</td>
<td>218.7004</td>
</tr>
<tr>
<td>[33]RK4(MQ)</td>
<td>1</td>
<td>26.93262</td>
<td>80.79787</td>
<td>218.15590</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>26.93259</td>
<td>80.79769</td>
<td>218.15515</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>26.93269</td>
<td>80.79774</td>
<td>218.15534</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>26.93141</td>
<td>80.79791</td>
<td>218.15595</td>
</tr>
<tr>
<td>[33]Ode113(MQ)</td>
<td>30</td>
<td>26.93310</td>
<td>80.80028</td>
<td>218.16659</td>
</tr>
</tbody>
</table>
Interaction of three solitary waves

In order to study the interaction of three solitary waves the Eq. (3.39) is combined with the boundary conditions taken as in case of two solitary wave and initial conditions given by

\[ \psi(x,0) = 3 \sum_{i=1}^{3} \omega_i \text{sech}^2(\rho_i(xL + a - y_i - \omega_i)) \quad i = 1, 2, 3. \]  

(3.49)

For numerical calculations values of different parameters involved are \( a = -10, \ b = 100, \ \rho_i = 0.5 \), \( \omega_1 = 4.5, \ \omega_2 = 1.5, \ \omega_3 = 0.5, \ J = 8, \ y_1 = 10, \ y_2 = 25 \) and \( y_3 = 35 \). The exact values of the invariants are \( \Upsilon_1 = 78, \ \Upsilon_2 = 655.2 \) and \( \Upsilon_3 = 5450.4 \). The computed values are presented in Table 3.13 and are matched with the ones present in the literature. The table shows good agreement with the exact as well as the previous results of different techniques. The solution profile is shown in Fig. 3.8 for various values of time. From the graphs it is clear that with the passage of time the waves having large amplitude seizes the wave of small amplitude and moves ahead.

### Table 3.13: Interaction of three solitary waves.

<table>
<thead>
<tr>
<th>Method</th>
<th>t</th>
<th>( \Upsilon_1 )</th>
<th>( \Upsilon_2 )</th>
<th>( \Upsilon_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present method</td>
<td>1</td>
<td>78.00078</td>
<td>654.99693</td>
<td>5447.56393</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>78.00078</td>
<td>654.33043</td>
<td>5436.62289</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>78.00076</td>
<td>654.97356</td>
<td>5447.55355</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>78.00075</td>
<td>654.99500</td>
<td>5447.74203</td>
</tr>
<tr>
<td>[33]RK4(MQ)</td>
<td>1</td>
<td>77.87006</td>
<td>654.17794</td>
<td>5441.96948</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>77.87007</td>
<td>654.16076</td>
<td>5441.72090</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>77.87007</td>
<td>654.12624</td>
<td>5441.26535</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>77.86967</td>
<td>654.09104</td>
<td>5440.78956</td>
</tr>
<tr>
<td>[33]ode113(MQ)</td>
<td>15</td>
<td>77.86957</td>
<td>656.92788</td>
<td>5478.64484</td>
</tr>
</tbody>
</table>
Chapter 3  Solution of (1+1)-dimensional PDEs

\[
\begin{align*}
t &= 1, & t &= 10, & t &= 15
\end{align*}
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures/chap3undularbore.png}
\caption{Interaction of three solitary Waves.}
\end{figure}

Undular bore

To examine the undular bore phenomena the boundary and initial conditions considered are

\[
\psi(0, t) = w_o, \quad \psi(1, t) = 0, \quad \psi(x, 0) = \frac{w_o}{2} \left( 1 - \tanh \left( \frac{Lx + a - y_0}{d} \right) \right). \tag{3.50}
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures/chap3undularbore2.png}
\caption{Undular bore (case 1).}
\end{figure}
The initial function shows the water surface elevations above the level of equilibrium at \( t = 0 \).
Two sets of parameters have been used for the development of undular bore. In one case \( w_0 = 0.1, \mu = 1, d = 2, y_o = 0 \) and space domain \([-20, 50]\) for different time up to \( t = 800 \).
In the second case the parameters used are \( w_0 = 0.1, \mu = 0.1666667, d = 5, y_o = 0 \). In both the cases \( J = 8 \). The solution graphs are shown in Figs. 3.9 and 3.6 respectively.

**Soliton collision**

In case of soliton collision the same initial and boundary conditions as that of two solitary waves have been used. The signs used are opposite to each other so that the waves can move in opposite directions. The parameter used in this case are \( a = -40, b = 40, \omega_1 = -1.2, \omega_2 = 1.2, J = 8, y_1 = 20, y_2 = -20, \rho_1 = \rho_2 = 0.5 \). The graphical solution is shown in Fig. 3.9. From the figures it can be observed that the two waves are moving towards each other, collides and generate the singularity. The singularity produce trains of smaller waves in both directions.
Figure 3.11: Soliton collision.
3.4 Conclusion

In this chapter we have applied Haar wavelet method combined with finite difference scheme for the study of (1+1)-dimensional nonlinear PDEs. The scheme is applied to solve Burger’s, Boussinesq and EW equations. The obtained solutions are compared with exact and those available in literature. In case of EW of solitary waves, undular bore and soliton collision have been considered. It has been observed that the proposed method is efficient and more accurate and can be applied to nonlinear problems.
Chapter 4

Solution of fourth order PDEs arising in Euler-Bernoulli Beam models

Small static and dynamic deflection problems can be observed properly in linear theory. For determination of large (dynamic and static) deflection linear theory is not beneficial and requires an accurate analysis. Linear theory apply inexact curvature in study of beam deflections. The beginning work in the field of thin beam theory was initiated by Bernoulli. Jacob Bernoulli studied elastic theory and showed that bending moment and curvature are both proportional. To explain the motion of thin vibrating beam Daniel Bernoulli [98] originated a (PDE). Later on Bernoulli theory was extended to loaded beams by Leonard Euler. The PDEs of slight and long beam is known as Euler-Bernoulli model. The solution of this model shows the shortest distance (transverse vibration) from the beginning position. For transverse vibration, stress and strain are linearly related. Mathematical form of Euler-Bernoulli model [99] is described as follows:

\[
\gamma_1(x)\psi_{tt}(x,t) + [C(x)\psi_{xx}(x,t)]_{xx} = E(x,t), \quad x \in \Phi, \quad t > 0, \quad (4.1)
\]

where \( \Phi = [0,1] \). The appropriate initial and boundary conditions with above equation are

\[
\psi(x,0) = \Upsilon_1(x), \quad \psi_t(x,0) = \Upsilon_2(x), \quad 0 \leq x \leq 1, \quad (4.2)
\]
Chapter 4  
Solution of fourth order PDEs arising in Euler-Bernoulli Beam models

\[
\begin{align*}
\psi(0,t) &= \varepsilon_o(t), & \psi_x(0,t) &= \delta_o(t), & t > 0 \\
\psi(1,t) &= \varepsilon_1(t), & \psi_x(1,t) &= \delta_1(t), & t > 0,
\end{align*}
\]

Eq. (4.3) represent variable coefficients fourth order partial differential equations (VCFOPDEs) in which \( \psi(x,t) \) represent the beam displacement, \( \gamma_1(x) \) is mass per unit length, \( C(x) \) bending stiffness of beam, \( E(x,t) \) is the source term and \( L \) is total length. This kind of equation has application in robotics designs and large flexible space structures [100,101]. Several analytical methods have been applied for analytical solution of governing equation. For example Wazwaz [102] used Adomian decomposition method to solve variable coefficients VCFOPDEs. Lieu [17] applied He’s variational iteration technique to explain free vibration in Euler-Bernoulli beam. For more generic cases of initial conditions, analytical solution are quite complicated. Therefore researchers turned toward numerical solution. Variety of numerical methods via finite difference schemes have been developed for solution of different forms of Eq. (4.1) by Jain et al. [103], Evans [1], Conte [104], Richter [105], Cranial [106]. Evans et al. [107] established a stable computational method using Hopscotch algorithm. Aziz et al. [108] proposed three level scheme using parametric quintic spline to solve VCFOPDEs. Rashidinia [109] implemented three level implicit scheme coupled with sextic spline to solve fourth order equations. In present chapter, the proposed method is used to find the numerical solution of Eq. (4.1) with homogeneous and non-homogeneous form.

4.1 Description of the method

In this part of the chapter, the proposed scheme is presented for Eq. (4.1) with boundary conditions in the form of Eq. (4.3). By rewriting Eq. (4.1) as

\[
\gamma_1(x)\psi_{tt} + C(x)\psi_{xxxx} + 2C'(x)\psi_{xxx} + C''(x)\psi_{xx} = E,
\]

Eq. (4.4)
where \( C'(x) = \frac{dC(x)}{dx} \). Approximating temporal part with finite difference and \( \theta \)-weighted scheme to spatial part, Eq. (4.5) reduces to

\[
\gamma_1(x) \frac{\psi^{j+1} - 2 \psi^j + \psi^{j-1}}{\tau^2} + \theta [C(x) \psi_{xxxx} + 2C'(x) \psi_{xxx} + C''(x) \psi_{xx}]^{j+1}
+(1 - \theta) [C(x) \psi_{xxxx} + 2C'(x) \psi_{xxx} + C''(x) \psi_{xx}]^j = \mathcal{E}^{j+1},
\]

(4.6)

where \( \psi^j = \psi(x, t^j) \), \( \mathcal{E}^j = \mathcal{E}(x, t^j) \). The associated boundary conditions Eq. (4.3) will transform to

\[
\begin{align*}
\psi^{j+1}(0) &= \varepsilon_o(t^{j+1}), & \psi_x^{j+1}(0) &= \delta_o(t^{j+1}), & t > 0 \\
\psi^{j+1}(1) &= \varepsilon_1(t^{j+1}), & \partial_x \psi_x^{j+1}(1) &= \delta_1(t^{j+1}), & t > 0.
\end{align*}
\]

(4.7)

In more simplified form Eq. (4.6) can be written as

\[
\gamma_1(x) \psi^{j+1} + \tau^2 \theta C(x) \psi_{xxxx} \psi^{j+1} + 2 \theta \tau^2 C'(x) \partial_{xxxx} \psi^{j+1} + \theta \tau^2 C''(x) \psi_{xx} = N^j + \tau^2 \mathcal{E}^{j+1},
\]

(4.8)

where

\[
N^j = (2 \psi^j - \psi^{j-1}) \gamma_1(x) + (\theta - 1) \tau^2 [C(x) \psi_{xxxx} + 2C'(x) \psi_{xxx} + C''(x) \psi_{xx}]^j.
\]

(4.10)

Next consider Haar wavelet approximation for highest order derivative as:

\[
\psi^{j+1}_{xxxx}(x) = \sum_{i=1}^{2M} \alpha_i R_i(x).
\]

(4.9)

Integration of Eq. (4.9) four times from 0 to \( x \) leads to

\[
\begin{align*}
\psi^{j+1}_{xx}(x) &= \sum_{i=1}^{2M} \alpha_i R_{i,1}(x) + \psi^{j+1}_{xx}(0), \\
\psi^{j+1}_x(x) &= \sum_{i=1}^{2M} \alpha_i R_{i,2}(x) + x \psi^{j+1}_{xx}(0) + \psi^{j+1}_x(0), \\
\psi^{j+1}(x) &= \sum_{i=1}^{2M} \alpha_i R_{i,3}(x) + \frac{x^2}{2!} \psi^{j+1}_{xx}(0) + x \psi^{j+1}_x(0) + \psi^{j+1}(0), \\
\psi^{j+1}(x) &= \sum_{i=1}^{2M} \alpha_i R_{i,4}(x) + \frac{x^3}{3!} \psi^{j+1}_{x}(0) + \frac{x^2}{2!} \psi^{j+1}_x(0) + x \psi^{j+1}(0) + \psi^{j+1}(0).
\end{align*}
\]

(4.10)
Using the boundary conditions $\psi_{j+1}^{+1}(1), \psi_{x}^{j+1}(1)$ in Eq. (4.10) the unknown terms can be computed as

\[ \psi_{xxx}^{j+1}(0) = \phi(t^{j+1}) + 12 \sum_{i=1}^{2M} \alpha_i R_{i,4}(1) - 6 \sum_{i=1}^{2M} \alpha_i R_{i,3}(1), \]

\[ \psi_{xx}^{j+1}(0) = \phi_{1}(t^{j+1}) - 6 \sum_{i=1}^{2M} \alpha_i R_{i,4}(1) + 2 \sum_{i=1}^{2M} \alpha_i R_{i,3}(1), \]

where

\[ \phi(t^{j+1}) = 6 \left[ \delta_1(t^{j+1}) + \delta_0(t^{j+1}) - 2 \varepsilon_1(t^{j+1}) + 2 \varepsilon_0(t^{j+1}) \right], \]

\[ \phi_{1}(t^{j+1}) = -2 \left[ \delta_1(t^{j+1}) + 2 \delta_0(t^{j+1}) \right] - 6 \left[ \varepsilon_0(t^{j+1}) - \varepsilon_1(t^{j+1}) \right]. \]

Making use of Eq. (4.11) in Eq. (4.10) we obtain

\[ \psi_{xxx}^{j+1}(x) = \sum_{i=1}^{2M} \alpha_i \left[ R_{i,1}(x) - 6 R_{i,3}(1) + 12 R_{i,4}(1) \right] + \phi(t^{j+1}), \]

\[ \psi_{xx}^{j+1}(x) = \sum_{i=1}^{2M} \alpha_i \left[ R_{i,2}(x) + (2 - 6x)R_{i,3}(1) + (12x - 6)R_{i,4}(1) \right] + x \phi(t^{j+1}) + \phi_{1}(t^{j+1}), \]

\[ \psi_{x}^{j+1}(x) = \sum_{i=1}^{2M} \alpha_i \left[ R_{i,3}(x) + (2x - 3x^2)R_{i,3}(1) + (6x^2 - 6x)R_{i,4}(1) \right] + \frac{x^2}{2} \phi(t^{j+1}) \]

\[ + x \phi_{1}(t^{j+1}) + \delta_0(t^{j+1}), \]

\[ \psi^{j+1}(x) = \sum_{i=1}^{2M} \alpha_i \left[ R_{i,4}(x) + (x^2 - x^3)R_{i,3}(1) + (2x^3 - 3x^2)R_{i,4}(1) \right] + \frac{x^3}{3!} \phi(t^{j+1}) \]

\[ + \frac{x^2}{2} \phi_{1}(t^{j+1}) + x \delta_0(t^{j+1}) + \varepsilon_0(t^{j+1}). \]

Putting Eqs. (4.9), (4.12) in Eq. (4.8) and replacing $x$ by $x_p$ (Eq. (1.10)) leads to the following system of linear equations

\[ \sum_{i=1}^{2M} \alpha_i \left[ \gamma_1(x_p) I(i, p) + \tau^2 \theta C(x_p) \Psi_i(x_p) + 2 \theta \tau^2 C'(x_p) K(i, p) + \tau^2 \theta C''(x_p) \Psi(i, p) \right] \]

\[ = \Lambda(p, j + 1) + \bar{\Omega}(p, j + 1), \]

52
where

\[ I(i, p) = \left[ R_{i,4}(x_p) + (x_p^2 - x^3)R_{i,3}(1) + (2x_p^3 - 3x^2)R_{i,4}(1) \right], \]

\[ K(i, p) = \left[ R_{i,1}(x_p) - 6R_{i,3}(1) + 12R_{i,4}(1) \right], \]

\[ \Psi(i, p) = \left[ R_{i,2}(x_p) + (2 - 6x_p)R_{i,3}(1) + (12x_p - 6)R_{i,4}(1) \right], \]

\[ \Lambda(p, t^{j+1}) = N(p, t^j) + E(p, t^{j+1}), \]

\[ \Phi(p, j + 1) = -\gamma_1(x_p)\left\{ \frac{x_p^3}{3!}\phi(t^{j+1}) + \frac{x_p^2}{2}\phi_1(t^{j+1}) + x_p\delta(t^{j+1}) + \epsilon(t^{j+1}) \right\} \]

\[ -2\tau^2\theta C'(x_p)\left\{ \phi(t^{j+1}) \right\} - \tau^2\theta\left\{ x_p\phi(t^{j+1}) + \phi_1(t^{j+1}) \right\}. \]

There are \( 2\mathcal{M} \) numbers of equations in Eq. (4.13) which has been solved for \( 2\mathcal{M} \) unknowns iteratively. After computation of the unknown constants the required solution has computed from Eq. (4.12).

### 4.2 Test problems

#### Problem 4.2.1

For comparison with previous work, we first consider Eq. (4.1) with \( \gamma_1(x) = C(x) = 1 \), in the following form

\[ \psi_{tt}u(x, t) + \psi_{xxxx}(x, t) = (\pi^4 - 1)\sin(\pi x)\cos t, \quad 0 \leq x \leq 1, \quad t > 0, \quad (4.14) \]

coupled with appropriate initial conditions

\[ \psi(x, 0) = \sin(\pi x), \quad \psi_t(x, 0) = 0, \quad 0 \leq x \leq 1, \quad (4.15) \]

and boundary conditions

\[ \psi(0, t) = u(1, t) = 0, \quad \psi_{xx}u(0, t) = \psi_{xx}(1, t) = 0, \quad t > 0. \quad (4.16) \]

Exact solution of this problem is given by

\[ \psi(x, t) = \psi(x, t) = \sin(\pi x)\cos(t). \]
Chapter 4  Solution of fourth order PDEs arising in Euler-Bernoulli Beam models

The problem Eq. (4.14)-(4.16) has been solved using the proposed method. The achieved results are matched with those exist in literature. In Table 4.1 point wise errors have been compared with [108–112] using parameters $\tau = 0.005$, $t = 0.02$, 0.05, 1.0, $2M = 64, 128$. From table it is clear that the our results are in good agreement with those available in literature. Graphical solution in the form of 2D and 3D plots with absolute error are displaced in Fig. 4.1 for time $t = 4$. It has been observed from figure that proposed method gives reasonable solution for small number of collocation points and promises well with exact solution.

![Graphical solution of problem 4.2.1](image)

Figure 4.1: Graphical solution of problem 4.2.1, (a): Exact an approximate at $t = 4$, $\tau = 0.001$ (b): Absolute error in (a), (c): Exact, (d): Approximate at $t = 4$, $\tau = 0.01$, $J = 4$. 
### Table 4.1: Absolute error at different points in problem 4.2.1.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Points</th>
<th>$t$</th>
<th>$\tau$</th>
<th>$x = 0.1$</th>
<th>$x = 0.2$</th>
<th>$x = 0.3$</th>
<th>$x = 0.4$</th>
<th>$x = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>64</td>
<td>0.02</td>
<td>0.005</td>
<td>6.54e-07</td>
<td>1.24e-06</td>
<td>1.71e-06</td>
<td>2.01e-06</td>
<td>2.11e-06</td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>0.05</td>
<td>0.005</td>
<td>5.22e-06</td>
<td>9.94e-06</td>
<td>1.36e-05</td>
<td>1.60e-05</td>
<td>1.69e-05</td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>1</td>
<td>0.005</td>
<td>7.14e-04</td>
<td>1.35e-03</td>
<td>1.87e-03</td>
<td>2.20e-03</td>
<td>2.31e-03</td>
</tr>
<tr>
<td></td>
<td>128</td>
<td>0.02</td>
<td>0.005</td>
<td>2.60e-07</td>
<td>4.94e-07</td>
<td>6.81e-07</td>
<td>8.00e-07</td>
<td>8.42e-07</td>
</tr>
<tr>
<td></td>
<td>128</td>
<td>0.05</td>
<td>0.005</td>
<td>2.59e-06</td>
<td>4.94e-06</td>
<td>6.80e-06</td>
<td>7.99e-06</td>
<td>8.40e-06</td>
</tr>
<tr>
<td></td>
<td>128</td>
<td>1</td>
<td>0.005</td>
<td>6.83e-04</td>
<td>1.29e-03</td>
<td>1.78e-03</td>
<td>2.10e-03</td>
<td>2.21e-03</td>
</tr>
<tr>
<td>Mittal [111]</td>
<td>91</td>
<td>0.02</td>
<td>0.005</td>
<td>3.20e-05</td>
<td>6.08e-05</td>
<td>8.37e-05</td>
<td>9.84e-05</td>
<td>1.04e-04</td>
</tr>
<tr>
<td></td>
<td>91</td>
<td>0.05</td>
<td>0.005</td>
<td>3.59e-05</td>
<td>6.83e-05</td>
<td>9.39e-05</td>
<td>1.10e-04</td>
<td>1.16e-04</td>
</tr>
<tr>
<td></td>
<td>91</td>
<td>1</td>
<td>0.005</td>
<td>6.32e-05</td>
<td>1.20e-04</td>
<td>1.65e-04</td>
<td>1.94e-04</td>
<td>2.04e-04</td>
</tr>
<tr>
<td></td>
<td>181</td>
<td>0.02</td>
<td>0.005</td>
<td>3.55e-06</td>
<td>6.76e-06</td>
<td>9.30e-06</td>
<td>1.09e-05</td>
<td>1.15e-05</td>
</tr>
<tr>
<td></td>
<td>181</td>
<td>0.05</td>
<td>0.005</td>
<td>3.99e-06</td>
<td>7.58e-06</td>
<td>1.04e-05</td>
<td>1.23e-05</td>
<td>1.29e-05</td>
</tr>
<tr>
<td></td>
<td>181</td>
<td>1</td>
<td>0.005</td>
<td>7.00e-06</td>
<td>1.33e-05</td>
<td>1.83e-05</td>
<td>2.16e-05</td>
<td>2.27e-05</td>
</tr>
<tr>
<td>Caglar [110]</td>
<td>121</td>
<td>0.02</td>
<td>0.005</td>
<td>4.80e-06</td>
<td>9.70e-06</td>
<td>1.40e-05</td>
<td>1.90e-05</td>
<td>2.40e-05</td>
</tr>
<tr>
<td></td>
<td>191</td>
<td>0.02</td>
<td>0.005</td>
<td>5.20e-06</td>
<td>2.10e-06</td>
<td>3.10e-06</td>
<td>4.20e-06</td>
<td>5.20e-06</td>
</tr>
<tr>
<td></td>
<td>521</td>
<td>0.02</td>
<td>0.005</td>
<td>4.90e-07</td>
<td>9.90e-07</td>
<td>1.40e-06</td>
<td>1.90e-06</td>
<td>2.40e-06</td>
</tr>
<tr>
<td>Aziz [108]</td>
<td>20</td>
<td>0.05</td>
<td>0.005</td>
<td>9.30e-06</td>
<td>8.00e-06</td>
<td>2.80e-06</td>
<td>1.00e-06</td>
<td>2.70e-06</td>
</tr>
<tr>
<td>Rashidinia [109]</td>
<td>20</td>
<td>0.05</td>
<td>0.005</td>
<td>2.91e-06</td>
<td>1.73e-06</td>
<td>1.60e-06</td>
<td>2.23e-06</td>
<td>2.60e-07</td>
</tr>
<tr>
<td>Mohammadi [112]</td>
<td>40</td>
<td>0.05</td>
<td>0.005</td>
<td>2.96e-06</td>
<td>1.77e-06</td>
<td>1.64e-06</td>
<td>2.28e-06</td>
<td>2.65e-07</td>
</tr>
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</table>
Table 4.2: Absolute error at $x = 0.5$ of problem 4.2.2

<table>
<thead>
<tr>
<th>Methods</th>
<th>Points</th>
<th>$\tau$</th>
<th>$t = 0.2$</th>
<th>$t = 0.4$</th>
<th>$t = 0.8$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>32</td>
<td>0.001</td>
<td>1.76e-13</td>
<td>5.72e-13</td>
<td>1.80e-12</td>
<td>2.14e-12</td>
<td>2.87e-12</td>
<td>2.39e-12</td>
</tr>
<tr>
<td></td>
<td>64</td>
<td>0.001</td>
<td>1.72e-13</td>
<td>5.75e-13</td>
<td>1.48e-12</td>
<td>1.90e-12</td>
<td>2.95e-12</td>
<td>6.13e-13</td>
</tr>
<tr>
<td>[112]</td>
<td>100</td>
<td>0.01</td>
<td>1.78e-05</td>
<td>5.85e-05</td>
<td>1.57e-04</td>
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<td>2.95e-04</td>
<td>1.60e-04</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>0.005</td>
<td>2.38e-06</td>
<td>7.80e-06</td>
<td>2.09e-05</td>
<td>2.67e-05</td>
<td>3.94e-05</td>
<td>2.57e-05</td>
</tr>
</tbody>
</table>

**Problem 4.2.2**

Consider the non-homogeneous problem of the form [112]

$$\psi_{tt}(x, t) + \left((1 + \sin \pi x) \psi_{xx}(x, t)\right)_{xx} = \mathcal{E}(x, t), \quad 0 \leq x \leq 1, \quad t > 0, \quad (4.17)$$

coupled with initial conditions

$$\psi(x, 0) = \psi_t(x, 0) = 0, \quad 0 \leq x \leq 1, \quad (4.18)$$

and the boundary conditions

$$\psi(0, t) = \psi(1, t) = 0, \quad \psi_x(0, t) = \psi_x(1, t) = 0, \quad t > 0. \quad (4.19)$$

The source term is easy to compute from exact solution

$$\psi(x, t) = x(1 - x)\exp(-t)t^2\sin(4\pi x).$$

We obtained the solution of this problem in the time domain $[0, 4]$. In Table 4.2 we computed the absolute error at point $x = 0.5$, using $t = 0.2$, $0.4$, $0.6$, $0.8$, $1$, $2$, $4$. From table it is obvious that our error is much smaller than those of Mohammadi [112]. In Table 4.3 different error norms have been calculated which identify that proposed method has better outcome at small resolution levels. Exact and approximate solution together with absolute error have been plotted at time $t = 4$ in Fig. 4.2. It is clear from figure that both solution have closed coincidence.
Table 4.3: Maximum error norms of problem 4.2.2 at different times with $\tau = 0.001$

<table>
<thead>
<tr>
<th>$J$</th>
<th>$t = 0.2$</th>
<th>$t = 0.5$</th>
<th>$t = 1$</th>
<th>$t = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L_{\infty}$</td>
<td>$L_2$</td>
<td>$L_{\infty}$</td>
<td>$L_2$</td>
</tr>
<tr>
<td>4</td>
<td>6.10e-05</td>
<td>3.38e-04</td>
<td>3.86e-04</td>
<td>2.11e-03</td>
</tr>
<tr>
<td>5</td>
<td>1.11e-05</td>
<td>7.36e-05</td>
<td>5.85e-05</td>
<td>3.29e-04</td>
</tr>
</tbody>
</table>

Figure 4.2: Graphical solution of problem 4.2.2, (a): Exact an approximate at $t = 4$, $\tau = 0.001$ (b): Absolute error in (a), (c): Exact 3D plot, (d): Approximate 3D plot at $t = 4$, $\tau = 0.01$, $J = 5$. 
Table 4.4: Maximum error norms of problem 4.2.3 at different times with $\tau = 0.001$.

<table>
<thead>
<tr>
<th></th>
<th>$t = 0.2$</th>
<th></th>
<th>$t = 0.5$</th>
<th></th>
<th>$t = 1$</th>
<th></th>
<th>$t = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J$</td>
<td>$L_{\infty}$</td>
<td>$L_2$</td>
<td>$L_{\infty}$</td>
<td>$L_2$</td>
<td>$L_{\infty}$</td>
<td>$L_2$</td>
<td>$L_{\infty}$</td>
</tr>
<tr>
<td>4</td>
<td>2.73e-03</td>
<td>6.11e-03</td>
<td>1.29e-02</td>
<td>2.86e-02</td>
<td>3.14e-02</td>
<td>6.95e-02</td>
<td>2.50e-02</td>
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<tr>
<td>5</td>
<td>2.72e-03</td>
<td>6.11e-03</td>
<td>1.27e-02</td>
<td>2.86e-02</td>
<td>3.08e-02</td>
<td>6.94e-02</td>
<td>2.29e-04</td>
</tr>
</tbody>
</table>

Problem 4.2.3

Now we consider the following equation

$$\psi_{tt}(x,t) + \psi_{xx}(1 + \sin \pi x) \psi_{xx}(x,t) = E(x,t), \quad 0 \leq x \leq 1, \quad t > 0,$$

(4.20)

with initial conditions

$$\psi(x,0) = \psi_t(x,0) = 0, \quad 0 \leq x \leq 1,$$

(4.21)

and the boundary conditions

$$\begin{aligned}
\psi(0,t) &= 0, \quad \psi_x(0,t) = t^2 \exp(-t), \quad t > 0, \\
\psi(1,t) &= t^2 \exp(-t), \quad \psi_x(1,t) = t^2 \exp(-t), \quad t > 0.
\end{aligned}$$

(4.22)

All conditions have been used from exact solution

$$\psi(x,t) = (\sin^3(\pi x)) t^2 \exp(-t).$$

The solution of this problem has been computed at different levels of resolutions. In Table 4.4 we recorded error norms for different times ($t=0.2, 0.5, 1, 4$) using $J = 4, 5$. It has been observed from the table that error norms are small which show that the method is efficient. Graphical solution and absolute error have shown in Fig. 4.3. From figure one can see that exact and approximated solutions agree mutually.
Figure 4.3: Graphical solution of problem 4.2.3 with exact (a): Exact an approximate at $t = 4$, $\tau = 0.001$ (b): Absolute error in (a), (c): Exact 3D plot, (d)= Approximate 3D plot at $t = 4$, $\tau = 0.01$, $J = 5$. 
4.3 Conclusion

In this chapter, we applied the proposed scheme to different VCFOPDEs. The method is tested with three different homogenous and nonhomogeneous problems. The obtained results are shown in tabulated and graphical form. The results are also compared with different results previously available. It has been noted from tabulated data and graphical solution that our results are much accurate.
Chapter 5

Solution of (1+2)-dimensional PDEs and coupled PDEs

Various physical phenomenon such as heat conduction, acoustic waves and modeling of dynamics can be modeled as PDEs. Disparity in temperature can be described in a region with the passage of time via heat equation. Diffusion equation is more generic form of heat equation arising in the study of chemical diffusion and relevant processes. Basic models of flow phenomena in transport problems involve the time dependent PDEs. In such studies Burgers’ equation is extensively using in turbulence. Sobolev equation, having extensive uses in thermodynamics [113], fissured rock [114], movement of moisture in soil [115]. Nonlinear evolution equations have also many applications in applied sciences and provide better information in the form of soliton solutions. Exact solution of nonlinear PDEs is not that simple to compute. Therefore, numerical techniques are the best choices for the solution of such problems.

In this chapter, we present a mixed numerical method based on finite differences and two dimensional Haar wavelets for two-dimensional PDEs. Convergence of the scheme has been discussed via asymptotic expansion. The scheme is tested with different two dimensional problems which show good results. The problems which be under considerations are described as follows:
Chapter 5  Solution of (1+2)-dimensional PDEs and coupled PDEs

Linear Heat equation

\[ \psi_t = \Delta \psi, \tag{5.1} \]

where \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is two dimensional Laplacian.

Linear Sobolev equation

\[ \psi_t - \Delta \psi_t - \Delta \psi = \mathcal{E}, \tag{5.2} \]

where \( \psi = \psi(x,y,t) \) and \( \mathcal{E} = \mathcal{E}(x,y,t) \) is the source term.

Nonlinear Burgers’ equation

\[ \psi_t + \psi \psi_x + \psi \psi_y = v \Delta \psi, \tag{5.3} \]

where \( v = \frac{1}{Re} \) represents coefficient of viscosity and \( Re \) Reynolds number.

Nonlinear Sobolev equation

\[ \psi_t - \Delta \psi_t - \nabla.(\psi \nabla \psi) + (\pi^2)\psi^2 = \mathcal{E}, \tag{5.4} \]

where \( \nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j \) is two dimensional gradient operator.

Nonlinear generalized Benjamin-Bona-Mahony-Burger’s equation (NGBBMB)

\[ \psi_t - \Delta \psi_t - \Delta \psi + \nabla.\psi = \psi \psi_x + \psi \psi_y + \mathcal{E}. \tag{5.5} \]

Nonlinear Burgers’ system

\[
\begin{align*}
\psi_t + \psi \psi_x + u \psi_y - \frac{1}{Re} (\psi_{xx} + \psi_{yy}) &= 0, \tag{5.6} \\
u_t + \psi u_x + uu_y - \frac{1}{Re} (u_{xx} + u_{yy}) &= 0. \tag{5.7}
\end{align*}
\]
5.1 Method description

In this section, the proposed scheme is discussed for linear PDEs. The same procedure can be extended to nonlinear and system of PDEs as well. For this purpose, let us consider two dimensional Heat equation (5.1) with initial condition and Dirichlet boundary conditions

\[
\begin{aligned}
\psi(x, y, 0) &= \sigma(x, y), \quad (x, y) \in \Phi, \\
\psi(x, y, t) &= \sigma_1(x, y, t), \quad (x, y) \in \partial \Phi, \quad t > 0,
\end{aligned}
\]  
(5.8)

where \( \Phi = [0, 1]^2 \), is domain and \( \partial \Phi \) is the boundary of domain. Applying \( \theta \)-weighted \( (0 \leq \theta \leq 1) \) scheme to spatial part and forward difference to temporal part Eq. (5.1) yields

\[
\frac{\psi^{j+1} - \psi^j}{\tau} - \theta (\psi_{xx}^{j+1} + \psi_{yy}^{j+1}) = (1 - \theta) (\psi_{xx}^j + \psi_{yy}^j),
\]  
(5.9)

where \( \psi = (x, y, t') \), and \( \tau \) is time mesh size. Simplification of Eq. (5.9) yields

\[
\psi^{j+1} - \tau \theta [\psi_{xx}^{j+1} + \psi_{yy}^{j+1}] = \psi^j + \tau (1 - \theta) [\psi_{xx}^j + \psi_{yy}^j],
\]  
(5.10)

with associated conditions

\[
\begin{aligned}
\psi(x, y, 0) &= \sigma(x, y), \quad (x, y) \in \Phi \\
\psi(x, y, t^{j+1}) &= \sigma_1(x, y, t^{j+1}), \quad (x, y) \in \partial \Phi, \quad t > 0.
\end{aligned}
\]  
(5.11)

Next we approximate the mixed order derivative \( \psi_{xxyy}^{j+1} \) by two dimensional truncated Haar wavelet series as:

\[
\psi_{xxyy}^{j+1}(x, y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} \alpha_{i,l} \mathcal{H}_i(x) \mathcal{H}_l(y),
\]  
(5.12)

where \( \alpha_{i,l} \) are unknowns to be determine. Integration of Eq. (5.12) in the domain \([0, y]\) leads to

\[
\psi_{xxyy}^{j+1}(x, y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} \alpha_{i,l} \mathcal{H}_i(x) \mathcal{R}_{i,1}(y) + \psi_{xxyy}^{j+1}(x, 0).
\]  
(5.13)

Integrating Eq. (5.13) w.r.t \( y \) from 0 to 1 the unknown term \( \psi_{xxyy}^{j+1}(x, 0) \) can be obtain as

\[
\psi_{xxyy}^{j+1}(x, 0) = \psi_{xx}^{j+1}(x, 1) - \psi_{xx}^{j+1}(x, 0) - \sum_{i=1}^{2M} \sum_{l=1}^{2M} \alpha_{i,l} \mathcal{H}_i(x) \mathcal{R}_{i,2}(1).
\]  
(5.14)
Chapter 5  Solution of (1+2)-dimensional PDEs and coupled PDEs

Substituting Eq. (5.14) in Eq. (5.13) the obtained result is given by

\[ \psi_{xxy}(x, y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} \alpha_{i,l} H_2(x) \left[ R_{i,1}(y) - R_{i,2}(1) \right] + \psi_{xx}^{j+1}(x, 1) - \psi_{xx}^{j+1}(x, 0). \]  

(5.15)

Again integrating Eq. (5.15) in the domain \([0, y]\), we get

\[ \psi_{xx}^{j+1}(x, y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} \alpha_{i,l} H_2(x) \left[ R_{i,2}(y) - yR_{i,2}(1) \right] + y\psi_{xx}^{j+1}(x, 1) + (1 - y)\psi_{xx}^{j+1}(x, 0). \]  

(5.16)

Repeating the same procedure one can easily derive the subsequent expressions

\[ \psi_{yy}^{j+1}(x, y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} \alpha_{i,l} \left[ R_{i,2}(x) - xR_{i,2}(1) \right] H_2(y) + x\psi_{yy}^{j+1}(1, y) + (1 - x) \psi_{yy}^{j+1}(0, y), \]  

(5.17)

\[ \psi_{x}^{j+1}(x, y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} \alpha_{i,l} \left[ R_{i,1}(x) - R_{i,2}(1) \right] H_2(y) + y\psi_{x}^{j+1}(x, 1) + (1 - y)\psi_{x}^{j+1}(0, 0), \]  

(5.18)

\[ \psi_{y}^{j+1}(x, y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} \alpha_{i,l} \left[ R_{i,2}(x) - xR_{i,2}(1) \right] \left[ R_{i,1}(y) - R_{i,2}(1) \right] + x\psi_{y}^{j+1}(1, y) + (1 - x)\psi_{y}^{j+1}(0, 1) + (1 - x)\psi_{y}^{j+1}(0, 0), \]  

(5.19)

\[ \psi^{j+1}(x, y) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} \alpha_{i,l} \left[ R_{i,2}(x) - xR_{i,2}(1) \right] \left[ R_{i,2}(y) - yR_{i,2}(1) \right] + y\psi^{j+1}(x, 1) + y\psi^{j+1}(0, 0), \]  

(5.20)

Putting values from Eqs. (5.16), (5.17) and (5.20) in Eq.(5.10) and using the collocation points,

\[ x_m = \frac{m - 0.5}{2M}, \quad y_n = \frac{n - 0.5}{2M}, \quad \text{where} \quad m, n = 1, 2, \ldots, 2M, \]  

(5.21)

64
give rise to the following system of algebraic equations

\[
2M \sum_{i=1}^{2M} \sum_{l=1}^{2M} \alpha_{i,l} \left[ \tilde{A}(i, l, m, n) - \theta \tau \left( \tilde{B}(i, l, m, n) + \tilde{C}(i, l, m, n) \right) \right] = \tilde{D}(m, n), \tag{5.22}
\]

where

\[
\begin{align*}
\tilde{A}(i, l, m, n) &= \left[ R_{i,2}(x_m) - x_m R_{i,2}(1) \right] \left[ R_{l,2}(y_n) - y_n R_{l,2}(1) \right], \\
\tilde{B}(i, l, m, n) &= H_i(x_m) \left[ R_{l,2}(y_n) - y_n R_{l,2}(1) \right], \\
\tilde{C}(i, l, m, n) &= \left[ R_{i,2}(x_m) - x_m R_{i,2}(1) \right] H_l(y_n), \\
\tilde{D}(m, n) &= \psi^j + (1 - \theta) \tau \left\{ \psi_{x x}^j + \psi_{y y}^j \right\} - \left[ y_n \psi_{x x}^{j+1}(x_m, 1) - y_n \psi_{x x}^{j+1}(0, 1) \right. \\
&\quad \left. + (1 - y_n) \left\{ \psi_{x x}^{j+1}(x_m, 0) - \psi_{x x}^{j+1}(0, 0) \right\} + x \psi_{x x}^{j+1}(1, y_n) - x_m \psi_{x x}^{j+1}(0, y_n) \right. \\
&\quad \left. - x_m y_n \left\{ \psi_{x x}^{j+1}(1, 1) - \psi_{x x}^{j+1}(0, 1) \right\} + x_m (y_n - 1) \psi_{x x}^{j+1}(1, 0) \right. \\
&\quad \left. + x_m (1 - y_n) \psi_{x x}^{j+1}(0, 0) + \psi_{x x}^{j+1}(0, y_n) \right] + \tau \theta \left[ y_n \psi_{x x}^{j+1}(x_m, 1) \right. \\
&\quad \left. + (1 - y_n) \psi_{x x}^{j+1}(x_m, 0) + x_m \psi_{y y}^{j+1}(1, y_n) + (1 - x_m) \psi_{y y}^{j+1}(0, y_n) \right].
\end{align*}
\]

In Eq. (5.22) there are \(2M \times 2M\) equations which has been solved to find wavelets coefficients. Once the unknown coefficients have been determined, then approximate solution can be computed from Eq. (5.20). To start the process, we take

\[
\begin{align*}
\psi(x, y, 0) &= \sigma(x, y), & \psi_x(x, y, 0) &= \sigma_x(x, y), & \psi_{xx}(x, y, 0) &= \sigma_{xx}(x, y), \\
\psi_y(x, y, 0) &= \sigma_y(x, y), & \psi_{yy}(x, y, 0) &= \sigma_{yy}(x, y).
\end{align*}
\]

### 5.2 Convergence analysis

For error analysis we need the following Lemma.

**Lemma 5.2.1** [63]

If \(f(x, y)\) satisfies a Lipschitz condition on \([0, 1] \times [0, 1]\), that is, there exists a positive \(L\) such that for all \((x_1, y), (x_2, y) \in [0, 1] \times [0, 1]\) we have \(|f(x_1, y) - f(x_2, y)| \leq L \ |x_1 - x_2|\) then

\[
\alpha_{i,l}^2 \leq \frac{L^2}{2^{4j+4m^2}}. \tag{5.23}
\]
Chapter 5  Solution of (1+2)-dimensional PDEs and coupled PDEs

**Theorem 5.2.2**

Assume $\psi(x, y)$ and $\psi_{2M}(x, y)$ be the exact and approximate solution, then

$$
\| E_{2M} \| \leq \frac{L}{4\sqrt{255244}}.
$$

(5.24)

**Proof**

To prove the theorem, we take the asymptotic expansion of Eq. (5.20) defined as follows:

$$
\psi^{j+1}(x, y) = \sum_{i=1}^{\infty} \sum_{l=1}^{\infty} \alpha_{i,l} [R_{i,2}(x) - xR_{i,2}(1)] [R_{l,2}(y) - yR_{l,2}(1)] + \Lambda(x, y),
$$

(5.25)

where

$$
\Lambda(x, y) = y\psi^{j+1}(x, 1) - y\psi^{j+1}(0, 1) + (1 - y) [\psi^{j+1}(x, 0) - \psi^{j+1}(0, 0)]
$$

$$
+ x\psi^{j+1}(1, y) - xy\psi^{j+1}(0, y) - xy [\psi^{j+1}(1, 1) - \psi^{j+1}(0, 1)] +
$$

$$
x(1 - y) \psi^{j+1}(1, 0) + x(1 - y) \psi^{j+1}(0, 0) + \psi^{j+1}(0, y).
$$

From Eqs. (5.25) and (5.20), at resolution level $J$, we can write

$$
|E_{2M}| = |\psi(x, y) - \psi_{2M}(x, y)| = \left| \sum_{i=2M+1}^{\infty} \sum_{l=2M+1}^{\infty} \alpha_{i,l} [R_{i,2}(x) - xR_{i,2}(1)] [R_{l,2}(y) - yR_{l,2}(1)] \right|.
$$

(5.26)

The $L^2$ – norm is given by

$$
\| E_{2M} \|^2 = \left| \int_{0}^{1} \int_{0}^{1} \left( \sum_{i=2M+1}^{\infty} \sum_{l=2M+1}^{\infty} \alpha_{i,l} [R_{i,2}(x) - xR_{i,2}(1)] [R_{l,2}(y) - yR_{l,2}(1)] \right)^2 \, dx \, dy \right|.
$$

(5.27)

$$
= \sum_{i,l=2M+1}^{\infty} \sum_{i',l'=2M+1}^{\infty} \alpha_{i,l} \alpha_{i',l'} \left| \int_{0}^{1} \int_{0}^{1} ([R_{i,2}(x) - xR_{i,2}(1)] [R_{l,2}(y) - yR_{l,2}(1)])^2 \, dx \, dy \right|.
$$

$$
= \sum_{i,l=2M+1}^{\infty} \alpha_{i,l}^2 \left| \int_{0}^{1} \int_{0}^{1} ([R_{i,2}(x) - xR_{i,2}(1)] [R_{l,2}(y) - yR_{l,2}(1)])^2 \, dx \, dy \right|.
$$
To evaluate the complex expression of integration, it is suitable to estimate the maximum bounds of Haar wavelets integrals which can be obtained from expression [91]

\[
R_{i,n}(x) = \frac{1}{n!} \sum_{k=2}^{n} \binom{n}{k} (x - \zeta_2)^{n-k} \left[ \left( \frac{1}{2j+1} \right)^k + \left( -\frac{1}{2j+1} \right)^k \right]
\]

(5.28)

\[
= \frac{1}{n!} \sum_{k=2}^{n} \binom{n}{k} (1 - \zeta_2)^{n-k} \left[ \left( \frac{1}{2j+1} \right)^k + \left( -\frac{1}{2j+1} \right)^k \right]
\]

(5.29)

From Eq. (5.28), one can deduce that

\[
R_{i,2}(1) \leq \frac{1}{(2j+1)^2}.
\]

(5.30)

Using Eq. (5.30) and Lemma 5.2.1 in Eq. (5.27), we can derive

\[
\| E_{2M} \|^2 \leq \sum_{i, l=2M+1}^{\infty} \frac{16L^2}{2^{4j+4}2^{2j}} \int_{\zeta_1}^{1} \int_{\zeta_1}^{1} \frac{1}{2^{4j+4}} dx dy
\]

(5.31)

\[
= \sum_{i, l=2M+1}^{\infty} \frac{16L^2}{2^{10j+8}} \int_{\zeta_1}^{1} (1 - \zeta_1) dy
\]

\[
\leq \sum_{i, l=2M+1}^{\infty} \frac{16L^2}{2^{10j+8}} \int_{\zeta_1}^{1} dy
\]

\[
\leq \sum_{i, l=2M+1}^{\infty} \frac{16L^2}{2^{10j+8}}
\]

\[
= \sum_{j=J+1}^{\infty} \left\{ \sum_{i=0}^{2^j-1} \sum_{l=0}^{2^j-1} \frac{16L^2}{2^{10j+8}} \right\}
\]

\[
= \sum_{j=J+1}^{\infty} \frac{16L^2}{2^{8j+8}}
\]

(5.32)

which gives

\[
\| E_{2M} \| \leq \frac{L}{4\sqrt{2552}4^j}.
\]

(5.33)

This completes proof of the theorem.
5.3 Test problems

The proposed technique is applied to solve some two dimensional test problems. Six test examples namely: linear Heat and Sobolev equations, non-linear Burgers Sobolev, NGBBMB and non-linear Burger’s system, have been studied. In all examples the obtained results are matched with those available in literature and exact solutions. To check validity of the scheme $L_2$, $L_\infty$, root mean square ($RMS$), and relative error $E_r$ have been computed. These error norms are defined as follows:

\[
L_2 = \sqrt{\sum_{i=0}^{2M} \sum_{j=0}^{2M} (\psi_{i,j}^{\text{ext}} - \psi_{i,j}^{\text{app}})^2}, \quad L_\infty = \max_{0 \leq i,j \leq 2M} ||\psi_{i,j}^{\text{ext}} - \psi_{i,j}^{\text{app}}||
\]

\[
RMS = \sqrt{\frac{\sum_{i=0}^{2M} \sum_{j=0}^{2M} (\psi_{i,j}^{\text{ext}} - \psi_{i,j}^{\text{app}})^2}{2M \times 2M}}, \quad E_r = \sqrt{\frac{\sum_{i=0}^{2M} \sum_{j=0}^{2M} (\psi_{i,j}^{\text{ext}} - \psi_{i,j}^{\text{app}})^2}{(\sum_{i=0}^{2M} \sum_{j=0}^{2M} (\psi_{i,j}^{\text{ext}}))^2}}
\]

(5.34)

where $\psi^{\text{ext}}$, $\psi^{\text{app}}$ are respectively exact and approximate solutions.

Example 5.3.1

Consider Eq. (5.1) together with boundary and initial conditions

\[
\psi(0, y, t) = \psi(1, y, t) = 0, \quad \psi(x, 0, t) = \psi(x, 1, t) = 0, \quad t > 0,
\]

(5.35)

\[
\psi(x, y, 0) = \sin(\pi x) \sin(\pi y), \quad (x, y) \in \Phi.
\]

(5.36)

The procedure discussed earlier is adopted to find the solution of the problem. Comparison of the obtained results with exact solutions at different points are given in Table 5.1 for $J = 4$. From table it is obvious that the results achieved using proposed scheme are well matched with exact solution. Graphical behaviour of exact and approximate solution together with absolute error are shown in Fig. 5.1 when $t = 0.2$ and $\tau = 0.001$. Error norms at different times are compared and presented in Table 5.1 also.
Example 5.3.2

Consider the linear case of Sobolev equation [40] with homogenous boundary conditions and initial condition . We take different cases of exact solutions as

\[
\text{case } (i) : \psi = \sin(\pi x) \sin(\pi y) \exp(-t) \\
\text{case } (ii) : \psi = \sin(\pi x) \sin(\pi y) \exp(x - y - t). \tag{5.37}
\]
All conditions and source term are deduced from exact solution. In Table 5.2 and 5.3 we recorded the computed error norms with increasing the resolution levels for both cases, which improve accuracy. The obtained results are also matched with the previous work which show that proposed method has better outcomes. Graphical solution and absolute error are displayed in Figs. 5.2 and 5.3. From table it is obvious that current scheme give good results with less number of collocation points.

Figure 5.2: Graphical solution and absolute error of example 5.3.2 at $t = 1, \tau = 0.01, \theta = 1/2$ for case (i).

Figure 5.3: Graphical solution and absolute error of example 5.3.2 at $t = 1, \tau = 0.01, \theta = 1/2$ for case (ii).
Example 5.3.3

We consider the following nonlinear two-dimensional Burgers’ equation

\[ \psi_t + \psi \psi_x + \psi \psi_y = \nu [\psi_{xx} + \psi_{yy}] . \tag{5.38} \]

Applying the scheme discussed earlier, one can write

\[ \frac{\psi^{j+1} - \psi^j}{\tau} + \theta (\psi \psi_x + \psi \psi_y)^{j+1} - \nu \theta (\psi_{xx} + \psi_{yy})^{j+1} = (\theta - 1) (\psi \psi_x + \psi \psi_y)^j + \nu(1 - \theta) (\psi_{xx} + \psi_{yy})^j. \tag{5.39} \]

In Eq. (5.2) nonlinear terms are linearized with the help of formula [116]

\[ (\psi \psi_x)^{j+1} \approx \psi^{j+1} (\psi_x)^j + (\psi_x)^{j+1} \psi^j - \psi^j (\psi_x)^j. \tag{5.40} \]

Using Eq. (5.40) in Eq. (5.39) gives

\[ \psi^{j+1} + \tau \theta (\psi_x^j + \psi_y^j) \psi^{j+1} + \tau \theta (\psi_x^{j+1} + \psi_y^{j+1}) - \nu \tau \theta (\psi_{xx}^{j+1} + \psi_{yy}^{j+1}) = \psi^j + \tau (2\theta - 1) \psi^j \psi_x^j + \tau (2\theta - 1) \psi^j \psi_y^j + \tau \nu (1 - \theta) (\psi_{xx}^j + \psi_{yy}^j) . \tag{5.41} \]

Putting values from Eqs. (5.16)-(5.20) in Eq. (5.41), we obtain required system of equation for determination of wavelets coefficients. Once these coefficient are known, solution at different time level can be computed from Eq. (5.20). The problem has been solved and numerical results for \( \nu = 1 \) are shown, at different points and times, in Table 5.4. The error norms are computed and compared with the results available in literature which illustrates the efficiency of the scheme (see Table 5.4). Graphical solution, absolute error and contour plots are shown in Fig. 5.4 when \( J = 4 \).
Figure 5.4: Exact, approximate solutions, error and contour plot of example 5.3.3 at $t = 0.5$, $\delta t = 0.0025$.

**Example 5.3.4**

Consider nonlinear Soblev equation of the form [117] with with initial and boundary conditions

$$
\begin{align*}
\psi(x, y, 0) &= \sin(\pi x) \sin(\pi y), \quad (x, y) \in \Phi \\
\psi(x, y, t) &= \exp(t) \sin(\pi x) \sin(\pi t), \quad (x, y), \quad (x, y) \in \partial \Phi, \quad t > 0.
\end{align*}
$$

(5.42)
Chapter 5  Solution of (1+2)-dimensional PDEs and coupled PDEs

Exact solution and source term for this problem are given by

\[ \psi = \exp(t) \sin(\pi x) \sin(\pi y), \]
\[ \xi = -\exp(2t)\pi^2 (\cos(\pi x))^2 (\sin(\pi y))^2 + \exp(t) \sin(\pi x) \sin(\pi y) + 2 \exp(t)\pi^2 \sin(\pi x) \sin(\pi y) 
- \exp(2t)\pi^2 (\cos(\pi x))^2 (\sin(\pi y))^2 + 3 \exp(2t)\pi^2 (\sin(\pi x))^2 (\sin(\pi y))^2. \]

The boundary and initial condition are used from exact solution. Different error norms have been determined and presented in Table 5.5. It is clear from table that increasing the value of \( J \) show decreasing in error norms. Therefore one can observe easily that proposed method has good accuracy at small resolution level. Graphical solution and absolute error are given in Fig. 5.5. Figure show that exact and numerical solution are quite matchable.

![Graphical solution and absolute error](image)

**Figure 5.5:** Graphical solution and absolute error of example 5.3.5 at \( t = 1, \tau = 0.01, \theta = 1/2. \)

**Example 5.3.5**

Consider NGBBMB equation (5.5) with associated initial and boundary conditions

\[
\begin{align*}
\psi(x, y, 0) &= 0, \quad (x, y) \in \Phi \\
\psi(x, y, t) &= t \sin(x + y), \quad (x, y) \in \partial \Phi, \ t > 0.
\end{align*}
\] (5.43)

The corresponding source term can be computed from exact solution

\[ \psi = t \sin(x + y). \]
Numerical solution has been computed for different resolution levels and matched with exact solution. In Table 5.5 we monitor the error norms at different times. It has been observed that in case of highly nonlinear problem the proposed method has good results. Exact verses numerical solution have been figured out and given in Fig. 5.6. From figure absolute error shows that exact and approximate solution are very near.

Example 5.3.6

Finally, we take two-dimensional Burgers system (5.6) with exact solution

\[ \psi(x, y, t) = \frac{3}{4} - \frac{1}{4} \left[ 1 + \exp \left( (-4x + 4y - t) \frac{R}{32} \right) \right]^{-1} \]  
\[ u(x, y, t) = \frac{3}{4} + \frac{1}{4} \left[ 1 + \exp \left( (-4x + 4y - t) \frac{R}{32} \right) \right]^{-1}. \]
Initial and boundary conditions are derived from the exact solution. Applying the proposed technique to system of equations (5.6) and (5.5), we obtain

\[
\sum_{i=1}^{2M} \sum_{l=1}^{2M} \left[ \alpha_{i,l} \left\{ \tilde{A}_{i,l}(m,n) + \theta \tau \left( \tilde{A}_{i,l}(m,n) \psi_{x}^{3} + \tilde{E}_{i,l}(m,n) \psi_{y}^{3} + \tilde{F}_{i,l}(m,n)u^{3} \right) \right\} - \frac{1}{Re} \left( \tilde{B}_{i,l}(m,n) + \tilde{C}_{i,l}(m,n) \right) \right] = \tilde{\Gamma}(m,n) \tag{5.46}
\]

\[
\sum_{i=1}^{2M} \sum_{l=1}^{2M} \left[ \tau \theta \alpha_{i,l} \left\{ \tilde{A}_{i,l}(m,n) \right\} + \tilde{\alpha}_{i,l} \left\{ \tilde{A}_{i,l}(m,n) + \tau \theta \left( \tilde{A}_{i,l}(m,n)u_{y}^{3} + \tilde{E}_{i,l}(m,n) \psi_{y}^{3} + \tilde{F}_{i,l}(m,n)u^{3} \right) \right\} \right] = \tilde{\Psi}(m,n) \tag{5.47}
\]

where \( \tilde{A}_{i,l}(m,n), \tilde{B}_{i,l}(m,n), \tilde{C}_{i,l}(m,n) \) are given in Eq. (5.22) and

\[
\tilde{E}_{i,l}(m,n) = \left[ \mathcal{R}_{i,1}(x_{m}) - \mathcal{R}_{i,2}(1) \right] \left[ \mathcal{R}_{i,2}(y_{n}) - \mathcal{R}_{i,1}(1) \right]
\]

\[
\tilde{F}_{i,l}(m,n) = \left[ \mathcal{R}_{i,2}(x_{m}) - x_{m} \mathcal{R}_{i,2}(1) \right] \left[ \mathcal{R}_{i,1}(y_{n}) - \mathcal{R}_{i,2}(1) \right].
\]

The expression for \( \tilde{\Gamma}(m,n), \tilde{\Psi}(m,n) \) can be derived in a similar way as \( \tilde{D}(m,n) \) in Eq. (5.22).

To compare the results with earlier work different values of \( Re \) have been used. In Table 5.7, the obtained results are matched with exact and those got by Zhu et al. [118]. The table shows that obtained result are more accurate. The error norms of both component are also presented in Table 5.8 using Reynolds numbers \( Re = 1, 10, 100 \) and \( t = 0.01, t = 2 \). In Figs. 5.7 and 5.8 the exact, approximate solutions, errors and contour plots are plotted which illuminate that proposed scheme guarantee good results with less number of collocation points.
Figure 5.7: Exact, approximate, error and contour plot of $\psi(x, y, t)$ when $t = 0.5$, $\tau = 0.001$, $R_e = 80$. 
Figure 5.8: Exact, approximate, error and contour plot of $u(x,y,t)$ when $t = 0.5$, $\tau = 0.001$, $R_e = 80$
5.4 Conclusion

In this chapter, we presented a mixed numerical method based on finite differences and two dimensional Haar wavelets for linear and nonlinear problems. To validate the results six test problems have been solved. The computed results have matched with exact solution. Further for efficiency of the proposed method various error norms have been calculated. For all problem the corresponding solution profiles are shown for different parameters. It is clearly identified that the suggested scheme is efficient and suitable for solution of two dimensional nonlinear PDEs and system of PDEs.
Chapter 5

Solution of (1+2)-dimensional PDEs and coupled PDEs

Exact vs approximate solution

\begin{table}[h]
\centering
\begin{tabular}{llllllll}
\hline
\((x, y)\) & \multicolumn{2}{c}{\textit{t} = 0.05} & \multicolumn{2}{c}{\textit{t} = 0.1} & \multicolumn{2}{c}{\textit{t} = 0.2} \\
& \(\psi^{ext}\) & \(\psi^{app}\) & \(\psi^{ext}\) & \(\psi^{app}\) & \(\psi^{ext}\) & \(\psi^{app}\) \\
\hline
\(0.1, 0.1)\) & 0.03559 & 0.03557 & 0.01326 & 0.01325 & 0.00184 & 0.00183 \\
\(0.2, 0.5)\) & 0.21907 & 0.21898 & 0.08164 & 0.08158 & 0.01134 & 0.01132 \\
\(0.3, 0.6)\) & 0.28676 & 0.28664 & 0.10688 & 0.10678 & 0.01484 & 0.01482 \\
\(0.4, 0.8)\) & 0.20835 & 0.20826 & 0.07765 & 0.07758 & 0.01078 & 0.01076 \\
\(0.5, 0.3)\) & 0.30152 & 0.30139 & 0.11238 & 0.11228 & 0.01561 & 0.01558 \\
\(0.6, 0.5)\) & 0.35446 & 0.35431 & 0.13211 & 0.13199 & 0.01835 & 0.01832 \\
\(0.7, 0.7)\) & 0.24394 & 0.24383 & 0.09091 & 0.09084 & 0.01262 & 0.01260 \\
\(0.8, 0.3)\) & 0.17723 & 0.17715 & 0.06605 & 0.06600 & 0.00917 & 0.00916 \\
\(0.9, 0.9)\) & 0.03559 & 0.03557 & 0.01326 & 0.01325 & 0.00184 & 0.00183 \\
\hline
\end{tabular}
\caption{Exact, approximate solutions and Error norms corresponding to example 5.3.1 when \(t = 0.2\), \(\tau = 0.001\) and \(J = 5\).}
\end{table}
### Case i

<table>
<thead>
<tr>
<th>$2M$</th>
<th>$L_{\infty}$</th>
<th>$L_2$</th>
<th>[119]</th>
</tr>
</thead>
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<td>2.2100e-02</td>
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<td>1.7223e-03</td>
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<td>9.5508e-04</td>
<td>4.2893e-04</td>
</tr>
</tbody>
</table>

Table 5.2: Error norms of example 5.3.2 at $t = 1, \tau = 0.01, \theta = 1/2$ case (i).

### Case ii

<table>
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<th>$L_{\infty}$</th>
<th>$L_2$</th>
<th>[119]</th>
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<td>4.9551e-05</td>
<td>6.8838e-04</td>
<td>1.1697e-04</td>
</tr>
</tbody>
</table>

Table 5.3: Error norms of example 5.3.2 at $t = 1, \tau = 0.01, \theta = 1/2$ case (ii).
### Exact vs approximate solution

<table>
<thead>
<tr>
<th>$(x, y)$</th>
<th>$t = 0.25$</th>
<th></th>
<th>$t = 0.5$</th>
<th></th>
<th>$t = 1$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\psi_{\text{ext}}$</td>
<td>$\psi_{\text{app}}$</td>
<td>$\psi_{\text{ext}}$</td>
<td>$\psi_{\text{app}}$</td>
<td>$\psi_{\text{ext}}$</td>
<td>$\psi_{\text{app}}$</td>
</tr>
<tr>
<td>(0.1,0.1)</td>
<td>0.506240</td>
<td>0.50624</td>
<td>0.53742</td>
<td>0.53742</td>
<td>0.59868</td>
<td>0.59868</td>
</tr>
<tr>
<td>(0.2,0.5)</td>
<td>0.44398</td>
<td>0.44396</td>
<td>0.47502</td>
<td>0.47497</td>
<td>0.53742</td>
<td>0.53733</td>
</tr>
<tr>
<td>(0.3,0.6)</td>
<td>0.41945</td>
<td>0.41944</td>
<td>0.45016</td>
<td>0.45011</td>
<td>0.51249</td>
<td>0.51237</td>
</tr>
<tr>
<td>(0.4,0.8)</td>
<td>0.38343</td>
<td>0.38343</td>
<td>0.41338</td>
<td>0.41334</td>
<td>0.47502</td>
<td>0.47490</td>
</tr>
<tr>
<td>(0.5,0.3)</td>
<td>0.43168</td>
<td>0.43161</td>
<td>0.46257</td>
<td>0.46253</td>
<td>0.52497</td>
<td>0.52499</td>
</tr>
<tr>
<td>(0.6,0.5)</td>
<td>0.39532</td>
<td>0.39526</td>
<td>0.42555</td>
<td>0.42551</td>
<td>0.48750</td>
<td>0.48748</td>
</tr>
<tr>
<td>(0.7,0.7)</td>
<td>0.36008</td>
<td>0.36005</td>
<td>0.38936</td>
<td>0.38933</td>
<td>0.45016</td>
<td>0.45013</td>
</tr>
<tr>
<td>(0.8,0.3)</td>
<td>0.39532</td>
<td>0.39521</td>
<td>0.42555</td>
<td>0.42549</td>
<td>0.48750</td>
<td>0.48753</td>
</tr>
<tr>
<td>(0.9,0.9)</td>
<td>0.31539</td>
<td>0.31539</td>
<td>0.34298</td>
<td>0.34298</td>
<td>0.40131</td>
<td>0.40130</td>
</tr>
</tbody>
</table>

### Error norms

<table>
<thead>
<tr>
<th>Method</th>
<th>t</th>
<th>Points</th>
<th>$L_\infty$</th>
<th>$L_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present method</td>
<td>0.5</td>
<td>$4 \times 4$</td>
<td>5.8158e-05</td>
<td>1.5352e-04</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>$8 \times 8$</td>
<td>6.3677e-05</td>
<td>2.7799e-04</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>$16 \times 16$</td>
<td>6.3042e-05</td>
<td>5.4247e-04</td>
</tr>
<tr>
<td>[120]</td>
<td>0.25</td>
<td>$10 \times 10$</td>
<td>7.8844e-04</td>
<td>4.0331e-03</td>
</tr>
<tr>
<td>[120]</td>
<td>0.25</td>
<td>$20 \times 20$</td>
<td>8.5596e-05</td>
<td>8.4763e-04</td>
</tr>
</tbody>
</table>

Table 5.4: Exact vs approximate solutions and error norms corresponding to example 5.3.3.
### Table 5.5: Error norms of example 5.3.5 at $t = 0.2$, $1.0, \tau = 0.01, \theta = 1$, $J = 3$.

<table>
<thead>
<tr>
<th>2M</th>
<th>$L_\infty$</th>
<th>$L_2$</th>
<th>$L_\infty$</th>
<th>$L_2$</th>
<th>$L_\infty$</th>
<th>$L_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2.3412e-02</td>
<td>5.7558e-02</td>
<td>5.6165e-02</td>
<td>1.6630e-01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>5.8451e-03</td>
<td>2.6368e-03</td>
<td>1.0653e-02</td>
<td>6.2990e-02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>8.0502e-04</td>
<td>7.4675e-03</td>
<td>7.8741e-04</td>
<td>9.3335e-03</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 5.6: Error norms of example 5.3.5 at $t = 0.2$, $0.5, \tau = 0.01, \theta = 1/2$, $J = 3$.

<table>
<thead>
<tr>
<th>2M</th>
<th>$L_\infty$</th>
<th>$L_2$</th>
<th>$L_\infty$</th>
<th>$L_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2.0642e-04</td>
<td>4.9922e-04</td>
<td>5.7942e-04</td>
<td>1.3820e-03</td>
</tr>
<tr>
<td>8</td>
<td>2.0581e-04</td>
<td>9.7197e-04</td>
<td>5.6414e-04</td>
<td>2.638e-03</td>
</tr>
<tr>
<td>16</td>
<td>2.0530e-04</td>
<td>1.9331e-03</td>
<td>5.6699e-04</td>
<td>5.2227e-03</td>
</tr>
</tbody>
</table>
Table 5.7: Comparison of exact and approximate solutions of example 5.3.6 at different times when $\tau = 0.0001$. 

<table>
<thead>
<tr>
<th>$(x, y)$</th>
<th>$u^{\text{ext}}$</th>
<th>$u^{\text{app}}$</th>
<th>$[118]$</th>
<th>$t = 0.2$</th>
<th></th>
<th>$u^{\text{ext}}$</th>
<th>$u^{\text{app}}$</th>
<th>$[118]$</th>
<th>$t = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0.1,0.1)$</td>
<td>0.90561</td>
<td>0.90565</td>
<td>0.88267</td>
<td>0.94432</td>
<td>0.94441</td>
<td>0.94399</td>
<td></td>
<td>$(0.1,0.1)$</td>
<td>0.90561</td>
</tr>
<tr>
<td>$(0.2,0.9)$</td>
<td>0.99986</td>
<td>0.99986</td>
<td>0.99980</td>
<td>0.99993</td>
<td>0.99995</td>
<td>0.99996</td>
<td></td>
<td>$(0.2,0.9)$</td>
<td>0.99986</td>
</tr>
<tr>
<td>$(0.3,0.8)$</td>
<td>0.99898</td>
<td>0.99898</td>
<td>0.99853</td>
<td>0.99951</td>
<td>0.99956</td>
<td>0.99971</td>
<td></td>
<td>$(0.3,0.8)$</td>
<td>0.99898</td>
</tr>
<tr>
<td>$(0.4,0.7)$</td>
<td>0.99267</td>
<td>0.99268</td>
<td>0.98954</td>
<td>0.99648</td>
<td>0.99656</td>
<td>0.99722</td>
<td></td>
<td>$(0.4,0.7)$</td>
<td>0.99267</td>
</tr>
<tr>
<td>$(0.5,0.9)$</td>
<td>0.99725</td>
<td>0.99723</td>
<td>0.99605</td>
<td>0.99869</td>
<td>0.99867</td>
<td>0.99913</td>
<td></td>
<td>$(0.5,0.9)$</td>
<td>0.99725</td>
</tr>
<tr>
<td>$(0.6,0.1)$</td>
<td>0.75274</td>
<td>0.75272</td>
<td>0.75190</td>
<td>0.75574</td>
<td>0.75574</td>
<td>0.75565</td>
<td></td>
<td>$(0.6,0.1)$</td>
<td>0.75274</td>
</tr>
<tr>
<td>$(0.6,0.8)$</td>
<td>0.98103</td>
<td>0.98107</td>
<td>0.97342</td>
<td>0.99066</td>
<td>0.99071</td>
<td>0.99078</td>
<td></td>
<td>$(0.6,0.8)$</td>
<td>0.98103</td>
</tr>
<tr>
<td>$(0.7,0.3)$</td>
<td>0.75732</td>
<td>0.75733</td>
<td>0.75509</td>
<td>0.76502</td>
<td>0.76501</td>
<td>0.76473</td>
<td></td>
<td>$(0.7,0.3)$</td>
<td>0.75732</td>
</tr>
<tr>
<td>$(0.9,0.9)$</td>
<td>0.90561</td>
<td>0.90546</td>
<td>0.88267</td>
<td>0.94432</td>
<td>0.94410</td>
<td>0.94399</td>
<td></td>
<td>$(0.9,0.9)$</td>
<td>0.90561</td>
</tr>
</tbody>
</table>
\[ \psi(x, y, t) \]

<table>
<thead>
<tr>
<th>Method</th>
<th>points</th>
<th>( R )</th>
<th>( \tau )</th>
<th>( L_\infty ) ( E_R )</th>
<th>( L_\infty ) ( E_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present Method</td>
<td>8 \times 8</td>
<td>1</td>
<td>0.001</td>
<td>6.6802e-13 3.1992e-13</td>
<td>3.0445e-12 1.6701e-12</td>
</tr>
<tr>
<td></td>
<td>8 \times 8</td>
<td>10</td>
<td>0.001</td>
<td>6.8382e-08 2.9995e-08</td>
<td>3.2447e-06 1.2519e-06</td>
</tr>
<tr>
<td></td>
<td>32 \times 32</td>
<td>100</td>
<td>0.001</td>
<td>2.3469e-05 5.9317e-06</td>
<td>3.8865e-04 1.0282e-04</td>
</tr>
<tr>
<td>[121]</td>
<td>10 \times 10</td>
<td>1</td>
<td>0.0005</td>
<td>6.7236e-07 5.0078e-07</td>
<td>9.5121e-07 1.3922e-06</td>
</tr>
<tr>
<td>[121]</td>
<td>20 \times 20</td>
<td>100</td>
<td>0.001</td>
<td>3.3603e-06 4.5221e-07</td>
<td>7.6757e-05 2.5477e-05</td>
</tr>
</tbody>
</table>

Table 5.8: Comparison of error norms of example 5.3.6 at different times and Reynolds numbers.
Chapter 6

Solution of time fractional diffusion wave equations

6.1 Introduction

Fractional calculus is an ancient topic and equally important like calculus of integer order. However practical work in this direction has been inspected recently (see [122–124]). Most of the physical phenomenon in physics, chemistry, engineering and other fields of sciences can be modeled using the parameters of fractional calculus [125,126], means fractional derivative and integral operators. Fractional derivative play vital role while studying different phenomena like electrolyte polarization [127], viscoelastic systems [128], dielectric polarization [129] etc. Time fractional diffusion wave equations (TFDWE) is an important model having widespread applications. The TFDWE is actually wave equation [130] with fractional time derivative which describe universal acoustic, electromagnetic and mechanical responses [131,132] in a better way. From past few decades extensive attention has been paid for the closed form solution of fractional partial differential equations (PDEs) and still is an open area of research. In present chapter, we apply the scheme discussed in chapter 3 and 5, to solve (1+1) and (1+2)-dimensional TFDWEs. The models which will be under consideration are characterized in the following types:
Chapter 6  Solution of time fractional diffusion wave equations

(1+1)-dimensional equation

\[ cD_t^\xi \psi(x, t) = -\psi_t(x, t) + \psi_{xx}(x, t) + \mathcal{E}(x, t), \quad x \in \Phi, \quad t > 0, \quad 1 < \xi \leq 2, \quad (6.1) \]

\[
\begin{cases}
\psi(x, 0) = f(x), \\
\psi_t(x, 0) = g(x) \\
\psi(x, t) = \kappa(x, t), \quad x \in \partial \Phi, \quad t > 0.
\end{cases}
\quad (6.2)
\]

(1+2)-dimensional equation

\[ cD_t^\xi \psi(x, y, t) = \Delta \psi(x, y, t) + \mathcal{E}(x, y, t), \quad (x, y) \in \Phi, \quad t > 0, \quad 1 < \xi \leq 2, \quad (6.3) \]

\[
\begin{cases}
\psi(x, y, 0) = f(x, y), \\
\psi_t(x, y, 0) = g(x, y), \\
\psi(x, y, t) = \kappa(x, y, t), \quad (x, y) \in \partial \Phi, \quad t > 0.
\end{cases}
\quad (6.4)
\]

In Eqs. (6.1)-(6.4) \( cD_t^\xi \psi \) denotes fractional derivative of \( \psi \) w.r.t to \( t \) in Caputo sense. \( \Delta \) is two dimensional Laplacian, \( \mathcal{E} \) is the source term, \( f, g, \kappa \) are known functions and \( \psi \) is unknown function. Eqs. (6.2) and (6.4) are the corresponding initial and boundary conditions. The symbols, \( \Phi \) and \( \partial \Phi \), represent domain and boundary of the domain. The fraction derivative, in Caputo sense, of order \( \xi > 0 \) is defined as [12]

\[ cD_t^\xi \psi = \begin{cases}
\frac{1}{\Gamma(2-\xi)} \int_0^t \frac{\psi_{xx}(x, \varphi)}{(t-\varphi)^{\xi-1}} d\varphi, & 1 < \xi \leq 2 \\
\frac{d^2\psi(x, t)}{dt^2}, & \xi = 2.
\end{cases} \quad (6.5) \]
6.2 Description of the method

This portion is devoted to discuss the scheme for Eqs. (6.1) and (6.3) separately. In both cases, the fractional order time derivative has been approximated by quadrature formula [133]

\[ cD^\xi_t \psi(x, t^{j+1}) = \frac{1}{\Gamma(2-\xi)} \int_0^t \psi^{(2)}(x, \varphi)(t^{j+1} - \varphi)^{-\xi-1} d\varphi \]

\[ = \frac{1}{\Gamma(2-\xi)} \sum_{\kappa=0}^{j} \int_{t^{\kappa}}^{t^{j+1}} \frac{\psi^{(j+1)} - 2\psi^{(j)} + \psi^{(j-1)}}{\tau^2} (t^{j+1} - \varphi)^{-\xi-1} d\varphi \]

\[ = \frac{1}{\Gamma(2-\xi)} \sum_{\kappa=0}^{j} \left[ \frac{\psi^{(j+1)} - 2\psi^{(j)} + \psi^{(j-1)}}{\tau^2} \right] \int_{t^{\kappa}}^{t^{j+1}} [(j+1)\tau - \varphi]^\xi d\varphi \]

\[ = \frac{\tau^{-\xi}}{\Gamma(3-\xi)} \sum_{\kappa=0}^{j} \left[ \psi^{(j+1)} - 2\psi^{(j)} - \psi^{(j-1)} \right] \left[ (k+1)^2 - \frac{(k-1)^2}{(2-\xi)(\tau^2)} \right] \]

\[ = A_\xi \left[ \psi^{(j+1)} - 2\psi^{(j)} - \psi^{(j-1)} \right] + A_\xi \sum_{\kappa=1}^{j} \left[ \psi^{(j-\kappa+1)} - 2\psi^{(j-\kappa)} + \psi^{(j-\kappa-1)} \right] B(\kappa), \]

where \( A_\xi = \frac{\tau^{-\xi}}{\Gamma(3-\xi)}, \) \( \tau \) is time step size and \( B(\kappa) = (k+1)^2 - (k-1)^2. \)

Case i

Using Eq. (6.6) and \( \theta \)-weighted scheme Eq. (6.1) reduces to

\[ A_\xi \left[ \psi^{(j+1)} - 2\psi^{(j)} + \psi^{(j-1)} \right] + A_\xi \sum_{\kappa=1}^{j} \left[ \psi^{(j-\kappa+1)} - 2\psi^{(j-\kappa)} + \psi^{(j-\kappa-1)} \right] B(\kappa) + \frac{1}{\tau} \left[ \psi^{(j+1)} - \psi^{(j)} \right] \]

\[ = \theta \psi^{(j+1)}_{xx} + (1 - \theta) \psi^{(j)}_{xx} + \mathcal{E}(x, t^{j+1}). \tag{6.7} \]

After simplification, the above equation transforms to

\[ (\tau A_\xi + 1)\psi^{(j+1)} + \tau \theta \psi^{(j+1)}_{xx} = 2\tau A_\xi \psi^{(j)} - \tau A_\xi \psi^{(j-1)} - \tau A_\xi \sum_{\kappa=1}^{j} \left[ \psi^{(j-\kappa+1)} - 2\psi^{(j-\kappa)} + \psi^{(j-\kappa-1)} \right] B(\kappa) \]

\[ + \psi^{(j)} + \tau(1 - \theta) \psi^{(j)}_{xx} + \tau \mathcal{E}(x, t^{j+1}). \tag{6.8} \]
Substituting values of \( \psi_{xx}^{j+1}, \psi_{yy}^{j+1} \) using Eq. (3.10) and Eq. (3.15) in domain \([0,1]\) and collocation points (see 1.10) Eq. (6.8), can be written as

\[
2M \sum_{i=1}^{2M} \alpha_i \left[ (\tau A_\xi + 1) \{ R_{i,2}(x) - x R_{i,2}(1) \} - \tau \theta \mathbb{H}_i(x) \right]_{x=x_p} = P(p), \tag{6.9}
\]

where

\[
P(p) = 2\tau A_\xi \psi^j - \tau A_\xi \psi^{j-1} - \tau A_\xi \sum_{\kappa=1}^{J} \left[ \psi^{j-\kappa+1} - 2\psi^{j-\kappa} + \psi^{j-\kappa-1} \right] B(\kappa) + \psi^j
\]

\[
+ \tau (1 - \theta) \psi_{xx}^j + \tau E(x_p, t^{j+1}) - (\tau A_\xi + 1) \left( x_p \left( \psi^{j+1}(1) - \psi^{j+1}(0) \right) + \psi^{j+1}(0) \right).
\]

Eq. (6.9) contains \(2M\) equations and \(2M\) unknown wavelet coefficients. After determination of these unknown constants, the required solution at each time can be calculated from Eq. (3.15).

**Case ii**

Following a similar approach, as discussed earlier, Eq. (6.3) gives

\[
A_\xi \psi^{j+1} - \theta \left[ \psi_{xx}^{j+1} + \psi_{yy}^{j+1} \right] = (1 - \theta) \left[ \psi_{xx}^j + \psi_{yy}^j \right] + \mathcal{E}(x, y, t^{j+1}) + 2A_\xi \psi^j - A_\xi \psi^{j-1}
\]

\[
- A_\xi \sum_{\kappa=1}^{J} \left[ \psi^{j-\kappa+1} - 2\psi^{j-\kappa} + \psi^{j-\kappa-1} \right] B(\kappa). \tag{6.10}
\]

Substituting values of \( \psi_{xx}^{j+1}, \psi_{yy}^{j+1}, \psi^{j+1} \) using Eqs. (5.16)-(5.17), Eq. (5.20) and collocation points (see 5.21) leads to

\[
2M \sum_{i=1}^{2M} \sum_{l=1}^{2M} \left[ A_\xi \mathcal{D}(i, l, m, n) - \theta \mathcal{F}(i, l, m, n) - \theta \mathcal{G}(i, l, m, n) \right] = \mathcal{L}(m, n) + \mathbb{M}(m, n), \tag{6.11}
\]

where

\[
\mathcal{D}(i, l, m, n) = \left[ R_{i,2}(x_m) - x_m R_{i,2}(1) \right] \left[ R_{l,2}(y_n) - y_n R_{l,2}(1) \right],
\]

\[
\mathcal{F}(i, l, m, n) = \mathbb{H}_i(x_m) \left[ R_{l,2}(y_n) - y_n R_{l,2}(1) \right],
\]

\[
\mathcal{G}(i, l, m, n) = \left[ R_{i,2}(x_m) - x_m R_{i,2}(1) \right] \mathbb{H}_l(y_n),
\]

88
Chapter 6  Solution of time fractional diffusion wave equations

\[ \mathcal{L}(m, n) = (1 - \theta) \left[ \psi_{xx}^j + \psi_{yy}^j \right] + E(x_m, y_n, t^{j+1}) + 2A_\xi \psi^j - A_\xi \psi^{j-1}, \]

\[ - A_\xi \sum_{\kappa=1}^{\mathcal{M}} \left[ \psi^{j-\kappa+1} - 2\psi^{j-\kappa} + \psi^{j-\kappa-1} \right] B(\kappa) \]

\[ \mathcal{M}(m, n) = -A_\xi \left[ y_n \psi_{xx}^{j+1}(x_m, 0) - y_n \psi_{xx}^{j+1}(0, 0) \right] \]

\[ + x_m (y_n - 1) \psi_{xx}^{j+1}(0, 0) + x_m (1 - y_n) \psi_{xx}^{j+1}(0, y_n) + \theta \left[ y_n \psi_{xx}^{j+1}(x_m, 1) \right] \]

Eq. (6.11) represents 2M x 2\mathcal{M} equations in so many unknowns. After calculation of these unknowns, approximate solution can be obtained from Eq. (5.20) for any time.

### 6.3 Test problems

In this part, we chose some test problem to confirm reliability and efficiency of the present scheme.

For validation of our results \( L_\infty \) and \( L_2 \) error norm are figured out.

**Example 6.3.1**

Let us take the following (1+1)-dimensional TFDWE with damping

\[ ^cD_t^\xi \psi(x, t) = -\psi_t(x, t) + \psi_{xx}(x, t) + \mathcal{E}(x, t), \quad x \in [0, 1], \quad t \in [0, 1], \quad 1 < \xi \leq 2, \quad (6.12) \]

where

\[ \mathcal{E}(x, t) = \frac{2x(1 - x)t^\xi}{\Gamma(3 - \xi)} + 2tx(1 - x) + 2t^2. \]

The initial and boundary condition are derived from exact solution

\[ \psi(x, t) = t^2x(1 - x). \]

For calculations values of the parameters used are, \( J = 4, \quad t = 0.01, \quad 0.1, \quad 1, \quad \xi = 1.01, \quad 1.1, \quad 1.3, \quad 1.5. \)

The obtained error norms are shown in Table 6.1. From tabulated data small values of error norms indicate that the scheme has good results. Solution profile has been shown in Fig. 6.1. The
Table 6.1: Error norms of problem 6.3.1 at different times and $\xi$.

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$L_\infty$</th>
<th>$L_2$</th>
<th>$L_\infty$</th>
<th>$L_2$</th>
<th>$L_\infty$</th>
<th>$L_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.01</td>
<td>5.2459e-07</td>
<td>2.1202e-06</td>
<td>4.7241e-05</td>
<td>1.8958e-04</td>
<td>2.0171e-03</td>
<td>8.3998e-03</td>
</tr>
<tr>
<td>1.1</td>
<td>9.4026e-06</td>
<td>3.8754e-05</td>
<td>5.2146e-04</td>
<td>2.1161e-03</td>
<td>8.9206e-04</td>
<td>3.9029e-03</td>
</tr>
<tr>
<td>1.3</td>
<td>1.9547e-05</td>
<td>8.0729e-05</td>
<td>1.2722e-03</td>
<td>5.1807e-03</td>
<td>5.0349e-04</td>
<td>1.7468e-03</td>
</tr>
<tr>
<td>1.5</td>
<td>2.3157e-05</td>
<td>9.5719e-05</td>
<td>1.7388e-03</td>
<td>7.1044e-03</td>
<td>1.9977e-04</td>
<td>8.6655e-04</td>
</tr>
</tbody>
</table>

graphical solution show that exact and numerical solutions are in good agreement. Absolute error of the obtained solution at final time $t = 1$ also noted in figure. The computed error validate that current method is effective for TFPDEs.

![Graphical behavior of problem 6.3.1 t=1, $\xi = 1.5$.](image)

**Figure 6.1:** Graphical behavior of problem 6.3.1 $t=1$, $\xi = 1.5$.

**Example 6.3.2**

Consider the following TFDWE with damping

$$^cD_t^\delta \psi(x,t) = -\psi_t(x,t) + \psi_{xx}(x,t) + \mathcal{E}(x,t), \quad x \in [0,1], \quad t > 0, \quad 1 < \delta \leq 2,$$  

(6.13)
Chapter 6  Solution of time fractional diffusion wave equations

Table 6.2: maximum error norms of problem 6.3.2 at different times and $\xi$ and $\tau = 0.001$.

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$t = 0.01$, $\tau = 0.001$</th>
<th>$t = 0.1$, $\tau = 0.001$</th>
<th>$t = 0.3$, $\tau = 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L_{\infty}$</td>
<td>$L_2$</td>
<td>$L_{\infty}$</td>
</tr>
<tr>
<td>1.3</td>
<td>5.3862e-07</td>
<td>2.1113e-06</td>
<td>4.4278e-04</td>
</tr>
<tr>
<td>1.5</td>
<td>6.5702e-07</td>
<td>2.5068e-06</td>
<td>8.7071e-04</td>
</tr>
<tr>
<td>1.7</td>
<td>7.0201e-07</td>
<td>2.6642e-06</td>
<td>1.0489e-03</td>
</tr>
<tr>
<td>1.9</td>
<td>7.2270e-07</td>
<td>2.7819e-06</td>
<td>5.1085e-04</td>
</tr>
</tbody>
</table>

coupled with initial and boundary conditions

$$
\begin{align*}
\psi(x, 0) &= 0, \quad \psi_t(x, 0) = 0 \quad x \in [0, 1] \\
\psi(0, t) &= t^3, \quad \psi(1, t) = e t^3, \quad t > 0.
\end{align*}
$$

(6.14)

The initial and boundary conditions are used from exact solution

$$
\psi(x, t) = e^x t^3.
$$

The corresponding source term is

$$
E(x, t) = \frac{6t^{3-\alpha}e^x}{\gamma(4 - \xi)} + 3t^2 e^x - t^3 e^x.
$$

The required solution has been calculated using different times $t = 0.01$, 0.05, 0.1, and $\xi = 1.3$, 1.5, 1.7, 1.9. Computed error norms have been recorded in Table 6.2. The solution profile and absolute error are displayed in Fig. 6.2 for final time $t = 0.3$, and $\xi = 1.1$. It is obvious from table and figure that achieved results are well matched with exact solution.
Chapter 6 Solution of time fractional diffusion wave equations

Figure 6.2: Graphical behavior of problem 6.3.2 at $t=0.3$, $\xi=1.1$.

**Example 6.3.3**

Now, we consider $(1+2)$-dimensional TFDWE

$$^cD_t^\xi \psi(x, y, t) = \Delta \psi(x, y, t) + \mathcal{E}(x, y, t), \quad (x, y) \in [0, 1] \times [0, 1], \quad t > 0, \quad 1 < \xi \leq 2, \quad (6.15)$$

with exact solution and source term as follows:

$$\psi(x, y, t) = \sin(\pi x) \sin(\pi y)t^\xi+3$$

$$\mathcal{E}(x, y, t) = \sin(\pi x) \sin(\pi y) \left[ \frac{\Gamma(\xi + 3)t^2}{2} + 2t^{\xi+2} \right].$$

All conditions are extracted from exact solution. The parameters for computation are $J = 4$, $t = 0.1, 0.2, 0.5, \xi = 1, 3, 1.5, 1.7, 1.9$. In Table 6.3 the error norms are recoded for different values of times $t$ and $\xi$. Absolute error at different collocation points $((x_m, y_n), m, n = 1, 2, \cdots 9)$, have been shown in Table 6.4. Calculated error norms and point wise error validate that proposed scheme has good results in case of $(1+2)$-dimensional fractional problems. Graphical solutions and absolute error of the problem have been plotted in Fig. 6.3. It is clear from figure that approximate solution has good agreement with exact solution.
Table 6.3: maximum error norms of problem 6.3.3, at different times and $\xi$.

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$t = 0.1$, $\tau = 0.001$</th>
<th>$t = 0.2$, $\tau = 0.01$</th>
<th>$t = 0.5$, $\tau = 0.05$.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L_\infty$</td>
<td>$E_r$</td>
<td>$L_\infty$</td>
</tr>
<tr>
<td>1.3</td>
<td>7.4150e-05</td>
<td>3.7164e-05</td>
<td>5.2686e-04</td>
</tr>
<tr>
<td>1.5</td>
<td>1.6049e-04</td>
<td>8.0439e-05</td>
<td>4.3534e-04</td>
</tr>
<tr>
<td>1.7</td>
<td>1.1635e-04</td>
<td>5.8320e-05</td>
<td>5.9673e-04</td>
</tr>
<tr>
<td>1.9</td>
<td>3.0965e-05</td>
<td>1.5519e-05</td>
<td>3.9390e-04</td>
</tr>
</tbody>
</table>

Figure 6.3: Graphical behavior of problem 6.3.3, at $t=0.5$, $\xi = 1.9$. 
Table 6.4: Absolute error at different points of 6.3.3 when \( t = 0.1, \ 0.5, \ \tau = 0.01, \ \xi = 1.9, \ J = 4 \) and \( \theta = \frac{1}{2} \).

<table>
<thead>
<tr>
<th>((x_m, y_n))</th>
<th>( t = 0.1 )</th>
<th>( t = 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x_1, y_1))</td>
<td>1.208838573635980e-07</td>
<td>4.518169928878084e-06</td>
</tr>
<tr>
<td>((x_2, y_2))</td>
<td>1.080980862169773e-06</td>
<td>4.040287373086797e-05</td>
</tr>
<tr>
<td>((x_3, y_3))</td>
<td>2.964278882315800e-06</td>
<td>1.107932522925619e-04</td>
</tr>
<tr>
<td>((x_4, y_4))</td>
<td>5.698403831052698e-06</td>
<td>2.129842428420887e-04</td>
</tr>
<tr>
<td>((x_5, y_5))</td>
<td>9.178284819915522e-06</td>
<td>3.430487029225764e-04</td>
</tr>
<tr>
<td>((x_6, y_6))</td>
<td>1.327019197404156e-05</td>
<td>4.959883282749794e-04</td>
</tr>
<tr>
<td>((x_7, y_7))</td>
<td>1.781687559626314e-05</td>
<td>6.659257348677165e-04</td>
</tr>
<tr>
<td>((x_8, y_8))</td>
<td>2.264360918478884e-05</td>
<td>8.463303234668984e-04</td>
</tr>
<tr>
<td>((x_9, y_9))</td>
<td>2.756490407467840e-05</td>
<td>1.030269246902e-03</td>
</tr>
</tbody>
</table>
6.4 Conclusion

In this chapter, the proposed method has been applied for solution of (1+1) and (1+2)-dimensional TFDWEs of constant order. The obtained solutions are compared with exact solution in all cases. Further, the efficiency of the method has been examined computing different error norms. In (1+2)-dimensional problem the computed solution matched with exact solution at different collocation points and shown absolute error. It has been observed from computation and graphical solution that the results of current method agree with exact solution.
Chapter 7

Solution of variable order time fractional dispersion and diffusion models

7.1 Introduction

In classical fractional calculus, the order of fractional derivative and integrations was an arbitrary non integral values but recently researchers have twisted their attention towards the topic of variable orders. This means that order may be function of time, space variable or both. Variable order fractional derivative (VOFD) can be used to model complex systems (134–138). The importance of VOFD versus constant order fractional derivative was showed by Sun et al. [139]. Numerical solution of variable order FDEs is currently an open forum for research because the complexity of analytical solution. In this direction, many authors have investigated numerical methods based on finite differences. Stability of explicit finite difference scheme was discussed by Lin [15] for variable order nonlinear diffusion equations. The stability and convergence of advection equations for Euler approximation was investigated by Zhuang et al. [140]. Anomalous variable sub-diffusion equation has been studied by Chen [141].
In current chapter, the suggested technique is implemented to obtain the solution of time fractional 
\((1+1)\)-dimensional advection dispersion and \((1+2)\)-dimensional advection diffusion equations. The 
problems solved are given as follows:

\textbf{(1+1)-dimensional equation}
\[
\eta_1 \psi_t(x,t) + \eta_2 cD_t^{\xi(x,t)} \psi(x,t) + \eta_3 \psi_x(x,t) - \eta_4 \psi_{xx}(x,t) = \mathcal{E}(x,t), \quad x \in \Phi, \ t > 0, \tag{7.1}
\]
with the following conditions
\[
\begin{cases}
\psi(x,0) = h(x), & x \in \Phi \\
\psi(x,t) = q(t), & x \in \partial\Phi, \ t > 0,
\end{cases} \tag{7.2}
\]
where \(\Phi, \partial\Phi\) represent domain and boundary of the domain, \(\eta_i > 0, \ i = 1, \ldots, 4, \ 0 \leq \xi(x,t) \leq 1\).

\textbf{(1+2)-dimensional equation}
\[
cD_t^{\xi(x,y,t)} \psi(x,y,t) = \Delta \psi(x,y,t) - \left[\psi_x(x,y,t) + \psi_y(x,y,t)\right] + \mathcal{E}(x,y,t), \ (x,y) \in \Phi, \ t > 0, \tag{7.3}
\]
with the following initial and boundary conditions
\[
\begin{cases}
\psi(x,y,0) = h(x,y), & (x,y) \in \Phi \\
\psi(x,y,t) = q(x,y,t), & (x,y) \in \partial\Phi, \ t > 0.
\end{cases} \tag{7.4}
\]
In both Eqs. (7.1) and (7.3), \(cD_t^{\xi(x,t)} \psi\) denote VOFD of \(\psi\) w.r.t \(t\) in Caputo sense [142,143] 
which is described as follows
\[
cD_t^{\xi(x,t)} \psi(x,t) = \begin{cases}
\frac{1}{\Gamma(1-\xi(x,t))} \int_0^t \frac{\psi_t(x,\varphi)}{(t-\varphi)^{1-\xi(x,t)}} d\varphi, & 0 \leq \xi(x,t) \leq 1, \\
\psi_t(x,t), & \xi(x,t) = 1.
\end{cases} \tag{7.5}
\]
Chapter 7  
Solution of variable order time fractional dispersion and diffusion models

7.2 Method description

In this section, we proceed to explain the proposed method for concern problems. For both types of equations the variable fractional time derivative can be approximated as [144]:

\[
\frac{cD_t^\lambda f(x,t)}{\Gamma(1-\xi(x,t))} = \frac{1}{\Gamma(1-\xi(x,t))} \int_0^{t+1} \psi_\varphi(x,\varphi) \left( (t+1-\varphi)^{-\xi(x,t+1)} \right) d\varphi
\]

where \( t^j = j\tau, \) \( j = 0, \cdots, N, \) and \( \tau \) is time step. Applying forward difference for the time derivative, Eq. (7.6) transforms to

\[
\frac{cD_t^\lambda f(x,t)}{\Gamma(1-\xi(x,t))} = \frac{1}{\Gamma(1-\xi(x,t))} \sum_{\lambda=0}^{J} \psi_\varphi(x,\varphi) \left( (t+1-\varphi)^{-\xi(x,t+1)} \right) d\varphi.
\]

Simplification of the above integral yields

\[
\int_{\lambda \tau}^{(\lambda+1)\tau} \left( (t+1-\varphi)^{-\xi(x,t+1)} \right) d\varphi = \left\{ \frac{\tau^{1-\xi(x,t+1)}}{1-\xi(x,t+1)} \left( (j-\lambda+1)^{1-\xi(x,t+1)} - (j-\lambda)^{1-\xi(x,t+1)} \right) \right\}.
\]

Let \( \omega_\lambda(x,t^{j+1}) = (\lambda + 1)^{1-\xi(x,t^{j+1})} - (\lambda)^{1-\xi(x,t^{j+1})}, \) and shifting the indices Eq. (7.7) reduces to

\[
\frac{cD_t^\lambda f(x,t)}{\Gamma(2-\xi(x,t^{j+1}))} = \frac{\tau^{-\xi(x,t^{j+1})}}{\Gamma(2-\xi(x,t^{j+1}))} \sum_{\lambda=0}^{J} \omega_\lambda(x,t^{j+1}) \left[ \psi_{j+1}^{\lambda+1} - \psi_{j}^{\lambda+1} \right] + O(\tau)^{2-\xi(x,t^{j+1})}
\]

(7.9)

where

\[
r(x,t^{j+1}) = \frac{\tau^{-\xi(x,t^{j+1})}}{\Gamma(2-\xi(x,t^{j+1}))}.
\]

Case i

Here we apply \( \theta \)-weighted \( (0 \leq \theta \leq 1) \) scheme to Eq. (7.1) in the following way:

\[
\eta_1 \left( \frac{\psi_{j+1}^{\lambda+1} - \psi_{j}^{\lambda+1}}{\tau} \right) + \eta_2 \frac{cD_t^\lambda f(x,t)}{\Gamma(1-\xi(x,t))} + \eta_3 \psi_\varphi(x,\varphi) \left( (t+1-\varphi)^{-\xi(x,t+1)} \right) \]

(7.10)

\[
= (\theta - 1) \left( \eta_3 \psi_\varphi(x,t) - \eta_4 \psi_\varphi(x,t^j) \right) + \mathcal{E}(x,t^{j+1}) + \mathcal{E}(x,t^{j+1}).
\]

98
Substituting Eq. (7.9) in Eq. (7.10), we obtain

\[ \eta_1 \psi_{j+1}^{r_{j+1}} + \eta_2 \tau r_{j+1} \psi_{j+1} - \tau \theta \psi_{j+1}^{r_{j+1}} \psi_{xx} = \eta_1 \psi_j + \eta_2 \tau r_{j+1} \psi_j \]

with boundary conditions

\[ \psi(x, t^{j+1}) = q(t^{j+1}), \quad x \in \partial \Phi, \quad t > 0. \] (7.12)

Now using values of \( \psi_{j+1}^{r_{j+1}}, \psi_{j+1}^{r_{j+1}}, \psi_{j+1} \) from Eqs. (3.10), (3.14), (3.15) in Eq. (7.11), the required system of equations is

\[ \sum_{i=1}^{2M} \alpha_i \left[ \eta_1 \tilde{N}(i, p) + \eta_2 \tau r_{j+1} \tilde{N}(i, p) + \tau \theta \tilde{O}(i, p) - \tau \theta \tilde{P}(i, p) \right] = \tilde{Z}(x_p), \] (7.13)

where

\[ \tilde{N}(i, m) = \left\{ \mathcal{R}_{i,2}(x_p) - x_p \mathcal{R}_{i,2}(1) \right\}, \quad \tilde{O}(i, p) = \left\{ \mathcal{R}_{i,1}(x_p) - \mathcal{R}_{i,2}(1) \right\}, \quad \tilde{P}(i, p) = \mathbb{H}(x_p), \]

\[ \tilde{Z}(x_p) = \eta_1 \psi_j + \eta_2 \tau r_{j+1} \psi_{j+1} - \eta_2 \tau r_{j+1} \sum_{\lambda=1}^j \omega_{\lambda}^{j+1} \left\{ \psi_{j+1}^{\lambda+1} - \psi_{j+1}^{\lambda} \right\} + \tau \theta \left\{ \psi_{j+1}^{ax} - \psi_{j+1}^{axx} \right\} \]

\[ + \tau \mathcal{E}_{j+1} - \eta_1 \left( x_p \psi_{j+1}^{r_{j+1}}(1) - \psi_{j+1}^{r_{j+1}}(0) \right) + \psi_{j+1}(0) \right\} - \eta_2 \tau r_{j+1} x_p \left\{ \psi_{j+1}^{r_{j+1}}(1) - \psi_{j+1}^{r_{j+1}}(0) \right\} \]

\[ - \eta_2 \tau r_{j+1} \psi_{j+1}(0) - \tau \theta \right\{ \psi_{j+1}^{r_{j+1}}(1) - \psi_{j+1}^{r_{j+1}}(0) \right\}. \]

The unknowns wavelet coefficients \( \alpha_i, \quad i = 1 \cdots 2M, \) can be obtained from Eq. (7.13). After determination of these constants the approximated solution can be computed from Eq. (3.15).

**Case ii**

Now consider Eq. (7.3) and applying the scheme leads to

\[ r_{j+1} \psi_{j+1} - \theta \left( \psi_{j+1}^{ax} + \psi_{j+1}^{axx} \right) - \left( \psi_{j+1}^{ax} + \psi_{j+1}^{axx} \right) \right] = r_{j+1} \left[ \psi_j - \sum_{\lambda=1}^j \omega_{\lambda}^{j+1} \left\{ \psi_{j+1}^{\lambda+1} - \psi_{j+1}^{\lambda} \right\} \right] \]

\[ + (1 - \theta) \left[ \psi_{j+1}^{ax} + \psi_{j+1}^{axx} - \psi_j^{ax} + \psi_j^{axx} + \mathcal{E}_{j+1} \right]. \] (7.14)
with boundary conditions

\[ \psi(x, y, t^{1+}) = h(x, y, t^{1+}), \quad (x, y) \in \partial \Phi, \; t > 0. \]  

Substituting values of \( \psi_{xx}^{j+1}, \psi_{yy}^{j+1}, \psi^{j+1} \) using Eqs. (5.16)-(5.17), Eq. (5.20) and collocation points (see 5.21) in Eq. (7.14), the resultant system of equations is given by

\[
\sum_{i=1}^{2M} \sum_{l=1}^{2M} a_{i,l} \left[ r^{j+1} \hat{Q}(i, l, m, n) - \theta \left\{ \hat{R}(i, l, m, n) - \hat{S}(i, l, m, n) \right\} \right] = \hat{W}(m, n),
\]

where

\[
\hat{Q}(i, l, m, n) = R_{i,2}(x_m) - x_m R_{i,2}(1) \left[ R_{l,2}(y_n) - y_n R_{l,2}(1) \right],
\]

\[
\hat{R}(i, l, m, n) = H_i(x_m) \left[ R_{i,2}(y_n) - y_n R_{i,2}(1) \right] + \left[ R_{i,2}(x_m) - x_m R_{i,2}(1) \right] \| H_i(y_n),
\]

\[
\hat{S}(i, l, m, n) = \left[ R_{i,1}(x_m) - R_{i,2}(1) \right] \left[ R_{l,2}(y_n) - y_n R_{l,2}(1) \right] + \left[ R_{l,2}(x_m) - x_m R_{l,2}(1) \right]
\]

\[
R_{l,1}(y_n) - R_{l,2}(1),
\]

\[
\hat{Z}(m, n) = r^{j+1} \left[ \psi^j - \sum_{\lambda=1}^{j} \omega^{j+1}_{\lambda} \left\{ \psi^{j-\lambda+1} - \psi^{j-\lambda} \right\} \right] + (1 - \theta) \left[ \psi_{xx}^j + \psi_{yy}^j - \psi_x^j - \psi_y^j \right]
\]

\[
+ \mathcal{E}^{j+1} - r^{j+1} \left[ y_n \psi_x^{j+1}(x_m, 1) - y_m \psi_x^{j+1}(0, 1) + (1 - y_m) \psi_x^{j+1}(x_m, 0) - \psi_x^{j+1}(0, 0) \right]
\]

\[
+ \left[ x_m \psi_y^{j+1}(1, y_n) - x_n \psi_y^{j+1}(0, y_n) - x_m y_n \psi_y^{j+1}(1, 1) - \psi_y^{j+1}(0, 1) \right]
\]

\[
+ \left[ x_m y_n \psi_y^{j+1}(1, 0) - x_n \psi_y^{j+1}(1, 0) + x_m (1 - y_n) \psi_y^{j+1}(0, 0) + \psi_y^{j+1}(0, y_n) \right]
\]

\[
+ \theta \left[ \left( y_n \psi_{xx}^{j+1}(x_m, 1) + (1 - y_n) \psi_{xx}^{j+1}(x_m, 0) \right) - \left( y_n \psi_{xx}^{j+1}(x_m, 1) + (1 - y_n) \psi_{xx}^{j+1}(x_m, 0) \right) \right]
\]

\[
+ \left[ y_n \psi_{yy}^{j+1}(x_m, 0) + \psi_{yy}^{j+1}(x_m, 0) - \psi_{yy}^{j+1}(0, y_n) - y_n \psi_{yy}^{j+1}(1, 1) \right]
\]

\[
+ \left[ y_n \psi_{yy}^{j+1}(0, y_n) + \psi_{yy}^{j+1}(0, 0) + (1 - y_n) \psi_{yy}^{j+1}(0, 0) \right]
\]

\[
+ \left[ x_m \psi_{yy}^{j+1}(1, y_n) + (1 - x_m) \psi_{yy}^{j+1}(0, y_n) + \psi_{yy}^{j+1}(x_m, 1) - \psi_{yy}^{j+1}(x_m, 0) \right]
\]

\[
- x_m \psi_{yy}^{j+1}(1, 1) + x_m \psi_{yy}^{j+1}(1, 0) + (x_m - 1) \psi_{yy}^{j+1}(0, 0) + (1 - x_m) \psi_{yy}^{j+1}(0, 0) \right] \right] .
\]

The required unknowns can be calculated from Eq. (7.16) recursively. After determination of these coefficients the approximate solution can be obtained from Eq. (5.20).
7.3 Test problems

To check the reliability of the proposed scheme some test example are discussed below.

Example 7.3.1

First consider the equation

\[ \eta_1 \psi_t(x,t) + \eta_2 C D_t^\xi \psi(x,t) + \eta_3 \psi_x(x,t) - \eta_4 \psi_{xx}(x,t) = \mathcal{E}(x,y), \quad x \in [0,1], \ t > 0, \quad (7.17) \]

where

\[ \mathcal{E}(x,t) = \mathcal{E}_1(x,t) + \mathcal{E}_2(x,t) + \mathcal{E}_3(x,t) + \mathcal{E}_4(x,t), \]
\[ \mathcal{E}_1(x,t) = 10\eta_1 x^2(1 - x)^2, \quad \mathcal{E}_2(x,t) = \frac{10\eta_2 x^2(1 - x)^2 t^{1 - \xi(x,t)}}{\Gamma(2 - \xi(x,t))}, \]
\[ \mathcal{E}_3(x,t) = 10\eta_3 (1 + t)(2x - 6x^2 + 4x^3), \quad \mathcal{E}_4(x,t) = -100\eta_4 (1 + t)(2 - 12x + 12x^2), \]

coupled with

\[
\begin{cases}
\psi(x,0) = 10x^2(1 - x)^2, & x \in [0,1] \\
\psi(0,t) = \psi(1,t) = 0, & t > 0.
\end{cases}
\]

(7.18)

The exact solution of this problem is

\[ \psi(x,t) = 10x^2(1 - x)^2(1 + t). \]

For numerical computation we have used parameters \( J = 6, \ \eta_1 = \eta_2 = \eta_3 = \eta_4 = 1, \ \xi(x,t) = 1 - \frac{1}{2} e^{-xt}. \) The results obtained matched with existing results in literature. In Table 7.1 the the absolute error at different points have been recorded and compared with the work presented in [145]. One can see that the proposed method has better results as compare to those available in literature. The solution at different times are plotted in Fig. 7.1. From table it is obvious that the results of the current scheme promises well with exact solution.
Table 7.1: Comparison of absolute error of example 7.3.1 with previous work for $J = 6$, $t = 1$, $\tau = 0.001$, $\theta = 1$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>Error</th>
<th>[145]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3.1933e-05</td>
<td>1.5629e-04</td>
</tr>
<tr>
<td>0.2</td>
<td>5.9210e-05</td>
<td>1.4006e-03</td>
</tr>
<tr>
<td>0.3</td>
<td>8.2203e-05</td>
<td>2.9751e-03</td>
</tr>
<tr>
<td>0.4</td>
<td>1.0032e-04</td>
<td>4.2976e-03</td>
</tr>
<tr>
<td>0.5</td>
<td>1.1202e-04</td>
<td>4.9721e-03</td>
</tr>
<tr>
<td>0.6</td>
<td>1.1514e-04</td>
<td>4.8034e-03</td>
</tr>
<tr>
<td>0.7</td>
<td>1.0830e-04</td>
<td>3.8152e-03</td>
</tr>
<tr>
<td>0.8</td>
<td>8.9421e-05</td>
<td>2.2746e-03</td>
</tr>
<tr>
<td>0.9</td>
<td>5.5022e-05</td>
<td>7.2075e-04</td>
</tr>
</tbody>
</table>

Figure 7.1: Solution profile of example 7.3.1 at $t = 1$, $\theta = 1$, with $\tau = \frac{1}{10}$.

Example 7.3.2

Consider the equation

$$
\eta_1 \psi_t(x,t) + \eta_2 c D_t^{\xi(x,t)} \psi(x,t) + \eta_3 \psi_x(x,t) - \eta_4 \psi_{xx}(x,t) = \mathcal{E}(x,y), \quad x \in [0,1], \quad t > 0,
$$

(7.19)
coupled with

\[
\begin{align*}
\psi(x, 0) &= 5x(1 - x), \quad x \in [0, 1] \\
\psi(0, t) &= \psi(1, t) = 0, \quad t > 0,
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{E}(x, t) &= \mathcal{E}_1(x, t) + \mathcal{E}_2(x, t) + \mathcal{E}_3(x, t), \\
\mathcal{E}_1(x, t) &= 5\eta_1x(1 - x), \quad \mathcal{E}_2(x, t) = \frac{5\eta_2x(1 - x)t^{1-\xi(x, t)}}{\Gamma(2 - \xi(x, t))}, \\
\mathcal{E}_3(x, t) &= 5\eta_4(1 + t)(1 - 2x).
\end{align*}
\]

Initial and boundary conditions are used from exact solution

\[
\psi(x, t) = x(1 - x)(1 + t).
\]

Calculations have been carried out using parameters \( J = 4, \ \eta_1 = \eta_2 = \eta_3 = \eta_4 = 1 \) and \( \xi(x, t) = 0.8 + 0.005\cos(xt)\sin(x) \). For validation of the technique different error norms have presented for various values \( \delta \) in Table 7.2, and compared with available results addressed in [145]. From table it has been observed that the obtained results are pretty much better. It is to be noted that suggested method seemly good for variable order time fraction partial differential equation. One dimensional solution and at 3D plots of exact and numerical solution at final time \( t = 1 \) shown in Fig. 7.2. Plotted solution validate that exact and numerical solution very near to each other.

![Exact](image1.png) ![Approximate](image2.png) ![Exact and approximate](image3.png)

Figure 7.2: Solution profile of example 7.3.2 at \( t = 1, \ \theta = 1, \) with \( \tau = \frac{1}{100} \).
Table 7.2: Comparison of maximum absolute error of example 7.3.2 with previous work for $J = 4$, $t = 1$, $\theta = 1$.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>Max Error</th>
<th>[145]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/50</td>
<td>1.7763e-15</td>
<td>2.1562e-02</td>
</tr>
<tr>
<td>1/100</td>
<td>4.8849e-15</td>
<td>1.0825e-02</td>
</tr>
<tr>
<td>1/200</td>
<td>1.9984e-15</td>
<td>5.4267e-03</td>
</tr>
<tr>
<td>1/400</td>
<td>4.4408e-15</td>
<td>2.7164e-03</td>
</tr>
</tbody>
</table>

Example 7.3.3

Consider the equation

$$\eta_1 \psi_t(x,t) + \eta_2 c D_t^{\xi(x,t)} \psi(x,t) + \eta_3 \psi_x(x,t) - \eta_4 \psi_{xx}(x,t) = 0, \quad (x,t) \in [0,1], t > 0,$$

(7.21)

with the following conditions

$$ \begin{cases} 
\psi(x,0) = \sin(\pi x), & 0 \leq x \leq 1 \\
\psi(0,t) = \psi(1,t) = 0, & t > 0. 
\end{cases} $$

(7.22)

The graphical solution of this example are given in Fig. 7.3 for parameters

$$\xi(x,t) = 0.8 + 0.05 \exp(-x) \sin(t),$$

and $J = 4$, $\eta_1 = 0.1$, $\eta_2 = 2$, $\eta_3 = 1$, $\eta_4 = 2$. The obtained solution in this case shows mobile-immobile behaviour and is matchable with the work of Zhang [145].
Example 7.3.4

Now we consider $(1+2)$-dimensional advection diffusion equation in the form

$$c D_t^{\xi(x,y,t)} \psi(x,y,t) = \psi_{xx}(x,y,t) + \psi_{yy}(x,y,t) - \left[ \psi_x(x,y,t) + \psi_y(x,y,t) \right] + \mathcal{E}(x,y,t), \quad (7.23)$$

with

$$\mathcal{E}(x,y,t) = \frac{2t^{2-\xi(x,y,t)}}{3-\Gamma(x,y,t)} + 2x + 2y - 4.$$

Boundary and initial conditions are extracted from exact solution

$$\psi(x,y,t) = x^2 + y^2 + t^2.$$

Numerical solution have computed for parameters $J = 4$, $\theta = \frac{1}{2}$ and

$$\xi(x,y,t) = 0.8 - 0.1 \cos(xt) \sin(x) - 0.1 \cos(yt) \sin(y).$$

In Table 7.3 the absolute error at collocation points are shown. From table it is evident that the reported error is minimum. In Fig. 7.4 graphical solution is presented. In figure the solution curve at $y = 0.984375$, $t = 0.1$, $\tau = 0.001$ also presented. From figure it is obvious that numerical results are in good agreement with exact solution. One can see from computation and solution profile that suggested method is efficient and easy to apply in case of time fraction PDEs of variable order.
Table 7.3: Absolute error at different points of example 7.3.4 when $t = 0.01$, $\tau = 0.001$, $J = 4$ and $\theta = \frac{1}{2}$.

<table>
<thead>
<tr>
<th>$(x_m, y_n)$</th>
<th>$t = 0.01$</th>
<th>$t = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$</td>
<td>\psi^\text{ext} - \psi^\text{app}</td>
</tr>
<tr>
<td>$(x_1, y_1)$</td>
<td>9.843750388101769e-05</td>
<td>9.843751123689e-03</td>
</tr>
<tr>
<td>$(x_2, y_2)$</td>
<td>9.53125349291536e-05</td>
<td>9.531260113205e-03</td>
</tr>
<tr>
<td>$(x_3, y_3)$</td>
<td>9.218759702544176e-05</td>
<td>9.218778092235e-03</td>
</tr>
<tr>
<td>$(x_4, y_4)$</td>
<td>8.906269016984922e-05</td>
<td>8.906305060780e-03</td>
</tr>
<tr>
<td>$(x_5, y_5)$</td>
<td>8.593781436230123e-05</td>
<td>8.593841018841e-03</td>
</tr>
<tr>
<td>$(x_6, y_6)$</td>
<td>8.281296960307299e-05</td>
<td>8.28138566417e-03</td>
</tr>
<tr>
<td>$(x_7, y_7)$</td>
<td>7.968815589193268e-05</td>
<td>7.968939903507e-03</td>
</tr>
<tr>
<td>$(x_8, y_8)$</td>
<td>7.65637322888007e-05</td>
<td>7.65602830114e-03</td>
</tr>
<tr>
<td>$(x_9, y_9)$</td>
<td>7.343862161401926e-05</td>
<td>7.344074746235e-03</td>
</tr>
</tbody>
</table>

Figure 7.4: Solution profile of example 7.3.4.
7.4 Conclusion

In this chapter, the suggested scheme is applied to study variable order time fractional advection dispersion and advection diffusion models. To check the reliability of the method three (1+1) and one (1+2)-dimension problems have been solved. Further various error norms have calculated and matched with available results in literature. It has been experiential that proposed method is easy to apply for such equations.
Chapter 8

Conclusion

In this work, we proposed Haar wavelets collocation method for solution of nonlinear ODEs, (1+1)-dimensional and (1+2)-dimensional PDEs of integer and fractional orders. In chapter 2, Quasilinearization is applied to linearize the nonlinear ODEs, then highest order derivative is approximated by Haar wavelets. To check the efficiency of the method three test problems have been solved namely: squeezing flow problem, MHD squeezing flow problem and problem of third grade fluid on inclined plan. The obtained solutions have been compared with OHAM and Runge Kutta method of order 4. From comparison it was observed that proposed method is easy to apply for such problems. In chapter 3 a mixed numerical scheme is presented for solution of nonlinear (1+1)-dimensional PDEs. The proposed method is applied for the solution of nonlinear Burgers’, Boussinesq and equal width equations. The convergence theorem for the scheme has been proved. For validation of the technique we compared the obtained solutions with exact and those presented before. It has been observed that suggested scheme has better outcome. In chapter 4, the scheme has been applied for the solution of variable coefficients higher order PDEs, arising in Euler-Bernoulli Beam models. The efficiency of the scheme has been observed by solving three test problems. The obtained results have been matched with those of exact and existing results in literature. In chapter 5 we presented the scheme for (1+2)-dimensional PDEs. In this case again the convergence has been proved, which show inverse relation with the resolution level. We solved
five linear and nonlinear problems and one nonlinear Burgers’ system. For efficiency of the scheme various error norms have been calculated and matched with previous work. Also the graphical solutions and absolute error have been plotted. It has been noted that proposed method is equal efficient for (1+2)-dimensional PDEs. In chapter 6, we modified the scheme for (1+1) and (1+2)-dimensional constant order time fractional diffusion wave models. It has been experiential that time fractional problems can be solved easily with Haar wavelets. Chapter 7 presents the scheme for (1+1)-dimensional variable order time fractional advection dispersion and (1+2)-dimensional advection diffusion problems. We concluded from chapters (6&7) that suggested method works will for time fractional PDEs. In all cases we got good results which show that Haar wavelets are effective for solution of nonlinear ODEs, PDEs and time fractional problems.
Bibliography


