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Chapter 1

Literature review about boundary layer flows and description of solution procedure

The study of boundary layer flows over moving surfaces has gained major interest in different industries and technologies. Mostly such flows occur in heat treatment of material traveling between a feed roll and wind-up roll or a conveyer belt, spinning of fibers, glass blowing, cooling of a large metallic plate in a bath, aerodynamic extrusion of plastic sheets, continuous moving of plastic films etc. The extrude from a die is usually drawn and at the same time stretched into a sheet which is then solidified through regular make cold by direct contact. In such processes the quality of final product greatly depends on the rate of cooling which is fixed by structure of the boundary layer near the moving strip. Sakiadis [1] initiated the pioneering work on boundary layer flows over a moving surface. After Sakiadis, Crane [2] found closed from solution of boundary layer flow of viscous incompressible fluid over a stretching surface. He considered elastic sheet that stretched with velocity linearly proportional to the distance from the origin. This idea was later used by various investigators for effects of suction/injection, porosity, slip, MHD and heat and mass transfer (see [3 – 6]). Mcleod and Rajagopal [7] discussed the boundary layer flow of Navier-Stokes fluid over a moving stretching surface. Wang [8] explored the rotating boundary layer flow generated by a stretching sheet. Andersson et al. [9] computed exact
solution of viscous fluid in the presence of slip effects. Axisymmetric boundary layer flow of viscous fluid over a stretching surface with partial slip was examined by Ariel [10]. Liu and Wang [11] explored the derivation of Prandtl boundary layer with small viscosity limit. The 2D Navier-Stokes equations settled in a curved domain. Boundary layer flow of Carreau fluid over a convectively heated surface has been described by Hayat et al. [12]. Rosali et al. [13] examined the time-dependent boundary layer stagnation point flow in porous medium. Boundary layer flow of micropolar fluid with second order slip velocity over a shrinking surface was studies by Rosca and Pop [14]. Analytical solution of MHD free convective boundary layer flow over a vertical porous surface has been investigated by Raju et al. [15]. They also considered thermal radiation, chemical reaction and constant suction. Zhang and Zhang [16] found similarity solutions of boundary layer flow of power law fluid. Awais et al. [17] reported 3D boundary layer flow of Maxwell fluid. Effect of uniform magnetic field on free convective boundary layer flow of nanofluid has been studied by Freidoonimehr et al. [18]. Sheikholeslami et al. [19] have explored the effects of radiation and MHD in flow of nanofluid. Hydrothermal behavior of nanofluid in the presence of variable magnetic field is examined by Sheikholeslami and Ganji [20].

It is now established argument that the fluids in numerous technological and biological applications do not follow the commonly assumed relationship between the stress and the rate of strain at a point. Such fluids have come to be known as the non-Newtonian fluids. Materials like molten plastics, polymers, shampoos, certain oils, personal care products, pulps, mud, ice, foods and fossil fuels, display non-Newtonian behavior. In recent times there has been a great deal of interest in understanding the behavior of viscoelastic fluids. Such fluids are of great interest to the applied mathematicians and engineers from the points of view of theory and simulation of differential equations. On the other hand in applied sciences such as rheology or physics of the atmosphere, the approach to fluid mechanics is in an experimental set up leading to the measurement of material coefficients. In theoretically studying how to predict the weather, the ordinary differential equations represent the main tool. Since the failures in the predictions are strictly related to a chaotic behavior, one may find it unessential to ask whether the fluids are really Newtonian. Constitutive equations describe the rheological properties of viscoelastic fluids. Most of the models or constitutive equations are empirical.
or semi-empirical. Also the extra rheological parameters in such relations add more nonlinear terms to the corresponding differential systems. The order of the differential systems involving non-Newtonian fluids is higher in general than the Navier-Stokes equations. For this reason, a variety of non-Newtonian fluid models (exhibiting different rheological effects) are available in the literature [21 – 30]. Boundary layer flow of Powell-Eyring fluid over a moving flat late was analyzed by Hayat et al. [31]. Mehmood et al. [32] studied the influence of heat transfer on peristaltic motion of Walters’ B fluid. Series solutions for flow of Jeffrey and Casson fluids over a surface with convective boundary condition have been computed by Hayat et al. [33, 34]. Time-dependent flow of Casson fluid over a moving surface is investigated by Mukhopadhyay et al. [35]. Steady flow of Powel- Eyring fluid by an exponentially stretching sheet was numerically investigated by Mushtaq et al. [36]. The flow and heat transfer of Powell- Eyring fluid thin film over unsteady stretching sheet are examined by Khader and Megahed [37]. Impact of uniform suction/injection in unsteady Couette flow of Powel-Eyring fluid is explored by Zaman et al [38]. Effect of variable thermal conductivity in boundary layer flow of Casson fluid is examined by Hayat et al. [39]. Shafiq et al. [40] presented the influence of MHD on third grade fluid between two rotating disk. Stagnation point flow of Jeffrey fluid towards a stretching/shrinking surface was reported by Turkyilmazoglu and Pop [41]. Hayat et al. [42] studied the combines effect of Joule heating and thermal radiation in flow of third grade fluid. Two-dimensional flow of viscoelastic fluid in a vertical channel was examined by Marinca and Herisanu [43]. Turkyilmazoglu [44] found the exact solution for flow of electrically conducting couple stress fluid. Effect of variable thermal conductivity in flow of couple stress fluid has been reported by Asad et al. [45]. Mathematical modelling for axisymmetric MHD flow of Jeffrey fluid in rotating channel was addressed by Alla and Dahab [46]. In melting process the steady flow of Burgers’ fluid over a stretching surface was examined by Awais et al. [47]. Effect of non-uniform external magnetic field in stagnation point flow of micropolar fluid was studied by Borrelli et al. [48]. Javed et al. [49] examined the effect of heat transfer in peristalsis of Burgers fluid through compliant channels walls. Pramanik [50] discussed boundary layer flow of incompressible Casson fluid by an exponentially stretching sheet. He also considered the effect of suction or blowing at the surface.

The heat transfer over a moving surface is one of the hot areas of research due to its extensive
applications in science and engineering disciplines especially in chemical engineering processes. In such processes the fluid mechanical properties of the end product mainly depend on two factors, one is the cooling liquid property and other the rate of stretching. Many practical situations demand for physical properties with variable fluid characteristics. Thermal conductivity is one of such properties which are assumed to vary linearly with the temperature \[ [51]. \] Chiam [52] investigated the heat transfer with variable conductivity in stagnation point flow towards a stretching sheet. Tsai et al. [53] investigated the flow and heat transfer over an unsteady stretching surface with non-uniform heat source. They considered the flow analysis subject to variable thermal conductivity, thermal radiation, chemical reaction, Ohmic dissipation and suction/injection. Recently the variable thermal conductivity effect in boundary layer flow of viscous fluid due to a circular cylinder is analyzed by Oyem and Koriko [54]. Turkyilmazoglu and Pop [55] considered the heat and mass transfer effects in the flow of viscous nanofluid past a vertical infinite flat plate with radiation effect. Also the radiative effects on heat transfer characteristics are important in space technology, physics and engineering disciplines. These effects cannot be neglected as gas cooled nuclear reactors and power plants. Su et al. [56] studied the MHD mixed convective flow and heat transfer of an incompressible fluid by a stretching permeable wedge. They analyzed the flow in the presence of thermal radiation and Ohmic heating. The effects of thermal radiation on the flow of micropolar fluid past a porous shrinking sheet is investigated by Bhattacharyya et al. [57]. Thermal radiation effects in magnetohydrodynamic free convection heat and mass transfer from a sphere has been reported by Prasad et al. [58]. Das [59] examined the effect of radiation on electrically conducting boundary layer flow over a moving sheet. Natural convection flow over a semi-infinite irregular surface with radiation effect has been studied by Siddiqa et al. [60]. Ara et al. [61] reported the influence of thermal radiation in flow of an Eyring-Powell by an exponentially stretching surface. Very few researchers investigated the flow and heat transfer analysis with nonlinear thermal radiation. For example, Mushtaq et al. [62] explored nonlinear radiation effects in the stagnation point flow of upper-convective Maxwell (UCM) fluid. Cortell [63] numerically investigated the fluid flow and radiative nonlinear heat transfer in flow of viscous fluid over a stretching sheet. Nonlinear radiative heat transfer in the flow of nanofluid has been discussed by Mushtaq et al. [64]. Three dimensional flow of Jeffrey fluid with non-linear thermal radiation is discussed by
The heat transfer analysis in the past has been analyzed extensively either through the prescribed surface temperature or surface heat flux. Now it is known that heat transfer under convective boundary conditions is useful in processes such as thermal energy storage, gas turbines, nuclear plants, heat exchangers etc. For convection heat transfer to have a physical meaning, there must be a temperature difference between the heated surface and moving fluid. Blasius and Sakiadis flows with convective boundary conditions have been described by Cortell [66]. Aziz [67] numerically investigated the viscous flow over a flat plate with convective boundary conditions. Ishak [68] provided the numerical solution for flow and heat transfer over a permeable stretching sheet with convective boundary condition. Makinde [69] investigated the hydromagnetic flow by a vertical plate with convective surface boundary condition. Makinde and Aziz [70] considered the MHD mixed convection from a vertical plate embedded in a porous medium. They considered convective boundary condition. Makinde [71] examined the MHD heat and mass transfer in flow over a moving vertical plate with a convective surface boundary condition. Series solution for flow of second grade fluid with convective boundary conditions has been computed by Hayat et al. [72]. Mustafa et al. [73] explored the axisymmetric flow of nanofluid over a convectively heated radially stretching sheet. Investigation of heat and mass transfer together with thermal convective condition has been made by Hamad et al. [74]. Rashad et al. [75] studied mixed convection boundary layer flow in the presence of the thermal convective condition. The flow of electrically conducting nanofluid over a heated stretching with convective condition has been numerically computed by Das et al. [76]. Ramesh and Gireesha [77] examined numerical solution for the flow of Maxwell fluid over a surface with convective boundary condition. Patil et al. [78] analyzed influence of convective condition in the two-dimensional double diffusive mixed convection flow. Hayat et al. [79] discussed MHD nanofluid flow with convective boundary condition over an exponentially stretching surface. Effects of magnetic field, hydrodynamic slip and convective condition in entropy generated flow was studied by Ibanez [80].

Although the flow and heat transfer due to stretching cylinder is important in wire drawing, hot rolling and fiber. However not much has been said about it yet through stretching cylinder. The study of steady flow of an incompressible viscous fluid outside the stretching
cylinder in an ambient fluid at rest is made by Wang [81]. He found exact similarity solution of nonlinear ordinary differential problem. Burded [82] investigated the exact solutions of the Navier-Stokes equations which describe the steady axisymmetric motion of an incompressible fluid near a stretching infinite circular cylinder. Later, Ishak et al. [83 – 85] extended the work of Wang [81] in different physical situation. They presented a numerical solution for flow and heat transfer due to stretching cylinder. The boundary layer flow past a stretching cylinder and heat transfer with variable thermal conductivity has been reported by Rangi and Ahmad [86]. They found that the curvature of stretching cylinder plays a key role in flow and temperature field. Mukhopadhyay and Ishak [87] studied the laminar boundary layer mixed convection flow of viscous incompressible fluid towards a stretching cylinder immersed in a thermally stratified medium. They concluded that the temperature gradient is smaller for flow in a thermally stratified medium when compared to that of an unstratified medium. Chemically reactive solute transfer in boundary layer slip flow along a stretching cylinder has been examined by Mukhopadhyay [88]. MHD flow with heat transfer characteristics at a non-isothermal stretching cylinder has been examined by Vajravelu et al. [89]. They also considered internal heat generation/absorption and temperature dependent thermal conductivity. Slip effects in flow by stretching cylinder was studies by Mukhopadhyay [90]. Si et al. [91] investigated flow and heat transfer analysis over an impermeable stretching cylinder. Time-dependent flow and heat transfer over a contracting cylinder is studied by Zaimi et al. [92]. Ahmed et al. [93] reported the influence of heat source/sink over a permeable stretching tube. Mishra and Singh [94] found the dual solutions of momentum and thermal slip at a shrinking cylinder. Hayat et al. [95] presented stagnation point flow of an electrically conducting second grade fluid over a stretching cylinder.

### 1.1 Solution procedure

Flow equations of non-Newtonian fluids in general are highly nonlinear. Therefore it is very difficult to find the exact solution of such equations. Usually perturbation, Adomian decomposition and homotopy perturbation methods are used to find the solution of nonlinear equations. But these methods have some drawback through involvement of large/small parameters in the
equations and convergence. Homotopy analysis method (HAM) [96 – 110] is one while is independent of small/large parameters. This method also gives us a way to adjust and control the convergence region (i.e. by plotting h-curve). It also provides exemption to choose different sets of base functions.
Chapter 2

Heat transfer analysis in flow of Walters’ B fluid with convective boundary condition

2.1 Introduction

This chapter addresses the radiative heat transfer in the steady two-dimensional flow of Walters’ B fluid. An incompressible fluid is bounded by a stretching porous surface. Convective boundary condition is used for the thermal boundary layer problem. Mathematical analysis is carried out in the presences of non-uniform heat source/sink. The relevant equations are first simplified under usual boundary layer assumptions and then transformed into similar form by suitable transformations. Convergent series solutions of velocity and temperature are derived by homotopy analysis method (HAM). The dimensionless velocity and temperature gradients at the wall are calculated and discussed.

2.2 Problem formulation

We consider the steady two-dimensional flow of an incompressible Walters’ B fluid over a porous stretching surface. In addition, heat transfer analysis is considered with radiation effects and non-uniform heat source/sink. Further we consider $x-$axis parallel to the stretching sheet and
The extra stress tensor $S$ for Walters’ B fluid is defined by
\begin{equation}
S = 2\eta_0 A_1 - 2k_0 \frac{dA_1}{dt},
\end{equation}
where $A_1$ is the rate of strain tensor, $V$ is the velocity field of fluid, $\frac{dA_1}{dt}$ is the covariant derivative of the rate of strain tensor in relation to the material in motion, and $\eta_0$ and $k_0$ are the limiting viscosities at small shear rate and short memory coefficient respectively. These are defined as follows:
\begin{equation}
\eta_0 = \int_0^{\infty} N(\tau)d\tau,
\end{equation}
\begin{equation}
k_0 = \int_0^{\infty} \tau N(\tau)d\tau,
\end{equation}
with $N(\tau)$ being as the distribution function with relaxation time $\tau$. By taking short memory into account the terms involving
\begin{equation}
\int_0^{\infty} \tau^n N(\tau)d\tau, \quad n \geq 2,
\end{equation}
are neglected in case of Walters’ B fluid [22].

The equations of continuity, momentum and energy are
\begin{equation}
\text{div } V = 0,
\end{equation}
\begin{equation}
\rho \frac{dV}{dt} = \text{div } \tau,
\end{equation}
\begin{equation}
\rho c_p \frac{dT}{dt} = k \nabla^2 T + \tau \cdot (\text{grad } V).
\end{equation}
Here $\rho$ is the fluid density, $t$ the time, $V$ the velocity, $T$ the fluid temperature, $\tau$ the Cauchy stress tensor, $c_p$ the specific heat, $k$ the thermal conductivity of the material and $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. The velocity and temperature fields under the boundary layer approximations are governed.
by the following equations

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\eta_0}{\rho} \left( \frac{\partial^2 u}{\partial y^2} \right) - \frac{k_0}{\rho} \left( \frac{u \frac{\partial^3 u}{\partial x \partial y^2} + v \frac{\partial^3 u}{\partial y^3} + \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} }{\nu} \right),
\]

\[
\rho c_p \left( \frac{u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} - \frac{\partial q_r}{\partial y} + q'',
\]

\[
u = u_w(x) = ax, \quad v = v_w, \quad -k \frac{\partial T}{\partial y} = h(T_f - T) \quad \text{at} \quad y = 0,
\]

\[
u \to 0, \quad T \to T_\infty \quad \text{as} \quad y \to \infty,
\]

where \( u \) and \( v \) represent the velocity components in the \( x \) and \( y \) directions, respectively, \( u_w \) is the velocity of the stretching sheet, \( v_w \) is the constant mass transfer velocity with \( v_w > 0 \) for injection and \( v_w < 0 \) for suction, \( T \) is the fluid temperature, \( T_\infty \) is the ambient fluid temperature, \( k \) is the thermal conductivity of the fluid, \( \nu = (\mu/\rho) \) is the kinematic viscosity, and \( q'' \) is the non-uniform heat generated \( (q'' > 0) \) or absorbed \( (q'' < 0) \) per unit volume. For non-uniform heat source/sink, \( q'' \) is modeled by the following expression:

\[
q'' = \frac{k u_w(x)}{\nu} \left[ A(T_f - T_\infty)f' + (T - T_\infty)B \right],
\]

in which \( A \) and \( B \) are the coefficients of space and temperature-dependent heat source/sink, respectively. Here two cases arise. For internal heat generation \( A > 0 \) and \( B > 0 \) and for internal heat absorption, we have \( A < 0 \) and \( B < 0 \), \( \rho \) is the density of the fluid, \( k \) is the thermal conductivity of fluid, \( h \) is the convective heat transfer coefficient, and \( T_f \) is the convective fluid temperature below the moving sheet.

We define the similarity transformations through

\[
\psi = x \sqrt{av} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty}, \quad \eta = y \sqrt{\frac{a}{\nu}},
\]

where prime denotes the differentiation with respect to \( \eta \), \( f \) is the dimensionless stream function,
θ is the dimensionless temperature, and ψ is the stream function given by

\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \]  

(2.16)

Now Eq. (2.9) is identically satisfied and Eqs. (2.10 – 2.13) yield

\[ f''' + f f'' - f'^2 + We(f'' - 2f' f'' + f f''') = 0, \]

(2.17)

\[ (1 + \frac{4}{3}R)\theta'' + Pr (f\theta') + Af' + B\theta = 0, \]

(2.18)

where \( f = S, \ f' = 1, \ \theta' = -Bi[1 - \theta(0)] \) at \( \eta = 0, \)

(2.19)

\[ f' = 0, \ \theta = 0 \] at \( \eta = \infty. \)

(2.20)

Here \( f', f'', f''', \) and \( f'''' \) are the derivatives of the stream function with respect to \( \eta, \) \( We = \frac{k_0a}{n_0} \) is the local Weissenberg number, \( S = -\frac{\omega}{\omega_{aw}} \) is the mass transfer parameter with \( S > 0 \) for suction and \( S < 0 \) for injection, \( Pr = \frac{c_p}{\nu} \) is the Prandtl number, \( R = \frac{4\sigma*\gamma^3}{k_k} \) is the radiation parameter and \( Bi = \frac{h}{k}\sqrt{\frac{T}{a}} \) is the Biot number.

The skin friction coefficient \( C_f \) and local Nusselt number \( Nu_x \) are

\[ C_f = \frac{\tau_w}{\frac{1}{2} \rho u_w^2}, \quad \frac{N u_x}{k(T_f - T_{\infty})}, \]

(2.21)

where the skin friction \( \tau_w \) and the heat transfer from the plate \( q_w \) are

\[ \tau_w = \mu_0 \left( \frac{\partial u}{\partial y} \right) + k_0 \left\{ u \frac{\partial^2 u}{\partial x \partial y} + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + v \frac{\partial^2 u}{\partial y^2} \right\} \bigg|_{y=0}, \]

(2.22)

\[ q_w = -k \frac{\partial T}{\partial y} \bigg|_{y=0} - \frac{16\sigma\gamma^3}{3k_k} \frac{\partial T}{\partial y} \bigg|_{y=0}. \]

(2.23)

Dimensionless form of skin friction coefficient \( C_f \) and local Nusselt number \( Nu_x \) are

\[ C_f / Re_x^{1/2} = (1 - We) f''(0) - S f'''(0), \quad \frac{N u_x}{Re_x^{1/2}} = -(1 + \frac{4}{3}R)\theta'(0). \]

(2.24)

In the above equations the Reynolds number \( Re_x = \sqrt{\frac{\nu}{x u_w}}. \)
2.3 Homotopy analysis solutions

Initial approximations and auxiliary linear operators are chosen below:

\[ f_0(\eta) = S + (1 - e^{-\eta}), \] (2.25)

\[ \theta_0(\eta) = \frac{Bi \exp(-\eta)}{1 + Bi}, \] (2.26)

\[ L_f = f''' - f', \] (2.27)

\[ L_\theta = \theta'' - \theta, \] (2.28)

with properties

\[ L_f(C_1 + C_2 e^\eta + C_3 e^{-\eta}) = 0, \] (2.29)

\[ L_\theta(C_4 e^\eta + C_5 e^{-\eta}) = 0, \] (2.30)

where \( C_i \) (\( i = 1 - 5 \)) are the arbitrary constants determined from the boundary conditions.

If \( p \in [0, 1] \) denotes an embedding parameter, \( h_f \) and \( h_\theta \) represent the non-zero auxiliary parameters then the zeroth order deformation problems are defined by

\[ (1 - p)L_f \left[ \hat{f}(\eta; p) - f_0(\eta) \right] = ph_fN_f \left[ \hat{f}(\eta; p) \right], \] (2.31)

\[ (1 - p)L_\theta \left[ \hat{\theta}(\eta; p) - \theta_0(\eta) \right] = ph_\theta N_\theta \left[ \hat{f}(\eta; p), \hat{\theta}(\eta; p) \right], \] (2.32)

\[ \hat{f}(0; p) = S, \quad \hat{f}'(0; p) = 1, \quad \hat{f}'(\infty; p) = 0, \quad \hat{\theta}'(0; p) = -Bi[1 - \theta(0; p)], \quad \hat{\theta}(\infty; p) = 0, \] (2.33)

where \( N_f \) and \( N_\theta \) are the nonlinear operators defined in the forms

\[ N_f[\hat{f}(\eta, p)] = \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} + \hat{f}(\eta, p) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right)^2 + \omega \left( \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right)^2 + \left( \frac{\partial^4 \hat{f}(\eta, p)}{\partial \eta^4} \right) - 2 \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} \right] (2.34) \]
\[
N_\theta[\hat{\theta}(\eta, p), \hat{f}(\eta, p)] = \left(1 + \frac{4}{3} R \right) \frac{\partial^2 \hat{\theta}(\eta, p)}{\partial \eta^2} + Pr \left( \hat{f}(\eta, p) \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} \right) \\
+ A \frac{\partial \hat{f}(\eta, p)}{\partial \eta} + B \hat{\theta}(\eta, p).
\] (2.35)

When \( p = 0 \) and \( p = 1 \) then one has

\[
\hat{f}(\eta; 0) = f_0(\eta), \quad \hat{\theta}(\eta, 0) = \theta_0(\eta) \text{ and } \hat{f}(\eta; 1) = f(\eta), \quad \hat{\theta}(\eta, 1) = \theta(\eta),
\] (2.36)

and when \( p \) variation is taken from \( 0 \) to \( 1 \) then \( f(\eta, p) \) and \( \theta(\eta, p) \) approach \( f(\eta) \) and \( \theta(\eta) \). Now \( f \) and \( \theta \) in Taylor’s series can be expanded as follows:

\[
f(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m,
\] (2.37)

\[
\theta(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m,
\] (2.38)

\[
f_m(\eta) = \frac{1}{m!} \frac{\partial^m f(\eta; p)}{\partial \eta^m} \bigg|_{p=0}, \quad \theta_m(\eta) = \frac{1}{m!} \frac{\partial^m \theta(\eta; p)}{\partial \eta^m} \bigg|_{p=0},
\] (2.39)

where the convergence depends on \( h_f \) and \( h_\theta \). By appropriately choosing \( h_f \) and \( h_\theta \) the series (2.37) and (2.38) converge for \( p = 1 \) and so

\[
f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta),
\] (2.40)

\[
\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta).
\] (2.41)

The \( m^{th} \)-order deformation problems can be provided as follows:

\[
L_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f R_f^m(\eta),
\] (2.42)

\[
L_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_\theta R_\theta^m(\eta),
\] (2.43)

\[
f_m(0) = f'_m(0) = f'_m(\infty) = 0, \quad \theta'_m(0) = B \theta_m(0) = \theta_m(\infty) = 0,
\] (2.44)
\[ R^m_f(\eta) = f''_{m-1}(\eta) + \sum_{k=0}^{m-1} \left[ f''_{m-1-k} f'_k - f'_{m-1-k} f''_k \right] \]
\[ + W e \sum_{k=0}^{m-1} (f''_{m-1-k} f''_k + f''_{m-1-k} f''_k - 2 f'_m f''_k), \quad (2.45) \]

\[ R^m_\theta(\eta) = (1 + \frac{4}{3} R) \theta''_{m-1} + Pr \sum_{k=0}^{m-1} \theta'_{m-1-k} f_k \]
\[ + A f'_{m-1} + B \theta_{m-1}, \quad (2.46) \]

\[ \chi_m = \begin{cases} 
0, & m \leq 1, \\
1, & m > 1. 
\end{cases} \quad (2.47) \]

The general solutions of Eqs. (2.42) and (2.43) can be expressed by the following equations:

\[ f_m(\eta) = f^*_m(\eta) + C_1 + C_2 \bar{\eta} + C_3 e^{-\eta}, \quad (2.48) \]

\[ \theta_m(\eta) = \theta^*_m(\eta) + C_4 \bar{\eta} + C_5 e^{-\eta}, \quad (2.49) \]

where \( f^*_m \) and \( \theta^*_m \) are the particular solutions of Eqs. (2.48) and (2.49) and constants \( C_i \) \( (i = 1 - 5) \) can be determined by the boundary conditions (2.44). They are given by

\[ C_2 = C_4 = 0, \quad C_3 = \frac{\partial f^*_m(\eta)}{\partial \eta} \bigg|_{\eta=0}, \quad C_1 = -C_3 - f^*(0), \quad (2.50) \]

\[ C_5 = \frac{1}{Bi + 1} \left[ \frac{\partial \theta^*_m(\eta)}{\partial \eta} \bigg|_{\eta=0} - B i \theta^*(0) \right]. \quad (2.51) \]

### 2.4 Convergence of the homotopy solutions

Series (2.40) and (2.41) have nonzero auxiliary parameters \( h_f \) and \( h_\theta \) that help us to control and adjust the convergence of series solutions. For admissible values of \( h_f \) and \( h_\theta \), we plotted the \( h \) - curves of \( f''(0) \) and \( \theta'(0) \). Figs. 2.1 and 2.2 show that the ranges of admissible values of \( h_f \) and \( h_\theta \) are \(-1.8 \leq h_f \leq -0.3 \) and \(-1.9 \leq h_\theta \leq -0.55 \) respectively. The series converge in the whole region of \( \eta \) when \( h_f = -1.5 \), and \( h_\theta = -1.3 \). Table 2.1 displays the convergence of homotopy solutions for different orders of approximations.
Fig. 2.1. $h$–curve for velocity field.

Fig. 2.2. $h$–curve for temperature field.
Table 2.1: Convergence of HAM solutions for different orders of approximations when \( S = 0.4, \ We = 0.2, \ A = B = 0.1, \ Pr = 1, \ Bi = 0.7 \) and \( R = 0.1 \).

<table>
<thead>
<tr>
<th>Order of approximation</th>
<th>(-f''(0))</th>
<th>(-\theta'(0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.51000</td>
<td>0.047359</td>
</tr>
<tr>
<td>5</td>
<td>1.54617</td>
<td>0.027114</td>
</tr>
<tr>
<td>10</td>
<td>1.54727</td>
<td>0.030756</td>
</tr>
<tr>
<td>15</td>
<td>1.54728</td>
<td>0.031432</td>
</tr>
<tr>
<td>20</td>
<td>1.54728</td>
<td>0.031568</td>
</tr>
<tr>
<td>25</td>
<td>1.54728</td>
<td>0.031601</td>
</tr>
<tr>
<td>30</td>
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<tr>
<td>35</td>
<td>1.54728</td>
<td>0.031611</td>
</tr>
<tr>
<td>40</td>
<td>1.54728</td>
<td>0.031611</td>
</tr>
<tr>
<td>50</td>
<td>1.54728</td>
<td>0.031611</td>
</tr>
</tbody>
</table>

2.5 Results and discussion

In this section we focus on the behaviors of different physical parameters entering into the series solutions. In particular the plots for dimensionless velocity gradient and heat transfer rate at the bounding surface are displayed and discussed. Here the velocity decreases with the increase of \( We \) which corresponds to a thinner momentum boundary layer. The viscoelasticity produces tensile stress which contracts the boundary layer and consequently the velocity decreases (see Fig. 2.3). An increase in suction parameter moves the velocity distribution towards the boundary, indicating a decrease in the thickness of boundary layer (see Fig. 2.4). Fig. 2.5 shows the influence of viscoelastic parameter \( We \) on temperature. We found that the temperature and the thermal boundary layer thickness decrease as viscoelastic effects strengthens. Influence of Biot number \( Bi \) is shown in Fig. 2.6. As expected, an increase in \( Bi \) results in the stronger convective heating which increases the surface temperature. This allows the thermal effect to penetrate more deeply into the quiescent fluid. The variation of temperature distribution with the Prandtl number \( Pr \) is presented in Fig. 2.7. From the physical point of view, the larger Prandtl number coincides with the weaker thermal diffusivity and thinner boundary layer. This is because a higher Prandtl number fluid has a relatively low thermal conductivity which re-
duces conduction and thereby increases the variations of thermal characteristics. Figure 2.8 shows that temperature $\theta$ appreciably rises when the thermal radiation effect is strengthened. However suction reduces the thermal boundary layer thickness as can be seen from Fig. 2.9. Figs. 2.10 and 2.11 show the influences of heat source/sink parameters on temperature $\theta$. As expected, the larger heat source parameter enhances the temperature distribution to a great extent and thus largely contributes to the growth of thermal boundary layer. Energy is also absorbed for smaller values of $A < 0$ and $B < 0$ and this causes the magnitude of temperature to decrease whereas non-uniform heat sink corresponding to $A < 0$ and $B < 0$ can contribute to quenching the heat from stretching sheet effectively.

Table 2.2 provides the numerical values of dimensionless velocity gradient at the wall for different values of suction parameter $S$ and viscoelastic parameter $We$. It is clear that increases of $S$ and $We$ reduce the dimensionless velocity gradient on the stretching surface. This is very useful result from the industrial point of view since the power generation involved in displacing the fluid over the sheet is reduced by assuming sufficiently large values of these parameters.

The numerical values of local Nusselt number for different values of embedded parameters have been tabulated (see Table 2.3). Here larger convective heating at the bounding surface generates larger heat flux from the surface which eventually increases the magnitude of local Nusselt number. It is clear from Fig. 2.4 that although thermal boundary layer thins but profiles become increasingly steeper with the increase of the value of $Pr$. The Nusselt number, being proportional to the increase of initial slope with the augment of the value of $Pr$. On the other hand the magnitude of local Nusselt number decreases as viscoelastic effect strengthens.
Fig. 2.3. Influence of $We$ on $f'_{\eta}$.

Fig. 2.4. Influence of $S$ on $f'_{\eta}$.
Fig. 2.5. Influence of $We$ on $\theta(\eta)$.

Fig. 2.6. Influence of $Bi$ on $\theta(\eta)$. 
Fig. 2.7. Influence of Pr on $\theta(\eta)$.

Fig. 2.8. Influence of $R$ on $\theta(\eta)$. 
Fig. 2.9. Influence of $S$ on $\theta(\eta)$.

Fig. 2.10. Influence of $A$ on $\theta(\eta)$.
Fig. 2.11. Influence of $B$ on $\theta(\eta)$.

Fig. 2.12. Influence of $Nu/Re_x^{1/2}$ on $Pr$. 
Fig. 2.13. Influence of $\frac{Nu}{Re_x^{1/2}}$ on $We$.  

**Table 2.2:** Values of skin friction coefficient $Re_x^{1/2} C_f$ for the parameters $S$ and $We$.  

<table>
<thead>
<tr>
<th>$S$</th>
<th>$We$</th>
<th>$-Re_x^{1/2} C_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.2</td>
<td>0.870726</td>
</tr>
<tr>
<td>0.2</td>
<td></td>
<td>0.849053</td>
</tr>
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<td>0.4</td>
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<td>0.6</td>
<td></td>
<td>0.364968</td>
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<tr>
<td>0.0</td>
<td>0.1</td>
<td>1.2198</td>
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<td>0.1</td>
<td>1.01937</td>
</tr>
<tr>
<td>0.2</td>
<td></td>
<td>0.712725</td>
</tr>
</tbody>
</table>
Table 2.3: Values of local Nusselt number $\frac{Nu}{Re_x^{1/2}}$ for the parameters $Bi$, $R$, $Pr$, $S$ and $We$. 

<table>
<thead>
<tr>
<th>$Bi$</th>
<th>$Pr$</th>
<th>$We$</th>
<th>$S$</th>
<th>$R$</th>
<th>$\frac{Nu}{Re_x^{1/2}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
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<td>0.223253</td>
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<td>0.7</td>
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<tr>
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<td>0.284298</td>
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2.6 Main points

Boundary layer flow of Walters’ B fluid over a porous stretching sheet with radiation and non-uniform heat source/sink is investigated here. The series solutions of the governing mathematical problems are obtained by using HAM and appropriately choosing the convergence control parameters. Thus the convergent solutions are obtained through plots of $h$–curves. It is noticed that velocity and wall shear stress decrease with the increase of Weissenberg number $We$ whereas temperature $\theta(\eta)$ is an increasing function of $We$. Increase in the suction parameter $S$ also reduces the momentum boundary layer. The temperature $\theta(\eta)$ and rate of heat transfer at the sheet increase for larger convective heating at the surface. For sufficiently large values of Biot number $Bi$, the results for the case of constant wall temperature ($\theta(0) = 1$) can be recovered. Moreover the present finding for the Newtonian fluid case can be retrieved by
choosing $We = 0$. 
Chapter 3

Flow of Walters’ B fluid by an inclined unsteady stretching sheet with thermal radiation

3.1 Introduction

This chapter deals with the boundary layer flow of Walters’ B liquid by an inclined unsteady stretching sheet. Simultaneous effects of heat and mass transfer are considered. Analysis has been carried out in the presence of thermal radiation and viscous dissipation. Convective condition is employed for the heat transfer process. Resulting problems are computed for the convergent solutions of velocity, temperature and concentration fields. The effects of different physical parameters on the velocity, temperature and concentration fields are discussed. Numerical values of skin friction coefficient, local Nusselt number and local Sherwood number are computed and analyzed.

3.2 Mathematical formulation

Here we investigate the flow of an incompressible Walters’ B fluid due to an inclined stretching sheet with convective boundary condition. The effects of viscous dissipation and thermal radiation are present. The sheet is inclined at an angle $\alpha$ with the vertical axis i.e. $y-axis$ and
In the $x$--axis is taken normal to it. The subjected boundary layer equations are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (3.1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\eta_0}{\rho} \left( \frac{\partial^2 u}{\partial y^2} \right) - \frac{k_0}{\rho} \left[ \frac{\partial^3 u}{\partial t \partial y^2} + \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 u}{\partial x^2 \partial y} \right] + \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + g_0 \beta_T (T - T_\infty) \cos \alpha, \quad (3.2)$$

$$\rho c_p \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} - \frac{\partial q_r}{\partial y} + \frac{\eta_0}{\rho} \left( \frac{\partial u}{\partial y} \right)^2 - 2k_0 \left[ \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial t \partial y} \right] + u \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial^2 u}{\partial y \partial y^2}, \quad (3.3)$$

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2}, \quad (3.4)$$

$$u = u_s(x), \quad v = 0, \quad -k \frac{\partial T}{\partial y} = h(T_f - T), \quad C = C_w \text{ at } y = 0, \quad (3.5)$$

$$u \to 0, \quad T \to T_\infty, \quad C \to C_\infty \text{ as } y \to \infty, \quad (3.6)$$

in which $u$ and $v$ represent the velocity components along and normal to the sheet, $u_s = \frac{br}{1-at}$ is the velocity of the stretching sheet, $T$ is the fluid temperature, $\nu = (\mu/\rho)$ is the kinematic viscosity, $\rho$ is the density of fluid, $g_0$ is the acceleration due to gravity, $\beta_T$ is the volumetric coefficient of thermal expansion, $k$ is the thermal conductivity of fluid, $h$ is the convective heat transfer coefficient, $T_f = T_\infty + T_{ref} \frac{br^2}{\nu} (1 - at)^{-3/2}$ is the surface temperature, $C$ is the concentration of fluid, $D$ is the effective diffusion coefficient and $C_w = C_\infty + \frac{cr}{1-at}$ is the fluid concentration.

We introduce

$$\psi = x \left( \frac{\nu b}{1-at} \right)^{-1/2} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty}, \quad \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}, \quad \eta = y \left( \frac{b}{\nu (1-at)} \right)^{1/2}, \quad (3.7)$$

and the velocity components

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad 31$$
where $\psi$ is the stream function. Now Eq. (3.1) is identically satisfied and Eqs. (3.2 – 3.6) are reduced into the following forms:

$$f'' + f'' - f'^2 + -\varepsilon \left( f' + \frac{1}{2} \eta f'' \right) + We \left( \varepsilon \left( 2f'' + \frac{1}{2} \eta f^{iv} \right) + f'^2 - f'f'' + f f'' \right)$$

$$+ G \theta \cos \alpha = 0,$$  

(3.8)

$$(1 + \frac{4}{3} R) \theta'' - Pr \left( \frac{1}{2} \varepsilon (3 \theta + \eta \theta') - f \theta' + 2f' \theta \right) + Pr E_c f''$$

$$- Pr E_c \left( \frac{1}{2} \varepsilon (3f'' + \eta f'') + f' f'' - f f'' \right) = 0,$$  

(3.9)

$$\phi'' + Sc \left( f \phi' - f' \phi \right) - \varepsilon Sc \left( \phi + \frac{1}{2} \eta \phi' \right) = 0,$$  

(3.10)

$$f = 0, \ f' = 1, \ \theta' = -Bi[1 - \theta(0)] \ \phi = 1 \ \text{at} \ \eta = 0,$$  

(3.11)

$$f' \to 0, \ \theta \to 0 \ \phi \to 0 \ \text{at} \ \eta \to \infty.$$  

(3.12)

Here $We = \frac{k q_0}{\eta_0 \sqrt{T_f - T_\infty}}$ is the Weissenberg number, $Pr = \frac{\nu}{\sigma}$ is the Prandtl number, $\varepsilon = \frac{u}{b}$ is the ratio parameter, $G = \frac{g_0 \beta_f (T_f - T_\infty) x^3 / \nu}{u_{\sqrt{x^2 / \nu}}}$ is the mixed convection parameter, $E_c = \frac{u_2}{c_p(T_f - T_\infty)}$ is the Eckert number, $R = \frac{\Delta \sigma \sqrt{T_f - T_\infty}}{k_{\phi}}$ is the radiation parameter, $Bi = \frac{h}{k} \sqrt{\frac{\nu (1 - \alpha)}{b}}$ is the Biot number, and $Sc = \frac{\nu}{\sigma}$ is the Schmidt number.

The skin friction coefficient $C_f$, local Nusselt number $Nu_x$ and local Sherwood number $Sh$ can be expressed in the following forms:

$$C_f = \frac{\tau_w}{\sqrt{T_f - T_\infty}}, \quad Nu_x = \frac{xq_w}{k(T_f - T_\infty)}, \quad Sh = \frac{xM_w}{D(C_w - C_\infty)},$$  

(3.13)

whence

$$\tau_w = \mu_0 \left. \frac{\partial u}{\partial y} \right|_{y=0} - k_0 \left. \left\{ \frac{\partial^2 u}{\partial t \partial y} + u \frac{\partial^2 u}{\partial x \partial y} - 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + v \left( \frac{\partial^2 u}{\partial y^2} \right) \right\} \right|_{y=0},$$

$$q_w = -k_0 \left. \frac{\partial T}{\partial y} \right|_{y=0} - \frac{16 \sigma^* T_\infty^3}{k_{\phi}} \left. \left( \frac{\partial T}{\partial r} \right) \right|_{y=0}, \quad M_w = -D \left. \frac{\partial C}{\partial y} \right|_{y=0}. \quad (3.14)$$

Dimensionless forms of skin friction coefficient $C_f$, local Nusselt number $Nu_x$ and local Sher-
wood number $Sh$ can be represented by the relations

$$C_f = 2 (Re_x)^{-1/2} \left[ (1 + 3W e) f''(\eta) - W e \frac{1}{2} (3f''(\eta) + \eta f'''(\eta)) \right] \bigg|_{\eta=0},$$

$$Nu/Re_x^{1/2} = -(1 + \frac{4}{3} R) \theta'(0) \quad Sh/Re_x^{1/2} = -\phi'(0), \quad (3.15)$$
in which $(Re_x)^{-1/2} = \sqrt{\frac{\nu}{x u_\infty}}$.

### 3.3 Homotopy analysis solutions

Initial approximations and auxiliary linear operators for homotopy analysis solutions can be put into the forms

$$f_0(\eta) = (1 - e^{-\eta}), \quad \theta_0(\eta) = \frac{Bi \exp(-\eta)}{1 + Bi}, \quad \phi_0 = e^{-\eta}, \quad (3.16)$$

$$\mathcal{L}_f = f''' - f', \quad \mathcal{L}_\theta = \theta'' - \theta, \quad \mathcal{L}_\phi = \phi'' - \phi, \quad (3.17)$$

with properties mentioned below:

$$\mathcal{L}_f(C_1 + C_2 e^\eta + C_3 e^{-\eta}) = 0, \quad (3.18)$$

$$\mathcal{L}_\theta(C_4 e^\eta + C_5 e^{-\eta}) = 0, \quad (3.19)$$

$$\mathcal{L}_\phi(C_6 e^\eta + C_7 e^{-\eta}) = 0, \quad (3.20)$$

where $C_i \ (i = 1 - 7)$ are the arbitrary constants. If $p \in [0, 1]$ denotes an embedding parameter and $h_f, h_\theta$ and $h_\phi$ the non-zero auxiliary parameters then the zeroth order deformation problems are

$$(1 - p) \mathcal{L}_f \left[ \hat{f}(\eta; p) - f_0(\eta) \right] = ph_f N_f \left[ \hat{f}(\eta; p) \right], \quad (3.21)$$

$$(1 - p) \mathcal{L}_\theta \left[ \hat{\theta}(\eta; p) - \theta_0(\eta) \right] = ph_\theta N_\theta \left[ \hat{\theta}(\eta; p) \right], \quad (3.22)$$

$$(1 - p) \mathcal{L}_\phi \left[ \hat{\phi}(\eta; p) - \phi_0(\eta) \right] = ph_\phi N_\phi \left[ \hat{\phi}(\eta; p) \right], \quad (3.23)$$
\[
\hat{f}(0; p) = 0, \quad \hat{f}'(0; p) = 1, \quad \hat{f}'(\infty; p) = 0, \quad \hat{\theta}'(0; p) = -Bi[1 - \theta(0; p)], \quad \hat{\theta}(\infty; p) = 0,
\]
\[
\hat{\phi}(0; p) = 1, \quad \hat{\phi}(\infty; p) = 0. \quad (3.24)
\]

The involved nonlinear operators \(N_f, N_\theta\) and \(N_\phi\) in above equations are

\[
N_f[\hat{f}(\eta, p)] = \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} + \hat{f}(\eta, p) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right)^2 - \epsilon \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} + \frac{1}{2} \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right)
\]
\[
+ We \left[ \epsilon \left( 2 \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} + \frac{1}{2} \frac{\partial^4 \hat{f}(\eta, p)}{\partial \eta^4} \right) + \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right)^2 \right] + G \theta \cos \alpha, \quad (3.25)
\]

\[
N_\theta[\hat{\theta}(\eta, p), \hat{f}(\eta, p)] = \left( 1 + \frac{4}{3} R \right) \frac{\partial^2 \hat{\theta}(\eta, p)}{\partial \eta^2} + Pr \left[ \frac{1}{2} \epsilon \left( 3 \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} + \eta \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} \right) \right]
\]
\[
- \hat{f}(\eta, p) \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} - 2 \hat{\theta}(\eta, p) \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right] + Pr E_c \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2}
\]
\[
- Pr E_c We \left\{ \frac{1}{2} \epsilon \left( 3 \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} + \eta \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^3} \right) + \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \left( \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right)^2 \right\}
\]
\[
\quad - \hat{f}(\eta, p) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \frac{\partial \hat{f}(\eta, p)}{\partial \eta^3} \}, \quad (3.26)
\]

\[
N_\phi[\hat{\phi}(\eta, p), \hat{f}(\eta, p)] = \frac{\partial^2 \hat{\phi}(\eta, p)}{\partial \eta^2} + Sc \left( \hat{f}(\eta, p) \frac{\partial \hat{\phi}(\eta, p)}{\partial \eta} - \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \hat{\phi}(\eta, p) \right)
\]
\[
- \epsilon Sc \left( \hat{\phi}(\eta, p) + \frac{1}{2} \eta \frac{\partial \hat{\phi}(\eta, p)}{\partial \eta} \right). \quad (3.27)
\]

For \(p = 0\) and \(p = 1\) we can write

\[
\hat{f}(\eta; 0) = f_0(\eta), \quad \hat{\theta}(\eta; 0) = \theta_0(\eta), \quad \hat{\phi}(\eta; 0) = \phi_0(\eta)
\]
\[
\hat{f}(\eta; 1) = f(\eta), \quad \hat{\theta}(\eta; 1) = \theta(\eta), \quad \hat{\phi}(\eta; 1) = \phi(\eta), \quad (3.28)
\]
and when \( p \) variation is taken from 0 to 1 then \( f(\eta, p), \theta(\eta, p) \) and \( \phi(\eta, p) \) approach \( f_0(\eta), \theta_0(\eta) \) and \( \phi_0(\eta) \) to \( f(\eta), \theta(\eta) \) and \( \phi(\eta) \). Now \( f, \theta \) and \( \phi \) in Taylor’s series are

\[
f(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \tag{3.29}\]

\[
\theta(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m, \tag{3.30}\]

\[
\phi(\eta, p) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta)p^m, \tag{3.31}\]

\[
f_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m f(\eta; p)}{\partial \eta^m} \right|_{p=0}, \quad \theta_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \theta(\eta; p)}{\partial \eta^m} \right|_{p=0}, \quad \phi_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \phi(\eta; p)}{\partial \eta^m} \right|_{p=0}, \tag{3.32}\]

where the convergence depends upon \( h_f, h_\theta \) and \( h_\phi \). By proper choices of \( h_f, h_\theta \) and \( h_\phi \) the series (3.29 – 3.31) converge for \( p = 1 \) and therefore

\[
f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{3.33}\]

\[
\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \tag{3.34}\]

\[
\phi(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta). \tag{3.35}\]

The \( m^{th} \)-order deformation problems are

\[
\mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f R^{m}_f(\eta), \tag{3.36}\]

\[
\mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_\theta R^{m}_\theta(\eta), \tag{3.37}\]

\[
\mathcal{L}_\phi[\phi_m(\eta) - \chi_m \phi_{m-1}(\eta)] = h_\phi R^{m}_\phi(\eta), \tag{3.38}\]

\[
f_m(0) = f'_m(0) = f'_m(\infty) = 0, \quad \theta'_m(0) - B_i \theta_m(0) = \theta_m(\infty) = 0, \quad \phi_m(0) = \phi_m(\infty) = 0, \tag{3.39}\]
with

\[
R_f^m(\eta) = f'''_{m-1}(\eta) + \sum_{k=0}^{m-1} \left[ f_{m-1-k} f''_{k} - f'_{m-1-k} f'_{k} \right] - \epsilon \left( 2f'_{m-1} + \frac{1}{2} \eta f''_{m-1} \right) \\
+ \epsilon We \left( 2f'''_{m-1} + \frac{1}{2} \eta f''_{m-1} \right) + We \sum_{k=0}^{m-1} \left( f'''_{m-1-k} f''_{k} \\
+ f_{m-1-k} f''_{k} - 2f'_{m-1-k} f'''_{k} \right) + G \theta_{m-1} \cos \alpha,
\]

(3.40)

\[
R_{\theta}^m(\eta) = (1 + \frac{4}{3} R) \theta'''_{m-1} + Pr \left( \frac{1}{2} \epsilon (3\theta_{m-1} + \eta \theta'_{m-1}) \right) \\
+ Pr \sum_{k=0}^{m-1} \left( \theta'_{m-1-k} f_k - 2f'_{m-1-k} \theta_k \right) + Pr E_c f'''_{m-1} \\
- Pr E_c We \sum_{k=0}^{m-1} \left( \frac{1}{2} \epsilon (3f'''_{m-1} + \eta f'''_{m-1}) + f'_{m-1-k} f'''_{k} - f_{m-l} \sum_{k=0}^{m-l} f'''_{k-l} f'''_{k} \right),
\]

(3.41)

\[
R_{\phi}^m(\eta) = \phi'''_{m-1} + Sc \sum_{k=0}^{m-1} \left( \phi'_{m-1-k} f_k - f'_{m-1-k} \phi_k \right) \\
- \epsilon Sc \left( \phi_{m-1} + \frac{1}{2} \eta \phi'_{m-1} \right),
\]

(3.42)

\[
\chi_m = \begin{cases} 
0, & m \leq 1, \\
1, & m > 1.
\end{cases}
\]

The general solutions of Eqs. (3.36 – 3.38) can be expressed in the forms:

\[
f_m(\eta) = f^*_m(\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta},
\]

(3.43)

\[
\theta_m(\eta) = \theta^*_m(\eta) + C_4 e^\eta + C_5 e^{-\eta},
\]

(3.44)

\[
\phi_m(\eta) = \phi^*_m(\eta) + C_6 e^\eta + C_7 e^{-\eta},
\]

(3.45)

where \(f^*_m, \theta^*_m\) and \(\phi^*_m\) are the particular solutions.
3.4 Convergence of the homotopy solutions

It is well known that the convergence of series (3.36 – 3.38) depends on the auxiliary parameters $h_f$, $h_\theta$ and $h_\phi$. These parameters are useful to control and adjust the convergence of series solutions. In order to find the admissible values of $h_f$, $h_\theta$ and $h_\phi$ the $h$ – curves of $f''(0)$, $\theta'(0)$ and $\phi'(0)$ are displayed. Figs. (3.1 – 3.3) depict that the range of admissible values of $h_f$, $h_\theta$ and $h_\phi$, are $-1.25 \leq h_f \leq -0.26$, $-1.23 \leq h_\theta \leq -0.27$ and $-1.26 \leq h_\phi \leq -0.4$. The series converge in the whole region of $\eta$ when $h_f = h_\theta = -0.8$ and $h_\phi = -0.9$. Table 3.1 displays the convergence of homotopy solutions for different order of approximations. Tabulated values clearly indicate that 25$^{th}$ order of approximations are enough for the convergent solutions.

Fig. 3.1. $h$–curve for velocity field.
Fig. 3.2. $h$–curve for temperature field.

\[ \epsilon = \text{We} = 0.3, G = 0.2, \alpha = \pi/4, R = 0.1, \]
\[ \text{Pr} = 1, Ec = 0.2, Bi = 0.4, Sc = 0.9 \]

Fig. 3.3. $h$–curve for concentration field.

\[ \epsilon = \text{We} = 0.3, G = 0.2, \alpha = \pi/4, R = 0.1, \]
\[ \text{Pr} = 1, Ec = 0.2, Bi = 0.4, Sc = 0.9 \]
Table 3.1: Convergence of HAM solutions for different order of approximations when \( \epsilon = We = 0.3, G = 0.2, \alpha = \pi/4, Pr = 1.0, Bi = 0.4, E_c = 0.2, Sc = 0.9 \) and \( R = 0.1 \).

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3.5 Discussion

The arrangement of this section is to disclose the impact of different physical parameters, including \( \epsilon \) ratio parameter, \( Bi \) the Biot number, \( We \) the Weissenberg number, \( Pr \) the Prandtl number, \( Sc \) the Sherwood number, \( Ec \) the Eckert number, \( G \) the mixed convection parameter, \( \alpha \) the angle of inclination and \( R \) the radiation parameter. The variation of aforementioned parameters are seen for the velocity, temperature and concentration fields. It is observed that the effects of \( \alpha \) and \( \epsilon \) on the velocity field are quite opposite (see Figs. 3.4 and 3.5). Velocity field and boundary layer thickness decay for larger values of \( \alpha \). Influence of Weissenberg number \( We \) on \( f'(\eta) \) is shown in Fig. 3.6. Here the velocity field is decreasing function of \( We \). For large values of mixed convection parameter \( G \) the velocity field increases (see Fig. 3.7). Effect of radiation parameter \( R \) and Eckert number \( Ec \) are qualitatively similar for the velocity field \( f'(\eta) \) (see Figs. 3.8 and 3.9). It is also noticed that the variation of \( R \) between 0 to 1 is insignificant on the temperature. Fig. 3.10 depicts the influence of \( \epsilon \) on the temperature field. Apparently both the temperature field and thermal boundary layer thickness are decreased via \( \epsilon \). Fig. 3.11 gives the influence of Weissenberg number \( We \) on the temperature field. Here the temperature field is increasing function of \( We \). Fig. 3.12 shows that the temperature field decays very slowly when mixed convection parameter \( G \) increases. Fig. 3.13 presents the influence of \( \alpha \) on the temperature field. Both the temperature field and thermal boundary layer thickness decrease.
when $\alpha$ increases. Fig. 3.14 witnesses that the temperature field is more pronounced when radiation effects strengthen. Fig. 3.15 depicts the variation of $Pr$ on temperature field. For larger values of $Pr$ the temperature and thermal boundary layer thickness decrease. This is due to the fact that enhancement in $Pr$ causes the reduction in thermal conductivity. Effect of Eckert number $E_c$ is displayed in Fig. 3.16. When we increase the Eckert number then the fluid kinetic energy increases and thus temperature field increases for larger values of Eckert number. Fig. 3.17 shows the influence of Biot number on temperature field. Larger Biot numbers correspond to more convection than conduction and this leads to increase in temperature as well as thermal boundary layer thickness. Influence of parameter $\epsilon$ on the concentration field is displayed in Fig. 3.18. It is noticed that the concentration field decreases when $\epsilon$ is increased. Fig. 3.19 shows the influence of Weissenberg number $We$ on the concentration field. It is clearly seen from this Fig. that the concentration field increases when there is an increase in Weissenberg number $We$. Fig. 3.20 gives the influence of $\alpha$ on the concentration field. The angle of inclination $\alpha$ increases the concentration field. Effect of $Sc$ on the concentration field is plotted in Fig. 3.21. The concentration field decreases when $Sc$ increases. Here larger $Sc$ number corresponds to lower molecular diffusivity.

Tables 3.2–3.4 include the numerical values of skin friction coefficient, local Nusselt number and Sherwood number respectively. The magnitude of skin friction coefficient decreases for larger values of $R$, $G$, $\epsilon$ and $E_c$. However it increases when $We$, $\alpha$ and $Pr$ are increased. It is noticed that the heat transfer at the wall $-\theta(0)$ increases for larger values of $\epsilon$, $G$, $Pr$, $Bi$ and it decreases for larger $We$, $\alpha$, $R$ and $E_c$. Table 3.3 shows that the local Sherwood number increases when radiation parameter $R$ and Schmidt number $Sc$ are increased.
Fig. 3.4. Influence of $\alpha$ on $f'(\eta)$.

Fig. 3.5. Influence of $\epsilon$ on $f'(\eta)$. 
Fig. 3.6. Influence of $We$ on $f'(\eta)$.

Fig. 3.7. Influence of $G$ on $f'(\eta)$. 

$\epsilon = 0.3$, $G = 0.2$, $\alpha = \pi/4$, $R = 0.1$, $Pr = 1$, $Ec = 0.2$, $Bi = 0.4$, $Sc = 0.9$

$We = 0.0$, $We = 0.3$, $We = 0.6$, $We = 0.9$

$G = 0.0$, $G = 0.5$, $G = 1.0$, $G = 1.5$
Fig. 3.8. Influence of $Ec$ on $f'(\eta)$.

Fig. 3.9. Influence of $R$ on $f'(\eta)$. 
Fig. 3.10. Influence of $\epsilon$ on $\theta(\eta)$.

Fig. 3.11. Influence of $We$ on $\theta(\eta)$. 
Fig. 3.12. Influence of $G$ on $\theta(\eta)$.

Fig. 3.13. Influence of $\alpha$ on $\theta(\eta)$. 

$$
\epsilon = \text{We} = 0.3, \alpha = \pi/4, R = 0.1, \text{Pr} = 1,
Ec = 0.2, Bi = 0.4, Sc = 0.9
$$

$$
G = 0.0, G = 0.7, G = 1.8, G = 3.0
$$

$$
\epsilon = \text{We} = 0.3, G = 0.2, R = 0.1, \text{Pr} = 1,
Ec = 0.2, Bi = 0.4, Sc = 0.9
$$

$$
\alpha = 0, \alpha = \pi/8, \alpha = \pi/4, \alpha = \pi/2
$$
**Fig. 3.14.** Influence of $R$ on $\theta(\eta)$.

**Fig. 3.15.** Influence of $Pr$ on $\theta(\eta)$. 

\[ \epsilon = We = 0.3, \alpha = \pi/4, G = 0.2, Pr = 1, \]
\[ Ec = 0.2, Bi = 0.4, Sc = 0.9 \]
\[ R = 0.0, R = 0.2, R = 0.4, R = 0.6 \]

\[ \epsilon = We = 0.3, \alpha = \pi/4, G = 0.2, R = 0.1, \]
\[ Ec = 0.2, Bi = 0.4, Sc = 0.9 \]
\[ Pr = 1.0, Pr = 1.2, Pr = 1.35, Pr = 1.5 \]
Fig. 3.16. Influence of $Ec$ on $\theta(\eta)$.

Fig. 3.17. Influence of $Bi$ on $\theta(\eta)$. 
Fig. 3.18. Influence of $\epsilon$ on $\phi(\eta)$.

Fig. 3.19. Influence of $We$ on $\phi(\eta)$. 

$\epsilon = 0.0$, $\epsilon = 0.3$, $\epsilon = 0.6$, $\epsilon = 0.9$

$We = 0.3$, $G = 0.2$, $\alpha = \pi/4$, $R = 0.1$, $Pr = 1$, $Ec = 0.2$, $Bi = 0.4$, $Sc = 0.9$

$We = 0.0$, $We = 0.3$, $We = 0.6$, $We = 0.9$
Fig. 3.20. Influence of $\alpha$ on $\phi(\eta)$.

Fig. 3.21. Influence of $Sc$ on $\phi(\eta)$.
Table 3.2: Values of skin friction coefficient $Re_x^{1/2}C_f$ for the parameters $\epsilon$, $We$, $G$, $\alpha$, $R$, $Pr$, $E_c$ and $Bi$.

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<th>$Pr$</th>
<th>$E_c$</th>
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Table 3.3: Values of local Nusselt number $Nu/Re_x^{1/2}$ for the parameters $\epsilon$, $We$, $G$, $\alpha$, $R$, $Pr$, $E_c$ and $Bi$.

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<th>$G$</th>
<th>$\alpha$</th>
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51
Table 3.4: Values of local Sherwood number Sh for the parameters R, We, α and Sc.

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<th>$R$</th>
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3.6 Main points

Boundary layer flow of Walters’ B liquid by an inclined stretching sheet is discussed in the presence of viscous dissipation and thermal radiation. The main observations can be mentioned below.

- Velocity field $f'(\eta)$ is decreasing function of parameter $\epsilon$.
- Effects of $\epsilon$ on the temperature and concentration fields are qualitatively similar.
- Weissenberg number $We$ decreases both the velocity and associated boundary layer thickness.
- Weissenberg number $We$ increases the temperature and concentration fields.
- Velocity field $f'$ is decreasing function through larger $\alpha$.
- Effects of $Ec$, $G$ and $R$ on velocity field are qualitatively similar.
• There is enhancement of temperature for larger values of Eckert number $E_c$, thermal radiation $R$ and Biot number $Bi$.

• Variations of $\epsilon$ and $Sc$ on the concentration field are qualitatively similar.

• Magnitude of skin friction coefficient is decreasing function of $\alpha$, $We$, $E_c$ and $R$.

• Influences of $\epsilon$, $G$ and $Bi$ on the temperature gradient at the surface are qualitatively similar.

• Temperature gradient at the surface decreases when $We$, $E_c$ and $R$ are enhanced.
Chapter 4

Flow of Powell-Eyring fluid by an exponentially stretching surface with nonlinear thermal radiation

4.1 Introduction

The present chapter concerns with the flow of Powell-Eyring fluid due to exponentially stretching sheet with thermal radiation effect. Here sheet is considered inclined and making an angle $\alpha$ with the vertical axis. By using the similarity transformations, the nonlinear partial differential equations are transformed into ordinary differential equations. The series solutions of ordinary differential equations are obtained by using homotopy analysis method (HAM). In order to analyze the variations of physical parameters on temperature field, the effect of heat transfer characteristics at wall are tabulated. Also the variations of parameters on the velocity and temperature fields are analyze graphically.

4.2 Mathematical formulation

We consider steady non-Newtonian two-dimensional incompressible flow by an exponentially stretching sheet. We assume that sheet stretch with velocity $U_0 e^\tau z$, and makes the angle $\alpha$ with $y – axis$ where $U_0$ is the reference velocity and $L$ is the characteristics length. Heat transfer
analysis are also investigated with non-linear thermal radiation effects. The stress tensor in the Eyring-Powell model is

$$\tau_{ij} = \mu \frac{\partial u_i}{\partial x_j} + \frac{1}{\beta^2} \sinh^{-1} \left( \frac{1}{C} \frac{\partial u_i}{\partial x_j} \right),$$  \hspace{1cm} (4.1)

where $\mu$ is the viscosity coefficient, $\beta$ and $C$ are the characteristics of Powell-Eyring Model. We take $x$ and $y$ axes along and perpendicular to the sheet respectively. Here the velocity components $u$ and $v$ depend on the $x$ and $y$ coordinates only. The resultant boundary layer equations with subjected boundary conditions are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \hspace{1cm} (4.2)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \left( \nu + \frac{1}{\rho \beta^2 C} \right) \frac{\partial^2 u}{\partial y^2} - \frac{1}{2 \rho \beta^2 C^3} \left( \frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} + g_0 \beta T (T - T_\infty) \cos \alpha, \hspace{1cm} (4.3)$$

$$\rho c_p \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left[ \left( k + \frac{16 \sigma^* T^3}{3 k^*} \right) \frac{\partial T}{\partial y} \right], \hspace{1cm} (4.4)$$

$$u = U_0 e^{x/L}, \quad v = 0, \quad T = T_\infty + T_0 e^{x/2L} \text{ at } y = 0, \hspace{1cm} (4.5)$$

$$u \to 0, \quad T \to 0 \text{ as } y \to \infty,$$

where $\nu = (\mu/\rho)$ is the kinematic viscosity, $k$ is the thermal conductivity of the fluid, $c_p$ is the specific heat, $k^*$ is the mean absorption coefficient, $\sigma^*$ is the Stefan-Boltzmann constant, $\rho$ is the fluid density, $T$ is the fluid temperature, $g_0$ is the acceleration due to gravity, $\beta T$ is the volumetric coefficient of thermal expansion, $c_p$ is the specific heat and $T_0$ and $T_\infty$ are the temperature at the plate and far away from the plate. Defining the dimensionless variables as

\begin{align*}
u &= U_0 e^{x/L} f'(\eta), \quad v = -\sqrt{\frac{\nu U_0}{2L}} e^{x/2L} \left[ f(\eta) + \eta f'(\eta) \right], \\
T &= T_0 e^{x/2L} \theta(\eta), \quad \eta = \sqrt{\frac{U_0}{2 \nu L}} e^{x/2L} y, \hspace{1cm} (4.6)
\end{align*}

Invoking Eq. (4.6) into Eqs. (4.2 – 4.5), Eq. (4.2) is identically satisfied and Eqs. (4.3 – 4.5) yield

$$(1 + \Gamma) f^{'''} - 2 f f^{''} + f f^{''} - \Gamma f^{'2} f^{''} + G \cos \alpha = 0, \hspace{1cm} (4.7)$$
\[
\left\{ 1 + R \left( 1 + (\theta_f - 1)\theta \right)^3 \right\}' + \text{Pr} \left[ f'\theta' - f'\theta \right] = 0, \quad (4.8)
\]

\[
f = 0, \quad f' = 1, \quad \theta = 1 \quad \text{at} \quad \eta = 0,
\]

\[
f' \to 0, \quad \theta \to 0 \quad \text{as} \quad \eta \to \infty, \quad (4.9)
\]

where prime denotes the differentiation with respect to \( \eta \), \( f \) is the dimensionless stream function, \( \theta \) is the dimensionless temperature and the dimensionless numbers are \( \Gamma = \frac{1}{\mu \beta c} \), \( R = \frac{4\pi^2 T^3_0}{k} \), \( G = \frac{G_{re}^*}{Re_x^*} \), \( \beta = \frac{\mu \beta c^3 \alpha_x}{4 \mu C_x^2 L} \), \( \theta_f = \text{and} \) \( \text{Pr} = \frac{\mu c_p}{k} \). Here \( \Gamma \) and \( \beta \) are the dimensionless material parameters, \( G \) is the mixed convection parameter, \( R \) is the radiation parameter, \( \theta_f \) is the temperature ratio parameter and \( \text{Pr} \) is the Prandtl number.

The local Nusselt number \( Nu_x \) is defined as

\[
Nu_x = \frac{x q_w}{k(T_0 - T_\infty)} \text{ with } q_w = -k \left( \frac{\partial T}{\partial y} \right)_{y=0} + (q_r), \text{ and so}
\]

\[
Nu_x = -T_0 e^{x/L} \sqrt{\frac{U_0}{2 \nu L}} (1 + R \theta_f^3) \theta'(0), \quad (4.10)
\]

where \( q_w \) is heat transfer from the plate.

### 4.3 Homotopy analysis solutions

Initial approximations and auxiliary linear operators are defined by

\[
f_0(\eta) = 1 - e^{-\eta}, \quad \theta_0(\eta) = e^{-\eta}. \quad (4.11)
\]

\[
\mathcal{L}_f = f''' - f', \quad \mathcal{L}_\theta = \theta'' - \theta, \quad (4.12)
\]

with properties

\[
\mathcal{L}_f(C_1 + C_2 e^\eta + C_3 e^{-\eta}) = 0, \quad \mathcal{L}_\theta(C_4 e^\eta + C_5 e^{-\eta}) = 0, \quad (4.13)
\]

where \( C_i \) (\( i = 1 - 5 \)) are the constants to be determined by the boundary conditions.
The zeroth order deformation problems are

\begin{equation}
(1 - p) \mathcal{L}_f \left[ \hat{f}(\eta; p) - f_0(\eta) \right] = ph_f \mathcal{N}_f \left[ \hat{f}(\eta; p) \right],
\end{equation}

\begin{equation}
(1 - p) \mathcal{L}_\theta \left[ \hat{\theta}(\eta; p) - \theta_0(\eta) \right] = ph_\theta \mathcal{N}_\theta \left[ \hat{f}(\eta; p), \hat{\theta}(\eta; p) \right],
\end{equation}

\begin{equation}
\hat{f}(0; p) = 0, \quad \hat{f}'(0; p) = 1, \quad \hat{f}'(\infty; p) = 0, \quad \hat{\theta}(0; p) = 1, \quad \hat{\theta}(\infty; p) = 0.
\end{equation}

If \( p \in [0, 1] \) denotes an embedding parameter, \( h_f \) and \( h_\theta \) the non-zero auxiliary parameters, then the nonlinear differential operators \( \mathcal{N}_f \) and \( \mathcal{N}_\theta \) are

\begin{equation}
\mathcal{N}_f[\hat{f}(\eta, p)] = (1 + \Gamma) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} + \hat{f}(\eta, p) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - 2 \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right)^2
\end{equation}

\begin{equation}
-\Gamma \beta \left( \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right)^2 \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} + G \cos \alpha,
\end{equation}

\begin{equation}
\mathcal{N}_\theta[\hat{\theta}(\eta, p), \hat{f}(\eta, p)] = \frac{\partial}{\partial \eta} \left[ \left\{ 1 + R \left( 1 + (\theta_f - 1)\hat{\theta}(\eta, p) \right) \right\} \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} \right]
\end{equation}

\begin{equation}
+ \Pr \left( \hat{f}(\eta, p) \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} - \hat{\theta}(\eta, p) \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right).
\end{equation}

Obviously when \( p = 0 \) and \( p = 1 \) then

\begin{equation}
\hat{f}(\eta; 0) = f_0(\eta), \quad \hat{\theta}(\eta, 0) = \theta_0(\eta),
\end{equation}

\begin{equation}
\hat{f}(\eta; 1) = f(\eta), \quad \hat{\theta}(\eta, 1) = \theta(\eta),
\end{equation}

and when \( p \) varies from 0 to 1 then \( f(\eta, p) \) and \( \theta(\eta, p) \) vary from \( f_0(\eta), \theta_0(\eta) \) to \( f(\eta) \) and \( \theta(\eta) \).

By using Taylor series expansion one has

\begin{equation}
f(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m,
\end{equation}

\begin{equation}
\theta(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m,
\end{equation}

57
\[ f_m(\eta) = \frac{1}{m!} \frac{\partial^m f(\eta; p)}{\partial \eta^m} \bigg|_{p=0}, \quad \theta_m(\eta) = \frac{1}{m!} \frac{\partial^m \theta(\eta; p)}{\partial \eta^m} \bigg|_{p=0}. \]  

(4.23)

The convergence of above series strongly depends upon \( h_f \) and \( h_\theta \). By proper choices of \( h_f \) and \( h_\theta \) the series (4.21) and (4.22) converge for \( p = 1 \) and so

\[ f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \]  

(4.24)

\[ \theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta). \]  

(4.25)

The resulting problems at \( m^{th} \) order are

\[ \mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f \mathcal{R}_f^m(\eta), \]  

(4.26)

\[ \mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_\theta \mathcal{R}_\theta^m(\eta), \]  

(4.27)

\[ f_m(0) = f'_m(0) = f'_m(\infty) = \theta_m(0) = \theta_m(\infty) = 0, \]  

(4.28)

\[ \mathcal{R}_f^m(\eta) = (1 + \alpha) f'''_{m-1}(\eta) + \sum_{k=0}^{m-1} \left( f'''_{m-1-k} f''_k - f''_{m-1-k} f'_k \right) - \alpha \beta f'''_{m-1} \sum_{k=0}^{m-1} f''_{k-1} f''_l \right) + G \theta_{m-1} \cos \alpha, \]  

(4.29)

\[ \mathcal{R}_\theta^m(\eta) = (1 + R) \theta''_{m-1}(\eta) + \text{Pr} \sum_{k=0}^{m-1} \left\{ \theta'_{m-1-k} f_k - f''_{m-1-k} \theta_k \right\} + \left( \theta_f - 1 \right) R \sum_{k=0}^{m-1} \left\{ \left( \theta_f - 1 \right) \theta_{m-1-k} \sum_{l=0}^{k} \theta_{k-l} \sum_{s=0}^{l} \theta_{l-s} \theta''_s \right. \right. \]  

\[ + 3 \left( \theta_f - 1 \right) \sum_{l=0}^{k} \theta''_{k-l} \theta_l + 3 \theta''_k \} \right \} + 3 \left( \theta_f - 1 \right) \sum_{k=0}^{m-1} \theta''_{m-1-k} \theta'_k \]  

\[ \left\{ 1 + \sum_{l=0}^{k} \theta_{k-l} \theta_l + 2 \left( \theta_f - 1 \right) \theta_k \right\} , \]  

(4.30)

\[ \chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \]
The general solutions are

\[ f_m(\eta) = f_m^*(\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta}, \]  

\[ \theta_m(\eta) = \theta_m^*(\eta) + C_4 e^\eta + C_5 e^{-\eta}, \]  

where \( f_m^* \) and \( \theta_m^* \) are the particular solutions and the constants \( C_i (i = 1 - 5) \) are given by

\[ C_2 = C_4 = 0, \quad C_3 = \left. \frac{\partial f^*(\eta)}{\partial \eta} \right|_{\eta=0}, \quad C_1 = -C_3 - f^*(0), \quad C_5 = f^*(\eta)|_{\eta=0}. \]

### 4.4 Convergence of the homotopy solutions

The convergent rate and region of the series (4.26) and (4.27) depend on the auxiliary parameters \( h_f \) and \( h_\theta \). So we plot \( h \)-curves for the functions \( f''(0) \) and \( \theta'(0) \) at 12th order of approximations. The range for the admissible values of \( h_f \) and \( h_\theta \) are \(-1.3 \leq h_f \leq -0.2 \) and \(-1.4 \leq h_\theta \leq -0.4 \). We find that series given in the equation (4.24) and (4.25) converge in the whole region of \( \eta \) for different values of pertinent parameters by choosing the admissible values of \( h_f = -0.6 \) and \( h_\theta = -0.7 \).
Fig. 4.2. \( h \)–curve for temperature field.

Table 4.1: Convergence of series solutions for different order of approximations when \( \alpha = \pi/4, \quad G = 0.4, \quad \Gamma = 0.5, \quad \beta = 0.3, \quad R = 0.2, \quad \theta_f = 1.02, \quad \Pr = 1.0, \quad h_f = -0.6 \) and \( h_\theta = -0.7 \).

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<th>(-\theta'(0))</th>
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<tr>
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<tr>
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<td>0.85770</td>
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4.5 Results and discussion

The variations of velocity and temperature profiles for different physical parameters are highlighted in this suction. Different parameters include $\Gamma$ and $\beta$ the fluid parameters, $\Pr$ the Prandtl number, $R$ the radiation parameter, $\theta_f$ the temperature ratio parameter and $G$ the mixed convection parameter. Figs. 4.3 and 4.4 are plotted for the non-Newtonian fluid parameter $\Gamma$ and $\beta$. Velocity profile $f'(\eta)$ is an increasing function of $\Gamma$. Slight decrease is observed for larger values of $\beta$. Fig. 4.5 shows the variation of angle of inclination $\alpha$ on $f'(\eta)$. As $\alpha$ increases, sheet moves from vertical to horizontal direction and buoyancy force decreases. Ultimate the velocity and momentum boundary layer thickness decay. Buoyancy parameter $G$ enhances the velocity profile (see Fig. 4.6). Here we considered $G > 0$ (Buoyancy aid flow) therefore temperature gradient at the wall is larger in comparison to the ambient. Fig. 4.7 depicts the effect of thermal radiation parameter $R$ on $f'(\eta)$. Near the wall the variation is negligible but at some distance from the wall small increment is observed in velocity profile. Fluid parameter $\Gamma$ decreases the thermal boundary layer thickness (see Fig. 4.8). Radiation parameter $R$ enhances the temperature field significantly (see Fig. 4.9). Larger values correspond to more surface heat flux. Effect of temperature ratio parameter $\theta_f$ is more pronounced near the wall (see Fig. 4.10). In fact its larger values corresponds to stronger heat transfer at the wall than far away. Variation in temperature profile for larger values of $\Pr$ is displayed in Fig. 4.11. Larger values of $\Pr$ decay the thermal diffusivity and hence the temperature and thermal boundary layer thickness decrease.

Table 4.2 gives the comparison between numerical and HAM solutions in special case ($\Gamma = \beta = 0$). Here we found an excellent agreement. Table 4.3 shows the effect of local Nusselt number for different values of physical parameters $\Gamma$, $\beta$, $R$, and $\Pr$. Here we observe that the magnitude of $\Gamma$ and $\beta$ on local Nusselt number are quite opposite. Local Nusselt number decreases for large values of $R$. 
Fig. 4.3. Influence of $\Gamma$ on velocity field.

Fig. 4.4. Influence of $\beta$ on velocity field.
Fig. 4.5. Influence of $\alpha$ on velocity field.

\[ \gamma = 0.5, \Gamma = 0.5, \beta = 0.3, \text{Pr} = 1.0, \]  
\[ G = 0.4, R = 0.3, \theta_f = 1.02 \]  
\[ \alpha = \pi/8, \alpha = \pi/4, \alpha = \pi/3, \alpha = \pi/2 \]

Fig. 4.6. Influence of $G$ on velocity field.

\[ \gamma = 0.5, \Gamma = 0.5, \beta = 0.3, \text{Pr} = 1.0, \]  
\[ \alpha = \pi/4, R = 0.3, \theta_f = 1.02 \]  
\[ G = 0.0, G = 0.4, G = 0.8, G = 1.2 \]
Fig. 4.7. Influence of $R$ on velocity field.

Fig. 4.8. Influence of $\Gamma$ on temperature field.
Fig. 4.9. Influence of $R$ on temperature field.

Fig. 4.10. Influence of $\theta_f$ on temperature field.
Table 4.2: Comparison between numerical and HAM solutions in a special case when $\Gamma = \beta = 0$.

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<td>-1.0735</td>
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Table 4.3: Values of heat transfer characteristics at wall $-\theta'(0)$ for the parameters $\alpha$, $\Gamma$, $\beta$, $R$ and $\text{Pr}$.

<table>
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<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\Gamma$</th>
<th>$\text{Pr}$</th>
<th>$R$</th>
<th>$-(1 + R \theta_0^3)\theta'(0)$</th>
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<tr>
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4.6 Conclusions

This chapter investigated the flow of Powell-Eyring fluid by an inclined exponentially stretching sheet. The major conclusions with nonlinear thermal radiation effects are summarized below:

- Influence of $\Gamma$ on the velocity and temperature fields are quite opposite.

- Velocity field $f'(\eta)$ is decreasing function of $\beta$.

- Temperature field $\theta(\eta)$ and thermal boundary layer thickness are increased for larger thermal radiation parameter $R$.

- Angle of inclination $\alpha$ decays the velocity profile $f'(\eta)$.

- Effects of $\beta$ and $R$ on temperature field are qualitatively similar.

- It is found that $\theta(\eta)$ decreases when $\text{Pr}$ increases.
• Temperature ratio parameter enhances the temperature and thermal boundary layer thickness.

• Behaviors of $\Gamma$ and Pr on the magnitude of local Nusselt number are qualitatively similar.

• Magnitude of local Nusselt number decreases for larger values of $\beta$ and $R$. 
Chapter 5

Radiation effects in the flow of Powell-Eyring fluid past an unsteady inclined stretching sheet with non-uniform heat source/sink

5.1 Introduction

This chapter addresses time-dependent flow of Powell-Eyring fluid past an inclined stretching sheet. Unsteadiness in the flow is due to the time-dependence of stretching velocity and wall temperature. Mathematical analysis is performed in the presence of thermal radiation and non-uniform heat source/sink. The relevant boundary layer equations are reduced into self-similar forms by using suitable transformations. The analytic convergent solutions are constructed in the series forms. The convergence interval of the auxiliary parameters is obtained. Graphical results displaying the impacts of interesting parameters are given. Numerical values of skin friction coefficient and local Nusselt number are computed and analyzed.
5.2 Mathematical modeling and analysis

We consider unsteady two-dimensional incompressible flow of Powell-Eyring fluid past a stretching sheet. The sheet makes an angle $\alpha$ with the vertical direction. The $x$- and $y$-axes are taken along and perpendicular to the sheet respectively. In addition the effects of thermal radiation and non-uniform heat source/sink are considered. The boundary layer equations comprising the balance laws of mass, linear momentum and energy can be written into the forms:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (5.1)
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \left( \nu + \frac{1}{\rho \beta C} \right) \frac{\partial^2 u}{\partial y^2} - \frac{1}{2 \rho \beta C^3} \left( \frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} + g_0 \beta T (T - T_\infty) \cos \alpha, \quad (5.2)
\]

\[
\rho c_p \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} - \frac{\partial q_r}{\partial y} + q'', \quad (5.3)
\]

In the above expressions $t$ is the time, $\nu = (\mu/\rho)$ is the kinematic viscosity, $k$ is the thermal conductivity of the fluid, $\rho$ is the fluid density, $T$ is the fluid temperature, $c_p$ is the specific heat, $g_0$ is the acceleration due to gravity, $\beta_T$ is the volumetric coefficient of thermal exponential, $q_r = -\frac{16 \sigma^* T_\infty^3}{3 k^*} \frac{\partial T}{\partial y}$ [56–65] is the linearized radiative heat flux, $k^*$ is the mean absorption coefficient, $\sigma^*$ is the Stefan-Boltzmann constant, $q''$ is the non-uniform heat generated ($q'' > 0$) or absorbed ($q'' < 0$) per unit volume. The non-uniform heat source/sink $q''$ is modeled by the following expression:

\[
q'' = \frac{k u_s(x, t)}{x \nu} \left[ A(T_s - T_\infty)f' + (T - T_\infty)B \right], \quad (5.4)
\]

in which $A$ and $B$ are the coefficients of space and temperature-dependent heat source/sink, respectively. Here two cases arise. For internal heat generation $A > 0$ and $B > 0$ and for internal heat absorption we have $A < 0$ and $B < 0$.

The surface velocity is denoted by $u_s(x, t) = \frac{b x}{(1 - at)}$ whereas the surface temperature $T_s(x, t) = T_\infty + T_{\text{ref}} b^2 \frac{b^2}{2 \nu} (1 - at)^{-3/2}$. Here $b$ (stretching rate) and $a$ are the positive constants with dimension time$^{-1}$. Also $T_{\text{ref}}$ is a constant reference temperature. We note that the temperature of stretching sheet is larger than the free stream temperature $T_\infty$.
The boundary conditions are taken as follows:

\[ u = u_s(x, t), \quad v = 0, \quad T = T_s(x, t) \text{ at } y = 0, \quad (5.5) \]

\[ u \to 0, \quad T \to T_\infty \text{ as } y \to \infty. \]

Introducing

\[ u = \frac{bx}{(1 - at)} f'(\eta), \quad v = -\sqrt{\frac{\nu b}{1 - at}} f(\eta), \]

\[ \theta = \frac{T - T_\infty}{T_s - T_\infty}, \quad \eta = \sqrt{\frac{b}{\nu(1 - at)}} y, \quad (5.6) \]

equation (5.1) is identically satisfied and Eqs. (5.2 – 5.5) become

\[ (1 + \Gamma) f''' - f'^2 + f f'' - \Gamma \beta f'^2 f''' - \varepsilon (f' + \frac{1}{2} \eta f'') + G \theta \cos \alpha = 0, \quad (5.7) \]

\[ \left( 1 + \frac{4}{3} R \right) \theta'' + \text{Pr} \left( f \theta' - 2 f' \theta - \frac{1}{2} \varepsilon (3 \theta + \eta \theta') \right) + Af' + B \theta = 0, \quad (5.8) \]

\[ f = 0, \quad f' = 1, \quad \theta = 1 \text{ at } \eta = 0, \]

\[ f' \to 0, \quad \theta \to 0 \text{ as } \eta \to \infty, \quad (5.9) \]

where prime denotes the differentiation with respect to \( \eta \), \( f \) is the dimensionless stream function, \( \theta \) is the dimensionless temperature and the dimensionless numbers are

\[ \Gamma = \frac{1}{\mu \beta C}, \quad R = \frac{4 \sigma^* T_\infty^3}{kk^*}, \quad \beta = \frac{\rho v_s^2}{\mu x C^2}, \quad G = \frac{g \beta r (T_s - T_\infty) x^3 / \nu^2}{u_s^2 x^2 / \nu^2} = \frac{Gr_x}{Re_x^2}, \quad \epsilon = \frac{a}{b}, \quad \text{Pr} = \frac{\mu c_p}{k}. \quad (5.10) \]

Here \( \Gamma \) and \( \beta \) are the dimensionless material fluid parameters, \( R \) is the radiation parameter, \( \epsilon \) is the unsteady parameter, \( G \) is the mixed convection parameter and \( \text{Pr} \) is the Prandtl number.

Local Nusselt number \( N_u x \) is defined by

\[ N_u x = \frac{x q_w}{k(T_s - T_\infty)}; \quad q_w = -k \left( \frac{\partial T}{\partial y} \right)_{y=0} + (q_r)_w, \quad (5.11) \]
\[ \text{Re}_x^{-1/2} Nu_x = - \left( 1 + \frac{4}{3} R \right) \theta'(0), \]

where \( \text{Re}_x = \frac{u_x}{v} \) is the local Reynolds number.

### 5.3 Series solutions

Initial approximations and auxiliary linear operators are taken as

\[ f_0(\eta) = 1 - e^{-\eta}, \quad \theta_0(\eta) = e^{-\eta}, \quad (5.12) \]

\[ \mathcal{L}_f = f'' - f', \quad \mathcal{L}_\theta = \theta'' - \theta, \quad (5.13) \]

subject to the properties

\[ \mathcal{L}_f(C_1 + C_2 e^\eta + C_3 e^{-\eta}) = 0, \quad \mathcal{L}_\theta(C_4 e^\eta + C_5 e^{-\eta}) = 0, \quad (5.14) \]

where \( C_i \) \((i = 1 - 5)\) are the constants.

The deformation problems at zeroth order are

\[ (1 - p) \mathcal{L}_f \left[ \hat{f}(\eta; p) - f_0(\eta) \right] = ph_f \mathcal{N}_f \left[ \hat{f}(\eta; p) \right], \quad (5.15) \]

\[ (1 - p) \mathcal{L}_\theta \left[ \hat{\theta}(\eta; p) - \theta_0(\eta) \right] = ph_\theta \mathcal{N}_\theta \left[ \hat{f}(\eta; p), \hat{\theta}(\eta; p) \right], \quad (5.16) \]

\[ \hat{f}(0; p) = 0, \quad \hat{f}'(0; p) = 1, \quad \hat{f}'(\infty; p) = 0, \quad \hat{\theta}(0; p) = 1, \quad \hat{\theta}(\infty; p) = 0. \quad (5.17) \]

If \( p \in [0, 1] \) indicates the embedding parameter, \( h_f \) and \( h_\theta \) the non-zero auxiliary parameters then the nonlinear differential operators \( \mathcal{N}_f \) and \( \mathcal{N}_\theta \) are given by

\[ \mathcal{N}_f[\hat{f}(\eta, p)] = (1 + \Gamma) \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} + \hat{f}(\eta, p) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right)^2 \]

\[ -\Gamma \beta \left( \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right)^2 \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} - \epsilon \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} + \frac{1}{2} \eta \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right) \]

\[ + G \hat{\theta}(\eta, p) \cos \alpha, \quad (5.18) \]
\[
\mathcal{N}_0[\hat{\theta}(\eta, p), \hat{f}(\eta, p)] = \left(1 + \frac{4}{3} R\right) \frac{\partial^2 \hat{\theta}(\eta, p)}{\partial \eta^2} + \left(\Pr \hat{f}(\eta, p) \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} - 2\hat{\theta}(\eta, p) \frac{\partial \hat{f}(\eta, p)}{\partial \eta}\right) \\
- \varepsilon \left(3\hat{\theta}(\eta, p) + \eta \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta}\right) + A \frac{\partial \hat{f}(\eta, p)}{\partial \eta} + B \hat{\theta}(\eta, p). \tag{5.19}
\]

We have for \( p = 0 \) and \( p = 1 \) the following equations

\[
\hat{f}(\eta; 0) = f_0(\eta), \quad \hat{\theta}(\eta, 0) = \theta_0(\eta), \tag{5.20}
\]

\[
\hat{f}(\eta; 1) = f(\eta), \quad \hat{\theta}(\eta, 1) = \theta(\eta). \tag{5.21}
\]

It is noticed that when \( p \) varies from \( 0 \) to \( 1 \) then \( f(\eta, p) \) and \( \theta(\eta, p) \) approach from \( \theta_0(\eta) \) and \( \theta_0(\eta) \) to \( f(\eta) \) and \( \theta(\eta) \). The series of \( f \) and \( \theta \) through Taylor’s expansion are chosen convergent for \( p = 1 \) and thus

\[
f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \quad f_m(\eta) = \frac{1}{m!} \frac{\partial^m f(\eta; p)}{\partial \eta^m}\bigg|_{p=0}, \tag{5.22}
\]

\[
\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \quad \theta_m(\eta) = \frac{1}{m!} \frac{\partial^m \theta(\eta; p)}{\partial \eta^m}\bigg|_{p=0}. \tag{5.23}
\]

The resulting problems at \( m^{th} \) order can be presented in the following forms:

\[
\mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f \mathcal{R}^m_f(\eta), \tag{5.24}
\]

\[
\mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_\theta \mathcal{R}^m_\theta(\eta), \tag{5.25}
\]

\[
f_m(0) = f'_m(0) = f'_m(\infty) = \theta_m(0) = \theta_m(\infty) = 0, \tag{5.26}
\]

\[
\mathcal{R}^m_f(\eta) = (1 + \Gamma) f''_{m-1}(\eta) + \sum_{k=0}^{m-1} \left[f_{m-1-k} f'_k - f'_{m-1-k} f''_k - \Gamma \beta f''_{m-1} \sum_{k=0}^{m-1} f''_{m-1-k} - \frac{1}{2} \eta f''_{m-1}\right] + G \theta_{m-1} \cos \alpha, \tag{5.27}
\]
\[ \mathcal{R}_y^m(\eta) = \left(1 + \frac{4}{3}R_d\right) \theta''_{m-1}(\eta) + Pr \sum_{k=0}^{m-1} \left[ \theta'_{m-1-k} f_k - 2f'_{m-1-k} \theta_k - \frac{1}{2} \epsilon (3\theta_{m-1} + \eta \theta'_{m-1}) \right] + A f'_{m-1} + B \theta_{m-1}, \] (5.28)

\[ \chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \]

The general solutions \((f_m, \theta_m)\) comprising the special solutions \((f^*_m, \theta^*_m)\) are

\[ f_m(\eta) = f^*_m(\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta}, \] (5.29)

\[ \theta_m(\eta) = \theta^*_m(\eta) + C_4 e^\eta + C_5 e^{-\eta}. \] (5.30)

### 5.4 Convergence of the homotopy solutions

It is now well established argument that the convergence of series solutions (5.24) and (5.25) depends upon the auxiliary parameters. The admissible range of values of \(h_f\) and \(h_\theta\) (for some fixed values of parameters) lie along the line segment parallel to \(h_f\) and \(h_\theta\)–axes. For example in Figs. 5.1 and 5.2 the permissible ranges of values of \(h_f\) and \(h_\theta\) are \(-1.26 \leq h_f \leq -0.24\) and \(-1.4 \leq h_\theta \leq -0.25\) respectively when \(\epsilon = 0.6\). The series solutions converge for the whole region of \(\eta\) when \(h_f = -0.9\) and \(h_\theta = -0.8\).
Fig. 5.1. $h-$ curves for the velocity field.

Fig. 5.2. $h-$ curves for the temperature field.
Table 5.1: Convergence of series solutions for different order of approximations when $\alpha = \pi/4$, $\beta = 0.5$, $\Gamma = 0.2$, $R = 0.2$, $\epsilon = 0.6$, $G = 0.3$, $Pr = 1.0$, $A = B = 0.1$, $hf = -0.8$ and $h_\theta = -0.7$.

<table>
<thead>
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<th>Order of approximation</th>
<th>$-f''(0)$</th>
<th>$-\theta'(0)$</th>
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<td>1.33250</td>
</tr>
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<td>30</td>
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</tr>
</tbody>
</table>

5.5 Discussion

This section is aimed to examine the effects of different physical parameters on the velocity and temperature fields. Hence Figs. (5.3 – 5.15) are plotted. Fig. 5.3 elucidates the behavior of inclination angle $\alpha$ on the velocity and boundary layer thickness. Here $\alpha = 0$ corresponds to velocity profiles in the case of vertical sheet for which the fluid experiences the maximum gravitational force. On the other hand when $\alpha$ changes from 0 to $\pi/2$ i.e when the sheet moves from vertical to horizontal direction then, the strength of buoyancy force decreases and consequently the velocity and the boundary layer thickness decrease. Fig. 5.4 indicates that the velocity field $f'$ is an increasing function of $G$. This is because of the fact that larger values of $G$ yield a stronger buoyancy force which leads to an increase in the $x-$component of velocity. The boundary layer thickness also increases by increasing $G$. Variation in $f'$ through $\beta$ can be seen from Fig. 5.5. It is noticed that $f'$ decreases and boundary layer thins when $\beta$ is increased. Influence of unsteady parameter $\epsilon$ on the velocity field is displayed in Fig. 5.6. Increasing values of $\epsilon$ indicates smaller stretching rate in the $x-$direction which eventually decreases the boundary layer thickness. Interestingly the velocity enhances by increasing $\epsilon$ at sufficiently large distance from the sheet. Variation in the $x-$ component of velocity with an increase in the fluid parameter $\Gamma$ can be described from Fig. 5.7. In accordance with Mushtaq et al. [36], the velocity field $f'$ increases when $\Gamma$ is increased. Radiation effects on the velocity and temperature distributions can be perceived through Figs. 5.8 and 5.9. An increase in $R$ enhances the heat
flux from the sheet which gives rise to the fluid’s velocity and temperature. Wall slope of the
temperature function therefore increases with an increase in $R$. Fig. 5.10 portrays the effect of
Prandtl number on the thermal boundary layer. From the definition of Pr given in Eq. (5.10),
it is obvious that increasing values of Pr decreases conduction and enhances pure convection
or the transfer of heat through unit area. That is why the temperature and thermal boundary layer
thickness decrease for larger Pr. Such reduction in the thermal boundary layer accompanies
with the larger heat transfer rate from the sheet. Temperature profiles for different values of
$\Gamma$ are shown in Fig. 5.11. It is seen that temperature $\theta$ is an increasing function of $\Gamma$. Fig.
5.12 indicates that an increase in the strength of buoyancy force due to temperature gradient
decreases the temperature and thermal boundary layer thickness. Influence of heat source/sink
parameter on the thermal boundary layer is shown in the Figs. 5.13 and 5.14. As expected the
larger heat source (corresponding to $A > 0$ and $B > 0$) rises the temperature of fluid above the
sheet. However the non-uniform heat sink corresponding to $A < 0$ and $B < 0$ can contribute in
quenching the heat from stretching sheet effectively. Fig. 5.18 depicts that temperature $\theta$ is a
decreasing function of the unsteady parameter $\epsilon$.

Table 5.2 shows the comparison of present work with Tsai et al. [53] in a special case. A
very good agreement is found for the results of wall temperature gradient. Table 5.3 shows the
effect of embedded parameters on heat transfer characteristics at the wall $-\theta'(0)$. Since in the
present case the sheet is hotter than the fluid i.e $T_w > T_\infty$ thus heat flows from the sheet to
the fluid and hence $\theta'(0)$ is negative. From this table we observe that with an increase in $\alpha$, $\beta$
and $R$ the wall heat transfer rate $|\theta'(0)|$ decreases. However it increases when $\Gamma$, $\epsilon$ and Pr are
increased.
Fig. 5.3. Variation of $\alpha$ on velocity field.

Fig. 5.4. Variation of $G$ on velocity field.
Fig. 5.5. Variation of $\beta$ on velocity field.

\[\Gamma = 0.2, \alpha = \pi/4, \epsilon = 0.6, G = 0.8, R = 0.2, Pr = 1.0, A = B = 0.1\]

$\beta = 0.0, \beta = 2.0, \beta = 4.0, \beta = 6.0$

Fig. 5.6. Variation of $\epsilon$ on velocity field.

$\beta = 0.5, \alpha = \pi/4, G = 0.3, \Gamma = 0.2, R = 0.2, Pr = 1.0, A = B = 0.1$

$\epsilon = 0.0, \epsilon = 0.6, \epsilon = 0.9, \epsilon = 1.5$
Fig. 5.7. Variation of $\Gamma$ on velocity field.

Fig. 5.8. Variation of $R$ on velocity field.
Fig. 5.9. Variation of $R$ on temperature field.

Fig. 5.10. Variation of $Pr$ on temperature field.
Fig. 5.11. Variation of $\Gamma$ on temperature field.

Fig. 5.12. Variation of $G$ on temperature field.
Fig. 5.13. Variation of $\mathcal{A}$ on temperature field.

Fig. 5.14. Variation of $\mathcal{B}$ on temperature field.
Fig. 5.15. Variation of $\epsilon$ on temperature field.

Table 5.2: Comparison between numerical solution Tsai et. al. [53] and HAM solution in a special case when $\alpha = \beta = \epsilon = \Gamma = G = R = 0$.

<table>
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<th>Pr</th>
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<th>$A$</th>
<th>Present study</th>
<th>Tsai et.[53]</th>
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Table 5.3: Values of heat transfer characteristics at wall $-\theta'(0)$ for different emerging parameters when $h_f = -0.8$ and $h_\theta = -0.7$.

<table>
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<tr>
<th>$\alpha$</th>
<th>$\Gamma$</th>
<th>$\beta$</th>
<th>$\epsilon$</th>
<th>$G$</th>
<th>$R$</th>
<th>Pr</th>
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85
5.6 Closing remarks

This chapter addressed the radiation effects in the unsteady boundary layer flow of Powell-Eyring fluid past an unsteady inclined stretching sheet with non-uniform heat source/sink. The important findings are listed below.

1. The strength of gravitational force can be varied by changing the inclination angle $\alpha$ which the sheet makes with the vertical direction. The velocity decreases with an increase in $\alpha$.

2. Velocity field $f'$ and temperature $\theta$ are decreasing function of the unsteady parameter $\epsilon$.

3. Velocity increases and temperature decreases when the fluid parameter $\Gamma$ is increased.

4. Increase in the radiation parameter $R$ enhances the heat flux from the plate. As a consequence the fluid velocity and temperature both increases.

5. The analysis for the case of viscous fluid can be obtained by choosing $\Gamma = \beta = 0$. Further the results for horizontal stretching sheet are achieved when $\alpha = \pi/2$. 


Chapter 6

MHD mixed convection flow of Burgers’ fluid in a thermally stratified medium

6.1 Introduction

This chapter explores the effects of nonlinear thermal radiation and magnetohydrodynamics (MHD) in mixed convection flow of Burgers’ fluid over a stretching sheet embedded in a thermally stratified medium. Transformation method has been employed to reduce the nonlinear PDE into the nonlinear ODE. The resulting nonlinear coupled ordinary differential equations are solved for the convergent series solutions. Convergence of derived series solutions is shown explicitly explored. Physical interpretation of different parameters through graphs and numerical values of local Nusselt number are discussed.

6.2 Flow equations

The extra stress tensor for Burgers’ fluid satisfies [49]:

\[
S + \lambda_1 \frac{DS}{Dt} + \lambda_2 \frac{D^2 S}{Dt^2} = \mu \left[ A_1 + \lambda_3 \frac{DA_1}{Dt} \right],
\]  

(6.1)
in which $\mu$ is the dynamic viscosity, $A_1$ is the rate of strain tensor, $\lambda_1$ and $\lambda_2$ are the relaxation times, $\lambda_3$ is the retardation time and the upper convected derivative $\frac{D}{Dt}$ is

$$\frac{DS}{Dt} = \frac{dS}{dt} - LS - SL^T,$$

(6.2)

in which $d/dt$ is the material time derivative and $L$ is the velocity gradient. It should be noted that Burgers’ fluid model reduces to the special cases of an Oldroyd-B model, Maxwell model and the Newtonian fluid model when $(\lambda_2 = 0)$, $(\lambda_2 = \lambda_3 = 0)$ and $(\lambda_1 = \lambda_2 = \lambda_3 = 0)$ respectively.

### 6.3 Problem statement

Here we consider MHD mixed convection flow of Burgers’ fluid in a thermally stratified medium. An incompressible fluid is electrically conducting in the presence of constant applied magnetic field $B_0$. Electric and induced magnetic fields are assumed negligible. The flow is because of linear stretching sheet along the $x-axis$. The $y-axis$ is taken normal to the surface with constant temperature. Also thermal radiation effects in heat transfer analysis are present.

The equations for velocity and temperature in boundary layer regime are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

(6.3)

$$\begin{align*}
&u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \lambda_1 \left[ u^2 \frac{\partial^2 u}{\partial x^2} + v^2 \frac{\partial^2 u}{\partial y^2} + 2uv \frac{\partial^2 u}{\partial x \partial y} \right] + \lambda_2 \left[ u^4 \frac{\partial^3 u}{\partial x^3} + v^4 \frac{\partial^3 u}{\partial y^3} \right] \\
&+ u^2 \left( \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 u}{\partial x \partial y} + 2 \frac{\partial v}{\partial y} \frac{\partial^2 u}{\partial x \partial y} \right) + 3v^2 \left( \frac{\partial v}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x^2} \right) \\
&+ 3uv \left( u \frac{\partial^3 u}{\partial x^2 \partial y} + v \frac{\partial^3 u}{\partial x \partial y^2} \right) + 2uv \left( 2 \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y} \frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial y} \frac{\partial^2 u}{\partial x \partial y} \right) \\
&= \nu \frac{\partial^2 u}{\partial y^2} + \nu \lambda_3 \left[ u \frac{\partial^3 u}{\partial x \partial y^2} + v \frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y} \frac{\partial^2 v}{\partial x \partial y} \right] + g_0 \beta_T (T - T_\infty) \\
&- \frac{\sigma B_0^2}{\rho} \left( u + \lambda_1 v \frac{\partial u}{\partial y} + \lambda_2 \left( u \frac{\partial u}{\partial y} - v \frac{\partial u}{\partial y} - w \frac{\partial u}{\partial y} + \frac{1}{4} \frac{\partial^2 u}{\partial y^2} \right) \right),
\end{align*}$$

(6.4)

$$\rho c_p \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left[ \left( k + \frac{16 \sigma T^3}{3k^*} \right) \frac{\partial T}{\partial y} \right],$$

(6.5)
with the subjected conditions

\[ u = u_w(x) = ax, \quad v = 0, \quad T = T_w, \quad \text{at} \quad y = 0, \]  \quad (6.6)

\[ u \to 0, \quad T \to T_\infty, \quad \text{as} \quad y \to \infty, \]  \quad (6.7)

where \( u \) and \( v \) represent the velocity components in the \( x \) and \( y \) directions respectively, \( T \) is the fluid temperature, \( T_w = T_0 + bx \) is the prescribed surface temperature, \( T_\infty = T_0 + cx \) is the variable ambient fluid temperature, \( \nu = (\mu/\rho) \) is the kinematic viscosity, \( \rho \) is the density of fluid, \( h \) is the convective heat transfer coefficient, \( D \) is the effective diffusion coefficient, \( c_p \) is the specific heat at constant pressure and \( k \) is the thermal conductivity of the fluid. Making use of following change of variables

\[ \psi = x (a \nu)^{-1/2} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_0}, \quad \eta = y \left( \frac{a}{\nu} \right)^{1/2}, \]  \quad (6.8)

\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \]  \quad (6.9)

equation (6.6) is identically satisfied and Eqs. (6.7–6.10) yield

\[ f''' + ff'' - f'^2 + \beta_1 \left(2ff'f'' - f'^2f''\right) + \beta_2 \left(f''f''' - 2f'^2f'' - 3f'^2f''ight) \]

\[ + \beta_3 \left(f'^2 - f''f'''\right) + G\theta - M^2 \left(f' - \beta_1 f f'' + \beta_2 f'^2 f''\right) = 0, \]  \quad (6.10)

\[ \left[1 + R(1 + (\theta_f - 1)\theta)^3\right]\theta' + Pr \left[f\theta' - f'\theta - \hat{S}f'\right], \]  \quad (6.11)

\[ f = 0, \quad f' = 1, \quad \theta = 1 - \hat{S} \quad \text{at} \quad \eta = 0, \]

\[ f' \to 0, \quad \theta \to 0 \quad \text{as} \quad \eta \to \infty, \]  \quad (6.12)

where \( \beta_1 = \lambda_1 a \) and \( \beta_2 = \lambda_2 a \) are the Deborah numbers in terms of relaxation times respectively, \( \beta_3 = \lambda_3 a \) is the Deborah number in term of retardation times, \( \psi \) is the stream function, \( M = \frac{aB^2}{\nu k} \) is the magnetic parameter, \( Pr = \frac{\nu c_p}{k_\infty} \) is the Prandtl number, \( G = \frac{b_2\beta_1}{\nu^2} \) is the mixed convection parameter, \( R = \frac{4\sigma T^3}{k_\infty k_n} \) is the radiation parameter, \( \theta_f \) is the temperature ratio parameter and \( \hat{S} = \frac{\xi}{\theta} \) is the stratification parameter.
The local Nusselt number $Nu_x$ is defined by

$$Nu_x = \frac{xq_w}{k(T_w - T_\infty)}, \quad \text{with} \quad q_w = -k \frac{\partial T}{\partial y}_{y=0} + (q_r)_w. \quad (6.13)$$

Dimensionless form of local Nusselt number $Nu_x$ is

$$Nu/Re_x^{1/2} = -(1 + R\theta^3) \theta'(0),$$

where $(Re_x)^{-1/2} = \sqrt{\frac{\nu}{xu_\infty}}$ is the local Reynolds number.

### 6.4 Development of series

Initial approximations and auxiliary linear operators for the HAM solution expressions are

$$f_0(\eta) = (1 - e^{-\eta}), \quad \theta_0(\eta) = (1 - \hat{S})e^{-\eta}, \quad (6.14)$$

$$L_f = f''' - f', \quad L_\theta = \theta'' - \theta, \quad (6.15)$$

$$L_f(C_1 + C_2 e^\eta + C_3 e^{-\eta}) = 0, \quad (6.16)$$

$$L_\theta(C_4 e^\eta + C_5 e^{-\eta}) = 0, \quad (6.17)$$

where $C_i \ (i = 1 - 7)$ are the arbitrary constants, $p \in [0,1]$ denotes an embedding parameter and $h_f$ and $h_\theta$ the non-zero auxiliary parameters. The zeroth order deformation problems are

$$(1 - p) L_f \left[ \hat{f}(\eta; p) - f_0(\eta) \right] = ph_f N_f \left[ \hat{f}(\eta; p) \right], \quad (6.18)$$

$$(1 - p) L_\theta \left[ \hat{\theta}(\eta; p) - \theta_0(\eta) \right] = ph_\theta N_\theta \left[ \hat{f}(\eta; p), \hat{\theta}(\eta; p) \right], \quad (6.19)$$

$$\hat{f}(0; p) = 0, \quad \hat{f}'(0; p) = 1, \quad \hat{f}(\infty; p) = 0, \quad \hat{\theta}(0; p) = 1 - \hat{S}, \quad \hat{\theta}(\infty; p) = 0. \quad (6.20)$$
\[ N_f[\hat{f}(\eta, p)] = \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} + \hat{f}(\eta, p) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right)^2 + \beta_1 \left[ 2\hat{f}(\eta, p) \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - \left( \hat{f}(\eta, p) \right)^2 \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} \right] + \beta_2 \left[ \left( \hat{f}(\eta, p) \right)^3 \frac{\partial^4 \hat{f}(\eta, p)}{\partial \eta^4} - 2\hat{f}(\eta, p) \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right)^2 \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right] - 3 \left( \hat{f}(\eta, p) \right)^2 \left( \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right)^2 + \beta_3 \left[ \left( \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right)^2 - \hat{f}(\eta, p) \frac{\partial^4 \hat{f}(\eta, p)}{\partial \eta^4} \right] + G\hat{\theta}(\eta, p) - M^2 \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} - \beta_1 \hat{f}(\eta, p) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} + \beta_2 \left( \hat{f}(\eta, p) \right)^2 \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right), \tag{6.21} \]

\[ N_0[\hat{\theta}(\eta, p), \hat{f}(\eta, p)] = \frac{\partial}{\partial \eta} \left[ \left\{ 1 + R \left( 1 + (\theta_f - 1)\hat{\theta}(\eta, p) \right)^3 \right\} \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} \right] + \text{Pr} \left( \hat{f}(\eta, p) \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} - \hat{\theta}(\eta, p) \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right). \tag{6.22} \]

When \( p = 0 \) and \( p = 1 \) then

\[ \hat{f}(\eta; 0) = f_0(\eta), \quad \hat{\theta}(\eta; 0) = \theta_0(\eta), \]
\[ \hat{f}(\eta; 1) = f(\eta), \quad \hat{\theta}(\eta; 1) = \theta(\eta). \tag{6.23} \]

When \( p \) variation is taken from 0 to 1 then \( f(\eta, p) \) and \( \theta(\eta, p) \) approach \( f_0(\eta) \) and \( \theta_0(\eta) \) to \( f(\eta) \) and \( \theta(\eta) \). Now \( f \) and \( \theta \) in Taylor's series can be expanded in the forms

\[ f(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) p^m, \tag{6.24} \]
\[ \theta(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) p^m, \tag{6.25} \]
\[ f_m(\eta) = \frac{1}{m!} \frac{\partial^m f(\eta; p)}{\partial \eta^m} \bigg|_{p=0}, \quad \theta_m(\eta) = \frac{1}{m!} \frac{\partial^m \theta(\eta; p)}{\partial \eta^m} \bigg|_{p=0}, \]  

where the convergence depends upon \( h_f \) and \( h_\theta \). For \( p = 1 \), the series (6.27) and (6.28) yield

\[ f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta). \]  

\[ \theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta). \]  

The deformation problems corresponding to \( m^{th} \)-order are

\[ \mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f \mathcal{R}^m_f(\eta), \]  

\[ \mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_\theta \mathcal{R}^m_\theta(\eta), \]  

\[ f_m(0) = f'_m(0) = f'_m(\infty) = 0, \quad \theta_m(0) = \theta_m(\infty) = 0, \]  

\[ \mathcal{R}^m_f(\eta) = f'''_{m-1}(\eta) + \sum_{k=0}^{m-1} \{ f''_{m-1-k} f''_{k} - f'_{m-1-k} f'_{k} \} \]

\[ + \beta_1 \sum_{k=0}^{m-1} \left\{ 2 f''_{m-1-k} \sum_{l=0}^{k} f''_{l} - f_{m-1-k} \sum_{l=0}^{k} f''_{l} \right\} \]

\[ + \beta_2 \sum_{k=0}^{m-1} \left\{ f''_{m-1-k} \sum_{l=0}^{k} f_{l} - 2 f''_{l} f''_{m-1-k} - 2 f_{m-1-k} \sum_{l=0}^{k} f''_{l} - 3 f_{m-1-k} \sum_{l=0}^{k} f''_{l} \right\} \]

\[ + \beta_3 \left\{ f'''_{m-1-k} \sum_{k=0}^{m-1} f''_{k} - f''_{m-1-k} \sum_{k=0}^{m-1} f'''_{k} \right\} + G \theta_{m-1} \]

\[ - M^2 \left( f'_{m-1} - \beta_1 \sum_{k=0}^{m-1} f_k - f'''_{m-1-k} + \beta_2 \sum_{k=0}^{m-1} f_{m-1-k} \sum_{l=0}^{k} f''_{l} \right), \]  

\[ 92 \]
\[ R^{''}_{\alpha}(\eta) = (1 + R) \theta^{''}_{m-1}(\eta) + Pr \sum_{k=0}^{m-1} \left\{ \theta^{'}_{m-1-k}f_k - f^{'}_{m-1-k}\theta_k - \ddot{S}f^{'}_{m-1} \right\} \]

\[ + (\theta_f - 1) R \sum_{k=0}^{m-1} \left\{ (\theta_f - 1)^2 \theta_{m-1-k} \left\{ \sum_{l=0}^{k} \theta_{k-l} \sum_{s=0}^{l} \theta_{l-s} \theta''_s \right\} + 3(\theta_f - 1) \sum_{k=0}^{m-1} \theta^{'}_{m-1-k}\theta''_k \right\} \]

\[ \left\{ 1 + \sum_{l=0}^{k} \theta_{k-l}\theta_l + 2(\theta_f - 1)\theta_k \right\} , \quad (6.33) \]

\[ \chi_m = \begin{cases} 
 0, & m \leq 1, \\
 1, & m > 1.
\end{cases} \]

The general solutions of Eqs. (6.32 – 6.33) are

\[ f_m(\eta) = f^*_m(\eta) + C_1 + C_2e^{\eta} + C_3e^{-\eta}, \quad (6.34) \]

\[ \theta_m(\eta) = \theta^*_m(\eta) + C_4e^{\eta} + C_5e^{-\eta}, \quad (6.35) \]

where \( f^*_m \) and \( \theta^*_m \) indicate the particular solutions.

### 6.5 Convergence of solutions

Obviously the series (6.32) and (6.33) consist of the auxiliary parameters \( \tilde{h}_f \) and \( \tilde{h}_\theta \). Such parameters have key role for the convergence of series solutions. We plot the \( \tilde{h} \)-curves at 17\(^{th}\) order of approximations (see Figs. 6.1 and 6.2). It is clearly seen from these Figs. that the admissible values of \( \tilde{h}_f \) and \( \tilde{h}_\theta \) are \([-1.3, -0.6]\) and \([-1.3, -0.6]\) respectively. Moreover Table 6.1 depicts that the series solutions are convergent up to five decimal places.
\[ \beta_1 = 0.2, \beta_2 = 0.2, \beta_3 = 0.3, M = 0.1, G = 0.2 \]

Fig. 6.1. \( h \)- curve for the velocity field.

\[ \beta_1 = 0.2, \beta_2 = \hat{S} = 0.3, \beta_3 = G = R = 0.5, \\
M = 0.1, \text{Pr} = 1.0, \theta_2 = 1.02 \]

Fig. 6.1. \( h \)- curve for the temperature field.
Table 6.1: Convergence of homotopic solution for different order of approximations.

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6.6 Discussion

To provide a physical insight into the computed analysis, the velocity and temperature fields are described for various values of pertinent variables. Here the influence of Deborah numbers on the velocity field is presented in the Figs. (6.3 – 6.5). Velocity field is found to decrease for larger values of \(\beta_1\) and \(\beta_2\) (see Fig. 6.3 and 6.4). As Deborah number exhibits both viscous and elastic effects therefore velocity always retards when we increase this parameter. Fig. 6.5 depicts the variation of \(\beta_3\) on the velocity field. A small increment is observed in the velocity field when \(\beta_3\) enhances. This is due to the fact that larger \(\beta_3\) corresponds to the large retardation time. Therefore velocity and associated boundary layer thickness increase. Impact of magnetic parameter \(M\) on the velocity field is displayed in Fig. 6.6. It is revealed that both the velocity and momentum boundary layer thickness decrease when magnetic parameter \(M\) is increased. In fact through magnetic field, the apparent viscosity enhances to the point of becoming viscoelastic solid. Also variation in magnetic field intensity is used to control the yield stress and boundary layer thickness. Consequently the fluid ability to transmit the force is controlled. Fig. 6.7 shows that an increase in mixed convection parameter \(G\) corresponds to an increase in the velocity field. This is because of the reason that buoyancy forces induce a pressure gradient in the boundary layer which acts to modify the flow field. Fig. 6.8 shows the variation of magnetic parameter \(M\) on the temperature field. Here the temperature increases when
\( M \) is increased. Physically larger magnetic field corresponds to much Lorentz force therefore temperature and thermal boundary layer thickness are increased. Temperature field decays for larger values of mixed convection parameter \( G \) (see Fig. 6.9). The influence of radiation parameter \( R \) is displayed in Fig. 6.10. Enhancement in radiation parameter \( R \) leads to increase in kinetic energy that rises the fluid temperature and thermal boundary layer thickness. Effect of temperature ratio parameter is displayed in Fig. 6.11. It is clearly seen from this Fig. that the temperature field increases when there is an increase in temperature ratio parameter. Thermal stratification parameter \( \hat{S} \) decays the temperature most rapidly near the wall but after some large distance from the wall the variation of \( \hat{S} \) is insignificant (see Fig. 6.12). Physically this is due to the fact that with the increase in the stratification parameter, the buoyancy factor reduces within the boundary layer and so the temperature decays. Fig. 6.13 illustrates the effect of Prandtl number on temperature field. Higher values of Prandtl number show a decrease in the thermal diffusivity. As a result both the temperature and thermal boundary layer thickness decrease.

Table 6.2 shows the effect of local Nusselt number for different values of physical parameters \( \beta_1, \beta_2, \beta_3, \hat{S}, R \) and \( \text{Pr} \). Here we observe that the effects of \( \beta_1 \) and \( \beta_3 \) on local Nusselt number are quite opposite. Local Nusselt number decreases for larger values of \( \beta_3, R \) and \( \text{Pr} \).

![Graph](image)

\( \beta_1 = 0.0, \beta_1 = 0.4, \beta_1 = 0.6, \beta_1 = 0.8, \beta_2 = 0.3, \beta_3 = 0.2, M = 0.1, G = 0.2 \)

Fig. 6.3. Variation of \( \beta_1 \) on velocity field.
Fig. 6.4. Variation of $\beta_2$ on velocity field.

Fig. 6.5. Variation of $\beta_3$ on velocity field.
Fig. 6.6. Variation of $M$ on velocity field.

Fig. 6.7. Variation of $G$ on velocity field.
Fig. 6.8. Variation of $M$ on temperature field.

$\beta_1 = 0.2$, $\beta_2 = \hat{S} = 0.3$, $\beta_3 = G = R = 0.5$, $Pr = 1.0$, $\theta_f = 1.02$

$M = 0.0$, $M = 0.4$, $M = 0.65$, $M = 0.9$

Fig. 6.9. Variation of $G$ on temperature field.

$\beta_1 = 0.2$, $\beta_2 = \hat{S} = 0.3$, $\beta_3 = 0.3$, $R = 0.5$, $M = 0.1$, $Pr = 1.0$, $\theta_f = 1.02$

$G = 0.0$, $G = 0.3$, $G = 0.6$, $G = 0.9$
Fig. 6.10. Variation of $R$ on temperature field.

$\beta_1 = 0.2, \beta_2 = 0.3, \beta_3 = 0.2, G = 0.2,$
$M = 0.1, S = 0.3, Pr = 1.0, \theta_f = 1.02$
$R = 0.0, R = 0.3, R = 0.6, R = 0.9$

Fig. 6.11. Variation of $\theta_f$ on temperature field.

$\beta_1 = 0.2, \beta_2 = 0.3, \beta_3 = 0.2, G = 0.2,$
$M = 0.1, S = 0.3, Pr = 1.0, R = 0.2$
$\theta_f = 1.01, \theta_f = 1.2, \theta_f = 1.35, \theta_f = 1.5$
Fig. 6.12. Variation of $\hat{S}$ on temperature field.

Fig. 6.13. Variation of Pr on temperature field.
Table 6.2: Values of local Nusselt number \( Nu/Re_x^{1/2} \) for different parameters.

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<th>( \beta_1 )</th>
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6.7 Main points

The effects of MHD, nonlinear radiation and viscous dissipation in mixed convection flow in a thermally stratified medium are studied. The following result are worth mentioning.

1. Velocity field decays slowly for larger values of martial parameters \( \beta_1 \) and \( \beta_2 \).
2. Deborah number corresponding to retardation time $\beta_3$ increases both the velocity and momentum boundary layer thickness.

3. The velocity field via mixed convection parameter and temperature via magnetic parameter behave in a similar fashion.

4. Temperature and thermal boundary layer thickness are enhanced for larger values of radiation parameter $R$ and temperature ratio parameter $\theta_f$.

5. Thermal stratification parameter $\hat{S}$ causes reduction in the temperature as well as thermal boundary layer thickness.

6. Local Nusselt number decreases when $\hat{S}$ and $\theta_f$ are increased.
Chapter 7

Flow of Burgers fluid through an inclined stretching sheet with heat and mass transfer

7.1 Introduction

Effects of heat and mass transfer in the flow of Burgers fluid by an inclined sheet are discussed in this chapter. Problems formulation and relevant analysis are given in the presence of thermal radiation and non-uniform heat source/sink. Thermal conductivity is taken temperature dependent. The nonlinear partial differential equations are simplified using boundary layer approximations. The resultant nonlinear ordinary differential equations are solved for the series solutions. The convergence of series solutions is obtained by plotting the $h$-curves for the velocity, temperature and concentration fields. Effects of various physical parameters on the velocity, temperature and concentration fields are studied. Numerical values of local Nusselt number and local Sherwood number are computed and analyzed.

7.2 Development of the problems

Here we have considered the effects of heat and mass transfer in the flow of Burgers fluid over an inclined stretching sheet. Thermal radiation, variable thermal conductivity and non-uniform
heat source/sink effects are taken into account. The stretching sheet makes an angle $\alpha$ with the vertical axis i.e. $y-axis$. The $x-axis$ is taken normal to the $y-axis$. Under the boundary layer approximation the velocity, temperature and the concentration fields are governed by the following equations

$$ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, $$

$$ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \lambda_1 \left[ u^2 \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial x \partial y} + 2uv \frac{\partial^2 u}{\partial x \partial y} \right] + \lambda_2 \left[ u \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial^2 u}{\partial y^2} \right] + 3u^2 \left( \frac{\partial u \partial^2 v}{\partial y^2} + 2 \frac{\partial v \partial^2 u}{\partial y \partial x} \right) + 3v^2 \left( \frac{\partial v \partial^2 u}{\partial y^2} + \frac{\partial u \partial^2 v}{\partial y \partial x} \right) + 2uv \left( \frac{\partial v \partial^2 u}{\partial y \partial x} + \frac{\partial v \partial^2 u}{\partial y \partial x} \right) = \nu \frac{\partial^2 u}{\partial y^2} + \nu \lambda_3 \left[ u \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^3} + \nu \frac{\partial^2 u}{\partial x \partial y^2} - \frac{\partial u}{\partial y} \frac{\partial^2 v}{\partial y^2} + \frac{\partial u}{\partial y} \frac{\partial^2 v}{\partial y^2} \right] + g_0 \beta_T (T - T_\infty) \cos \alpha, $$

$$ \rho c_p \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) - \frac{\partial q_r}{\partial y} + q'', $$

$$ u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \frac{\partial^2 C}{\partial y^2}, $$

$$ u = u_w(x) = ax, \quad v = 0, \quad T = T_w, \quad C = C_w \quad \text{at} \quad y = 0, $$

$$ u \to 0, \quad T \to T_\infty, \quad C \to C_\infty \quad \text{as} \quad y \to \infty, $$

where $u$ and $v$ represent the velocity components in the $x$ and $y$ directions respectively, $T$ is the fluid temperature, $c_p$ is the specific heat, $T_\infty$ is the ambient fluid temperature, $C_\infty$ is the ambient fluid concentration, $g_0$ is the acceleration due to gravity, $\beta_T$ is the volumetric coefficient of thermal expansion, $\nu = (\mu/\rho)$ is the kinematic viscosity, $\rho$ is the density of fluid, $h$ is the convective heat transfer coefficient, $C$ is the concentration of fluid, $D$ is the effective diffusion coefficient and the variable thermal conductivity of the fluid $k$ is adopted in the form:

$$ k = k_\infty \left( 1 + \hat{\epsilon} \frac{T - T_\infty}{\Delta T} \right), $$

in which $\hat{\epsilon}$ is the small parameter, $k_\infty$ is the thermal conductivity of the fluid far away from the surface, $\Delta T = T_w - T_\infty$ and $q''$ is the non-uniform heat generated ($q'' > 0$) or absorbed.
(q'' < 0) per unit volume. The non-uniform heat source/sink q'' is modeled by the following expression

$$q'' = \frac{k u_s(x, t)}{x \nu} \left[ A(T_w - T_\infty) f' + (T - T_\infty) B \right], \quad (7.8)$$

in which A and B are the coefficients of space and temperature-dependent heat source/sink, respectively. It should be noted that for internal heat generation \(A > 0\) and \(B > 0\) and for internal heat absorption we have \(A < 0\) and \(B < 0\). We introduce the change of variables as follows:

$$\psi = x (a \nu)^{1/2} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}, \quad \eta = y \left( \frac{a \nu}{\nu} \right)^{1/2} \quad (7.9)$$

and the velocity components are

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$ 

Here \(\psi\) is the stream function. Now Eq. (7.1) is identically satisfied and Eqs. (7.2 - 7.6) yield

$$f''' + f f'' - f'^2 + \beta_1 (2 f f' f'' - f^2 f'''') + \beta_2 (f^3 f'''' - 2 f f'' f'' - 3 f^2 f'''') + \beta_3 (f'''' - f f''') + G \theta \cos \alpha = 0, \quad (7.10)$$

$$1 + \frac{4}{3} R + \xi \theta \theta'' + Pr f f' + \xi \theta \theta' + (1 + \xi \theta) \left( A f' + B \theta \right) = 0, \quad (7.11)$$

$$\phi'' + Sc (f \phi' - f' \phi) = 0, \quad (7.12)$$

$$f(0) = 0, \quad f'(0) = 1, \quad \theta = 1 \quad \phi = 1 \quad \text{at} \ \eta = 0, \quad (7.13)$$

$$f' \to 0, \quad \theta \to 0 \quad \phi \to 0 \quad \text{as} \ \eta \to \infty, \quad (7.14)$$

where \(\beta_1 = \lambda_1 \alpha\) and \(\beta_2 = \lambda_2 \alpha\) are the Deborah numbers in terms of relaxation times respectively, \(\beta_3 = \lambda_3 \alpha\) is the Deborah number in terms of retardation times, \(Pr = \frac{\nu \rho c_w}{k_\infty}\) is the Prandtl number, \(G = \frac{\alpha \beta_3 (T_w - T_\infty) x^3 / \nu}{a \nu x^2 / \nu}\) is the mixed convection parameter, \(R = \frac{4 \alpha_x T_\infty}{k_\infty k^2}\) is the radiation parameter and \(Sc = \frac{\nu}{\nu}\) is the Schmidt number.
The local Nusselt number $N_{u_x}$ and local Sherwood number $Sh$ are

$$N_{u_x} = \frac{xq_w}{k(T_w - T_\infty)}, \quad Sh = \frac{xM_w}{D(C_w - C_\infty)},$$

(7.15)

with

$$q_w = -k \frac{\partial T}{\partial y} \bigg|_{y=0} + (q_r)_w, \quad M_w = -D \frac{\partial C}{\partial y} \bigg|_{y=0}.$$ 

Dimensionless form of local Nusselt number $Nu_{x}^{1/2}$ and local Sherwood number $Sh_{x}^{1/2}$ are

$$Nu/Re_{x}^{1/2} = -\left(1 + \frac{4}{3}R\right) \theta'(0), \quad Sh/Re_{x}^{1/2} = -\phi'(0),$$

in which the local Reynolds number $(Re_x)^{-1/2} = \sqrt{\frac{\nu}{xw}}$.

### 7.3 Homotopy analysis solutions

Initial approximations and auxiliary linear operators are chosen as follows:

$$f_0(\eta) = (1 - e^{-\eta}), \quad \theta_0(\eta) = e^{-\eta}, \quad \phi_0 = e^{-\eta},$$

(7.16)

$$\mathcal{L}_f = f''' - f', \quad \mathcal{L}_\theta = \theta'' - \theta, \quad \mathcal{L}_\phi = \phi'' - \phi,$$

(7.17)

with properties

$$\mathcal{L}_f(C_1 + C_2 e^\eta + C_3 e^{-\eta}) = 0,$$

(7.18)

$$\mathcal{L}_\theta(C_4 e^\eta + C_5 e^{-\eta}) = 0, \quad \mathcal{L}_\phi(C_6 e^\eta + C_7 e^{-\eta}) = 0,$$

(7.19)

where $C_i$ ($i = 1 - 7$) are the arbitrary constants. If $p \in [0, 1]$ denotes an embedding parameter and $h_{f}, h_{\theta}$ and $h_{\phi}$ the non-zero auxiliary parameters then the zeroth order deformation problems are

$$(1 - p) \mathcal{L}_f \left[ \hat{f}(\eta; p) - f_0(\eta) \right] = ph_f N_f \left[ \hat{f}(\eta; p) \right],$$

(7.20)

$$(1 - p) \mathcal{L}_\theta \left[ \hat{\theta}(\eta; p) - \theta_0(\eta) \right] = ph_\theta N_\theta \left[ \hat{\theta}(\eta; p), \hat{\varphi}(\eta; p) \right],$$

(7.21)
\[
(1 - p) \mathcal{L}_\phi \left[ \hat{\phi}(\eta; p) - \phi_0(\eta) \right] = p h_\phi \mathcal{N}_\phi \left[ \hat{f}(\eta; p), \hat{\phi}(\eta; p) \right],
\]
(7.22)

\[
\hat{f}(0; p) = 0, \quad \hat{f}'(0; p) = 1, \quad \hat{f}'(\infty; p) = 0, \quad \hat{\theta}(0; p) = 1, \quad \hat{\theta}(\infty; p) = 0,
\]
(7.23)

where the nonlinear operators \( \mathcal{N}_f \), \( \mathcal{N}_\theta \) and \( \mathcal{N}_\phi \) are

\[
\mathcal{N}_f[\hat{f}(\eta, p)] = \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} + \hat{f}(\eta, p) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right)^2 + \beta_1 \left[ 2 \hat{f}(\eta, p) \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - \left( \hat{f}(\eta, p) \right)^2 \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} \right]
\]
\[
+ \beta_2 \left[ \left( \hat{f}(\eta, p) \right)^3 \frac{\partial^4 \hat{f}(\eta, p)}{\partial \eta^4} - 2 \hat{f}(\eta, p) \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right)^2 \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right] - 3 \left( \hat{f}(\eta, p) \right)^2 \left( \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right)^2 + \beta_3 \left[ \left( \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right)^2 - \hat{f}(\eta, p) \frac{\partial^4 \hat{f}(\eta, p)}{\partial \eta^4} \right] + G \hat{\theta}(\eta, p) \cos \alpha,
\]
(7.24)

\[
\mathcal{N}_\theta[\hat{\theta}(\eta, p), \hat{f}(\eta, p)] = \left( 1 + \frac{4}{3} R + \xi \theta \right) \frac{\partial^2 \hat{\theta}(\eta, p)}{\partial \eta^2} + Pr \left( \hat{f}(\eta, p) \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} \right) + \epsilon \left( \hat{\theta}(\eta, p) \right)^2 + (1 + \epsilon \theta) (Af' + B\theta),
\]
(7.25)

\[
\mathcal{N}_\phi[\hat{\phi}(\eta, p), \hat{f}(\eta, p)] = \frac{\partial^2 \hat{\phi}(\eta, p)}{\partial \eta^2} + Sc \left( \hat{f}(\eta, p) \frac{\partial \hat{\phi}(\eta, p)}{\partial \eta} - \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \hat{\phi}(\eta, p) \right).
\]
(7.26)

For \( p = 0 \) and \( p = 1 \) one has

\[
\hat{f}(\eta; 0) = f_0(\eta), \quad \hat{\theta}(\eta; 0) = \theta_0(\eta), \quad \hat{\phi}(\eta; 0) = \phi_0(\eta),
\]
\[
\hat{f}(\eta; 1) = f(\eta), \quad \hat{\theta}(\eta; 1) = \theta(\eta), \quad \hat{\phi}(\eta; 1) = \phi(\eta).
\]
(7.27)
When $p$ variation is taken from 0 to 1 then $f(\eta, p)$, $\theta(\eta, p)$ and $\phi(\eta, p)$ approach $f_0(\eta)$, $\theta_0(\eta)$ and $\phi_0(\eta)$ to $f(\eta)$, $\theta(\eta)$ and $\phi(\eta)$. Now, $f$, $\theta$ and $\phi$ in Taylor’s series can be expanded in the forms

\[ f(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \quad (7.30) \]

\[ \theta(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m, \quad (7.31) \]

\[ \phi(\eta, p) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta)p^m, \quad (7.32) \]

where the convergence depends upon $h_f$, $h_\theta$ and $h_\phi$. By proper choices of $h_f$, $h_\theta$ and $h_\phi$ the series (7.30 – 7.32) converge for $p = 1$ and hence

\[ f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \quad (7.34) \]

\[ \theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \quad (7.35) \]

\[ \phi(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta). \quad (7.36) \]

The $m^{th}$-order deformation problems can be provided in the forms

\[ L_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f R_f^m(\eta), \quad (7.37) \]

\[ L_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_\theta R_\theta^m(\eta), \quad (7.38) \]

\[ L_\phi[\phi_m(\eta) - \chi_m \phi_{m-1}(\eta)] = h_\phi R_\phi^m(\eta), \quad (7.39) \]

\[ f_m(0) = f'_m(0) = f''_m(\infty) = 0, \quad \theta_m(0) = \theta_m(\infty) = 0, \quad \phi_m(0) = \phi_m(\infty) = 0, \quad (7.40) \]
in which

\[ R^m_f(\eta) = f''_{m-1}(\eta) + \sum_{k=0}^{m-1} \left\{ f'_{m-1-k} f''_k - f'_{m-1-k} f'_k \right\} + \beta_1 \sum_{k=0}^{m-1} \left\{ 2f''_{m-1-k} f''_{l} - f''_{m-1-k} \sum_{l=0}^{k} f_{k-l} f''_l \right\} \]

\[ + \beta_2 \sum_{k=0}^{m-1} \left\{ f'_{m-1-k} \sum_{l=0}^{k} f''_{l} - 2f''_{m-1-k} \sum_{l=0}^{k} f_{k-l} f''_l - 3f''_{m-1-k} \sum_{l=0}^{k} f_{k-l} f''_l \right\} \]

\[ + \beta_3 \left\{ f''_{m-1-k} \sum_{k=0}^{m-1} f''_k - f''_{m-1-k} \sum_{k=0}^{m-1} f''_k \right\} + G\theta_{m-1} \cos \alpha, \quad (7.41) \]

\[ R^m_\theta(\eta) = (1 + \frac{4}{3} R+) \theta''_{m-1} + \ddot{\epsilon} \theta_{m-1-k} \sum_{k=0}^{m-1} \theta'_k + Pr \left\{ \sum_{k=0}^{m-1} \theta'_k \right\} \]

\[ + \ddot{\epsilon} \theta_{m-1-k} \sum_{k=0}^{m-1} \theta_k + (A f'_m - B \theta_{m-1}) + \ddot{\epsilon} A \theta_{m-1-k} \sum_{k=0}^{m-1} f'_k \]

\[ + \ddot{\epsilon} B \theta_{m-1-k} \sum_{k=0}^{m-1} \theta_k, \quad (7.42) \]

\[ R^m_\phi(\eta) = \phi''_{m-1} + S c \sum_{k=0}^{m-1} (\phi'_{m-1-k} f_k - f'_{m-1-k} \phi_k), \quad (7.43) \]

\[ \chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \]

The general solutions of Eqs. (7.37 – 7.39) can be written as follows:

\[ f_m(\eta) = f^*_m(\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta}, \quad (7.44) \]

\[ \theta_m(\eta) = \theta^*_m(\eta) + C_4 e^\eta + C_5 e^{-\eta}, \quad \phi_m(\eta) = \phi^*_m(\eta) + C_6 e^\eta + C_7 e^{-\eta}, \quad (7.45) \]

where \( f^*_m, \theta^*_m \) and \( \phi^*_m \) are the particular solutions.
7.4 Convergence of the solutions

It is fairly understandable that the resultant series solutions contain the auxiliary parameters $h_f$, $h_\theta$ and $h_\phi$. Such parameters have key role for the convergence of series solutions. For permissible values of auxiliary parameters, we plot the $h$–curves at 14$^{th}$ order of approximations (see Figs. 7.1 – 7.3). It is clearly seen from these Figs. that the meaningful values of $h_f$, $h_\theta$ and $h_\phi$ are $[-1.3, -0.6]$, $[-1.25, -0.3]$ and $[-1.3, -0.6]$ respectively. Moreover Table 7.1 depicts that the series solutions are convergent up to six decimal places.

Fig. 7.1. $h_f$– curve for the velocity field.
Fig. 7.2. $h_\theta$— curve for the temperature field.

$\beta_1 = \beta_2 = \beta_3 = R = 0.2$, $\alpha = \pi/4$, $G = 0.2$, $Pr = 1.0$, $\varepsilon = 0.3$, $A = B = 0.1$

Fig. 7.3. $h_\phi$— curve for the concentration field.

$\beta_1 = \beta_2 = \beta_3 = 0.2$, $\alpha = \pi/4$, $G = 0.2$, $Sc = 1.1$
Table 7.1: Convergence of HAM solution for different order of approximations.

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7.5 Discussion

Considered problem contains different physical parameters like $\beta_1$, $\beta_2$, $\beta_3$, $\dot{\epsilon}$, $G$, $\alpha$, Pr, $R$, $A$, $B$ and $Sc$. Graphical results of such parameters are displayed and discussed in the following subsections.

7.5.1 Dimensionless velocity field

Fig. 7.4 presents the influence of $\alpha$ on the velocity field. It is seen that larger values of $\alpha$ cause a reduction in the velocity profile. In fact an increase in $\alpha$ corresponds to more gravity effect. Figs. (7.5 – 7.7) are plotted for various values of Deborah numbers ($\beta_1$, $\beta_2$ and $\beta_3$). An increase in $\beta_1$ and $\beta_2$ causes a reduction in velocity profile and associated boundary layer thickness. Effect of $\beta_3$ on the fluid motion is quite opposite to that of $\beta_1$ and $\beta_2$. It is also clear from Fig. 7.7 that the velocity field is more pronounced near the wall but after some large distance it vanishes. Fig. 7.8 illustrates that an increase in $G$ enhances in the velocity and boundary layer thickness.
7.5.2 Dimensionless temperature field

Fig. 7.9 depicts the variation of $\dot{\varepsilon}$ on the temperature profile. In fact the increase in variable thermal conductivity parameter $\dot{\varepsilon}$ result in the increase of the temperature and thermal boundary layer thickness. Figs. 7.10 and 7.11 illustrate the effects of $\beta_1$ and $\beta_2$ on temperature profile. Deborah numbers related to relaxation times ($\beta_1$ and $\beta_2$) increase the temperature profile and thermal boundary layer thickness. Influence of $\beta_3$ (Deborah number related to retardation times) and $G$ (mixed convection parameter) are qualitatively similar (see Figs. 7.12 and 7.13). Both parameters decrease the temperature and thermal boundary layer thickness. Fig.7.14 illustrates the effect of thermal radiation $R$ on the temperature profile. Temperature profile decreases rapidly when $R$ increases. Physically the larger values of $R$ imply larger surface heat flux and thereby making the fluid warmer which enhances the temperature profile. The effect of Prandtl number $Pr$ on the temperature distribution is displayed in Fig. 7.15. It can be seen that the temperature decreases with an increase of $Pr$. It implies that the momentum boundary layer is thicker than the thermal boundary layer. This is due to the fact that for higher Prandtl number, the fluid has a relatively low thermal conductivity which reduces conduction. For large values of heat generation/absorption parameters $A$ and $B$, the temperature distribution across the boundary layer increases (see Figs. 7.16 and 7.17).

7.5.3 Dimensionless concentration field

Figs. (7.18 – 7.20) show the variation of Deborah numbers ($\beta_1$, $\beta_2$, and $\beta_3$) on the concentration field. Here concentration field shows gradual increase when $\beta_1$ and $\beta_2$ are increased. However the concentration field is decreasing function of $\beta_3$ (see Fig. 7.20). Fig. 7.21 illustrates the effect of mixed convection parameter $G$ on the concentration field. For larger values of $G$ the concentration field decreases. Fig. 7.22 depicts the influence of $\alpha$ on the concentration field. It is seen that larger values of $\alpha$ decrease the concentration field. Fig. 7.23 shows the variation of $Sc$ on $\phi(\eta)$. Concentration field decreases rapidly when $Sc$ increases.
Fig. 7.4. Influence of $\alpha$ on velocity field.

Fig. 7.5. Influence of $\beta_1$ on velocity field.
Fig. 7.6. Influence of $\beta_2$ on velocity field.

Fig. 7.7. Influence of $\beta_3$ on velocity field.
Fig. 7.8. Influence of $G$ on velocity field.

Fig. 7.9. Influence of $\varepsilon$ on velocity field.
Fig. 7.10. Influence of $\beta_1$ on temperature field.

$$\beta_2 = R = 0.2, \ G = 0.3, \ \alpha = \pi/4, \ \bar{\varepsilon} = 0.2, \ \beta_3 = 0.3, \ Pr = 1.0, \ A = B = 0.1,$$

$$\beta_1 = 0.0, \ \beta_1 = 0.4, \ \beta_1 = 0.8, \ \beta_1 = 1.$$ 

Fig. 7.11. Influence of $\beta_2$ on temperature field.

$$\beta_1 = R = 0.2, \ G = 0.3, \ \alpha = \pi/4, \ \bar{\varepsilon} = 0.2, \ \beta_3 = 0.3, \ Pr = 1.0, \ A = B = 0.1,$$

$$\beta_2 = 0.0, \ \beta_2 = 0.3, \ \beta_2 = 0.6, \ \beta_2 = 0.5.$$
Fig. 7.12. Influence of $\beta_3$ on temperature field.

Fig. 7.13. Influence of $G$ on temperature field.
Fig. 7.14. Influence of $R$ on temperature field.

Fig. 7.15. Influence of $Pr$ on temperature field.
Fig. 7.16. Influence of \( A \) on temperature field.

\[
\begin{align*}
\beta_1 = \beta_2 &= R = 0.2, \ G = 0.3, \ \alpha = \pi/4, \ \bar{\varepsilon} = 0.1, \\
\beta_3 &= 0.3, \ \text{Pr} = 1.0, \ B = 0.1, \\
A &= 0.0, \ A = 0.2, \ A = 0.3, \ A = 0.3
\end{align*}
\]

Fig. 7.17. Influence of \( B \) on temperature field.

\[
\begin{align*}
\beta_1 = \beta_2 &= R = 0.2, \ G = 0.3, \ \alpha = \pi/4, \ \bar{\varepsilon} = 0.2, \\
\beta_3 &= 0.3, \ \text{Pr} = 1.0, \ A = 0.1, \\
B &= 0.0, \ B = 0.2, \ B = 0.3, \ B = 0.3
\end{align*}
\]
Fig. 7.18. Influence of $\beta_1$ on concentration field.

Fig. 7.19. Influence of $\beta_2$ on concentration field.
Fig. 7.20. Influence of $\beta_3$ on concentration field.

Fig. 7.21. Influence of $G$ on concentration field.
Fig. 7.22. Influence of $\alpha$ on concentration field.

Fig. 7.23. Influence of $Sc$ on concentration field.
Table 7.2: Values of local Nusselt number $Nu/Re_x^{1/2}$ for different parameters.

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Table 7.3: Values of local Sherwood number $Sh$ for different parameters.

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7.6 Concluding remarks

Here series solutions are developed to solve the nonlinear problem of Burgers fluid flow by an inclined stretching sheet with heat and mass transfer. Results presented describe the role of different physical parameters involved in the problem. The Deborah numbers corresponding to relaxation times ($\beta_1$ and $\beta_2$) and angle of inclination ($\alpha$) decrease the fluid velocity and concentration field. Deborah number corresponding to retardation times ($\beta_3$) has opposite behavior on the velocity and temperature fields. Concentration field decays as ($\beta_3$) and mixed convection parameter ($\Gamma$) increase. Local Nusselt number decreases when $\dot{e}$, $\beta_1$, $\beta_2$ and $R$ are larger. It is also found that $\theta'(0)$ increases with an increase in ($\beta_3$).
Chapter 8

Stretched flow of Casson fluid with nonlinear thermal radiation and viscous dissipation

8.1 Introduction

The purpose of this chapter is to examine the steady flow of Casson fluid over an inclined exponential stretching surface. The heat transfer analysis is considered in the presence of nonlinear thermal radiation and viscous dissipation. The governing partial differential equations are reduced to ordinary differential equations by employing adequate transformations. Series solutions of the resulting problems are computed. The effects of physical parameter $\zeta$, Prandtl number $Pr$, temperature ratio parameter $\theta_f$, Eckert number $E_c$ and radiation parameter $R$ on the velocity, temperature and Nusselt number are investigated. Comparison of the present analysis is made with the case of viscous fluid.

8.2 Flow equations

We consider the flow of an incompressible non-Newtonian fluid by an exponentially stretching sheet which makes an angle $\alpha$ with $y$–axis. The flow is confined to $y > 0$. Also the $y$–axis is perpendicular to $x$–axis. The heat transfer is considered in the presence of viscous dissipation.
and nonlinear thermal radiation. The extra stress tensor in Casson model is given by [34]:

\[
\tau_{ij} = \begin{cases} 
2 \left( \mu_B + \tau_r / \sqrt{2\pi} \right) e_{ij}, & \pi > \pi_c \\
2 \left( \mu_B + \tau_r / \sqrt{2\pi} \right) e_{ij}, & \pi < \pi_c 
\end{cases} \tag{8.1}
\]

where \(\mu_B\) is the plastic dynamic viscosity, \(\tau_r\) is the yield stress of fluid, \(\pi\) is the product of the component of deformation rate with itself, \(\pi_c\) is a critical value of this product based on the non-Newtonian model, \(\pi = e_{ij}e_{ij}\) and \(e_{ij}\) is the \((i,j)th\) component of the deformation rate.

The governing boundary layer equations are

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{8.2}
\]

\[
u \frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial y} = \nu \left( 1 + \frac{1}{\zeta} \right) \frac{\partial^2 u}{\partial y^2} + g_0 \beta_T (T - T_\infty) \cos \alpha, \tag{8.3}
\]

\[
\rho c_p \left( \frac{\partial T}{\partial x} + \nu \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left[ \left( k + \frac{16\sigma^* T^3}{3k^*} \right) \frac{\partial T}{\partial y} \right] - \nu \frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p} \left( 1 + \frac{1}{\zeta} \right) \left( \frac{\partial \nu}{\partial y} \right)^2, \tag{8.4}
\]

where \(u\) and \(v\) represent the velocity components along the flow \((x-)\) direction and normal to the flow \((y-)\) direction, \(\nu = (\mu/\rho)\) is the kinematic viscosity, \(\rho\) is the fluid density, \(g_0\) is the acceleration due to gravity, \(\beta_T\) is the volumetric coefficient of thermal expansion, \(\zeta = \frac{\mu_B \tau_r}{\tau_r} \) is the Casson fluid parameter, \(T\) is the fluid temperature, \(k\) is the thermal conductivity of fluid, \(k^*\) is the mean absorption coefficient, \(\sigma^*\) is the Stefan-Boltzmann constant and \(c_p\) is the specific heat. The boundary conditions for the velocity and temperature fields are

\[
u = U_0 e^{x/L}, \quad v = 0, \quad T = T_\infty + T_w e^{x/2L} \quad \text{at} \quad y = 0, \tag{8.5}
\]

\[
u \rightarrow 0, \quad T \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty,
\]

where \(T_w\) and \(T_\infty\) are the temperatures at the plate and far away from the plate while \(L\) is a constant and \(U_0\) is the reference velocity. Defining variables
\[ u = U_0 e^{x/L} f'(\eta), \quad v = -\sqrt{\frac{\nu U_0 e^{x/2L}}{2L}} [f(\eta) + \eta f'(\eta)], \]
\[ \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \eta = \sqrt{\frac{U_0}{2\nu L}} e^{x/2L} y, \quad (8.6) \]

where prime denotes the differentiation with respect to \( \eta \), \( f \) is the dimensionless stream function, \( \theta \) is the dimensionless temperature with \( T = T_\infty (1 + (\theta_f - 1) \theta) \) and \( \theta_f = \frac{T_w}{T_\infty} \), Eq. (8.2) is identically satisfied and Eqs. (8.3 - 8.5) yield

\[ \left(1 + \frac{1}{\zeta}\right) f''' - 2f'' + f f'' + G \cos \alpha = 0, \quad (8.7) \]
\[ \left\{ 1 + R (1 + (\theta_f - 1) \theta)^3 \right\} \theta' + \text{Pr} \left[ f \theta' - f' \theta + \left(1 + \frac{1}{\zeta}\right) E_c f'' \right] = 0, \quad (8.8) \]

\[ f = 0, \quad f' = 1, \quad \theta = 1 \quad \text{at} \quad \eta = 0, \]
\[ f' \to 0, \quad \theta \to 0 \quad \text{as} \quad \eta \to \infty, \quad (8.9) \]

in which \( R = \frac{16 \sigma^* T_\infty^3}{4\kappa k^2} \) is the radiation number, \( E_c = \frac{U_0^2}{T_w^2} \) is the Eckert number, \( G = \frac{q_0 (T_w - T_\infty) x^3/\nu}{\text{Pr} \nu^2 x^2 / \nu} \) is the mixed convection parameter and \( \text{Pr} = \frac{\mu c_p}{k} \) is the Prandtl number.

The local Nusselt number \( Nu_x \) is defined below:

\[ Nu_x = \frac{x q_w}{k(T_w - T_\infty)} \quad \text{with} \quad q_w = -k \left( \frac{\partial T}{\partial y} \right)_{y=0} + (q_r)_w, \quad \text{and so} \quad Nu_x = -\left(1 + R \theta_f^3\right) \theta'(0), \quad (8.10) \]

where \( q_w \) is heat transfer from the plate.

### 8.3 Construction of solutions

Initial approximations and auxiliary linear operators are defined below

\[ f_0(\eta) = 1 - e^{-\eta}, \quad \theta_0(\eta) = e^{-\eta}. \quad (8.11) \]
\[ L_f = f''' - f', \quad L_\theta = \theta'' - \theta, \quad (8.12) \]
with
\[ L_f(C_1 + C_2e^y + C_3e^{-y}) = 0, \quad L_\theta(C_4e^y + C_5e^{-y}) = 0, \] (8.13)
where \( C_i \) (\( i = 1 - 5 \)) are the integral constants to be determined by the boundary conditions.

The zeroth order deformation problems are
\[ (1 - \pi) L \phi = \eta \phi_0 (\pi); \quad \pi = 0; \quad \phi_0 (\pi) \] (8.14)
\[ (1 - \pi) \mu \phi = \eta _1 + \phi_0 (\pi); \quad \pi = 0; \quad \phi_0 (\pi) \] (8.15)
\[ \hat{\phi} (0; \pi) = 0, \quad \hat{\phi}' (0; \pi) = 1, \quad \hat{\phi}' (\infty; \pi) = 0, \quad \hat{\phi} (\pi; 0) = 1, \quad \hat{\phi} (\pi; \infty) = 0. \] (8.16)

If \( \pi \in [0, 1] \) denotes an embedding parameter, \( h_f \) and \( h_\theta \) the non-zero auxiliary parameters, then the nonlinear differential operators \( N_f \) and \( N_\theta \) are
\[ N_f[\hat{\phi}(\pi, \eta)] = \left(1 + \frac{1}{\zeta} \right) \frac{\partial^2 \hat{\phi}(\pi, \eta)}{\partial \eta^2} + \hat{\phi}(\pi, \eta) \frac{\partial^2 \hat{\phi}(\pi, \eta)}{\partial \eta^2} - 2 \left( \frac{\partial \hat{\phi}(\pi, \eta)}{\partial \eta} \right)^2 + C \hat{\phi}(\pi, \eta) \cos \alpha, \] (8.17)
\[ N_\theta[\hat{\theta}(\eta, \pi), \hat{\phi}(\eta, \pi)] = \frac{\partial}{\partial \eta} \left[ \left(1 + R \left(1 + (\theta_f - 1)\hat{\theta}(\eta, \pi) \right) \right) \frac{\partial \hat{\theta}(\eta, \pi)}{\partial \eta} \right] \]
\[ + \Pr \left( \hat{\phi}(\pi, \eta) \frac{\partial \hat{\phi}(\eta, \pi)}{\partial \eta} - \hat{\phi}(\eta, \pi) \frac{\partial \hat{\phi}(\eta, \pi)}{\partial \eta} \right) \]
\[ + \Pr E_c \left(1 + \frac{1}{\zeta} \right) \left( \frac{\partial^2 \hat{\phi}(\pi, \eta)}{\partial \eta^2} \right)^2. \] (8.18)

Obviously when \( p = 0 \) and \( p = 1 \) then
\[ \hat{\phi}(\pi, 0) = \phi_0 (\pi), \quad \hat{\theta}(\eta, 0) = \theta_0 (\eta), \] (8.19)
\[ \hat{\phi}(\eta, 1) = \phi (\eta), \quad \hat{\theta}(\eta, 1) = \theta (\eta), \]
and when \( p \) varies from 0 to 1 then \( f(\eta, \pi) \) and \( \theta(\eta, \pi) \) vary from \( f_0 (\eta), \theta_0 (\eta) \) to \( f(\eta) \) and \( \theta(\eta) \). According to Taylor series
\[ f(\eta, \pi) = f_0 (\eta) + \sum_{m=1}^{\infty} f_m (\eta) \pi^m, \] (8.20)
\[ \theta(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m, \quad (8.21) \]

\[ f_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m f(\eta; p)}{\partial \eta^m} \right|_{p=0}, \quad \theta_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \theta(\eta; p)}{\partial \eta^m} \right|_{p=0}, \quad (8.22) \]

and convergence of above series strongly depends upon \( h_f \) and \( h_\theta \). By proper choices of \( h_f \) and \( h_\theta \) the series (8.20) and (8.21) converge for \( p = 1 \) and so

\[ f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \quad (8.23) \]

\[ \theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta). \quad (8.24) \]

The resulting problems at \( m^{th} \) order are

\[ \mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f \mathcal{R}_f^m(\eta), \quad (8.25) \]

\[ \mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_\theta \mathcal{R}_\theta^m(\eta), \quad (8.26) \]

\[ f_m(0) = f'_m(0) = f'_m(\infty) = \theta_m(0) = \theta_m(\infty) = 0, \quad (8.27) \]

\[ \mathcal{R}_f^m(\eta) = \left( 1 + \frac{1}{\zeta} \right) f''_{m-1}(\eta) + \sum_{k=0}^{m-1} \left[ f_{m-1-k} f''_k - f''_{m-1-k} f'_k \right] + G \cos \alpha \theta_{m-1}, \quad (8.28) \]

\[ \mathcal{R}_\theta^m(\eta) = \left( 1 + R \right) \theta''_{m-1}(\eta) + \Pr \sum_{k=0}^{m-1} \left\{ \theta'_{m-1-k} f_k - f'_{m-1-k} \theta_k \right\} \]

\[ + \Pr Ec \left( 1 + \frac{1}{\zeta} \right) f''_{m-1-k} f'_k \]

\[ + (\theta_f - 1) R \sum_{k=0}^{m-1} \left[ (\theta_f - 1)^2 \theta_{m-1-k} \left\{ \sum_{l=0}^{k} \theta_{k-l} \sum_{s=0}^{l} \theta_{l-s} \theta''_s \right\} \right] \]

\[ + 3(\theta_f - 1) \sum_{l=0}^{k} \theta''_{k-l} \theta_l + 3\theta''_k \left[ \sum_{k=0}^{m-1} \theta'_{m-1-k} \theta_k \right] \]

\[ \left\{ 1 + \sum_{l=0}^{k} \theta_{k-l} \theta_l + 2(\theta_f - 1) \theta_k \right\}, \quad (8.29) \]

\[ \chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \]

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The general solutions are

\[ f_m(\eta) = f_m^*(\eta) + C_1 + C_2 e^{\eta} + C_3 e^{-\eta}, \quad (8.30) \]

\[ \theta_m(\eta) = \theta_m^*(\eta) + C_4 e^{\eta} + C_5 e^{-\eta}, \quad (8.31) \]

where \( f_m^* \) and \( \theta_m^* \) are the particular solutions and the constants \( C_i (i = 1 - 5) \) are given by

\[ C_2 = C_4 = 0, \quad C_3 = \left. \frac{\partial f_m^*(\eta)}{\partial \eta} \right|_{\eta=0}, \quad C_1 = -C_3 - f^*(0), \quad C_5 = f^*(0). \]

### 8.4 Convergence of the homotopy solutions

Our series solutions (8.25) and (8.26) contain the auxiliary parameters \( h_f \) and \( h_\theta \), which influence the convergence rate and regions of these two series. We choose proper values of \( h_f \) and \( h_\theta \) from \( h \)-curves at 14th order approximations. It is noticed that meaningful values of \( h_f \) and \( h_\theta \) are \(-0.2 \leq h_f \leq -0.9 \) and \(-0.1 \leq h_\theta \leq -0.7 \). We find that series given in the equations (8.25) and (8.26) converge in the whole region of \( \eta \) for different values of pertinent parameters when \( h_f = -0.6 \) and \( h_\theta = -0.5 \).

![Fig. 8.1. \( h \)- curve for the velocity field.](image-url)
Table 8.1: Convergence of homotopy solutions for different order of approximations when $E_c = 0.2$, $\alpha = \pi/4$, $G = 0.4$, $\theta_f = 1.02$, $R = 0.3$, $\Pr = 1.0$, $\zeta = 0.7$, $h_f = -0.6$ and $h_\theta = -0.5$.

<table>
<thead>
<tr>
<th>Order of approximation</th>
<th>$-f''(0)$</th>
<th>$-\theta'(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.953530</td>
<td>0.705974</td>
</tr>
<tr>
<td>5</td>
<td>0.948613</td>
<td>0.752558</td>
</tr>
<tr>
<td>10</td>
<td>0.948896</td>
<td>0.752877</td>
</tr>
<tr>
<td>15</td>
<td>0.948876</td>
<td>0.752846</td>
</tr>
<tr>
<td>20</td>
<td>0.948877</td>
<td>0.752850</td>
</tr>
<tr>
<td>25</td>
<td>0.948877</td>
<td>0.752849</td>
</tr>
<tr>
<td>30</td>
<td>0.948877</td>
<td>0.752849</td>
</tr>
<tr>
<td>35</td>
<td>0.948877</td>
<td>0.752849</td>
</tr>
<tr>
<td>40</td>
<td>0.948877</td>
<td>0.752849</td>
</tr>
<tr>
<td>50</td>
<td>0.948877</td>
<td>0.752849</td>
</tr>
</tbody>
</table>

Fig. 8.2. $h-$ curve for the temperature field.
8.5 Interpretation of results

This section discloses the variations of emerging parameters on the velocity and temperature fields. In particular, the variations of Casson fluid parameter $\zeta$, mixed convection parameter $G$, angle of inclination $\alpha$, radiation parameter $R$, temperature ratio parameter $\theta_f$, Prandtl number $Pr$ and Eckert number $E_c$ are studied. In order to analyze the salient features of the emerging parameters on the velocity and temperature profiles, we have plotted the Figs. (8.3 – 8.13). The effects of Casson fluid parameter have been depicted in the Figs. 8.3 and 8.4. These Figs. show that both velocity components $f$ and $f'$ decrease with an increase in $\zeta$. Fig. 8.5 presents the variation of angle of inclination $\alpha$ on the velocity profile. Angle of inclination decreases the velocity and corresponding boundary layer thickness. Fig. 8.6 demonstrates the effect of mixed convection parameter $G$ on $f'(\eta)$. Mixed convection parameter enhances the velocity profile. This is due to fact that the larger $G$ is associated with a stronger buoyancy force. A boundary layer is laminar for smaller $G$. Fig. 8.7 shows the influence of radiation parameter $R$ on the velocity profile. As larger values of $R$ correspond to more heat flux from the wall therefore velocity and momentum boundary layer thickness increase. Fig. 8.8 is prepared to examine the profiles of $\theta(\eta)$ for different values of $\zeta$. We find that $\theta$ increases with an increase in $\zeta$. Fig. 8.9 depicts the influence of $G$ on the temperature profile. Mixed convection parameter decreases the temperature and thermal boundary layer thickness. When $G >> 1$, the viscous force is negligible in comparison to the buoyancy and inertial forces. When buoyant forces overcome the viscous forces, the flow starts a transition to the turbulent regime. The effect of radiation parameter $R$ on $\theta(\eta)$ has been shown in Fig. 8.10. It clearly shows that by increasing $R$, $\theta(\eta)$ increases. Fig. 8.11 is prepared to perceive the effect of ratio parameter on temperature profile. Clearly the temperature increases for larger values of $\theta_f$ because wall temperature is larger than ambient fluid temperature. Fig. 8.12 shows the change in $\theta(\eta)$ with respect to the Eckert number $E_c$. It is noted that $\theta(\eta)$ increases for large $E_c$. Because larger values of Eckert number enhances the kinetic energy that heat up the fluid. To observe the variations of temperature profile for different values of $Pr$ we have prepared Fig. 8.13. As Prandtl number is used to adjust the rate of cooling in conducting flows therefore thermal boundary layer thickness decreases by increasing Prandtl number $Pr$.

Tables 8.2 and 8.3 show the effect of heat transfer characteristics at wall $-\theta'(0)$ for different
values of the parameters $\zeta$, $R$, $G$, $\theta_f$, $E_c$ and $Pr$. In Table 8.2 the values of heat transfer characteristics $-\theta'(0)$ at the wall are compared with the existing numerical solution in the special case ($\zeta \to \infty$) i.e for viscous fluid. We found an excellent agreement in the HAM and numerical solutions. Table 8.3 shows that the influences of $\zeta$, $G$, $\theta_f$, $R$ and $E_c$ on $-\theta'(0)$ are qualitatively similar. Increase in $\zeta$ decreases the heat transfer coefficient at the wall $-\theta'(0)$.

Fig. 8.3. Influence of $\zeta$ on $f(\eta)$. 
Fig. 8.4. Influence of $\zeta$ on $f'(\eta)$.

Fig. 8.5. Influence of $\alpha$ on $f''(\eta)$. 

$\alpha = \pi/4, G = 0.4, Pr = 1.0, R = 0.3,$
$Ec = 0.3, \theta_f = 1.02$

$\zeta = 0.7, \zeta = 1.0, \zeta = 1.5, \zeta = 2.0$

$\zeta = 2.0, G = 0.8, Pr = 1.0, R = 0.3,$
$Ec = 0.3, \theta_f = 1.02$

$\alpha = \pi/8, \alpha = \pi/4, \alpha = \pi/3, \alpha = \pi/2$
Fig. 8.6. Influence of $G$ on $f'(\eta)$.

Fig. 8.7. Influence of $R$ on $f'(\eta)$. 

$\alpha = \pi/4, \, \zeta = 2.0, \, \text{Pr} = 1.0, \, R = 0.3, \, \text{Ec} = 0.3, \, \theta_f = 1.02$

$G = 0.0, \, G = 0.4, \, G = 0.8, \, G = 1.2$

$\alpha = \pi/4, \, \zeta = 2.0, \, \text{Pr} = 1.0, \, G = 0.4, \, \text{Ec} = 0.3, \, \theta_f = 1.02$

$R = 0.0, \, R = 0.5, \, R = 1.0, \, R = 1.5$
Fig. 8.8. Influence of $\zeta$ on $\theta(\eta)$.

\[ \alpha = \pi/4, \ G = 0.4, \ Pr = 1.0, \ R = 0.3, \ Ec = 0.3, \ \theta_f = 1.02 \]
\[ \zeta = 0.5, \ \zeta = 1.0, \ \zeta = 2.0, \ \zeta = 3.0 \]

Fig. 8.9. Influence of $G$ on $\theta(\eta)$.

\[ \alpha = \pi/4, \ \zeta = 2.0, \ Pr = 1.0, \ R = 0.3, \ Ec = 0.3, \ \theta_f = 1.02 \]
\[ G = 0.0, \ G = 0.5, \ G = 1.0, \ G = 1.5 \]
Fig. 8.10. Influence of $R$ on $\theta(\eta)$.

Fig. 8.11. Influence of $\theta_f$ on $\theta(\eta)$. 

$\alpha = \pi/4, \ G = 0.4, \ \zeta = 2.0, \ Pr = 1.0, \ Ec = 0.3, \ \theta_f = 1.02$

$R = 0.1, \ R = 0.3, \ R = 0.5, \ R = 0.7$

$\alpha = \pi/4, \ G = 0.4, \ \zeta = 2.0, \ Pr = 1.0, \ Ec = 0.3, \ R = 0.3$

$\theta_f = 1.1, \ \theta_f = 1.2, \ \theta_f = 1.3, \ \theta_f = 1.4$
Fig. 8.12. Influence of $E_c$ on $\theta(\eta)$.

Fig. 8.13. Influence of Pr on $\theta(\eta)$.
Table 8.2: Comparison between numerical solutions [6] and HAM solutions in the special case ($\zeta \to \infty$).

<table>
<thead>
<tr>
<th>$R$</th>
<th>$E_c = 0$</th>
<th>$E_c = 0.2$</th>
<th>$E_c = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pr = 1</td>
<td>Pr = 2</td>
<td>Pr = 1</td>
</tr>
<tr>
<td>0.0</td>
<td>[6]</td>
<td>0.9548  1.4714</td>
<td>0.8622  1.3055</td>
</tr>
<tr>
<td>0.0</td>
<td>HAM</td>
<td>0.9548  1.4715</td>
<td>0.8623  1.03055</td>
</tr>
<tr>
<td>0.5</td>
<td>[6]</td>
<td>0.6765  1.0735</td>
<td>0.6177  0.9656</td>
</tr>
<tr>
<td>0.5</td>
<td>HAM</td>
<td>0.6765  1.0735</td>
<td>0.6173  0.9654</td>
</tr>
<tr>
<td>1.0</td>
<td>[6]</td>
<td>0.5315  0.8627</td>
<td>0.4877  0.7818</td>
</tr>
</tbody>
</table>
Table 8.3: Values of heat transfer characteristics at wall for the parameters $\zeta$, $R$, $E_c$ and $Pr$.

<table>
<thead>
<tr>
<th>$\zeta$</th>
<th>$R$</th>
<th>$E_c$</th>
<th>$Pr$</th>
<th>$\theta_f$</th>
<th>$-(1 + R\theta_f^2)\theta''(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>0.3</td>
<td>0.2</td>
<td>1.0</td>
<td>1.0059</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td></td>
<td></td>
<td></td>
<td>0.99776</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td></td>
<td></td>
<td></td>
<td>0.99438</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>0.0</td>
<td></td>
<td></td>
<td>0.86813</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td></td>
<td></td>
<td></td>
<td>0.99254</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td></td>
<td></td>
<td></td>
<td>1.0986</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td></td>
<td></td>
<td></td>
<td>1.1332</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td></td>
<td></td>
<td></td>
<td>0.99253</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td></td>
<td></td>
<td></td>
<td>1.1310</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>1.0</td>
<td></td>
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<td>1.5</td>
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<tr>
<td>1.8</td>
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<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>1.0</td>
<td>1.01</td>
<td></td>
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<td>0.99637</td>
<td></td>
</tr>
<tr>
<td>1.02</td>
<td></td>
<td></td>
<td></td>
<td>0.99254</td>
<td></td>
</tr>
<tr>
<td>1.04</td>
<td></td>
<td></td>
<td></td>
<td>0.98475</td>
<td></td>
</tr>
</tbody>
</table>

8.6 Concluding remarks

The flow of Casson fluid through nonlinear thermal radiation and viscous dissipation has been studied. The major conclusions have been summarized as follows.

- Velocity field is decreasing function of non-Newtonian fluid parameter $\zeta$.

- Angle of inclination $\alpha$ and mixed convection parameter $G$ have opposite behavior on the velocity field $f'(\eta)$.

- Temperature field $\theta(\eta)$ increases for larger values of $\zeta$.

- The effect of radiation parameter $R$ and Eckert number $E_c$ on temperature field $\theta(\eta)$ are qualitatively similar.
• Temperature profile decreases for larger values of $G$ whereas it increases with temperature ratio parameter $\theta_f$.

• Increase in Pr decreases the temperature profile $\theta(\eta)$.

• Influences of $\zeta$, $R$ and $E_c$ on $-\theta'(0)$ are similar in a qualitative sense.
Chapter 9

Thermal radiation and viscous dissipation effects in flow of Casson fluid due to a stretching cylinder

9.1 Introduction

The purpose of present chapter is to examine the boundary layer flow by a stretching cylinder. An incompressible Casson fluid is considered. The flow analysis is formulated through convective boundary condition. Effects of thermal radiation and viscous dissipation are present. The ordinary differential equations are computed for the convergent series solutions of velocity and temperature. The velocity and temperature fields are analyzed for Casson fluid parameter, curvature parameter, Prandtl number, Biot number, radiation parameter and Eckert number. Skin friction coefficient and local Nusselt number are analyzed through numerical values.

9.2 Problem formulation and flow equations

We consider steady and two-dimensional incompressible flow of Casson fluid by a stretching cylinder. The $x-$ and $r -$ axes are taken parallel and perpendicular to the cylinder respectively. In addition, the effects of thermal radiation and viscous dissipation are considered. The boundary layer equations comprising the balance laws of mass, linear momentum and energy
can be written as follows:

\begin{equation}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} = 0, \tag{9.1}
\end{equation}

\begin{equation}
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = \nu \left( 1 + \frac{1}{\zeta} \right) \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right), \tag{9.2}
\end{equation}

\begin{equation}
\rho c_p \left[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial r} \right] = k \frac{\partial^2 T}{\partial r^2} + \frac{k}{r} \frac{\partial T}{\partial r} - \frac{1}{r} \frac{\partial}{\partial r} (rq_r) + \nu \left( 1 + \frac{1}{\zeta} \right) \left( \frac{\partial u}{\partial r} \right)^2. \tag{9.3}
\end{equation}

In the above expressions \( \nu = (\mu/\rho) \) is the kinematic viscosity, \( k \) is the thermal conductivity of the fluid, \( \rho \) is the fluid density, \( T \) is the fluid temperature, \( c_p \) is the specific heat, \( q_r = -\frac{4\sigma^* T^3}{3k^*} \frac{\partial T}{\partial r} \) is the radiative heat flux, \( k^* \) is the mean absorption coefficient and \( \sigma^* \) is the Stefan-Boltzmann constant. The boundary conditions are taken as follows:

\begin{equation}
u = xU_0 \frac{r^2}{l}, \quad -k \frac{\partial T}{\partial r} = h(T_f - T) \text{ at } r = a, \tag{9.4}
\end{equation}

\begin{equation}
u \to 0, \quad T \to T_\infty \text{ as } r \to \infty. \tag{9.5}
\end{equation}

Introducing

\begin{equation}
\nu = \frac{xU_0}{l} f'(\eta), \quad v = -\frac{a}{r} \left( \frac{\nu U_0}{l} \right)^{1/2} f(\eta), \tag{9.6}
\end{equation}

\begin{equation}
\frac{T - T_\infty}{T_w - T_\infty}, \quad \frac{T - T_\infty}{T_w - T_\infty} = \eta = \frac{r^2 - a^2}{2a} \left( \frac{U_0}{\nu l} \right)^{1/2}, \tag{9.7}
\end{equation}

equation (9.1) is identically satisfied and Eqs. (9.2 – 9.5) become

\begin{equation}
\left( 1 + \frac{1}{\zeta} \right) \left( (1 + 2\eta\gamma) f''' + 2\gamma f'' \right) - f'^2 + ff'' = 0, \tag{9.8}
\end{equation}

\begin{equation}
(1 + \frac{4}{3} R) ((1 + 2\gamma\eta) \theta'' + 2\gamma \theta') + \text{Pr} f \theta' + \text{Pr} E_c (1 + 2\gamma\eta) \left( 1 + \frac{1}{\zeta} \right) f'^2 = 0, \tag{9.9}
\end{equation}

\begin{equation}
f = 0, \quad f' = 1, \quad \theta' = -Bi[1 - \theta(0)] \text{ at } \eta = 0, \tag{9.10}
\end{equation}

\begin{equation}
f' = 0, \quad \theta = 0 \text{ as } \eta \to \infty, \tag{9.11}
\end{equation}

where prime denotes the differentiation with respect to \( \eta \), \( f \) is the dimensionless stream function, \( \theta \) is the dimensionless temperature and the dimensionless numbers are \( \zeta = \frac{\mu \nu \tau_\infty}{\tau_r}, R = \frac{4\sigma^* T^3}{kk^*} \),

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\[ \gamma = \sqrt{\frac{u}{a U_0}}, \quad Bi = \frac{h_\infty}{k} \sqrt{\frac{U_0}{a}}, \quad Pr = \frac{\mu\gamma}{k}, \quad \text{and} \quad E_e = \frac{U_0^2 (x/l)^2}{c_p(T_f - T_\infty)}. \]

Here \( \zeta \) is the dimensionless material parameters, \( R \) is the radiation parameter, \( \gamma \) is the curvature parameter, \( Bi \) is the Biot number, \( Pr \) is the Prandtl number and \( E_e \) is the Eckert number.

The skin friction coefficient \( C_f \) is defined as

\[ C_f = \frac{\tau_w}{2 \rho U_w^2}, \quad \tau_w = \mu \left( \frac{\partial u}{\partial r} \right)_{r=a}, \quad \text{and} \quad \text{Re}^{1/2} C_f = \left( 1 + \frac{1}{\zeta} \right) f''(0). \]

Local Nusselt number \( Nu_x \) is given by

\[ Nu_x = \frac{x q_w}{k(T_f - T_\infty)}, \quad q_w = -k \left( \frac{\partial T}{\partial r} \right)_{r=a} + (q_r)_{w} \quad \text{and} \quad \text{Re}^{1/2} Nu_x = - \left( 1 + \frac{4}{3} R \right) \theta'(0). \quad (9.12) \]

### 9.3 Solutions

Initial approximations and auxiliary linear operators are chosen in the forms

\[ f_0(\eta) = 1 - e^{-\eta}, \quad \theta_0(\eta) = \frac{Bi e^{-\eta}}{1 + Bi}, \quad (9.13) \]

\[ \mathcal{L}_f = f''' - f', \quad \mathcal{L}_\theta = \theta'' - \theta, \quad (9.14) \]

with the properties

\[ \mathcal{L}_f(C_1 + C_2 e^\eta + C_3 e^{-\eta}) = 0, \quad \mathcal{L}_\theta(C_4 e^\eta + C_5 e^{-\eta}) = 0, \quad (9.15) \]

where \( C_i \) \((i = 1 - 5)\) are the constants.

The deformation problems at zeroth order are

\[ (1 - p) \mathcal{L}_f \left[ \hat{f}(\eta; p) - f_0(\eta) \right] = ph_f N_f \left[ \hat{f}(\eta; p) \right], \quad (9.16) \]

\[ (1 - p) \mathcal{L}_\theta \left[ \hat{\theta}(\eta; p) - \theta_0(\eta) \right] = ph_\theta N_\theta \left[ \hat{\theta}(\eta; p) \right], \quad (9.17) \]

\[ \hat{f}(0; p) = 0, \quad \hat{f}'(0; p) = 1, \quad \hat{f}'(\infty; p) = 0, \quad \hat{\theta}'(0; p) = -Bi[1 - \hat{\theta}(0; p)], \quad \hat{\theta}(\infty; p) = 0. \quad (9.18) \]

If \( p \in [0, 1] \) indicates the embedding parameter, \( h_f \) and \( h_\theta \) the non-zero auxiliary parameters
then the nonlinear differential operators $\mathcal{N}_f$ and $\mathcal{N}_\theta$ are given by

\[
\mathcal{N}_f[\hat{f}(\eta, p)] = \left(1 + \frac{1}{\zeta}\right) \left(1 + 2\gamma \eta\right) \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} + 2\gamma \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} + \hat{f}(\eta, p) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - \left(\frac{\partial \hat{f}(\eta, p)}{\partial \eta}\right)^2.
\]

\[\mathcal{N}_\theta[\hat{\theta}(\eta, p), \hat{f}(\eta, p)] = \left(1 + \frac{4}{3} \frac{R}{\zeta}\right) \left(1 + 2\gamma \eta\right) \frac{\partial^2 \hat{\theta}(\eta, p)}{\partial \eta^2} + 2\gamma \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} + \text{Pr} \hat{f}(\eta, p) \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} + \text{Pr} \frac{Ec}{1} \left(1 + \frac{1}{\zeta}\right) \left(1 + 2\gamma \eta\right) \left(\frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2}\right)^2.
\]

We have for $p = 0$ and $p = 1$ the following equations:

\[
\hat{f}(\eta; 0) = f_0(\eta), \quad \hat{\theta}(\eta; 0) = \theta_0(\eta),
\]

\[
\hat{f}(\eta; 1) = f(\eta), \quad \hat{\theta}(\eta; 1) = \theta(\eta).
\]

It is noticed that when $p$ varies from 0 to 1 then $f(\eta, p)$ and $\theta(\eta, p)$ approach from $f_0(\eta), \theta_0(\eta)$ to $f(\eta)$ and $\theta(\eta)$. The series of $f$ and $\theta$ through Taylor’s expansion are chosen convergent for $p = 1$ and thus

\[
f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \quad f_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m f(\eta, p)}{\partial \eta^m} \right|_{p=0},
\]

\[
\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \quad \theta_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \theta(\eta, p)}{\partial \eta^m} \right|_{p=0}.
\]

The resulting problems at $m^{th}$ order can be presented in the following forms:

\[
\mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f R_m^f(\eta),
\]

\[
\mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_\theta R_m^\theta(\eta),
\]

\[
f_m(0) = f'_m(0) = f'_m(\infty) = \theta'_m(0) - Bi \theta(0) = \theta_m(\infty) = 0,
\]
\[ R_f^m(\eta) = \left( 1 + \frac{1}{\zeta} \right) \left( (1 + 2\gamma \eta) f''_{m-1} + 2\gamma f'''_{m-1} \right) + \sum_{k=0}^{m-1} \left( f_{m-1-k} f_k'' - f'_{m-1-k} f_k' \right) \]  

9.28

\[ R_\theta^m(\eta) = \left( 1 + \frac{4}{3} R \right) \left( (1 + 2\gamma \eta) \theta''_{m-1}(\eta) + 2\gamma \theta'_{m-1}(\eta) \right) + Pr \sum_{k=0}^{m-1} \theta'_{m-1-k} f_k + Pr Ec \left( 1 + \frac{1}{\zeta} \right) (1 + 2\gamma \eta) \sum_{k=0}^{m-1} f''_{m-1-k} f_k'' , \]  

9.29

\[ \chi_m = \begin{cases} 
0, & m \leq 1, \\
1, & m > 1. 
\end{cases} \]

The general solutions \((f_m, \theta_m)\) comprising the special solutions \((f^*_m, \theta^*_m)\) are

\[ f_m(\eta) = f^*_m(\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta} , \]  

9.30

\[ \theta_m(\eta) = \theta^*_m(\eta) + C_4 e^\eta + C_5 e^{-\eta} , \]  

9.31

\[ C_1 = -C_3 - f^*(0), \quad C_2 = C_4 = 0, \quad C_3 = \left. \frac{\partial f^*(\eta)}{\partial \eta} \right|_{\eta=0}, \quad C_5 = \left. \frac{\partial \theta^*(\eta)}{\partial \eta} \right|_{\eta=0} - \frac{Bi \theta^*(\eta)|_{\eta=0}}{1 + Bi}. \]

9.4 Convergence analysis

It is now well established argument that the convergence of series solutions (9.25) and (9.26) depend upon the auxiliary parameters \(h\). It is clear from Figs. 9.1 and 9.2 that the admissible values of \(h_f\) and \(h_\theta\) are \(-1.26 \leq h_f \leq -0.24\) and \(-1.1 \leq h_\theta \leq -0.45\) respectively. The series solutions converge for the whole region of \(\eta\) when \(h_f = -0.9\) and \(h_\theta = -0.8\).
Fig. 9.1. $h-$ curve for the velocity field.

Fig. 9.2. $h-$ curve for the temperature field.
Table 9.1: Convergence of series solutions for different order of approximations when $\zeta = 2.0$, $Bi = 0.3$, $R = 0.1$, $\gamma = 0.1$, $Pr = 1.0$, $Ec = 0.4$, $h_f = -0.65$ and $\theta_0 = -0.8$.

<table>
<thead>
<tr>
<th>Order of approximation</th>
<th>$f''(0)$</th>
<th>$\theta'(0)$</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
<td>5</td>
<td>0.85415</td>
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<tr>
<td>45</td>
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<td>0.36134</td>
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</tbody>
</table>

9.5 Results and discussion

This section is aimed to examine the effects of different physical parameters on the velocity and temperature fields. Hence the Figs. (9.3 - 9.10) are plotted. Fig. 9.3 represents the variation of $\zeta$ on the velocity field. This Fig. shows that the velocity field decreases when Casson fluid parameter $\zeta$ is large. Also the velocity vanishes at some large distance from the sheet (at $\eta = 12$). Fig. 9.4 depicts the effect of $\gamma$ on $f'(\eta)$. Here $f'(\eta)$ is increasing function of curvature parameter $\gamma$. The boundary layer thickness decreases as $\gamma$ increases. The velocity component approaches to zero asymptotically when $\eta = 0$. Influence of parameter $\zeta$ on the temperature field is displayed in Fig. 9.5. It is noticed that $\theta(\eta)$ decreases near the wall, when $\zeta$ is increased but at some large distance from the wall it vanishes. Fig. 9.6 presents the influence of $\gamma$ on the temperature field. We see that the temperature profile decreases when $\eta < 1$, thereafter, for large values of curvature parameter the temperature $\theta(\eta)$ increases. Also when $\gamma = 0$, physically the outer surface of cylinder behaves like a flat surface. Fig. 9.7 shows that temperature field is decreasing function of Prandtl number $Pr$ when $\eta > 1$ and increasing function of $Pr$ when $\eta < 1$. Effect of Biot number $Bi$ on temperature field is plotted in Fig.
9.8. Temperature field increases when $Bi$ increases. Here we considered the Biot number less than one due to uniformity of the temperature field in the fluid. A lower Biot number means the conductive heat transfer is much faster than the convective heat transfer. Fig. 9.9 depicts the variation of $R$ on $\theta(\eta)$. For large values of $R$ the temperature field increases. Fig. 9.10 gives the influence of Eckert number $Ec$ on temperature field. It is clearly seen from this Fig. that temperature field increases when there is an increase in the Eckert number $Ec$. Physically, the larger Eckert number leads to enhancement in kinetic energy which enhances the temperature rises. When $\eta > 6$ the temperature field vanishes asymptotically. Fig. 9.11 shows the effect of Biot number on temperature and temperature gradient. Temperature field increases near the wall most rapidly and it vanishes far away the wall where as temperature gradient has opposite behavior for large values of Biot number.

The study of Table 9.2 indicates the effects of $\zeta$ and $\gamma$ on wall shear stress. From this table it is clear that shear stress at wall is negative here. Physically, negative sign shows that surface exerts a dragging force on the fluid. Table 9.3 shows the effect of physical parameters on heat transfer characteristics at the wall $-\theta(0)$. We observe through tabular values that for large values of $\zeta, \gamma, Bi$ and $Pr$ the magnitude of heat transfer characteristics at the wall $-\theta(0)$ increases. However it decreases for $R$ and $Ec$.

![Graph](image)

Fig. 9.3. Influence of $\zeta$ on $f'(\eta)$.
Fig. 9.4. Influence of $\gamma$ on $f'(\eta)$.

Fig. 9.5. Influence of $\zeta$ on $\theta(\eta)$. 
Fig. 9.6. Influence of \( \gamma \) on \( \theta(\eta) \).

Fig. 9.7. Influence of \( \text{Pr} \) on \( \theta(\eta) \).
Fig. 9.8. Influence of $Bi$ on $\theta(\eta)$.

Fig. 9.9. Influence of $R$ on $\theta(\eta)$. 
Fig. 9.10. Influence of Ec on $\theta(\eta)$.

Fig. 9.11. Influence of Bi on $\theta(\eta)$ and $\theta'(\eta)$.
Table 9.2: Values of skin friction coefficient $Re_x^{1/2}C_f$ for the parameters $\zeta$ and $\gamma$.

<table>
<thead>
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<th>$\zeta$</th>
<th>$\gamma$</th>
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Table 9.3: Values of heat transfer characteristics at wall $-\theta'(0)$ for different emerging parameters when $h_f = -0.65$ and $h_\theta = -0.8$.

<table>
<thead>
<tr>
<th>$\zeta$</th>
<th>$\gamma$</th>
<th>Pr</th>
<th>Bi</th>
<th>$R$</th>
<th>$Ec$</th>
<th>$-(1 + \frac{1}{4}R)\theta'(0)$</th>
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9.6 Conclusions

Here the radiative effect in the flow of Casson fluid due to stretching cylinder is modeled. Convective boundary condition is used. The important findings are listed below.

- Effects of $\gamma$ and $\zeta$ on the velocity field are quite opposite.
- Temperature field $\theta(\eta)$ is decreasing function of $\zeta$.
- Prandtl number $Pr$ and curvature parameter $\gamma$ have monotonic behavior on the temperature field.
- Influences of radiation parameter $R$ and Eckert number $Ec$ on temperature field are qualitatively similar.
- Temperature field enhances when Biot number $Bi$ is increased.
- Behaviors of $\zeta$ and curvature parameter $\gamma$ on skin friction coefficient are quite opposite.
- Magnitude of temperature gradient increases when curvature parameter $\gamma$ and Biot number $Bi$ are increased.
Chapter 10

Flow of variable thermal conductivity fluid due to inclined stretching cylinder with viscous dissipation and thermal radiation

10.1 Introduction

The aim of present chapter is to investigate the flow of variable thermal conductivity Casson fluid by an inclined stretching cylinder. Heat transfer analysis has been carried out in the presence of thermal radiation and viscous dissipation effects. The relevant equations are first simplified under usual boundary layer assumptions and then transformed into ordinary differential equations by suitable transformations. The transformed ordinary differential equations are computed for the series solutions of velocity and temperature. Convergence analysis is shown explicitly. Velocity and temperature fields are discussed for different physical parameters through graphs and numerical values. It is found that the velocity decreases when angle of inclination increases but it increases for larger values of mixed convection parameter. Further an enhancement in the thermal conductivity and radiation effects corresponds to higher fluid temperature. It is also found that heat transfer is more pronounced in a cylinder when
compared to a flat plate. Thermal boundary layer thickness enhances with increasing values of Eckert number. Radiation and variable thermal conductivity decreases the heat transfer rate at the surface.

### 10.2 Flow description

We consider steady two-dimensional incompressible flow of Casson fluid by an inclined stretching cylinder. The stretching cylinder makes an angle $\alpha$ with the vertical axis i.e. $r-axis$ and $x-axis$ is taken normal to it. In addition, the energy equation is considered with combined effects of thermal radiation and viscous dissipation. The thermal conductivity is taken temperature dependent. The boundary layer equations through the balance laws of mass, linear momentum and energy can be written as follows:

\[
\frac{\partial (ru)}{\partial x} + \frac{\partial (rv)}{\partial r} = 0, \quad (10.1)
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = \nu \left( 1 + \frac{1}{\zeta} \right) \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) + g_0 \beta_T (T - T_\infty) \cos \alpha, \quad (10.2)
\]

\[
\rho c_p \left[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial r} \right] = \frac{1}{r} \frac{\partial}{\partial r} \left( k \frac{\partial T}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial r} (r q_r) + \mu_B \left( 1 + \frac{1}{\zeta} \right) \left( \frac{\partial u}{\partial r} \right)^2. \quad (10.3)
\]

In the above expressions $\nu$ is the kinematic viscosity, $\rho$ is the fluid density, $T$ is the fluid temperature, $c_p$ is the specific heat, $q_r = -\frac{16 \sigma^* T^3}{3k^*} \frac{\partial T}{\partial r}$ is the radiative heat flux, $\sigma^*$ is the mean absorption coefficient, $g_0$ is the acceleration due to gravity, $\beta_T$ is the volumetric coefficient of thermal exponential, $\sigma^*$ is the Stefan-Boltzmann constant and $k$ is the variable thermal conductivity of the fluid defined as

\[
k = k_\infty \left( 1 + \tilde{\epsilon} \frac{T - T_\infty}{\Delta T} \right), \quad (10.4)
\]

in which $\tilde{\epsilon}$ is the small parameter, $k_\infty$ is the thermal conductivity of the fluid far away from the surface and $\Delta T = T_w - T_\infty$. The boundary conditions are expressed as follows:

\[
u = U_0 \frac{x}{T}, \quad v = 0, \quad T = T_w \quad \text{at} \quad r = a, \quad (10.5)
\]
\[ u \to 0, \quad T \to T_\infty \quad \text{as} \quad r \to \infty. \quad (10.6) \]

Introducing the relations

\[ u = \frac{x U_0}{l} f'(\eta), \quad v = -\frac{a}{r} \left( \frac{\nu U_0}{l} \right) f(\eta), \quad (10.7) \]
\[ \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \eta = \frac{r^2 - a^2}{2a} \left( \frac{U_0}{\nu l} \right), \quad (10.8) \]

equation (10.1) is identically satisfied and Eqs. (10.2 - 10.6) become

\[ \left( 1 + \frac{1}{\zeta} \right) \left( (1 + 2\eta \gamma) f''' + 2\gamma f'' \right) - f' + f f'' + G \theta \cos \alpha = 0, \quad (10.9) \]
\[ (1 + \frac{4}{3} R) \left[ (1 + 2\gamma \eta) \theta'' + 2\gamma \theta' \right] + \dot{\varepsilon} \left[ (1 + 2\gamma \eta) (\theta \theta'' + \theta'^2) + 2\gamma \theta \theta' \right] + \Pr f \theta' \]
\[ + \Pr E_c (1 + 2\gamma \eta) \left( 1 + \frac{1}{\zeta} \right) f''' = 0, \quad (10.10) \]
\[ f = 0, \quad f' = 1, \quad \theta = 1 \quad \text{at} \quad \eta = 0, \quad (10.11) \]
\[ f' = 0, \quad \theta = 0 \quad \text{as} \quad \eta \to \infty, \quad (10.12) \]

where prime denotes the differentiation with respect to \( \eta \), \( f \) is the dimensionless stream function, \( \theta \) is the dimensionless temperature and the dimensionless numbers are \( \zeta = \frac{\mu \sqrt{T^3}}{\tau_r} \), \( R = \frac{4 \sigma^2 T_\infty^3}{k k^2} \), \( \gamma = \sqrt{\frac{\nu l}{\alpha^2 U_0}} \), \( G = \frac{\rho \beta_T (T_w - T_\infty)^2}{\alpha^2 / \nu^2} \), \( \Pr = \frac{\mu \rho \alpha^2}{k} \) and \( E_c = \frac{T_w^2 (x/l)^2}{c_p (T_w - T_\infty)} \). Here \( \zeta \) is the Casson fluid parameter, \( R \) is the radiation parameter, \( \gamma \) is the curvature parameter, \( \Pr \) is the Prandtl number, \( G \) is the mixed convection parameter and \( Ec \) is the Eckert number.

The skin friction coefficient \( C_f \) is

\[ C_f = \frac{\tau_w}{\frac{1}{2} \rho U_0^2}, \quad \tau_w = \mu \left( \frac{\partial u}{\partial r} \right)_{r=a} \quad \text{and} \quad \text{Re}_x^{1/2} C_f = \left( 1 + \frac{1}{\zeta} \right) f''(0). \quad (10.13) \]

Local Nusselt number \( Nu_x \) is

\[ Nu_x = \frac{x q_w}{k (T_w - T_\infty)}, \quad q_w = -k \left( 1 + \frac{16 \sigma^2 T_\infty^3}{3 k^2 k} \right) \left( \frac{\partial T}{\partial r} \right)_{r=a} \quad \text{and} \quad \text{Re}_x^{1/2} Nu_x = -(1 + \frac{4}{3} R) \theta'(0). \quad (10.14) \]
10.3 Analytical solutions

Here we select the following values of initial approximations and auxiliary linear operators:

\[ f_0(\eta) = 1 - e^{-\eta}, \quad \theta_0(\eta) = e^{-\eta}, \quad (10.15) \]

\[ \mathcal{L}_f = f'' - f', \quad \mathcal{L}_\theta = \theta'' - \theta, \quad (10.16) \]

subject to the properties

\[ \mathcal{L}_f(C_1 + C_2 e^\eta + C_3 e^{-\eta}) = 0, \quad \mathcal{L}_\theta(C_4 e^\eta + C_5 e^{-\eta}) = 0, \quad (10.17) \]

where \( C_i \) \( (i = 1 - 5) \) are the constants.

If \( p \in [0, 1] \) indicates the embedding parameter, the zeroth order deformation problems are constructed as follows:

\[ (1 - p) \mathcal{L}_f \left[ \hat{f}(\eta; p) - f_0(\eta) \right] = ph_f \mathcal{N}_f \left[ \hat{f}(\eta; p) \right], \quad (10.18) \]

\[ (1 - p) \mathcal{L}_\theta \left[ \hat{\theta}(\eta; p) - \theta_0(\eta) \right] = ph_\theta \mathcal{N}_\theta \left[ \hat{\theta}(\eta; p) \right], \quad (10.19) \]

\[ \hat{f}(0; p) = 0, \quad \hat{f}'(0; p) = 1, \quad \hat{f}'(\infty; p) = 0, \quad \hat{\theta}(0; p) = 1, \quad \hat{\theta}(\infty; p) = 0. \quad (10.20) \]

Here \( h_f \) and \( h_\theta \) are the non-zero auxiliary parameters and the nonlinear operators \( \mathcal{N}_f \) and \( \mathcal{N}_\theta \) are given by

\[ \mathcal{N}_f[\hat{f}(\eta, p)] = \left( 1 + \frac{1}{\zeta} \right) \left( (1 + 2\gamma) \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} + 2\gamma \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right) + \hat{f}(\eta, p) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \]

\[ - \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right)^2 + G \cos \alpha \hat{\theta}(\eta, p). \]
\[ N_0[\dot{\theta}(\eta, p), \dot{f}(\eta, p)] = \left( 1 + \frac{4}{3} R \right) \left( (1 + 2\gamma \eta) \frac{\partial^2 \dot{\theta}(\eta, p)}{\partial \eta^2} + 2\gamma \frac{\partial \dot{\theta}(\eta, p)}{\partial \eta} \right) \\
\quad + \epsilon \left[ (1 + 2\gamma \eta) \left( \frac{\partial \dot{\theta}(\eta, p)}{\partial \eta} \right)^2 + \dot{\theta}(\eta, p) \frac{\partial^2 \dot{\theta}(\eta, p)}{\partial \eta^2} \right] + 2\gamma \dot{\theta}(\eta, p) \frac{\partial \dot{\theta}(\eta, p)}{\partial \eta} \\
\quad + \text{Pr} \dot{f}(\eta, p) \frac{\partial \dot{\theta}(\eta, p)}{\partial \eta} + \text{Pr} E \left( 1 + \frac{1}{\zeta} \right) (1 + 2\gamma \eta) \left( \frac{\partial^2 \dot{f}(\eta, p)}{\partial \eta^2} \right)^2. \tag{10.21} \]

For \( p = 0 \) and \( p = 1 \) the following equations are obtained:

\[ \dot{f}(\eta; 0) = f_0(\eta), \quad \dot{\theta}(\eta, 0) = \theta_0(\eta), \tag{10.22} \]

\[ \dot{f}(\eta; 1) = f(\eta), \quad \dot{\theta}(\eta, 1) = \theta(\eta). \]

It is noticed that when \( p \) varies from 0 to 1 then \( f(\eta, p) \) and \( \theta(\eta, p) \) approach from \( f_0(\eta), \theta_0(\eta) \) to \( f(\eta) \) and \( \theta(\eta) \). The series of \( f \) and \( \theta \) through Taylor’s expansion are chosen convergent for \( p = 1 \) and thus

\[ f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \quad f_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m f(\eta, p)}{\partial \eta^m} \right|_{p=0}, \tag{10.23} \]

\[ \theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \quad \theta_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \theta(\eta, p)}{\partial \eta^m} \right|_{p=0}. \tag{10.24} \]

The resulting problems at \( m^{th} \) order can be presented in the following forms

\[ L_f [f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f R^n_f(\eta), \tag{10.25} \]

\[ L_\theta [\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_\theta R^n_\theta(\eta), \tag{10.26} \]

\[ f_m(0) = f'_m(0) = f'_m(\infty) = \theta_m(0) = \theta_m(\infty) = 0, \tag{10.27} \]

\[ R^n_f(\eta) = \left( 1 + \frac{1}{\zeta} \right) \left( (1 + 2\gamma \eta) f''_{m-1} + 2\gamma f'''_{m-1} \right) + \sum_{k=0}^{m-1} (f_{m-1-k} f''_k - f'_{m-1-k} f'_k) + G \cos \alpha \theta_{m-1}, \tag{10.28} \]
\[ R^m_\theta(\eta) = \left(1 + \frac{4}{3} R\right) \left[(1 + 2\gamma \eta) \theta''_{m-1}(\eta) + 2\gamma \theta'_{m-1}\right] \\
+ \hat{\epsilon} \left[(1 + 2\gamma \eta) \sum_{k=0}^{m-1} \left(\theta'_{m-1-k} \theta'_k + \theta_{m-1-k} \theta''_k\right) + 2\gamma \sum_{k=0}^{m-1} \theta_{m-1-k} \theta'_k\right] \\
+ \Pr \sum_{k=0}^{m-1} \theta'_{m-1-k} f_k + \Pr Ec \left(1 + \frac{1}{\zeta}\right) (1 + 2\gamma \eta) \sum_{k=0}^{m-1} f''_{m-1-k} f''_k, \quad (10.29) \]

\[ \chi_m = \begin{cases} 
0, & m \leq 1, \\
1, & m > 1.
\end{cases} \]

The general solutions \((f_m, \theta_m)\) consisting of the special solutions \((f^*_m, \theta^*_m)\) are

\[ f_m(\eta) = f^*_m(\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta}, \quad (10.30) \]

\[ \theta_m(\eta) = \theta^*_m(\eta) + C_4 e^\eta + C_5 e^{-\eta}. \quad (10.31) \]

### 10.4 Convergence of the developed solutions

It is well recognized fact that the convergence of series solutions (10.25) and (10.26) depends upon the non-zero auxiliary parameter \(h\). To find values of auxiliary parameters ensuring the convergence we have plotted the \(h\)-curves for velocity and temperature profiles. It is clear from Figs. 10.1 and 10.2 that the admissible values of \(h_f\) and \(h_\theta\) are \(-1.26 \leq h_f \leq -0.24\) and \(-1.1 \leq h_\theta \leq -0.45\) respectively. These series solutions converge for the whole region of \(\eta\) when \(h_f = -0.7\) and \(h_\theta = -0.7\). Table 10.1 shows the convergence of homotopy solutions. It is obvious that 25th-order approximations are enough for the convergent series solutions.
Fig. 10.1. $h-$ curve for the velocity field.

Fig. 10.2. $h-$ curve for the temperature field.
Table 10.1: Convergence of series solutions for different order of approximations when \( \zeta = 2.0, \dot{e} = R = \gamma = 0.1, \text{Pr} = 1.0, G = Ec = 0.4, h_f = -0.7 \) and \( h_\theta = -0.7 \).

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10.5 Discussion

Our intention in this section is to analyze the velocity and temperature profiles for different physical parameters Casson fluid parameter, curvature parameter, angle of inclination, mixed convection parameter, Eckert number, Prandtl number, radiation parameter and variable thermal conductivity parameter. The salient features of Casson fluid and curvature parameters are displayed in the Figs. 10.3 and 10.4. It is observed that the velocity field is decreasing function of \( \zeta \). On the other hand, the velocity is found to increase for larger curvature parameter \( \gamma \). Also it shows that the rate of transport decreases with the increasing distance from the cylinder and vanishes at some large distance from the cylinder. Fig. 10.5 exhibits the variation of angle of inclination. As expected, an increase in the values of \( \alpha \) decreases the velocity. This is due to the fact that the angle of inclination increases the influence of the buoyancy force due to thermal decrease by the factor of \( \cos(\alpha) \). Fig. 10.6 presents the effects of mixed convection parameter \( G \) on velocity profile. For mixed convection parameter, velocity decays very slowly to the ambient but the fluid motion can infiltrate quite deeply into the ambient fluids. Interestingly if \( G \) is positive, it means heating of fluid or cooling of the boundary surface whereas for
negative values of $G$ means cooling of the fluid or heating of the boundary surface and when $G = 0$ it corresponds to the absence of free convection current. When $\eta > 9$ the temperature field vanishes asymptotically. The effect of curvature parameter $\gamma$ on the temperature profile is displayed in Fig. 10.7. When $\gamma = 0$ the outer surface of the cylinder behaves like a flat plate. Temperature profile increases most rapidly when curvature parameter $\gamma$ increases. Also for $0 \leq \eta \leq 1$ no variation can be observed for various values of $\gamma$. Temperature and thermal boundary layer thickness decrease for large values of Pr. The fluid with higher values of Prandtl number implies more viscous fluid and lower Prandtl number corresponds to less viscous fluid. Fluids with higher viscosity have lower temperature and fluids with lower viscosity have higher temperature. This leads to decrease in temperature and boundary layer thickness (see Fig. 10.8). Fig 10.9 illustrates the effect of $\dot{\varepsilon}$ on the temperature profile. Temperature profile is larger when thermal conductivity varies linearly with the temperature whereas for constant thermal conductivity ($\dot{\varepsilon} = 0$) the temperature profile decays. As we assume that the thermal conductivity varies linearly with temperature which causes reduction in the magnitude of the transverse velocity, therefore temperature profile increases with an increase in variable thermal conductivity parameter. The enhancement in $R$ accelerates the temperature as well as boundary layer thickness (see Fig. 10.10). Because larger values of $R$ imply higher surface heat flux and thereby making the fluid more warmer which enhances the temperature profile. Here we noted that the temperature increases most rapidly in the region $0 \leq \eta \leq 4$. Fig. 10.11 plots the temperature distribution versus the Eckert number $Ec$. It is seen that fluid temperature rises when the Eckert number becomes pronounced. Physically the Eckert number depends on the kinetic energy. When we increase the values of Eckert number then the kinetic energy enhances. This enhancement in the kinetic energy leads to an increase in the temperature and thermal boundary layer thickness. In all cases the velocity is maximum at the surface of the cylinder and it starts decreasing as $\eta \to \infty$. Figs (10.12 – 10.15) show the influences of $\dot{\varepsilon}$, $R$, $Ec$ and Pr on the temperature distribution, for the flat plate ($i.e \gamma = 0$) and stretching cylinder ($i.e \gamma = 0.7$) respectively. It is noticed that the temperature profile increases remarkably when $\gamma = 0.7$ in comparison to flat plate when $\gamma = 0$ (see Figs. 10.12 – 10.14). The magnitude of boundary layer thickness in case of stretching cylinder is larger than the flat plate. Fig. 10.15 illustrates that for larger values of Pr the temperature profile decays. Also the thermal bound-
ary layer thickness reduces. It is also noted that the influence of Pr is much more prominent for stretching flat plate when compared with stretching cylinder. Fluid with lower Prandtl number possess higher thermal conductivity so that heat can diffuse from the cylinder faster than for higher Pr fluid.

The magnitude of skin friction coefficient is enhanced when curvature parameter $\gamma$ and angle of inclination $\alpha$ are increased (see Table 10.2). Table 10.3 shows the local Nusselt number for different set of values of the involved parameters. The effects of $\zeta$ and $\gamma$ on local Nusselt number are similar. The magnitude of local Nusselt number decreases for larger values of Eckert number $Ec$, radiation parameter $R$ and variable thermal conductivity parameter $\dot{c}$.

![Graph](image)

Fig. 10.3. Influence of $\zeta$ on $f'(\eta)$. 

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Fig. 10.4. Influence of $\gamma$ on $f'(\eta)$.

Fig. 10.5. Influence of $\alpha$ on $f'(\eta)$.
Fig. 10.6. Influence of $G$ on $f'(\eta)$. 

Fig. 10.7. Influence of $\gamma$ on $\theta(\eta)$. 

$\zeta = 2.0, \gamma = 0.1, \alpha = \pi/6$

$G = 0.0, G = 0.4, G = 0.8, G = 1.2$

$\zeta = 2.0, \gamma = R = 0.1, G = Ec = 0.4,$

$\Pr = 1.0, \alpha = \pi/6, \dot{\varepsilon} = 0.1$

$\gamma = 0.0, \gamma = 0.15, \gamma = 0.25, \gamma = 0.4$
Fig. 10.8. Influence of $\Pr$ on $\theta(\eta)$. 

\[ \zeta = 2.0, \gamma = R = 0.1, Ec = G = 0.4, \alpha = \pi/6, \dot{\epsilon} = 0.1 \]

$\Pr = 1.0, \Pr = 1.2, \Pr = 1.8, \Pr = 2.2$

Fig. 10.9. Influence of $\dot{\epsilon}$ on $\theta(\eta)$. 

\[ \zeta = 2.0, \gamma = R = 0.1, Ec = G = 0.4, \]

$\Pr = 1.0, \alpha = \pi/6$

$\dot{\epsilon} = 0.0, \dot{\epsilon} = 0.2, \dot{\epsilon} = 0.4, \dot{\epsilon} = 0.6$
Fig. 10.10. Influence of $R$ on $\theta(\eta)$. 

Fig. 10.11. Influence of $Ec$ on $\theta(\eta)$. 

$\beta = 2.0, \gamma = 0.1, Ec = G = 0.4, Pr = 1.0, \alpha = \pi/6, \epsilon = 0.1$ 

$R = 0.0, R = 0.1, R = 0.2, R = 0.3$ 

$\beta = 2.0, \gamma = R = 0.1, G = 0.4, \alpha = \pi/6, Pr = 1.0, \epsilon = 0.1$ 

$Ec = 0.0, Ec = 0.3, Ec = 0.6, Ec = 0.9$ 

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Fig. 10.12. Influence of \( \dot{\varepsilon} \) on \( \theta(\eta) \).

Fig. 10.13. Influence of \( R \) on \( \theta(\eta) \).
Fig. 10.14. Influence of $Ec$ on $\theta(\eta)$.

Fig. 10.15. Influence of $Pr$ on $\theta(\eta)$. 
Table 10.2: Values of skin friction coefficient $Re_x^{1/2}C_f$ for different parameters.

<table>
<thead>
<tr>
<th>$\zeta$</th>
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<th>$\alpha$</th>
<th>$G$</th>
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</tr>
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10.6 Concluding remarks

An analysis in this attempt has been carried out for flow and heat transfer by a stretching cylinder in presence of variable thermal conductivity and thermal radiation effects. The present study indicated that due to increase in curvature parameter ($\gamma$), the velocity and temperature are increased. Variable thermal conductivity enhances the temperature profile and associated boundary layer thickness. The magnitude of temperature in case of stretching cylinder rises when Eckert number ($Ec$), radiation parameter ($R$) and variable thermal conductivity parameter ($\hat{\epsilon}$) are increased. Such magnitude of temperature is smaller for flat plate. The growth of the boundary layers with distance from the leading edge for stretching cylinder is greater than the flat plate. The local Nusselt number decreases when radiation ($R$) and variable thermal conductivity parameters ($\hat{\epsilon}$) are increased.
Chapter 11

MHD stagnation-point flow of Jeffrey fluid over a convectively heated stretching sheet

11.1 Introduction

This chapter deals with two-dimensional stagnation-point flow of Jeffrey fluid over an exponentially stretching sheet. Convective boundary condition is used for the analysis of thermal boundary layer. In addition the combined effects of thermal radiation and applied magnetic field are taken into consideration. The developed nonlinear problems have been solved for the series solution. The convergence of the series solutions is carefully analyzed. The behaviors of various physical parameters such as viscoelastic parameter ($\beta_3$), magnetic field parameter ($M$), radiation parameter ($R$), Biot number ($Bi$) and velocity ratio parameter ($\alpha^*$) are examined through graphical and numerical results of velocity and temperature distributions.

11.2 Flow equations

The extra stress tensor for Jeffrey fluid is [41]

$$S = \frac{\mu}{1 + \lambda} \left[ A_1 + \lambda_3 \frac{dA_1}{dt} \right].$$  \hspace{1cm} (11.1)
In above expressions \( \mu \) is the dynamic viscosity, \(-pI\) is the indeterminate part of the stress tensor, \( \lambda \) is the ratio of relaxation to retardation times, \( \lambda_3 \) is the retardation time, \( A_1 \) is the first Rivlin-Erickson tensor, \( d/dt \) is the material derivative define as

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + (V \cdot \nabla),
\]

in which Eq. (11.1) reduces to a Newtonian fluid when \( \lambda_1 = \lambda_2 = 0 \).

11.3 Problem formulation

We consider the steady two-dimensional MHD stagnation point flow of an incompressible Jeffrey fluid by an exponentially stretching sheet. Constant magnetic field \( B_0 \) is applied normal to the plate. Induced magnetic field is assumed negligible in comparison to applied magnetic field for small magnetic Reynolds number. In addition heat transfer analysis is considered with radiation effects. Further we consider \( x - axis \) parallel to the sheet and \( y - axis \) normal to it (see Fig. 11.1).

Fig. 11.1. Physical model and coordinate system.
The velocity and temperature fields subject to boundary layer approximations are governed by the following equations

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{11.3}
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U_0 \frac{dU_0}{dx} + \frac{v}{1 + \lambda} \left[ \frac{\partial^2 u}{\partial y^2} + \lambda_3 \left( \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + u \frac{\partial^3 u}{\partial x \partial y^2} \right) \right] - \frac{\sigma B_0^2}{\rho} (u - U_0), \tag{11.4}
\]

\[
\rho c_p \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} - \frac{\partial q_v}{\partial y}, \tag{11.5}
\]

\[u = U_w(x) = ce^{\xi x}, \ v = 0, \ -k \frac{\partial T}{\partial y} = h(T_f - T) \text{ at } y = 0, \tag{11.6}\]

\[u = U_0(x) = ae^{\xi x}, \ T = T_\infty \text{ as } y \to \infty, \tag{11.7}\]

where \(u\) and \(v\) represent the velocity components along the \(x\) and \(y\)-axes respectively, \(U_0\) is the strain velocity, \(\lambda\) is the ratio of relaxation to retardation times, \(\lambda_3\) is the retardation time, \(U_w\) is the stretching/shrinking velocity, \(T\) is the fluid temperature, \(\sigma\) is the thermal diffusivity of the fluid, \(\nu = (\mu/\rho)\) is the kinematic viscosity, \(\rho\) is the density of the fluid, \(k\) is the thermal conductivity of fluid, \(h\) is the convective heat transfer coefficient and \(T_f\) is the convective fluid temperature below the moving sheet.

We define the following transformations as

\[
\psi = \sqrt{2avLF(\eta)e^{x/2L}}, \ \theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty}, \ \eta = y\sqrt{\frac{a}{2vL}}e^{x/2L}, \tag{11.8}
\]

where \(a\) and \(L\) are constants and prime denotes the differentiation with respect to \(\eta\), \(f\) is the dimensionless stream function, \(\theta\) is the dimensionless temperature and \(\psi\) is the stream function given by

\[
\frac{\partial u}{\partial y} = \frac{\partial \psi}{\partial x}, \quad v = -\frac{\partial \psi}{\partial x}.
\]

Now Eq. (11.3) is identically satisfied and (11.5 - 11.8) yield

\[
f'''' + (1 + \lambda)(ff'' - 2f'^2 + 2) + \beta_3 \left( \frac{3}{2}f'^2 + f'f'' - \frac{1}{2}f f''\right) - 2(1 + \lambda)M^2(f' - 1) = 0, \tag{11.9}
\]
(1 + \frac{4}{3} R)\theta'' + \text{Pr} \left( f\theta' \right) = 0, \quad (11.10)

f = 0, \quad f' = \alpha^* = c/a, \quad \theta' = -Bi[1 - \theta(0)] \quad \text{at } \eta = 0, \quad (11.11)

f' = 1, \quad \theta = 0 \quad \text{at } \eta = \infty, \quad (11.12)

where \( \beta_3 = \frac{\lambda_3 \alpha^* L}{\Gamma} \) is a local dimensionless parameter, \( \alpha^* = \frac{c}{a} \) is a ratio parameter with \( \alpha^* > 1 \) for assisting flow and \( \alpha^* < 1 \) for opposing flow (i.e. when free stream velocity exceeds the stretching velocity), \( M = \frac{\sigma B_0^2 L}{\rho U_w} \) is the magnetic parameter, \( \text{Pr} = \frac{\nu}{\sigma} \) is the Prandtl number, \( R = \frac{4 \alpha^* r^2}{k k'} \) is a radiation parameter and \( Bi = \frac{b}{T_{\infty} \sqrt{\alpha}} \) is the Biot number. The skin friction coefficient \( C_f \) and local Nusselt number \( Nu_x \) are

\[ C_f = \frac{\tau_w}{\rho U_w^2}, \quad Nu_x = \frac{xq_w}{k(T_f - T_{\infty})}, \quad (11.13) \]

where the skin friction \( \tau_w \) and the heat transfer from the plate \( q_w \) are

\[
\tau_w = \frac{\mu}{1 + \lambda} \left[ \left( \frac{\partial u}{\partial y} \right) + \lambda_3 \left\{ u \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} + v \frac{\partial^2 u}{\partial y^2} \right\} \right]_{y=0},
\]

\[
q_w = -k \left( 1 + \frac{16 \sigma^* T_{\infty}^3}{3 k' k} \right) \left( \frac{\partial T}{\partial y} \right)_{y=0}.
\]

Dimensionless form of skin friction coefficient \( C_f \) and local Nusselt number \( Nu_x \) are

\[ C_f = (2 \text{Re}_x)^{-1/2} \alpha^*^{-2} \left[ \frac{1}{1 + \lambda} \left\{ f''(0) + \beta_3 \left( \frac{3}{2} \alpha f''(0) \right) \right\} \right], \quad (11.14) \]

\[ Nu/\text{Re}_x^{1/2} = -(1 + \frac{4}{3} R)\theta'(0). \quad (11.15) \]

### 11.4 Homotopy analysis solutions

The definitions of initial guesses and auxiliary linear operators are

\[ f_0(\eta) = (\alpha^* - 1) + \eta + (1 - \alpha^*)e^{-\eta}, \quad (11.16) \]

\[ \theta_0(\eta) = \frac{Bi \exp(-\eta)}{1 + Bi}, \quad (11.17) \]
\[\mathcal{L}_f = f''' - f',\quad (11.18)\]
\[\mathcal{L}_\theta = \theta'' - \theta,\quad (11.19)\]
\[\mathcal{L}_f(C_1 + C_2e^\eta + C_3e^{-\eta}) = 0,\quad (11.20)\]
\[\mathcal{L}_\theta(C_4e^\eta + C_5e^{-\eta}) = 0,\quad (11.21)\]

where \(C_i\) \((i = 1 - 5)\) are the arbitrary constants. The problems at zeroth order are

\[(1 - \pi)\mathcal{L}_f \left[ \hat{f}(\eta; p) - f_0(\eta) \right] = ph_f\mathcal{N}_f \left[ \hat{f}(\eta; p) \right],\quad (11.22)\]
\[(1 - \pi)\mathcal{L}_\theta \left[ \hat{\theta}(\eta; p) - \theta_0(\eta) \right] = ph_\theta\mathcal{N}_\theta \left[ \hat{f}(\eta; p), \hat{\theta}(\eta, p) \right],\quad (11.23)\]

\[\hat{f}(0; p) = 0, \quad \hat{f}'(0; p) = c/a = \alpha^*, \quad \hat{f}'(\infty; p) = 1, \quad \hat{\theta}'(0, p) = -Bi[1 - \theta(0, p)], \quad \hat{\theta}(\infty, p) = 0,\quad (11.24)\]

\[\mathcal{N}_f[\hat{f}(\eta, p)] = (1 + \lambda) \left[ \hat{f}(\eta, p) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - 2 \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right)^2 + 2 \right] + \beta_3 \left[ \frac{3}{2} \left( \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right)^2 - \frac{1}{2} \left( \hat{f}(\eta, p) \frac{\partial^4 \hat{f}(\eta, p)}{\partial \eta^4} \right) + \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} \right] - 2(1 + \lambda)M^2 \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} - 1 \right),\quad (11.25)\]

\[\mathcal{N}_\theta[\hat{\theta}(\eta, p), \hat{f}(\eta, p)] = \left( 1 + \frac{4}{3} \right) \frac{\partial^2 \hat{\theta}(\eta, p)}{\partial \eta^2} + Pr \left( \hat{f}(\eta, p) \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} \right),\quad (11.26)\]

When \(p = 0\) and \(p = 1\) then one has

\[\hat{f}(\eta; 0) = f_0(\eta), \quad \hat{\theta}(\eta; 0) = \theta_0(\eta) \quad \text{and} \quad \hat{f}(\eta; 1) = f(\eta), \quad \hat{\theta}(\eta; 1) = \theta(\eta)\quad (11.27)\]

and when \(p\) variation is taken from 0 to 1 then \(f(\eta, p)\) and \(\theta(\eta, p)\) approach \(f_0(\eta), \theta_0(\eta)\) to \(f(\eta)\) and \(\theta(\eta)\). Now \(f\) and \(\theta\) in Taylor’s series can be expanded in the series

\[f(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m,\]
\[ \theta(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m, \quad (11.28) \]

\[ f_m(\eta) = \frac{1}{m!} \frac{\partial^m f(\eta; p)}{\partial \eta^m} \bigg|_{p=0}, \quad \theta_m(\eta) = \frac{1}{m!} \frac{\partial^m \theta(\eta; p)}{\partial \eta^m} \bigg|_{p=0}, \quad (11.29) \]

where the convergence depends upon \( h_f \) and \( h_\theta \). By proper choices of \( h_f \) and \( h_\theta \) the series (11.28) and (11.29) converge for \( p = 1 \) and so

\[ f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \quad (11.30) \]

\[ \theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta). \quad (11.31) \]

The \( m^{th} \)-order deformation problems are

\[ \mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f \mathcal{R}^m_f(\eta), \quad (11.32) \]

\[ \mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_\theta \mathcal{R}^m_\theta(\eta), \quad (11.33) \]

\[ f_m(0) = f'_m(0) = f'_m(\infty) = 0, \quad \theta'_m(0) - B_i \theta_m(0) = \theta_m(\infty) = 0, \quad (11.34) \]

\[ \mathcal{R}^m_f(\eta) = f''_{m-1}(\eta) + (1 + \lambda) \sum_{k=0}^{m-1} \left[ f''_{m-1-k} f_k'' - 2 f'_{m-1-k} f_k' + 2 \right] \\
+ \beta_3 \sum_{k=0}^{m-1} \left( \frac{3}{2} f''_{m-1-k} f_k'' - \frac{1}{2} f''_{m-1-k} f_k'' + f'_{m-1-k} f_k'' \right) \\
- 2M^2(1 + \lambda)(f'_m - 1), \quad (11.35) \]

\[ \mathcal{R}^m_\theta(\eta) = (1 + \frac{4}{3} R) \theta''_{m-1} + Pr \sum_{k=0}^{m-1} \theta'_{m-1-k} f_k, \quad (11.36) \]

\[ \chi_m = \begin{cases} 0, & m \leq 1, \\
1, & m > 1. \end{cases} \]

The general solutions of Eqs. (11.32) and (11.33) can be expressed by the following equations

\[ f_m(\eta) = f_m^*(\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta}, \quad (11.37) \]
\[ \theta_m(\eta) = \theta^*_m(\eta) + C_4 e^{\eta} + C_5 e^{-\eta}, \]

(11.38)

where \( f^*_m \) and \( \theta^*_m \) are the particular solutions and the constants \( C_i \) \( (i = 1 - 5) \) can be determined by the boundary conditions (11.34). These are given by

\[
\begin{align*}
C_2 &= C_4 = 0, \\
C_3 &= \left. \frac{\partial f^*(\eta)}{\partial \eta} \right|_{\eta=0}, \\
C_1 &= -C_3 - f^*(0), \\
C_5 &= \frac{1}{Bi + 1} \left[ \left. \frac{\partial \theta^*(\eta)}{\partial \eta} \right|_{\eta=0} - Bi \theta^*(0) \right].
\end{align*}
\]

11.5 Convergence of the series solutions

As pointed out by Liao [96], the non-zero auxiliary parameter \( \beta \) can adjust and accelerate the convergence of the series solutions. To obtain an appropriate value of such parameters the so-called \( \beta \)–curves have been plotted (see Figs. 11.2 and 11.3). The admissible ranges of \( \beta \) can be obtained from the line segment parallel to \( \beta \)–axis. It is found that the interval of convergence is \([-1.2, -0.5]\). The range for \( \beta \)–curve shrinks as \( \beta_3 \) is increased. This is because larger values of \( \beta_3 \) yields strongly nonlinear differential equation. Obviously one requires a higher order approximation for reasonably convergent series solution. Table 11.1 guarantees the convergence of homotopy solutions. It is clear that 20th-order approximations are sufficient for convergent series solution.
Fig. 11.2. The $h-$curves of $f''(0)$ and $\theta'(0)$.

$$\alpha^* = 1.2, \varepsilon_3 = 0.2, \lambda = 0.1, Bi = 0.3, Pr = 1.0, R = 0.1, M = 0.2$$

Fig. 11.3. The $h-$curves of $f''(0)$ and $\theta'(0)$.

$$\alpha^* = 0.9, \varepsilon_3 = 0.2, \lambda = 0.1, Bi = 0.3, Pr = 1.0, R = 0.1, M = 0.2$$
Table 11.1: Convergence of HAM solution for different order of approximations when $M = 0.2, \alpha^* = 0.9, \beta_3 = 0.2, \lambda = 0.1, \text{Pr} = 1, Bi = 0.3$ and $R = 0.2$.

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<th>$\theta'(0)$</th>
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</tbody>
</table>

Table 11.2: Convergence of HAM solutions for different order of approximations when $M = 0.2, \alpha^* = 1.2, \beta_3 = 0.3, \lambda = 0.1, \text{Pr} = 1, Bi = 0.3$ and $R = 0.1$.

<table>
<thead>
<tr>
<th>Order of approximation</th>
<th>$-f''(0)$</th>
<th>$\theta'(0)$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2754</td>
<td>0.2281</td>
</tr>
<tr>
<td>5</td>
<td>0.3694</td>
<td>0.2206</td>
</tr>
<tr>
<td>10</td>
<td>0.3614</td>
<td>0.2158</td>
</tr>
<tr>
<td>15</td>
<td>0.3823</td>
<td>0.2136</td>
</tr>
<tr>
<td>20</td>
<td>0.3822</td>
<td>0.2126</td>
</tr>
<tr>
<td>30</td>
<td>0.3822</td>
<td>0.2124</td>
</tr>
<tr>
<td>35</td>
<td>0.3822</td>
<td>0.2124</td>
</tr>
</tbody>
</table>

11.6 Results and discussion

The presented series solutions can be easily used to explore the effects of various interesting parameters on the velocity and temperature distributions.
11.6.1 Dimensionless velocity profile

The parameter $M$ measures the effect of applied magnetic field. It can take values in the range $0 < M < \infty$. The larger $M$, the stronger will be magnetic field strength. Figs. 11.4 and 11.5 indicate that presence of magnetic field creates a bulk known as Lorentz force which causes a huge restriction to the fluid velocity and, therefore decreases the momentum boundary layer. The parameter $\alpha^*$ compares the ratio of the stretching sheet velocity to the velocity of external stream. The larger values of $\alpha^*$ indicates larger stretching sheet velocity compared to the free stream velocity. It is clear from Figs. 11.6 and 11.7 that velocity field $f'$ is a strong function of ratio parameter $\alpha^*$ and it increases with an increase in $\alpha^*$. The boundary layer layer thickness decreases with an increase in $\alpha^*$ when $\alpha^* < 1$. The case $\alpha^* > 1$ represents the situation when velocity of the stretching sheet $U_w(x)$ exceeds the external stream velocity $U_\infty(x)$ and the flow has an inverted boundary layer structure. Moreover the boundary layer is not formed when $\alpha^* = 1$. Fig. 11.8 shows that velocity decreases and profiles move away from the bounding surface indicating an increase in the momentum boundary layer when $\alpha^* < 1$. On the other hand the boundary layer thins when $\beta_3$ is increased and $\alpha^* > 1$. This outcome is similar to that accounted in the case of second grade fluid. Irrespective of the chosen value of $\alpha^*$, the horizontal velocity increases by increasing the ratio $\lambda$. The boundary layer thickness decreases when $\alpha^* < 1$ and it increases for $\alpha^* > 1$ (see Fig. 11.9).

11.6.2 Dimensionless temperature profile

Fig. 11.10 displays the effect of Prandtl number Pr for a fixed value of convective heating parameter (Biot number) $Bi$. Specifically, Prandtl number $Pr = 0.72, 1.0$ and $7.0$ correspond to air, electrolyte solution such as salt water and water respectively. We have arbitrarily selected $Pr = 1$ to retrieve the graphical results. It is seen that temperature $\theta$ is a decreasing function of $Pr$. Physically a larger Prandtl number fluid has a relatively lower thermal conductivity. Thus an increase in Pr decreases conduction and thereby increases the variations of thermal characteristics. This results in the reduction of the thermal boundary layer thickness and an increase in the heat transfer rate at the bounding surface. As expected, the temperature significantly rises when the thermal radiation effect strengthens. The thermal boundary layer also grows when $R$ is increased (see Fig. 11.11). Fig. 11.12 plots the temperature distribution
versus the ratio parameter $\alpha^*$. It is seen that fluid’s temperature rises when the ratio parameter $\alpha^*$ becomes pronounced. The variation of temperature $\theta$ with an increase in $Bi$ can be seen in Fig. 11.13. Here a gradual increase in $Bi$ results in the larger convection at the stretching sheet which rises the temperature.

11.6.3 Dimensionless skin friction coefficient

The numerical values of dimensionless skin friction coefficient for various parametric values have been given in the Table 11.3. We noticed that skin friction coefficient increases with an increase in $M$ while it decreases when $\alpha^*$, $\lambda$ and $\beta_3$ are increased. In other words the power generation involved in stretching the sheet can be reduced by considering the fluids which display strong viscoelastic effects.

11.6.4 Dimensionless heat transfer rate

In Table 11.4 the values of dimensionless heat transfer rate at the boundary are computed. We have already witnessed in the graphical results that profiles become increasingly steeper by increasing $Pr$ and $Bi$. Thus local Nusselt number $\theta'(0)$, being proportional to the slope of temperature at $\eta = 0$, increases with an increase in $Pr$ and $Bi$. It is found that $|\theta'(0)|$ also increases with an increase in $\alpha^*$. This outcome leads to the conclusion that heat transfer rate at the sheet is enhanced by increasing the velocity of the stretching sheet. A slight variation in $|\theta'(0)|$ is noticed by increasing the viscoelastic parameter $\beta_3$. It can be clearly seen from Fig. 11.11 that slope of temperature function continuously decreases when the radiation effect intensifies. Thus magnitude of local Nusselt number increases when $R$ is increased.
Fig. 11.4. Influence of $M$ on $f'(\eta)$ (for opposing flow).

Fig. 11.5. Influence of $M$ on $f'(\eta)$ (for assisting flow).
Fig. 11.6. Influence of $\alpha^*$ on $f'(\eta)$ (for opposing flow).

Fig. 11.7. Influence of $\alpha^*$ on $f'(\eta)$ (for assisting flow).
Fig. 11.8. Influence of $\beta_3$ on $f'(\eta)$.

Fig. 11.9. Influence of $\lambda$ on $f'(\eta)$.
Fig. 11.10. Influence of Pr on $\theta(\eta)$.

Fig. 11.11. Influence of $R$ on $\theta(\eta)$. 
Fig. 11.12. Influence of $\alpha^*$ on $\theta(\eta)$. 

Fig. 11.13. Influence of $Bi$ on $\theta(\eta)$. 

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Table 11.3: Values of skin friction coefficient $Re_x^{1/2} C_f$ for the parameters $M$, $\alpha^*$, $\lambda$ and $\beta_3$.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$\alpha^*$</th>
<th>$\lambda$</th>
<th>$\beta_3$</th>
<th>$Re_x^{1/2} C_f$</th>
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Table 11.4: Values of local Nusselt number $Nu/Re_x^{1/2}$ for the parameters $Bi$, $\alpha^*$, $\lambda$, $Pr$, $S$ and $\beta_3$.

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<th>$Bi$</th>
<th>$\alpha^*$</th>
<th>$\lambda$</th>
<th>$Pr$</th>
<th>$\beta_3$</th>
<th>$M$</th>
<th>$R$</th>
<th>$Nu/Re_x^{1/2}$</th>
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11.7 Concluding remarks

Heat transfer characteristics in the MHD stagnation-point flow of Jeffrey fluid toward a stretching surface with convective condition are studied. The convergent expressions of velocity and temperature are constructed. It is seen that for sufficiently large Biot number the analysis for constant wall temperature case can be recovered. The velocity ratio ($\alpha^*$) has a dual behavior
on the momentum boundary layer and the skin friction coefficient. We found that temperature \( \theta \) decreases with an increase in Pr. This decrease is associated with the larger rate of heat transfer at the sheet. Surface heat transfer rate also enhances by increasing convective heating at the sheet. The present findings for the case of Newtonian fluid can be recovered by choosing \( \beta_3 = 0 \).
Chapter 12

Radiative flow of Jeffrey fluid through a convectively heated stretching cylinder

12.1 Introduction

This chapter studied the two-dimensional flow of Jeffrey fluid by an inclined stretching cylinder with convective boundary condition. In addition the combined effects of thermal radiation and viscous dissipation are taken into consideration. The developed nonlinear partial differential equations are reduced into the ordinary differential equations. The governing equations are solved for the series solutions. The convergence of the series solutions for velocity and temperature fields are carefully analyzed. The effects of various physical parameters such as ratio of relaxation to retardation times, Deborah number, radiation parameter, Biot number, curvature parameter, mixed convection parameter, Prandtl number, Eckert number and angle of inclination are examined through graphical and numerical results of the velocity and temperature distributions.
12.2 Flow equations

We consider the steady two-dimensional flow of an incompressible Jeffrey fluid by an inclined stretching cylinder. We also considered thermal radiation and viscous dissipation effects. The convective boundary condition for a cylinder is employed. The stretching cylinder makes an angle $\alpha$ with vertical axis i.e. r-axis. Here x-axis is taken normal to the r-axis.

The relevant boundary layer equations are

$$\frac{\partial (ru)}{\partial x} + \frac{\partial (rv)}{\partial r} = 0,$$  \hspace{1cm} (12.1)

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = \frac{v}{1+\lambda} \left[ \frac{\partial^2 u}{\partial x^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} \frac{\partial^2 u}{\partial x \partial r} + \frac{\partial^3 u}{\partial x^2 \partial r} \right] + g_0 \beta_T (T - T_\infty) \cos \alpha,$$ (12.2)

$$\rho c_p \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial r} \right) = \frac{k}{1+\lambda} \left[ \frac{\partial^2 u}{\partial r^2} + \frac{16 \sigma T^3}{3 k^*} \left( \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} \right) \right] + \frac{\mu}{1+\lambda} \left[ \frac{\partial^2 u}{\partial r^2} + \frac{\partial^3 u}{\partial x \partial r^2} \right],$$ (12.3)

$$u = U_w = \frac{U_0 x}{l}, \quad v = 0, \quad -k \frac{\partial T}{\partial r} = h(T_f - T) \quad \text{at} \quad r = a,$$ (12.4)

$$u \to 0, \quad T \to T_\infty \quad \text{as} \quad r \to \infty,$$ (12.5)

in which $u$ and $v$ represent the velocity components along the $x$ and $r$ directions respectively, $U_0$ is the reference velocity, $\lambda$ is the ratio of relaxation to retardation times, $\lambda_3$ is the retardation time, $l$ is the characteristic length, $g_0$ is the acceleration due to gravity, $\beta_T$ is the volumetric coefficient of thermal exponential, $T$ is the fluid temperature, $\sigma$ is the thermal diffusivity of the fluid, $\nu = (\mu/\rho)$ is the kinematic viscosity, $\rho$ is the density of fluid, $k$ is the thermal conductivity of fluid, $k^*$ is the mean absorption coefficient, $h$ is the convective heat transfer coefficient and $T_f$ is the convective fluid temperature. Note that the thermal radiation term in Eq. (12.3) is written using Rosselands’ approximation. It should be pointed out that convection condition in Eq. (12.4) represents the heat transfer rate through the surface being proportional to the local
difference in temperature with the ambient conditions. Such configuration occurs in several engineering devices for instance in heat exchangers, where the conduction in the solid tube wall is greatly influenced by the condition in the fluid flowing over it.

We introduce the similarity transformations in the forms

\[
\begin{align*}
  u &= \frac{xU_0}{l} f'(\eta), \quad v = -\frac{a}{r} \left( \frac{\nu U_0}{l} \right)^{1/2} f(\eta), \\
  \theta(\eta) &= \frac{T - T_\infty}{T_f - T_\infty}, \quad \eta = \frac{r^2 - a^2}{2a} \left( \frac{U_0}{\nu l} \right)^{1/2}.
\end{align*}
\]

(12.6)

With the help of above transformations, Eq. (12.1) is identically satisfied and Eqs. (12.2 – 12.5) yield

\[
\begin{align*}
  (1 + 2\gamma\eta) f''' + 2\gamma f'' + (1 + \lambda)(ff'' - f'^2) + \beta_3\gamma (f' f'' - 3f f''') \\
  + \beta_3(1 + 2\gamma\eta)(f''^2 - f f''') + G \theta \cos \alpha = 0, \\
  (1 + \frac{4}{3}R) \left( (1 + 2\gamma\eta) \theta'' + 2\gamma \theta' \right) + \text{Pr} f \theta' + \text{Pr}E_c \frac{1}{1 + \lambda} \left[ (1 + 2\gamma\eta) f''^2 \\
  + \beta_3 \left\{ (1 + 2\gamma\eta) \left( ff'' - f f''^2 - \gamma f f''' \right) \right\} \right] = 0, \\
  f = 0, \quad f' = 1, \quad \theta' = -Bi[1 - \theta(0)] \quad \text{at} \ \eta = 0, \\
  f' = 0, \quad \theta = 0 \quad \text{at} \ \eta = \infty,
\end{align*}
\]

(12.7)

(12.8)

where \( \beta_3 = \frac{\lambda l U_0}{a^2} \) is a Deborah number, \( \gamma = \sqrt{\frac{\nu}{a^2 U_0}} \) is a curvature parameter, \( \text{Pr} = \frac{k}{\sigma} \) is the Prandtl number, \( R = \frac{4a^3}{kT_{\infty}^2} \) is a radiation parameter, \( E_c = \frac{U_0^2 (a/l)^2}{c_p (T_f - T_{\infty})} \) is the Eckert number, \( G = \frac{\beta_3 a (T_f - T_{\infty}) a^3}{U_0^2 (a/l)^2 a^2/\nu} \) is the mixed convection parameter and \( Bi = \frac{h_r}{k} \sqrt{\frac{U_0}{\nu a}} \) is the Biot number.

The skin friction coefficient \( C_f \) and local Nusselt number \( Nu_x \) are

\[
\begin{align*}
  C_f &= \frac{\tau_w}{\frac{1}{2} \rho U_{aw}^2}, \quad Nu_x = \frac{x q_w}{k (T_f - T_{\infty})},
\end{align*}
\]

(12.10)
where $\tau_w$ and $q_w$ are
\begin{align*}
\tau_w &= \frac{\mu}{1 + \lambda} \left[ \left( \frac{\partial u}{\partial r} \right) + \lambda_3 \left\{ u \frac{\partial^2 u}{\partial x \partial r} + v \frac{\partial^2 u}{\partial r^2} \right\} \right]_{r=a}, \\
q_w &= -k \left( \frac{\partial T}{\partial r} \right)_{r=a} - \frac{16\sigma T^4}{3k^*} \left( \frac{\partial T}{\partial r} \right)_{r=a}.
\end{align*}

Dimensionless form of skin friction coefficient $C_f$ and local Nusselt number $Nu_x$ are
\begin{equation}
C_f = 2 \left( Re_x \right)^{-1/2} \left[ \frac{1}{1+\lambda} (1 + \beta_4)f''(0) \right], \quad Nu_x = \frac{1}{Re_x^{1/2}} = -(1 + \frac{4}{3} R) \hat{\theta}'(0).
\end{equation}

### 12.3 Homotopy analysis solutions

Initial approximations and auxiliary linear operators are chosen as follows:
\begin{align*}
f_0(\eta) &= 1 - e^{-\eta}, \\
\theta_0(\eta) &= Bi \exp(-\eta) + Bi, \\
L_f &= f''' - f', \\
L_\theta &= \theta'' - \theta.
\end{align*}

The operators satisfy the following properties
\begin{align*}
L_f(C_1 + C_2 e^\eta + C_3 e^{-\eta}) &= 0, \\
L_\theta(C_4 e^\eta + C_5 e^{-\eta}) &= 0,
\end{align*}

where $C_i (i = 1 - 5)$ are the arbitrary constants determined from the boundary conditions. If $p \in [0, 1]$ denotes an embedding parameter, $h_f$ and $h_\theta$ the non-zero auxiliary parameters then the zeroth order deformation problems are
\begin{align*}
(1 - p) L_f \left[ \hat{f}(\eta; p) - f_0(\eta) \right] &= ph_f N_f \left[ f(\eta; p) \right], \\
(1 - p) L_\theta \left[ \hat{\theta}(\eta; p) - \theta_0(\eta) \right] &= ph_\theta N_\theta \left[ f(\eta; p), \hat{\theta}(\eta; p) \right],
\end{align*}
\[ \hat{f}(0; p) = 0, \hat{f}'(0; p) = 1, \hat{f}'(\infty; p) = 0, \hat{\theta}'(0; p) = -Bi[1 - \theta(0; p)], \hat{\theta}(\infty; p) = 0, \quad (12.20) \]

\[
N_f[\hat{f}(\eta, p)] = (1 + 2\gamma \eta) \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} + 2\gamma \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} + (1 + \lambda) \left[ \hat{f}(\eta, p) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right)^2 \right] \\
+ \beta_3 \gamma \left[ \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - 3\hat{f}(\eta, p) \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} \right] \\
+ \beta_3 (1 + 2\gamma \eta) \left( \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - \hat{f}(\eta, p) \frac{\partial^4 \hat{f}(\eta, p)}{\partial \eta^4} \right) + G\theta \cos \alpha, \quad (12.21) \]

\[
N_\theta[\hat{\theta}(\eta, p), \hat{f}(\eta, p)] = \left( 1 + \frac{4}{3} R \right) \left( 1 + 2\gamma \eta \right) \frac{\partial^2 \hat{\theta}(\eta, p)}{\partial \eta^2} + 2\gamma \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} + Pr \left( \hat{f}(\eta, p) \frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} \right) \\
+ Pr Ec \frac{1}{1 + \lambda} \left[ (1 + 2\gamma \eta) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} + \beta_3 \left\{ (1 + 2\gamma \eta) \left( \hat{f}(\eta, p) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right) \\
- \hat{f}(\eta, p) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \left( \frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right) \right\} \right] - \gamma \hat{f}(\eta, p) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right], \quad (12.22) \]

When \( p = 0 \) and \( p = 1 \) then one can easily write

\[ \hat{f}(\eta; 0) = f_0(\eta), \hat{\theta}(\eta, 0) = \theta_0(\eta) \quad \text{and} \quad \hat{f}(\eta; 1) = f(\eta), \hat{\theta}(\eta, 1) = \theta(\eta). \quad (12.23) \]

When \( p \) variation is taken from 0 to 1 then \( f(\eta, p) \) and \( \theta(\eta, p) \) approach \( f_0(\eta), \theta_0(\eta) \) to \( f(\eta) \) and \( \theta(\eta) \). Now \( f \) and \( \theta \) in Taylor’s series can be expanded as follows:

\[ f(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \quad (12.24) \]

\[ \theta(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m, \quad (12.25) \]

\[ f_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m f(\eta; p)}{\partial \eta^m} \right|_{p=0}, \quad \theta_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \theta(\eta; p)}{\partial \eta^m} \right|_{p=0}, \quad (12.26) \]
where the convergence depends upon $h_f$ and $h_\theta$. By proper choices of $h_f$ and $h_\theta$ the series (12.29) and (12.30) converge for $p = 1$ and hence

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta),$$  \hspace{1cm} (12.27)

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta).$$  \hspace{1cm} (12.28)

The $m^{th}$-order deformation problems are

$$\mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f \mathcal{R}^m_f(\eta),$$  \hspace{1cm} (12.29)

$$\mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_\theta \mathcal{R}^m_\theta(\eta),$$  \hspace{1cm} (12.30)

$$f_m(0) = f'_m(0) = f'_m(\infty) = 0, \quad \theta'_m(0) - B \theta_m(0) = \theta_m(\infty) = 0, \hspace{1cm} (12.31)$$

$$\mathcal{R}^m_f(\eta) = (1 + 2\gamma \eta) f'''_{m-1}(\eta) + 2\gamma f''_{m-1} - (1 + \lambda) \sum_{k=0}^{m-1} \left[ f_{m-1-k} f'_k - f'_{m-1-k} f'_k \right]$$

$$+ \beta_3 \gamma \sum_{k=0}^{m-1} (f''_{m-1-k} f''_k - 3 f_{m-1-k} f'''_k)$$

$$+ \beta_3 (1 + 2\gamma \eta) \sum_{k=0}^{m-1} (f'''_{m-1-k} f'_k - f'_{m-1-k} f'''_k) + G \theta_{m-1} \cos \alpha, \hspace{1cm} (12.32)$$

$$\mathcal{R}^m_\theta(\eta) = (1 + \frac{4}{3} R) \left[ (1 + 2\gamma \eta) \theta'''_{m-1} + 2\gamma \theta''_{m-1} \right] + \text{Pr} \left( \sum_{k=0}^{m-1} \theta'_{m-1-k} f_k \right)$$

$$+ \text{Pr} \frac{E_i}{1 + \lambda} \left[ (1 + 2\gamma \eta) \sum_{k=0}^{m-1} (f''_{m-1-k} f'_k) + \beta_3 \left\{ (1 + 2\gamma \eta) \sum_{k=0}^{m-1} (f''_{m-1-k} f''_k) \right\} \right.$$

$$\left. - f_{m-1-k} \sum_{l=0}^{k} f'_{k-l} f''_p \right) - \gamma \sum_{k=0}^{m-1} (f_{m-1-k} f'''_k), \hspace{1cm} (12.33)$$

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

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The general solutions of Eqs. (12.33) and (12.34) are given by

\[
\begin{align*}
f_m(\eta) &= f^*_m(\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta}, \\
\theta_m(\eta) &= \theta^*_m(\eta) + C_4 e^\eta + C_5 e^{-\eta},
\end{align*}
\]

where \( f^*_m \) and \( \theta^*_m \) are the particular solutions.

### 12.4 Convergence analysis

Convergence of series solution strongly depends upon the non-zero auxiliary parameters \( h_f \) and \( h_\theta \). For this purpose we plot the \( h \) – curves for the velocity \( f''(0) \) and temperature \( \theta'(0) \) fields. Figs. 12.1 and 12.2 depict the \( h \) – curves for different values of physical parameters. Here admissible range of \( h_f \) thus is \(-1.3 < h_f < -0.1\) and for \( h_\theta \) it is \(-1.2 < h_\theta < -0.5\). The auxiliary parameter \( h \) can be regarded as an iteration factor that is usually used in numerical computations. It is well known that a properly chosen iteration factor can ensure the convergence of iteration. In fact the mentioned convergence interval are the values for which the plots of physical quantities \( f''(0) \) and \( \theta'(0) \) are parallel to horizontal axis [96]. It is found that the series solutions converge in the whole region of \( \eta \) when \( h_f = -1.1 \) and \( h_\theta = -0.9 \). Table 12.1 guarantees the convergence of HAM solutions. It is obvious that 25th-order approximations are enough for the convergent series solutions.
Fig. 12.1. $h-$ curve for the velocity field.

Fig. 12.2. $h-$ curve for the temperature field.
Table 12.1: Convergence of HAM solution for different order of approximations when $\beta_3 = 0.2$, $\lambda = 0.3$, $\alpha = \pi/6$, $G = 0.1$, $Pr = 1, \gamma = R = 0.1, Ec = 0.2$ and $Bi = 0.3$.

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<th>$-\theta'(0)$</th>
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12.5 Result and discussion

Here the nonlinear analysis is computed for the homotopy solutions. The presented analytically solutions can be easily used to explore the influence of various physical parameters on the velocity and temperature fields.

12.5.1 Dimensionless velocity profile

The velocity field is illustrated in Fig. 12.3 for different values of $\lambda$ (ratio of relaxation to retardation times). The horizontal velocity decreases for large values of $\lambda$ with the increasing distance ($\eta$) of the cylinder. It is because of the fact that $\lambda$ being the viscoelastic parameter exhibits both viscous and elastic characteristics. Thus the fluids always retard whenever viscosity or elasticity increases. The velocity increases with increasing values of $\beta_3$ (see Fig. 12.4). As in the case of polymers, a high Deborah number means the flow is strong enough that the polymer becomes highly oriented in one direction and stretched. For most flows this occurs when the time it takes for the polymer to relax is long compared to the rate at which the flow is deforming it. A very small Deborah number can be obtained for a fluid with extremely small relaxation time if the Deborah number is very large then fluid behaves like a solid. With the
increasing $\alpha$, the horizontal velocity is found to decrease because the effect of buoyancy force increases due to thermal decrease (see Fig. 12.5). The graph of mixed convection parameter $G$ on the velocity field $f'(\eta)$ is displayed in Fig. 12.6. It is clearly seen from this Fig. that for large values of $G$ shows that the velocity field increases. The positive values of $G$ cool the wall or heat the fluid. When $G \gg 1$, the viscous force is negligible when compared to the buoyancy and inertial forces. When buoyant forces overcome the viscous forces, the flow starts a transition to the turbulent regime. In Fig. 12.7 the horizontal velocity profile is shown for different values of curvature parameter $\gamma$. The effect of curvature parameter $\gamma$ near the cylinder wall is negligible but far away it increases for larger values of $\gamma$. This is due to the fact that larger values of curvature parameter $\gamma$ leads to increase the radius of cylinder.

### 12.5.2 Dimensionless temperature profile

Fig. 12.8 depicts the influence of $\lambda$ on temperature field $\theta(\eta)$. Here temperature field increases rapidly for larger values of $\lambda$ (ratio of relaxation to retardation times). Fig. 12.9 depicts the variations of Deborah number $\beta_3$ on $\theta$. The temperature profile $\theta$ is decreasing function of $\beta_3$. In fact small Deborah numbers correspond to situations where the material has time to relax while high Deborah numbers represents the cases where the material behaves rather elastic. Fig. 12.10 illustrates the effect of curvature parameter $\gamma$ on temperature field. Through an increase in $\gamma$ the temperature increases and maximum temperature is found near the wall. Figs. 12.11 and 12.12 show the effect of Prandtl number $Pr$ and Eckert number $Ec$ on temperature field. The temperature $\theta(\eta)$ and thermal boundary layer thickness are decreased via increase in $Pr$. The higher Prandtl number fluid has a lower thermal conductivity (or a higher viscosity) which results in thinner thermal boundary layer and hence higher heat transfer rate at the surface. Whereas opposite behavior is noted for Eckert number. Physically this is due to the reason that the fluid takes the kinetic energy from the motion of the fluid and transforms it into internal energy. Therefore the fluid temperature rises. Obviously this process is more pronounced when the fluid is more viscous or it is flowing at high speed. Fig. 12.13 demonstrates the effect of thermal radiation parameter $R$ on temperature field $\theta(\eta)$. It is noted that thermal radiation increases the temperature field and associated thermal boundary layer thickness. Physically the larger values of $R$ imply larger surface heat flux and thereby the fluid become warmer which
enhances the temperature profile. Also the effect of thermal radiation in thermal boundary layer is equivalent to that an increased thermal diffusivity in Prandtl number. Fig 12.14 depicts the effect of Biot number $Bi$ on $\theta(\eta)$. Biot number is a ratio of the temperature fall in the solid material and the temperature fall the solid and the fluid. So for smaller Biot number, most of the temperature drop is in the fluid and the solid may be considered isothermal. Therefore Biot number enhances both the temperature field and thermal boundary layer thickness. Here we considered the Biot number less than one due to uniformity of the temperature field in the fluid. A lower Biot number means that the conductive heat transfer is much faster than the convective heat transfer.

12.5.3 Dimensionless skin friction coefficient

Table 12.2 shows the numerical values of dimensionless skin friction coefficient for various physical parameters. We notice that skin friction coefficient decreases with an increase in $\lambda$ and $\beta_3$. We see that when $\gamma$ increases then the viscous forces start decreasing and consequently the skin friction coefficient increases. Skin friction coefficient increases for large values of angle of inclination $\alpha$.

12.5.4 Dimensionless Nusselt number

In Table 12.3 the values of dimensionless Nusselt number are computed. It is seen that Nusselt number is positive therefore heat transfer is from hot fluid to the sheet. Fluid parameters $\lambda$ and $\beta_3$ increase the local Nusselt number at the wall. Thus local Nusselt number increases with an increase in Pr and $R$. Hence Prandtl number can be used to adjust the rate of cooling in conducting flows. It is found that Nusselt number also increases when $\alpha$ is increased. A slight variation in local Nusselt number is noticed by increasing the mixed convection parameter $G$. It is found that slope of temperature function continuously increases when the convective heat transfer intensifies. Thus local Nusselt number enhances when $Bi$ is increased. As curvature parameter $\gamma$ increases, the local Nusselt number is increased.
Fig. 12.3. Influence of $\beta_3$ on $f'(\eta)$.

$\beta_3 = 0.2, \alpha = \pi/6, G = G = 0.1$

$\lambda = 0.0, \lambda = 0.4, \lambda = 0.8, \lambda = 1.0$

Fig. 12.4. Influence of $\beta_3$ on $f'(\eta)$.

$\lambda = 0.3, \alpha^* = \pi/6, G = G = 0.1$

$\beta_3 = 0.0, \beta_3 = 0.5, \beta_3 = 0.9, \beta_3 = 1.5$

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Fig. 12.5. Influence of $\alpha$ on $f'(\eta)$.

Fig. 12.6. Influence of $G$ on $f'(\eta)$.
Fig. 12.7. Influence of $\gamma$ on $f'(\eta)$.

Fig. 12.8. Influence of $\lambda$ on $\theta(\eta)$.
Fig. 12.9. Influence of $\beta_3$ on $\theta(\eta)$.

Fig. 12.10. Influence of $\gamma$ on $\theta(\eta)$. 

$\lambda = 0.4$, $\alpha = \pi/6$, $G = \gamma = R = 0.1$, $Pr = 1.0$, $Ec = 0.2$, $Bi = 0.3$

$\beta_3 = 0.0$, $\beta_3 = 0.4$, $\beta_3 = 0.8$, $\beta_3 = 1.0$

$\gamma = 0.0$, $\gamma = 0.1$, $\gamma = 0.2$, $\gamma = 0.3$
Fig. 12.11. Influence of $Pr$ on $\theta(\eta)$.

Fig. 12.12. Influence of $Ec$ on $\theta(\eta)$.
Fig. 12.13. Influence of $R$ on $\theta(\eta)$.

Fig. 12.14. Influence of $Bi$ on $\theta(\eta)$.  

$\lambda = 0.3, \beta_3 = 0.2, \alpha = \pi/6, G = \gamma = 0.1, $  
$Pr = 1.0, Ec = 0.2, Bi = 0.3$  

$R = 0.0, R = 0.1, R = 0.2, R = 0.3$  

$\lambda = 0.3, \beta_3 = 0.2, \alpha = \pi/6, G = R = 0.1, $  
$\gamma = 0.1, Pr = 1.0, Ec = 0.2$  

$Bi = 0.0, Bi = 0.3, Bi = 0.6, Bi = 1.0$
Table 12.2: Values of skin friction coefficient $Re_2^{1/2} C_f$ for the parameters $\lambda$, $\beta_3$, $\alpha$, and $\gamma$.

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Table 12.3: Values of local Nusselt number $Nu/Re_x^{1/2}$ for the parameters $\beta_3 = 0.2$, $\lambda = 0.3$, $\alpha = \pi/6$, $G = 0.1$, $Pr = 1$, and $\gamma = R = 0.1$.

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12.6 Final remarks

Heat transfer characteristics in the flow of Jeffrey fluid with convective boundary condition are explored. The effects of thermal radiation and viscous dissipation are taken into account. It is seen that the curvature of stretching cylinder is essential parameter for the velocity and temperature fields. Curvature parameter $\gamma$ enhances the velocity as well as the temperature of fluid. Fluid parameters $\lambda$ (ratio of relaxation to retardation times) and $\beta_3$ (Deborah number) have opposite behavior on the velocity and temperature fields. The local Nusselt number increases for sufficiently large Biot number $Bi$. Radiation parameter $R$ and Eckert number $Ec$ also accelerate the temperature field. Thermal boundary layer thickness reduces for larger values of Prandtl number (Pr).
Bibliography


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