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List of Abbreviations

\begin{itemize}
  \item \( p \) Acoustic pressure
  \item \( \rho_0 \) Density in equilibrium state
  \item \( \phi(x, y) \) Potential function
  \item \( \vec{u} \) The velocity vector
  \item \( \omega \) Angular frequency
  \item \( C \) Speed of sound
  \item \( k \) Wave number
  \item \( \nabla^2 \) Two dimensional Laplacian operator
  \item \( a, b, c \) and \( d \) Dimensions of ducts
  \item \( B_n^\pm, C_n^\pm \) and \( A_n, B_n \ldots F_n \) Unknown amplitudes
  \item \( A_n^s/a \) and \( B_n^s/a \) Symmetric/antisymmetric unknown amplitudes
  \item \( R \) Reflection amplitude
  \item \( l \) Length of inner duct
  \item \( \phi(s/a)(x, y) \) Symmetric and antisymmetric Potential function
  \item \( \alpha_n \) Eigenvalues
  \item \( \beta_n \) Eigenvalues
  \item \( \gamma_n \) Eigenvalues
  \item \( \omega_n \) Eigenvalues
  \item \( \beta_n^\pm \) Eigenvalues
  \item \( \alpha_n \) Associative Eigenvalues
  \item \( \beta_n \) Associative Eigenvalues
  \item \( \gamma_n \) Associative Eigenvalues
  \item \( \omega_n \) Associative Eigenvalues
  \item \( \beta_n^\pm \) Associative Eigenvalues
  \item \( \psi_n(y) \) Eigenfunctions
  \item \( \xi_n(y) \) Eigenfunctions
  \item \( \eta_n(y) \) Eigenfunctions
  \item \( \sigma_n(y) \) Eigenfunctions
  \item \( \zeta_n(y) \) Eigenfunctions
  \item \( \epsilon_n(y) \) Eigenfunctions
  \item \( \xi_n^\pm(y) \) Eigenfunctions
  \item \( L_{mn} \) Inner product between eigenfunctions \( \psi_m(y) \) and \( \epsilon_n(y) \)
  \item \( Q_{mn} \) Inner product between eigenfunctions \( \xi_m(y) \) and \( \epsilon_n(y) \)
  \item \( R_{mn} \) Inner product between eigenfunctions \( \sigma_m(y) \) and \( \epsilon_n(y) \)
  \item \( S_{mn} \) Inner product between eigenfunctions \( \eta_m(y) \) and \( \epsilon_n(y) \)
  \item \( T_{mn} \) Inner product between eigenfunctions \( \psi_m(y) \) and \( \xi_2m(y) \)
  \item \( l_{mn} \) Inner product between eigenfunctions \( \psi_m(y) \) and \( \xi_2m(y) \)
  \item \( p_{mn} \) Inner product between eigenfunctions \( \psi_m(y) \) and \( \xi_2m(y) \)
  \item \( P_{mn}^+ \) Inner product between eigenfunctions \( \psi_m(y) \) and \( \xi_2m(y) \)
  \item \( R_0^m(k) \) The far field reflection for the first mode
  \item \( T_0^m(k) \) The far field transmission for the first mode
\end{itemize}
Chapter 1

General Introduction

Noise pollution is an unwanted or disturbing sound which can interfere with normal activities for humans and wildlife, such as sleeping, conversation, reproduction, communication, or disrupt or diminish one’s quality of life. Noise is a common issue of modern world and is a grave hazard to healthy environment [19]. Frequent sources of unwanted noise such as vehicle, aero-engines, heating, ventilation, and air conditioning systems contribute a lot in environmental nuisance. Study of acoustic exhaustion has manifested an increasing keen attention during the last decades. Numerous kind of methods have been proposed for noise attenuation as well as vast literature is available on the issue. Amongst these developments, the waveguide diffraction is one of the most popular method. Factory machines performing at optimum speeds having huge sound force are inclined to produce notable noise even in areas away from source through air ducts which behave as a waveguide for propagation of sound. Sound absorptive duct linings and proper geometrical duct patterns can be beneficial to reduce the intensity of this unwanted sound. Mathematical determination of the wave propagation has application in many disciplines of physical importance in the field of acoustics. Here we mention some of the distinct works that took place in the field of propagation and scattering of acoustic waves.

Scattering of acoustic waves was first investigated mathematically in 1877 by Lord Rayleigh with the assumption that scatterers are small as compared to wavelength. The solution of scattering by rigid, immovable circular cylinder and spheres, not necessarily small as compared to wave-length was given by Morse [11]. Bowmen et. al. [1] presented a comprehensive collection of results for electromagnetic and acoustic waves dispersion by finite, semi finite and infinite frames. McIver [2] has tackled the two dimensional wave scattering model with application to water waves. He discovered that for sufficiently closely spaced barriers each zero of reflection is accompanied by a zero of transmission at a nearby frequency. Ammari et al. [21, 22] studied the electromagnetic waves scattering by a thin planar model. Nawaz and Ayub [23] have solved wave diffraction problem in the complex domain. They analyzed the problem of wave scattering of an electromagnetic field in a homogeneous bi-isotropic medium by a perfectly conducting strip. Acoustic wave scattering/diffraction by a half plane is one of the significant and simple problem which has been studied by many scientists [3, 7, 4, 6]. Jones [8] was the first to study the problem of $3 \times 3$ matrix Wiener-Hopf equations which are resulted
by the scattering of a wave in three equally spaced half planes. Nilsson and Brander [9, 10] presented the solution of diffraction problem of sound in a cylindrical duct having bulk-reacting lining, consisting of sound-absorbing material behind a perforate. Their model incorporates a uniform flow of the gas. Tayyar et al. [16] have used a rigorous Wiener-Hopf approach to investigate the band-stop filter characteristics of a parallel plate waveguide with finite length impedance loading. The representation of the solution to the boundary-value problem in terms of Fourier integrals leads to two simultaneous modified Wiener-Hopf equations which are uncoupled by using the pole removal technique.

Electromagnetic plane wave (harmonic in time) which is incident on a cavity has been studied by Ammari et al. [25, 26] by using integral and variational techniques. Ammari et al. [29] examined boundaries to rebuild the spatial aid of noise sources by using method of cross correlation. Same approach has been used by Wahab and Rab Nawaz [31] for elastic media.

Many researchers have used different types of parallel plates to attenuate the sound within the waveguide. Rawlins [6] and Hassan [3] studied bifurcated and trifurcated duct problems respectively. Numerous researchers like [50, 4, 5, 6, 8, 9, 10, 16] applied Wiener Hopf technique. Wiener Hopf technique is based on the application of integral transforms, analytic continuation of complex valued functions and residue theory. In this method boundary value problem is transformed to Weiner Hopf functional equations/matrix by using integral transforms (formulas for matrix factorization are very limited). Wiener Hopf equation involves two unknown functions of a complex variable which are analytic in two overlapping half planes. This provides sufficient information to apply the theory of analytic continuation. An important step in this method is to decompose the kernel functions into a product and sum of two functions in appropriate half planes. These decomposed/split functions involve infinite products which are very difficult to cope with. There exists an alternate simple and direct method known as mode matching technique/eigenfunction expansion method which is practically more advantageous. In this method, we first find eigenmodes of each region and these eigenmodes are then matched at all junction discontinuities. This method can be employed to the geometries in which method of separation of variables is applicable. In this method formula for reflection field amplitude is simple which is difficult to obtain and generalize otherwise. This method has been applied in several diffraction problems, [13, 14, 15, 17, 18, 20, 30, 32] and has been discovered simple to generalize, implement, and to give precise results.

Here in our work, we present mathematical modeling for different geometries using eigenfunction expansion. In first chapter, we give the basic definitions, mathematical preliminaries and fundamental equations which would be beneficial to understand the contents in subsequent chapters. We divide rest of thesis into two parts:

**Part 1:** Pentafurcated Duct Problems.

**Part 2:** Extraordinary Acoustic Transmission (EAT) and Helmholtz Resonators.

In part 1, we consider the acoustic diffraction of a fundamental plane wave mode which propagates out of the open end of a middle semi-infinite duct. This semi-infinite duct is symmetrically located in the infinite duct. The whole system forms a pentafurcated duct whose solution is presented by taking different nature
of plates. We further divide part 1 into two chapters 3 and 4. In chapter 3 a pentafurcated duct having all hard plates is taken while in chapter 4 a pentafurcated duct having soft outer lining is considered. We divide both pentafurcated ducts into six regions. By using separation of variables, potentials in these regions are calculated. We then apply mode matching technique to get an infinite system of linear equations involving unknown coefficients which are determined by MATLAB programming. In the end we will compare the results of both ducts with each other and with related existing work [48, 56, 57]. This comparison would help one to design the practically efficient acoustic exhaust models. The results will be helpful for sound attenuating gadgets (e.g., silencers, band-stop filters etc.) We also present energy balance formula and the impact of the spacing of duct on reflection field amplitude.

In part 2, the phenomenon of extraordinary acoustic transmission (EAT) in a Helmholtz resonator, which has recently been investigated experimentally, is studied theoretically. Extraordinary transmission is a phenomenon in which anomalous transmission occurs, sometimes perfect transmission, for a geometry which should be strongly reflecting. It occurs only for specific frequencies and it is associated with resonances in the response.

Extraordinary optical transmission (EOT) was discovered first in [33] and has been the subject of significant study, e.g. [34, 35]. Recently the phenomena of extraordinary acoustic transmission (EAT) has been examined and unsurprisingly, given the similar nature of the equations which govern electromagnetic waves and acoustic waves, many examples of EAT have been found, for example [36, 37, 38, 39, 40]. Lately the problem of EAT for the case of a Helmholtz resonator has been studied experimentally [41, 42]. These experiments are the starting point for our current investigation.

A Helmholtz resonator is a resonant cavity with a small connection to the outside. There is no strict definition about what constitutes small in this context and the definition is somewhat vague. However, it seems clear that the phenomena of EAT is closely associate with Helmholtz type resonators. For our current purposes we think of a Helmholtz resonator as a perturbed self-adjoint operator in which, through the connection with infinity, the real eigenvalue of the self-adjoint operator has become a complex resonance, that is a singularity of the analytic extension of the resolvent, with small imaginary part. This definition is not geometric and allows us to consider very simple geometries as Helmholtz resonators because they have complex resonances close to the real axis. Such methods have found wide spread application in many areas of wave scattering and they are known as singularity expansion methods [43, 44, 45].

The idea of computing the analytic extension to understand wave characteristics for real frequencies has recently been applied to investigate absorption by subwavelength resonances [44, 45, 46]. In this work the solution was visualized in the complex frequency plane and the position of the singularities and zeros examined. We perform here a similar analysis. In the case considered by [44, 45, 46] the analytic extension was more straightforward because a single mode approximation was made. For our problem we use bespoke computer code which allows the construction of the analytic extension of the solution to complex wavenumbers or complex frequencies without making a single mode approximation. This
analytic extension is actually one of the major challenges of using this method of analysis and we will discuss in detail how we accomplish this for our current problem. To develop our solution we use the eigenmode matching method [47], which will restrict the geometries but is simple to code and numerically efficient. It also allow consideration of problem close to those reported by [41, 42]. As we mentioned earlier eigenmode matching method is a straight forward method and is widely used in electromagnetism and acoustics to analyze waveguides. It can also be used to solve complex problems. It has been applied to the problem of scattering by trifurcated and pentafurcated ducts [48], to scattering by floating elastic plates [18], submerged elastic plate [13] and even to predict wave scattering in the marginal ice zone [49].

The outline of Part 2 is as follows:
This part consists of three chapters 5, 6 and 7. In Chapter 5 we consider the simplest case which consists of a waveguide with hard walls and a finite inner duct symmetrically (along y-axis) located within an infinite duct. We solve this problem using mode matching exploiting symmetry to decompose the solution. We explore in detail the consequences of this decomposition and show how this leads to EAT.
In Chapter 6 we consider a similar problem where the cavity is not symmetrically placed in the waveguide. We show in this case the EAT exists but that the cut-off frequency is halved. Significant numerical results are given for both problems including movies (supplementary material). These movies show the analytic extension of the solution for complex wavenumber which is key to our analysis.
Chapter 7 consists of a more complicated problem in which there is no longer symmetry and we give brief results which show that the lack of symmetry destroys EAT. In last, we present summary of part 2.
In appendix, we present solutions for two trifurcated waveguide problems with still air by using the mode matching method which have been previously tackled by Hassan and Rawlins [5] using Wiener-Hopf technique. We compare our results with exiting results and obtain identical reflection coefficient as found in [5].
Chapter 2

Preliminaries

Introduction
In this chapter, some mathematical preliminaries are defined which would be useful in understanding the stuff in this work. These can be found in standard books such as: [12], [27],[28], [53] and [54] etc.

2.1 Boundary Value Problem
When solving ordinary or partial differential equations in the presence of a boundary, there needs to be a boundary condition on the solution. To find a mathematical solution of the given differential equation which is subjected to the particular boundary conditions is referred as boundary value problem (BVP).

There are usually three types of boundary conditions. Dirichlet (or first-type) boundary condition (named after Peter Gustav Lejeune Dirichlet) is a type of boundary condition which specifies the values that a solution requires to take on along the boundary of the domain. Neumann (or second-type) boundary condition (named after Carl Neumann) specifies the normal derivative of the function on a surface. Robin or mixed boundary condition specifies a linear combination of a field value and its normal derivative.

2.2 Eigenvalue Problem
There exists a set of functions for a self-adjoint linear operator $L$ such that

$$L\phi(x) = \lambda \phi(x).$$

where $\lambda$ and $\phi$ are eigenvalue and eigenfunction respectively.

2.3 Inner Product
We define inner product of two real valued continuous functions $\phi_m(x)$ and $\phi_n(x)$ on the interval $[a, b]$ as

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)dy$$

2.4 Orthonormal Functions
Two functions $\phi_m$ and $\phi_n$ are orthonormal over the interval $[a, b]$ if
\[
\int_{a}^{b} \phi_m(x)\phi_n(x)w(x)dy = \delta_{mn}
\]
where \(\delta_{mn}\) is the Kronecker delta
\[
\delta_{mn} = \begin{cases} 
1 & m = n, \\
0 & m \neq n
\end{cases}
\]
and \(w(x)\) is weight function.

### 2.5 Sturm Liouville’s Problem (SLP)

The SL differential equation on a finite interval \([a, b]\) subjected to homogeneous mixed boundary conditions, that is,
\[
\frac{d}{dx}[p(x)\phi'] + [q(x) + \lambda w(x)]\phi = 0, \quad x \in (a, b) \tag{2.5.1}
\]
\[
\alpha \phi(a) + \beta \phi'(a) = 0, \quad \alpha^2 + \beta^2 > 0
\]
\[
\gamma \phi(b) + \delta \phi'(b) = 0, \quad \gamma^2 + \delta^2 > 0
\]
where \(p(x) > 0\) and \(w(x) > 0\) for \(x \in (a, b)\), called as regular Sturm-Liouville system (or problem).

#### 2.5.1 Properties of Sturm Liouville’s Problem

Some properties of SLP are given as following.

- Eigenvalues of SLP are always real.
- Eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function.
- Eigenvalues of a regular Sturm Liouville’s Problem are distinct. Thus an eigenfunction that corresponds to an eigenvalue is unique up to a constant multiple.
- Eigenfunction corresponding to different eigenvalues are linearly independent.

### 2.6 Waves

A wave can be defined as a variation/disturbance which travels through a medium from one location to another. The waves are classified mainly into two types that are longitudinal and transverse waves.

Transverse waves show displacement in the perpendicular direction of the field intensity vector (e.g., water and electromagnetic waves).

If the vibration of the particles of the medium are in the direction of wave propagation then this type of wave is called longitudinal wave (e.g., sound and earthquake waves).
2.6.1 Acoustic Waves

Acoustic is defined as the study of the sound waves. Acoustic wave is a longitudinal wave of energy that we hear as sound. It is defined as a pressure stimulated by a mechanical disturbance of an elastic medium at some source and propagated by the action of perturbed particles on neighboring particles. The scope of acoustics ranges from fundamental physical acoustics (e.g. music, bio acoustics, psycho acoustics etc.) to technical fields (e.g. transducer technology, sound recording, design of theaters and concert halls, and noise control).

2.6.2 Plane Waves

The plane wave is a basic concept in acoustics. Plane wave is a wave in which any acoustic variable at a given time remains constant on any plane which is perpendicular to the direction of propagation. Such waves can propagate in a waveguide/duct. The general form of a plane wave is

$$\Phi(x, y) = Ae^{i(k \cdot r - \omega t)}$$

where $A$ is a positive constant called the amplitude, $r$ is position vector, $k$ is propagating vector and $\omega$ is angular frequency.

2.6.3 Reflection and Transmission of Waves

When a wave strikes a boundary which is neither rigid nor soft (but some where in between), some part of the wave is reflected back while some part is transmitted. Exact behaviour of this transmission and the reflection is decided by material properties on both sides of boundary.

2.6.4 Extraordinary and Perfect Transmission

Extraordinary acoustic transmission (EAT) is the phenomenon of greatly enhanced transmission of sound wave. If all of the wave is transmitted and no reflection takes place then this is called perfect transmission.

2.6.5 Wave Equation

Following equation represents a two dimensional wave equation.

$$\frac{\partial^2 \Phi}{\partial t^2} = C^2 \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right)$$

(2.6.2)

where $C^2 = \frac{T_0 \rho_0}{\rho}$.

2.6.6 Helmholtz Equation

Helmholtz equation named after a German physicist Hermann Ludwig Ferdinand von Helmholtz (1821-1894) is the partial differential equation

$$\nabla^2 \phi + k^2 \phi = 0,$$

(2.6.3)

where $\nabla^2$ is laplacian, $k$ is wave number and $\phi$ is wave function. Helmholtz equation often arises in the study of physical problem involving PDEs in both space and time. Equation 2.6.3 which represents a time independent form of wave equation 2.6.2 results from applying the technique of separation of variable
to reduce the complexity of the analysis. It’s an eigenvalue problem. Helmholtz pitch notation is a system for naming musical notes of the western chromatic scale developed by Helmholtz.

### 2.7 Separation of Variables

By using the method of separation of variables, we can easily find the solutions of heat, wave and Laplace equation etc. This method helps in reducing PDE into a system of ODE’s. This method requires homogeneous boundary conditions. We may convert inhomogeneous boundary conditions to homogeneous boundary conditions by a transformation formula. The shape of the boundary determines the choice of coordinate system. Consider the following Laplace equation

\[ \nabla^2 \phi = 0 \]

Assume

\[ \phi(x, y, z) = f_1(x)f_2(y)f_3(z) \]

then we have following three ODEs

\[ \frac{d^2 f_3(z)}{dx^2} + \alpha f_3(z) = 0, \]

\[ \frac{d^2 f_2(y)}{dy^2} + \beta f_2(y) = 0, \]

and

\[ \frac{d^2 f_1(x)}{dz^2} + (\alpha + \beta)f_1(x) = 0, \]

where \( \alpha \) and \( \beta \) are constants.

### 2.8 Radiation Condition

Radiation conditions are conditions at infinity for uniqueness of a solution. Their physical meaning is the selection of the solution of BVP describing divergent waves with sources located in a bounded domain. If \( \phi \) represents an outgoing wave at infinity and if the time factor is \( \exp(-i\omega t) \) then

\[ \lim_{kx \to \pm \infty} \left( \frac{\partial \phi}{\partial x} \mp ik\phi \right) = 0, \]

and if the time factor is taken as \( \exp(+i\omega t) \) then

\[ \lim_{kx \to \pm \infty} \left( \frac{\partial \phi}{\partial x} \pm ik\phi \right) = 0. \]

### 2.9 Analytic Functions of a Complex Variable

A function \( f(z) \) is said to be analytic in a region \( R \) of the complex plane if \( f(z) \) has a derivative at each point of \( R \) and if \( f(z) \) is single valued.

#### 2.9.1 Singularities of Analytic Functions

Points at which a function \( f(z) \) is not analytic are called singular points or singularities of \( f(z) \).
2.9.2 Types of Singularities

Poles
In the Laurent series
\[ f(z) = \sum_{n=1}^{\infty} a_n(z - z_0)^{-n} + \sum_{n=0}^{\infty} b_n(z - z_0)^n, \]
if the principal part contains finite number of non zero terms then \( z_0 \) is called a pole of \( f(z) \).

Removable singularity
The singular point \( z_0 \) is called removable singularity of \( f(z) \) if there is no principal part in Laurent series.

Essential singularity
A point \( z = z_0 \) is called an essential singularity if there are infinite number of terms in principal part.

2.9.3 Branch points
A branch point of an complex function is a point in the complex plane whose complex argument can be mapped from a single point in the domain to multiple points in the range.

2.9.4 Zeros of Complex Functions
Zero of \( f(z) \) is the point \( z_0 \) where \( f(z_0) = 0 \). A zero is of order \( n \) if
\[ f'(z_0) = f''(z_0) = f^{(n-1)}(z_0) = 0, \]
but
\[ f^n(z_0) \neq 0 \]

2.9.5 Blaschke Product
A regular analytic function of a complex variable, which is defined in the disc \( k = \{ z : |z| \leq 1 \} \) can be expressed in form of finite or infinite product
\[ B(z) = z^n \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a}_k z} \]
known as Blaschke product, where \( n \) is a positive integer, and \( \{a_k\}, k = 1, 2, ... \) is a sequence of points such that the product on the right side of above equation is convergent. This product was first presented by W. Blaschke [55].

2.10 Helmholtz Resonator
A German physicist Hermann Ludwig Ferdinand von Helmholtz (1821-1894) designed Helmholtz resonator in the 1860s. A Helmholtz resonator or Helmholtz oscillator is a container of gas (usually air) with an open hole (or neck or port). Since ancient Greece, the Helmholtz resonator has been used to intensify and dampen sound. Theory of Helmholtz resonance have been of interest over the years to model and inspect different systems. A benefit of the Helmholtz resonator is that it has the property of powerful sound attenuation, even though it has comparatively simple geometry. When it is properly tuned, it has the ability to lessen considerable amount of noise over the low frequency domain.
2.10.1 Helmholtz system

Helmholtz system is based upon Helmholtz resonance which is generally known as the phenomenon of air resonance in the container. The highest amplitude sound is produced near and at the resonant frequency, which is chosen by the volume and the neck dimensions of the Helmholtz resonator. A well-known geometry of such a resonator is the musical tone which is the product of Helmholtz resonance produced when we blow air across the top of an empty bottle. In a Helmholtz resonator, the air in its neck performs as a discrete mass, while the air in the cavity behaves like a spring. During oscillation, the gas in resonator is alternately compressed and expanded. Because of the inertia of the mass, the mass passes through the equilibrium. However, again the restoring force of the air in the body compresses the air and again, the mass of air in the neck is pushed upwards, and this cycle gets repeated. This process can be considered as a simple harmonic motion, producing a resonance.

2.10.2 Uses

Many researchers and engineers have employed the Helmholtz resonator for different applications including many types of geometries, some of them as part of complex configurations. Helmholtz resonators are around us in many different forms. They are in audio equipment and instruments. These help us in amplifying sound and in eliminating noise in a room. Another usage of Helmholtz resonators is in architectural acoustics where they help to cut down undesirable sounds. This is accomplished by building a resonator adjusted to the same frequency as the unwanted tone. This is normally used for low frequency ranges.
PART 1
Pentafurcated Duct Problems
Chapter 3

Wave Scattering in Rigid Pentafurcated Duct

3.1 Introduction

In this chapter we will discuss the diffraction of the lowest plane wave propagating out of the opening of a semi-infinite hard duct which is symmetrically placed inside an infinite hard duct. The whole system forms a five spaced (pentafurcated) waveguide whose solution is given by eigenfunction expansion method. In appendix-A same method has been used to validate the results of a rigid trifurcated duct problem previously tackled by [5] using the standard Wiener-Hopf technique. Here it would be interesting to see the impact on scattering behaviour with insertion of extra hard plates at $y = \pm b$ in trifurcated duct. Some graphical results showing influence of waveguide spacing on the reflection coefficient are also presented. Most of the contents of this chapter have been published in the journal of Mathematical Methods in the Applied Science [48].

3.2 Formulation of the Problem

We investigate the acoustic reflection of a lowest wave mode which comes out from a semi-infinite pipe $|y| < a, x < 0$. Figure 3.1 shows the schematic geometry of the problem. The configuration of the pentafurcated problem is such that a semi infinite hard trifurcated duct is placed symmetrically inside the infinite hard duct. The sound source field, which propagates modes along $|y| < a$ is located at $(x_0, y_0), (x_0 < 0, |y_0| < a)$. The acoustic pressure and velocity are defined by

$$p = -\rho_0 \frac{\partial \Phi}{\partial t}$$

and

$$\vec{u} = \text{grad} (\Phi)$$

respectively, where $\Phi(x, y, t)$ is the scalar potential function and $\rho_0$ is the equilibrium state density. The time variant can be removed by writing

$$\Phi(x, y, t) = \text{Re} [\phi(x, y) e^{-iwt}]$$

(3.2.1)

where the angular frequency is $w = kC$, ($k$ and $C$ are the wave number and speed of sound respectively). Then the wave equation

$$\nabla^2 \Phi = (1/C^2) \Phi_{tt}$$
Figure 3.1: Geometry of rigid pentafulcated waveguide problem.

reduces to two dimensional Helmholtz equation

\[ \nabla^2 \phi(x, y) + k^2 \phi(x, y) = 0, \quad x \in \Lambda, \quad y \in D. \tag{3.2.2} \]

We solve (3.2.2) for \( \phi(x, y) \) subject to BCS:

\[ \phi_y(x, y) = 0, \tag{3.2.3} \]

where \( D \in (-d, d), y = \pm a, \pm b \) for \( x \notin \Lambda \), \( y = \pm d \) for \( x \in \Lambda \) and \( \Lambda \in (-\infty, \infty) \). In addition, we have the radiation conditions which guarantee a bounded solution.

3.3 Solution of the Problem

Here we will split considered problem into 6 regions. The resultant solutions in these regions will be calculated by separation of variables. These solutions will be matched across the boundary to model an infinite system of equations which will be solved by standard way of truncation.

3.3.1 Region 1 \( \{-d \leq y \leq -b, \quad x < 0\} \)

Using the method of separation of variables, we can write the solution of (3.2.2) as

\[ \phi(x, y) = \sum_{n=1}^{\infty} A_n e^{-i\tilde{\alpha}_n x} \psi_n(y) \tag{3.3.4} \]

which satisfies (3.2.2), (3.2.3) and the radiation conditions. Where eigenvalues \( \alpha_n \), associated eigenvalues \( \tilde{\alpha}_n \) and eigenfunctions \( \psi_n \) are

\[ \alpha_n = \frac{(n-1)\pi}{d-b}, \quad n = 1, 2, 3, ... \]

\[ \tilde{\alpha}_n = \sqrt{k^2 - \left(\frac{(n-1)\pi}{d-b}\right)^2}, \quad n = 1, 2, 3, ... \]
\[ \psi_n(y) = \begin{cases} \sqrt{\frac{2}{d-b}} \cos \alpha_n(y + d), & n \neq 1, \\ \sqrt{\frac{1}{d-b}}, & n = 1, \end{cases} \quad (3.3.5) \]

respectively with \(0 < \text{Im} \tilde{\alpha}_1 < \text{Im} \tilde{\alpha}_2,\ldots\). The eigenvalues \(\alpha_n\) are the solutions of

\[ \sin \alpha_n(d - b) = 0 \]

The eigenfunctions \(\psi_n(y)\) satisfy the relation

\[ \int_{-d}^{-b} \psi_m(y) \psi_n(y) dy = \delta_{mn} \quad m, n = 1, 2, \ldots \quad (3.3.6) \]

where Kronecker delta \(\delta\) is defined by

\[ \delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \]

#### 3.3.2 Region 2 \{-b \leq y \leq -a, \ x < 0\}

Eigenfunctions in region 2 are given by

\[ \xi_n(y) = \begin{cases} \sqrt{\frac{2}{b-a}} \cos \beta_n(y + b), & n \neq 1, \\ \sqrt{\frac{1}{b-a}}, & n = 1, \end{cases} \quad (3.3.7) \]

which satisfy the orthonormal relation

\[ \int_{-a}^{-b} \xi_m(y) \xi_n(y) dy = \delta_{mn}. \quad (3.3.8) \]

Here \(\beta_n = \frac{(n-1)\pi}{b-a}\) and \(\tilde{\beta}_n\) are eigenvalues and the associated eigenvalues respectively. The general solution for the potential in the region 2 is

\[ \phi(x, y) = \sum_{n=1}^{\infty} B_n e^{-i\tilde{\beta}_n x} \xi_n(y) \quad (3.3.9) \]

which satisfies (3.2.2), (3.2.3) and the radiation conditions.

#### 3.3.3 Region 3 \{-a \leq y \leq a, \ x < 0\}

In region 3, the general potential is

\[ \phi(x, y) = \sum_{n=1}^{\infty} C_n e^{-i\gamma_n x} \eta_n(y) + e^{i\gamma_1 x} \eta_1(y) \quad (3.3.10) \]

which satisfies (3.2.2), (3.2.3) and the radiation conditions where \(\gamma_n\) are eigenvalues given by

\[ \gamma_n = \frac{(n-1)\pi}{2a}, \quad n = 1, 2, 3, \ldots \quad (3.3.11) \]
and the incident wave ($e^{i\tilde{\gamma}_1 x} \eta_1(y)$) is propagating towards right. The eigenvalues $\gamma_n$ are the solutions of $\sin 2\gamma_n a = 0$ and eigenfunctions $\eta_n$ are given by

$$\eta_n(y) = \begin{cases} \sqrt{\frac{1}{a}} \cos \gamma_n(y + a), & n \neq 1, \\ \sqrt{\frac{1}{2a}}, & n = 1 \end{cases}$$

satisfying the relation

$$\int_{-a}^{a} \eta_m(y) \eta_n(y) dy = \delta_{mn}, \quad m, n = 1, 2, 3, ....$$

(3.3.12)

### 3.3.4 Region 4 \{a \leq y \leq b, \quad x < 0\}

Here the general solution satisfying (3.2.2), (3.2.3) and the radiation conditions is

$$\phi(x, y) = \sum_{n=1}^{\infty} D_n e^{-i\tilde{\beta}_n x} \zeta_n(y)$$

(3.3.13)

where orthonormal eigenfunctions $\zeta_n(y)$ are given by

$$\zeta_n(y) = \begin{cases} \sqrt{\frac{2}{b-a}} \cos \beta_n(y - b), & n \neq 1, \\ \sqrt{\frac{1}{b-a}}, & n = 1 \end{cases}$$

(3.3.14)

### 3.3.5 Region 5 \{b \leq y \leq d, \quad x < 0\}

The general solution in this region which satisfies (3.2.2), (3.2.3) and the radiation conditions is

$$\phi(x, y) = \sum_{n=1}^{\infty} E_n e^{-i\tilde{\alpha}_n x} \sigma_n(y)$$

(3.3.15)

The orthonormal eigenfunctions $\sigma_n$ are defined by

$$\sigma_n(y) = \begin{cases} \sqrt{\frac{2}{d-b}} \cos \alpha_n(y - d), & n \neq 1, \\ \sqrt{\frac{1}{d-a}}, & n = 1 \end{cases}$$

(3.3.16)

### 3.3.6 Region 6 \{-d \leq y \leq d, \quad x > 0\}

Here the general solution for the potential is given by

$$\phi(x, y) = \sum_{n=1}^{\infty} F_n e^{i\tilde{\omega}_n x} \epsilon_n(y)$$

(3.3.17)

which satisfies (3.2.2), (3.2.3) and the radiation conditions where $\omega_n = \frac{n\pi}{2d}$ and $\tilde{\omega}_n$ are the eigenvalues and associated eigenvalues respectively. The orthonormal eigenfunctions $\epsilon_n(y)$ are given by

$$\epsilon_n(y) = \begin{cases} \sqrt{\frac{1}{d}} \cos \omega_n(y - d), & n \neq 1, \\ \sqrt{\frac{1}{2d}}, & n = 1 \end{cases}$$

(3.3.18)

Now we will exploit the continuity of pressure and its velocity across the boundary $x = 0$ which will lead us to an infinite system of equations.
3.4 Continuity of Pressure

The continuity of the pressure across region 1 and region 6 at \( x = 0 \) gives

\[
\sum_{n=1}^{\infty} A_n \psi_n(y) = \sum_{n=1}^{\infty} F_n \epsilon_n(y).
\]

By taking the inner product with \( \psi_m(y) \), integrating over \([-d,-b]\) and using (3.3.6), the above equation becomes

\[
A_m = \sum_{n=1}^{\infty} F_n L_{mn}, \quad m = 1, 2, \ldots \quad (3.4.19)
\]

where

\[
L_{mn} = \int_{-d}^{-b} \psi_m(y) \epsilon_n(y) dy
\]

\[
= \begin{cases} 
\sqrt{(d - b)} & \text{if } m = n = 1, \\
\frac{1}{2d} \sin \omega_n (b + d), & \text{if } m = 1, \\
-\sqrt{2} \omega_n \sin \omega_n (b + d) \cos \alpha_m (d - b), & \text{if } m \neq 1,
\end{cases}
\]

\[
= \frac{\sqrt{d(d - b)} (\omega_n^2 - \alpha_m^2)}{2 \omega_n} + (d - b) \cos \omega_n d, \quad \omega_n = \alpha_m.
\]

The continuity of the pressure across region 2 and region 6 at \( x = 0 \) gives

\[
\sum_{n=1}^{\infty} B_n \xi_n(y) = \sum_{n=1}^{\infty} F_n \epsilon_n(y).
\]

Taking the inner product with \( \xi_m(y) \) and integrating over \([-b,-a]\) and using (3.3.8), the above equation becomes

\[
B_m = \sum_{n=1}^{\infty} F_n Q_{mn}, \quad m = 1, 2, \ldots \quad (3.4.21)
\]

The continuity of the pressure across region 3 and region 6 at \( x = 0 \) gives

\[
\sum_{n=1}^{\infty} C_n \eta_n(y) + \eta_1(y) = \sum_{n=1}^{\infty} F_n \epsilon_n(y).
\]

Taking the inner product with \( \eta_m(y) \), integrating over \([-a,a]\) and using (3.3.12), we get

\[
\delta_{1m} + C_m = \sum_{n=1}^{\infty} F_n T_{mn}, \quad m = 1, 2, \ldots \quad (3.4.22)
\]
where
\[ T_{mn} = \int_{-a}^{a} \eta_m(y) \epsilon_n(y) dy \]  
\[ = \begin{cases} 
\sqrt{\frac{a}{d}}, & m = n = 1, \\
\sqrt{\frac{2}{ad}} \cos \omega_n d \sin \omega_n a, & m = 1, n \neq 1, \\
\omega_n \left[ \sin \omega_n (a + d) + \sin \omega_n (a - d) \cos 2a\gamma_m \right], & \omega_n \neq \gamma_m, \\
\frac{1}{2\sqrt{ad}} \sin \omega_n (a + d) + \sin \omega_n (3a - d) + 2a \cos \omega_n (a + d) \right], & \omega_n = \gamma_m.
\end{cases} \] 

The continuity of the pressure across region 4 and region 6 at \( x = 0 \) gives
\[ \sum_{n=1}^{\infty} D_n \zeta_n(y) = \sum_{n=1}^{\infty} F_n \epsilon_n(y). \]

Taking the inner product with \( \zeta_m(y) \), and integrating over \([a, b]\), we get
\[ \sum_{n=1}^{\infty} D_n \int_a^b \zeta_n(y) \zeta_m(y) dy = \sum_{n=1}^{\infty} F_n \int_a^b \epsilon_n(y) \zeta_n(y) dy. \]

By using orthonormality of eigenfunctions, we get
\[ D_m = \sum_{n=1}^{\infty} F_n R_{mn}, \quad m = 1, 2... \]  
\[ (3.4.24) \]

where
\[ Q_{mn} = \int_{-b}^{-a} \epsilon_n(y) \xi_m(y) dy \]  
\[ = \begin{cases} 
\sqrt{\frac{b - a}{d}}, & m = n = 1, \\
\frac{1}{2\sqrt{d} (b - a)} \left[ \sin \omega_n (b + d) - \sin \omega_n (a + d) \right] \omega_n, & m = 1, \\
\frac{2}{d (b - a)} \sin \omega_n (b + d) - \sin \omega_n (a + d) \cos \beta_m (a - b) \omega_n^2 - \beta_m^2, & m \neq 1, \\
\frac{1}{2\sqrt{2d} (b - a)} \left[ \sin \omega_n (b - d - 2a) + \sin \omega_n (b + d) \right] \omega_n + (b - a) \cos \omega_n (b + d) \right], & \omega_n = \beta_m.
\end{cases} \]
\[ R_{mn} = \int_{a}^{b} \epsilon_n(y) \zeta_m(y) \, dy \]

\[
\left\{ \begin{array}{ll}
\sqrt{b - a} \frac{1}{d}, & m = n = 1, \\
\frac{\omega_n}{d(b - a)} \left[ \frac{\sin \omega_n (b - d) - \sin \omega_n (a - d)}{\omega_n} \right], & m = 1, \\
\frac{2}{d(b - a)} \omega_n \left[ \frac{\sin \omega_n (b - d) - \sin \omega_n (a - d) \cos \beta_m (a - b)}{\omega_n^2 - \beta_m^2} \right], & m \neq 1, \\
\frac{1}{2d(b - a)} \left[ (b - a) \cos \omega_n (b - d) + \frac{\sin \omega_n (b - d) - \sin \omega_n (2a - b - d)}{2\omega_n} \right] \omega_n = \beta_m. 
\end{array} \right.
\]

(3.4.26)

The continuity of the pressure across region 5 and region 6 at \(x = 0\) gives

\[ \sum_{n=1}^{\infty} E_n \sigma_n(y) = \sum_{n=1}^{\infty} F_n \epsilon_n(y). \]

Integrating over the interval \([b, d]\) after multiplying by \(\sigma_m(y)\), we obtain

\[ \sum_{n=1}^{\infty} E_n \int_{b}^{d} \sigma_n(y) \sigma_m(y) \, dy = \sum_{n=1}^{\infty} F_n \int_{b}^{d} \epsilon_n(y) \sigma_m(y) \, dy, \]

which implies

\[ E_m = \sum_{n=1}^{\infty} F_n S_{mn}, \quad m = 1, 2, \ldots \]  

(3.4.27)

where

\[ S_{mn} = \int_{b}^{d} \epsilon_n(y) \sigma_m(y) \, dy \]

\[
\left\{ \begin{array}{ll}
\sqrt{d - b} \frac{1}{2d}, & m = n = 1, \\
-\frac{1}{d(b - d)} \left[ \frac{\sin \omega_n (b - d)}{\omega_n} \right], & m = 1, \\
-\frac{2}{d(b - d)} \omega_n \left[ \frac{\sin \omega_n (b - d) \cos \alpha_m (b - d)}{\omega_n^2 - \alpha_m^2} \right], & m \neq 1, \\
\sqrt{\frac{d - b}{2d}}, & \omega_n = \alpha_m.
\end{array} \right.
\]

(3.4.28)

### 3.5 Continuity of Velocity

We will follow the same procedure as we used in section 3.4 to match velocity of the potentials across the boundary at \(x = 0\) which yields the following system of equations:

\[ -\bar{\alpha}_m A_m = \sum_{n=1}^{\infty} \bar{\omega}_n F_n L_{mn}, \quad m = 1, 2, \ldots \]  

(3.5.29)
\[-\tilde{\beta}_m B_m = \sum_{n=1}^{\infty} \tilde{\omega}_n F_n Q_{mn}, \quad m = 1, 2, \ldots \quad (3.5.30)\]
\[\tilde{\gamma}_1 \delta_{1m} - \tilde{\gamma}_m C_m = \sum_{n=1}^{\infty} \tilde{\omega}_n F_n T_{mn}, \quad m = 1, 2, \ldots \quad (3.5.31)\]
\[-D_m \tilde{\beta}_m = \sum_{n=1}^{\infty} \tilde{\omega}_n F_n R_{mn}, \quad m = 1, 2, \ldots \quad (3.5.32)\]
\[-E_m \tilde{\alpha}_m = \sum_{n=1}^{\infty} \tilde{\omega}_n F_n S_{mn}, \quad m = 1, 2, \ldots \quad (3.5.33)\]

### 3.6 System of Equations

We need to solve unknown coefficients in different regions. For this we compile the above system of equations and get

\[\sum_{n=1}^{\infty} F_n L_{mn} (\tilde{\alpha}_m + \tilde{\omega}_n) = 0, \quad m = 1, 2, 3, \ldots \quad (3.6.34)\]
\[\sum_{n=1}^{\infty} F_n Q_{mn} (\tilde{\beta}_m + \tilde{\omega}_n) = 0, \quad m = 1, 2, 3, \ldots \quad (3.6.35)\]
\[\sum_{n=1}^{\infty} F_n T_{mn} (\tilde{\gamma}_m + \tilde{\omega}_n) = \tilde{\gamma}_1 \delta_{1m} + \tilde{\gamma}_m \delta_{1m}, \quad m = 1, 2, 3, \ldots \quad (3.6.36)\]
\[\sum_{n=1}^{\infty} F_n R_{mn} (\tilde{\beta}_m + \tilde{\omega}_n) = 0, \quad m = 1, 2, 3, \ldots \quad (3.6.37)\]
\[\sum_{n=1}^{\infty} F_n S_{mn} (\tilde{\alpha}_m + \tilde{\omega}_n) = 0, \quad m = 1, 2, 3, \ldots \quad (3.6.38)\]

Now we truncate equations (3.6.34) to (3.6.38) by standard way of truncation to find \(F_n\). The rest of coefficients are determined from previous system of equations.

### 3.7 Energy Balance

We apply Green’s theorem [24] to derive an relationship for energy balance relationship. We start with

\[\int_{\Omega} (\phi^* \nabla^2 \phi - \phi \nabla^2 \phi^*) \, dv = \int_{\partial \Omega} \left( \phi^* \frac{\partial \phi}{\partial n} - \phi \frac{\partial \phi^*}{\partial n} \right) \, ds \quad (3.7.39)\]

where the * denotes the complex conjugate and \(\Omega\) is the region of the plane \(-d < y < d, -\infty < x < \infty\) with cuts for the semi-infinite duct. It is then easy to get the following relationship

\[\sum_{n: \tilde{\alpha}_n \text{ is real}} \tilde{\alpha}_n |A_n|^2 + \sum_{n: \tilde{\beta}_m \text{ is real}} \tilde{\beta}_m |B_n|^2 + \sum_{n: \tilde{\gamma}_n \text{ is real}} \tilde{\gamma}_n |C_n|^2 + \]
\[\sum_{n: \tilde{\beta}_m \text{ is real}} \tilde{\beta}_m |D_n|^2 + \sum_{n: \tilde{\alpha}_n \text{ is real}} \tilde{\alpha}_n |E_n|^2 + \sum_{n: \tilde{\omega}_n \text{ is real}} \tilde{\omega}_n |F_n|^2 = \tilde{\gamma}_1 \quad (3.7.40)\]

where the sum is over the modes which are propagating.
3.8 Numerical Outcomes

We are concerned with the acoustic energy in the incident mode $\exp(i \gamma_1 x) \eta_1(y)$ which is scattered among the different regions of waveguide. The energy of the incident wave which is spread in different modes in the rest of duct can be calculated by $|R| = |C_1|$. We present numerical outcomes by solving the truncated infinite system of equations. We then plot the absolute reflection coefficients versus the wave number for given dimensions which would be helpful to investigate the radiated acoustic energy.

3.8.1 Case: 1

Here $b = \frac{2a}{3}$, $d = \frac{3a}{2}$ and the frequency range is $0 < ka < \pi$. Figure 3.2 represents $|R|$ versus wave number $ka$. Here the dashed lines is for the trifurcated while solid line is for pentafurcated waveguide. Identical behavior for both ducts can be seen for a given frequency range.

![Figure 3.2: Reflection coefficient against wave number for $d = 3a/2$.](image)

3.8.2 Case: 2

In figure 3.3, dimensions of problem are such that $b = 2a$, $d = 3a$ and the frequency range is $0 < ka < \pi$. We notice that from 0 to $\pi/3$, the reflection has little reduced for pentafurcated duct whereas from $\pi/3$ to $2\pi/3$, there is nearly no propagating wave and reflection has boosted. After this we see similar behavior as was obtained in trifurcated duct. In figures 3.4 and 3.5, we take the higher range of frequency $0 < ka < 2\pi$ for $d = 1.5a$, $b = 1.25a$ and $d = 3a$, $b = 2a$ respectively. Figure 3.6 is drawn for (i) $a = 1$ (solid line) and (ii) $a = 0.5$ (dotted line) for set values of $d = 1.5$ and $b = 1.25$. From figures 3.7 and 3.8 we can observe the effect of altering duct spacing on potential and velocity of potential respectively. The system of infinite equations has been truncated for $n = 1, 2, 3, ..., N$ and $m = 1, 2, 3, ..., M$. Figure 3.9 shows the change in reflection when number of equations are increased. For trifurcated problem the duct dimensions are such that $a = 1$, $b = 3$, $k = \pi$ while for pentafurcated duct we have chosen $a = 1, b = 1.25, d = 1.5, k = \pi$ and $N = 5M$. One can observe that the graph becomes unresponsive for $M \geq 10$ and $M \geq 35$ for tri and penta furcated waveguides respectively.
Figure 3.3: Reflection coefficient against the wave number for $b = 2a$ and $d = 3a$.

Figure 3.4: Reflection coefficient against the wave number for $b = 1.25a$ and $d = 1.5a$.

Figure 3.5: Reflection coefficient against the wave number for $b = 2a$ and $d = 3a$. 
Figure 3.6: Reflection coefficient against the wave number for $b = 1.25$ and $d = 1.5$.

Figure 3.7: Potential against duct spacing.
Figure 3.8: Velocity potential against duct spacing.

Figure 3.9: Reflection against number of equations.
3.9 Conclusion

We have solved the trifurcated (appendix-A) and pentafurcated exhaust models for still air by using mode matching technique. We obtain same results for the reflection field amplitudes for trifurcated duct as were calculated by Hassan and Rawlins [5] using the Wiener-Hopf technique. The mode matching method does not contain complex functions (which appear in the Wiener-Hopf method). We demonstrate numerical outcomes by assuming that only the lowest mode propagates since it contains most of the energy and is of practical concern. We extend the mode matching technique to a more complex pentafurcated duct. The absolute value of reflection versus wave number for various duct spacing has been plotted. Figures 3.2 and 3.3 give comparison of the pentafurcated problem with trifurcated problem for different spacings of duct. From figure 3.4 we observe that the onsets of wave propagating onward are at $ka = \frac{2\pi}{3}$ and $ka = \frac{4\pi}{3}$ while the onset of waves propagating backward is at $ka = \pi$. In figure 3.5, there are more abrupt changes in reflection coefficient with higher frequency range which show activation of additional modes in different regions. Here one can observe that onsets of waves propagating in forward and backward directions are at $k = \frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$ and $k = \pi$, respectively. Figure 3.6 show that value of $|R|$ increases with decrease of width of inner most region from $a = 1$ to $a = 0.5$ for fixed $b$ and $d$. From figures 3.7 and 3.8, we can observe that potential and velocity of potential across the boundary are nearly matched. In figure 3.8, there are few oscillations which are not crucial and are non singular in nature so they don’t effect the precision of our solution. This comparison proves the accuracy of our calculations for pentafurcated duct. The solutions we have presented would be useful in constructing practically efficient exhaust systems. Future work can be extended to the case of fluid flow.
Chapter 4

Wave Scattering in Outer Soft Pentafurcated Duct

4.1 Introduction

In this chapter we will give the reflection analysis of a plane fundamental mode which comes out from any sound source located at \((x_0, y_0)\) in \(|y| < a, x < 0\) and propagating towards right. This region is symmetrically placed inside an infinite soft duct to form a pentafurcated waveguide as shown in figure 4.1. The solution of this pentafurcated problem is given by eigenfunction expansion method. Some graphical results showing the influence of waveguide spacing on the reflection coefficient are presented. In appendix-B same method has been applied to outer-soft trifurcated duct problem which was previously tackled by Hassan and Rawlins \[5\] by Wiener-Hopf technique. It is shown that same numerical results are obtained which shows the validity of this current method.

We have also presented the reflection field amplitude comparison among trifurcated (appendix B), current pentafurcated and rigid pentafurcated duct \[48\] problems. This comparison would be helpful to analyze acoustic radiated energy for considered geometries. Most of the contents of this chapter have been published in Canadian Journal of Physics \[59\].

4.2 Formulation of Problem

To find the solution, we will split our duct into different regions and present the solutions in each region by method of separation of variables. These solutions will be matched across the boundary \(x = 0\) to get an infinite set of algebraic equations. Here we solve (3.2.2) for \(\phi(x, y)\) subject to BCS:

\[\phi(x, y) = 0, \ y = D_1, \ x \in (-\infty, \infty),\]  
(4.2.1)
\[\phi_y(x, y) = 0, \ y = D_2, \ x \in (-\infty, 0].\]  
(4.2.2)

where \(D_1 = -d, d\), and \(D_2 = -b, -a, a, b\).
4.3 Solution of the Problem

4.3.1 Region 1 \{-d \leq y \leq -b, \ x < 0\}

Here the solution of (3.2.2) is given by

\[ \phi(x, y) = \sum_{n=1}^{\infty} A_n e^{-\tilde{\alpha}_n x} \psi_n(y) \]  

which satisfies (4.2.1), (4.2.2) and radiation condition. Eigenvalues, associated eigenvalues and eigenfunctions in this region are defined by

\[ \alpha_n = \frac{(2n - 1) \pi}{2(d - b)}, \quad n = 1, 2, \ldots \]

\[ \tilde{\alpha}_n = \sqrt{k^2 - \alpha_n^2}, \quad n = 1, 2, \ldots \]

and

\[ \psi_n(y) = \sqrt{\frac{2}{(d - b)}} \cos \alpha_n (y + b), \]  

respectively with \( 0 < \mathrm{Im} \tilde{\alpha}_1 < \mathrm{Im} \tilde{\alpha}_2, \ldots \). The eigenvalues \( \alpha_n \) are the solutions of

\[ \cos \alpha_n (d - b) = 0 \]

while the eigenfunctions \( \psi_n(y) \) satisfy orthonormality relation

\[ \int_{-d}^{-b} \psi_m(y) \psi_n(y) dy = \delta_{mn}, \quad m, n = 1, 2, \ldots \]  

4.3.2 Region 2 \{-b \leq y \leq -a, \ x < 0\}

Here the vertical eigenfunctions are given by

\[ \xi_n(y) = \begin{cases} \sqrt{\frac{2}{b-a}} \cos \beta_n (y + b), & n \neq 1, \\ \sqrt{\frac{1}{b-a}}, & n = 1. \end{cases} \]
The eigenvalues \( \beta_n = \frac{(n-1)\pi}{b-a} \) are the solutions of

\[
sin \beta_n (b - a) = 0, \quad n = 1, 2, 3, \ldots
\]

The eigenfunctions \( \xi_n(y) \) satisfy

\[
\int_{-a}^{-b} \xi_m(y) \xi_n(y) dy = \delta_{mn}. \tag{4.3.7}
\]

The general potential is given by

\[
\phi(x, y) = \sum_{n=1}^{\infty} B_n e^{-\tilde{\beta}_n x} \xi_n(y) \tag{4.3.8}
\]

which satisfies (3.2.2), (4.2.2) and radiation conditions where

\[
\tilde{\eta}_n = \sqrt{k^2 - \beta_n^2}. \tag{4.3.9}
\]

### 4.3.3 Region 3 \(|y| \leq a, \ x < 0\)

Here general potential is

\[
\phi(x, y) = \sum_{n=1}^{\infty} C_n e^{-\tilde{\gamma}_n x} \eta_n(y) + e^{\tilde{\gamma}_1 x} \eta_1(y) \tag{4.3.10}
\]

which satisfies (3.2.2), (4.2.2) and radiation conditions. The associated eigenvalues are

\[
\tilde{\gamma}_n = \sqrt{k^2 - \gamma_n^2}, \quad n = 1, 2, \ldots \tag{4.3.11}
\]

and incident wave \( e^{\tilde{\gamma}_1 x} \eta_1(y) \) is excited in the lowest mode propagating from \( x = -\infty \). The eigenvalues \( \gamma_n = \frac{(n-1)\pi}{2a} \) satisfy

\[
sin 2\gamma_n a = 0.
\]

The eigenfunctions \( \eta_n \) are given by

\[
\eta_n(y) = \begin{cases} 
\sqrt{\frac{1}{a}} \cos \gamma_n (y + a), & n \neq 1 \\
\sqrt{\frac{1}{2a}}, & n = 1 
\end{cases} \tag{4.3.12}
\]

which satisfy

\[
\int_{-a}^{a} \eta_m(y) \eta_n(y) dy = \delta_{mn}, \quad m, n = 1, 2, 3, \ldots \tag{4.3.13}
\]

### 4.3.4 Region 4 \(a \leq y \leq b, \ x < 0\)

Here the solution satisfying (3.2.2), (4.2.2) is given by

\[
\phi(x, y) = \sum_{n=1}^{\infty} D_n e^{-\tilde{\beta}_n x} \zeta_n(y). 
\]

The eigenfunctions \( \zeta_n(y) \)

\[
\zeta_n(y) = \begin{cases} 
\sqrt{\frac{2}{b-a}} \cos \beta_n (y - b), & n \neq 1, \\
\frac{1}{\sqrt{b-a}}, & n = 1 
\end{cases} \tag{4.3.14}
\]

satisfy the orthonormal relation

\[
\int_{a}^{b} \zeta_m(y) \zeta_n(y) dy = \delta_{mn}, \quad m, n = 1, 2, 3, \ldots \tag{4.3.15}
\]
4.3.5 Region 5 \{b \leq y \leq d, \ x < 0\}

The general potential which satisfies (3.2.2), (4.2.1), (4.2.2) and radiation conditions is

\[ \phi(x, y) = \sum_{n=1}^{\infty} E_n e^{-\bar{\alpha}_n x} \sigma_n(y). \]  \hspace{1cm} (4.3.16)

The eigenfunctions \(\sigma_n\) in section 5 are defined by

\[ \sigma_n(y) = \sqrt{\frac{2}{d-b}} \cos \alpha_n (y - b) \]  \hspace{1cm} (4.3.17)

which satisfy

\[ \int_b^d \sigma_n(y)\sigma_m(y)dy = \delta_{mn}, \quad m, n = 1, 2, \ldots. \]  \hspace{1cm} (4.3.18)

4.3.6 Region 6 \{-d \leq y \leq d, \ x > 0\}

Here general potential which satisfies (3.2.2), (4.2.1) and radiation condition is given by

\[ \phi(x, y) = \sum_{n=1}^{\infty} F_n e^{\bar{\omega}_n x} \epsilon_n(y). \]  \hspace{1cm} (4.3.19)

where

\[ \epsilon_n(y) = \sqrt{\frac{1}{d}} \sin \omega_n (y + d), \]

\[ \bar{\omega}_n = \sqrt{k^2 - \omega_n^2}, \quad n = 1, 2, 3, \ldots. \]  \hspace{1cm} (4.3.20)

and

\[ \omega_n = \frac{n\pi}{2d} \]

which are the roots of \(\sin 2\omega_n d = 0\).

**Formulation of the Infinite Set of Equations**

We will exploit the continuity of potential and velocity of potential across \(x = 0\) in the next two subsections.

4.4 Continuity of Pressure

The continuity of the pressure across regions 6 and 1 at \(x = 0\) gives

\[ \sum_{n=1}^{\infty} A_n \psi_n = \sum_{n=1}^{\infty} F_n \epsilon_n. \]

Multiplying by \(\psi_m\), integrating over \([-d, -b]\) and using (4.3.5), the above equation becomes

\[ A_m = \sum_{n=1}^{\infty} F_n L_{mn}, \quad m = 1, 2, \ldots \]  \hspace{1cm} (4.4.21)
where

\[ L_{mn} = \int_{-d}^{-b} \psi_m(y) \epsilon_n(y) dy \]

\[ = \begin{cases} \sqrt{2} \omega_n \left( \cos \omega_n (d-b) - \cos \alpha_m (d-b) \right) / \left( \omega_n^2 - \alpha_m^2 \right), & \omega_n \neq \alpha_m \\ \sqrt{d-b} \sin \alpha_m (d-b), & \omega_n = \alpha_m. \end{cases} \]

The continuity of the pressure across region 6 and 2 at \( x = 0 \) gives

\[ \sum_{n=1}^{\infty} B_n \xi_n = \sum_{n=1}^{\infty} F_n \epsilon_n. \]

Integrating over \(-b \leq y \leq -a\) after multiplying by \( \xi_m \) and using (4.3.7), the above equation becomes

\[ B_m = \sum_{n=1}^{\infty} F_n Q_{mn}, \quad m = 1, 2, \ldots \quad (4.4.22) \]

where

\[ Q_{mn} = \int_{-b}^{-a} \epsilon_n(y) \xi_m(y) dy \]

\[ = \begin{cases} \frac{1}{\sqrt{d(b-a)}} \left[ \cos \omega_n (d-b) - \cos \omega_m (d-a) \right], & m = 1 \\ - \frac{2}{\sqrt{d(b-a)}} \omega_n \left[ \cos \beta_m (b-a) \cos \omega_n (d-a) - \cos \omega_n (d-b) \right], & \omega_n \neq \beta_m \\ \frac{1}{\sqrt{2d(b-a)}} \left[ - \cos \beta_m (b+b-2a) + \cos \beta_m (d-b) \right] + (b-a) \sin \beta_m (d-b), & \omega_n = \beta_m. \end{cases} \]

The continuity of the pressure across region 6 and 3 at \( x = 0 \) gives

\[ \sum_{n=1}^{\infty} C_n \eta_n + \eta_1 = \sum_{n=1}^{\infty} F_n \epsilon_n. \]

Integrating over \(|y| \leq a\) after multiplying by \( \eta_m \) and using (4.3.13), the above equation becomes

\[ \delta_{1m} + C_m = \sum_{n=1}^{\infty} F_n T_{mn}, \quad m = 1, 2, \ldots \quad (4.4.23) \]

where

\[ T_{mn} = \int_{-a}^{a} \epsilon_n(y) \eta_m(y) dy \]

\[ = \begin{cases} \sqrt{2} \sin \omega_n a \sin \omega_m d / \omega_n, & m = 1 \\ - \omega_n / \sqrt{ad} \left[ - \cos \omega_n (d-a) + \cos \omega_n (a+d) \cos 2 \gamma_m a \right], & \omega_n \neq \gamma_m \\ \frac{1}{2 \sqrt{ad}} \left[ - \cos \gamma_m (3a+d) + \cos \gamma_m (d-a) / \gamma_m - 2 a \sin \gamma_m (a-d) \right], & \omega_n = \gamma_m. \end{cases} \]

The continuity of the pressure across regions 6 and 4 at \( x = 0 \) gives

\[ \sum_{n=1}^{\infty} D_n \zeta_n = \sum_{n=1}^{\infty} F_n \epsilon_n. \]
Integrating over $a \leq y \leq b$ after multiplying by $\zeta_m$ and using (4.3.15), we get

$$D_m = \sum_{n=1}^{\infty} F_n R_{mn}, \quad m = 1, 2... \quad (4.4.24)$$

where

$$R_{mn} = \int_a^b \epsilon_n(y) \zeta_m(y) dy$$

$$= \begin{cases} \sqrt{\frac{2}{d(b-a)}} \frac{\cos \omega_n(b+d) - \cos \omega_n(a+d)}{\omega_n}, & \omega_n \neq \alpha_m \\
- \sqrt{\frac{1}{d(b-a)}} \omega_n \left[ \frac{\cos \omega_n(b+d) - \cos \omega_n(a+d) \cos \beta_m(b-a)}{\omega_n - \beta_m} \right], & m = 1 \\
\sqrt{\frac{1}{2d(b-a)}} \left[ (b-a) \sin \omega_n (b+d) - \frac{\cos \omega_n(b+d) - \cos \omega_n(2b+d-b)}{2\omega_n} \right], & \omega_n = \beta_m. \end{cases}$$

The continuity of the pressure across region 6 and 5 at $x = 0$ gives

$$\sum_{n=1}^{\infty} E_n \sigma_n = \sum_{n=1}^{\infty} F_n \epsilon_n.$$

Integrating over $b \leq y \leq d$ after taking inner product with $\sigma_m$ and using (4.3.18), we obtain

$$E_m = \sum_{n=1}^{\infty} F_n S_{mn}, \quad m = 1, 2,... \quad (4.4.25)$$

where

$$S_{mn} = \int_b^d \epsilon_n(y) \sigma_m(y) dy$$

$$= \begin{cases} \sqrt{\frac{2}{d(d-b)}} \frac{\cos \omega_n d + b}{\omega_n}, & \omega_n \neq \alpha_m \\
\sqrt{\frac{1}{d(d-b)}} \left[ (d-b) \sin \omega_n (b+d) + \frac{\cos \omega_n(b+d) - \cos \omega_n(3d-b)}{2\omega_n} \right], & \omega_n = \alpha_m. \end{cases}$$

### 4.5 Continuity of Velocity

Following the above procedure, the continuities of the velocities across different regions at $x = 0$ give the following set of equations

$$-\tilde{\alpha}_m A_m = \sum_{n=1}^{\infty} \tilde{\omega}_n F_n L_{mn}, \quad m = 1, 2,... \quad (4.5.26)$$

$$-\tilde{\beta}_m B_m = \sum_{n=1}^{\infty} \tilde{\omega}_n F_n Q_{mn}, \quad m = 1, 2,... \quad (4.5.27)$$

$$\tilde{\gamma_1} \delta_{1m} - \tilde{\gamma_m} C_m = \sum_{n=1}^{\infty} \tilde{\omega}_n F_n T_{mn}, \quad m = 1, 2,... \quad (4.5.28)$$

$$-D_m \tilde{\beta}_m = \sum_{n=1}^{\infty} \tilde{\omega}_n F_n R_{mn}, \quad m = 1, 2,... \quad (4.5.29)$$
and

\[ -E_m \tilde{c}_m = \sum_{n=1}^{\infty} \tilde{\omega} F_n S_{mn}, \quad m = 1, 2, \ldots \]  \hspace{1cm} (4.5.30) 

Now we find unknown coefficients in region 1 to region 6 by solving the infinite system of equations (4.4.21) to (4.5.30). For this we limit ourselves to \( m = 1, 2, \ldots M \) and \( n = 1, 2, \ldots 5M \). In figure 4.2, subfigures 1, 3 and 5 are plotted to observe the absolute error value on log scale against \( M \) by using Richardson extrapolation formula while subfigures 2, 4 and 6 present the convergence in reflection as a function of truncation number \( M \). Here the duct dimensions are such that \( a = 1, b = 2, d = 3 \) for \( k = \pi/4, \pi/3 \) and \( 3\pi/2 \). We can see that for \( M \geq 110 \), the reflection becomes insensitive which indicates that the error lies within the line width.

![Graphs showing absolute error and reflection amplitude versus number of equations](image)

Figure 4.2: Absolute error and reflection amplitude versus number of equations for \( a = 1, b = 2 \) and \( d = 3 \)

### 4.6 Numerical Outcomes

In this section we plot \(|R| = |C_1| \) (the coefficient of the lowest mode \( \exp(-i\gamma_1 x) \)) against \( k \) (wave number). This would be of concern to determine the quantity of acoustic energy which is dispersed among the different regions. The energy \((1 - |R|^2)\) coming out from the region \(|y| < a, x < 0\) is distributed among the other waveguide regions. We draw some graphs for \(|R|\) for different dimensions of the duct spacing which would be helpful to predict the acoustic radiated power.

#### 4.6.1 Case: 1

Here the duct spacing is such that \( a = 1, b = \frac{5}{4}, \) and \( d = \frac{3}{2} \) in the frequency range \( 0 < k < \pi \). Figure 4.3 depicts the situation graphically. There is no wave before
which is onset of wave propagating in forward direction for both tri and current pentafurcated problems while for rigid pentafurcated problem there is no mode before $k = \frac{2\pi}{3}$.

Figure 4.3: Absolute reflection versus wave number for $a=1$, $b=1.25$ and $d=1.5$

### 4.6.2 Case: 2

Figure 4.4 is plotted for duct spacing $a = 1$, $b = 2$, $d = 3$ and the frequency range is $0 < k < \pi$. Here the cut on (off) frequencies are at $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$, $\frac{3\pi}{4}$ and $\frac{5\pi}{6}$. Figure 4.5 is plotted to see the effect of reflection for extended frequency range ($0 < k < 2\pi$). Here we observe more abrupt changes as compared to case 1. This shows excitation of more modes in different regions. Similarly figure 4.6 is plotted to have comparison with pentafurcated problems having different boundary conditions; current, rigid [48], outer soft inner hard [56] and soft hard [57].

Figure 4.4: Absolute reflection versus wave number for $a=1$, $b=2$ and $d=3$

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Figure 4.5: Absolute reflection versus wave number for $a=1$, $b=2$ and $d=3$

Figure 4.6: Absolute reflection versus wave number for $a = 1$, $b = 2$ and $d = 3$

4.6.3 Case: 3

In case 3 $|R|$ versus $k$ is plotted for current outer-soft pentafurcated waveguide to observe the effect of varying duct spacing on reflection. Figures 4.7, 4.8 and 4.9 are plotted for reflection field analysis against wave number for pentafurcated duct with frequency range $0 < k < \pi$. In Figure 4.7 the duct spacing is such that 

1) $a = 0.5$ (solid line) 
2) $a = 1$ (dotted line) for fixed width $b = 2$ and $d = 3$.

Figure 4.8 is plotted for

1) $b = 2$ (dotted line) 
2) $b = 2.5$ (solid line) for fixed $a = 1$ and $d = 3$ while figure 4.9 is drawn for

1) $d = 3$ (dotted line) 
2) $d = 4$ (solid line) for set values of $a = 1$ and $b = 2$. 
Figure 4.7: Absolute reflection versus wave number for fixed \( b=2 \) and \( d=3 \)

Figure 4.8: Absolute reflection versus wave number for fixed \( a=1 \) and \( d=3 \)

Figure 4.9: Absolute reflection versus wave number for fixed \( a=1 \) and \( b=2 \)
4.7 Energy Conservation

We can derive an energy conservation equation by using the Green’s theorem \[24\] for one propagating mode in each of the regions 1-5 and consequently five modes in region 6 as follow

\[
\tilde{\alpha}_1 |A_1|^2 + \tilde{\beta}_1 |B_n|^2 + \tilde{\gamma}_1 |C_1|^2 + \tilde{\beta}_1 |D_1|^2 + \tilde{\alpha}_1 |E_1|^2 + \sum_{n=1}^{5} \tilde{\omega}_n |F_n|^2 = \tilde{\gamma}_1. \quad (4.7.31)
\]

4.8 Conclusion

We have worked out the problems of tri (Appendix-B) and penta furcated exhaust models having soft outer lining by using the eigenfunction expansion method, assuming that only lowest mode propagates in region \( y \leq |a|, \ x < 0 \). The same trifurcated problem has been earlier studied in \[5\] by the Wiener-Hopf technique. We have obtained the same results for the absolute reflection as were obtained in \[5\] (see figures B.1 and B.2) which is the confirmation of the precision of our work. Figure 4.3 compares the reflection field behaviour among trifurcated, current and rigid pentafurcated ducts \[48\] when no mode propagates in coaxial regions. We can observe that behaviour of reflection for both tri and outer-soft pentafurcated ducts is exactly the same for \( 0 \leq k \leq \pi \). For \( 0 \leq k \leq 1.2 \) reflection coefficient for rigid pentafurcated duct is less as compared to other two ducts while afterwards this mechanism reverses. Hence acoustic energy \((1 - |R|^2)|\) coming out from outer-soft pentafurcated duct is smaller than that of rigid duct in region \( 0 \leq k \leq 1.2 \). Figure 4.4 shows the reflection field behaviour when two modes propagate in backward direction and three modes travel in forward direction for tri and outer-soft penta pentafurcated ducts while for rigid pentafurcated duct there are two forward and one backward mode. Figures 4.7-4.9 show the effect on reflection for outer-soft pentafurcated duct when one of the width dimension changes while other two are kept fixed. From figures 4.7, we notice that as the width of region \(|y| \leq a\) decreases, the value of \(|R|\) increases which agrees with the physical situation.

We have observed that when there is no wave in the coaxial regions (figure 4.3), the accumulative value of reflection for outer-soft pentafurcated duct (1307.8) is greater than accumulated value of reflection for hard pentafurcated duct (803.3268). While for extended frequency range we can find that accumulated reflection for tri, hard penta and outer-soft penta ducts are 1227.2, 1144.8 and 1375.1 respectively (see figure 4.5). Thus the reflection for outer soft pentafurcated duct is greatest among all considered ducts which results in lesser transmitted acoustic energy. We have also noticed that the value 1375.1 is greater among the other existing pentafurcated problems (\[56\] and \[57\]). Hence the insertion of soft outer lining in pentafurcated duct provides more active noise attenuation which improves the performance of duct.
PART 2
Extraordinary Acoustic Transmission (EAT) and Helmholtz Resonator
Chapter 5

Helmholtz Resonator 1: Symmetry About Both Axes

5.1 Introduction

In this chapter we will study extraordinary transmission in a Helmholtz resonator which has symmetry about both the $x$ and $y$ direction. This is the simplest case of a Helmholtz resonator and is similar to the experimental cases considered by [41, 42], although they had a narrow neck bounding the two regions and their problem was three-dimensional. However, the symmetry properties of both problems are identical. We will show shortly that we can decompose the problem into a symmetric and an anti-symmetric problem. The solution to the original problem is then found by adding these two problems together using the superposition principle. Most of the contents of this chapter have been published in the journal of Wave Motion [58].

![Schematic diagram of the waveguide for the Helmholtz Resonator 1.]

Figure 5.1: Schematic diagram of the waveguide for the Helmholtz Resonator 1.

5.2 Formulation of Problem

Figure 5.1 shows the schematic geometry of the problem. The problem consists of a finite hard inner duct symmetrically placed in an infinite duct with all hard plates. We begin with the Helmholtz equation (3.2.2) to be solved for $\phi(x,y)$
subject to the following boundary conditions

\[ \phi_y = 0, \ y = b, \ -\infty < x < \infty, \] (5.2.1a)
\[ \phi_y = 0, \ y = a, \ -l < x < l, \] (5.2.1b)
\[ \phi_y = 0, \ y = -a, \ -l < x < l, \] (5.2.1c)
\[ \phi_y = 0, \ y = -b, \ -\infty < x < \infty, \] (5.2.1d)
\[ \phi_x = 0, \ x = \pm l, \ -b < y < -a, \] (5.2.1e)
\[ \phi_x = 0, \ x = \pm l, \ a < y < b. \] (5.2.1f)

In addition, we have radiation conditions which guarantee a bounded solution.

### 5.3 Decomposition using Symmetry

The problem is symmetric about \( x = 0 \) and this means that we can decompose the problem into one which is symmetric and one which is anti-symmetric about \( x = 0 \). We will show that this decomposition has important consequences for EAT. We solve the problem using bespoke code, because we are interested in computing the solution for complex frequencies, which is not possible with most numerical codes. We use mode matching, and we only need to consider the solution for \( x < 0 \) (the solution for \( x > 0 \) following from appropriate symmetry). This problem has a further symmetry about \( y = 0 \) which means that if the incident mode is even about \( y = 0 \) (which is the case for the fundamental mode) then only even modes will be excited.

We break the solution into the symmetric and anti-symmetric solutions, which we denote by \( \phi^{(s)}(x,y) \) and \( \phi^{(a)}(x,y) \) respectively. The solution is written only for \( x < 0 \) and it is extended to \( x > 0 \) using the appropriate symmetry. We divide the solution into two regions, region 1 where \( \{ -b \leq y \leq b, \ -\infty < x < -l \} \) and region 2 where \( \{ -a \leq y \leq a, \ -l < x < 0 \} \). In each of these regions we expand the solution using mode matching. We also impose the appropriate condition from the symmetry at \( x = 0 \).

### 5.4 Symmetric Solution

#### 5.4.1 Region 1 \( \{ -b \leq y \leq b, \ -\infty < x < -l \} \)

For \( x < -l \) the symmetric solution is given by

\[ \phi^{(s)}(x,y) = \sum_{n=0}^{\infty} A_n^{(s/a)} e^{-i\tilde{\alpha}_n(x+l)} \psi_{2n}(y) + e^{i\tilde{\alpha}_0(x+l)} \psi_0(y), \] (5.4.2)

which satisfies equations (3.2.2), (5.2.1a), (5.2.1d) and the radiation condition. We define eigenvalues \( \alpha_n \) and associated eigenvalues \( \tilde{\alpha}_n \) by

\[ \alpha_n = \frac{n\pi}{2b}, \quad n = 0, 1, 2, \ldots \] (5.4.3)

and

\[ \tilde{\alpha}_n = \sqrt{(k^2 - \alpha_n^2)}, \quad n = 0, 1, 2, \ldots \] (5.4.4)

respectively. Note that we retain the odd modes here because we will use them subsequently. The sign of the square root in equation (5.4.4) is chosen to be
positive real or to have positive imaginary part so that the solution (5.4.2) satisfies the radiation conditions. The eigenfunctions \( \psi_n(y) \) are defined by

\[
\psi_n(y) = \begin{cases} 
\sqrt{\frac{1}{b}} \cos \alpha_n(y - b), & n \neq 0, \\
\sqrt{\frac{1}{2b}}, & n = 0,
\end{cases}
\] (5.4.5)

which are orthonormal, i.e.

\[
\int_{-b}^{b} \psi_m(y) \psi_n(y) y = \delta_{mn}.
\] (5.4.6)

**5.4.2 Region 2 \( \{-a \leq y \leq a, \ -l < x < 0\} \)**

The solution for \(-l < x < 0\) is very similar. The vertical eigenfunctions are given by

\[
\xi_n(y) = \begin{cases} 
\sqrt{\frac{1}{a}} \cos \beta_n(y - a), & n \neq 0, \\
\sqrt{\frac{1}{2a}}, & n = 0,
\end{cases}
\] (5.4.7)

The eigenvalues and the associated eigenfunctions are given by

\[
\beta_n = \frac{n\pi}{2a},
\]

and

\[
\tilde{\beta}_n = \sqrt{k^2 - \beta_n^2}, \quad n = 0, 1, 2, \ldots
\]

They satisfy the equation

\[
\int_{-a}^{a} \xi_n(y) \xi_m(y) y = \delta_{mn}.
\] (5.4.8)

The symmetric solution in region 2 can be written as

\[
\phi^{(s)}(x, y) = \sum_{n=0}^{\infty} B_n^{(s)} e^{-i\tilde{\alpha}_2 n x} \psi_{2n}(y) + e^{i\tilde{\alpha}_0 (x+l)} \psi_{2}(y),
\] (5.4.9)

which satisfies equations (3.2.2), (5.2.1b) and (5.2.1c) at the appropriate condition of symmetry or antisymmetry respectively.

**5.5 Antisymmetric Solution**

**5.5.1 Region 1 \( \{-b \leq y \leq b, \ -\infty < x < -l\} \)**

Antisymmetric solution in this region is same as we found in symmetric case and is given by

\[
\phi^{(a)}(x, y) = \sum_{n=0}^{\infty} A_n^{(a)} e^{-i\tilde{\alpha}_2 n (x+l)} \psi_{2n}(y) + e^{i\tilde{\alpha}_0 (x+l)} \psi_{0}(y),
\] (5.5.10)

which satisfies equations (3.2.2), (5.2.1a), (5.2.1d) and the radiation condition.
5.5.2 Region 2 \( \{-a \leq y \leq a, \quad -l < x < 0\} \)

The general antisymmetric solution for the potential in the region 2 can be written

\[
\phi^{(a)}(x, y) = \sum_{n=0}^{\infty} B_n^{(a)} \frac{\sin \beta_{2n} x}{-\sin \beta_{2n} l} \xi_{2n}(y) \tag{5.5.11}
\]

The total solution for an incident wave traveling from the left is found by combining the solutions and is given by

\[
\phi = \begin{cases} \frac{1}{2} \left( \phi^{(s)}(x, y) + \phi^{(a)}(x, y) \right), & x < 0, \\ \frac{1}{2} \left( \phi^{(s)}(-x, y) - \phi^{(a)}(-x, y) \right), & x > 0. \end{cases} \tag{5.5.12}
\]

5.6 Formulation of the System of Equations

We derive the system of equations which arises from mode matching. The presentation focuses on the symmetric case which we consider first. The continuity of the pressure across \( x = -l \) gives

\[
\sum_{n=0}^{\infty} A_n^{(s)} \psi_{2n}(y) + \psi_0(y) = \sum_{n=0}^{\infty} B_n^{(s)} \xi_{2n}(y), \quad -a \leq y \leq a. \tag{5.6.13}
\]

By taking the inner product with \( \xi_{2m}(y) \) and integrating over \([-a, a]\) we obtain

\[
\sum_{n=0}^{\infty} A_n^{(s)} \int_{-a}^{a} \xi_{2m}(y) \psi_{2n}(y) dy + \int_{-a}^{a} \xi_{2m}(y) \psi_0(y) dy = \sum_{n=0}^{\infty} B_n^{(s)} \int_{-a}^{a} \xi_{2n}(y) \xi_{2m}(y) dy.
\]

Using equation (5.4.8), this can be written as

\[
\sum_{n=0}^{\infty} l_{mn} A_n^{(s)} + l_{m0} = B_m^{(s)}, \quad m = 0, 1, 2, \ldots \tag{5.6.15}
\]

where

\[
l_{mn} = \int_{-a}^{a} \psi_{2n}(y) \xi_{2m}(y) dy, \tag{5.6.16}
\]

\[
l_{mn} = \begin{cases} \frac{a}{b}, & m = n = 0, \\ \frac{\sin \alpha_{2n}(a - b) + \sin \alpha_{2n}(a + b)}{\sqrt{2ab\alpha_{2n}}}, & m = 0, n \neq 0, \\ 0, & n = 0, m \neq 0, \\ \frac{\sin \alpha_{2n}(a - b) + \sin \alpha_{2n}(3a + b) + 4\alpha_{2n} \cos \alpha_{2n}(b - a) - 4\sqrt{ab\alpha_{2n}\alpha_{2n}^2 + \beta_{2m}^2}}{\sqrt{ab(\alpha_{2n}^2 - \beta_{2m}^2)}}, & \alpha_{2n} = \beta_{2m}, \\ \frac{\sin \alpha_{2n}(a - b) + \sin \alpha_{2n}(a + b) \cos 2\alpha_{2n}a}{\sqrt{ab(\alpha_{2n}^2 - \beta_{2m}^2)}}, & \text{other cases}. \end{cases}
\]

The continuity of the velocity of the potential across \( x = -l \)

\[
-\sum_{n=0}^{\infty} iA_n^{(s)} \bar{\alpha}_{2n} \psi_{2n}(y) + i\bar{\alpha}_0 \psi_0(y) = \begin{cases} 0, & -b \leq y \leq -a, \\ \sum_{n=0}^{\infty} B_n^{(s)} \xi_{2n}(y) \bar{\beta}_{2n} \tan \bar{\beta}_{2n} l, & -a \leq y \leq a, \\ 0, & a \leq y \leq b. \end{cases}
\]
Multiplying by $\psi_{2m}(y)$ and integrating over $[-b, b]$, we get

$$-\sum_{n=0}^{\infty} i\tilde{\alpha}_{2n} A_n^{(s)} \int_{-b}^{b} \psi_{2m}(y)\psi_{2n}(y) dy$$

$$+ i\tilde{\alpha}_0 \int_{-b}^{b} \psi_{2m}(y)\psi_0(y) dy = \sum_{n=0}^{\infty} B_n^{(s)} \tilde{\beta}_{2n} \tan \tilde{\beta}_{2n} l, \quad m = 0, 1, ...$$

By using equation (5.4.6) and (5.6.16) the above equation becomes

$$-i\tilde{\alpha}_{2m} A_m^{(s)} + i\tilde{\alpha}_0 \delta_{m0} = \sum_{n=0}^{\infty} B_n^{(s)} l_{nm} \tilde{\beta}_{2n} \tan \tilde{\beta}_{2n} l, \quad m = 0, 1, ...$$

We therefore have the following system of equations to be solved for the expansion constants

$$\sum_{n=0}^{\infty} A_n^{(s)} l_{mn} + l_{m0} = B_m^{(s)}, \quad m = 0, 1, 2, ...$$

$$-i\tilde{\alpha}_{2m} A_m^{(s)} + i\tilde{\alpha}_0 \delta_{m0} = \sum_{n=0}^{\infty} B_n^{(s)} l_{nm} \tilde{\beta}_{2n} \tan \tilde{\beta}_{2n} l, \quad m = 0, 1, 2, ...$$

The derivation for the antisymmetric solution is almost identical and we are lead to the following system of equations

$$\sum_{n=0}^{\infty} A_n^{(a)} l_{mn} + l_{m0} = B_m^{(a)}, \quad m = 0, 1, 2, ...$$

and

$$-i\tilde{\alpha}_{2m} A_m^{(a)} + i\tilde{\alpha}_0 \delta_{m0} = -\sum_{n=0}^{\infty} B_n^{(a)} l_{nm} \tilde{\beta}_{2n} \cot \tilde{\beta}_{2n} l, \quad m = 0, 1, 2, ...$$

Note that the symmetric and antisymmetric solutions are almost identical so that only the smallest change to the numerical code for the symmetric problem is required to solve the antisymmetric problem.

To solve these equations numerically, we restrict ourselves to a finite number of modes. We do not need to have the same number of modes on each side and the finite system of equations is, for the symmetric case with $M$ modes for $x < -l$ and $N$ modes for $-l < x < 0$,

$$\sum_{n=0}^{M} A_n^{(s)} l_{mn} + l_{m0} = B_m^{(s)}, \quad m = 0, 1, 2, ..., N$$

and

$$-i\tilde{\alpha}_{2m} A_m^{(s)} + i\tilde{\alpha}_0 \delta_{m0} = \sum_{n=0}^{N} B_n^{(s)} l_{nm} \tilde{\beta}_{2n} \tan \tilde{\beta}_{2n} l, \quad m = 0, 1, 2, ..., M.$$

We validate our code by ensuring that the solution satisfies the matching conditions as well as energy conservation.
5.7 Reflection and Transmission

We are interested here primarily in the far field reflection and transmission for a wave incident from the left. From equation (??) the solution to the problem of a wave of unit amplitude incident from the left propagating through the Helmholtz resonator is given by averaging the symmetric and antisymmetric solutions. The far field reflection for the first mode is

\[ R_0(k) = \frac{A_0^{(s)} + A_0^{(a)}}{2}, \]  

and the far field transmission for the first mode is

\[ T_0(k) = \frac{A_0^{(s)} - A_0^{(a)}}{2}. \]

5.8 Energy Conservation

Conservation of energy tells us that, provided there is only one propagating mode, which for our current example due to the symmetry requires that \( k < \pi/b \)

\[ |A_0^{(s)}(k)| = |A_0^{(a)}(k)| = 1, \quad k < \frac{\pi}{b}. \]  

There are many other energy balance relations which can be derived and further examples can be found in [51]

5.9 Numerical Results

The absolute value of the reflection coefficient \(|R_0|\) versus the truncation number \(M\) (with \(N = M/2\)) and the absolute error are show in figure 5.2 for \(l = 2, a = 1, b = 2\) and \(k = 6\). We can see that the absolute error in the reflection coefficient becomes linear and the absolute value of the reflection becomes accurate to line width after \(M\) is ten.
Figure 5.2: The convergence of the reflection coefficient $R_0$ as a function of $M$. Subfigure a shows the absolute value of the reflection coefficient and subfigure b shows the absolute error. The parameters are $l = 2, a = 1, b = 2$ and $k = 6$.

Figures 5.3 to 5.6 show the absolute value of the reflection $|R_0|$ and transmission $|T_0|$ versus wavenumber $k$. The values are $b = 2$ for all figures and $a = 0.5$, $l = 5$, (figure 5.3) $a = 0.5$, $l = 20$, (figure 5.4), $a = 0.125$, $l = 5$, (figure 5.5) and $a = 0.125$, $l = 20$, (figure 5.6). We see a pattern of EAT below $k = \pi/2$. There are regions of enhanced transmission above $k = \pi/2$ but it is nothing like the perfect transmission obtained below $k = \pi/2$. 
Figure 5.3: Reflection and Transmission against wavenumber $k$ for $l = 5$, $a = 0.5$, $b = 2$ for frequency range $0 < k < \pi$
Figure 5.4: Reflection and Transmission against wavenumber $k$ for $l = 20$, $a = 0.5$, $b = 2$ for frequency range $0 < k < \pi$
Figure 5.5: Reflection and Transmission against wavenumber for $l = 5$, $a = 0.125$, $b = 2$ for frequency range $0 < k < \pi$
We now offer an explanation of what is causing the EAT. We obtain perfect transmission when $A_0^{(a)} = -A_0^{(a)}$ (provided that there is only one propagating mode). Since they both have modulus one this requirement is that they are in anti-phase. We now show how the presence of a Helmholtz resonator coupled with this symmetry gives rise to exactly this anti-phase relation.

Figure 5.7 shows the $\arg(A_0^{(a)}(k))$ and $\arg(A_0^{(a)}(k))$ for $b = 2$ and $l = 5$ for two values of $a$, $a = 0.125$ and $a = 0.5$. Note that we have plotted the argument as a continuous function. We notice abrupt changes of phase by $2\pi$ at certain values. Exactly this phase change has been observed to be associated with EAT experimentally [41, 42]. We will see shortly that mathematically the changes in phase are associated with both resonances and perfect transmission. We also note that above a frequency of $\pi/2$ we no longer have this abrupt change of phase. Figure 5.8 is the same plot but with $l = 20$. 

Figure 5.6: Reflection and Transmission against wavenumber for $l = 20$, $a = 0.125$, $b = 2$ for frequency range $0 < k < \pi$
Figure 5.7: $\arg(A_0^{(s)})$ and $\arg(A_0^{(a)})$ versus wavenumber $k$ for $b = 2$ and $l = 5$. The solid line is for $a = 0.125$, and the dashed line is for $a = 0.5$. 
Figure 5.8: \(\arg(A_{0}^{s})\) and \(\arg(A_{0}^{a})\) versus wavenumber \(k\) for \(b = 2\) and \(l = 20\). The solid line is for \(a = 0.125\), and the dashed line is for \(a = 0.5\).

5.10 Solution for complex \(k\)

So far, while we have solved for the configuration, we have not considered the condition of a Helmholtz resonator. There are many ways of thinking about what creates a Helmholtz resonator but the key idea is that there is a mode of vibration which leaks a small amount of energy into the surrounding system. If there was no leaking of energy the system would be self-adjoint and there would be an eigenvalue on the real axis. This means that the solution is not invertible at this point which appears as a singularity in the resolvent which is on the real axis. When there is leaking of energy this real eigenvalue becomes a complex resonance. In some very real sense the singularity cannot be created or destroyed. It simply moves into the complex plane. If the system is close to resonant the singularity will remain close to the real axis and this is precisely what gives rise to a Helmholtz resonator. We will see shortly that the singularities move around the complex plane but they are never created or destroyed (although they can appear from a Riemann surface).

We need to develop a method to find this complex resonance and we do this by extending the functions \(|A_{n}^{s}(k)|\) and \(|A_{n}^{a}(k)|\) to complex \(k\) values analytically. When we look at the system of equations we need to calculate the reflection coefficients we see that the only place where the parameter \(k\) appears is in the computation of the roots \(\tilde{\alpha}_{n}\) and \(\tilde{\beta}_{n}\). We therefore just need to compute the roots for complex \(k\), the only difficulty being that we need to choose the roots consistently with the choice for real \(k\); otherwise the analyticity will be broken.
To achieve the analyticity extension from real $k$ to complex $k$ we find roots of $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ using a homotopy method, which is an iterative method to solve nonlinear systems. We define

$$\tilde{\alpha}_n(\theta) = \sqrt{|k|^2 e^{2i\theta} - \alpha_n^2}, \quad n = 0, 1, 2, \ldots$$

Note that this equation has roots in plus and minus pairs. In our problem for real $k$ we only choose one from each pair and we need to insure that for complex $k$ we make the same choice. We solve this equation first for $\theta = 0$ and choose the roots according to the rules for real $k$. We then slowly vary the angle, using the previously computed solutions as the initial guess to solve for the roots. We terminate when $\theta = \arg(k)$. This essentially allows us to track the roots as they move in the complex plane and insures we always have made the correct choice to preserve analyticity. We can validate the analyticity numerically by taking a numerical derivative in different complex directions and checking we have the same result (to numerical error).

It is not easy to visualize a complex function but we do so using the method developed by Wegert [52] and the numerical tools which accompany this book. Figure 5.9 is a frame from Movie 1 which is in the supplementary material. The top picture is $|R_0|$ versus real $k$. The inset shows the geometry with $a = 0.5$. The bottom picture is a visualization of the analytic extension of $R_0$ versus complex $k$. The colour shows the phase information and the height is the shade. Also shown are the positions of the zeros (green) and poles (black).

These figures and the movie given in the supplementary material are the key to understanding EAT. We see that there is a branch cut at $k = \pi/2$. Below this cut-off the poles and zeros are complex conjugates of each other. The poles and zeros move around and but they cannot be created or destroyed, although one can be seen appearing from the Riemann surface. Note that the strange behaviour we see in the reflection coefficient at $\pi/2$, sometimes called the Wood anomaly, is explained exactly by this branch cut. The slight flickering seen in the movies we believe is real as it is not caused by numerical convergence.
Figure 5.9: The top picture is $|R_0|$ versus real $k$. The inset shows the geometry with $a = 0.5$. The bottom picture is a visualization of the analytic extension of $R_0$ versus complex $k$. The colour shows the phase information and the height is the shade. Also shown are the positions of the zeros (green) and poles (black). The full animation can be seen in Movie 1 in the supplementary material.
Figure 5.10: The top picture is $|R_0|$ versus real $k$. The inset shows the geometry with $a = 0.25$. The bottom picture is a visualization of the analytic extension of $R_0$ versus complex $k$. The colour shows the phase information and the height is the shade. Also shown are the positions of the zeros (green) and poles (black). The full animation can be seen in Movie 1 in the supplementary material.
5.11 Blaschke product

We know that, provided that there is only a single propagating mode (i.e. \( k < \pi/2 \)) \(|A^0_s(k)| = |A^0_a(k)| = 1\). This means that the poles and zeros of the analytic extension of these function must be at complex conjugate values. Such an expansion must take the form of a product of an exponential of an function and a Blaschke product. This allows us to approximate our functions by

\[
A^0_s(k) \approx e^{if(k)} \prod_i \left( \frac{k - k_i}{k - \kappa_i} \right),
\]

where the bar denotes complex conjugate. Note that there are two versions of the Blaschke product, one on the unit circle and one on the real axis, related via a Cayley transform. Here we use the one defined on the unit circle. The conjugation insures that the function has modulus one for real values of \( k \). There is no requirement that the function \( f(k) \) have special properties except that it takes real values for real \( k \). However, in practice this function is usually slowly varying and for our example well approximated by \( f(k) = 0 \). As the function passes between the pole and zero on the real axis there is a change of phase by \( 2\pi \).

Figures 5.11 and 5.12 compare the exact solution and the approximate solution using Blaschke products for \( b = 2 \), and \( L = 5 \) with \( a = 0.5 \) and \( 0.125 \) respectively. The absolute value of the reflection coefficient \(|R_0|\) is given as a black solid line. Note that this curve corresponds to exactly the same one which has appeared in figures 5.3 and 5.9 (figure 5.11) and figures 5.5 and 5.10 (figure 5.12). Also shown is the approximation as a sum of Blaschke products using only the poles and zeros with real part below \( k = \pi/2 \) (red dashed line), for which the poles and zeros are complex conjugate. We also show the approximation using all the poles and zeros (green chained line) written as

\[
A^0_s(k) \approx \prod_i \left( \frac{k - k_i}{k - \kappa_i} \right),
\]

where \( k_i \) are the zeros and \( \kappa_i \) are the poles. Note that for real part greater than \( k = \pi/2 \) the poles and zero are no longer complex conjugates and that while the conjugacy requires energy conservation the approximation does not. Obviously perfect transmission would be lost without energy conservation. We can see that the Blaschke product with the conjugate poles and zeros very well approximates the exactly solution above \( \pi/2 \). Including the subsequent poles and zeros actually makes the comparison worse in the case when \( a = 0.5 \) below \( k = \pi/2 \) because of the effect of the branch cut at \( k = \pi/2 \). However we get good agreement above \( k = \pi/2 \) especially in the case \( a = 0.125 \) when including all the poles and zeros.
Figure 5.11: Absolute value of the reflection coefficient $|R_0|$ as a function of $k$ for $a = 0.5$, $b = 2$, and $L = 5$ (black solid line). Also shown is the approximation as a sum of Blaschke products using only the poles and zeros with real part below $\pi/2$ (red dashed line) and with all the poles and zeros (green chained line).
Figure 5.12: Absolute value of the reflection coefficient $|R_0|$ as a function of $k$ for $a = 0.125$, $b = 2$, and $L = 5$ (black solid line). Also shown is the approximation as a sum of Blaschke products using only the poles and zeros with real part below $\pi/2$ (red dashed line) and with all the poles and zeros (green chained line).

5.12 Conclusion

In this chapter the phenomenon of extraordinary acoustic transmission (EAT) in a Helmholtz resonator, which has recently been investigated experimentally, has been studied theoretically. It is shown that the combination of a single propagating mode and a symmetry orthogonal to the direction of propagation for a Helmholtz resonator leads to EAT. This has been accomplished by decomposing the problem using symmetry, the Blaschke product and the properties of functions of a single complex variable which have modulus one on the real axis. The conditions of a Helmholtz resonator requires that the solution has singularities in the analytic extension to complex frequencies (resonances) and it is precisely near these resonances that we observe EAT. The condition of a Blaschke product requires that there is a zero at the complex conjugate of the singularity and EAT occurs when the solution on the real axis passes between these complex conjugate pairs of poles and zeros. A detailed numerical study of the problem is conducted.
Chapter 6

Helmholtz Resonator 2: Symmetry in the \( x \) direction only

6.1 Introduction

We now slightly change the geometry of the problem discussed in last chapter to remove the symmetry about the line \( y = 0 \) but retain the symmetry about \( x = 0 \). This still allows the decomposition into symmetric and antisymmetric solutions and leads to perfect transmission as we will see. However, we cannot use the decomposition in even and odd modes in the \( y \) direction and this means that the cut-off frequency for perfect transmission is altered. We will show that EAT still occurs but branch cut will be shifted to \( \pi/4 \). A schematic diagram of this problem is shown in Figure 6.1. Most of the contents of this chapter have been published in the journal of *Wave Motion* [58].

![Figure 6.1: Schematic diagram of the waveguide for the Helmholtz Resonator 2.](image-url)
6.2 Formulation of Problem

We solve the Helmholtz equation (3.2.2) for $\phi(x, y)$ subject to the following boundary conditions

$$
\begin{align*}
\phi_y &= 0, \quad y = b, \quad -\infty < x < \infty, \\
\phi_y &= 0, \quad y = a, \quad -l < x < l, \\
\phi_y &= 0, \quad y = c, \quad -l < x < l, \\
\phi_y &= 0, \quad y = -b, \quad -\infty < x < \infty, \\
\phi_x &= 0, \quad x = \pm l, \quad -b < y < c, \\
\phi_x &= 0, \quad x = \pm l, \quad a < y < b.
\end{align*}
$$

In addition, we impose the appropriate radiation conditions.

6.3 Solution of Problem

We split waveguide into two regions for convenience. Potential functions in these regions will be found by using separation of variables.

6.3.1 Region 1 $\{-b \leq y \leq b, \quad -\infty < x < -l\}$

In this region the solution is similar to the one we found previously. We focus on the symmetric solution as the anti-symmetric solution is almost identical. The symmetric solution is given by

$$
\phi^{(s)}(x, y) = \sum_{n=0}^{\infty} A_n^{(s)} e^{-i\tilde{\alpha}_n(x+l)} \psi_n(y) + e^{i\tilde{\alpha}_0(x+l)} \psi_0(y).
$$

We define eigenvalues $\alpha_n$ and associated eigenvalues $\tilde{\alpha}_n$ by

$$
\alpha_n = \frac{n\pi}{2b}, \quad n = 0, 1, 2, \ldots
$$

and

$$
\tilde{\alpha}_n = \sqrt{(k^2 - \alpha_n^2)}, \quad n = 0, 1, 2, \ldots
$$

respectively.

The eigenfunctions $\psi_n(y)$ are defined by

$$
\psi_n(y) = \begin{cases} 
\sqrt{\frac{1}{b}} \cos \alpha_n(y - b), & n \neq 0, \\
\sqrt{\frac{1}{2b}}, & n = 0,
\end{cases}
$$

which are orthonormal, i.e.

$$
\int_{-b}^{b} \psi_m(y) \psi_n(y) y = \delta_{mn}.
$$
6.3.2 Region 2 \( \{ c \leq y \leq b, \ -l < x < 1 \} \)

Vertical orthonormal eigenfunctions in this region are

\[
\xi_n(y) = \begin{cases} 
\sqrt{\frac{2}{a-c}} \cos \beta_n(y-c), & n \neq 0, \\
\sqrt{\frac{1}{a-c}}, & n = 0,
\end{cases}
\] (6.3.7)

which satisfy the relation

\[
\int_c^a \xi_n(y) \xi_m(y) dy = \delta_{mn}.
\] (6.3.8)

Here eigenvalues and the associated eigenvalues are given by

\[
\beta_n = \frac{n\pi}{a-c},
\]

and

\[
\tilde{\beta}_n = \sqrt{k^2 - \beta_n^2}, \quad n = 0, 1, 2, \ldots
\]

respectively.

As before the general solution for the symmetric and antisymmetric potentials in the region 2 can be written as

\[
\phi(s)(x,y) = \sum_{n=0}^{\infty} B_n(s) \cos \beta_n x \xi_n(y),
\] (6.3.9)

\[
\phi(a)(x,y) = \sum_{n=0}^{\infty} B_n(a) \sin \beta_n x \xi_n(y)
\] (6.3.10)

respectively. The total solution for an incident wave traveling from the left is found by combining the solutions and is given by

\[
\phi = \begin{cases} 
\frac{1}{2} (\phi(s)(x,y) + \phi(a)(x,y)), & x < 0, \\
\frac{1}{2} (\phi(s)(-x,y) - \phi(a)(-x,y)), & x > 0.
\end{cases}
\] (6.3.11)

6.4 System of Equations

We will match potential and velocity of potential to get one infinite set of unknown coefficients which will be determined by MATLAB programming.

6.4.1 Continuity of Potential

We first focus on symmetric solution. The continuity of the pressure across \( x = -l \) gives

\[
\sum_{n=0}^{\infty} A_n(s) \psi_n(y) + \psi_0(y) = \sum_{n=0}^{\infty} B_n(s) \xi_n(y)
\] (6.4.12)

By taking the inner product with \( \xi_m(y) \), integrating over \( [c,a] \) and using (6.3.8), we obtain

\[
\sum_{n=0}^{\infty} A_n(s) p_{mn} + p_{m0} = B_m(s), \quad m = 0, 1, 2, \ldots
\] (6.4.13)
where
\[ p_{mn} = \int_{c}^{a} \psi_n(y) \xi_m(y) dy \] (6.4.14)

\[
\begin{cases}
\sqrt{\frac{a-c}{2b}}, & m = n = 0, \\
\frac{\sin \alpha_n(a-b) - \sin \alpha_n(c-b)}{\sqrt{b(a-c)}}, & m = 0, n \neq 0, \\
0, & n = 0, m \neq 0, \\
-\sin \alpha_n(c-b) + \sin \alpha_n(2a-b-c) + 2\alpha_n(a-c) \cos \alpha_n(c-b), & \alpha_n = \beta_m, \\
-\sqrt{2\alpha_n}[\sin \alpha_n(c-b) - \sin \alpha_n(a-b) \cos \beta_m(a-c)], & \text{other cases.}
\end{cases}
\]

\[ m \leq n \leq 975 \]

### 6.4.2 Continuity of velocity of Potential

The continuity of the velocity of the potential across \( x = -l \) gives the following equation

\[-\sum_{n=0}^{\infty} iA_n^{(s)} \tilde{\alpha}_n \psi_n(y) + i\tilde{\alpha}_0 \psi_0(y) = \begin{cases}
0, & -b \leq y \leq -a, \\
\sum_{n=0}^{\infty} B_n^{(s)} \tilde{\xi}_n(y) \tilde{\beta}_n \tan \tilde{\beta}_n l, & -a \leq y \leq a, \\
0, & a \leq y \leq b.
\end{cases} \]

Taking the inner product with \( \psi_m(y) \) and integrating over \([-b, b]\), we obtain

\[-\sum_{n=0}^{\infty} \tilde{\alpha}_n A_n^{(s)} \int_{-b}^{b} \psi_m(y) \psi_n(y) dy + i\tilde{\alpha}_0 \int_{-b}^{b} \psi_m(y) \psi_0(y) dy = \sum_{n=0}^{\infty} B_n^{(s)} \tilde{\beta}_n \tan \tilde{\beta}_n l \int_{-a}^{a} \xi_n(y) \psi_m(y) dy. \] (6.4.15)

By using orthogonality of eigenfunctions

\[-i\tilde{\alpha}_m A_m^{(s)} + i\tilde{\alpha}_0 \delta_{m0} = \sum_{n=0}^{\infty} B_n^{(s)} p_{nm} \tilde{\beta}_n \tan \tilde{\beta}_n l, \quad m = 0, 1, 2, \ldots \] (6.4.16)

As before the antisymmetric problem is nearly identical and is given by

\[ \sum_{n=0}^{\infty} A_n^{(a)} p_{mn} + p_{m0} = B_m^{(a)}, \quad m = 0, 1, 2, \ldots \] (6.4.17)

\[-i\tilde{\alpha}_m A_m^{(a)} + i\tilde{\alpha}_0 \delta_{m0} = -\sum_{n=0}^{\infty} B_n^{(a)} p_{nm} \tilde{\beta}_n \cot \tilde{\beta}_n l, \quad m = 0, 1, 2, \ldots \] (6.4.18)

### 6.5 Numerical Results

We can see the same perfect transmission for the Helmholtz resonator 2 as we have seen for the Helmholtz resonator 1 except the cutoff frequency is halved. Figures 6.2 and 6.3 show the absolute value of the reflection and transmission for and \( b = 2 \) and \( L = 5 \) for \( a = 1.5, c = 1 \) (Figure 6.2) and for \( a = 1.125, c = 0.975 \) (Figure 6.3). The first cutoff frequency is now at \( k = \pi/4 \) because we no longer have symmetry about \( y = 0 \).
Figure 6.2: The reflection and transmission against wavenumber $k$ for Helmholtz resonator 2 for and $b = 2$ and $L = 5$ for $a = 1.5$, $c = 1$
Figure 6.3: The reflection and transmission against wavenumber $k$ for Helmholtz resonator 2 for $a = 1.125$, $c = 0.875$.

We show the solution in the complex plane as a movie as we vary the position of the duct. Figures 6.4 and 6.5 are taken from Movie 2 in the supplementary material. For this case we set $b = 2$, and keep the distance $a - c$ to be a constant $a - c = 0.5$. This movie shows the appearance of the new branch cut at $k = \pi/4$ which appears when the symmetry is broken.
Figure 6.4: As in Figure 5.9 except for the Helmholtz resonator 2 with \( l = 5, \ a = 1, \ b = 2, \ c = 0.5 \) for frequency range \( 0 < k < \pi \). The full animation can be seen in Movie 2 in the supplementary material
Figure 6.5: As in Figure 5.9 except for the Helmholtz resonator 2 with $l = 5$, $a = 1.5$, $b = 2$, $c = 1$ for frequency range $0 < k < \pi$. The full animation can be seen in Movie 2 in the supplementary material.

### 6.6 Conclusion

In this chapter we considered the Helmholtz resonator 2 which consists of similar finite hard duct as we considered in last chapter but it is not symmetrically placed within an infinite hard duct. We used mode matching technique to find the solution of the problem. We found that once the symmetry along x-axis is broken EAT still exists (provided that there is only one propagating mode) but now cut-off frequency is halved. Significant numerical results are presented including movie 2 which is included as supplementary material.
Chapter 7

Helmholtz Resonator 3:
No symmetry along either axis

7.1 Introduction
We now consider a problem in which we have removed both symmetries. The problem is shown in Figure 7.1. This situation can be thought of as being formed by gluing two Helmholtz resonators of type 2 together. We outline the solution method rather briefly and the details of the matching equations can be derived from the previous chapters. In this case we cannot use the decomposition into symmetric and anti-symmetric solutions which means that the numerical solution is more difficult and we need to match at each boundary. Most of the contents of this chapter have been published in the journal of Wave Motion [58].

7.2 Formulation of the Problem
Helmholtz equation (3.2.2) is to be solved for $\phi(x, y)$ subject to following boundary conditions

$$\phi_y = 0, \quad y = \pm b, \quad -\infty < x < -l^{-}, \quad (7.2.1a)$$
$$\phi_y = 0, \quad y = c^{-}, \quad -l^{-} < x < l, \quad (7.2.1b)$$
$$\phi_y = 0, \quad y = a^{-}, \quad -l^{-} < x < l, \quad (7.2.1c)$$
$$\phi_y = 0, \quad y = \pm b, \quad -l < x < l, \quad (7.2.1d)$$
$$\phi_y = 0, \quad y = c^{+}, \quad l^{-} < x < l^{+}, \quad (7.2.1e)$$
$$\phi_y = 0, \quad y = a^{+}, \quad l < x < l^{+}, \quad (7.2.1f)$$
$$\phi_y = 0, \quad y = \pm b, \quad l^{+} < x < \infty, \quad (7.2.1g)$$

$$\phi_x = 0, \quad x = \pm l^{\pm}, \quad -b < y < c^{\pm}, \quad (7.2.1h)$$
$$\phi_x = 0, \quad x = \pm l, \quad -b < y < c^{\pm}, \quad (7.2.1i)$$
$$\phi_x = 0, \quad x = \pm l^{\pm}, \quad a^{\pm} < y < b, \quad (7.2.1j)$$

In addition, we have to impose the radiation conditions.
7.3 Solution of the Problem

The mode matching problem is now more complicated and we need to expand the solution in five different regions and match at each boundary.

7.3.1 Region 1\{-b \leq y \leq b, \ -\infty < x < -l^-\}

Here the solution is

\[ \phi(x, y) = \sum_{n=0}^{\infty} A_n e^{-\tilde{\alpha}_n(x+l^-)} \psi_n(y) + e^{i\tilde{\alpha}_0(x+l^-)} \psi_0(y), \] (7.3.2)

which satisfies equation (3.2.2), (7.2.1a), and radiation condition. We define eigenvalues \( \alpha_n \) and associated eigenvalues \( \tilde{\alpha}_n \) by

\[ \alpha_n = \frac{n\pi}{2b}, \quad n = 0, 1, 2, \ldots \] (7.3.3)

and

\[ \tilde{\alpha}_n = \sqrt{(k^2 - \alpha_n^2)}, \quad n = 0, 1, 2, \ldots \] (7.3.4)

respectively.

The eigenfunctions \( \psi_n(y) \) are defined by

\[ \psi_n(y) = \begin{cases} \sqrt{\frac{1}{b}} \cos \alpha_n(y - b), & n \neq 0, \\ \sqrt{\frac{1}{2b}}, & n = 0, \end{cases} \] (7.3.5)

which are orthonormal, i.e.

\[ \int_{-b}^{b} \psi_m(y) \psi_n(y) y = \delta_{mn}. \] (7.3.6)
7.3.2 Region 2\{c^- \leq y \leq a^-, \ -l^- < x < -l^-\}

The general solution in the region 2 can be written as

\[ \phi(x, y) = \sum_{n=0}^{\infty} \left[ B_n e^{i\tilde{\beta}_n(x + l^-)} + C_n e^{-i\tilde{\beta}_n(x + l^-)} \right] \xi_n^-(y), \quad (7.3.7) \]

which satisfies equations (3.2.2), (7.2.1b) and (7.2.1c) where

\[ \xi_n^-(y) = \begin{cases} \sqrt{\frac{2}{a^- - c^-}} \cos \beta_n^-(y - c^-), & n \neq 0, \\ \sqrt{\frac{1}{a^- - c^-}}, & n = 0. \end{cases} \quad (7.3.8) \]

Here eigenvalues and the associated eigenvalues are given by

\[ \beta_n^- = \frac{n\pi}{a^- - c^-}, \]

and

\[ \tilde{\beta}_n^- = \sqrt{k^2 - (\beta_n^-)^2}, \quad n = 0, 1, 2, \ldots \]

respectively.

7.3.3 Region 3\{-b \leq y \leq b, \ -l < x < l\}

In this region the solution can be written as

\[ \phi(x, y) = \sum_{n=0}^{\infty} \left[ D_n e^{i\tilde{\alpha}_n(x + l)} + E_n e^{-i\tilde{\alpha}_n(x - l)} \right] \psi_n(y), \quad (7.3.9) \]

which satisfies equations (3.2.2) and (7.2.1d).

7.3.4 Region 4\{c^+ \leq y \leq a^+, \ l < x < l^+\}

The general solution in this region is

\[ \phi(x, y) = \sum_{n=0}^{\infty} \left[ B_n^+ e^{i\tilde{\beta}_n^+(x - l)} + C_n^+ e^{-i\tilde{\beta}_n^+(x - l)} \right] \xi_n^+(y), \quad (7.3.10) \]

which satisfies equations (3.2.2), (7.2.1e) and (7.2.1f).

\[ \xi_n^+(y) = \begin{cases} \sqrt{\frac{2}{a^+ - c^+}} \cos \beta_n^+(y - c^+), & n \neq 0, \\ \sqrt{\frac{1}{a^+ - c^+}}, & n = 0. \end{cases} \quad (7.3.11) \]

Here eigenvalues and the associated eigenvalues are given by

\[ \beta_n^+ = \frac{n\pi}{a^+ - c^+}, \]

and

\[ \tilde{\beta}_n^+ = \sqrt{k^2 - (\beta_n^+)^2}, \quad n = 0, 1, 2, \ldots \]

respectively.
7.3.5 Region $5\{-b \leq y \leq b, \ l^+ < x < \infty\}$

In this region the solution can be written as

$$\phi(x, y) = \sum_{n=0}^{\infty} \left[ F_n e^{i\alpha_n(x-l^+)} \right] \psi_n(y),$$

(7.3.12)

which satisfies equations (3.2.2), (7.2.1g), and the radiation conditions.

7.4 Formulation of System of Equations

7.4.1 Continuity of Pressure and Velocity of Pressure

We now have four boundaries ($x = l^-, \pm l, l^+$) over which we need to match the potential and its normal derivative. We will follow a very similar procedure as we did previously.

The continuity of the pressure across $x = -l^-$ gives

$$\sum_{n=0}^{\infty} A_n \psi_n(y) + \psi_0(y) = \sum_{n=0}^{\infty} \left[ B_n^- + C_n^- e^{i\beta_n (l^- - l)} \right] \xi_n^-(y).$$

Taking the inner product with $\xi_m^-(y)$ and integrating over $[c^-, a^-]$ gives

$$\sum_{n=0}^{\infty} A_n p_{m}^- + p_{m0}^- = B_m^- + C_m^- e^{i\beta_m (l^- - l)}, \quad m = 0, 1, 2, \ldots$$

(7.4.13)

where

$$p_{mn}^- = \int_{c^-}^{a^-} \psi_n(y) \xi_m(y) dy,$$

(7.4.14)

$$= \begin{cases}
\sqrt{\frac{a^- - c^-}{2b}}, & m = n = 0, \\
\frac{[\sin \alpha_n(a^- - b) - \sin \alpha_n(c^- - b)]}{\sqrt{b(a^- - c^-)\alpha_n}}, & m = 0, n \neq 0, \\
0, & n = 0, m \neq 0, \\
\frac{-\sin \alpha_n(c^- - b) + \sin \alpha_n(2a^- - b - c^-) + 2\alpha_n(a^- - c^-) \cos \alpha_n(c^- - b)}{\sqrt{8b(a^- - c^-)\alpha_n}}, & \alpha_n = \beta_m, \\
\frac{-\sqrt{2}\alpha_n[\sin \alpha_n(c^- - b) - \sin \alpha_n(a^- - b) \cos \beta_m(a^- - c^-)]}{\sqrt{b(a^- - c^-)(\alpha_n^2 - \beta_m^2)}}, & \text{other cases}.
\end{cases}$$

In a similar manner we equate normal derivative across $x = -l^-$ to obtain

$$-\sum_{n=0}^{\infty} A_n \bar{\alpha}_n \psi_n(y) + \bar{\alpha}_0 \psi_0(y) = \sum_{n=0}^{\infty} \bar{\beta}_n \left[ B_n^- - C_n^- e^{i\beta_n (l^- - l)} \right] \xi_n^-(y).$$

Taking the inner product with $\psi_n(y)$ and integrating over $[-b, b]$, we get

$$-\bar{\alpha}_m A_m + \bar{\alpha}_0 \delta_m = \sum_{n=0}^{\infty} \bar{\beta}_n p_{nm}^- \left( B_n^- - C_n^- e^{i\beta_n (l^- - l)} \right), \quad m = 0, 1, 2, \ldots$$

(7.4.15)
We obtain further six equations from matching at $x = -l, l$ and $l^+$ which are

$$\sum_{n=0}^{\infty} p_{mn}^- (D_n + E_n e^{i\alpha_m (2l)}) = B_m^- e^{i\beta_n^- (l^- - l)} + C_m^-, \quad m = 0, 1, 2, \ldots (7.4.16)$$

$$\tilde{\alpha}_m D_m - \tilde{\alpha}_m E_m e^{i\alpha_m (2l)} = \sum_{n=0}^{\infty} \tilde{\beta}_n^- p_{mn}^- \left( B_n^- e^{i\beta_n^- (l^- - l)} - C_n^- \right), \quad m = 0, 1, 2, \ldots (7.4.17)$$

$$\sum_{n=0}^{\infty} p_{mn}^+ (D_n e^{i\alpha_m (2l)} + E_n) = B_m^+ e^{i\beta_n^+ (l^+ - l)} + C_m^+, \quad m = 0, 1, 2, \ldots$$

$$\tilde{\alpha}_m D_m e^{i\alpha_m (2l)} - \tilde{\alpha}_m E_m = \sum_{n=0}^{\infty} \tilde{\beta}_n^+ p_{mn}^+ \left( B_n^+ - C_n^+ e^{i\beta_n^+ (l^+ - l)} \right), \quad m = 0, 1, 2, \ldots (7.4.18)$$

$$\sum_{n=0}^{\infty} p_{mn}^+ F_n = B_m^+ e^{i\beta_n^+ (l^+ - l)} + C_m^+, \quad m = 0, 1, 2, \ldots (7.4.19)$$

where $p_{mn}^+$ is the just the same expression as $p_{mn}^-$ except we replace $a^-$ and $c^-$ by $a^+$ and $c^+$ respectively.

### 7.5 Numerical Outcomes

Figure 7.2 plots the absolute value of transmission coefficient $|R_0|$ against $k$ for $b = 2$, with $a^- = 0.5$, $c^- = -0.5$, $a^+ = 0.5$, $c^+ = -0.5$ (subfigure a) and $a^- = 0.5$, $c^- = -0.5$, $a^+ = 1$, $c^+ = -1$ (subfigure b). Figure 7.3 is the same as Figure 7.2 except $a^- = 1.5$, $c^- = -0.5$, $a^+ = 1.5$, $c^+ = -0.5$ (subfigure a) and $a^- = 0.5$, $c^- = -0.5$, $a^+ = 1.5$, $c^+ = -0.5$ (subfigure b). In both figures it can be seen that as soon as the symmetry about $x = 0$ is broken we no longer have perfect transmission.
Figure 7.2: The absolute value of transmission coefficient $|T_0|$ against $k$ for $b = 2$, $l_1 = l_3 = 1$, $l_2 = 5$ with $a_1 = 0.5$, $c_1 = -0.5$, $a_2 = 0.5$, $c_2 = -0.5$ (subfigure a) and $a_1 = 0.5$, $c_1 = -0.5$, $a_2 = 1$, $c_2 = -1$ (subfigure b)
Figure 7.3: The absolute value of transmission coefficient $|T_0|$ against $k$ for $b = 2$, $l_1 = L_3 = 1$, $l_2 = 5$ with $a_1 = 1.5$, $c_1 = -0.5$, $a_2 = 1.5$, $c_2 = -0.5$ (subfigure a) and $a_1 = 0.5$, $c_1 = -0.5$, $a_2 = 1.5$, $c_2 = -0.5$ (subfigure b)

7.6 Conclusion

Above chapter consists of a more complicated Helmholtz resonator in which there is no symmetry along either axis. A detailed numerical study of the problem is conducted and we show that once the symmetry orthogonal to the direction of propagation is broken then EAT (at least perfect transmission) no longer holds generally.
Chapter 8

Final Summary

Final conclusions of the thesis are given as follows.

- In chapter 1, we give the basic definitions, mathematical preliminaries and fundamental equations which would be beneficial to understand the contents in subsequent chapters.

We divide rest of thesis into two parts:
- **Part 1**: Pentafurcated Duct Problems.
- **Part 2**: Extraordinary Acoustic Transmission (EAT) and Helmholtz Resonators.

In part 1, we consider the acoustic diffraction of a fundamental plane wave mode which propagates out of the open end of a middle semi-infinite duct. This semi-infinite duct is symmetrically located in the infinite duct. The whole system forms a pentafurcated duct whose solution is presented by taking different nature of plates. We further divide part 1 into two chapters 3 and 4.

- In chapter 3 a pentafurcated duct having all hard plates is taken. We divide this pentafurcated duct into six regions. By using separation of variables, potentials in these regions are calculated. We then apply mode matching technique. In this technique potential solutions satisfying boundary conditions are presented in form of eigen modes which are matched across the boundary of each continuity to get an infinite system of linear equations involving unknown coefficients which are determined by MATLAB programming. We also present energy balance formula and the impact of the spacing of duct on reflection field amplitude.

- In chapter 4 we have introduced soft lining on outer plates of above pentafurcated duct and follow the same steps as done in chapter 3. We have presented the reflection field amplitude comparison between outer-soft pentafurcated and rigid pentafurcated duct. It is shown that the soft lining on outer plates has better impact as compared to that of hard boundary in pentafurcated duct, thus resulting in reduction of sound. We have also given comparison of reflection field behavior with related existing results of tri and pentafurcated duct problems. WE have observed that insertion of soft lining on outer plates provides more efficient noise attenuation, hence improves the performance of the model.
In part 2, the phenomenon of extraordinary acoustic transmission (EAT) in a Helmholtz resonator, which has recently been investigated experimentally, is studied theoretically. This part consists of three chapters 5, 6 and 7.

- In Chapter 5 we have considered the simplest case which consists of a waveguide with hard walls and a finite inner duct symmetrically (along $y$-axis) located within an infinite duct. We have solved this problem using mode matching exploiting symmetry to decompose the solution. We have explored in detail the consequences of this decomposition and show how this leads to EAT.

- In Chapter 6 we have considered a similar problem where the cavity is not symmetrically placed in the waveguide. We have shown in this case the EAT exists but that the cut-off frequency is halved. Significant numerical results are given for both problems including movies (supplementary material). These movies show the analytic extension of the solution for complex wavenumber which is key to our analysis.

- Chapter 7 consists of a more complicated problem in which there is no longer symmetry and we have shown that the lack of symmetry destroys EAT. A detailed numerical study of the problem is conducted and we show that once the single mode of propagation or the symmetry is broken then EAT (at least perfect transmission) no longer holds generally.

- In appendix A and B, solution for a rigid and outersoft trifurcated waveguide problems using the mode matching method has been presented respectively. These particular problems have been solved previously by Hassan and Rawlins using the Wiener-Hopf technique. We get identical results for specific dimensions of the ducts.

Future work would be to extend the model to the case of fluid flow or for elastic plates.

It is worthy to mention that the following papers from this thesis have been published from this thesis.


Bibliography


Appendix A

Eigenfunction Expansion for Rigid Trifurcated Duct

Here we briefly study the acoustic wave scattering in rigid trifurcated duct by using Eigenfunction expansion method. Hassan and Rawlins [5] solved the same problem by Wiener-Hopf method. It will be shown that same results are obtained for reflection field amplitude as were obtained in [5].

A.1 Solution of the problem

This particular problem is analogous to pentafurcated problem given in chapter (3) with the removal of the semi infinite plate at \( y = \pm b \). This problem is given by (3.2.2)and (3.2.3).

The potential in region 1 \( \{-d \leq y \leq -a, \ x < 0\} \) is given by

\[
\phi(x, y) = \sum_{n=1}^{\infty} B_n e^{-i\tilde{\alpha}_n x} \psi_n(y) \tag{A.1.1}
\]

which satisfies (3.2.2), (3.2.3) and the radiation conditions. We define eigenvalues \( \alpha_n \), associated eigenvalues \( \tilde{\alpha}_n \) and orthonormal eigenfunctions \( \psi_n \)

\[
\alpha_n = \frac{(n - 1) \pi}{d - a}, \quad n = 1, 2, 3, ..., \quad \tilde{\alpha}_n = \sqrt{k^2 - \alpha^2_n}, \quad n = 1, 2, 3, ...
\]

and

\[
\psi_n(y) = \begin{cases} 
\sqrt{\frac{2}{(d-a)}} \cos \alpha_n (y + d), & n \neq 1, \\
\sqrt{\frac{1}{(d-a)}} & n = 1,
\end{cases} \tag{A.1.2}
\]

respectively with \( 0 < \text{Im} \tilde{\alpha}_1 < \text{Im} \tilde{\alpha}_2, ... \).

The eigenfunction expansion in region 2 \( \{|y| < a, \ x < 0\} \) is

\[
\phi(x, y) = \sum_{n=1}^{\infty} C_n e^{-i\tilde{\gamma}_n x} \eta_n(y) + e^{i\tilde{\gamma}_1 x} \eta_1(y) \tag{A.1.3}
\]

where \( \tilde{\gamma}_n \) and \( \eta_n(y) \) are same as given in chapter 3.
The eigenfunction in region 3 \( \{a < y < d, \, x < 0\} \) is given by
\[
\phi(x, y) = \sum_{n=1}^{\infty} E_n e^{-i\tilde{\alpha}_n x} \sigma_n(y),
\] (A.1.4)
where orthonormal eigenfunctions \( \sigma_n \) are defined by
\[
\sigma_n(y) = \begin{cases} 
\sqrt{\frac{2}{d-a}} \cos \alpha_n(y-d), & n \neq 1, \\
\sqrt{\frac{1}{d-a}}, & n = 1.
\end{cases}
\] (A.1.5)
The potential function in region 4 \( \{|y| < d, \, -\infty < x < \infty\} \) satisfying (3.2.2), (3.2.3) and the radiation conditions is given by
\[
\phi(x, y) = \sum_{n=1}^{\infty} F_n e^{i\tilde{\omega}_n x} \epsilon_n(y)
\] (A.1.6)
where \( \omega_n = \frac{n\pi}{2d} \) and \( \epsilon_n(y) \) is given by equation (3.3.18).

A.2 System of equations

We equate the potentials and their velocities across \( x = 0 \). Taking the inner products by multiplying with appropriate eigenfunctions and integrating over respective dimensions, we get the following compiled system of equations.
\[
\sum_{n=1}^{\infty} F_n L_{mn} (\tilde{\omega}_n + \tilde{\alpha}_m) = 0, \quad m = 1, 2, 3, \ldots
\] (A.2.7)
\[
\delta_{1m} (\tilde{\gamma}_1 + \tilde{\gamma}_m) = \sum_{n=1}^{\infty} F_n T_{mn} (\tilde{\omega}_n + \tilde{\gamma}_m), \quad m = 1, 2, 3, \ldots
\] (A.2.8)
\[
\sum_{n=1}^{\infty} F_n S_{mn} (\tilde{\omega}_n + \tilde{\alpha}_m) = 0, \quad m = 1, 2, 3, \ldots
\] (A.2.9)
where \( L_{mn}, \, T_{mn} \) and \( S_{mn} \) are same as given in chapter 3 when \( b \) is replaced by \( a \).

A.3 Numerical Results

We present numerical outcomes by solving the truncated infinite system of equations\( (n = 1, 2, 3, \ldots N, \, m = 1, 2, 3, \ldots M \) and \( N = 3M + 2) \). The absolute reflection against the wave number for the same dimensions as considered by Hassan and Rawlins [5] has been plotted.

A.3.1 Case (i)

In this case, we consider the waveguide dimensions such that \( d = 3a/2 \) and the frequency range is \( 0 < ka < \pi \). Figure A.1 shows \(|R|\) versus \( ka \) where dashed line represents \(|R|\) against \( ka \) using the Wiener Hopf technique (WH) while solid line shows the graph of \(|R|\) by using Eigenfunction expansion method (EF).

In figure A.2 \(|R|\) is plotted for fixed \( d = 3/2 \) and (i) \( a = 1 \) and (ii) \( a = 0.5 \). The data 1 (solid line) is for \( a = 1 \) and data 3 (dotted line ) is for \( a = 0.5 \).
A.3.2 Case (ii)

In this case, we take $d = 3a$ and $\pi/6 < ka < 3\pi/4$. Figure A.2 represents the graph of $|R|$ against $ka$ by both techniques.

Figure A.1: Reflection coefficient against wave number for $d = 3a/2$.

Figure A.2: Reflection coefficient against wave number for $d = 3a$. 
From above we can observe that in figure A.1, the onset of wave propagating forward is at $ka = \frac{2\pi}{3}$ and there is no mode before $ka = \frac{2\pi}{3}$. Similarly in figure A.2 onset of wave propagating forward are at $ka = \frac{\pi}{3}$ and $ka = \frac{2\pi}{3}$ while onset of wave propagating backward is at $ka = \frac{\pi}{2}$. We can see that as more modes are excited in distinct regions, the value of the reflection coefficient encounters an abrupt change.

**A.4 Conclusion**

We have given the solution of the rigid trifurcated exhaust problem with no fluid flow using the matched eigenfunction expansion technique. The same problem has been previously solved by Hassan and Rawlins [5] by using the Wiener-Hopf technique which needs splitting of complicated functions while mode matching method is much easier to deal with. We have proved that we get the identical numerical results for reflection coefficient by both methods (see figures A.1 and A.2).
Appendix B

Eigenfunction Expansion for Outer-Soft Trifurcated Duct

We give a brief description of outer soft trifurcated problem previously solved in [5] using Wiener Hopf technique. We will use the same procedure as we used for outer-soft pentafurcated problem in chapter 4. This trifurcated problem is given by (3.2.2), (4.2.1), (4.2.2) and radiation conditions while plate b is removed.

B.1 Solution of the Problem

The potential in region 1 \{-d \leq y \leq -a, \ x < 0\} is given by

\[ \phi(x, y) = \sum_{n=1}^{\infty} a_n e^{-\tilde{\alpha}_n x} \psi_n(y), \quad (B.1.1) \]

where

\[ \tilde{\alpha}_n = \sqrt{k^2 - \alpha_n}, \]
\[ \alpha_n = \frac{(2n - 1)\pi}{2(d - a)} \]

which are roots of \( \cos \alpha_n (d - a) = 0 \) and the orthonormal function \( \psi_n(y) \) are given by

\[ \psi_n(y) = \sqrt{\frac{2}{d - a}} \sin \alpha_n (y + d). \]

The eigenfunction expansion in region 2 \{ |y| < a, \ x < 0 \} is

\[ \phi(x, y) = e^{i\tilde{\beta}_n x} \eta_1(y) + \sum_{n=1}^{\infty} b_n e^{-\tilde{\beta}_n x} \eta_n(y) \]

where

\[ \tilde{\beta}_n = \sqrt{k^2 - \beta_n}, \quad n = 1, 2, ... \]
\[ \beta_n = \frac{(n - 1)\pi}{2a} \]

which are roots of \( \sin2\beta_n a = 0 \) and orthonormal function \( \eta_n(y) \) are given by

\[ \eta_n(y) = \begin{cases} \sqrt{\frac{1}{a}} \cos \beta_n (y - a), & n \neq 1, \\ \frac{1}{2a}, & n = 1. \end{cases} \]

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The incident wave $e^{i\tilde{\beta}_1 x} \eta_1(y)$ is traveling from left to right. To have one fundamental incident and one reflected wave mode, we restrict $0 < ka < \frac{\pi}{2}$ and
\[
\tilde{\beta}_n = i \sqrt{\frac{(n-1)^2 \pi^2}{4a^2} - k^2}
\]
so that $\text{Re} \tilde{\beta}_n = 0$ and $\text{Im} \tilde{\beta}_n > 0$ for $n > 2$.

The eigenfunction in region 3 \{a < y < d, x < 0\} is given by
\[
\phi(x, y) = \sum_{n=1}^{\infty} c_n e^{-\tilde{\alpha}_n x i} \sigma_n(y).
\] (B.1.5)

where eigenfunctions $\sigma_n(y)$ are
\[
\sigma_n(y) = \sqrt{\frac{2}{d-a}} \sin \alpha_n(y - d).
\]

The eigenfunction in region 4 \{|y| < d, -\infty < x < \infty\} is given by
\[
\phi(x, y) = \sum_{n=1}^{\infty} d_n e^{\tilde{\omega}_n x i} \epsilon_n(y)
\] (B.1.6)

where
\[
\epsilon_n(y) = \sqrt{\frac{1}{d}} \sin \omega_n(y + d),
\]

$\tilde{\omega}_n$ is given by equation (4.3.20)

**B.2 System of equations**

Equating the potentials and their velocities across $x = 0$, we get the following compiled system of equations.
\[
\sum_{n=1}^{\infty} F_n L_{mn} (\tilde{\omega}_n + \tilde{\alpha}_m) = 0, \quad m = 1, 2, 3,....
\] (B.2.7)
\[
\delta_{1m} (\tilde{\gamma}_1 + \tilde{\gamma}_m) = \sum_{n=1}^{\infty} F_n T_{mn} (\tilde{\omega}_n + \tilde{\gamma}_m), \quad m = 1, 2, 3,....
\] (B.2.8)
\[
\sum_{n=1}^{\infty} F_n S_{mn} (\tilde{\omega}_n + \tilde{\alpha}_m) = 0, \quad m = 1, 2, 3,....
\] (B.2.9)

where $L_{mn}$, $T_{mn}$ and $S_{mn}$ are same as given in chapter (4) where $b$ is replaced by $a$.

**B.3 Numerical Results**

The graphs B.1 and B.2 are plotted to have the comparison between eigenfunction expansion (dotted line) and Wiener-Hopf methods (solid line) for $d = \frac{3}{2}$ and $d = 3$ respectively where $a = 1$ and Mach number $M = 0$. This shows that eigenfunction expansion method gives same numerical results for outer soft trifurcated problem as found in [5].
Figure B.1: $|R|$ versus $k$ for $a = 1$ and $d = 1.5$

Figure B.2: $|R|$ versus $k$ for $a = 1$ and $d = 3$