A THESIS TITLED

SOME CLASSES OF MULTIVALUED DYNAMICAL SYSTEMS

Submitted to GC University Lahore
in partial fulfillment of the requirements
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Doctor of Philosophy

IN

MATHEMATICS

By

RIZWAN AHMED

Registration No.

2014-PHD-ASSMS-16

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Date

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Abstract

In this thesis we study three different problems.

First, we study a class of a multivalued perturbations of m-dissipative evolution inclusions with nonlocal initial condition in arbitrary Banach spaces. We prove the existence of solutions when the multivalued right hand side is Lipschitz and admits nonempty closed bounded but, in general case, neither convex nor compact values. Illustrative example is provided.

Second, we prove two variants of the well known lemma of Filippov–Pliss in case of dynamical inclusions on time scale. The first variant is when the right-hand side is Lipschitz continuous on the state variable. Afterward we introduce one sided Perron conditions for multifunctions on time scale and prove the second variant of that lemma. Some discussions on relaxed systems is provided.

Third, we investigate fuzzy fractional integral inclusions under compactness type conditions. We prove the existence of solutions when the right-hand side is almost upper semicontinuous. We also show that the solution set is connected. Finally, an application to fuzzy fractional differential inclusions is given.

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Introduction

This thesis is devoted to three different problems studied in [6, 7, 74].

In **chapter I** we study a class of multivalued perturbations of m-dissipative evolution inclusions with nonlocal initial condition in arbitrary Banach spaces. When somebody studies periodic problems then conditions at several different points are imposed. The nonlocal models cotain all such kinds of problems. These models are more adequate in some reaction diffusion problems, population dynamics, thermodynamics etc. We refer the interested reader to [27].

Let $\mathfrak{X}$ be a Banach space, $I = [t_0, T] \subset \mathbb{R}$, $A : D(A) \subset \mathfrak{X} \rightrightarrows \mathfrak{X}$ an m-dissipative operator, let $F : I \times \mathfrak{X} \rightrightarrows \mathfrak{X}$ be a multifunction with nonempty closed bounded values, and let $g : C(I, \mathfrak{X}) \to \overline{D(A)}$ be a given function.

Consider the nonlinear evolution inclusion having the form

$$\begin{cases} \dot{x} \in Ax + F(t, x) \\ x(t_0) = g(x(\cdot)). \end{cases} \tag{0.0.1}$$

The problem of existence of solutions of (0.0.1) has been extensively studied by many authors under compactness or dissipative type assumptions, using appropriate approaches. We recall the paper [66], where it is assumed that A generates a compact semigroup and $F(\cdot, \cdot)$ has convex, weakly compact values. In this case $g(\cdot)$ maps the bounded subsets of $C(I, \mathfrak{X})$, which are precompact in $C([t_0 + \delta, T], \mathfrak{X})$ for every $\delta > 0$, into precompact sets in $\mathfrak{X}$. We refer the reader also to [81, 82]. Another type of assumptions used to prove the existence of solutions of (0.0.1) is that A generates an equicontinuous semigroup, while $F(\cdot,\cdot)$ satisfies appropriate compactness type conditions. In this case the dual space $\mathfrak{X}^*$ is commonly assumed to be uniformly convex. We refer the reader to [9, 87, 88]. When A is a general m-dissipative operator, $F(\cdot,\cdot)$ should commonly satisfy some dissipative type conditions (see, e.g., [88]). The theory of m-dissipative differential equations is presented among others in [17, 75, 78].

Our main goal is to prove the existence of solutions of (0.0.1) when A is a general m-dissipative operator and $F(t,\cdot)$ is Lipschitz continuous, which is the same assumption as in [88]; however, our Lipschitz constant is, in general, bigger than the one in [88]. Moreover, in our case, the Banach space $\mathfrak{X}$ is arbitrary and we do not require $F(\cdot,\cdot)$ to have convex values as in [88]. Similar problem is studied in [8] in case of non-autonomous semilinear system. The result is similar to our, however, the proof is not correct.

We first investigate the following local Cauchy problem

$$\begin{cases} \dot{x} \in Ax + F(t,x) \\ x(t_0) = x_0 \in \overline{D(A)} \end{cases}$$

and prove a variant of Filippov-Plis lemma, which will be the key tool in proving the existence of solutions of (0.0.1). The reader is referred to [26, 42, 54] where local Cauchy problem is also studied. In what follows we will assume every time that the initial conditions are in $\overline{D(A)}$ and will commonly not write explicitly that $x_0 \in \overline{D(A)}$. Lipschitz assumptions used here are in some sense strong (Theorem 1.2.3), however, we do not require any conditions on the dissipative operator A and on the state space.

In **chapter II** we discusses theory of dynamical inclusions on time scale. We study the following dynamical inclusion on time scale:

$$x^{\Delta} \in F(t, x(t)),\ \Delta\ - a.e.\ t \in \mathbb{I}\ x(t_0) = x_0.$$

where $\mathbb{I} \subset \mathbb{R}$ is closed, bounded set. Timescale theory is introduced in [52, 53] in order to unify the continuous and discrete systems. We refer the reader to [23, 24] for the theory of dynamical equations on time scale and [3, 22], where some applications are given. Among others notice [61] where theory of dynamical systems in measure chains are studied. In [49] the integration on time scale is investigated and its connections with standard Lebesgue integral is considered in [28].

In the last years different optimization problems on time scale are studied. We refer the reader to [63, 65, 67, 68]. The theory of dynamical inclusions on time scale is presented [12, 48, 76]. In the later paper [77] the authors prove analogy of Filippov's selection theorem, which shows that the optimal control of Caratheodory controlled systems can be written equivalently as a differential inclusions.

One of the most useful result in optimal control is the significant lemma of Filippov–Pliss (see e.g. [44]). It was proved first by Filippov in case of Lipschitzian right-hand side in [47] and afterward extend by Pliss in [69] under much weaker conditions. We refer the reader to [13, 62] for the main applications of this result. Notice also [21] where the history and review of this lemma is presented. This result has been extended to the case of one sided Lipschitz differential inclusions in [43]

We refer the reader to [54, 79] for the needed facts in set-valued analysis and differential inclusions.

In the **chapter II** we prove a variant of Filippov–Pliss lemma in case of Lipschitz dynamical inclusion. Afterward one sided Perron condition is introduced in case of time scale and a variant of Filippov–Pliss lemma is proved in case of one sided Perron inclusions on time scale. One of the most important theorems in the optimal control is that that under some additional hypotheses closure of the solution set of the original system is the solution set of the convexified one. We discuss about its extension on dynamical inclusions.

In **chapter III** we discuss fuzzy fractional integral and differential inclusions. The modeling of the real problems often requires some kinds of uncertainty and it is one of the main reasons to investigate the fuzzy systems and multivalued differential equations. Starting with the work of Kaleva [56], the theory of fuzzy differential and integral equations was rapidly developed. We refer the reader to [56, 60, 70, 71, 73, 86]. However, the differential inclusions (multivalued differential equations) give more adequate models of the real processes and they are extensively used in optimal control (see, e.g. [37, 55]).

In the last years several directions of the theory of fractional differential equations and inclusions has rapidly developed due to their applications in physics, biological sciences, engineering, chemistry etc. (see, e.g. [16, 51]). The main properties of the fractional differential equations are studied in [40, 57, 59, 72]. Fractional differential inclusions are considered, among others, in [2, 46, 83].

Recently, the interest for fractional differential and integral equations with uncertainty has increased (see [1, 4, 10, 11, 83] and the references therein). To the authors knowledge, there are a few papers devoted to fuzzy fractional differential and integral inclusions (see, e.g. [5]).

Our main target is to investigate the properties of the solution set of (3.3.1) under compactness type conditions. We prove a variant of Kneser's theorem with the aid of the locally Lipschitz approximations. This technique is commonly used to derive some topological properties of the solution set. We refer the reader to [26, 37, 55], where the results are restricted to reflexive Banach spaces.

Chapter 1

Nonlocal m-dissipative evolution inclusions in Banach spaces

In this chapter we discuss a class of a multivalued perturbations of m-dissipative evolution inclusions with nonlocal initial condition in arbitrary Banach spaces. We prove the existence of solutions when the multivalued right hand side is Lipschitz and admits nonempty closed bounded but, in general case, neither convex nor compact values. We extend an existence result from [88] and give correct proof of a result of [8]. Illustrative example is provided.

1.1 Preliminaries

In this section we give the basic definitions, notations and the results used here.

Definition 1.1.1. Let $\mathfrak{X}$ be a Banach space and $\|\cdot\|$ be the norm on $\mathfrak{X}$. Let $x, y \in \mathfrak{X}$, then the right directional derivative of the norm calculated at x in the direction y, denoted by $[x, y]_+$, is defined by

$$[x, y]_+ = \lim_{h \to 0^+} \frac{\|x + hy\| - \|x\|}{h}$$

and the right directional derivative of $\frac{1}{2}\|\cdot\|^2$ calculated at x in the direction y, denoted by $(x,y)_+$, is defined by

$$(x,y)_+ = \lim_{h\to 0^+} \frac{\|x+hy\|^2 - \|x\|^2}{2h}$$

Analogously, the left directional derivative of the norm calculated at x in the direction y, denoted by $[x,y]_-$, is defined by

$$[x,y]_- = \lim_{h\to 0^-} \frac{\|x+hy\| - \|x\|}{h}$$

and the left directional derivative of $\frac{1}{2}\|\cdot\|^2$ calculated at x in the direction y, denoted by $(x,y)_-$, is defined by

$$(x,y)_- = \lim_{h\to 0^-} \frac{\|x+hy\|^2 - \|x\|^2}{2h}$$

The mappings $[x,y]_\pm$ are called the normalized semi-inner products on $\mathfrak{X}$, while $(x,y)_\pm$ are called the semi-inner products on $\mathfrak{X}$.

Proposition 1.1.1. *The mappings $[x,y]_\pm$ and $(x,y)_\pm$ satisfy the following properties:*

(i) $(x,y)_\pm = \|x\|[x,y]_\pm$

(ii) $|[x,y]_\pm| \leq \|y\|$

(iii) $|[x,y]_\pm - [x,z]_\pm| \leq \|y-z\|$

(iv) If $a,b>0$, *then* $[ax,by]_\pm = b[x,y]_\pm$

(v) $[x,y]_+ = -[-x,y]_- = -[x,-y]_-$

(vi) For $a\in\mathbb{R}$, $[x,y+ax]_\pm = [x,y]_\pm + a\|x\|$

(vii) $[x,y+z]_+ \leq [x,y]_+ + [x,z]_+$ *and* $[x,y+z]_- \geq [x,y]_- + [x,z]_-$

(viii) $[x,y+z]_+ \geq [x,y]_+ + [x,z]_-$ *and* $[x,y+z]_- \leq [x,y]_- - [x,z]_+$

(ix) $[\cdot,\cdot]_\pm$ *is upeer semicontinuous and* $(\cdot,\cdot)_\pm$ *is lower semicontinuous.*

Definition 1.1.2. Let A be operator. We define the domain and range of A as it follows:

$$D(A) := \{x \in \mathfrak{X} : Ax \neq \emptyset\},\ R(A) := \bigcup_{x \in D(A)} Ax.$$

We say that the operator $A : D(A) \subset \mathfrak{X} \rightrightarrows \mathfrak{X}$ is dissipative if $(x_1 - x_2, y_1 - y_2)_- \leq 0$ for every $x_1, x_2 \in D(A)$ and $y_1 \in Ax_1, y_2 \in Ax_2$. The operator A is m-dissipative if it is dissipative and for each $\lambda > 0$, $R(I - \lambda A) = \mathfrak{X}$.

Definition 1.1.3. A set $\{S(t) : t \geq 0\}$ of operators on $\mathfrak{X}$ is said to be a semigroup if:

(i) $S(0) = I$.

(ii) $S(t+s) = S(t)S(s),\ \forall s, t \geq 0$.

It is said to be semigroup of contraction if it is semigroup and $\|S(t)\| \leq 1,\ \forall t \geq 0$.

Remark 1.1.1. Every m-dissipative operator generates a semigroup of contraction. If the operator A generates semigroup of contraction $S(\cdot)$, then the unique solution to

$$\dot{x} \in Ax, x(t_0) = x_0 \in \overline{D(A)}$$

is $x(t) = S(t - t_0)x_0$.

Definition 1.1.4. Let $f(\cdot)$ be a Bochner integrable function and consider the problem

$$\begin{cases} \dot{x}(t) \in Ax(t) + f(t), \\ x(t_0) = x_0. \end{cases} \tag{1.1.1}$$

A continuous function $x : [t_0, T] \to \overline{D(A)}$ is said to be an *integral solution* of (1.1.1) on $[t_0, T]$ if $x(t_0) = x_0$ and for every $u \in D(A)$, $v \in Au$ and $t_0 \leq \tau < t \leq T$ the following inequality holds

$$|x(t) - u| \leq |x_0 - u| + \int_{\tau}^{t} [x(s) - u, f(s) + v]_+ ds.$$

The function $f(\cdot)$ will be called the *pseudoderivative* of $x(\cdot)$ and will be denoted by $f_x(\cdot)$.

The following Theorem is well known (see, e.g., [20, 58]).

Theorem 1.1.2. *Let $f(\cdot)$ be Bochner integrable function then the evolution equation* (1.1.1) *has a unique integral solution.*

Let $\mathfrak{X}$ be a general Banach space, $I = [t_0, T] \subset \mathbb{R}$, $A : D(A) \subset \mathfrak{X} \rightrightarrows \mathfrak{X}$ an m-dissipative operator, let $F : I \times \mathfrak{X} \rightrightarrows \mathfrak{X}$ be a multifunction with nonempty closed bounded values, and let $g : C(I, \mathfrak{X}) \to \overline{D(A)}$ be a given function.

Consider the nonlinear evolution inclusion having the form

$$\begin{cases} \dot{x} \in Ax + F(t, x) \\ x(t_0) = g(x(\cdot)). \end{cases} \tag{1.1.2}$$

The problem of existence of solutions of (1.1.2) has been extensively studied by many authors under compactness or dissipative type assumptions, using appropriate approaches. We proved the existence of solutions of (1.1.2) when A is a general m-dissipative operator and $F(t, \cdot)$ is Lipschitz continuous, which is the same assumption as in [88]; however, our Lipschitz constant is, in general, bigger than the one in [88]. Moreover, in our case, the Banach space $\mathfrak{X}$ is arbitrary and we do not require $F(\cdot, \cdot)$ to be convex valued as in [88]. We first investigated the following local Cauchy problem

$$\begin{cases} \dot{x} \in Ax + F(t, x) \\ x(t_0) = x_0 \in \overline{D(A)} \end{cases} \tag{1.1.3}$$

and proved a variant of Filippov-Plis lemma, which will be the key tool in proving the existence of solutions of (1.1.2).

Definition 1.1.5. The continuous function $x(\cdot)$ is said to be a solution of (1.1.3) if its pseudoderivative $f_x(t) \in F(t, x(t))$. Notice that the pseudoderivative $f_x(\cdot)$ of $x(\cdot)$ is associated to A, i.e. $f_x(\cdot)$ depends also on A. In this thesis the operator A is fixed and the dependence of A is not written explicitly.

Theorem 1.1.3. *([20]) If $x(\cdot), y(\cdot)$ are solutions of* (1.1.1) *with the pseudoderivatives $f_x(\cdot)$, $f_y(\cdot)$ and $x(t_0) = x_0$, $y(t_0) = y_0$, then*

$$|x(t) - y(t)| \leq |x_0 - y_0| + \int_{t_0}^{t} |f_x(\tau) - f_y(\tau)| d\tau \tag{1.1.4}$$

for any $t \in [t_0, T]$.

Remark 1.1.2. Let $\{x_n(\cdot)\}_{n=1}^{\infty}$ be a sequence of integrable solutions to the problem

$$\dot{x}_n(t) \in Ax_n(t) + f_n(t),\ x(t_0) = x_0.$$

As it is shown in [25], it is possible that $x_n(\cdot)$ converges uniformly to $x(\cdot)$ and $f_n(\cdot)$ converges L_1-weakly to $f(\cdot)$, but $x(\cdot)$ is not a solution to

$$\dot{x}(t) \in Ax(t) + f(t),\ x(t_0) = x_0.$$

even in finite dimensional space. Consequently the local problem (1.1.3) may have no solution even if A generates compact semigroup.

Lemma 1.1.4. *Let $(f_k(\cdot))_k$ be a sequence of Bochner integrable functions with $f_n(\cdot) \to f(\cdot)$ strongly in $L_1(I, \mathfrak{X})$. If $x_n(\cdot)$ is the unique solution of* (1.1.1) *with $f(\cdot)$ replaced by $f_n(\cdot)$ then $x_n(\cdot) \to x(\cdot)$ uniformly on I and $x(\cdot)$ is the solution of* (1.1.1).

Proof. Let $x(\cdot)$ be the unique solution of (1.1.1) exists. Due to (1.1.4) we have that

$$|x(t) - x_n(t)| \leq \int_{t_0}^{t} |f(\tau) - f_n(\tau)| d\tau$$

for any $t \in I$, which, under our hypotheses, leads to $x_n(\cdot) \to x(\cdot)$ uniformly on I. The proof is complete. □

Definition 1.1.6. The Hausdorff distance between the bounded subsets B and C of $\mathfrak{X}$ is defined by $D_H(B;C) = \max\{e(B;C), e(C;B)\}$, where $e(B;C)$ is the excess of B to C, defined by $e(B;C) = \sup_{x \in B} \operatorname{dist}(x;C)$.

1.2 Main results

In this section we prove the main results. We will use the following standing hypotheses.

F1. a) For every $x \in \overline{D(A)}$ there exists a strongly measurable function $f_x(\cdot)$ such that $f_x(t) \in F(t,x)$ for a.e. $t \in I$.

b) Let $x_0 \in \overline{D(A)}$. There exists a Lebesgue integrable function $\alpha(\cdot)$ such that $\|F(t,x_0)\| := \max\{|v|;\ v \in F(t,x_0)\} \leq \alpha(t)$ for any $t \in I$.

F2. There exists a Lebesgue integrable function $L : I \to \mathbb{R}_+$ such that for every $\varepsilon > 0$, every $y, z \in \overline{D(A)}$ and every strongly measurable function $f_y(\cdot)$ with $f_y(t) \in F(t,y)$ a.e. on I, there exists a strongly measurable selection $f_z(t) \in F(t,z)$ such that

$$|f_y(t) - f_z(t)| \leq L(t)|y - z| + \varepsilon,$$

for a.a. $t \in I$.

Proposition 1.2.1. *Suppose that there exists a Lebesgue integrable function L : $I \to \mathbb{R}_+$ such that $D_H(F(t,x), F(t,y)) \leq L(t)|x-y|$ for any $x, y \in \mathfrak{X}$ and any $t \in I$. Then **F2** is satisfied if one of the following conditions holds:*

(i) $\mathfrak{X}$ *is separable and* $F(\cdot, x)$ *is measurable for any* $x \in \overline{D(A)}$*;*

(ii) $F(\cdot, x)$ *is strongly measurable for any* $x \in \overline{D(A)}$*.*

Proof. It is enough to prove in both cases that for every $y \in \overline{D(A)}$, every strongly measurable $f_x(\cdot)$ with $f_x(t) \in F(t, x)$ and every $\varepsilon > 0$ there exists a strongly measurable $f_y(\cdot)$ such that $f_y(t) \in F(t, y)$ and $|f_x(t) - f_y(t)| < dist(f_x(t), F(t, y)) + \varepsilon$.

Let $g(t) := dist(f_x(t), F(t, y))$ and define the multifunction $G(t) = f_x(t) + (g(t) + \varepsilon)\mathbb{B}$. Notice that in both cases the function $g(\cdot)$ is strongly measurable (see, e.g., [13, Corollary 8.2.13] for the separable case see). For the nonseparable case, since $g(\cdot)$ is a step function when $f_x(\cdot)$ and $F(\cdot, y)$ are, then $g(\cdot)$ is strongly measurable. Moreover, the multifunction $G(\cdot)$ is strongly measurable.

Let us consider the first case. The intersection $G(t) \cap F(t, y)$ is graph measurable (see Theorem 8.2.4 in [13]) and hence there exists a measurable selection $f_y(t)$ (see Theorem III.6 in [35]), which completes the proof in the case of separable $\mathfrak{X}$.

Let $F(\cdot, x)$ be strongly measurable. Then $F(\cdot, x)$ and $G(\cdot)$ are limits of step (simple) functions. The nonempty intersection of step functions is also a step function. Thus, $H(t) := G(t) \cap F(t, x) \neq \emptyset$ is strongly measurable and hence there exists a strongly measurable selection $h(\cdot)$ of $H(\cdot)$. □

Definition 1.2.1. Let $\varepsilon > 0$. The continuous function $x(\cdot)$ is said to be ε-solution of (1.1.3) if its pseudoderivative $f_x(t) \in F(t, x(t) + \varepsilon\mathbb{B})$ for a.e. $t \in I$, where $\mathbb{B}$ is the closed unit ball in $\mathfrak{X}$. If $\varepsilon = 0$ then it is a solution of (1.1.3).

The following variant of Filippov-Pliss lemma is crucial in order to prove the existence of solution of (1.1.2).

Theorem 1.2.2. *Under* ***F1–F2****, for every* $\varepsilon > 0$ *and* $\delta > 0$*, every* $x_0, y_0 \in \overline{D(A)}$ *and every solution* $x(\cdot)$ *of* (1.1.3) *with* $x(t_0) = x_0$*, there exists a solution* $y(\cdot)$ *of*

(1.1.3) *with* $y(t_0) = y_0$ *such that*

$$|x(t) - y(t)| \leq \left(\exp\left(\int_{t_0}^{t} L(s)ds\right) + \varepsilon\right)|x_0 - y_0|, \tag{1.2.1}$$

and, moreover,

$$|f_x(t) - f_y(t)| \leq L(t)\left(\exp\left(\int_{t_0}^{t} L(s)ds\right) + \varepsilon\right)|x_0 - y_0| + \delta. \tag{1.2.2}$$

Proof. Let $x(\cdot)$ be a solution of (1.1.3) with the pseudoderivative $f_x(\cdot)$. Fix $\nu > 0$ and denote $C := \int_{t_0}^{T} L(s)ds$.

Let $\delta > 0$. We will construct a δ-solution $y_1(\cdot)$ of (1.1.3) with $y_1(t_0) = y_0$ such that

$$|x(t) - y_1(t)| \leq \left(\exp\left(\int_{t_0}^{t} L(s)ds\right) + \nu\right)|x_0 - y_0| \tag{1.2.3}$$

and

$$|f_x(t) - f_1(t)| \leq L(t)\left(|x(t) - y_1(t)| + \frac{\nu}{2C}\right) + \frac{\nu}{2T}, \tag{1.2.4}$$

where $f_1(\cdot)$ is the pseudoderivative of $y_1(\cdot)$. Let $y_1(\cdot)$ be already defined on some interval $[t_0, \tau]$, $\tau < T$. Let $\mu > 0$. Due to **F2**, there exists a strongly measurable function $f_1(\cdot)$ such that $f_1(t) \in F(t, y_1(\tau))$ and

$$|f_x(t) - f_1(t)| \leq L(t)|x(t) - y_1(\tau)| + \mu$$

for a.e. $t \in I$. There exists $\lambda > 0$ such that if an extension of $y_1(\cdot)$ on $[\tau, \tau + \lambda]$, denoted also by $y_1(\cdot)$, such that $\dot{y}_1(t) \in Ay_1(t) + f_1(t)$, then $|y_1(t) - y_1(\tau)| \leq \delta$ for any $t \in [\tau, \tau + \lambda]$. Therefore,

$$\begin{aligned} |f_x(t) - f_1(t)| &\leq L(t)(|x(t) - y_1(t)| + |y_1(t) - y_1(\tau)|) + \mu \\ &\leq L(t)(|x(t) - y_1(t)| + \delta) + \mu, \end{aligned}$$

for a.e. $t \in [\tau, \tau + \lambda]$. Then, using Theorem 1.1.3, we derive

$$|x(t) - y_1(t)| \leq |x_0 - y_0| + \int_{t_0}^{t} [L(s)(|x(s) - y_1(s)| + \delta) + \mu] ds,$$

$$\leq \int_{t_0}^{t} L(s)|x(s) - y_1(s)| ds + |x_0 - y_0| + \delta C + \mu T, \ i.e.,$$

$$|x(t) - y_1(t)| \leq \left(\exp\left(\int_{t_0}^{t} L(s) ds\right) + \nu\right) (|x_0 - y_0| + \delta C + \mu T).$$

Clearly, we can choose $\delta > 0$ and $\mu > 0$ such that $(\delta C + \mu T)\exp(C) < \nu|x_0 - y_0|$. By Zorn's lemma, we can define the δ-solution $y_1(\cdot)$ on the whole I with $y_1(t_0) = y_0$ satisfying (1.2.3) and (1.2.4).

Let $\alpha > 0$. Using the same method, we construct a sequence of $(\alpha/(2^k C))$-solutions $y_k(\cdot)$ of (1.1.3), for $k \geq 2$, with the pseudoderivatives $f_k(\cdot)$ such that $y_k(t_0) = y_0$,

$$|y_k(t) - y_{k+1}(t)| \leq \frac{\alpha}{2^k} \exp\left(\int_{t_0}^{t} L(s) ds\right) \ and$$

$$|f_k(t) - f_{k+1}(t)| \leq \frac{\alpha}{2^k} \left[L(t) \left(\exp\left(\left(\int_{t_0}^{t} L(s) ds\right) + \frac{1}{2C}\right) + \frac{1}{2T}\right].$$

Consequently $y_k(t) \to y(t)$ uniformly on I and $f_k(t) \to f(t)$ in $L_1(I, \mathfrak{X})$. Thus, by Lemma 1.1.4, $y(\cdot)$ is the solution of

$$\begin{cases} \dot{y}(t) \in Ay + f(t), \ f(t) \in F(t, y(t)) \\ y(t_0) = y_0. \end{cases} \tag{1.2.5}$$

Furthermore,

$$\begin{aligned} |x(t) - y(t)| \leq |x(t) - y_1(t)| + |y_1(t) - y_2(t)| + ... + |y_k(t) - y_{k+1}(t)| \\ +|y_{k+1}(t) - y(t)| \leq \left(\exp\left(\int_{t_0}^{t} L(s) ds\right) + \nu\right) |x_0 - y_0| \\ +\alpha \exp\left(\int_{t_0}^{t} L(s) ds\right) \sum_{k=1}^{\infty} \frac{1}{2^k} + |y_{k+1}(t) - y(t)| \end{aligned}$$

and hence

$$|x(t)-y(t)| \leq (\exp(\left(\int_{t_0}^{t} L(s)ds\right) + \nu)|x_0-y_0| + \alpha \exp\left(\int_{t_0}^{t} L(s)ds\right)$$
$$\leq (\exp\left(\int_{t_0}^{t} L(s)ds\right) + \nu + \bar{\alpha})|x_0-y_0|,$$

where $\bar{\alpha} = \alpha \exp\left(\left(\int_{t_0}^{t} L(s)ds\right)/|x_0-y_0|\right.$. Moreover,

$$|f_x(t)-f(t)| \leq L(t)[(\exp\left(\int_{t_0}^{t} L(s)ds\right) + \nu + \bar{\alpha})|x_0-y_0| + \frac{\nu+\alpha}{2C}] + \frac{\nu+\alpha}{2T}.$$

Clearly, we can choose $\nu > 0$ and $\alpha > 0$ such that (1.2.1) and (1.2.2) hold. □

Remark 1.2.1. We have also proved that the problem (1.1.3) has a solution.

Theorem 1.2.3. *Assume that* ***F1*–*F2*** *are satisfied. Let* $g : C(I, \mathfrak{X}) \to \overline{D(A)}$ *be a Lipschitz continuous function with the Lipschitz constant* K. *If*

$$K \exp(C) < 1 \tag{1.2.6}$$

then the nonlocal problem (1.1.2) *has a solution.*

Proof. Let $x_0(\cdot) : I \to \overline{D(A)}$ be a continuous function. We consider the evolution inclusion

$$\begin{cases} \dot{x}(t) \in Ax(t) + F(t, x(t)) \\ x(t_0) = g(x_0(\cdot)). \end{cases} \tag{1.2.7}$$

Let $x(\cdot)$ be a solution of (1.2.7) with the pseudoderivative $f_x(\cdot)$.

Let $\varepsilon > 0$ be such that $\alpha := K(\varepsilon + \exp(C)) < 1$. Let $\delta_n = \dfrac{\varepsilon}{2^n}$. Due to Theorem 1.2.2 there exists a solution $y_1(\cdot)$ of

$$\begin{cases} \dot{y}(t) \in Ay(t) + F(t, y(t)) \\ y(t_0) = g(x(\cdot)) \end{cases}$$

such that

$$|x(t) - y_1(t)| \leq \left(\varepsilon + \exp\left(\int_{t_0}^{t} L(s)ds\right)\right) |g(x_0(\cdot)) - g(x(\cdot))|$$

$$|f_x(t) - f_1(t)| \leq L(t)\left(\varepsilon + \exp\left(\int_{t_0}^{t} L(s)ds\right)\right) |g(x_0(\cdot)) - g(x(\cdot))| + \delta_1,$$

where $f_1(\cdot)$ is the pseudoderivative of $y_1(\cdot)$. Since $g(\cdot)$ is K–Lipschitz, then

$$|x(t) - y_1(t)| \leq K\left(\varepsilon + \exp\left(\int_{t_0}^{t} L(s)ds\right)\right) |x_0(\cdot) - x(\cdot)|_\infty \; and$$

$$|f_x(t) - f_1(t)| \leq L(t)K\left(\varepsilon + \exp\left(\int_{t_0}^{t} L(s)ds\right)\right) |x_0(\cdot) - x(\cdot)|_\infty + \delta_1.$$

We continue by induction and define a sequence $(y_n(\cdot))$ of solutions of the problem

$$\begin{cases} \dot{y}(t) \in Ay(t) + F(t, y(t)), \\ y(t_0) = g(y_{n-1}(\cdot)), \end{cases} \tag{1.2.8}$$

with pseudoderivatives $f_n(\cdot)$, for any $n \geq 2$,

$$|y_n(t) - y_{n+1}(t)| \leq \left(\varepsilon + \exp\left(\int_{t_0}^{t} L(s)ds\right)\right) |g(y_{n-1}(\cdot)) - g(y_n(\cdot))|$$

$$\leq K\left(\varepsilon + \exp\left(\int_{t_0}^{t} L(s)ds\right)\right) |y_{n-1}(\cdot) - y_n(\cdot)|_\infty \leq \alpha|y_{n-1}(\cdot) - y_n(\cdot)|_\infty$$

and

$$|f_n(t) - f_{n+1}(t)| \leq L(t)K\left(\varepsilon + \exp\left(\int_{t_0}^{t} L(s)ds\right)\right) |y_{n-1}(\cdot) - y_n(\cdot)|_\infty + \delta_{n+1}$$

$$\leq L(t)\alpha|y_{n-1}(\cdot) - y_n(\cdot)|_\infty + \delta_{n+1},$$

for any $n \geq 1$.

Since $\alpha < 1$, we get that $\lim_{n\to\infty} y_n(t) = y(t)$ uniformly on I and $f_n(\cdot) \to f(\cdot)$ strongly $L_1(I, \mathfrak{X})$. Lemma 1.1.4 then applies and we get the conclusion. □

Remark 1.2.2. The proof of Theorem 1.2.3 is more complicated than the proof of the corresponding result in [88]. However, our conditions are weaker due to the following lemma.

Lemma 1.2.4. *Let $K + C < 1$. Then $Ke^C \le K + C$.*

Proof. Let $K = \mu - C$ with $\mu < 1$, i.e. $Ke^C = (\mu - C)e^C$. Consider the function $h(C) = (\mu - C)e^C$, $C \in [0, \mu]$. Clearly $h(0) = \mu$, $h(\mu) = 0$. Furthermore, $\dot{h}(C) = (\mu - C - 1)e^C < 0$. Thus $h(C) \le \mu = K + C$. □

Remark 1.2.3. Notice that it is impossible to prove existence of solutions under the assumptions in this chapter by using multivalued contraction fixed points. Indeed let $Sol(x(\cdot)) = \{y(\cdot) \in C(I, \mathfrak{X}) : \dot{y} \in Ax + F(t, x(t)), y(t_0) = g(x_0)\}$. This operator is set valued contraction, however its graph is not closed. In [8] the author does not take in account the last fact and the result (Theorem 3.5) is not true. Notice that standard modification of the proof of Theorem 1.2.2 gives correct proof of Theorem 3.5 of [8].

Example 1.2.5. *Let $\Omega \subset \mathbb{R}^n$, with $n \ge 4$, be a domain with smooth boundary $\partial\Omega$. Define $\varphi(r) = |r|^{\gamma-1}r$ for $r \ne 0$ and $0 < \gamma < \dfrac{n-2}{n}$. We consider the following system:*

$$\begin{cases} \begin{pmatrix} u_t \\ \dot{v} \end{pmatrix} \in \begin{pmatrix} \Delta\varphi(u) \\ 0 \end{pmatrix} + G(t, u, v) \ \text{on } (0, T) \times \Omega \\ -\frac{\partial\varphi(u)}{\partial\nu} \in \beta(u) \ \text{on } (0, T) \times \partial\Omega \\ u(0, x) = \int_\Omega \int_0^T h(s, x, \lambda, u(s, \lambda)) ds d\lambda, \ x \in \Omega \\ v(0) = v_0. \end{cases} \tag{1.2.9}$$

Here $\beta(\cdot)$ is a maximal monotone graph in $\mathbb{R}$ with $\beta(0) \ni 0$ and $G : [0, T] \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is a given multifunction.

Define the operator B *in* $L_1(\Omega)$ *as follows:*

$$Bu = \Delta\varphi(u), \quad \textit{for } u \in D(B), \quad \textit{where}$$

$$D(B) = \{u \in L_1(\Omega) : \ \varphi(u) \in W^{1,1}(\Omega), \ \Delta\varphi(u) \in L_1(\Omega), \\ -\frac{\partial\varphi(u)}{\partial\nu} \in \beta(u) \quad \textit{on } \partial\Omega\}.$$

Here, the derivatives are understood in the sense of distributions.

As it is shown in [26, p. 97], the operator $B = \Delta\varphi(u)$ *is m-dissipative in* $L_1(\Omega)$ *and generates a noncompact semigroup. Notice that in [26] the author works with m-accretive operators, however* B *is m-dissipative iff* $-B$ *is m-accretive. The operator* 0 *is linear continuous and generates a semigroup* $S(t) = I$*, where* I *is the identity operator in* $L_1([0,T],\mathbb{R})$*. Consequently, the operator* $A := (B, 0)$ *is also m-dissipative and generates a noncompact semigroup.*

Let $(u, v) \in \mathfrak{X} := L_1(\Omega) \times W^{1,1}[0,T]$*. The considered problem* (1.2.9) *can be rewritten in the abstract form* (1.1.2)*. Namely, we set*

$$g(u,v)(x) = \left(\int_\Omega \int_0^T h(s,x,\lambda,u(s)(\lambda))dsd\lambda, v_0 \right), \quad \textit{for } x \in \Omega,$$

and

$$F(t,u(\cdot),v) = \{(y_1(\cdot),y_2) \in \mathfrak{X}; (y_1(x),y_2) \in G(t,u(x),v) \quad \textit{for} \quad \textit{a.e. } x \in \Omega\}.$$

We suppose that:

G. *the multifunction* G *has nonempty closed values,* $G(\cdot,u,v)$ *is measurable,* $\|G(\cdot,u,v)\|$ *is Lebesgue integrable and there exists a Lebesgue integrable* $L(\cdot)$ *such that* $D_H(G(t,z_1),G(t,z_2)) \leq L(t)|z_1 - z_2|$*, where* $z_i = (u_i, v_i) \in \mathbb{R}^2$*, for* $i = 1, 2$*.*

H. *the function* $h(t,x,\lambda,r)$ *is measurable in* (t,x,λ) *for all* $r \in \mathbb{R}$*, there exist a function* $H(\cdot) \in C(\Omega,\mathbb{R}_+)$ *and a positive Lebesgue integrable function* $\nu(\cdot)$ *such that*

$|h(t,x,\lambda,r)| \leq \nu(t)H(\lambda)$ *for any* $(t,x,\lambda,r) \in [0,T] \times \Omega \times \Omega \times \mathbb{R}$, *and* $|h(t,x,\lambda,u) - h(t,x,\lambda,v)| \leq \frac{K}{T\mu(\Omega)}|u-v|$ *for all* $(t,x,\lambda,u),(t,x,\lambda,v) \in [0,T] \times \Omega \times \Omega \times \mathbb{R}$.

It is standard to prove, in view of hypothesis ***G****, that the multifunction* F *satisfies* ***F1****–****F2****. From* ***H*** *it follows that* $g(\cdot)$ *is well defined and*

$$|g(u_1) - g(u_2)|_{L^2(\Omega)} \leq K|u_1(\cdot) - u_2(\cdot)|_\infty,$$

for any $u_1, u_2 \in C([0,T]; L^2(\Omega))$.

Then, due to Theorem 1.2.3, we have the following result.

Theorem 1.2.6. *Under the assumptions* ***G*** *and* ***H****, the nonlocal problem* (1.2.9) *has a solution when*

$$K \exp\left(\int_0^T L(s)ds\right) < 1.$$

Remark 1.2.4. Most of the results of this chapter are published in [7].

Chapter 2

Filippov-Pliss lemma for dynamical inclusions on time scale

In this chapter we study the dynamical inclusions on time scale and their solutions. We refer the interested reader to the fundamental book [23, 24], where dynamical equations on time scale are comprehensively studied. We prove two variants of the well known lemma of Filippov–Pliss in case of dynamical inclusions on time scale. The first variant is when the right-hand side is Lipschitz continuous on the state variable. Afterward we introduce one sided Perron conditions for multifunctions on time scale and prove the second variant of that lemma. Some discussions on relaxed systems is provided.

2.1 Preliminaries

In this section we give the basic definitions, notations and the results used here. Every nonempty closed set $\mathbb{T} \subset \mathbb{R}$ is called time scale. Therefore the time scale $\mathbb{T}$ is a complete metric space with the usual metric on $\mathbb{R}$. Furthermore, the intersection of $\mathbb{T}$ with any closed bounded interval is a compact set. The calculus of time scales was initiated by Stefan Hilger [52] in order to unify the continuous and discrete analysis.

Definition 2.1.1. Let $\mathbb{T}$ be any time scale then for any $t \in \mathbb{T}$ the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as follows

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined as follows

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition we define $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M and $\rho(m) = m$ if $\mathbb{T}$ has a minimum m.

We classify the points in time scale as follows:

i) $t \in \mathbb{T}$ is left dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$.

ii) $t \in \mathbb{T}$ is right dense if $t > \sup \mathbb{T}$ and $\sigma(t) = t$.

iii) $t \in \mathbb{T}$ is left scattered if $\rho(t) < t$.

iv) $t \in \mathbb{T}$ is right scattered if $\sigma(t) > t$.

v) $t \in \mathbb{T}$ is dense if it is both right dense and left dense.

vi) $t \in \mathbb{T}$ is isolated if it is both right scattered and left scattered.

Definition 2.1.2. For any $t \in \mathbb{T}$ the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. Clearly $\mu(t) = 0$ if t is right dense point, the point $t \in \mathbb{T}$ is right scattered if $\mu(t) > 0$.

Denote by $\mathbb{T}_{rd}$ the right dense and by $\mathbb{T}_{rs}$ the right scattered points of $\mathbb{T}$, then $\mathbb{T}_{rd} \bigcap \mathbb{T}_{rs} = \emptyset$. The set $\mathbb{T}^{\kappa}$ is defined as follows:

If $\mathbb{T}$ has a left scattered maximum S then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{S\}$, otherwise it coincides with $\mathbb{T}$.

Now consider the function $f : \mathbb{T} \longrightarrow \mathbb{R}^n$ and we will define the Δ-derivative (Hilger derivative).

Definition 2.1.3. Let $f : \mathbb{T} \longrightarrow \mathbb{R}^n$ and let $t \in \mathbb{T}$. Suppose there exists $A \in \mathbb{R}^n$ with the property: for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(\sigma(t)) - f(s) - A(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|$$

for every $s \in (t - \delta, t + \delta) \cap \mathbb{T}$. The vector A is called $\triangle$-derivative and is denoted by $f^\Delta(t)$.

Some properties of the Δ-derivative are:

Proposition 2.1.1. *Let $f : \mathbb{T} \to \mathbb{R}^n$ and $t \in \mathbb{T}^k$.*

a) If f is Δ- differentiable at t, then f is continuous at t.

b) If f is continuous at $t \in \mathbb{T}_{rs}$ then

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

c) The map f is Δ- differentiable at $t \in \mathbb{T}^k \backslash \mathbb{T}_{rs}$ if and only if

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

d) Let $f, g : \mathbb{T} \to \mathbb{R}^n$ be Δ-differentiable, then the scalar product $\langle f, g\rangle$ is also Δ-differentiable and

$$\langle f(t), g(t)\rangle^\Delta = \langle f(t), g^\Delta(t)\rangle + \langle f^\Delta(t), g(t + \mu(t))\rangle.$$

In particular,

$$\langle f^2(t)\rangle^\Delta = \langle f(t) + f(t + \mu(t)), f^\Delta(t)\rangle.$$

It follows from Proposition 2.1.1 that $f(\sigma(t)) = f(t)+\mu(t)f^{\Delta}(t)$. Evidently vector valued function $f(\cdot)$ is Δ-differentiable at t iff every coordinate function $f_i(.)$ is Δ-differentiable at t and $f^{\Delta}(t) = (f_1^{\Delta}(t), \ldots, f_n^{\Delta}(t))$.

(i) if $\mathbb{T} = \mathbb{Z}$, then in this case by proposition 2.1.1(b) we have

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = f(t+1) - f(t) = \Delta f(t)$$

(ii) if $\mathbb{T} = \mathbb{R}$, then in this case by proposition 2.1.1(c) we have

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s} = f'(t).$$

Definition 2.1.4. Denote the interval $\mathbb{I} = [t_0, T)_{\mathbb{T}} = \{\tau \in \mathbb{T} : \ a \leq \tau < b\}$. The outer measure of the set $A \subset \mathbb{T}$ is

$$m^*(A) = \inf\{\sum_{i=1}^{\infty}(b_i - a_i) : \ A \subset \bigcup_i [a_i, b_i)\}.$$

The set $A \subset \mathbb{T}$ is said to be Δ-measurable if

$$m^*(A) = m^*(A \bigcap B) + m^*(A \bigcap (\mathbb{T} \setminus B))$$

for every subset $B \subset \mathbb{T}$.

Definition 2.1.5. The function $f : \mathbb{T} \to \mathbb{R}^n$ is said to be Δ-measurable if for every open set $B \subset \mathbb{R}^n$ the set

$$f^{-1}(B) = \{t \in \mathbb{T} : \ f(t) \in B\}$$

is Δ-measurable.

The multifunction $F : \mathbb{T} \rightrightarrows \mathbb{R}^n$ is said to be Δ-measurable if for every compact set $B \subset \mathbb{R}^n$

$$F^{-1}(B) = \{t \in \mathbb{T} : \ F(t) \bigcap B \neq \emptyset\}$$

is Δ-measurable.

We notice the following property:

If $A_1, A_2$are Δ-measurable with $A_1 \bigcap A_2 = \emptyset$ and $A = A_1 \bigcup A_2$, then the multifunction $H : A \rightrightarrows \mathbb{R}^n$ is Δ-measurable if and only if H is Δ-measurable as a map from A_i into $\mathbb{R}^n$ for $i = 1, 2$.

Furthermore, every Δ-measurable function $f(\cdot)$ satisfies Lusin's property, i.e. there exists sequence of pairwise disjoint closed sets $\mathbb{I}_n \subset \mathbb{I}$ such that Δ measure of $\mathbb{I} \setminus \bigcup_{m=1}^{\infty} \mathbb{I}_m$ is 0 and f is continuous on $\mathbb{I}_k \times \mathbb{R}^n$ for every $k \geq 1$.

Proposition 2.1.2. *(Proposition 2.13 in [48]). If the function $f : \mathbb{I} \to \mathbb{R}^n$ is absolutely continuous, then the Δ-measure of the set $\{t \in \mathbb{I}_{rd} : f(t) = 0$ and $f^{\Delta}(t) \neq 0\}$ is zero.*

As it is well known every Δ-measurable multifunction $F(\cdot)$ admits Δ-measurable selection $f(t) \in F(t)$. (see e.g. [79])

Definition 2.1.6. The multifunction $F : \mathbb{I} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be

– Upper semi continuous (USC) at (τ, y) if for every $\varepsilon > 0$ there exists δ such that $F((\tau - \delta, \tau + \delta) \bigcap \mathbb{I}, x + \delta\mathbb{B}) \subset F(\tau, y) + \varepsilon\mathbb{B}$, where $\mathbb{B}$ is the closed unit ball.

– Lower semi continuous (LSC) at (τ, y) if for every $\mathbb{I} \ni t_i \to \tau$, $x_i \to y$ and $f \in F(\tau, y)$ there exists $f_i \in F(t_i, x_i)$ with $f_i \to f$.

– Continuous if it is simultaneously USL and LSC.

2.2 System description

We study the following dynamical inclusion on time scale:

$$\begin{cases} x^{\Delta} \in F(t, x(t)), \ \Delta \ - a.e. \ t \in \mathbb{I} \\ x(t_0) = x_0. \end{cases} \tag{2.2.1}$$

where $\mathbb{I}$ is bounded time scale, $F : \mathbb{I} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is with nonempty convex compact values. Notice that $\mathbb{I} = \mathbb{I}_{rd} \bigcup \mathbb{I}_{rs}$.

Definition 2.2.1. The Δ-absolute continuous function $x(\cdot)$ is said to be a solution of (2.2.1) if it satisfies the inclusion (2.2.1) for Δ a.a. $t \in \mathbb{I}$.

Denote by $\mathcal{B}$ the Borel σ algebra on $\mathbb{R}^n$.

Definition 2.2.2. The multivalued map $F : \mathbb{I} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be $\Delta \times \mathcal{B}$ measurable if for every open set (and hence $\mathcal{B}$ measurable) $B \subset \mathbb{R}^n$ the set

$$F^{-1}(B) = \{(t,u) \in \mathbb{I} \times \mathbb{R}^n : \ F(t,u) \bigcap B \neq \emptyset\} \text{ is } \Delta \times \mathcal{B} \text{ measurable.}$$

F1. $|F(t,x)| \leq \lambda(1 + |x|)$ (sub linear growth), where λ is positive constant.

Remark 2.2.1. *It is standard to show that under* ***F1*** *there exist constants M and N such that $|x(t)| \leq M$ and $\|F(t, x(t) + \mathbb{B}) + \mathbb{B}\| \leq N$ for every $t \in \mathbb{T}$ and for every solution $x(\cdot)$ of*

$$x^{\Delta}(t) \in F(t, x(t) + \mathbb{B}) + \mathbb{B}, x(t_0) = x_0$$

when such a solution exists. Here $\mathbb{B}$ is the unit ball in $\mathbb{R}^n$.

F2. The map $F(\cdot,\cdot)$ is $\Delta \times \mathcal{B}$ measurable and for every $t \in \mathbb{I}$ the map $F(t,\cdot)$ is USC.

We prove the existence theorem, because we didn't see the following variant of it. For other existence results we refer the reader to [48, 76].

Theorem 2.2.2. *Under* ***F1, F2*** *the system (2.2.1) has a solution. The solution set is $C(I_{\mathbb{I}}, \mathbb{R}^n)$ compact.*

Proof. Fix $h_n > 0$. For $t > t_0$ we define the approximate solution $x_n(\cdot)$ as it follows:

$$x_n(t) = x(t_0) + \int_{t_0}^{t} f_n(s)ds \tag{2.2.2}$$

Where $f_n(t) \in F(t, x_n(t_0))$. If $t_0 + h_n \in \mathbb{I}$, then we replace t_0 by $t_0 + h_n$ when it is less than T and continue as in (2.2.2) for $t > t_0 + h_n$. If $t_0 + h_n \notin \mathbb{I}$, then there exists $\tau \in \mathbb{I}$ such that $t_0 + h_n \in (\tau, \sigma(\tau))$. then we take $x_n(\sigma(\tau)) = x_n(\tau) + x_n^{\Delta}(\tau) \cdot \mu(\tau)$, where $x_n^{\Delta} \in F(\tau, x_n(\tau))$. Then we replace t_0 by $\sigma(\tau)$ and continue as in (2.2.2) for $t > \sigma(\tau)$. Clearly one can define $x_n(\cdot)$ on the whole interval $\mathbb{I}$, thanks to the growth condition **F1**. The sequence $\{x_n(\cdot)\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous. In fact we have $x_n^{\Delta}(t) \in F(t, x_n(t) + \mathbb{B})$ and $\|x_n^{\Delta}(t)\| \le N$ (see Remark 2.2.1 mentioned above). We claim that $x_n^{\Delta} \in F(t, x_n(t) + \varphi_n(t)\mathbb{B})$, where $\varphi_n \to 0$ Δ a.e. on $[t_0, T)_{\mathbb{T}}$. Due to our construction $t \in [\tau, \tau')_{\mathbb{T}}$

$$x_n(t) = x_n(\tau) + \int_{\tau}^{t} f_n(s)ds$$

where $f_n(s) \in F(s, x_n(\tau))$. We have two cases:

Case(i): If $\tau \in \mathbb{T}_{rd}$, then $\tau' - \tau \le h_n$, i.e. $|x(t) - x(\tau)| \le Nh_n$, where N is the constant from Remark 2.2.1. Therefore $x_n^{\Delta}(t) \in F(t, x_n(t) + Nh_n)$

Case(ii): If $\tau \in \mathbb{T}_{rs}$, then $\tau' = \sigma(\tau)$. In this case $[\tau, \tau') \bigcap \mathbb{T} = \tau$ and hence $t = \tau$. However $x_n^{\Delta}(t) \in F(t, x_n^{\Delta}(t))$. The claim is proved, because $\lim_{h_n \to 0} Nh_n = 0$.

From Theorem 3.5 of [77] we know that there exists a subsequence $x_{n_k}(\cdot)$ which converge uniformally to a solution $x(\cdot)$ of (2.2.1). The proof is therefore complete. □

Definition 2.2.3. The multimap $F(\cdot, \cdot)$ is said to be almost USC (LSC, continuous) if for every $\varepsilon > 0$ there exists a set $N_\varepsilon \subset \mathbb{I}$ with Δ measure less than ε and such that $F(\cdot, \cdot)$ is USC (LSC, continuous) on $(\mathbb{I} \setminus I_\varepsilon) \times \mathbb{R}^n$.

Proposition 2.2.3. *Let $F(\cdot,\cdot)$ have nonempty convex compact values and let it be almost USC, then F is $\Delta \times \mathcal{B}$ measurable and $F(t,\cdot)$ is USC for Δ a.a. $t \in \mathbb{I}$.*

Proof. It is easy to see that $F(\cdot,\cdot)$ is almost USC iff there exists sequence of pairwise disjoint closed sets $\mathbb{I}_n \subset \mathbb{I}$ such that Δ measure of $\mathbb{I} \setminus \bigcup_{m=1}^{\infty} \mathbb{I}_m$ is 0 and F is USC on $\mathbb{I}_k \times \mathbb{R}^n$ for every $k \geq 1$. Therefore F is $\Delta \times \mathfrak{B}$ measurable on $\mathbb{I}_k \times \mathbb{R}^n$ for every $k \geq 1$ and hence on $\mathbb{I} \times \mathbb{R}^n$. Also $F(t,\cdot)$ is USC for Δ a.e. t. □

Now we are to prove that the dynamical inclusion (2.2.1) has a solution if the right-hand side is almost LSC.

Proposition 2.2.4. *Let $F(\cdot,\cdot)$ be almost LSC with closed nonempty values. Under **F1** the dynamical inclusion* (2.2.1) *admits a solution.*

Proof. Consider the compacts $\mathbb{I}_k$ as in the previous proof, where $F(\cdot,\cdot)$ is LSC on $\mathbb{I}_k \times \mathbb{R}^n$. Let N be from Remark 2.2.1. Consider the cone $K_N = \{(t,x) \in \mathbb{I} \times \mathbb{R}^k : |x| \leq (N+1)t\}$.

It is well known that every LSC multifunction has a Γ^{N+1} continuous selection, i.e. selection $f(t,x) \in F(t,x)$ such that $f_k(t_i,x_i) \to f_k(t,x)$ if $\mathbb{I}_k \ni t_i \to t$ and $|x_i - x| \leq (N+1)(t_i - t)$ (see e.g. Lemma 6.2 in [37]).

Define the multifunction $G_k(t,x) = \bigcap_{\varepsilon>0} \overline{co}\ f_k((t-\varepsilon, t+\varepsilon) \bigcap \mathbb{I}_k, x + \varepsilon\mathbb{B})$. Clearly $G_k(\cdot,\cdot)$ is USC on $\mathbb{I}_k \times \mathbb{R}^n$. Let $G(t,x) = G_k(t,x)$ as $t \in \mathbb{I}_k$, $k = 1,2\ldots$ and $G(t,x) = \{0\}$ otherwise (see e.g.[36]). Consequently $G(\cdot,\cdot)$ is almost USC and hence the dynamical inclusion

$$x^{\Delta}(t) \in G(t,x(t)),\ \Delta - a.e.\ t \in \mathbb{I}\ x(t_0) = x_0.$$

has a solution $y(\cdot)$.

Dealing as in the proof of Lemma 6.1 of [37] (see also [36]) that $y(\cdot)$ is also a solution (2.2.1). $\square$

2.3 Lemma of Filippov-Pliss on time scale

In this section we prove two variants of the lemma of Filippov–Pliss which have many applications in optimal control (cf. [44]). The first variant deals with Lipschitz condition on the right-hand side. In second variant we relax Lipschitz condition to the much weaker one sided Perron condition.

We need the following result, which is a particular case of Proposition 1.43 of [54].

Proposition 2.3.1. *Let $F, G : \mathbb{I} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be $\Delta \times \mathcal{B}$ measurable and at least one with compact values. Then the map $H(t,x) = F(t,x) \bigcap G(t,x)$ is also $\Delta \times \mathcal{B}$ measurable.*

Now we will prove two variant of theorem of Filippov–Pliss for dynamical inclusion on time scale. The first proof deals with Lipschitz right hand side.

Theorem 2.3.2. *Let $F(\cdot,\cdot)$ satisfy **F1, F2** and let $F(t,\cdot)$ be L–Lipschitz. If $f(\cdot)$ is Δ-integrable function on $\mathbb{I}$ and if $y(\cdot)$ is AC function with*

$$dist(y^{\Delta}(t), F(t, y(t))) \le f(t),$$

then there exists a solution $x(\cdot)$ of (2.2.1) such that $|x(t) - y(t)| \le r(t)$, where $r^{\Delta}(t) = Lr(t) + f(t)$ and $r(t_0) = |x_0 - y_0|$. Furthermore $|x^{\Delta}(t) - y^{\Delta}(t)| \le Lr(t) + f(t)$.

Proof. Define the map

$$G(t,u) = \{v \in F(t,u) : |y^{\Delta}(t) - v| \le L|y(t) - u| + f(t)\}$$

We claim that $G(\cdot,\cdot)$ satisfies **F1** and **F2**.

Namely $G(t,u)$ admits nonempty values because $F(t,\cdot)$ is Lipschitz and it is with nonempty convex compact values. We are going to prove that $G(t,u)$ is closed and convex. Indeed if $v_i \in F(t,u)$ and $v_i \to v$, we know that $|v - y^\Delta(t)| = \lim_{i\to\infty} |v_i - y^\Delta(t)|$. Therefore if $v_1,\, v_2 \in G(t,u)$ then we have that $|y^\Delta(t) - \lambda v_1 - (1-\lambda)v_2| \leq \lambda|y^\Delta(t) - v_1| + (1-\lambda)|y^\Delta(t) - v_2| \leq L|y(t) - u| + f(t), \quad \forall\lambda \in (0,1)$. Also $G(t,u)$ is USC. For this it is enough to see that $G(t,\cdot)$ has closed graph. Let $v_i \in G(t,u_i)$, $u_i \to u$ and $v_i \to v$. Since $F(t,.)$ is USC, one has that $\lim_{i\to\infty} v_i = v \in F(t,u)$. Furthermore, $|v_i - y^\Delta(t)| \leq L|y(t) - u_i| + f(t)$ and hence $|v - y^\Delta(t)| \leq L|y(t) - u| + f(t)$.

Now, we have to show that $G(\cdot,\cdot)$ is $\Delta \times \mathcal{B}$ measurable. Let $X \in \mathbb{R}^n$. Since $y(.)$ is AC, then $y^\Delta(.)$ is Δ-measurable and hence the multimap $H(t,X) = \{(t,z) \in [t_0,T)_\mathbb{T} \times \mathbb{R}^n : \ |z - y^\Delta(t)| \leq L|y(t) - X| + f(t)\}$ is $\Delta \times \mathcal{B}$ measurable. Due to Proposition 2.3.1 the map $G(t,X) = H(t,X) \cap F(t,X)$ is also $\Delta \times \mathcal{B}$ measurable. The claim is therefore proved. It follows from Theorem 2.2.2 that $x^\Delta \in G(t,x(t))$, $x(t_0) = x_0$ admits a solution $x(\cdot)$.

From Theorem 1.67 of [23] we know that $|x(t) - y(t)| = r(t)$, where $r^\Delta(t) \leq Lr(t) + f(t)$ for Δ-a.e. t and $r(t_0) = |x_0 - y_0|$.

The definition of $G(\cdot,\cdot)$ then implies the last statement of the theorem. □

Now we will prove a Filippov– Pliss type theorem under much weaker condition, which gives the estimation only of the difference between $x(\cdot)$ and $y(\cdot)$ but not between their derivatives.

Definition 2.3.1. The multivalued map $F : \mathbb{I} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be OSP (one-sided Perron) (on the state variable) if there exists a Perron function $w(\cdot,\cdot)$ such that:

For every $x, y \in \mathbb{R}^n$, almost every $t \in \mathbb{I}$ and every $f_x \in F(t,x)$ there exists $f_y \in F(t,y)$ such that

$$\langle x-y, f_x - f_y\rangle \le \frac{1}{2} w(t, |x-y|)|x-y|, \text{ if } t \in \mathbb{T}_{rd}$$
$$|f_x - f_y| \le w(t, |x-y|), \text{ if } t \in \mathbb{T}_{rs}.$$

$F(t,\cdot)$ is called full Perron w.r.t $w(\cdot,\cdot)$ if $D_H(F(t,x), F(t,y)) \le w(t, |x-y|)$. Clearly every full Perron $F(t,\cdot)$ is OSP.

Recall that the function $v(\cdot,\cdot)$ is said to be Perron function if

– $v(\cdot,\cdot)$ is $\Delta \times \mathcal{B}$ measurable, $v(t,\cdot)$ is continuous.

– $v(\cdot,\cdot)$ is Δ-integrally bounded on the bounded sets and $v(t,0)=0$

– the unique solution of $r^\Delta(t) = v(t, r(t))$, $r(t_0) = 0$ is $r(t) = 0$.

$v(\cdot,\cdot)$ is called module if it satisfies only the first two conditions, but not necessarily the third one.

The definition of OSP condition on time scale is different than in ordinary differential inclusions. Here it depends also on the point t.

Now we extend the previous theorem to the case of OSP multifunctions.

Theorem 2.3.3. *Let $F(\cdot,\cdot)$ satisfy **F1, F2** and let $F(t,\cdot)$ be OSP w.r.t. a Perron function $w(\cdot,\cdot)$. If $f(\cdot)$ is a Δ-integrable function on $\mathbb{I}$ and if $y(\cdot)$ is AC function with $dist(y^\Delta(t), F(t,y(t))) \le f(t)$, then there exists a solution $x(\cdot)$ of (2.2.1) such that $|x(t) - y(t)| \le r(t)$, where $r^\Delta(t) = w(t, r(t)) + f(t)$ and $r(t_0) = |x_0 - y_0|$.*

Proof. Clearly the set valued map $t \to y^\Delta(t) + f(t)\mathbb{B}$ is Δ-measurable. Therefore $H(t) = F(t, y(t)) \bigcap (y^\Delta(t) + f(t)\mathbb{B})$ is also $\Delta - measurable$ and hence $t \to F(t, y(t))$ is $\Delta - measurable$ Thus there exists a Δ- measurable selection $h(t) \in H(t)$. Evidently $h(t) \in F(t, y(t))$.

Now we define the following multifunction:

$$G(t,u) = \{v \in F(t,u)\} \text{ such that} \tag{2.3.1}$$

$$\begin{cases} \langle y(t) - u, h(t) - v\rangle \leq w(t, |x-y|)|x-y|, \ t \text{ is right dense} \\ |h(t) - v| \leq w(t, |x-y|), \ t \text{ is right scattered.} \end{cases}$$

We claim that $G(t, \cdot)$ is upper- semi- continuous for every $t \in \mathbb{I}$. Indeed we have to prove that the graph of $G(t, \cdot)$ is compact. However, the graph is bounded and hence it remains to show that it is closed.

Let $u_i \to u$, $v_i \in G(t, u_i)$ and $v_i \to v$ We have to show that $v \in G(t, u)$. Clearly $v \in F(t, u)$ because $F(t, \cdot)$ is USC. If t is right dense, then $\langle y(t) - u_i, h(t) - v_i\rangle \to \langle y(t) - u, h(t) - v\rangle$, $|y(t) - u_i| \to |y(t) - u|$ and $w(|y(t) - u_i|) \to w(t, |y(t) - u|)$. Thus $\langle y(t) - u, h(t) - v\rangle \leq w(t, |y(t) - u|)|y(t) - u|$, i.e $v \in G(t, u)$ If t is right scattered, then $|h(t) - v| \leq w(t, |y(t) - u|)$. because $\lim_{i\to\infty} |h(t) - v_i| = |h(t) - v|$.

We have to prove that $G(\cdot, \cdot)$ is $\Delta \times \mathcal{B}$ measurable.

Consider first the case $G : \mathbb{T}_{rs} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Since $w(t, \cdot)$ and $y(\cdot)$ are continuous as well as $h(\cdot)$ is Δ-measurable, one has that $S(t, u) = \{v \in \mathbb{R}^n : |h(t) - v| \leq w(t, |y(t) - u|)\}$ is $\Delta \times \mathcal{B}$ measurable. Then $\overline{G}(t, u) = F(t, u) \bigcap \overline{S}(t, u)$ is $\Delta \times \mathcal{B}$ measurable.

Let $G : \mathbb{T}_{rd} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. It is easy to see that the map:

$$S(t,u) = \{v \in \mathbb{R}^n : \langle y(t) - u, h(t) - v\rangle \leq w(t, |y(t) - u|)|y(t) - u|\}$$

is $\Delta \times \mathcal{B}$ measurable.

Therefore $\overline{G}(t, u) = F(t, u) \bigcap \overline{S}(t, u)$ is also Δ-measurable. Consequently $G(\cdot, \cdot)$ is $\Delta \times \mathcal{B}$ measurable.

Due to Theorem 2.2.2 the system

$$\dot{x} \in G(t, x(t)), x(t_0) = x_0$$

has a solution $x(\cdot)$. Therefore

$$\langle y(t) - x(t), y^{\Delta}(t) - x^{\Delta}(t)\rangle$$
$$\leq |y(t) - x(t)||y^{\Delta}(t) - h(t)| + w(t, |y(t) - x(t)|)|y(t) - x(t)|,$$

$\forall t \in \mathbb{T}_{rd}$, i.e $(|y(t) - x(t)|^2)^{\Delta} = 2|x(t) - y(t)| \cdot (|x(t) - y(t)|)^{\Delta} \leq 2|y(t) - x(t)|(f(t) + w(t, |y(t) - x(t)|)$. Clearly $t \to |x(t) - y(t)|$ is Δ-AC.

From Proposition 2.1.2 we know that the intersection of the sets $\{t \in \mathbb{T} : |x(t) - y(t)| = 0\}$ and $\{t \in \mathbb{T} : |x^{\Delta}(t) - y^{\Delta}(t)| \neq 0\}$ have Δ-measure zero. Thus $|x(t) - y(t)|^{\Delta} \leq w(t, |y(t) - x(t)|) + f(t)$ for Δ almost every $t \in \mathbb{T}_{rd}$.

If $t \in \mathbb{T}_{rs}$ then $|y^{\Delta}(t) - x^{\Delta}(t)| \leq |y^{\Delta}(t) - h(t)| + |h(t) - x^{\Delta}(t)| \leq w(t, |y(t) - x(t)|)$. However, $|x(t) - y(t)|^{\Delta} \leq |x^{\Delta}(t) - y^{\Delta}(t)|$ and hence $|x(t) - y(t)|^{\Delta} \leq w(t, |x(t) - y(t)|) + f(t)$.

Consequently

$$|x(t) - y(t)| \leq r(t), \text{ where } r^{\Delta}(t) = w(t, r(t)) + f(t), \; r(t_0) = |x_0 - y_0|.$$

The proof is therefore complete. □

Remark 2.3.4. *It is easy to see that Theorem 2.3.2 remains true also when $w(\cdot, \cdot)$ is only module.*

As it is well known

$$(\|x(t)\|^2)^{\Delta} = \langle x(t), x(t)\rangle^{\Delta} = \langle x(t) + x(\sigma(t)), x^{\Delta}(t)\rangle$$
$$= \langle x(t) + x(t) + \mu(t)x^{\Delta}(t), x^{\Delta}(t)\rangle = 2\langle x(t), x^{\Delta}(t)\rangle + \mu(t)\|x^{\Delta}(t)\|^2.$$

The above conclusion leads us to another definition of OSP on time scale. Namely: $F(t,\cdot)$ is said to be OSP if there exists a Perron function $V(\cdot,\cdot)$ such that for every $f_x \in F(t,x)$ there exists $f_y \in F(t,y)$ such that

$$2\langle x-y, f_x - f_y\rangle + \mu(t)|f_x - f_y|^2 \le V(t, |x-y|^2)$$

In this case, however, it is difficult to prove meaningful version of the lemma of Filippov–Pliss, although the following theorem is true.

Theorem 2.3.5. *Let $x(\cdot)$ be a solution of (2.2.1) with $x(t_0) = x_0$. Then for any y_0 there exists a solution $y(\cdot)$ of (2.2.1) with $y(t_0) = y_0$ such that:*

$$|x(t) - y(t)|^2 \le r(t), \text{ where } r(t_0) = |x_0 - y_0|^2 \text{ and } r^{\Delta}(t) = V(t, r(t)).$$

Example 2.3.6. *Let $\mathbb{T}$ be a time scale on $[0,1]$. Let $\{x_i\}_i$ be a dense subset of $\mathbb{B}$. Define the multifunction $H(x) = \overline{co}\,\{f_i(t,x)\}_{i=1}^k + \sum_{i=k}^{\infty} \frac{f_i(t,x)}{2^i}$, where $k \ge 5$, $f_i(t,x) = c(t)g_i(x)$ and*

$$g_i(x) = \begin{cases} -\frac{x-x_i}{\sqrt{|x-x_i|}}, & x \ne x_i \\ 0, x = x_i \end{cases}$$

While $c = \max\{\tau - t : \ [t,\tau] \subset \mathbb{T}\}$. Clearly every $g_i(\cdot)$ is one sided Lipschitz with a constant 0. Therefore $H(t,\cdot)$ is OSL with a constant 0 on any point (t,x) with $t \in \mathbb{T}_{rd}$ and it is 0 on any point (t,x) with $t \in \mathbb{T}_{rs}$.

Let $G : \mathbb{T} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be bounded full Perron. Define $F(t,x) = H(x) + G(t,x)$. Then $F(\cdot,\cdot)$ satisfies all the assumptions of Theorem 2.3.2.

Then clearly the system (2.2.1) with $t_0 = 0$, $T = 1$ and $x_0 = 0$ satisfies the conditions of Theorem 2.3.3.

2.3.1 Relaxation

In this section we discuss about the closure of the solution set for the system (2.2.1) when $F(\cdot,\cdot)$ is almost continuous and not necessarily convex valued.

Clearly the closure of the solution set of (2.2.1) is not the solution set of

$$x^{\Delta}(t) \in \overline{co}\ F(t,x),\ x(t_0) = x_0.$$

The following theorem holds true:

Theorem 2.3.7. *Let $F(\cdot,\cdot)$ be almost continuous with compact values. Suppose that **F1** holds. The closure of the solution set of* (2.2.1) *is a subset of the solution set of*

$$x^{\Delta}(t) \in H(t,x(t)),\ x(t_0) = x_0, \tag{2.3.2}$$

where

$$H(t,x) = \begin{cases} \overline{co}\ F(t,x), & t \in \mathbb{I}_{rd} \\ F(t,x), & t \in \mathbb{I}_{rs}. \end{cases}$$

Proof. Let $x_m(\cdot)$ be sequence of solutions of (2.2.1) such that $x_m(t) \to x(t)$ uniformly on $\mathbb{I}$. As in [28] we can extend every $x_k^{\Delta}(\cdot)$ on I as a Lebesgue integrable function $g_k(\cdot)$. Due to Diestel criterion (see e.g. [39]) the sequence $\{g_k(\cdot)\}_k$ is weakly L_1 precompact and passing to subsequences if necessary $g_k(t) \to g(t)$ L_1 weakly. Due to Mazur's lemma there exists a convex combination $\sum_{i=k}^{k_i} \alpha_i g_i(t)$ converging to $g(t)$ L_1 strongly and passing to subsequences for a.a. $t \in I$. Clearly its restriction to $\mathbb{I}$ $g_{\mathbb{I}}(t) \in \overline{co}\ F(t,x(t))$ for Δ a.e. t. Notice that every $g_(\cdot)$ is constant on $[t,\sigma(t)]$ the latter s not a single point in case $t \in \mathbb{I}_{rs}$. Since $\mathbb{I}_{rs}$ is countable, then it is easy to show that in the $g(t) \in F(t,x(t))$ for Δ a.a. $t \in \mathbb{I}_{rs}$. Consequently $g(t) \in H(t,x(t))$. □

It will be interesting to prove or disprove the following conjecture, which is analogue of the very important in the optimal control relaxation theorem.

Conjecture 1. *Let $F(\cdot,\cdot)$ be almost continuous with compact values. Suppose that **F1** holds and $F(t,\cdot)$ is OSP. Then the closure of the solution set of* (2.2.1) *is the solution set of* (2.3.2).

One can try the following modification of the proof in case of differential inclusions on time intervals (see e.g. [45]). Fix $\varepsilon > 0$ and define the map

$$G(t,u) = \overline{\{v \in F(t,u) : \langle x(t)-u, x^{\Delta}(t)-v\rangle < v(t,|x(t)-u|)+\varepsilon)|x(t)-u|\}}$$

when $t \in \mathbb{I}_{rd}$ and $G(t,u) = Proj_{F(t,u)}x^{\Delta}(t)$, when $t \in \mathbb{I}_{rs}$.

If the system

$$x^{\Delta}(t) \in G(t,x(t)),\ \Delta - a.e.\ t \in \mathbb{I}\ x(t_0) = x_0.$$

has a solution $y(\cdot)$, then $|x(t)-y(t)| \le r(t)$, where $r(0)=0$ and $r^{\Delta}(t) \le w(t,r(t))+\varepsilon$ Δ a.e. due to the definition of $G(\cdot,\cdot)$.

Therefore it is enough to prove only that $G(\cdot,\cdot)$ is almost LSC. Here we have difficulties.

For example it is possible $\bar{t}$ to be right dense, however, to be a limit of a sequence of right scatter points. Clearly in that case $G(\cdot,\cdot)$ is not LSC in general at $\bar{t}$.

2.4 Conclusions

In this chapter we studied dynamical inclusions on time scale. We proved two variants of Flippov-Pliss lemma, which has many applications in optimal control and in stability theory of differential inclusions. It is clear from the proof that there is significant difference between the used assumptions in case of right dense and right scattered points. In the theory of time scale one can also define the so called ∇-derivative, which is more appropriate to study backward behaviour of the solutions to dynamical equations (inclusions). We refer the reader to [23, 24]. Most of the results of this chapter are published in [74].

Chapter 3

On the solution set of multivalued fuzzy fractional systems

In this chapter we study fuzzy fractional integral inclusions and their solutions. We investigate fuzzy fractional integral inclusions under compactness type conditions. We prove the existence of solutions when the right-hand side is almost upper semicontinuous. We also show that the solution set is connected. Finally, an application to fuzzy fractional differential inclusions is given.

3.1 Preliminaries

In this section we give the basic definitions, notations and some basic results used here. The modelling of the real problems often requires some kinds of uncertainty and is one of the main reasons to investigate the fuzzy systems and multivalued differential equations. The differential inclusions give more adequate models of the real processes and they are extensively used in optimal control.

Definition 3.1.1. A fuzzy set is a pair (U, x), where U is a nonempty set and $x : U \to [0, 1]$ a membership function. For each $u \in U$, the value $x(u)$ is called the membership grade of u.

Definition 3.1.2. Let $\mathbb{E} = \{x : \mathbb{R}^n \to [0,1];\ x$ satisfies (1) – (4)$\}$ be the space of fuzzy numbers, where

(1) x is normal, i.e. there exists $v_0 \in \mathbb{R}^n$ such that $x(v_0) = 1$;

(2) x is fuzzy convex, i.e. $x(\lambda v+(1-\lambda)w) \geq \min\{x(v), x(w)\}$ whenever $v, w \in \mathbb{R}^n$ and $\lambda \in [0,1]$;

(3) x is upper semicontinuous;

(4) the closure of the set $\{v \in \mathbb{R}^n;\ x(v) > 0\}$ is compact.

Definition 3.1.3. For $x \in \mathbb{E}$ and $\alpha \in (0,1]$, the set $[x]^\alpha = \{v \in \mathbb{R}^n;\ x(v) \geq \alpha\}$ is called α-level set of x. It follows from (1) – (4) that the α-level sets $[x]^\alpha$ are nonempty, convex and compact subsets of $\mathbb{R}^n$ for all $\alpha \in (0,1]$.

Definition 3.1.4. The fuzzy zero is defined by

$$\hat{0}(v) = \begin{cases} 0 \text{ for } v \neq 0, \\ 1 \text{ for } v = 0. \end{cases}$$

The set $\mathbb{E}$ is a semilinear space with the following operations:

$$(x+y)(v) = \sup_{v_1+v_2=v} \min\{x(v_1), y(v_2)\},$$

$$\lambda(v) = \begin{cases} x(v/\lambda) \text{ at } \lambda \neq 0 \\ \chi_0(v) \text{ at } \lambda = 0, \end{cases}$$

where $x, y \in \mathbb{E}$ and $\lambda \in \mathbb{R}$. The metric in $\mathbb{E}$ is

$$D(x,y) = \sup_{\alpha \in (0,1]} D_H([x]^\alpha, [y]^\alpha),$$

where $D_H(A,B) = \max\{\max_{a\in A}\min_{b\in B}|a-b|, \max_{b\in B}\min_{a\in A}|a-b|\}$ is the Hausdorff distance between the convex compact subsets of $\mathbb{R}^n$. Then, $\mathbb{E}$ is a complete semilinear metric space with respect to the metric D (see [73]). This space is not locally compact and it is nonseparable. The metric D satisfies the following properties:

(i) $D(x+y, z+y) = D(x,z)$,

(ii) $D(\lambda x, \lambda y) = \lambda D(x,y)$,

(iii) $D(x+\overline{x}, y+\overline{y}) \leq D(x,y) + D(\overline{x}, \overline{y})$, for any $\lambda \geq 0$ and $x, y, z, \overline{x}, \overline{y} \in \mathbb{E}$.

The distance from $x \in \mathbb{E}$ to the closed bounded set $A \subset \mathbb{E}$ is defined by $dist(x, A) = \inf_{y\in A} D(x,y)$ and the Hausdorff distance between two closed and bounded subsets A, B of $\mathbb{E}$ is defined by $d_H(A,B) = \max\{\sup_{x\in A}\inf_{y\in B} D(x,y), \sup_{y\in B}\inf_{x\in A} D(x,y)\}$.

Definition 3.1.5. The function $f : I \to \mathbb{E}$ is said to be continuous if it is continuous with respect to the metric D.

Definition 3.1.6. The map $f : I \to \mathbb{E}$ is said to be strongly measurable if there exists a sequence $\{f_m\}_{m=1}^{\infty}$ of step functions $f_m : I \to \mathbb{E}$ such that $\lim_{m\to\infty} D(f_m(t), f(t)) = 0$ for a.a. $t \in I$. Clearly, $[f]^{\alpha}(\cdot)$ are strongly measurable for every $\alpha \in (0,1]$ if $f(\cdot)$ is strongly measurable. The converse does not necessarily holds (see [56]).

Definition 3.1.7. If $f : I \to \mathbb{E}$ is strongly measurable and $D(f(t), \hat{0}) \leq \lambda(t)$, where $\lambda(\cdot)$ is Lebesgue integrable real valued function, then f is Bochner integrable and

$$\int_{t_0}^{t} f(s)ds = \lim_{m\to\infty} \int_{t_0}^{t} f_m(s)ds,$$

where $f_m(\cdot)$ are the step functions with $f_m(t) \to f(t)$ for a.a. $t \in I$.

In the fuzzy set literature starting from [73], the integral of fuzzy functions is defined levelwise, i.e. there exists $g(t) \in \mathbb{E}$ such that $[g]^{\alpha}(t) = \int_{t_0}^{t} [f]^{\alpha}(s)ds$.

As it is shown in [56], there are levelwise integrable functions which are not almost everywhere separably valued, i.e. not Bochner integrable.

Definition 3.1.8. The function $g : I \to \mathbb{E}$ is called absolutely continuous if there exists a strongly measurable function $f : I \to \mathbb{E}$ such that $g(t) = \int_{t_0}^{t} f(s)ds$.

The space $\mathbb{E}$ can be embedded as a closed convex cone in a Banach space $\mathbb{X}$ (see Theorem 2.1 of [56]). The embedding map $j : \mathbb{E} \to \mathbb{X}$ is isometry and isomorphism. Hence, $f : I \to \mathbb{E}$ is continuous if and only if $j(f)(\cdot)$ is continuous. Furthermore, $j(\cdot)$ preserves differentiation and integration. Namely, if $\dot{f}(t)$ exists, then $\frac{d}{dt}j(f)(t)$ also exists and $j\left(\dot{f}\right)(t) = \frac{d}{dt}j(f)(t)$, where $\frac{d}{dt}$ is the usual differential operator. Now, if $g : I \to \mathbb{E}$ is strongly measurable and integrable, then $j(g)(\cdot)$ is strongly measurable and Bochner integrable, and

$$j\left(\int_{t_0}^{t} g(s)ds\right) = \int_{t_0}^{t} j\left(g\right)(s)ds \text{ for all } t \in I. \tag{3.1.1}$$

The multifunction F is said to be almost USC when for every $\delta > 0$ there exists a compact set $I_\delta \subset I$ with Lebesgue measure $meas(I \setminus I_\delta) < \delta$ such that $F|_{I_\delta \times \mathbb{E}}$ is USC.

3.2 Measures of noncompactness

Let Y be a complete metric space and denote by $\mathcal{B}(Y)$ the family of all bounded subsets of Y. We recall that the Hausdorff measure of noncompactness $\beta_Y : \mathcal{B}(Y) \to \mathbb{R}$ for the bounded subset A of Y is defined by

$$\beta_Y(A) := \inf\{d > 0; A \text{ can be covered by finite number of balls with radius } d \text{ and centers in } Y\}.$$

The Kuratowski measure of noncompactness $\rho_Y : \mathcal{B}(Y) \to \mathbb{R}$ for the bounded subset A of Y is defined by

$$\rho(A) := \inf\{d > 0; A \text{ can be covered by finite number of sets with diameter } \leq d\}.$$

For any bounded set $A \subset Y$ we denote $\operatorname{diam}(A) = \sup_{a,b\in A} \varrho_Y(a,b)$, where $\varrho_Y(\cdot,\cdot)$ is the distance in Y. Then, it is easy to see that A is a subset of a ball with radius equal to $\operatorname{diam}(A)$.

Remark 3.2.1. *It is known that if $A \subset L \subset Y$ then $\beta_Y(A) \leq \beta_L(A) \leq 2\beta_Y(A)$ and $\rho(A)$ does not depend on the subspace Y. Furthermore, $\rho(A) \leq 2\beta(A) \leq 2\rho(A)$.*

Let $\gamma(\cdot)$ represents both $\rho(\cdot)$ and $\beta(\cdot)$. Then some properties of $\gamma(\cdot)$ are listed below. Let $A, B \in \mathcal{B}(Y)$.

(i) $\gamma(A) = 0$ if and only if $\bar{A}$ is compact.

(ii) $\gamma(\overline{co}\ A) = \gamma(A)$.

(iii) $\gamma(A + B) \leq \gamma(A) + \gamma(B)$.

(iii) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.

(iv) $\gamma(A \bigcup B) = \max\{\gamma(A), \gamma(B)\}$.

(v) $\gamma(\cdot)$ is continuous with respect to the Hausdorff distance.

The following result regarding the imbedding map $j : \mathbb{E} \to \mathbb{X}$ will be used later in this chapter.

Theorem 3.2.2. *Let A be a bounded subset in $\mathbb{E}$. Then $\beta(j(A)) \leq \beta(A) \leq 2\beta(j(A))$.*

Proof. We know that $j(\cdot)$ is an isometry and isomorphism, which implies that $\beta_{\mathbb{E}}(A) = \beta_{j(\mathbb{E})}(j(A))$. Since $j(\mathbb{E}) \subset \mathbb{X}$ then $\beta_{j(\mathbb{E})}(j(A)) \geq \beta_{\mathbb{X}}(j(A))$. Consequently, $\beta_{\mathbb{E}}(A) \geq \beta_{\mathbb{X}}(j(A))$. The latter can be written as $\beta(A) \geq \beta(j(A))$.

Now, we will prove the other part of the inequality. Since $j(\cdot)$ preserves the diameter, one has that $\rho(j(A)) = \rho(A)$. Using $\rho(j(A)) \leq 2\beta(j(A))$ (see [38] page 42) we have $\rho(A) \leq 2\beta(j(A))$. But $\beta(A) \leq \rho(A)$. Therefore $\beta(A) \leq 2\beta(j(A))$. □

Remark 3.2.3. *From Theorem* 3.2.2 *we also get that* $\beta(A) \leq 2\beta(j(A)) \leq 2\beta(A)$.

The following result is interesting itself because it proves the most important property of $\beta(\cdot)$ in the case of fuzzy sets. For the proof see [46].

Theorem 3.2.4. *Let* $\{f_m(\cdot)\}_{m=1}^{\infty}$ *be an integrally bounded sequence of strongly measurable fuzzy functions from* I *to* $\mathbb{E}$. *Then the function* $t \to \beta(\{f_m(t), m \geq 1\})$ *is measurable and*

$$\beta\left(\int_t^{t+h}\left\{\bigcup_{m=1}^{\infty} j(f_m(s))\right\} ds\right) \leq 2\int_t^{t+h} \beta\left\{\bigcup_{m=1}^{\infty} f_m(s)\right\} ds. \tag{3.2.1}$$

Definition 3.2.1. The Carathéodory function $g : I \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be Perron function if for every $r > 0$ there exists $m_r \in L^{\frac{1}{q_2}}(I, \mathbb{R}^+)$, $q_2 \in [0, q), q \in (0, 1)$, such that $|g(t, w)| \leq m_r(t)$ for all $w \in r\mathbb{B}$, $g(t, \cdot)$ is monotone nondecreasing, $g(t, 0) = 0$ for a.a. $t \in I$ and $u(t) \equiv 0$ is the unique solution of $u(t) \leq \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t - s)^{q-1} g(s, u(s))ds$, with $u(0) = 0$.

Definition 3.2.2. The multifunction $G : I \times \mathbb{E} \rightrightarrows \mathbb{E}$ is said to satisfy compactness type condition (CTC) if there exists a Perron function $w(\cdot, \cdot)$ such that $\beta\left(G(t, A)\right) \leq \frac{1}{2} w(t, \beta(A))$ for any bounded set $A \subset \mathbb{E}$.

3.3 Fuzzy fractional integral inclusions

Let $\mathbb{E}$ be the space of fuzzy sets and $I = [t_0, T]$. Let $F : I \times \mathbb{E} \rightrightarrows \mathbb{E}$ be a given multifunction. Consider the following fuzzy fractional integral inclusion

$$x(t) \in x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} F(s, x(s)) ds, \ t \in I, \tag{3.3.1}$$

where $0 < q < 1$. As usual, $x(\cdot)$ is said to b e a solution of (3.3.1) if there exists a strongly measurable selection $f(s) \in F(s, x(s))$ such that

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s) ds. \tag{3.3.2}$$

Our main target is to investigate the properties of the solution set of (3.3.1) under compactness type conditions. We prove a variant of Kneser's theorem with the aid of the locally Lipschitz approximations. This technique is commonly used to derive some topological properties of the solution set.

It is suggested in [10] that the solutions of the following fuzzy fractional differential inclusion

$$D_c^q x(t) \in F(t, x(t)), \ t \in I = [t_0, T], \ x(t_0) = x_0, \tag{3.3.3}$$

where D_c^q is the Caputo derivative of order q, can be defined as the solutions of (3.3.1). Hence, our results are applicable also to (3.3.3).

In this section we will study the properties of the solution set of the fuzzy integral inclusion (3.3.1). To this aim, we need the following result regarding the fuzzy fractional integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds, \ t \in (t_0, T]. \tag{3.3.4}$$

Theorem 3.3.1. *Assume that there exists a constant $\eta > 0$ such that $D(f(t,x), \hat{0}) \leq \eta(1 + D(x, \hat{0}))$, that $f(\cdot, x)$ is strongly measurable and $f(t, \cdot)$ is locally Lipschitz, i.e. for every $x \in \mathbb{E}$ there exists a neighborhood $U_x \ni x$ and a constant L_x such that $D(f(t,y), f(t,z)) \leq L_x D(y,z)$ for every $y, z \in U_x$. Then the integral equation*

(3.3.4) admits an unique solution on the interval I, which depends continuously on x_0.

Proof. Suppose first that the needed unique solution $x(\cdot)$ exists on some subinterval $[t_0, \tau] \subset I$. By hypothesis, there exists $\delta > 0$ such that $f(t, \cdot)$ is Lipschitz with a constant $L > 0$ on $x(\tau) + \delta\mathbb{B}$. Suppose that $\tau < T$ and let $y, z \in C(I, \mathbb{E})$ with $y(t) = z(t) = x(t)$ on $[t_0, \tau]$. Since y and z are continuous in τ, there exists $\mu > 0$ such that $y(t)$ and $z(t)$ belong to $x(\tau) + \delta\mathbb{B}$ for every $t \in [\tau, \tau + \mu]$. For $w \in C(I, \mathbb{E})$ we define the operator

$$\mathfrak{L}(w(t)) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, w(s)) ds.$$

Applying $\mathfrak{L}$ to the functions $y(\cdot)$ and $z(\cdot)$, for $t \in [\tau, \tau + \mu]$ we have

$$\begin{aligned} D(\mathfrak{L}(y(t)), \mathfrak{L}(z(t))) &= \frac{1}{\Gamma(q)} D\left(\int_{t_0}^{t} (t-s)^{q-1} f(s, y(s)) ds, \int_{t_0}^{t} (t-s)^{q-1} f(s, z(s)) ds. \right) \\ &\leq \frac{1}{\Gamma(q)} \int_{\tau}^{t} (t-s)^{q-1} D(f(s, y(s)), f(s, z(s))) ds \\ &\leq \frac{L}{\Gamma(q)} \int_{\tau}^{t} (t-s)^{q-1} D(y(s), z(s)) ds \leq \frac{L\mu^q}{\Gamma(q+1)} \max_{t \in [\tau, \tau+\mu]} D(y(s), z(s)). \end{aligned}$$

If μ is sufficiently small, then $\mathfrak{L}$ is a contraction and hence the unique solution $x(\cdot)$ of (3.3.4) can be extended to $[t_0, \tau + \mu]$. A standard a pplication of Zorn's lemma proves the existence and uniqueness on the whole interval I. Furthermore, $D(y(x_0, t), y(y_0, t)) \leq r(t)$, where $D_c^q r(t) = Lr(t)$ and $r(t_0) = D(x_0, y_0)$. □

Proposition 3.3.2. *Assume that there exists a positive constant $\lambda > 0$ such that*

$$\max_{v \in F(t,x)} D(v, \hat{0}) \leq \lambda(1 + D(x, \hat{0})). \tag{3.3.5}$$

Then the solution set of

$$D_c^q x(t) \in \overline{co}\ F(t, x(t) + \mathbb{B}),\ \ x(t_0) = x_0 \tag{3.3.6}$$

is Hölderian of degree q (if nonempty).

Proof. Suppose that there exists a solution $x(\cdot)$ of (3.3.6). Then,

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s) ds,$$

where the function g is a strongly measurable selection $g(s) \in \overline{co}\ F(s, x(s) + \mathbb{B})$. First, we will prove that $x(\cdot)$ is bounded. For this purpose observe that it follows from (3.3.5) that

$$\max_{v \in \overline{co}\ F(t, x+\mathbb{B})} D(v, \hat{0}) \leq \lambda(1 + D(x + \mathbb{B}, \hat{0})) \leq \lambda(2 + D(x, \hat{0}))$$

and we obtain that

$$D(x(t), \widehat{0}) \leq D(x_0, \widehat{0}) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \lambda(2 + D(x\,(s)\,, \widehat{0})) ds.$$

Due to Lemma 7.1.1 in [50], and taking into account (3.3.5), we conclude that there exists a constant $M_1 > 0$ such that $D(x(t), \widehat{0}) \leq M_1$. Using again (3.3.5), we derive that

$$\max_{v \in F(t, x(t)+\mathbb{B})} D(v, \hat{0}) \leq M,$$

where $M = \lambda(2 + M_1)$.

Let $t_0 \leq \tau < t \leq T$. Then one has that

$$\begin{aligned}
D(y(t), y(\tau)) &= \frac{1}{\Gamma(q)} D\left(\int_{\tau}^{t} (t-s)^{q-1} g_y(s) ds, \int_{t_0}^{\tau} \left((\tau - s)^{q-1} - (t-s)^{q-1} \right) g_y(s) ds \right) \\
&\leq \frac{M}{\Gamma(q)} \left\{ \left| \int_{\tau}^{t} (t-s)^{q-1} ds \right| + \left| \int_{t_0}^{\tau} \left((t-s)^{q-1} - (\tau - s)^{q-1} \right) ds \right| \right\} \\
&= \frac{M}{\Gamma(q+1)} \left(|(t-s)^q|_{\tau}^{t}| + |\left((t-s)^q - (\tau - s)^q \right)|_{t_0}^{\tau}| \right) \\
&= \frac{M}{\Gamma(q+1)} \left(2(t-\tau)^q - (t-t_0)^q + (\tau - t_0)^q \right) \leq \frac{2M}{\Gamma(q+1)} (t-\tau)^q.
\end{aligned}$$

Therefore the solution set of (3.3.6) is Hölderian of degree q with a constant less than $\dfrac{2N}{\Gamma(q+1)}$. ∎

We will use the following **standing hypotheses**:

H1. $F(\cdot,\cdot)$ is almost USC with nonempty convex compact values and there exists $\lambda > 0$ such that (3.3.5) holds.

H2. $F(\cdot,\cdot)$ satisfies (CTC).

The following topological definitions are used in Theorem 3.3.3.

Definition 3.3.1. a) The set $Y \subset \mathbb{E}$ is said to be connected if for any two open disjoint sets $\mathbb{O}_1,\ \mathbb{O}_2$ such that $Y \subset \mathbb{O}_1 \bigcup \mathbb{O}_2$ either $Y \bigcap \mathbb{O}_1$ or $Y \bigcap \mathbb{O}_2$ is empty.

b) The set $Y \subset \mathbb{E}$ is said to be contractible (contractible to a point) if there exist a point $a \in Y$ and a continuous function $H : [0,1] \times Y \to Y$ such that $H(0,x) = x$, and $H(1,x) = a$ for all $x \in Y$.

Now we give the main result of the present chapter.

Theorem 3.3.3. *Under the standing hypotheses, the solution set* $\mathbb{S}$ *of* (3.3.1) *is nonempty, compact and connected.*

Proof. **Step 1.** We prove that the solution set $\mathbb{S}$ is nonempty and compact.

To this aim, we shall use the locally Lipschitz approximations of F (see section 2.4 of [37]), defined by

$$F_k(t,x) = \sum_{\lambda \in \Lambda} \varphi_\lambda(x) C_\lambda^k(t) \text{ with } C_\lambda^k(t) = \overline{co}\ F(t, x_\lambda + 2r_k\mathbb{B}).$$

Recall that $(\varphi_\lambda)_{\lambda\in\Lambda}$ is a locally Lipschitz partition of unity subordinate to some locally finite refinement $(U_\lambda)_{\lambda\in\Lambda}$ of $\{x + r_k\mathbb{B};\ x \in \mathbb{E}\}$ with $r_k = 3^{-k}$ and $x_\lambda \in U_\lambda \subset x_\lambda + r_k\mathbb{B}$. We then have

$$F(t,x) \subset F_{k+1}(t,x) \subset F_k(t,x) \subset \overline{co}\ F(t, x + 3r_k\mathbb{B}) \tag{3.3.7}$$

on $I \times \mathbb{E}$. Since $F(\cdot,\cdot)$ is almost USC, one has that for every fixed $\bar{x} \in \mathbb{E}$ there exists a strongly measurable selection $f_{\bar{x}}(\cdot)$ of $F(\cdot,\bar{x})$ (see [37] page 29). Thus, there exists a strongly measurable selection $\varsigma_\lambda(\cdot)$ of $F(\cdot,x_\lambda)$, hence $\varsigma_\lambda(t) \in C_\lambda^k(t)$ for a.a. $t \in I$. Define $f^k : I \times \mathbb{E} \to \mathbb{E}$ by

$$f^k(t,x) = \sum_{\lambda\in\Lambda} \varphi_\lambda(x)\varsigma_\lambda(t) \in F_k(t,x). \tag{3.3.8}$$

Since $(U_\lambda)_{\lambda\in\Lambda}$ is a locally finite refinement, one has that $f^k(\cdot,x)$ is strongly measurable and $f^k(t,\cdot)$ is locally Lipschitz.

Consider equation (3.3.4) with f replaced by f^k. Due to Theorem 3.3.1, it admits an unique solution $x^k(\cdot)$ given by

$$x^k(t) = x_0 + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} f^k(s,x^k(s))ds. \tag{3.3.9}$$

Denote $M(t) = \bigcup_{m=1}^{\infty}\{x^m(t)\}$. Clearly, $\beta(M(t)) = \beta\left(\bigcup_{m=k}^{\infty}\{x^m(t)\}\right)$ for any $k \geq 1$. Using the properties of β, Theorem 3.2.4 and the hypothesis (CTC), we have that

$$\beta(M(t)) = \frac{1}{\Gamma(q)}\beta\left(\int_{t_0}^t (t-s)^{q-1}\left(\bigcup_{m=k}^{\infty}(f^m(s,x^m(s))\right)ds\right)$$

$$\leq \frac{2}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}\beta(F_k(s,M(s)))ds \leq \frac{2}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}\beta(\overline{co}\ F(s,M(s)+3r_k\mathbb{B}))ds$$

$$= \frac{2}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}\beta(F(s,M(s)+3r_k\mathbb{B}))ds \leq \frac{2}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}\frac{1}{2}w(s,\beta(M(s))+3r_k)ds.$$

Furthermore, $\lim_{k\to\infty} w(s,\beta(M(s))+3r_k) = w(s,\beta(M(s))$ for a.a. $s \in I$. Due to Lebesgue dominate convergence theorem,

$$\beta(M(t)) \leq \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} w(s,\beta(M(s)))ds.$$

Since $w(\cdot,\cdot)$ is a Perron function, we get that $\beta\left(M(t)\right) = 0$ and hence $M(t)$ is relatively compact. Moreover, due to Proposition 3.3.2, the set $\{x^k(\cdot)\}_k$ is equicontinuous. Then, by Arzela-Ascoli's theorem, passing to subsequences, we obtain that $\lim_{k\to\infty} x^k(t) = x(t)$ uniformly on I.

Let us remark that, by (3.3.7), for any $n \geq 1$ and any $k \geq n$ we have that

$$f^k(t,x) \in \overline{co}\ F(t, x + 3r_n\mathbb{B})$$

on $I \times \mathbb{E}$. It follows that

$$\{f^k(t, x^k(t))\}_{k\geq n} \subseteq \overline{co}\ F(t, M(t) + 3r_n\mathbb{B}) \tag{3.3.10}$$

for any natural n. Then, using (3.3.10) and (CTC), we obtain that

$$\beta(\bigcup_{k=1}^{\infty} f^k(t, x^k(t))) = \beta(\bigcup_{k=n}^{\infty} f^k(t, x^k(t))) \leq \beta(\overline{co}\ F(t, M(t) + 3r_n\mathbb{B})$$

$$\leq \frac{1}{2} w(t, \beta(M(t) + 3r_n\mathbb{B})) \leq \frac{1}{2} w(t, \beta(M(t)) + 3r_n) = \frac{1}{2} w(t, 3r_n)$$

for any natural n. Since $r_n \to 0$ and $w(\cdot,\cdot)$ is a Perron function, we get that

$$\beta(\bigcup_{k=1}^{\infty} f^k(t, x^k(t))) = 0.$$

Furthermore, $\mathbb{E}$ can be embedded as closed convex cone $j(\mathbb{E})$ in a Banach space $\mathbb{X}$. Then, using Theorem 3.2.2,

$$\beta(j(\bigcup_{k=1}^{\infty} f^k(t, x^k(t)))) \leq \beta(\bigcup_{k=1}^{\infty} f^k(t, x^k(t))) = 0,$$

so $j(\bigcup_{k=1}^{\infty} f^k(t, x^k(t)))$ is relatively compact in $\mathbb{X}$. Due to Diestel theorem (see Proposition 1.4 in [80]) we get that the sequence $\{j(f^k(\cdot, x^k(\cdot)))\}_k$ is weakly relatively compact in $L_1(I, \mathbb{X})$. Therefore, $\{j(f^k(\cdot, x^k(\cdot)))\}$ is weakly convergent (on a subsequence) in

$L_1(I, \mathbb{X})$ to $j(f(\cdot))$. Then it is standard to prove with the help of Mazur's lemma that $x(\cdot)$ is a s olution of (3.3.1), so $\mathbb{S}$ is nonempty.

Denote by $\mathbb{S}_n$ the solution set of (3.3.1) with F_n instead of F. Clearly, $\mathbb{S} \subset \bigcap_{n\geq 1} \mathbb{S}_n$. Using the same arguments as in the first part of the proof, we can show that if $x_n \in \mathbb{S}_n$ for any $n \geq 1$ then there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ that converges uniformly to some $x \in \mathbb{S}$. We get that $\mathbb{S} = \bigcap_{n\geq 1} \bar{\mathbb{S}}_n$ is $C(I, \mathbb{E})$ compact and $\lim_{n\to\infty} \beta(\mathbb{S}_n) = 0$. Moreover, $\lim_{n\to\infty} D_H(\mathbb{S}_n, \mathbb{S}) = 0$.

Step 2. We prove that the solution set $\mathbb{S}$ is connected.

First, we prove that $\mathbb{S}_k$ are contractible. To this end, we take $\tau \in [0, 1]$ and denote $a_\tau = t_0 + \tau(T - t_0)$. Let $u \in \mathbb{S}_k$, i.e.

$$u(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g_u(s) ds,$$

where $g_u(s) \in F_k(s, u(s))$ is strongly measurable.

Next we define the map $H : [0, 1] \times \mathbb{S}_k \to \mathbb{S}_k$ as follows

$$H(\tau, u)(t) = \begin{cases} u(t) & \text{on } [t_0, a_\tau], \\ \tilde{x}(t) & \text{on } [a_\tau, T], \end{cases} \tag{3.3.11}$$

where

$$\tilde{x}(t) = x_0 + \frac{1}{\Gamma(q)} \left[\int_{t_0}^{a_\tau} (t-s)^{q-1} g_u(s) ds + \int_{a_\tau}^{t} (t-s)^{q-1} f^k(s, \tilde{x}(s)) ds \right].$$

Clearly $H(0, u)(t) = x^k(t)$, where $x^k(\cdot)$ is given by (3.3.9), and $H(1, u)(t) = u(t)$.

Since $\mathbb{S}_k$ is equicontinuous, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $D(u(a_\tau), u(a_\tau + \Delta)) < \frac{\varepsilon}{35}$, and $D((\tilde{x}(a_\tau), \tilde{x}(a_\tau + \Delta)) < \frac{\varepsilon}{35}$ for every $\Delta < \delta$. Furthermore,

$$\frac{1}{\Gamma(q)} \left| \int_{t_0}^{a_\tau} [(a_\tau - s)^{q-1} - (a_\tau + \Delta - s)^{q-1}] g_u(s) ds \right|$$

$$\leq \frac{1}{\Gamma(q)} \int_{t_0}^{a_\tau} [(a_\tau - s)^{q-1} - (a_\tau + \Delta - s)^{q-1}] |g_u(s)| ds$$
$$\leq \frac{M_1}{\Gamma(q+1)} [(a_\tau + \Delta - t_0)^q - (a_\tau - t_0)^q].$$

Consequently, $D(H(\tau, u)(t), H(\tau + \Delta, u)(t)) \leq r(t)$, where

$$r(t) = \frac{2\varepsilon}{35} + \frac{M_1}{\Gamma(q+1)} [(a_\tau + \Delta - t_0)^q - (a_\tau - t_0)^q] + \frac{L}{\Gamma(q)} \int_{a_\tau + \Delta}^{t} (t-s)^{q-1} r(s) ds,$$

because of $D(f^k(t, x), f^k(t, y)) \leq LD(x, y)$. From Lemma 6.11 of [41] we know that

$$r(t) < \left(\frac{2\varepsilon}{35} + \frac{M_1}{\Gamma(q+1)} [(a_\tau + \Delta - t_0)^q - (a_\tau - t_0)^q] \right) E_q(L(t - a_\tau - \Delta)),$$

where $E_q(\cdot)$ is the Mittag–Leffler function. The last fact together with Theorem 3.3.1 implies that $H(\cdot, \cdot)$ is continuous and hence $\mathbb{S}_k$ are contractible. The latter implies that $\mathbb{S}_k$ are connected.

Suppose that $\mathbb{S}$ is not connected. Then there exist two disjoint open sets A, B such that $\mathfrak{A} = \mathbb{S} \bigcap A \neq \emptyset$, $\mathfrak{B} = \mathbb{S} \bigcap B \neq \emptyset$ and $\mathbb{S} = \mathfrak{A} \bigcup \mathfrak{B}$. We consider first the case when $d(\mathfrak{A}, \mathfrak{B}) := \min_{a \in \mathfrak{A}, b \in \mathfrak{B}} |a - b| = 0$. Consequently, there exists $\hat{x} \in \mathbb{S}$ which is point of density of $\mathfrak{A}$ and $\mathfrak{B}$ simultaneously, which is a contradiction. If $d(\mathfrak{A}, \mathfrak{B}) > 0$, then using the fact that $\lim_{n \to \infty} D_H(\mathbb{S}_n, \mathbb{S}) = 0$, we obtain a contradiction with the connectedness of $\mathbb{S}_n$. Consequently, the set $\mathbb{S}$ is connected. □

Definition 3.3.2. A set $A \subset \mathbb{E}$ is said to be compact R_δ if there exists a decreasing sequence of compact contractible sets A_k such that $A = \bigcap_{k=1}^{\infty} A_k$ (see e.g. [50]).

The lack of the semigroup property of the solutions of (3.3.1) and the fact that we have to know the Caputo derivative from t_0 to a_τ in (3.3.11) doesn't permit us to prove that the solution set of (3.3.1) is R_δ, in contrast with the ordinary fuzzy differential equations (see e.g. [46]).

The obstacle to show that the solution set of (3.3.1) is compact R_δ is the fact that the solution set of

$$D_c^q x(t) \in F_k(t,x),\ x(t_0) = x_0 \tag{3.3.12}$$

is not closed and not precompact in general. Moreover we can not prove that its closure is contractible. To overcome that difficulty one can assume that $F(t,\cdot)$ maps bounded sets into relatively compact sets, and, clearly, in this case, the solution set of (3.3.12) is compact. The following theorem is then valid:

Theorem 3.3.4. *Let F map bounded sets into relatively compact sets. Under the standing hypotheses, the solution set $\mathbb{S}$ of* (3.3.1) *is nonempty compact R_δ.*

Proof. Clearly in this case the solution set of (3.3.12) is compact. Indeed f $\{x_n(\cdot)\}_n$ is a sequence of solutions of (3.3.12), then $\beta\left(\bigcup_{k=1}^{\infty}\right) = 0$. Due to Arzela Ascoli theorem passing to subsequences if necessary $x_n(\cdot) \to x(\cdot)$ uniformly on I. Using Diestel criterion $\{\dot{x}_n(\cdot)\}_n$ is weakly L_1 precompact. Then we can show as it the proof of Theorem 3.3.3 that $x(\cdot)$ is a solution of (3.3.12) and hence the solution set of (3.3.12) is nonempty compact and contractible. Therefore the solution set of (3.3.1) is compact R_δ. □

3.4 Fuzzy fractional differential inclusions

In this section we apply the previous result to the fuzzy fractional differential inclusion (3.3.3).

Definition 3.4.1. Let $u, v \in \mathbb{E}$. If there exists $w \in \mathbb{E}$ such that $u = v \oplus w$ then w is denoted by $u \ominus v$ and it is called the H-difference between u and v.

Definition 3.4.2. Let $f : I \to \mathbb{E}$ and $t_0 \in I$. We say that f is differentiable (H-differentiable) at τ if there exists an element $f'(\tau) \in \mathbb{E}$ such that for all $h > 0$ sufficiently small, there are $f(\tau + h) \ominus f(\tau)$, $f(\tau) \ominus f(\tau - h)$, and

$$\lim_{h \to 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0). \tag{3.4.1}$$

Definition 3.4.3. Let $u : [a, b] \to \mathbb{E}$ be Bochner integrable. The Riemann-Liouville fractional integral $I^q u(\cdot)$ of order $q > 0$ is defined by

$$I^q u(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} u(s) ds, \ a < t < b.$$

Definition 3.4.4. The Riemann-Liouville fractional derivative $D^q u$ of order $0 < q < 1$ of u is defined by

$$D^q u(t) = \frac{d}{dt} I^{1-q} u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_a^t (t-s)^{-q} u(s) ds, \ a < t < b,$$

provided that the expression on right-hand side is defined.

Definition 3.4.5. The Caputo fractional derivative $D_c^q u$ of order $0 < q < 1$ of u is defined by

$$D_c^q u(t) = \frac{1}{\Gamma(1-q)} \int_a^t (t-s)^{-q} \dot{u}(s) ds, \ a < t < b,$$

provided that the expression on right-hand side is defined. For example, $u(\cdot)$ should be absolutely continuous with Bochner integrable derivative.

Theorem 3.3.3 can be reformulated as follows.

Theorem 3.4.1. *Under the standing hypotheses, the solution set $\mathbb{S}$ of (3.3.3) is nonempty compact and connected.*

The given definition of (Hukuhara) derivative has some bad properties. To avoid these bad properties, B. Bede and his coauthors defined generalized derivative (see [18, 19]).

Definition 3.4.6. We say that f is Bede differentiable (B-differentialble) at τ if there is an element $f'_B(\tau) \in \mathbb{E}$, such that for all $h < 0$, sufficiently nearby 0, there are $f(\tau+h) \ominus f(\tau)$, $f(\tau) \ominus f(\tau - h)$, and limits

$$\lim_{h\to 0^-} \frac{f(\tau+h)\ominus f(\tau)}{h} \text{ and } \lim_{h\to 0^-} \frac{f(\tau)\ominus f(-h)}{h},$$

which are equal to $f'_B(\tau)$.

Recall that the embedding map $j : \mathbb{B} \to \mathbb{X}$ is isometry and isomorphism. Clearly $j(f'_H(t)) = \frac{d}{dt} j(f(t)) \neq j(f'_B(t))$, when $f'_H(t) \neq f'_B(t)$.

Definition 3.4.7. The function f is said to have generalized derivative at τ if it is H-differentiable or B-differentiable at τ.

Notice that if $f'_H(\tau) = f'_B(\tau)$ then $f(\tau)$ is crisp.

The generalized derivative has many applications in the case of one dimensional fuzzy numbers. However, in the case of multidimensional fuzzy numbers (fuzzy vectors), Bede's derivative poses some problems.

Consider now the fuzzy fractional differential inclusion (with respect to Bede derivative)

$$D_c^{q_B} x(t) \in F(t, x(t)),\ x(t_0) = x_0,\ t \in I = [t_0, T], \tag{3.4.2}$$

where $D_c^{q_B} x(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} x'_B(s) ds$. The function $y : I \to \mathbb{E}$ will be a solution of (3.4.2) if there exists a strongly measurable selection $f(t) \in F(t, x(t))$, such that

$$y(t) = y_0 \ominus \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s) ds.$$

The solution will exist only if the difference $\ominus$ exists.

Clearly, if $f(\cdot,\cdot)$ is a Carathéodory single valued fuzzy function, Lipschitz on the state variable and there exists a solution with respect to B-derivative, then the solution of the fractional differential equation

$$D_c^q x(t) = f(t, x(t)),\ x(t_0) = x_0,\ t \in [t_0, T] \tag{3.4.3}$$

does not have unique solution. In this case we are not able to prove that the solution set of (3.3.1) is connected.

3.5 Conclusions

In this chapter we study the main properties of the solution set of a class of fuzzy fractional differential inclusions with Caputo derivative. We show that the solution set is nonempty, compact and connected under compactness type assumptions. That is we extend the classical Kneser's theorem to the case of fuzzy fractional systems. Notice that all these results can be proved if Hausdorff measure of noncompactness β is replaced by the Kuratowski one ρ.

We prove that if F maps bounded sets into relative compact then the solution set of (3.3.1) is a compact R_δ. The last result can be used to prove existence of solutions to fuzzy differential inclusions with nonlocal condition. Most of the results of this chapter are published in [6].

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