# Total and Entire Labeling of Graphs

Since 1864

Name: Maria Naseem

Year of Admission: 2013

Registration No.: 60–GCU–PHD–SMS–13

Abdus Salam School of Mathematical Sciences

GC University, Lahore, Pakistan

# Total and Entire Labeling of Graphs

## Submitted to

Abdus Salam School of Mathematical Sciences

GC University Lahore, Pakistan

in the partial fulfillment of the requirements for the award of degree of

## Doctor of Philosophy

**in**

## Mathematics

**By**

| | |
|---|---|
| **Name:** | **Maria Naseem** |
| **Year of Admission:** | **2013** |
| **Registration No.:** | **60–GCU–PHD–SMS–13** |

**Abdus Salam School of Mathematical Sciences**

**GC University, Lahore, Pakistan**

# DECLARATION

I, **Ms Maria Naseem** Registration No. **60–GCU–PHD–SMS–13** student at **Abdus Salam School of Mathematical Sciences GC University** in the subject of **Mathematics**, hereby declare that the matter printed in this thesis titled ***Total and Entire Labeling of Graphs***

is my own work and that

(i) I am not registered for the similar degree elsewhere contemporaneously.

(ii) No direct major work had already been done by me or anybody else on this topic; I worked on, for the Ph.D. degree.

(iii) The work, I am submitting for the Ph.D. degree has not already been submitted elsewhere and shall not in future be submitted by me for obtaining similar degree from any other institution.

Dated: ______________________ ______________________

Signature of Deponent

# RESEARCH COMPLETION CERTIFICATE

Certified that the research work contained in this thesis titled

"***Total and Entire Labeling of Graphs***"

has been carried out and completed by Ms **Maria Naseem**

Registration No. **60–GCU–PHD–SMS–13** under my supervision.

______________________ ______________________

Date Supervisor

Prof. Dr. Martin Bača

Submitted Through

**Prof. Dr. Shahid S. Siddique** ______________________

Director General

Abdus Salam School of Mathematical Sciences

GC University, Lahore, Pakistan

Controller of Examination

GC University, Lahore

Pakistan

**Acknowledgement**

In the name of ALLAH, the most Gracious and the entire source of knowledge and wisdom and all due respect for Holy Prophet (Peace be upon Him) who is forever a light of guidance for humanity.

I would like to thank **Prof. Dr. Alla Ditta Raza Choudary**, Founding Director General of Abdus Salam School of Mathematical Sciences (ASSMS), for giving me the opportunity to work in a professional research environment and under the supervision of world known professors. I am also grateful to Director General of ASSMS, **Prof. Dr. Shahid S. Siddiqui** for his efforts and support.

My gratitude also goes to my respected supervisor **Prof. Dr. Martin Bača**, for the continuous support of my PhD research, for his patience, motivation, enthusiasm and immense knowledge. His able guidance, valuable suggestions and continuous encouragement helped me a lot in order to complete this work. I am also thankful to **Dr. Andrea S. Feňovčíková**, **Dr. Ali Ahmad** for their help and valuable suggestions.

I would like to extend my gratitude to the entire ASSMS faculty, especially to those professors whom I have learnt from, whose teaching encouragement is an invaluable remembering asset for me. I would like to thanks administrative staff for their help and support. I am also thankful to my friends helping and sustaining me all the time.

Last but not the least, I would also like to thanks my parents for their kindness, affections, prayers and supporting me spiritually throughout my life...

**Dedicated to my beloved**

**Parents**

**Abstract**

A plane graph is a particular drawing of a planar graph on the Euclidean plane. Let $G(V, E, F)$ be a plane graph with vertex set $V$, edge set $E$ and face set $F$. A *proper entire $t$-colouring* of a plane graph is a mapping

$$\alpha : \; V(G) \cup E(G) \cup F(G) \longrightarrow \{1, 2, \dots, \; t\}$$

such that any two adjacent or incident elements in the set $V(G) \cup E(G) \cup F(G)$ receive distinct colours. The *entire chromatic number*, denoted by $\chi_{vef}(G)$, of a plane graph $G$ is the smallest integer $t$ such that $G$ has a proper entire $t$-colouring.
The proper entire $t$-colouring of a plane graph have been studied extensively in the literature.

There are several modification on entire $t$-colouring. We focus on a *face irregular entire $k$-labeling* of a 2-connected plane graph as a labeling of vertices, edges and faces of $G$ with labels from the set $\{1, 2, \dots, k\}$ in such a way that for any two different faces their weights are distinct. The *weight* of a face under a $k$-labeling is the sum of labels carried by that face and all the edges and vertices incident with the face. The minimum $k$ for which a plane graph $G$ has a face irregular entire $k$-labeling is called the *entire face irregularity strength* .

Another variation to entire $t$-colouring is a *$d$-antimagic* labeling as entire labeling of a plane graph with the property that for every positive integer $s$, the weights of $s$-sided faces form an arithmetic sequence with a common difference $d$.

In the thesis, we estimate the bounds of the entire face irregularity strength for disjoint union of multiple copies of a plane graph and prove the sharpness of the lower bound in two cases. Also we study the existence of $d$-antimagic labelings for

the Klein-bottle fullerene that is for a finite trivalent graph embedded on the Klein-bottle with each face is a hexagon. In last chapter we investigate the 3-total edge product cordial labeling of hexagonal grid (honeycomb) that is the planar graph with $m$ rows and $n$ columns of hexagons .

**List of publications arising from this thesis**

[1] M. Bača, **M. Naseem** and A. Shabbir, *Face labelings of Klein-bottle fullerenes*, **Acta Math. Appl. Sinica**, to appear.

[2] M. Bača, M. Lascsáková, **M. Naseem** and A. Semaničová-Feňovčíková, *On entire face irregularity strength of disjoint union of plane graphs*, **Applied Mathematics and Computation**, DOI: 10.1016/j.amc.2017.02.051.

[3] A. Ahmad, M. Bača, **M. Naseem** and A. Semaničová-Feňovčíková, *On 3-total edge product cordial labeling of honeycomb*, **AKCE Internat. J. Graphs Combin.**, DOI: 10.1016/j.akcej.2017.02.002. http://www.sciencedirect.com/science/article/pii/S097286001630055X

**Further publications produced during my PhD candidature**

[1] A. Ahmad, M. Bača, **M. Naseem**, *On entire face irregularity strength of grid.* preprint.

[2] M. Bača, **M. Naseem** and A. Semaničová-Feňovčíková, *3-cordial labeling of triangular grid.* manuscript in process.

[3] A. Ahmad, R. Hasni, M. Irfan, M. Naseem, M. K. Siddiqui, *On 3-total edge product cordial labeling of grid.* **Acta Mathematica Universitatis Comenianae**, submitted.

## Introduction

Graph labeling is a mapping from graph elements i.e. edges, vertices and faces (for planar graphs), into a set of numbers. This thesis deals with two types of labelling, one is called *totallablelling* and another is called entire lableling. In many cases we consider sums of labels associated with a graph elements. we call these sums weights of the elements. In the thesis we consider an entire $k$-labeling of plane graph as mapping from set of edges, vertices and faces into the set of integers $\{1, 2, \ldots, k\}$ and the weight of a face is the sum of labels carried by that face and all edges and vertices surrounding it. Dependent on restrictions or conditions for face weights we can obtain various entire irregular labelings.

An entire $k$-labeling is defined to be a *face irregular entire $k$-labeling* of the plane graph if for every two different faces there are distinct face weights. The *entire face irregularity strength* of a plane graph is the smallest integer $k$ such that the graph admits a face irregular entire $k$-labeling. This graph parameter, entire face irregularity strength, was introduced by Bača, Jendroľ, Kathiresan and Muthugurupackiam in [34]. They estimated the lower bounds of this parameter and also obtained the precise values for some families of graphs. They obtained exact values of the face irregularity strength prove that the lower bounds are tight. In the thesis we estimate the lower and upper bounds of the entire face irregularity strength for disjoint union of multiple copies of a plane graph and determined the precise values for two classes of graphs that prove the sharpness of the lower bound .

An entire labeling (or labeling of type $(1, 1, 1)$) of a plane graph $G$ assigns labels from the set $\{1, 2, \ldots, |V(G)| + |E(G)| + |F(G)|\}$ into the vertices, edges and faces in such a way that each vertex, edge and face receives exactly one label and each number is used exactly once as a label. The *weight* of a face under an entire labeling is the sum of the labels carried by that face and the edges and vertices surrounding it. An entire labeling of a graph is called *d-antimagic* if the weights of all faces form an arithmetic sequence starting from $a$ and having common difference $d$, where $a > 0$ and $d \geq 0$ are two given integers.

The concept of the $d$-antimagic labeling of the plane graphs was introduced in [25], where labelings of type $(1, 1, 1)$ were described for prism $D_n$. The $d$-antimagic labelings of type $(1, 1, 1)$ for the generalized Petersen graph $P(n, 2)$, the hexagonal planar maps and the grids can be found in [26, 27, 32]. In particular for $d = 0$, Lih in [84] calls such labeling *magic* and describes magic (0-antimagic) labeling of type $(1, 1, 0)$ for wheels, friendship graphs and prisms. The magic (0-antimagic) labelings for grid graphs and honeycombs are given in [33] and [16], respectively. A $d$-antimagic labeling is called *super* if the smallest possible labels appear on the vertices.

In this thesis we have studied the existence of super $d$-antimagic labelings of Klein-bottle polyhex. We showed that for $n$ even, $n \geq 2$, $m \geq 1$, the non-bipartite Klein-bottle polyhex admits a super $d$-antimagic entire labeling for differences $d = 1, 3$.

The last part of the thesis deals with investigation the existence of an 3-total edge product cordial labeling of hexagonal grid. We prove that the hexagonal grid $H_n^m$ admits a 3-total edge product cordial labeling for every $n \geq 1$, $m \geq 1$, $(n, m) \neq (1, 1)$.

# Outline of the thesis

In Chapter 1, we provide the basic terminology of graph theory, recall some definitions and notations of certain families of graphs that will be used throughout the thesis and we present some preliminary results of the graph theory.

In Chapter 2, we give an overview of certain types of labelings, namely magic types of labelings, antimagic types of labelings and lastly irregular labelings and cordial labelings. Moreover we present certain known results regarding these labelings.

In Chapter 3, we estimate lower and upper bounds of the entire face irregularity strength for the disjoint union of multiple copies of a plane graph and determined the precise values for two classes of graphs. These two cases prove the sharpness of the lower bound of the entire face irregularity strength for disjoint union of multiple copies of a plane graph.

In Chapter 4, we study the existence of super $d$-antimagic labelings of Klein-bottle polyhex. We show that for $n$ even, $n \geq 2$, $m \geq 1$, the non-bipartite Klein-bottle polyhex $\mathbb{KB}^n_{m+\frac{1}{2}}$ admits a super $d$-antimagic labeling of type $(1, 1, 1)$ with $d = 1, 3$.

In Chapter 5, we describe a construction to obtain 3-total edge product cordial labeling of the hexagonal grid $H^m_n$ from a smaller hexagonal grid.

At the end of the thesis, in Appendix A, we provide a list of graph-theoretic symbols used in the thesis .

# Contents

# List of Figures

# Chapter 1

# Basic Terminology

In this chapter we will define the basic terminology and some preliminary results of the graph theory. In this thesis, a graph means a finite undirected simple graphs without multiple edges or loops, unless otherwise stated.

A *graph* $G = (V(G), E(G))$ consists of two sets. The vertex set $V(G)$ is a nonempty set of elements called *vertices*. The edge set $E(G)$ is a set of elements called *edges*. Each edge $e \in E$ is an unordered pair of vertices $\{u, v\}$ called the end vertices of $e$. We simply denote $e = uv$. We also say that $e$ is incident to $u$ and $v$ .

In graph $G = (V, E)$ the vertices are often called *nodes* and the edges are called *arcs or links*. It is possible to have a vertex joined to itself by an edge and such an edge is called a *loop*. If two or more edges of $G$ have the same end vertices, we say that these are *parallel edges*. A graph is called *simple* if it has no loops and no parallel edges. The *order* of $G$ is the cardinality of its vertex set, usually denoted by $n$ and the *size* of graph $G$ is the cardinality of its edge set, denoted by $m$.

*Degree* of a vertex $x$ of any graph $G$ is defined as the number of edges incident with $x$, where the number of loops are counting twice. It is denoted by $d_G(x)$ or $d(x)$. The maximum degree of a graph $G$ is denoted by $\Delta(G)$ and the minimum degree of the graph $G$ is denoted by $\delta(G)$. If $d(x)$ is even (odd) for some $x \in V(G)$ then the vertex $x$ is called an even vertex (odd vertex ). The vertex of degree 0 is called *isolated vertex* and the vertex of degree 1 is called *leaf* or *pendent*. Graphs are finite or infinite according to their order. The graph of order 1 is called *trivial* graph. Two vertices $x$ and $y$ are called adjacent if $xy \in E(G)$. Two edges $e_1$ and $e_2$ are adjacent if they have a common vertex. Two adjacent vertices are also known as *neighbors* of each other, the set of all neighbors of the vertex $x$ is denoted by $N(x)$. There is known theorem which associate the number of edges and the degrees of the vertices of a graph $G$, known as *First Theorem of Graph Theory*.

**Theorem 1.** [50] *For any graph $G$,*

$$\sum_{x \in V(G)} d(x) = 2 \mid E(G) \mid$$

*where $d(x)$, denotes the degree of the vertex $x$.*

When we sum the degrees of all vertices, each edge gets counted twice (once in $d(x)$, once in $d(y)$). So the resulting sum is twice the number of edges.

**Corollary 1.** [50] *In every graph the number of vertices with odd degree is even.*

There are many ways to represent a graph. However, traditionally a graph is represented by a *diagram*. A *dot* represents a vertex and a *curve* (usually a line segment) represents an edge.

A graph $H$ is a *s*ubgraph of graph $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $H$ is a subgraph of $G$, we can write $H \subset G$ and we can say that $H$ is contained in $G$ or

$G$ contains $H$. A subgraph $H$ is said to be $p$roper subgraph of $G$ if either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$. For a given graph $G$, there are two natural ways to get smaller graphs from $G$ [50]. If $v \in V(G)$, the graph $G-v$, vertex-deleted subgraph, is obtained by deleting the vertex and remove all the edges incident to that vertex. If $e \in E(G)$, the graph $G-e$, edge-deleted subgraph, is obtained by deleting the edge. Vertex-deleted subgraph and edge-deleted subgraph of $G$ are the proper subgraphs of the graph $G$. A $s$panning subgraph is a subgraph that is obtained only by edge deletion. In other words, the vertex set of the subgraph is the entire vertex set of the original graph. A subgraph obtained only by vertex deletion is called $i$nduced subgraph. Specifically, if we have a graph $G$ and $X$ is a set of vertices that are deleted, the resulting graph is $G \setminus X$. In induced subgraph we are interested in the set of vertices $Y = V \setminus X$, so for induced subgraph we used the notation $G[Y]$ and called the graph induced by the vertices in $Y$.

A *path*, denoted by $P_n$, is a graph whose vertices can be arranged in a sequence so that two vertices are adjacent if they are consecutive in the sequence and not adjacent otherwise. A graph is said to be *connected* if there is a path between every pair of vertices. A graph which is not connected is called *disconnected* graph, which consists of two or more connected components .

If we join all the vertices of path $P_n$ to a further vertex then the obtained graph is called *fan graph*. It is denoted by $F_n$. The *cycle* $C_n$, $n \geq 3$, consists of $n$ vertices, say $x_1, x_2, \ldots, x_n$ and $n$ edges, say $x_i x_{i+1}$, $1 \leq i \leq n-1$ and $x_n x_1$.

Figure 1.1 represents the fan $F_8$ and the cycle $C_8$.

A graph is $r$-*regular* if every vertex of graph has degree $r$. In $r$- regular graphs, $\delta(G) = \Delta(G) = r$. Figure 1.2 illustrates a 4-regular graph. The graphs that are

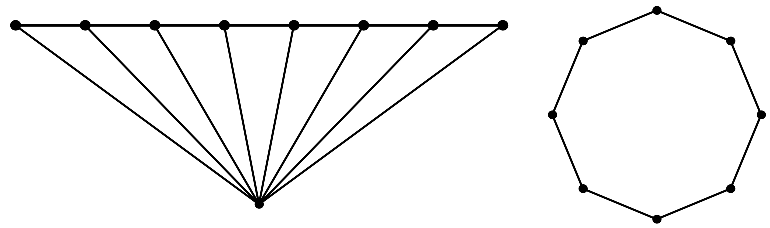

Figure 1.1: Fan graph $F_8$ and cycle graph $C_8$.

3-regular are called *cubic*.

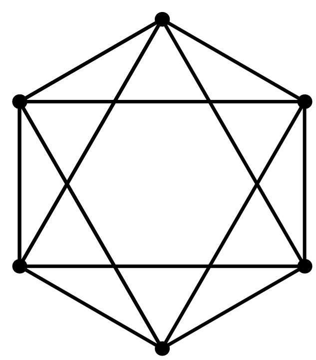

Figure 1.2: A 4-regular graph.

*Complete graph* on $n$ vertices, denoted by $K_n$, is a simple graph with an edge between every pair of vertices. So, complete graph is $(n-1)$-regular and have $n(n-1)/2$ edges. Figure 1.3 shows the complete graphs on 5 and 7 vertices.

A *sun graph*, denoted by $S_n$, is obtained by attaching a pendent edge to every vertex of an $n$-cycle. $S_8$ is shown in Figure 1.4. A graph whose vertex set can be partitioned into two sets $X$ and $Y$ so that every edge in the graph has one end vertex in $X$ and other end vertex in $Y$, is either called *bipartite* or called *bipartite graph.* A graph $G$ is called *complete bipartite* if we can partition the vertex set into two subsets i.e $V = X \cup Y$ in such a way that every vertex of the partitioned set $X$ is connected to every vertex in $Y$, it is denoted by $K_{s,t}$, where $|X| = s$ and $|Y| = t$,

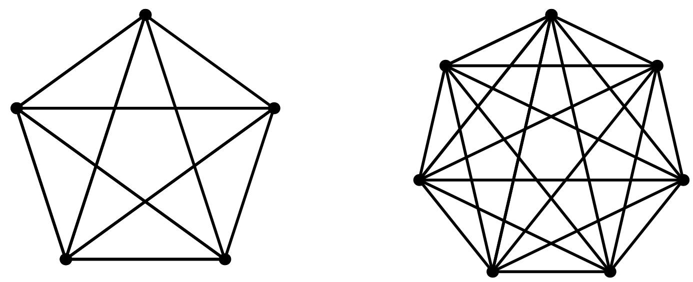

Figure 1.3: Complete graph $K_5$ and $K_7$.

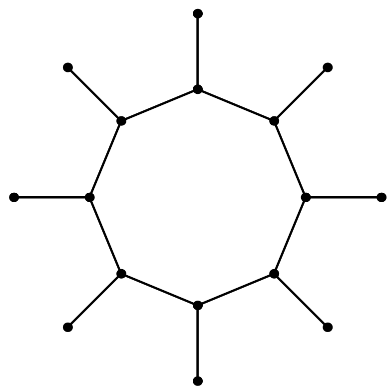

Figure 1.4: Sun graph $S_8$.

with $s + t = n$ and $|E(K_{s,t})| = st$. Figure 1.5 shows the complete bipartite graph $K_{4,4}$.

In general a graph $G$ is called $k$-partite ($k \geq 2$), if $V(G)$ can be partitioned into $k$ disjoint nonempty subsets such that each edge has its end vertices in different subsets. A $k$-partite graph in which every two vertices from different partite classes are adjacent is called *complete $k$-partite* graph.

Next theorem gives a characterization for bipartite graphs.

**Theorem 2.** [62] *A graph $G$ is bipartite if and only if $G$ contains no odd cycle.*

A graph is *acyclic* if it has no cycle. A *forest* is an acyclic graph and a connected acyclic graph is called *tree*. Star graph $K_{1,n-1}$ which consists of a center and $n - 1$

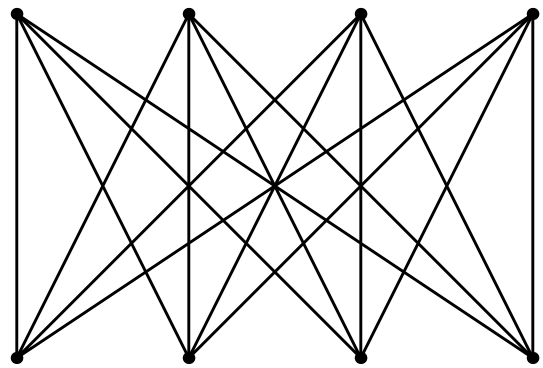

Figure 1.5: Complete bipartite graph $K_{4,4}$.

pendant vertices is an example of tree. A bistar which is also known as double star and it is denoted by $B_{r,s}$, where $r + s = n - 2$, is obtained by joining the central vertices of $K_{1,r}$ and $K_{1,s}$ by an edge.

Figure 1.6 and 1.7 shows a star graph $K_{1,12}$ and bistar $B_{5,5}$ respectively.

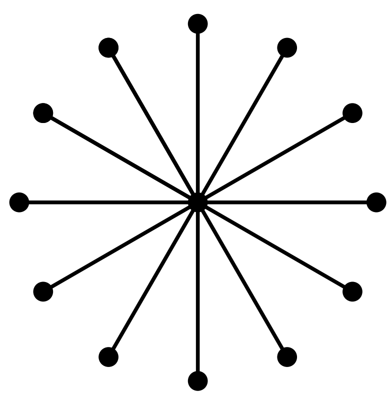

Figure 1.6: Star graph $K_{1,12}$.

A *friendship graph* $f_n$, consists of $n$ triangles, having a common central vertex. In a friendship graph $|V(f_n)| = 2n + 1$ and $|E(f_n)| = 3n$. Figure 1.8 represents the friendship graph $f_6$.

A graph is called *planar* graph if it can be drawn on the plane in such a way that

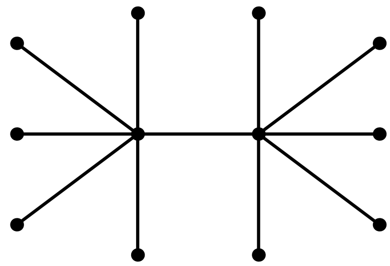

Figure 1.7: Bistar $B_{5,5}$.

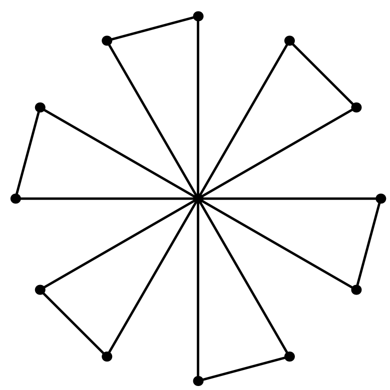

Figure 1.8: Friendship graph $f_6$.

its edges intersect only at their end points, i.e. no edge cross each other. Any such drawing is called a plane drawing of graph $G$ or a plane graph. A plane graph divides the plane into regions. The regions are called the *faces* of the graph. Every plane graph has an unbounded region called the exterior face. The boundary of the face in the plane graph is the set of vertices and edges that outline it. Every graph that can be drawn on a plane can be drawn on a sphere as well. In addition a graph can be embedded on a torus and Klein bottle as well. Figure 1.9 shows a plane graph.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The *union of two graphs* is denoted by $G \cup H$, the vertex set of $G \cup H$ is $V(G) \cup V(H)$ and the edge set of $G \cup H$ is $E(G) \cup E(H)$.

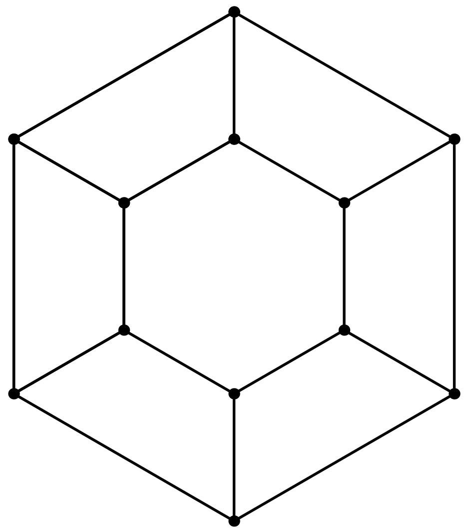

Figure 1.9: A plane graph.

The *join of two graphs* $G$ and $H$ is denoted by $G+H$, with the vertex set $V(G)\cup V(H)$ and the edge set $E(G)\cup E(H)\cup\{xy : x\in V(G), y\in V(H)\}$.

A wheel on $n+1$ vertices denoted by $W_n$, is obtained by joining a single vertex to all vertices of a cycle $C_n$, i.e. $W_n = K_1 + C_n$. Thus $W_n$ contains $n+1$ vertices and $2n$ edges. Figure 1.10 illustrates the wheel graph on 8 vertices.

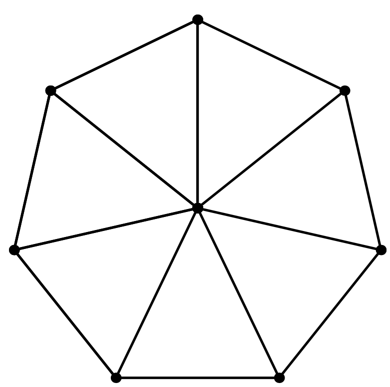

Figure 1.10: Wheel graph $W_7$.

Let $G=(V,E)$ be a graph of order $n$. The *complement graph* $\overline{G}$ is the graph

with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = E(K_n) \setminus E(G)$. The cardinality of the edge set is $E(\overline{G}) = \frac{n(n-1)}{2} - m$, where $m$ is the size of the graph $G$.
Suppose $G$ and $H$ are graphs with the vertex sets $V(G) = \{x_1,\ x_2,\ \ldots,\ x_n\}$ and $V(H) = \{y_1,\ y_2,\ \ldots,\ y_m\}$. Then the *Cartesian product of two graphs* is denoted by $G \Box H$. Its vertex set is $V(G \Box H) = V(G) \times V(H)$ and $(x_i,\ y_j)(x_k,\ y_l) \in E(G \Box H)$ if and only if $i = k$ and $y_j y_l \in E(H)$ or $j = l$ and $x_i x_k \in E(G)$.

For $n \geq 2$ and $m \geq 2$, the grid graph denoted by $G_m^n$ is defined as the Cartesian product $P_m \Box P_n$ of a path on $m$ vertices with a path on $n$ vertices embedded in the plane.

A *prism graph*, denoted by $D_n$, is defined as the Cartesian product of the path on two vertices and the cycle on $n$ vertices, i.e. $P_2 \Box C_n$. It is a 3-regular graph.

An *antiprism graph* consists of two cycles, an inner cycle on $n$ vertices $u_1 u_2 \ldots u_n$, an outer cycle on $n$ vertices $v_1 v_2 \ldots v_n$ and the set of $2n$ spokes i.e. $\{u_i v_i, u_{i+1} v_i : i = 1, 2, \ldots, n-1\} \cup \{u_n v_n, u_1 v_n\}$. It is denoted by $A_n$. Antiprism is a 4-regular graph. The prism $D_9$ is depicted in Figure 1.11 and the antiprism $A_9$ is shown in Figure 1.12.

A *generalized Petersen graph* $P(n, m)$, $n \geq 3$ and $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, consists of an outer $n$-cycle $u_0 u_1 \ldots u_{n-1}$, a set of $n$ spokes $u_i v_i$, $0 \leq i \leq n-1$ and $n$ edges $u_i u_{i+m}$, $0 \leq i \leq n-1$, with indices taken modulo $n$. Figure 1.13 illustrates the generalized Petersen graph $P(8, 3)$.

For other concept which are not provided in this chapter see [55], [113], [111] .

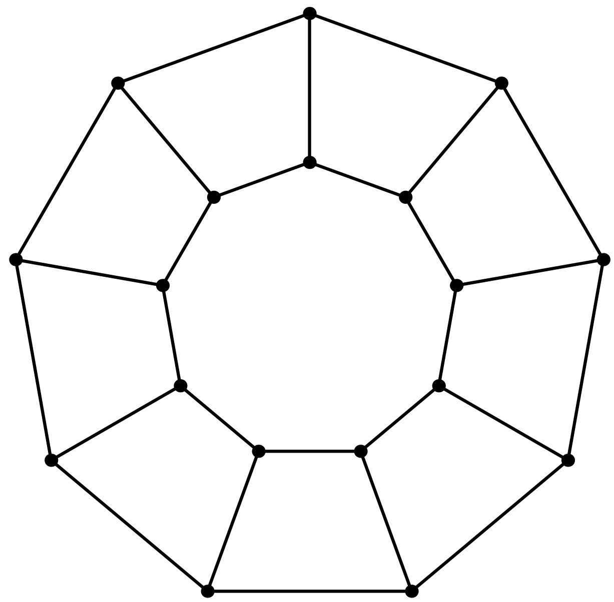

Figure 1.11: Prism $D_9$.

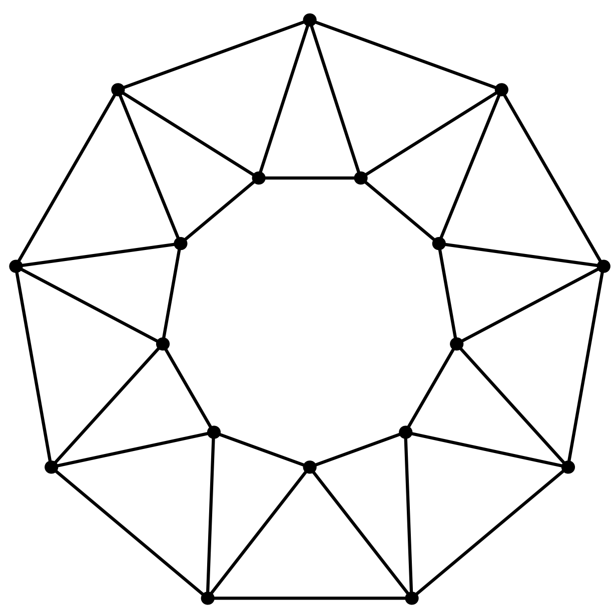

Figure 1.12: Antiprism $A_9$.

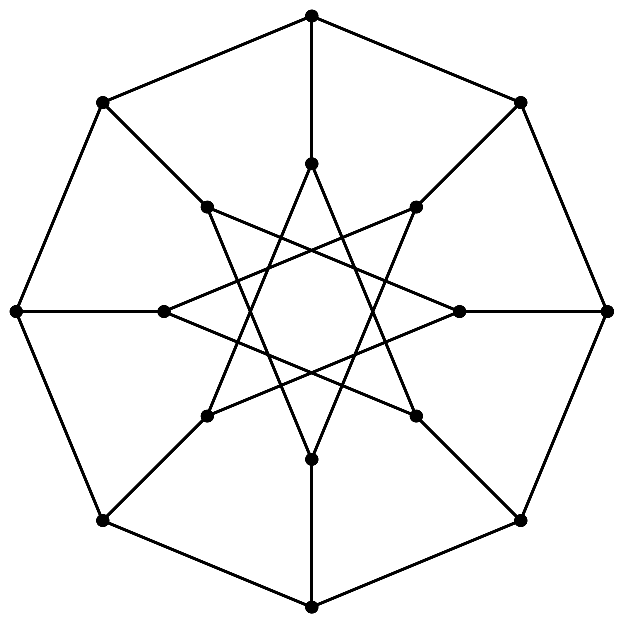

Figure 1.13: Generalized Petersen graph $P(8,3)$.

# Chapter 2

# Graph Labelings

The labeling of discrete structures is also a field which possess the same characteristic. The problems arising from the study of a variety of labeling schemes of the elements of a graph and any discrete structure is a potential area of challenge as it cuts across wide range of disciplines of human understanding. Graph labeling problems are actually not of recent origin. For instance, coloring the vertices of a graph arose in connection with the four color theorem, which remained for a long time known by the name four color conjecture, took more than 150 years for its solution in 1976. The concept of labeling of graphs has recently gained a lot of popularity in the area of graph theory. This popularity is due not only to the mathematical challenges of graph labeling but also to the wide range of application that graph labeling offer to the other branches of science, for instance, X-ray, crystallography, coding theory, cryptography, astronomy, circuit design and communication networks design [45, 44]. Graph labelings were first introduced in the late 1960s. Among the huge diversity of concepts that appear while studying this subject one that has gained a lot of popularity is the concept of labelings of graphs with more than thirteen hundred papers in the literature and a very complete dynamic survey by Gallian [60]. These labelings can be distinguished by their conditions which are mostly described using the weights of the elements of graph namely vertex weights or edge weights. Informally, by graph labeling we mean an assignment of numbers to the graph elements, such as vertices, edges, or both, with respect to some specified conditions. If the domain is the vertex set or edge set the labeling is called *vertex labeling* or *edge labeling*, respectively. Moreover, if the domain is both the vertex and edge sets then the labeling is called a *total labeling*. For plane graph it make sense to consider their faces, so we can also assign labels to the faces. Thus, if domain is the set of vertices, edges and faces, then the labeling is called *supertotal* or *entire*. There are different kinds of labeling according to some specified conditions that are satisfied.

## 2.1 Magic Graphs

The concept of magic labeling was first introduced by Sedláček [100]. He gets the motivation from the notion of magic squares in number theory. In his terminology, a graph is said to be magic if there exist a function $\phi : E(G) \to Z^+$ such that sums of the edge labels around any vertex equal to a constant (called the magic constant), independent of the choice of the vertex. Figure 2.1 shows the magic labeling of the complete bipartite graph $K_{4,4}$, where the magic constant is 34.

For the existence of magic graphs, some sufficient conditions are given in [15, 90, 100, 104]. Necessary and sufficient conditions for the existence of magic graphs were

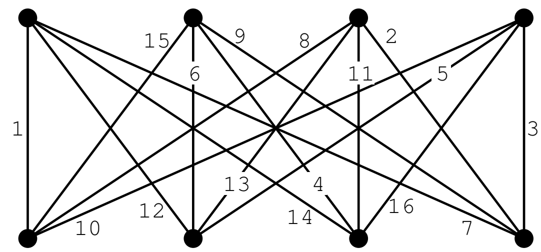


Figure 2.1: Magic labeling of $K_{4,4}$.

given by Jeurissen in [72, 73] and independently by Jezný and Trenkler in [74].

Stewart [104] proved the following: $K_n$ is magic for $n = 2$ and all $n \geq 5$, $K_{n,n}$ is magic for all $n \geq 3$, fans $F_n$ are magic if and only if $n$ is odd and $n \geq 3$, wheels $W_n$ are magic for $n \geq 4$, and $W_n$ with one spoke deleted is magic for $n = 4$ and for $n \geq 6$.

A magic labeling is called supermagic if the edge labels are consecutive positive integers. Stewart [104] introduced the notion of supermagic graphs. In Figure 2.2 the supermagic labeling of octahedron is shown.

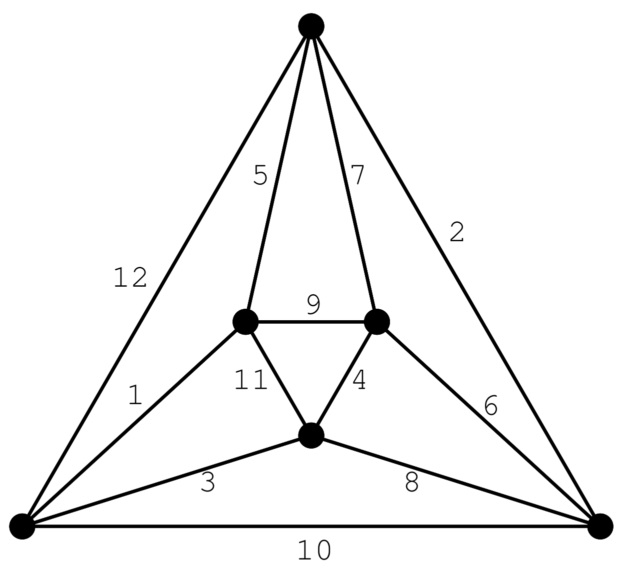


Figure 2.2: Supermagic labeling of octahedron.

There is no known characterization of all supermagic graphs. Only some special classes of supermagic graphs have been characterized. Stewart characterized the supermagic complete graphs as follows:

**Theorem 3.** [104] *A complete graph* $K_n$ *is supermagic if and only if either* $n \geq 6$ *and* $n \not\equiv 0 \pmod 4$ *or* $n = 2$.

Ivančo in [68] gives the characterization of supermagic regular complete multipartite graphs. He also proved that $Q_n$ is supermagic if and only if $n = 1$ or $n$ is even and greater than 4 and that $C_n \square C_n$ and $C_{2m} \square C_{2n}$ are supermagic.

Shiu, Lam and Cheng [103] proved that for $n \geq 2$, $mK_{n,n}$ (disjoint union of $m$ copies of complete bipartite graph) is supermagic if and only if $n$ is even, or both $m$ and $n$ are odd.

Sedláček [101] showed that Möbius ladders $M_n$ are supermagic when $n \geq 3$ and $n$ is odd and that $C_n \square P_2$ is magic, but not supermagic, when $n \geq 4$ and $n$ is even.

If the edge labels in a magic graph are prime numbers then the graph is called *prime magic*. The terminology of prime magic graphs was defined by Stewart [104]. An illustration of a prime magic graph is shown in Figure 2.3 .

In 1970, Kotzig and Rosa [80] defined a magic valuation of a graph $G(V, E)$ as a bijective mapping $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that for all edges $xy$, the weight of $xy$ i.e. $wt(xy) = f(x) + f(y) + f(xy)$ is constant (called the magic constant). This type of labeling is called *edge-magic total* (EMT) *labeling* of $G$. The edge magic total labeling of $K_{4,3}$ is shown in Figure 2.4.

This notion was rediscovered by Ringel and Lladó [98] in 1996 who called this labeling *edge-magic*.

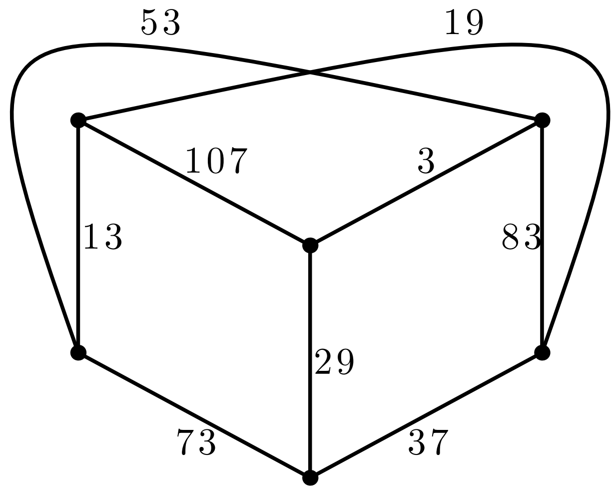


Figure 2.3: Prime magic labeling of graph.

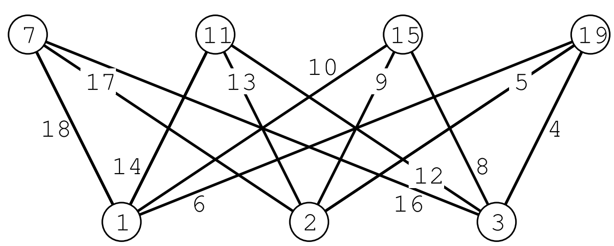


Figure 2.4: Edge magic total labeling of $K_{4,3}$.

In [80] Kotzig and Rosa proved that $K_{n,m}$ has an edge-magic total labeling for all $n$ and $m$; $C_n$ has an edge-magic total labeling for all $n \geq 3$, see also [43, 61, 99], the disjoint union of $n$ copies of $P_2$ has an edge-magic total labeling if and only if $n$ is odd.

Wallis, Baskoro, Miller and Slamin in [110] stated that $K_n$ has an edge-magic total labeling if and only if $n = 1, 2, 3, 5, 6$. Ringel and Lladó proved the following result:

**Theorem 4.** [98] *Let $G$ be a graph of order $n$ and size $m$ with the property that the degree of every vertex is odd, $m$ is even and $n + m \equiv 2 \pmod 4$. Then $G$ is not* *edge-magic total.*

Enomoto, Lladó, Nakamigawa and Ringel [58] defined a *super edge-magic total* (SEMT) *labeling* where the smallest possible labels appear on the vertices. Wallis [111] calls the super edge-magic total labeling *strongly edge-magic*. Enomoto, Lladó, Nakamigawa and Ringel [23] conjectured that every tree is SEMT. Lee and Shan [42] have verified this conjecture for trees with up to 17 vertices by computer search.

Other results on $EMT$ and $SEMT$ labelings are proved in [60, 91, 106, 111].

MacDougall, Miller, Slamin, and Wallis [85, 86] introduced the notion of a *vertex-magic total labeling* (VMT) in 1999. For a graph $G(V, E)$ an injective mapping $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ is a vertex-magic total labeling if there is a constant $k$, called the *magic constant*, such that for every vertex $v$, the weight of the vertex $v$ is $k$, that is

$$wt(v) = f(v) + \sum_{uv \in E(G)} f(vu) = k.$$

In [85] it was proved that the graphs cycle $C_n$, path $P_n$, $n > 2$, $K_{n,n}$, $n > 1$, $K_{n,n} - \{e\}$, $n > 2$, and $K_n$, $n$ odd, all have $VMT$ labelings. MacDougall, Miller and Wallis [85] also proved that when $m > n+1$ then $K_{n,m}$ does not have a vertex-magic total labeling. They conjectured that $K_{n,n+1}$ has a vertex-magic total labeling for all $n$ and that $K_n$ has vertex-magic total labeling for all $n \geq 3$.

More details and other results on $VMT$ labelings can be found in [60] and [111].

Lih [84] in 1983 introduced a magic-type method for labeling the vertices, edges and faces of a plane graph. He presented that for a planar graph $G(V, E, F)$ a labeling of type type $(\alpha, \beta, \gamma)$, where $\alpha, \beta, \gamma \in \{0, 1\}$, is a mapping $\rho : \{1, 2, \ldots, \alpha|V| + \beta|E| + \gamma|F|\} \to V \cup E \cup F$ defined in such a way that each vertex receives $\alpha$ labels, each edge receives $\beta$ labels and each face receives $\gamma$ labels and each number is used exactly once as a label. The weight of face is the sum of the labels of the face, vertices and the edges surrounding that face. A labeling of type $(\alpha, \beta, \gamma)$ is said to be face-magic, if for every positive integer $s$ all $s$-sided faces have the same weight. We allow different weights for different $s$.

Lih [84] described face-magic labelings of type $(1, 1, 0)$ for the wheels, the friendship graphs, the prisms and for certain platonic polyhedra. For $m \geq 2$, $n \geq 3$, $n \neq 4$, the cylinder $C_n \times P_m$ admits a face-magic labeling of type $(1, 1, 0)$, see [20]. The face magic labelings of type $(1, 1, 1)$ for fans, ladders, planar bipyramids, grids, hexagonal lattices, Möbius ladders and certain classes of convex polytopes are described in [21, 22, 23, 20, 24, 33, **?**, 38, 40, 39].

Kathiresan and Gokulakrishnan in [75] provided the face-magic labelings of type $(1, 1, 1)$ for the families of planar graphs with 3-sided faces, 5-sided faces, 6-sided faces and one external infinite face.

## 2.2 Antimagic Graphs

A graph $G = (V, E)$ with order $n$ and size $m$ is called antimagc if we define the labeling on edges, i.e. $f : E(G) \rightarrow \{1, 2, \ldots, m\}$ such that the weights of all pairwise distinct vertices are distinct, where the weight of a vertex $v$ is the sum of the labels of all edges incident with the vertex $v$. An antimagic labeling of a tree is shown in Figure 2.5.

The concept of antimagic labeling was first introduced by Hartsfield and Ringel in [64]. They showed that the path $P_n$, $n \geq 3$, cycles, wheels and complete graphs

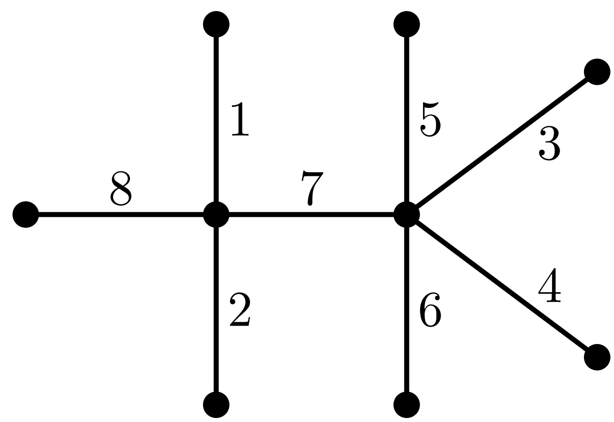


Figure 2.5: Antimagic labeling of the tree

$K_n$, $n \geq 3$, are antimagic. They gave the following conjecture:

**Conjecture 1.** [64] *Every connected graph, except the complete graph on* 2 *vertices, is antimagic.*

Alon, Kaplan, Lev, Roditty and Yuster [9] showed that this conjecture is true for all graphs having minimum degree $\Omega(log|V(G)|)$. Bodendiek and Walther [46] present some extra limitations on the vertex-weights and defined the concept of an $(a, d)$-antimagic labeling. In our terminology this labeling is called an $(a, d)$-vertex-antimagic edge labeling.

### 2.2.1 Vertex-antimagic Edge Labelings

Under an edge labeling $f : E(G) \rightarrow \{1, 2, \dots, m\}$, the weight of a vertex $x$ is the sum of labels $g(e)$ assigned to all the edges incident with the vertex $x$.

A connected $(n, m)$ graph $G$ is said to be $(a, d)$*-vertex-antimagic edge* or $(a, d) - VAE$ if there exist integers $a > 0$, $d \geq 0$ and a bijection $g : E(G) \rightarrow \{1, 2, \dots, m\}$

such that the induced mapping $f_g : V(G) \longrightarrow W$ is also bijection, where

$$W = \{\omega(x) \mid x \in V(G)\} = \{a, a+d, a+2d, \dots, a+(n-1)d\}$$

is the set of the weights of the vertices of $G$. If a graph $G$ is $(a,d)$-VAE and $g : E(G) \to \{1, 2, \dots, m\}$ is a corresponding bijective mapping of $G$, then $g$ is said to be an $(a,d)$-VAE labeling of $G$.

As an illustration, Figure 2.6 provides an example of a $(15,3)$-vertex antimagic edge labeling for the generalized Petersen graph $P(8,2)$, where the vertex labels mean vertex-weights.

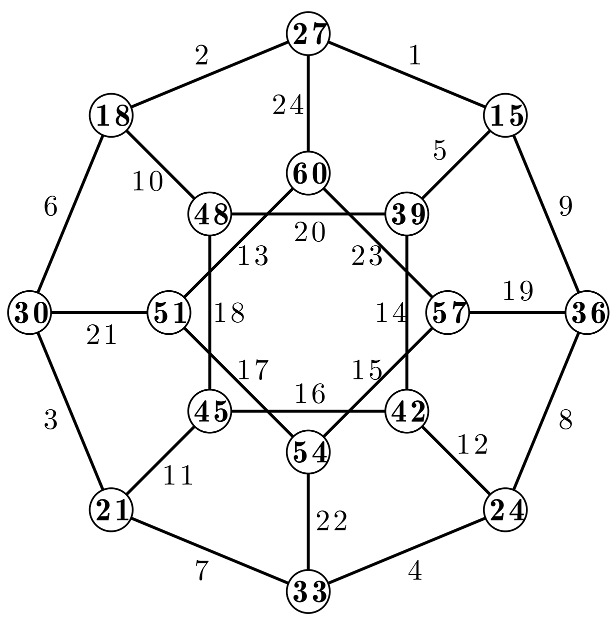


Figure 2.6: $(15,3)$- vertex antimagic edge labeling of $P(8,2)$.

$(a,d)$-VAE labelings for some classes of graphs, for instance paths, cycles, complete graphs, are mentioned in [47], [48].

In [19] there are characterized all $(a,d)$-vertex-antimagic edge graphs of prism $C_n \square P_2$, when $n$ is even and there is shown that if $n$ is odd then the prisms $C_n \square P_2$

are $(\frac{5n+5}{2}, 2)$-VAE. For generalized Petersen graph, in [14] it was proved that:

**Theorem 5.** [14] *Generalized Petersen graph $P(n, m)$ has an $(\frac{7n+4}{2}, 1)$-VAE labeling if and only if $n$ is even, $n \geq 4$ and $1 \leq m \leq \frac{n}{2} - 1$.*

In [93] Nicholas, Samasundaram and Vilfred got several results about $(a, d)$-VAE labelings for special trees (caterpillars), unicyclic graphs and complete bipartite graphs.

### 2.2.2 Vertex-antimagic Total Labelings

Bača, Bertault, MacDougall, Miller, Simanjuntak and Slamin [18] putting restriction on the weights of vertices and introduced a new concept of vertex-antimagic total labeling as a natural extension of the vertex-magic total labeling defined by MacDougall, Miller, Slamin and Wallis [85] .

For a total labeling $h : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| \cup |E(G)|\}$ the associated vertex-weight of a vertex $x \in V(G)$ is

$$wt_h(x) = h(x) + \sum_{xy \in E(G)} h(xy).$$

An $(a, d)$-*vertex-antimagic total labeling* of graph $G$ is a bijection $h$ from $V(G) \cup E(G)$ onto the integers $\{1, 2, \ldots, |V(G)| \cup |E(G)|\}$ with the condition that the set of vertex-weights form an arithmetic sequence with first term $a$ and having common difference $d$, where $a > 0$ and $d \geq 0$ are two given integers. We denote a vertex-antimagic total labeling as VAT labeling. If $d = 0$ then the $(a, d)$-VAT labeling is called a *vertex-magic total* (VMT) *labeling.* An $(a, d)$-VAT labeling $h$ is called *super* if we assign smallest possible labels to the vertices. A graph is said to be *(super)* $(a, d)$-*VAT* if it admits a (super) $(a, d)$-VAT labeling.

Constructions of the super $(a, d)$-VAT labelings for certain families of graphs, including complete graphs, complete bipartite graphs, cycles, paths and generalized Petersen graphs are shown in [105] .

In [105], it was also proved that every tree with an even number of vertices and every cycle with at least one tail and an even number of vertices, has no a super $(a, 1)$- VAT labeling. Also for star $S_n,\ n \geq 3$, there is no $(a, d)$-VAT labeling, for any $d$. In [8], there are some new results on existence of super $(a, d)$-VAT labelings for disconnected graphs, namely for a disjoint union of $m$ copies of a regular graph. Other results on $(a, d)$-VAT graphs can be found in [79] and in the general survey on graph labeling by Gallian [60].

### 2.2.3 d-antimagic Labelings of Graphs

Consider a labeling of type $(\alpha, \beta, \gamma)$ with $\alpha, \beta, \gamma \in \{0, 1\}$. A labeling of type $(\alpha, \beta, \gamma)$ of a plane graph $G$ is called *d-antimagic*, if for every positive integer $s$, the set of $s$-sided face-weights is $W_s = \{a_s, a_s + d, a_s + 2d, \ldots, a_s + (f_s - 1)d\}$ for some integers $a_s$ and $d \geq 0$, where $f_s$ is the number of $s$-sided faces. We allow different sets $W_s$ for different $s$. If $d = 0$ then the $d$-antimagic labeling is *face-magic labeling.*

The concept of $d$-antimagic labeling of plane graphs was introduced in [25], where are described $d$-antimagic labelings of type $(1, 1, 1)$ for prism $D_n$. The $d$-antimagic labelings of type $(1, 1, 1)$ for the generalized Petersen graph $P(n, 2)$, the hexagonal planar maps and the grids can be found in [26, 27, 32].

Bača, Lin and Miller in [32] proved that for $m, n > 8$, $P_m \Box P_n$ has no $d$-antimagic labeling of type $(1, 1, 1)$ with $d \geq 9$. For $m \geq 2,\ n \geq 2$, and $(m, n) \neq (2, 2)$, $P_m \Box P_n$ has $d$-antimagic labelings of type $(1, 1, 1)$ for $d = 1, 2, 3, 4, 6$. They conjecture that the same is true for $d = 5$.

A $d$-antimagic labeling is called *super* if the smallest possible labels appear on the vertices. Super $d$-antimagic labelings of type $(1, 1, 1)$ for antiprisms and for $d \in \{1, 2, 3, 4, 5, 6\}$ are described in [28] and for disjoint union of prisms and for $d \in \{0, 1, 2, 3, 4, 5\}$ are given in [1]. The existence of super $d$-antimagic labeling of type $(1, 1, 1)$ for plane graphs containing a special Hamilton path is examined in [29] and super $d$-antimagic labelings of type $(1, 1, 1)$ for disconnected plane graphs are investigated in [31].

## 2.3 Irregular Graphs

The notion of the irregularity strength was introduced by Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba in [54]. It was motivated by a well known fact that a simple graph of order at least 2 cannot be completely irregular: it always has at least two vertices of equal degree. For a survey of known results and many open questions about the irregularity strength, see Lehel [22].

Chartrand et al. [54] introduced an edge $k$-labeling $\phi : E(G) \to \{1, 2, \dots, k\}$ of a graph $G$ such that the weights of vertices are distinct, i.e.

$$w_\phi(x) = \sum_{xy \in E(G)} \phi(xy) \neq w_\phi(x') = \sum_{x'y' \in E(G)} \phi(x'y').$$

for all vertices $x, x' \in V(G)$, $x \neq x'$. Such labelings were called irregular assignments and the *irregularity strength* $s(G)$ of a graph $G$ is known as the minimum $k$ for which $G$ has an irregular assignment using labels at most $k$. In Figure 2.7 there is an irregular assignment of fan graph $F_6$ with irregularity strength 3.

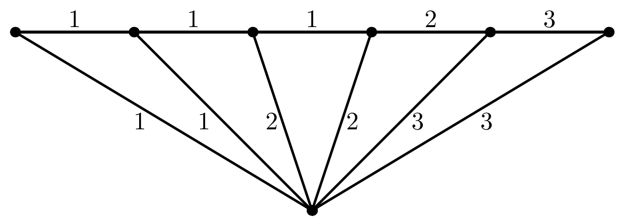


Figure 2.7: Irregular assignment of fan $F_6$ with $s(G) = 3$.

Suppose for graph $G$ we are able to construct a multigraph $G'$ such that each edge of $G$ is replaced by a set of at most $k$ parallel edges and degrees of the vertices in resulting multigraph $G'$ are all different. Then the smallest integer $k$ is the irregularity strength $s(G)$.

The question which energized the study of the irregularity strength of regular graphs is due to Jacobson as mentioned in [83].

Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see [83].

### 2.3.1 Edge Irregular Labeling

A vertex $k$-labeling $\varphi : V(G) \to \{1, 2, \ldots, k\}$ is called an *edge irregular $k$-labeling* of the graph $G$ if for every two different edges $uv$ and $u'v'$ there is

$$w_\varphi(uv) = \varphi(u) + \varphi(v) \neq \varphi(u') + \varphi(v') = w_\varphi(u'v').$$

The minimum integer $k$ for which the graph $G$ has an edge irregular $k$-labeling is called the *edge irregularity strength* of $G$. This parameter is denoted by *es(G)*. Ahmad, Al-Mushayt and Bača [3] defined the concept of the edge irregularity strength. They estimate the lower bound of parameter $es(G)$ for simple graph $G$ and determined the precise values for several families of graphs namely, paths, stars, double stars and Cartesian product of two paths. The lower bound for simple graph $G$ is estimated as follows:

**Theorem 6.** [3] *Let $G$ be a simple graph with maximum degree $\Delta(G)$. Then*

$$es(G) \geq max\left\{\left\lceil\frac{|E(G)|+1}{2}\right\rceil, \Delta(G)\right\}.$$

The next two theorems show that the lower bound in Theorem 6 is tight.

**Theorem 7.** [3] *Let $P_n$ be a path on $n$ vertices, $n \geq 2$. Then*

$$es(P_n) = \left\lceil\frac{n}{2}\right\rceil.$$

**Theorem 8.** *[3] Let $K_{1,n}$ be a star on $n+1$ vertices, $n \geq 1$. Then*

$$es(K_{1,n}) = n.$$

An upper bound on the edge irregularity strength is given in the following theorem:

**Theorem 9.** [3] *Let $G$ be a graph of order $n$. Let the sequence $F_n$ of Fibonacci numbers be defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n \geq 3$, where $F_1 = 1$ and $F_2 = 2$. Then $es(G) \leq F_n$.*

In [3], it is proved that $es(S_{n,m}) = m$, for $3 \leq n \leq m$ and for Cartesian product of two paths $P_m$ and $P_n$, $m, n \geq 2$, $es(P_m \Box P_n) = \left\lceil\frac{2mn-m-n+1}{2}\right\rceil$.

### 2.3.2 Irregular Total Labeling

Bača, Jendrol', Miller and Ryan in [36] defined a *vertex irregular total $k$-labeling* of a graph $G$ to be a total labeling, $\psi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$, such that the total vertex-weights are all different for any $u, v \in V(G)$, $u \neq v$, i.e. $wt_\psi(u) \neq wt_\psi(v)$, where

$$wt(u) = \psi(u) + \sum_{uv \in E(G)} \psi(uv).$$

The *total vertex irregularity strength* of $G$, denoted by *tvs($G$)*, is the minimum $k$ for which $G$ has a vertex irregular total $k$-labeling. Figure 2.8 illustrates a vertex irregular total 2-labeling of the complete graph $K_5$. It is easy to see that the irregularity

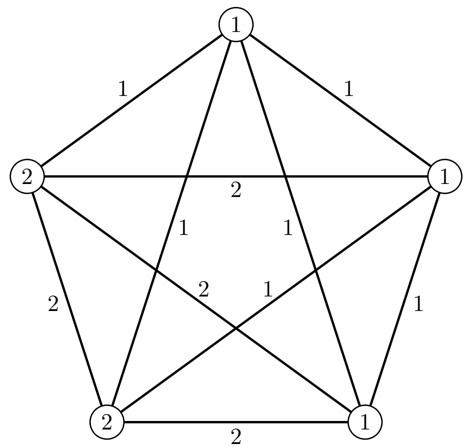

Figure 2.8: Vertex irregular total 2-labeling of $K_5$.

strength $s(G)$ of a graph $G$ is defined only for graphs containing at most one isolated vertex and no connected component of order 2. On the other hand, the total vertex irregularity strength *tvs($G$)* is defined for every graph $G$. Thus for graphs with no component of order less or equal to 2, $tvs(G) \leq s(G)$.

Nierhoff [94] recently proved that for all graphs $G$ with no component of order at most 2 and $G \neq K_3$, the irregularity strength $s(G)$ of $G$ is at most $|V(G)| - 1$. The bounds for the total vertex irregularity strength are given by the following theorem.

**Theorem 10.** [36] *Let $G$ be a graph of order $n$ and size $m$ with minimum degree $\delta$*

*and maximum degree* $\Delta$*. Then*

$$\left\lceil \frac{n+\delta}{\Delta+1} \right\rceil \leq tvs(G) \leq n + \Delta - 2\delta + 1.$$

**Corollary 2.** [36] *Let* $G$ *be an* $r$*-regular graph of order* $n$ *and size* $m$*. Then*

$$\left\lceil \frac{n+r}{1+r} \right\rceil \leq tvs(G) \leq n - r + 1.$$

For a regular Hamiltonian graph $G$ of order $n$ and size $m$ in [36] is proved that $tvs(G) \leq \lceil \frac{n+2}{3} \rceil$. Moreover there is proved the following upper bound.

**Theorem 11.** [36] *Let* $G$ *be a graph of order* $n$ *and size* $m$ *with maximum degree* $\Delta$ *and no component of order less or equal to* 2*. Then*

$$tvs(G) \leq n - 1 - \left\lfloor \frac{n-2}{\Delta+1} \right\rfloor.$$

In [96] it is proved that $tvs(G) < \frac{32n}{\delta} + 8$ in general and $tvs(G) < \frac{8n}{r} + 3$ for $r$-regular graphs. Recently Anholcer, Kalkowski and Przybylo imposed the upper bound by the following form.

**Theorem 12.** [11] *Let* $G$ *be a graph of order* $n$ *and with minimum degree* $\delta > 0$*. Then*

$$tvs(G) \leq 3\left\lceil \frac{n}{\delta} \right\rceil + 1 \leq \frac{3n}{\delta} + 4.$$

Based on a random ordering of the vertices Majerski and Przybylo proved in [87] the following upper bound.

**Theorem 13.** [87] *Let* $G$ *be a graph of order* $n$ *and with minimum degree* $\delta > \sqrt{n}\ln n$*. Then*

$$tvs(G) \leq \frac{(2+\mathcal{O}(1))n}{\delta} + 4.$$

Bača, Jendrol', Miller and Ryan in [36] defined the total labeling $\zeta : V(G) \cup E(G) \longrightarrow \{1, 2, 3, \ldots, k\}$ to be an *edge irregular total k-labeling* of the graph $G$ if for every two different edges $uv$ and $u'v'$ of $G$, the weights of $uv$ and $u'v'$ are different, i.e. $wt_\zeta(uv) \neq wt_\zeta(u'v')$, where $wt_\zeta(uv) = \zeta(u) + \zeta(uv) + \zeta(v)$.

The *total edge irregularity strength* is defined as the minimum $k$ for which $G$ has an edge irregular total $k$-labeling and is denoted by $tes(G)$. Figure 2.9 shows the edge irregular total 6-labeling of the star $K_{1,10}$.

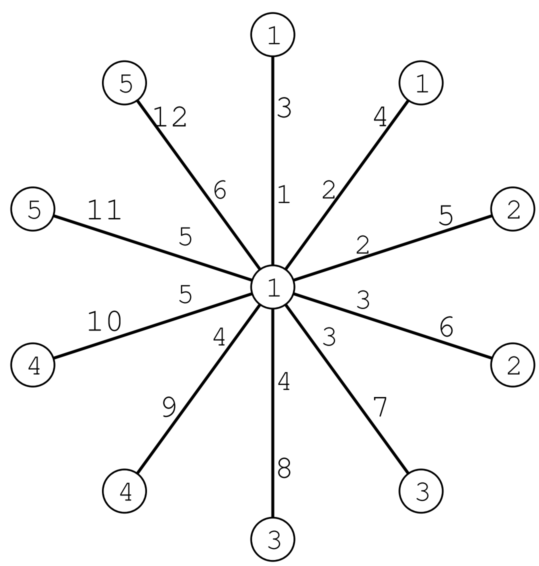

Figure 2.9: Edge irregular total 6-labeling of $K_{1,10}$.

The bounds for the total edge irregularity strength are given by the following theorems:

**Theorem 14.** [36] *Let $G = (V, E)$ be a graph with vertex set $V(G)$ and a non-empty edge set $E(G)$. Then*

$$\left\lceil \frac{|E(G)| + 2}{3} \right\rceil \leq tes(G) \leq |E(G)|.$$

**Theorem 15.** [36] *Let $G = (V, E)$ be a graph with maximum degree $\Delta = \Delta(G)$. Then $tes(G) \geq \lceil \frac{\Delta+1}{2} \rceil$ and $tes(G) \leq |E| - \Delta$ if $\Delta \leq \frac{|E|-1}{2}$.*

Ivančo and Jendrol' [69] proposed a conjecture that says that, for arbitrary graph $G$ different from $K_5$ and maximum degree $\Delta(G)$,

$$tes(G) = \max\left\{ \left\lceil \frac{|E|+2}{3} \right\rceil, \left\lceil \frac{\Delta+1}{2} \right\rceil \right\}.$$

This conjecture has been verified for complete graphs and complete bipartite graphs in [70] and [71], for the Cartesian, the categorical and the strong products of two paths in [4],[7],[89], for the categorical product of two cycles in [6], for generalized Petersen graphs in [63], for generalized prisms in [42], for the corona product of a path with certain graphs in [95] and for large dense graphs with $\frac{|E(G)|+2}{3} \leq \frac{\Delta(G)+1}{2}$ in [51].

Combining previous variations of the irregularity strength, Marzuki, Salman and Miller [88] introduced a new irregular total $k$-labeling of a graph $G$ called *totally irregular total $k$-labeling*, which is required to be vertex irregular total and also edge irregular total at the same time. The minimum $k$ for which a graph $G$ has a totally irregular total $k$-labeling is called the *total irregularity strength* of $G$ and is denoted by $ts(G)$. Upper and lower bounds for the parameter $ts(G)$ are given in [88]. Ramdani and Salman in [97] determined the exact values of the total irregularity strength for several Cartesian product graphs. Namely, they proved that for $n \geq 3$, $ts(P_n \Box P_2) = n$, $ts(C_n \Box P_2) = n + 1$ and $ts(S_n \Box P_2) = n + 1$.

### 2.3.3 Face Irregular Entire Labeling

Motivated by irregularity strengths and a recent paper on entire colouring of plane graphs [112], irregular labelings of plane graphs with conditions on the weights of faces were studied in [34].

For a 2-connected plane graph $G = (V, E, F)$ with the face set $F$ in [34] is defined a labeling $\phi : V \cup E \cup F \to \{1, 2, \dots, k\}$ to be an entire $k$-labeling. The *weight* of a face $f$ under an entire $k$-labeling $\phi$, $w_\phi(f)$, is the sum of labels carried by that face and all the edges and vertices surrounding it. An entire $k$-labeling $\phi$ is defined to be a *face irregular entire $k$-labeling* of the plane graph $G$ if for every two different faces $f$ and $g$ of $G$ there is $w_\phi(f) \neq w_\phi(g)$. The *entire face irregularity strength*, denoted by efs($G$), of a plane graph $G$ is the smallest integer $k$ such that $G$ has a face irregular entire $k$-labeling.

Bača *et al.* in [34] proved that for every 2-connected plane graph $G = (V, E, F)$ with $n_i$ $i$-sided faces, $i \geq 3$,

$$\mathrm{efs}(G) \geq \lceil (2a + n_3 + n_4 + \cdots + n_b)/(2b + 1) \rceil, \tag{2.1}$$

where $a = \min\{i |\ n_i \neq 0\}$ and $b = \max\{i |\ n_i \neq 0\}$.

In [35] is described a face irregular entire 2-labeling of octahedron, see Figure 2.10 and it shows that the lower bound in (2.1) is tight.

In the case if a 2-connected plane graph contains only one largest face, $n_b = 1$, and $c = \max\{i |\ n_i \neq 0,\ i < b\}$, then the lower bound on efs is estimated as follows:

$$\mathrm{efs}(G) \geq \lceil (2a + |F| - 1)/(2c + 1) \rceil. \tag{2.2}$$

The exact value of the entire face irregularity strength for ladder $L_n \simeq P_n \square P_2$, $n \geq 3$, is determined in [34] and was proved that the lower bound in (2.2) is tight.

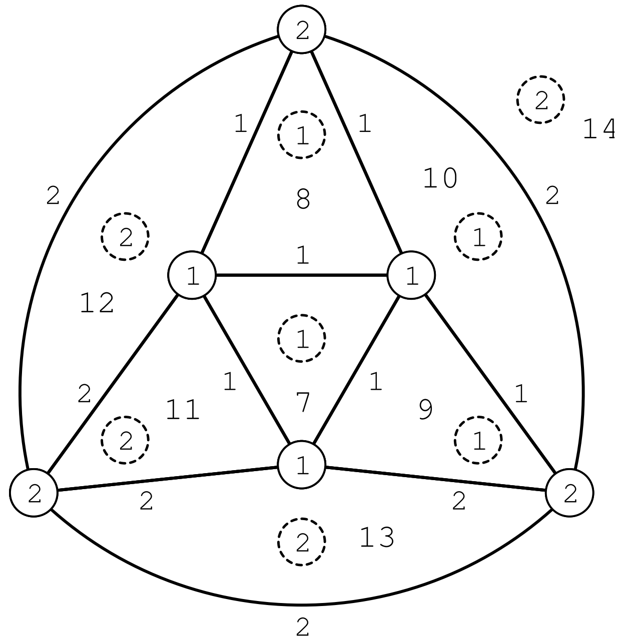


Figure 2.10: Face irregular entire 2-labeling of octahedron.

Another variant of the ladder graph $L_n$ is the graph $B_n$ depicted in Figure 2.11.

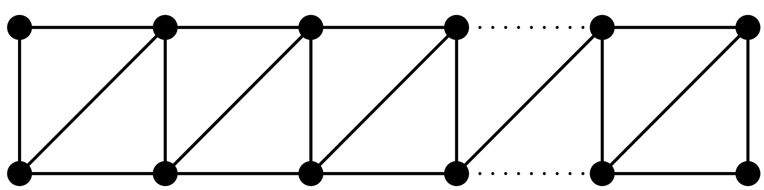

Figure 2.11: Graph $B_n$.

The graph $B_n$ contains $(2n-2)$ 3-sided faces, $(n-1)$ 4-sided faces and the external $(2n+2)$-sided face. The next theorem also proves that the lower bound in (2.2) is tight.

**Theorem 16.** [34] *For $B_n$, $n \geq 3$, we have*

$$efs(B_n) = \left\lceil \frac{n+1}{3} \right\rceil.$$

If the maximum degree of the 2-connected plane graph is considered then the lower bound of entire face irregularity strength is estimated as follows:

$$\text{efs}(G) \geq \left\lceil (2\overline{a} + \Delta(G) - 1)/(2\overline{b}) \right\rceil, \tag{2.3}$$

where an $\overline{a}$-sided face and a $\overline{b}$-sided face are the smallest face and the biggest face incident with a vertex of degree $\Delta$.

Sharpness of the lower bound in (2.3) is proved for wheels on $n+1$ vertices, $n \geq 3$, that is $efs(W_n) = \lceil (n+5)/6 \rceil$.

## 2.4 Cordial Graphs

The notion of a cordial labeling was first introduced by Cahit [53] as a weaker version of graceful labeling .

A vertex labeling $\phi : V(G) \to \{0,1\}$ induces an edge labeling $\phi^* : E(G) \to \{0,1\}$ defined by $\phi^*(uv) = |\phi(u) - \phi(v)|$. For a vertex labeling $\phi$ and $i \in \{0,1\}$, a vertex $v$ is an $i$-vertex if $\phi(v) = i$ and an edge is an $i$-edge if $\phi^*(e) = i$. Denote the numbers of 0-vertices, 1-vertices, 0-edges, and 1-edges of $G$ under $\phi$ and $\phi^*$ by $v_\phi(0)$, $v_\phi(1)$, $e_{\phi^*}(0)$, and $e_{\phi^*}(1)$, respectively. A vertex labeling $\phi$ is called *cordial* if $|v_\phi(0) - v_\phi(1)| \leq 1$ and $|e_{\phi^*}(0) - e_{\phi^*}(1)| \leq 1$.

Cordial labeling for certain families of graphs were studied in [65], [76], [81].

A binary vertex labeling $\phi : V(G) \to \{0,1\}$ with induced edge labeling $\phi^* : E(G) \to \{0,1\}$ defined by $\phi^*(uv) = \phi(u)\phi(v)$ is called a *product cordial labeling* if $|v_\phi(0) - v_\phi(1)| \leq 1$ and $|e_{\phi^*}(0) - e_{\phi^*}(1)| \leq 1$. The concept of a product cordial labeling was introduced by Sundaram et al. [107]. Some labelings with variations in cordial theme, namely an edge product cordial labeling and a total edge product cordial labeling have been introduced by Vaidya and Barasara in [108, 109].

Let $k$ be an integer, $2 \leq k \leq |E(G)|$. An edge labeling $\varphi : E(G) \to \{0, 1, \dots, k-1\}$ with induced vertex labeling $\varphi^* : V(G) \to \{0, 1, \dots, k-1\}$ defined by $\varphi^*(v) = \varphi(e_1) \cdot \varphi(e_2) \cdot \ldots \cdot \varphi(e_n) \pmod{k}$, where $e_1, e_2, \dots, e_n$ are the edges incident to the vertex $v$, is called a *$k$-total edge product cordial labeling* of $G$ if $|(e_\varphi(i) + v_{\varphi^*}(i)) - (e_\varphi(j) + v_{\varphi^*}(j))| \leq 1$ for every $i, j$, $0 \leq i < j \leq k-1$.

The concept of a $k$-total edge product cordial labeling was introduced by Azaizeh et al. in [13]. A graph $G$ with a $k$-total edge product cordial labeling is called *$k$-total edge product cordial graph.* Figure 2.12 illustrates a 3-total edge product cordial labeling of the cycle $C_{10}$.

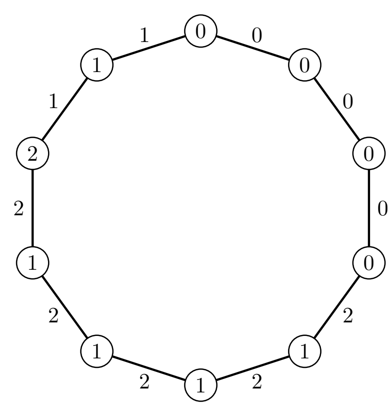

Figure 2.12: 3-total edge product cordial labeling of $C_{10}$.

**Theorem 17.** *Let $G$ be a $(n,m)$ $k$-edge product cordial graph. If $m$ or $n \equiv 0 \pmod{k}$, then $G$ is $k$-total product cordial.*

It was proved that path graph $P_n$, $n \geq 4$, cycle $C_n$, $n \neq 3, 6$ and star graph $K_{1,n}$, $n \equiv 0, 2 \pmod{3}$ are 3-total product cordial.

Sundaram, Ponraj and Somasundaram [15] have introduced the notion of prime cordial labeling [15]. A prime cordial labeling of a graph $G$ with vertex set $V(G)$ is a bijection $f : V(G) \to \{1, 2, \ldots, |V|\}$ defined by

$$f(e = uv) = \begin{cases} 1, & \gcd(f(u), f(v)) = 1 \\ 0, & \text{otherwise} \end{cases}$$

and $|e_f(0) - e_f(1)| \leq 1$. See [66, 77, 82] for related results and [52, 56, 67] for generalizations of cordial labeling .

# Chapter 3

# Entire face irregularity strength of disjoint union of plane graphs

In this chapter, we study the entire face irregularity strength of a disjoint union of $m$ copies of an arbitrary plane graph $G$, denoted by $mG$. We estimate the bounds for the parameter $\mathrm{efs}(mG)$ and prove the sharpness of the lower bound in two cases.

## 3.1 Lower Bounds of the Parameter $\mathrm{efs}(mG)$

The first theorem gives a lower bound of the entire face irregularity strength of a disjoint union of $m$ copies of an arbitrary plane graph $G$.

**Theorem 18.** [30] *Let $G = (V, E, F)$ be a 2-connected plane graph with $n_i$ $i$-sided faces, $i \geq 3$. Let $a = \min\{i|\ n_i \neq 0\}$ and $b = \max\{i|\ n_i \neq 0\}$. Then the entire face irregularity strength of the disjoint union of $m \geq 1$ copies of the plane graph $G$*

*satisfies the following inequality.*

$$\text{efs}(mG) \geq \left\lceil \frac{2a + (|F(G)| - 1)m + 1}{2b + 1} \right\rceil.$$

*Proof.* Let $G = (V(G), E(G), F(G))$ be a 2-connected plane graph with $n_i$ $i$-sided faces, $i \geq 3$. Let $mG$ be the disjoint union of $m$ copies of the graph $G$. Then $|V(mG)| = m|V(G)|$, $|E(mG)| = m|E(G)|$ and $|F(mG)| = m|F(G)| - m + 1$. Let $\phi$ be a face irregular entire $K$-labeling of $mG$ with $\text{efs}(mG) = K$. The smallest weight of a face under the $K$-labeling is at least $2a + 1$ and the largest weight of a face has value at most $(2b + 1)K$. It means that for every face $f \in F(G)$ we have

$$2a + 1 \leq w_\phi(f) \leq (2b + 1)K.$$

Since

$$|F(mG)| = (n_a + \cdots + n_b)m - m + 1 = m|F(G)| - m + 1,$$

thus $2a + (|F(G)| - 1)m + 1 \leq (2b + 1)K$ and

$$K \geq \left\lceil \tfrac{2a+(|F(G)|-1)m+1}{2b+1} \right\rceil.$$

□

If a 2-connected plane graph $G$ contains only one face of the largest size then using similar arguments as in the proof of Theorem 18 we have the following lower bound on the parameter $\text{efs}(mG)$.

**Theorem 19.** [30] *Let $G = (V(G), E(G), F(G))$ be a 2-connected plane graph with $n_i$ $i$-sided faces, $i \geq 3$. Let $a = \min\{i|\ n_i \neq 0\}$ and $b = \max\{i|\ n_i \neq 0\}$, $n_b = 1$, and $c = \max\{i|\ n_i \neq 0,\ i < b\}$. Let the $b$-sided face be the external face. Then the entire* *face irregularity strength of the disjoint union of $m \geq 1$ copies of the plane graph $G$ is*

$$\text{efs}(mG) \geq \left\lceil \frac{2a+(|F(G)|-1)m}{2c+1} \right\rceil.$$

The lower bound in Theorem 19 is tight as it can be seen from the next theorem.

**Theorem 20.** [30] *Let $n \equiv 1 \pmod 9$, $n \geq 10$, $m \geq 1$, and $L_n \simeq P_n \Box P_2$ be a ladder. Then*

$$\text{efs}(mL_n) = \left\lceil \frac{8+m(n-1)}{9} \right\rceil.$$

*Proof.* Let $L_n \simeq P_n \Box P_2$, $n \geq 10$, be a ladder with $V(L_n) = \{u_i, v_i | 1 \leq i \leq n\}$ and $E(L_n) = \{u_i u_{i+1}, v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{u_i v_i | 1 \leq i \leq n\}$. The ladder $L_n$ contains $(n-1)$ 4-sided faces and one external $2n$-sided face, say $f_{ext}$. Let us denote the 4-sided faces by the symbols $f_1, f_2, \ldots, f_{n-1}$ such that the face $f_i$ is surrounded by vertices $u_i, u_{i+1}, v_i, v_{i+1}$ and edges $u_i u_{i+1}, v_i v_{i+1}, u_i v_i, u_{i+1} v_{i+1}$, for $1 \leq i \leq n-1$.

According to Theorem 19 we have that $\text{efs}(mL_n) \geq \lceil (8+(n-1)m)/9 \rceil = K$. To prove the equality, it suffices to show the existence of an optimal face irregular entire $\lceil (8+(n-1)m)/9 \rceil$-labeling.

It was proved in [34] that $\text{efs}(L_n) = \lceil \frac{n+7}{9} \rceil = k$ and there is described the corresponding face irregular entire $k$-labeling $\phi$ as follows:

$$\begin{aligned}
\phi(u_i) &= \left\lfloor \tfrac{i+4}{9} \right\rfloor + 1, && \text{for } 1 \leq i \leq n \\
\phi(v_i) &= \left\lfloor \tfrac{i+2}{9} \right\rfloor + 1, && \text{for } 1 \leq i \leq n \\
\phi(u_i u_{i+1}) &= \left\lfloor \tfrac{i+7}{9} \right\rfloor + 1, && \text{for } 1 \leq i \leq n-1 \\
\phi(v_i v_{i+1}) &= \left\lfloor \tfrac{i+6}{9} \right\rfloor + 1, && \text{for } 1 \leq i \leq n-1 \\
\phi(u_i v_i) &= \left\lfloor \tfrac{i}{9} \right\rfloor + 1, && \text{for } 1 \leq i \leq n \\
\phi(f_i) &= \left\lceil \tfrac{i}{9} \right\rceil, && \text{for } 1 \leq i \leq n-1.
\end{aligned}$$

It was proved that the weights of 4-sided faces $f_i$, $1 \leq i \leq n-1$, are all distinct and

$$w_\phi(f_i) = i + 8. \tag{3.1}$$

For every vertex $v$ in $L_n$, we denote by the symbol $v^j$ the corresponding vertex of $v$ in the $j$-th copy of $L_n$ in $mL_n$, $1 \leq j \leq m$. Analogously, let $u^j v^j$ denote the edge corresponding to edge $uv$ in the $j$-th copy of $L_n$ in $mL_n$, $1 \leq j \leq m$ and let $f^j$ denote the face corresponding to the internal face $f$ in the $j$-th copy of $L_n$ in $mL_n$, $1 \leq j \leq m$. Moreover, there is one external $2nm$-sided unbounded face $f_{EXT} \in F(mL_n)$.

Now we define a labeling $\psi$ of $mL_n$ in the following way:

$$\psi(f_{EXT}) = K,$$

$$\psi(x^j) = \phi(x) + (k-1)(j-1),$$

for every element $x$, where $x \in V(L_n) \cup E(L_n) \cup (F(L_n) - \{f_{ext}\})$ and $1 \leq j \leq m$. If $n = 9s + 1$, $s \geq 1$, then the parameter $k = \lceil (n+7)/9 \rceil = s + 1$ and thus $n = 9k - 8$ in this case.

As $\phi$ is a $k$-labeling, thus for every vertex, edge and face $y \in V(mL_n) \cup E(mL_n) \cup (F(mL_n) - \{f_{EXT}\})$ we have

$$\begin{aligned}\psi(y) \leq & k + (k-1)(m-1) = (k-1)m + 1 = sm + 1 = \left\lceil sm + \tfrac{8}{9} \right\rceil \\ & = \left\lceil \tfrac{8+m(9s+1-1)}{9} \right\rceil = \left\lceil \tfrac{8+m(n-1)}{9} \right\rceil = K.\end{aligned}$$

As $\psi(f_{EXT}) = K$, it implies that all vertex, edge and face labels are at most $K$ and the labeling $\psi$ is an entire $K$-labeling.

Moreover, using (3.1) we get that the weights of 4-sided faces $f_i^j$, $1 \leq i \leq n-1$ and $1 \leq j \leq m$, attain the values

$$\begin{aligned} w_\psi(f_i^j) = & \psi(f_i^j) + \sum_{v \sim f_i} \psi(v^j) + \sum_{e \sim f_i} \psi(e^j) \\ = & \big(\phi(f_i) + (k-1)(j-1)\big) + \sum_{v \sim f_i} \big(\phi(v) + (k-1)(j-1)\big) \\ & + \sum_{e \sim f_i} \big(\phi(e) + (k-1)(j-1)\big) \\ = & \phi(f_i) + (k-1)(j-1) + \sum_{v \sim f_i} \phi(v) + 4(k-1)(j-1) \\ & + \sum_{e \sim f_i} \phi(e) + 4(k-1)(j-1) \\ = & \big(\phi(f_i) + \sum_{v \sim f_i} \phi(v) + \sum_{e \sim f_i} \phi(e)\big) + 9(k-1)(j-1) \\ = & w_\phi(f_i) + 9(k-1)(j-1) = 9(k-1)(j-1) + i + 8. \end{aligned}$$

Since

$$\begin{aligned} w_\psi(f_{n-1}^j) & = 9(k-1)(j-1) + (n-1) + 8 = 9(k-1)(j-1) + 9k - 9 + 8 \\ & = 9(k-1)j + 8 \end{aligned}$$

for every $1 \leq j \leq m-1$ and as

$$w_\psi(f_1^{j+1}) = 9(k-1)(j+1-1)+1+8 = 9(k-1)j+9$$

for every $1 \leq j \leq m-1$, then the 4-sided face-weights are different for all pairs of distinct faces and they form the arithmetic sequence of difference 1 from 9 up to $(n-1)m+8$. It is easy to see that under the labeling $\psi$ the weight of the external $2nm$-sided face $f_{EXT}$ is at least $4nm+K$. Thus, the labeling $\psi$ is desired face irregular entire $K$-labeling . □

Figure 3.1 illustrates the face irregular entire 6-labeling in the proof of Theorem 20 for disjoint union of 5 copies of the ladder $L_{10}$.

If we consider the maximum degree $\Delta$ of the 2-connected plane graph $G$, then we obtain the following lower bound:

**Theorem 21.** [30] *Let $G=(V,E,F)$ be a 2-connected plane graph with maximum degree $\Delta(G)$. Let $x$ be a vertex of degree $\Delta(G)$ and let the smallest face and the biggest face incident with $x$ be an $\overline{a}$-sided face and a $\overline{b}$-sided face, respectively. Then the entire face irregularity strength of the disjoint union of $m \geq 1$ copies of the plane graph $G$ is*

$$\text{efs}(mG) \geq \left\lceil \frac{2\overline{a}+m\Delta(G)}{2\overline{b}+1} \right\rceil.$$

*Proof.* Let $f_1, f_2, \ldots, f_{\Delta(G)}$ be the faces incident with a fixed vertex $x$ of maximum degree $\Delta(G)$ in a 2-connected plane graph $G$. We denote by symbol $x^j$ the corresponding vertex of $x$ in the $j$-th copy of $G$ in $mG$. Analogously, we denote by symbols $f_1^j, f_2^j, \ldots, f_{\Delta(G)}^j$ the corresponding faces $f_1, f_2, \ldots, f_{\Delta(G)}$ in the $j$-th copy of $G$ in $mG$.

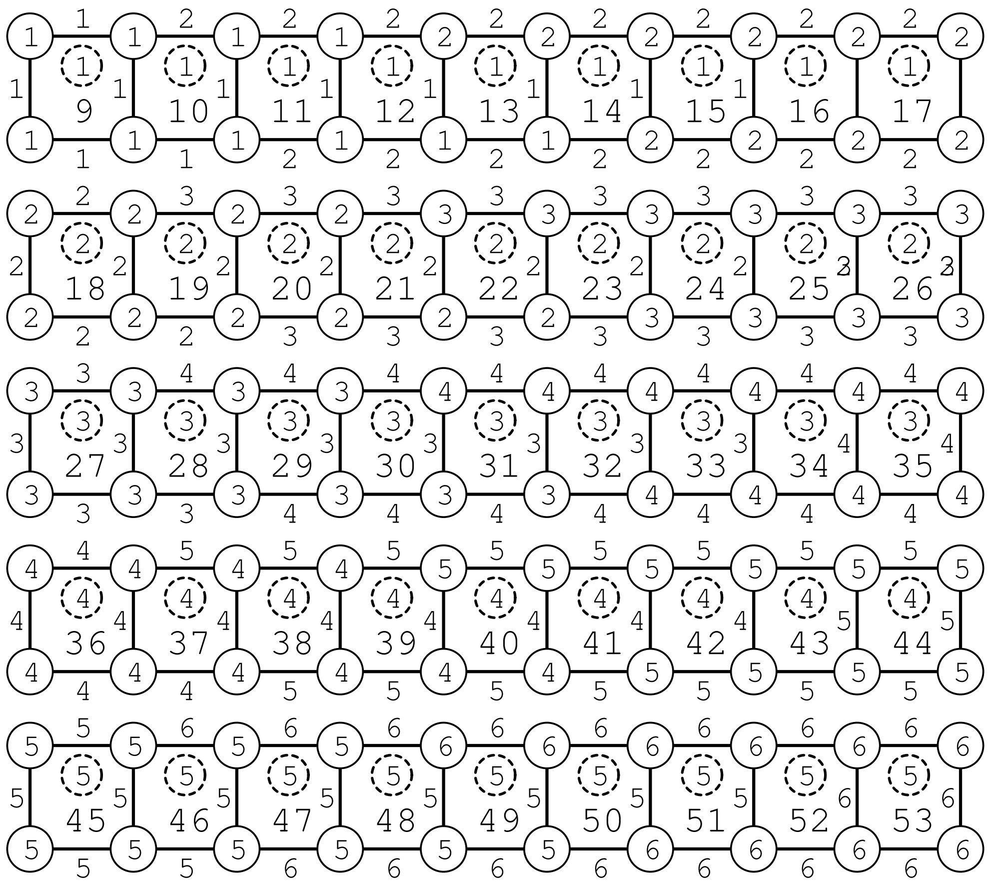


Figure 3.1: Face irregular entire 6-labeling of $5L_{10}$.

Suppose that $\vartheta$ is a face irregular entire $K$-labeling of $mG$ with $\text{efs}(mG) = K$. Then the face-weights $w_\vartheta(f_i^j)$, $1 \leq i \leq \Delta(G)$ and $1 \leq j \leq m$, are all distinct. The smallest face-weight among all of them is at least $2\overline{a} + 1$ and the largest face-weight among all of them cannot be larger than $(2\overline{b} + 1)K$. Thus

$$2\overline{a} + m\Delta(G) \leq (2\overline{b} + 1)K$$

and

$$K \geq \left\lceil \frac{2\overline{a} + m\Delta(G)}{2\overline{b} + 1} \right\rceil .$$

$\square$

The sharpness of the lower bound in Theorem 21 can be seen from the following lemma.

**Lemma 1.** [30] *Let* $W_7$ *be the wheel on* 8 *vertices and let* $m \geq 1$*. Then*

$$\text{efs}(mW_7) = m + 1.$$

*Proof.* The wheel $W_7$ may be visualized as a plane graph obtained by joining all vertices of cycle $C_7$ to a vertex $v$. Let $V(W_7) = \{v\} \cup \{v_i | 1 \leq i \leq 7\}$ and $E(W_n) = \{vv_i | 1 \leq i \leq 7\} \cup \{v_i v_{i+1} | 1 \leq i \leq 6\} \cup \{v_7 v_1\}$ be the vertex-set and the edge-set of the wheel $W_7$. Let $F(W_7) = \{f_i | 1 \leq i \leq 7\} \cup \{f_{ext}\}$ be the set of all 3-sided faces of the wheel $W_7$ and one 7-sided unbounded external face. The 3-sided face $f_i$, $1 \leq i \leq 6$, is surrounded by vertices $v, v_i, v_{i+1}$ and edges $vv_i, v_i v_{i+1}, vv_{i+1}$, and $f_7$ is the 3-sided face surrounded by vertices $v, v_1, v_7$ and edges $vv_7, v_1 v_7, vv_1$.

We construct a face irregular entire 2-labeling of $W_7$ as follows:

$$\rho(v) = 1,$$

$$\rho(v_1v_7) = 1,$$

$$\rho(v_i) = \begin{cases} 1, & \text{for } i = 1, 2 \\ 2, & \text{otherwise} \end{cases}$$

$$\rho(vv_i) = \begin{cases} 2, & \text{for } i = 3, 4, 5 \\ 1, & \text{otherwise} \end{cases}$$

$$\rho(f_i) = \begin{cases} 2, & \text{for } i = 4, 5 \\ 1, & \text{otherwise} \end{cases}$$

$$\rho(v_iv_{i+1}) = \begin{cases} 1, & \text{for } i = 1, 2, 3 \\ 2, & \text{for } i = 4, 5, 6. \end{cases}$$

We can see that

$$w_\rho(f_i) = \begin{cases} 2i + 5, & \text{for } 1 \leq i \leq 4 \\ 22 - 2i, & \text{for } 5 \leq i \leq 7, \end{cases}$$

i.e., the weights of 3-sided faces successively attain the values $7, 8, \ldots, 13$. The smallest weight is obtained for the face $f_1$ and the largest one for the face $f_4$. For every $x \in V(W_7) \cup E(W_7) \cup (F(W_7) - \{f_{ext}\})$, we denote by the symbol $x^j$ the corresponding element of $x$ in the $j$-th copy of $W_7$ in $mW_7$, $1 \leq j \leq m$. Moreover, the graph $mW_7$ contains one external $7m$-sided unbounded face $f_{EXT}$.

For $m \geq 1$, we construct an entire labeling $\vartheta$ of $mW_7$ such that:

$$\vartheta(f_{EXT}) = m + 1,$$

$$\vartheta(x^j) = \rho(x) + j - 1$$

for $1 \leq j \leq m$.

Since $\rho(x) \leq 2$ for every $x \in V(W_7) \cup E(W_7) \cup (F(W_7) - \{f_{ext}\})$ then

$$\vartheta(x^j) \leq \rho(x) + j - 1 \leq m + 1.$$

The weights of 3-sided faces $f_i^j$, $1 \leq i \leq 7$ and $1 \leq j \leq m$, attain the values

$$\begin{aligned} w_\vartheta(f_i^j) =& \vartheta(f_i^j) + \sum_{v \sim f_i} \vartheta(v^j) + \sum_{e \sim f_i} \vartheta(e^j) \\ =& \big(\rho(f_i) + j - 1\big) + \sum_{v \sim f_i} \big(\rho(v) + j - 1\big) + \sum_{e \sim f_i} \big(\rho(e) + j - 1\big) \\ =& \rho(f_i) + j - 1 + \sum_{v \sim f_i} \rho(v) + 3(j-1) + \sum_{e \sim f_i} \rho(e) + 3(j-1) \\ =& \big(\rho(f_i) + \sum_{v \sim f_i} \rho(v) + \sum_{e \sim f_i} \rho(e)\big) + 7(j-1) = w_\rho(f_i) + 7(j-1). \end{aligned}$$

Moreover,

$$w_\vartheta(f_4^j) = w_\rho(f_4) + 7(j-1) = 13 + 7(j-1) = 6 + 7j$$

for $1 \leq j \leq m$ and

$$w_\vartheta(f_1^{j+1}) = w_\rho(f_1) + 7j = 7 + 7j$$

for $1 \leq j \leq m-1$. In fact, the labeling $\vartheta$ has been chosen in such a way that the weights of 3-sided faces are different for all pairs of distinct faces and they successively attain the values from 7 up to $7m+6$. One can see that under the labeling $\vartheta$ the weight of the external $7m$-sided face is at least $15m+1$.

On the other side, using Theorem 21 we have

$$\text{efs}(mW_7) \geq \lceil \tfrac{2 \cdot 3 + 7m}{2 \cdot 3 + 1} \rceil = m + 1.$$

This concludes the proof . □

Figure 3.2 shows a face irregular entire 5-labeling described in the proof of Lemma 1 for disjoint union of 4 copies of the wheel $W_7$.

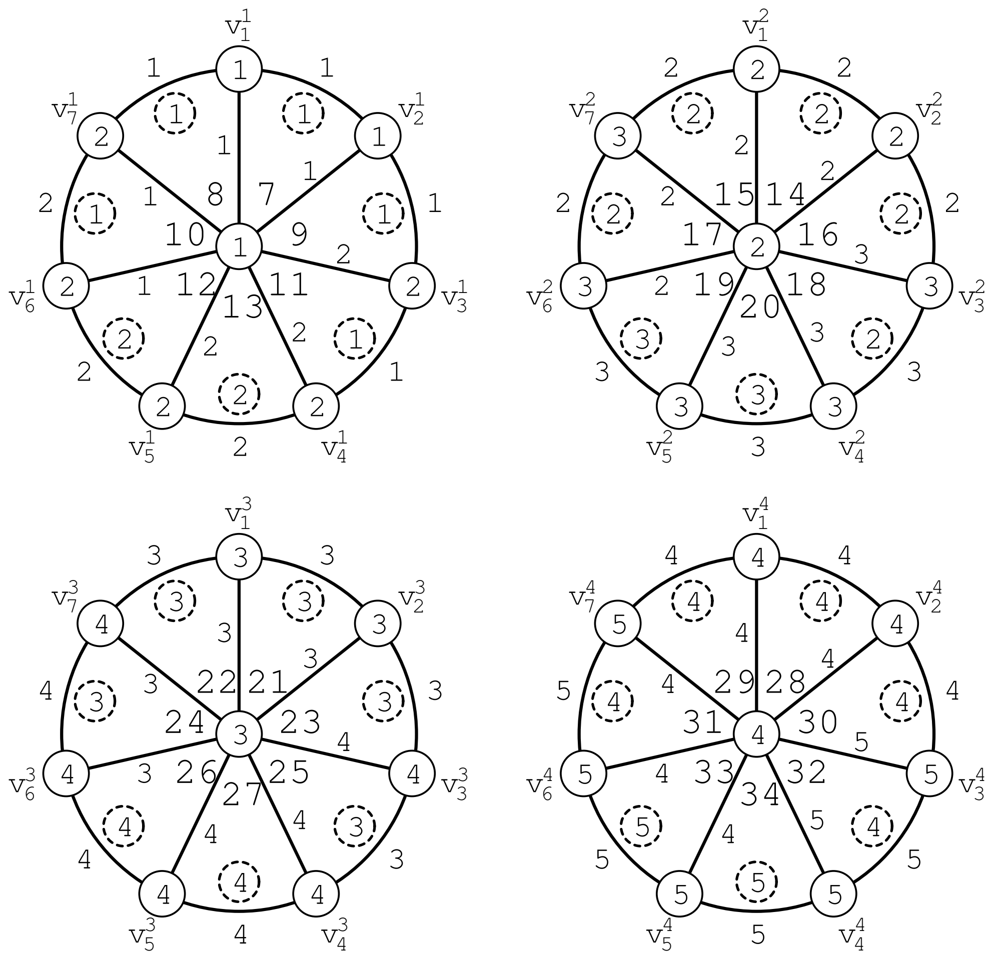


Figure 3.2: Face irregular entire 5-labeling of $4W_7$.

## 3.2 Upper Bound of the Parameter efs($mG$)

The next theorem provides an upper bound of the entire face irregularity strength for disjoint union of $m$ copies of a plane graph $G$.

**Theorem 22.** [30] *Let $G$ be a 2-connected plane graph with entire face irregularity strength* $\mathrm{efs}(G) = k$ *and $b$-sided external face. Let $\varphi$ be a face irregular entire $k$-labeling of $G$ and let the $c$-sided face be the internal face of the biggest face-weight. If $b > ck$ then the entire face irregularity strength for the disjoint union of $m \geq 1$ copies of $G$ satisfies the following inequality:*

$$\mathrm{efs}(mG) \leq m\,\mathrm{efs}(G).$$

*Proof.* Let $G$ be a 2-connected plane graph such that $\mathrm{efs}(G) = k$ and let $\varphi : V(G) \cup E(G) \cup F(G) \to \{1, 2, \ldots, k\}$ be a face irregular entire $k$-labeling such that for all pairs of faces $f_i, f_j \in F(G)$, $i \neq j$, the face-weights are distinct.

We will not consider the external face of $G$ and let us denote the remaining faces of $G$ by the symbols $f_1, f_2, \ldots, f_{|F(G)|-1}$ such that

$$w_\varphi(f_i) < w_\varphi(f_{i+1}), \tag{3.2}$$

for $i = 1, 2, \ldots, |F(G)| - 2$.

For every vertex $v$ in $G$, we denote by symbol $v^j$ the corresponding vertex of $v$ in the $j$-th copy of $G$ in $mG$. Analogously, let $u^j v^j$ denote the edge corresponding to edge $uv$ in the $j$-th copy of $G$ in $mG$ and let $f^j$ denote the face corresponding to face $f$ in the $j$-th copy of $G$ in $mG$, $1 \leq j \leq m$. Moreover, there is one external $2bm$-sided unbounded face $f_{EXT} \in F(mG)$.

We define an entire labeling $\phi$ of $mG$ in the following way:

$$\begin{aligned}
\phi(v^j) &= m\varphi(v) && \text{for } v \in V(G),\ 1 \le j \le m,\\
\phi(e^j) &= m\varphi(e) && \text{for } e \in E(G),\ 1 \le j \le m,\\
\phi(f^j) &= m\varphi(f) - m + j && \text{for } f \in F(G),\ 1 \le j \le m.\\
\phi(f_{EXT}) &= mk
\end{aligned}$$

It is easy to see that under the labeling $\phi$ the vertices, edges and faces of $mG$ are labeled with numbers $1, 2, \dots, mk$.

For the face-weight of $f_i^j \in F(mG)$, $1 \le j \le m$, $1 \le i \le |F(G)| - 1$, under the labeling $\phi$ we have

$$\begin{aligned}
w_\phi(f_i^j) &= \phi(f_i^j) + \sum_{v\sim f_i} \phi(v^j) + \sum_{e\sim f_i} \phi(e^j)\\
&= \big(m\varphi(f_i) - m + j\big) + \sum_{v\sim f_i} m\varphi(v) + \sum_{e\sim f_i} m\varphi(e)\\
&= m\varphi(f_i) - m + j + m\sum_{v\sim f_i} \varphi(v) + m\sum_{e\sim f_i} \varphi(e)\\
&= m\big(\varphi(f_i) + \sum_{v\sim f_i} \varphi(v) + \sum_{e\sim f_i} \varphi(e)\big) - m + j\\
&= m\, w_\varphi(f_i) - m + j.
\end{aligned}$$

Let us consider the face-weights of the faces $f_i^1, f_i^2, \dots, f_i^m$, $1 \le i \le |F(G)| - 1$, under the labeling $\phi$:

$$\begin{aligned}
w_\phi(f_i^1) &= m\, w_\varphi(f_i) - m + 1,\\
w_\phi(f_i^2) &= m\, w_\varphi(f_i) - m + 2,\\
&\vdots\\
w_\phi(f_i^m) &= m\, w_\varphi(f_i) - m + m = m\, w_\varphi(f_i).
\end{aligned}$$

Using the inequality (3.2) we get that for $1 \leq i \leq |F(G)| - 2$ it holds

$$w_\varphi(f_i) \leq w_\varphi(f_{i+1}) - 1.$$

Multiplying this inequality by $m$, $m \geq 1$, we get

$$m\, w_\varphi(f_i) \leq m\, w_\varphi(f_{i+1}) - m < m\, w_\varphi(f_{i+1}) - m + 1,$$

which means that

$$w_\phi(f_i^m) < w_\phi(f_{i+1}^1),$$

for $1 \leq i \leq |V(G)| - 2$. It suffices to prove that the biggest face-weight of the internal $c$-sided face $f_{|F(G)|-1}^m$ is less than the face-weight of $f_{EXT}$. Since the biggest face-weight of the internal $c$-sided face $f_{|F(G)|-1}^m$ is at most $2cmk+mk$ and the smallest face-weight of $f_{EXT}$ is at least $2bm + mk$ then for $b > ck$ holds

$$w_\phi(f_{|F(G)|-1}^m) < w_\phi(f_{EXT}).$$

This proves that $\mathrm{efs}(mG) \leq m\,\mathrm{efs}(G)$ . □

# Chapter 4

# Entire Labeling of Klein-bottle Fullerenes

In this chapter we deal with the problem of entire labeling of the Klein-bottle fullerene in such a way that the weights of all 6-sided faces constitute an arithmetic progression of difference $d$. We study the existence of such labelings for several differences $d$.

## 4.1 Klein-bottle Polyhexes

The discovery of the fullerene molecules and related forms of carbon such as nanotubes has generated an explosion of activity in chemistry, physics, and materials science. Classical fullerene is an all-carbon molecule in which the atoms are arranged on a pseudospherical framework made up entirely of pentagons and hexagons. Its molecular graph is a finite trivalent graph embedded on the surface of a sphere with only hexagonal and (exactly 12) pentagonal faces. Deza et al. [57] considered fullerene's extension to other closed surfaces and showed that only four surfaces are possible, namely sphere, torus, Klein bottle and projective plane. Unlike spherical fullerenes, toroidal and Klein bottle's fullerenes have been regarded as tessellations of entire hexagons on their surfaces since they must contain no pentagons, see [57, 78].

Let $L$ be a regular hexagonal lattice and $P_m^n$ be an $m \times n$ quadrilateral section (with $m \geq 2$ hexagons on the top and bottom sides and $n \geq 2$ hexagons on the lateral sides, $n$ is even) cut from the regular hexagonal lattice $L$, (see Figure 4.1). First identify two lateral sides of $P_m^n$ to form a cylinder. If we identify the top and bottom sides of $P_m^n$ such that we identify the vertices $u_i^0$ and the vertices $v_i^{n-1}$, for $i = 0, 1, 2, \ldots, m-1$, we obtain the *Klein-bottle fullerene* (Klein-bottle polyhex) $\mathbb{KB}_m^n$ with $mn$ hexagons. In this case $\mathbb{KB}_m^n$ is a cubic bipartite graph of order $2mn$ and size $3mn$ embedded on the Klein-bottle and contains only hexagons, where $V(\mathbb{KB}_m^n) = \{u_i^j, v_i^j : 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\}$ is its vertex set, $E(\mathbb{KB}_m^n) = \{u_i^j v_i^j : 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\} \cup \{u_i^j v_i^{j-1} : 0 \leq i \leq m-1,\ 2 \leq j \leq n-2,\ j \text{ even}\} \cup \{u_i^0 v_{m-1-i}^{n-1} : 0 \leq i \leq m-1\} \cup \{v_i^j u_i^{j+1} : 0 \leq i \leq m-1,\ 0 \leq j \leq n-2,\ j \text{ even}\} \cup \{u_i^j v_{i+1}^{j-1} : 0 \leq i \leq m-1,\ 1 \leq j \leq n-1,\ j \text{ odd}\} \cup \{v_i^j u_{i+1}^{j+1} : 0 \leq i \leq m-1,\ 1 \leq j \leq n-3,\ j \text{ odd}\} \cup \{u_i^0 v_{m-i}^{n-1} : 0 \leq i \leq m-1\}$ is its edge set with the indices $i$ (respectively $j$) taken modulo $m$ (respectively $n$), and $F(\mathbb{KB}_m^n) = \{z_i^j : 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\}$ is its face set .

Now, let $L$ be a regular hexagonal lattice and $P_{m+1/2}^n$ be a quadrilateral section (with $m+1/2$ hexagons on the top and bottom sides, $m \geq 1$, and $n \geq 2$ hexagons on the lateral sides, $n$ is even) cut from the regular hexagonal lattice $L$, (see Figure 4.2). We identify the top and bottom sides of $P_{m+1/2}^n$ to form a cylinder. Then we identify the lateral sides of cylinder such that we identify the vertices $u_0^j$ and the vertices

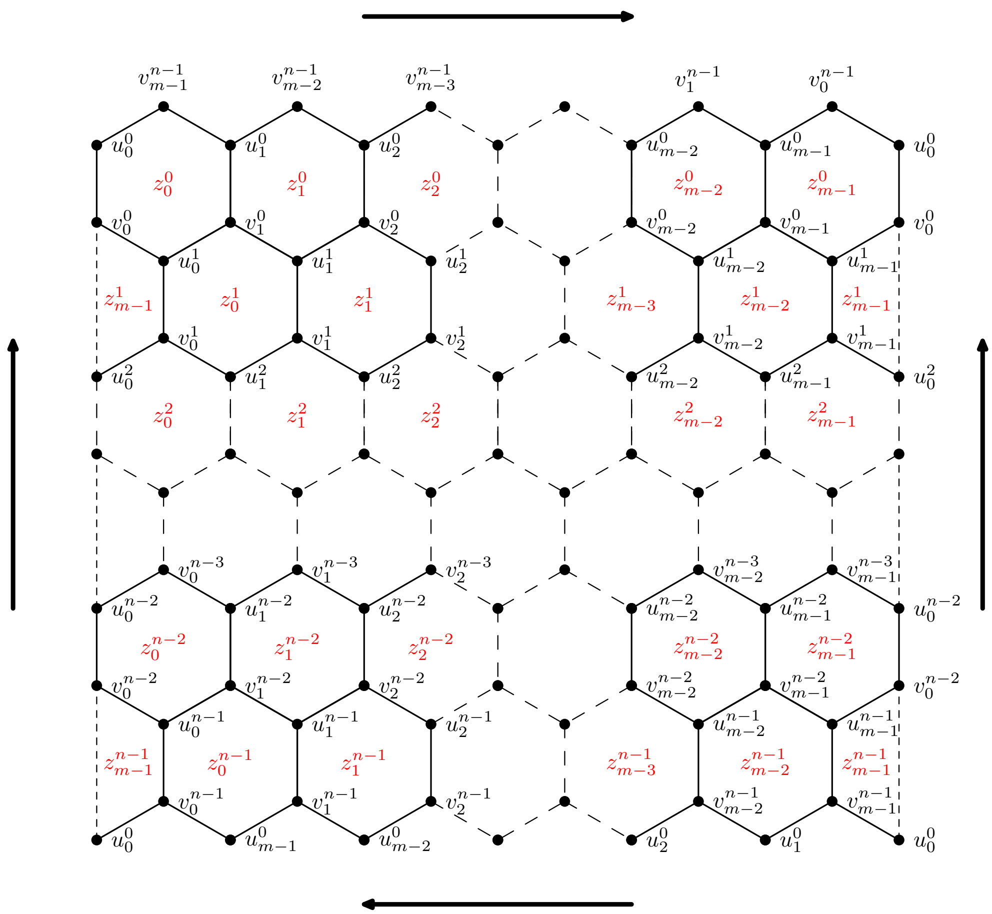

Figure 4.1: Quadrilateral section $P_m^n$ cuts from the regular hexagonal lattice.

$v_0^j$, for $j = 0, 2, 4, \ldots, n-2$, to obtaining the Klein-bottle polyhex $\mathbb{KB}_{m+1/2}^n$. We can see that $\mathbb{KB}_{m+1/2}^n$ is a cubic non-bipartite graph of order $2n(m+1/2)$ and size $3n(m+1/2)$ embedded on the Klein-bottle and contains $n(m+1/2)$ hexagons, where the vertex set is $V(\mathbb{KB}_{m+\frac{1}{2}}^n) = \{u_i^j, v_i^j : 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\} \cup \{u_m^j, v_m^j : 0 \leq j \leq n-2,\ j \text{ even}\}$ and the edge set is $E(\mathbb{KB}_{m+\frac{1}{2}}^n) = \{u_i^j v_i^j : 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\} \cup \{u_m^j v_m^j : 0 \leq j \leq n-2,\ j \text{ even}\} \cup \{u_i^j v_i^{j-1} : 0 \leq i \leq m-1,\ 0 \leq j \leq n-2,\ j \text{ even}\} \cup \{v_i^j u_i^{j+1} : 0 \leq i \leq m-1,\ 0 \leq j \leq n-2,\ j \text{ even}\} \cup \{u_m^j u_0^{n-j} : 0 \leq j \leq n-2,\ j \text{ even}\} \cup \{v_m^j v_0^{n-2-j} : 0 \leq j \leq n-2,\ j \text{ even}\} \cup \{u_i^j v_{i+1}^{j-1} : 0 \leq i \leq$

$m-1,\ 1 \leq j \leq n-1,\ j \text{ odd}\} \cup \{v_i^j u_{i+1}^{j+1} : 0 \leq i \leq m-1,\ 1 \leq j \leq n-1,\ j \text{ odd}\}$ with the indices $i$ (respectively $j$) taken modulo $m$ (respectively $n$), and the face set is $F(\mathbb{KB}^n_{m+\frac{1}{2}}) = \{z_i^j : 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\} \cup \{z_m^j : 0 \leq j \leq n-2,\ j \text{ even}\}$.

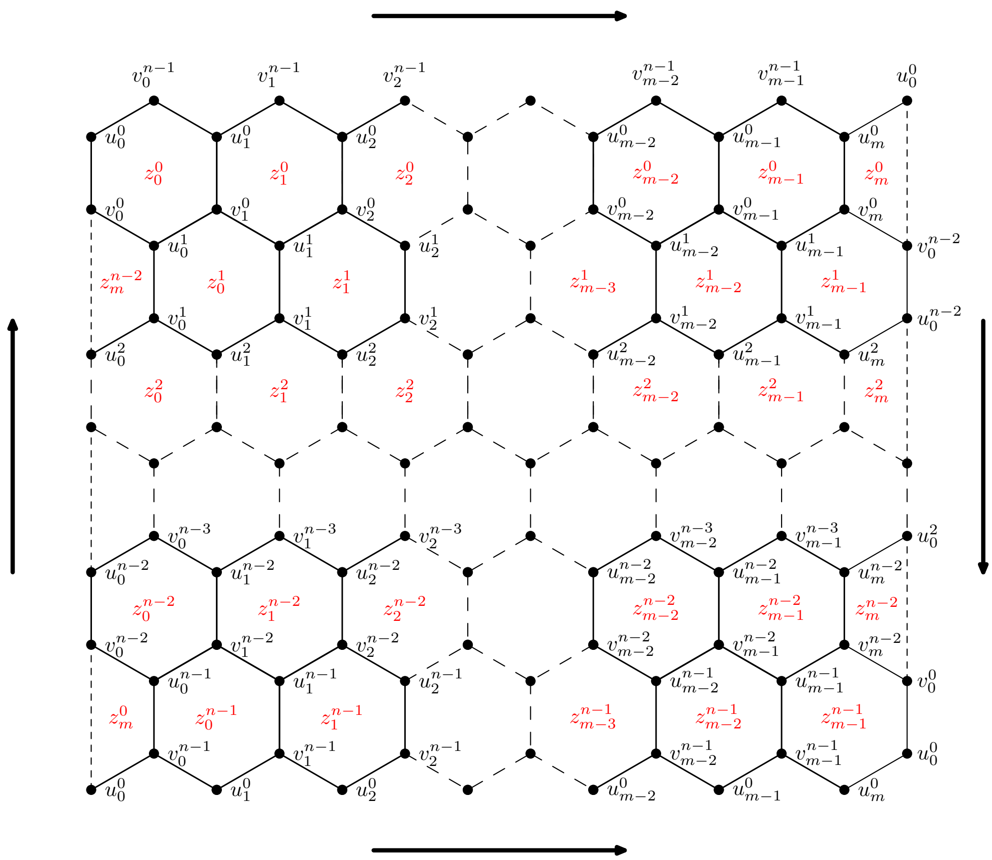


Figure 4.2: Quadrilateral section $P^n_{m+1/2}$ cuts from the regular hexagonal lattice.

The next theorem gives an upper bound for the difference $d$ and we present it without proof.

**Theorem 23.** [41, 92] *For every Klein-bottle polyhex there is no super $d$-antimagic labeling of type $(1,1,1)$ with $d \geq 41$.*

## 4.2 Results for Bipartite Klein-bottle Polyhex

The results on super $d$-antimagic labelings of the bipartite Klein-bottle polyhex $\mathbb{KB}_m^n$ have been presented in [92]. Only for completeness we recall the following theorems.

**Theorem 24.** [41, 92] *For $n$ even, $m, n \geq 2$, the Klein-bottle polyhex $\mathbb{KB}_m^n$ has a super 1-antimagic labeling and a super 3-antimagic labeling of type $(1,1,1)$.*

**Theorem 25.** [41, 92] *For $n$ even, $m, n \geq 2$, the Klein-bottle polyhex $\mathbb{KB}_m^n$ has a super 5-antimagic labeling of type $(1,1,1)$ .*

## 4.3 Results for Non-bipartite Klein-bottle Polyhex

In this section, we will focus in super d-antimagic labelings of type $(1,1,1)$ for the non-bipartite Klein-bottle Polyhex. We start by introducing a partial labeling of this graph. Consider the graph of $\mathbb{KB}_{m+\frac{1}{2}}^n$ for $n$ even, $n \geq 2$, $m \geq 1$ and we construct a vertex labeling in the following way:

$$g(u_i^j) = \begin{cases} (m+\frac{1}{2})j+i+1, & \text{if } 0 \leq i \leq m,\ 0 \leq j \leq n-2,\ j \text{ even} \\ \frac{(2m+1)j+3}{2}+i, & \text{if } 0 \leq i \leq m-1,\ 1 \leq j \leq n-1,\ j \text{ odd} \end{cases}$$

$$g(v_i^j) = \begin{cases} (m+\frac{1}{2})(n+j)+i+1, & \text{if } 0 \leq i \leq m,\ 0 \leq j \leq n-2,\ j \text{ even} \\ \frac{(2m+1)(n+j)+3}{2}+i, & \text{if } 0 \leq i \leq m-1,\ 1 \leq j \leq n-1,\ j \text{ odd.} \end{cases}$$

For $0 \leq i \leq m-1$, we define an edge labeling as follows:

$g(v_i^j u_i^{j+1}) = (m+\frac{1}{2})(4n-j)-i$, if $0 \leq j \leq n-2$, $j$ is even,

$g(v_i^j u_{i+1}^{j+1}) = \frac{(2m+1)(4n-j)-1}{2} - i$, if $1 \leq j \leq n-1$, $j$ is odd,

$g(u_i^j v_i^{j-1}) = (m+\frac{1}{2})(5n-j) - i$, if $0 \le j \le n-2$, $j$ is even,

$g(u_i^j v_{i+1}^{j-1}) = \frac{(2m+1)(5n-j)-1}{2} - i$, if $1 \le j \le n-1$, $j$ is odd

and for $0 \le j \le n-2$ we put:

$g(v_m^j v_0^{n-2-j}) = (m+\frac{1}{2})(4n-j) - m$, if $j$ is even,

$g(u_m^j u_0^{n-j}) = (m+\frac{1}{2})(5n-j) - m$, if $j$ is even.

According to this partial labeling, we introduce the notion of partial weights that is stated blow:

Let $w(z_i^j) = g(u_i^j) + g(u_i^j v_i^{j-1}) + g(v_i^{j-1}) + g(v_i^{j-1} u_{i+1}^j) + g(u_{i+1}^j) + g(v_i^j) + g(v_i^j u_i^{j+1}) + g(u_i^{j+1}) + g(u_i^{j+1} v_{i+1}^j) + g(v_{i+1}^j)$ be a partial weight of the face $z_i^j$ for $j$ even, $0 \le j \le n-2$ and $0 \le i \le m-1$,

let $w(z_m^j) = g(u_m^j) + g(u_m^j u_0^{n-j}) + g(u_0^{n-j}) + g(u_0^{n-j} v_0^{n-j-1}) + g(v_0^{n-j-1}) + g(v_m^j) + g(v_m^j v_0^{n-j-2}) + g(v_0^{n-j-2}) + g(v_0^{n-j-2} u_0^{n-j-1}) + g(u_0^{n-j-1})$ be a partial weight of the face $z_m^j$ for $j$ even, $0 \le j \le n-2$,

$w(z_i^j) = g(u_i^j) + g(u_i^j v_{i+1}^{j-1}) + g(v_{i+1}^{j-1}) + g(v_{i+1}^{j-1} u_{i+1}^j) + g(u_{i+1}^j) + g(v_i^j) + g(v_i^j u_{i+1}^{j+1}) + g(u_{i+1}^{j+1}) + g(u_{i+1}^{j+1} v_{i+1}^j) + g(v_{i+1}^j)$ be a partial weight of the face $z_i^j$ for $j$ odd, $1 \le j \le n-1$ and $0 \le i \le m-2$,

and let $w(z_{m-1}^j) = g(u_{m-1}^j) + g(u_{m-1}^j v_m^{j-1}) + g(v_m^{j-1}) + g(v_m^{j-1} v_0^{n-j-1}) + g(v_0^{n-j-1}) + g(v_{m-1}^j) + g(v_{m-1}^j u_m^{j+1}) + g(u_m^{j+1}) + g(u_m^{j+1} u_0^{n-j-1}) + g(u_0^{n-j-1})$ be a partial weight of the face $z_{m-1}^j$ for $j$ odd, $1 \le j \le n-1$.

With the vertex and edge labeling $g$ in the hand, we prove the following lemma.

**Lemma 2.** [41] *The partial weights of the face $z_i^j$, for $0 \le i \le m$ and $0 \le j \le n-1$, constitute an arithmetic progression of difference 2.*

*Proof.* By a direct computation for the partial weights of the face $z_i^j$ we obtain:

$w_g(z_i^j) =$

$$\begin{cases} (2m+1)(\frac{21n}{2}+j)+2i+8, & \text{for } 0 \le i \le m-1,\ 0 \le j \le n-2,\ j \text{ even} \\ (2m+1)(\frac{21n}{2}+j)+2i+9, & \text{for } 0 \le i \le m-2,\ 1 \le j \le n-1,\ j \text{ odd} \\ (2m+1)(\frac{23n}{2}-j)-2m+5, & \text{for } i = m-1,\ 1 \le j \le n-1,\ j \text{ odd} \\ (2m+1)(\frac{23n}{2}-j)-2m+6, & \text{for } i = m,\ 0 \le j \le n-2,\ j \text{ even.} \end{cases}$$

One can see that the partial weights of the faces $z_i^j$ form an arithmetic progression of difference 2, namely $(2m+1)\frac{21n}{2}+6, (2m+1)\frac{21n}{2}+8, \ldots, (2m+1)\frac{23n}{2}+4$. □

By using the previous lemma we get:

**Theorem 26.** [41] *If $n$ is even, $n \ge 2$, $m \ge 1$, then the Klein-bottle polyhex $\mathbb{KB}^n_{m+\frac{1}{2}}$ has a super $d$-antimagic labeling of type $(1,1,1)$ with $d = 1, 3$.*

*Proof.* Let us consider two cases.

*Case 1.* $d = 1$. Define a labeling for faces $z_i^j$ and edges $u_i^j v_i^j$ of $\mathbb{KB}^n_{m+\frac{1}{2}}$ in the following way:

$h_1(z_i^j) =$

$$\begin{cases} (m+\frac{1}{2})(5n+j)+i+1, & \text{if } 0 \le i \le m,\ 0 \le j \le n-2,\ j \text{ even} \\ \frac{(2m+1)(5n+j)+3}{2}+i, & \text{if } 0 \le i \le m-1,\ 1 \le j \le n-1,\ j \text{ odd} \end{cases}$$

$h_1(u_i^j v_i^j) =$

$$\begin{cases} (m+\frac{1}{2})(3n-j)-i, & \text{if } 0 \le i \le m,\ 0 \le j \le n-2,\ j \text{ even} \\ \frac{(2m+1)(3n-j)-1}{2}-i, & \text{if } 0 \le i \le m-1,\ 1 \le j \le n-1,\ j \text{ odd.} \end{cases}$$

It is easy to see that the partial sums of the face label $z_i^j$ and edge labels $u_i^j v_i^j$ and $u_{i+1}^j v_{i+1}^j$ are

$w_{h_1}(z_i^j) = h_1(z_i^j) + h_1(u_i^j v_i^j) + h_1(u_{i+1}^j v_{i+1}^j) =$

$$\begin{cases} (m+\frac{1}{2})(11n-j)-i, & \text{if } 0 \le i \le m-1,\ 0 \le j \le n-2,\ j \text{ even} \\ \frac{(2m+1)(11n-j)+1}{2}-i-1, & \text{if } 0 \le i \le m-2,\ 1 \le j \le n-1,\ j \text{ odd} \\ \frac{(2m+1)(10n+j+1)}{2}+1, & \text{if } i=m-1,\ 1 \le j \le n-1,\ j \text{ odd} \\ \frac{(2m+1)(10n+j+1)+1}{2}, & \text{if } i=m,\ 0 \le j \le n-2,\ j \text{ even} \end{cases}$$

and they successively attain the values $5n(2m+1)+2, 5n(2m+1)+3, \dots, 11n(m+\frac{1}{2})+1$.

If we join the labelings $g$ and $h_1$, then the resulting labeling is a bijective function from $V(\mathbb{KB}^n_{m+\frac{1}{2}}) \cup E(\mathbb{KB}^n_{m+\frac{1}{2}}) \cup F(\mathbb{KB}^n_{m+\frac{1}{2}})$ onto the set $\{1, 2, \dots, 3n(2m+1)\}$ with vertex labels from 1 up to $n(2m+1)$. Combining the partial weights of the faces $z_i^j$ described above and in Lemma 1 we obtain that

$wt(z_i^j) = w_g(z_i^j) + w_{h_1}(z_i^j) =$

$$\begin{cases} (m+\frac{1}{2})(32n+j)+i+8, & \text{if } 0 \le i \le m-1,\ 0 \le j \le n-2,\ j \text{ even} \\ \frac{(2m+1)(32n+j)-1}{2}+i+9, & \text{if } 0 \le i \le m-2,\ 1 \le j \le n-1,\ j \text{ odd} \\ \frac{(2m+1)(33n-j)+1}{2}-m+6, & \text{if } i=m-1,\ 1 \le j \le n-1,\ j \text{ odd} \\ \frac{(2m+1)(33n-j)}{2}-m+7, & \text{if } i=m,\ 0 \le j \le n-2,\ j \text{ even} \end{cases}$$

and these weights of faces constitute the consecutive integers from $16n(2m+1)+7, 16n(2m+1)+8, \dots, \frac{33n}{2}(2m+1)+6$, which implies that the resulting labeling is a super 1-antimagic labeling of type $(1,1,1)$.

*Case 2.* $d=3$. In this case we define the labeling $h_2$ for faces $z_i^j$ and edges $u_i^j v_i^j$ of $\mathbb{KB}^n_{m+\frac{1}{2}}$ as follows:

$h_2(z_i^j) =$

$$\begin{cases} (m+\frac{1}{2})(6n-j)-i, & \text{if } 0 \le i \le m,\ 0 \le j \le n-2,\ j \text{ even} \\ \frac{(2m+1)(6n-j)-1}{2}-i, & \text{if } 0 \le i \le m-1,\ 1 \le j \le n-1,\ j \text{ odd} \end{cases}$$

$h_2(u_i^j v_i^j) =$

$$\begin{cases} (m+\frac{1}{2})(2n+j)+i+1, & \text{if } 0\le i\le m,\ 0\le j\le n-2,\ j \text{ even}\\ \frac{(2m+1)(2n+j)+3}{2}+i, & \text{if } 0\le i\le m-1,\ 1\le j\le n-1,\ j \text{ odd.}\end{cases}$$

The partial sums of the face label $z_i^j$ and edge labels $u_i^jv_i^j$ and $u_{i+1}^jv_{i+1}^j$ are following:

$$w_{h_2}(z_i^j) = h_2(z_i^j)+h_2(u_i^jv_i^j)+h_2(u_{i+1}^jv_{i+1}^j) =$$
$$\begin{cases} (m+\frac{1}{2})(10n+j)+i+3, & \text{if } 0\le i\le m-1,\ 0\le j\le n-2,\ j \text{ even}\\ \frac{(2m+1)(10n+j)+7}{2}+i, & \text{if } 0\le i\le m-2,\ 1\le j\le n-1,\ j \text{ odd}\\ \frac{(2m+1)(11n-j-1)}{2}+2, & \text{if } i=m-1,\ 1\le j\le n-1,\ j \text{ odd}\\ \frac{(2m+1)(11n-j-1)+5}{2}, & \text{if } i=m,\ 0\le j\le n-2,\ j \text{ even}\end{cases}$$

and successively attain the consecutive integers $5n(2m+1)+2, 5n(2m+1)+3, \ldots, 11n(m+\frac{1}{2})+1$.

By combining the labelings $g$ and $h_2$ gives a bijection from the set of vertices, edges and faces of $\mathbb{KB}^n_{m+\frac{1}{2}}$ onto the set $\{1,2,\ldots,3n(2m+1)\}$ with the smallest possible labels on the vertices and for sum of the partial weights of the face $z_i^j$ we have

$$wt(z_i^j) = w_g(z_i^j)+w_{h_2}(z_i^j) =$$
$$\begin{cases} (m+\frac{1}{2})(31n+3j)+3i+11, & \text{if } 0\le i\le m-1,\ 0\le j\le n-2,\ j \text{ even}\\ \frac{(2m+1)(31n+3j)-1}{2}+3i+13, & \text{if } 0\le i\le m-2,\ 1\le j\le n-1,\ j \text{ odd}\\ \frac{(2m+1)(34n-3j-1)}{2}-2m+7, & \text{if } i=m-1,\ 1\le j\le n-1,\ j \text{ odd}\\ \frac{(2m+1)(34n-3j-1)+1}{2}-2m+8, & \text{if } i=m,\ 0\le j\le n-2,\ j \text{ even.}\end{cases}$$

It is a routine procedure to verify that these weights of faces constitute an arithmetic sequence with the first term $31n(m+\frac{1}{2})+8$ and common difference $d=3$. This completes the proof . $\square$

# Chapter 5

# On 3-total Edge Product Cordial Labeling of Honeycomb

For $n \geq 1$, $m \geq 1$ we denote by $H_n^m$ the hexagonal grid (honeycomb) as the planar graph with $m$ rows and $n$ columns of hexagons. Thus the hexagonal grid contains $2mn+2(m+n)$ vertices, $3mn+2(m+n)-1$ edges, $mn$ 6-sided faces and one external infinite face. Figure 5.1 illustrates the honeycomb $H_n^m$ for $n$ even.

Next theorem shows that $H_1^m$ and $H_n^1$ admit 3-total edge product cordial labeling for $m, n \geq 2$. Let us note that by *open edge* we will mean the edge with only one end vertex.

**Theorem 27.** [5] *The graphs $H_1^m$ and $H_n^1$ are 3-total edge product cordial for $m \geq 2$, $n \geq 2$.*

*Proof.* Let $m = 1$.

Let us consider that $H_1^1$ admits 3-total edge product cordial labeling $\varphi$ and let $\varphi^*$ be the induced vertex labeling. Then, as $H_1^1$ is of order 6 and size 6, then in the set of

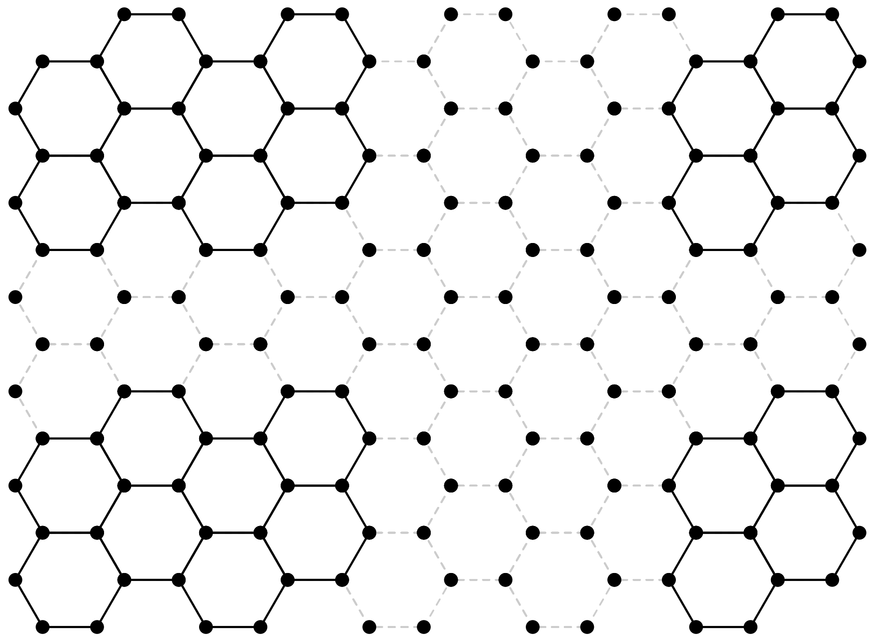

Figure 5.1: The honeycomb $H_n^m$.

all edge labels and induced vertex labels every number of 0, 1 and 2 must be used exactly 4 times. Vertex label of a vertex $v$ can be 0 only if at least one edge incident with $v$ is labeled with 0. However, the label of the other end vertex of this edge will be labeled with 0 in this case. This leads to contradiction that number of elements labeled with 0 must be 4.

Let $m \geq 2$.

A 3-total edge product cordial labeling of $H_1^2 \cong H_2^1$ is depicted in Figure 5.2. Note that this labeling has property that every number of 0, 1 and 2 is used exactly 7 times.

Let us consider the labeled segment $V_1$ illustrated in Figure 5.3. It is possible to glue this segment to the graph $H_1^2$ such that the resulting graph is $H_1^3$. As the open edges in the segment $V_1$ are labeled with number 1, by gluing these two graphs we do

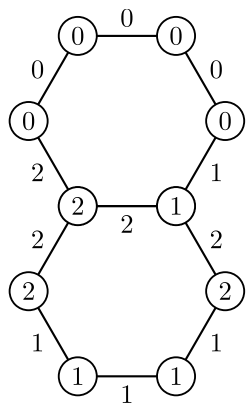


Figure 5.2: The 3-total edge product cordial labeling of $H_1^2 \cong H_2^1$.

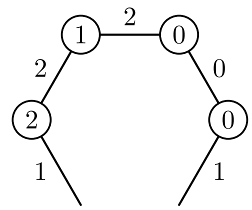


Figure 5.3: The labeled segment $V_1$.

not change the vertex labels in the graph $H_1^2$. Moreover, the segment $V_1$ is labeled with exactly 3 zeros, 3 ones and 3 twos. Thus the resulting labeling of the graph $H_1^3$ is 3-total edge product cordial.

Using the same arguments, we get that it is possible to glue the segment $V_1$ to the graph $H_1^2$ arbitrarily and the resulting graph $H_1^m$ will be also 3-total edge product cordial. Moreover, we can generalize this operation. It is possible to glue arbitrarily the labeled segment $V_1$ to any 3-total edge product cordial graph and the resulting graph will be again 3-total edge product cordial.

Thus we get that the graphs $H_1^m$ and $H_n^1$ are 3-total edge product cordial for $m \geq 2$, $n \geq 2$. Symbolically we can get

$$H_1^m = H_1^2 + (m-2)V_1.$$

Moreover, the constructed labeling uses every number of 0, 1 and 2 exactly $3m+1$ times as a label . □

Next we extend the 3-total edge product cordial labeling of $H_1^m$ and $H_n^1$ by gluing the special segments which preserves the requested labeling.

**Theorem 28.** [5] *The graph* $H_n^m$ *is* 3*-total edge product cordial for* $m \equiv 1 \pmod 3$*,* $n \geq 1$*,* $(n, m) \neq (1, 1)$*.*

*Proof.* Let $m = 1$.

By Theorem 27, The graph $H_n^1$, $n \geq 2$, is a 3-total edge product cordial.

Now let us consider the labeled segment $H_1$, which is isomorphic to segment $V_1$, and the labeled segment $H_3$ illustrated in Figures 5.4 and 5.5. The labeled segment

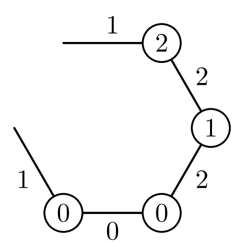


Figure 5.4: The labeled segment $H_1$.

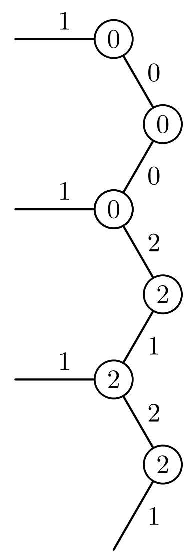


Figure 5.5: The labeled segment $H_3$.

$H_3$ contains every number of 0, 1 and 2 exactly 5 times as the label. Moreover, its open edges are labeled with 1 thus by gluing this segment to any 3-total edge product cordial graph/segment the resulting graph/segment will be again 3-total edge product cordial. We construct a segment $H_{1+3k}$ by gluing vertically one segment $H_1$ and $k$ segments $H_3$. The resulting segment $H_{1+3k}$ will have the following properties:

1. every number of 0, 1 and 2 is used exactly $3 + 5k$ times as a label,

2. every open edge is labeled by 1.

Let $m \geq 4$, $m \equiv 1 \pmod 3$. Then $m = 1 + 3k$, $k \geq 1$. Using Theorem 27 we get that by gluing the graph $H_1^{1+3k}$ with the segment $H_{1+3k}$, the resulting graph $H_2^{1+3k}$ is 3-total edge product cordial. Moreover, the constructed labeling uses every number of 0, 1 and 2 exactly $5m + 2 - \frac{m-1}{3}$ times as a label.

It is easy to see that by gluing $H_2^m$ with a vertically reflected segment $H_{1+3k}$ we can get that the graph $H_3^m$, where $m \geq 4$, $m \equiv 1 \pmod 3$, is 3-total edge product cordial and every number of 0, 1 and 2 is used $7m + 3 - \frac{2(m-1)}{3}$ times as a label.

Continuing by the same manner we can obtain that the graph $H_n^m$ is 3-total edge product cordial for $m \geq 4$, $m \equiv 1 \pmod 3$, $n \geq 1$, and every number of 0, 1 and 2 is used exactly $m(2n+1) + n - \frac{(n-1)(m-1)}{3}$ times as a label. □

In the next theorem we prove that the graph $H_n^m$ is 3-total edge product cordial also for $m \equiv 2 \pmod 3$, $n \geq 1$. We are able to prove it by constructing desired labeling by gluing the graph $H_n^{m-1}$ with a gluing segment $V_n$.

**Theorem 29.** [5] *The graph $H_n^m$ is 3-total edge product cordial for $m \equiv 2 \pmod 3$, $n \geq 1$.*

*Proof.* Let us consider the labeled segments $P_1$, $P_3$, $P_5$, $P_6$ and $P_3^E$ illustrated in Figures from 5.6 through 5.10.

Every open edge of these segments is labeled by 1. In Table 5.1 it is given how many times the numbers 0, 1 and 2 are used as edge or vertex labels in these segments.

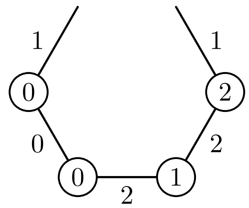

Figure 5.6: The labeled segment $P_1$.

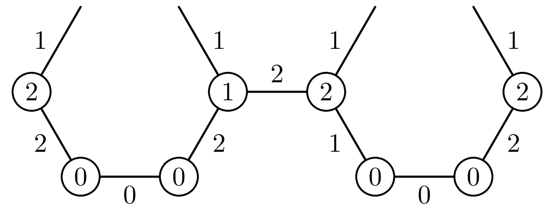

Figure 5.7: The labeled segment $P_3$.

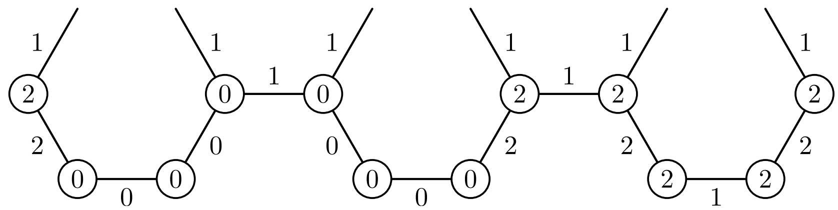

Figure 5.8: The labeled segment $P_5$.

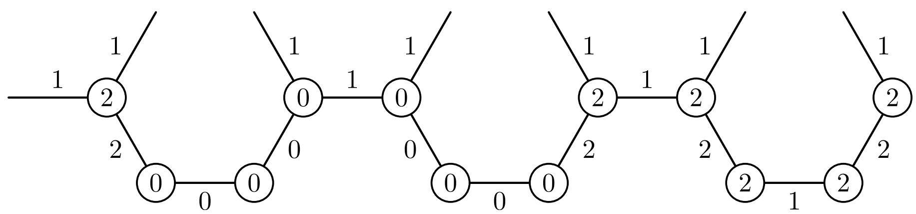

Figure 5.9: The labeled segment $P_6$.

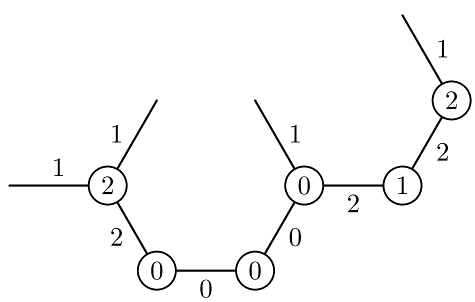

Figure 5.10: The labeled segment $P_3^E$.

Let $\oplus_v$ be the symbol denoting the operation of gluing two segments/graphs in vertical direction and analogously let $\oplus_h$ be the symbol denoting the operation of

| segment | # of 0s | # of 1s | # of 2s |
|---|---|---|---|
| $P_1$ | 3 | 3 | 3 |
| $P_3$ | 6 | 6 | 7 |
| $P_5$ | 10 | 9 | 10 |
| $P_6$ | 10 | 10 | 10 |
| $P_3^E$ | 5 | 5 | 5 |

Table 5.1: Numbers of 0s, 1s and 2s in the segments $P_1$, $P_3$, $P_5$, $P_6$ and $P_3^E$.

gluing two segments/graphs in horizontal direction.

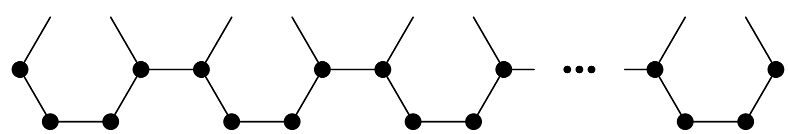

Figure 5.11: The segment $V_n$ for $n$ odd.

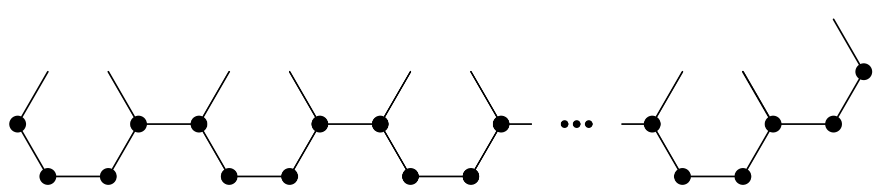

Figure 5.12: The segment $V_n$ for $n$ even.

Table 5.2 illustrates how to construct a big segment $V_n$ by gluing small segments in horizontal direction. Figure 5.11 and 5.12 shows the segment $V_n$ for $n$ odd and $n$ even, repectively.

| $n$ | construction | # of 0s | # of 1s | # of 2s |
|---|---|---|---|---|
| $n = 6t + 1,\ t \geq 0$ | $V_n = P_1 \oplus_v t \cdot P_6$ | $10t + 3$ | $10t + 3$ | $10t + 3$ |
| $n = 6t + 2,\ t \geq 1$ | $V_n = P_5 \oplus_v (t - 1) \cdot P_6 \oplus_v P_3^E$ | $10t + 5$ | $10t + 4$ | $10t + 5$ |
| $n = 6t + 3,\ t \geq 0$ | $V_n = P_3 \oplus_v t \cdot P_6$ | $10t + 6$ | $10t + 6$ | $10t + 7$ |
| $n = 6t + 4,\ t \geq 0$ | $V_n = P_1 \oplus_v t \cdot P_6 \oplus_v P_3^E$ | $10t + 8$ | $10t + 8$ | $10t + 8$ |
| $n = 6t + 5,\ t \geq 0$ | $V_n = P_5 \oplus_v t \cdot P_6$ | $10t + 10$ | $10t + 9$ | $10t + 10$ |
| $n = 6t + 6,\ t \geq 0$ | $V_n = P_3 \oplus_v t \cdot P_6 \oplus_v P_3^E$ | $10t + 11$ | $10t + 11$ | $10t + 12$ |

Table 5.2: Construction of the segment $V_n$.

Moreover, for $n = 2$ the segment $V_2$ is illustrated in Figure 5.13.

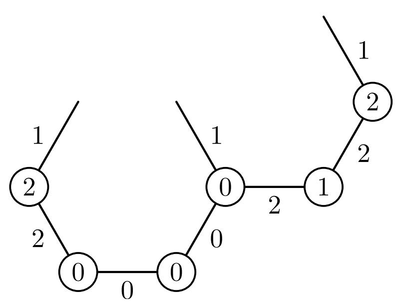

Figure 5.13: The labeled segment $V_2$.

Thus, as every open edge of $V_n$, $n \geq 1$, is labeled by 1, according to Theorem 28 we get that if for $m \equiv 2 \pmod 3$ we glue horizontally the graph $H_n^{m-1}$ with a gluing segment $V_n$ then the resulting graph $H_n^m$ will be 3-total edge product cordial. This concludes the proof. □

Now we prove that the graph $H_n^m$ is 3-total edge product cordial also for $m \equiv 0 \pmod 3$, $n \geq 1$.

**Theorem 30.** [5] *The graph $H_n^m$ is 3-total edge product cordial for $m \equiv 0 \pmod 3$, $n \geq 1$.*

*Proof.* Let us consider the labeled segments $P_3^2$ and $P_5^2$ illustrated in Figures 5.14 and 5.15.

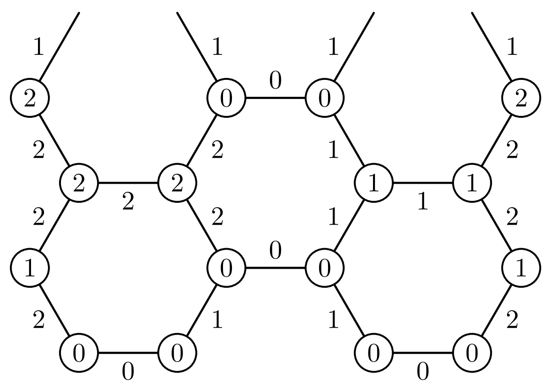


Figure 5.14: The labeled segment $P_3^2$.

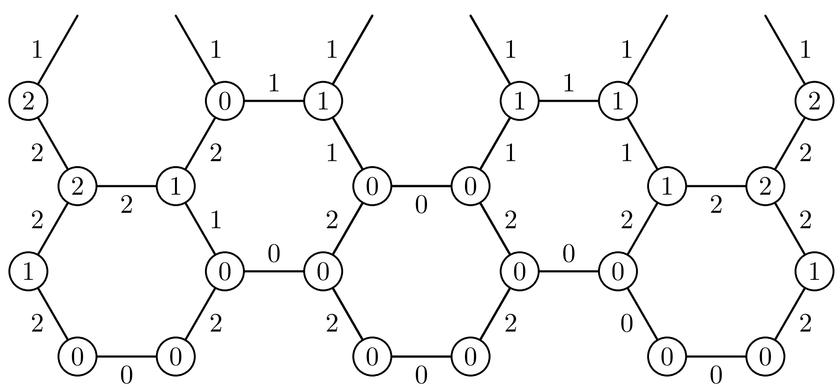


Figure 5.15: The labeled segment $P_5^2$.

Every open edge of these segments is labeled by 1. In Table 5.3 it is shown how many times the numbers 0, 1 and 2 are used as edge and vertex labels in these segments.

| segment | # of 0s | # of 1s | # of 2s |
|---|---|---|---|
| $P_3^2$ | 12 | 13 | 13 |
| $P_5^2$ | 20 | 19 | 19 |

Table 5.3: Numbers of 0s, 1s and 2s in the segments $P_3^2$ and $P_5^2$.

By gluing small segments we will construct a big segment $V_n^2$ in the following way, see Table 5.4.

Moreover, for $n = 2$ the segment $V_2^2$ is illustrated in Figure 5.16.

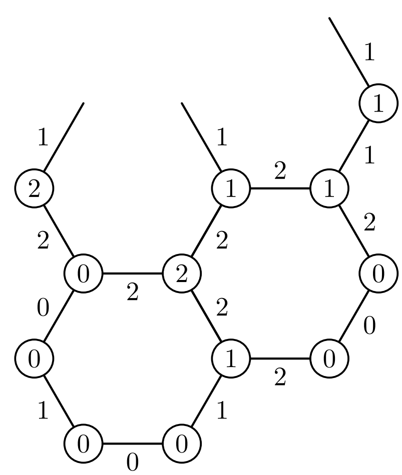

Figure 5.16: The labeled segment $V_2^2$.

Thus, as every open edge of $V_n^2$, $n \geq 1$, is labeled by 1, with respect to Theorem 28 we get that if for $m \equiv 0 \pmod 3$ we glue horizontally the graph $H_n^{m-2}$ with a gluing segment $V_n^2$ then the resulting graph $H_n^m$ will be 3-total edge product cordial. □

Combining the previous theorems we immediately obtain the main theorem.

**Theorem 31.** [5] *The graph* $H_n^m$ *is* 3*-total edge product cordial for every* $n \geq 1$, $m \geq 1$, $(n, m) \neq (1, 1)$ .

| $n$ | construction | # of 0s | # of 1s | # of 2s |
|---|---|---|---|---|
| $n = 6t+1,\ t \geq 0$ | $V_n^2 = \begin{bmatrix} V_n \\ \oplus_h \\ V_n \end{bmatrix}$ | $20t+6$ | $20t+6$ | $20t+6$ |
| $n = 6t+2,\ t \geq 1$ | $V_n^2 = P_5^2 \oplus_v \begin{bmatrix} (t-1) \cdot P_6 \oplus_v P_3^E \\ \oplus_h \\ (t-1) \cdot P_6 \oplus_v P_3^E \end{bmatrix}$ | $20t+10$ | $20t+9$ | $20t+9$ |
| $n = 6t+3,\ t \geq 0$ | $V_n^2 = P_3^2 \oplus_v \begin{bmatrix} t \cdot P_6 \\ \oplus_h \\ t \cdot P_6 \end{bmatrix}$ | $20t+12$ | $20t+13$ | $20t+13$ |
| $n = 6t+4,\ t \geq 0$ | $V_n^2 = \begin{bmatrix} V_n \\ \oplus_h \\ V_n \end{bmatrix}$ | $20t+16$ | $20t+16$ | $20t+16$ |
| $n = 6t+5,\ t \geq 0$ | $V_n^2 = P_5^2 \oplus_v \begin{bmatrix} t \cdot P_6 \\ \oplus_h \\ t \cdot P_6 \end{bmatrix}$ | $20t+20$ | $20t+19$ | $20t+19$ |
| $n = 6t+6,\ t \geq 0$ | $V_n^2 = P_3^2 \oplus_v \begin{bmatrix} t \cdot P_6 \oplus_v P_3^E \\ \oplus_h \\ t \cdot P_6 \oplus_v P_3^E \end{bmatrix}$ | $20t+22$ | $20t+23$ | $20t+23$ |

Table 5.4: Construction of the segment $V_n^2$.

# Chapter 6

# Conclusion

In Chapter 1, we provided the basic terminology of graph theory, recalled some definitions and notations of certain families of graphs that are used throughout the thesis and presented some preliminary results from the graph theory.

In Chapter 2, we gave an overview of certain types of labelings, namely magic types of labelings, antimagic types of labelings and lastly irregular labelings and cordial labelings. Moreover we reviewed some known results regarding these labelings.

In Chapter 3, we estimated the lower and upper bounds of the entire face irregularity strength for disjoint union of multiple copies of a plane graph and determined the precise value for disjoint union of $m$ copies of the ladder $L_n$, $n \equiv 1 \pmod 9$, and for disjoint union of $m$ copies of the wheel $W_7$. These two cases prove the sharpness of the lower bound of the entire face irregularity strength for disjoint union of multiple copies of a plane graph .

In Chapter 4, we studied the existence of super $d$-antimagic labelings of Klein-bottle polyhex. We have shown that for $n$ even, $n \geq 2$, $m \geq 1$, the non-bipartite

Klein-bottle polyhex $\mathbb{KB}^n_{m+\frac{1}{2}}$ admits a super $d$-antimagic labeling of type $(1,1,1)$ with $d = 1, 3$. We have tried to find a super $d$-antimagic labeling of type $(1,1,1)$ also for differences $d = 0, 2$ but so far without success. So, we propose the following open problem.

**Open Problem 1.** *For the non-bipartite Klein-bottle polyhex* $\mathbb{KB}^n_{m+\frac{1}{2}}$*, $n$ even, $n \geq 2$, $m \geq 1$, determine if there is a super $d$-antimagic labeling of type* $(1,1,1)$ *with* $d = 0, 2$.

In Chapter 5, we described a construction how to obtain 3-total edge product cordial labeling of the hexagonal grid $H^m_n$ from a smaller hexagonal grid. The used method of construction of labeling has several advantages. Firstly, we are able to obtain the 3-total edge product cordial labeling of any hexagonal grid. Secondly, it is possible to use this method also for another grids and also for another $k$-total edge product cordial labelings.

# Appendix

# Graph-theoretic symbols

| | |
|---|---|
| $V(G)$ | vertex set of $G$ |
| $E(G)$ | edge set of $G$ |
| $F(G)$ | face set of $G$ |
| $\|V(G)\| = n$ | order of $G$ |
| $\|E(G)\| = m$ | size of $G$ |
| $\|F(G)\| = f$ | number of faces of a plane graph $G$ |
| $d_G(x)$ | degree of the vertex $x$ (in $G$) |
| $N(x)$ | set of all neighbors of the vertex $x$ |
| $\delta(G)$ | minimum degree of $G$ |
| $\Delta(G)$ | maximum degree of $G$ |
| $P_n$ | path on $n$ vertices |
| $C_n$ | cycle on $n$ vertices |
| $K_n$ | complete graph on $n$ vertices |
| $S_n$ | sun graph on $2n$ vertices |
| $K_{s,t}$ | complete bipartite graph with partite sets of cardinalities $s$ and $t$ |
| $K_{1,s}$ | star graph |
| $B_{r,s}$ | bistar or double star |

| | |
|---|---|
| $f_n$ | friendship graph on $n$ triangles |
| $P(n,m)$ | generalized Petersen graph |
| $W_n$ | wheel on $n+1$ vertices |
| $F_n$ | fan graph on $n+1$ vertices |
| $mG$ | disjoint union of $m$ copies of a graph $G$ |
| $G\Box H$ | Cartesian product of graphs $G$ and $H$ |
| $D_n$ | prism $C_n\Box P_2$ |
| $A_n$ | antiprism |
| $H \subseteq G$ | graph $H$ is a subgraph of a graph $G$ |
| $L_n$ | Ladder |
| $mL_n$ | disjoint union of $m$ copies of $L_n$ |
| $\mathbb{KB}_m^n$ | the bipartite Klein-bottle polyhex |
| $\mathbb{KB}_{m+\frac{1}{2}}^n$ | the non-bipartite Klein-bottle polyhex |
| $H_n^m$ | hexagonal grid(honeycomb graph) |

# Bibliography


[1] G. Ali, M. Bača, F. Bashir and A. Semaničová-Feňovčíková, On face antimagic labelings of disjoint union of prisms, *Utilitas Math.* **85** (2011), 97–112.

[2] M. Aigner and E. Triesch, Irregular assignments of trees and forests, *SIAM J. Discrete Math.* **3** (1990), 439-449.

[3] A. Ahmad, O. B. S. Al-Mushayt and M. Bača, On edge irregularity strength of graphs, *Appl. Math. Comput.* **243** (2014), 607-610.

[4] A. Ahmad and M. Bača, Total edge irregularity strength of a categorical product of two paths, *Ars Combin.* **114** (2014), 203-212.

[5] A. Ahmad, M. Bača, M. Naseem and A. Semaničová-Feňovčíková, On 3-total edge product cordial labeling of honeycomb, *AKCE Internat. J. Graphs Combin*, submitted.

[6] A. Ahmad, M. Bača and M. K. Siddiqui, On edge irregular total labeling of categorical product of two cycles, *Theory Comp. Systems.* **54**(1) (2014), 1-12.

[7] A. Ahmad, M. Bača, Y. Bashir and M. K. Siddiqui, Total edge irregularity strength of strong product of two paths, *Ars Combin.* **106** (2012), 449-459.

[8] G. Ali, M. Bača, Y. Lin and A. Semaničová-Feňovčíková, Super vertex-antimagic labelings of disconnected graphs, *Discrete Math.* **309** (2009), 6048–6054.

[9] N. Alon, G. Kaplan, A. Lev, Y. Roditty and R. Yuster, Dense graphs are antimagic, *J. Graph Theory.* **47** (2004), 297-309.

[10] D. Amar and O. Togni, Irregularity strength of trees, *Discrete Math.* **190** (1998), 15-38.

[11] M. Anholcer, M. Kalkowski and J. Przybylo, A new upper bound for the total vertex irregularity strength of graphs, *Discrete Math.* **309** (2009), 6316-6317.

[12] M. Anholcer and C. Palmer, Irregular labellings of Circulant graphs, *Discrete Math.* **312** (2012), 3461-3466.

[13] A. Azaizeh, R. Hasni, A. Ahmad and G.C. Lau, 3-total edge product cordial labeling of graphs, *Far East J. Math. Sci.* **96(2)** (2015), 193–209.

[14] M. Bača, Consecutive-magic labeling of generalized Petersen graphs, *Utilitas Math.* **58** (2000), 237–241.

[15] M. Bača, On certain properties of magic graphs, *Utilitas Math.* **37** (1990), 259-264.

[16] M. Bača, On magic labelings of honeycomb, *Discrete Math.* **105** (1992), 305–311.

[17] M. Bača, E.T. Baskoro, S. Jendrol̆ and M. Miller, Antimagic labelings of hexagonal planar maps, *Utilitas Math.* **66** (2004) 231–238.

[18] M. Bača, F. Bertault, J.A. MacDougall, M. Miller, R. Simanjuntak and Slamin, Vertex-antimagic total labelings of graphs, *Discuss. Math. Graph Theory.* **23**(2003), 67–83.

[19] M. Bača and I. Holländer, On (a, d)-antimagic prisms, *Ars Combin.* **48** (1998), 297-306.

[20] M. Bača, Labelings of m-antiprisms, *Ars Combin.* **28** (1989), 242–245.

[21] M. Bača, On magic and consecutive labelings for the special classes of plane graphs, *Utilitas Math.* **32** (1987), 59–65.

[22] M. Bača, Labelings of two classes of convex polytopes, *Utilitas Math.* **39** (1988), 24–31.

[23] M. Bača, On magic labelings of Möbius ladders, *J. Franklin Inst.* **326** (1989), 885–888.

[24] M. Bača, On magic labelings of type (1,1,1) for three classes of plane graphs, *Math. Slovaca.* **39** (1989), 233–239.

[25] M. Bača and M. Miller, On d-antimagic labelings of type (1,1,1) for prisms, *J.Combin. Math. Combin. Comput.* **44** (2003), 199–207.

[26] M. Bača, S. Jendrol', M. Miller and J. Ryan, Antimagic labelings of generalized Petersen graphs that are plane, *Ars Combin.* **73** (2004), 115–128.

[27] M. Bača, E.T. Baskoro, S. Jendrol' and M. Miller, Antimagic labelings of hexagonal planar maps, *Utilitas Math.* **66** (2004), 231–238.

[28] M. Bača, F. Bashir and A. Semaničová-Feňovčíková, Face antimagic labelings of antiprisms, *Utilitas Math.* **84** (2011), 209–224.

[29] M. Bača, L. Brankovic and A. Semaničová-Feňovčíková, Labelings of plane graphs containing Hamilton path, *Acta Math. Sinica (Engl. Ser.)* **27** (4) (2011),701–714.

[30] M. Bača, M. Lascsáková, M. Naseem and A. Semaničová-Feňovčíková, On entire face irregularity strength of disjoint union of plane graphs, it Applied Mathematics and Computation, submitted.

[31] M. Bača, M. Miller, O. Phanalasy and A. Semaničová-Feňovčíková, Super $d$-antimagic labelings of disconnected plane graphs, *Acta Math. Sinica - English Series.* **26** (12) (2010), 2283–2294.

[32] M. Bača, Y. Lin, and M. Miller, Antimagic labelings of grids, *Util. Math.* **72** (2007)65–75.

[33] M. Bača, On magic labelings of grid graphs, *Ars Combin.* **33** (1992), 295–299.

[34] M. Bača, S. Jendroľ, K. Kathiresan and K. Muthugurupackiam, On the face irregularity strength, *Appl. Math. Inf. Sci.* **9**(1) (2015), 263-267.

[35] M. Bača, S. Jendroľ, K. Kathiresan, K. Muthugurupackiam, A. Semaničová-Feňovčíková, A survey of irregularity strength, *Electronic Notes Discrete Math.* **48** (2015) 19-26.

[36] M. Bača, S. Jendroľ, M. Miller and J. Ryan, On irregular total labellings, *Discrete Math.* **307** (2007), 1378-1388.

[37] M. Bača, Y. Lin and M. Miller, Antimagic labelings of grids, *Utilitas Math.* **72** (2007) 65–75.

[38] M. Bača, On magic labelings of convex polytopes, *Annals Discrete Math.* **51** (1992), 13–16.

[39] M. Bača, On magic labelings of type (1,1,1) for the special class of plane graphs, *J. Franklin Inst.* **329** (1992), 549–553.

[40] M. Bača, Labelings of two classes of plane graphs, *Acta Math. Appl. Sinica 9.* **1** (1993), 82–87.

[41] M. Bača, M. Naseem and A. Shabbir, Face labelings of Klein-bottle fullerenes, *Acta Math. Appl. Sinica*, appear.

[42] M. Bača and M.K. Siddiqui, Total edge irregularity strength of generalized prism, *Applied Math. Comput.* **235** (2014), 168-173.

[43] O. Berkman, M.Parnas and Y. Roditty, All cycle are edge magic, *Ars Combin.* **59** (2001), 145–151.

[44] G.S Bloom and S.W.Golomb, Numbered complete graphs,usual rules and assorted applications, In: *Theory and application of graphs, Lecture Notes in Math.* **642**. Springer-Verlag (1978), 53–65

[45] G.S Bloom and S.W.Golomb, Application of numbered undirected graphs, *Proc. IEEE* **65***(1977)*, 562–570.

[46] R. Bodendiek and G. Walther, Arithmatisch antimagische Graphen, In: *K. Wagner and R. Bodendiek, eds. Graphentheorie III, BI-Wiss. Verl. Mannheim,* 1993.

[47] R. Bodendiek and G. Walther, On number theoretical methods in graph labeling. *Res. Exp. Math.* **21** 3–25.

[48] R. Bodendiek and G. Walther, On arithmetic antimagic edge labelings of graphs. *Mitt. Math. Ges. Hamburg.* **17**(1998),85–99.

[49] T. Bohman and D. Kravitz, On the irregularity strength of trees, *J. Graph Theory.* **45** (2004), 241-254.

[50] J. A. Bondy, U. S. R. Murty, Graph Theory, Springer, 2008.

[51] S. Brandt, J. Miskuf and D. Rautenbach, On a conjecture about edge irregular total labellings, *J. Graph Theory.* **57** (2008), 333-343.

[52] I. Cahit, On cordial and 3-equitable labelings of graphs, *Utilitas Math.* **37** (1990), 189-198.

[53] I. Cahit, Cordial graphs: a weaker version of graceful and harmonious graphs, *Ars Combin.* **23** (1987), 201–207.

[54] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz and F. Saba, Irregular networks, *Congr. Numer.* **64** (1988), 187-192.

[55] G. Chartrand and P. Zhang, Introduction to graph theory, McGraw-Hill, New York, 2005.

[56] K.L. Collins and M. Hovey, Most graphs are edge-cordial, *Ars Combin.* **30** (1990), 289-295.

[57] M. Deza, P.W. Fowler, A. Rassat, K.M. Rogers, Fullerenes as tilings of surfaces, *J. Chem. Inf. Comput. Sci.* **40** (2000) 550–558.

[58] H. Enomoto, A.S. Lladó, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (1998), 105-109.

[59] R.J. Faudree, M. S. Jacobson, J. Lehel and R.H. Schlep, Irregular networks, regular graphs and integer matrices with distinct row and column sums, *Discrete Math.* **76** (1988), 223-240.

[60] J. Gallian, A dynamic survey of graph labeling. *The Electronic Journal of Combinatorics.* **14**(2007).

[61] R.D. Godbold and P.J. Slater, All cycles are edge magic. *Bull. Inst. Combin. Appl.* **22**(1998) 93–97.

[62] J. L. Gross, J. Yellen, Graph theory and its applications, Chapman- Hall, 2006.

[63] K.M.M. Haque, Irregular total labellings of generalized Petersen graphs, *Theory Comp. Systems.* **50** (2012), 537-544.

[64] N. Hartsfield and G. Ringel, Pearls in Graph Theory, *Academic Press, Boston - San Diego - New York - London*, 1990.

[65] Y.S. Ho, S.M. Lee and S.C. Shee, Cordial labelings of the Cartesian product and composition of graphs, *Ars Combin.* **29** (1990), 169-180.

[66] Y.S. Ho and S.C. Shee, The cordiality of one-point union of $n$ copies of a graph, *Discrete Math.* **117** (1993), 225-243.

[67] M. Hovey, A-cordial graphs, *Discrete Math.* **93** (1991), 183-194.

[68] J. Ivančo , On supermagic regular graphs, *Math. Bohemica.* **125** (2000), 99–114.

[69] J. Ivančo and S. Jendroľ, Total edge irregularity strength of trees, *Discussiones Math. Graph Theory.* **26** (2006), 449-456.

[70] S. Jendroľ, J. Miškuf and R. Soták, Total edge irregularity strength of complete and complete bipartite graphs, *Electron. Notes Discrete Math.* **28** (2007), 281-285.

[71] S. Jendroľ, J. Miškuf and R. Soták, Total edge irregularity strength of complete graphs and complete bipartite graphs, *Discrete Math.* **310** (2010), 400-407.

[72] R.H. Jeurissen, Magic Graphs, a Characterization, Mathematisch Instituut Universiteit Toernooiveld, 6525 ED Nijmegen (1982), The Netherlands.

[73] R.H. Jeurissen, Magic graphs, a characterization, *Europ. J. Combin.* **9** (1988), 363-368.

[74] S. Jezný and M. Trenkler, Characterization of magic graphs, Czechoslovak *Math. J.* **33** (1983), 435-438.

[75] K. Kathiresan and S. Gokulakrishnan, On magic labelings of type (1,1,1) for the special classes of plane graphs, *Utilitas Math.* **63** (2003), 25–32.

[76] W.W. Kirchherr, On the cordiality of some specific graphs, *Ars Combin.* **31** (1991), 127-137.

[77] W.W. Kirchherr, NEPS operations on cordial graphs, *Discrete Math.* **115** (1993), 201-209.

[78] D.J. Klein, Elemental benzenoids, *J. Chem. Inf. Comput. Sci.* **34** (1994) 453–459.

[79] P.Kovar, Magic Labeling of a Graphs, P.hD Thesis, Faculty of Electrical Engineering and Computer Science, VSB-Technical University of Ostrava, 2004.

[80] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970) 451–461.

[81] D. Kuo, G.J. Chang and Y.H.H. Kwong, Cordial labeling of $mK_n$, *Discrete Math.* **169** (1997), 121-131.

[82] S.M. Lee and A. Liu, A construction of cordial graphs from smaller cordial graphs, *Ars Combin.* **32** (1991), 209-214.

[83] J. Lehel, Facts and quests on degree irregular assignments, Graph Theory, Combinatorics and Applications, Willey, New York, 1991, pp. 765–782.

[84] K.W Lih, On magic and consecutive labelings of plane graphs, *Util. Math.* **24** (1983) 165–197.

[85] J.A. MacDougall, M. Miller, Slamin and W.D. Wallis, Vertex-magic total labelings of graphs, *Utilitas Math.* **61** (2002), 3-21.

[86] J.A. MacDougall, M. Miller and W.D. Wallis, Vertex-magic total labelings of wheels and related graphs, *Utilitas Math.* **62** (2002), 175-183.

[87] P. Majerski, J. Przybylo, Total vertex irregularity strength of dense graphs, *J. Graph Theory.* **76** (1) (2014) 34–41.

[88] C.C. Marzuki, A. N. M. Salman and M. Miller, On the total irregularity strength on cycles and paths, preprint.

[89] J. Miškuf and S. Jendrol, On total edge irregularity strength of the grids, *Tatra Mt. Math. Publ.* **36** (2007), 147151.

[90] J. Muhlbacher, Magische Quadrate und ihre Verallgemeinerung: ein Graphen-theoretisches Problem, Graph, Data Structures, Algorithms, Hansen Verlag1979, München.

[91] F.A. Muntaner-Batle, Magic Graphs, *Ph.D Thesis*, Department de Matemática Aplicada IV, Universitat Politécnica de Catalunya, Barcelona, 2001

[92] M. Naseem, Colourings and labelings of graphs on the torus and the Klein-bottle, *MPhill Thesis*, ASSMS GC University Lahore (2014).

[93] T.Nicholas, S. Samasundaram and V. Vilfred, On (a,d)-antrimagic special trees, unicyclic graphs and complete bipartite graphs, *Ars Cobin.* **70**(2004). 207–220.

[94] T. Nierhoff, A tight bound on the irregularity strength of graphs, *SIAM J. Discrete Math.* **13** (2000), 313-323.

[95] Nurdin, A.N.M. Salman and E.T. Baskoro, The total edge-irregular strengths of the corona product of paths with some graphs, *J. Combin. Math. Combin. Comput.* **65** (2008), 163-175.

[96] J. Przybylo, Linear bound on the irregularity strength and the total vertex irregularity strength of graphs, *SIAM J. Discrete Math.* **23** (2009), 511-516.

[97] R. Ramdani and A.N.M. Salman, On the total irregularity strength of some Cartesian product graphs, *AKCE Int. J. Graphs Comb.* **10**(2) (2013), 199-209.

[98] G. Ringel and A. Llado, Another tree conjecture, *Bull. Inst. Combin. Appl.* **18** (1996) 83–85.

[99] Y. Roditty, A note on edge magic cycles. *Bull. Inst. Combin. Appl.* **29**(2000) 94–96.

[100] J. Sedláček, Problem 27, In: *Theory and Its Applications, Proc. Symp. Smolenice.* 1963, 163-169.

[101] J. Sedláček, On magic graphs, *Math. Slov.* **26** (1976) 329–335.

[102] R. Simanjuntak, F. Bertault and M. Miller, Two new (a, d)-antimagic graph labelings, *Proc. of the Eleventh Australasian Workshop on Combinatorial Algorithms.* (2000), 179-189.

[103] W.C.Shiu, P.C.B. Lam and H.L. Cheng, Supermagic labeling of an s-duplicate of $K_{n,n}$, *Congress. Numer.* **146**(2000), 119–124.

[104] B.M. Stewart, Magic graphs, *Can. J. Math.* **18** (1966), 1031-1059.

[105] K.A. Sugeng, M.Miller, Y.Lin and M.Bača, Super (a,d)-vertex antimagic total labelings, *J. Combin, Math. Combin. Comput.* **55**(2005), 91–102.

[106] K.A. Sugeng, Magic and antimagic labeling of graphs, *Ph.D Thesis*, university of Ballarat, Ballarat, 2005.

[107] M. Sundaram, R. Ponraj and S. Somasundaram, Product cordial labeling of graphs, *Bull. Pure Appl. Sci. Sect. E Math. Stat.* **23** (2004), 155–163.

[108] S.K. Vaidya and C.M. Barasara, Total edge product cordial labeling of graphs, *Malaya J. Mathematik.* **3**(1) (2013), 55–63.

[109] S.K. Vaidya and C.M. Barasara, Edge product cordial labeling of graphs, *J. Math. Comput. Sci.* **2**(5) (2012), 1436–1450.

[110] W.D. Wallis, E.T. Baskoro, M.Miller and Slamin, Edge-magic total labeling, *Austral. J. Combin.* **22**(2000), 177–190.

[111] W.D. Wallis, Magic Graphs, Birkhäuser, Boston - Basel - Berlin, 2001

[112] W. Wang and X. Zhu, Entire colouring of plane graphs, *J. Combin Theory Ser.B.* **101** (2011), 490-501.

[113] D.B. West, Introduction to Graph Theory, 2nd Edition, Prentice-Hall, New Jersey, USA, (2003).