Conformal Symmetries of the Ricci Tensor for Certain Spacetimes

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By

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Dedicated To
My Parents
My Wife
and
My beloved Son
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Abstract

In this thesis, we have investigated conformal symmetries of the Ricci tensor, also known as conformal Ricci collineations (CRCs), for certain physically important spacetimes including Kantowski-Sachs spacetimes, static spacetimes with maximal symmetric transverse spaces, non-static spherically symmetric spacetimes and locally rotationally symmetric Bianchi type I and V spacetimes. For each of these spacetimes, the CRC equations are solved in degenerate as well as non-degenerate cases.

When the Ricci tensor is degenerate, it is observed that for all the above mentioned spacetimes, the Lie algebra of CRCs is infinite-dimensional. For non-degenerate Ricci tensor, it is shown that the spacetimes under consideration always admit a finite-dimensional Lie algebra of CRCs.

For Kantowski-Sachs and locally rotationally symmetric Bianchi type V metrics, we obtain 15-dimensional Lie algebras of CRCs, which is the maximum dimension of conformal algebra for a spacetime. In case of static spacetimes with maximal symmetric transverse spaces, the dimension of Lie algebra of CRCs turned out to be 6, 7 or 15. Similarly, it is observed that non-static spherically symmetric spacetimes may possess 5, 6 or 15 CRCs for non-degenerate Ricci tensor. Finally, the dimension of Lie algebra of CRCs for locally rotationally symmetric Bianchi type I spacetimes is shown to be 7- or 15-dimensional.

For all the above mentioned spacetimes, the CRCs are found subject to some highly non-linear differential constraints. In order to show that the classes of CRCs are non-empty, some examples of exact form of the corresponding metric satisfying these constraints are provided.
List of Publications

Out of the research work presented in this thesis, the following papers have been published/submitted in different journals.


- T. Hussain and **F. Khan**, Conformal Ricci Collineations of Static Spacetimes with Maximal Symmetric Transverse Spaces (Submitted Manuscript).

- **F. Khan** and T. Hussain, Non-Static Spherically Symmetric Spacetimes and their Conformal Ricci Collineations (Submitted Manuscript).
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Chapter 1

Preliminaries

1.1 Introduction

Albert Einstein introduced the theory of relativity in the earlier twentieth century, which is considered as one of the towering achievements in science. The theory of relativity describes two theories, special relativity (SR) and general relativity (GR). In 1905, Albert Einstein published the theory of space and time, known as the theory of SR. Einstein published three articles in 1905, which includes light quanta, Brownian motion and the theory of SR, each of these is worthy of a Nobel prize.

Some important consequences of SR are length contraction, time dilation and the dependence of space and time. This theory also defines an equivalence relation between mass and energy which can be stated by the equation $E = mc^2$.

Following 10 years, Einstein finished his efforts to enhance and generalize the theory of relativity in the presence of gravitational forces and exhibited
the theory of GR in 1916. This theory emerged as one of the best accomplishments in science.

In GR, any substance that contributes to the energy-momentum tensor, that is, which can produce a gravitational field, is called matter. In principle of GR, Einstein concluded that enormous items (matter) cause bending in a spacetime, called curvature. A standout amongst the most essential results of Einstein’s theory of GR is the substructure of connection between geometry and physics of spacetime, given by the following system of non-linear coupled partial differential equations (PDEs), called the Einstein’s field equations (EFEs) [69]:

\[ G_{ab} = R_{ab} - \frac{R}{2} g_{ab} = k T_{ab}, \]  

(1.1.1)

where \( G_{ab}, R_{ab}, T_{ab} \) and \( g_{ab} \) represent Einstein, Ricci, energy-momentum and metric tensors respectively, \( R \) the Ricci scalar and \( k \) defines the gravitational coupling between geometry and matter. The EFEs associate the curvature to the energy-momentum tensor of the matter in a spacetime.

The EFEs constitute a system of ten non-linear PDEs. As these equations are highly non-linear, it is very hard to find their exact solutions. Even in the vacuum case, that is when \( T_{ab} = 0 \), Eq. (1.1.1) becomes \( R_{ab} = 0 \), which may be very difficult to solve. However if some geometric constrains are assumed to be possessed by the metric, Ricci or energy-momentum tensors, then these equations can be simplified to obtain their exact solutions. In the literature, few exact solutions of EFEs are available [45,48,69] which are sought out using the assumption that they possess some specific symmetries. Some of the known exact solutions of EFEs include Friedmann solution for cosmology, Schwarzschild and Kerr solutions for black holes and the plane
wave solution.

The remaining portion of this chapter discusses some basic concepts of GR which will be frequently used in this thesis. These include manifolds, spacetime, Lie algebra, tensors, covariant derivative, Lie derivative and spacetime symmetries.

1.2 Manifolds

The theory of manifolds is considered as essential theory not only in pure mathematics but also in the fields of genetics, econometrics, biomedical imaging, theoretical physics, robotics etc [39]. Generally speaking, manifolds are like curves and surfaces except that they could be of higher dimension. The number of independent parameters needed to specify a point on a manifold is called its dimension. The simplest examples of manifolds are circle, ellips, parabola and hyperbola, which are one-dimensional manifolds, whereas torus, sphere, ellipsoid, paraboloid and hyperboloid are the examples of 2−dimensional manifolds.

Mathematically, an \( n \)-dimensional real smooth manifold \( M \) is a set along with a collection of subsets \( \{S_k\} \) satisfying the following conditions [72]:

\( i. \) The collection \( \{S_k\} \) covers \( M \).

\( ii. \) For every \( k \), we can define a bijective mapping \( \psi_k : S_k \rightarrow V_k \), where \( V_k \) is an open subset of \( \mathbb{R}^n \).

\( iii. \) If \( S_k \cap S_j \neq \emptyset \), then we can consider the map

\[
\psi_j \circ \psi_k^{-1} : \psi_k[S_k \cap S_j] \subset V_k \subset \mathbb{R}^n \rightarrow \psi_j[S_k \cap S_j] \subset V_j \subset \mathbb{R}^n
\]
such that $V_k$ and $V_j$ are open subsets of $\mathbb{R}^n$ and this map is smooth. For each $k$, the mapping $\psi_k$ in above definition is called a chart on $M$.

An $n$–dimensional complex smooth manifold is defined in the same way by replacing $\mathbb{R}^n$ by $\mathbb{C}^n$.

### 1.3 Topological Manifold

Before defining a topological manifold, recall the definition of a topological space. A topological space is a pair $(X, \tau)$ of a non-empty set $X$ and a collection $\tau$ of subsets of $X$ such that the following conditions are satisfied:

i. $\phi, X \in \tau$

ii. The union of any number of elements of $\tau$ belongs to $\tau$.

iii. The intersection of a finite number of elements of $\tau$ belongs to $\tau$.

A topology on a manifold $M$ can be defined by assuming that all the charts $\psi_k$, used in the definition of manifold, are homeomorphisms. The manifold $M$ along with this topology is called a topological manifold.

If the charts $\psi_k$ of a topological manifold $M$ are diffeomorphisms, that is for each $k$, $\psi_k$ and $\psi_k^{-1}$ are smooth, then $M$ is called a smooth manifold.

### 1.4 Tangent Vectors

If $\tau$ is a collection of all smooth mappings from a manifold $M$ to $\mathbb{R}$, then a tangent vector $v$ at the point $p \in M$ is a map $v : \tau \rightarrow \mathbb{R}$, satisfying the following conditions, respectively known as Linearity and Leibnitz rule [72]:

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i. \( v(\alpha_1 f + \alpha_2 g) = \alpha_1 v(f) + \alpha_2 v(g) \), for all \( f, g \in \tau \) and \( \alpha_1, \alpha_2 \in \mathbb{R} \).

ii. \( v(fg) = f(p) v(g) + g(p) v(f) \), for all \( f, g \in \tau \).

The set of all tangent vectors at a point \( p \) on a manifold \( M \), denote by \( V_p \) or \( T_p M \) gives the structure of a vector space, where addition and scalar multiplication are respectively defined as:

\[
(v_1 + v_2)(f) = v_1(f) + v_2(f),
\]

\[
(\alpha v)(f) = \alpha v(f).
\]

The vector space \( V_p \) or \( T_p M \) is known as a tangent space to \( M \) at the point \( p \) such that \( \text{dim } V_p = \text{dim } M \).

A vector field \( X \) on a manifold \( M \) is a mapping \( X : M \rightarrow TM \), which associates every point \( p \in M \) to a tangent vector \( v_p \) in \( T_p M \). If the vector field \( X \) is defined on the whole manifold \( M \), it is called a global vector field and in the case when it is defined on some open subset of \( M \), the vector field \( X \) is called a local vector field.

For two vector fields \( X \) and \( Y \) on \( M \), their commutator \( [X, Y] \) is another vector field which assigns, to each point \( p \in M \), a tangent vector \( [X, Y]_p \) such that \( [X, Y]_p(f) = X_p Y(f) - Y_p X(f) \).

## 1.5 Spacetime

Before the development of Einstein’s theory of relativity, it was assumed that time is a universal quantity and is independent of space. Thus if an object is in motion or static, the time will remains uniform. In a prerelativistic space,
the position of a point is designated by three numbers (spatial coordinates). For example, in the polar coordinate system, these are \( r, \theta \) and \( \phi \), while in the cartesian coordinate system, these are \( x, y \) and \( z \). Einstein presented his theory in terms of kinematics and shown how time varies for objects in different frame of references. He postulated that space and time depend on each other and in this way a continuum of space and time is formed, called spacetime. Thus a point in a spacetime requires four coordinates, three are spatial coordinates and the other is time.

Mathematically, a spacetime is a four-dimensional smooth, Hausdorff manifold \( M \) together with a metric \( g \), denoted by \((M, g)\), of signature \((+, -, -, -)\) or \((-+, +, +, +)\).

A spacetime is called flat, if the curvature in the spacetime vanishes. When the energy-momentum tensor vanishes, that is, \( T_{ab} = 0 \), then \( M \) is called the vacuum spacetime. In this case, Eq. (1.1.1) becomes, \( G_{ab} = 0 \), which is the simplest form of the EFEs.

A spacetime \( M \) is conformally flat if its associate Weyl tensor vanishes identically on \( M \). In such a case, for each \( m \in M \) there is an open neighborhood \( U \), a flat metric \( h \) on \( U \) and a smooth function \( \sigma : U \to \mathbb{R} \) such that \( g = \sigma h \) [21].

### 1.6 Lie Algebra

If \( V \) is a vector space over a field \( F \), then it is called an algebra if left and right distributive laws of multiplication over addition hold in \( V \). That is, if for all \( v_1, v_2, v_3 \) in \( V \), we have:
\[ v_1(v_2 + v_3) = v_1v_2 + v_1v_3, \]
\[ (v_1 + v_2)v_3 = v_1v_3 + v_2v_3. \]

An algebra \( V \) over a field \( F \) is said to be a Lie algebra if it satisfies the additional properties of anti-commutative and the Jacobi identity. That is, for all \( v_1, v_2, v_3 \in V \):

\[ v_1v_2 + v_2v_1 = 0, \]
\[ v_1(v_2v_3) + v_2(v_3v_1) + v_3(v_1v_2) = 0. \]

### 1.7 Tensors

For a vector space \( V \) over a field \( F \), a linear function \( T : V \to F \) is called a linear functional on \( V \). The set consisting of all linear functionals on \( V \) is denoted by \( V^* \) and it gives the structure a vector space over \( F \). The vector space \( V^* \) is known as dual space of \( V \).

A real-valued multilinear map \( T : V^* \times V^* \times \ldots V^* \times V \times V \times \ldots V \to \mathbb{R} \) is called a tensor of type \((k, l)\). The number \( k \) is called the covariant degree, while \( l \) is known as the contravariant degree of \( T \). The rank or order of a tensor \( T \) is defined to be the sum of its covariant and contravariant degrees.

An ordinary vector is a type \((0, 1)\) tensor which is also called a covariant vector, while a dual vector is a type \((1, 0)\) tensor and is known as a contravariant vector. A tensor of type \((0, 0)\) is regarded as a scalar, while a mix tensor is defined to be a tensor of type \((k, l)\). Similarly, tensors of type \((0, l)\) and \((k, 0)\) are known as covariant and contravariant tensors respectively.
Under the usual addition and scalar multiplication of mappings, the set of all tensors of type \((k, l)\) on a finite-dimensional vector space \(V\) of dimension \(n\) forms a vector space of dimension \(n^{k+l}\), known as tensor space over \(V\). Thus, if two tensors of type \((k, l)\) are added, the resultant tensor is a tensor of the same type. Similarly, a scalar can be multiplied to a tensor to obtain another tensor of the same type.

The contraction \(C\) with respect to \(i\)th dual vector and \(j\)th ordinary vector is a mapping from the vector space of type \((k, l)\) tensors to the vector space of type \((k-1, l-1)\) tensors, which is defined as:

\[
CT = \sum_{r=1}^{n} T(\ldots, v^r, \ldots; \ldots, v_r, \ldots)
\]

where \(\{v_r\}\) is a basis of \(V\) and \(\{v^r\}\) is its dual basis.

A tensor \(T\) is said to be symmetric if its value remains unchanged by interchanging any two of its contravariant (or covariant) indices.

As the EFEs are tensor equations, therefore tensors play a key role in GR. Here we briefly define some of the important tensors used in this thesis.

### 1.7.1 Metric Tensor

The metric tensor, denoted by \(g_{ab}\), is an important type of tensor which expresses all the geometric structure of a spacetime and is used to define certain notions in a spacetime, such as time, distance, volume and angle. It is a covariant symmetric and non-degenerate tensor of type \((0, 2)\) which takes a pair of tangent vectors \(a\) and \(b\) at a point on a manifold and maps it to a real number in a similar way of the dot product on the Euclidean space.
A metric tensor is usually denoted by $ds^2$ and in terms of its components $g_{ab}$, it can be expressed as:

$$ds^2 = \sum_{a,b} g_{ab} \, dx^a dx^b. \tag{1.7.1}$$

The metric tensor on an $n-$dimensional manifold of the signature $(1, n-1)$ or $(n-1, 1)$ is called Lorentzian. If the signature of metric tensor $g$ is $(n, 0)$, it is known as Riemannian metric. The metric of a spacetime is always Lorentzian.

### 1.7.2 Riemann Curvature Tensor

The Riemann curvature tensor (RCT) associates a tensor to each point of a Riemannian manifold and it is used to measure the curvature of the Riemannian manifold. In terms of Christoffel symbols, it is defined as \[44\]:

$$R^a_{\ bcd} = \Gamma^a_{\ bd,c} - \Gamma^a_{\ bc,d} + \Gamma^e_{\ ec} \Gamma^a_{\ bd} - \Gamma^a_{\ ed} \Gamma^e_{\ bc}. \tag{1.7.2}$$

In a totally covariant form, the RCT can be written as:

$$R_{abcd} = g_{ae} R^e_{\ bcd} = \Gamma^a_{\ bda,c} - \Gamma^a_{\ bca,d} + \Gamma^e_{\ ade} \Gamma^a_{\ bc} - \Gamma^a_{\ ace} \Gamma^e_{\ bd}. \tag{1.7.3}$$

Some well known properties of the Riemann curvature tensors are:

(i) $R_{abcd} = -R_{bacd} = -R_{abdc}$ (Antisymmetry)

(ii) $R_{abcd} = R_{cdab}$ (Interchange Symmetry)

(iii) $R_{abcd} + R_{acdb} + R_{adbc} = 0$ (Algebraic Bianchi Identity)

(iv) $\nabla_a R_{debc} + \nabla_b R_{deca} + \nabla_c R_{deab} = 0$ (Differential Bianchi Identity)
For an $n$-dimensional manifold $M$, the RCT has $\frac{n^2}{12}(n^2 - 1)$ non-zero components. Thus there are 20 non-zero independent components of the RCT for a 4-dimensional spacetime manifold.

1.7.3 Ricci Tensor

The Ricci tensor is the part of curvature of a spacetime that determines the degree to which matter will tend to converge in time. It is a covariant symmetric tensor of order 2, that is $R_{ab} = R_{ba}$, and is obtained from the Riemann curvature tensor given in Eq. (1.7.2) by contracting the first and third indices:

$$R_{ab} = R_{acb}^c.$$  

(1.7.4)

If we contract the Ricci tensor with the metric, we obtain the Ricci scalar, that is:

$$g^{ab}R_{ab} = R.$$  

(1.7.5)

1.7.4 Energy-Momentum Tensor

The energy-momentum tensor, denoted by $T_{ab}$ and defined by Eq. (1.1.1), is a second order symmetric tensor which elaborates the density, stress and flux of energy and momentum in a spacetime. Just like matter, $T_{ab}$ is also a source of the gravitational field in EFEs and it relates the structure of a spacetime to the matter it contains.

Depending upon the matter source, the energy-momentum tensor takes a particular form. For example, if the source is a perfect fluid, the energy-momentum tensor is given by $T_{ab} = (p + \rho)u_{a}u_{b} - pg_{ab}$, where $p, \rho$ denote
the pressure and density and \( u^a \) is the four velocity of the perfect fluid.

1.7.5 Weyl Tensor

The Weyl tensor is an important type of tensor which is derived from the metric tensor and it expresses the tidal forces that is felt by a body while moving along a geodesic. The Weyl tensor is not concerned with the change in volume of a body but with the shape of the body distorted by the tidal forces. For a spacetime manifold, the \((0, 4)\) type Weyl tensor is defined as:

\[
C_{abcd} = R_{abcd} + \frac{1}{2} \left\{ g_{ad} R_{bc} + g_{bc} R_{ad} - g_{ac} R_{bd} - g_{bd} R_{ac} \right\} + \frac{1}{6} R \left\{ g_{ac} g_{bd} - g_{ad} g_{bc} \right\}.
\] (1.7.6)

The \((1, 3)\) type Weyl tensor can be obtained by simply contracting the above \((0, 4)\) form of the Weyl tensor with the inverse of the metric. Under the conformal changes to the metric, the Weyl tensor in its \((1, 3)\) form remains constant. For this reason, the Weyl tensor is also called the conformal tensor.

Following are some important properties of the Weyl tensor:

(i) \( C^a_{cab} \equiv g^{ad} C_{acdb} = 0 \)

(ii) \( C_{abcd} = -C_{bacd} = -C_{abdc} = C_{cdab} \)

(iii) \( C_{abcd} + C_{acdb} + C_{adbc} = 0 \)

1.8 Covariant Derivative

If the usual partial derivative is applied to a tensor, the result may not be a tensor. The idea to differentiate a tensor, on a manifold, to get another
tensor has a vital role in GR. Such a derivative is called covariant derivative and is denoted by $\nabla$ or a semicolon.

A covariant derivative $\nabla$ is a map from a tensor field $T$ of type $(k,l)$ on a manifold $M$ to another tensor field of type $(k,l + 1)$ on $M$ [72]:

(i) $\nabla$ is linear, that is, $\nabla(\alpha T_1 + \beta T_2) = \alpha \nabla T_1 + \beta \nabla T_2$

(ii) $\nabla$ satisfies the Leibnitz rule, that is, $\nabla[T_1 T_2] = [\nabla T_1] T_2 + T_1 [\nabla T_2]$

(iii) On scalar, it reduce to usual partial derivative, that is, $t(f) = t^a \nabla_a f$, for a smooth real valued function $f$ on $M$

(iv) It commutes with the contraction.

(v) $\nabla$ is torsion free. Thus if $f$ is a smooth function on $M$, then

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f$$

The covariant derivative is sometime called the semicolon derivative.

### 1.9 Derived and Pullback Maps

Let $\phi : M \to M'$ be a smooth map between the smooth manifolds $M$ and $M'$, and $p \in M$ such that $\phi(p) = p' \in M'$. The mapping $\phi_* : T_p M \to T_{p'} M'$ defined by $\phi_* v(f) = v(f \circ \phi)$, for $v \in T_p M$ is called the derived map of $\phi$ at $p$. Here $T_p M$ and $T_{p'} M'$ are the sets of tangent vectors on $M$ and $M'$ at points $p$ and $p'$ respectively, known as tangent spaces.

If $\phi : M \to M'$ is a smooth map and $f : M' \to \mathbb{R}$ is some function of $M'$, then the composition mapping $f \circ \phi$ is called the pullback of $f$ under $\phi$, denoted by $\phi^* f$ [72].
1.10 Lie Derivative

Let $T$ and $X$ respectively be global smooth tensor field and global smooth vector field on a smooth manifold $M$. For some appropriate $t$, let $\phi_t$ be the local diffeomorphisms associated with $X$. The Lie derivative of $T$ along $X$ at a point $q \in M$, denoted by $\mathcal{L}_X(q)$, is a global smooth tensor field on $M$ which has the same type as that of $T$ and is defined as [44,72]:

$$\mathcal{L}_X T(q) = \lim_{t \to 0} \left( (f^*_t T)(q) - T(q) \right) / t,$$

where $f^*$ denotes the pullback of $f$. If $X, Y$ are some smooth vector fields, $\phi$ is a smooth real valued function and $T, S$ are smooth tensor fields on a manifold $M$, then the following relations hold:

(i.) $\mathcal{L}_X(aT + bS) = a \mathcal{L}_X T + b \mathcal{L}_X S$, which shows that $\mathcal{L}$ is linear

(ii.) $\mathcal{L}_X Y = [X,Y]$

(iii.) $\mathcal{L}_{(aX + bY)} T_1 = a \mathcal{L}_X T_1 + b \mathcal{L}_Y T_1$

(iv.) $\mathcal{L}_X \phi = X(\phi)$

(v.) $\mathcal{L}_{[X,Y]} T = \mathcal{L}_X (\mathcal{L}_Y T) - \mathcal{L}_Y (\mathcal{L}_X T)$

1.11 Spacetime Symmetries

Let $M$ be a spacetime and $U, V$ be open subsets of $M$. A smooth local diffeomorphism $\phi$ from $U$ to $V$ preserving some geometrical or physical features of $M$ is called a local symmetry of $M$. To define the symmetry on $M$ globally, one needs to assume the existence of a smooth vector field on $M$
such that its associated local diffeomorphisms preserve some geometrical or physical features of $M$. These features include geodesics, affine parameter, metric tensor, curvature tensor, Ricci tensor and energy-momentum tensor of the spacetime. Depending upon the features preserved by the symmetry vector field, spacetime symmetries are given names like projective vector fields, affine vector fields, Killing vectors, curvature collineations, Ricci collineations and matter collineations. In the forthcoming sections, we briefly discuss these symmetries.

1.11.1 Projective Vector Fields

A global smooth vector field $X$ on a spacetime manifold $M$ is called a projective vector field if its local diffeomorphisms preserve the geodesics of $M$. Mathematically, a projective vector field $X$ on a spacetime manifold $M$ satisfies the relation [21]:

$$X_{a;bc} \equiv R_{abcd}X^d + 2g_{a(b\psi c)},$$

(1.11.1)

where $\psi$ is a one-form.

The collection $P(M)$ of all projective vector fields on $M$ constitutes a finite-dimensional Lie algebra such that $\text{dim} P(M) \leq 24$, where the binary operation is Lie bracket.

1.11.2 Affine Vector Fields

Let $M$ be a spacetime and $X$ be a global smooth vector field on $M$, then $X$ is said to be an affine vector field if each of the smooth local diffeomorphisms of
$X$ preserves the geodesics of $M$ and their affine parameters. Mathematically, such vector fields satisfy the relation [21]:

$$\nabla_c(\mathcal{L}_X g_{ab} = 0),$$

(1.11.2)

where $\mathcal{L}_X$ represents the Lie derivative operator and $\nabla_c$ is the covariant derivative.

The set of all affine vector field, denoted by $A(M)$, is a subset of $P(M)$ and forms a Lie algebra. In other words, $A(M)$ is a Lie subalgebra of $P(M)$ and its dimension is less than or equal to 20.

### 1.11.3 Conformal Killing Vectors

Let $X$ be a global smooth vector field on a spacetime $M$, then $X$ is known as a conformal Killing vector (CKV) if it satisfies the relation [21]:

$$\mathcal{L}_X g_{ab} = 2\psi g_{ab},$$

(1.11.3)

where $\psi$ is a scalar valued function of all spacetime coordinates, known as conformal factor. The CKVs preserve the metric of spacetime up to a conformal factor. If the conformal factor becomes constant, the CKVs are called homothetic vectors (HVs) and these vectors preserve the metric up to a constant conformal factor. Similarly, if the conformal factor vanishes, the CKVs are reduced to Killing vectors (KVs). The KVs preserve the metric of spacetime.

Under the Lie bracket operation, the collection $C(M)$ of all CKVs on a manifold $M$ forms a finite-dimensional Lie algebra such that $\dim C(M) \leq 15$. A spacetime $M$ is conformally flat if and only if $\dim C(M) > 7$ [21]. In fact, for a conformally flat spacetime, we have $\dim C(M) = 15$. 

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The sets $K(M)$ and $H(M)$ of all KVs and HVs are Lie subalgebras of $C(M)$ and their dimensions are less than or equals to 10 and 11 respectively.

1.11.4 Curvature Collineations

A global smooth vector field $X$ on a spacetime $M$ is known as a curvature collineation (CC) if it preserves the Riemann tensor. Equivalently, we have [21]:

$$\mathcal{L}_X R^a_{bcd} = 0. \quad (1.11.4)$$

The set $CC(M)$ of all smooth CCs on $M$ forms a Lie algebra under the binary operation of Lie bracket, which may be infinite-dimensional.

The condition of smoothness for CCs to form a Lie algebra is mandatory. If this condition is dropped, the collection of CCs may not form a Lie algebra.

It is to be noted that $K(M)$ and $A(M)$ are subalgebras of $CC(M)$. A curvature collineation, if not affine, is called proper.

1.11.5 Conformal Ricci Collineations

A smooth vector field $X$ on a spacetime $M$ is said to be a conformal Ricci collineation (CRC) if [19]:

$$\mathcal{L}_X R_{ab} = 2\psi R_{ab}, \quad (1.11.5)$$

where $\psi$ is a scalar valued function of all spacetime coordinates, known as the inheriting factor. More specifically, if the inheriting factor becomes a constant, the CRCs reduce to homothetic Ricci collineations (HRCs) and in case when $\psi = 0$, the CRCs are called Ricci collineations (RCs).
The Ricci tensor $R_{ab}$ may be degenerate or non-degenerate according as $\det(R_{ab}) = 0$ or $\det(R_{ab}) \neq 0$ respectively. In case of non-degenerate $R_{ab}$, it may be considered as a metric and hence finding the CRCs of a metric is same as finding the CKVs for that metric. Thus the collection of all CRCs forms a Lie algebra under the Lie bracket operation and its dimension is at most 15. When the Ricci tensor is degenerate, the smooth CRCs form a Lie algebra but there is no guarantee of its finite dimensionality.

The collections of HRCs and RCs form Lie subalgebras of the algebra of CRCs, provided that the Ricci tensor is non-degenerate and the dimensions of these subalgebras are at most 11 and 10 respectively.

### 1.11.6 Conformal Matter Collineations

A smooth vector field $X$ on a spacetime $M$ is said to be a conformal matter collineation (CMC) if it satisfies the following relation:

$$\mathcal{L}_X T = 2\psi T,$$

(1.11.6)

where $\psi$ is the scalar valued function of all spacetime coordinates and $T$ represents the energy-momentum tensor. In particular, CMCs are called homothetic matter collineations (HMCs) and matter collineations (MCs) for $\psi = \text{constant}$ and $\psi = 0$ respectively.

It is remarkable here that the matter collineations can be defined in three different ways depending on the position of indices on the energy-momentum tensor. These three definitions may give different results. As an example, one can see the work of Sharif and Ismaeel [66] on MCs of static spherically symmetric spacetimes where they explored the MCs by using $\mathcal{L}_X T_{ab} = 0$. 


\[ L_x T_{ab} = 0 \] and \[ L_x T^a_b = 0 \] separately and then compared the results to conclude that these three ways of defining MCs give different Lie algebra of MCs in some cases.

Like the Ricci tensor, the energy-momentum tensor \( T_{ab} \) may be degenerate or non-degenerate depending upon whether \( \det(T_{ab}) = 0 \) or \( \det(T_{ab}) \neq 0 \). In the former case, the smooth CMCs form a Lie algebra which may be of infinite dimension. In the later case, that is when \( \det(T_{ab}) \neq 0 \), the collection of CMCs forms a finite-dimensional Lie algebra of dimension less than or equal to 15. Like the case of HRCs and RCs, the sets containing HMCs and MCs also form Lie subalgebras of the algebra of CMCs whose dimensions are at most 11 and 10 respectively, provided that \( \det(T_{ab}) \neq 0 \).

1.12 Literature Review

In general relativity, the interest in spacetime symmetries is long-standing. In particular, the conformal symmetries are of particular interest as they provide a deeper insight into the spacetime geometry and are physically significant. Sometimes, conformal symmetries give rise to extra information which are not given by invariant symmetries. For example, the conservation laws in a spacetime are usually given by KVs but in some cases, the conformal transformations are employed which may provide additional conservation laws, not given by KVs. As an example, if we consider the Friedman metric, there is no energy conservation law in it but after an appropriate conformal transformation, \( g \rightarrow \bar{g} = a^{-2}(t) \), it provides a CKV which gives a conformal analogue of energy conservation.
In order to understand the physical and geometric structure of spacetimes, a large body of the literature is devoted to study different types of spacetime symmetries and their interrelationship. In particular, the symmetries of Ricci tensor and their physical interpretations are extensively studied. Llosa [40] studied the infinitesimal transformations that leave invariant a two-covariant symmetric tensor. The author determined that in most cases, the set of infinitesimal generators constitute a finite-dimensional Lie algebra. However, in some cases a higher degree of degeneracy is also exhibited. Akbar [1] classified static spacetimes with maximal symmetric transverse spaces according to their RCs for non-degenerate Ricci tensor and concluded that there are only four, six, seven or ten RCs admitted by these spacetimes. Camci and Barnes [7] explored the Ricci and conformal Ricci collineations in Friedmann-Robertson-Walker spacetimes. The authors concluded that these spacetimes always admit a 15-dimensional Lie algebra of CRCs when the Ricci tensor is non-degenerate. Moreover, these spacetimes possess six RCs, which are same as KVs. In case of degenerate Ricci tensor, the group of Ricci and conformal Ricci collineations turned out to be infinite-dimensional. Qadir et. al. [50] provided classification of cylindrically symmetric static Lorentzian manifolds via RCs and compared the results with Killing and HVs. The authors stated that in case of non-degenerate Ricci tensor, the dimension of the Lie algebra of RCs is 3, 4, 5, 6, 7 or 10. In most of the degenerate Ricci tensor cases, they obtained an infinite-dimensional group of RCs, while some cases produced three, four, five and ten RCs. Camci et. al. [9] studied the CRCs in static spherically symmetric spacetimes. The authors observed that the dimension of Lie algebra of CRCs for these spacetimes is 15 in non-degenerate case and
infinite in degenerate case. Considering a perfect fluid source, Camci and Türkyılmaz [13] explored RCs in Bianchi type V spacetimes and determined that these spacetimes possess four, five, six or seven RCs for the choice of non-degenerate Ricci tensor, while the degenerate Ricci tensor produce 5– or infinite-dimensional group of RCs. Oliver and Davis [18] used RCs to obtain conservation expressions for perfect fluids. Tsamparlis and Mason [70] explored certain properties of fluid spacetimes admitting RCs. Recently, Hussain et. al. studied the HRCs in locally rotationally symmetric Bianchi type I, Bianchi type II and Kantowski-Sachs spacetimes [25, 27, 31].

Parallel to the symmetries of the Ricci tensor, the other spacetime symmetries are also widely studied in the literature. Moopanar and Maharaj [46] studied the conformal and inheriting conformal symmetries of the metric tensor in spherically symmetric spacetimes and shown that, in an exact solution of the EFEs, there is a hypersurface orthogonal CKV for a relativistic fluid which is expanding, accelerating and shearing. Saifullah and Yazdan [53] classified plane symmetric static spacetimes according to conformal motions. The authors stated that unless otherwise the plane symmetric static spacetimes become conformally flat, they do not admit any proper CKV. Later on, Hussain et. al. [30] argued that these spacetimes may admit proper CKVs in the case when they are non-conformally flat. The authors presented some specific non-conformally flat static plane symmetric metrics admitting proper CKVs. In [16], the authors studied the exact perfect fluid solutions of EFEs admitting CKVs and a particular class of perfect fluid models possessing a 3–dimensional Lie algebra of CKVs acting on 2–dimensional timelike surfaces was investigated. Coley and Tupper [17] explored the CKVs for spher-
ically symmetric spacetimes in the presence of a perfect fluid. The authors discussed some known examples of these spacetimes in which the CKVs are inheriting. Khan et. al. [37] gave a complete classification of LRS Bianchi type-V spacetimes according to their CKVs and inheriting CKVs. During the classification, they obtained finite-dimensional Lie algebras of dimensions 4, 5, 6, 7 and 15 in different cases. The same authors investigated the CKVs in non-static plane symmetric Lorentzian manifolds [36] and obtained proper CKVs in these spacetimes for some particular values of the metric functions. The CKVs of some other spacetimes are studied by different authors, the details of which can be seen in [4, 20, 22–24, 38, 41–43, 47, 52, 71]. The special case of CKVs, known as HVs, are also explored for different spacetimes. Some of such spacetimes are Bianchi types I, IV, V and static cylindrically symmetric spacetimes [2, 3, 57, 58].

The curvature collineations, defined by Eq. (1.11.4), are studied for some physically important spacetimes. Bokhari et. al. [5] classified the plane symmetric static spacetimes according to their curvature collineations and found some examples of plane symmetric static spacetimes admitting proper CCs. In 2009, Camci et. al. [8] discussed the curvature and Weyl collineations of Bianchi type V spacetimes using rank argument of curvature and the Weyl matrices respectively. The authors concluded that, for Bianchi type V spacetimes, the rank of curvature matrix is 3, 4, 5 or 6. In case when the rank of curvature matrix is 3, these spacetimes admit infinite-dimensional Lie algebra of CCs. For the study of CCs in some other spacetimes, we refer [51, 55, 56, 59].

The symmetries of the energy-momentum tensors are extensively studied
in the literature. Salti et al. [54] studied the Bianchi-Kantowski-Sachs type spacetimes admitting matter collineations in both degenerate and non-degenerate energy-momentum tensor cases. They concluded that in case of degenerate $T_{ab}$, the dimension of Lie algebra is infinite, whereas there exist finite number of MCs when the energy-momentum tensor is non-degenerate.

Camci and Sharif [11] explored MCs for Kantowski-Sachs, Bianchi types I and III spacetimes in degenerate and non-degenerate energy-momentum tensor cases. For non-degenerate $T_{ab}$, the dimension of Lie algebra turned out to be ten, six or four, while for degenerate $T_{ab}$, the obtained Lie algebra is almost infinite dimensional. In [10], Camci and Sahin classified the Bianchi type II spacetimes via their MCs. The authors observed that in most cases of the degenerate energy-momentum tensor, the dimension of Lie algebra is infinite. However, these spacetimes always possess finite number of MCs that can be 3, 4, or 5, for the choice of non-degenerate $T_{ab}$. A complete classification of the Bianchi type V spacetimes according to their MCS is given by Camci [6]. The author found infinite MCs in these spacetimes in most cases, when $T_{ab}$ is considered to be degenerate, while the dimension of the Lie algebra of the MCs was found to be 4, 5, 6 or 7, when the energy-momentum tensor is non-degenerate. The classification of some other spacetimes via MCs are given in [12, 15, 35, 49, 60–68]

Though the spacetime symmetries are widely studied in the literature, the direction of studying the conformal symmetries of the Ricci tensor is still open. To the best of our knowledge, only Friedmann-Robertson-Walker and static spherically symmetries spacetimes have been classified through CRCs [7, 9]. With the hope that the classification of some other spacetimes via
CRCs may give some extra information about spacetimes, not given by RCs, we intend to investigate the CRCs of some other spacetimes. In particular, we will focus on the CRCs of locally rotationally symmetric Bianchi type I and V, Kantowski-Sachs, static spacetimes with maximal symmetric transverse spaces and non-static spherically symmetric spacetimes.

1.13 Outlines of Work

The organization of the thesis is as follows:

1. In chapter 1, some basics of general relativity are discussed, which will be helpful to understand this thesis.

2. Chapter 2 provides a study of the conformal Ricci collineations of Kantowski-Sachs metric. This work has been published in [29].

3. In chapter 3, the conformal Ricci collineations of non-static spherically symmetric spacetimes are studied. This work is submitted for possible publication [33].

4. Chapter 4 deals with the classification of static spacetimes with the metric
\[ ds^2 = e^{A(r)} dt^2 - e^{B(r)} dr^2 - r^2 \left[ d\theta^2 + f_k^2(\theta) d\phi^2 \right] \] via their conformal Ricci collineations. This work is also submitted for publication in a reputed journal [28].

5. In chapter 5, classification of LRS Bianchi type I Spacetimes using conformal Ricci collineations is presented. This work has been published in a reputed journal [26].
6. In chapter 6, the conformal Ricci collineations of LRS Bianchi type V spacetimes with perfect fluid matter are investigated. Out of this work, one research paper [34] has been published.

7. In the last chapter, a brief conclusion of the work is given.
Chapter 2

Conformal Ricci Collineations
of Kantowski-Sachs Spacetimes

In this chapter, we have calculated CRCc for Kantowski-Sachs spacetimes. The set of CRC equations are solved in two cases, according as the Ricci tensor is degenerate or non-degenerate. When the Ricci tensor is non-degenerate, the CRC equations are solved analytically to get a vector field generating CRCs, subject to some integrability conditions. These integrability conditions are solved in eight different cases, giving the explicit form of CRCs. When the Ricci tensor is degenerate, the CRC equations are directly integrated to get the closed form of CRCs. The exact forms of some Kantowski-Sachs metrics admitting non-trivial CRCs along with their physical implications are also provided.
2.1 CRC Equations

The Kantowski-Sachs spacetime metric describes a spatially homogeneous and anisotropic universe model which admits an isometry group $G_4$ acting on homogeneous spacelike hypersurfaces. This metric serves the purpose of studying effects of anisotropies in the evolution of the universe and at the same time describes the interior of Schwarzschild black hole in the vacuum case. The Kantowski-Sachs metric is given by [32]:

$$ds^2 = dt^2 - \alpha^2(t)dr^2 - \beta^2(t)\left[d\theta^2 + \sin^2\theta d\phi^2\right],$$

where $\alpha$ and $\beta$ are non-zero functions of $t$ only. The minimal set of KVs possessed by the above metric is given by:

$$X_{(1)} = \partial_r, \quad X_{(2)} = \partial_\phi, \quad X_{(3)} = \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi,$$
$$X_{(4)} = \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi.$$  \hfill (2.1.2)

The Ricci tensor for Kantowski-Sachs metric has the following non-vanishing components:

$$R_{00} = -\left[\frac{\alpha''}{\alpha} + 2\frac{\beta''}{\beta}\right] = A(t),$$
$$R_{11} = \alpha^2 \left[\frac{\alpha''}{\alpha} + 2\frac{\alpha'\beta'}{\alpha\beta}\right] = B(t),$$
$$R_{22} = \beta^2 \left[\frac{\beta''}{\beta} + \frac{\alpha'\beta'}{\alpha\beta} + \left(\frac{\beta'}{\beta}\right)^2 + \frac{1}{\beta^2}\right] = C(t),$$
$$R_{33} = \sin^2\theta C(t).$$ \hfill (2.1.3)

Using $R = g^{ab}R_{ab}$, the Ricci scalar $R$ for the Kantowski-Sachs metric becomes:

$$R = 2\left(\frac{\alpha''}{\alpha} + 2\frac{\beta''}{\beta} + 2\frac{\alpha'\beta'}{\alpha\beta} + \frac{\beta'^2}{\beta^2} + \frac{1}{\beta^2}\right).$$ \hfill (2.1.4)
Using the EFEs, given in Eq. (1.1.1), we have:

\[
T_{00} = 2\frac{\alpha' \beta'}{\alpha \beta} + \frac{\beta'^2}{\beta^2} + \frac{1}{\beta^2},
\]

\[
T_{11} = -\alpha^2 \left(2\frac{\beta''}{\beta} + \frac{\beta'^2}{\beta^2} + \frac{1}{\beta^2}\right),
\]

\[
T_{22} = -\beta^2 \left(\frac{\alpha''}{\alpha} + \frac{\beta''}{\beta} + \frac{\alpha' \beta'}{\alpha \beta}\right),
\]

\[
T_{33} = \sin^2 \theta \ T_{22}.
\]  

(2.1.5)

As discussed in chapter 1, a CRC satisfies the relation:

\[
\mathcal{L}_X R_{ab} = 2\psi R_{ab},
\]  

(2.1.6)

which can be written in the explicit form as:

\[
R_{ab,c} X^c + R_{ac} X^c_{,b} + R_{bc} X^c_{,a} = 2\psi R_{ab}.
\]  

(2.1.7)

Using the Ricci tensor components from Eq.(2.1.3) in the above equation, we get the following CRC equations:

\[
A' X^0 + 2A X^0_0 = 2\lambda A,
\]  

(2.1.8)

\[
A X^0_1 + B X^1_0 = 0,
\]  

(2.1.9)

\[
A X^0_2 + C X^2_0 = 0,
\]  

(2.1.10)

\[
A X^0_3 + C \sin^2 \theta X^3_0 = 0,
\]  

(2.1.11)

\[
B' X^0 + 2B X^1_0 = 2\psi B,
\]  

(2.1.12)

\[
B X^1_0 + C X^2_0 = 0,
\]  

(2.1.13)

\[
B X^1_3 + C \sin^2 \theta X^3_0 = 0,
\]  

(2.1.14)

\[
C' X^0 + 2C X^2_0 = 2\psi C,
\]  

(2.1.15)
\[ C \ X^2_3 + C \ \sin^2 \theta \ X^3_2 = 0, \]  
\[ C' \ X^0 + 2C \ (\cot \theta X^2 + X^3_3) = 2\psi C. \]

The solution of the above system of equations would give the explicit form of CRCs admitted by Kantowski-Sachs metric. In the forthcoming sections, we solve these equations in both degenerate and non-degenerate cases.

### 2.2 CRCs for Degenerate Ricci Tensor

In degenerate case, we have \( \text{det}R_{ab} = 0 \). This means that either \( A = B = C = 0 \), which yields a trivial solution of Eqs. (2.1.8)-(2.1.17), or one of the following possibilities holds:

- **(D1)** \( A \neq 0, B \neq 0, C = 0 \)
- **(D2)** \( A = 0, B \neq 0, C \neq 0 \)
- **(D3)** \( A \neq 0, B = 0, C \neq 0 \)
- **(D4)** \( A \neq 0, B = C = 0 \)
- **(D5)** \( A = C = 0, B \neq 0 \)
- **(D6)** \( A = B = 0, C \neq 0 \)

In case **(D1)**, the system of equations (2.1.8)-(2.1.17) reduces to:

\[
A' \ X^0 + 2A \ X^0_0 = 2\psi A, \tag{2.2.1}
\]
\[
A \ X^0_1 + B \ X^1_0 = 0, \tag{2.2.2}
\]
\[
X^0_2 = X^0_3 = X^1_2 = X^1_3 = 0, \tag{2.2.3}
\]
\[
B' \ X^0 + 2B \ X^1_1 = 2\psi B. \tag{2.2.4}
\]

From Eq. (2.2.1), the inheriting factor \( \psi \) is obtained as:

\[
\psi = \frac{A'}{2A}X^0 + X^0_0. \tag{2.2.5}
\]

Integrating Eq. (2.2.3), we get \( X^0 = f^1(t, r) \) and \( X^1 = f^2(t, r) \), where \( f^1 \) and \( f^2 \) are functions of integration. Using these values in Eq. (2.2.2) and then
integrating it with respect to $r$, we get $f^1 = \frac{-B}{A} f^2_t(t, r) + G^1(t)$; $G^1$ being a function of integration. Putting the value of $\psi$, $X^0$ and $X^1$ in Eq. (2.2.4), we get:

$$\frac{1}{2} \left( \frac{B}{A} \right)' f^2_t + \frac{A}{2B} \left( \frac{B}{A} \right)' G^1 + \frac{B}{A} f^2_{tt} + f^2_{rr} - G^1_t = 0. \tag{2.2.6}$$

The above equation is highly non-linear and cannot be solved as it stands. We assume that $f^2_{rr} = 0 \Rightarrow f^2 = rG^2(t) + G^3(t)$. Here the functions $G^2$ and $G^3$ arise during the process of integration. Putting back this value of $f^2$ in Eq. (2.2.6) and differentiating the resulting equation with respect to $r$, we get $G^1 = \frac{B}{A} G^3_t + c_3 \sqrt{\frac{B}{A}}$ and $G^2 = c_1 \int \sqrt{\frac{A}{B}} dt + c_2$. Thus we have:

$$f^1 = \sqrt{\frac{B}{A}} (c_3 - c_1 r),$$

$$f^2 = c_1 r \int \sqrt{\frac{A}{B}} dt + G^3 + c_2 r.$$

Hence the complete solution of Eqs. (2.2.1)-(2.2.4) is given by:

$$X^0 = \sqrt{\frac{B}{A}} (c_3 - c_1 r),$$

$$X^1 = c_1 \int \sqrt{\frac{A}{B}} dt + c_2,$$

$$X^i = X^i(x^a), \text{ for } i = 2, 3,$$

$$\psi = \frac{B'}{2\sqrt{AB}} (c_3 - c_1 r). \tag{2.2.7}$$

Since $\partial_\phi$ is a KV for the Kantowski-Sachs spacetimes, so it is a RC and hence a CRC. Also, from the system (2.2.7), we see that $X^3(t, r, \theta, \phi)\partial_\phi$ is a CRC, where $X^3(t, r, \theta, \phi)$ is any function of all spacetime coordinates. One can check that because of the arbitrary function $X^3$, the Lie algebra of CRCs is infinite-dimensional. To prove this, let us choose $X^3(t, r, \theta, \phi) = f(t)$, that
is $X^3$ is a function of $t$ only. Then $\partial_\phi, \ t\partial_\phi, \ t^2\partial_\phi, \ldots, \ t^n\partial_\phi$ are all CRCs, for any positive integer $n$. These CRCs are linearly independent because if $a_0, \ a_1, \ldots, \ a_n$ are scalars such that:

$$a_0\partial_\phi + a_1t\partial_\phi + a_2t^2\partial_\phi + \cdots + a_nt^n\partial_\phi = 0$$

$$\Rightarrow (a_0 + a_1t + a_2t^2 + \cdots + a_nt^n)\partial_\phi = 0.$$ 

Since $\partial_\phi$ is nowhere zero, so we must have:

$$a_0 + a_1t + a_2t^2 + \cdots + a_nt^n = 0,$$

which gives $a_0 = a_1 = \cdots = a_n = 0$. Thus the CRCs in this case are infinite-dimensional.

One can similarly solve the system of equations (2.1.8)-(2.1.17) in the remaining degenerate cases. We omit to write the basic calculations and present the final results in the following tables. One can easily check that the dimension of Lie algebra of CRCs in all these cases is infinite.

<table>
<thead>
<tr>
<th>Case</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>D2</td>
<td>$X^0 = \frac{2C_1}{n(B)} \left[-c_3 \cos \theta \ln</td>
<td>\csc \theta - \cot \theta + c_5 \cos \theta + f'(r) - c_3\right], \quad X^1 = f(r)$, $X^2 = -c_1 \sin \phi + c_2 \cos \phi + c_3 \sin \theta \ln</td>
</tr>
<tr>
<td>D3</td>
<td>$X^0 = c_1 \sqrt{\frac{n}{B}}, \quad X^1 = X^1(u^a)$, $X^2 = c_2 \sin \phi - c_3 \cos \phi$, $X^3 = \cot \theta (c_1 \cos \phi + c_2 \sin \phi) + c_4$.</td>
<td>$\psi = \frac{c_1 C_1}{2\sqrt{BC_4}}$</td>
</tr>
</tbody>
</table>

Table 2.1: CRCs for Non-Degenerate Ricci Tensor
### Table 2.2: CRCs for Degenerate Ricci Tensor

<table>
<thead>
<tr>
<th>Case</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>D4</td>
<td>$X^0 = f(t)$, $X^i = X^i(x^a)$, for $i = 1, 2, 3.$</td>
<td>$\psi = \frac{d^2}{dt^2} f(t) + f'(t)$</td>
</tr>
<tr>
<td>D5</td>
<td>$X^1 = f(r)$, $X^i = X^i(x^a)$, for $i = 0, 2, 3.$</td>
<td>$\psi = \frac{d^2}{dr^2} X^0 + f'(r)$</td>
</tr>
<tr>
<td>D6</td>
<td>$X^2 = -c_1 \sin \phi + c_2 \cos \phi + c_3 \sin \theta \ln \mid \csc \theta - \cot \theta \mid + c_5 \sin \theta,$ $X^3 = -\cot \theta (c_1 \cos \phi + c_2 \sin \phi) + c_3 \phi + c_4,$ $X^i = X^i(x^a)$, for $i = 0, 1$</td>
<td>$\psi = \frac{d^2}{dr^2} X^0 + X^2$</td>
</tr>
</tbody>
</table>

#### 2.3 CRCs for Non-degenerate Ricci Tensor

In this section, the CRCs for Kantowski-Sachs metric are calculated in the case when Ricci tensor is non-degenerate, that is, $\text{det}R_{ab} \neq 0$. Thus $A \neq 0$, $B \neq 0$ and $C \neq 0$. In this case, the procedure of solving the system of CRC equations, given by Eqs. (2.1.8)-(2.1.17), is given below.

Differentiating Eqs. (2.1.13) and (2.1.14) with respect to $\phi$ and $\theta$ respectively and then substracting, we get:

$$X_{13}^2 - \sin^2 \theta X_{12}^3 - 2 \sin \theta \cos \theta X_{11}^3 = 0.$$  \hspace{1cm} (2.3.1)

Now, differentiating Eq. (2.1.16) with respect to $r$ and substracting the resultant from Eq. (2.3.1), we obtain $\sin \theta X_{12}^3 + \cos \theta X_{11}^3 = 0$, solving which gives $X^3 = \csc \theta P^1_\phi(t, r, \phi) + P^2(t, \theta, \phi)$. Here $P^1$ and $P^2$ are functions of integrations.

Similarly, differentiating Eqs. (2.1.10) and (2.1.11) with respect to $\phi$ and $\theta$ respectively and then substracting, we have:

$$X_{03}^2 - \sin^2 \theta X_{02}^3 - 2 \sin \theta \cos \theta X_{01}^3 = 0.$$  \hspace{1cm} (2.3.2)
Differentiating Eq. (2.1.16) with respect to $t$ and subtracting the resultant from Eq. (2.3.2), we obtain
\[ \sin \theta P_t^2 + \cos \theta P_r^2 = 0 \Rightarrow P^2 = \csc \theta F_\phi^1(t, \phi) + F_\phi^2(t, \phi) \]
and hence:
\[ X^3 = \csc \theta P_\phi^1 + \csc \theta F_\phi^1 + F_\phi^2. \]

The functions $F^1$ and $F^2$ appearing above are the functions of integrations. Solving Eqs. (2.1.10), (2.1.13) and (2.1.16) by substituting the above value of $X^3$, we have:
\begin{align*}
X^0 &= -\frac{C}{A} \left[ \sin \theta P_t^1 + \sin \theta F_t^1 + P_r^3(t, \theta) \right] + P^4(t, r, \phi), \\
X^1 &= -\frac{C}{B} \left[ \sin \theta P_t^1 + P_r^3 \right] + P^5(t, r, \phi), \\
X^2 &= \cos \theta P^1 + \cos \theta F^1 - \sin^2 \theta F^2_\theta + P^3(t, r, \theta), \\
\end{align*}
(2.3.3)
where $P^3$, $P^4$ and $P^5$ are unknown functions which arise during integration.

Solving Eqs. (2.1.11) and (2.1.14), we get $P^4 = F^3(t, r)$ and $P^5 = F^4(t, r)$; $F^3$ and $F^4$ being functions of integration. Putting these values of $P^4$ and $P^5$ and setting $P^1 + F^1 = P^6(t, r, \phi)$, the system (2.3.3) becomes:
\begin{align*}
X^0 &= -\frac{C}{A} \left[ \sin \theta P_t^6 + P_r^3 \right] + F^3, \\
X^1 &= -\frac{C}{B} \left[ \sin \theta P^6_r + P_r^3 \right] + F^4, \\
X^2 &= \cos \theta P^6 - \sin^2 \theta F^2_\theta + P^3, \\
X^3 &= \csc \theta P^6_\phi + F^2_\phi. \\
\end{align*}
(2.3.4)

Subtracting Eq. (2.1.15) from Eq. (2.1.17) and simplifying the difference using the above system, we get:
\[ P^6 + \sin^2 \theta F^2_\theta + \sin^2 \theta \cos \theta F^2_\theta - \sin \theta P^3_{\theta\theta} + \cos \theta P^3_\theta + P^6_{\phi\phi} + \sin \theta F^2_{\phi\phi} = 0. \] (2.3.5)
The following values of $P^3$ and $P^6$ are obtained by differentiating the above equation with respect to $r$ and $\phi$ and then integrating the resulting equations:

$$
P^3 = -\sin \theta F^8(t, r) - \cos \theta F^9(t, r) + F^{10}(t, \theta) + F^{11}(t, r),
$$
$$
P^6 = \sin \phi F^5(t, r) - \cos \phi F^6(t, r) + F^7(t, \phi) + F^8(t, r),
$$

(2.3.6)

where $F^5, ..., F^{11}$ are functions of integration. Putting back these values in Eq. (2.3.5), we get:

$$
F^7 + F^7_{\phi\phi} - \sin \theta F^{10}_{t\theta} + \cos \theta F^{10}_{t\theta} + \sin^3 \theta F^{10}_{t\theta} + \sin^2 \phi \cos \theta F^2_{t\phi} + \sin \theta F^2_{t\phi} = 0 \quad (2.3.7)
$$

Next, we differentiate Eq. (2.3.7) with respect to $t$ and $\phi$ and simplify to obtain the following values:

$$
F^7 = \sin \phi G^1(t) - \cos \phi G^2(t) + G^3(\phi) + G^4(t),
$$
$$
F^{10} = -\sin \theta G^4(t) - \cos \theta G^5(t) + G^6(\theta) + G^7(t).
$$

(2.3.8)

The functions $G^1, ..., G^7$ appearing in Eqs. (2.3.8) are the integration functions. Substituting back these values of $F^7$ and $F^{10}$ in Eq. (2.3.7) and simplifying it by using some basic algebraic manipulation, we obtain:

$$
X^0 = -\frac{C}{A} \left[ \sin \theta \left( \sin \phi F^{12}_t(t, r) - \cos \phi F^{13}_t(t, r) \right) - \cos \theta F^{14}_t(t, r) \right] + F^{15}(t, r),
$$
$$
X^1 = -\frac{C}{B} \left[ \sin \theta \left( \sin \phi F^{12}_r(t, r) - \cos \phi F^{13}_r(t, r) \right) - \cos \theta F^{14}_r(t, r) \right] + F^{16}(t, r),
$$
$$
X^2 = \cos \theta \left( \sin \phi F^{12}(t, r) - \cos \phi F^{13}(t, r) \right) + \sin \theta F^{14}(t, r) + c_1 \sin \phi
$$
$$
- c_2 \cos \phi + c_3 \sin \theta \ln | \csc \theta - \cot \theta |,
$$
$$
X^3 = \csc \theta \left( \cos \phi F^{12}(t, r) + \sin \phi F^{13}(t, r) \right) + \cot \theta (c_1 \cos \phi + c_2 \sin \phi)
$$
$$
+ c_3 \phi + c_4,
$$
$$
\psi = \frac{C'}{2C} X^0 + X^2.
$$

(2.3.9)
This system satisfies Eqs. (2.1.10), (2.1.11), (2.1.13)-(2.1.16) and (2.1.17). We substitute the above values of $X^0$, $X^1$, $X^2$, $X^3$ and $\psi$ in Eqs. (2.1.8), (2.1.9) and (2.1.12) and then compare the like terms on both sides of the resulting equations. During this procedure, the constant $c_3$ vanishes and we obtain the following twelve integrability conditions:

\[
\begin{align*}
\left( \frac{C}{A} \right)' F^{12}_t + \frac{2C}{A} F^{12}_{tt} - 2F^{12} &= 0, \\
\left( \frac{C}{A} \right)' F^{13}_t + \frac{2C}{A} F^{13}_{tt} - 2F^{13} &= 0, \\
\left( \frac{C}{A} \right)' F^{14}_t + \frac{2C}{A} F^{14}_{tt} - 2F^{14} &= 0, \\
\left( \frac{A}{C} \right)' F^{15} + \frac{2A}{C} F^{15}_t &= 0, \\
\left( \frac{C}{B} \right)' F^{12}_r + \frac{2C}{B} F^{12}_{tr} &= 0, \\
\left( \frac{C}{B} \right)' F^{13}_r + \frac{2C}{B} F^{13}_{tr} &= 0, \\
\left( \frac{C}{B} \right)' F^{14}_r + \frac{2C}{B} F^{14}_{tr} &= 0, \\
AF^{15}_r + BF^{16}_t &= 0, \\
\left( \frac{C}{B} \right)' F^{12}_r - \frac{2AC}{B^2} F^{12}_{rr} + \frac{2A}{B} F^{12} &= 0, \\
\left( \frac{C}{B} \right)' F^{13}_r - \frac{2AC}{B^2} F^{13}_{rr} + \frac{2A}{B} F^{13} &= 0, \\
\left( \frac{C}{B} \right)' F^{14}_r - \frac{2AC}{B^2} F^{14}_{rr} + \frac{2A}{B} F^{14} &= 0, \\
\left( \frac{B}{C} \right)' F^{15} + \frac{2B}{C} F^{16}_r &= 0.
\end{align*}
\]

If we could find the unknown functions appearing in the system (2.3.9) by solving the above integrability conditions, we will find the explicit form...
of CRCs admitted by Kantowski-Sachs spacetimes. For a complete classification, we solve these integrability conditions in the following cases:

(ND1) $A' = B' = C' = 0$  
(ND2) $A' = C' = 0$, $B' \neq 0$

(ND3) $B' = C' = 0$, $A' \neq 0$  
(ND4) $A' = B' = 0$, $C' \neq 0$

(ND5) $B' \neq 0$, $A' = 0$, $C' \neq 0$  
(ND6) $B' = 0$, $A' \neq 0$, $C' \neq 0$

(ND7) $A' \neq 0$, $B' \neq 0$, $C' = 0$  
(ND8) $A' \neq 0$, $B' \neq 0$, $C' \neq 0$

**Case ND1:** In this case, we have $A' = B' = C' = 0$. Let us denote $A = a$, $B = b$ and $C = c$, where $a$, $b$ and $c$ are non-zero constants. Thus the system of equations (2.3.10)-(2.3.21) takes the following form:

$$c F_{tt}^{12} - a F_{tt}^{12} = 0,$$  \hspace{1cm} (2.3.22)

$$c F_{tt}^{13} - a F_{tt}^{13} = 0,$$  \hspace{1cm} (2.3.23)

$$c F_{tt}^{14} - a F_{tt}^{14} = 0,$$  \hspace{1cm} (2.3.24)

$$F_{tt}^{12} = F_{tr}^{13} = F_{tr}^{14} = 0,$$  \hspace{1cm} (2.3.25)

$$F_{tr}^{15} = F_{tr}^{16} = 0,$$  \hspace{1cm} (2.3.26)

$$a F_{r}^{15} + b F_{r}^{16} = 0,$$  \hspace{1cm} (2.3.27)

$$c F_{rr}^{12} - b F_{t}^{12} = 0,$$  \hspace{1cm} (2.3.28)

$$c F_{rr}^{13} - b F_{t}^{13} = 0,$$  \hspace{1cm} (2.3.29)

$$c F_{rr}^{14} - b F_{t}^{14} = 0.$$  \hspace{1cm} (2.3.30)

From Eq. (2.3.25), the value of $F^{12}$ turns out to be $F^{12} = G^1(t) + G^2(r)$, where $G^1$ and $G^2$ are functions of integration. When we put this value of $F^{12}$ in Eq. (2.3.28), it gives $\frac{c}{b} G_{rr}^2 - G^2 = G^1 = c_5$, where $c_5$ is a separation constant. This shows that the function $F^{12}$ depends only on $r$. Consequently, Eq. (2.3.22) gives $F^{12} = 0$. A similar procedure may be followed to show...
that $F^{13} = F^{14} = 0$.

Integrating Eq. (2.3.26), one can see that $F^{15} = G^3(r)$ and $F^{16} = G^4(t)$; $G^3$ and $G^4$ being functions of integration. Using these values in Eq. (2.3.27) and integrating it, we find that $G^3 = \frac{c_6 b}{a} r + c_7$ and $G^4 = -c_6 t + c_8$. Hence the system (2.3.9) becomes:

\begin{align*}
X^0 &= \frac{c_6 b}{a} r + c_7, \\
X^1 &= -c_6 t + c_8, \\
X^2 &= c_1 \sin \phi - c_2 \cos \phi, \\
\xi^3 &= \cot \theta (c_1 \cos \phi + c_2 \sin \phi) + c_4, \\
\psi &= 0. \tag{2.3.31}
\end{align*}

From above, it can be seen that the inheriting factor vanishes and hence the CRCs reduce to RCs in this case, the dimension of Lie algebra of RCs turned out to be 6. We can see that the set of basic KVs of Kantowski-Sachs spacetimes, given in (2.1.2), is contained in the algebra of RCs.

**Case ND2:** In this case, we have $A' = C' = 0$ and $B' \neq 0$. We denote $A = a$ and $C = c$, where $a$ and $c$ are non-zero constants. Thus the integrability conditions, Eqs. (2.3.10)-(2.3.21), become:

\begin{align*}
c F_{tt}^{12} - a F^{12} &= 0, \tag{2.3.32} \\
c F_{tt}^{13} - a F^{13} &= 0, \tag{2.3.33} \\
c F_{tt}^{14} - a F^{14} &= 0, \tag{2.3.34} \\
F_t^{15} &= 0, \tag{2.3.35} \\
B' F_r^{12} - 2 B F_{tt}^{12} &= 0. \tag{2.3.36}
\end{align*}
\[ B'F_{r}^{13} - 2BF_{tr}^{13} = 0, \quad (2.3.37) \]
\[ B'F_{r}^{14} - 2BF_{tr}^{14} = 0, \quad (2.3.38) \]
\[ aF_{r}^{15} + BF_{t}^{16} = 0, \quad (2.3.39) \]
\[ cB'F_{r}^{12} + 2acF_{rr}^{12} - 2aBF_{12} = 0, \quad (2.3.40) \]
\[ cB'F_{r}^{13} + 2acF_{rr}^{13} - 2aBF_{13} = 0, \quad (2.3.41) \]
\[ cB'F_{r}^{14} + 2acF_{rr}^{14} - 2aBF_{14} = 0, \quad (2.3.42) \]
\[ B'F_{15} + 2BF_{r}^{16} = 0. \quad (2.3.43) \]

Integrating Eq. (2.3.36), we get \( F_{12} = \sqrt{B} G_{1}^{1}(r) + G_{2}^{2}(t); \) \( G_{1} \) and \( G_{2} \) being functions of integration. Putting this value in Eq. (2.3.40) and then differentiating it with respect to \( r \), we get \( G_{1} = \frac{c_{5}}{2} r^{2} + c_{6} r + c_{7} \), \( G_{2} = \frac{c_{8}}{2} e^{-\sqrt{\frac{a}{c}} t} \) and \( B(t) = e^{2\sqrt{\frac{a}{c}} t}. \) Thus \( F_{12} = \sqrt{\frac{a}{c}} t \left( \frac{c_{5}}{2} r^{2} + c_{6} r + c_{7} \right) + \frac{c_{8}}{2} e^{-\sqrt{\frac{a}{c}} t}. \) This value of \( F_{12} \) identically satisfies Eq. (2.3.32).

In a similar way, the solution of Eqs. (2.3.33), (2.3.34), (2.3.37), (2.3.38), (2.3.41) and (2.3.42) yield \( F_{13} = \sqrt{\frac{a}{c}} t \left( \frac{c_{11}}{2} r^{2} + c_{10} r + c_{13} \right) + \frac{c_{14}}{2} e^{-\sqrt{\frac{a}{c}} t} \) and \( F_{14} = \sqrt{\frac{a}{c}} t \left( \frac{c_{15}}{2} r^{2} + c_{12} r + c_{13} \right) + \frac{c_{14}}{2} e^{-\sqrt{\frac{a}{c}} t}. \)

It remains to find the values of the functions \( F_{15} \) and \( F_{16}. \) Integrating Eq. (2.3.35), we obtain \( F_{15} = G_{3}(r), \) where \( G_{3} \) is a function of integration. Using this value in Eq. (2.3.39) and integrating it with respect to \( t, \) we get \( F_{16} = \frac{e^{-2\sqrt{\frac{a}{c}} t}}{2\sqrt{\frac{a}{c}}} G_{3}^{2} + G_{4}(r), \) where \( G_{4} \) is a function of integration. Putting these values of \( F_{15} \) and \( F_{16} \) in Eq. (2.3.43) and solving it, we have \( G_{3} = c_{14} r + c_{15} \) and \( G_{4} = -\sqrt{\frac{a}{c}} \left( \frac{c_{14}}{2} r^{2} + c_{15} r \right) + c_{16}. \) Finally, we have \( F_{15} = c_{14} r + c_{15} \) and \( F_{16} = \frac{c_{14}}{2\sqrt{\frac{a}{c}}} e^{-2\sqrt{\frac{a}{c}} t} \) \( - \sqrt{\frac{a}{c}} \left( \frac{c_{14}}{2} r^{2} + c_{15} r \right) + c_{16}. \) Hence the system (2.3.9)
becomes:

\[ X^0 = -\sin \theta \sin \phi \left[ e^{\sqrt{\frac{c}{a}t}} \left( \frac{c_5}{2} r^2 + c_6 r + c_7 \right) - \frac{c}{2} c_5 e^{-\sqrt{\frac{c}{a}t}} \right] \\
+ \sin \theta \cos \phi \left[ e^{\sqrt{\frac{c}{a}t}} \left( \frac{c_8}{2} r^2 + c_9 r + c_{10} \right) - \frac{c}{2} c_8 e^{-\sqrt{\frac{c}{a}t}} \right] \\
+ \cos \theta \left[ e^{\sqrt{\frac{c}{a}t}} \left( \frac{c_1}{2} r^2 + c_{12} r + c_{13} \right) - \frac{c}{2} c_{11} e^{-\sqrt{\frac{c}{a}t}} \right] + c_{14} r + c_{15}, \]

\[ X^1 = -e^{-\sqrt{\frac{c}{a}t}} \sin \theta \left[ \sin \phi \left( c_5 r + c_6 \right) - \cos \phi \left( c_8 r + c_9 \right) \right] \\
+ e^{-\sqrt{\frac{c}{a}t}} \cos \theta \left( c_{11} r + c_{12} \right) + \frac{c}{2a} \sqrt{\frac{c}{a}} e^{-2\sqrt{\frac{c}{a}t}} \left( r^2 + c_{15} r \right) + c_{16}, \]

\[ X^2 = \cos \theta \sin \phi \left[ e^{\sqrt{\frac{c}{a}t}} \left( \frac{c_5}{2} r^2 + c_6 r + c_7 \right) + \frac{c}{2} c_8 e^{-\sqrt{\frac{c}{a}t}} \right] \\
- \cos \theta \cos \phi \left[ e^{\sqrt{\frac{c}{a}t}} \left( \frac{c_8}{2} r^2 + c_9 r + c_{10} \right) + \frac{c}{2} c_8 e^{-\sqrt{\frac{c}{a}t}} \right] \\
+ \sin \theta \left[ e^{\sqrt{\frac{c}{a}t}} \left( \frac{c_1}{2} r^2 + c_{12} r + c_{13} \right) + \frac{c}{2} c_{11} e^{-\sqrt{\frac{c}{a}t}} \right] + c_1 \sin \phi - c_2 \cos \phi, \]

\[ X^3 = \cot \theta \left( c_1 \cos \phi + c_2 \sin \phi \right) + c_4, \]

\[ \psi = X^0. \]  

From above, we can see that in this case the Lie algebra of CRCs is 15—dimensional.

The dimension of Lie algebra of RCs is 6 and these RCs can be obtained by setting \( \psi = 0 \). Moreover, the set of minimum KVs of Kantowski-Sachs space-times, given in (2.1.2), is contained in the set of obtained RCs in this case.

In the remaining non-degenerate Ricci tensor cases, namely ND3-ND8, a similar procedure may be followed to solve Eqs. (2.3.10)-(2.3.21). Since the calculation involved is basic and similar to the above cases, we avoid the
repetition and summarize the obtained results in the following tables:

<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>ND3</td>
<td>$B = b$</td>
<td>$X^0 = \frac{1}{a^2} (c_5 br + c_6)$, $X^1 = -c_5 \int \sqrt{r} , dt + c_7$, $X^2 = c_1 \sin \phi - c_2 \cos \phi$, $X^3 = \cot \theta (c_1 \cos \phi + c_2 \sin \phi) + c_4$</td>
<td>$\psi = 0$</td>
</tr>
<tr>
<td></td>
<td>$C = c$</td>
<td>$X^0 = \frac{1}{a^2} (c_5 br + c_6)$</td>
<td></td>
</tr>
</tbody>
</table>

| ND4  | $C = \sqrt{a^2}$ | $X^0 = \frac{1}{\sqrt{a}} \sin \theta \sin \phi \left[ \frac{c_2}{a^2} \left( r^2 - \frac{t^2}{2} \right) + c_6 \gamma + c_7 \right] - \frac{b}{a} \sin \theta \cos \phi \left[ \frac{c_2}{a^2} \left( r^2 - \frac{t^2}{2} \right) + c_6 \gamma + c_7 \right] + c_8 \psi = 0$ | $X^0$, $X^1$, $X^2$, $X^3$ |
|      | $A = a$     | $X^1 = -\frac{b}{a} \sin \theta \sin \phi (c_6 \gamma + c_7) + \frac{b}{a} \sin \theta \cos \phi (c_6 \gamma + c_7)$ | |
|      | $B = b$     | $X^2 = \cos \theta \sin \phi \left[ \frac{c_2}{a^2} \left( r^2 + \frac{t^2}{2a^2} \right) + c_6 \gamma + c_7 \sqrt{a} \right] - \frac{b}{a} \cos \theta \cos \phi \left[ \frac{c_2}{a^2} \left( r^2 + \frac{t^2}{2a^2} \right) + c_6 \gamma + c_7 \sqrt{a} \right] + c_9 \psi$ | |
|      |             | $X^3 = \cos \theta \cos \phi \left[ \frac{c_2}{a^2} \left( r^2 + \frac{t^2}{2a^2} \right) + c_6 \gamma + c_7 \sqrt{a} \right] + \frac{b}{a} \cos \theta \sin \phi \left[ \frac{c_2}{a^2} \left( r^2 + \frac{t^2}{2a^2} \right) + c_6 \gamma + c_7 \sqrt{a} \right] + c_4$ | |

Table 2.3: CRCs for Non-Degenerate Ricci Tensor
### Constraints

<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>ND5</td>
<td>$\frac{B}{\alpha } = e^{-2\alpha}$ where $\alpha = \int \sqrt{F} dt$</td>
<td>$X^0 = \sqrt{F} \sin \theta \sin \phi \left( \frac{4a}{2} r^2 + c_6 r + c_7 - \frac{4b}{2} C \right) - \cos \theta \left( \frac{4a}{2} r^2 + c_9 r + c_{10} \right)$</td>
<td>$\psi = X^0_0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X^1 = \sqrt{F} \sin \theta \left[ -\sin \phi \left( c_5 r + c_9 \right) + \cos \theta \left( c_7 r + c_6 \right) \right]$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ \cos \theta \sqrt{F} \left( c_7 r + c_6 \right) \left( r^2 - \frac{2}{r} \right) + c_{15} \sqrt{F} + c_{16}$,</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X^2 = \sqrt{F} \cos \theta \left( \frac{4a}{2} r^2 + c_6 r + c_7 + \frac{4b}{2} C \right) - \cos \theta \left( \frac{4a}{2} r^2 + c_9 r + c_{10} \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ \frac{4a}{2} C \right) + \sin \theta \sqrt{F} \left( c_6 r + c_7 + \frac{4b}{2} C \right) + c_1 \sin \phi - c_2 \cos \phi$,</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X^3 = \sqrt{F} \csc \theta \left[ \cos \phi \left( c_5 r^2 + c_6 r + c_7 + \frac{4b}{2} C \right) + \sin \phi \left( \frac{4a}{2} r^2 + c_9 r + c_{10} \right) \right] + \cos \phi \left( c_1 \cos \phi + c_2 \sin \phi \right) + c_4$.</td>
<td></td>
</tr>
<tr>
<td>ND6</td>
<td>$C = \alpha^2$ where $\alpha = \int \sqrt{F} dt$</td>
<td>$X^0 = \sin \theta \sin \phi \left[ \frac{c_7 r^2 + c_6 r + c_7 - c_8 C}{c_6 C} \right] - \sin \theta \cos \phi \left( \frac{4a}{2} r^2 + c_9 r + c_{10} \right)$</td>
<td>$\psi = X^0_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$- \frac{c_8 C}{c_6 C} \right] - \cos \theta \left( \frac{4a}{2} r^2 + c_9 r + c_{10} \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ \cos \theta \sqrt{F} \left( c_7 r + c_6 \right) \left( r^2 - \frac{2}{r} \right) + c_{15} \sqrt{F} + c_{16}$,</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X^1 = \frac{4a}{2} \sin \theta \sin \phi \left( c_5 r + c_9 \right) + \frac{4a}{2} \sin \theta \cos \phi \left( c_6 r + c_7 \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ \cos \theta \sqrt{F} \left( c_7 r + c_6 \right) \left( r^2 - \frac{2}{r} \right) + c_{15} \sqrt{F} + c_{16}$,</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X^2 = \cos \theta \sin \phi \left( c_5 r^2 + c_6 r + c_7 + \frac{4b}{2} C \right) - \cos \theta \cos \phi \left( \frac{4a}{2} r^2 + c_9 r + c_{10} \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ \frac{4a}{2} C \right) + \sin \theta \sqrt{F} \left( c_6 r + c_7 + \frac{4b}{2} C \right) + c_1 \sin \phi - c_2 \cos \phi$,</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X^3 = \frac{4a}{2} \cos \theta \cos \phi \left( \frac{4a}{2} r^2 + c_9 r + c_{10} \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+ \cos \theta \sin \phi \left( \frac{4a}{2} r^2 + c_9 r + c_{10} \right) \right] + \cos \theta \left( c_1 \cos \phi + c_2 \sin \phi \right) + c_4$.</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.4: CRCs for Non-Degenerate Ricci Tensor
Table 2.5: CRCs for Non-Degenerate Ricci Tensor
2.4 Some Kantowski-Sachs Metrics admitting CRCs

In the previous section, we have completely classified the Kantowski-Sachs spacetimes according to their CRCs. However, the obtained CRCs are found subject to some differential constraints to be satisfied by the Ricci tensor components. In order to show that the obtained classes of CRCs are non-empty, one needs to solve these differential constraints, so that we can get the exact form of the Kantowski-Sachs metric admitting these CRCs. Due to the high non-linearity of these constraints, it is not easy to solve them generally. However, we may choose some specific values of the metric functions satisfying these constraints. In this section, we present some of such metrics.

Example 2.4.1: Taking $\alpha(t) = e^t$ and $\beta(t) = \xi$, where $\xi$ is a non-zero constant, the Kantowski-Sachs metric, given in Eq. (2.1.1), becomes:

$$ds^2 = dt^2 - e^{2t} dr^2 - \xi^2 \left[d\theta^2 + \sin^2 \theta d\phi^2\right].$$

(2.4.1)

The above metric satisfies the differential constraints of case ND2. The CRCs for this metric are given in (2.3.44). The components of energy-momentum tensor for this metric are $T_{00} = \frac{1}{\xi^2}$, $T_{11} = -\frac{e^{2t}}{\xi^2}$, $T_{22} = -\xi^2$ and $T_{33} = -\xi^2 \sin^2 \theta$.

Example 2.4.2: If we choose $\beta(t) = \xi$, with $\xi$ a non-zero constant, and restrict $\alpha(t)$ to satisfy $\alpha \alpha'' = 1$, then the Kantowski-Sachs metric reduces to:

$$ds^2 = dt^2 - \alpha^2(t) dr^2 - \xi^2 \left[d\theta^2 + \sin^2 \theta d\phi^2\right].$$

(2.4.2)
which satisfies the constraints of case \textbf{ND3}. The RCs admitted by the above metric are listed in Table 2.2 and the components of energy-momentum tensor are $T_{00} = \frac{1}{\xi^2}$, $T_{11} = -\frac{\alpha^2}{\xi^2}$, $T_{22} = -\xi^2 \frac{\alpha''}{\alpha}$ and $T_{33} = -\xi^2 \sin^2 \theta \frac{\alpha''}{\alpha}$.

**Example 2.4.3:** The following metric satisfies the constraints of case \textbf{D3}:

$$ds^2 = dt^2 - e^{-4t} dr^2 - e^{-2t} \left[ d\theta^2 + \sin^2 \theta d\phi^2 \right].$$  \hspace{1cm} (2.4.3)

This metric admits infinite-dimensional Lie algebra of CRCs, presented in Table 2.1 and the corresponding components of energy-momentum tensor are $T_{00} = e^{-2t} - 3$, $T_{11} = -3e^{-4t} - e^{-6t}$, $T_{22} = -3e^{2t}$ and $T_{33} = -3e^{2t} \sin^2 \theta$.

In a similar way, one may find some other metrics corresponding to the remaining cases.

### 2.5 Summary and Discussion

In this chapter, a complete classification of Katowski-Sachs spacetimes via their CRCs is presented in both degenerate as well as non-degenerate cases. For degenerate Ricci tensor, six different cases are discussed and it is observed that in each case the Katowski-Sachs spacetimes possess infinite-dimensional Lie algebra of CRCs. When the Ricci tensor is non-degenerate, the CRC equations are solved in eight different cases depending upon one, two or three components of the Ricci tensor are constant. In these cases, it is shown that the Kantowski-Sachs spacetimes either admit a 15-dimensional Lie algebra of CRCs or otherwise the CRCs reduce to RCs, the dimension of Lie algebra of RCs being 6. Setting $\psi = 0$, it can be seen that the dimension of Lie algebra of RCs for Kantowski-Sachs metric is 4, 6 or 10.
In almost all the cases considered in our classification, the CRCs are found subject to some non-linear differential constraints. In order to show that the classes of CRCs are non-empty, we have presented some examples of Kantowski-Sachs metrics satisfying these constraints.

Moreover, during the classification we have not mentioned any source of the energy-momentum tensor. However, until we specify the source of the energy-momentum tensor, the study of EFEs with non-zero energy-momentum tensor may not give physically interesting results. For physical implications, we assume that the matter source is a perfect fluid, then the energy-momentum tensor has the form $T_{ab} = (p + \rho)u_au_b - pg_{ab}$; $p, \rho$ and $u^a$ being pressure, density and four velocity of the perfect fluid. For Kantowski-Sachs metric, given in Eq. (2.1.1), we have $u^a = \delta^a_0$, so that $T_{00} = \rho$, $T_{11} = p\alpha^2$, $T_{22} = p\beta^2$ and $T_{33} = p\beta^2 \sin^2 \theta$. Using these values, the Ricci tensor components, given in Eq. (2.1.3), get the form:

$$A(t) = \frac{\rho + 3p}{2}, \quad B(t) = \frac{\alpha^2}{2} (\rho - p), \quad C(t) = \frac{\beta^2}{2} (\rho - p). \quad (2.5.1)$$

In case when the Ricci tensor is degenerate, a perfect fluid is allowed in only two cases, namely D2 and D4. In case D2, Eq. (2.5.1) gives $\rho + 3p = 0$, which violates the strong energy condition. Following is an example of Kantowski-Sachs metric satisfying the constraints of this case:

$$ds^2 = dt^2 - \left(\cos \sqrt{2}t + \sin \sqrt{2}t\right)^2 dr^2 - e^{2t} \left[d\theta^2 + \sin^2 \theta \, d\phi^2\right]. \quad (2.5.2)$$

The energy-momentum tensor components for the above metric are:

$$T_{00} = \frac{2\sqrt{2}(\cos \sqrt{2}t - \sin \sqrt{2}t)}{\cos \sqrt{2}t + \sin \sqrt{2}t} + e^{-2t} + 1, \quad T_{11} = -(1 + 2 \cos \sqrt{2}t \sin \sqrt{2}t)(3 + e^{-2t}),$$
\[ T_{22} = -e^{2t} \left( -1 + \sqrt{2} \cos \sqrt{2t} - \sin \sqrt{2t} \right), \]
\[ T_{33} = \sin^2 \theta T_{22}. \] (2.5.3)

Similarly, the constraints of case D4 yields \( p = \rho \), which denotes a stiff matter. The following metric satisfies the constraints of this case:

\[ ds^2 = dt^2 - dr^2 - (2 + 2t - t^2) \left[d\theta^2 + \sin^2 \theta \ d\phi^2\right]. \] (2.5.4)

For the above metric, the energy-momentum tensor has the following components:

\[ T_{00} = \frac{4t^2 + 8t + 5}{(2 + 2t - t^2)^2}, \]
\[ T_{11} = -\frac{8t^2 - 16t - 3}{(2 + 2t - t^2)^2}, \]
\[ T_{22} = -2t^2 + 4t + 4, \]
\[ T_{33} = (-2t^2 + 4t + 4) \sin^2 \theta. \] (2.5.5)

In all the sub cases of the non-degenerate Ricci tensor case, a perfect fluid can be considered as a matter source. Eq. (2.4.1) is an example of perfect fluid kantowski-Sachs metric admitting 15-dimensional Lie algebra of CRCs in non-degenerate case, where the pressure \( p \) and the energy density \( \rho \) of the fluid are related as \( \rho + 3p = 2 \) and \( \rho - p = \frac{2}{\xi^2} \). Similarly the condition of perfect fluid establishes relations between pressure and energy density in the remaining non-degenerate cases. However, since the system of equations (2.1.3) are highly non-linear, it is quite difficult to find the exact form of the metric satisfying the constraints of the remaining non-degenerate cases.
Chapter 3

Conformal Ricci Collineations of the Metric $ds^2 = e^{A(r)}dt^2 - e^{B(r)}dr^2 - r^2[d\theta^2 + f_k^2(\theta)d\phi^2]$

In this chapter, we explore CRCs of the following metric:

$$ds^2 = e^{A(r)}dt^2 - e^{B(r)}dr^2 - r^2[d\theta^2 + f_k^2(\theta)d\phi^2], \quad (3.0.1)$$

where $A$ and $B$ are any functions of $r$ and

$$f_k(\theta) = \begin{cases} 
\sin \theta, & k = 1 \\
\theta, & k = 0 \\
\sinh \theta, & k = -1.
\end{cases}$$

For the above metric, the CRC equations are solved in degenerate and non-degenerate cases. In degenerate case, the CRC equations are directly integrated to obtain the explicit form of CRCs. In non-degenerate case, these
equations are solved to get a vector field generating CRCs in the form of some unknown functions of \( t \) and \( r \), subject to some constraints on the Ricci tensor components. These constraint are then solved for different choices of the Ricci tensor components to get the explicit form of CRCs.

It can be observed that \( k = -\frac{f_{k,2}}{f_k}, f_{k}^{2k+2}(f_{k,2})_{,2} = -1 \) and \( (f_{k,2})_{,2} = 2f_{k,2}^2 - 1 \). These relations will be frequently used in the forthcoming calculations.

### 3.1 CRC Equations

The non-vanishing Ricci tensor components for the metric (3.0.1) are:

\[
R_{00} = -e^{A-B} \left( \frac{A''}{2} - \frac{A'B'}{4} + \frac{A'^2}{4} + \frac{A'}{r} \right) = R_0(r),
\]

\[
R_{11} = -\frac{A''}{2} + \frac{A'B'}{4} - \frac{A'^2}{4} + \frac{B'}{r} = R_1(r),
\]

\[
R_{22} = e^{-B} \left( \frac{rB'}{2} - \frac{rA'}{2} - 1 \right) + k = R_2(r),
\]

\[
R_{33} = f_k^2(\theta)R_2(r).
\] (3.1.1)

Using these Ricci tensor components in Eq. (1.11.5), we obtain:

\[
R'_{0} X^1 + 2R_0 X^0_{,0} = 2\psi R_0,
\] (3.1.2)

\[
R_0 X^0_{,1} + R_1 X^1_{,1} = 0,
\] (3.1.3)

\[
R_0 X^0_{,2} + R_2 X^2_{,0} = 0,
\] (3.1.4)

\[
R_0 X^0_{,3} + R_2 f_k^2 X^3_{,0} = 0,
\] (3.1.5)

\[
R'_1 X^1 + 2R_1 X^1_{,1} = 2\psi R_1,
\] (3.1.6)

\[
R_1 X^1_{,2} + R_2 X^2_{,1} = 0,
\] (3.1.7)

\[
R_1 X^1_{,3} + R_2 f_k X^3_{,1} = 0,
\] (3.1.8)

\[
R'_2 X^1 + 2R_2 X^2_{,2} = 2\psi R_2,
\] (3.1.9)
\[
R_2 \left( X^2_3 + f_k^2 X^3_2 \right) = 0, \quad (3.1.10)
\]
\[
R'_2 \ X^1 + 2R_2 \left( \frac{f_k^2}{f_1} X^2 + X^3_3 \right) = 2\psi R_2. \quad (3.1.11)
\]
In the above system of equations, prime denotes derivative with respect to \( r \) and the commas in subscript denote the partial derivatives with respect to a spatial coordinate. To obtain the closed form of CRCs, we solve the above equations in both degenerate and non-degenerate cases.

### 3.2 CRCs for Degenerate Ricci Tensor

When the Ricci tensor is considered to be degenerate, then \( \det R_{ab} = 0 \). Thus either \( R_0 = R_1 = R_2 = 0 \), giving trivial solution, or the following possibilities arise:

- **(D1)** \( R_0 = R_1 = 0, \ R_2 \neq 0 \)
- **(D2)** \( R_0 = R_2 = 0, \ R_1 \neq 0 \)
- **(D3)** \( R_1 = R_2 = 0, \ R_0 \neq 0 \)
- **(D4)** \( R_2 = 0, \ R_0 \neq 0, \ R_1 \neq 0 \)
- **(D5)** \( R_0 = 0, \ R_1 \neq 0, \ R_2 \neq 0 \)
- **(D6)** \( R_1 = 0, \ R_0 \neq 0, \ R_2 \neq 0 \)

In case D1, we have \( R_0 = R_1 = 0, \) and \( R_2 \neq 0 \). Therefore, the system of equations (3.1.2)-(3.1.11) becomes:

\[
X^2_{,\theta} = X^3_{,\theta} = X^2_{,\phi} = X^3_{,\phi} = 0 \quad (3.2.1)
\]
\[
R'_2 \ X^1 + 2R_2 \ X^2_2 = 2\psi R_2, \quad (3.2.2)
\]
\[
X^2_3 + f_k^2 X^3_2 = 0, \quad (3.2.3)
\]
\[
R'_2 \ X^1 + 2R_2 \left( \frac{f_k^2}{f_1} X^2 + X^3_3 \right) = 2\psi R_2. \quad (3.2.4)
\]

From Eq. (3.2.1), we get \( X^2 = f^1(\theta, \phi) \) and \( X^3 = f^2_\phi(\theta, \phi) \), where \( f^1 \) and \( f^2 \) are functions of integration. We put these values in Eq. (3.2.3) and then integrate it with respect to \( \phi \), it gives \( f^1 = -\theta^2 f^2_\phi + G^1(\theta) \); \( G^1 \) being a function of integration.
of integration. From Eq. (3.2.2), we obtain the value of the inheriting factor as
\[ \psi = \frac{R'_2}{2R_2} X^1 + X^2. \]
Substituting all these values in Eq. (3.2.4), we have the following relation:
\[ \theta^2 f_\theta^2 + \theta^3 f_\theta^2 + \theta f_\phi^2 - \theta G_\theta^1 + G_1^1 = 0. \]  
(3.2.5)
Due to the non-linearity of the above equation, it is not possible to solve it generally. However, we may simplify it by certain assumption on the functions it involves. For example, if we set \( [\theta f_\phi^2]_\theta = 0 \), it gives \( f^2 = \frac{1}{\theta} G^2(\phi) + \phi G^3(\theta) + G^4(\theta) \), where \( G^2, G^3 \) and \( G^4 \) are functions of integration.

Using this value of \( f^2 \) in Eq. (3.2.5) and differentiating it twice with respect to \( \phi \), we obtain \( G^1 = -c_4 + \theta^2 G^3 \theta + c_7 \theta, \)
\( G^2 = -c_1 \cos \phi - c_2 \sin \phi + c_3 \phi + c_4 \)
and \( G^3 = -\frac{c_5}{\theta} + c_5 \ln \theta + c_6. \) Hence the solution of Eqs. (3.2.1)-(3.2.4) becomes:
\[ X^i = X^i(x^a), \text{ for } i = 0, 1, \]
\[ X^2 = -c_1 \cos \phi - c_2 \sin \phi - c_5 \theta \phi + c_7 \theta, \]
\[ X^3 = \frac{1}{\theta} (c_1 \sin \phi - c_2 \cos \phi) + c_5 \ln \theta + c_6, \]
\[ \psi = \frac{R'_2}{2R_2} X^1 + X^2. \]  
(3.2.6)
Since \( X^0 \) and \( X^1 \) depend on all spacetime coordinates, one can proceed as we did in chapter 1 to prove that the Lie algebra of CRCs is infinite-dimensional in this case.

Proceeding in a similar way, the remaining degenerate cases can be solved easily to get the final form of CRCs. Since the procedure is quite simple and similar to the above case, we avoid the repeated calculations and the obtained results of all the remaining degenerate cases are listed in the following table.
<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>D2</td>
<td>—</td>
<td>$X^1 = f(r)$, $X^i = X^i(x^n)$, for $i = 0, 2, 3$.</td>
<td>$\psi = \frac{R_0}{\sqrt{f(r)}} f'(r)$</td>
</tr>
<tr>
<td>D3</td>
<td>—</td>
<td>$X^0 = f(t)$, $X^i = X^i(x^n)$, for $i = 1, 2, 3$.</td>
<td>$\psi = \frac{R_0}{\sqrt{f(t)}} X^1$</td>
</tr>
<tr>
<td>D4</td>
<td>—</td>
<td>$X^0 = \left(c_1 t + c_2\right) f \sqrt{\frac{df}{dt}} dr + c_3 t + c_4$.</td>
<td>$\psi = \frac{R_0}{\sqrt{f(t)}} X^1$</td>
</tr>
<tr>
<td>D5</td>
<td>—</td>
<td>$X^0 = X^0(x^n)$, $X^1 = -\sqrt{\frac{R_0}{R_2}} \theta \left(c_1 \sin \phi - c_2 \cos \phi\right) - \sqrt{\frac{R_0}{R_2}} c_3$.</td>
<td>$\psi = \frac{R_0}{\sqrt{f(t)}} X^1$</td>
</tr>
<tr>
<td>D6(a)</td>
<td>$R_0 \neq R_2$</td>
<td>$X^0 = f(t)$, $X^1 = \frac{2R_0}{R_2} \left[-f_x(t) - \frac{2}{3\phi^3} \left(c_1 \cos 3\phi + c_2 \sin 3\phi\right) + c_3 \phi + c_4\right]$.</td>
<td>$\psi = \frac{R_0}{\sqrt{f(t)}} X^1$</td>
</tr>
<tr>
<td>D6(b)</td>
<td>$R_0 = R_2$</td>
<td>$X^0 = \frac{2}{3\phi^3} \phi^2 - \frac{4}{3\phi^2} t^2 - c_2 t - c_3 - \theta \left(c_4 \sin \phi - c_5 \cos \phi\right)$.</td>
<td>$\psi = \frac{R_0}{\sqrt{f(t)}} X^1$</td>
</tr>
</tbody>
</table>

Table 3.1: CRCs for Degenerate Ricci Tensor
3.3 CRCs for Non-degenerate Ricci Tensor

For non-degenerate Ricci tensor, that is $detR_{ab} \neq 0$, all the three Ricci tensor components $R_0$, $R_1$ and $R_2$ are non-zero. As in the previous chapter, first we integrate Eqs. (3.1.2)-(3.1.11) to get a solution involving unknown functions of $t$ and $r$ only.

Differentiating Eqs. (3.1.7) and (3.1.8) with respect to $\phi$ and $\theta$ respectively and subtracting, we obtain:

$$X_{,13}^2 - f_k^2 X_{,12}^3 - 2 f_k f_{k,2} X_{,1}^3 = 0.$$  \hfill (3.3.1)

Next, we differentiate Eq. (3.1.10) with respect to $t$ and subtracting the obtained result from Eq. (3.3.1). It gives $X^3 = \frac{1}{f_k} P_1(t, r, \phi) + P_2(t, \theta, \phi)$, where the functions $P_1$ and $P_2$ arise during integration. Similarly, differentiating Eqs. (3.1.4) and (3.1.5) with respect to $\phi$ and $\theta$ and then subtracting, we get:

$$X_{,03}^2 - f_k^2 X_{,02}^3 - 2 f_k f_{k,2} X_{,0}^3 = 0.$$  \hfill (3.3.2)

Next, differentiating Eq. (3.1.10) with respect to $r$ and subtracting the obtained result from Eq. (3.3.2), we have $P_2 = \frac{1}{f_k} F_1(t, \phi) + F_2(t, \theta, \phi)$; $F_1$ and $F_2$ being functions of integration. If we merge the function $F_1$ in $P_1$, the value of $X^3$ reduces to $X^3 = \frac{1}{f_k} P_1 + F_2$. With this value of $X^3$, Eq. (3.1.10) gives $X^2 = f_{k,2} P_1 - f_k^2 F_2 + P_3(t, r, \theta)$, where $P_3$ is a function of integration. Substituting this value of $X^2$ in Eqs. (3.1.4) and (3.1.7), we get:

$$X^0 = -\frac{R_2}{R_0} [f_k P_1 + P_3^1] + P_4(t, r, \phi),$$  \hfill (3.3.3)

$$X^1 = -\frac{R_2}{R_1} [f_k P_1 + P_3^1] + P_5(t, r, \phi).$$  \hfill (3.3.4)
where $P^4$ and $P^5$ arise during integration process. Simplifying Eqs. (3.1.5) and (3.1.8) using these values, we obtain $P^4 = F^3(t, r)$ and $P^5 = F^4(t, r)$ respectively, where $F^3$ and $F^4$ are integrating functions. Finally, we have:

\[ X^0 = -\frac{R_2}{R_0} \left[ f_k P^1_t + P^3_t \right] + F^3, \]
\[ X^1 = -\frac{R_2}{R_1} \left[ f_k P^1_r + P^3_r \right] + F^4, \]
\[ X^2 = f_{k,2} P^1 - f_k^2 F_\theta^2 + P^3_\theta, \]
\[ X^3 = \frac{1}{f_k} P^1_\phi + F^2_\phi. \] (3.3.5)

Subtracting Eq. (3.1.11) from Eq. (3.1.9) and using the above values, we get:

\[ P^1_t + f_k^2 f_{k,2} F_\theta^2 - f_k P^3_\theta + f_{k,2} P^3_\theta + f_k^3 F_\theta^2 + P^3_\phi + f_k F^2_\phi = 0. \] (3.3.6)

The following values are obtained by differentiating the above equation with respect to $r$ and $\phi$:

\[ P^1 = \sin \phi F^5(t, r) - \cos \phi F^6(t, r) + F^7(t, \phi) + F^8(t, r), \]
\[ P^3 = F^8(t, r) \int \left( f_k \int \frac{d\theta}{f_k^2} \right) d\theta + F^9(t, r) \int f_k d\theta + F^{10}(t, \theta) + F^{11}(t, r). \]

The functions $F^5, ..., F^{11}$ appearing above arise during the process of integration. Putting back these values of $P^1$ and $P^3$ in Eq. (3.3.6), we have:

\[ F^7 + f_k^2 f_{k,2} F_\theta^2 - f_k F_{\theta\theta}^{10} + f_{k,2} F_{\theta\theta}^{10} + f_k F_\theta^{10} + F^7_\phi + f_k F^2_\phi = 0. \] (3.3.7)

Differentiating the above equation with respect to $t$ and $\phi$ and doing some simple algebraic calculations, we get the values of the functions $F^2, F^7$ and $F^{10}$. We put back these values and obtain the following system satisfying
Eqs. (3.1.4), (3.1.5), (3.1.7), (3.1.8), (3.1.10) and the difference of (3.1.9) and (3.1.11).

\[ X^0 = -\frac{R_2}{R_0} \left[ f_k F_5^5 \sin \phi - f_k F_6^5 \cos \phi + F_5^9 \int f_k d\theta \right] + F^3, \]

\[ X^1 = -\frac{R_2}{R_1} \left[ f_k F_5^5 \sin \phi - f_k F_6^5 \cos \phi + F_5^9 \int f_k d\theta \right] + F^4, \]

\[ X^2 = f_{k,2} F_5^5 \sin \phi - f_{k,2} F_6^6 \cos \phi + f_k F_9^5 + \frac{a_1}{2} f_k \left( \int \frac{d\theta}{f_k} \right)^2 \]
\[ + \ a_3 f_k \int \frac{d\theta}{f_k} - f_k \left( \frac{a_1}{2} \phi^2 + a_2 \phi \right) + b_1 \sin \phi - b_2 \cos \phi, \]

\[ X^3 = \frac{1}{f_k} F_5^5 \cos \phi + \frac{1}{f_k} F_6^6 \sin \phi + \frac{f_{k,2}}{f_k} \left( b_1 \cos \phi + b_2 \sin \phi \right) \]
\[ + \ (a_1 \phi + a_2) \int \frac{d\theta}{f_k} + a_3 \phi + b_3. \]  

(3.3.8)

The inheriting factor is obtained from Eq. (3.1.9) as \( \psi = \frac{R_2'}{2R_2} X^1 + X^2 \).

Inserting the above values of \( X^a \) and the inheriting factor in Eqs. (3.1.2), (3.1.3) and (3.1.6) and comparing the like terms on both sides of the resulting equations, we get \( a_1 = a_2 = a_3 = 0 \) and the following twelve integrability conditions are generated:

\[ \left( \frac{R_2}{R_0} \right)' F_i^5 + \frac{2R_2}{R_0} F_i^6 = 0, \]  

(3.3.9)

\[ \left( \frac{R_2}{2R_2} \right)' f_k F_r^i - \frac{R_0}{R_0} f_k F_t^i - \frac{R_1}{R_2} f_{k,22} F_i^5 = 0, \]  

(3.3.10)

\[ \left( \frac{R_2}{2R_1} \right)' f_k F_r^i + \frac{R_2}{R_1} f_k F_t^i + f_{k,22} F_i^5 = 0, \]  

(3.3.11)

\[ R_0 F_r^3 + R_1 F_t^4 = 0, \]  

(3.3.12)

\[ \left( \frac{R_2'}{2R_0} - \frac{R_2'}{2R_2} \right)' F^4 + F^3 = F^j, \]  

(3.3.13)

\[ \frac{1}{2} \left( \frac{R_1'}{R_1} - \frac{R_2'}{R_2} \right)' F^4 + F^4 = F^j, \]  

(3.3.14)
where \( i = 5, 6, 9 \) and \( F^j = F^9 \), if \( f_k = \theta \) and \( F^j = 0 \), otherwise. It can be seen that for \( f_k = \sin \theta \), the CRC equations (3.1.2)-(3.1.11) for the spacetimes under consideration reduce to the CRC equations of static spherically symmetric spacetimes [9]. The above integrability conditions are integrated in [9] for \( f_k = \sin \theta \) to obtain the explicit form of CRCs and it is concluded that static spherically symmetric spacetimes possess 15-dimensional Lie algebra of CRCs. Probably, the CRCs for \( f_k = \sinh \theta \) will be same as those for \( f_k = \sin \theta \) except that the trigonometric functions will be replaced by hyperbolic. Therefore, we exclude these two cases from our discussion and solve the integrability conditions, Eqs. (3.3.9)-(3.3.14), only in the case when \( f_k = \theta \). In order to have a complete classification, we consider the following cases:

(ND1) \( R_0' = R_1' = R_2' = 0 \)

(ND2) \( R_0' = R_2' = 0, R_1' \neq 0 \)

(ND3) \( R_1' = R_2' = 0, R_0' \neq 0 \)

(ND4) \( R_0' = R_1' = 0, R_2' \neq 0 \)

(ND5) \( R_2' = 0, R_0' \neq 0, R_1' \neq 0 \)

(ND6) \( R_0' = 0, R_1' \neq 0, R_2' \neq 0 \)

(ND7) \( R_1' = 0, R_0' \neq 0, R_2' \neq 0 \)

(ND8) \( R_2' \neq 0, R_1' \neq 0, R_2' \neq 0 \)

In the case when \( R_0, R_1 \) and \( R_2 \) are all constants, say \( R_0 = a, R_1 = b \) and \( R_2 = c \), that is case ND1, the integrability conditions (3.3.9)-(3.3.14) become:

\[
F_{tr}^5 = F_{tr}^6 = F_{tr}^9 = 0, \quad (3.3.15)
\]

\[
F_{tt}^5 = F_{tt}^6 = F_{tt}^9 = 0, \quad (3.3.16)
\]

\[
F_{rr}^5 = F_{rr}^6 = F_{rr}^9 = 0, \quad (3.3.17)
\]

\[
aF_r^3 + bF_t^4 = 0, \quad (3.3.18)
\]
\[ F_t^3 = F^9, \quad F_r^3 = F^9. \] (3.3.19)

Solving Eqs. (3.3.15), (3.3.16) and (3.3.17), we obtain:

\[ F^5 = c_1 t + c_8 r, \] (3.3.21)
\[ F^6 = c_2 t + c_9 r, \] (3.3.22)
\[ F^9 = c_3 t + c_4 r + c_5. \] (3.3.23)

Using these values in Eqs. (3.3.18), (3.3.19) and (3.3.20) and integrating, we have:

\[ F^3 = \frac{c_3}{2} \left( t^2 - \frac{b}{a} r^2 \right) + c_4 tr + c_5 t + \frac{c_6}{a} r + c_7, \] (3.3.24)
\[ F^4 = \frac{c_4}{2} \left( r^2 - \frac{a}{b} t^2 \right) + c_3 tr - \frac{c_6}{b} t + c_5 r + c_{10}. \] (3.3.25)

Hence the complete solution of the integrability conditions gives:

\[ X^0 = \theta (-c_1 \sin \phi + c_2 \cos \phi) - \frac{c_3}{2} \left( \theta^2 - t^2 + \frac{b}{a} r^2 \right) + c_4 tr + c_5 t + \frac{c_6}{a} r + c_7 \\
- t\theta(c_{14} \cos \phi - c_{15} \sin \phi), \]
\[ X^1 = \theta (-c_8 \sin \phi + c_9 \cos \phi) - \frac{c_4}{2} \left( \theta^2 + \frac{a}{b} t^2 - r^2 \right) + c_3 tr - \frac{c_6}{b} t + c_5 r + c_{10} \\
- r\theta(c_{14} \cos \phi - c_{15} \sin \phi), \]
\[ X^2 = \sin \phi(c_1 t + c_8 r + c_{11}) - \cos \phi(c_2 t + c_9 r + c_{12}) + \theta(c_3 t + c_4 r + c_5) \\
+ \frac{1}{2} (r^2 - t^2 - \theta^2) (c_{14} \cos \phi - c_{15} \sin \phi), \]
\[ X^3 = \frac{1}{\theta} \cos \phi(c_1 t + c_8 r + c_{11}) + \frac{1}{\theta} \sin \phi(c_2 t + c_9 r + c_{12}) + c_{13} \\
+ \frac{1}{2} (t^2 - r^2 - \theta^2) (c_{14} \sin \phi - c_{15} \cos \phi), \]
\[ \psi = \xi_2^2. \] (3.3.26)
From above, we can see that the metric under consideration admits a 15-dimensional Lie algebra of CRCs.

The remaining cases of non-degenerate Ricci tensor can be solved by a similar procedure as followed in case ND1. To avoid the repetition, we present the obtained results of the cases ND2-ND8 in the following tables.

<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>ND2</td>
<td>$R_3 = a$</td>
<td>$X^0 = \sqrt{R_1} \sin \phi \left( c_1 t + c_2 \right)$</td>
<td>$\psi = X^2_3$</td>
</tr>
<tr>
<td></td>
<td>$R_2 = b$</td>
<td>$X^1 = \sqrt{R_1} \left( c_1 t + c_2 \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X^2 = \sqrt{R_1} \left( c_1 t + c_2 \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X^3 = \sqrt{R_1} \left( c_1 t + c_2 \right)$</td>
<td></td>
</tr>
<tr>
<td>ND3(a)</td>
<td>$R_0 = \left( c_3 r + c_4 \right)^a$, $R_1 = a, R_2 = b$</td>
<td>$X^0 = c_1 t + c_2, X^1 = c_3 r + c_4$</td>
<td>$\psi = X^2_3$</td>
</tr>
</tbody>
</table>

Table 3.2: CRCs for Non-Degenerate Ricci Tensor
<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>ND3(b)</td>
<td>$R_0 = r^2$, $R_1 = a$, $R_2 = b$</td>
<td>$X^0 = \frac{b}{\sqrt{36}} \left[ \theta \sin \phi \left( c_1 \sin \frac{\theta}{\sqrt{3}} + c_2 \cos \frac{\theta}{\sqrt{3}} \right) - \theta \cos \phi \left( c_3 \sin \frac{\theta}{\sqrt{3}} - c_4 \cos \frac{\theta}{\sqrt{3}} \right) \right]$</td>
<td>$\psi = X^0_2$</td>
</tr>
<tr>
<td>ND4(a)</td>
<td>$R_2 = r^2$, $R_0 = a$, $R_1 = b$</td>
<td>$X^0 = \frac{\sqrt{3}}{2} (t^2 - \frac{b^2}{2}) + c_2 t + c_3$, $X^1 = \frac{\sqrt{3}}{2} (c_1 t + c_2)$, $X^2 = c_4 \sin \phi - c_5 \cos \phi$, $X^3 = \frac{1}{2} (c_4 \cos \phi + c_5 \sin \phi) + c_6$</td>
<td>$\psi = X^0_0$</td>
</tr>
<tr>
<td>ND4(b)</td>
<td>$R_2 = e^{2r}$, $R_0 = a$, $R_1 = b$</td>
<td>$X^0 = -\frac{\sqrt{3}}{2} \left[ \theta \sin \phi \left( c_1 \sin \frac{\sqrt{3} \phi}{\sqrt{3}} + c_2 \cos \frac{\sqrt{3} \phi}{\sqrt{3}} \right) - \theta \cos \phi \left( c_3 \sin \frac{\sqrt{3} \phi}{\sqrt{3}} - c_4 \cos \frac{\sqrt{3} \phi}{\sqrt{3}} \right) \right]$</td>
<td>$\psi = X^0_2$</td>
</tr>
<tr>
<td>ND5(a)</td>
<td>$R_0 = e^{2\alpha}$, $\alpha = \frac{1}{\sqrt{4\pi d r}}$, $R_2 = a$</td>
<td>$X^0 = \frac{1}{\sqrt{4\pi d r}} \left[ t^2 - 2c_2 t + c_3 \right]$, $X^1 = \frac{1}{\sqrt{4\pi d r}} (c_1 t + c_2)$, $X^2 = c_4 \sin \phi - c_5 \cos \phi$, $X^3 = \frac{1}{2} (c_4 \cos \phi + c_5 \sin \phi) + c_6$</td>
<td>$\psi = 0$</td>
</tr>
</tbody>
</table>

Table 3.3: CRCs for Non-Degenerate Ricci Tensor
<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
</table>
| ND5(b) | $R_0 = (\alpha)^2$ | $X^0 = -\frac{1}{\sqrt{R_0}} \left[ \theta \sin \phi (c_1 \cos t + c_2 \sin t) - \theta \cos \phi (c_3 \cos t + c_4 \sin t) 
+ \frac{\phi^2}{2 \sqrt{R_0}} (c_5 \cos t + c_6 \sin t) - \frac{\sqrt{R_0}}{2 \sqrt{R_1}} (c_7 \cos t + c_8 \sin t), 
- \frac{R_0}{\sqrt{R_1}} (c_{11} \sin \phi - c_{12} \cos \phi) \right]$ | $\psi = X_2^2$ |
|        | $\alpha = \int \sqrt{R_1} dr$ | $R_2 = a$                                                                |                   |
|        | $X^1 = -\frac{1}{\sqrt{R_1}} \left[ \theta \sin \phi (c_1 \sin t - c_2 \cos t) - \theta \cos \phi (c_3 \sin t - c_4 \cos t) 
+ \frac{\phi^2}{2 \sqrt{R_1}} (c_5 \sin t - c_6 \cos t) + \frac{\sqrt{R_0}}{2 \sqrt{R_1}} (c_7 \sin t - c_8 \cos t) 
+ \frac{\sqrt{R_0}}{2 \sqrt{R_1}} (c_{11} \sin \phi - c_{12} \cos \phi) \right]$ |                   |
|        | $X^2 = \sqrt{R_0} \left[ \sin \phi (c_1 \sin t - c_2 \cos t) - \cos \phi (c_3 \sin t - c_4 \cos t) + \theta (c_5 \sin t - c_6 \cos t) + b_1 \sin \phi - b_2 \cos \phi + \theta, \right.$ |
|        | $+ (\frac{2}{\sqrt{R_1}} - \frac{\phi^2}{2 \sqrt{R_1}}) (c_{11} \sin \phi - c_{12} \cos \phi) \right]$ |                   |
|        | $X^3 = \sqrt{R_0} \left[ \cos \phi (c_1 \sin t - c_2 \cos t) + \sin \phi (c_3 \sin t - c_4 \cos t) 
+ \frac{\phi^2}{2 \sqrt{R_1}} (c_{11} \sin \phi + c_{12} \cos \phi) + b_3 \right]$ |                   |
|        |                                                      |                   |
|        | Same as given in case ND4(a) with $R_2 \rightarrow \frac{R_0}{\sqrt{R_1}}$ in $X^1$ |                                                      | $\psi = X_0^0$ |
|        |                                                      |                   |
| ND6(a) | $R_2 = (\int \sqrt{R_1} dr)^2$ | $X^0 = -\frac{1}{\sqrt{R_0}} \left[ \theta \sin \phi (c_1 \cos \sqrt{\pi t} + c_2 \sin \sqrt{\pi t}) - \theta \cos \phi (c_3 \cos \sqrt{\pi t} + c_4 \sin \sqrt{\pi t}) 
+ \frac{\phi^2}{2 \sqrt{R_0}} (c_5 \cos \sqrt{\pi t} + c_6 \sin \sqrt{\pi t}) - \frac{\sqrt{R_0}}{2 \sqrt{R_1}} (c_7 \cos \sqrt{\pi t} + c_8 \sin \sqrt{\pi t}), 
- \frac{R_0}{\sqrt{R_1}} (c_{11} \sin \phi - c_{12} \cos \phi) \right]$ | $\psi = X_0^0$ |
|        |                                                      |                   |
|        |                                                      |                   |
| ND6(b) | $R_2 = e^2 / \sqrt{R_1} dr$ | $X^0 = -\frac{1}{\sqrt{R_0}} \left[ \theta \sin \phi (c_1 \cos \sqrt{\pi t} + c_2 \sin \sqrt{\pi t}) - \theta \cos \phi (c_3 \cos \sqrt{\pi t} + c_4 \sin \sqrt{\pi t}) 
+ \frac{\phi^2}{2 \sqrt{R_0}} (c_5 \cos \sqrt{\pi t} + c_6 \sin \sqrt{\pi t}) - \frac{\sqrt{R_0}}{2 \sqrt{R_1}} (c_7 \cos \sqrt{\pi t} + c_8 \sin \sqrt{\pi t}), 
- \frac{R_0}{\sqrt{R_1}} (c_{11} \sin \phi - c_{12} \cos \phi) \right]$ | $\psi = X_0^0$ |
|        | $R_0 = a$ |                                                      |                   |
|        |                                                      |                   |
|        | $X^1 = \sqrt{R_0} \left[ \theta \sin \phi (c_1 \sin \sqrt{\pi t} - c_2 \cos \sqrt{\pi t}) - \theta \cos \phi (c_3 \sin \sqrt{\pi t} - c_4 \cos \sqrt{\pi t}) 
+ \frac{\phi^2}{2 \sqrt{R_0}} (c_5 \sin \sqrt{\pi t} - c_6 \cos \sqrt{\pi t}) - \frac{\sqrt{R_0}}{2 \sqrt{R_1}} (c_7 \sin \sqrt{\pi t} - c_8 \cos \sqrt{\pi t}), 
+ \frac{\sqrt{R_0}}{2 \sqrt{R_1}} (c_{11} \sin \phi - c_{12} \cos \phi) \right]$ |                   |
|        |                                                      |                   |
|        | $X^2 = \frac{1}{\sqrt{R_0}} \left[ \sin \phi (c_1 \sin \sqrt{\pi t} - c_2 \cos \sqrt{\pi t}) - \cos \phi (c_3 \sin \sqrt{\pi t} - c_4 \cos \sqrt{\pi t}) + \theta (c_5 \sin \sqrt{\pi t} - c_6 \cos \sqrt{\pi t}) + b_1 \sin \phi - b_2 \cos \phi + \theta, \right.$ |                   |
|        | $+ (\frac{2}{\sqrt{R_1}} - \frac{\phi^2}{2 \sqrt{R_1}}) (c_{11} \sin \phi - c_{12} \cos \phi) \right]$ |                   |
|        | $X^3 = \frac{1}{\sqrt{R_0}} \left[ \cos \phi (c_1 \sin \sqrt{\pi t} - c_2 \cos \sqrt{\pi t}) + \sin \phi (c_3 \sin \sqrt{\pi t} - c_4 \cos \sqrt{\pi t}) 
+ \frac{\phi^2}{2 \sqrt{R_1}} (c_{11} \sin \phi + c_{12} \cos \phi) + b_3 \right]$ |                   |

Table 3.4: CRCs for Non-Degenerate Ricci Tensor
<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCS</th>
<th>Inheriting Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>ND7(a)</td>
<td>$\frac{R_0}{\sqrt{2}} = e^{-2t} \frac{\sqrt{R_2}}{\sqrt{R_2}}$</td>
<td>Same as in case ND4(a) with $R_2 \leftrightarrow \frac{R_2}{R_0}$ in $X^0$</td>
<td>$\psi = X_1^1$</td>
</tr>
<tr>
<td>ND7(b)</td>
<td>$R_0 = R_2$</td>
<td>Same as in case ND2 with $R_1 \leftrightarrow \frac{R_1}{\sqrt{2}}$</td>
<td>$\psi = \frac{R_1}{\sqrt{2}} X_2^1$</td>
</tr>
<tr>
<td>ND8(a)</td>
<td>$\frac{R_0}{\sqrt{2}} = e^{2a}$ where $\alpha = \int \sqrt{\frac{R_1}{R_0}} , dr$</td>
<td>Same as in case ND5(a) with $R_1 \leftrightarrow \frac{R_1}{\sqrt{2}}$</td>
<td>$\psi = \frac{R_1}{\sqrt{2}} X_2^1$</td>
</tr>
<tr>
<td>ND8(b)</td>
<td>$\frac{R_0}{\sqrt{2}} = e^{2a}$ where $\alpha = \int \sqrt{\frac{R_1}{R_0}} , dr$</td>
<td>$X^0 = -\sqrt{\frac{R_2}{R_0}} \left( \frac{1}{2} \left( \frac{R_1}{R_0} \right)^2 (c_1 \sin t + c_2 \sin t) - \frac{1}{2} \left( \frac{R_1}{R_0} \right)^2 (c_3 \cos t + c_4 \sin t) \right)$</td>
<td>$\psi = \frac{R_1}{\sqrt{2}} X_2^1$</td>
</tr>
</tbody>
</table>

Table 3.5: CRCS for Non-Degenerate Ricci Tensor
3.4 Summary

Solving the CRC equations for the metric given in Eq. (3.0.1), it is shown that this metric possesses 6-, 7- or 15-dimensional Lie algebra of CRCs in non-degenerate case, while the Lie algebra of CRCs is infinite-dimensional for degenerate case. In case when \( f_k = \sin \theta \), the chosen metric reduces to the metric of static spherically symmetric spacetimes, whose complete classification via CRCs was already presented in [9]. Probably, the same arguments are true for \( f_k = \sinh \theta \). We have focused only on the case when \( f_k = \theta \).

It is worth mentioning that in almost all the cases considered here, the CRCs are obtained subject to some highly non-linear differential constraints. In order to show that the classes of CRCs found here are non-empty, we need to solve these constraints. Due to the highly non-linear nature of these constraints, we have not been able to solve them presently. However, our classification shows that the CRCs exist in principle.
Chapter 4

Conformal Ricci Collineations of Non-Static Spherically Symmetric Spacetimes

In this chapter, the CRCs are investigated for non-static spherically symmetric spacetimes in degenerate as well as non-degenerate cases. The source of energy-momentum tensor is assumed to be a perfect fluid. In case when the Ricci tensor is degenerate, the CRC equations are directly integrated to obtain the explicit form of CRCs. For non-degenerate Ricci tensor, these equations are solved to get a vector field generating CRCs in the form of some unknown functions of $t$ and $r$. This procedure generates a list of differential constraints on the Ricci tensor components, which are then solved for different choices of the Ricci tensor components to get the explicit form of CRCs.
4.1 CRC Equations

A spacetime is said to be spherically symmetric if its isometry group contains a subgroup isomorphic to the group \( SO(3) \) and the orbits of this subgroup (that is the collection of points resulting from the action of the subgroup on a given point) are \( 2 \)-dimensional spheres. The \( SO(3) \) isometries may be interpreted physically as rotations. Thus a spherically symmetric spacetime is one whose metric remains invariant under rotation [72].

As these spacetimes are of great interest and there is a great deal of information about them scattered throughout the literature, it would be useful to have a survey of the symmetries of these spacetimes. The line element of non-static spherically symmetric spacetimes is given by [69]:

\[
ds^2 = e^\nu(t,r) \, dt^2 - e^\mu(t,r) \, dr^2 - e^\lambda(t,r) \left[ d\theta^2 + \sin^2 \theta \, d\phi^2 \right], \tag{4.1.1}
\]

where \( \nu, \mu \) and \( \lambda \) are any functions of \( t \) and \( r \). The minimum three KVs admitted by the above metric are:

\[
X^{(1)} = \sin \phi \, \frac{\partial}{\partial \theta} + \cos \phi \, \cot \theta \, \frac{\partial}{\partial \phi}, \quad X^{(2)} = -\cos \phi \, \frac{\partial}{\partial \theta} + \sin \phi \, \cot \theta \, \frac{\partial}{\partial \phi}, \quad X^{(3)} = \frac{\partial}{\partial \phi}. \tag{4.1.2}
\]

The non-zero components of Ricci tensor for this metric are:

\[
R_{00} = -\frac{1}{4} \left( 2\ddot{\mu} + \dot{\mu}^2 - 2\dot{\nu} \dot{\lambda} + 4\ddot{\lambda} + 2\ddot{\lambda}^2 - \dot{\mu} \dot{\nu} \right)
+ \frac{e^{\nu-\mu}}{4} \left( 2\nu'' + \nu'^2 - \mu' \nu' + 2\lambda' \nu' \right) = A(t,r),
\]

\[
R_{11} = \frac{e^{\mu-\nu}}{4} \left( 2\ddot{\mu} + \dot{\mu}^2 + 2\ddot{\lambda} \dot{\mu} - \dot{\nu} \dot{\mu} \right)
- \frac{1}{4} \left( 2\nu'' + \nu'^2 + 4\lambda'' - \mu' \nu' + 2\lambda'^2 - 2\lambda \mu' \right) = B(t,r),
\]

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\[ R_{22} = 1 + \frac{e^{\lambda-\nu}}{4} \left( 2\ddot{\lambda} + 2\dot{\lambda}^2 - \dot{\lambda} \dot{\nu} + \dot{\lambda} \dot{\mu} \right) \]
\[ \quad - \frac{e^{\lambda-\mu}}{4} \left( 2\lambda'' + 2\lambda'^2 - \lambda' \mu' + \lambda' \nu' \right) = C(t, r), \]
\[ R_{33} = \sin^2 \theta R_{22}, \]
\[ R_{01} = -\frac{1}{2} \left( 2\ddot{\lambda} + \dot{\lambda} \dot{\nu} - \dot{\lambda} \dot{\mu} \right) = D(t, r), \quad (4.1.3) \]

where the dot and prime on the metric functions represent their derivatives with respect to \( t \) and \( r \) respectively. If we assume the matter field to be a perfect fluid, then the energy-momentum tensor has the form \( T_{ab} = (p + \rho)u_a u_b - pg_{ab} \), where \( u^a \) signifies the four velocity and \( p, \rho \) respectively represent pressure and density of the perfect fluid. Using this form of \( T_{ab} \) for the metric given in Eq. (4.1.1), we have \( T_{00} = \rho e^\nu, \ T_{11} = p e^\mu, \ T_{22} = p e^\lambda, \ T_{33} = \sin^2 \theta T_{22} \) and \( T_{01} = 0 \). Consequently, the Ricci tensor components given in (4.1.3) take the form:

\[ A = \frac{e^\nu}{2} (\rho + 3p), \ B = \frac{e^\mu}{2} (\rho - p), \ C = \frac{e^\lambda}{2} (\rho - p), \ D = 0. \quad (4.1.4) \]

Using the above Ricci tensor components in Eq.(1.11.5), we get the following system of coupled CRC equations:

\[ A_{,0} X^0 + A_{,1} X^1 + 2A X^0_{,0} = 2\psi A, \quad (4.1.5) \]
\[ A X^0_{,1} + B X^1_{,0} = 0, \quad (4.1.6) \]
\[ A X^0_{,2} + C X^2_{,0} = 0, \quad (4.1.7) \]
\[ A X^0_{,3} + C \sin^2 \theta X^3_{,0} = 0, \quad (4.1.8) \]
\[ B_{,0} X^0 + B_{,1} X^1 + 2B X^1_{,1} = 2\psi B, \quad (4.1.9) \]
\[ B X^1_{,2} + C X^2_{,1} = 0, \quad (4.1.10) \]
\[ B X^3_{,3} + C \sin^2 \theta X^3_{,1} = 0, \quad (4.1.11) \]
\[ C_{,0} X^0 + C_{,1} X^1 + 2C X^2_{,2} = 2\psi C, \quad (4.1.12) \]
\[ C \left( X^2_{,3} + \sin^2 \theta X^3_{,2} \right) = 0, \quad (4.1.13) \]
\[ C_{,0} X^0 + C_{,1} X^1 + 2C X^3_{,2} + 2C \cot \theta X^2 = 2\psi C. \quad (4.1.14) \]

In the above set of equations, \( X = (X^0, X^1, X^2, X^3) \) is the collineation vector field generating CRCs and the commas in the subscripts denote partial derivatives with respect to spacetime coordinates. To obtain the explicit form of CRCs in non-static spherically symmetric spacetimes, one needs to solve these equations. In the forthcoming sections, we solve these equations in degenerate and non-degenerate cases.

### 4.2 CRCs for Degenerate Ricci Tensor

In case when the Ricci tensor is degenerate, we have \( \text{det } R_{ab} = 0 \). It means that one, two or three of the Ricci tensor components \( A, B \) and \( C \) vanish. Thus we have the following six possibilities:

- **(D1)** \( A \neq 0, B = C = 0 \)
- **(D2)** \( B \neq 0, A = C = 0 \)
- **(D3)** \( C = 0, A \neq 0, B \neq 0 \)
- **(D4)** \( C \neq 0, A = B = 0 \)
- **(D5)** \( A = 0, B \neq 0, C \neq 0 \)
- **(D6)** \( B = 0, A \neq 0, C \neq 0 \)

**Case D1:** Here \( A \neq 0 \) and \( B = C = 0 \). In this case, the system of CRC equations \((4.1.5)-(4.1.14)\) get the form:

\[ A_{,0} X^0 + A_{,1} X^1 + 2A X^0_{,0} = 2\psi A, \quad (4.2.1) \]
\[ X^0_{,1} = X^0_{,2} = X^0_{,3} = 0. \quad (4.2.2) \]
These equations can be easily solved to get $X^0 = g(t)$ and $\psi = \frac{A_0}{2A}g(t) + \frac{A_1}{2A}X^1 + g_1(t)$, where $g(t)$ is an arbitrary function. This gives an infinite-dimensional Lie algebra of CRCs.

Similarly, the solution of CRC equations (4.1.5)-(4.1.14) in case D2 produce the following CRCs:

\[
\begin{align*}
X^1 &= g(r), \\
X^i &= X^i(x^a); \ i = 0, 2, 3 \\
\psi &= \frac{B_0}{2B}X^0 + \frac{B_1}{2B}g(r) + g_1(r),
\end{align*}
\]

where $g(r)$ is an arbitrary function.

**Case D3:** In this case, we take $C = 0$, $A \neq 0$ and $B \neq 0$. Here we have to deal with the following equations:

\[
\begin{align*}
A_0 X^0 + A_1 X^1 + 2A X^0_{,0} &= 2\psi A, \hspace{1cm} (4.2.3) \\
A X^0_1 + B X^1_0 &= 0, \hspace{1cm} (4.2.4) \\
X^0_2 &= X^0_3 = 0, \hspace{1cm} (4.2.5) \\
B_0 X^0 + B_1 X^1 + 2B X^1_0 &= 2\psi B, \hspace{1cm} (4.2.6) \\
X^1_{,2} &= X^1_{,3} = 0. \hspace{1cm} (4.2.7)
\end{align*}
\]

Solving Eqs. (4.2.5) and (4.2.7), we obtain $X^0 = F^1(t, r)$ and $X^1 = F^2(t, r)$, where $F^1$ and $F^2$ are unknown functions which arise during integration. Substituting these values in Eq. (4.2.4) and then integrating it with respect to $r$, we have $F^1 = -\int \frac{B}{A}F^2 r dr + g^1(t)$. The function $g^1$ is due to integration. Moreover, the inheriting factor $\psi$ can be obtained form Eq. (4.2.6) as:

\[
\psi = -\frac{B_0}{2B} \int \frac{B}{A}F^2 dr + \frac{B_0}{2B}g^1 + \frac{B_1}{2B}F^2 + F^2_r.
\]
Simplifying Eq. (4.2.3) with the help of all these values, we get:

\[
\left(\frac{A}{B}\right),_0 \int \frac{B}{A} F_t^2 dr + 2\frac{A}{B} \int \left(\frac{B}{A} F_t^2\right),_0 dr - \left(\frac{A}{B}\right),_1 F^2 + 2\frac{A}{B} F_t^2 - \left(\frac{A}{B}\right),_0 g^1 - 2\frac{A}{B} g_t^1 = 0.
\]

(4.2.8)

It can be observed that the above equation is highly non-linear and cannot be solved as it stands. One may impose some more conditions on \(A\) and \(B\) to get the values of \(F^2\) and \(g^1\). However, the two components of the collineation vector \(X\), namely \(X^2\) and \(X^3\) are arbitrary. Consequently, we have infinite-dimensional Lie algebra of CRCs in this case.

Similar to the above cases, the remaining three cases also produce infinite-dimensional families of CRCs. To avoid the repetition, we summarize the obtained results of the remaining three cases in the following tables:

<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>D4</td>
<td>—</td>
<td>(X^i = X^i(x^a)), for (i = 0, 1) (X^2 = a_1 \sin \phi - a_2 \cos \phi,) (X^3 = \cot \theta (a_1 \cos \phi + a_2 \sin \phi) + a_3.)</td>
<td>(\psi = \frac{C_0}{4\omega} X^0 + \frac{C_1}{4\omega} X^1)</td>
</tr>
<tr>
<td>D5</td>
<td>(\left(\frac{B}{A}\right),_0 \neq 0)</td>
<td>(X^0 = -\left(\frac{B}{A}\right),_0 G(r) - \frac{2B}{A} G_t(r),) (X^1 = G(r),) (X^2 = a_1 \sin \phi - a_2 \cos \phi,) (X^3 = \cot \theta (a_1 \cos \phi + a_2 \sin \phi) + a_3.)</td>
<td>(\psi = \frac{C_0}{4\omega} X^0 + \frac{C_1}{4\omega} X^1)</td>
</tr>
</tbody>
</table>

Table 4.1: CRCs for Degenerate Ricci Tensor
4.3 CRCs for Non-degenerate Ricci Tensor

In this section, we solve the CRC equations (4.1.5)-(4.1.14) for non-degenerate Ricci tensor, that is $A \neq 0$, $B \neq 0$ and $C \neq 0$. Like the previous two chapters, our first try is to decouple and integrate the system of CRC equations and to get a system comprising $X^0$, $X^1$, $X^2$ and $X^3$ in terms of some unknown functions of $t$ and $r$ which satisfy some of the CRC equations. The procedure of getting such a solution is exactly similar to that we have presented in the previous two chapters. Consequently, we obtain:

$$X^0 = -\frac{C}{A} \left[ \sin \theta \sin \phi F^1_t(t,r) - \sin \theta \cos \phi F^2_t(t,r) - \cos \theta F^3_t(t,r) \right] + F^4(t,r),$$

$$X^1 = -\frac{C}{B} \left[ \sin \theta \sin \phi F^1_r(t,r) - \sin \theta \cos \phi F^2_r(t,r) - \cos \theta F^3_r(t,r) \right] + F^5(t,r),$$

$$X^2 = \cos \theta \left[ \sin \phi F^1(t,r) - \cos \phi F^2(t,r) \right] + \sin \theta F^3(t,r) + a_1 \sin \phi - a_2 \cos \phi + a_4 \sin \theta \ln(\csc \theta - \cot \theta),$$

$$X^3 = \csc \theta \left[ \cos \phi F^1(t,r) + \sin \phi F^2(t,r) \right] + \cot \theta (a_1 \cos \phi + a_2 \sin \phi) + a_4 \phi + a_3.$$

(4.3.1)
The inheriting factor $\psi$ is found to be:

$$\psi = \frac{C_0}{2C}X^0 + \frac{C_1}{2C}X^1 + X^2_{,2}. \quad (4.3.2)$$

In the above solution, the angular dependence in $\theta$ and $\phi$ is known explicitly. One can easily verify that some of the CRC equations, Eqs. (4.1.5)-(4.1.14), are identically satisfied by the above values of $X^a$. Inserting these values of $X^a$ in the remaining CRC equations, we see that the constant $a_4$ vanishes and we get the following integrability conditions which place restrictions on the Ricci tensor components:

$$A \left( \frac{C}{A} \right)_{,1} F^i + B \left( \frac{C}{B} \right)_{,0} F^i + 2CF^i_{,tt} = 0, \quad (4.3.3)$$

$$\left( \frac{C}{A} \right)_{,0} F^i - \frac{A}{B} \left( \frac{C}{A} \right)_{,1} F^i - 2F^i + \frac{2C}{A} F^i_{,t} = 0, \quad (4.3.4)$$

$$\frac{B}{A} \left( \frac{C}{B} \right)_{,0} F^i - \left( \frac{C}{B} \right)_{,1} F^i + 2F^i - \frac{2C}{B} F^i_{,r} = 0, \quad (4.3.5)$$

$$AF^4 + B F^5 = 0, \quad (4.3.6)$$

$$\left( \frac{C}{A} \right)_{,0} F^4 + \left( \frac{C}{A} \right)_{,1} F^5 - \frac{2C}{A} F^4_{,t} = 0, \quad (4.3.7)$$

$$2C^2 \left( \frac{B}{C} \right)_{,0} F^4 + \left( 2B,1C - BC,1 \right) F^5 + 4BCF^5_{,r} = 0, \quad (4.3.8)$$

where $i = 1, 2, 3$. Thus the problem of solving the CRC equations (4.1.5)-(4.1.14) is now reduced to the solution of the above integrability conditions. Due to the high non-linearity of these equations, it is not possible to solve them generally. However, they can be simplified and completely solved by taking some specific forms of the Ricci tensor components. As an example, here we consider the following simple cases:

(I) $A = A(t), \quad B' = C' = 0$ 

(II) $A = B = C = A(t, r)$
Case (I): In this case, we consider $A = A(t)$, $B = b$ and $C = c$, where the constants $b$ and $c$ are non-zero. The integrability conditions (4.3.3)-(4.3.8) reduce to:

\[
F^i_{tr} = 0, \quad (4.3.9)
\]
\[
A_0cF^i_t + 2A^2F^i - 2AcF^i_{tt} = 0, \quad (4.3.10)
\]
\[
bF^i - cF^i_{rr} = 0, \quad (4.3.11)
\]
\[
AF^4 + bF^5_t = 0, \quad (4.3.12)
\]
\[
A_0F^4 + 2AF^4_t = 0, \quad (4.3.13)
\]
\[
F^5_r = 0, \quad (4.3.14)
\]

where $i = 1, 2, 3$. Taking $i = 1$ and integrating Eq. (4.3.9), we get $F^1 = G^1(t) + G^2(r)$, where $G^1$ and $G^2$ are functions of integration. Substituting this value in Eq. (4.3.11), we have $bG^1 = cG^2_{rr} - bG^2 = c_1$, where $c_1$ is a separation constant. This yields $G^1 = \frac{c_1}{b}$ and $G^2 = \frac{c_2}{b}e^r + \frac{c_3}{b}e^{-r} - \frac{c_1}{b}$. Thus $F^1 = c_2e^r + c_3e^{-r}$. Using this value in Eq. (4.3.10), we obtain $c_2 = c_3 = 0$ and hence $F^1 = 0$. Similarly, $F^2 = F^3 = 0$.

As far as the values of the functions $F^4$ and $F^5$ are concerned, they can be found using Eqs. (4.3.12)-(4.3.14). Integrating Eqs. (4.3.14) and (4.3.13), we have $F^5 = G^3(t)$ and $F^4 = \frac{1}{\sqrt{A}}G^4(r)$, where the functions $G^3$...
and $G^4$ arise during the process of integration. Simplifying Eq. (4.3.12), we get $G^4_t = - \frac{b}{\sqrt{A}} G^3_t = c_4$, where $c_4$ is a separation constant. Solving the last equation, we obtain $G^4 = c_4 r + c_5$ and $G^3 = - \frac{c_4}{b} \int \sqrt{A} dt + c_6$. Therefore, $F^4 = \frac{1}{\sqrt{A}} (c_4 r + c_5)$ and $F^5 = - \frac{c_4}{b} \int \sqrt{A} dt + c_6$. Substituting back all these values in the system (4.3.1), we obtain:

$$X^0 = \frac{1}{\sqrt{A}} (c_4 r + c_5),$$
$$X^1 = - \frac{c_4}{b} \int \sqrt{A} dt + c_6,$$
$$X^2 = a_1 \sin \phi - a_2 \cos \phi,$$
$$X^3 = \cot \theta (a_1 \cos \phi + a_2 \sin \phi) + a_3. \quad (4.3.15)$$

In this case, $\psi = 0$ and the CRCs reduce to RCs, the dimension of Lie algebra of RCs being 6. One can see that the set of minimum KVs of spherically symmetric spacetimes, given in (4.1.2) is contained in the set of RCs.

**Case (II):** Here we assume that $A = B = C = A(t, r)$, then the integrability conditions (4.3.3)-(4.3.8) become:

$$F^i_{tr} = 0, \quad (4.3.16)$$
$$F^i - F^i_{tt} = 0, \quad (4.3.17)$$
$$F^i - F^i_{rr} = 0, \quad (4.3.18)$$
$$F^4_t + F^5_t = 0, \quad (4.3.19)$$
$$F^4_t = 0, \quad (4.3.20)$$
$$A_1 F^5 + 4A F^5_r = 0, \quad (4.3.21)$$

where $i = 1, 2, 3$. For $i = 1$, solving Eqs. (4.3.16) and (4.3.18) by using the same steps as in the above case, we find that $F^1 = c_1 e^r + c_2 e^{-r}$. This
shows that $F^1$ depends on $r$ only. Consequently, Eq. (4.3.17) gives $F^1 = 0$. Similarly, $F^2 = F^3 = 0$.

Next, if we integrate Eq. (4.3.20) with respect to $t$, it gives $F^4 = G^1(r)$; $G^1$ being a function of integration. Putting this value in Eq. (4.3.19) and integrating it with respect to $t$, we have $F^5 = -tG^1_r(r) + G^2(r)$. We use these values of $F^4$ and $F^5$ in Eq. (4.3.21) and simplify it, this yields $F^4 = c_3$ and $F^5 = c_4$. Hence the system (4.3.1) becomes:

\[
\begin{align*}
X^0 &= c_3, \\
X^1 &= c_4, \\
X^2 &= a_1 \sin \phi - a_2 \cos \phi, \\
X^3 &= \cot \theta (a_1 \cos \phi + a_2 \sin \phi) + a_3, \\
\psi &= \frac{c_3 A_0}{2A} + \frac{c_4 A_1}{2A}.
\end{align*}
\] (4.3.22)

From above, we can see that in this the Lie algebra of CRCs is 5-dimensional. As in the previous case, the set of minimum KVs of spherically symmetric spacetimes is a subset of the algebra of CRCs.

**Case (III):** In this case, we take $A = C = A(r)$, $B = B(t)$. Under these conditions, the system of equations (4.3.3)-(4.3.8) reduces to:

\[
\begin{align*}
B_0 F^i_r - 2BF^i_{tr} &= 0, \\
F^i - F^i_{tt} &= 0, \\
B_0 F^i_t + A_1 F^i_r - 2BF^i_t + 2AF^i_{rr} &= 0, \\
AF^4_r + BF^5_t &= 0, \\
F^4_t &= 0,
\end{align*}
\]
\[ 4AF^4 - A_1F^5 + 4AF^5_r = 0, \quad (4.3.28) \]

where \( i = 1, 2, 3 \). For \( i = 1 \), the solution of Eq. (4.3.24) yields \( F^1 = e^t G^1(r) + e^{-t} G^2(r) \). Here \( G^1 \) and \( G^2 \) are unknown functions which arise due to integration. With this value of \( F^1(t, r) \), Eq. (4.3.23) takes the form:

\[ B_0 \left[ e^t G^1_r + e^{-t} G^2_r \right] - 2B \left[ e^t G^1_r - e^{-t} G^2_r \right] = 0. \quad (4.3.29) \]

To be more specific, we choose \( B(t) = e^{2t} \), then the above equation gives \( G^2 = c_1 \). Substituting all these values in Eq. (4.3.25) and doing some simple algebraic manipulations, we obtain \( G^1 = c_1 \left( \int \frac{dr}{\sqrt{A}} \right)^2 + c_2 \int \frac{dr}{\sqrt{A}} + c_3 \). Finally, we have \( F^1 = c_1 e^t \left( \int \frac{dr}{\sqrt{A}} \right)^2 + c_2 e^t \int \frac{dr}{\sqrt{A}} + c_3 e^t + c_4 e^{-t} \). Dealing with Eqs. (4.3.23), (4.3.24) and (4.3.25) in a similar manner for \( i = 2, 3 \), we obtain:

\[
\begin{align*}
F^2 &= c_4 e^t \left( \int \frac{dr}{\sqrt{A}} \right)^2 + c_5 e^t \int \frac{dr}{\sqrt{A}} + c_6 e^t + c_7 e^{-t}, \\
F^3 &= c_7 e^t \left( \int \frac{dr}{\sqrt{A}} \right)^2 + c_8 e^t \int \frac{dr}{\sqrt{A}} + c_9 e^t + c_7 e^{-t}. \quad (4.3.30)
\end{align*}
\]

Moreover, Eqs. (4.3.26) and (4.3.27) can be easily solved to get \( F^4 = G^3(r) \) and \( F^5 = \frac{A}{2} e^{-2t} G^3_r(r) + G^4(r) \), where \( G^3 \) and \( G^4 \) are functions of integration. Putting these values in Eq. (4.3.28) and then differentiating it with respect to \( t \), we get \( G^3 = c_{10} \int A^{-\frac{3}{2}} dr + c_{11} \) and \( G^4 = -c_{10} A^\frac{1}{2} \int (A^{-\frac{1}{2}} \int A^{-\frac{3}{2}} dr) dr - c_{11} A^\frac{1}{2} \int A^{-\frac{3}{2}} dr + c_{12} A^\frac{1}{2} \). Thus we have:

\[
\begin{align*}
F^4 &= c_{10} \int A^{-\frac{3}{2}} dr + c_{11}, \\
F^5 &= \frac{c_{10}}{2} e^{-2t} A^\frac{1}{2} - c_{10} A^\frac{1}{2} \int (A^{-\frac{1}{2}} \int A^{-\frac{3}{2}} dr) dr \\
&\quad - c_{11} A^\frac{1}{2} \int A^{-\frac{3}{2}} dr + c_{12} A^\frac{1}{2}. \quad (4.3.31)
\end{align*}
\]
Using the values from (4.3.30) and (4.3.31) in (4.3.1), we have:

\[
X^0 = -e^t \sin \theta \sin \phi \left[ c_1 \left( \int \frac{dr}{\sqrt{A}} \right)^2 - c_1 e^{-2t} + c_2 \int \frac{dr}{\sqrt{A}} + c_3 \right]
\]

\[
+ e^t \sin \theta \cos \phi \left[ c_4 \left( \int \frac{dr}{\sqrt{A}} \right)^2 - c_4 e^{-2t} + c_5 \int \frac{dr}{\sqrt{A}} + c_6 \right]
\]

\[
+ e^t \cos \theta \left[ c_7 \left( \int \frac{dr}{\sqrt{A}} \right)^2 - c_7 e^{-2t} + c_8 \int \frac{dr}{\sqrt{A}} + c_9 \right] + c_{10} \int A^{-\frac{3}{2}} dr + c_{11},
\]

\[
X^1 = -e^{-t} \sqrt{A} \left[ \sin \theta \sin \phi \left( 2c_1 \int \frac{dr}{\sqrt{A}} + c_2 \right) - \sin \theta \cos \phi \left( 2c_4 \int \frac{dr}{\sqrt{A}} + c_5 \right) \right]
\]

\[
+ \cos \theta \left[ 2c_7 \int \frac{dr}{\sqrt{A}} + c_8 \right] + c_{10} A^{\frac{1}{2}} \left[ e^{-2t} - \int \left( A^{-\frac{3}{2}} \int A^{-\frac{3}{2}} dr \right) dr \right]
\]

\[- c_{11} A^{\frac{1}{2}} \int A^{-\frac{3}{2}} dr + c_{12} A^{\frac{1}{2}},
\]

\[
X^2 = e^t \cos \theta \sin \phi \left[ c_1 \left( \int \frac{dr}{\sqrt{A}} \right)^2 + c_1 e^{-2t} + c_2 \int \frac{dr}{\sqrt{A}} + c_3 \right]
\]

\[- e^t \cos \theta \cos \phi \left[ c_4 \left( \int \frac{dr}{\sqrt{A}} \right)^2 + c_4 e^{-2t} + c_5 \int \frac{dr}{\sqrt{A}} + c_6 \right]
\]

\[+ e^t \sin \theta \left[ c_7 \left( \int \frac{dr}{\sqrt{A}} \right)^2 + c_7 e^{-2t} + c_8 \int \frac{dr}{\sqrt{A}} + c_9 \right] + a_1 \sin \phi - a_2 \cos \phi,
\]

\[
X^3 = e^t \csc \theta \cos \phi \left[ c_1 \left( \int \frac{dr}{\sqrt{A}} \right)^2 + c_1 e^{-2t} + c_2 \int \frac{dr}{\sqrt{A}} + c_3 \right]
\]

\[+ e^t \csc \theta \sin \phi \left[ c_4 \left( \int \frac{dr}{\sqrt{A}} \right)^2 + c_4 e^{-2t} + c_5 \int \frac{dr}{\sqrt{A}} + c_6 \right]
\]

\[+ \cot \theta (a_1 \cos \phi + a_2 \sin \phi) + a_3,
\]

\[
\psi = \frac{A_{11}}{2A} X^1 + X^2, \quad (4.3.32)
\]

which shows that in this case, the non-static spherically symmetric space-times possess a 15-dimensional Lie algebra of CRs.

In the remaining cases, the procedure of solving the integrability conditions (4.3.3)-(4.3.8) is straightforward and similar to the above three cases. Therefore, we skip the repeated calculations and summarize the final results.
for all these cases in the following tables.

It is to mention here that, in addition to the cases which we have discussed in this chapter, a number of similar more cases can be considered by imposing restrictions on the Ricci tensor components. Some of such cases are already considered in [9], where the spherically symmetric spacetime was assumed to be static.

<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
</table>
| IV   | $C = at^2$  | $X^0 = -\frac{2 \sin \theta \sin \phi}{\sqrt{at}} \left[ c_1 \left( \frac{at^2}{b} - r^2 \right) - 2c_2r - 2c_3 \right] + \frac{\sin \theta \cos \phi}{\sqrt{at}} \left[ c_4 \left( \frac{at^2}{b} - r^2 \right) 
- 2c_5r - 2c_6 \right] + \frac{\sin \theta}{\sqrt{at}} \left[ c_7 \left( \frac{at^2}{b} + r^2 \right) + 2c_8r + 2c_9 \right] + t(c_10r + c_{11}) \right]$ | $\psi = C \phi X^0$ |
| $A = a$ | $B = b$ | $X^1 = \frac{\sqrt{at}}{2} \left[ -\sin \theta \sin \phi (c_1 \Gamma + c_2 \tau) + \sin \theta \cos \phi (c_4 \Gamma + c_3 \tau) \right.$ | $+ X^2 \frac{\sqrt{at}}{2}$ |
| & | $\left. + \cos \theta (c_7 \Gamma + c_8 \tau) \right] + \frac{\sqrt{at}}{2} (r^2 - \frac{at^2}{b}) + c_{11}r + c_{12}$ | & |
| $X^2 = \frac{\cos \theta \sin \phi}{\sqrt{at}} \left[ c_1 \left( \frac{at^2}{b} + r^2 \right) + 2c_2r + 2c_3 \right] + \frac{\cos \theta \cos \phi}{\sqrt{at}} \left[ c_4 \left( \frac{at^2}{b} + r^2 \right) 
+ 2c_5r + 2c_6 \right] + \frac{\cos \theta}{\sqrt{at}} \left[ c_7 \left( \frac{at^2}{b} + r^2 \right) + 2c_8r + 2c_9 \right] + a_1 \sin \phi - a_2 \cos \phi$, | & |
| $X^3 = \frac{\cos \theta \cos \phi}{\sqrt{at}} \left[ c_1 \left( \frac{at^2}{b} + r^2 \right) + 2c_2r + 2c_3 \right] + \frac{\cos \theta \sin \phi}{\sqrt{at}} \left[ c_4 \left( \frac{at^2}{b} + r^2 \right) 
+ 2c_5r + 2c_6 \right] + \cot \theta (a_1 \cos \phi + a_2 \sin \phi) + a_3$. | & |

Table 4.3: CRCs for Non-Degenerate Ricci Tensor
<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>$B = e^{2t \sqrt{\eta}}$</td>
<td>$X^0 = -e^{\sqrt{\eta} \rho} \sin \phi \left[ c_1 r + c_2 - \frac{c_4}{4} \left( r^2 - e^{-2\sqrt{\eta} r} \right) \right] + e^{\sqrt{\eta} \rho} \cos \phi \left[ c_4 r + c_5 - \frac{c_4}{4} \left( r^2 - e^{-2\sqrt{\eta} r} \right) \right] + e^{\sqrt{\eta} \rho} \cos \theta \left[ c_7 r + c_8 - \frac{c_4}{4} \left( r^2 - e^{-2\sqrt{\eta} r} \right) \right] + c_{10} \theta + c_{11}$</td>
<td>$\psi = X_{X_2}^2$</td>
</tr>
<tr>
<td></td>
<td>$A = a$</td>
<td>$X^1 = -e^{\sqrt{\eta} \rho} \sin \phi \left( c_1 r - c_3 r \right) + e^{\sqrt{\eta} \rho} \cos \phi \left( c_4 - c_6 \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$C = c$</td>
<td>$X^2 = e^{\sqrt{\eta} \rho} \cos \theta \cos \phi \left[ c_4 r + c_5 - \frac{c_4}{4} \left( r^2 + e^{-2\sqrt{\eta} r} \right) \right] + e^{\sqrt{\eta} \rho} \sin \theta \left[ c_4 r - c_8 + \frac{c_4}{4} \left( r^2 - e^{-2\sqrt{\eta} r} \right) \right] + a_1 \sin \phi - a_2 \cos \phi$, $X^3 = e^{\sqrt{\eta} \rho} \csc \theta \cos \phi \left[ c_4 r + c_5 - \frac{c_4}{4} \left( r^2 + e^{-2\sqrt{\eta} r} \right) \right] + e^{\sqrt{\eta} \rho} \csc \theta \sin \phi \left[ c_4 r + c_5 - \frac{c_4}{4} \left( r^2 + e^{-2\sqrt{\eta} r} \right) \right] + \cot \theta \left( a_1 \cos \phi + a_2 \sin \phi \right) + a_3$.</td>
<td></td>
</tr>
<tr>
<td>VI</td>
<td>$-$</td>
<td>$X^0 = c_1 r + c_2$, $X^1 = -c_1 t + c_3$, $X^2 = a_1 \sin \phi - a_2 \cos \phi$, $X^3 = \cot \theta \left( a_1 \cos \phi + a_2 \sin \phi \right) + a_3$.</td>
<td>$\psi = \frac{A_0 \sqrt{\eta}}{\pi} X^0$</td>
</tr>
<tr>
<td>VII</td>
<td>$C = \frac{1}{4} \alpha^2$ where $\alpha = \int \sqrt{\mathcal{A}} dt$</td>
<td>$X^0 = \sqrt{\mathcal{A}} \left( c_1 r + c_2 \right)$, $X^1 = \frac{1}{2} \left( \frac{1}{4} \alpha^2 r^2 + c_2 r \right) - \frac{c_4}{4} \alpha^2 \mathcal{A} + c_3$, $X^2 = a_1 \sin \phi - a_2 \cos \phi$, $X^3 = \cot \theta \left( a_1 \cos \phi + a_2 \sin \phi \right) + a_3$.</td>
<td>$\psi = \frac{C_0 \sqrt{\mathcal{A}}}{\pi} X^0$</td>
</tr>
<tr>
<td>VIII</td>
<td>$B = e^\alpha$ where $\alpha = \int \sqrt{\mathcal{A}} dt$</td>
<td>$X^0 = \frac{1}{2} \alpha \left( c_1 r + c_2 \right)$, $X^1 = c_1 \left( \frac{1}{2} - \frac{1}{2} \alpha^2 \right) - \frac{c_4}{4} \alpha^2 r + c_3$, $X^2 = a_1 \sin \phi - a_2 \cos \phi$, $X^3 = \cot \theta \left( a_1 \cos \phi + a_2 \sin \phi \right) + a_3$.</td>
<td>$\psi = 0$</td>
</tr>
<tr>
<td>IX</td>
<td>$B = e^\alpha$ where $\alpha = \int \sqrt{\mathcal{A}} dt$</td>
<td>$X^0 = \sqrt{\mathcal{A}} \left( c_1 r + c_2 \right)$, $X^1 = -c_1 \left( a \int \sqrt{\mathcal{A}} dt + \frac{1}{2} \alpha^2 \right) - \frac{c_4}{4} \alpha^2 r + c_3$, $X^2 = a_1 \sin \phi - a_2 \cos \phi$, $X^3 = \cot \theta \left( a_1 \cos \phi + a_2 \sin \phi \right) + a_3$.</td>
<td>$\psi = \frac{C_0 \sqrt{\mathcal{A}}}{\pi} X^0$</td>
</tr>
</tbody>
</table>

Table 4.4: CRCs for Non-Degenerate Ricci Tensor
<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>$A = e^{-2t}$</td>
<td>$X^0 = -e^t \sin \theta \sin \phi \left[ c_1 \left( \int (f \sqrt{c_2 r^2 + c_3})^2 - e^{-2t} \right) + c_2 \int \sqrt{c_2 r^2 + c_3} \right] + e^t \sin \theta \cos \phi \left[ c_3 \left( \int (f \sqrt{c_2 r^2 + c_3})^2 - e^{-2t} \right) + c_4 \int \sqrt{c_2 r^2 + c_3} + c_9 \right] + e^t \cos \theta \left[ c_6 \left( \int (f \sqrt{c_2 r^2 + c_3})^2 - e^{-2t} \right) + c_8 \int \sqrt{c_2 r^2 + c_3} + c_12 \right] + 2c_8 \int \sqrt{c_2 r^2 + c_3} + c_9, \right.$</td>
<td>$\psi = \frac{A}{2 \pi} X^0 + X^2_2$</td>
</tr>
<tr>
<td>XI</td>
<td>$A = e^{2r}$</td>
<td>$X^0 = -\sqrt{c_2 r^2 + c_3} \sin \theta \sin \phi \left( 2c_1 \int \frac{dt}{\sqrt{c_2 r^2 + c_3}} + c_2 \right) + \sin \theta \cos \phi \left( 2c_3 \int \frac{dt}{\sqrt{c_2 r^2 + c_3}} + c_1 \right) - \cos \theta \left( 2c_5 \int \frac{dt}{\sqrt{c_2 r^2 + c_3}} + c_6 \right) - c_7 \sqrt{c_2 r^2 + c_3} \left[ \int \left( \frac{dt}{\sqrt{c_2 r^2 + c_3}} \right)^2 - e^{-2r} \right] + c_8 \int \frac{dt}{\sqrt{c_2 r^2 + c_3}} + c_9 \sqrt{c_2 r^2 + c_3}, \right.$</td>
<td>$\psi = \frac{A}{2 \pi} X^0 + X^2_2$</td>
</tr>
</tbody>
</table>

Table 4.5: CRCs for Non-Degenerate Ricci Tensor

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4.4 Summary

Considering a perfect fluid source, we have investigated the CRCs in non-static spherically symmetric spacetimes. The CRC equations are solved in both degenerate as well as non-degenerate cases. Each of the degenerate case yields an infinite-dimensional Lie algebra of CRCs. However, Eq. (4.1.4) shows that $B = 0$ if and only if $C = 0$. Thus a perfect fluid source is allowed only in two cases, namely $D1$ and $D5$. In case $D1$, Eq. (4.1.4) gives $p = \rho$, which denotes a stiff matter, while in case $D5$ the pressure and density of the fluid are related as $p + 3\rho = 0$.

For non-degenerate Ricci tensor, we have found the general form of collineation vector generating CRCs in terms of some unknown functions of $t$ and $r$, while the angular dependence in $\theta$ and $\phi$ is known explicitly. It generates a list of differential constraints on the Ricci tensor components. Further, the Ricci tensor components are restricted to satisfy specific conditions and the differential constraints are completely solved to get the closed form of CRCs. For the cases considered here, we see that the dimension of Lie algebra of CRCs for the spacetimes under consideration may be 5, 6 or 15. In some cases, the CRCs reduce to RCs, having dimension 6 (cases I and VIII).

It is to be mention here that the CRCs in static spherically symmetric spacetimes were already investigated by Camci et. al [9]. They concluded with the remarks that these spacetimes possess 15—dimensional Lie algebra of CRCs for the choice of non-degenerate Ricci tensor, while the dimension of Lie algebra of CRCs in degenerate case is infinite. Extending their results to the non-static spherically symmetric spacetimes, we have observed that our results are similar to the results presented in [9], for degenerate Ricci
tensor. However, when the Ricci tensor is non-degenerate, these spacetimes may possess 5—or 6—dimensional Lie algebra of CRCs.

In order to show that the classes of CRCs found here are non-empty, we need to solve the constraints imposed on the Ricci tensor components. In this way, one may get the explicit form of the non-static spherically symmetric spacetimes metric admitting these CRCs. Due to the highly non-linear nature of these constraints, we have not been able to solve them presently. However, our classification shows that the CRCs exist in principle.
Chapter 5

Conformal Ricci Collineations of LRS Bianchi type I Spacetimes

This chapter contains a complete classification of LRS Bianchi type I spacetimes via CRCs. Similar to the previous chapters, we have classified these spacetimes in two cases according as the Ricci tensor is degenerate or non-degenerate. When the Ricci tensor is considered to be degenerate, the CRC equations are directly solved to obtain the explicit form of CRCs. In case of non-degenerate Ricci tensor, the CRC equations for LRS Bianchi type I spacetimes are solved in eight different cases by imposing some specific restrictions on the Ricci tensor components. Like the previous chapters, first the CRC equations are integrated to obtain a vector field producing CRCs in terms of some unknown functions of $t$ and $x$ only. This process leads to twelve differential constraints, which are solved for different choices of the
Ricci tensor components, giving the final form of CRCs.

5.1 CRC Equations

The Bianchi type I, II,...,IX cosmological models are spatially homogeneous spacetimes of dimension $1 + 3$ admitting a group of motions $G_3$ acting on spacelike hypersurfaces. Out of these spacetimes, the Bianchi type I are the simplest cosmological models for which $G_3$ is the Abelian group of translations of the three dimensional Euclidian space. The metric of Bianchi type I spacetimes is given by [69]:

$$ds^2 = -dt^2 + h^2(t) \ dx^2 + k^2(t) dy^2 + l^2(t) dz^2,$$

(5.1.1)

where $h(t)$, $k(t)$ and $l(t)$ are functions of time coordinate only. If any two of the functions $h(t)$, $k(t)$ and $l(t)$ are equal, the above equation represents the metric of locally rotationally symmetric Bianchi type I spacetimes.

The locally rotationally symmetric spacetimes are the important solutions of EFEs which have been widely studied in the literature. They admit a group of motions $G_4$ acting multiply transitively on 3–dimensional orbits spacelike or timelike, the isotropy group being a spatial rotation.

For our study, we choose $h = k$ in Eq. (5.1.1), so that the metric of LRS Bianchi type I spacetimes becomes:

$$ds^2 = -dt^2 + h^2(t) \ dx^2 + k^2(t) \ [\ dy^2 + dz^2].$$

(5.1.2)

The four basic KVs admitted by the above metric are:

$$X_{(1)} = \partial_x, \ X_{(2)} = \partial_y, \ X_{(3)} = \partial_z, \ X_{(4)} = z\partial_y - y\partial_z.$$

(5.1.3)
The surviving components of the Ricci tensor for this metric are:

\[
\begin{align*}
R_{00} &= -\frac{h'' k + 2hk''}{hk} = A(t), \\
R_{11} &= \frac{hh''k + 2h'k'}{k} = B(t), \\
R_{22} &= R_{33} = \frac{hkk'' + h'kk' + hk'^2}{h} = C(t),
\end{align*}
\]  

(5.1.4)

where the primes on the metric functions denote their derivatives with respect to \( t \). The Ricci curvature scalar \( R \) becomes:

\[
R = 2 \left( \frac{h''}{h} + 2 \frac{k''}{k} \right) + 2 \left( \frac{k'^2}{k^2} + \frac{2h'k'}{hk} \right).
\]  

(5.1.5)

Using the EFEs (1.1.1) with the gravitational constant to be equal to unity, we get the following components of stress-energy tensor:

\[
\begin{align*}
T_{00} &= 2 \frac{h'k'}{hk} + \frac{k'^2}{k^2}, \\
T_{11} &= -h^2 \left( \frac{k'^2}{k^2} + \frac{2k''}{k} \right), \\
T_{22} &= T_{33} = -k^2 \left( \frac{h''}{h} + \frac{k''}{k} + \frac{h'k'}{hk} \right).
\end{align*}
\]  

(5.1.6)

Using the Ricci tensor components from (5.1.4) in Eq. (1.11.5), we have the following system of ten coupled non-linear CRC equations:

\[
\begin{align*}
A' X^0 + 2A X^0_0 &= 2\psi A, \\
A X^0_1 + B X^1_0 &= 0, \\
A X^0_2 + C X^2_0 &= 0, \\
A X^0_3 + C X^3_0 &= 0, \\
B' X^0 + 2B X^1_1 &= 2\psi B.
\end{align*}
\]  

(5.1.7-5.1.11)
\[
B X^1_2 + C X^2_1 = 0, \quad (5.1.12)
\]
\[
B X^1_3 + C X^3_1 = 0, \quad (5.1.13)
\]
\[
C' X^0 + 2C X^2_2 = 2\psi C, \quad (5.1.14)
\]
\[
C (X^2_3 + X^3_2) = 0, \quad (5.1.15)
\]
\[
C' X^0 + 2C X^3_3 = 2\psi C. \quad (5.1.16)
\]

Similar to the previous chapters, we solve the above equations in degenerate and non-degenerate Ricci tensor cases.

### 5.2 CRCs for Degenerate Ricci Tensor

If the Ricci tensor is degenerate, that is \( \text{det}R_{ab} = ABC^2 = 0 \), then either \( A = B = C = 0 \) where every direction is a CRC or one of the following possibilities hold.

- \( (D1) \) \( A \neq 0, B \neq 0, C = 0 \)
- \( (D2) \) \( A = 0, B \neq 0, C \neq 0 \)
- \( (D3) \) \( A \neq 0, B = 0, C \neq 0 \)
- \( (D4) \) \( A \neq 0, B = C = 0 \)
- \( (D5) \) \( A = C = 0, B \neq 0 \)
- \( (D6) \) \( A = B = 0, C \neq 0 \)

In all the above cases, we solve the system of Eqs. \((5.1.7)-(5.1.16)\).

In case \( D1 \), the system of CRC equations becomes:

\[
A' X^0 + 2A X^0_0 = 2\psi A, \quad (5.2.1)
\]
\[
A X^0_1 + B X^1_0 = 0, \quad (5.2.2)
\]
\[
X^0_2 = X^0_3 = X^1_2 = X^1_3 = 0, \quad (5.2.3)
\]
\[
B' X^0 + 2B X^1_1 = 2\psi B. \quad (5.2.4)
\]
Solving Eqs. (5.2.2) and (5.2.3), we have

\[ X^0 = -\frac{B}{A} f_t(t, x) + g(t), \quad X^1 = f_x(t, x), \quad (5.2.5) \]

where \( f \) and \( g \) are arbitrary functions of integration. Using the above value of \( X^0 \) in Eq. (5.2.1), we get:

\[ \psi = \left( A' B - 2B'A \right) \frac{f_t - A' f_{tt} + A g}{2A} g + g_t. \quad (5.2.6) \]

Simplifying Eq. (5.2.4) after using Eqs. (5.2.5) and (5.2.6), we have:

\[ \frac{1}{2} \left( \frac{B'}{A} \right) f_t + \frac{B}{A} f_{tt} + f_{xx} + \left( \frac{B'}{2B} - \frac{A'}{2A} \right) g - g_t = 0, \quad (5.2.7) \]

which is a highly non linear equation and cannot be solved generally. However, one can choose some specific values of the functions \( f \) and \( g \) such that it holds true. With these values of \( f \) and \( g \), we would be able to write the two components \( X^0 \) and \( X^1 \) of CRCs presented in (5.2.5) in the final form. The other two components \( X^2 \) and \( X^3 \) are arbitrary functions of \( t, x, y \) and \( z \). Hence the algebra of CRCs is infinite-dimensional, subject to the differential constraint given in Eq. (5.2.7).

It is straightforward to solve the CRC Eqs. (5.1.7)-(5.1.16) in the remaining five cases and to see that each case yields infinite-dimensional Lie algebra of CRCs. We exclude the basic calculations and present the obtained results of all these cases in the following table.
When the Ricci tensor $R_{ab}$ is supposed to be non-degenerate, then $det R_{ab} \neq 0$. This means that the Ricci tensor components $A$, $B$ and $C$ are all non-zero. Like the previous chapters, our first try is to get a solution of Eqs. (5.1.7)-(5.1.16) in terms of some unknown functions of $t$ and $x$. The procedure is explained below:

5.3 CRCs for Non-degenerate Ricci Tensor

When the Ricci tensor $R_{ab}$ is supposed to be non-degenerate, then $det R_{ab} \neq 0$. This means that the Ricci tensor components $A$, $B$ and $C$ are all non-zero. Like the previous chapters, our first try is to get a solution of Eqs. (5.1.7)-(5.1.16) in terms of some unknown functions of $t$ and $x$. The procedure is explained below:

Table 5.1: CRCs for Degenerate Ricci Tensor

<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor ($\psi$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>D2(a)</td>
<td>$B = C$</td>
<td>$X^0 = X^0(x^a)$, $X^1 = \frac{\sqrt{2}}{2} (x^2 - y^2 - z^2) - c_2 x z - c_3 z - c_4 x y - c_5 y - c_6 x - c_7$, $X^2 = \frac{\sqrt{2}}{2} (x^2 - y^2 + z^2) - c_2 y z - c_6 y + c_8 z + c_1 x y + c_5 x + c_9$, $X^3 = \frac{\sqrt{2}}{2} (x^2 + y^2 - z^2) + c_1 x z + c_3 x - c_6 z - c_4 y z - c_8 y + c_10$</td>
<td>$\frac{\partial^2}{\partial t^2} X^0 + X^0_2$</td>
</tr>
<tr>
<td>D2(b)</td>
<td>$B \neq C$</td>
<td>$X^0 = X^0(x^a)$, $X^1 = h(x)$, $X^2 = c_1 (c_3 x + c_4)$, $X^3 = -c_3 (c_1 y + c_2)$</td>
<td>$\frac{\partial^2}{\partial t^2} X^0 + X^0_2$</td>
</tr>
<tr>
<td>D3</td>
<td>—</td>
<td>$X^0 = -\sqrt{2} \left[ \frac{\sqrt{2}}{2} (y^2 + z^2) + c_2 y + c_3 z + (c_4 y - c_5 z) f \sqrt{2} \right] dt = c_1 f \left( \frac{\sqrt{2}}{2} \right) dt - c_6 f \sqrt{2} dt + c_7$, $X^1 = X^1(x^a)$, $X^2 = c_1 y f \sqrt{2} dt + c_4 f \left( \frac{\sqrt{2}}{2} \right) dt - c_2 f \sqrt{2} dt + c_8 y + c_9$, $X^3 = c_1 z f \sqrt{2} dt - c_5 f \left( \frac{\sqrt{2}}{2} \right) dt + c_3 f \sqrt{2} dt - c_4 y z - c_8 y + c_7$</td>
<td>$\frac{\partial^2}{\partial t^2} X^0 + X^0_2$</td>
</tr>
<tr>
<td>D4</td>
<td>—</td>
<td>$X^0 = h$, $X^1 = X^1(x^a)$; $i = 1, 2, 3$, where $h = h(t)$</td>
<td>$\frac{\partial^2}{\partial t^2} X^0 + X^0_0$</td>
</tr>
<tr>
<td>D5</td>
<td>—</td>
<td>$X^1 = h$, $X = X^1(x^a)$; $i = 0, 2, 3$, where $h = h(x)$</td>
<td>$\frac{\partial^2}{\partial t^2} X^0 + X^0_1$</td>
</tr>
<tr>
<td>D6</td>
<td>—</td>
<td>$X^1 = X^1(x^a)$; $i = 0, 1$, $X^2 = c_1 (c_3 x + c_4)$, $X^3 = -c_3 (c_1 y + c_2)$</td>
<td>$\frac{\partial^2}{\partial t^2} X^0$</td>
</tr>
</tbody>
</table>
Differentiating Eqs. (5.1.9) and (5.1.10) with respect to $z$ and $y$ respectively and then subtracting, we get:

\[ X^2_{,03} - X^3_{,02} = 0. \] (5.3.1)

Next, we differentiate Eq. (5.1.15) with respect to $t$ and then subtract it from Eq. (5.3.1), it gives:

\[ X^0_{,23} = X^2_{,03} = X^3_{,02} = 0. \] (5.3.2)

Similarly, differentiating Eqs. (5.1.12), (5.1.13) and (5.1.15) with respect to $z$, $y$ and $x$ respectively and subtracting, we obtain:

\[ X^1_{,23} = X^2_{,13} = X^3_{,12} = 0. \] (5.3.3)

Subtracting Eqs. (5.1.11) and (5.1.14), after dividing by $2B$ and $2C$ respectively, and then differentiating the difference with respect to $y$ and $z$, we obtain $X^2_{,223} = 0$. This gives:

\[ X^2_{,3} = yE^1(t, x, z) + E^2(t, x, z), \] (5.3.4)

where $E^1$ and $E^2$ are functions of integration. Putting the above value in Eq. (5.1.15) and integrating it with respect to $y$, we obtain:

\[ X^3 = -\frac{y^2}{2}E^1 - yE^2 + E^3(t, x, z), \] (5.3.5)

where $E^3$ is a function of integration. Next, we subtract Eq. (5.1.16) from Eq. (5.1.14) to obtain:

\[ X^2_2 = X^3_3. \] (5.3.6)
Using the values of \( X^2 \) and \( X^3 \) in the above equation and differentiating it with respect to \( z \) and \( y \), we get \( E^1_{zz} = E^2_{zz} = 0 \) and \( E^3_{zz} = E^1 \). Simplifying these relations, we get:

\[
E^1 = zF^1(t,x) + F^2(t,x),
E^2 = zF^3(t,x) + F^4(t,x),
E^3 = \frac{z^3}{6}F^1(t,x) + \frac{z^2}{2}F^2(t,x) + zF^5(t,x) + F^6(t,x),
\]

(5.3.7)

where the functions \( F^i \), for \( i = 1, 2, \cdots, 6 \) are generated during the process of integration. Using these values in Eqs. (5.3.4) and (5.3.5), we get:

\[
X^2 = y \left[ \frac{z^2}{2}F^1 + zF^2 \right] + \frac{z^2}{2}F^3 + zF^4 + E^4(t,x,y),
X^3 = -\frac{y^2}{2} \left[ zF^1 + F^2 \right] - y \left[ zF^3 + F^4 \right] + \frac{z^3}{6}F^1 + \frac{z^2}{2}F^2 + zF^5 + F^6.
\]

(5.3.8)

With these values, Eq. (5.3.6) gives \( E^4 = -\frac{y^3}{6}F^1 - \frac{y^2}{2}F^3 + yF^5 + F^7(t,x) \), where \( F^7 \) is a function of integration. Substituting the value of \( X^2 \) from above system in Eq. (5.1.9) and integrating it with respect to \( y \), we have:

\[
X^0 = -\frac{C}{A} \left[ \frac{y^2 z^2}{4}F^1_1 + \frac{y^2 z^2}{2}F^2_1 + \frac{yz^2}{2}F^3_1 + yzF^4_1 - \frac{y^4}{24}F^1_1 - \frac{y^3}{6}F^3_1 + \frac{y^2}{2}F^5_1 + yF^7_1 \right] + E^5(t,x,z).
\]

(5.3.9)

The function \( E^5 \) appearing above is a function of integration. Putting the values of \( X^0 \) and \( X^3 \) in Eq. (5.1.10) and differentiating it with respect to \( y \) and \( z \), we see that \( F^i_1 = 0 \), for \( i = 1, \ldots, 4 \) and \( E^5 = -\frac{C}{A} \left[ \frac{z^2}{2}F^5_1 + zF^6_1 \right] + F^8(t,x) \); \( F^8 \) being a function of integration. Therefore the value of \( X^0 \), given in Eq. (5.3.9) becomes:

\[
X^0 = -\frac{C}{A} \left[ \frac{y^2}{2}F^5_1 + yF^7_1 + \frac{z^2}{2}F^5_1 + zF^6_1 \right] + F^8.
\]

(5.3.10)
Moreover, using the value of $X^2$ in Eq. (5.1.12) and integrating it with respect to $y$, we have:

$$X^1 = -\frac{C}{B} \left[ \frac{y^2 z^2}{4} F^1_x + \frac{y^2 z}{2} F^2_x + \frac{yz^2}{2} F^3_x + yZF^1_x \right. \left. - \frac{y^3}{24} F^1_x - \frac{yz^2}{6} F^3_x + \frac{y^2}{2} F^5_x + yF^7_x \right] + E^6,$$

where $E^6 = E^6(t, x, z)$ appearing in above equation is a function of integration. Simplifying Eq. (5.1.13) and differentiating it with respect to $y$ and $z$, we obtain $F^i_x = 0$, for $i = 1, ..., 4$ and $E^6 = -\frac{C}{B} \left[ \frac{z^2}{2} F^5_x + z F^6_x \right] + F^9(t, x)$; $F^9$ being a function of integration. Hence we have $F^i = a_i$, for $i = 1, ..., 4$.

Renaming the unknown functions $F^5 = P^1$, $F^6 = P^2$, $F^7 = P^3$, $F^8 = P^0$ and $F^9 = P^4$, we have the following vector field generating CRCs, in terms of some unknown functions of $t$ and $x$:

$$X^0 = -\frac{C}{A} \left[ \frac{y^2 + z^2}{2} P^1_t + z P^2_t + y P^3_t \right] + P^0,$$

$$X^1 = -\frac{C}{B} \left[ \frac{y^2 + z^2}{2} P^1_x + z P^2_x + y P^3_x \right] + P^4,$$

$$X^2 = a_2 yz + a_3 \left( \frac{z^2 - y^2}{2} \right) + a_4 z + y P^1 + P^3,$$

$$X^3 = a_2 \left( \frac{z^2 - y^2}{2} \right) - a_3 yz - a_4 y + z P^1 + P^2. \quad (5.3.12)$$

The inheriting factor can be obtained from Eq. (5.1.14) as $\psi = \frac{C'}{2C} X^0 + X^2$. We substitute the above values in the system of Eqs. (5.1.7)-(5.1.16) one by one and observe that some of these equations are identically satisfied and the rest of them give the following integrability conditions:

$$\frac{1}{2} \left( \frac{C'}{C} - \frac{B'}{B} \right) P^1_t - \frac{A}{B} P^1_{xx} = 0, \quad (5.3.13)$$

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\[
\frac{1}{2} \left( \frac{C'}{C} - \frac{B'}{B} \right) P_t^2 - \frac{A}{B} P_{xx}^2 - a_2 \frac{A}{C} = 0, \quad (5.3.14)
\]
\[
\frac{1}{2} \left( \frac{C'}{C} - \frac{B'}{B} \right) P_t^3 - \frac{A}{B} P_{xx}^3 + a_3 \frac{A}{C} = 0, \quad (5.3.15)
\]
\[
2C P_{tx}^1 + B \left( \frac{C}{B} \right)' P_x^1 = 0, \quad (5.3.16)
\]
\[
2C P_{tx}^2 + B \left( \frac{C}{B} \right)' P_x^2 = 0, \quad (5.3.17)
\]
\[
2C P_{tx}^3 + B \left( \frac{C}{B} \right)' P_x^3 = 0, \quad (5.3.18)
\]
\[
\frac{C}{A} P_{tt}^1 + \frac{1}{2} \left( \frac{C}{A} \right)' P_t^1 = 0, \quad (5.3.19)
\]
\[
\frac{C}{A} P_{tt}^2 + \frac{1}{2} \left( \frac{C}{A} \right)' P_t^2 + a_2 = 0, \quad (5.3.20)
\]
\[
\frac{C}{A} P_{tt}^3 + \frac{1}{2} \left( \frac{C}{A} \right)' P_t^3 - a_3 = 0, \quad (5.3.21)
\]
\[
\frac{1}{2} \left( \frac{B'}{B} - \frac{C'}{C} \right) P^0 + P^4_x - P^1_x = 0, \quad (5.3.22)
\]
\[
\frac{1}{2} \left( \frac{A'}{A} - \frac{C'}{C} \right) P^0 + P^0_t - P^1_t = 0. \quad (5.3.23)
\]
\[A P^0_x + B P^4_t = 0 \quad (5.3.24)
\]

To have a complete solution of Eqs. (5.1.7)-(5.1.16), we require to solve the above system. For a complete classification, we solve these equations in the following eight cases:

1. \( A' = B' = C' = 0 \)
2. \( A' = C' = 0, B' \neq 0 \)
3. \( B' = C' = 0, A' \neq 0 \)
4. \( A' = B' = 0, C' \neq 0 \)
5. \( B' = 0, A' \neq 0, C' \neq 0 \)
6. \( B' \neq 0, A' = 0, C' \neq 0 \)
7. \( B' \neq 0, A' \neq 0, C' = 0 \)
8. \( A' \neq 0, B' \neq 0, C' \neq 0 \)

Case (1): In the first case, we assume that all the Ricci tensor compo-
ponents $A$, $B$ and $C$ are non-zero constant, say $A = a$, $B = b$ and $C = c$. Then the system of equations (5.3.13)- (5.3.24) takes the form:

\[ P_{1xx} = 0, \quad (5.3.25) \]
\[ cP_{xx} + ba_2 = 0, \quad (5.3.26) \]
\[ cP_{xx} - ba_3 = 0, \quad (5.3.27) \]
\[ P_{tx}^1 = P_{tx}^2 = P_{tx}^3 = 0, \quad (5.3.28) \]
\[ P_{tt}^1 = 0, \quad (5.3.29) \]
\[ \frac{c}{a} P_{tt}^2 + a_2 = 0, \quad (5.3.30) \]
\[ \frac{c}{a} P_{tt}^3 - a_3 = 0, \quad (5.3.31) \]
\[ P_x - P^1 = 0, \quad (5.3.32) \]
\[ P_t^0 - P^1 = 0, \quad (5.3.33) \]
\[ aP_x^0 + bP_t^4 = 0. \quad (5.3.34) \]

From Eq. (5.3.28), we have $P^1 = G^1(t) + G^2(x)$, where $G^1$ and $G^2$ are functions of integration. Putting this value in Eq. (5.3.25), we have $G^2_{xx} = 0 \Rightarrow G^2 = c_1 x + c'_2$. Similarly, Eq. (5.3.29) implies $G^1_{tt} = 0 \Rightarrow G^1 = c_2 t + c'_3$. Thus $P^1 = c_1 x + c_2 t + c_3$, where $c'_2 + c'_3 = c_3$. Similarly, by integrating Eq. (5.3.28) with respect to $t$ and $x$, we get $P^2 = G^3(t) + G^4(x)$; $G^3$ and $G^4$ being functions of integration. Using this value of $P^2$ in Eq. (5.3.26) and integrating it twice with respect to $x$, it gives $G^4 = -\frac{b}{2c} x^2 + c_4 x + c'_5$. Next, we use the value of $P^2(t, x)$ in Eq. (5.3.30), integrate it twice with respect to $t$ to obtain $G^3 = -\frac{a}{2c} t^2 + c_5 t + c'_6$. Thus we have $P^2 = -\frac{a}{2c}(at^2 + bx^2) + c_4 x + c_5 t + c_6$, where $c'_5 + c'_6 = c_6$. In a similar way, Eqs. (5.3.27), (5.3.28) and (5.3.31) gives $P^3 = \frac{a}{2c}(at^2 + bx^2) + c_7 x + c_8 t + c_9$. 

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Next, integrating Eqs. (5.3.32) and (5.3.33) with respect to \( x \) and \( t \) respectively, we have \( P^0 = G^5(x) + c_1 t + \frac{c_2}{2} t^2 + c_3 t \) and \( P^4 = G^6(t) + \frac{c_5}{2} t^2 + c_2 t + c_3 x \). Substituting these values of \( P^0 \) and \( P^4 \) in Eq. (5.3.34) and solving it, we have \( G^5 = -\frac{c_{10}}{a} x - \frac{b c_2}{2a} x^2 + c_{11} \) and \( G^6 = \frac{c_{10}}{b} t - \frac{a c_1}{2b} t^2 + c_{12} \). Hence the system (5.3.12) gets the form:

\begin{align*}
X^0 & = \frac{c_2}{2a} (at^2 - bx^2 - cy^2 - cz^2) + a_2 z - \frac{c}{a} c_5 z - a_3 y t - \frac{c}{a} c_8 y + c_1 t + c_3 t \\
& \quad - \frac{c_{10}}{a} x + c_{11}, \\
X^1 & = \frac{c_1}{2b} (b x t^2 - at^2 - cy^2 - cz^2) - a_2 x z + \frac{c}{b} c_4 z - a_3 y x - \frac{c}{b} c_7 x + c_2 t + c_3 x \\
& \quad + \frac{c_{10}}{b} t + c_{12}, \\
X^2 & = a_2 y z + \frac{a_3}{2c} (a t^2 + b x^2 + c z^2 - cy^2) + a_4 z + y (c_1 x + c_2 t + c_3) + c_7 x \\
& \quad + c_8 t + c_9, \\
X^3 & = \frac{a_2}{2c} (cz^2 - at^2 - bx^2 - cy^2) - a_3 y z - a_4 y + z (c_1 x + c_2 t + c_3) \\
& \quad + c_4 x + c_5 t + c_6. \quad (5.3.35)
\end{align*}

The inheriting factor in this case turned out to be \( \psi = a_2 z - a_3 y + c_1 x + c_2 t + c_3 \).

It can be easily seen that in this case, the Lie algebra of CRCs is 15-dimensional. Out of these 15 CRCs, five are proper CRCs and the remaining ten are RCs.

The remaining non-degenerate cases can be solved by using the similar procedure as that of the above case. We omit the repetition and the finally obtained results for all the remaining cases are summarized in the following tables.
<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCS</th>
<th>Inheriting Factor ($\psi$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2(a)</td>
<td>$B = (c_1 t + c_2)^{\alpha}$, $A = a$, $C = c$</td>
<td>$X^0 = -\frac{c}{2} c_1 \left[ \frac{1}{c^2} (c_2 \cos c_1 \frac{a}{c^2} + c_3 \sin c_1 \frac{a}{c^2}) + y \left( C_4 \cos c_1 \frac{a}{c^2} + c_5 \sin c_1 \frac{a}{c^2} \right) + z \left( C_6 \cos c_1 \frac{a}{c^2} + c_7 \sin c_1 \frac{a}{c^2} \right) + \frac{x}{c^2} \left( C_8 \sin c_1 \frac{a}{c^2} - c_9 \cos c_1 \frac{a}{c^2} \right) + \frac{x}{c^2} \left( C_10 \sin c_1 \frac{a}{c^2} + c_11 \cos c_1 \frac{a}{c^2} \right) \right]$</td>
<td>$X^2$</td>
</tr>
<tr>
<td>2(b)</td>
<td>$B = (c_1 t + c_2)^{\alpha}$, where $\alpha = 2 \left( 1 - \frac{4}{9} \right)$</td>
<td>$X^0 = c_1 t + c_2$, $X^1 = c_3 x + c_4$, $X^2 = a_4 x + c_5 y + c_6$, $X^3 = -a_4 y + c_1 z + c_6$.</td>
<td>$X^2$</td>
</tr>
<tr>
<td>3</td>
<td>$B = b$, $C = c$</td>
<td>$X^0 = -\frac{1}{c^2} \left[ \frac{1}{c^2} (b t^2 + c_9 y^2 + c_3 z^2) + c_2 t y + c_3 s z + c_4 x + \left( a_3 y - c_2 z - c_5 x - c_6 \right) f \sqrt{\mathcal{A}} dt - \frac{1}{c} \left( f \sqrt{\mathcal{A}} dt \right)^2 dt - c_7 \right]$, $X^1 = \frac{c}{c^2} \left( b t^2 - \frac{c_9}{c^2} y^2 - \frac{c_3}{c^2} z^2 \right) - a_3 x y + a_2 t x - \frac{c_9}{c^2} c_9 z + c_6 x + \left( c_1 x + c_2 \right) f \sqrt{\mathcal{A}} dt - \frac{1}{c^2} \left( f \sqrt{\mathcal{A}} dt \right)^2 + c_4$, $X^2 = \frac{c}{c^2} \left( b t^2 - \frac{c_9}{c^2} y^2 + \frac{c_3}{c^2} z^2 \right) + a_2 t y + c_5 x y + c_6 c_9 x + \left( c_1 y + c_2 \right) f \sqrt{\mathcal{A}} dt + \frac{1}{c^2} \left( f \sqrt{\mathcal{A}} dt \right)^2 + c_{11}$, $X^3 = \frac{c}{c^2} \left( -b t^2 - \frac{c_9}{c^2} y^2 + \frac{c_3}{c^2} z^2 \right) - a_3 y^2 + c_5 x z + c_6 z - a_4 y + c_9 x + \left( c_1 y + c_2 \right) f \sqrt{\mathcal{A}} dt + \frac{1}{c^2} \left( f \sqrt{\mathcal{A}} dt \right)^2 + c_{12}$</td>
<td>$X^2$</td>
</tr>
</tbody>
</table>

Table 5.2: CRCS for Non-Degenerate Ricci Tensor

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<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
</table>
| 4(a) | $C = (c_5 t + c_7)^2$  
  $A = a$  
  $B = b$ | $X^0 = c_1 x t + c_2 t + c_3 x + c_4$,  
  $X^1 = \frac{1}{\sqrt{2}} (x^2 - t^2) + c_2 x - c_3 t + c_5$,  
  $X^2 = a_4 z + c_7$,  
  $X^3 = -a_4 y + c_6$. | $\frac{c_1^2}{4} X^0$  
  $+ X_2^2$ |
| 4(b) | $C = (c_1 t + c_2)^\alpha$,  
  where  
  $\alpha = 2 \left(1 - \frac{1}{4}\right)$ | $X^0 = c_1 t + c_2$,  
  $X^1 = c_4 x + c_4$,  
  $X^2 = a_4 z + c_4 y + c_4$,  
  $X^3 = -a_4 y + c_4 x + c_4$. | $\frac{c_1^2}{4} X^0$  
  $+ X_2^2$ |
| 4(c) | $C = e^t$  
  $A = a$  
  $B = b$ | $X^0 = 2 a_3 y - 2 a_2 z - \frac{w}{\sqrt{2}} \left(\frac{c_1}{\sqrt{2}} \cos \frac{\pi}{4} + c_2 \sin \frac{\pi}{4}\right)$  
  $- \sqrt{\frac{e}{\pi}} \left(\frac{c_3}{\sqrt{2}} \cos \frac{\pi}{4} + c_4 \sin \frac{\pi}{4}\right) + \sqrt{\frac{e}{\pi}} \left(c_5 \cos \frac{\pi}{4} + c_6 \sin \frac{\pi}{4}\right)$  
  $- \sqrt{\frac{e}{\pi}} \left(\frac{c_7}{\sqrt{2}} \cos \frac{\pi}{4} + c_8 \sin \frac{\pi}{4}\right) \left[\frac{1}{2} (y^2 + z^2) - 2 t \sqrt{\frac{e}{\pi}}\right]$  
  $- 2 c_9$,  
  $X^1 = \frac{e^t}{2} \left[ y \left(-c_1 \sin \frac{\pi}{4} \cos \sqrt{\frac{e}{\pi}} + c_2 \cos \sqrt{\frac{e}{\pi}}\right)\right.$  
  $+ c_3 \sin \frac{\pi}{4} \cos \sqrt{\frac{e}{\pi}} + c_4 \cos \sqrt{\frac{e}{\pi}}\left.\right] + 2 \sqrt{\frac{e}{\pi}} \left(c_5 \sin \frac{\pi}{4} - c_6 \cos \frac{\pi}{4}\right)$  
  $+ \left(c_7 \sin \frac{\pi}{4} - c_8 \cos \frac{\pi}{4}\right) \left[\frac{e^t}{2} (y^2 + z^2) + 2 t \sqrt{\frac{e}{\pi}}\right] + c_{10}$,  
  $X^2 = a_2 y z + a_3 \left(z^2 - y^2\right) + a_4 x + c_4 y + 2 a_3 x e^{-t}$  
  $- 2 \sqrt{\frac{e}{\pi}} \left(\frac{c_1}{\sqrt{2}} \cos \frac{\pi}{4} + c_2 \sin \frac{\pi}{4}\right)$  
  $+ \sqrt{\frac{e}{\pi}} \left(\frac{c_3}{\sqrt{2}} \cos \frac{\pi}{4} + c_4 \sin \frac{\pi}{4}\right) + c_{11}$,  
  $X^3 = -a_3 y z + a_2 \left(z^2 - y^2\right) - a_4 x + c_4 y - 2 a_3 x e^{-t}$  
  $- 2 \sqrt{\frac{e}{\pi}} \left(\frac{c_1}{\sqrt{2}} \cos \frac{\pi}{4} + c_2 \sin \frac{\pi}{4}\right)$  
  $+ \sqrt{\frac{e}{\pi}} \left(\frac{c_3}{\sqrt{2}} \cos \frac{\pi}{4} + c_4 \sin \frac{\pi}{4}\right) + c_{12}$. | $\frac{c_1^2}{4} X^0$  
  $+ X_2^2$ |

Table 5.3: CRCs for Non-Degenerate Ricci Tensor
<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>5(a)</td>
<td>$A = \frac{b}{e^{2}}$, $B = b$</td>
<td>$X^0 = -\frac{\mathcal{F}}{e^{2}} \left[ z^2 \cdot \left( e_1 \cos \frac{z}{2} - e_2 \sin \frac{z}{2} \right) + y \left( e_3 \cos \frac{z}{2} + e_4 \sin \frac{z}{2} \right) + z \left( e_5 \cos \frac{z}{2} + e_6 \sin \frac{z}{2} \right) \right] + \frac{2b a y}{e^{2}} + \frac{2b a z}{e^{2}} + \frac{2b a z}{e^{2}} - \sqrt{z} \left( e_7 \sin \frac{z}{2} + e_8 \cos \frac{z}{2} \right) - e_9$,</td>
<td>$\frac{C' X^0}{X^1} + \frac{X^0}{X^2}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X^1 = -\frac{\mathcal{F}}{e^{2}} \left[ z^2 \cdot \left( e_1 \sin \frac{z}{2} + e_2 \cos \frac{z}{2} \right) + y \left( e_3 \sin \frac{z}{2} - e_4 \cos \frac{z}{2} \right) + z \left( e_5 \sin \frac{z}{2} - e_6 \cos \frac{z}{2} \right) \right] - \frac{2b a y}{e^{2}} - \frac{2b a z}{e^{2}} - \frac{2b a z}{e^{2}} - e_6$,</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X^2 = a_3 g y + \frac{a y}{e^{2}} \left( x^2 - y^2 \right) + a_4 z + b_9 y + \frac{2b a z}{e^{2}} - \frac{2b a z}{e^{2}} - \frac{2b a z}{e^{2}} - \frac{2b a z}{e^{2}} - e_6$,</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$X^3 = -a_3 g y + \frac{a y}{e^{2}} \left( x^2 - y^2 \right) - a_4 y - b_9 z - \frac{2b a z}{e^{2}} - \frac{2b a z}{e^{2}} - \frac{2b a z}{e^{2}} - \frac{2b a z}{e^{2}} - e_6$,</td>
<td></td>
</tr>
</tbody>
</table>

| 5(b) | $A = \frac{-c^{2}}{c}$ | $X^0 = -\frac{\mathcal{F}}{e^{2}} \left[ z^2 \cdot \left( e_1 \cosh \frac{z}{2} + e_2 \sinh \frac{z}{2} \right) + y \left( e_3 \cosh \frac{z}{2} + e_4 \sinh \frac{z}{2} \right) + z \left( e_5 \cosh \frac{z}{2} + e_6 \sinh \frac{z}{2} \right) \right] + \frac{2b a y}{e^{2}} + \frac{2b a z}{e^{2}} + \frac{2b a z}{e^{2}} - \sqrt{z} \left( e_7 \sinh \frac{z}{2} + e_8 \cosh \frac{z}{2} \right) - e_9$, | $\frac{C' X^0}{X^1} + \frac{X^0}{X^2}$ |
|      |             | $X^1 = -\sqrt{z} \left[ z^2 \cdot \left( e_1 \sinh \frac{z}{2} + e_2 \cosh \frac{z}{2} \right) + y \left( e_3 \sinh \frac{z}{2} - e_4 \cosh \frac{z}{2} \right) + z \left( e_5 \sinh \frac{z}{2} - e_6 \cosh \frac{z}{2} \right) \right] + \frac{2b a y}{e^{2}} + \frac{2b a z}{e^{2}} + \frac{2b a z}{e^{2}} + e_6$, |                         |
|      |             | $X^2 = a_3 g y + \frac{a y}{e^{2}} \left( x^2 - y^2 \right) + a_4 z + b_9 y + \frac{2b a z}{e^{2}} + \frac{2b a z}{e^{2}} + \frac{2b a z}{e^{2}} + \frac{2b a z}{e^{2}} + e_6$, |                         |
|      |             | $X^3 = -a_3 g y + \frac{a y}{e^{2}} \left( x^2 - y^2 \right) - a_4 y - b_9 z - \frac{2b a z}{e^{2}} - \frac{2b a z}{e^{2}} - \frac{2b a z}{e^{2}} - \frac{2b a z}{e^{2}} - e_6$, |                         |

Table 5.4: CRCs for Non-Degenerate Ricci Tensor
<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$\frac{B}{\gamma} = \left( f \frac{1}{\sqrt{\gamma}} dt + \xi \right)^2$ where $\xi$ is a constant $A = a$</td>
<td>$X^0 = -\frac{\sqrt{2}}{2\pi} \left( \frac{y^2 + z^2}{x^2 + y^2 + z^2} \right)$ $\left( c_1 \cos \frac{y}{x} + c_2 \sin \frac{y}{x} \right)$ $+ y \left( c_3 \cos \frac{y}{x} + c_4 \sin \frac{y}{x} \right) + z \left( c_5 \cos \frac{y}{x} + c_6 \sin \frac{y}{x} \right)$ $- \frac{a}{\gamma} \left( c_1 \cos \frac{y}{x} + c_2 \sin \frac{y}{x} \right) \left( c_7 \sin \frac{y}{x} - c_6 \cos \frac{y}{x} \right)$ $+ a \sqrt{\pi} \left( a_3 y - a_2 z - a_0 \right)$</td>
<td>$\frac{C}{\sqrt{2}} X^0 + X_2^2$</td>
</tr>
<tr>
<td>7</td>
<td>$A = \frac{B^2}{\gamma}$ $C = e$</td>
<td>$X^0 = -\frac{\sqrt{2}}{2\pi} \left( \frac{y^2 + z^2}{x^2 + y^2 + z^2} \right)$ $\left( c_1 \cos \frac{y}{x} + c_2 \sin \frac{y}{x} \right)$ $+ y \left( c_3 \cos \frac{y}{x} + c_4 \sin \frac{y}{x} \right) + z \left( c_5 \cos \frac{y}{x} + c_6 \sin \frac{y}{x} \right)$ $+ \frac{a}{\gamma} \left( c_1 \cos \frac{y}{x} + c_2 \sin \frac{y}{x} \right) \left( c_7 \sin \frac{y}{x} - c_6 \cos \frac{y}{x} \right)$ $+ \frac{a}{\gamma} \left( a_3 y - a_2 z - a_0 \right)$</td>
<td>$X_2^2$</td>
</tr>
</tbody>
</table>

Table 5.5: CRCs for Non-Degenerate Ricci Tensor
<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor</th>
</tr>
</thead>
</table>
| 8(a) | \(B = C\) | \(X^0 = -\sqrt{\frac{3}{\pi}} \left[ \frac{2}{3} (x^2 + y^2 + z^2) \\
+ (a_3 y - a_2 z - c_2 x) \int \sqrt{\frac{3}{2}} dt - c_4 x + c_5 y \\
+ c_6 z - \frac{a_5}{2} \left( \int \sqrt{\frac{3}{2}} dt \right) - c_7 \right] \)
\(X^1 = \frac{2}{\sqrt{3}} (x^2 - y^2 - x^2) + c_4 x - c_4 y - c_9 z - a_3 y \\
+ a_2 z + c_1 x \int \sqrt{\frac{2}{2}} dt - \frac{a_1}{2} (c_2 \int \sqrt{\frac{2}{2}} dt + c_4)^2 + c_{10}. \)
\(X^2 = \frac{a_1}{2} (x^2 - y^2 + x^2) + a_2 y z + a_4 x + c_2 y x + c_6 x \\
+ c_3 y + (c_1 y + c_8) \int \sqrt{\frac{2}{2}} dt + \frac{a_5}{2} (f \int \frac{3}{2} dt)^2 + c_{11}. \)
\(X^3 = \frac{a_5}{2} (-x^2 - y^2 + z^2) - a_3 y z - a_4 y + c_2 x z + c_3 \\
+ c_9 x + (c_2 z + c_6) \int \sqrt{\frac{2}{2}} dt - \frac{a_5}{2} (f \int \frac{3}{2} dt)^2 + c_{12}. \) | \(\frac{c_1}{\sqrt{3}} X^0 + X^2_{\frac{n}{2}}\) |
| 8(b) | \(\frac{a_1 y z}{c_2} = (\frac{1}{\sqrt{3}})^2\) | \(X^0 = -\frac{2}{\sqrt{3}} \left[ \frac{2}{\sqrt{3}} (c_1 \sin x - c_2 \cos x) \\
+y (c_3 \sin x - c_4 \cos x) + z (c_5 \sin x - c_6 \cos x) \\
- \frac{1}{\sqrt{3}} (c_1 \sin x - c_2 \cos x) + c_4 \sin x - c_6 \cos x \\
+ \sqrt{\frac{3}{2}} (a_1 y - a_2 z - c_9) \right] \)
\(X^1 = -\frac{2}{\sqrt{3}} \left[ \frac{2}{\sqrt{3}} (c_1 \cos x + c_2 \sin x) \\
+y (c_3 \cos x + c_4 \sin x) + z (c_5 \cos x + c_6 \sin x) \\
+ \frac{1}{\sqrt{3}} (c_1 \cos x + c_2 \sin x) + c_7 \cos x + c_8 \sin x \right] + c_{10}. \)
\(X^2 = a_2 y z + \frac{a_5}{2} (x^2 - y^2) + a_4 x + c_9 y + a_3 \frac{a_5}{2} \\
+ \sqrt{\frac{3}{2}} y (c_1 \sin x - c_2 \cos x) + \sqrt{\frac{3}{2}} (c_3 \sin x - c_4 \cos x) + c_{11}. \)
\(X^3 = -a_3 y z + \frac{a_5}{2} (x^2 - y^2) - a_4 y + c_9 x - a_2 \frac{a_5}{2} \\
+ \sqrt{\frac{3}{2}} z (c_1 \sin x - c_2 \cos x) + \sqrt{\frac{3}{2}} (c_3 \sin x - c_4 \cos x) + c_{12}. \) | \(\frac{c_1}{\sqrt{3}} X^0 + X^2_{\frac{n}{2}}\) |

Table 5.6: CRCs for Non-Degenerate Ricci Tensor
5.4 Some LRS Bianchi Type I Metrics Admitting CRCs

In the previous section, we have presented a complete classification of LRS Bianchi type I spacetimes via their CRCs. However, the obtained CRCs are found subject to some differential constraints to be satisfied by the Ricci tensor components. For showing that the obtained classes of CRCs are non-empty, we require to solve these differential constraints to get the exact form of the LRS Bianchi type I spacetimes metric admitting these CRCs. As the differential constraints are high non-linear, it is quite difficult to solve them generally. However, they may be simplified by choosing some specific values of the metric functions. In this section, we present some of such metrics.

**Example 5.4.1:** Setting $h(t) = t$ and $k(t) = t^2$, we get the following LRS Bianchi type I spacetime metric:

$$ds^2 = -dt^2 + t^2 \, dx^2 + t^4 \left(dy^2 + dz^2\right).$$  \hfill (5.4.1)

The above metric satisfies all the constraints of case 5(b) and hence it admits 15-dimensional Lie algebra of CRCs, as given in Table 5.5. Out of these 15 CRCs, seven are RCs and remaining eight are proper CRCs. The Ricci scalar $R$ for the above metric is found as $R = \frac{24}{t^2}$ and the components of stress-energy tensor are $T_{00} = \frac{8}{t^2}$, $T_{11} = -8$ and $T_{22} = T_{33} = -4t^2$, which yield a non-degenerate Lorentzian matter tensor metric.

**Example 5.4.2:** The conditions of case D2 implies $h''k + 2hkk'' = 0$. Choosing $h(t) = c_1t + c_2$ and $k(t) = c_3t + c_4$, we have the following metric
satisfying this equation.

\[ ds^2 = -dt^2 + (c_1 t + c_2)^2 \, dx^2 + (c_3 t + c_4)^2 \, [dy^2 + dz^2]; \ c_1 \neq 0, c_3 \neq 0. \quad (5.4.2) \]

The components of stress-energy tensor for the above metric are

\[
\begin{align*}
T_{00} &= \frac{c_3}{c_3 t + c_4} \left( \frac{2c_1}{c_1 t + c_2} + \frac{c_3}{c_3 t + c_4} \right), \\
T_{11} &= -c_3^2 \left( \frac{c_1 t + c_2}{c_3 t + c_4} \right)^2, \\
T_{22} &= T_{33} = -c_1 c_3 \left( \frac{c_3 t + c_4}{c_1 t + c_2} \right). \quad (5.4.3)
\end{align*}
\]

The Ricci scalar \( R \) in this case becomes \( R = \frac{2c_3}{c_3 t + c_4} \left( \frac{c_3}{c_3 t + c_4} + \frac{2c_1}{c_1 t + c_2} \right) \).

**Example 5.4.3:** The constraints in case D4 give \( h''k + 2h'k' = 0 \) and \( hkk'' + h'kk' + hk'^2 = 0 \). Following is one of the LRS Bianchi type I metrics satisfying these equations:

\[ ds^2 = -dt^2 + e^{2t} \, dx^2 + e^{-t} \, [dy^2 + dz^2]. \quad (5.4.4) \]

Here the Ricci scalar gets the value \( R = \frac{3}{2} \). The stress-energy tensor components are \( T_{00} = -\frac{3}{4}, T_{11} = -\frac{3}{4} e^{2t} \) and \( T_{22} = T_{33} = -\frac{3}{4} e^{-t} \).

One may work on finding other such metrics in the remaining cases.

### 5.5 Summary

In this chapter, we have presented a complete classification of LRS Bianchi type I spacetimes via CRCs. This work may be considered as an extension of the previously published work about HRCs in the same spacetimes [27].
Moreover, by setting the inheriting factor $\psi = 0$, one may get the classification of LRS Bianchi type I spacetimes according to their RCs.

To achieve a complete classification of LRS Bianchi type I spacetimes according to their CRCs, we have considered both degenerate and non-degenerate Ricci tensor cases. Our analysis shows that when the Ricci tensor is degenerate, the LRS Bianchi type I spacetimes admit infinite-dimensional Lie algebra of CRCs. For the choice of non-degenerate Ricci tensor, these spacetimes produce 7- or 15-dimensional Lie algebra of CRCs. In most of the cases considered here, the CRCs are found subject to some differential constraints to be satisfied by the Ricci tensor components. To show that such LRS Bianchi type I metrics exist, we have presented some examples in Sec. 5.4.

During our classification, we have not mentioned any form of the stress-energy tensor $T_{ab}$. For the physical implications of the obtained results, if we assume a perfect fluid as a source of the stress-energy tensor, then $T_{ab}$ has the form:

$$T_{ab} = (p + \rho)u_a u_b + p g_{ab}, \quad (5.5.1)$$

where $p, \rho$ and $u^a$ are the pressure, energy density and the four velocity of the perfect fluid. We choose the fluid velocity as $u^a = \delta^a_0$, then $T_{00} = \rho$, $T_{11} = ph^2$ and $T_{22} = T_{33} = pk^2$. Also, the system (5.1.6) gives:

$$2 \frac{h' k'}{hk} + \frac{k'^2}{k^2} = \rho,$$

$$\frac{k'^2}{k^2} + 2 \frac{k''}{k} = -p,$$

$$\frac{h''}{h} + \frac{k''}{k} + \frac{h' k'}{hk} = -p. \quad (5.5.2)$$

With the help of (5.5.2), we can write the Ricci tensor components given in
(5.1.4) as follows:

\[ A(t) = \frac{3p + \rho}{2}, \quad B(t) = \frac{h^2}{2} (\rho - p), \quad C(t) = \frac{k^2}{2} (\rho - p). \quad (5.5.3) \]

The above expressions show that a perfect fluid source is not allowed in some of the degenerate cases, namely D1, D3, D5 and D6. Therefore, we may choose some other matter for these cases. In case D2, we have the equation of state \( 3p + \rho = 0 \), which violates the strong energy condition. A perfect fluid metric satisfying the constraints of this case is given in Eq. (5.4.2). In case D4, the pressure and energy density of the perfect fluid are obtained as \( p = \rho = -\frac{3}{4} \), which violate the energy conditions. An example of a perfect fluid metric satisfying the constraints of this case is given in Eq. (5.4.4).

In case 1 of non-degenerate Ricci tensor, the conditions of perfect fluid imply \( 3p + \rho = 2, \rho = \frac{2}{h^2} + p \) and \( h = k \). Thus the LRS Bianchi type I spacetime metric reduces to the well known conformally flat FRW metric with \( K=0 \). A more detailed discussion about RCs and CRCs in FRW metric is given in [7]. Similar remarks follow for the case 3, where the perfect fluid conditions yield \( h = k \) and \( p = \rho - \frac{2}{h^2} \). In this case, the system (5.1.4) gives \( A = -\frac{3h''}{h} \) where \( h \) satisfies the differential constraint \( hh'' + 2h'^2 = 1 \).

In the remaining non-degenerate cases, the conditions of a perfect fluid establish relation between pressure \( p \) and energy density \( \rho \) of the fluid, however the system (5.1.4) produces some highly non-linear differential constraints involving \( h \) and \( k \) which cannot be solved easily. One may choose some particular values of \( h \) and \( k \) satisfying these differential constraints to form the exact form of perfect fluid LRS Bianchi type I spacetime metrics.
Chapter 6

Conformal Ricci Collineations of LRS Bianchi type V Spacetimes

In this chapter, we assume a perfect fluid source and investigate the CRCs for LRS Bianchi type V spacetimes. Like the previous chapters, the CRC equations are solved in two cases depending upon whether the Ricci tensor is degenerate or non-degenerate. In degenerate case, the CRC equations are directly integrated to find the final form of CRCs. In non-degenerate case, the collineation vector is found subject to some differential constraints. For different choices of the Ricci tensor components, these differential constraints are solved to get the explicit form of CRCs.
6.1 CRC Equations

The Bianchi type V universe is the natural generalization of Friedmann-Robertson-Walker model with negative curvature. The metric of LRS Bianchi type V spacetimes is given by [69]:

$$ds^2 = dt^2 - \alpha^2(t) \, dx^2 - e^{2q x} \beta^2(t) \left[ dy^2 + \, dz^2 \right], \quad (6.1.1)$$

where $\alpha(t)$ and $\beta(t)$ are functions of $t$ only. For $q = 0$, the above metric reduces to the metric of LRS Bianchi type I spacetimes whose complete classification via CRCs was presented in the previous chapter. Throughout this chapter, we assume that $q \neq 0$. The above metric admits the following four KVs:

$$X_{(1)} = \partial_y, \ X_{(2)} = \partial_z, \ X_{(3)} = z \partial_y - y \partial_z, \ X_{(4)} = \partial_x - qy \partial_y - qz \partial_z. \quad (6.1.2)$$

Following are its non-vanishing components of the Ricci tensor:

$$R_{00} = - \left( \frac{\alpha''}{\alpha} + 2 \frac{\beta''}{\beta} \right) = R_0,$$
$$R_{11} = \alpha^2 \left( \frac{\alpha''}{\alpha} + 2 \frac{\alpha' \beta}{\alpha \beta} - 2 \frac{q^2}{\alpha^2} \right) = R_1,$$
$$R_{22} = R_{33} = e^{2q x} R_2,$$
$$R_{01} = 2q \left( \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} \right), \quad (6.1.3)$$

where the primes on the metric functions denote their derivatives with respect to $t$ and $R_2 = \beta^2 \left( \frac{\beta''}{\beta} + \frac{\alpha' \beta'}{\alpha \beta} + \frac{\beta^2}{\beta^2} - 2 \frac{q^2}{\alpha^2} \right)$. The Ricci curvature scalar $R = g^{ab} R_{ab}$ becomes:

$$R = 2 \left( \frac{\alpha''}{\alpha} + 2 \frac{\beta''}{\beta} \right) + 2 \left( \frac{\beta'^2}{\beta^2} + \frac{2 \alpha' \beta'}{\alpha \beta} \right) - \frac{6q^2}{\alpha^2}. \quad (6.1.4)$$
Using the EFEs (1.1.1) with unit gravitational constant, we get the components of energy-momentum tensor as:

\begin{align*}
T_{00} &= T_0 = 2\frac{\alpha'\beta'}{\alpha\beta} + \frac{\beta'^2}{\beta^2} - \frac{3q^2}{\alpha^2}, \\
T_{11} &= T_1 = -\alpha^2 \left( \frac{\beta'^2}{\beta^2} + \frac{2\beta''}{\beta} - \frac{q^2}{\alpha^2} \right), \\
T_{22} &= T_{33} = e^{2qx}T_2, \\
T_{01} &= 2q \left( \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} \right),
\end{align*}

(6.1.5)

where \(T_2 = -\beta^2 \left( \frac{\alpha''}{\alpha} + \frac{\beta''}{\beta} + \frac{\alpha'\beta'}{\alpha\beta} - \frac{q^2}{\alpha^2} \right)\).

Considering the perfect fluid as a source of energy-momentum tensor, we have \(T_{ab} = (p + \rho)u_au_b - pg_{ab}\), where \(p, \rho\) and \(u^a\) are the pressure, energy density and the four velocity of the perfect fluid. We choose the fluid velocity as \(u^a = \delta^a_0\), then one can obtain \(T_{00} = \rho, T_{11} = p\alpha^2, T_{22} = T_{33} = pe^{2qx}\beta^2\) and \(T_{01} = 0\). The relation \(T_{01} = 0\) implies \(\alpha = \beta\) and hence \(R_{01} = 0\). Moreover, the system (6.1.3) yields \(R_{00} = R_0 = -3\frac{2''}{\alpha}, R_{11} = R_1 = \alpha\alpha'' + 2\alpha'^2 - 2q^2\) and \(R_{22} = R_{33} = e^{2qx}R_1\). In terms of \(p\) and \(\rho\), the components \(R_0\) and \(R_1\) can be easily expressed as \(R_0 = \frac{1}{2} (\rho + 3p)\) and \(R_1 = \frac{\alpha^2}{2} (\rho - p)\). Using the Ricci tensor components in Eq. (1.11.5), we have the following CRC equations:

\begin{align*}
R'_0X^0 + 2R_0X^0_{,0} &= 2\psi R_0, \\
R_0X^0_0 + R_1X^1_0 &= 0, \\
R_0X^0_2 + e^{2qx}R_1X^2_0 &= 0, \\
R_0X^0_3 + e^{2qx}R_1X^3_0 &= 0, \\
R'_1X^0 + 2R_1X^1_0 &= 2\psi R_1, \\
R_1 \left( X^1_2 + e^{2qx}X^2_1 \right) &= 0, \\
R_1 \left( X^1_3 + e^{2qx}X^3_1 \right) &= 0,
\end{align*}

(6.1.6) \quad (6.1.7) \quad (6.1.8) \quad (6.1.9) \quad (6.1.10) \quad (6.1.11) \quad (6.1.12)
In the forthcoming sections, we solve the above system of equations in two cases depending upon whether the Ricci tensor is degenerate or non-degenerate.

6.2 CRCs for Degenerate Ricci Tensor

In this section, we solve the CRC equations (6.1.6)-(6.1.15) in case of degenerate Ricci tensor, that is when \( \det R_{ab} = 0 \). Thus two cases arise according as \( R_0 = 0 \), \( R_1 \neq 0 \) and \( R_0 \neq 0 \), \( R_1 = 0 \). Here we discuss both the cases in detail.

Case (I): In this case, the CRC equations (6.1.6)-(6.1.15) take the form:

\[
R'_1 X^0 + 2q R_1 X^1 + 2 R_1 X^2_2 = 2 \psi R_1, \quad (6.1.13)
\]
\[
R_1 (X^2_3 + X^3_2) = 0, \quad (6.1.14)
\]
\[
R'_1 X^0 + 2 R_1 (q X^1 + X^3_3) = 2 \psi R_1. \quad (6.1.15)
\]

Eq. (6.2.1) gives \( X^1 = P^1(x, y, z) \), \( X^2 = P^2_y(x, y, z) \) and \( X^3 = P^3(x, y, z) \), where \( P^1 \), \( P^2 \) and \( P^3 \) are functions of integration. Putting these values in Eq. (6.2.6) and integrating it with respect to \( y \), we get \( P^3 = -P^2_z + F^1_z \);
\( \frac{1}{2} G^6 + e^{2qx} G^3 = G^8(x) \) and \( G^4 + F^4 = F^5(y, z) \), then the
system (6.2.9) gets the following form:

\[
\begin{align*}
X^1 &= \frac{1}{2} \left[ z^2 e^{2q x} G_1^1 + z G_5^1 \right] + e^{2q x} \left[ \frac{y^2}{2} G_1^1 + y G_2^1 \right] + G^8, \\
X^2 &= -y G^1 - G^2 - F^5_y, \\
X^3 &= -z G^1 - \frac{1}{2} \int e^{-2q x} G_5^5 dx + F^5_x.
\end{align*}
\] (6.2.11)

Thus Eq. (6.2.10) becomes:

\[
F^5_{yy} + F^5_{zz} = 0.
\] (6.2.12)

Subtracting Eq. (6.2.5) from Eq. (6.2.2) and using the above values, we have:

\[
\begin{align*}
&\frac{e^{2q x}}{2} \left( y^2 + z^2 \right) G_{xx}^1 + \frac{e^{2q x}}{2} \left( y^2 + z^2 \right) G_x^1 + y e^{2q x} G_{xx}^2 + q y e^{2q x} G_x^2 \\
&\quad + \frac{1}{2} G_5^5 - \frac{q}{2} G_5^x + G_8^x - q G^8 + G^1 + F^5_y = 0.
\end{align*}
\] (6.2.13)

The following values are obtained by differentiating the above equation twice with respect to \( z \) and by some simple algebraic manipulation:

\[
\begin{align*}
F^5 &= -\frac{c_1}{4} y^2 z^2 + z G^9(y) + G^{10}(y) + y G^{11}(z) + G^{12}(z), \\
G^1 &= \frac{c_1}{2q^2} e^{-2q x} - \frac{c_2}{q} e^{-q x} + c_3, \\
G^9 &= -\frac{c_4}{2} y^2 + c_5 y + c_6, \\
G^5 &= -\frac{2c_4}{q} x + \frac{c_7}{q} e^{q x} + c_8.
\end{align*}
\] (6.2.14)

Thus Eq. (6.2.13) becomes:

\[
\frac{c_1}{2} y^2 + y e^{2q x} G_{xx}^2 + q y e^{2q x} G_x^2 + G_8^x - q G^8 + \frac{c_1}{2q^2} e^{-2q x} - \frac{c_2}{q} e^{-q x} + c_3 + G^{10}_{yy} = 0
\] (6.2.15)
Differentiating the above equation with respect to $y$, we obtain:

$$
G^2 = \frac{c_9}{2q^2} e^{-2qx} - \frac{c_{10}}{q} e^{-qx} + c_{11},
$$

$$
G^{10} = -\frac{c_1}{24} y^4 - \frac{c_9}{6} y^3 + \frac{c_{12}}{2} y^2 + c_{13} y + c_{14},
$$

$$
G^8 = \frac{c_1}{6q^3} e^{-2qx} - \frac{c_2}{2q^2} e^{-qx} + \frac{c_3}{q} + \frac{c_{12}}{q} + c_{15} e^{qx}. \quad (6.2.16)
$$

Using all these values in Eq. (6.2.12) and differentiating it twice with respect to $y$, we get

$$
c_1 = 0, \quad G^{11} = \frac{c_9}{2} z^2 + z c_{16} + c_{17} \quad \text{and} \quad G^{12} = \frac{c_4}{6} z^3 - \frac{c_{14}}{2} z^2 + c_{18} z + c_{19}.
$$

Thus a complete solution of the Eqs. (6.2.1)-(6.2.7) provide infinite-dimensional CRCs, which are given below:

$$
X^0 = X^0(x^a),
$$

$$
X^1 = \frac{c_2}{2} e^{qx} (y^2 + z^2) - \frac{c_9}{q} y + \frac{c_7}{2} y^2 - \frac{c_{11}}{2} y e^{-qx} - \frac{c_2}{2q^2} e^{-qx} + \frac{c_{20}}{q} + c_{15} e^{qx},
$$

$$
X^2 = \frac{c_2 y}{q} e^{-qx} - c_2 y - \frac{c_{10}}{2q^2} e^{-2qx} + \frac{c_2}{2q^2} e^{-qx} + c_{11} y e^{qx} - \frac{c_9}{2} y^2 + \frac{c_{23}}{2} (y^2 - z^2) - c_{21},
$$

$$
X^3 = \frac{c_2 z}{q} e^{-qx} - c_2 y - \frac{c_4}{2q^2} e^{-2qx} + \frac{c_4}{2} (y^2 - z^2) + \frac{c_7}{2q} e^{-qx} + c_{12} y + c_9 y z + c_{22},
$$

$$
\psi = \frac{R'_1}{2R_1} X^0 + X^1, \quad (6.2.17)
$$

where $c_20 = c_3 + c_{12}$.

**Case (II):** Here we take $R_0 \neq 0$, $R_1 = 0$, then the CRC equations get the form:

$$
R'_0 X^0 + 2R_0 X^0 = 2\psi R_0, \quad (6.2.18)
$$

$$
X^0, X^1, X^2, X^3 = 0. \quad (6.2.19)
$$

Solving Eq. (6.2.19), we have $X^0 = f(t)$; $f(t)$ being a function of integration. The value of the inheriting factor $\psi$ can be found from Eq. (6.2.18) as
\[ \psi = \frac{R_0'}{2R_0} f(t) + f'(t). \]

Hence the solution set in this case takes the form:

\[ X^0 = f(t), \]
\[ X^i = X^i(x^a), i = 1, 2, 3, \]
\[ \psi = \frac{R_0'}{2R_0} f(t) + f'(t). \]  \hspace{1cm} (6.2.20)

The above two cases shows that when the Ricci tensor is degenerate, the LRS Bianchi type V spacetimes with perfect fluid matter admit an infinite-dimensional Lie algebra of CRCs.

### 6.3 CRCs for Non-degenerate Ricci Tensor

In case when the Ricci tensor is non-degenerate, we have \( \det R_{ab} \neq 0 \). This shows that \( A \neq 0, B \neq 0 \) and \( C \neq 0 \). Like the previous chapters, our first try is to get a solution of some of the Eqs. (6.1.6)-(6.1.15) in terms of some unknown functions of \( t \) and \( x \). The following steps are followed to get such a solution.

Eqs. (6.1.11) and (6.1.12) are differentiated with respect to \( z \) and \( y \) respectively and then subtracted to obtain:

\[ X^2_{,13} - X^3_{,12} = 0. \]  \hspace{1cm} (6.3.1)

Next, differentiating Eq. (6.1.14) with respect to \( x \) and subtracting the obtained result from Eq. (6.3.1), we have \( X^1_{,23} = X^2_{,13} = X^3_{,12} = 0 \), which gives \( X^1 = P^1(t, x, y) + P^2(t, x, z), \)
\( X^2 = P^3_y(t, x, y) + P^4(t, y, z) \)
\( X^3 = P^5_z(t, x, z) + P^6(t, y, z) \), where the functions \( P^i \), for \( i = 1, \ldots, 6 \) arise during integration. Using these values in Eqs. (6.1.11) and (6.1.12), we get \( P^1 = \ldots \)
\(-e^{2qx}P_x^3 + F^1(t, x)\) and \(P^2 = -e^{2qx}P_x^5 + F^2(t, x)\), where \(F^1\) and \(F^2\) are functions of integration. Thus we have:

\[
\begin{align*}
X^1 &= -e^{2qx} \left[ P_x^3 + P_x^5 \right] + F^3, \\
X^2 &= P_y^3 + P_y^4, \\
X^3 &= P_z^5 + P_z^6,
\end{align*}
\]

(6.3.2)

where \(F^3 = F^3(t, x) = F^1(t, x) + F^2(t, x)\). Similarly, differentiating Eqs. (6.1.8), (6.1.9) and (6.1.14) with respect to \(z, y\) and \(t\) respectively and then subtracting, we obtain \(X^2_{,03} = X^3_{,02} = 0\), which gives \(P^4 = F^4_y(t, y) + F^5_y(y, z)\) and \(P^6 = F^6_z(y, z) + F^7_z(t, z)\); \(F^4, ..., F^7\) being functions of integration. Thus (6.3.2) becomes:

\[
\begin{align*}
X^1 &= -e^{2qx} \left[ P_x^3 + P_x^5 \right] + F^3, \\
X^2 &= P_y^3 + F_y^4 + F^5, \\
X^3 &= P_z^5 + F_z^6 + F_z^7.
\end{align*}
\]

(6.3.3)

Using the above system in Eqs. (6.1.8) and (6.1.14), we get \(F^5 = -F_y^6 + G^1_y(y)\), and \(X^0 = -e^{2qx} \frac{R_3}{R_0}\left[ P_t^3 + P_t^4 \right] + P^7(t, x, z)\), where \(P^7\) and \(G^1_y\) are functions of integration. Therefore the system (6.3.3) takes the form:

\[
\begin{align*}
X^0 &= -e^{2qx} \frac{R_1}{R_0} \left[ P_t^3 + P_t^4 \right] + P^7, \\
X^1 &= -e^{2qx} \left[ P_x^3 + P_x^5 \right] + F^3, \\
X^2 &= P_y^3 + F_y^4 + F^5, \\
X^3 &= P_z^5 + F_z^6 + F_z^7.
\end{align*}
\]

(6.3.4)

Putting these values in Eq. (6.1.9) and integrating it with respect to \(z\), we get \(P^7 = -e^{2qx} \frac{R_1}{R_0} \left[ P_t^5 + P_t^7 \right] + F^8(t, x)\), where \(F^8\) is a function of integration.
Thus the system (6.3.4) becomes:

\[
\begin{align*}
X^0 &= -e^{2qx} \frac{R_1}{R_0} \left[ P^3_t + P^5_t + F^4_t + F^7_t \right] + F^8, \\
X^1 &= -e^{2qx} \left[ P^3_x + P^5_x \right] + F^3, \\
X^2 &= P^3_y + F^4_y - F^6_y + G^1_y, \\
X^3 &= P^5_z + F^6_z + F^7_z.
\end{align*}
\] (6.3.5)

Subtracting Eq. (6.1.15) from Eq. (6.1.13) and using the above values, we have:

\[
P^3_{yy} - P^5_{zz} + F^4_{yy} - F^6_{zz} - F^7_{zz} + G^1_{yy} = 0 \tag{6.3.6}
\]

By differentiating Eq. (6.3.6) with respect to \(x\) and \(y\), we get:

\[
\begin{align*}
P^3 &= \frac{y^2}{2} F^9(t, x) + y F^{10}(t, x) + F^{11}(t, x) + F^{12}(t, y), \\
P^5 &= \frac{z^2}{2} F^9(t, x) + z F^{13}(t, x) + F^{14}(t, x) + F^{15}(t, z).
\end{align*}
\]

Using the values of \(P^3\) and \(P^5\) in the system (6.3.5) and considering \(F^{15} + F^7 = F^{16}(t, z), F^{12} + F^4 = F^{17}(t, y), -\frac{B}{A} \left[ F^{11}_t + F^{14}_t \right] + F^8 = F^{18}(t, x), -F^2_x + F^4_x + F^3 = F^{19}(t, x)\) and \(F^6 - G^1 = F^{20}(y, z)\), we obtain:

\[
\begin{align*}
X^0 &= -e^{2qx} \frac{R_1}{R_0} \left[ \frac{(y^2 + z^2)}{2} F^9_t + y F^{10}_t + z F^{13}_t + F^16_t + F^{17}_t \right] + F^{18}, \\
X^1 &= -e^{2qx} \left[ \frac{(y^2 + z^2)}{2} F^9_x + y F^{10}_x + z F^{13}_x \right] + F^{19}, \\
X^2 &= y F^9 + F^{10} + F^{17}_y - F^{20}_y, \\
X^3 &= z F^9 + F^{13} + F^16_z + F^{20}_z. \tag{6.3.7}
\end{align*}
\]

Simplifying the difference of Eqs. (6.1.15) and (6.1.13), after using the above system, we have:

\[
F^{17}_{yy} - F^{20}_{yy} - F^{16}_{zz} - F^{20}_{zz} = 0. \tag{6.3.8}
\]
If we differentiate the above equation with respect to $t$ and $y$, we get:

\[
F^{17} = \frac{y^2}{2}G^1(t) + yG^2(t) + G^3(t) + G^4(y),
\]
\[
F^{16} = \frac{z^2}{2}G^1(t) + zG^5(t) + G^6(t) + G^7(z),
\]

where the functions $G^1, \ldots, G^7$ arise during the process of integration. Moreover, we denote $F^9 + G^1 = F^{21}(t,x)$, $F^{10} + G^2 = F^{22}(t,x)$, $F^{13} + G^5 = F^{23}(t,x)$, $F^{18} - \frac{B}{A}G^8 = F^{24}(t,x)$ and $G^3 + G^6 = G^8(t)$. With all these values, Eq. (6.3.8) becomes:

\[
F^{20} + F^{20} - G^4_{yy} + G^7_{zz} = 0.
\] (6.3.9)

We differentiate Eq. (6.3.9) with respect to $y$ and $z$, it gives:

\[
F^{20} = G^8(y) + G^9(z) + \frac{y^2}{2} \left( \frac{a_1}{2} z^2 + a_2 z \right) + \frac{a_3}{2} yz^2 + a_4 yz,
\]
\[
G^4 = G^8(y) + \frac{a_1}{24} y^4 + \frac{a_3}{6} y^3 + \frac{a_5}{2} y^2 + a_6 y + a_7,
\]
\[
G^7 = -G^9(z) + \frac{a_5}{2} z^2 - \frac{a_1}{24} z^3 - \frac{a_2}{6} z^2 + a_8 z + a_9.
\]

Here we use $F^{21}(t,x) + a_5 = H^1(t,x)$, $F^{23}(t,x) = H^2(t,x)$, $F^{22}(t,x) + a_6 = H^3(t,x)$, $F^{24}(t,x) = H^0(t,x)$ and $F^{19}(t,x) = H^4(t,x)$. Finally, we have the following collineation vector generating CRCs in terms of unknown functions of $t$ and $x$:

\[
X^0 = -e^{2xy} \frac{R_1}{R_0} \left[ \frac{1}{2} (y^2 + z^2) H^1_t + z H^2_t + y H^3_t \right] + H^0,
\]
\[
X^1 = -e^{2xy} \left[ \frac{1}{2} (y^2 + z^2) H^1_x + z H^2_x + y H^3_x \right] + H^4,
\]
\[
X^2 = y H^1 + H^3 - \frac{a_1}{2} yz^2 - a_2 y + \frac{a_3}{2} (y^2 - z^2) - a_4 z + \frac{a_1}{6} y^3,
\]
\[
X^3 = z H^1 + H^2 + \frac{a_1}{2} y^2 z + \frac{a_2}{2} (y^2 - z^2) + a_3 yz + a_4 y - \frac{a_1}{6} z^3.
\] (6.3.10)
The inheriting factor $\psi$ can be obtained from Eq. (6.1.13) as:

$$\psi = \frac{R'_1}{2R_1} X^0 + qX^1 + X^2_2.$$ 

By substituting these values in the system of equations (6.1.6)-(6.1.15) one by one, we see that some of these equations are identically satisfied, while the remaining equations give $a_1 = 0$ and the following integrability equations:

\[ e^{2qx} \left[ \frac{1}{2} \left( \frac{R_1}{R_0} \right)' H^i_t + \frac{R_1}{R_0} H^i_{tt} - qH^i_x \right] - k_i = 0, \quad (6.3.11) \]

\[ H^i_{tx} + qH^i_t = 0, \quad (6.3.12) \]

\[ e^{2qx} \left[ H^i_{xx} + qH^i_x \right] + k_i = 0, \quad (6.3.13) \]

\[ \frac{1}{2} \left( \frac{R'_0}{R_0} - \frac{R'_1}{R_1} \right) H^0 + H^0_t - qH^4 - H^1 = 0, \quad (6.3.14) \]

\[ \frac{R_0}{R_1} H^0_x + H^4_t = 0, \quad (6.3.15) \]

\[ H^1 + qH^4 - H^4_x = 0, \quad (6.3.16) \]

where $k_i = 0, -a_2, a_3$ for $i = 1, 2, 3$ respectively.

In order to find the explicit form of CRCs, given in the system (6.3.10), we solve the above system of equations for the following choices of the Ricci tensor components.

(I) $R'_0 \neq 0, \ R'_1 = 0$, \hspace{1cm} (II) $R'_0 = 0, \ R'_1 \neq 0$,

(III) $R'_0 = 0, \ R'_1 = 0$, \hspace{1cm} (IV) $R'_0 \neq 0, \ R'_1 \neq 0$,

Case I: In the first case, we consider $R'_0 \neq 0$ and $R_1 = a$, where $a$ is a non-zero constant. Thus the system of Eqs. (6.3.11)-(6.3.16) becomes:

\[ e^{2qx} \left[ R'_0 H^i_t - 2R_0 H^i_{tt} + 2\frac{q}{a} R^2_0 H^i_x \right] + 2R^2_0 \frac{k_i}{a} = 0, \quad (6.3.17) \]
\[ H_i^{xx} + qH_i^x = 0, \]  
\[ e^{2qx} \left[ H_i^{xx} + qH_i^x \right] + k_i = 0, \]  
\[ H^0 + 2R_0 H^0_t - 2qR_0 H^4 - 2R_0 H^1 = 0, \]  
\[ R_0 H^0_x + aH_i^1 = 0, \]  
\[ H^1 + qH^4 - H^4_x = 0, \]

where \( k_i = 0, -a_2, a_3 \) for \( i = 1, 2, 3 \) respectively. Solving Eq. (6.3.18), for \( i = 1, 2, 3 \) separately, we get:

\[ H^1 = e^{-qx} h^1(t) + h^2(x), \]
\[ H^2 = e^{-qx} h^3(t) + h^4(x), \]
\[ H^3 = e^{-qx} h^5(t) + h^6(x), \]  

where \( h^i \), for \( i = 1, 2, ..., 6 \) are the functions of integration. Using these values in Eq. (6.3.19) and integrating the resulting equation, we obtain:

\[ h^2(x) = -\frac{c_4'}{q} e^{-qx} + c_1, \]
\[ h^4(x) = \frac{a_2}{2q^2} e^{-2qx} - \frac{c_6'}{q} e^{-qx} + c_2, \]
\[ h^6(x) = -\frac{a_3}{2q^2} e^{-2qx} - \frac{c_8'}{q} e^{-qx} + c_3. \]

Thus the above system becomes:

\[ H^1 = e^{-qx} h^1(t) - \frac{c_4'}{q} e^{-qx} + c_1, \]
\[ H^2 = e^{-qx} h^3(t) + \frac{a_2}{2q^2} e^{-2qx} - \frac{c_6'}{q} e^{-qx} + c_2, \]
\[ H^3 = e^{-qx} h^5(t) - \frac{a_3}{2q^2} e^{-2qx} - \frac{c_8'}{q} e^{-qx} + c_3. \]
To find the functions $h^i(t)$, for $i = 1, 2, 3$, we substitute the above values of $H^i$ in Eq. (6.3.17). Some simple algebraic calculations yield $R_0 = t$ and the above values of $H^i$ are reduced to:

\[
H^1(t, x) = e^{-qx} \left[ c_4 \cos \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right) + c_5 \sin \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right) \right] + c_1,
\]

\[
H^2(t, x) = e^{-qx} \left[ c_6 \cos \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right) + c_7 \sin \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right) \right] + \frac{a_2}{2q^2}e^{-2qx} + c_2,
\]

\[
H^3(t, x) = e^{-qx} \left[ c_8 \cos \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right) + c_9 \sin \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right) \right] - \frac{a_3}{2q^2}e^{-2qx} + c_3.
\]

(6.3.27)

Using the above value of $P^1(t, x)$ in Eq. (6.3.22) and integrating it with respect to $x$, we obtain:

\[
H^4 = -\frac{e^{-qx}}{2q} \left[ c_4 \cos \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right) + c_5 \sin \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right) \right] - \frac{c_1}{q} + e^{qx}h^7(t),
\]

(6.3.28)

where $h^7(t)$ is a function of integration. We put the above value in Eq. (6.3.21) and integrate it with respect to $x$ to get:

\[
H^0 = -\frac{\sqrt{ae^{-qx}}}{2q\sqrt{t}} \left[ -c_4 \sin \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right) + c_5 \cos \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right) \right] - \frac{ae^{qx}}{qt}h^7(t) + h^8(t).
\]

(6.3.29)

The function $h^8(t)$ appearing in the above expression is a function of integration. Finally, we put these values of $H^0$ and $H^4$ in Eq. (6.3.20) and simplify it to get $h^7(t) = c_{10} \cos \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right) + c_{11} \sin \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right)$ and $h^8(t) = \frac{c_{12}}{\sqrt{t}}$. Therefore, we have:

\[
H^0 = -\frac{\sqrt{ae^{-qx}}}{2q\sqrt{t}} \left[ -c_4 \sin \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right) + c_5 \cos \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right) \right] - \frac{\sqrt{ae^{qx}}}{\sqrt{t}} \left[ -c_{10} \sin \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right) + c_{11} \cos \left( \frac{2q}{3\sqrt{a}} t^\frac{3}{2} \right) \right] + \frac{c_{12}}{\sqrt{t}},
\]

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\[ H^4 = -\frac{e^{-q_x}}{2q} \left[ c_4 \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) + c_5 \sin \left( \frac{2q}{\sqrt{a}} t^2 \right) \right] + e^{q_x} \left[ c_{10} \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) + c_{11} \sin \left( \frac{2q}{\sqrt{a}} t^2 \right) \right] - \frac{c_1}{q}. \] (6.3.30)

Thus in this case, the vector field \( X \) generating the CRCs have the form:

\[
X^0 = \frac{q\sqrt{a}e^{q_x}}{\sqrt{t}} \left[ \frac{1}{2}(y^2 + z^2) \right] \left\{ c_4 \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) + c_5 \sin \left( \frac{2q}{\sqrt{a}} t^2 \right) \right\} 
+ z \left\{ c_6 \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) + c_7 \sin \left( \frac{2q}{\sqrt{a}} t^2 \right) \right\} + \frac{c_9}{q} \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) 
+ \frac{\sqrt{a}e^{-q_x}}{2q} \left[ c_4 \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) + c_5 \sin \left( \frac{2q}{\sqrt{a}} t^2 \right) \right] 
+ \frac{\sqrt{a}e^{q_x}}{\sqrt{t}} \left[ c_{10} \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) + c_{11} \sin \left( \frac{2q}{\sqrt{a}} t^2 \right) \right] + \frac{c_{12}}{\sqrt{t}},
\]

\[
X^1 = qe^{q_x} \left[ \frac{1}{2}(y^2 + z^2) \right] \left\{ c_4 \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) + c_5 \sin \left( \frac{2q}{\sqrt{a}} t^2 \right) \right\} 
+ z \left\{ c_6 \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) + c_7 \sin \left( \frac{2q}{\sqrt{a}} t^2 \right) \right\} + \frac{c_9}{q} \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) 
+ \frac{e^{-q_x}}{2q} \left[ c_4 \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) + c_5 \sin \left( \frac{2q}{\sqrt{a}} t^2 \right) \right] 
+ e^{q_x} \left[ c_{10} \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) + c_{11} \sin \left( \frac{2q}{\sqrt{a}} t^2 \right) \right] - \frac{c_1}{q} - \frac{a_2}{q}z + \frac{a_3}{q}y,
\]

\[
X^2 = ye^{-q_x} \left[ c_4 \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) + c_5 \sin \left( \frac{2q}{\sqrt{a}} t^2 \right) \right] + e^{-q_x} \left[ c_8 \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) \right] 
+ c_9 \sin \left( \frac{2q}{\sqrt{a}} t^2 \right) \right\} + c_1 y + \frac{a_3}{2q^2} e^{-2q_{x}} + a_2 yz - \frac{a_3}{2} (y^2 - z^2) + a_4 z + c_3,
\]

\[
X^3 = ze^{-q_x} \left[ c_4 \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) + c_5 \sin \left( \frac{2q}{\sqrt{a}} t^2 \right) \right] + e^{-q_x} \left[ c_6 \cos \left( \frac{2q}{\sqrt{a}} t^2 \right) \right] 
+ c_7 \sin \left( \frac{2q}{\sqrt{a}} t^2 \right) \right\} + c_1 z - \frac{a_2}{q^2} e^{-2q_{x}} - a_3 yz - \frac{a_2}{2} (y^2 - z^2) - a_4 y + c_2,
\]

\[ \psi = X^1. \] (6.3.31)
Thus in this case, the LRS Bianchi type V spacetimes admit a 15-dimensional Lie algebra of CRCs. Out of these 15 CRCs, five are proper CRCs and the remaining ten are RCs.

The remaining three non-degenerate cases can be solved by a similar procedure. We present the obtained results for the remaining three cases in the following tables.

<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor (ψ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(II)</td>
<td>( R_0 = a, ) ( R_1 = t )</td>
<td>( X^0 = q\sqrt{a^2 + z^2} \left[ \frac{1}{2} (y^2 + z^2) \left( c_1 \sin(2\sqrt{vt}) - c_2 \cos(2\sqrt{vt}) \right)</td>
<td></td>
</tr>
</tbody>
</table><p>ight. ) ( + x \left( c_3 \sin(2\sqrt{vt}) - c_4 \cos(2\sqrt{vt}) \right) + y \left( c_5 \sin(2\sqrt{vt}) - c_6 \cos(2\sqrt{vt}) \right) ) ( + \sqrt{\frac{-q^2}{2a}} \left( c_1 \sin(2\sqrt{vt}) - c_2 \cos(2\sqrt{vt}) \right) ) ( + \sqrt{\frac{q^2}{2a}} \left( c_7 \sin(2\sqrt{vt}) - c_8 \cos(2\sqrt{vt}) \right) + c_9 \sqrt{a} \right] ) | ( \frac{R^2}{a^2} X^0 + X^1 ) |
| | | | |
| | | ( X^1 = qe^{-q\sqrt{a^2 + z^2}} \left[ \frac{1}{2} (y^2 + z^2) \left( c_1 \cos(2\sqrt{vt}) + c_2 \sin(2\sqrt{vt}) \right) \right. ) ( + x \left( c_1 \cos(2\sqrt{vt}) + c_4 \sin(2\sqrt{vt}) \right) + y \left( c_5 \cos(2\sqrt{vt}) + c_6 \sin(2\sqrt{vt}) \right) ) ( - \frac{e^{-q\sqrt{a^2 + z^2}}}{2a} \left( c_1 \cos(2\sqrt{vt}) + c_2 \sin(2\sqrt{vt}) \right) ) ( + e^{-q\sqrt{a^2 + z^2}} \left( c_7 \sin(2\sqrt{vt}) + c_8 \cos(2\sqrt{vt}) \right) ) | |
| | | ( \frac{a^2}{2x^2}e^{-2q\sqrt{a^2 + z^2}} - \frac{a_2}{2} (y^2 - z^2) + a_3 z + c_{10} y + c_{11} ) |
| | | ( - \frac{a^2}{2y^2}e^{-2q\sqrt{a^2 + z^2}} - a_2 y z - \frac{a_2}{2} (y^2 - z^2) + c_{10} z - a_3 y + c_{12}. ) |
| | | ( X^2 = ye^{-q\sqrt{a^2 + z^2}} \left( c_1 \cos(2\sqrt{vt}) + c_2 \sin(2\sqrt{vt}) \right) ) ( + e^{-q\sqrt{a^2 + z^2}} \left( c_5 \cos(2\sqrt{vt}) + c_6 \sin(2\sqrt{vt}) \right) + \frac{a^2}{2x^2}e^{-2q\sqrt{a^2 + z^2}} + a_1 y z ) | |
| | | ( \frac{a^2}{2y^2}e^{-2q\sqrt{a^2 + z^2}} - \frac{a_2}{2} (y^2 - z^2) + a_3 z + c_{10} y + c_{11}. ) |
| | | ( X^3 = ze^{-q\sqrt{a^2 + z^2}} \left( c_1 \cos(2\sqrt{vt}) + c_2 \sin(2\sqrt{vt}) \right) ) ( + e^{-q\sqrt{a^2 + z^2}} \left( c_3 \cos(2\sqrt{vt}) + c_4 \sin(2\sqrt{vt}) \right) ) | |
| | | ( - \frac{a^2}{2x^2}e^{-2q\sqrt{a^2 + z^2}} - a_2 y z - \frac{a_2}{2} (y^2 - z^2) + c_{10} z - a_3 y + c_{12}. ) |</p>

Table 6.1: CRCs for Non-Degenerate Ricci Tensor
<table>
<thead>
<tr>
<th>Case</th>
<th>Constraints</th>
<th>CRCs</th>
<th>Inheriting Factor (ψ)</th>
</tr>
</thead>
</table>
| (III) | \( R_0 = a \) \( R_1 = b \) | \[ X^0 = q \sqrt{q} e^{\psi \varphi} \left( \frac{1}{2} (y^2 + z^2) \left( c_1 \sin(q \sqrt{q} t) - c_2 \cos(q \sqrt{q} t) \right) + z \left( c_3 \sin(q \sqrt{q} t) - c_4 \cos(q \sqrt{q} t) \right) \right) + \frac{1}{\sqrt{q}} e^{\psi \varphi} \left( c_7 \sin(q \sqrt{q} t) - c_8 \cos(q \sqrt{q} t) \right) + c_9, \]  
\[ X^1 = q e^{\psi \varphi} \left( \frac{1}{2} (y^2 + z^2) \left( c_1 \cos(q \sqrt{q} t) + c_2 \sin(q \sqrt{q} t) \right) + z \left( c_3 \cos(q \sqrt{q} t) + c_4 \sin(q \sqrt{q} t) \right) \right) - \frac{a - \sqrt{q} \psi}{2} \left( c_1 \cos(q \sqrt{q} t) + c_2 \sin(q \sqrt{q} t) \right) - \frac{\sqrt{q} \psi}{2} - \frac{a t}{q} + \frac{q \psi}{a} y, \]  
\[ X^2 = e^{-\psi \varphi} \left[ y \left( c_1 \cos(q \sqrt{q} t) + c_2 \sin(q \sqrt{q} t) \right) + \frac{a}{q} e^{-2\psi \varphi} \left( a_1 y z - \frac{q}{a} \right) (y^2 - z^2) + a_3 z + c_{11}, \right] \]  
\[ X^3 = e^{-\psi \varphi} \left[ z \left( c_1 \cos(q \sqrt{q} t) + c_2 \sin(q \sqrt{q} t) \right) + \frac{a}{q} \left( c_1 \cos(q \sqrt{q} t) + c_2 \sin(q \sqrt{q} t) \right) \right] + \frac{c}{q} z - \frac{a}{2q} e^{-2\psi \varphi} + a_1 y z - \frac{a}{q^2} (y^2 - z^2) - a_3 y + c_{12}. \] | \( q X^1 \) \( +X^2 \) \( +X^1 X^2 \) |
| (IV) | \( \frac{R_0^4}{R_1^4} = t^2 \) | \[ X^0 = t e^{\psi \varphi} \left[ c_1 \sin(q \ln t) - c_2 \cos(q \ln t) \right] + c_3, \]  
\[ -c_4 e^{\psi \varphi} \left[ q \cos(q \ln t) + \sin(q \ln t) \right] - c_5 e^{\psi \varphi} \left[ q \sin(q \ln t) + \cos(q \ln t) \right] + t e^{\psi \varphi} \left[ c_6 \sin(q \ln t) - c_7 \cos(q \ln t) \right] + t e^{\psi \varphi} \left[ c_8 \sin(q \ln t) - c_9 \cos(q \ln t) \right] \]  
\[ X^1 = e^{\psi \varphi} \left[ c_1 \cos(q \ln t) + c_2 \sin(q \ln t) \right] - \frac{a t}{q} \left( c_1 \cos(q \ln t) + c_2 \sin(q \ln t) \right) - \frac{\sqrt{q} \psi}{2} + \frac{a}{2q} e^{-2\psi \varphi} + a_1 y z + a_2 z + c_{10} y + c_{11} \]  
\[ X^2 = \frac{a}{q^2} \left( y^2 - y^2 \right) + \frac{a}{q^2} e^{-2\psi \varphi} + a_1 y z + a_2 z + c_{10} y + c_{11} \]  
\[ X^3 = \frac{a}{q^2} \left( y^2 - y^2 \right) - \frac{a}{q^2} e^{-2\psi \varphi} - a_2 y z - a_3 y + c_{10} z + c_{12} \] | \( \frac{R_0^4}{R_1^4} X^0 \) \( +q X^1 \) \( +X^2 \) |

Table 6.2: CRCs for Non-Degenerate Ricci Tensor
6.4 Summary and Discussion

For a perfect fluid matter, we have presented a complete classification of LRS Bianchi type V spacetimes via CRCs. The CRC equations are solved in both degenerate as well as non-degenerate Ricci tensor cases. Our classification shows that the LRS Bianchi type V spacetimes admit infinite-dimensional Lie algebra of CRCs for degenerate Ricci tensor, while they admit a 15-dimensional Lie algebra of CRCs when the Ricci tensor is non-degenerate.

Considering \( \psi = 0 \) in all the non-degenerate cases, one can observe that the Lie algebra of RCs is either 6– or 7–dimensional. In cases (I) and (III), we have seven RCs out of which four are the KVs as mentioned in Eq. (6.1.2) and the remaining three are proper RCs. Similarly, the cases (II) and (IV) give six RCs out of which four are the KVs and two are proper RCs.

It is worth mentioning that the obtained CRCs in each case are found subject to some non-linear differential constraints which are to be satisfied by the Ricci tensor components. The solution of these constraints would give the exact form of the LRS Bianchi type V metric admitting these CRCs.
Chapter 7

Conclusion

In this thesis, we have dealt with the problem of calculating conformal Ricci collineations for some well known spacetimes in general relativity. For each of the spacetimes considered, the conformal Ricci collineation equations are solved in degenerate as well as non-degenerate Ricci tensor cases. The obtained results of every problem are summarized at the end of each chapter. However an overall conclusion of the thesis is stated below.

In chapter 2, we have presented a complete classification of Kantowski-Sachs spacetimes via CRCs. When the Ricci tensor is non-degenerate, we obtained a 15-dimensiona algebra of CRCs, while in some cases the CRCs are reduced to RCs. Moreover, we have observed that when we take the inheriting factor to be zero, the dimension of algebra of RCs for these spacetimes turns out to be 4, 6 or 10. These results are same as already stated by Camci et. al. [14]. Though the Lie algebra of RCs in Kantowski-Sachs metric for degenerate Ricci tensor is shown to be finite-dimensional in some special cases [14], our analysis show that the algebra of CRCs for the same metric
is always infinite-dimensional.

In chapter 3, we have calculated CRCs for static spacetimes with the metric \( ds^2 = e^{A(r)} dt^2 - e^{B(r)} dr^2 - r^2 [d\theta^2 + f_k^2(\theta) d\phi^2] \) and concluded that these spacetimes admit 6-, 7- or 15-dimensional Lie algebra of CRCs in non-degenerate case. Taking the vanishing inheriting factor, we can see that the dimension of Lie algebra of RCs can be 4, 6, 7 or 10, which is same as already stated by Akbar [1]. Moreover, it is shown that the degenerate Ricci tensor produce infinite-dimensional algebra of CRCs.

In chapter 4, we have found a vector field generating CRCs of non-static spherically symmetric spacetimes in terms of some unknown functions. These functions have to satisfy some highly non-linear differential constraints which cannot be solved like we have solved the integrability conditions in the previous chapters. However, some simple cases are considered to solve the integrability conditions. For these cases, the dimension of Lie algebra of CRCs turned out to be 5, 6 and 15.

In chapter 5, we have classified LRS Bianchi type I spacetimes according to their CRCs and obtained infinite-dimensional algebra of CRCs for degenerate Ricci tensor and 7- or 15-dimensional Lie algebra of CRCs when the Ricci tensor is non-degenerate. It is worth mentioning here that there are some mistakes in our published paper [26] about the dimension of Lie algebra of CRCs, which have been corrected in this chapter.

In chapter 6, the CRCs are calculated for perfect fluid LRS Bianchi type V spacetimes. Similar to the previous chapters, an infinite-dimensional Lie algebra of CRCs is obtained in degenerate case, while a 15-dimensional Lie algebra of CRCs is found for different choices of the non-degenerate Ricci
tensor. In our recent paper [34], we have wrongly stated that these spacetimes may admit 9 CRCs. This has been corrected in this chapter.

After studying the CRCs for five different spacetimes, we conclude with the remarks that while calculating CRCs for any spacetime, we face some highly non-linear differential constraints to be satisfied by the Ricci tensor components. To show that the classes of CRCs are non-empty and to get the exact form of the metric admitting these CRCs, one needs to solve these constraints. Because of the non-linearity of these constraints, it is not easy to solve them generally, however in some cases we have provided some such examples.

Finally, we would like to mention here that the conformal matter collineations in a spacetime can be calculated by a similar manner. The mathematical similarity between conformal Ricci and conformal matter collineation equations suggest that we do not need to solve conformal matter collineation equations separately. Probably, the same steps will be repeated just by replacing the Ricci tensor with the energy-momentum tensor. However, the differential constraints satisfied by the components of Ricci tensor will be replaced by those of energy-momentum tensor, which may be solved to get the exact form of the corresponding spacetime metric admitting the obtained conformal matter collineations.
References


