Oscillatory Channel Flow for Newtonian and Non-Newtonian Fluids

By

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CIIT/SP09-PMT-001/ISB

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I, Aamir Ali, CIIT/SP09-PMT-001/ISB, hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

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Certificate

It is certified that Aamir Ali, registration number CIIT/SP09-PMT-001/ISB has carried out all the work related to this thesis under my supervision at the department of Mathematics, COMSATS Institute of Information Technology, Islamabad and the work fulfills the requirement for award of PhD degree.

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Dedicated

to

my family

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ABSTRACT

Oscillatory Channel Flow for Newtonian and Non-Newtonian Fluids

The flow between parallel porous plates may be driven by the uniform suction/injection at the permeable walls or by the movement of the walls. Sometimes, this flow can also be induced due to the pressure gradient. In this thesis, we have examined the flow between parallel plates and cylindrical tubes due to the uniform suction/injection at the walls with oscillatory pressure gradient. The oscillatory flow inside a channel/tube has many applications in industrial and engineering processes. Understanding the physics of oscillatory or transient flows in small channels is of fundamental interest for many biological and industrial applications. For example, the quasi-periodic blood flow in the cardiovascular diseases are described by the frequency components of the pressure and flow rate pulses, and many vascular diseases are associated with disturbances of the local flow conditions in the blood vessels. Furthermore, it has applications in modeling of respiratory functions in lungs, modeling of chemical/blood dispensing in biochemistry/clinical labs etc. The oscillatory channel flow have special relevance in vibrating media with applications in oil drilling, control of blood flow during surgical operations, manufacturing and processing of foods and paper, oil exploration and paper industry. Some other applications of value are to detect the intensity of underground explosions, chemicals and material processing, isotope separation, irrigation systems, rocket propulsion, filtration mechanism, sweat cooling, cooling of electronic device, heat exchanger and many others.

Most of the theoretical work undertaken in oscillatory channels and tubes is for viscous fluids. Nothing or very little has been said for non-Newtonian fluids.

Motivated by these considerations, we extend the analysis of oscillatory channel and tube flows from clear Newtonian fluid to Newtonian fluid in porous medium and non-Newtonian second grade and Jeffery fluids. The flow in the channel is driven by suction at the permeable walls while time harmonic pressure waves are responsible for oscillations in the velocity field. The analytic solutions of the corresponding boundary value problems for Newtonian (porous medium) and non-Newtonian fluids are established. The combined effects of porosity of the medium and the fluid oscillations on the oscillatory axial velocity between porous channels are investigated for the case of a Newtonian fluid. Further investigations are made for a second grade fluid in a channel and a cylindrical tube. The effects of wall suction, the second grade fluid parameter on the amplitude and penetration depth of the oscillatory axial velocity are determined. The oscillatory channel flow is also discussed for Jeffery fluid and the effects of relaxation/retardation time parameters and Deborah number are examined.

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$\mathbf{V}$	Velocity vector (L/T)
$\mathbf{V}_o$	Time-independent velocity vector (L/T)
$\mathbf{V}_1$	Time-dependent velocity vector (L/T)
u^*, v^*	Dimensional axial and normal velocity components (L/T)
u, v	Non-dimensional axial and normal velocity components
x^*, y^*	Dimensional axial and normal distance (L)
x, y	Non-dimensional axial and normal distance
h	Channel height from the centre (L)
L	Length of the channel (L)
w	Width of the channel (L)
v_w	Velocity at the wall (L/T)
μ	Dynamic viscosity of fluid (M/LT)
ρ	Fluid density (M/L^3)
ν	Kinematic viscosity of fluid (L^2/T)
a_s	Stagnation speed of sound (L/T)
p_s	Stagnation pressure (M/LT^2)
ρ_s	Stagnation density (M/L^3)
κ	Ratio of specific heats
ω_s	Frequency of the longitudinal pressure oscillation ($1/T$)
t	Time (T)
ω	Non-dimensional wave frequency
S_t	Strouhal number
D_a	Darcy number
γ	Darcy inverse parameter
ϕ	Porosity of the medium
$\mathbf{q}$	Flow rate of fluid (M^3/T)

K	Permeability of porous medium (L^2)
D/Dt	Material time derivative ($1/T$)
M	Mach number
R_e	Reynolds number
ε	Reciprocal of the Reynolds number
δ	Pressure wave amplitude
ψ	Stream function (L^2/T)
F	Similarity function
$\mathbf{A}_1$	First Rivlin-Ericksen kinematic tensor ($1/T$)
$\mathbf{A}_2$	Second Rivlin-Ericksen kinematic tensor ($1/T^2$)
$\mathbf{T}$	Cauchy stress tensor (M/LT^2)
$\mathbf{I}$	Identity tensor
α_1, α_2	Material constants
$\boldsymbol{b}$	Body force (ML/T^2)
α	Second grade fluid parameter
λ_1	Ratio of relaxation to retardation time
λ_2	Relaxation time
$\dot{\gamma}$	Shear rate
D_e	Deborah number
∇	$\partial/\partial x\mathbf{i}+\partial/\partial y\mathbf{j}$
∇^2	$\partial^2/\partial x^2+\partial^2/\partial y^2$, Laplacian operator ($1/L^2$)

LIST OF PUBLICATIONS FROM THESIS

[1] Aamir Ali and Saleem Asghar, *Oscillatory channel flow for non-Newtonian fluid*, International Journal of the Physical Sciences, **6 (36)**, (2011), 8035-8043.

[2] Aamir Ali and S. Asghar, *Analytic solution for oscillatory flow in a channel for Jeffery fluid*, Journal of Aerospace Engineering, (2012), doi: 10.1061/ (ASCE) AS.1943-5525.0000298.

[3] Aamir Ali and Saleem Asghar, *Oscillatory flow of a second grade fluid in a cylindrical tube,* Applied Mathematics and Mechanics (English Edition), (2013), doi: 10.1007/s10483-013-1730-8.

[4] Aamir Ali and S. Asghar, *Oscillatory flow in a porous channel with porous medium and small suction*, Journal of Mechanics, (Accepted).

Chapter 01

Introduction

The history of channel flows is quite rich and extensive. Pfitzner [1] was first to study one-dimensional steady laminar incompressible flow in an infinite channel with constant pressure gradient. This was extended to parallel flows in an infinite channel with top suction and bottom injection. The unsteady flow in an infinite channel with oscillating pressure is also visited in the literature. These studies were undertaken for one-dimensional flows.

The extension to two-dimensional flow between parallel porous plates was first addressed by Berman [2]. Since then a number of studies were undertaken to address various situations of small and large suction/injection velocities for different values of Reynolds number on the channel walls [3-12]. Terrill and Shrestha [13, 14] discussed electrically conducting viscous and incompressible fluid in a two-dimensional channel with suction/ injection. They found the boundary layer behaviour for small and large values of Reynolds number.

The unsteady channel flows are of immense importance in fluid mechanics for their great mathematical interest and wide range of applications in engineering, industry and biological sciences. Mathematical interest lies in the solution of two-dimensional full Navier-Stokes equations to help understand the flow behaviour in the channel. The applications have a long list and we will only mention a few of these: inkjet printing, blood dialysis in artificial kidney, blood flow in capillaries, flow in blood oxygenators and design of porous pipe. Further applications of oscillatory channel flows are in oil drilling, control of blood flow during surgical operations, manufacturing and processing of foods and paper industry. The study and understanding of oscillatory transient flows in small channels is of fundamental interest for many biological and industrial processes. For example, the quasi-periodic blood flow in the cardiovascular diseases is described by the frequency components of the pressure and flow rate pulses. In biology, it has applications in modeling of respiratory functions in lungs, modeling of blood dispensing in clinical labs etc. Some other applications of value are to detect the intensity of underground explosions, chemicals and material processing, isotope separation, irrigation systems, rocket propulsion, filtration mechanism, sweat cooling, cooling of electronic device, heat exchanger and many others. The underlying concepts of all these applications require a modeling and understanding of the fluid behaviour in channel flows.

Physically interesting situation arises when small oscillations are introduced in the channel flow. This practical situation has a number of engineering applications, such as flows inside enclosures with transpiring walls, the design of hydraulic line transmissions, the modeling of the respiratory functions in lungs, flow separation processes and many more. The harmonic time dependent disturbances arising inside a rectangular channel with porous/transpiring walls develop an irrotational field and an acoustic field. The existence of such velocity oscillations in the channels and tubes have been established experimentally.

Motivated by these considerations, the pioneering studies of Majdalani [15] related the idea of oscillatory flow in a two-dimensional laminar viscous flow generated by sidewall injection in a planar channel. The full Navier-Stokes equations are used to model the problem. The steady and unsteady parts of the velocity field are separated and solved analytically. The unsteady part gives rise to an acoustic velocity and a vortical velocity. The solution of the acoustic component is obtained easily, while the solution of the vortical component is found using different analytic techniques.

Later on, Majdalani and his coworkers published a series of papers on oscillatory flow in channel for different cases of suction and injection. Jankowski and Majdalani [16] study the "*Acoustical and vortical interactions inside a channel with wall suction*". "*The oscillatory channel flow with large wall injection*" was studied by Jankowski and Majdalani [17], arbitrary wall injection by Majdalani [18]. "*The oscillatory channel flow with hard blowing*", was also studied by Majdalani and Roh [19]. Jankowski and Majdalani [20] discussed oscillatory flow in a channel for viscous fluid for arbitrary wall suction. "*Symmetric solutions for the oscillatory channel flow with arbitrary suction*" was studied by Jankowski and Majdalani [21]. We, however observe that all these studies for the oscillatory channel flows have been undertaken for the viscous fluids only. Nothing has been said on the oscillatory channel flows for Non- Newtonian fluids. The consideration of oscillatory non-Newtonian fluids is, however important from the view of understanding and applications.

Equally important is the case of oscillatory tube flows. The flows inside a cylindrical tube whose walls are porous have a number of applications, such as: isotope separation, flow filtration, pulmonary circulation, surface ablation and arterial blood flow modeling. Yuan & Finkelstein [49] studied tube flows and presented an asymptotic solution for the cases of small suction and large suction/ injection. Terrill and Thomas [22] used the method of asymptotic analysis to discuss the different cases. Terrill [23] obtained more accurate results by inclusion of exponentially small terms. Brady and Acrivos [24] found the asymptotic solution for the tube problem with accelerating surface velocity. Some further developments for laminar flows in cylindrical tube are undertaken for a viscous fluid [25-28].

All these studies for the oscillatory channel and tube flows have been undertaken for the viscous fluids only. Nothing has been said on the oscillatory channel flows for Non-Newtonian fluids both in channels and tubes.

It is now well established that most of the fluids in engineering and technological processes are non-Newtonian in nature. The physiological fluids are also considered as non-Newtonian. Blood, ketchup, nail polish, lotions and creams, shampoo, toothpaste, some oils, borer mud and paints are some examples of such fluids. The non- Newtonian fluids are now considered as a rule rather than an exception. Therefore for any realistic application in industry, engineering and physiological flows, it is imperative to discuss the oscillating channel flows for non- Newtonian fluids.

Motivated by these considerations, this thesis describes an analytical solution of oscillatory flows of an incompressible **non-Newtonian fluid** in channels and tubes. The flow in the channel is driven by suction at the permeable walls, whereas small amplitude time harmonic pressure waves are responsible for oscillations in the velocity field. The time independent axial velocity and the time dependent oscillatory axial velocity are calculated analytically. The important physical quantities like the velocity profile amplitude of the oscillation and the penetration depth are given special emphasis. The results are compared with those of viscous fluids. The geometry and the flow configuration remains the same throughout the thesis. Primary goal is thus; to extend the analysis of oscillatory channel and tube flows from viscous to non-Newtonian fluids. In addition, the porosity effects are also seen for viscous fluid in a channel flows. This is a first such analysis for second grade fluid and a Jeffrey fluid for oscillatory channel flows. It is hoped that this analysis can be further extended to other types of non-Newtonian fluids.

Mathematically, the problems are formulated in terms of the governing momentum equation and the appropriate boundary conditions for non-Newtonian fluids in channels and tubes. The full Navier-Stokes equations are first linearized and analytic solution is sought using separation of variable, perturbation method, variation of parameter, WKB approximation and Multiple-Scale method. To achieve this end, we first decompose the velocity field into a time independent and a time dependent parts. The solution for the time independent part is readily obtained using perturbation method. The time dependent velocity distribution is split into acoustic and vortical parts. The axial solution for the acoustic part is obtained using separation of variable method. The solution of the vortical part is given using WKB approximations.

The thesis consists of six chapters. **Chapter 01** contains introduction, history and applications of the oscillatory flows.

In **Chapter 02**, we present some basic definitions of the fluid mechanics, the governing equations of motion, the dimensionless numbers and the analytical methods used to solve the problems presented in this thesis.

In **Chapter 03**, we extend the work of Jankowski and Majdalani [21] to a **porous medium** in place of a clear viscous fluid. The introduction of porosity provides a better understanding of energy and momentum transport in an oscillatory channel. Some studies of porous medium can be seen in the literature [29-36]. Brinkman extended Darcy law is used to add the effects of porous medium in the governing Navier-Stokes equation. We investigate the combined effects of porosity and oscillations on the oscillatory axial velocity between the porous channels. A careful mathematical analysis is made to solve the boundary value problem analytically and the effects of porosity on the oscillatory axial velocity are presented graphically. We observe that the amplitude of the oscillatory axial velocity decreases significantly; whereas the penetration depth decreases slightly as the inverse Darcy parameter γ increases. In other words, the oscillations in the channel are minimized by decreasing the permeability of the medium. These observations are expected from the physics of porous medium and are of great physical importance. The content of this work has been accepted for publication in **"Journal of Mechanics"**

In **Chapter 04,** we describe the oscillatory flow in a porous channel for a **non-Newtonian second grade fluid**. The resulting governing equation for second grade fluid adds additional complexities to already nonlinear Navier-Stokes equations. Some studies of second grade fluid are presented in [37-48]. We consider the oscillatory channel flow for non-Newtonian second grade fluid and find an analytic solution for small second grade and wall suction parameters. The effects of second grade fluid parameter on the oscillatory axial velocity, its amplitude and penetration depth are analyzed graphically. As we increase the second grade parameter the amplitude of the oscillatory velocity increases and the penetration depth of the velocity decrease. As we increase the suction parameter, we observe a decrease in the amplitude of the oscillatory velocity for the second grade fluid. The decrease in amplitude of the velocity is more apparent in second grade fluid than in viscous fluid. The results for viscous fluid can be recovered as $\alpha \to 0$ in this study. We see that, this is the first attempt to consider the effects of non-Newtonian second grade fluid in the theory of oscillatory flows in channel and the discussion can be further extended to other types of non-Newtonian fluids with different rheological properties, different governing equations and different industrial applications. The work given in this chapter has been published in **"International Journal of the Physical Sciences".**

Starting from the study of Yuan & Finkelstein [49], a number of investigations for oscillatory flow in a tube are undertaken for viscous fluid [50-52]. A number of experimental studies have supported the existence of velocity oscillations inside tube with transpiring walls. We thus take up the discussion of non-Newtonian channel flow to **non-Newtonian second grade cylindrical tube flow**. This will extend the analysis of channel flow to the tube flow and viscous fluid to second grade fluid. Some studies of tube flow for second grade fluid can be seen in the literature [53-58]. For the tube flow, the problem is formulated in terms of cylindrical coordinates, which adds extra mathematical complications on top of the channel flow. The main focus of this study remains the oscillatory axial velocity in terms of finding its profile, amplitude and penetration depth. The effects of the second grade fluid parameter on oscillatory flow field, its amplitude and penetration depth are investigated graphically. It is observed that the amplitude of the oscillatory velocity increases and penetration depth remains the same by increasing values of the second grade fluid parameter. Increase of suction parameter decreases the amplitude of the oscillatory axial velocity. The results for viscous fluid Jankowski and Majdalani [52] can be recovered as $\alpha \rightarrow 0$. These observations are presented in **Chapter 05** has been published in **"Applied Mathematics and Mechanics" (English Edition)**

Chapter 06 contains the analysis of oscillatory channel flow in **non-Newtonian Jeffrey fluid** with small suction. It is important to note that measurement of non-Newtonian characteristics is frequently used to ascertain the state of structure in a fluid. In reverse the desired rheological properties can be conceived from suitably engineered structures. We consider the Jeffrey model because it is relatively simple and predicts relaxation/retardation time effects. This fluid model represents a different rheological property from that of a Newtonian fluid. The degree of non-Newtonian nature or viscoelasticity of the fluid is measured through non-dimensional parameters called ratio of relaxation time to retardation time parameter and Deborah number. These time effects are particularly important while studying the viscoelastic effects in polymer industry. Another important reason for consideration of this problem is that oscillatory shear motion is normally used to understand the behaviour of viscoelastic fluids. In our case, these strains are generated from fluctuations in the suction rate at the walls of the channel. It is well known that in response to applied oscillatory strain, the stress will be in phase for a purely elastic body and out of phase by 90^o in case of purely viscous fluids. In between these values lies the behaviour of a viscoelastic fluid determined by the values of Deborah number. It is rather important to mention that most of the instruments designed to measure the viscoelasticity of the fluids are for an oscillatory strain. Many studies have been done in the literature to discuss Jeffrey fluid model in a channel [59-67].

The appropriately modeled problem is solved analytically using small oscillatory velocity amplitude and small Deborah number assumptions. The quantities of interest are the amplitude and penetration depth of the oscillatory velocity in the channel. The effects of viscoelastic parameters on these quantities are analyzed graphically. It is observed that by increasing the values of the ratio of relaxation/retardation time parameter decreases the amplitude as well as the penetration depth of the oscillatory velocity. Also, the increase of the non-dimensional Deborah number decreases the amplitude of the oscillatory velocity. The results for the viscous fluid can be recovered as a special case of this study by taking $\lambda_1 = 0$ and $D_e = 0$. The theoretical investigation of the effects of viscoelastic parameter helps understand the behaviour and determine the degree of viscoelasticity of the non-Newtonian fluids. The work of this chapter has been published in "**ASCE Journal of Aerospace Engineering**".

Chapter 02

Preliminaries

2.1 Fluid

The fluid is defined as an isotropic substance that continuously deforms under an applied shear stress. The fluid comprises the liquids and gases. A fluid, in which there is no friction, i.e., its viscosity is zero is called an ideal fluid or an inviscid fluid. Such fluids do not exist in reality. Gases are usually treated as ideal fluids for engineering problems. The fluids, for which the viscosity is not zero, are known as real fluids. Real fluids are also called viscous fluids and are compressible or incompressible. A fluid is said to be incompressible if the density of the fluid remains constant throughout the motion. Flow of such fluids may be laminar or turbulent. A flow in which paths taken by the individual fluid particles do not cross one another and each particle moves along a well defined path is known as a laminar flow. A flow in which each fluid particle does not have a definite path and it moves randomly is called turbulent flow. The real fluids are divided into two main classes: Newtonian fluid and non-Newtonian fluid.

2.1.1 Newtonian Fluid

Newton's law of viscosity states that "*shear stress is linearly proportional to the rate of angular deformation/velocity gradient*." The fluids which obey the Newton's law of viscosity are called Newtonian fluids. Mathematically,

$$\text{Shear stress} \propto \frac{d\mathbf{V}}{dy} \quad \text{or} \quad \text{Shear stress}=\mu \frac{d\mathbf{V}}{dy} \tag{2.1}$$

where μ is the proportionality constant and is called absolute or dynamic viscosity. Navier-Stokes equations represents the motion of the fluid. The Cauchy stress tensor $\mathbf{T}$ satisfies the following relation for Newtonian fluids

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 \tag{2.2}$$

where, $\mathbf{I}$ is identity tensor, p is hydrostatic pressure, μ is dynamic viscosity of the fluid, $\mathbf{A}_1$ is first Rivlin-Ericksen tensor defined by

$$\mathbf{A}_1 = grad\mathbf{V} + \left(grad\ \mathbf{V}\right)^T \tag{2.3}$$

where $\mathbf{V}$ is the velocity vector.

2.1.2 Non-Newtonian Fluid

There are a large number of fluids which do not obey Newton's law of viscosity; such fluids are called non-Newtonian fluids. When the relation between applied shear stress and rate of shear strain is not linear, the fluid is known as a non-Newtonian fluid. Mathematically,

$$\text{Shear stress} = \sigma_{yx} = k\left(\frac{d\mathbf{V}}{dy}\right)^n, \quad n \neq 1 \tag{2.4}$$

where k is the consistency index and n is flow behaviour index. For $n = 1$ and $k = \mu$, we obtain Newtonian fluids. The substances exhibiting non-Newtonian fluid behaviour are given in the table below:

• Adhesives (wall paper paste, carpet adhesive, for instance) • Ales (beer, liqueurs etc.) • Animal waste slurries from cattle farms • Biological fluids (blood, synovial fluid, salvia etc.) • Bitumen • Cement paste and slurries • Chalk slurries • Chocolates • Coal slurries • Cosmetic and personal care products (nail polish, lotions and creams, lipsticks, shampoos, shaving foams and creams, toothpaste etc.) • Dairy products and dairy waste streams (cheese, butter, yogurt, fresh cream, whey, for instance) • Drilling muds • Fire fighting foams	• Food stuffs (fruit/vegetable purees and concentrates, sauces salad dressings, mayonnaise, jams and marmalades, ice-cream, soups, cake mixes and cake top-pings, egg white, bread mixes, snakes etc) • Greases and lubricating oils • Mine tailings and mineral suspensions • Molten lava and magmas • Paints, polishes and varnishes • Paper pulp suspensions • Peat and lignite slurries • Polymer melts and solutions, reinforced plastics, rubber • Printing colors and inks • Pharmaceutical products (creams, foams, suspensions, for instance) • Sewage sludge • Wet beach sand • Waxy crude oils

Table 2.1 Examples of some non-Newtonian fluids

There are several categories of non-Newtonian fluids based on relationship of shear stress and rate of shear strain and graphically it can be shown as:

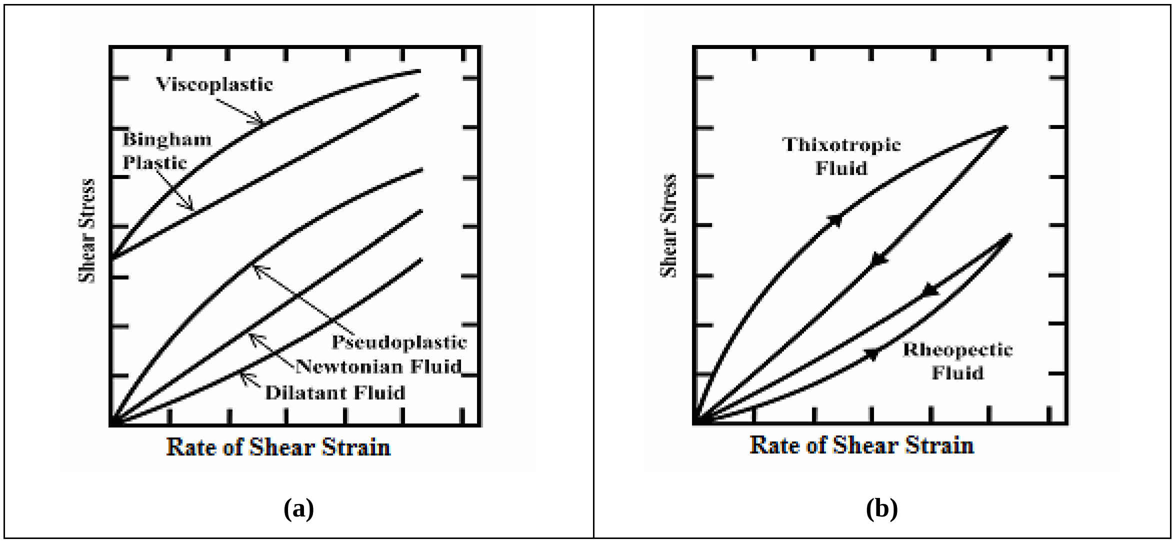


Figure 2.1 Relationship of shear stress and rate of shear strain

2.1.2.1 Time Independent non-Newtonian Fluids

The non-Newtonian fluids are further subdivided into dilatants, pseudoplastics and yield stress fluids such as Bingham plastics and Carson plastics. Bingham plastics are idealized materials which behave partly like solids and partly like fluids. These fluids would not flow until a yield stress is exceeded and the relation between shear stress and rate of shear strain is linear (Fig 2.1a). Toothpaste, mayonnaise and tomato ketchup are examples of such fluids. Pseudoplastic fluids are characterized by a decreasing slope with increasing stress. Most non-Newtonian fluids belong to this category. These fluids are also called shear-thinning fluids. If the apparent viscosity is increasing with increasing rate of shear strain, the fluid is known as a dilatant. Some of these fluids can become almost solid within a pump or pipe line. Cream becomes butter and candy compounds, clay, slurries and similar heavily filled fluids do the same. These fluids are also called shear-thickening fluids.

2.1.2.2 Time Dependent non-Newtonian Fluids

Time dependent non-Newtonian fluids are the fluids whose behaviour may vary with time (Fig 2.1b). If the rate of shear strain is kept constant, the shear stress may vary and vice versa. If the shear stress decreases with time as the fluid is sheared, the fluid is known as thixotropic and the opposite effect is rheopectic. Many fluids in this category lose their rheopectic property at large rate of shear strain to become thixotropic fluids.

Ketchup and mayonnaise are examples of thixotropic materials. They appear thick or viscous but pump quite easily.

2.1.2.3 Viscoelastic Fluids

These fluids are characterized by the fact that their shear stress is non-linear function of rate of shear strain, e.g. bitumen and flour dough. In the deformation of an elastic solid, in general, some energy of a viscoelastic fluid may be recoverable. This contrasts with perfect fluids in which all energy of deformation is dissipated. A viscoelastic fluid combines both elastic and viscous characteristics. These fluids have attracted many researchers because of their interesting properties and industrial applications.

Due to many applications, the generalizations of the highly non-linear Navier-Stokes equations to constitutive laws have been discussed in the past years. In a number of fields e.g. drilling operations, bio-engineering, food industry, these fluids are either natural or synthetic and are mixtures of different constituents e.g. water, oil, red cells and other long chain molecules. The viscosity varies non-linearly with the rate of shear strain and the elasticity is felt through elongation effects and time-dependent effects. These fluids are termed as non-Newtonian fluids. The Navier-Stokes equations are highly non-linear partial differential equations, and because of the nonlinear inertial terms, there are only a limited number of exacts solutions. It is known that Navier-Stokes equations are inadequate for several flows in industry and technology, for that reason the generalizations of the Navier-Stokes equations to highly non-linear constitutive equations are continuously proposed. The fluids of differential type are introduced to deal several non-standard features e.g. rod climbing, normal stress effects, shear thickening and shear thinning.

2.2 Porosity and Porous Medium

2.2.1 Porous Medium

A material which consists of a solid matrix with interconnected pores is called a porous medium. The pores are filled with fluid which allows the fluid to flow from the material. The pores are irregular in natural porous medium and are distributed with different shapes and sizes. A porous medium is characterized by its porosity. Examples of porous media are: Sandstone, beach sand, wood, rye bread, limestone, human lungs, some natural materials, rocks, soil, biological tissues, cements, ceramics and many more.

2.2.2 Porosity

The porosity of a medium is the ratio of the connected void area to the total area of the material. It is denoted by ϕ. Thus, $1-\phi$ is the fraction occupied by the material or solid. In defining ϕ, we assume that all the pore space is connected. If all pore space is not connected then the porosity is effective porosity which is defined as ratio of connected void to the total volume.

2.2.3 Permeability

Permeability of a porous material is defined as the measure of the ability to allow fluids to pass through it. By Darcy experiment for unidirectional flow, there exists a relationship between the flow rate and applied pressure gradient. Mathematically it is written as

$$\mathbf{q} = -\frac{K}{\mu}\nabla p \tag{2.5}$$

where μ is the dynamic viscosity of fluid, $\frac{\partial p}{\partial x}$ is pressure gradient and $\mathbf{q}$ is flow rate of the fluid, K is coefficient of proportionality and is called intrinsic permeability or specific permeability of a porous medium. Permeability coefficient depends on the geometry of medium and is independent of the nature of the fluid.

Various physical laws have been applied in porous medium and the interaction between fluid and solid structure is neglected very often. It means that the porous medium is considered to be rigid and fluid flow through the porous medium depends on the porosity, the permeability of porous medium and fluid properties. The elastic properties are not considered. The fluid flow in porous domain is modeled by Darcy’s law or the Brinkman equations.

2.2.4 Darcy's Law

The constitutive equation that is used to describe the flow of a fluid in a porous medium is called "*Darcy's law*". It is used to illustrate water, oil and gas flows in petroleum reservoirs. It is named after a French engineer Henry Darcy. A one-dimensional, steady flow in uniform medium disclosed a proportionality relation between the flow rate $\mathbf{q}$ of the water and pressure difference, i.e.

$$\mathbf{q} = -\frac{K}{\mu}\nabla p \tag{2.6}$$

where $\frac{\partial p}{\partial x}, \mu$ and K are as defined above. Darcy's law in three dimensions can be written as:

$$\mathbf{q} = -\frac{\mathbf{K}}{\mu}\nabla p \tag{2.7}$$

where $\mathbf{K}$ is now a general second-order tensor. Sometimes, we note that $\mathbf{q}$ denotes the relative velocity of the fluid through the porous medium called seepage velocity. One needs to divide $\mathbf{q}$ by the porosity ϕ to get the physical velocity $\mathbf{V}$ of the fluid, i.e.

$$\mathbf{q} = \phi\mathbf{V}$$

This relation is called "*Dupuit - Forchheimer assumption*".

2.2.5 Brinkman Extended Darcy's Law

Darcy's law is used for materials with low porosity, since the no-slip boundary condition cannot be applied, for infinite domains. For highly porous media ($\phi > 90\%$) or in order to impose a no-slip condition along the boundary, "*Brinkman's equation*" is used, which is an extension of Darcy's law, which is frequently used to model the fluid flow in a porous medium. Since the equation is of the same type as the Navier-Stokes or Stokes equations, which describes the behaviour in the surrounding fluid domain, the two systems of equations, Brinkman's equation and Navier-Stokes equation are then written as the coupled Navier-Stokes-Brinkman system. i.e. if we simply add the Darcy's law to Navier-Stokes equation and retain all terms, then it is called Brinkman Extended Darcy's law equation.

2.3 Basic Equations

2.3.1 Continuity Equation

The continuity equation is the law of conservation of mass which states that the mass is neither created nor destroyed inside a control volume. Equation of continuity is expressed as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0. \tag{2.8}$$

For a homogeneous and an incompressible fluid, the density ρ remains constant throughout the entire fluid, and the equation of continuity becomes

$$\nabla \cdot \mathbf{V} = 0. \tag{2.9}$$

2.3.1.1 Continuity Equation in Cartesian Coordinates

In cartesian coordinates $(x,\ y,\ z)$, the velocity vector is defined as $\mathbf{V} = (u,\ v,\ w)$, and Eq. (2.8) becomes:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \tag{2.10}$$

For steady flow, we have:

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \tag{2.11}$$

For incompressible flow, we have:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.12}$$

2.3.1.2 Continuity Equation in Cylindrical Coordinates

In cylindrical coordinates $(r,\ \theta,\ z)$, the velocity vector is defined as $\mathbf{V} = (u_r,\ u_\theta,\ u_z)$, and Eq. (2.8) can be written as:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (\rho r u_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho u_\theta)}{\partial \theta} + \frac{\partial (\rho u_z)}{\partial z} = 0 \tag{2.13}$$

For steady incompressible flow, we have:

$$\frac{1}{r}\frac{\partial\left(ru_r\right)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}=0 \tag{2.14}$$

2.3.2 Momentum Equation

The Navier-Stokes equation and the continuity equations provide a complete mathematical description of the flow fluid.

The Cauchy differential equation of motion for an incompressible fluid is:

$$\rho\frac{D\mathbf{V}}{Dt}=\nabla\cdot\mathbf{T}+\rho g \tag{2.15}$$

In vector notation, the Navier-Stokes equation can be expressed in the form:

$$\frac{\partial \mathbf{V}}{\partial t}+\left(\mathbf{V}\cdot\nabla\right)\mathbf{V}=-\frac{1}{\rho}\nabla P+\upsilon\nabla^2\mathbf{V}+\boldsymbol{F} \tag{2.16}$$

2.3.2.1 Navier-Stokes Equation in Cartesian Coordinates

The Navier-Stokes equation in component form is given as follows.

The x-component is: (2.17)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+g_x$$

The $y-$component is:

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+g_y \tag{2.18}$$

The $z-$component is:

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\upsilon\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+g_z \tag{2.19}$$

2.3.2.2 Navier-Stokes Equation in Cylindrical Coordinates

In cylindrical coordinates $\left(r,\ \theta,\ z\right)$, the velocity vector is defined as $\mathbf{V}=\left(u_r,\ u_\theta,\ u_z\right)$ and the Navier-Stokes equations in cylindrical coordinates are:

The $r-$component:

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta{}^2}{r}+u_z\frac{\partial u_r}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial r}$$
$$+\upsilon\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)-\frac{u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right)+g_r \qquad (2.20)$$

The $\theta-$component:

$$\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}=-\frac{1}{\rho r}\frac{\partial P}{\partial \theta}$$
$$+\upsilon\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right)+g_\theta \qquad (2.21)$$

The $z-$component:

$$\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}$$
$$+\upsilon\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right)+g_z \qquad (2.22)$$

2.3.3 The Constitutive Equations

A constitutive equation is a mathematical statement or relation of the mechanical behaviour of a group of materials. It is not possible to describe the non-Newtonian fluids by a single constitutive equation because of their diverse physical structures. Therefore, different models are proposed to describe non-Newtonian fluids in literature. There are three types of constitutive equations, namely, the differential type, the integral type and the rate type. A class of equations whose stress tensor is a function of differential kinematics at the moment of observation is called a constitutive equation of differential type. These are equations of fluid memory because higher order deformation tensors are involved. In integral type, the stress is given by one or more integrals of deformation history. In rate type, there is at least one time derivative of the stress tensor. Both differential and integral type constitutive equations are explicit in the stress tensor but rate type equations are not explicit in the stress. The rate of change of the stress that appears in these equations gives the name of this category. The differential type fluids have been properly studied by the researchers in the literature.

Rivlin and Eriksen [68] and Truesdell and Noll [69] devised the earliest method of classifying viscoelastic fluids. They presented a constitutive equation for a group of fluids that have come to be known as Rivlin-Eriksen fluids or fluids of a differential type. The constitutive equations for these fluids present a complexity in momentum equations where the order of magnitude of the equations of motion are higher than the order of the Navier-Stokes equation, Rajagopal and Kaloni [70], Rajagopal [71].

2.3.3.1 Second Grade Fluid

A second grade fluid is a subclass of fluid called differential type fluid and Cauchy stress tensor $\mathbf{T}$ for second grade fluid is defined by:

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^{\,2} \tag{2.23}$$

where μ is viscosity, p is pressure, α_1 and α_2 are material constants, $\mathbf{A_1}$ and $\mathbf{A_2}$ are first two Rivlin-Ericksen tensors and are defined as:

$$\begin{aligned} \mathbf{A}_1 &= \left(grad\,\mathbf{V}\right) + \left(grad\,\mathbf{V}\right)^{\mathrm{T}} \\ \mathbf{A}_2 &= \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1\left(grad\,\mathbf{V}\right) + \left(grad\,\mathbf{V}\right)^{\mathrm{T}}\mathbf{A}_1 \end{aligned} \tag{2.24}$$

where $\dfrac{d}{dt}$ is material time derivative and $\mathbf{V}$ is velocity field. For a fluid of differential type modeled by Eq. (2.23), we require the following conditions:

$$\mu \ge 0,\ \alpha_1 \ge 0 \ \text{ and } \ \alpha_1 + \alpha_2 = 0. \tag{2.25}$$

The details can be found in [72-78]. The detailed discussion about the sign of α_1 and the stability or instability of the fluid motions can also be found in Dunn and Rajagopal [77]. The momentum equation for second grade fluid in non-dimensional vector form, after using Eq. (2.25), can be written as:

$$\begin{aligned} \rho\left[\omega\frac{\partial\mathbf{V}}{\partial t} + \left(\mathbf{V}\cdot\nabla\right)\mathbf{V}\right] &= -\nabla p + M\varepsilon\left[\frac{4}{3}\nabla\left(\nabla\cdot\mathbf{V}\right) - \nabla\times\left(\nabla\times\mathbf{V}\right)\right] \\ &\quad -\alpha\nabla\left[\mathbf{V}\bullet\nabla^2\mathbf{V} + \frac{1}{4}\left|\mathbf{A}_1\right|^2\right] + \alpha\left[\omega\nabla^2\mathbf{V}_t + \nabla^2\left(\nabla\times\mathbf{V}\right)\times\mathbf{V}\right] \end{aligned} \tag{2.26}$$

where $\alpha = \dfrac{\alpha_1}{\rho_s h^2}$ is the non-dimensional second grade fluid parameter, $\omega = \dfrac{\omega_s h}{a_s}$ is the dimensionless wave frequency, $M = \dfrac{v_w}{a_s}$ is the Mach number and $\varepsilon = \dfrac{1}{R_e} = \dfrac{\upsilon}{v_w h}$ is the reciprocal of the Reynolds number (suction parameter).

2.3.3.2 Jeffrey Fluid

The Jeffrey fluid model is a relatively simple model. In this model, simple material derivatives are used. This model represents a rheology different from that of a Newtonian fluid. The Jeffrey fluid model predicts relaxation/retardation time effects. Cauchy stress tensor $\mathbf{T}$ for Jeffrey fluid is defined as:

$$\mathbf{T} = -p\boldsymbol{I} + \mathbf{S} \tag{2.27}$$

where p is pressure, $\boldsymbol{I}$ is the identity tensor and $\mathbf{S}$ can be defined as:

$$\mathbf{S} = \frac{\mu}{1+\lambda_1}\left(\dot{\gamma} + \lambda_2 \ddot{\gamma}\right) \tag{2.28}$$

In the above expression, μ is dynamic viscosity of fluid, λ_1 is ratio of the relaxation to retardation times, λ_2 is relaxation time and $\dot{\gamma}$ is shear rate. The dot over the quantities denotes the material differentiation and defined as,

$$\begin{aligned} \dot{\gamma} &= \left(grad\,\mathbf{V}\right) + \left(grad\,\mathbf{V}\right)^T \\ \ddot{\gamma} &= \frac{d}{dt}\left(\dot{\gamma}\right) \end{aligned} \tag{2.29}$$

The momentum equation for Jeffrey fluid in non-dimensional form can be written as:

The x^{th} momentum equation

$$\begin{aligned} &\omega \frac{\partial(\rho u)}{\partial t} + u\frac{\partial(\rho u)}{\partial x} + v\frac{\partial(\rho u)}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\varepsilon M}{1+\lambda_1}\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right\} \\ &+ \frac{D_e}{1+\lambda_1}\left[\omega\frac{\partial}{\partial t}\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right\} + 2\frac{\partial}{\partial x}\left(v\frac{\partial^2 u}{\partial x \partial y} + u\frac{\partial^2 u}{\partial x^2}\right)\right. \\ &\left. + \frac{\partial}{\partial y}\left\{v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right\}\right] \end{aligned} \tag{2.30}$$

The y^{th} momentum equation

$$\omega\frac{\partial(\rho v)}{\partial t}+u\frac{\partial(\rho v)}{\partial x}+v\frac{\partial(\rho v)}{\partial y}=-\frac{\partial p}{\partial y}+\frac{\varepsilon M}{1+\lambda_1}\left\{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}$$
$$+\frac{D_e}{1+\lambda_1}\left[\omega\frac{\partial}{\partial t}\left\{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}+2\frac{\partial}{\partial y}\left(u\frac{\partial^2 v}{\partial x\partial y}+v\frac{\partial^2 v}{\partial x^2}\right)\right. \tag{2.31}$$
$$\left.+\frac{\partial}{\partial x}\left\{u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\}\right]$$

where $D_e=\frac{\upsilon\lambda_2}{h^2}$ is the non-dimensional Deborah number.

2.4 Dimensionless Parameters

2.4.1 Reynolds Number

Reynolds number is a dimensionless number defined as, the ratio of inertial (dynamic) forces to viscous (shearing) forces denoted by R_e. Mathematically, it is defined as,

$$R_e=\frac{v_w h\rho}{\mu}$$

It is used to characterize different flow regimes.

Laminar flow when $R_e<2300$, in which viscous forces are dominant.

Transient flow when $2300<R_e<4000$.

Turbulent flow when $4000<R_e$ in which inertial forces are dominant.

2.4.2 Strouhal Number

Strouhal number describes the oscillatory flow mechanism. It is the ratio of the dimensionless wave frequency ω to dimensionless Mach number M and is denoted by S_t. Mathematically, it is defined as

$$S_t=\frac{\omega}{M}.$$

It is important to analyze unsteady oscillatory flow problems. It defines the fraction of inertial forces due to unsteadiness in flow and inertial forces due to change in velocity field. For large Strouhal number, viscosity dominates and describes the collective oscillatory movement of the fluid while for small Strouhal number, the steady state portion of the movement dominates the oscillations.

2.4.3 Mach Number

Mach number is the dimensionless number defined as the fraction of the speed of an object moving through air with a velocity v_w and the speed of sound in air a_s. It is denoted by M and mathematically, it is defined as:

$$M = \frac{v_w}{a_s}$$

2.4.4 Darcy Number

Darcy number denoted by D_a is the dimensionless number defined as the ratio of the medium permeability K to the square of the domain thickness h^2. Mathematically, it is defined as:

$$D_a = \frac{K}{h^2}$$

2.4.5 Deborah Number

Deborah number is the non-dimensional number defined as the ratio of the characteristic time of the fluid, it takes to adjust to applied stresses/deformations and the characteristic time of the flow system. It is denoted by D_e and is mathematically defined as:

$$D_e = \frac{t_{fluid}}{t_{flow}}$$

It includes both the elasticity and the viscosity of the material. For small Deborah number, the material behaves more like Newtonian viscous flow. At higher Deborah number, the material behaviour changes to non-Newtonian regime increasingly dominated by elasticity. For very high Deborah number, the material behaves like a solid.

2.5 Solution Methodologies

2.5.1 Perturbation Method

The perturbation method is one of the essential tools of applied mathematics and theoretical physics and is used to find the analytic solution of linear and nonlinear differential equation. This method may solve a problem that may otherwise be impossible to solve using standard analytical techniques and numerical methods to obtain reasonable accuracy. The finite difference method is, for example, widely used in numerical analysis due to its efficiency, accuracy and stability but it may fail to solve stiff equations. Perturbation method has been used by a number of researchers and suggested the use of this method when solving problems with insufficient boundary conditions. The perturbation method has the advantage of reducing the order of the equations and treating a singular perturbation problem like a regular perturbation problem, Beard and Walters [79], Verma and Sharma [80]. In addition, it has been suggested by Simmonds and Mann [81] that the approximate analytic solutions obtained using the perturbation method often reveal the important dependence of the exact solution on the flow or material parameters in a manner that is not possible with a full numerical solution. Perturbation and asymptotic methods consist of expanding the solution in an asymptotic series in terms of a small parameter. The classic perturbation method of Poincar´e [82] consists in expanding the solution in an asymptotic series of initial or boundary value problem using a single parameter or functions of such a parameter. When the series converges or is expected to converge, the method is known as a perturbation method. If the series is diverging asymptotically, the method is called an asymptotic method [83-87]. In physical sciences, it has been applied to different problems, e.g. in optics, the boundary layer theory of viscous flows, shock waves, reaction diffusion equations, celestial mechanics and nonlinear oscillations. These reasons provide sufficient motivation and justification for using perturbation methods to investigate the problem of oscillatory flow in a channel for Newtonian and non-Newtonian fluids. The classical perturbation method of Poincar´e has been extended by Nowinski and Ismail [88] to many parameters. This generalization is called the multi-parameter perturbation technique. To apply this method, the parameters must be independent of each another and must be of the same order. In addition, these parameters must describe different physical and fluid properties such as the material and dynamic properties and must be small so that their higher powers and products can be neglected Nowinski and Ismail [88].

2.5.2 WKB Approximation Method

It is an approximate method to find the asymptotic approximation of a linear differential equation in which a small parameter is multiplied with the highest derivative. It is named after the three physicists Wentzel, Kramers and Brillouin (WKB). It is similar to the dominant balance method in that the solution $y(x,\varepsilon)$ is expressed as an exponential of an asymptotic expansion. In this case, the asymptotic sequence is a set of functions of ε which we denote by $\lambda_n(\varepsilon)$. Each term in the expansion has a coefficient function $Q_n(x)$. The form of the asymptotic sequence is not assumed at the onset. The functions $\lambda_n(\varepsilon)$ are determined by using a dominant balance approach.

Local breakdown

A local break down of WKB occurs when the approximate solution is exponentially increasing/decreasing. To construct an approximate solution of a differential equation containing a small parameter $\varepsilon,$ there is a requirement in boundary layer theory to match gradually changing outer solution to a rapidly changing inner solution. In the limit as ε approaches $0+$, the outer solution remains smooth but an inner solution becomes discontinuous at the boundary layer because the thickness of the boundary layer tends to zero. In this situation, the solution experiences local breakdown and this kind of behaviour is called dissipative because the rapidly changing component of the solution decays exponentially away from the point of local breakdown.

Global breakdown

A global breakdown occurs when the solution behaves rapidly oscillatory. e.g. the solution of boundary value problem with small parameter ε

$$\varepsilon y'' + y = 0, \quad y(0) = 0, \; y(1) = 1.$$

exhibits a global breakdown, because the exact solution of this boundary value problem given by:

$$y(x) = \frac{\sin\left(\frac{x}{\sqrt{\varepsilon}}\right)}{\sin\left(\frac{1}{\sqrt{\varepsilon}}\right)}, \quad \varepsilon \neq \frac{1}{(\mathrm{n}\pi)^2},$$

becomes highly oscillatory for small values of ε and discontinuous when $\varepsilon \to 0+$. This breakdown occurs throughout the finite interval $0 \le x \le 1.$ This solution is also called a dispersive solution. The dispersive solution is a wavelike solution in which the small wavelength and amplitude is changing very slowly. Boundary layer techniques are not powerful enough to handle dispersive phenomenon. Both dispersive and dissipative phenomena are characterized by exponential behaviour. The exponent is imaginary in dispersive and real in dissipative case. Thus, the approximate solution of a differential equation which exhibits either or both kinds of behaviour must be of the following form:

$$y(x) \sim A(x) e^{\frac{Q(x)}{\lambda(\varepsilon)}}, \quad \lambda \to 0+.$$

This approximation is conventionally known as a WKB approximation, in which the phase $Q(x)$ is considered to be non constant and gradually varying in the breakdown region. There is a boundary layer of thickness λ when $Q(x)$ is real, while there is a region of rapid oscillations of waves with wavelength λ when $Q(x)$ is imaginary. For the case of constant $Q(x)$, the behaviour of $y(x)$ which is characteristic of an outer solution in boundary layer theory is expressed by the gradually varying amplitude function $A(x)$.

The exponential approximation given above is not in a form most suitable for deriving asymptotic approximations because the amplitude function $A(x)$ and phase function $Q(x)$ depend implicitly on δ. It is best to represent the dependence of these functions on λ explicitly by expanding $A(x)$ and $Q(x)$ as series in powers of λ. We can then combine these two series in a single exponential power series of the form,

$$y(x) \sim \exp\left(\frac{1}{\lambda}\sum_{n=0}^{\infty} \lambda^n(\varepsilon) Q_n(x)\right), \qquad \lambda \to 0.$$

This is the general expression for WKB approximation Bender [89], Bush [89]. WKB approximation is a useful tool to obtain the global approximation for the solution of linear differential equation where a small parameter is multiplied with its highest derivative. It contains a boundary layer theory as a special case. The WKB approximation to a solution of differential equation gives a simple structure. Exact solution may be some unknown function, WKB approximation order by order in powers of parameter consists of exponentials of algebraic functions and well-known special functions such as the Airy function or parabolic cylinder function. WKB approximation is suitable for linear differential equations of any order, for initial-value and boundary-value problems and for eigenvalues problems. It may also be used to evaluate integrals of the solution of differential equation.

Chapter 03

Oscillatory flow in a porous channel with porous medium and small suction

In this chapter, we study the oscillatory flow inside a porous channel for viscous fluid. We investigate the combined effects of porosity and oscillations on the flow between two parallel porous plates. Brinkman-extended Darcy's law relationship is used to add the effects of porous medium in the governing Navier-Stokes equations. We find the analytic solution of the oscillatory flow in a channel saturated with porous medium. The effects of the non-dimensional inverse Darcy parameter on the amplitude and penetration depth of the oscillatory axial velocity are examined.

3.1 Formulation of the Problem

We consider the laminar and viscous fluid in a porous channel. The length of the channel is L, width w and walls of the channel are separated by a distance $2h$. The suction is taking place from the porous walls with velocity v_w. The space between the walls of the channel is saturated with viscous fluid. The Brinkman-extended Darcy's law relationship is used for the description of porous medium. The flow is generated by suction at the permeable walls. The geometry and flow configuration of the problem is shown in Figure 3.1.

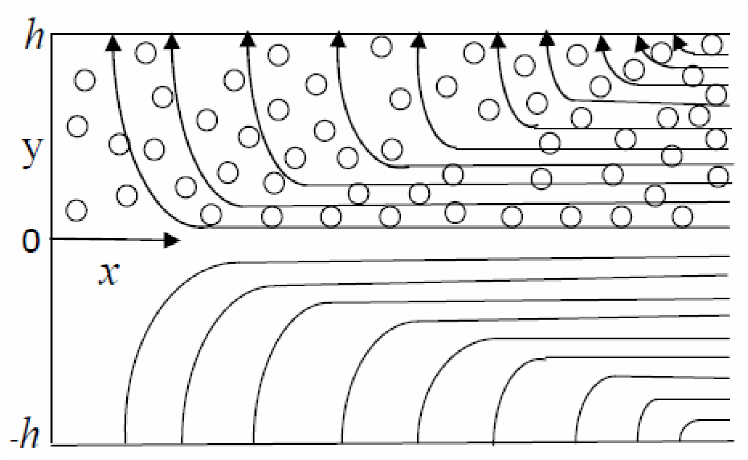


Figure 3.1 Geometry and flow configuration of the problem

The governing equations of motion for viscous fluid in a porous medium are:

The continuity equation

$$\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot \left(\rho^* \mathbf{V}^* \right) = 0 \tag{3.1}$$

and the momentum equation

$$\rho^*\left[\frac{\partial \mathbf{V}^*}{\partial t^*}+\left(\mathbf{V}^*\cdot\nabla^*\right)\mathbf{V}^*\right]=-\nabla^* p^*+\mu\left[\frac{4}{3}\nabla^*(\nabla^*\cdot\mathbf{V}^*)-\nabla^*\times\left(\nabla^*\times\mathbf{V}^*\right)\right]-\frac{\mu}{K}\mathbf{V}^* \tag{3.2}$$

with the following boundary conditions

$$\begin{aligned} &v^*=0,\quad \frac{\partial u^*}{\partial y^*}=0; \qquad \text{at} \quad y^*=0,\\ &v^*=v_w,\quad u^*=0; \qquad \text{at} \quad y^*=h, \end{aligned} \tag{3.3}$$

where $\mathbf{V}^*$ is the velocity vector, u^* and v^* are the axial and normal components of the velocity vector $\mathbf{V}^*$, ρ^* is the density, p^* is the pressure, μ is the dynamic viscosity, K is the permeability of free space and (*) denotes the dimensional variables. In order to present the dimensionless flow formulation, we introduce the following quantities:

$$x=\frac{x^*}{h},\quad y=\frac{y^*}{h},\quad \mathbf{V}=\frac{\mathbf{V}^*}{a_s},\quad p=\frac{p^*}{\kappa p_s},\quad \rho=\frac{\rho^*}{\rho_s},\quad t=t^*\omega_s \tag{3.4}$$

where a_s, p_s and ρ_s are stagnation speed of sound, stagnation pressure and stagnation density respectively while κ is the ratio of specific heats and ω_s is frequency of the longitudinal pressure oscillations. Using Eq. (3.4), the non-dimensional form of Eqs. (3.1) - (3.3) are:

$$\omega\frac{\partial \rho}{\partial t}+\nabla\cdot\left(\rho\mathbf{V}\right)=0 \tag{3.5}$$

$$\rho\left[\omega\frac{\partial \mathbf{V}}{\partial t}+\left(\mathbf{V}\cdot\nabla\right)\mathbf{V}\right]=-\nabla p+M\varepsilon\left[\frac{4}{3}\nabla\left(\nabla\cdot\mathbf{V}\right)-\nabla\times\left(\nabla\times\mathbf{V}\right)-\gamma\mathbf{V}\right] \tag{3.6}$$

$$\begin{aligned} &v=0,\quad \frac{\partial u}{\partial y}=0; \qquad \text{at} \quad y=0,\\ &v=M,\quad u=0; \qquad \text{at} \quad y=1, \end{aligned} \tag{3.7}$$

where γ is the non-dimensional inverse Darcy parameter, ω is the dimensionless wave frequency, M is the Mach number and ε is the reciprocal of the Reynolds number (suction parameter) and are defined as:

$$\gamma = \frac{1}{D_a} = \frac{h^2}{K},\ \ \omega = \frac{\omega_s h}{a_s},\ \ M = \frac{v_w}{a_s},\ \ \varepsilon = \frac{1}{R_e} = \frac{\upsilon}{v_w h}$$

The expressions for pressure, density and velocity in terms of small pressure wave amplitude δ are perturbed as follows:

$$p(x,\ y,\ t) = p_0(x,\ y) + \delta p_1(x,\ y)\exp(-it), \tag{3.8}$$

$$\rho(x,\ y,\ t) = 1 + \delta\rho_1(x,\ y)\exp(-it), \tag{3.9}$$

$$\mathbf{V}(x,\ y,\ t) = M\mathbf{V}_0(x,\ y) + \delta\mathbf{V}_1(x,\ y)\exp(-it). \tag{3.10}$$

where $\delta = \dfrac{A}{\kappa p_s} \ll 1$ is the dimensionless wave amplitude. In the subsequent analysis, we will be concerned with finding the unperturbed time independent and perturbed time dependent parts of the velocity field by calculating $\mathbf{V}_0$ and $\mathbf{V}_1$ respectively.

3.2 Time Independent Part of Velocity Field

Utilizing Eqs. (3.8) – (3.10) in Eqs. (3.5) and (3.6), taking the leading order terms in δ, we arrive at:

$$\nabla \cdot \mathbf{V}_0 = 0, \tag{3.11}$$

$$M^2(\mathbf{V}_0 \cdot \nabla)\mathbf{V}_0 = -\nabla p_0 + M^2\varepsilon\left[\frac{4}{3}\nabla(\nabla \cdot \mathbf{V}_0) - \nabla \times (\nabla \times \mathbf{V}_0) - \gamma\mathbf{V}_0\right]. \tag{3.12}$$

To reduce the system of partial differential equations into an ordinary differential equation, we use the stream function defined by Jankowski and Majdalani [21].

$$\Psi = -xF(y). \tag{3.13}$$

The velocity components in terms of the stream function are then given by:

$$\mathbf{V}_0 = u_0\hat{i} + v_0\hat{j} = -xF'\hat{i} + F\hat{j}, \tag{3.14}$$

where $\hat{i}$ and $\hat{j}$ are the unit vectors parallel to x- and y-axes respectively. Using Eq. (3.14), Eq. (3.11) is identically satisfies and Eq. (3.12) takes the following form:

$$M^2\left[\left(xF'^2 - xFF''\right)\hat{i} + FF'\hat{j}\right] = -\frac{\partial p_0}{\partial x}\hat{i} - \frac{\partial p_0}{\partial y}\hat{j} + M^2\varepsilon\left[x\left(-F''' + \gamma F'\right)\hat{i} + \left(F'' - \gamma F\right)\hat{j}\right],$$

Taking cross-differentiation of i^{th} and j^{th} components, of the above equation, with respect to y and x respectively and subtracting the resulting equations, we get:

$$\left(F'''' - \gamma F''\right) + R\left(F'F'' - FF'''\right) = 0 \tag{3.15}$$

In Eq. (3.15), F satisfies the boundary conditions

$$F'(1) = F(0) = F''(0) = 0\ ,\quad F(1) = 1. \tag{3.16}$$

In the subsequent analysis, we restrict the domain of R as $1 < R < 20$. This give rise to two parameters, $\varepsilon = \frac{1}{R} < 1$ and $\frac{R}{100} < 1$. We further assume that the permeability of the medium is small and propose $\gamma = \left(\frac{R}{100}\right)^2$. Expanding Eqs. (3.15) and (3.16) in the powers of $\left(\frac{R}{100}\right)$, the solution of the leading order terms can easily be expressed as:

$$F(y) = \frac{1}{2}y\left(3 - y^2\right) + O\left(\frac{R_e}{100}\right),\qquad R_e < 20 \tag{3.17}$$

We observe that the porosity is taken small enough not to contribute in the leading order. This sacrifice is made to ensure its contribution in the transient part while discussing the time dependent oscillatory velocity: the main purpose of this paper. The time dependent part of the velocity distribution is now calculated in the next section.

3.3 Time Dependent Part of Velocity Field

To find the time dependent oscillatory velocity, we make use of Eqs. (3.8) - (3.10) in Eqs. (3.5) and (3.6) and taking the terms at $O(\delta)$, the spatial part of the time dependent velocity field $\mathbf{V}$, can be written as:

$$-i\omega\rho_1 + \nabla\cdot\mathbf{V}_1 = -M\nabla\cdot\left(\rho_1\mathbf{V}_0\right), \tag{3.18}$$

$$\begin{aligned}-i\omega\mathbf{V}_1 = -M\left[\nabla\left(\mathbf{V}_{0.}\mathbf{V}_1\right) - \mathbf{V}_0\times\left(\nabla\times\mathbf{V}_1\right) - \mathbf{V}_1\times\left(\nabla\times\mathbf{V}_0\right)\right] - \nabla p_1 \\ + M\varepsilon\left[\frac{4}{3}\nabla\left(\nabla\cdot\mathbf{V}_1\right) - \nabla\times\left(\nabla\times\mathbf{V}_1\right) - \gamma\mathbf{V}_1\right].\end{aligned} \tag{3.19}$$

The boundary conditions satisfied by $\mathbf{V}_1$ are the no-slip condition at the wall $\mathbf{V}_1(x,1) = 0$ and symmetry about the midsection of the channel $\frac{\partial\mathbf{V}_1(x,0)}{\partial t} = 0$. We proceed to decompose $\mathbf{V}_1(x,y)$ as:

$$\mathbf{V}_1 = \widehat{\mathbf{V}} + \widetilde{\mathbf{V}}, \tag{3.20}$$

where $\widehat{\mathbf{V}}$ is the curl-free, pressure-driven, acoustic velocity and $\widetilde{\mathbf{V}}$ is the divergence-free, vorticity-driven, vortical velocity. These velocities have thus the properties:

$$\nabla\times\widehat{\mathbf{V}} = 0\ ,\ \ \nabla\cdot\widetilde{\mathbf{V}} = 0\ ,\ \ p_1 = \widehat{p}\ ,\ \ \rho_1 = \widehat{\rho}, \tag{3.21}$$

This decomposition is based on the fundamental theorem of vector analysis and was first addressed by Stokes [90] and proved by Blumenthal in 1905. This theorem also makes its appearance in the work of Helmholtz's 1958 on vortex motion. Similar decomposition of a small disturbance into pressure and vorticity modes has been given by Carrier & Carlson [91] and Chu & Kovasznay [92]. The acoustic and vortical parts of Eqs. (3.18) and (3. 19) are separated with the help of Eqs. (3.20) and (3.21).

The acoustic set:

$$-i\omega\widehat{\rho} + \nabla\cdot\widehat{\mathbf{V}} = -M\nabla\cdot\left(\widehat{\rho}\mathbf{V}_0\right) \tag{3.22}$$

$$-i\omega\widehat{\mathbf{V}} = -\nabla\widehat{p} + M\varepsilon\left[\frac{4}{3}\nabla\left(\nabla\cdot\widehat{\mathbf{V}}\right) - \gamma\widehat{\mathbf{V}}\right] - M\left[\nabla\left(\widehat{\mathbf{V}}\cdot\mathbf{V}_0\right) - \widehat{\mathbf{V}}\times\left(\nabla\times\mathbf{V}_0\right)\right] \tag{3.23}$$

The vortical set:

$$\nabla\cdot\widetilde{\mathbf{V}} = 0 \tag{3.24}$$

$$-i\omega\widetilde{\mathbf{V}} = -M\varepsilon\left[\nabla\times\left(\nabla\times\widetilde{\mathbf{V}}\right)+\gamma\widetilde{\mathbf{V}}\right]-M\left[\nabla\left(\widetilde{\mathbf{V}}\cdot\mathbf{V}_0\right)-\widetilde{\mathbf{V}}\times\left(\nabla\times\mathbf{V}_0\right)-\mathbf{V}_0\times\left(\nabla\times\widetilde{\mathbf{V}}\right)\right]. \tag{3.25}$$

3.3.1 Solution of the Acoustic Part ($\widehat{\mathbf{V}}$)

The axial acoustic pressure and velocity are dominant for the purpose of oscillatory channel flow under consideration. For perfect gas undergoing isentropic oscillations, we know that $\widehat{\rho}=\widehat{p}$. The Eqs. (3.22) and (3.23) are thus coupled equations in $\widehat{p}$ and $\widehat{\mathbf{V}}$. Eliminating $\widehat{\mathbf{V}}$, from these equations and solving the resulting equation for $\widehat{p}$ upto $O(M)$. This gives the axial acoustic pressure and velocity as:

$$\widehat{p}=\cos(\omega x)+O(M), \tag{3.26}$$

$$\widehat{\mathbf{V}}=i\sin(\omega x)+O(M). \tag{3.27}$$

3.3.2 Solution of the Vortical Part ($\widetilde{\mathbf{V}}$)

We now write the vortical velocity $\widetilde{\mathbf{V}}$ in components form as $\widetilde{\mathbf{V}}=\left(\tilde{u},\tilde{v}\right)$, and assuming that the ratio of the horizontal to vertical part is of $O(M)$, i.e. $\frac{\tilde{v}}{\tilde{u}}\sim O(M)$ Jankowski and Majdalani [16]. This assumption can be justified in view of the argument presented in Flandro [93] and Majdalani & Moorhem [94]. Neglecting $\tilde{v}$ and using $\mathbf{V}_0=\left(-xF',\ F\right)$ and $\widetilde{\mathbf{V}}=X(x)Y(y)$ in Eq. (3.25), we obtain:

$$\varepsilon\frac{d^2Y}{dy^2}-F\frac{dY}{dy}+\left[iS_t+\left(1+x\frac{X'}{X}\right)F'-\gamma\varepsilon\right]Y=0. \tag{3.28}$$

where $S_t=\frac{\omega}{M}$ is the dimensionless Strouhal number. For the solution of $X(x)$, we write:

$$x\frac{X'}{X}=k_n$$

where k_n are the unknown eigenvalues which will be determined using no slip boundary condition at the wall. Integrating the above equation, we have $X(x) = x^{k_n}$. Using this solution, the axial component of vortical velocity can be expressed as:

$$\widetilde{\mathbf{V}}(x, y) = \tilde{u}(x, y) = \sum_n c_n x^{k_n} Y_n(y) \tag{3.29}$$

where c_n is an integration constant associated with k_n. At $y = 1$, the no-slip condition $\mathbf{V}_1(x, 1) = 0$ implies that the vortical velocity is equal to the negative of the acoustic velocity:

$$\widetilde{\mathbf{V}}(x,\ 1) = -\widehat{\mathbf{V}}(x,\ 1),$$

Substituting Eq. (3.29) in the above condition, we conclude that

$$k_n = 2n + 1\ ,\ \ c_n = -i\frac{(-1)^n \omega^{2n+1}}{(2n+1)!}\ ,\ \ Y_n(1) = 1$$

Using these values of k_n and c_n, Eqs. (3.29) and (3.28) takes the form:

$$\widetilde{\mathbf{V}}(x, y) = -i\sum_{n=0}^{\infty} \frac{(-1)^n (\omega x)^{2n+1}}{(2n+1)!} Y_n(y). \tag{3.30}$$

$$\varepsilon \frac{d^2 Y_n}{dy^2} - F\frac{dY_n}{dy} + \left[iS_t + (2n+2)F' - \gamma\varepsilon\right] Y_n = 0 \tag{3.31}$$

with

$$Y_n(1) = 1\ ,\ \ Y_n^{'}(0) = 0. \tag{3.32}$$

In order to determine the complete solution of the vortical part, we proceed to find Y appearing in Eq. (3.30) and defined by Eqs. (3.31) and (3.32). The method adopted is the WKB approximation.

3.3.3 The Solution for ($\mathbf{Y}$)

The solution of Eq. (3.31) subject to the boundary conditions (3.32) can be found by using WKB approximation. The WKB approximation assumes that the solution exhibits an exponential behaviour. The general expression for the WKB approximation is written as:

$$Y_n = \exp\left[\frac{1}{\lambda}\sum_{j=0}^{\infty}\lambda^j Q_j\right]. \tag{3.33}$$

Upon making use of Eq. (3.33) and its derivatives into Eq. (3.31) and expanding upto Q_1 term we have

$$\frac{1}{\varepsilon}Q_0'^2 + 2Q_0'Q_1' + Q_o'' - \frac{F}{\varepsilon}Q_0' - FQ_1' + iS_t + (2n+2)F' - \gamma\varepsilon + O(\varepsilon) = 0,$$

where we assume $S_t = O\left(\frac{1}{\varepsilon}\right)$, $\lambda = \varepsilon$ and $\gamma = \left(\frac{1}{\varepsilon^2}\right)$. Collecting the terms up to $O\left(\frac{1}{\varepsilon}\right)$ in the above equation, we have

$$Q_0'^2 - FQ_0' + iS_t\varepsilon - \gamma\varepsilon^2 = 0. \tag{3.34}$$

Eq. (3.34) is recognized as eikonal equation and its solution can be expressed as:

$$Q_0 = \frac{1}{2}\int_1^y \left(F \pm \sqrt{F^2 - 4\left(i\varepsilon S_t - \gamma\varepsilon^2\right)}\right)d\eta \tag{3.35}$$

and hence the one term WKB solution is given by

$$\begin{aligned} Y_n = c_1 \exp\left[\frac{1}{2\varepsilon}\int_1^y \left(F + \sqrt{F^2 - 4\left(i\varepsilon S_t - \gamma\varepsilon^2\right)}\right)d\eta\right] \\ + c_2 \exp\left[\frac{-1}{2\varepsilon}\int_1^y \left(-F + \sqrt{F^2 - 4\left(i\varepsilon S_t - \gamma\varepsilon^2\right)}\right)d\eta\right] \end{aligned} \tag{3.36}$$

where c_1 and c_2 are integration constants. Using the boundary conditions (3.32) into the above equation and the observation that the second term is of no physical significance (being wave propagating into the surface), we have $c_1 = 1$ and $c_2 = 0$. Thus, Eq. (3.36) becomes

$$Y_n = \exp\left[\frac{1}{2\varepsilon}\int_1^y \left(F + \sqrt{F^2 - 4\left(i\varepsilon S_t - \gamma\varepsilon^2\right)}\right)d\eta\right]. \tag{3.37}$$

Collecting the terms of $O(1)$, we have

$$2Q_0^{'}Q_1^{'}+Q_o^{''}-FQ_1^{'}+\left(2n+2\right)F^{'}=0. \tag{3.38}$$

Eq. (3.38) is recognized as the transport equation and has the solution of the form

$$Q_1=-\frac{1}{2}\int_1^y\left(\frac{(4n+5)F^{'}}{\sqrt{F^2-4\left(i\varepsilon S_t-\gamma\varepsilon^2\right)}}+\frac{FF^{'}}{F^2-4(i\varepsilon S_t-\gamma\varepsilon^2)}\right)d\eta. \tag{3.39}$$

Thus, the two terms WKB solution can be constructed by combining both eikonal (3.37) and transport (3.39) solutions. Hence one can write

$$\begin{aligned}Y_n&=\exp\left[\frac{1}{2\varepsilon}\int_1^y\left(F+\sqrt{F^2-4\left(i\varepsilon S_t-\gamma\varepsilon^2\right)}\right)d\eta\right]\\&\times\left(\frac{1+\sqrt{1-4\left(i\varepsilon S_t-\gamma\varepsilon^2\right)}}{F+\sqrt{F^2-4\left(i\varepsilon S_t-\gamma\varepsilon^2\right)}}\right)^{(2n+\frac{5}{2})}\times\left(\frac{1-4\left(i\varepsilon S_t-\gamma\varepsilon^2\right)}{F^2-4\left(i\varepsilon S_t-\gamma\varepsilon^2\right)}\right)^{\frac{1}{4}}.\end{aligned} \tag{3.40}$$

Substitution of Eq. (3.40) into Eq. (3.30) gives the following expression of vortical velocity:

$$\begin{aligned}\widetilde{\mathbf{V}}\left(x,\ y\right)=-i\sum_{n=0}^{\infty}\frac{(-1)^n\left(\omega x\right)^{2n+1}}{\left(2n+1\right)!}&\left[\exp\left(\frac{1}{2\varepsilon}\int_1^y\left(F+\sqrt{F^2-4\left(i\varepsilon S_t-\gamma\varepsilon^2\right)}\right)d\eta\right)\right.\\&\left.\times\left(\frac{1+\sqrt{1-4\left(i\varepsilon S_t-\gamma\varepsilon^2\right)}}{F+\sqrt{F^2-4\left(i\varepsilon S_t-\gamma\varepsilon^2\right)}}\right)^{(2n+\frac{5}{2})}\times\left(\frac{1-4\left(i\varepsilon S_t-\gamma\varepsilon^2\right)}{F^2-4\left(i\varepsilon S_t-\gamma\varepsilon^2\right)}\right)^{\frac{1}{4}}\right].\end{aligned} \tag{3.41}$$

After calculating both acoustic and vortical velocities, we revert back to the time dependent oscillatory velocity. The time dependent oscillatory velocity is expressed with the help of Eqs. (3.20), (3.27) and (3.41):

$$\mathbf{V}_1 = i\left[\sin(\omega x) - \left(\frac{1+\sqrt{1-4\left(i\varepsilon S_t - \gamma\varepsilon^2\right)}}{F+\sqrt{F^2-4\left(i\varepsilon S_t - \gamma\varepsilon^2\right)}}\right)^{\frac{3}{2}} \times \left(\frac{1-4\left(i\varepsilon S_t - \gamma\varepsilon^2\right)}{F^2-4\left(i\varepsilon S_t - \gamma\varepsilon^2\right)}\right)^{\frac{1}{4}}\right.$$
$$\left.\times \exp\left(\frac{1}{2\varepsilon}\int_1^y \left(F+\sqrt{F^2-4\left(i\varepsilon S_t - \gamma\varepsilon^2\right)}\right)d\eta\right) \times \sin\left(\frac{\omega x\left(1+\sqrt{1-4\left(i\varepsilon S_t - \gamma\varepsilon^2\right)}\right)}{F+\sqrt{F^2-4\left(i\varepsilon S_t - \gamma\varepsilon^2\right)}}\right)\right]. \tag{3.42}$$

The integrals in this solution cannot be calculated analytically and hence numerical integration is performed. Mathematically, the results for viscous fluid are recovered as $\gamma \to 0$, and are found to match with Jankowski and Majdalani [21].

3.4 Results and Discussion

We recall that the main focus of this study is to find the oscillatory axial velocity in the channel. For that, we will discuss the behaviour of the oscillatory axial velocity $\mathbf{V}_1 \exp(-it)$. We first establish the results for clear (non-porous medium) fluid as a special case ($\gamma \to 0$) of the results. In Figure 3.2, we draw the graph of the real part of $\mathbf{V}_1 \exp(-it)$ against y for clear fluid at four equiphased timelines $t = 90^0, 180^0, 270^0$ and 360^0 keeping the other parameters fixed and compare our results with the existing work Jankowski and Majdalani [21] for non-porous medium case. The doted lines represent our results while solid lines represent those of Jankowski and Majdalani [21]. A perfect agreement is found to exist mathematically and hence graphically.

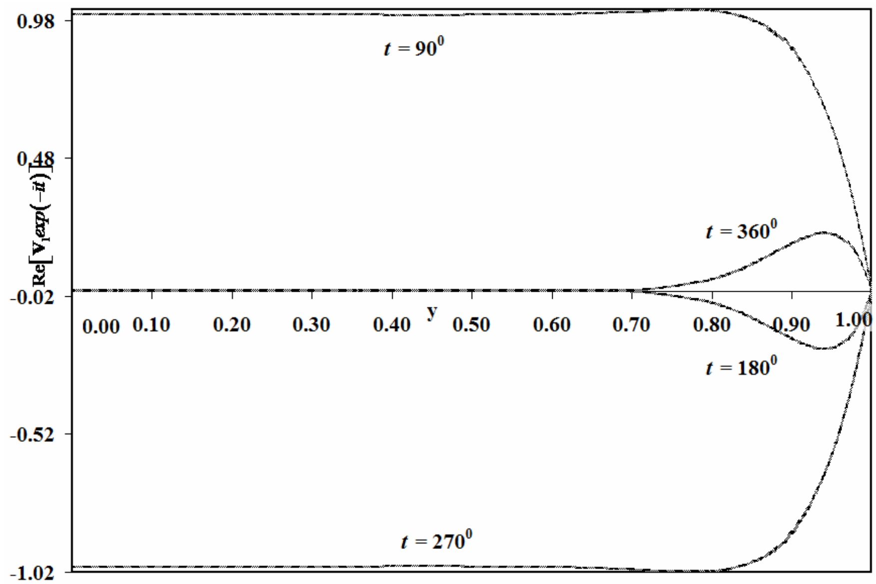
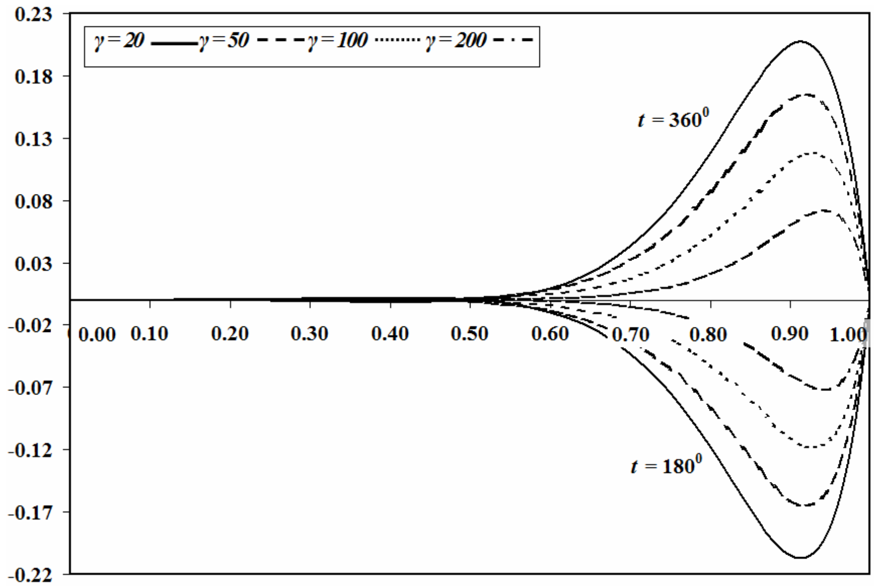
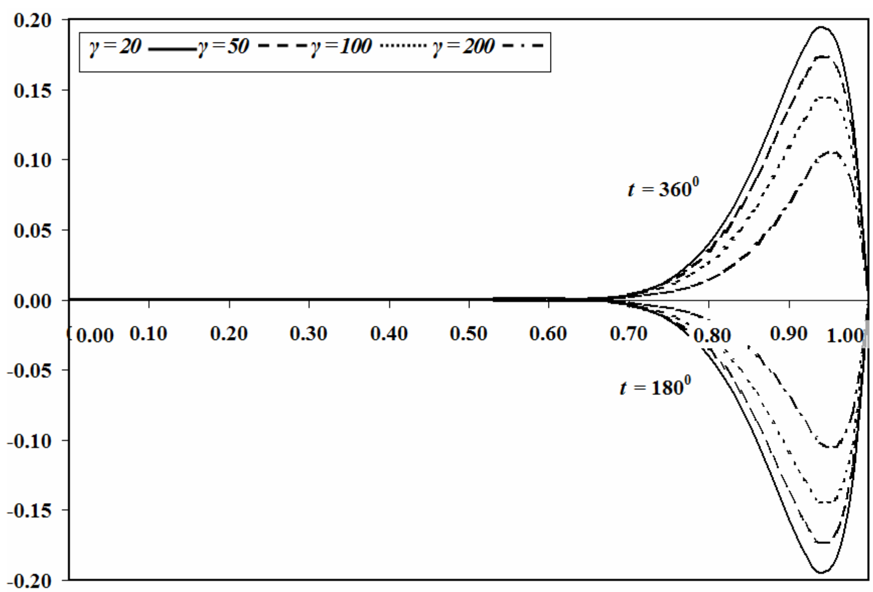


Figure 3.2 Comparison of $Re\left(\mathbf{V}_1 \exp(-it)\right)$ **against** y **for** $\frac{x}{l}=1, m=1, S_t=20, R_e=10$ **and different** t.

In Figures 3.3, we plot the effects of γ on $\mathbf{V}_1 \exp(-it)$. This clearly shows that the amplitude of oscillation decreases and the penetration depth of the oscillatory velocity also decrease as the values of γ increases. This means that with the increase in the values of γ, the permeability of the medium decreases and size of the rigid matrix increases which reduces the space for pure fluid. With increase of the value of suction parameter in Figure 3.3 (b) and plotting the effects of γ on $\mathbf{V}_1 \exp(-it)$, the same behaviour is observed for amplitude and penetration depth of the oscillatory velocity but the rate of decrease is faster by increasing values of γ. Thus by decreasing the permeability of the medium, we reduce the oscillations in the velocity field.



(a) $R_e = 5$



(b) $R_e = 10$

Figure 3.3 Effects of γ **on** $Re\left(\mathbf{V}_1 \exp(-it)\right)$ **for** $\frac{x}{l} = 1, m = 1, S_t = 20$ **at time phase**

$t = 180^o$ and $t = 360^o$

In Figure 3.4, we plot the effects of R on $\mathbf{V}_1 \exp(-it)$ for a fixed value of γ . This clearly shows that the amplitude and penetration depth of oscillation increases by increasing values of R. This means that by increasing the Reynolds number, the inertial effects dominates on the velocity profile.

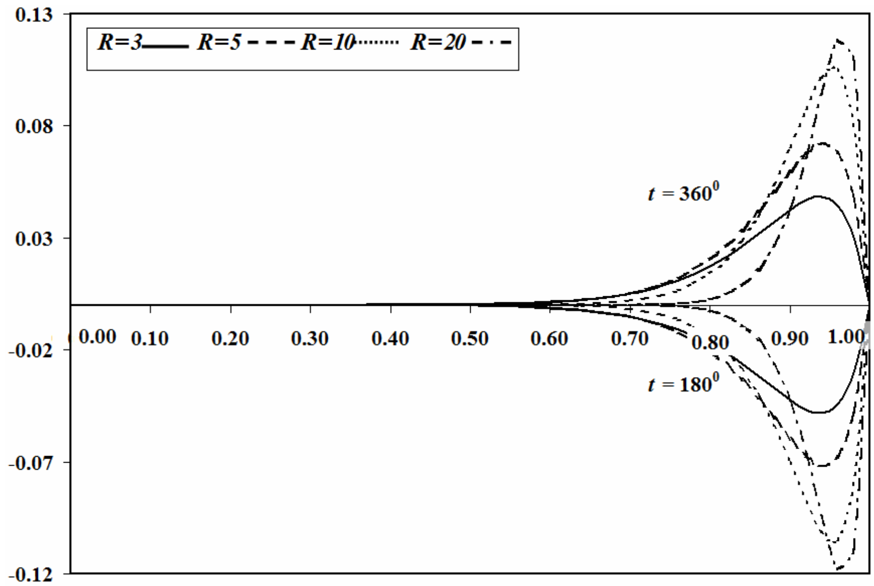


Figure 3.4 Effects of suction parameter R **on** $Re\left(\mathbf{V}_1 \exp(-it)\right)$ **for** $\frac{x}{l}=1, m=1, S_t=20$ **at time phase** $t=180^0$ and $t=360^0$.

3.5 Closing Remarks

The problem of unsteady two-dimensional oscillatory channel flow of a viscous fluid in a porous medium is studied analytically. The main results can be summarized as follows:

- The amplitude of the oscillatory axial velocity decreases by increasing inverse Darcy parameter.
- The penetration depth of the velocity decreases with an increase of inverse Darcy parameter.
- The oscillations in the velocity field can be minimized by decreasing the permeability of the porous medium.

Chapter 04

Oscillatory channel flow for non-Newtonian fluid

In this chapter, we study the unsteady oscillatory flow of a non-Newtonian second grade fluid in a porous channel. The geometry of the problem remains the same as in Chapter 3. Cauchy stress tensor for the second grade fluid is used to describe the effects of non-Newtonian fluid. The analytic solution is found by considering both second grade fluid and suction parameters small. The effects of non-dimensional second grade fluid parameter and suction parameter on the oscillatory axial velocity, its amplitude and penetration depth are found and presented graphically. The results for viscous fluid Jankowski and Majdalani [21] can be recovered in the limit $\alpha \to 0$.

4.1 Mathematical Description of the Problem

We consider the flow of a non-Newtonian second grade fluid between a long and narrow channel of length L and width w. The plates are porous and the distance between the two plates is $2h$. One end of the channel is open and the other end is closed. The $x-$axis is taken along the walls of the channel and $y-$axis normal to the walls. The suction is taking place from the porous walls with a uniform velocity v_w. The assumption $w >> h$ enables us to treat the problem as a two-dimensional one. The symmetric condition at the midsection of the channel reduces the domain to one half of its original size. The geometrical description is shown in Figure 4.1.

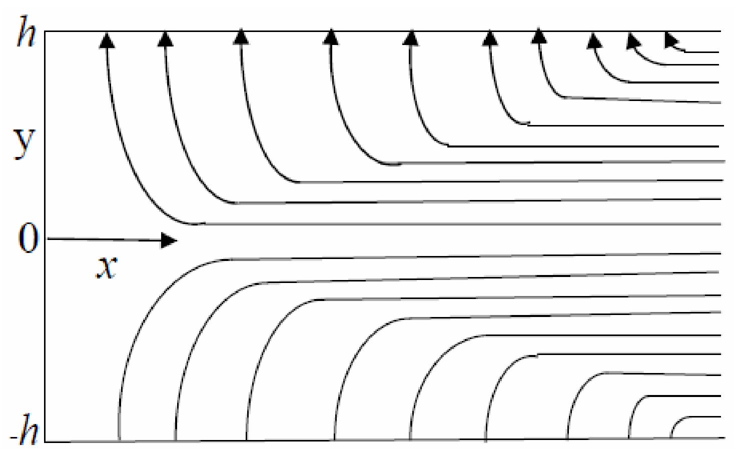


Figure 4.1 Geometry and flow configuration of the problem

The equations that govern the flow of a second grade fluid are:

The continuity equation

$$\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot \left(\rho^* \mathbf{V}^* \right) = 0 \tag{4.1}$$

and the momentum equation

$$\operatorname{div} \mathbf{T}^* + \rho^* \boldsymbol{b} = \rho^* \frac{d\mathbf{V}^*}{dt^*} \tag{4.2}$$

where $\mathbf{V}^*, \rho^*, \boldsymbol{b}$ are the velocity vector, the density, the body force respectively and $\mathbf{T}^*$ is the Cauchy stress tensor for the second grade fluid which is defined as:

$$\mathbf{T}^* = -p^* \boldsymbol{I} + \mu \mathbf{A}_1^* + \alpha_1 \mathbf{A}_2^* + \alpha_2 \mathbf{A}_1^{2*} \tag{4.3}$$

where $p^*, \boldsymbol{I}, \mu$ are pressure, identity tensor and dynamic viscosity respectively. α_1 and α_2 are material constants, $\mathbf{A}_1^*$ and $\mathbf{A}_2^*$ are first two Rivlin-Ericksen tensors and are given by:

$$\begin{aligned} \mathbf{A}_1^* &= \left(grad\, \mathbf{V}^* \right) + \left(grad\, \mathbf{V}^* \right)^T \\ \mathbf{A}_2^* &= \frac{d\mathbf{A}_1^*}{dt} + \mathbf{A}_1^* \left(grad\, \mathbf{V}^* \right) + \left(grad\, \mathbf{V}^* \right)^T \mathbf{A}_1^* \end{aligned} \tag{4.4}$$

with thermodynamics conditions:

$$\mu \geq 0, \alpha_1 \geq 0, \alpha_1 + \alpha_2 = 0$$

where $\frac{d}{dt}$ is material time derivative, *grad* is the gradient operator and T in superscript denotes the transpose. Substitution of Eqs. (4.3) and (4.4) in Eq. (4.2) gives:

$$\begin{aligned} & grad \left[\frac{1}{2} \rho^* \left| \mathbf{V}^* \right|^2 + p^* - \alpha_1 \left(\mathbf{V}^* \cdot \nabla^* \mathbf{V}^* + \frac{1}{4} \left| \mathbf{A}_1^* \right|^* \right) \right] + \rho^* \left[\mathbf{V}_{t^*}^* - \mathbf{V}^* \times \left(\nabla^* \times \mathbf{V}^* \right) \right] \\ & = \mu \nabla^{*2} \mathbf{V}^* + \alpha_1 \left[\nabla^{*2} \mathbf{V}_{t^*}^* + \nabla^{*2} \left(\nabla^* \times \mathbf{V}^* \right) \times \mathbf{V}^* \right] + \left(\alpha_1 + \alpha_2 \right) \operatorname{div} \mathbf{A}_1^{*2} + \rho^* b, \end{aligned} \tag{4.5}$$

where the subscript t^* denotes the partial derivative with respect to time, ∇^2 is the Laplacian operator and $(*)$ denotes the dimensional variables. To non-dimensionalize the governing Eqs. (4.1) and (4.5), we introduce the following dimensionless quantities:

$$x = \frac{x^*}{h}, \quad y = \frac{y^*}{h}, \quad \mathbf{V} = \frac{\mathbf{V}^*}{a_s}, \quad p = \frac{p^*}{\kappa p_s}, \quad \rho = \frac{\rho^*}{\rho_s}, \quad t = t^* \omega_s. \tag{4.6}$$

In Eq. (4.6), a_s is the stagnation speed of sound, p_s is the stagnation pressure, ρ_s is the stagnation density while κ is ratio of the specific heats and ω_s is the frequency of the longitudinal pressure oscillation. Making use of Eq. (4.6) in Eqs. (4.1) and (4.5), we arrive at the following non -dimensional boundary value problem:

$$\omega \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{4.7}$$

$$\begin{aligned} \rho\left[\omega \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right] &= -\nabla p + M\varepsilon\left[\frac{4}{3}\nabla(\nabla\cdot\mathbf{V}) - \nabla\times(\nabla\times\mathbf{V})\right] \\ &- \alpha\nabla\left[\mathbf{V}\bullet\nabla^2\mathbf{V} + \frac{1}{4}|\mathbf{A}_1|^2\right] + \alpha\left[\omega\nabla^2\mathbf{V}_t + \nabla^2(\nabla\times\mathbf{V})\times\mathbf{V}\right] \end{aligned} \tag{4.8}$$

$$\begin{aligned} &v = 0, \quad \frac{\partial u}{\partial y} = 0; \qquad \text{at} \quad y = 0, \\ &v = M, \quad u = 0; \qquad \text{at} \quad y = 1, \end{aligned} \tag{4.9}$$

where M is dimensionless Mach number, ε is reciprocal of cross flow Reynolds number (suction parameter), ω is dimensionless wave frequency and α is dimensionless second grade fluid parameter. These quantities are defined as:

$$M = \frac{v_w}{a_s}, \quad \varepsilon = \frac{1}{R} = \frac{\nu}{v_w h}, \quad \omega = \frac{\omega_s h}{a_s}, \quad \alpha = \frac{\alpha_1}{\rho_s h^2}$$

The oscillations are produced due to the small amplitude time harmonic pressure wave. So, we perturb the pressure, density and velocity in terms of small pressure wave amplitude δ:

$$p(x, y, t) = p_0(x, y) + \delta p_1(x, y) e^{(-it)}, \tag{4.10}$$

$$\rho(x, y, t) = 1 + \delta\rho_1(x, y) e^{(-it)}, \tag{4.11}$$

$$\mathbf{V}(x, y, t) = M\mathbf{V_0}(x, y) + \delta\mathbf{V_1}(x, y) e^{(-it)}. \tag{4.12}$$

In what follows, we will be concerned with finding leading order time independent velocity and the time dependent perturbed velocity.

4.2 Leading Order Time Independent Velocity

We first determine the time independent part of the velocity that corresponds to the leading order system in δ. For that, we make use of Eqs. (4.10) - (4.12) in Eqs. (4.7) and (4.8) and taking the terms of $O(\delta)^0$ to arrive at the following system of equations:

$$\nabla \cdot \mathbf{V_0} = 0, \tag{4.13}$$

$$\begin{gathered}M^2\left(\mathbf{V_0}\cdot\nabla\right)\mathbf{V_0} = -\nabla p_0 + M^2\varepsilon\left[\frac{4}{3}\nabla\left(\nabla\cdot\mathbf{V_0}\right) - \nabla\times\left(\nabla\times\mathbf{V_0}\right)\right] \\ -\alpha M^2\left[\nabla\left[\mathbf{V_0}\cdot\nabla^2\mathbf{V_0} + \frac{1}{4}M^2\left\{4\frac{\partial u_0}{\partial x}\frac{\partial v_0}{\partial y} - \left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\right)^2\right\}^2\right] - \nabla^2\left\{\left(\nabla\times\mathbf{V_0}\right)\times\mathbf{V_0}\right\}\right].\end{gathered} \tag{4.14}$$

where u_0 and v_0 are the two components of $\mathbf{V}_0$. The solution of this system is obtained by using the similarity parameter used by Jankowski and Majdalani [21]:

$$\Psi = -xF\left(y\right). \tag{4.15}$$

With this value of stream function, the continuity equation (4.13) is identically satisfied and momentum equation (4.14) takes the following form:

$$\begin{gathered}M^2\left[x\left(F'^2 - FF''\right)\hat{i} + FF'\hat{j}\right] = -\frac{\partial p_0}{\partial x}\hat{i} - \frac{\partial p_0}{\partial y}\hat{j} + M^2\varepsilon\left[-xF'''\hat{i} + F''\hat{j}\right] \\ +\alpha M^2\left[x\left\{M^2\left(4F'^2F''^2 + x^2F''^4\right) - FF'^{v} - F''^2\right\}\hat{i}\right] \\ +\alpha M^2\left[\left\{-2x^2F''F''' + FF''' - F'F'' + M^2\left(4F'F'' + x^2F''F'''\right)\left(4F'^2 + x^2F''^2\right)\right\}\hat{j}\right]\end{gathered} \tag{4.16}$$

where $'$ in the superscript denotes the derivatives with respect to y. Splitting Eq. (4.16) into component form, differentiating the i^{th} and the j^{th} components with respect to y and x respectively and eliminating the pressure by subtracting the resulting equations, we obtain:

$$F^{iv} + R\left(F'F'' - FF'''\right) + \overline{\alpha}\left(FF^{v} + F'F^{iv} - 2F''F'''\right) = 0 \tag{4.17}$$

where $\overline{\alpha} = \alpha R$ and R is the dimensionless Reynolds number (suction parameter). The boundary conditions in F takes the form:

$$F'\left(1\right) = F\left(0\right) = F''\left(0\right) = 0\ ,\ \ F\left(1\right) = 1. \tag{4.18}$$

Eq. (4.17) is the 5th order nonlinear ordinary differential equation. The perturbation solution of Eq. (4.17) subject to Eq. (4.18) in which $\overline{\alpha}$ is the perturbation parameter, given by:

$$F(y)=\frac{1}{2}y\left(3-y^{2}\right)+\alpha\left[\frac{3}{20}y\left(y^{2}-1\right)^{2}\right]+O\left(\alpha^{2}\right) \tag{4.19}$$

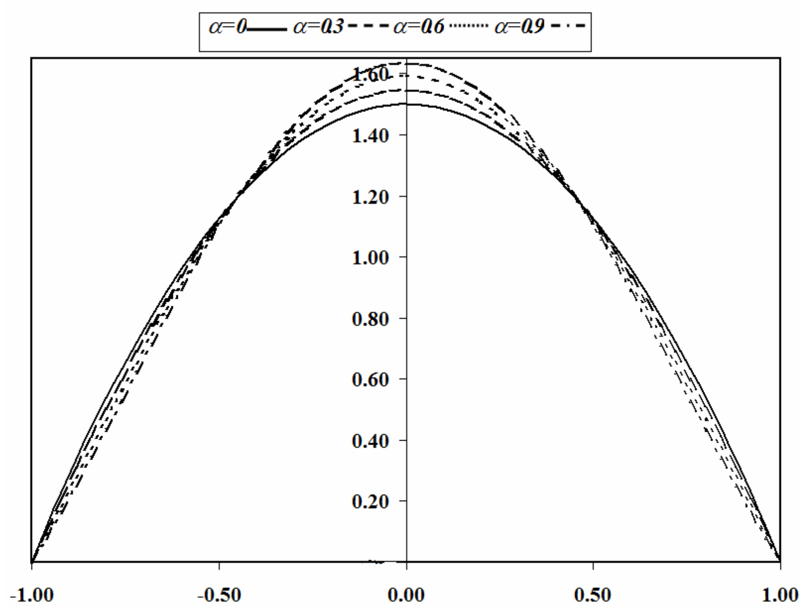


Figure 4.2 Effects of second grade parameter α on time independent velocity profile $\mathbf{V_0}$

The effects of the second grade fluid parameter on the time independent part of the velocity $\mathbf{V_0}$ computed from Eq. (4.17) are shown in Figure 4.2. For $\overline{\alpha}=0$, the result matches with Jankowski and Majdalani [21] and an increase in the values of $\overline{\alpha}$ shows an increase in the velocity distribution $\mathbf{V_0}$. After having found the time independent part, we proceed to find time dependent part of the velocity field subsequently.

4.3 Time Dependent Oscillatory Velocity

Using Eqs. (4.10) - (4.12) in Eqs. (4.7) and (4.8) and taking the terms at $O(\delta)$, the time dependent part of the velocity field $\mathbf{V}$, can be written as:

$$-i\omega\rho_1+\nabla\cdot\mathbf{V_1}=-M\nabla\cdot\left(\rho_1\mathbf{V_0}\right), \tag{4.20}$$

$$\begin{aligned}-i\omega\mathbf{V_1}&=-M\left[\nabla\left(\mathbf{V_0}\cdot\mathbf{V_1}\right)-\mathbf{V_0}\times\left(\nabla\times\mathbf{V_1}\right)-\mathbf{V_1}\times\left(\nabla\times\mathbf{V_0}\right)\right]\\&-\nabla p_1+M\varepsilon\left[\frac{4}{3}\nabla\left(\nabla\cdot\mathbf{V_1}\right)-\nabla\times\left(\nabla\times\mathbf{V_1}\right)\right]+\alpha M\left[\nabla\left\{\mathbf{V_0}\cdot\nabla\left(\nabla\cdot\mathbf{V_1}\right)\right\}\right.\end{aligned} \tag{4.21}$$

$$-\nabla\left\{\mathbf{V_0}\cdot\nabla\times\left(\nabla\times\mathbf{V_1}\right)\right\}+\nabla\left\{\mathbf{V_1}\cdot\nabla\left(\nabla\cdot\mathbf{V_0}\right)\right\}-\nabla\left\{\mathbf{V_1}\cdot\nabla\times\left(\nabla\times\mathbf{V_0}\right)\right\}\Big]$$
$$-\alpha i\omega\left[\nabla\left(\nabla\cdot\mathbf{V_1}\right)-\nabla\times\left(\nabla\times\mathbf{V_1}\right)\right]+\alpha M\nabla\left[\nabla\cdot\left\{\left(\nabla\times\mathbf{V_1}\right)\times\mathbf{V_0}+\left(\nabla\times\mathbf{V_0}\right)\times\mathbf{V_1}\right\}\right]$$
$$-\alpha M\nabla\times\left[\nabla\times\left\{\left(\nabla\times\mathbf{V_1}\right)\times\mathbf{V_0}+\left(\nabla\times\mathbf{V_0}\right)\times\mathbf{V_1}\right\}\right]$$

The velocity $\mathbf{V}_1$ satisfies no-slip boundary condition at the wall $\mathbf{V}_1(x,\ 1)=0$ and symmetry at the midsection of the channel $\frac{\partial \mathbf{V}_1(x,\ 0)}{\partial y}=0$. To proceed further, we decompose $\mathbf{V}_1(x,\ y)$ into an acoustic, pressure-driven, irrotational part $\widehat{\mathbf{V}}$ and vortical, vorticity-driven, rotational part $\widetilde{\mathbf{V}}$ as,

$$\mathbf{V_1}=\widehat{\mathbf{V}}+\widetilde{\mathbf{V}}, \tag{4.22}$$

with the properties

$$\nabla\times\widehat{\mathbf{V}}=0\ ,\ \ \nabla\cdot\widetilde{\mathbf{V}}=0\ ,\ \ p_1=\widehat{p}\ ,\ \ \rho_1=\widehat{\rho}, \tag{4.23}$$

This is a fundamental theorem of vector analysis that was first tackle by Stokes [90] and thoroughly proven by Blumenthal in 1905 and is at the root of Helmholtz's work in 1958 on vortex motion, which is of great importance in both fluid dynamic and electromagnetic theories. Using Eq. (4.22) in Eqs. (4.20) and (4.21), we get the following two sets of equations.

4.3.1 The Acoustic Part:

$$-i\omega\widehat{\rho}+\nabla\cdot\widehat{\mathbf{V}}=-M\nabla\cdot\left(\widehat{\rho}\mathbf{V_0}\right) \tag{4.24}$$

$$\begin{aligned}-i\omega\widehat{\mathbf{V}}=&-\nabla\widehat{p}+\frac{4}{3}M\varepsilon\nabla\left(\nabla\cdot\widehat{\mathbf{V}}\right)-M\left[\nabla\left(\widehat{\mathbf{V}}\cdot\mathbf{V_0}\right)-\widehat{\mathbf{V}}\times\left(\nabla\times\mathbf{V_0}\right)\right]\\&+\alpha M\left[\nabla\left\{\mathbf{V_0}\cdot\nabla\left(\nabla\cdot\widehat{\mathbf{V}}\right)\right\}+\nabla\left\{\widehat{\mathbf{V}}\cdot\nabla\left(\nabla\cdot\mathbf{V_0}\right)\right\}-\nabla\left\{\widehat{\mathbf{V}}\cdot\nabla\times\left(\nabla\times\mathbf{V_0}\right)\right\}\right]\\&-\alpha i\omega\nabla\left(\nabla\cdot\widehat{\mathbf{V}}\right)+kM\nabla\left[\nabla\cdot\left(\nabla\times\mathbf{V_0}\right)\times\widehat{\mathbf{V}}\right]-\alpha M\nabla\times\left[\nabla\times\left(\nabla\times\mathbf{V_0}\right)\times\widehat{\mathbf{V}}\right]\end{aligned} \tag{4.25}$$

For the purpose of oscillatory flow in a channel, the axial acoustic pressure and velocity are dominant. We know that $\widehat{\rho}=\widehat{p}$ (perfect gas undergoing isentropic oscillations). Eliminating $\widehat{\mathbf{V}}$, from Eqs. (4.24) and (4.25), the resulting equation in $\widehat{p}$ can be readily solved upto $O(M)$. Thus the expressions for axial acoustic pressure $\widehat{p}$ and velocity $\widehat{\mathbf{V}}$ are:

$$\hat{p} = \cos\left(\frac{\omega x}{\sqrt{1-\alpha\omega^2}}\right) + O(M), \tag{4.26}$$

$$\hat{\mathbf{V}} = i\sqrt{1-\alpha\omega^2}\,\sin\left(\frac{\omega x}{\sqrt{1-\alpha\omega^2}}\right) + O(M). \tag{4.27}$$

4.3.2 The Vortical Part:

$$\nabla \cdot \tilde{\mathbf{V}} = 0 \tag{4.28}$$

$$\begin{aligned}
-i\omega\tilde{\mathbf{V}} = {} & -M\varepsilon\nabla\times\left(\nabla\times\tilde{\mathbf{V}}\right) - M\left[\nabla\left(\tilde{\mathbf{V}}\cdot\mathbf{V_0}\right) - \tilde{\mathbf{V}}\times\left(\nabla\times\mathbf{V_0}\right) - \mathbf{V_0}\times\left(\nabla\times\tilde{\mathbf{V}}\right)\right] \\
& -\alpha M\left[\nabla\left\{\mathbf{V_0}\cdot\nabla\times\left(\nabla\times\tilde{\mathbf{V}}\right)\right\} + \nabla\left\{\tilde{\mathbf{V}}\cdot\nabla\left(\nabla\cdot\mathbf{V_0}\right)\right\} - \nabla\left\{\tilde{\mathbf{V}}\cdot\nabla\times\left(\nabla\times\mathbf{V_0}\right)\right\}\right] \\
& +\alpha i\omega\nabla\times\left(\nabla\times\tilde{\mathbf{V}}\right) + \alpha M\nabla\left[\nabla\cdot\left(\nabla\times\tilde{\mathbf{V}}\right)\times\mathbf{V_0} + \nabla\cdot\left(\nabla\times\mathbf{V_0}\right)\times\tilde{\mathbf{V}}\right] \\
& -\alpha M\nabla\times\left[\nabla\times\left(\nabla\times\tilde{\mathbf{V}}\right)\times\mathbf{V_0} + \nabla\times\left(\nabla\times\mathbf{V_0}\right)\times\tilde{\mathbf{V}}\right]
\end{aligned} \tag{4.29}$$

Writing $\tilde{\mathbf{V}}$ in component form as $\tilde{\mathbf{V}} = \left(\tilde{u},\ \tilde{v}\right)$, let us assume that the ratio of the horizontal to vertical component of the vortical velocity is of $O(M)$, we can conveniently neglect the vertical component of the vortical velocity. Now, using $\mathbf{V}_0 = \left(-xF',\ F\right)$ in Eq. (4.29), dropping $\tilde{v}$ and writing $\tilde{u} = X(x)Y(y)$, we obtain:

$$\begin{aligned}
& \varepsilon\frac{d^2Y}{dy^2} - F\frac{dY}{dy} + \left[iS_t + \left(1 + x\frac{X'}{X}\right)F'\right]Y \\
& = \alpha\left[-F\frac{d^3Y}{dy^3} + \left\{iS_t + \left(x\frac{X'}{X} - 1\right)F'\right\}\frac{d^2Y}{dy^2} - F''\frac{dY}{dy} + \left(1 + x\frac{X'}{X}\right)F'''Y\right].
\end{aligned} \tag{4.30}$$

where $S_t = \dfrac{\omega}{M}$ is the dimensionless Strouhal number. The x dependent function in Eq. (4.30) can be written as:

$$x\frac{X'}{X} = k_n,$$

where k_n are so far unknown eigenvalues and will be determined using no slip boundary condition. Solving this equation, we get the eigenfunctions $X(x) = x^{k_n}$. The axial component of acoustic velocity can be expressed in terms of these eigenfunctions as:

$$\widetilde{\mathbf{V}}(x,\ y) = \tilde{u}(x,\ y) = \sum_n c_n x^{k_n} Y_n(y). \tag{4.31}$$

From no slip boundary condition $\mathbf{V}_1(x,\ 1) = 0,$ we have:

$$\widehat{\mathbf{V}}(x,\ 1) = -\widetilde{\mathbf{V}}(x,\ 1),$$

Using the values of acoustic velocity from Eq. (4.27) and vortical velocity from Eq. (4.31) in the no-slip condition, we conclude that:

$$k_n = 2n+1\ ,\ \ c_n = -i\sqrt{1-\alpha\omega^2}\,\frac{(-1)^n}{(2n+1)!}\left(\frac{\omega}{\sqrt{1-\alpha\omega^2}}\right)^{2n+1}\ ,\ \ Y_n(1) = 1$$

Using the aforementioned values of k_n and c_n, Eqs. (4.31) and (4.30) take the form:

$$\widetilde{\mathbf{V}}(x,y) = -i\sqrt{1-\alpha\omega^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\omega x}{\sqrt{1-\alpha\omega^2}}\right)^{2n+1} Y_n(y). \tag{4.32}$$

$$\begin{aligned}&\varepsilon\frac{d^2Y}{dy^2} - F\frac{dY}{dy} + \left[iS_t + (2n+2)F'\right]Y \\ &= \alpha\left[-F\frac{d^3Y}{dy^3} + \left(iS_t + 2nF'\right)\frac{d^2Y}{dy^2} - F''\frac{dY}{dy} + (2n+2)F'''Y\right].\end{aligned} \tag{4.33}$$

in which α is a dimensionless second grade fluid parameter and Y satisfies the following boundary conditions.

$$Y(1) = 1\ ,\ \ Y'(0) = 0. \tag{4.34}$$

We find the solution of Eq. (4.33) subject to the conditions (4.34) for complete solution of the vortical part.

In Eq. (4.33), α is a small perturbation parameter. We express the solution of Eq. (4.33) in regular perturbation expansion in α as:

$$Y(y,k) \approx Y_0(y) + \alpha Y_1(y) + O(\alpha^2). \tag{4.35}$$

Using Eq. (4.35) into Eqs. (4.33) and (4.34) and equating like powers of α, the leading order system in α is given by:

$$\varepsilon \frac{d^2Y_0}{dy^2} - F\frac{dY_0}{dy} + \left[iS_t + (2n+2)F' \right] Y_0 = 0 \tag{4.36}$$

$$Y_0(1) = 1\,,\quad Y_0'(0) = 0. \tag{4.37}$$

This system is now solved using WKB approximation for small ε. For that, we write the general expression for the WKB approximation as:

$$Y_0 = e^{\frac{1}{\lambda}\sum_{j=0}^{\infty}\lambda^j Q_j}. \tag{4.38}$$

Making use of aforementioned expression into Eq. (4.36) and expanding up to Q_1, we have:

$$\frac{1}{\varepsilon}Q_0'^2 + 2Q_0'Q_1' + Q_0'' - \frac{F}{\varepsilon}Q_0' - FQ_1' + iS_t + (2n+2)F' + O(\varepsilon) = 0,$$

where we assume $S_t = O\left(\frac{1}{\varepsilon}\right)$ and $\lambda = \varepsilon$. Collecting the terms of $O\left(\frac{1}{\varepsilon}\right)$ in the aforementioned equation, we write:

$$Q_0'^2 - FQ_0' + iS_t\varepsilon = 0. \tag{4.39}$$

which has solution of the form:

$$Q_0 = \frac{1}{2}\int_1^y \left(F \pm \sqrt{F^2 - 4i\varepsilon S_t} \right) d\eta \tag{4.40}$$

The one term WKB solution is thus written as:

$$Y_0 = c_1 e^{\frac{1}{2\varepsilon}\int_1^y (F + \sqrt{F^2 - 4i\varepsilon S_t})d\eta} + c_2 e^{\frac{-1}{2\varepsilon}\int_1^y (-F + \sqrt{F^2 - 4i\varepsilon S_t})d\eta} \tag{4.41}$$

The boundary conditions (Eq. (4.37)) and the observation that the second term is of no physical interest (being wave propagating into the surface) implies $c_1 = 1$ and $c_2 = 0$. Thus one term WKB solution of Eq. (4.36) becomes:

$$Y_0 = e^{\frac{1}{2\varepsilon}\int_1^y (F + \sqrt{F^2 - 4i\varepsilon S_t})d\eta}. \tag{4.42}$$

Using Eq. (4.35) into Eqs. (4.33) and (4.34) and taking the terms of $O(\alpha)$, we have:

$$\frac{d^2Y_1}{dy^2} - \frac{F}{\varepsilon}\frac{dY_1}{dy} + \left[iS_t + (2n+2)F' \right]\frac{Y_1}{\varepsilon} = g(y) \tag{4.43}$$

where

$$g(y) = -\frac{F}{\varepsilon}\frac{d^3Y_0}{dy^3} + \frac{iS_t + 2nF'}{\varepsilon}\frac{d^2Y_0}{dy^2} - \frac{F''}{\varepsilon}\frac{dY_0}{dy} + (2n+2)\frac{F'''}{\varepsilon}Y_0$$

and the corresponding boundary conditions are:

$$Y_1(1) = 0\ ,\quad Y_1'(0) = 0. \tag{4.44}$$

The Eq. (4.43) is non homogeneous differential equation whose solution can be found using variation of parameter method. For that purpose, we find the complementary solution and particular solution. For the complementary solution, we write the homogeneous equation:

$$\varepsilon\frac{d^2Y_1}{dy^2} - F\frac{dY_1}{dy} + \left[iS_t + (2n+2)F'\right]Y_1 = 0. \tag{4.45}$$

The solution of this differential equation is readily found to be:

$$Y_{1c} = c_3 e^{\frac{1}{2\varepsilon}\int_1^y (F+\sqrt{F^2-4i\varepsilon S_t})d\eta} + c_4 e^{\frac{-1}{2\varepsilon}\int_1^y (-F+\sqrt{F^2-4i\varepsilon S_t})d\eta}. \tag{4.46}$$

Now to find the particular solution, we substitute Y_0 from Eq. (4.42) in $g(y)$ and simplify it by taking the terms of $O\left(\frac{1}{\varepsilon}\right)$. Thus we arrived

$$g(y) = -3(2n+2)e^{\frac{1}{2\varepsilon}\int_1^y \left(F+\sqrt{F^2-4i\varepsilon S_t}\right)d\eta} \tag{4.47}$$

Using the numerical integration and omitting the straight forward details, the particular solution of Eq. (4.43) is written as:

$$Y_{1p} = \int_1^y \left[-\frac{6(n+1)\left(1+\frac{F^2}{8i\varepsilon S_t}\right)}{(-4i\varepsilon S_t)^{\frac{1}{2}}}\right]d\eta . e^{\frac{1}{2\varepsilon}\int_1^y \left(F+\sqrt{F^2-4i\varepsilon S_t}\right)d\eta}$$
$$-\int_1^y \left[\frac{6(n+1)\left(1+\frac{F^2}{8i\varepsilon S_t}\right)}{(-4i\varepsilon S_t)^{\frac{1}{2}}}\right]d\eta . e^{\frac{-1}{2\varepsilon}\int_1^y \left(-F+\sqrt{F^2-4i\varepsilon S_t}\right)d\eta} \tag{4.48}$$

After calculating the complementary solution and the particular solution, the general solution of Eq. (4.43) is thus given by:

$$Y_1 = Y_{1c} + Y_{1p}$$

Using Eqs. (4.46) and (4.48), the solution of Eq. (4.43) becomes:

$$\begin{aligned} Y_1 &= c_1 e^{\frac{1}{2\varepsilon}\int_1^y (F+\sqrt{F^2-4i\varepsilon S_t})d\eta} + c_2 e^{\frac{-1}{2\varepsilon}\int_1^y (-F+\sqrt{F^2-4i\varepsilon S_t})d\eta} \\ &+ \int_1^y \left[-\frac{6(n+1)\left(1+\frac{F^2}{8i\varepsilon S_t}\right)}{(-4i\varepsilon S_t)^{\frac{1}{2}}} \right] d\eta . e^{\frac{1}{2\varepsilon}\int_1^y \left(F+\sqrt{F^2-4i\varepsilon S_t}\right)d\eta} \\ &- \int_1^y \left[\frac{6(n+1)\left(1+\frac{F^2}{8i\varepsilon S_t}\right)}{(-4i\varepsilon S_t)^{\frac{1}{2}}} \right] d\eta . e^{\frac{-1}{2\varepsilon}\int_1^y \left(-F+\sqrt{F^2-4i\varepsilon S_t}\right)d\eta} \end{aligned} \tag{4.49}$$

Using boundary conditions (Eq. (4.44)), and the argument as given for Y_0, we have:

$$Y_1 = \int_1^y \left[-\frac{6(n+1)\left(1+\frac{F^2}{8i\varepsilon S_t}\right)}{(-4i\varepsilon S_t)^{\frac{1}{2}}} \right] d\eta . e^{\frac{1}{2\varepsilon}\int_1^y \left(F+\sqrt{F^2-4i\varepsilon S_t}\right)d\eta} \tag{4.50}$$

Thus, the two term perturbation solution of Eq. (4.33) is given by:

$$Y = Y_0 + \alpha Y_1$$

$$Y = \left[1+\alpha\int_1^y \left[-\frac{6(n+1)\left(1+\frac{F^2}{8i\varepsilon S_t}\right)}{(-4i\varepsilon S)^{\frac{1}{2}}} \right] d\eta \right] e^{\frac{1}{2\varepsilon}\int_1^y \left(F+\sqrt{F^2-4i\varepsilon S_t}\right)d\eta} \tag{4.51}$$

Now after using Eq. (4.51), the expression for vortical velocity Eq. (4.32) can be written as:

$$\begin{aligned} \widetilde{\mathbf{V}}(x,\ y) &= -i\sqrt{1-\alpha\omega^2}\,\sin\left(\frac{\omega x}{\sqrt{1-\alpha\omega^2}} \right) \\ &. \left[1+\alpha\int_1^y \left(-\frac{6(n+1)\left(1+\frac{F^2}{8i\varepsilon S_t}\right)}{(-4i\varepsilon S_t)^{\frac{1}{2}}} \right) d\eta \right] e^{\frac{1}{2\varepsilon}\int_1^y (F+\sqrt{F^2-4i\varepsilon S_t})d\eta} \end{aligned} \tag{4.52}$$

After calculating both acoustic velocity Eq. (4.27) and vortical velocity Eq. (4.52), we can write the expression for the time dependent oscillatory velocity Eq. (4.22) as:

$$\mathbf{V}_1 = i\left[\sin(\omega x)(1-Y_0) - \alpha\left\{\sin(\omega x)Y_1 - \left(\frac{1}{2}\omega^3 x - \frac{1}{2}\omega^2\sin(\omega x)\right)(1-Y_0)\right\}\right] \quad (4.53)$$

where Y_0 and Y_1 are given by Eqs. (4.42) and (4.50) respectively. The integrals in these solutions cannot be calculated analytically and hence numerical integration is performed. Mathematically, the viscous fluid results are obtained for $\alpha = 0$, and are found to match with Jankowski and Majdalani [21].

4.4 Graphical Results and Description

Here, we would like to discuss the behaviour of time dependent part, which is the crux of our discussion for oscillatory channel flow. To add credibility and to substantiate our results for the second grade fluid (Eq. (4.53)), we first plot the real part of oscillatory axial velocity $\mathbf{V}_1 \exp(-it)$ for viscous fluid ($\alpha = 0$). Figure 4.3 represents the velocity distribution at four time lines $t = 90^0, 180^0, 270^0$ and 360^0 while Figure 4.4 shows the amplitude and penetration depth of the oscillatory velocity for different values of suction parameter. The two graphs are found to match identically with Jankowski and Majdalani [21].

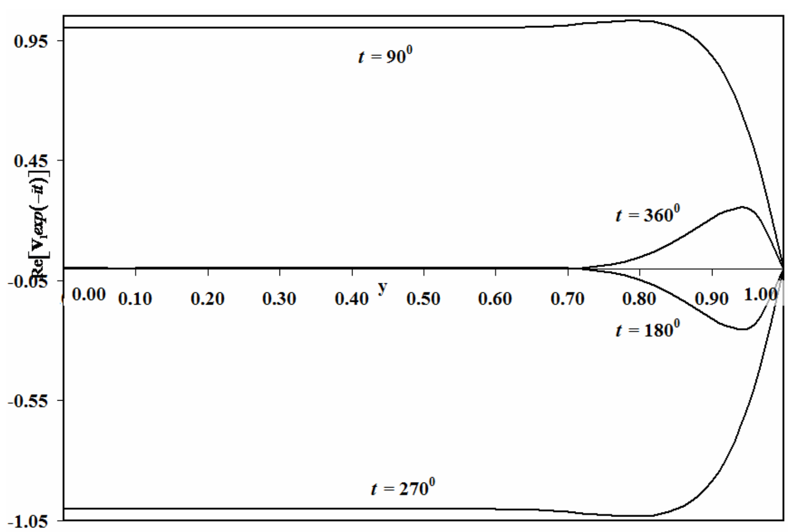


Figure 4.3 Plot of $\mathrm{Re}(\mathbf{V}_1 \exp(-it))$ **against** y **for viscous fluid** $\alpha = 0$ **and** $\frac{x}{l} = 1,\ m = 1,\ S_t = 20,$ $R = 10$ **at different dimensionless time** t.

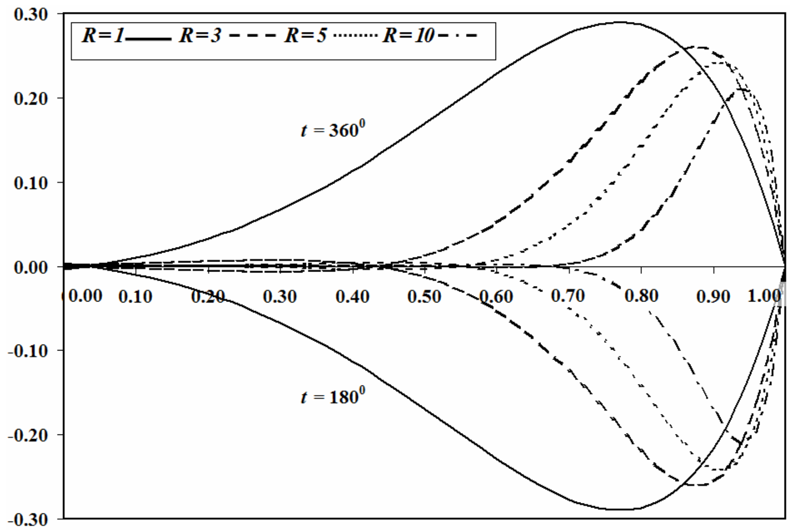


Figure 4.4 Effects of suction parameter R **on** $\mathrm{Re}\left(\mathbf{V}_1 \exp(-it)\right)$ **for viscous fluid** $\alpha = 0$ **and**

$\frac{x}{l} = 1, m = 1, S_t = 20$ **at time phase** $t = 180^0$ **and** $t = 360^0$

The effects of suction parameter on the oscillatory axial velocity for the second grade fluid ($\alpha = 0.6$) are as shown in Figure 4.5. The graph shows the amplitude of oscillation and the penetration depth decreases as the suction parameter R increases. However, the rate of decrease of the amplitude with respect to suction parameter is more in the second grade fluid than in the viscous fluid.

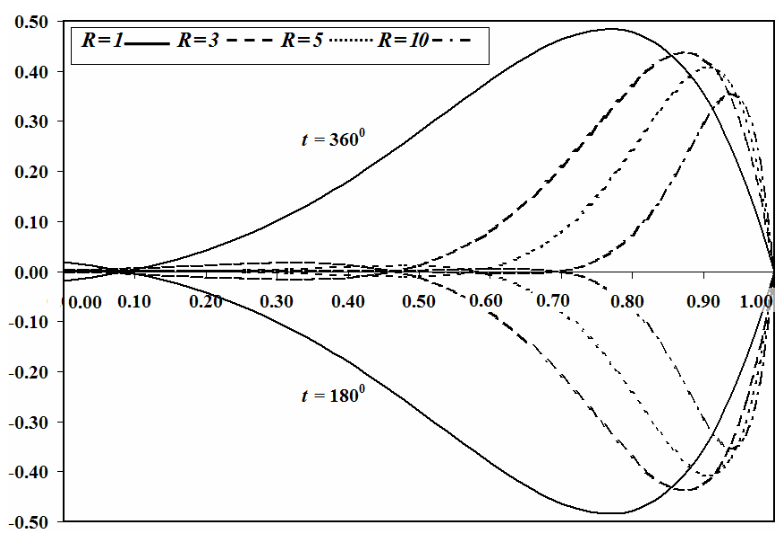
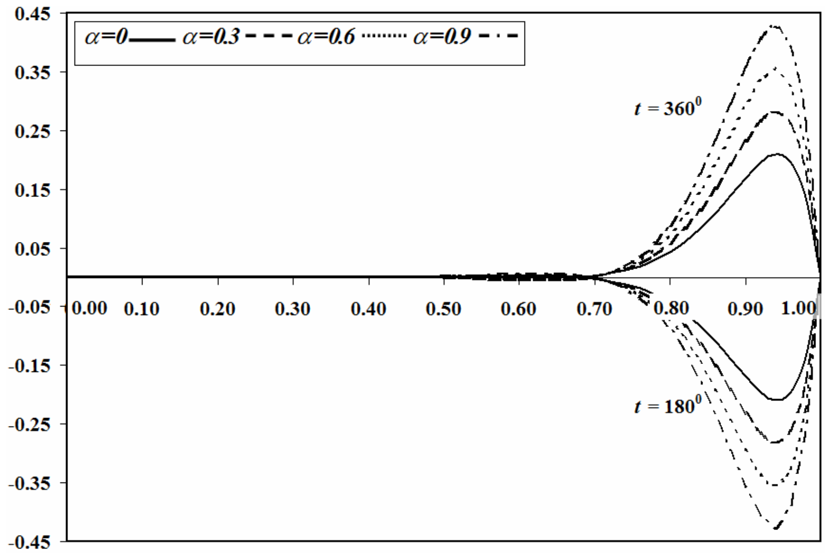


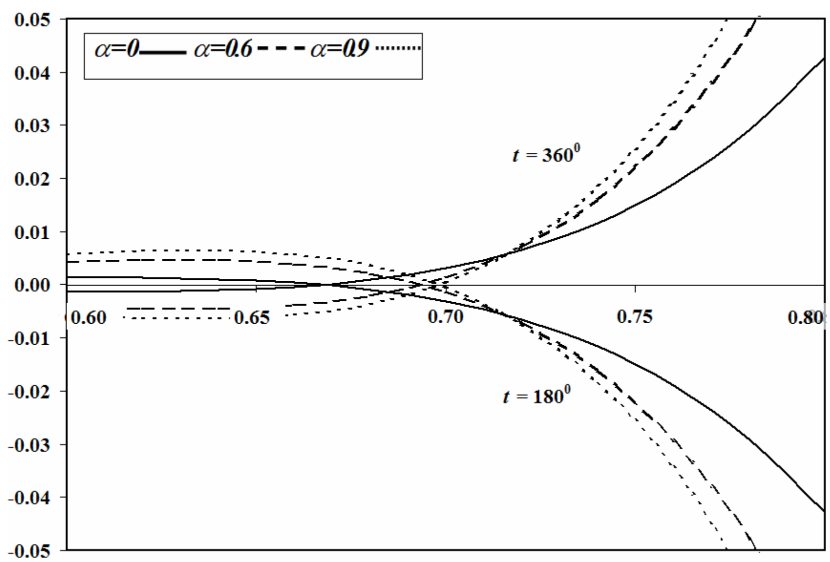
Figure 4.5 Effects of suction parameter R **on** $\mathrm{Re}\left(\mathbf{V}_1 \exp(-it)\right)$ **for** $\frac{x}{l} = 1, m = 1, S_t = 20$ **and**

$\alpha = 0.6$ **at time phase** $t = 180^0$ **and** $t = 360^0$

The effects of second grade fluid on the oscillatory axial velocity are as shown in Figure 4.6. Starting from $\alpha = 0$, the amplitude of the wave is found to increase substantially as the second grade parameter α increases (Figure 4.6 (a)). The penetration depth of the wave decreases nominally with the increase in second grade parameter. This fact is separately highlighted in (Figure 4.6 (b)).



(a)



(b)

Figure 4.6 Effects of second grade parameter α on $\mathrm{Re}\left(\mathbf{V}_1 \exp(-it)\right)$ for

$\frac{x}{l} = 1, m = 1, S_t = 20, R = 10$ **at time phase** $t = 180^0$ **and** $t = 360^0$**.**

4.5 Conclusion

The problem of unsteady oscillatory flow of a non-Newtonian second grade fluid in a channel has been studied. The analytical solution of the two-dimensional problem is obtained. The main findings are given below:

- The velocity profile for time independent part is found to increase with the increase of second grade fluid parameter.
- The amplitude of the oscillatory axial velocity increases significantly as the second grade fluid parameter increases.
- The penetration depth of the velocity decreases with increasing values of second grade fluid parameter.
- Suction decreases the amplitude of the oscillatory velocity.

Chapter 05

Oscillatory flow of second grade fluid in a cylindrical tube

After studying the oscillatory flow in a rectangular channel for non-Newtonian fluid, we extend our analysis of oscillatory flow from rectangular to cylindrical domain. In this chapter, we study the flow of a non-Newtonian oscillatory fluid inside a cylindrical tube with large suction. The flow in the tube is generated by uniform suction at the permeable walls and the oscillations in the velocity field are due to small amplitude time harmonic pressure waves. The problem is modeled in cylindrical coordinates using the equations of motion for second grade fluid. The effects of the non-dimensional second grade fluid parameter and suction parameter on the physical quantities of interest like amplitude of the oscillatory velocity and penetration depth are calculated and analyzed graphically.

5.1 Mathematical Formulation

We consider the laminar flow of a non-Newtonian second grade fluid inside a cylindrical tube. The length of the tube is L, width w and of radius a. One side of the tube is closed and the other side is open. The walls of the tube are permeable and suction is taking place from the porous walls of the tube with wall velocity v_w. We impose the symmetry condition about the axis of the tube and an axisymmetric flow can be realized by ignoring the variations in θ-direction. The basic geometry and flow configuration are shown in Figure 5.1.

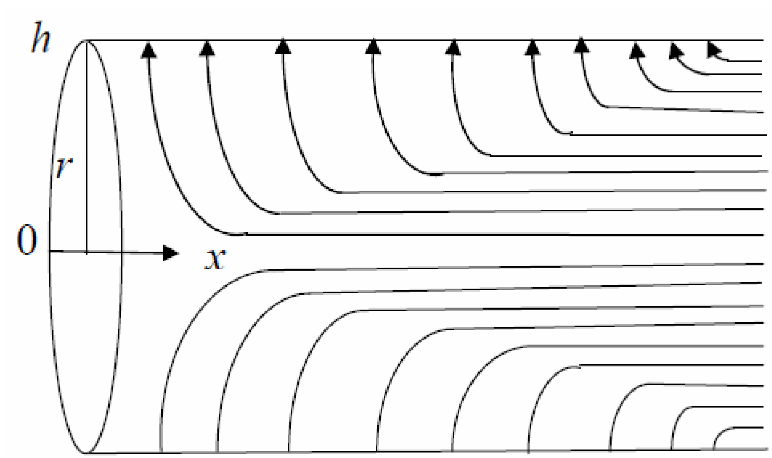


Figure 5.1 Geometry of the Problem

The governing equations of motion for unsteady, homogeneous second grade fluid in a non-dimensional form are adopted directly from section 4.1 of chapter 04:

$$\omega \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{5.1}$$

$$\begin{aligned} \rho\left[\omega \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right] &= -\nabla p + M\varepsilon\left[\frac{4}{3}\nabla(\nabla\cdot\mathbf{V}) - \nabla\times(\nabla\times\mathbf{V})\right] \\ &-\alpha\nabla\left[\mathbf{V}\cdot\nabla^2\mathbf{V} + \frac{1}{4}|\mathbf{A}_1|^2\right] + \alpha\left[\omega\nabla^2\mathbf{V}_t + \nabla^2(\nabla\times\mathbf{V})\times\mathbf{V}\right] \end{aligned} \tag{5.2}$$

and the boundary conditions for this problem are:

$$\begin{aligned} &u_r(x,1) = 1, \quad u_x(x,1) = 0; && \text{at} \quad r = 1, \\ &u_r(x,0) = 0, \quad \left(\frac{\partial u_x(x,r)}{\partial r}\right)_{r=0} = 0; && \text{at} \quad \mathrm{r} = 0, \end{aligned} \tag{5.3}$$

where $\mathbf{V}$ is the velocity field, u_r and u_x are the radial and axial components of the velocity vector $\mathbf{V}$, ρ is the density, p is the pressure, t in the subscript denotes the partial derivative with respect to time, ∇^2 is the Laplacian operator. The dimensionless numbers are, M Mach number, ε reciprocal of cross flow Reynolds number (suction parameter), ω the dimensionless wave frequency and α the dimensionless second grade fluid parameter. Mathematically, these quantities are defined as:

$$M = \frac{v_w}{a_s}, \ \ \varepsilon = \frac{1}{R} = \frac{\nu}{v_w a}, \ \ \omega = \frac{\omega_s a}{a_s}, \ \alpha = \frac{\alpha_1}{\rho_s a^2}$$

where a_s, ρ_s, v_w are the stagnation speed of the sound, the stagnation density and velocity at the wall respectively and ω_s is the frequency of the longitudinal pressure oscillation. Time harmonic pressure waves are responsible for oscillations in the velocity field; the pressure, density and velocity are perturbed in terms of small pressure wave amplitude δ as:

$$p(x, r, t) = p_0(x, r) + \delta p_1(x, r) e^{(-it)}, \tag{5.4}$$

$$\rho(x, r, t) = 1 + \delta\rho_1(x, r) e^{(-it)}, \tag{5.5}$$

$$\mathbf{V}(x, r, t) = M\mathbf{V_0}(x, r) + \delta\mathbf{V_1}(x, r) e^{(-it)}. \tag{5.6}$$

where $\delta = \dfrac{A}{\kappa p_s} \ll 1$ is dimensionless wave amplitude.

5.2 Steady (unperturbed) Part of the Velocity

To find the steady or unperturbed part of the velocity field, we make use of Eqs. (5.4) - (5.6) into Eqs. (5.1) and (5.2) and writing the leading order terms in δ, we get:

$$\nabla\cdot\mathbf{V_0}=0, \tag{5.7}$$

$$\begin{aligned}M^2\left(\mathbf{V_0}\cdot\nabla\right)\mathbf{V_0}&=-\nabla p_0+M^2\varepsilon\left[\frac{4}{3}\nabla\left(\nabla\cdot\mathbf{V_0}\right)-\nabla\times\left(\nabla\times\mathbf{V_0}\right)\right]\\&-\alpha M^2\left[\nabla\left[\mathbf{V_0}\cdot\nabla^2\mathbf{V_0}+\frac{1}{4}M^2\left\{4\frac{\partial u_r}{\partial r}\frac{\partial u_x}{\partial x}-\left(\frac{\partial u_x}{\partial r}+\frac{\partial u_r}{\partial x}\right)^2\right\}^2\right]-\nabla^2\left\{\left(\nabla\times\mathbf{V_0}\right)\times\mathbf{V_0}\right\}\right].\end{aligned} \tag{5.8}$$

where u_r and u_x are the axial and normal components of $\mathbf{V}_0$. We solve this system using similarity transformation defined by Jankowski and Majdalani [50]:

$$\Psi=-xF\left(r\right). \tag{5.9}$$

The continuity equation (5.7) is identically satisfied by using Eq. (5.9), and the momentum equation (5.8) can be transformed in the following form:

$$\begin{aligned}&M^2\left[\left(-\frac{F^2}{r^3}+\frac{FF'}{r^2}\right)\widehat{e_r}+\left(\frac{x}{r^3}FF'-\frac{x}{r^2}FF''+\frac{x}{r^2}F'^2\right)\widehat{e_x}\right]=-\frac{\partial p_0}{\partial r}\widehat{e_r}-\frac{\partial p_0}{\partial x}\widehat{e_x}\\&-M^2\varepsilon\left[\left(\frac{F'}{r^2}-\frac{F''}{r}\right)\widehat{e_r}+\left(\frac{x}{r^3}F'-\frac{x}{r^2}F''+\frac{x}{r}F'''\right)\widehat{e_x}\right]+\alpha M^2\left[\left(-\frac{4F^2}{r^5}+\frac{5FF'}{r^4}\right.\right.\\&-\frac{3FF''}{r^3}+\frac{FF'''}{r^2}-\frac{F'^2}{r^3}+\frac{F'F''}{r^2}-\frac{4x^2}{r^5}F'^2+\frac{5x^2}{r^4}F'F''-\frac{3x^2}{r^3}F'F'''+\frac{x^2}{r^2}F'F'^{v}\\&\left.\left.-\frac{x^2}{r^3}F''^2+\frac{x^2}{r^2}F''F'''\right)\widehat{e_r}+\left(\frac{2x}{r^4}F'^2-\frac{2x}{r^3}F'F''+\frac{2x}{r^2}F'F'''\right)\widehat{e_x}\right]+\alpha M^2\left[\left(\frac{9x^2}{r^5}F'^2\right.\right.\\&+\alpha M^2\left[\left(-\frac{14x^2}{r^4}F'F''+\frac{5x^2}{r^3}F'F'''-\frac{x^2}{r^2}F'F'^{v}+\frac{5x^2}{r^3}F''^2-\frac{3x^2}{r^2}F''F'''\right.\right.\\&\left.+\frac{2}{r^3}F'^2-\frac{2}{r^2}F'F''\right)\widehat{e_r}+\left(\frac{9x}{r^5}FF'-\frac{9x}{r^4}FF''+\frac{4x}{r^3}FF'''-\frac{x}{r^2}FF'^{v}\right.\\&\left.\left.-\frac{5x}{r^4}F'^2+\frac{6x}{r^3}F'F''-\frac{2x}{r^2}F'F'''-\frac{x}{r^2}F''^2\right)\widehat{e_x}\right]\end{aligned} \tag{5.10}$$

where $'$ in the superscript denotes the derivatives with respect to r. Cross differentiating the r^{th} and x^{th} components of Eq. (5.10) with respect to x and r respectively, and subtracting the resulting equations, we find:

$$F^{iv} + R\left(F'F'' - FF'''\right) + K\left(FF^{v} + F'F^{iv} - 2F''F'''\right) = 0 \tag{5.11}$$

and the boundary equations are:

$$F'(1) = F(0) = F''(0) = 0\ , \quad F(1) = 1. \tag{5.12}$$

The perturbation solution of Eq. (5.11) subject to Eq. (5.12) is given by

$$F(r) = r^2 + \bar{\alpha}\left[\frac{1}{2}r^2\left(1 - r^2\right)\right] + O\left(\bar{\alpha}^2\right). \tag{5.13}$$

where $\bar{\alpha} = \alpha R.$ We note that the results of Jankowski and Majdalani [50] for viscous fluid are recovered in the limit $\bar{\alpha}$ goes to zero.

5.3 Unsteady (perturbed) Part of the Velocity

Now we find the unsteady or perturbed part of the velocity filed. For this, we make use of Eqs. (5.5) and (5.6) into Eqs. (5.1) and (5.2) and collecting the terms of $O(\delta)$, the perturbed part of the velocity field can be written as:

$$-i\omega\rho_1 + \nabla\cdot\mathbf{V_1} = -M\nabla\cdot\left(\rho_1\mathbf{V_0}\right) \tag{5.14}$$

$$\begin{aligned} -i\omega\mathbf{V_1} = &-M\left[\nabla\left(\mathbf{V_0}\cdot\mathbf{V_1}\right) - \mathbf{V_0}\times\left(\nabla\times\mathbf{V_1}\right) - \mathbf{V_1}\times\left(\nabla\times\mathbf{V_0}\right)\right] \\ &-\nabla p_1 + M\varepsilon\left[\frac{4}{3}\nabla\left(\nabla\cdot\mathbf{V_1}\right) - \nabla\times\left(\nabla\times\mathbf{V_1}\right)\right] + \alpha M\left[\nabla\left\{\mathbf{V_0}\cdot\nabla\left(\nabla\cdot\mathbf{V_1}\right)\right\}\right. \\ &\left.-\nabla\left\{\mathbf{V_0}\cdot\nabla\times\left(\nabla\times\mathbf{V_1}\right)\right\} + \nabla\left\{\mathbf{V_1}\cdot\nabla\left(\nabla\cdot\mathbf{V_0}\right)\right\} - \nabla\left\{\mathbf{V_1}\cdot\nabla\times\left(\nabla\times\mathbf{V_0}\right)\right\}\right] \\ &-\alpha i\omega\left[\nabla\left(\nabla\cdot\mathbf{V_1}\right) - \nabla\times\left(\nabla\times\mathbf{V_1}\right)\right] + \alpha M\nabla\left[\nabla\cdot\left\{\left(\nabla\times\mathbf{V_1}\right)\times\mathbf{V_0} + \left(\nabla\times\mathbf{V_0}\right)\times\mathbf{V_1}\right\}\right] \\ &-\alpha M\nabla\times\left[\nabla\times\left\{\left(\nabla\times\mathbf{V_1}\right)\times\mathbf{V_0} + \left(\nabla\times\mathbf{V_0}\right)\times\mathbf{V_1}\right\}\right] \end{aligned} \tag{5.15}$$

The boundary conditions satisfied by the velocity $\mathbf{V_1}$ are the no-slip boundary condition $\mathbf{V}_1(x,\ 1) = 0$ at the wall and symmetry condition $\frac{\partial\mathbf{u}_1(x,\ 0)}{\partial r} = 0$ at the midsection of the channel. To proceed further, we decompose the perturbed velocity field $\mathbf{V}_1(x,\ r)$ into an acoustic, irrotational, pressure-driven part and vortical, rotational, vorticity-driven part by using a fundamental theorem vector analysis that was first studied by Stokes [90] and demonstrated by Blumenthal in 1905 as:

$$\mathbf{V_1} = \widehat{\mathbf{V}} + \widetilde{\mathbf{V}}, \tag{5.16}$$

with the properties

$$\nabla \times \widehat{\mathbf{V}} = 0\ ,\ \ \nabla \cdot \widetilde{\mathbf{V}} = 0\ ,\ \ p_1 = \widehat{p},\ \ \rho_1 = \widehat{\rho}, \tag{5.17}$$

Using Eqs. (5.16) and (5.17), the acoustic and vortical parts of Eqs. (5.14) and (5.15) can now be separated as follows:

5.3.1 Pressure-driven Part ($\widehat{\mathbf{V}}$)

$$-i\omega\widehat{\rho} + \nabla \cdot \widehat{\mathbf{V}} = -M\nabla \cdot \left(\widehat{\rho}\mathbf{V_0}\right) \tag{5.18}$$

$$\begin{aligned} -i\omega\widehat{\mathbf{V}} &= -\nabla\widehat{p} + \frac{4}{3}M\varepsilon\nabla\left(\nabla \cdot \widehat{\mathbf{V}}\right) - M\left[\nabla\left(\widehat{\mathbf{V}} \cdot \mathbf{V_0}\right) - \widehat{\mathbf{V}} \times \left(\nabla \times \mathbf{V_0}\right)\right] \\ &+ \alpha M\left[\nabla\left\{\mathbf{V_0} \cdot \nabla\left(\nabla \cdot \widehat{\mathbf{V}}\right)\right\} + \nabla\left\{\widehat{\mathbf{V}} \cdot \nabla\left(\nabla \cdot \mathbf{V_0}\right)\right\} - \nabla\left\{\widehat{\mathbf{V}} \cdot \nabla \times \left(\nabla \times \mathbf{V_0}\right)\right\}\right] \\ &- \alpha i\omega\nabla\left(\nabla \cdot \widehat{\mathbf{V}}\right) + \alpha M\nabla\left[\nabla \cdot \left(\nabla \times \mathbf{V_0}\right) \times \widehat{\mathbf{V}}\right] - \alpha M\nabla \times \left[\nabla \times \left(\nabla \times \mathbf{V_0}\right) \times \widehat{\mathbf{V}}\right] \end{aligned} \tag{5.19}$$

The axial part of acoustic pressure and velocity are central in our discussion of oscillatory flow. For perfect gas undergoing isentropic oscillations, we know that $\widehat{\rho} = \widehat{p}$. Using this fact, the velocity $\widehat{\mathbf{V}}$ u can be eliminated from Eqs. (5.18) and (5.19). The resulting equation in $\widehat{p}$ can readily be solved up to $O\left(M\right)$ by using separation of variables. The axial acoustic pressure $\widehat{p}$ and velocity $\widehat{\mathbf{V}}$ are presented as:

$$\widehat{p} = \cos\left(\frac{\omega x}{\sqrt{1-\alpha\omega^2}}\right) + O\left(M\right), \tag{5.20}$$

$$\widehat{\mathbf{V}} = i\sqrt{1-\alpha\omega^2}\,\sin\left(\frac{\omega x}{\sqrt{1-\alpha\omega^2}}\right) + O\left(M\right). \tag{5.21}$$

We note that the effect of the second grade parameter is to modify the frequency ω and the amplitude unity of the acoustic wave in the viscous fluid. The modified frequency and amplitude are now $\dfrac{\omega}{\sqrt{1-\alpha\omega^2}}$ and $\sqrt{1-\alpha\omega^2}$ respectively.

5.3.2 Vorticity-driven Part ($\widetilde{\mathbf{V}}$)

$$\nabla\cdot\widetilde{\mathbf{V}}=0 \tag{5.22}$$

$$\begin{aligned}
-i\omega\widetilde{\mathbf{V}} &= -M\varepsilon\nabla\times\left(\nabla\times\widetilde{\mathbf{V}}\right)-M\left[\nabla\left(\widetilde{\mathbf{V}}\cdot\mathbf{V_0}\right)-\widetilde{\mathbf{V}}\times\left(\nabla\times\mathbf{V_0}\right)-\mathbf{V_0}\times\left(\nabla\times\widetilde{\mathbf{V}}\right)\right]\\
&-\alpha M\left[\nabla\left\{\mathbf{V_0}\cdot\nabla\times\left(\nabla\times\widetilde{\mathbf{V}}\right)\right\}+\nabla\left\{\widetilde{\mathbf{V}}\cdot\nabla\left(\nabla\cdot\mathbf{V_0}\right)\right\}-\nabla\left\{\widetilde{\mathbf{V}}\cdot\nabla\times\left(\nabla\times\mathbf{V_0}\right)\right\}\right]\\
&+\alpha i\omega\nabla\times\left(\nabla\times\widetilde{\mathbf{V}}\right)+\alpha M\nabla\left[\nabla\cdot\left(\nabla\times\widetilde{\mathbf{V}}\right)\times\mathbf{V_0}+\nabla\cdot\left(\nabla\times\mathbf{V_0}\right)\times\widetilde{\mathbf{V}}\right]\\
&-\alpha M\nabla\times\left[\nabla\times\left(\nabla\times\widetilde{\mathbf{V}}\right)\times\mathbf{V_0}+\nabla\times\left(\nabla\times\mathbf{V_0}\right)\times\widetilde{\mathbf{V}}\right]
\end{aligned} \tag{5.23}$$

Writing $\widetilde{\mathbf{V}}$ in components $\widetilde{\mathbf{V}}=\left(\widetilde{u_r},\ \widetilde{u_x}\right)$, assuming that the ratio of the normal to axial velocity is of $O(M)$, the component $\widetilde{u_r}$ can be neglected at the leading order. Now, using $\mathbf{V_0}=\left(\frac{F}{r},-\frac{x}{r}F'\right)$ into Eq. (5.23), considering axial component ($\widetilde{u_r}=0$) and writing $\widetilde{u_x}=X(x)Y(r)$, we obtain

$$\begin{aligned}
&\varepsilon\frac{d^2Y}{dr^2}-\left(\frac{F}{r}-\frac{\varepsilon}{r}\right)\frac{dY}{dr}+\left[iS_t+\left(1+x\frac{X'}{X}\right)\frac{F'}{r}\right]Y=\alpha\left[-\frac{F}{r}\frac{d^3Y}{dr^3}\right.\\
&+\left\{iS_t+\frac{F}{r^2}+\left(1-x\frac{X'}{X}\right)\frac{F'}{r}\right\}\frac{d^2Y}{dr^2}+\left\{\frac{iS_t}{r}-\frac{F}{r^3}+\frac{F'}{r^2}\left(2+x\frac{X'}{X}\right)-\frac{F''}{r}\right\}\frac{dY}{dr}\\
&\left.+\left(1+x\frac{X'}{X}\right)\left\{\frac{F'}{r^3}-\frac{F''}{r^2}+\frac{F'''}{r}\right\}Y\right].
\end{aligned} \tag{5.24}$$

where $S_t=\frac{\omega}{M}$ is the dimensionless Strouhal number. The x dependent function in Eq. (5.24) is written as:

$$x\frac{X'}{X}=k_n\ ,$$

where k_n are so far unknown eigenvalues and will be determined using no slip boundary condition. Integrating the above equation, we get the eigenfunctions $X(x)=x^{k_n}$. The axial component of acoustic velocity can be expressed in terms of these eigenfunctions as

$$\widetilde{\mathbf{V}}(x,\ r)=\widetilde{u_x}(x,\ r)=\sum_n c_n x^{k_n} Y_n(r). \tag{5.25}$$

At $r=1,$ the vortical solution is equal to the negative of the acoustic solution giving:

$$\widehat{\mathbf{V}}(x,\ 1)=-\widetilde{\mathbf{V}}(x,\ 1),$$

Substituting Eq. (5.25) in the above condition, we concludes that

$$k_n=2n+1\ ,\ \ c_n=-i\sqrt{1-k\omega^2}\,\frac{(-1)^n}{(2n+1)!}\left(\frac{\omega}{\sqrt{1-k\omega^2}}\right)^{2n+1}\ ,\ \ Y_n(1)=1$$

Using these values of k_n and c_n, Eqs. (5.25) and (5.24) reduce to the following form:

$$\widetilde{\mathbf{V}}(x,r)=-i\sqrt{1-\alpha\omega^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\omega x}{\sqrt{1-\alpha\omega^2}}\right)^{2n+1}Y_n(r). \tag{5.26}$$

$$\varepsilon\frac{d^2Y}{dr^2}-\left(\frac{F}{r}-\frac{\varepsilon}{r}\right)\frac{dY}{dr}+\left[iS_t+(2n+2)\frac{F'}{r}\right]Y=\alpha\left[-\frac{F}{r}\frac{d^3Y}{dr^3}\right. \tag{5.27}$$

$$+\left\{iS_t+\frac{F}{r^2}-(2n)\frac{F'}{r}\right\}\frac{d^2Y}{dr^2}+\left\{\frac{iS_t}{r}-\frac{F}{r^3}+\frac{F'}{r^2}(2n+3)-\frac{F''}{r}\right\}\frac{dY}{dr}$$

$$\left.+(2n+2)\left\{\frac{F'}{r^3}-\frac{F''}{r^2}+\frac{F'''}{r}\right\}Y\right].$$

where Y satisfies the following boundary conditions

$$Y(1)=1\ ,\ \ Y'(0)=0. \tag{5.28}$$

In order to determine the complete solution of the vortical part, we need to find the solution of Y from Eq. (5.27) using the boundary conditions (5.28). The perturbation method and WKB approximations are used to reach the solution. Expressing the solution in terms of regular perturbation expansion in α, solving the leading order term using standard WKB approximation, taking two terms of perturbation expansion and omitting the details of calculation, we arrive at:

$$Y=\left[1+\alpha\int_1^y\left[\frac{(2n+2)\left\{\frac{F'}{r^2}-\frac{F''}{r}+F'''\right\}\left(1+\frac{F^2}{8iS_t\varepsilon}\right)}{\left(-4iS_t\varepsilon\right)^{\frac{1}{2}}}\right]d\eta\right]e^{\frac{1}{2\varepsilon}\int_1^r\frac{\left(\sqrt{F^2-4i\varepsilon S_t\eta^2}+F\right)}{\eta}d\eta} \tag{5.29}$$

Substitution of Eq. (5.29) into Eq. (5.26) gives the following expression for vortical velocity:

$$\widetilde{\mathbf{V}}(x,\ y)=-i\sqrt{1-\alpha\omega^2}\sin\left(\frac{\omega x}{\sqrt{1-\alpha\omega^2}}\right)$$
$$.\left[1+\alpha\int_1^y\left[\frac{(2n+2)\left\{\frac{F'}{r^2}-\frac{F''}{r}+F'''\right\}\left(1+\frac{F^2}{8iS_t\varepsilon}\right)}{\left(-4iS_t\varepsilon\right)^{\frac{1}{2}}}\right]d\eta\right]e^{\frac{1}{2\varepsilon}\int_1^r\frac{\left(\sqrt{F^2-4i\varepsilon S_t\eta^2}+F\right)}{\eta}d\eta} \tag{5.30}$$

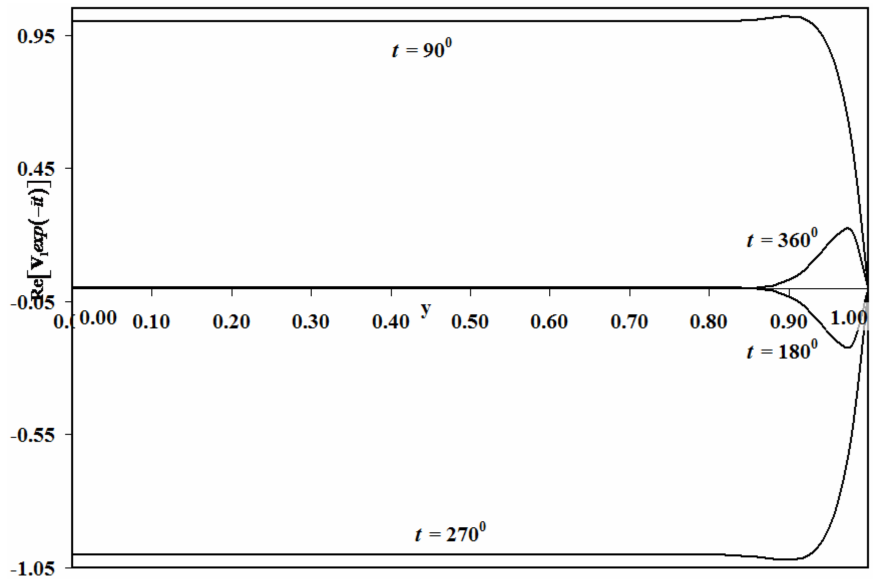
After calculating both acoustic and vortical velocities, we revert back to Eq. (5.16). Thus, Eqs. (5.16), (5.21) and (5.30) together gives the time dependent oscillatory velocity up to $O(\alpha)$ as:

$$\mathbf{V}_1=i\left[\sin(\omega x)(1-Y_0)-\alpha\left\{\sin(\omega x)Y_1-\left(\frac{1}{2}\omega^3x-\frac{1}{2}\omega^2\sin(\omega x)\right)(1-Y_0)\right\}\right] \tag{5.31}$$

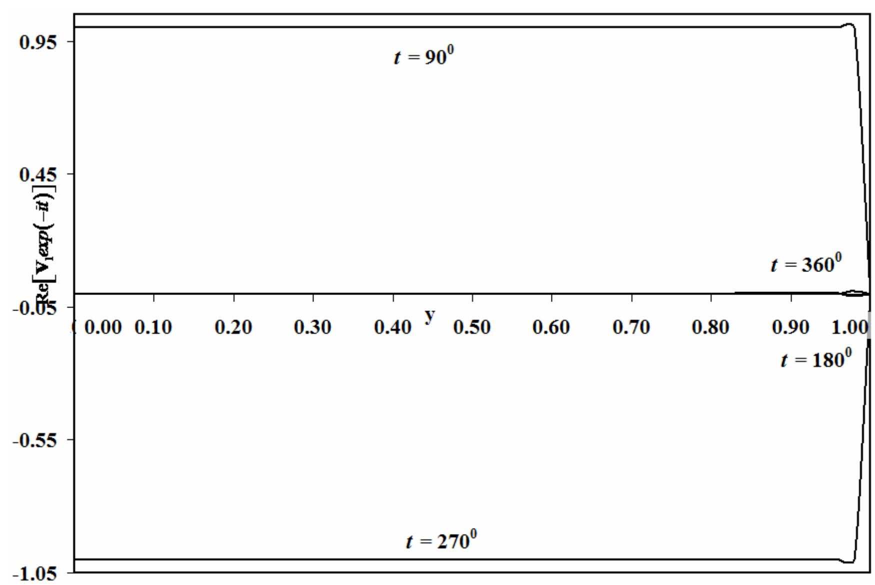
where Y_0 and Y_1 are the leading order and first order solutions of Eq. (5.27). The integrals in this solution cannot be calculated analytically and hence numerical integration is performed. Mathematically, the results for viscous fluid are recovered by taking $\alpha=0$ which matches with Jankowski and Majdalani [50].

5.4 Discussion

We recall that the essence of the oscillatory flow is the investigation of oscillation effects on the fluid flow. The time independent part being of lesser interest, we will concentrate on the time dependent part of the velocity field established in Eq. (5.31). We plot the real part of oscillatory axial velocity $\mathbf{V}_1\exp(-it)$ for viscous fluid $\alpha=0$ in Figures 5.2 (a) and (b) at four equiphased timelines. We find that by increasing the values of suction parameter and decreasing the values of Strouhal number, the amplitude and penetration depth of the wave dampens. The graphs and observations match well with Jankowski and Majdalani [50].



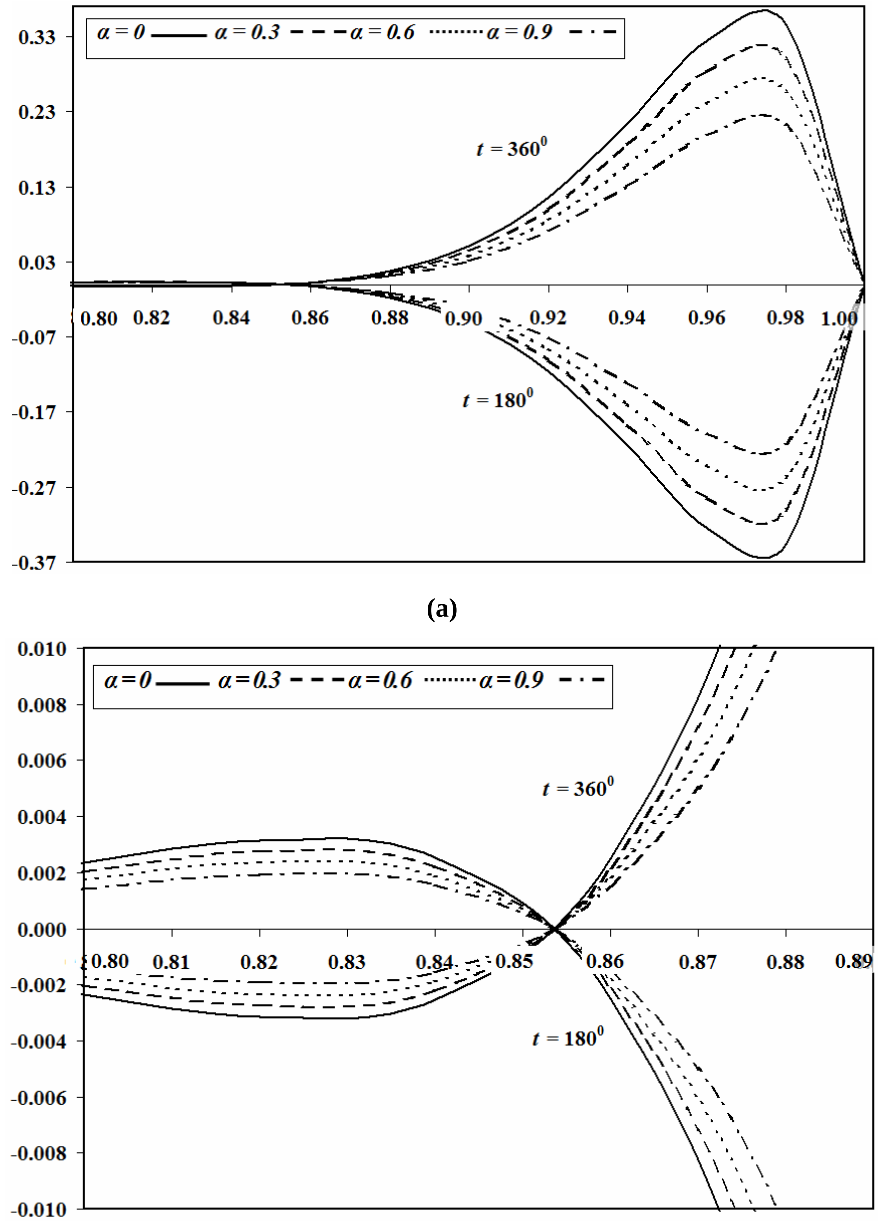
(a) $R = 20$ and $S_t = 50$



(b) $R = 200$ and $S_t = 5$

Figure 5.2 Plot of $\mathrm{Re}\left(\mathbf{V}_1 \exp(-it)\right)$ **against** y **for viscous fluid** $\alpha = 0$ **and** $\frac{x}{l} = 1, m = 1$ **at dimensionless time** t.

The effects of second grade fluid parameter on the oscillatory axial velocity are presented in Figure 5.3. We clearly observe that the amplitude of oscillations decreases and the penetration depth remains the same as the second grade parameter α increases. These observations are highlighted in Figures 5.3 (a) and (b).



(a)

(b)

Figure 5.3 Effects of second grade parameter α on $\mathrm{Re}\left(\mathbf{V}_1 \exp(-it)\right)$ for

$$\frac{x}{l}=1, m=1, S_t=50, R=20 \text{ at time } t=180^0 \text{ and } t=360^0$$

The effects of second grade fluid parameter on the oscillatory axial velocity for large value of suction parameter and small value of Strouhal number are plotted in Figure 5.4. Starting from $\alpha = 0$, we conclude that as the second grade parameter α increases, the amplitude of the wave increases while the penetration depth remains the same. The amplitude decreases faster in this case by increasing the values of second grade parameter than from viscous fluid.

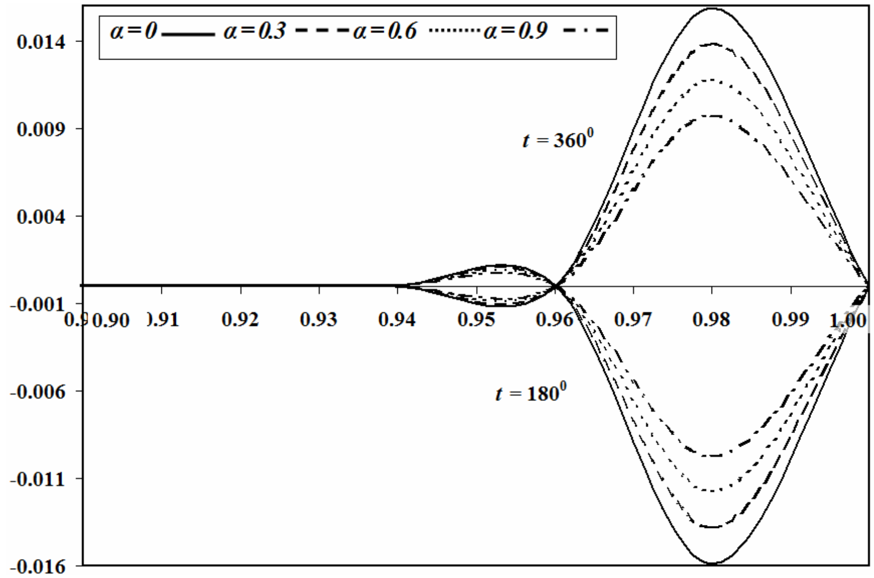


Figure 5.4 Effects of second grade parameter α **on** $\mathrm{Re}\left(\mathbf{V}_1 \exp(-it)\right)$ **for** $\frac{x}{l}=1, \quad m=1, \quad S_t=5,$ $R=200$ **at time** $t=180^0$ **and** $t=360^0$

5.5 Conclusions

In this chapter, the problem of oscillatory flow of a non-Newtonian second grade fluid inside a cylindrical tube has been discussed. The main conclusions of this chapter are given below:

- The amplitude of the oscillatory axial velocity decreases with the increase of second grade fluid parameter.
- The penetration depth of the velocity remains the same by increasing values of second grade fluid parameter.
- The amplitude and penetration depth of the velocity dampen as the suction parameter increases.

Chapter 06

Analytic solution for oscillatory flow in a channel for Jeffrey fluid

In this chapter, we discuss oscillatory flow in non-Newtonian Jeffrey fluid inside rectangular channel with small suction. The main objective is to investigate the viscoelastic behaviour of non-Newtonian fluid subject to time harmonic oscillations. This fluid model is considered because it uses the simple material derivatives and predicts relaxation/retardation time effects which are incorporated through the ratio of relaxation to retardation time parameter and Deborah number. These properties are particularly important while studying the viscoelastic effects in polymer industry. Another important reason for the consideration of this model is that the oscillatory shear motion is normally used to understand the behaviour of viscoelastic fluids. In our case, these strains are generated from fluctuations in the suction rate at the walls of the channel. It is further well known that in response to applied oscillatory strain, the stress will be in phase for purely elastic body and out of phase by 90^o in case of purely viscous fluids. In between these values, lies the viscoelastic fluid, determined by the value of Deborah number. It is rather important to mention that the most of the instruments designed to measure the viscoelasticity of the fluids are for oscillatory strain. The governing equations of motion for Jeffrey fluid are modeled and analytical solution of the time independent axial velocity, and time dependent oscillatory axial velocity in the channel is obtained by using small amplitude and small Deborah number assumptions. To understand the characteristics of non-Newtonian fluids, the effects of retardation time and Deborah number on the oscillatory velocity, its amplitude and penetration depth are analyzed. The effects of viscoelastic parameters on these quantities are analyzed.

6.1 Mathematical formulation

We consider the flow of a non-Newtonian Jeffrey fluid between two infinite parallel porous plates of length L and width w, separated by a distance $2h$. One side of the channel is open and the other side is close. Suction is taking place from the porous plates with a uniform velocity v_w. With the assumption $w >> h$, we can treat the problem as two-dimensional. The symmetry at the centre of the channel reduces the solution domain to one half of its original size. The small variation in the suction give rise to longitudinal pressure oscillations of amplitude A. The geometry of the problem is shown in Figure 6.1.

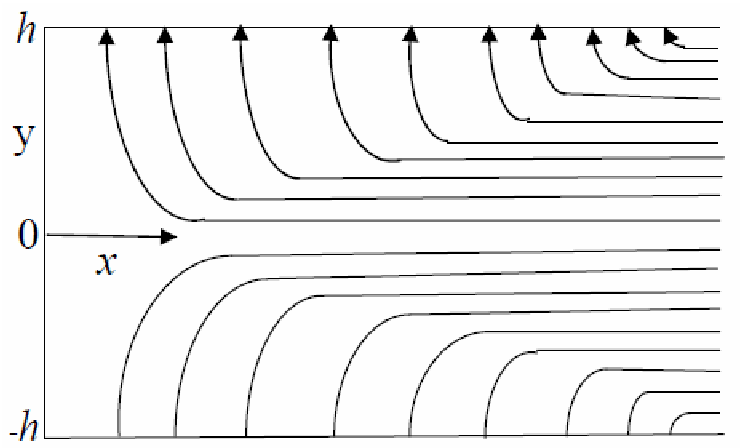


Figure 6.1 Geometry and flow configuration of the problem

The equations of motion that governs the flow of a non-Newtonian Jeffrey fluid are:

The continuity equation

$$\frac{\partial \rho^*}{\partial t^*} + \operatorname{div}\left(\rho^* \mathbf{V}^*\right) = 0 \tag{6.1}$$

The momentum equation

$$\rho^* \frac{d\mathbf{V}^*}{dt^*} = \operatorname{div}\mathbf{T}^* + \rho^* \mathbf{b} \tag{6.2}$$

where $\mathbf{V}^*\left(u^*, v^*\right)$ is the velocity vector, ρ^* is the density, $\mathbf{b}$ is the body force and $\mathbf{T}^*$ is the Cauchy stress tensor for Jeffrey fluid. The constitutive equation for Jeffrey fluid is:

$$\mathbf{T}^* = -p^* \boldsymbol{I} + \mathbf{S}^* \tag{6.3}$$

where,

$$\mathbf{S}^* = \frac{\mu}{1+\lambda_1}\left(\dot{\gamma} + \lambda_2 \ddot{\gamma}\right) \tag{6.4}$$

In these expressions, p^* is pressure, $\boldsymbol{I}$ is identity tensor, μ is dynamic viscosity. λ_1 is ratio of the relaxation to retardation time, λ_2 is the relaxation time and $\dot{\gamma}$ is shear rate. The dot over the quantities denotes the material differentiation and defined as:

$$\begin{aligned} \dot{\gamma} &= \left(grad\,\mathbf{V}^*\right) + \left(grad\,\mathbf{V}^*\right)^T \\ \ddot{\gamma} &= \frac{d}{dt}\left(\dot{\gamma}\right) \end{aligned} \tag{6.5}$$

Eqs. (6.1) and (6.2) after using Eqs. (6.3) - (6.5), in components form can be written as:

$$\frac{\partial \rho^*}{\partial t^*}+\frac{\partial\left(\rho^* u^*\right)}{\partial x^*}+\frac{\partial\left(\rho^* v^*\right)}{\partial y^*}=0 \tag{6.6}$$

The x^{th} momentum equation

$$\begin{aligned}
&\frac{\partial\left(\rho^* u^*\right)}{\partial t^*}+u^* \frac{\partial\left(\rho^* u^*\right)}{\partial x^*}+v^* \frac{\partial\left(\rho^* u^*\right)}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{\mu}{1+\lambda_1}\left\{\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}+\frac{1}{3} \frac{\partial}{\partial x^*}\left(\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}\right)\right\} \\
&+\frac{\mu \lambda_2}{1+\lambda_1}\left[\frac{\partial}{\partial t^*}\left\{\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}+\frac{\partial}{\partial x^*}\left(\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}\right)\right\}+2 \frac{\partial}{\partial x^*}\left(v^* \frac{\partial^2 u^*}{\partial x^* \partial y^*}+u^* \frac{\partial^2 u^*}{\partial x^{*2}}\right)\right. \\
&\left.+\frac{\partial}{\partial y^*}\left\{v^* \frac{\partial}{\partial y^*}\left(\frac{\partial v^*}{\partial x^*}+\frac{\partial u^*}{\partial y^*}\right)+u^* \frac{\partial}{\partial x^*}\left(\frac{\partial v^*}{\partial x^*}+\frac{\partial u^*}{\partial y^*}\right)\right\}\right]
\end{aligned} \tag{6.7}$$

The y^{th} momentum equation

$$\begin{aligned}
&\frac{\partial\left(\rho^* v^*\right)}{\partial t^*}+u^* \frac{\partial\left(\rho^* v^*\right)}{\partial x^*}+v^* \frac{\partial\left(\rho^* v^*\right)}{\partial y^*}=-\frac{\partial p^*}{\partial y^*}+\frac{\mu}{1+\lambda_1}\left\{\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial y^{*2}}+\frac{1}{3} \frac{\partial}{\partial y^*}\left(\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}\right)\right\} \\
&+\frac{\mu \lambda_2}{1+\lambda_1}\left[\frac{\partial}{\partial t^*}\left\{\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial y^{*2}}+\frac{\partial}{\partial y^*}\left(\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}\right)\right\}+2 \frac{\partial}{\partial y^*}\left(u^* \frac{\partial^2 v^*}{\partial x^* \partial y^*}+v^* \frac{\partial^2 v^*}{\partial x^{*2}}\right)\right. \\
&\left.+\frac{\partial}{\partial x^*}\left\{u^* \frac{\partial}{\partial x^*}\left(\frac{\partial v^*}{\partial x^*}+\frac{\partial u^*}{\partial y^*}\right)+v^* \frac{\partial}{\partial y^*}\left(\frac{\partial v^*}{\partial x^*}+\frac{\partial u^*}{\partial y^*}\right)\right\}\right]
\end{aligned} \tag{6.8}$$

where (*) denotes the dimensional variables. The Eqs. (6.6) - (6.8) are in dimensional form and can be made dimensionless by introducing the following dimensionless variables:

$$u=\frac{u^*}{a_s},\ v=\frac{v^*}{a_s},\ x=\frac{x^*}{h},\ \ y=\frac{y^*}{h},\ \ p=\frac{p^*}{\kappa p_S},\ \ \rho=\frac{\rho^*}{\rho_S},\ \ t=t^* \omega_S. \tag{6.9}$$

where a_s, p_s, ρ_s, κ and ω_s are the stagnation speed of sound, stagnation pressure, stagnation density, the ratio of specific heats and frequency of the longitudinal pressure oscillations respectively and $\mathbf{V}(u,v)$ is the non-dimensional velocity vector. The Eqs. (6.6) – (6.8) after using Eq. (6.9) in non-dimensional form can be expressed as:

The continuity equation

$$\omega \frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0 \tag{6.10}$$

The x^{th} momentum equation

$$\omega\frac{\partial(\rho u)}{\partial t}+u\frac{\partial(\rho u)}{\partial x}+v\frac{\partial(\rho u)}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\varepsilon M}{1+\lambda_1}\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}$$
$$+\frac{D_e}{1+\lambda_1}\left[\omega\frac{\partial}{\partial t}\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}+2\frac{\partial}{\partial x}\left(v\frac{\partial^2 u}{\partial x\partial y}+u\frac{\partial^2 u}{\partial x^2}\right)\right. \tag{6.11}$$
$$\left.+\frac{\partial}{\partial y}\left\{v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\}\right]$$

The y^{th} momentum equation

$$\omega\frac{\partial(\rho v)}{\partial t}+u\frac{\partial(\rho v)}{\partial x}+v\frac{\partial(\rho v)}{\partial y}=-\frac{\partial p}{\partial y}+\frac{\varepsilon M}{1+\lambda_1}\left\{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}$$
$$+\frac{D_e}{1+\lambda_1}\left[\omega\frac{\partial}{\partial t}\left\{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}+2\frac{\partial}{\partial y}\left(u\frac{\partial^2 v}{\partial x\partial y}+v\frac{\partial^2 v}{\partial x^2}\right)\right. \tag{6.12}$$
$$\left.+\frac{\partial}{\partial x}\left\{u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\}\right]$$

The boundary conditions for this problem are:

$$\begin{aligned} v=0,\quad \frac{\partial u}{\partial y}=0; &\qquad \text{at}\quad y=0,\\ v=M,\quad u=0; &\qquad \text{at}\quad y=1, \end{aligned} \tag{6.13}$$

where M,ε,ω and D_e are Mach number, the reciprocal of cross flow Reynolds number (suction parameter), the dimensionless wave frequency and the dimensionless Deborah number respectively. These quantities are defined as:

$$M=\frac{v_w}{a_s},\ \ \varepsilon=\frac{1}{R_e}=\frac{\upsilon}{v_w h},\ D_e=\frac{\upsilon\lambda_2}{h^2},\omega=\frac{\omega_s h}{a_s}$$

The oscillations are produced due to small amplitude time harmonic pressure waves. We perturb the velocity, pressure and density in terms of small pressure wave amplitude δ as:

$$u(x,\ y,\ t)=Mu_0(x,\ y)+\delta u_1(x,\ y)e^{(-it)}, \tag{6.14}$$

$$v(x,\ y,\ t)=Mv_0(x,\ y)+\delta v_1(x,\ y)e^{(-it)}, \tag{6.15}$$

$$p(x,\ y,\ t)=p_0(x,\ y)+\delta p_1(x,\ y)e^{(-it)}, \tag{6.16}$$

$$\rho(x,\ y,\ t) = 1 + \delta\rho_1(x,\ y)e^{(-it)}, \tag{6.17}$$

where $\delta = \dfrac{A}{\kappa p_s} \ll 1$ is the dimensionless wave amplitude. The leading order time independent velocity and time dependent oscillatory velocity are calculated in the subsequent analysis.

6.2 Time Independent Solution

Making use of Eqs. (6.14) - (6.17) in Eqs. (6.10) - (6.12) and writing leading order terms in δ, we get:

$$M\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) = 0, \tag{6.18}$$

$$\begin{aligned} M^2\left(u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y}\right) &= -\frac{\partial p_0}{\partial x} + \frac{\varepsilon M^2}{1+\lambda_1}\left\{\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right)\right\} \\ &+ \frac{D_e M^2}{1+\lambda_1}\left[2\frac{\partial}{\partial x}\left(u_0\frac{\partial^2 u_0}{\partial x^2} + v_0\frac{\partial^2 u_0}{\partial x\partial y}\right) + \frac{\partial}{\partial y}\left\{u_0\frac{\partial}{\partial x}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\right) + v_0\frac{\partial}{\partial y}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\right)\right\}\right] \end{aligned} \tag{6.19}$$

$$\begin{aligned} M^2\left(u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y}\right) &= -\frac{\partial p_0}{\partial y} + \frac{\varepsilon M^2}{1+\lambda_1}\left\{\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2} + \frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right)\right\} \\ &+ \frac{D_e M^2}{1+\lambda_1}\left[\frac{\partial}{\partial x}\left\{u_0\frac{\partial}{\partial x}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\right) + v_0\frac{\partial}{\partial y}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\right)\right\} + 2\frac{\partial}{\partial y}\left(u_0\frac{\partial^2 v_0}{\partial x\partial y} + v_0\frac{\partial^2 v_0}{\partial y^2}\right)\right] \end{aligned} \tag{6.20}$$

We solve this system using similarity transformation defined by Jankowski and Majdalani [21]:

$$\Psi = -xF(y). \tag{6.21}$$

Using this stream function given in Eq. (6.21), the continuity equation (6.18) is identically satisfied. Cross differentiating Eq. (6.19) and Eq. (6.20) with respect to y and x respectively, subtract the resulting equations, we arrive at the following equation:

$$F'''' + R(1+\lambda_1)\left(F'F'' - FF'''\right) + \overline{D_e}\left(FF^{v} + F'F'''' - 2F''F'''\right) = 0 \tag{6.22}$$

and the boundary conditions

$$F'(1) = F(0) = F''(0) = 0\ ,\ \ F(1) = 1. \tag{6.23}$$

where $\overline{D_e} = RD_e$. The two terms perturbation solution of Eqs. (6.22) and (6.23) is easily expressed as:

$$F\left(y\right)=\frac{1}{2}y\left(3-y^{2}\right)+\overline{D_{e}}\left[\frac{3}{20}y\left(y^{2}-1\right)^{2}\right]+O\left(\overline{D_{e}}^{2}\right): \tag{6.24}$$

The $O\left(\overline{D_{e}}\right)$ term on the right side gives the correction term for the Jeffrey fluid.

6.3 Time Dependent Oscillatory Velocity

Using Eqs. (6.14) - (6.17) in Eqs. (6.10) - (6.12) and collecting the terms of $O\left(\delta\right)$, the time dependent part of the velocity field can be expressed as:

$$-i\omega\rho_{1}+M\left(\frac{\partial\left(\rho_{1}u_{0}\right)}{\partial x}+\frac{\partial\left(\rho_{1}v_{0}\right)}{\partial y}\right)+\frac{\partial u_{1}}{\partial x}+\frac{\partial v_{1}}{\partial y}=0, \tag{6.25}$$

$$\begin{aligned}&-i\omega u_{1}+M\left(u_{0}\frac{\partial u_{1}}{\partial x}+u_{1}\frac{\partial u_{0}}{\partial x}+v_{0}\frac{\partial u_{1}}{\partial y}+v_{1}\frac{\partial u_{0}}{\partial y}\right)=-\frac{\partial p_{1}}{\partial x}+\frac{\varepsilon M}{1+\lambda_{1}}\left\{\frac{\partial^{2}u_{1}}{\partial x^{2}}+\frac{\partial^{2}u_{1}}{\partial y^{2}}+\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u_{1}}{\partial x}+\frac{\partial v_{1}}{\partial y}\right)\right\}\\&-\frac{D_{e}i\omega}{1+\lambda_{1}}\left\{\frac{\partial^{2}u_{1}}{\partial x^{2}}+\frac{\partial^{2}u_{1}}{\partial y^{2}}+\frac{\partial}{\partial x}\left(\frac{\partial u_{1}}{\partial x}+\frac{\partial v_{1}}{\partial y}\right)\right\}+\frac{D_{e}M}{1+\lambda_{1}}\left[2\frac{\partial}{\partial x}\left(u_{0}\frac{\partial^{2}u_{1}}{\partial x^{2}}+u_{1}\frac{\partial^{2}u_{0}}{\partial x^{2}}+v_{0}\frac{\partial^{2}u_{1}}{\partial x\partial y}+v_{1}\frac{\partial^{2}u_{0}}{\partial x\partial y}\right)\right.\\&\left.+\frac{\partial}{\partial y}\left\{u_{0}\frac{\partial}{\partial x}\left(\frac{\partial v_{1}}{\partial x}+\frac{\partial u_{1}}{\partial y}\right)+u_{1}\frac{\partial}{\partial x}\left(\frac{\partial v_{0}}{\partial x}+\frac{\partial u_{0}}{\partial y}\right)+v_{0}\frac{\partial}{\partial y}\left(\frac{\partial v_{1}}{\partial x}+\frac{\partial u_{1}}{\partial y}\right)+v_{1}\frac{\partial}{\partial y}\left(\frac{\partial v_{0}}{\partial x}+\frac{\partial u_{0}}{\partial y}\right)\right\}\right]\end{aligned} \tag{6.26}$$

$$\begin{aligned}&-i\omega v_{1}+M\left(u_{0}\frac{\partial v_{1}}{\partial x}+u_{1}\frac{\partial v_{0}}{\partial x}+v_{0}\frac{\partial v_{1}}{\partial y}+v_{1}\frac{\partial v_{0}}{\partial y}\right)=-\frac{\partial p_{1}}{\partial y}+\frac{\varepsilon M}{1+\lambda_{1}}\left\{\frac{\partial^{2}v_{1}}{\partial x^{2}}+\frac{\partial^{2}v_{1}}{\partial y^{2}}+\frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial u_{1}}{\partial x}+\frac{\partial v_{1}}{\partial y}\right)\right\}\\&-\frac{D_{e}i\omega}{1+\lambda_{1}}\left\{\frac{\partial^{2}v_{1}}{\partial x^{2}}+\frac{\partial^{2}v_{1}}{\partial y^{2}}+\frac{\partial}{\partial y}\left(\frac{\partial u_{1}}{\partial x}+\frac{\partial v_{1}}{\partial y}\right)\right\}+\frac{D_{e}M}{1+\lambda_{1}}\left[2\frac{\partial}{\partial y}\left(u_{0}\frac{\partial^{2}v_{1}}{\partial x\partial y}+u_{1}\frac{\partial^{2}v_{0}}{\partial x\partial y}+v_{0}\frac{\partial^{2}v_{1}}{\partial x^{2}}+v_{1}\frac{\partial^{2}v_{0}}{\partial x^{2}}\right)\right.\\&\left.+\frac{\partial}{\partial x}\left\{u_{0}\frac{\partial}{\partial x}\left(\frac{\partial v_{1}}{\partial x}+\frac{\partial u_{1}}{\partial y}\right)+u_{1}\frac{\partial}{\partial x}\left(\frac{\partial v_{0}}{\partial x}+\frac{\partial u_{0}}{\partial y}\right)+v_{0}\frac{\partial}{\partial y}\left(\frac{\partial v_{1}}{\partial x}+\frac{\partial u_{1}}{\partial y}\right)+v_{1}\frac{\partial}{\partial y}\left(\frac{\partial v_{0}}{\partial x}+\frac{\partial u_{0}}{\partial y}\right)\right\}\right]\end{aligned} \tag{6.27}$$

The boundary conditions satisfied by time dependent part of the velocity are the no-slip condition at the wall $u_{1}\left(x,\ 1\right)=v_{1}\left(x,1\right)=0$ and symmetry about the midsection of the channel $\frac{\partial u_{1}(x,\ 0)}{\partial y}=\frac{\partial v_{1}(x,\ 0)}{\partial y}=0$. The time dependent velocities can now be further decomposed as:

$$u_{1}=\hat{u}+\tilde{u},\quad v_{1}=\hat{v}+\tilde{v}, \tag{6.28}$$

The quantities with hat and tilde represent the acoustic part and the vortical part respectively with the properties:

$$\frac{\partial\hat{v}}{\partial x}-\frac{\partial\hat{u}}{\partial y}=0\ ,\quad \frac{\partial\tilde{u}}{\partial x}+\frac{\partial\tilde{v}}{\partial y}=0\ ,\quad p_1=\hat{p}\,,\quad \rho_1=\hat{\rho},\tag{6.29}$$

The decomposition, based on the fundamental theorem of vector analysis, was first addressed by Stokes [90] and proved by Blumenthal in 1905. This theorem is also at the root of Helmholtz's work 1958 on vortex motion. The acoustic and vortical parts can be separated from Eqs. (6.25) – (6.27) by using Eqs. (6.28) and (6.29) and presented below:

6.3.1 Acoustic Part:

The acoustic part takes the following form:

$$-i\omega\hat{\rho}+\frac{\partial\hat{u}}{\partial x}+\frac{\partial\hat{v}}{\partial y}=-M\left(\frac{\partial\left(\hat{\rho}u_0\right)}{\partial x}+\frac{\partial\left(\hat{\rho}v_0\right)}{\partial y}\right),\tag{6.30}$$

$$\begin{aligned}&-i\omega\hat{u}+M\left(u_0\frac{\partial\hat{u}}{\partial x}+\hat{u}\frac{\partial u_0}{\partial x}+v_0\frac{\partial\hat{u}}{\partial y}+\hat{v}\frac{\partial u_0}{\partial y}\right)=-\frac{\partial\hat{p}}{\partial x}+\frac{\varepsilon M}{1+\lambda_1}\left\{\frac{\partial^2\hat{u}}{\partial x^2}+\frac{\partial^2\hat{u}}{\partial y^2}+\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial\hat{u}}{\partial x}+\frac{\partial\hat{v}}{\partial y}\right)\right\}\\&-\frac{D_e i\omega}{1+\lambda_1}\left\{\frac{\partial^2\hat{u}}{\partial x^2}+\frac{\partial^2\hat{u}}{\partial y^2}+\frac{\partial}{\partial x}\left(\frac{\partial\hat{u}}{\partial x}+\frac{\partial\hat{v}}{\partial y}\right)\right\}+\frac{D_e M}{1+\lambda_1}\left[2\frac{\partial}{\partial x}\left(u_0\frac{\partial^2\hat{u}}{\partial x^2}+\hat{u}\frac{\partial^2 u_0}{\partial x^2}+v_0\frac{\partial^2\hat{u}}{\partial x\partial y}+\hat{v}\frac{\partial^2 u_0}{\partial x\partial y}\right)\right.\\&\left.+\frac{\partial}{\partial y}\left\{u_0\frac{\partial}{\partial x}\left(\frac{\partial\hat{v}}{\partial x}+\frac{\partial\hat{u}}{\partial y}\right)+\hat{u}\frac{\partial}{\partial x}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)+v_0\frac{\partial}{\partial y}\left(\frac{\partial\hat{v}}{\partial x}+\frac{\partial\hat{u}}{\partial y}\right)+\hat{v}\frac{\partial}{\partial y}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)\right\}\right]\end{aligned}\tag{6.31}$$

$$\begin{aligned}&-i\omega\hat{v}+M\left(u_0\frac{\partial\hat{v}}{\partial x}+\hat{u}\frac{\partial v_0}{\partial x}+v_0\frac{\partial\hat{v}}{\partial y}+\hat{v}\frac{\partial v_0}{\partial y}\right)=-\frac{\partial\hat{p}}{\partial y}+\frac{\varepsilon M}{1+\lambda_1}\left\{\frac{\partial^2\hat{v}}{\partial x^2}+\frac{\partial^2\hat{v}}{\partial y^2}+\frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial\hat{u}}{\partial x}+\frac{\partial\hat{v}}{\partial y}\right)\right\}\\&-\frac{D_e i\omega}{1+\lambda_1}\left\{\frac{\partial^2\hat{v}}{\partial x^2}+\frac{\partial^2\hat{v}}{\partial y^2}+\frac{\partial}{\partial y}\left(\frac{\partial\hat{u}}{\partial x}+\frac{\partial\hat{v}}{\partial y}\right)\right\}+\frac{D_e M}{1+\lambda_1}\left[2\frac{\partial}{\partial y}\left(u_0\frac{\partial^2\hat{v}}{\partial x\partial y}+\hat{u}\frac{\partial^2 v_0}{\partial x\partial y}+v_0\frac{\partial^2\hat{v}}{\partial x^2}+\hat{v}\frac{\partial^2 v_0}{\partial x^2}\right)\right.\\&\left.+\frac{\partial}{\partial x}\left\{u_0\frac{\partial}{\partial x}\left(\frac{\partial\hat{v}}{\partial x}+\frac{\partial\hat{u}}{\partial y}\right)+\hat{u}\frac{\partial}{\partial x}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)+v_0\frac{\partial}{\partial y}\left(\frac{\partial\hat{v}}{\partial x}+\frac{\partial\hat{u}}{\partial y}\right)+\hat{v}\frac{\partial}{\partial y}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)\right\}\right]\end{aligned}\tag{6.32}$$

The axial acoustic pressure and velocity are dominant for the purpose of oscillatory flow under consideration. Knowing that $\hat{\rho}=\hat{p}$ for perfect gas, the Eqs. (6.31) and (6.32) are coupled in pressure and velocity. Eliminating the velocity from these equations and solving the resultant equations for acoustic pressure and velocity up to $O\left(M\right)$, we finally arrive at:

$$\hat{u} = i\sqrt{1-\frac{2D_e\omega^2}{1+\lambda_1}}\sin\left(\frac{\omega x}{\sqrt{1-\frac{2D_e\omega^2}{1+\lambda_1}}}\right)+O(M). \tag{6.33}$$

$$\hat{p} = \cos\left(\frac{\omega x}{\sqrt{1-\frac{2D_e\omega^2}{1+\lambda_1}}}\right)+O(M), \tag{6.34}$$

6.3.2 Vortical Part:

The vortical part is given by:

$$\frac{\partial\tilde{u}}{\partial x}+\frac{\partial\tilde{v}}{\partial y}=0 \tag{6.35}$$

$$\begin{aligned}
&-i\omega\tilde{u}+M\left(u_0\frac{\partial\tilde{u}}{\partial x}+\tilde{u}\frac{\partial u_0}{\partial x}+v_0\frac{\partial\tilde{u}}{\partial y}+\tilde{v}\frac{\partial u_0}{\partial y}\right)=\frac{\varepsilon M}{1+\lambda_1}\left\{\frac{\partial^2\tilde{u}}{\partial x^2}+\frac{\partial^2\tilde{u}}{\partial y^2}+\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial\tilde{u}}{\partial x}+\frac{\partial\tilde{v}}{\partial y}\right)\right\}\\
&-\frac{D_e i\omega}{1+\lambda_1}\left\{\frac{\partial^2\tilde{u}}{\partial x^2}+\frac{\partial^2\tilde{u}}{\partial y^2}+\frac{\partial}{\partial x}\left(\frac{\partial\tilde{u}}{\partial x}+\frac{\partial\tilde{v}}{\partial y}\right)\right\}+\frac{D_e M}{1+\lambda_1}\left[2\frac{\partial}{\partial x}\left(u_0\frac{\partial^2\tilde{u}}{\partial x^2}+\tilde{u}\frac{\partial^2 u_0}{\partial x^2}+v_0\frac{\partial^2\tilde{u}}{\partial x\partial y}+\tilde{v}\frac{\partial^2 u_0}{\partial x\partial y}\right)\right.\\
&\left.+\frac{\partial}{\partial y}\left\{u_0\frac{\partial}{\partial x}\left(\frac{\partial\tilde{v}}{\partial x}+\frac{\partial\tilde{u}}{\partial y}\right)+\tilde{u}\frac{\partial}{\partial x}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)+v_0\frac{\partial}{\partial y}\left(\frac{\partial\tilde{v}}{\partial x}+\frac{\partial\tilde{u}}{\partial y}\right)+\tilde{v}\frac{\partial}{\partial y}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)\right\}\right]
\end{aligned} \tag{6.36}$$

$$\begin{aligned}
&-i\omega\tilde{v}+M\left(u_0\frac{\partial\tilde{v}}{\partial x}+\tilde{u}\frac{\partial v_0}{\partial x}+v_0\frac{\partial\tilde{v}}{\partial y}+\tilde{v}\frac{\partial v_0}{\partial y}\right)=\frac{\varepsilon M}{1+\lambda_1}\left\{\frac{\partial^2\tilde{v}}{\partial x^2}+\frac{\partial^2\tilde{v}}{\partial y^2}+\frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\partial\tilde{u}}{\partial x}+\frac{\partial\tilde{v}}{\partial y}\right)\right\}\\
&-\frac{D_e i\omega}{1+\lambda_1}\left\{\frac{\partial^2\tilde{v}}{\partial x^2}+\frac{\partial^2\tilde{v}}{\partial y^2}+\frac{\partial}{\partial y}\left(\frac{\partial\tilde{u}}{\partial x}+\frac{\partial\tilde{v}}{\partial y}\right)\right\}+\frac{D_e M}{1+\lambda_1}\left[2\frac{\partial}{\partial y}\left(u_0\frac{\partial^2\tilde{v}}{\partial x\partial y}+\tilde{u}\frac{\partial^2 v_0}{\partial x\partial y}+v_0\frac{\partial^2\tilde{v}}{\partial x^2}+\tilde{v}\frac{\partial^2 v_0}{\partial x^2}\right)\right.\\
&\left.+\frac{\partial}{\partial x}\left\{u_0\frac{\partial}{\partial x}\left(\frac{\partial\tilde{v}}{\partial x}+\frac{\partial\tilde{u}}{\partial y}\right)+\tilde{u}\frac{\partial}{\partial x}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)+v_0\frac{\partial}{\partial y}\left(\frac{\partial\tilde{v}}{\partial x}+\frac{\partial\tilde{u}}{\partial y}\right)+\tilde{v}\frac{\partial}{\partial y}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)\right\}\right]
\end{aligned} \tag{6.37}$$

Considering axial component of vortical velocity (Eq. (6.36)), assuming that the ratio of the horizontal to vertical component is of $O(M)$, Majdalani [15]. This assumption can also be justified in view of the argument presented by Flandro [93], Majdalani & Moorhem [94]. Thus substituting $(u_0, v_0)=\left(-xF',\ F\right)$ and separating the variables $\tilde{u}=X(x)Y(y)$, we obtain:

$$\frac{\varepsilon}{1+\lambda_1}\frac{d^2Y}{dy^2}-F\frac{dY}{dy}+\left(iS_t+\left(1+x\frac{X'}{X}\right)F'\right)Y$$
$$=-\frac{D_e}{1+\lambda_1}\left[F\frac{d^3Y}{dy^3}-\left(iS_t-\left(1-x\frac{X'}{X}\right)F'\right)\frac{d^2Y}{dy^2}-\left(1+x\frac{X'}{X}\right)F''\frac{dY}{dy}-F'''Y\right] \tag{6.38}$$

The x dependent function in Eq. (6.38) is written as:

$$x\frac{X'}{X}=k_n$$

where k_n are so far unknown eigenvalues and will be determined by using no slip boundary condition. Solving this equation, we get the eigenfunctions $X(x)=x^{k_n}$, and the axial component of acoustic velocity Eq. (6.38) can be expressed as:

$$\tilde{u}(x,\ y)=\sum_n c_n x^{k_n} Y_n(y). \tag{6.39}$$

No-slip condition at $y=1$, requires that the vortical velocity is equal to the negative of the acoustic velocity giving:

$$\hat{u}(x,\ 1)=-\tilde{u}(x,\ 1),$$

The above condition together with Eq. (6.39) gives:

$$k_n=2n+1\ ,\quad c_n=-i\sqrt{1-\frac{2D_e\omega^2}{1+\lambda_1}}\frac{(-1)^n}{(2n+1)!}\left(\frac{\omega}{\sqrt{1-\frac{2D_e\omega^2}{1+\lambda_1}}}\right)^{2n+1}\quad ,\ Y_n(1)=1$$

Using this value of k_n and c_n in Eqs. (6.38) and (6.39), we get:

$$\tilde{u}(x,y)=-i\sqrt{1-\frac{2D_e\omega^2}{1+\lambda_1}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\omega x}{\sqrt{1-\frac{2D_e\omega^2}{1+\lambda_1}}}\right)^{2n+1}Y_n(y). \tag{6.40}$$

$$\frac{\varepsilon}{1+\lambda_1}\frac{d^2Y}{dy^2} - F\frac{dY}{dy} + \left(iS_t + (2n+2)F'\right)Y$$
$$= -\frac{D_e}{1+\lambda_1}\left[F\frac{d^3Y}{dy^3} - \left(iS_t + 2nF'\right)\frac{d^2Y}{dy^2} - (2n+2)F''\frac{dY}{dy} - F'''Y\right] \tag{6.41}$$

with

$$Y(1) = 1\ ,\quad Y'(0) = 0. \tag{6.42}$$

In order to determine the complete vortical solution [Eq. (6.39)], we proceed to find the solution of Eq. (6.41) subject to the boundary conditions given in Eq. (6.42). The perturbation method works well for small parameter assumption and yields good approximation even for two terms. The perturbation method and WKB approximation is used to solve Eq. (6.41). For that, we express the solution of Eq. (6.41) in regular perturbation expansion in D_e :

$$Y(y, D_e) \approx Y_0(y) + D_e Y_1(y) + O\left(D_e^{\,2}\right) \tag{6.43}$$

Using Eq. (6.43) into Eqs. (6.41) and (6.42) and equating like powers of D_e , the leading order system is given by:

$$\frac{\varepsilon}{1+\lambda_1}\frac{d^2Y_0}{dy^2} - F_0\frac{dY_0}{dy} + \left(iS_t + (2n+2)F_0'\right)Y_0 = 0 \tag{6.44}$$

$$Y_0(1) = 1\ ,\quad Y_0'(0) = 0. \tag{6.45}$$

This system is now solved using WKB approximation for small ε . The one term WKB solution is written as:

$$Y_0 = e^{\frac{1+\lambda_1}{2}\int_1^y \left(\sqrt{F^2 - \frac{4i\varepsilon S_t}{1+\lambda_1}} + F\right)d\eta}. \tag{6.46}$$

The system of order $O(D_e)$ is written as:

$$\frac{\varepsilon}{1+\lambda_1}\frac{d^2Y_1}{dy^2} - F_0\frac{dY_1}{dy} + \left(iS_t + (2n+2)F_0'\right)Y_1 = g(y) \tag{6.47}$$

where

$$g(y) = -\frac{1}{1+\lambda_1}\left[F_0\frac{d^3Y_0}{dy^3} - \left(iS_t + 2nF_0'\right)\frac{d^2Y_0}{dy^2} - (2n+2)F_0''\frac{dY_0}{dy} - F_0'''Y_0\right] + \frac{F_1}{\varepsilon}\frac{dY_0}{dy} - (2n+2)$$

and the corresponding boundary conditions are:

$$Y_1(1)=0\ ,\quad Y_1^{'}(0)=0. \tag{6.48}$$

The solution of this system is provided by the variation of parameter method and use numerical integration. The expression for axial vortical velocity is finally written as:

$$\tilde{u}(x,y)=-i\sqrt{1-\frac{2D_e\omega^2}{1+\lambda_1}}\sin\left(\frac{\omega x}{\sqrt{1-\frac{2D_e\omega^2}{1+\lambda_1}}}\right)\left[Y_0(y)+D_eY_1(y)\right]. \tag{6.49}$$

After having calculated both acoustic and vortical velocities, we revert back to the time dependent oscillatory axial velocity given by Eq. (6.28). Thus, using Eqs. (6.28), (6.33) and (6.49) together gives the time dependent oscillatory axial velocity as:

$$u_1=i\left[\sin(\omega x)(1-Y_0)-D_e\left\{\sin(\omega x)Y_1-(\omega x-\sin(\omega x))\frac{\omega^2(1-Y_0)}{1+\lambda_1}\right\}\right] \tag{6.50}$$

where Y_0 and Y_1 are given by Eqs. (6.44) and (6.47) respectively. The integrals in these solutions cannot be calculated analytically and hence numerical integration is performed. Mathematically, the viscous fluid results are obtained for $\lambda_1=0$ and $D_e=0$, and are found to match with Jankowski and Majdalani [21]. The second term on right side of Eq. (6.50) provides the correction due to the Jeffrey fluid through the Deborah number D_e .

6.4 Discussion on Results

We would first compare our results with the existing work. In the special case of $\lambda_1=0$ and $D_e=0$ (viscous fluid), the real part of oscillatory axial velocity $u_1\exp(-it)$ [Eq. (6.50)] at four equiphased timelines ($t=90^0, 180^0, 270^0$ and 360^0) is plotted in Fig. 6.2. The velocity represents a spatially damped wave traveling in time which exhibits a large inviscid core that stretches about the symmetry plane and a rotational boundary layer in the direct vicinity of the wall. The no-slip condition at the wall $y=1$ is satisfied. These results are compared with Jankowski and Majdalani [21] for viscous fluid: the solid lines represent our results while doted lines represent those of Jankowski and Majdalani [21]. A perfect agreement is found to exist.

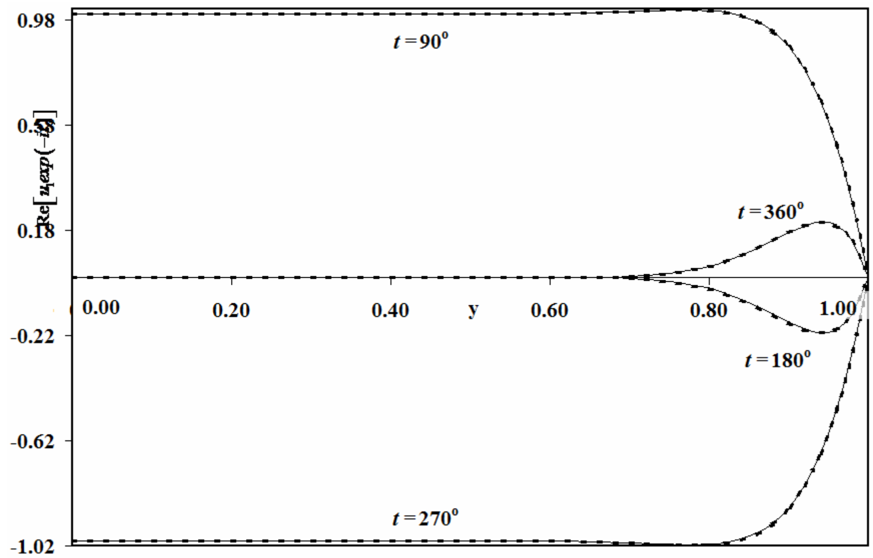
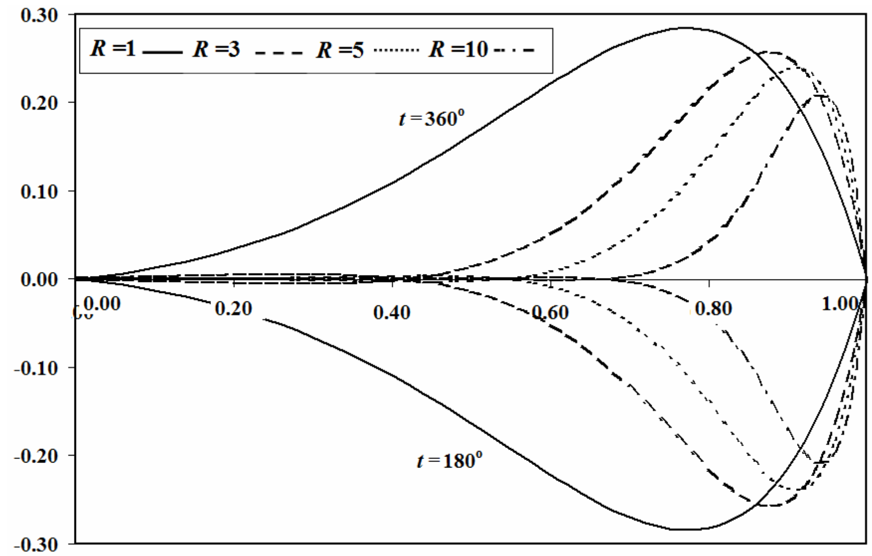
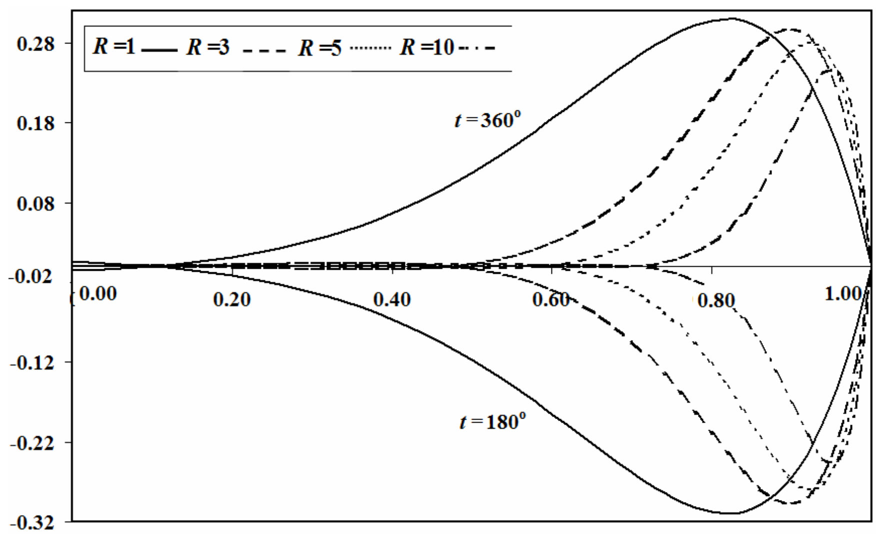


Figure 6.2 Comparison of $\mathrm{Re}\left(u_1 \exp(-it)\right)$ **against** y **for** $\frac{x}{l}=1, m=1, S_t=20, R=10$ **at four equiphased timelines** t.

The effects of suction parameter on the oscillatory axial velocity have been shown in Figure 6.3. For sake of comparison, we draw these effects for both viscous fluid [Figure 6.3 (a)] and Jeffrey fluid [Figure 6.3 (b)]. It is clearly observed that the amplitude and penetration depth decrease with the increase of suction parameter for both the cases. The comparison of the wave and the decrease of the penetration depth (velocity layer) are higher in Jeffrey fluid than in the viscous fluid, with increasing suction on the walls. These quantities (amplitude and penetration depth) further undergo compression. However, these quantities are smaller in Jeffrey fluid. For the smallest value of the suction level $R=1$, the viscosity is more significant and the rotational layer penetrates deeper into the channel than higher Reynolds number. As the suction increase, the relative effects of viscosity becomes less significant and a rotational layer undergoes a progressive compression and being pulled closer to the wall. This compression is higher in Jeffrey fluid than in viscous fluid.



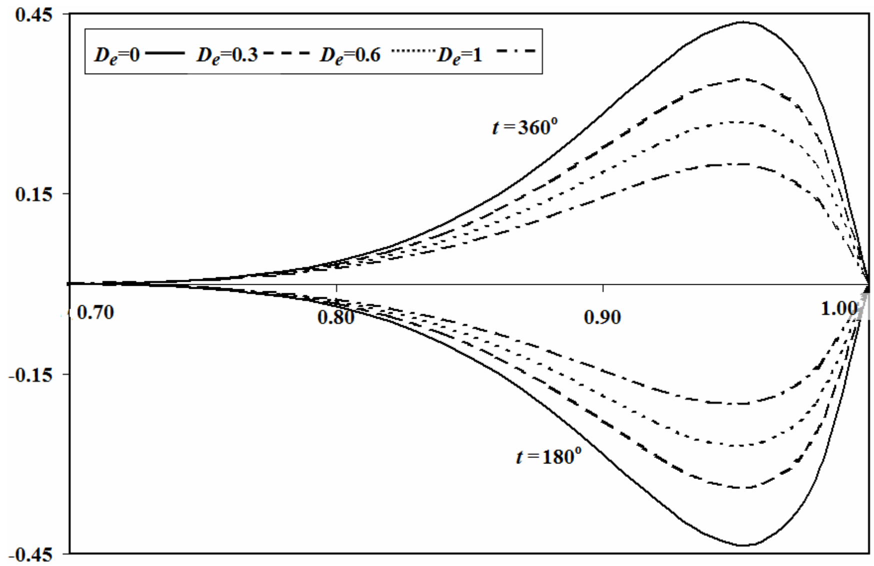
(a)



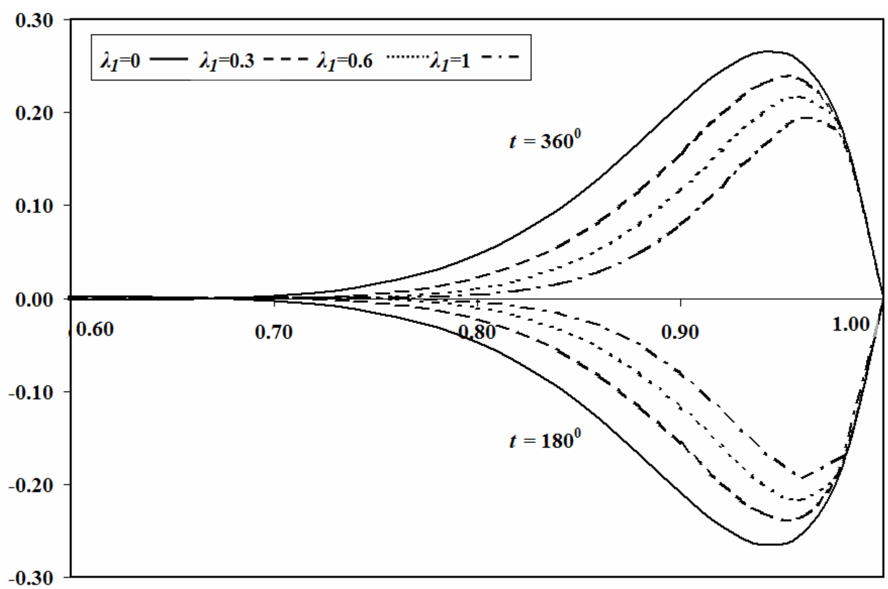
(b)

Figure 6.3 Effects of suction parameter R **on** $\mathrm{Re}\left(u_1 \exp(-it)\right)$ **for** $\frac{x}{l}=1, m=1, S_t=20$ **at time** $t=180^0$ **and** $t=360^0$.

Figure 6.4 (a) represents the effects of λ_1 on the oscillatory axial velocity. The amplitude of oscillatory velocity and the penetration depth decrease as the values of λ_1 increases. The effects of the Deborah number D_e on the oscillatory velocity are shown in Figure 6.4 (b). It is observed that as the Deborah number increases, the amplitude of the oscillatory axial velocity decreases; whereas there is only a little effect on the penetration depth. These observations signify that the velocity undergoes a compression in the Jeffrey fluid and this compression increases with the increase of the Jeffrey fluid parameter. The penetration depth or the velocity layer decreases that is pulled near the wall, as the Jeffrey fluid parameter increases. We recall that the increase of suction parameter has the similar effects. Thus the Jeffrey fluid enhances the compression and reduction of the penetration depth of the oscillatory velocity between the channels.



(a)



(b)

Figure 6.4 Effects of (a) λ_1 **and (b)** D_e **on** $\mathrm{Re}\left(u_1 \exp(-it)\right)$ **against** y **for** $\frac{x}{l}=1,\quad m=1,$

$S_t = 20,\quad R=10$ **at time phase (** $t=180^0$ **and** $t=360^0$ **).**

6.5 Main Observations

The problem of unsteady oscillatory flow for non-Newtonian Jeffrey fluid is discussed. The analytical solution for time independent and time dependent oscillatory velocity has been found. The main observations of this study are given below:

- The amplitude of the oscillatory axial velocity decreases with the increase of ratio of relaxation time to retardation time parameter.
- The increase in the value of Deborah number also decreases the amplitude of the velocity.
- The increase in the degree of elasticity of the fluid decreases the amplitude of the oscillations.
- The penetration depth of the velocity decreases with increasing values of Deborah number and ratio of relaxation time to retardation time parameter.

Chapter 07

Conclusions

In this thesis, we study oscillatory channel and tube flows in non-Newtonian fluids. To begin with channel flow in porous viscous medium is discussed. A number of experimental studies have supported the existence of velocity oscillations inside channels and tubes with transpiring walls. The physical quantities of interest are the oscillatory axial velocity profile, its amplitude and penetration depth. The effect of suction on the wall has profound effect on the velocity field and is throughout reckoned with. Besides its mathematical appeal of finding the solution of full Navier-Stokes equations in Newtonian and non- Newtonian fluids, this study has enormous applications in engineering, industry and biological sciences. Further, the study has a beautiful mix of the fluid oscillations coupled with the generation of acoustic field due to the vibrations of the channel walls.

The introduction of porosity, through Darcy parameter, in the viscous medium reduces the amplitude of the oscillatory axial velocity significantly and penetration depth slightly with an increase of the inverse Darcy parameter. Thus, reducing permeability, the oscillations are minimized. From here onward, we take the discussion to oscillatory channel and tube flows to non-Newtonian second grade fluid and Jaffrey fluid.

The effects of second grade fluid parameter on the oscillatory axial velocity, its amplitude and penetration depth are calculated analytically and analyzed graphically. As we increase the second grade parameter, the amplitude of the oscillatory velocity increases and the penetration depth of the velocity decrease. As we increase the suction parameter, we observe a decrease in the amplitude of the oscillatory velocity for the second grade fluid. The decrease in amplitude of the velocity is more apparent in second grade fluid than in viscous fluid. The results for viscous fluid can be recovered as $\alpha \to 0$.

Our next attempt has been the consideration of oscillatory tube flow. This study has a great relevance in biological fluid flows. The effects of the second grade fluid parameter on oscillatory flow field in a tube, its amplitude and penetration depth are calculated analytically and analyzed graphically. It is observed that the amplitude of the oscillatory velocity increases and penetration depth remains the same by increasing the values of the second grade fluid parameter. An increase in suction parameter decreases the amplitude. The results for viscous fluid Jankowski and Majdalani [52] are again recovered as $\alpha \to 0$.

In the end, we have discussed oscillatory flow in a channel to non-Newtonian Jeffrey fluid with small suction. Measurement of the non-Newtonian characteristic is frequently used to determine the structure of a fluid. In return the desired rheological properties can be conceived from suitably engineered structures. Further that, Jeffrey fluid predicts relaxation/retardation time effects and presents different rheology from that of Newtonian fluids. The degree of viscoelacticity is determined through relaxation/retardation time parameter called Deborah number. Further importance lies in that the oscillatory shear motion helps to understand the behaviour of viscoelastic fluids, which in our instance are generated from fluctuations in the suction rate. Most of the instruments designed to measure viscoelacity are for oscillating strain.

The appropriately modeled problem is solved analytically using small oscillatory velocity amplitude and small Deborah number assumptions. The effects of viscoelastic parameters on these quantities are analyzed graphically. It is observed that by increasing the values of the ratio of relaxation/retardation time parameter decreases the amplitude as well as penetration depth of the oscillatory velocity. Also, the increase of the non-dimensional Deborah number decreases the amplitude of the oscillatory velocity. The results of the viscous fluid can be recovered as a special case of this study by taking $\lambda_1 = 0$ and $D_e = 0$. The theoretical investigation of the effects of viscoelastic parameter helps to understand the behaviour and determine the degree of viscoelasticity of the non-Newtonian fluids.

We see that, this is the first attempt to consider the effects of non-Newtonian second grade fluid in the theory of oscillatory flows in channel and the discussion can further be extended to other types of non-Newtonian fluids with different rheological properties, different governing equations and different industrial applications.

Chapter 08

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