SAMPLED-DATA LTV REGULATION USING RECONSTRUCTION OBSERVER

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In the name of Allah, the most Merciful and the most Beneficent
ABSTRACT

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Sampled-data output feedback regulation for continuous time varying system is discussed in this dissertation. The continuous time varying system is generally referred as plant. The plant output asymptotically tracks a continuous reference signal generated by an exogenous system. It is assumed that only the sampled-output of both plant and reference signals are available for measurement. The problem is to design a continuous-time (non-impulsive) reconstruction observer for both plant and exogenous system states estimation followed by a continuous-time controller to achieve ripple free smooth regulation. The design of reconstruction observer is based on two proposed observer schemes named mainly in accordance of their functionality that is current impulsive observer and prediction impulsive observer. This insight to the problem leads to an innovative idea of continuous (non-impulsive) reconstruction observer with the fusion of two impulsive observer designed under well-defined weighting function. A comprehensive convergence analysis of the proposed observer is presented for stable, unstable and highly unstable continuous-time linear time varying systems.

The application of reconstruction (non-impulsive) observer for sampled data regulation without ripple is explored for linear time varying system. Augmented stability analysis is proposed for closed loop system. The overall scheme is
demonstrated with the help of linear time varying system. Subsequently, single input single output feedback linearizable nonlinear system regulation is investigated in the frame work of suggested theory of non-impulsive observers. The proposed novel observer design not only provides state estimates but also performs feedback linearization for a nonlinear system. This ultimately leads to a non-impulsive continuous-time sampled-data regulation without ripples for a nonlinear system. A stability analysis is carried out while considering the model uncertainties of a nonlinear system as a non-vanishing perturbation. An example of a third order nonlinear system illustrates the efficacy of proposed design methodology.
To

My family and my G-4 teacher Miss Safia
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CHAPTER 1

INTRODUCTION

1.1 Output Regulation

One of the most important problems in linear multivariable classical control theory is that of controlling a fixed plant such that its output tracks (or rejects) a reference (or disturbance) signals. It is assumed that only output is available for controlling the plant. The plant states satisfy certain observability conditions for states estimation. The reference (or disturbance) signal to be tracked (or rejected) is generated by some external generator. Such signal generator is commonly known as exogenous system or simply exosystem. Exosystem output constitutes both the reference and disturbance signals. It is assumed that reference is available, whereas rests of the exosystem states are generally assumed to be observable. Such problem in control theory is termed as output regulation problem.

Output regulation theory is widely discussed and covered in literature for linear and nonlinear systems. Output regulation problem for linear systems is being addressed by a number of researchers[1],[2] in which multivariable regulator problem is discussed and feed-forward control for time-invariant systems is covered in [3]. Different conditions require for the tracking of constant reference signals in output regulation are briefly discussed in [4]. The output regulation with periodic exosystem for linear time varying system is covered in [5]. Differential regulator equations are proposed as solution for time varying regulation. The same approach has been employed with time varying exosystem to develop a solution for a class of minimum phase linear time varying systems [6]. Discrete-time output regulation is proposed in
[7] and the same was extended by using reconstruction realizable filter with discrete controller for output regulation [8]–[10].

Output regulation for nonlinear systems has also received due attention in literature. In [11], a solution based on differentiable manifolds is presented for the regulation of nonlinear systems. Tracking and disturbance rejection of multiple input multiple output for nonlinear system is discussed in [12]. Moreover, a comprehensive review on output regulation theory for nonlinear systems along with outline of the major problems is presented in [13]. The research work proposed in [14] is presented a complete extension to the theory established for nonlinear systems. Time varying reference / disturbance signals as an extension for nonlinear systems is discussed in [15]. The following sections review relevant details about output regulation of linear and nonlinear systems.

1.2 Output Regulation for Linear Time Varying System

Consider a following multiple input multiple output linear time varying system

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) + P(t)w(t) \\
y(t) &= C(t)x(t),
\end{align*}
\]

where \( x \in \mathbb{R}^n \) are the plant states and \( u \in \mathbb{R}^m \) are continuous inputs. The plant disturbances are represented by \( P(t)w(t) \) and \( y \in \mathbb{R}^p \) are linear system outputs. The signal to be tracked or rejected is modeled by a system, termed as exosystem written as

\[
\begin{align*}
\dot{w}(t) &= S(t)w(t) \\
y_r(t) &= Q(t)w(t).
\end{align*}
\]
where $w \in \mathbb{R}^n$ are the exosystem states and the tracking error between the actual plant output $y(t)$ and the reference signal $y_r(t)$ is expressed as

$$e_r(t) = C(t)x(t) + Q(t)w(t), \quad (1.2.3)$$

where $e_r(t)$ is the plant tracking error, subscript $t$ stands for tracking and $C(t)$ and $Q(t)$ are selected with appropriate dimensions. Output regulation tracking error with all initial conditions $x(0), w(0)$ for a closed-loop system is given as

$$e_r(t) \to 0 \quad \text{as} \quad t \to \infty \quad \forall \quad t$$

1.2.1 Output Regulation for Linear Time Invariant System

Linear time varying regulation is the prime focus of this dissertation however; some part is also come under discussion for linear time invariant systems as well. Linear time invariant system is expressed as

$$\dot{x}(t) = Ax(t) + Bu(t) + Pw(t)$$

$$y(t) = Cx(t), \quad (1.2.4)$$

where $P$ is used for mapping disturbance into the system and output regulation problem is defined in terms of tracking a reference signal $y_r(t)$ generated by time invariant exosystem such that the tracking error signal is written as

$$e_r(t) = Cx(t) + Qw(t), \quad (1.2.5)$$
where tracking error asymptotically reduces to zero that is \( e_i(t) \rightarrow 0 \) as \( t \rightarrow \infty \). The following two discussed problems provide a solution to for continuous-time output regulation for linear time varying and linear time invariant systems [16].

1.2.2 Output Regulation via State Feedback

The solution for an output regulation problem with state feedback is to find a state feedback law

\[
    u(t) = F(t)x(t) + G(t)w(t),
\]

such that:-

1. The system \( \dot{x}(t) = (A(t) + B(t)F(t))x(t) \) is asymptotically stable.

2. The solution of a closed loop system for all initial condition satisfies:-

\[
    \lim_{t \rightarrow \infty} e_i(t) = 0,
\]

1.2.3 Output Regulation via Output Feedback

The solution for an output regulation problem with output feedback is expressed as [16]

\[
    \begin{align*}
    \dot{x}(t) &= M(t)\hat{x}(t) + G(t)e(t) \\
    u(t) &= F(t)\hat{x}(t),
    \end{align*}
\]

where \( \hat{x}(t) \) is the observed state vector. The closed loop system is expressed by control input (1.2.7) along with (1.2.1), (1.2.2) and (1.2.5)
\[ \dot{x}(t) = A(t)x(t) + B(t)F(t)\dot{x}(t) + P(t)w(t) \]
\[ \dot{\dot{x}}(t) = G(t)C(t)x(t) + M(t)\dot{x}(t) + G(t)Q(t)w(t) \]
\[ \dot{w}(t) = S(t)w(t). \]  

(1.2.8)

It is assumed that the response for \((x(t), \dot{x}(t))\) for all initial conditions \((x(0), \dot{x}(0), w(0))\)

1. is bounded.

2. The solution of the closed loop system satisfies \(\lim_{t \to \infty} e_j(t) = 0\), for all initial conditions.

Regulation problem solution is based on the following assumptions [2], [16]

A1. \((A(t), B(t))\) is stabilizable.

A2. The pair \(\begin{pmatrix} C(t) & Q(t) \end{pmatrix} \begin{pmatrix} A(t) & P(t) \\ 0 & S(t) \end{pmatrix}\) is detectable.

A3. \(|W(t_0, t)| \leq d_i\) for any \(t \geq t_0\), where \(W(t, s)\) is the state transition matrix generated by \(S(t)\).

Asymptotic stabilization of the closed loop system is guaranteed for both state and output feedback closed loop systems by first assumption. The second assumption deals with the stabilization for output feedback. It is a stronger assumption than observability of pair \((A(t), C(t))\).

If above assumptions hold, than state and output feedback problems is given by the following assumption [16].

**Proposition 1.1**

If A1-A3 holds, then output regulation problem via state feedback is solvable if and only if following assumptions hold
A4. There exist time varying matrices $\Pi(t)$ and $\Gamma(t)$, which satisfy the following differential regulator equations:

\[
\dot{\Pi}(t) = A(t)\Pi(t) - \Pi(t)S(t) + P(t) + B(t)\Gamma(t)
\]

\[
\lim_{t \to \infty}[C(t)\Pi(t) + Q(t)]W(t, t_0) = 0.
\] (1.2.9)

An appropriate feedback is then given by $u(t) = F(t)(x(t) - \Pi(t)w(t)) + \Gamma(t)w(t)$, such that $(A(t) + B(t)F(t))$ is stable. Moreover (1.2.9) implies

\[
\lim_{t \to \infty}[C(t)\Pi(t) + Q(t)] = 0
\] (1.2.10)

The condition (1.2.9) can be replaced by (1.2.10) if the following condition holds

A5. $\|W(t, t_0)\| \leq d_2$ for any $t \geq t_0$.

**Proposition 1.2**

Suppose assumptions A1- A3 holds. Output regulation problem with state feedback is solvable if and only if assumption A4 is satisfied. Then a suitable feedback is given by

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{\hat{w}}(t)
\end{bmatrix} =
\begin{bmatrix}
A(t) & P(t) \\
0 & S(t)
\end{bmatrix}
\begin{bmatrix}
\dot{x}(t) \\
\hat{w}(t)
\end{bmatrix} +
\begin{bmatrix}
B(t) \\
0
\end{bmatrix}u(t)
\]

\[+H(t)
\begin{bmatrix}
y(t) - [C(t) & Q(t)]
\end{bmatrix}
\begin{bmatrix}
\dot{x}(t) \\
\hat{w}(t)
\end{bmatrix},
\]

\[u(t) =
\begin{bmatrix}
F(t) & \Gamma(t) - \Pi(t)
\end{bmatrix}
\begin{bmatrix}
\dot{x}(t) \\
\hat{w}(t)
\end{bmatrix},
\] (1.2.11)

such that $(A(t) + B(t)F(t))$ is stable and observer gain $H(t)$ is designed to keep

$$
\begin{bmatrix}
A(t) & P(t) \\
0 & S(t)
\end{bmatrix} + H(t)[C(t) & Q(t)]
$$
stable [16].
1.3 Output Regulation for Nonlinear System

An output regulation for a class of following nonlinear system is discussed in this section [17]

\[
\begin{align*}
\dot{x}(t) &= A_{c}x(t) + B_{c}\phi(x(t),\eta(t),u(t)) \triangleq f(x(t),\eta(t),u(t)), \\
\dot{\eta}(t) &= \psi(x(t),\eta(t),u(t)), \\
y_{p}(t) &= C_{c}x(t), \\
\zeta(t) &= \Theta(x(t),\eta(t)),
\end{align*}
\tag{1.3.1}
\]

where \(u(t) \in R\) is the control input, \(y_{p}(t) \in R\) and \(\zeta(t) \in R^{d}\) are the measured outputs, \(x(t) \in R^{n}\) and \(\eta(t) \in R^{i}\) constitute the state vector. Order of the system is \(n\) whereas \(\rho \leq n\) is called the relative degree of a system. For feedback linearizable nonlinear system (1.3.1) can be represented in normal form. The \(n \times n\) matrix \(A_{c}\), the \(n \times 1\) matrix \(B_{c}\) and \(1 \times n\) matrix \(C_{c}\) is written as [17], [18]

\[
A_{c} = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \quad B_{c} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}
\]

\[
C_{c} = (1 \ 0 \ \cdots \ \cdots \ 0),
\]

and it represents a chain of \(n\) integrators that can be modeled in normal form for input output linearizable nonlinear system as
\[ \dot{\eta}(t) = \psi(x(t), \eta(t), u(t)) \]
\[ \dot{x}(t) = A_c x(t) + B_c \frac{1}{\beta(x, \zeta)} \left( u(t) - \alpha(x, \zeta) \right) \]
\[ y_p(t) = C_c x(t) \]
\[ \zeta(t) = \Theta(x(t), \eta(t)) \]

where \( \beta(x, \zeta) = \gamma^{-1}(x, \zeta) \neq 0 \) and \( \alpha(x, \zeta) \) are the scalar functions of \( x \) and \( \zeta \), very well defined over the domain of interest. The existence and uniqueness of the solution can be ensured with assumption that \( \alpha(x, \zeta) \) and \( \beta(x, \zeta) \) are locally Lipchitz over the domain of interest. It is assumed that pair \( (A_c, C_c) \) is observable. Global stabilization/ tracking can be achieved if input \( u \) in (1.3.2) is assumed to be input-to-state stable [17]. The local stabilization only requires \( \dot{\eta} = f_o(\eta, 0) \) to be asymptotically stable. Then the output regulation problem for feedback linearizable system is to find a control law of the form

\[ u(t) = \alpha_o(x, \zeta) + \beta_o(x, \zeta)v(t), \]

where \( \beta(x, \zeta) = \gamma^{-1}(x, \zeta) \neq 0 \) and \( \alpha(x, \zeta) \) are the scalar functions of \( x \) and \( \zeta \), very well defined over the domain of interest. The existence and uniqueness of the solution can be ensured with assumption that \( \alpha(x, \zeta) \) and \( \beta(x, \zeta) \) are locally Lipchitz over the domain of interest. It is assumed that pair \( (A_c, C_c) \) is observable. Global stabilization/ tracking can be achieved if input \( u \) in (1.3.2) is assumed to be input-to-state stable [17]. The local stabilization only requires \( \dot{\eta} = f_o(\eta, 0) \) to be asymptotically stable. Then the output regulation problem for feedback linearizable system is to find a control law of the form

\[ u(t) = \alpha_o(x, \zeta) + \beta_o(x, \zeta)v(t), \]

It can be concluded that provided a suitable feedback (1.3.3) exists, the non-linear system (1.3.2) can be represented as LTI system (1.2.1) given by

\[ \dot{\eta}(t) = \psi(x(t), \eta(t), u(t)), \]
\[ \dot{x}(t) = A_c x(t) + B_c v(t), \]
\[ y_p(t) = C_c x(t) \]
\[ \zeta(t) = \Theta(x(t), \eta(t)) \]

Then asymptotic tracking can be achieved by selecting the input \( v(t) \) to be

\[ v(t) = [F \quad \Gamma - F\Pi] \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \]

such that \( (A_c + B_c F) \) is Hurwitz.
1.4 Discrete Control Design for Sampled-data System

Sampled-data theory has gained more attraction for researchers with the increase in use of digital electronics and microcontrollers [19]–[22]. Application of control systems is generally based on output feedback wherein, observer is used to estimate the system states. Mostly in sampled-data systems, only sampled-data output is available for controlling the continuous-time systems [23]–[31]. In case of sampled output, two different classes of observers are used. The first one is discrete-time observer and the second is continuous-time observer (impulsive in nature) that estimates system states at the sampling points using sampled outputs and discrete / continuous input. A controller design based on such observers are termed as sampled-data control [32]–[37]. Consider a following sampled-data linear time invariant system for (1.2.4) (without external disturbance)

\[
\dot{x}(t) = Ax(t) + Bu(t), \\
y[k] = Cx[k],
\]

A discrete controller design for (1.4.1) (typically a microcontroller) is fed with a discrete observer sampled output. Discrete input and sampled output is used for providing discrete-time states estimation with the help of discrete-time observer. The discrete controller then generates the control input by using the observed states. Two types of discrete-time observers can be found in literature; current and prediction discrete observers. Observer is termed as current estimate, \(\bar{x}(k)\), if based on measurement \(y[k]\) up to and including the \(k\) th instant ; and it is called the predictor estimate \(\hat{x}[k]\), if based on measurement up to \(y[k-1]\). Historically, pure discrete-time observers have been the core of output feedback sampled-data systems. Their simple nature allows ease of implementation. However, their inability to reconstruct continuous system states forces a designer to rely on certain approximations like zero order hold / generalized old device
that may compromise system performance [38], [39]. The concept of discrete control design for sampled-data system is illustrated in the Figure 1-1 [40].

In the preceding section both observer designs are discussed with discrete-time linear time invariant systems for better understanding and relating the two observer schemes as covered in literature [41], [42].

1.4.1 **Discrete-time Prediction Observer**

There are different methods to estimate the system state vector, one is to construct the system states with the model of plant dynamics [42]

\[
\hat{e}[k + 1] = \Phi \hat{e}[k] + Gu[k],
\]  

(1.4.2)

---

1 Portion of this Section appears in [42]
where \( \Phi = e^{AT} \), \( G = \int_{kT}^{(k+1)T} e^{A[(k+1)T-\tau]} Bu(\tau)d(\tau) \) and \( \hat{e}[k] = x[k] - \hat{x}[k] \) is the prediction observer error. The plant states and observer states will equal if plant initial value \( x(0) \) and observer initial value \( \hat{x}(0) \) will be set similar. After simplification the open loop observer error can be written as

\[
\hat{e}[k+1] = \Phi \hat{e}[k].
\] (1.4.3)

The observer error dynamics will equal to the uncompensated plant, if the initial value of \( \hat{e} \) is zero. Observer error will never decrease form the initial value for a marginally stable or unstable plant. An initial error for an asymptotically stable plant will reduce only, if the plant output and observer output will both approach to zero. As the observer is running in open loop and is not using any continuous measurement of the plant states and for the same it is likely expected that it would diverge from actual plant states. However, divergence can be minimized if the observer output error is feed backed and the model is constantly corrected with this error signal. The basic idea is to construct a feedback system integrated with the open loop estimator while using the estimated output error as the feedback. This scheme is illustrated in Figure 1-2 and mathematically it is expressed as

\[
\hat{x}[k+1] = \Phi \hat{x}[k] + Gu[k] + H_p \left( y[k] - C\hat{x}[k] \right),
\] (1.4.4)

where \( H_p \) is the feedback gain matrix, (1.4.4) is called a discrete-time prediction observer. In this observer design, a measurement at time \( k \) results in an estimate of the state vector that is valid at time \( k+1 \). This estimation is predicted one cycle in advance for the future.

A difference equation describing the behavior of the observer error is expressed as

\[
\hat{e}[k+1] = [\Phi - H_p C] \hat{e}[k].
\] (1.4.5)
1.4.2 **Discrete-time Current Observer**

Discrete-time prediction observer (1.4.4) calculates the state vector estimate $\hat{x}$ after receiving measurement up to $y[k-1]$. This means that the current value of control does not depend on the most current value of the observation and thus might not be as accurate as it could be. For high order systems controlled with a slow computer or any time the sample periods are comparable to the computation time, this delay between the observation instant and the validity time of the control output can be a blessing because it allows time for the computation required to evaluate (1.4.4) and is quite short compared to the sample period. But on the other hand, this delay of almost a cycle between the measurement and the proper time to apply the resulting control calculation represents an unnecessary waste. Therefore, it is useful to construct an

---

2 Portion of this Section appears in [42]
alternative estimator formulation that provides a current estimate $\bar{x}$ based on the current sampled measurement $y[k]$. Modifying (1.4.4) to yield this feature, we obtain [42]

$$\bar{x}[k] = \hat{x}[k] + H_c (y[k] - C\hat{x}[k]),$$

(1.4.6)

where $\hat{x}[k]$ is the predicted estimate based on a model prediction from the previous time estimate, that is

$$\bar{x}[k] = \Phi\hat{x}[k-1] + Gu[k-1].$$

(1.4.7)

A control design calculated form this estimator cannot be implemented exactly because it is impossible to sample, perform calculation and output with absolutely no time lapsed. However, the calculation of $u[k]$ based on (1.4.6) can be arranged to minimize computational delays by performing all calculations before the sample instant that do not directly depend on $y[k]$.

To develop a relation between two observers gains, after simplification of (1.4.4) and (1.4.6) thus result in

$$\hat{x}[k + 1] = \Phi\hat{x}[k] + Gu[k] + \Phi H_c (y[k] - C\hat{x}[k]).$$

(1.4.8)

Furthermore, the estimation-error equation for $\hat{x}[k]$ is rewritten as

$$\hat{e}[k + 1] = (\Phi - \Phi H_c C)\hat{e}[k].$$

(1.4.9)

By comparing (1.4.4) and (1.4.8), similarly (1.4.5) and (1.4.9), relation between two observers gains is written as

$$H_p = \Phi H_c.$$  

(1.4.10)

This unique relationship is very useful in design of both time invariant observer gains. It can be extended for time varying observer gain design. The same relationship between the two estimates is further illustrated by Figure 1-3, while by showing that $\hat{x}$ and $\bar{x}$ represent different outputs of the same observer system.
These two discrete-time observers can be used for control purposes. The estimator block diagram for both observers schemes are illustrated in Figure 1-3. The current estimate is the obvious choice because it is based on the most current value of the measurement of sampled output. Its disadvantage is that it is out of date before the computer can complete the computation of (1.4.6) and (1.4.8), thus creating a delay that is not accounted for in the design process. The use of the predicated estimate for control eliminates the modeling error form the latency because it can be calculated using the measurement $y[k-1]$ thus providing an entire sample period to complete the necessary calculations on $u[k]$ before its value is required. Current observer is more appropriate choice because it provides the fastest response to unknown disturbances or measurements errors and thus better regulation of the desired output. Any deficiencies in the system response due to the latency form the computation lag that is found by simulation or experiment can be patched up with additional iterations on the desired pole locations or accounted for exactly by including computing delay in the plant model [42].
Prediction and current discrete-time observers structures are discussed in literature for discrete linear time invariant systems [42]. However to extend the same concept for observer design for linear time varying systems, discrete model for linear time varying system (1.2.1) (without disturbance) is discussed as under [43].

1.4.3 Discrete-time Model of Linear Time varying System

Consider the following linear time varying system

\[ x(t) = A(t)x(t) + B(t)u(t). \] (1.4.11)

The exact discrete-time equivalent model for (1.4.11) with sampling time \( T \) is expressed as [43]

\[ x((k+1)T) = \Phi_A\left((k+1)T, kT\right)x(kT) + \int_{kT}^{(k+1)T} \Phi_A\left((k+1)T, \sigma\right)B(\sigma)d\sigma u(kT), \] (1.4.12)

where discrete indices representing the sampling points denoted by \( k = k_o, k_o + 1... \) and parentheses (.) are used for continuous-time (CT) whereas square brackets [.] are used for discrete-time (DT) indices throughout this thesis. Let \( \Phi_A(t_f, t_i) \) be the state transition matrix associated with \( A(t) \) in its usual sense [43], where \( t_o \) and \( t_f \) are the initial and final times respectively. Discrete-time equivalent of state transition matrix can be defined as \( \Phi_A[k_f, k_o] = \Phi_A(k_f, T, k_o, T) \). In case of single step transition from \( k \) to \( k + 1 \), it is convenient to define

\[ A[k] = \Phi_A\left((k+1)T, kT\right), \]
\[ = \Phi_A[k+1, k]. \] (1.4.13)

With the identification from (1.4.12) and (1.4.13)
where \( x[k] = x(kT) \) and \( B[k] = \int_{kT}^{(k+1)T} \Phi_A((k+1)T, \sigma))B(\sigma) d\sigma \). The subsequent control law to stabilize (1.4.14) can be conveniently designed [43].

### 1.5 Continuous Control Design for Sampled-data System

Continuous control design for sampled-data system has an obvious difference with the presence of CT controller from DT control. The main requirement for the CT controller is the availability of CT observer states. This availability of continuous-time states is only possible with a continuous-time sampled data observer. Such observers scheme takes input of continuous-time control signal \( u(t) \) and sampled output \( y[k] \) [31], [44]–[48]. The concept of continuous control design for sampled-data system is illustrated in Figure 1-4.

![Schematic of sampled-data system with continuous-time control](image)

**Figure 1-4  Schematic of sampled-data system with continuous-time control**

This class of observer is mostly known as continuous impulsive observers, which overcome the limitations faced by discrete-time observers discussed in Section 1.4. In this
observer design, reconstruction of continuous-time system states are carried out by using sampled output and continuous-time input [49]. The functionality and structure of impulsive observers typically discussed in literature are conceptually that of current observers. Even though, impulsive observes can also be categorized into current and prediction observers but in literature only current impulsive observer is discussed because of its peculiar advantages [49], [50]. The probable reason for this is the intuitively appealing structure of current impulsive observers. However from an application perspective the estimates cannot be computed within infinitesimally small time, as it is not possible to sample and perform all the computations in such a short duration[37], [50]–[55].

In literature current impulsive observers for a class of linear impulsive system have been dealt in [50], [56], [57]. In these contributions, the system dynamics were also impulsive with discrete-time measurements of the output. In [58] an impulsive observer is proposed for the stabilization of uncertain non-impulsive system. These authors designed impulsive observer by taking the samples just before the sampling instant i.e. $t_k^- = kT^-$. This philosophy is not in line with the standard sampling theory wherein the samples are obtained at the exact sampling instants. In [59], an impulsive observer is proposed making use of continuous-time and discrete-time outputs. The same concept of current observer is also extended to the sampled-data nonlinear systems as discussed in [9], [60], [61]. Furthermore, system output at sampling point is used to introduce impulsive correction at $t_k^+$. This leads to lack of differentiability of the open-loop state estimation dynamic equation from the left hand side. Semi-global or local stability can be achieved by choosing a considerably large observer gain or small enough sampling time. Different techniques for observation of system states and estimation of model uncertainties for nonlinear systems are discussed in literature. The design of discrete controller using adaptive
algorithms [62]–[65] or sampled-data based fuzzy models [66]–[68] for estimation are also available possibilities.

1.6 Output Regulation for Sampled-data Systems

Output regulation theory for sampled-data system is one of the significant areas of control system research. It deals with tracking of reference signals and rejection of disturbance signals. A feedback controller is designed such that asymptotic tracking is achieved, while maintaining closed loop stability for a class of reference signals and rejection of disturbances [51]–[54] [73]. Different control schemes have been discussed in literature such as linear quadratic optimal control, proportional-integral-derivative control, feedback linearization, Kalman filtering control, and sliding mode control [74]–[79] and references therein. The sampled-data regulation control theory in particular has been studied for constant reference tracking with zero order hold and the same is discussed in [80], [81]. Researchers attempted to address the sampled-data regulation problem with discrete-time controllers. Controller design in discrete-time was explored in [82]–[85]

In literature, the controller design of sampled-data nonlinear system has been designed in [86]–[90]. Sampled-data nonlinear design schemes such as continuous-time controller redesigns and control system for a class of nonlinear uncertain systems are considered in [86], [87]. Integrator back-stepping for sampled-data nonlinear systems stabilization [88], and controller designs by receding horizon methods [27] also discussed for stabilization of nonlinear systems. In [89], filtering on sampled-data with parametric uncertainty is discussed and multirate output feedback controls are discussed for stabilization of nonlinear systems in [90].
Furthermore, Reference [91] gives a power series representation of the sampled-data system in form of vector fields of original continuous-time system. In [91], [92], it is investigated how certain properties such as controllability, observability, and linearizability are preserved in regard with sampling for sampled-data systems. Quantization effects including local stability analysis are presented in [93]. In [94]–[96], certain conditions are discussed for sampled-data controller under which it recovers the performance of continuous-time controller for sufficiently large sampling frequency.

Feedback linearization is also a very effective design strategy to address the regulation problem for nonlinear systems. An exact linear representation of original nonlinear model is produced by cancelling the non-linearities, thus the control design works over the large set of operating conditions [17]. This control design requires the information of estimates used in feedback linearization.

Sampled-data regulators for linear systems have been presented in [8], [9], [58], [97]–[100]. Amongst these, reconstruction of the continuous-time states from sampled output using a realizable reconstruction filter is proposed in [8]. The same technique was further extended for regulation of non-liner systems with sampled output in [9]. In [97], a memoryless generalized hold device has been introduced that generates the continuous-time control signal. Sampled-data output regulation problem with constant exogenous signals has been addressed in [98]. The regulation is achieved by considering constant exogenous signals. The impulsive observer based stabilization of uncertain system has been discussed in [58].

Formally, output regulation for sampled-data linear time varying systems is stated as

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t) + P(t)w(t)
\]

\[
y[k] = C[k]x[k].
\]

(1.6.1)
The estimation of states / disturbances is obtained through an observer as a standard solution for output feedback problems. Continuous-time plant tracks an output reference signal generated by an exosystem of the form (1.2.2). A following continuous-time control is generated by solving regulator equations (assumption A4) such that

\[ u(t) = [F(t) \quad \Gamma - F\Pi(t)] \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}. \]  

(1.6.2)

The main challenge for controlling a continuous plant output (1.6.1) is the availability of continuous-time observer states \( \hat{x}(t) \) for control input \( u(t) \) calculations. An impulsive observer is thus required to produce continuous-time states estimates with the help of continuous-time input and sampled output such that the tracking error \( e_i(t) \rightarrow 0 \) as \( t \rightarrow \infty \).

### 1.7 Output Regulation with Impulsive Observer

State estimation for continuous-time linear systems with mixed continuous-discrete [101] or only discrete [56], [102]–[104] measurements naturally leads to an observer with an impulsive structure. These structures are used for regulation purpose in different configurations. The stabilization of impulsive sampled-data linear time varying systems is discussed in [50], [57], [50], [104]. In these contributions arbitrarily spacing of impulses and possible singular state transition matrices has been considered. In these contributions, authors discussed the stability of impulsive observers by only considering a stable closed-loop system. Stability analysis was carried out with the geometric approach, whereas time varying Lyapunov function is defined for establishing the stability of the closed-loop system in [58] for sampled-data uncertain system.

The discussion in [101], [105], [106] leads to the opportunity for use of impulsive observers for sampled-data nonlinear systems regulation. The same idea is used for only
linearization of nonlinear systems in [31], [9] and regulation is achieved by converting discrete control input to continuous control signal with realizable reconstruction filter. Impulsive observer is used for feedback linearization. This design strategy has the limitation that discrete-time control scheme is used for the regulation of sampled-data nonlinear system with discrete-time state observations [9]. In feedback linearization there is always some mismatch between actual model and nominal model of nonlinear system. This difference appears as a model uncertainty. In [107], a model uncertainties estimation of a nonlinear system converts the problem of unmatched disturbance rejection into a matched one. Disturbance is estimated first by using a nominal model and control design incorporates the same estimated disturbance for plant out regulation.

In these approaches convergence analysis of state estimation error mainly depends upon discrete-time convergence whereas, inter-sample behavior of the observer has not been discussed in stability analysis. Stability requirement does not meet by neglecting the effect of inter-sample behavior of unstable system.

1.7.1 Effects of Inter-sampling Behavior in Sampled-data Regulation

In literature, various methods are adopted to discuss the inter-sampling effects and stability analysis of closed-loop system [55]–[58], [43], [83]. The stability of impulsive observers by only considering a stable closed loop system is considered in [50], [56], [104]. Stability analysis was carried out with the geometric approach, whereas time varying Lyapunov function is defined for establishing the stability of the closed-loop system in [58]. In these approaches convergence analysis of state estimation error merely depends upon discrete-time convergence and inter-sample behavior of the observer has not been discussed.
1.8 **Requirement of New Developments**

The impulsive action in control design is not considered appropriate for regulation and efforts are required to smooth it at the maximum. The inability to produce a continuous (ripple free) output regulation with sampled output of linear or non-linear continuous-time systems is the key problem encountered in the sampled-data regulation. Impulsive observers discussed in the literatures are used for regulation provides regulation with ripples. This is because of the impulsive corrective action of the observer at sampling time instant. Some mathematical ambiguities as highlighted in Section 1.5 in present structure of so-called current impulsive observers is also required to be addressed for clarity. The construction of continuous-time (non-impulsive) reconstruction observer is the solution for the ripple free regulation. It is only possible to review the complete impulsive observer design strategy and suggest a non-impulsive observer scheme. In literature, inter-sample behavior of impulsive observers is covered only for stable systems. There is a requirement to design a robust stability analysis which covers stable, unstable and highly unstable systems convergence stability analysis.

The continuous (non-impulsive) reconstruction observer presented in this dissertation has the advantages of reconstruction of system states without impulsive jumps at sampling instants. This feature makes it an excellent contender for non-impulsive control signal and further used for smooth ripple free regulation. Reconstruction observer is proposed as a sampled-data observer which in turns used in sampled-data regulation (without ripples) for three important classes of systems: linear time invariant, linear time varying and feedback linearizable nonlinear system.
1.9 Overview of the Thesis

This thesis is organized into five chapters including the introduction. The emphasis is to develop a novel sampled-data observer without jumps for continuous linear time varying system. The proposed observer is based on states estimates from two different sampled-data observers with jumps, combined in such a way that overall estimated states become continuous. Proposed reconstruction observer is further used in design of a ripple free regulation control.

Chapter 2 discusses the major contribution of this thesis that is a development of (non-impulsive) reconstruction observer for linear time varying systems. Current impulsive observer and prediction impulsive observer are proposed. Prediction impulsive observer is not discussed in literature as per the proposed design. Both the proposed impulsive observers’ schemes are fused together with a mathematical relation resulting in a design of reconstruction observer. This observer combines the state estimate of current and prediction observer by using appropriate time dependent weighting function. The advantage of non-impulsive reconstruction observer is eradication of jumps in state estimation, with very obvious advantages in closed loop control. A comprehensive convergence analysis of the proposed observer is presented for stable, unstable and highly unstable continuous-time linear time varying systems.

A linear time varying regulator for sampled-data system has been presented in Chapter 3. Sampled-data linear time varying regulator has been proposed with non-impulsive state reconstruction observer for sampled data linear time varying systems. A regulation is achieved for constant and time varying sampled-data reference signals along with disturbance rejection. Furthermore hybrid reconstruction observer based on the combination of current and prediction observers enables non-impulsive output regulation. The closed-loop stability analysis is performed to show uniform exponential stability of the sampled-data linear time varying
regulator with reconstruction observer.

Chapter 4 discusses the design methodology for regulating a feedback linearizable sampled-data nonlinear system. Feedback linearization along with state estimation is carried out with continuous reconstruction observer (without jumps). The novel proposed observer design not only provides state estimates but also performs feedback linearization. This is being carried out with the help of two different sampled-data observers with jumps, combined in such a way that overall estimated states become continuous and linear. This leads to a non-impulsive continuous-time sampled-data regulation. A stability analysis is carried out while considering the model uncertainties of a nonlinear system as a non-vanishing perturbation. An example of a third order nonlinear system illustrates the efficacy of proposed design methodology.

The last chapter is about conclusions and suggestions and it also concludes the thesis.
CHAPTER 2

3 CONTINUOUS RECONSTRUCTION OBSERVER

In this chapter, a continuous observer without jumps for sampled-data continuous linear time varying system is discussed. The proposed observer is based on states estimates from two different sampled-data impulsive observers, fused in a way that overall estimated states become continuous (without jumps). A mathematical relationship is developed between the two newly designed observers and ultimately this relationship is exploited to develop a continuous (non-impulsive) reconstruction observer. A comprehensive convergence analysis of the proposed observer is presented for stable, unstable and highly unstable continuous-time linear time varying systems. It guarantees uniform exponential stability on discrete sampling points and explicitly covers the inter-sample behavior of continuous-time systems.

In literature impulsive observers for a class of linear impulsive system have been dealt in [3], [9], [10]. The estimation of reconstruction of the impulsive system states using impulsive observer is carried out in two steps. In the first steps, open-loop estimation of continuous-time system states is carried out within the sampling interval by integrating. In the second step impulsive correction is introduced in the estimated states. In these contributions, the system dynamics were also impulsive with discrete-time measurements of the output. In this thesis, prediction impulsive observer design is also presented which to the best of authors’ knowledge has not been discussed in literature.

To summarize, this chapter cover the following motives: i) to design a prediction impulsive observer, ii) address the ambiguities in the mathematical framework of existing current

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3 Portion of this Chapter appears in [113]
observer design, iii) to introduce mathematically correct notation for discontinuous nature of impulsive observers, iv) to develop a relation between prediction and current impulsive observer schemes, v) using this relation, design a continuous (non-impulsive) reconstruction filter vi) to carry out a comprehensive stability analysis for both stable unstable and highly unstable systems for all the discussed observers.

The reconstruction observer design is the main focus of this chapter. Sampled-data system with the brief account of continuous-time and discrete-time stability along with some elementary definitions is discussed in Section 2.1. Section 2.2 presents the design of proposed reconstruction observer in relation with the design of proposed current and prediction impulsive observers, which is further used in Section 2.3 to present observer error dynamics. Stability analysis in Section 2.4 is started with a motivating example; this section discusses the stability for stable, unstable and highly divergent (unstable) systems. Finally a second order system example is illustrated in Section 2.5

### 2.1 Sampled-data Continuous-time System

Consider the following sampled-data linear time varying system

\[
x(t) = A(t)x(t) + B(t)u(t) \quad x(t_o) = x_o.
\]

\[
y[k] = C[k]x[k],
\]

where \( x(t) \in \mathbb{R}^n \) is the continuous-time state vector and \( u(t) \in \mathbb{R}^m \) is the continuous-time plant input. Control input \( u(t) \) may be discontinuous but is assumed to be non-impulsive in nature. Only sampled output \( y[k] \in \mathbb{R}^n \) is available for measurement, where \( y[k] = y(kT) \cdot T \) is assumed to be non-pathological fixed sampling time [108]. \( A(t), B(t) \) and \( C[k] \) are time varying bounded
system matrices of appropriate dimensions. The objective is to design an observer that reconstructs the continuous-time states estimate from the discrete-time observations. It may be noted that the contribution in this thesis can conveniently be extended for non-uniform sampling time, however, that case has not been discussed for clarity of presentation.

It is assumed that the continuous-time state estimate \( \hat{x}(t) \) is generated by some observer that makes use of the sampled output \( y[k] \) and the continuous-time input \( u(t) \). It is also assumed that the observer has exact knowledge of system matrices \( A(t), B(t) \) and \( C[k] \). The case of uncertain system matrices and presence of any disturbances is not the scope of this thesis. The state estimation error may be expressed as

\[
e(t) = x(t) - \hat{x}(t). \tag{2.1.3}
\]

The initial error corresponding to initial guess \( \hat{x}(t_o) \) is

\[
e_o = e(t_o) = x(t_o) - \hat{x}(t_o).
\]

### 2.1.1 Uniform exponential stability

The exponential convergence of estimation error is mathematically expressed as follows

**Definition 2.1:** An observer is uniformly exponentially stable with rate \( \lambda \), if there exist a positive constant \( \gamma \) such that for any \( t_o \) and \( e_o \) estimation \( e(t) \) satisfies

\[
\|e(t)\| \leq \gamma e^{-\lambda(t-t_o)} \|e_o\| \quad \forall \quad t \geq t_o, \tag{2.1.4}
\]

where \( \lambda > 1 \) and \( \gamma \geq 1 \).
The process of continuous-time states estimation generally relies on impulsive observers [50]. A combination of two types of equations is required to model such systems. One is continuous-time and the other one is impulsive in nature i.e. an equation which incorporates jumps. Related detailed mathematical discussion will follow in subsequent sections. In this section, we only mention the fact that design of impulsive observer is performed by using an associated discrete-time system. Accordingly, a signal of interest is the discrete-time estimation error \( e[k] \), which may be defined as [41]

\[
e[k] = e(kT) = x(kT) - \hat{x}(kT)
\]

\[
e[k_o] = e(t_o) = x(t_o) - \hat{x}[k_o] = (t_o)
\]

(2.1.5)

The exponential convergence for discrete-time sequence is mathematically defined as follows

**Definition 2.2**: The discrete-time sequence \( e[k] \) is called UES with rate \( \tilde{\lambda} \), if there exist a constant \( \tilde{\gamma} \) such that for any \( k_o \) and \( e_o \), the observer error \( e[k] \) satisfies the following

\[
\|e[k]\| \leq \tilde{\gamma}^{(k-k_o)T} \|e_o\|
\]

(2.1.6)

where \( \tilde{\lambda} > 1 \) and \( \tilde{\gamma} \geq 1 \).

\[
\text{It may be noted that uniform exponential stability of the observer is associated with the uniform exponential stability of the continuous-time estimation error } e(t) \text{ and not with uniform exponential stability of discrete-time estimation error } e[k]. \text{ The significance of this fact will become clear later.}
\]
2.1.2 State transition matrix

Let $\Phi(t_f, t_o)$ be the state transition matrix associated with $A(t)$ in its usual sense [43], where $t_o$ and $t_f$ are the initial and final times respectively. Discrete-time equivalent of state transition matrix can be defined as $\Phi[k_f, k_o] = \Phi(k_f, T, k_o T)$. In case of single step transition from $k$ to $k+1$, it is convenient to define

$$A[k] = \Phi((k+1)T, kT),$$
$$\quad = \Phi[k + 1, k]. \tag{2.1.7}$$

By virtue of state transition matrix being full rank, $A[k]$ is always invertible. An important assumption for observer design is the $l$-step observability of the pair $(A[k], C[k])$ [43]. This condition may be relaxed to the detectability of the same pair. However, this case is not discussed in this thesis.

2.2 Sampled-Data Impulsive Observers

In literature state estimation of dynamic systems based on sampled observations has been extensively discussed. The most typical approach of reconstruction is the introduction of impulsive correction in state estimates on the basis of sampled output immediately after the observation has been obtained. Following the impulsive correction, the state estimates are constructed based on system dynamics for the rest of the sampling interval. This concept can be found in [50][57] and from here on in, shall be referred to as current impulsive observer in this thesis.
2.2.1 Current Impulsive Observer

The current impulsive observer for (2.1.1) is proposed as follows

\[
\bar{x}(t^+) = \bar{x}(t) + H_r[k](y[k] - \bar{y}[k]) \quad t = kT,
\]

\[
\dot{x}(t) = A(t)\bar{x}(t) + B(t)u(t) \quad kT < t < (k+1)T,
\]

\[
D^-\bar{x}(t) = A(t)\bar{x}(t) + B(t)u(t) \quad t = (k+1)T, \\
k = k_o, k_o + 1, \ldots,
\]

where \( \bar{x} \in R^n \) is the current observer state and \( H_r[k] \in R^{nxp} \) is a time varying current observer gain matrix. Let the initial state of the observer be \( \bar{x}(t_o) = \bar{x}_o \). We use the notation \( \bar{x}(t^+) = \lim_{h \to 0^+} \bar{x}(t+h) = \bar{x}(kT^+) \). The significance of strict inequalities may be noted. The second term on the right hand side of (2.2.1) is the impulsive correction. Equation (2.2.2) predicts the states for time \( kT < t < (k+1)T \) based on the estimate at \( t = kT \). Due to impulsive nature of the observer the derivative from the left has been adopted in (2.2.3), as discussed in preceding section. The discrepancy in structure of current observer typically found in literature i.e. definition of \( x(t^-) \) creates a mathematical ambiguity with regards to availability of sample point at \( t = kT \) [3], has been addressed. Due to impulsive nature of the observer, the derivative of \( \bar{x}(t) \) does not exist at \( t = kT \) for \( k = k_o + 1, k_o + 2, \ldots \). Instead, we have used the derivative from below, which does not face mathematical difficulties as the state estimation \( \bar{x}(t) \) is continuous from left. Equations (2.2.2) and (2.2.3) have to be integrated over \((kT, (k + 1)T]\) in order to obtain \( \bar{x}(t) \).

Differentiability from left at the initial point does not cause difficulty for open-loop integration due to Riemann-Lebesgue Theorem [109].
It is illustrated in Figure 2-1 and mathematically expressed by (2.2.1) that the states estimate experiences a jump because of the impulsive correction. The discontinuous nature of the estimated signal is undesirable from application perspective in a closed loop system.

The jump in the state estimate can be circumvented, if during the interval of time (significantly smaller than the sampling time) in which the jump occurs, the estimates are obtained from an alternate algorithm which provides continuous state estimates during this interval. However, the alternate algorithm must coincide with the current impulsive observer at the sample points. Therefore the alternate algorithm would also be impulsive in nature but its impulsive correction would take place at a different point as compared to the current impulsive observer during the sampling interval.

The alternate observer which is continuous immediately following the sampling point and coincide with the current observer at the sampling point must experience a jump just prior to the sampling point, such an observer is termed as prediction observer.
2.2.2 Prediction Impulsive Observer

The prediction impulsive observer for (2.1.1) is proposed as follows

\[
D^+ \hat{x}(t) = A(t)\hat{x}(t) + B(t)u(t) \quad t = kT, \quad (2.2.4)
\]

\[
\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) \quad kT < t < (k+1)T, \quad (2.2.5)
\]

\[
\hat{x}[k+1] = \hat{x}(t^-) + H_p[k](y[k] - \hat{y}[k]) \quad t = (k+1)T \quad k = k_o, k_o + 1, \ldots, \quad (2.2.6)
\]

where \( \hat{x} \in R^n \) is the prediction observer state and \( H_p[k] \in R^{nxp} \) is a time varying prediction observer gain matrix. Let the initial state of the observer be \( \hat{x}(t_o) = \hat{x}_o \). We also use the notation \( \hat{x}(t^-) = \lim_{h \to 0} \hat{x}(t-h) \). Equation (2.2.5) predicts the states for time \( kT < t < (k+1)T \) based on the estimate at \( t = kT \). The significance of strict inequalities may be noted. Having a closer look at (2.2.6), \( \hat{x}(t^-) = \hat{x}((k + 1)T^-) \) may be recognized as the predicted state. The second term of right hand side of (2.2.6) may be construed as the impulsive correction, which is applied within an infinitesimally small duration from \( \hat{x}((k + 1)T^-) \) to \( \hat{x}((k+1)T) \). Left hand side of (2.2.6) is in fact \( \hat{x}[k + 1] \). The observer thus falls within the well-known predictor-corrector form. For the next iteration \( k+1 \) becomes \( k \) and the process continues. Due to impulsive nature of the observer, the derivative of \( \hat{x}(t) \) does not exist at \( t = kT \) for \( k = k_o + 1, k_o + 2, \ldots \). Instead, we have used the derivative from above, which does not face mathematical difficulties as the state estimation \( \hat{x}(t) \) is continuous from right. Equations (2.2.4) and (2.2.5) have to be integrated over \([kT, (k + 1)T)\) in order to obtain \( \hat{x}(t) \). Differentiability from right at the initial point does not cause difficulty for open-loop integration due to Riemann-Lebesgue Theorem [109].
2.2.3 Relation between Current and Prediction Observer

A relationship between current and prediction observer gains is established on the basis of the following expression as discussed for discrete-time observers in [42]

\[ \bar{x}(t) = \hat{x}(t), \quad t = kT, \quad k = k_0, k_0 + 1... \]

(2.2.7)

The following analysis reveals an interesting relationship between the current and prediction impulsive observers. The solution of (2.2.2), (2.2.3) is written as

\[ \bar{x}(t) = \Phi(t, kT^+) \bar{x}(kT^+) + \int_{kT^-}^{t} \Phi(t, \sigma) \beta(\sigma) u(\sigma) d(\sigma), \quad kT < t \leq (k+1)T. \]

(2.2.8)

Evaluating (2.2.8) at \( t = (k+1)T \) with the help of (2.1.7), (2.2.1) and (2.2.7) where the assumption on \( B(t), u(t) \) and continuity of states have been kept under consideration, we get
\[
\bar{x}[k + 1] = A[k]\hat{x}[k] + \int_{kT}^{(k+1)T} \Phi((k+1)T, \sigma)B(\sigma)u(\sigma)d(\sigma)
+ A[k]H_p[k]C[k](\hat{x}[k] - \hat{x}[k]).
\] (2.2.9)

Similarly for prediction observer

\[
\hat{x}[k + 1] = A[k]\hat{x}[k] + \int_{kT}^{(k+1)T} \Phi((k+1)T, \sigma)B(\sigma)u(\sigma)d(\sigma)
+ H_p[k]C[k](\hat{x}[k] - \hat{x}[k]).
\] (2.2.10)

Comparing (2.2.9) and (2.2.10), we get

\[
H_e[k] = A^{-1}[k]H_p[k],
\] (2.2.11)

and \(\bar{x}(k_oT) = \hat{x}(k_oT)\), guarantee the following

\[
\bar{x}(t) = \hat{x}(t), \quad t = kT,
\]

\[
k = k_o, k_o + 1, ...
\] (2.2.12)

Relation (2.2.12) signifies that the current and prediction impulsive observer estimates coincide at the sampling points. This important property will be exploited in the development of proposed continuous observer in the following section.

The expression (2.2.12) implies \(\hat{e}[k] = \bar{e}[k]\). The respective DT equations representing error dynamics for current and prediction impulsive observers are as follows

\[
\bar{e}[k + 1] = (A[k] - H_p[k]C[k])\bar{e}[k].
\] (2.2.13)

The property of equality of estimation errors for current and prediction impulsive observers is exploited in the development of proposed continuous sampled-data observer in the following section.
2.2.4 A Continuous Reconstruction Observer

A combination of the current and prediction observers formed in such a way that the weightage of estimate from the current observer is approximately zero following the sample point and the weightage of the prediction observer immediately before the sampling point is close to zero results in a continuous-time reconstruction observer from sampled states as illustrated in Figure 2-3.

This relationship between prediction and current observers can be exploited to develop a continuous (non-impulsive) reconstruction observe by fusing the outputs of the two observer’s schemes, using a following weighting function as illustrated in Figure 2-4.
Figure 2-4  Fusion function $\psi(t) = 1 - 0.5 \text{tanh}(5\mu(t)) + 0.5$, where $\mu(t)$ is a sawtooth function with $t^* = \left( k + \frac{1}{2} \right) T$

\[
\Psi(t) = \begin{cases} 
0 & t = kT \\
1 & t = (k+1)T \\
\frac{1}{2} & t = t^* \\
\Psi(t) & t \in \Delta_2 \\
0 & t \in \Delta_1, \Delta_3,
\end{cases}
\]

where $t^*$ is generally the midpoint i.e. $\left( k + \frac{1}{2} \right) T$ but may have a different value if required and $\varphi(t)$ is monotonically increasing continuous function. Reconstruction observer output depends upon current, prediction and weighting function outputs and is mathematically expressed as

\[
\tilde{x}(t) = \bar{x}(t)\psi(t) + \hat{x}(t)\left(1 - \psi(t)\right). \tag{2.2.14}
\]

Once the sampled output is available, maximum weight is given to the prediction observer output and the least weight is given to the current observer output for some obvious reasons as explained earlier. Outputs from both the observers are added to get a continuous-time
reconstruction observer output. Both outputs are given equal weights at \( t' \) and hence the output of the reconstruction filter is rightly in the middle of the current and prediction observer. More weight is given to the current observer output for the next half of sampling time period. Maximum weight is accorded to current observer output just before the next sampling point, hence reconstruction observer output follows exactly of the current observer output. Finally the reconstruction observer output is concluded at a common point of current and prediction observer. Figure 2-5 illustrates the design of proposed reconstruction observer with two impulsive observers.

The smoothness properties of the proposed Continuous Sampled-data reconstruction observer with necessary conditions are discussed in the following two Lemmas. Continuity of the estimated states is established in Lemma 2.2.5. Conditions for continuous differentiability of the state estimates are summarized in Lemma 2.2.6

![Block diagram of continuous reconstruction observer](image)

**Figure 2-5** Block diagram of continuous reconstruction observer

### 2.2.5 Lemma

Under the stated assumptions, the state estimates using the proposed continuous sampled-data reconstruction observer (2.2.14) are continuous.
Proof:

The continuity of the state estimates outside the neighborhood of sampling points is guaranteed by continuity properties of addition and multiplication of continuous function [43][109][110]. The continuity of the estimates \( \tilde{x}(t) \) at the sampling instants is proved as follows.

From continuity of system states, current impulsive observer is continuous from left that is \( \tilde{x}(t) = \tilde{x}(t^-) \) and prediction impulsive observer is continuous from right that is \( \hat{x}(t) = \hat{x}(t^+) \) at \( t = kT \). From (2.2.7), \( \tilde{x}(t^-) = \hat{x}(t^+) \), which proves the continuity of reconstruction observer at sampled point and its neighborhood.

2.2.6 Lemma

The state estimates (2.2.14) are continuously differentiable under the additional assumption of continuous differentiability of \( u(t) \) and \( \psi(t) \). \( \Delta_1 \) and \( \Delta_2 \) are assumed to be sufficiently large to contain the jump.

Proof:

The continuous differentiability of state estimates of continuous reconstruction observer outside the intervals \( \Delta_1 \) and \( \Delta_3 \), again follows trivially under the assumptions of continuous differentiability of \( u(t) \) and \( \psi(t) \). In the proximity of a sample point, the dynamics of the continuous reconstruction observer are represented by the following differential equations.

Immediately prior to sample point \( t = kT \) in the interval \((k-1)T, kT)\)

\[
\dot{x}(t) = A(t)\tilde{x}(t) + B(t)u(t) \quad t \in \Delta_3,
\]

\[
D^-\dot{x}(t) = A(t)\tilde{x}(t) + B(t)u(t) \quad t = kT,
\]

and immediately following the sample point \( t = kT \) the dynamics in the interval \((kT, (k+1)T)\)

\[
D^+\dot{x}(t) = A(t)\hat{x}(t) + B(t)u(t) \quad t = kT,
\]

\[
\dot{x}(t) = A(t)\hat{x}(t) + B(t)u(t) \quad t \in \Delta_1.
\]
From (2.2.12), 2\textsuperscript{nd} equation of (2.2.15) and 1st equation of (2.2.16)

\[ D^\tau \ddot{x}(t) = D^\tau \ddot{x}(t) \quad t = kT, \]

which guarantees the existence of \( \frac{d}{dt} \ddot{x} \) and proves the assertion.

### 2.3 Observers Error Dynamics

#### 2.3.1 Prediction Impulsive observer

Error state vector of prediction impulsive observer is written as

\[ \dot{e}(t) = x(t) - \ddot{x}(t), \quad (2.3.1) \]

derivative of (2.3.1) is

\[ D^\tau \dot{e}(t) = A(t)\dot{e}(t) \quad t = kT, \quad (2.3.2) \]

\[ \dot{e}(t) = A(t)\dot{e}(t) \quad kT < t < (k+1)T, \quad (2.3.3) \]

where (2.3.2) signifies the discontinuous nature of error \( \dot{e}(t) \). DT error dynamics using prediction observer is written as

\[ \dot{e}[k+1] = x(t) - \left[ \dot{x}(t^-) + H_p[k](y[k] - \hat{y}[k]) \right] \quad t = (k+1)T, \quad (2.3.4) \]

continuity of system states [19] imply \( x(t) = x(t^-) \) leading to

\[ \dot{e}[k + 1] = \dot{e}(t^-) - H_p[k](y[k] - \hat{y}[k]) \quad t = (k+1)T. \quad (2.3.5) \]

Associated discrete equation is expressed as
\[ \dot{e}[k+1] = (A[k] - H_p[k]C[k])\dot{e}[k]. \] (2.3.6)

The gain \( H_p[k] \) is to be designed such that (2.3.6) is uniformly exponential stable (UES). One such design as discussed in [19] is presented in the following theorem. The theorem involves \( M[k-l+1, k+1] \) and this also connects to the notation of \( l \)-step observability, whereas conditions for convergence for prediction impulsive observer is covered in detail in preceding sections.

2.3.2 Theorem

For linear state equation (2.2.6), suppose \( A[k] \) is invertable at each \( k \), and suppose there exists a positive integer \( l \) and positive constant \( \epsilon_1 \) and \( \epsilon_2 \) such that [43]

\[ \epsilon_1 I \leq \Phi^T[k-l+1,k+1]M[k-l+1,k+1]\Phi[k-l+1,k+1] \leq \epsilon_2 I \quad \forall \, k, \]

for a given a constant \( \tilde{\lambda} > 1 \) the observer gain

\[ H_p[k] = [\Phi^T[k-l+1,k+1]M[k-l+1,k+1]\Phi[k-l+1,k+1]]^{-1} A^{-T}[k]C^T[k], \] (2.3.7)

is such that the resulting observer error state equation (2.3.6) is uniform exponential stable with rate \( \tilde{\lambda} \) and \( M_\lambda(k_o,k_f) \) is the discrete-time observability Gramian

\[ M_\lambda(k_o,k_f) = \sum_{j=k_o}^{k_f-1} \lambda^{\alpha(j-k_o)} \Phi^T(j,k_o)C^T(j)C(j)\Phi(j,k_o). \]

2.3.3 Current Impulsive Observer

Error state vector of current impulsive observer is written as
\begin{equation}
\bar{e}(t) = x(t) - \bar{x}(t) \quad \forall \ t,
\end{equation}

The derivative of (2.3.8) is

\begin{equation}
D^e \bar{e}(t) = A(t) - \bar{e}(t) \quad t = kT,
\end{equation}

\begin{equation}
\bar{e}(t) = A(t) \bar{e}(t) \quad kT < t < (k+1)T.
\end{equation}

Discrete-time error dynamics using current observer

\begin{equation}
\bar{e}(t^*) = x(t^*) - \bar{x}(t^*) \quad t = kT,
\end{equation}

\begin{equation}
= x(t^*) - [\bar{x}(t) + H_c[k](y[k] - \bar{y}[k])],
\end{equation}

continuity of system states [43] imply \(x(t) = x(t^*)\) leading to

\begin{equation}
\bar{e}(t^*) = \bar{e}(t) - H_c[k]C[k](x[k] - \bar{x}[k]) \quad t = kT,
\end{equation}

after simplification (2.3.12) is expressed as

\begin{equation}
\bar{e}(t^*) = (I - H_c[k]C[k])\bar{e}[k] \quad t = kT,
\end{equation}

where gain \(H_c[k]\) is required to be designed such that (2.3.13) is uniformly exponentially stable.

\subsection*{2.3.4 Reconstruction Observer}

Reconstruction observer error is written in terms of prediction and current observer errors with relation to weighting function as expressed in (2.2.14)

\begin{equation}
\hat{e}(t) = \bar{e}(t)\psi(t) + \hat{e}(t)(1 - \psi(t)),
\end{equation}

where \(\hat{e}(t)\) is reconstruction observer error.
2.4 Stability Analysis

The stability analyses of current and prediction impulsive observers and continuous reconstruction filter are presented in this section. The convergence of continuous time estimation error signal $e(t)$ is established for both prediction and current observers. Three different cases are considered for each observer depending upon system dynamics. The stability of continuous-time reconstruction observer follows directly from the stability of prediction and current observer.

Stability of reconstruction observer is discussed in conjunction with the stability of proposed prediction and current impulsive observers. Overall stability of reconstruction observer depends upon the stability of current and prediction impulsive observers for all time. Stability of reconstruction observer is guaranteed only by ensuring the stability at discrete points with assurance of inter-sample behavior convergence as well. In literature, stability analysis for only stable systems are discussed in design of impulsive observers [50][104]. A following motivating example of an unstable system will illustrate the fact that merely guaranteed stability at discrete sampled points does not guarantee the overall stability of impulsive observers.

2.4.1 Motivating Example

Consider the following linear time varying system

$$\dot{x}(t) = \begin{bmatrix} 2ct & 0 \\ 0 & -2\kappa t \end{bmatrix} + Bu(t),$$

$$y[k] = C[k]x[k].$$
where \( B = [0 \ 1]^T \), \( C = [1 \ 2] \) and \( u(t) = \sin(2\pi t) \) with initial conditions \( x_0 = [1 \ 0.5]^T \). The system is unstable by selecting \( \varepsilon = \frac{1}{3} \) and \( \kappa = -\frac{1}{30} \). Prediction impulsive observer is designed for the system \( \tilde{\lambda} > 1 \), which guarantees the convergence of associated DT estimation error dynamics (2.3.6) [43].

![Graph](image-url)

**Figure 2-6** Estimation error with prediction impulsive observer, \( \tilde{\lambda}_i = 1.11 \) (a). Convergence at discrete points (b). Divergent inter sample behavior
State Estimation error with two different convergence rates $\lambda_1 = 1.11$ and $\lambda_2 = 1.60$ for prediction impulsive observers are plotted in Figure 2-6 and Figure 2-7 respectively. Figure 2-6(a) shows convergence of estimation error at discrete points in time for $\lambda_1 = 1.11$, however Figure 2-6(b) shows that prediction estimation error within the sampling points described by equation (2.2.4) and (2.2.5) diverges. In Figure 2-7, estimation error for convergence rate $\lambda_2 > \lambda_1$ is plotted for $\lambda_2 = 1.60$. In this case estimation error is convergent at discrete points in time and also within the sampling interval.

It turns out that in unstable systems, divergence inter-sample behavior and convergent impulsive corrections occurs simultaneously. For the overall asymptotic stability of estimation error, convergence must dominate divergence. It shows that it is must to incorporate the inter-sample behavior of the states in stability analysis discussion.

The stability of reconstruction observer is only guaranteed when the stability of both current and prediction impulsive observers are ensured for all times. A following novel stability
analysis is carried out, which covers both the aspect of discrete convergence at sampled points
and inter-sample behavior of the states for proposed observer’s schemes.

2.4.2 Continuous Exponential Bound for Current Impulsive Observer

We define \( \alpha_i \geq 1 \) as the following bound for STM over the sampling interval

\[
\| \Phi_{A}(kT, t) \| \leq \alpha_i \quad (k+1)T \geq t > kT. \tag{2.4.1}
\]

The order of the two time argument of STM indicates integration backward in time,
where \( i=0,1,2,\ldots \). Such a bound \( \alpha_i \) always exists in linear systems on STM over a closed
intervals \( (k+1)T \geq t > kT \), where \( k \) is the discrete time index. A sequence of such bounds
termed as State Transition Matrix Backward Sequence (STMBS) over the sampling interval \( T \) is
given by

\[
\{\alpha_0, \alpha_1, \alpha_2, \ldots\}. \tag{2.4.2}
\]

The STMBS diverges with less than or equal to exponential rate if

\[
\alpha_i \leq \eta_i \alpha^i \quad \forall \ i \geq 0, \tag{2.4.3}
\]

where \( \alpha = \frac{\alpha_i}{\eta_i} \) and \( \eta_i = \alpha_0 \).

With the above definitions, the following theorem establishes sufficient conditions for
continuous exponential bound for estimation error of current impulsive observers.

2.4.3 Theorem

The CT estimation error for the current impulsive observer is uniformly exponentially
bounded as follows

\[
\| \bar{e}(t) \| \leq \gamma_{e} e^{\gamma_{e} (t-t_0)} \| e_0 \|,
\]
with positive constant \( \gamma_c \) and \( \lambda_c = \left( \frac{-1}{T} \right) \ln \left( \frac{\alpha}{\bar{\lambda}} \right) \), if the observer gain \( H_c[k] \) is designed such that the DT observer error dynamics (2.2.13) exponentially convergence with convergence rate \( \bar{\lambda} \) satisfying \( \alpha < \bar{\lambda} \).

**Proof:**

The exponential convergence of (2.2.13) with convergence rate \( \bar{\lambda} \) implies

\[
\| \hat{e}(k) \| \leq \tilde{\gamma} \left( \frac{1}{\bar{\lambda}} \right)^{k-k_0} \| \hat{e}(k_0) \|.
\]

where \( \tilde{\gamma} > 1 \) and \( \bar{\lambda} \geq 1 \), both independent of \( k_0 \). The maximum current observer error in between the sampled intervals is expressed as

\[
\| \bar{e}(t) \| \leq \alpha_i \| \bar{e}[k] \| \quad (k + 1)T \geq t > kT,
\]

iteratively (2.4.4) and (2.4.5) are written while considering the left continuity of states for current observer i.e. \( \Phi(t,kT) = \Phi(t,kT^-) \)

\[
\| \bar{e}[k] \| \leq \frac{\tilde{\gamma}}{\bar{\lambda}} \| \bar{e}[k_0] \| \quad \| \bar{e}(t) \| \leq \| \Phi(t,1) \| \| \bar{e}[k_1] \| \quad \forall \quad kT \geq t > k_0 T,
\]

\[
\| \bar{e}[k] \| \leq \frac{\tilde{\gamma} \alpha_0}{\bar{\lambda}^{k-1}} \| \bar{e}[k_0] \|,
\]

\[
\| \bar{e}[k_1] \| \leq \frac{1}{\bar{\lambda}} \| \bar{e}[k_1] \| \quad \| \bar{e}(t) \| \leq \alpha_i \| \bar{e}[k_1] \| \quad \forall \quad k_1 T \geq t > k_0 T,
\]

\[
\| \bar{e}[k_1] \| \leq \frac{\tilde{\gamma} \alpha_1}{\bar{\lambda}^{k_1-1}} \| \bar{e}[k_0] \|,
\]

\[
\vdots \quad \vdots \quad \vdots
\]

\[
\| \bar{e}[k] \| \leq \frac{1}{\bar{\lambda}^{k-1}} \| \bar{e}[k-1] \| \quad \| \bar{e}(t) \| \leq \alpha_i \| \bar{e}[k-1] \| \quad \forall \quad (k+1)T \geq t > kT,
\]

\[
\| \bar{e}[k] \| \leq \frac{\tilde{\gamma} \alpha_k}{\bar{\lambda}^{k-k_0}} \| \bar{e}[k_0] \|.
\]

Incorporating (2.4.3), considering the starting point of the STMBS as \( k_0 \), we get
\[
\| \bar{e}(t) \| \leq \frac{\bar{\alpha}^{k-1}}{\lambda^{k-1}} \| \bar{e}[k_0] \|,
\]  
(2.4.7)

hence from (2.4.7), convergence of (2.4.6) is guaranteed only if
\[
\alpha < \tilde{\lambda}.
\]  
(2.4.8)

Following uniform exponential bound is defined by choosing \( \gamma_\epsilon = \frac{\tilde{\eta}_\epsilon}{\alpha} \) and \( \lambda_\epsilon = \left( \frac{-1}{T} \right) \ln \left( \frac{\alpha}{\tilde{\lambda}} \right) \) for any \( t_o \) and \( e_o \) for the solution of linear state equation (2.3.9) and (2.3.10)

\[
\| \bar{e}(t) \| \leq \gamma_\epsilon e^{-\lambda_\epsilon (t-t_o)} \| e_o \|,
\]  
(2.4.9)

A special case of the above result if a supremum of (2.4.2) exists, with \( \alpha \) defined as
\[
\alpha = \sup \{ \alpha_0, \alpha_1, \alpha_2,... \},
\]  
(2.4.10)

is presented in the following corollary.

### 2.4.4 Corollary

The CT state estimation error for current impulsive observer is uniformly exponentially bounded with convergence rate \( \lambda_e = \left( \frac{1}{T} \right) \ln(\tilde{\lambda}) \) if there exists \( \alpha \) defined in (2.4.10) and observer gain \( H_e[k] \) designed such that the DT estimation error (2.2.13) exponentially converges with rate \( \tilde{\lambda} \).

**Proof**

Under these condition, \( \alpha = 1 \) in (2.4.3). The rest of the proof is trivial.

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2.4.5 Corollary

UES of DT estimation error (25) for linear time invariant (LTI) systems implies UES of current observer.

Proof

Supremum of (2.4.10) always exists for LTI systems.

2.4.6 Continuous Exponential Bound for Prediction Impulsive Observer

We define $\beta_i \geq 1$ as the following bound on the STM

$$\|\Phi(t,kT)\| \leq \beta_i, \quad kT \leq t < (k+1)T.$$  \hfill (2.4.11)

A sequence of such bounds termed as State Transition Matrix Forward Sequence (STMFS) over the sampling interval $T$ is given by

$$\{\beta_0, \beta_1, \beta_2, \ldots\}.$$ \hfill (2.4.12)

The sequence STMFS diverges with less than or equal to exponential rate, then

$$\beta_i \leq \eta_p \beta^i,$$ \hfill (2.4.13)

where $\eta_p = \beta_0$ and $\beta = \frac{\beta_1}{\eta_p}$.

With the above definitions, we present the following theorem which establishes sufficient conditions for exponential convergence of CT state estimation error of the prediction impulsive observer.
2.4.7 Theorem

The CT estimation error for the prediction impulsive observer is uniformly exponentially bounded as follows

\[ \|\hat{e}(t)\| \leq \gamma_p e^{-\lambda_p (t-t_0)} \|e_o\|, \]

with positive constants \( \gamma_p \) and \( \lambda_p = \left( \frac{1}{T} \right) \ln \left( \frac{\beta}{\lambda} \right) \). If the observer gain \( H_p[k] \) is designed such that the DT estimation error dynamics (2.2.13) exponentially converge with convergence rate \( \bar{\lambda} \) satisfying \( \beta < \bar{\lambda} \).

Proof

The exponential convergence of (2.2.13) with convergence rate \( \bar{\lambda} \) implies (2.4.4). Maximum prediction observer error in between the sampled intervals is expressed as

\[ \|\hat{e}(t)\| \leq \beta_i \|\hat{e}[k]\| \quad kT \leq t < (k+1)T, \] \hspace{1cm} (2.4.14)

Iteratively (2.4.4) and (2.4.14) can be combined as

\[ \|\hat{e}(t)\| \leq \beta_0 \|\hat{e}[k_0]\| \quad \forall \ k_0 T \leq t < k_1 T, \]

\[ \|\hat{e}(t)\| \leq \beta_1 \|\hat{e}[k_1]\| \quad \forall \ k_1 T \leq t < k_2 T \quad \|\hat{e}[k_1]\| \leq \frac{\gamma_p}{\lambda_p} \|\hat{e}[k_0]\|, \]

\[ \leq \frac{\gamma_p}{\lambda_p} \|\hat{e}[k_0]\|, \]

\[ \vdots \]

\[ \|\hat{e}(t)\| \leq \beta_k \|\hat{e}[k]\| \quad \forall \ kT \leq t < (k+1)T \quad \|\hat{e}[k]\| \leq \frac{\gamma_p}{\lambda_p^{k-k_0}} \|\hat{e}[k_0]\|, \]

\[ \leq \frac{\gamma_p}{\lambda_p^{k-k_0}} \|\hat{e}[k_0]\|. \] \hspace{1cm} (2.4.15)

Incorporating (41) by considering the starting point for the STMFS sequence as \( k_0 \), we get

\[ \|\hat{e}(t)\| \leq \frac{\gamma_p}{\lambda_p^{k-k_0}} \|\hat{e}[k_0]\|. \]
It can be seen that the convergence of the above is guaranteed if

\[ \beta < \bar{\lambda}. \]  

(2.4.16)

Consequently, the following uniform exponential bound is defined by choosing

\[ \gamma_p = \frac{\hat{\eta}p}{\beta} \quad \text{and} \quad \lambda_p = \left( -\frac{1}{T} \right) \ln\left( \frac{\beta}{\bar{\lambda}} \right) \]

for any \( t_o \) and \( e_o \) for the solution of linear state equation (2.3.2) and (2.3.3)

\[ \| \hat{e}(t) \| \leq \gamma_p e^{-\lambda_p(t-t_o)} \| e_o \|. \]  

(2.4.17)

Analogous to the case of current impulsive observers if the supremum of (2.4.12) exists, which is defined as

\[ \beta = \sup\{ \beta_0, \beta_1, \beta_2, \ldots \}. \]  

(2.4.18)

Then a special case of Theorem 2.4.7 for prediction impulsive observers is stated as follows

2.4.8 Corollary

The CT state estimation error for prediction impulsive observer is uniformly exponentially bounded with convergence rate \( \lambda_p = \left( \frac{1}{T} \right) \ln(\bar{\lambda}), \) if there exists \( \beta \) defined in (2.4.18) and observer gain \( H_p[k] \) designed such that the DT estimation error (2.2.13) exponentially converges with rate \( \bar{\lambda}. \)

Proof

Under these condition, \( \beta = 1 \) in (2.4.18). The rest of the proof is trivial.
2.4.9 **Corollary**

UES of DT estimation error (2.2.13) for LTI systems implies UES of prediction observer.

**Proof**

Supremum of (2.4.18) always exists in LTI systems.

2.4.10 **Exponential Bound for Continuous Sampled-data Reconstruction Observer**

The estimation error for the continuous reconstruction observer is defined as

\[ \ddot{e}(t) = \ddot{\theta}(t)\psi(t) + \dot{\theta}(t)(1 - \psi(t)). \]  

(2.4.19)

The exponential convergence of the estimation error (2.4.19) follows naturally if the continuous estimation errors of the current and prediction impulsive observers are exponentially bounded. The result is summarized in the following Lemma.

2.4.11 **Lemma**

The estimation error of the continuous reconstruction observer (2.4.19) is exponentially bounded if the CT estimation errors of current and prediction impulsive observers are exponentially bounded.

The proof of the Lemma trivially follows from Theorem 2.4.3 and Theorem 2.4.7.

**Proof:**

The proof of the Lemma trivially follows from Theorem 2.4.3 and Theorem 2.4.7.
To explain the discussed situations of prediction and current observers, consider a first order plant \( \dot{x}(t) = 2atx(t) \) for \( t \geq t_o \). For \( a < 0 \), system will be stable. Prediction impulsive observer will be discussed under the case if supremum of \( \alpha \) exists and current observer is discussed when supremum of \( \beta \) does not exists. STM \( e^{\alpha(t-t_o)} \) guarantees the existence of \( \beta = \sup\{\beta_0, \beta_1, \beta_2, \ldots\} \) for prediction observer. Whereas due to backward calculation in time from \( t \) to \( t_o \), current observer will be divergent and the supremum of \( \{\alpha_0, \alpha_1, \alpha_2, \ldots\} \) does not exist. Likewise, for \( a > 0 \), system will be divergent, prediction and current observer will be discussed under different and opposite scenarios.

\section*{2.5 Example}

The prediction, current and reconstruction observers are illustrated for state observations with following linear time varying system. The peculiar reason for selecting following second order system to clearly demonstrate the discussion results for unstable system

\begin{equation}
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} 2\varepsilon t & 0 \\ 0 & -2\kappa t \end{bmatrix} + Bu(t), \\
y[k] &= C[k]x[k],
\end{align*}
\end{equation}

where \( B = [0 \ 1]^T \) and \( C = [1 \ 2] \). The constant \( \varepsilon = 0.05 \) and \( \kappa = 0.05 \) results in an unstable (divergent) system. The simulation results with \( T = 1 \) sec and sinusoidal input \( u(t) = sin(2\pi t) \) are discussed for initial conditions \( x_o = [1 \ 0.5]^T \).

The estimation error for second order system with prediction, current and reconstruction observer is illustrated in Figure 2-8 for discrete convergence rate \( \lambda = 1.11 \) and \( l \) step.
observability with $l=3$ [43]. Prediction estimate is the continuation of current estimate from the common point of both observers. Reconstruction observer utilizes current and prediction impulsive estimates to provide continuous (non-impulsive) output for sampled-data system.

Figure 2-8  Observer error for $x_0=[1 \ 0.5]^T$ and $\lambda = 1.11$ (a) state $x_1$  (b) state $x_2$

Impulsive jumps can be observed at sampling time instants in prediction and current estimated error states. The reconstruction observer error states are available at sampling instants without impulsive jumps. Reconstruction observer error output follows prediction observer for $\Delta_i$
duration and current observer for $\Delta_3$ duration. Reconstruction observer error for $\Delta_2$ duration is constructed with current, prediction and fusion function outputs.

The error convergence rate for reconstruction observer is flexible to change in between the prediction and current observer as per defined function $\Psi(t)$. In this example $t^* = (k + \frac{1}{2})T$ is considered for simulation results.

![Observer errors norm for $x_o = [1, 0.5]^T$](image)

**Figure 2-9** Observer errors norm for $x_o = [1, 0.5]^T$ (a) Prediction deadbeat observer (b) Current deadbeat observer

The open-loop dead beat observer for prediction and current observers are simulated for with $\varepsilon = 1$ and $\kappa = 0.1$. Linear time varying system diverges at the rate greater than the observer convergence rate. A solution to this case is a dead beat observer, which converges within finite
time as shown in Figure 2-9 for prediction and current observer for \( l = 2 \). It is noted that prediction observer achieved deadbeat convergence in \( t = 2 \) sec, whereas current observer achieved the same in \( t = 1 \) sec.

### 2.6 Chapter Summary

In this chapter, a continuous (non-impulsive) reconstruction observer is proposed for sampled-data system. Its construction based on the fusion of two proposed impulsive observers. Prediction and current impulsive observer designs are based on associated discrete-time dynamical equation for estimation error. For a current observer, impulsive correction is applied immediately after receiving the output sample, whereas the same is done just before the next observation for a prediction observer. The relationship between the current and prediction impulsive observers exhibits a commonality of estimation at the sampled point. This relationship is exploited to develop a continuous (non-impulsive) reconstruction observer by fusing the estimates of the two observers. The comprehensive stability analyses carried out in this chapter deals proof the UES of proposed observer schemes. It covers for stable, unstable and highly unstable LTV systems. Certain bounds on state transition matrix play an important role in this context.
CHAPTER 3

OUTPUT REGULATION FOR LINEAR TIME VARYING SAMPLED-DATA SYSTEMS USING RECONSTRUCTION OBSERVER

In this chapter, a continuous (without ripples) regulator for linear sampled-data continuous time varying system is discussed. A reconstruction observer discussed in Chapter one is used for regulation for sampled-data systems. Ripple free regulator output is achieved due to non-impulsive nature of reconstruction observer. The stability analysis of closed-loop system is carried out to establish uniform exponential stability for a closed loop system. A numerical example of linear time varying system illustrates the efficacy of proposed design methodology.

The regulator scheme for tracking a reference signal and rejection of disturbance signal for linear time varying system is proposed in [16]. A feedback controller is proposed to achieve asymptotic tracking for a class of reference inputs and rejection of disturbances, while maintaining overall closed loop stability of the system. The same approach is used as an extension for linear time varying sampled-data systems regulation for tracking a reference signal generated by an exosystem.

The non-impulsive ripple free regulation is achieved and is also the main focus of this chapter. Continuous-time linear time varying regulator is discussed with some basic definitions of linear time varying system in Section 3.1. Section 3.2 covers the details of augmented prediction, current and continuous (non-impulsive) reconstruction observers, which is further used in Section 3.3 to present sampled data regulator for augmented system. Stability analysis is
discussed in Section 3.4. Finally a second order system linear time varying example is illustrated in Section 3.5. The whole scheme is illustrated in Figure 3-1 for clarity purpose.

![Block Diagram of sampled-data regulator]

**Figure 3-1  Block Diagram of sampled-data regulator**

### 3.1 Sampled-data Regulation with Reconstruction Observer

Consider the following sampled-data linear time varying system

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t) + P(t)w(t) \quad x(t_0) = x_0, \tag{3.1.1}
\]

\[
\dot{w}(t) = S(t)w(t) \quad w(t_0) = w_0, \tag{3.1.2}
\]

\[
y[k] = C_t[k]x[k] + Q_t[k]w[k], \tag{3.1.3}
\]
\[ e_i(t) = C(t)x(t) + Q(t)w(t), \]  

(3.1.4)

where \( x(t) \in \mathbb{R}^n \) is the continuous-time state vector and \( u(t) \in \mathbb{R}^n \) is the continuous-time plant input. Control input \( u(t) \) may be discontinuous but is assumed to be non-impulsive in nature. Only sampled output \( y[k] \in \mathbb{R}^p \) is available for measurement, where \( y[k] = y(kT) \) is assumed to be non-pathological fixed sampling time [108]. \( x_0 \) is the initial condition at time \( t = t_o \). The matrices \( A(t), B(t), S(t), C_i[k], Q_i[k] \) and \( Q[k] \) are time varying bounded matrices of appropriate dimensions. An exosystem system with continuous-time state vector \( w(t) \in \mathbb{R}^r \). It models the class of reference or/ and disturbance signals such as \( S(t) = \begin{bmatrix} S_r(t) & 0 \\ 0 & S_d(t) \end{bmatrix} \), where \( S_r(t) \) and \( S_d(t) \) are reference and disturbance signal matrices respectively. Regulator tracking error \( e_i(t) \in \mathbb{R}^p \) is a difference between plant output and the reference signal. It may be noted that the contribution in this paper can conveniently be extended for non-uniform sampling time, however, that case is not discussed for brevity purpose. Some basic discussion is carried out in next subsection for clarity purpose.

### 3.1.1 Definitions and Notations

Sampled-data system (3.1.1)-(3.1.4) can be expressed as augmented system matrices

\[
\begin{align*}
\dot{z}(t) &= \tilde{A}(t)z(t) + \tilde{B}(t)u(t) \\
y[k] &= \tilde{C}[k]z[k],
\end{align*}
\]

(3.1.5)

where \( \tilde{A}(t) = \begin{bmatrix} A(t) & P(t) \\ 0 & S_d(t) \end{bmatrix}, \tilde{B}(t) = \begin{bmatrix} B(t) \\ 0 \end{bmatrix}, \tilde{C}[k] = \begin{bmatrix} C_i[k] & 0 \end{bmatrix} \) and augmented state vector is
represented as \( z(t) = \begin{bmatrix} x(t) \\ w_d(t) \end{bmatrix} \), where \( z \in \mathbb{R}^q \). In the absence of full states information, uniform exponential stability of the regulator is also dependent on states observer error convergence. Observer error for augmented estimated state is defined as

\[
e(t) = z(t) - \hat{z}(t).
\]  

(3.1.6)

The initial error corresponds to initial guess \( \hat{z}(t_o) \) is

\[
e_o = e(t_o) = z(t_o) - \hat{z}(t_o).
\]

In output regulation problem, an output feedback controller is designed to make a closed-loop system uniformly exponentially stable, such that

\[
\lim_{t \to \infty} e_r(t) = 0 \quad \forall \ x_o, w_o,
\]  

(3.1.7)

where \( e_r(t) \) is the regulator tracking error. Uniform exponential stability of a regulator is only guaranteed while ascertaining the exponential stability of observer error.

### 3.1.2 Augmented State Transition Matrix

Let \( \Phi(t_f, t_o) \) be the state transition matrix associated with \( \tilde{A}(t) \) in its usual sense [43], where \( t_o \) and \( t_f \) are the initial and final times respectively. Discrete-time equivalent of state transition matrix can be defined as \( \Phi(k_f, k_o) = \Phi(k_f T, k_o T) \). In case of single step transition from \( k \) to \( k + 1 \), it is convenient to define
\[
\tilde{A}[k] = \Phi((k,T,k,T)T,kT) = \Phi[k + 1, k]. \tag{3.1.8}
\]

By virtues of state transition matrix being full rank, \( \tilde{A}[k] \) is always invertible.

### 3.2 Reconstruction Observer for Augmented System

The construction of the continuous reconstruction observer is same as discussed in (2.2.14), however reconstruction observer states for augmented system is written as

\[
\dot{z}(t) = \bar{z}(t)\psi(t) + \hat{z}(t)(1 - \psi(t)), \tag{3.2.1}
\]

where \( \hat{z} \) and \( \bar{z} \) are augmented prediction and current impulsive observer states and the same is discussed in the preceding section.

#### 3.2.1 Prediction Impulsive Observer for Augmented System

The prediction impulsive observer for (3.1.5) is expressed as

\[
D^*\hat{z}(t) = \bar{A}(t)\hat{z}(t) + \bar{B}(t)u(t) \quad t = kT, \tag{3.2.2}
\]

\[
\hat{z}(t) = \bar{A}(t)\hat{z}(t) + \bar{B}(t)u(t) \quad kT < t < (k + 1)T, \tag{3.2.3}
\]

\[
\hat{z}[k + 1] = \hat{z}(t^-) + H_p[k]\left(y[k] - \hat{y}[k]\right) \quad t = (k + 1)T, k = k_0, k_0 + 1, \ldots, \tag{3.2.4}
\]

where \( \hat{z} \in \mathbb{R}^p \) is the prediction observer state and \( H_p[k] \in \mathbb{R}^{p \times p} \) is a time varying prediction observer gain matrix, may also be represented as \( H_p[k] = \begin{bmatrix} H_{pu}[k] \\ H_{pu}[k] \end{bmatrix} \). Error state vector of prediction impulsive observer is written as
\[ \hat{e}(t) = z(t) - \hat{z}(t), \quad (3.2.5) \]

derivative of (3.2.5) is

\[ \hat{e}(t) = \tilde{A}(t)\hat{e}(t) \quad kT < t < (k+1)T, \quad (3.2.6) \]

\[ D^+\hat{e}(t) = \tilde{A}(t)\hat{e}(t) \quad t = kT, \quad (3.2.7) \]

where (3.2.6) signifies the discontinuous nature of error \( \hat{e}(t) \). Discrete-time error dynamics using prediction impulsive observer is written as

\[ \hat{e}[k+1] = z(t) - \left( \hat{z}(t^-) + H_p[k](y[k] - \hat{y}[k]) \right) \quad t = (k+1)T, \quad (3.2.8) \]

continuity of system state [43] imply \( z(t) = z(t^-) \) leading to

\[ \hat{e}[k+1] = \hat{e}(t^-) - H_p[k](y[k] - \hat{y}[k]) \quad t = (k+1)T. \quad (3.2.9) \]

Associated discrete equation is expressed as

\[ \hat{e}[k+1] = \left( \hat{A}[k] - H_p[k]\hat{C}[k] \right)\hat{e}[k]. \quad (3.2.10) \]

The gain \( H_p[k] \) is to be designed such that (3.2.10) is uniformly exponentially stable.

### 3.2.2 Current Impulsive Observer for Augmented System

The current impulsive observer for (3.1.5) is proposed as follows

\[ \tilde{z}(t^+) = \tilde{z}(t) + H_r[k](y[k] - \hat{y}[k]) \quad t = kT, \quad (3.2.11) \]

\[ \tilde{z}(t) = \tilde{A}(t)\tilde{z}(t) + \tilde{B}(t)u(t) \quad kT < t < (k+1)T, \quad (3.2.12) \]
\[ D^r \bar{z}(t) = \tilde{A}(t)\bar{z}(t) + \tilde{B}(t)u(t) \quad t = (k+1)T \]
\[ k = k_o, k_o + 1, \ldots, \]

where \( \bar{z} \in \mathbb{R}^n \) is the current observer state and \( H_{e}[k] \in \mathbb{R}^{r \times p} \) is a time varying current observer gain matrix, may also be represented as \( H_{e}[k] = \begin{bmatrix} H_{e_{x}}[k] \\ H_{e_{n}}[k] \end{bmatrix} \), error state vector of current impulsive observer is expressed as

\[
\bar{e}(t) = z(t) - \bar{z}(t) \quad \forall \ t,
\]
derivative of (3.2.14) is

\[
\dot{\bar{e}}(t) = \tilde{A}(t)\bar{e}(t) \quad kT < t < (k+1)T
\]
\[ D^r \bar{e}(t) = \tilde{A}(t)\bar{e}(t) \quad t = (k+1)T. \]

Discrete-time error dynamics using current observer are

\[
\bar{e}(t^+) = z(t^+) - \bar{z}(t^+) \quad t = (k+1)T,
\]
\[ = z(t^+) - [\bar{z}(t) + H_{e}[k](z[k] - \bar{z}[k])],\]
continuity of system states [43] imply \( z(t) = z(t^+) \) leading to

\[
\bar{e}(t^+) = \bar{e}(t) - H_{e}[k]C[k](z[k] - \bar{z}[k]) \quad t = kT,
\]
after simplification (3.2.18) is expressed as

\[
\bar{e}(t^+) = (I - H_{e}[k]C[k])\bar{e}[k] \quad t = kT,
\]
where gain \( H_{e}[k] \) is required to be designed such that (3.2.19) is uniformly exponentially stable.
3.2.3 Augmented Reconstruction Observer Formulation

A relationship between current and prediction observer gains is established on the basis as explained in Section 2.2.3. The two impulsive observers outputs are fused together with the help of weighting function. Once the sampled output is available, maximum output is given to the prediction observer and the least weight is given to the current observer output for obvious reason as explained in Section 2.2.4. Likewise for the next half of the sampling period more weight is given to the output of the current observer and finally the reconstruction filter output is concluded at a common point of current and prediction observer resulting non-impulsive continuous output of the reconstruction observer.

Detailed stability analysis is carried out in Section 2.4. Uniform exponential stability for reconstruction observer is proved with exponential bounds. The advantage of the same is used for establishing the uniform exponential stability for sampled-data regulation problem as discussed in preceding section.

3.3 Stability Analysis of Augmented System for Regulation

In this section sampled-data regulation of linear time varying system with state estimation using reconstruction observer is discussed. The linear time varying system with augmented prediction impulsive observer is expressed as

\[
\dot{\hat{z}}(t) = \hat{A}(t)\hat{z}(t) + \hat{B}(t)u(t) \quad \forall t,
\]

\[
D^+ \dot{\hat{z}}(t) = \hat{A}(t)\dot{\hat{z}}(t) + \hat{B}(t)u(t) \quad t = kT,
\]

\[
\dot{\hat{z}}(t) = \hat{A}(t)\hat{z}(t) + \hat{B}(t)u(t) \quad kT < t < (k+1)T,
\]
\[
\hat{z}[k+1] = \hat{z}(t^+) + H_p[k](y[k] - \hat{y}[k])
\]
\[
t = (k + 1)T,
\]
\[
k = k_0, k_0 + 1, ...
\]

Similarly, linear time varying system with current impulsive observer is given as

\[
\dot{z}(t) = \tilde{A}(t)z(t) + \tilde{B}(t)u(t)
\]
\[
\forall t,
\]

\[
\tilde{z}(t^+) = \tilde{z}(t) + H_p[k](y[k] - \bar{y}[k])
\]
\[
t = kT,
\]

\[
\tilde{z}(t) = \tilde{A}(t)\tilde{z}(t) + \tilde{B}(t)u(t)
\]
\[
kT < t < (k + 1)T,
\]

\[
D^\top\tilde{z}(t) = \tilde{A}(t)\tilde{z}(t) + \tilde{B}(t)u(t)
\]
\[
t = (k + 1)T,
\]
\[
k = k_0, k_0 + 1, ...
\]

The solution for sampled-data output regulation problem is considered with following assumptions

**A-1** The pair \((A(t), B(t))\) is stabilizable.

**A-2** \(\|W(t_0, t)\| \leq d_t\) for any \(t \geq t_0\), where \(W(t, s)\) is the state transition matrix generated by \(S(t)\)

**A-3** The Pair \((\tilde{A}[k], \tilde{C}[k])\) is detectable.

### 3.3.1 Theorem

Suppose the assumptions A-1-A-3 hold. Then the output regulation problem is solvable by a bounded controller if there exists continuous bounded matrices \(\Pi(t)\) and \(\Gamma(t)\) which satisfy the following differential regulation equations (3.2.5)
\[ \dot{\Pi}(t) = A(t)\Pi(t) - \Pi(t)S(t) + P(t)\Gamma(t), \quad (3.3.9) \]

\[ \lim_{t \to \infty} [C(t)\Pi(t) + Q(t)]W(t,t_0) = 0. \quad (3.3.10) \]

The admissible control law based on the (3.3.9) and (3.3.10) with state estimates using impulsive observers is given as

\[ u(t) = [F(t)\Gamma(t) - F(t)\Pi(t)]\bar{z}(t), \quad (3.3.11) \]

where \( F(t) \) is any matrix such that \( A + BF \) is exponentially stable and \( \bar{z}(t) = \begin{bmatrix} \bar{x} \\ \bar{\eta} \end{bmatrix}(t) \) is the observed reconstruction state vector. Moreover, (3.3.10) implies

\[ \lim_{t \to \infty} [C(t)\Pi(t) + Q(t)] = 0. \quad (3.3.12) \]

The condition (3.3.10) can be replaced by (3.3.12) if the following inequality holds

\[ A-4 \quad \|W(t,t_0)\| \leq d_2 \quad \text{for any} \quad t \geq t_0. \]

**Proof:**

The condition (3.3.9) and (3.3.10), implies uniform exponential stability of the closed loop system using (full information feedback) control law (3.3.11) with full information feedback i.e. \( y = z \) which can be expressed as

\[ u(t) = [F(t)\Gamma(t) - F(t)\Pi(t)]z(t), \quad (3.3.13) \]

where \( F(t) \) is bounded and stabilizing. Controller (3.3.11) is a realization of this controller via an observer and is stabilizing. Regulator equation unknown can be calculated by solving a set of
differential equations simultaneously. In time varying case, the values of these two unknown will keep changing with time. The closed loop linear time varying system with control law (3.3.11) and the augmented prediction impulsive observer is expressed as

\[
\dot{z}(t) = \tilde{A}(t)z(t) + \tilde{B}(t)\left[ F(t) \Gamma(t) - F(t)\Pi(t) \right] z(t) - \tilde{B}(t)\left[ F(t) \Gamma(t) - F(t)\Pi(t) \right] \hat{e}(t),
\]

where prediction observer error is augmented \( \forall t \) as

\[
D^+ \hat{e}(t) = \tilde{A}(t)\hat{e}(t) \quad t = kT,
\]

\[
\dot{\hat{e}}(t) = \tilde{A}(t)\hat{e}(t) \quad kT < t < (k+1)T,
\]

\[
\hat{e}[k+1] = (\tilde{A}[k] - H_i[k]\tilde{C}[k])\hat{e}[k].
\]

Since the observer error is uniformly exponentially sable i.e \( \hat{e}(t) \to 0 \) as \( t \to \infty \). The asymptotic regulation is achieved. Uniform exponential stability of (3.3.14)-(3.3.16) with exponential bounds for prediction impulsive observer is discussed in Section 2.4.6, where first part of (3.3.14) is already uniformly exponentially stable under full state feedback hence guarantees the overall uniform exponential stability of (3.3.14). Similarly the closed loop linear time varying system with control law (3.3.11) and current impulsive observer is expressed as

\[
\dot{z}(t) = \tilde{A}(t)z(t) + \tilde{B}(t)\left[ F(t) \Gamma(t) - F(t)\Pi(t) \right] z(t) - \tilde{B}(t)\left[ F(t) \Gamma(t) - F(t)\Pi(t) \right] \bar{e}(t),
\]

where current observer error is augmented \( \forall t \) as

\[
\bar{e}(t^+) = (I - H_i[k]C[k])\bar{e}[k] \quad t = kT
\]
\[
\dot{e}(t) = \tilde{A}(t)e(t) \quad kT < t < (k+1)T,
\]
(3.3.20)

\[
D \cdot \bar{e}(t) = \tilde{A}(t)\bar{e}(t) \quad t = (k+1)T,
\]
(3.3.21)

since the estimation error is uniformly exponential stable i.e \( \bar{e}(t) \to 0 \) as \( t \to \infty \). The asymptotic regulation is achieved. Uniform exponential stability of (3.3.19)-(3.3.21) with exponential bounds for current impulsive observer is discussed in Section 2.4 where first part of (3.3.18) is already uniformly exponentially stable under full state feedback hence guarantees the overall uniform exponential stability of (3.3.18).

3.3.2 Remark

Closed-loop uniform exponential stability of prediction and current non-linear impulsive observers guarantees the uniform exponential stability of reconstruction observer.

3.4 Example

Consider the following linear time varying system

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t),
\]
(3.4.1)

where

\[
A = \begin{bmatrix}
-1 & 0 \\
e^{-\cos(t)} & \sin(t)
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

For linear time varying system (3.4.1), we design a sampled-data output feedback controller such that \( e_i(t) \to 0 \) as \( t \to \infty \), we consider the following exosystem

\[
\dot{w}(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}w(t), \quad w_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]
The system (3.4.1) is written in the form of (3.1.1)-(3.1.4) as

\[ \dot{x}(t) = A(t)x(t) + Bu(t), \]
\[ \dot{w}(t) = Sw(t), \]
\[ y[k] = C_1x[k] + Q_1w[k], \]
\[ e_r(t) = Cx(t) + Qw(t), \]

with \( C = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \) and \( Q_1 = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}. \) The solution \( \Pi \)

**Figure 3-2**  Plant observer error for state (a) \( x_1 \), (b) \( x_2 \)
and \( \Gamma(t) \) of the regulator equation is explicitly given by

\[
\Pi = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \Gamma(t) = \begin{bmatrix} -\sin(t) + 1 & -\sin(t) - 1 \end{bmatrix}.
\]

The observer is designed with reconstruction observer gain \( \lambda = 1.2 \) for sampling time \( T = 1 \text{ sec} \) and controller gains are chosen as \( F(t) = [e^{-\cos(t)} - \sin(t) - 1] \). The Eigen values of \( A(t) + B(t)F(t) \) is -1. The simulation results for sampled-data output regulation of (3.4.2) is simulated with \( x_0 = [0 \ -0.3]^T \).

Figure 3-2 shows the observer error of plant states \( x_1 \) and \( x_2 \). Three observer errors for prediction, current and reconstruction observers are simulated. Impulsive jumps are observed in

![Figure 3-3 Exosystem observer states (a) w1 (b) w2](image-url)
case of prediction and current observers, whereas reconstruction observer output is smooth (without jumps). Reconstruction observer error output follows prediction observer for $\Delta_1$ duration and current observer for $\Delta_3$ duration. Reconstruction observer error for $\Delta_2$ duration is measured with current, prediction and fusion function outputs.

Similarly Figure 3-3 shows the observed exosystem states $w_1$ and $w_2$ with prediction, current and reconstruction observers. Reconstruction observer output is non-impulsive and smooth, whereas impulsive jumps are observed in case of prediction and current observers at sampled points.

![Sampled-data output regulation with reconstruction observer](image)

**Figure 3-4** Sampled-data output regulation with reconstruction observer (a) Sine tracking (b) Regulation error
Figure 3-4, a sampled-data regulation of sine tracking is simulated for $\omega_o = 1 \text{ rad/sec}$ with reconstruction observer. Smooth sine tracking (without ripples) at sampling point is simulated with reconstruction observer.

### 3.5 Chapter Summary

In this chapter the output regulation problem for sampled-data linear time varying system has been discussed. The main contribution is the design of ripple free output regulation. This is only possible by optimally use the prediction and current impulsive observer outputs under well-defined weighting function. A closed loop stability analysis is carried out to show uniform exponential stability of a sampled-data linear time varying regulator. For illustration a sine tracking problem is presented and regulation result with prediction, current and reconstruction observer are also simulated.
CHAPTER 4

SAMPLED-DATA REGULATION OF FEEDBACK LINEARIZABLE NONLINEAR SYSTEM WITH RECONSTRUCTION OBSERVER

This chapter deals with the design strategy for regulating a feedback linearizable sampled-data nonlinear system. The reconstruction observer (without jumps) discussed in Chapter one is used for two purposes: state estimation and feedback linearization of a nonlinear system. This is being carried out with the help of two different sampled-data observers with jumps, combined in such a way that overall estimated states become continuous and linear. This leads to a ripple free continuous-time sampled-data regulator as discussed in Chapter three. A stability analysis is carried out while considering the model uncertainties of a nonlinear system as a non-vanishing perturbation. An example of a third order nonlinear system illustrates the efficacy of proposed design methodology.

Researchers attempted to address the sampled-data regulation for non-linear system problem with different approaches. In [31], an impulsive observer is used for cancellation of nonlinear terms, whereas rest of the regulation problem is being dealt in discrete-time domain. The proposed scheme in this chapter addresses the issue of ripple due to non-impulsive action of the reconstruction observer during regulation. Due to availability of continuous-time states, a continuous control designed is used for regulation purpose.

The ripple free regulation for feedback linearizable sampled-data nonlinear system is mainly discussed and is the prime focus of this chapter. Sampled-data nonlinear regulator is
discussed in Section 4.1. Section 4.2 covers the discussion of feedback linearization with prediction, current and continuous-time (non-impulsive) reconstruction observer. Stability analysis is discussed in Section 4.3. Finally a second order nonlinear sampled-data system is illustrated to visualize and authenticate the theoretical discussion in Section 4.2. The overall regulator scheme is illustrated in Figure 4-1.

![Block diagram of sampled-data regulator with reconstruction observer](image)

Figure 4-1. Block diagram of sampled-data regulator with reconstruction observer

### 4.1 Sampled-data Nonlinear System

Consider the following single input single output sampled-data nonlinear system

\[
\begin{align*}
\dot{x}(t) &= A_c x(t) + B_c \phi(x(t), \eta(t), u(t)) + f(x(t), \eta(t), u(t)), \\
\dot{\eta}(t) &= \psi(x(t), \eta(t), u(t)), \\
y_\nu[k] &= C_c x[k], \\
\zeta(t) &= \Theta(x(t), \eta(t)),
\end{align*}
\]  

(4.2.1)
where \( u(t) \in R \) is the control input, \( y_p[k] \in R \) and \( \zeta(t) \in R^q \) are the measured outputs, and \( x(t) \in R^n \) and \( \eta(t) \in R^l \) constitute the state vector. The \( n \times n \) matrix \( A_c \), the \( n \times 1 \) matrix \( B_c \) and \( 1 \times n \) matrix \( C_c \) is given by

\[
A_c = \begin{bmatrix}
0 & 1 & \cdots & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 \\
0 & \cdots & \cdots & \cdots & 0
\end{bmatrix},
B_c = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix},
C_c = (1 \ 0 \ \cdots \ \cdots \ 0),
\]

represents a chain of \( n \) integrator can be modeled in normal form for input output linearizable nonlinear system, while considering \( \zeta = \eta \) [111]

\[
x(t) = A_c(t)+B_c \frac{1}{\beta(x, \zeta)}[u(t)-\alpha(x, \zeta)],
\]

\[
y_p[k] = C_c x[k].
\]

where \( \beta(x, \zeta) = \gamma^{-1}(x, \zeta) \neq 0 \) and \( \alpha(x, \zeta) \) are scalar functions of \( x \) and \( \zeta \). These scalar functions are well defined over the domain of interest. The existence and uniqueness of (4.2.2) solution can be ensured by assumption that \( \alpha(x, \zeta) \) and \( \beta(x, \zeta) \) are locally Lipchitz over the domain of interest.

Exact knowledge of the \( \alpha(x, \zeta) \) and \( \beta(x, \zeta) \) is required for the cancellation of nonlinear terms in state feedback linearization. This is not possible because of practical shortcomings like model simplification, parameter uncertainties and computational errors. Continuous-time state estimates are used for carrying out linearization and the same is expressed in control input

\[
u(t) = \alpha_v(\hat{x}, \zeta) + \beta_v(\hat{x}, \zeta)v(t),
\]

(4.2.3)
where $\alpha_o(\hat{x}, \zeta)$ and $\beta_o(\hat{x}, \zeta)$ are the observed values of the nominal system, $\hat{x}(t)$ are the estimated states and $v(t) = -K \hat{x}(t)$ with controller gain $K$. The closed loop system under this feedback control is expressed as

$$\dot{x}(t) = A_c x(t) + B_c v(t) + B_c \delta(x), \quad (4.2.4)$$

where $\delta(x)$ is the perturbation term appears due to the difference between the actual and nominal plant parameters. Open loop system for sufficiently small $\delta(x)$ can be conveniently written as [111]

$$\dot{x}(t) = A_c x(t) + B_c v(t) + d(t), \quad (4.2.5)$$

$$y_p[k] = C_c x[k], \quad (4.2.6)$$

where (4.2.5) and (4.2.6) appeared to be a typical sampled-data regulation problem, with assumption that reference signal $y_r[k]$ to be tracked and disturbance $d(t)$ to be rejected, Both reference and disturbance signals are modeled by the exosystem is given as

$$\dot{w}(t) = \begin{bmatrix} w_r(t) \\ w_\delta(t) \end{bmatrix} = Sw(t), \quad (4.2.7)$$

$$y_r[k] = Qw[k], \quad (4.2.8)$$

$$d(t) = Pw(t), \quad (4.2.9)$$

where $S = \begin{bmatrix} S_r & 0 \\ 0 & S_\delta \end{bmatrix}$. $Q$ and $P$ are appropriately selected matrices. It is assumed that sampled reference signal $y_r[k]$ is only available for measurement, where $y_r[k] = y_r(kT)$. 

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Systems (4.2.5)-(4.2.9) is written as

\[ \dot{x}(t) = A_x x(t) + B_x v(t) + P w(t) \quad x(t_o) = x_o, \quad (4.2.10) \]

\[ \dot{w}(t) = S w(t) \quad \quad w(t_o) = w_o, \quad (4.2.11) \]

\[ y(t) = C_i x[k] + Q_i[k], \quad (4.2.12) \]

\[ e(t) = C_x x(t) + Q w(t), \quad (4.2.13) \]

where \( C_i \) and \( Q_i \) are appropriately selected matrices for representing output for sampled-data system. Equation (4.2.10) can be expressed with augmented system matrices as

\[ \bar{A}(t) = \begin{bmatrix} A_x & P \\ 0 & S \end{bmatrix}, \quad \bar{B}(t) = \begin{bmatrix} B_x \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C_i & Q_i \end{bmatrix} \]

and \( z(t) = \begin{bmatrix} x(t) \\ w_o(t) \end{bmatrix} \) is an augmented state vector. The objective is to design a continuous-time regulator for feedback linearizable sampled-data nonlinear system such that

\[ \lim_{t \to \infty} e(t) = 0, \quad (4.2.14) \]

where \( e(t) \) is the tracking error state vector.

The solution of proposed regulator scheme is based on first linearizing the nonlinear system such that the same can be written as an linear time invariant system [112].

### 4.2 Feedback Linearization with Reconstruction Observer

The proposed regulator scheme used nonlinear impulsive observers for two purposes: for feedback linearization and continuous-time (non-impulsive) state observations, while using sampled nonlinear system output and continuous-time input for regulation. Dual purpose
nonlinear reconstruction (non-impulsive) output is obtained by fusing the outputs of two nonlinear impulsive current and prediction observers. Following nonlinear current and prediction impulsive observers are proposed, as the availability of continuous states estimate is required for converting the nonlinear system (4.2.2) into its corresponding linear version (4.2.10). A relationship between current and prediction observer gains as discussed in Section 2.2.3 is used for the design dual purpose nonlinear reconstruction observer.

4.2.1 Nonlinear Current Impulsive Observer

The nonlinear current impulsive observer for (4.2.2) is proposed as follows.

\[
\dot{x}(t^+) = \bar{x}(t) + H_c(y[k] - \bar{y}[k]) \quad t = kT, \tag{4.3.1}
\]

\[
\dot{\bar{x}}(t) = A_c \bar{x}(t) + B_c \gamma(\bar{x})[u(t) - \alpha(\bar{x})] \quad kT < t < (k+1)T, \tag{4.3.2}
\]

\[
D^{-} \bar{x}(t) = A_c(t) \bar{x}(t) + B_c \gamma(\bar{x})[u(t) - \alpha(\bar{x})] \quad t = (k+1)T \\
\quad \quad \quad \quad k = k_o, k_o + 1, \ldots, \tag{4.3.3}
\]

where \( \bar{x} \in \mathbb{R}^n \) is the current observer state and \( H_c \in \mathbb{R}^{m \times n} \) is a current observer gain vector. Linearization is being carried out by taking (4.2.3) as input, Thus (4.3.2) and (4.3.3) are written as

\[
\dot{\bar{x}}(t) = A_c \bar{x}(t) + B_c v(t) + \bar{d}(t) \quad kT < t < (k+1)T, \tag{4.3.4}
\]

\[
D^{-} \bar{x}(t) = A_c(t) \bar{x}(t) + B_c v(t) + \bar{d}(t) \quad t = (k+1)T. \tag{4.3.5}
\]

Error state vector of nonlinear current impulsive observer is expressed as

\[
\bar{x}(t) = x(t) - \bar{x}(t) \quad \forall \ t, \tag{4.3.6}
\]
The derivative of (4.3.6) is
\[
\dot{\epsilon}(t) = A_x \epsilon(t) + g(\bar{\epsilon}) \quad kT < t < (k+1)T,
\]
\[
D^\epsilon \epsilon(t) = A_x \epsilon(t) + g(\bar{\epsilon}) \quad t = (k+1)T,
\]
where (4.3.8) signifies the discontinuous nature of error \( \bar{\epsilon}(t) \). Observer error would not diminish asymptotically as \( g(\bar{\epsilon}) \) is considered to be a bounded non-vanishing perturbation. Discrete-time error dynamics using current observer are expressed as
\[
\bar{\epsilon}(t^+) = x(t^+) - \bar{x}(t^+) \quad t = kT,
\]
\[
= x(t^+) - [\bar{x}(t) + H_x (y[k] - y[k])],
\]
continuity of system states [43] imply \( x(t) = x(t^+) \) leading to
\[
\bar{\epsilon}(t^+) = \bar{\epsilon}(t) - H_x C_x (x[k] - \bar{x}[k]) \quad t = kT,
\]
after simplification (4.3.10) is expressed as
\[
\bar{\epsilon}(t^+) = (I - H_x C_x) \bar{\epsilon}[k] \quad t = kT,
\]
where \( H_x \) is required to be designed such that (4.3.11) is uniformly exponentially stable.

### 4.2.2 Nonlinear Prediction Impulsive Observer

The nonlinear prediction impulsive observer for (4.2.2) is proposed as follows
\[
D^\epsilon \hat{x}(t) = A_x \hat{x}(t) + B_x \gamma(\hat{x})[u(t) - \alpha(\bar{x})] \quad t = kT,
\]
\[
\dot{\hat{x}}(t) = A_x \hat{x}(t) + B_x \gamma(\hat{x})[u(t) - \alpha(\bar{x})] \quad kT < t < (k+1)T,
\]
\[
\hat{x}[k+1] = \hat{x}(t^-) + H_p (y[k] - \hat{y}[k]) \quad t = (k+1)T \\
k = k_0 + k_0 + 1, \ldots, 
\]

where \( \hat{x} \in \mathbb{R}^n \) is the prediction observer state and \( H_p \in \mathbb{R}^{m \times 1} \) is a predication observer gain vector,

Linearization is being carried out by selecting (4.2.3) as input, thus (4.3.12) and (4.3.13) are written as

\[
D^t \dot{x}(t) = A_c \hat{x}(t) + B_c v(t) + \dot{d}(t) \quad t = kT, 
\]

(4.3.15)

\[
\dot{x}(t) = A_c \hat{x}(t) + B_c v(t) + \dot{d}(t) \quad kT < t < (k+1)T. 
\]

(4.3.16)

Error state vector of prediction impulsive observer is written as

\[
\hat{e}(t) = x(t) - \hat{x}(t) \quad \forall \ t, 
\]

(4.3.17)

derivative of (4.3.17) is

\[
D^t \dot{\hat{e}}(t) = A_c \hat{e}(t) + g(\hat{e}) \quad t = kT, 
\]

(4.3.18)

\[
\dot{\hat{e}}(t) = A_c \hat{e}(t) + g(\hat{e}) \quad kT < t < (k+1)T, 
\]

(4.3.19)

where (4.3.18) signifies the discontinuous nature of error \( \hat{e}(t) \). Observer error would not diminish asymptotically as \( g(\bar{e}) \) is considered to be a bounded non-vanishing perturbation.

Discrete-time error dynamics using prediction observer is written as

\[
\hat{e}[k+1] = x(t) - [\hat{x}(t^-) + H_p (y[k] - \hat{y}[k])] \quad t = (k+1)T, 
\]

(4.3.20)

continuity of system states \[43\] imply \( x(t) = x(t^-) \) leading to
\[
\dot{e}[k+1] = \dot{e}(t^-) - H_p(y[k] - \hat{y}[k]) \quad t = (k+1)T. 
\] (4.3.21)

Associated discrete equation is expressed as

\[
\hat{e}[k+1] = (\Phi - H_p C_e)\hat{e}[k], 
\] (4.3.22)

where \( \Phi = e^{A_T} \) and \( H_p[k] \) is to be designed such that (4.3.22) is uniformly exponentially stable.

### 4.2.3 A Nonlinear (Non-Impulsive) Reconstruction Observer

The construction of nonlinear continuous observer is similar to the theory discussed in Chapter one for linear time varying systems. The relationship between the nonlinear prediction and current impulsive observers are used for the design of nonlinear (non-impulsive) reconstruction observer. Outputs from both the observers are fused using a weighing function to make the output of the reconstruction observer non-impulsive. Further details can be referred from Section 2.2.4.

### 4.3 Stability Analysis

A stability discussion for a system with non-vanishing perturbations cannot be studied by only emphasising the stability of equilibrium point at origin. Hence, it is sufficient to prove that the stability solution be ultimately bounded by a small bound, while considering perturbation term \( g(\bar{e}) \) is small in some sense \([17][45]\). Stability proof of nonlinear current impulsive observer is discussed as under.

The nominal system is expressed in generic form for the sake of clarity as

\[
\dot{\bar{e}}(t) = A_e \bar{e}(t) \quad t \neq kT, 
\] (4.4.1)
\( \bar{e}(t^+) = (I - H_c C_e) \bar{e}[k] \quad t = kT, \quad (4.4.2) \)

Equation (4.3.7), (4.3.8) and (4.3.11) is also written as

\[
\dot{\bar{e}}(t) = A_i \bar{e}(t) + g(\bar{e}) \quad t \neq kT, \quad (4.4.3)
\]

\[
\bar{e}(t^+) = A_i \bar{e}[k] \quad t = kT, \quad (4.4.4)
\]

where \( A_i = (I - H_c C_e) \) and \( H_c \) is to be designed such that (4.4.4) is uniformly exponentially stable.

### 4.3.1 Theorem

The closed loop perturbed system (4.4.3), (4.4.4) is uniformly exponentially stable if there exist a symmetric matrix \( \tilde{P} \) for finite positive constants \( c_1 \) and \( c_2 \) for all times \( t \), such that [43][104]

\[
c_1 I \leq \tilde{P} \leq c_2 I, \quad (4.4.5)
\]

\[
A_i^T \tilde{P} + \tilde{PA}_i + Q_c \leq 0 \quad t \neq kT, \quad (4.4.6)
\]

\[
A_i^T \tilde{P}A_i - \tilde{P} + Q_l \leq 0 \quad t = kT, \quad (4.4.7)
\]

where \( Q_c \) and \( Q_l \) are semi positive definite matrices, with assumption that there exists a finite positive constant \( c_3 \) such that \( Q_c \geq c_3 I \) for all \( t \).

**Proof:**

For first part of stability proof for \( t \neq kt \), consider the following Lyapunov function
\[ V(\bar{e}(t)) = \bar{e}^T(t) \bar{P} \bar{e}(t), \quad (4.4.8) \]

such that \( V(\bar{e}(t)) \) for the nominal system (4.4.1) satisfy

\[ \left\| \frac{\delta V(\bar{e}(t))}{\delta \bar{e}(t)} \right\| \leq c_4 \left\| \bar{e}(t) \right\|, \quad (4.4.9) \]

It is assumed that perturbation term satisfy the following uniform bound

\[ \|g(\bar{e}(t))\| \leq \delta_x \quad \forall \quad t \geq 0. \quad (4.4.10) \]

Derivative of (4.4.8) after simplification from (4.4.3) is expressed as

\[ \dot{V}(\bar{e}(t)) = \dot{\bar{e}}^T(t) \bar{P} \dot{\bar{e}}(t) + \bar{e}^T(t) \bar{P} \dot{\dot{\bar{e}}}(t) \]
\[ = \bar{e}^T(t) \left[ A^T \bar{P} + \bar{P} A \right] \bar{e}(t) + g^T \left( \bar{e}(t) \right) \bar{P} \bar{e}(t) + \bar{e}^T(t) \bar{P} g(\bar{e}(t)), \quad (4.4.11) \]

with inequalities (4.4.9) and (4.4.10), (4.4.11) is expressed as

\[ \dot{V}(\bar{e}(t)) \leq -c_3 \| \bar{e}(t) \|^2 + \left\| \frac{\delta V(\bar{e}(t))}{\delta \bar{e}} \right\| \| g(\bar{e}(t)) \|, \]
\[ \leq -c_3 \| \bar{e}(t) \|^2 + c_4 \delta_x \| \bar{e}(t) \|, \]
\[ \leq -(1-\theta)c_3 \| \bar{e}(t) \|^2 - \theta c_3 \| \bar{e}(t) \|^2 + c_4 \delta_x \| \bar{e}(t) \| \quad 0 < \theta < 1, \quad (4.4.12) \]

thus defining bound on (4.4.12), results in

\[ \dot{V}(\bar{e}(t)) \leq -(1-\theta)c_3 \| \bar{e}(t) \|^2 \quad \forall \quad \| \bar{e}(t) \| \geq b, \quad (4.4.13) \]

where \( b = \frac{c_4 \delta_x}{\theta c_3} \), using (4.4.5) it is written as
\[ V(\bar{e}(t)) \leq c_2 \|\bar{e}(t)\|^2, \]
\[ \|\bar{e}(t)\|^2 \leq \frac{V(\bar{e}(t))}{c_2}, \] (4.4.14)

equation (4.4.13) is written as

\[ \dot{V}(\bar{e}(t)) \leq -\gamma V(\bar{e}(t)) \quad \forall \quad \|\bar{e}(t)\| \geq b, \] (4.4.15)

where \( \gamma = \frac{(1-\theta)c_1}{c_2} \), this implies that for \( t \in (kT, (k+1)T] \), we have the bound

\[ V(\bar{e}(t)) \leq e^{-\gamma t} V(\bar{e}) \quad \forall \quad \|\bar{e}(t)\| \geq b, \] (4.4.16)

For second part of stability proof for \( t = kT \), Lyapunov function is defined as

\[ \Delta V(\bar{e}(t)) = V(\bar{e}(t+\tau)) - V(\bar{e}(t)) \quad t = kT, \] (4.4.17)

simplify (4.4.17) by (4.4.4) results

\[ \Delta V(\bar{e}(t)) = \bar{e}^T[k] \left[ A^T[k] \bar{P} - \bar{P} \right] \bar{e}[k], \] (4.4.18)

using inequality (4.4.7)

\[ \Delta V(\bar{e}(t)) \leq -\bar{e}^T[k] Q \bar{e}[k], \] (4.4.19)

for \( Q \geq 0 \), (4.4.19) is expressed as

\[ \Delta V(\bar{e}(t)) \leq 0. \] (4.4.20)

Thus (4.4.20) shows that \( \Delta V(\bar{e}(t)) \) for \( t = kT \) cannot increase across any sample time, and hence satisfies the exponential decreasing bound (4.4.16), such that
\[ V(\bar{e}(t)) \leq e^{-\gamma t} V(\bar{e}_0) \quad \forall \|\bar{e}(t)\| \geq b, \quad t \geq t_0, \quad (4.41) \]

from (4.45)

\[ \|\bar{e}(t)\|^2 \leq \frac{1}{c_1} V(\bar{e}(t)) \leq \frac{1}{c_1} e^{-\gamma t} V(\bar{e}_0), \quad (4.42) \]

for \( t = t_0 \), from (4.45) and (4.8)

\[ V(\bar{e}_0) \leq c_2 \|\bar{e}_0\|^2, \quad (4.43) \]

using (4.42) and (4.43) in (4.41)

\[ \|\bar{e}(t)\| \leq \sqrt{\frac{c_2}{c_1}} e^{-\gamma /2} \|\bar{e}(t_0)\|, \quad \forall \|e\| \geq b \quad t \geq t_0. \quad (4.44) \]

This implies that closed loop perturbed system (4.43),(4.44) is ultimately bounded with a bound \( b \).

\[
\text{□□□}
\]

The stability proof of nonlinear prediction impulsive observer may also be discussed on the same lines of nonlinear current impulsive observer with only difference of \( A_t = \Phi - H_p C \) and \( t \in [kT, (k+1)T) \). The stability of nonlinear reconstruction observer is similar as discussed in chapter one with the linear sum of bounds of two impulsive observers.
4.4 Example

Consider the following sampled-data nonlinear system

\[ \dot{\eta}(t) = (-\eta(t) + x_2(t) + \tan^{-1} x_2(t)) \left( 1 + \frac{2 + x_2^2(t)}{1 + x_2^2(t)} x_2(t) \right), \]
\[ x_1(t) = x_2(t), \]
\[ \dot{x}_2(t) = (-\eta(t) + x_2(t) + \tan^{-1} x_2(t))x_2(t) + u(t), \]
\[ y_p[k] = x_1[k]. \]

(4.5.1)

Nonlinear system (4.5.1) is expressed in normal form as

\[ \dot{x}(t) = A_c(t) + B_c \frac{1}{\beta(x, \zeta)} [u(t) - \alpha(x, \zeta)] \]
\[ y_p[k] = C_c x[k]. \]

(4.5.2)

where \( u(t) = \alpha_o(\hat{x}, \zeta) + \beta_o(\hat{x}, \zeta)v(t), \) \( \alpha_o(\hat{x}, \zeta) = -\eta(t) + \dot{x}_2(t) + \tan^{-1} \dot{x}_2(t) \) and \( \beta_o(\hat{x}, \zeta) = 1 \) are the observed values of the nominal system (4.5.1). The nonlinear prediction, current and reconstruction observers are used in this example for two purposes: linearization of the nonlinear system through feedback linearization and system observed states are used for regulation.

The estimation error for (4.5.2) with prediction, current and reconstruction observer is illustrated in Figure 4-2 for \( \lambda = 1.11 \) and \( l = 3 \) with initial conditions \( x_0 = [1 \ 2]^T \) and \( T = 1 \text{sec} \) [43]. Prediction estimate is the continuation of current estimate from the common point of both observers. Reconstruction observer utilizes prediction and current estimates as per defined weighting function explained in Section 2.2. The error convergence rate for reconstruction
observer is flexible to change in between the prediction and current observer as per defined weighing function.

![Graph showing observer error for nonlinear plant](image)

**Figure 4-2 Nonlinear plant observer error for state (a) $x_1$ (b) $x_2$**

For (4.5.2), a sampled-data output feedback controller is design such that $e(t) \to 0$ as $t \to \infty$, a following exosystem is considered for regulation

$$\dot{w} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} w(t), \quad w_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$ (4.5.3)
Prediction, current and continuous observer’s output for second order exosystem (4.5.3) is shown in Figure 4-3. Reconstruction observer output is non-impulsive and smooth, whereas impulsive jumps are observed in case of prediction and current observers at sampled points, where

\[ C = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad Q_i = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}. \]

The solution \( \Pi \) and \( \Gamma \) of the regulator equation is calculated for given matrices as

\[ \Pi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Gamma(t) = \begin{bmatrix} -1 & 0 \end{bmatrix}. \]
and controller gains $F$ are chosen such as the Eigen values of $A(t) + B(t)F$ is at -7. The simulation results for sampled-data output regulation for (4.5.2) is illustrated in Figure 4-4 for prediction, current and reconstruction regulators. Ripples can be seen in prediction and current regulator outputs at sampling time that is $T = 1$ sec, whereas in reconstruction based regulator

**Figure 4-3** Exosystem observer states (a) $w_1$ (b) $w_2$

**Figure 4-4** Nonlinear plant output for (a) Prediction observer (b) Current observer (c) Reconstruction observer
output is ripple free at sampling time instant. Impulsive jumps are observed in case of prediction
and current observers, whereas reconstruction observer output is smooth (without jumps).
Reconstruction observer utilizes current and prediction impulsive estimates to provide continuous
(non-impulsive) output for sampled-data system. Ripples can be observed at sampling time
instants in prediction and current based regulators. The reconstruction observer based regulators
are ripple free at sampling instants. It is because that reconstruction observer output follows
prediction observer for $\Delta_1$ duration and current observer for $\Delta_3$ duration. Reconstruction
observer output for $\Delta_2$ duration is constructed with current, prediction and fusion function
outputs. Therefore no ripples are observed in regulator output based on reconstruction observer at
sampling time instants.

4.5 Chapter Summary

In this chapter a ripple free regulation solution is proposed for nonlinear sampled-data
feedback linearizable system. Feedback linearization and states observations are carried with
non-impulsive nonlinear reconstruction observer. The design of nonlinear reconstruction
observer (without jumps) is based on fusion of two impulsive nonlinear prediction and current
observers. Both the nonlinear impulsive observers are fused together, while incorporating
mathematical requirements to get non-impulsive reconstruction observer. The ripple free
regulation is achieved which also cater for disturbances / model uncertainties. The sampled-
data regulation is achieved by designing the continuous-time controller/observer due to
availability of continuous-time (non-impulsive) estimated system states. An example of third
order nonlinear plant is given to demonstrate the utility of the proposed scheme.
CHAPTER 5

CONCLUSIONS AND FUTURE SUGGESTIONS

5.1 Conclusions

This dissertation proposed a smooth and ripple free regulator solution for linear time varying system. The main focus is to design a continuous-time regulator based on continuous-time controller for following three important classes of sampled-data systems

- Linear time invariant systems
- Linear time varying systems
- Feedback linearizable nonlinear systems

A continuous-time controller is used as contrary to common practice of using discrete-time controller for sampled-data systems. It helps in avoidance of approximation of discrete equivalent models and thus making a practical continuous-time input available. It makes overall design convenient and simple. The plant to be regulated is assumed to be a physical system and is continuous in nature with sampled output. A continuous input is required for steering the output to the desired trajectory. Hence, a reconstruction observer plays an important role in providing ripple free regulation at sampling instants.

A continuous reconstruction observer without jumps for sampled-data continuous linear time varying system is based on states estimates from two different sampled-data impulsive observers. Impulsive observers are named basing on their functionality that is prediction observer and current observer. Such classification of impulsive observers is rarely discussed and found in literature. Outputs from both impulsive observers are fused in a way that overall estimated states become continuous (without jumps). A mathematical relationship is developed between the two new designed observers and ultimately this relationship is exploited to develop a continuous
(non-impulsive) reconstruction observer. Stability discussion of impulsive observers covered in literature is mainly focused for stable systems only. In this dissertation a comprehensive convergence analysis of the proposed observer is discussed for stable, unstable and highly unstable linear time varying systems. It appeals to be more practical approach for stability discussion. It guarantees uniform exponential stability on discrete sampling points and explicitly covers the inter-sample behavior of continuous-time systems. The non-impulsive nature of proposed observer makes it more practicable in use as compared to different impulsive observer designs.

Tracking of a time varying signal in sampled-data environment with impulsive observers is not a feasible and practical solution. The non-impulsive design of reconstruction observer has the all potentials and ability to provide non-impulsive (without jumps) solution at sampling instances. This makes it a worthy replacement for tracking a constant and time varying signals. The example presented shows the ripple free simulations results of the augmented system (plant and exosystem) in continuous-time. The stability analysis proves that the sampled-data based regulation can be effectively carried out with reconstruction observer for both linear time invariant and time varying systems.

The availability of sampled output measurements resulted in design of dual purpose non-impulsive observer for feedback linearizable nonlinear system. Dual purpose observer observes the unavailable states and subsequently used them for feedback linearization of a nonlinear system as a linearizing control The proposed non-impulsive observer / controller is also designed to accommodate model mismatch and model uncertainties. Keeping in view all mentioned advantages the proposed scheme ultimately provides ripple free regulation for a class of a nonlinear system. The stability analysis in Chapter 4 provides ultimate bounds on error convergence for reconstruction non-impulsive observer. The extension of the presented theory to multiple input multiple output systems may establishes its effectiveness with reconstruction
observer even in higher order environments and can provide an alternative to computationally extensive high gain observers.

5.2 Future Suggestions

It is recommended that the realization of proposed methodology with reconstruction observer is carried out on actual physical systems. The hardware implementation would open a wide range of applications since it is practically more efficient technique as compared to the impulsive observers.

A discussion on model uncertainties was carried out in the context feedback linearizable systems. In this dissertation, it is assumed that the model of exogenous system is known. However, the same can be extended to discuss the possibility of incorporating model error in the exogenous system. This may lead to some very interesting results.

The presented work is focused on the sampled-data regulation for only linear time varying systems. More detail can be studied and discussed in realm dead beat observers in contrast to the concept of receding horizon.
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