Applications of hyperstructures and soft sets in ideal theory of BCK/BCI-algebras

By

Muhammad Touqeer

Supervised By

Dr. Muhammad Aslam Malik

and

Prof. Dr. Wieslaw A. Dudek

UNIVERSITY OF THE PUNJAB

LAHORE- PAKISTAN

May, 2015

Applications of hyperstructures and soft sets in ideal theory of BCK/BCI-algebras

By

Muhammad Touqeer

A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

AT

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF THE PUNJAB

LAHORE- Pakistan

May, 2015

Declaration

I, Muhammad Touqeer s/o Mr. Muhammad Farid hereby declare that the dissertation has been written by me and is my original work. The work in this dissertation does not contain any material that has been submitted for the award of any other degree nor has it been submitted as part of the partial requirements for a degree in this university or any other university, except the information sources and literature that has been fully acknowledged and due reference is made in the text, to the best of my knowledge.

Muhammad Touqeer

Certificate

The undersigned hereby certify that the work presented in this thesis is original work of **Mr. Muhammad Touqeer s/o Mr. Muhammad Farid** and is carried out under my supervision and endorse evaluation of the thesis titled **Applications of hyperstructures and soft sets in ideal theory of BCK/BCI-algebras** by **Muhammad Touqeer** for the award of **Ph.D.** degree, through the official procedure of the **University of the Punjab**.

Research Supervisors:

Dr. Muhammad Aslam Malik

Prof. Dr. Wieslaw A. Dudek

Dedicated to My Family

Table of Contents

Abstract

The aim of this thesis is to study the applications of hyperstructures and soft sets to the ideals in BCK/BCI-algebras. The applications of intuitionistic fuzzy sets to six different ideals in BCK/BCI-algebras are described. Characterizations of these ideals using level subsets and Transfer principle and with respect to some more aspects are discussed. Connections between these ideals with the help of different examples are considered. A detail view is taken on the applications of hyperstructures to the ideal theory of BCK-algebras. Different types of fuzzy hyper ideals are presented and their relations are discussed. These fuzzy hyper ideals are characterized using the idea of strongest fuzzy relations, level sets and hyper homomorphism. Also the products of these fuzzy hyper ideals are conferred and their connections are discussed.

Soft sets and their applications in ideal theory of BCI-algebras are discussed. The concept of fuzzy soft α-ideals of BCI-algebras is presented and their basic properties are proved. Connections between various types of fuzzy soft α-ideals and fuzzy soft ideals are described, and characterizations of some types of fuzzy α-ideals by $\in$-soft sets are discussed. By considering the idea of intuitionistic fuzzy sets, we extend the study of applications of soft sets in α-ideals of BCI-algebras and introduce the concept of intuitionistic fuzzy soft α-ideals and prove their basic properties. Intuitionistic fuzzy soft α-ideals are connected with intuitionistic fuzzy soft ideals. We explore useful facts on various operations on intuitionistic fuzzy soft α-ideals and characterize intuitionistic fuzzy α-ideals by soft (δ, η)-level sets.

Acknowledgements

In the name of **Almighty Allah**, the Most Beneficient, the Most Merciful, **Who** bestowed on me an opportunity of taking over this work and blessed me with enough ability to finish it. All my praises are devoted to the Holy Prophet **Hazrat Muhammad** (S.A.W) who is a blessing for mankind.

My thesis would have not been possibly completed without the help and guidance of my supervisors. I acknowledge, appreciate, and return the love and support of my family especially my father **Muhammad Farid** who has been my emotional anchor not only throughout the completion of my present assignment but also my entire life.

I am extremely fortunate to have **Dr. Muhammad Aslam Malik** and **Prof. Dr. Wieslaw A. Dudek** as my supervisors. This work would never have been concluded without their inspirational guidance and continuous support. I am very grateful to both of them for their supervision of my thesis.

Lahore **Muhammad Touqeer**
May, 2015

Introduction

The operations of union, intersection and the set difference are the most elementary operations of set theory. The study of these operations lead to the creation of a number of branches of algebra, for instance the notion of Boolean algebra is a result of generalization of these three operations and their properties. Also the algebraic structures of distributive lattices, semi-rings, upper and lower semi-lattices are introduced on the basis of properties of intersection and union. Till 1966, different algebraic structures were discussed using the properties of intersection and union but the operation of set difference and its properties remained unexplored. Imai and Iséki [24], in 1966, considered the properties of set difference and presented the idea of a BCK-algebra. Iséki, in the same year, generalized BCK-algebras and presented the notion of BCI-algebras. BCK-algebras are inspired by BCK logic, i.e., an implicational logic based on modus ponens and the following axioms scheme:

Axiom **B**	$\mathcal{A} \supset \mathcal{B}. \supset .(\mathcal{C} \supset A) \supset (\mathcal{C} \supset B)$
Axiom **C**	$\mathcal{A} \supset (\mathcal{B} \supset C). \supset .\mathcal{B} \supset (\mathcal{A} \supset \mathcal{C})$
Axiom **K**	$\mathcal{A} \supset (\mathcal{B} \supset A)$

Similarly, BCI-algebras are inspired by BCI logic.

Some branches of Mathematics are expressed and developed in terms of other branches such as Boolean algebras that have laid basis for ring theory, projective geometries, characterized by Birkhoff [7] as lattices of special type, projective, descriptive and spherical geometries represented by Prenowitz [59, 60, 61], as multigroups, linear geometries and convex sets presented by Prenowitz and Jantosciak [62] as join spaces. Likewise, the theory of Hyperstructures presented by Marty [53] is used in a variety of mathematical disciplines.

At the very beginning, Marty applied the theory of Hyperstructures to groups, rational fractions and algebraic functions. Afterwards, Eaton, Drbohlav, Ore, Utumi, Harrison, Mockor, Roth, Haddad and Sureau [17, 15, 21, 56, 58, 65, 20, 74] discussed a number of new applications of Hyperstructure theory in groups. Later different researchers discussed the connections of this theory with Fields, Lattices, Rings, Quasigroups and Groupoids, Semigroups, Ordered Structures, Combinatorics, Vector Spaces, Ternary Algebras and Topology. Prenowitz, in 1940's [59, 60, 61], introduced different types of Geometries (Spherical, Descriptive, Projective) as hypergroups and later with Jantosciak [62] presented geometries on Join Spaces, a special type of hypergroups, which upto decades have been a useful tool for the study of various matters in particular, rough sets, fuzzy sets, binary relations, graphs and hypergraphs. Another link, called Steiner hypergroups between a type of hypergroups and geometries was established in 1978 by Tallini [67].

Rosenberg [64], in 1996, for the first time considered, in the most general meanings relations between Binary Relations and Hyperstructures. In 1994, Chvalina [10] discussed the connections of Hyperstuctures with binary relations, especially with ordered relations. In 1996, Corsini [11] associated the Hyperstructure theory with hypergraphs. Hyperstructures are also used for constructing sophisticated cryptographic systems. L. Berardi, F. Eugeni, St. Innamorati, R. Migliorato and G. Gentile [6, 55] discussed the applications of hyperstructures in algebraic cryptography. A. Maturo [54], using a particular non-standard algebraic hyperstructure, verified that difficulties in comprehensive calculations of probability and its solutions are expressed in a very simple manner.

Corsini, Corsini-Tofan and Ameri-Zahedi [12, 14, 3] considered the relations between Fuzzy Sets and Hyperstructures. An application of hyperstructures in the setting of Fuzzy Set Theory and in particular of Decision Making is that one to Factor Spaces. Borzooei et al. [8] studied hyper BCC-algebras, which are a generalization of BCC-algebras. Jun et al. [34] first time related hyperstructures with BCK-algebras and presented the idea of a hyper BCK-algebra. Long [39] related hyperstructures with BCI-algebras by presenting the idea of a hyper BCI-algebra. The study of hyper BCK-algebras is important with respect to their application to the theory of automaton. Such applications was firstly described by Corsini and Leoreanu [13]. In this monograph are described also many other applications of hyper structures in real world. In [18] it is proved that, on the states of a deterministic finite automaton, the set of all equivalence classes of an equivalence relation defines a hyper BCK-algebra which may be used to characterization of states. Here we are interested in application of hyperstructures to different ideals in BCK-algebras.

In the present era, uncertainty is one of the definitive changes in science. The traditional view is that uncertainty is objectionable in science and science should endeavor for certainty through all conceivable means. At the present it is believed that uncertainty is vigorous for science that is not only an inevitable epidemic but also has great effectiveness. The statistical methods particularly the probability theory was the first type of this approach to study the physical process at the molecular level as the existing computational approaches were not able to meet the enormous number of units involved in Newtonian Mechanics. Till mid-twentieth century, Probability theory was the only tool for handling certain type of uncertainty called Randomness. But there are several other kinds of uncertainties, one such type is called “vagueness” or “imprecision” which is inherent in our natural languages.

During the world war II, the development of computer technology assisted quite effectively in overcoming many complicated problems. But later it was realize that complexity can be handled up to a certain limit, that is, there are complications which cannot be overcome by human skills or any computer technology. Then the problem was to deal with such type of complications where no computational power is effective. Zadeh in 1965, [75] put forward his idea of fuzzy set theory which is considered to be the most suitable tool in overcoming the uncertainties. The concept of fuzzy set was suggested to achieve a simplified modeling of complex systems. The application of basic operations as direct generalization of complement, intersection and union for characteristic function was also proposed as a result of this idea. This theory is considered as a substitute of probability theory and is widely used in solving decision making problems. Later this “Fuzziness” concept lead to the highly acclaimed theory of Fuzzy Logic. This theory has been applied with a good deal of success to many areas of engineering, economics, medical science etc., to name a few, with great efficiency.

After the invention of fuzzy sets many other hybrid concepts begun to develop. K. Atanassov [4], in 1983, generalized the fuzzy sets by presenting the idea of Intuitionistic fuzzy sets, a set with each member having a degree of belongingness as well as a degree of non-belongingness. Although Fuzzy set theory is very successful in handling uncertainties arising from vagueness or partial belongingness of an element in a set, it cannot model all sorts of uncertainties prevailing in different real physical problems. Thus search for new theories has been continued. As a result two new theories; rough set theory and theory of interval mathematics were also introduced to cope with uncertainties. In daily life, conventional methods are not efficacious for solving difficult problems. Molodtsov [57] pointed out that due to insufficiency of parametrization tool, the theories like, the "probability theory", the "fuzzy set theory", the "theory of interval mathematics" are difficult to apply. He solved this problem by presenting the idea of soft set theory. This theory is extensively used in many different fields.

Soft set theory was primarily based on "parametrization of tools". In dealing with uncertain situations, fuzzy set theory was perhaps the most appropriate theory till then. But the main difficulty with fuzzy sets is to frame a suitable membership function for a specific problem. The reason behind this is the inability of the parametrization tool of the theory. Soft set theory is considered to be the one of the most reliable method for dealing with uncertainties. This theory is a classification of elements of the universe with respect to some given set of parameters. It has been proved that soft set is more general in nature and has more capabilities in handling uncertain information. A fuzzy set or a rough set is also considered as a special case of soft sets. Research involving soft sets and its application in various fields of science and technology are currently going on in a rapid pace.

Sezgin and Atagun [66], by using different operation, demonstrated that certain De Morgans law is valid in soft set theory. Qin and Hong [63] described the lattice structure of soft sets and constructed the algebraic structures of soft sets. They also presented the idea of soft equality and derived related properties. Some basic properties of soft sets were also considered by Aktaş and Çağman [1] and their connection with fuzzy and rough sets. Soft groups were also presented and their basic properties were derived. Babitha and Sunil [5], conferred soft set relations. They also explored related concepts like composition, partition, equivalent soft set relation and function. Kharal and Ahmad [37] considered images and inverse images of soft sets and the mappings on soft classes, which have many applications in medical field. Ali et al. [2] modified certain definitions and results discussed by Maji et al. in [41] and defined some new operations. Jun [26] was the first who applied the idea of soft sets to BCK/BCI-algebras. He presented the idea of soft subalgebras and soft BCK/BCI-algebras. In [27], he further explored the union-soft sets and considered their applications in BCK/BCI-algebras.

This thesis comprises six chapters. **Chapter 1** is a brief view of basic theory of BCK/BCI-algebras and different ideals in this theory. In **chapter 2**, we discuss the applications of intuitionistic fuzzy sets in six different ideals of BCK/BCI-algebras. Characterization of these ideals using level subsets and Transfer principle and with respect to some more aspects is discussed. Lastly, We try to explore the connections between these ideals with the help of different examples. **Chapter 3** provides a detail view of applications of hyperstructures to the ideal theory of BCK-algebras. Different types of fuzzy hyper ideals are presented and their relations are discussed. These fuzzy hyper ideals are characterized using the idea of strongest fuzzy relations, level sets and hyper homomorphism. Also the products of these fuzzy hyper ideals are conferred and their connections are discussed.

In **Chapter 4**, we present some new ideas about soft sets by applying them to ideals in BCI-algebras. The concept of fuzzy soft α-ideals of BCI-algebras is presented and their basic properties are proved. Connections between various types of fuzzy soft α-ideals and fuzzy soft ideals are described, and characterizations of some types of fuzzy α-ideals by $\in$-soft sets are discussed. By considering the idea of intuitionistic fuzzy sets by Atanassov [4], in **chapter 5**, we extend the study of applications of soft sets in α-ideals of BCI-algebras and introduce the concept of intuitionistic fuzzy soft α-ideals and prove their basic properties. We also describe connections between various types of intuitionistic fuzzy soft α-ideals and intuitionistic fuzzy soft ideals. We explore useful facts on various operations on intuitionistic fuzzy soft α-ideals and characterize intuitionistic fuzzy α-ideals by soft (δ, η)-level sets. In **chapter 6**, we conclude our previous discussion and try to guide towards further study in this field.

Chapter 1

Preliminaries

In this chapter, we discuss the basic properties *BCK/BCI-algebras* and define the common ideals in *BCK/BCI-algebras*. Fuzzy ideals of *BCI-algebras* will also be defined with the help of examples. The concepts defined in this chapter will be frequently used in the subsequent chapters.

1.1 Basic Concepts

BCK/BCI-algebras: An algebra $(\Omega, \cdot, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satiates the following axioms:

1.2.1. $((\imath \cdot \jmath) \cdot (\imath \cdot \ell)) \cdot (\ell \cdot \jmath) = 0$

1.2.2. $(\imath \cdot (\imath \cdot \jmath)) \cdot \jmath = 0$

1.2.3. $\imath \cdot \imath = 0$

1.2.4. $\imath \cdot \jmath = 0$ and $\jmath \cdot \imath = 0$ imply $\imath = \jmath$

1.2.5. $\imath \cdot 0 = 0 \Rightarrow \imath = 0$

for any $\imath, \jmath, \ell \in \Omega$.

In a *BCI-algebra*, a partial ordering " $\leq$ " is demarcated as, $\imath \leq \jmath \iff \imath \cdot \jmath = 0$. In a *BCI-algebra* Ω, the set $M = \{\imath \in \Omega \mid 0 \cdot \imath = 0\}$ is a subalgebra and is called the *BCK-part* of Ω. Ω is called proper if $\Omega - M \neq \Phi$. Otherwise it is improper. Moreover, in a *BCI-algebra* the succeeding axioms hold:

1.2.6. $(\imath \cdot \jmath) \cdot \ell = (\imath \cdot \ell) \cdot \jmath$

1.2.7. $\imath \cdot 0 = \imath$

1.2.8. $\imath \leq \jmath$ implies $\imath \cdot \ell \leq \jmath \cdot \ell$ and $\ell \cdot \jmath \leq \ell \cdot \imath$

1.2.9. $0 \cdot (\imath \cdot \jmath) = (0 \cdot \imath) \cdot (0 \cdot \jmath)$

1.2.10. $0 \cdot (0 \cdot (\imath \cdot \jmath)) = 0 \cdot (\jmath \cdot \imath)$

1.2.11. $(\imath \cdot \ell) \cdot (\jmath \cdot \ell) \leq \imath \cdot \jmath$

for any $\imath, \jmath, \ell \in \Omega$.

A mapping $\theta : X \to Y$ of BCI -algebras is called a homomorphism if $\theta(\imath \cdot \jmath) = \theta(\imath) \cdot \theta(\jmath)$, for any $\imath, \jmath \in X$.

1.2 Ideal theory of BCK/BCI-algebras

Let Ω be a *BCI-algebra*. A non-empty subset $I \subseteq \Omega$ containing 0 is called

- an *ideal* of Ω if $\imath \cdot \jmath \in I$ and $\jmath \in I$ implies $\imath \in I$.
- a *p-ideal* of Ω if $(\imath \cdot \ell) \cdot (\jmath \cdot \ell) \in I$ and $\jmath \in I$ implies $\imath \in I$.
- an *h-ideal* of Ω if $\imath \cdot (\jmath \cdot \ell) \in I$ and $\jmath \in I$ implies $\imath \cdot \ell \in I$.
- an *α-ideal* of Ω if $(\imath \cdot \ell) \cdot (0 \cdot \jmath) \in I$ and $\ell \in I$ implies $\jmath \cdot \imath \in I$.
- a *BCI-commutative ideal* of Ω if $(\imath \cdot \jmath) \cdot \ell \in I$ and $\ell \in I$ implies $\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))) \in I$.
- a *BCI-implicative ideal* of Ω if $(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell \in I$ and $\ell \in I$ implies $\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))) \in I$.
- a *BCI-positive implicative ideal* of Ω if $((\imath \cdot \ell) \cdot \ell) \cdot (\jmath \cdot \ell) \in I$ and $\jmath \in I$ implies $\imath \cdot \ell \in I$.

An ideal I of a *BCI-algebra* Ω is termed as closed if $0 \cdot \imath \in I$, for all $\imath \in I$.

By a fuzzy set ϖ in a nonempty set Ω we mean a function $\varpi : \Omega \to [0,1]$ and the complement of ϖ, denoted by $\bar{\varpi}$, is the fuzzy set in Ω delineated as $\bar{\varpi}(\imath) = 1 - \varpi(\imath)$ for all $\imath \in \Omega$. If ϖ and ξ are two fuzzy sets in a *BCI-algebra* Ω then by $\varpi \leq \xi$ we mean that $\varpi(\imath) \leq \xi(\imath)$, for any $\imath \in \Omega$. A fuzzy set ϖ in a *BCI-algebra* Ω is called a *fuzzy Subalgebra* of Ω if
$\varpi(\imath \cdot \jmath) \geq \min\{\varpi(\imath), \varpi(\jmath)\}$, for any $\imath, \jmath \in \Omega$.

1.3 Fuzzy ideals of BCI-algebras

A fuzzy set ϖ of a *BCI-algebra* Ω satisfying $\varpi(0) \geq \varpi(\imath)$, for all $\imath \in \Omega$, is called

- a *fuzzy ideal* of Ω if

$$\varpi(\imath) \geq \min\{\varpi(\imath \cdot \jmath), \varpi(\jmath)\}.$$

- a *fuzzy p-ideal* of Ω if

$$\varpi(\imath) \geq \min\{\varpi((\imath \cdot \ell) \cdot (\jmath \cdot \ell)), \varpi(\jmath)\}.$$

- a *fuzzy h-ideal* of Ω if

$$\varpi(\imath \cdot \ell) \geq \min\{\varpi(\imath \cdot (\jmath \cdot \ell)), \varpi(\jmath)\}.$$

- a *fuzzy α-ideal* of Ω if

$$\varpi(\jmath \cdot \imath) \geq \min\{\varpi((\imath \cdot \ell) \cdot (0 \cdot \jmath)), \varpi(\ell)\}.$$

- a *fuzzy BCI-commutative ideal* of Ω if

$$\varpi(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \min\{\varpi((\imath \cdot \jmath) \cdot \ell), \varpi(\ell)\}.$$

- a *fuzzy BCI-implicative ideal* of Ω if

$$\varpi(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \min\{\varpi((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \varpi(\ell)\}.$$

- a *fuzzy BCI-positive implicative ideal* of Ω if

$$\varpi(\imath\cdot\ell)\geq\min\{\varpi(((\imath\cdot\ell)\cdot\ell)\cdot(\jmath\cdot\ell)),\varpi(\jmath)\}.$$

A fuzzy ideal ϖ of a *BCI-algebra* Ω is said to be a closed if for all $\imath\in\Omega$, $\varpi(0\cdot\imath)\geq\varpi(\imath)$.

1.4 Examples

In order to clarify the above definitions we present the following examples.

Example 1.4.1. Consider the *BCI-algebra* $\Omega=\{0,\imath,\jmath\}$ with the succeeding cayley table:

$\cdot$	0	$\imath$	$\jmath$
0	0	$\jmath$	$\imath$
$\imath$	$\imath$	0	$\jmath$
$\jmath$	$\jmath$	$\imath$	0

Delineate a fuzzy set ϖ in Ω as,

$$\varpi(0)=0.9,\ \varpi(\imath)=\varpi(\jmath)=0.7.$$

Then ϖ is a *fuzzy p-ideal* of Ω.

Define a fuzzy set ϖ in Ω as,

$$\varpi(0)=s_1,\ \varpi(\imath)=\varpi(\jmath)=s_2,$$

where $s_1,s_2\in[0,1]$ and $s_1>s_2$. Then ϖ is a *fuzzy BCI-implicative ideal* of Ω.

Example 1.4.2. Consider the *BCI-algebra* $\Omega = \{0, \imath, \jmath\}$ with the succeeding cayley table:

$\cdot$	0	$\imath$	$\jmath$
0	0	0	$\jmath$
$\imath$	$\imath$	0	$\jmath$
$\jmath$	$\jmath$	$\jmath$	0

Define a fuzzy set ϖ in Ω as,

$$\varpi(0) = 0.9,\ \varpi(\imath) = \varpi(\jmath) = 0.6.$$

Then ϖ is a *fuzzy h-ideal* of Ω.

Delineate a fuzzy set ϖ in Ω as,

$$\varpi(0) = \varpi(\imath) = 0.7,\ \varpi(\jmath) = 0.3.$$

Then ϖ is a *fuzzy α-ideal* of Ω.

Example 1.4.3. Consider the *BCI-algebra* $\Omega = \{0, \imath, \jmath\}$ with the following cayley table:

$\cdot$	0	$\imath$	$\jmath$
0	0	0	0
$\imath$	$\imath$	0	0
$\jmath$	$\jmath$	$\imath$	0

Define a fuzzy set ϖ in Ω as,

$$\varpi(0) = 0.8,\ \varpi(\imath) = \varpi(\jmath) = 0.5.$$

Then ϖ is a *fuzzy BCI-commutative ideal* of Ω.

Example 1.4.4. Consider the *BCI-algebra* $\Omega = \{0, \imath, \jmath\}$ with the succeeding cayley table:

$\cdot$	0	$\imath$	$\jmath$
0	0	0	0
$\imath$	$\imath$	0	0
$\jmath$	$\jmath$	$\jmath$	0

Define a fuzzy set ϖ in Ω as,

$$\varpi(0) = 0.8,\ \varpi(\imath) = \varpi(\jmath) = 0.3.$$

Then ϖ is a *fuzzy BCI-positive implicative ideal* of Ω.

1.5 Positive implicative BCK-algebra

A *BCK-algebra* Ω is said to be *positive implicative* if it satiates for all $\imath, \jmath, \ell \in \Omega$, $(\imath \cdot \ell) \cdot (\jmath \cdot \ell) = (\imath \cdot \jmath) \cdot \ell$.

1.6 Fuzzy ideal extensions

Let ϖ be a fuzzy set in a BCK-algebra Ω and $a \in \Omega$. Then the fuzzy set, $< \varpi, a >: \Omega \mapsto [0, 1]$ defined by $< \varpi, a > (\imath) = \varpi(\imath \cdot a)$ is called the extension of ϖ by a.

1.7 Upper δ-level cut and Lower η-level cut

For any $\delta, \eta \in [0, 1]$ and a fuzzy set ϖ in a non-empty set Ω, the set $U(\varpi; \delta) = \{\imath \in \Omega \mid \varpi(\imath) \geq \delta\}$ is called an upper δ-level cut of ϖ and the set $L(\varpi; \eta) = \{\imath \in \Omega \mid \varpi(\imath) \leq \eta\}$ is called a lower η-level cut of ϖ.

Chapter 2

Intuitionistic fuzzy ideals in BCK/BCI-algebras

2.1 Introduction

In this chapter, we present the notions of intuitionistic fuzzy p-ideals, intuitionistic fuzzy h-ideals and intuitionistic fuzzy α-ideals in BCI-algebras and confer apposite properties. We also derive some characterization theorems on these intuitionistic fuzzy ideals and discuss their relationship. Moreover, we will also introduce the notions of intuitionistic fuzzy BCI-implicative, intuitionistic fuzzy BCI-positive implicative and intuitionistic fuzzy BCI-commutative ideals in BCI-algebras and elaborate their properties. We also derive some characterization theorems on these intuitionistic fuzzy ideals and confer their relation. Lastly, we will interrelate all these different types of intuitionistic fuzzy ideals. In the sequel, Ω will be a "BCI-algebra" and IFS, IFI, IF_pI, IF_hI, $IF_\alpha I$, $IF_{BCI}PII$, $IF_{BCI}II$, $IF_{BCI}CI$ will be an "intuitionistic fuzzy set", "intuitionistic fuzzy ideal", "intuitionistic fuzzy p-ideal", "intuitionistic fuzzy h-ideal", "intuitionistic fuzzy α-ideal", "intuitionistic fuzzy BCI-positive implicative ideal", "intuitionistic fuzzy BCI-implicative ideal" and "intuitionistic fuzzy BCI-commutative ideal" respectively.

2.2 Intuitionistic fuzzy set

An "intuitionistic fuzzy set" (IFS) Δ in a non-empty set Ξ is an object having the form $\Delta = \{(\imath, \varpi_\Delta(\imath), \xi_\Delta(\imath)) \mid \imath \in \Xi\}$, where the mappings $\varpi_\Delta : \Xi \mapsto [0,1]$ and $\xi_\Delta : \Xi \mapsto [0,1]$ signify the "degree of membership" and the "degree of non-membership" and $0 \leq \varpi_\Delta(\imath) + \xi_\Delta(\imath) \leq 1$ for all $\imath \in \Xi$ [28].

An IFS $\Delta = \{(\imath, \varpi_\Delta(\imath), \xi_\Delta(\imath)) \mid \imath \in \Xi\}$ in Ξ can be identified to an ordered pair $(\varpi_\Delta, \xi_\Delta)$ in $I^\Xi \times I^\Xi$. In the sequel, $\Delta = (\varpi_\Delta, \xi_\Delta)$ will be used instead of the notation $\Delta = \{(\imath, \varpi_\Delta(\imath), \xi_\Delta(\imath)) \mid \imath \in \Xi\}$.

2.3 Intuitionistic fuzzy subalgebra

An IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω is called an "intuitionistic fuzzy subalgebra" of Ω if it satisfies [28]:

$\varpi_\Delta(\imath \cdot \jmath) \geq \min\{\varpi_\Delta(\imath), \varpi_\Delta(\jmath)\}$ and $\xi_\Delta(\imath \cdot \jmath) \leq \max\{\xi_\Delta(\imath), \xi_\Delta(\jmath)\}$

for all $\imath, \jmath \in \Omega$.

Proposition 2.3.1. *Any "intuitionistic fuzzy subalgebra" $\Delta = (\varpi_\Delta, \xi_\Delta)$ of Ω satisfies the inequalities [28]:*

$\varpi_\Delta(0) \geq \varpi_\Delta(\imath)$ *and* $\xi_\Delta(0) \leq \xi_\Delta(\imath)$ *for all* $\imath \in \Omega$.

2.4 Intuitionistic Fuzzy Ideal

An IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω is called an "intuitionitstic fuzzy ideal" (IFI) of Ω if it satisfies the following inequalities [28]:

$(IFI-1)$ $\varpi_\Delta(0) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0) \leq \xi_\Delta(\imath)$

$(IFI-2)$ $\varpi_\Delta(\imath) \geq \min\{\varpi_\Delta(\imath \cdot \jmath), \varpi_\Delta(\jmath)\}$

$(IFI-3)$ $\xi_\Delta(\imath) \leq \max\{\xi_\Delta(\imath \cdot \jmath), \xi_\Delta(\jmath)\}$

for all $\imath, \jmath \in \Omega$.

An IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω is called an "intuitionistic fuzzy closed ideal" of Ω if it satisfies $(IFI-2)$, $(IFI-3)$ and

$(IFI-4)$ $\varpi_\Delta(0 \cdot \imath) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0 \cdot \imath) \leq \xi_\Delta(\imath)$ for all $\imath \in \Omega$.

Proposition 2.4.1. *Any IFI of a BCK-algebra Ξ is an intuitionistic fuzzy subalgebra of Ξ [28].*

Lemma 2.4.2. *Let IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ be an IFI of Ω. If the inequality $\imath \cdot \jmath \leq \ell$ holds in Ω, then, $\varpi_\Delta(\imath) \geq \min\{\varpi_\Delta(\jmath), \varpi_\Delta(\ell)\}$ and $\xi_\Delta(\imath) \leq \max\{\xi_\Delta(\jmath), \xi_\Delta(\ell)\}$ [28].*

Lemma 2.4.3. *Let IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ be an IFI of Ω. If the inequality $\imath \leq \jmath$ holds in Ω, then $\varpi_\Delta(\imath) \geq \varpi_\Delta(\jmath)$ and $\xi_\Delta(\imath) \leq \xi_\Delta(\jmath)$, that is ϖ_Δ is order reversing while ξ_Δ is order preserving [28].*

Definition 2.4.4. Let $(\varpi_\Delta, \xi_\Delta)$ be an IFS in a BCK-algebra Ξ and $a, b \in \Xi$. Then the IFS $< (\varpi_\Delta, \xi_\Delta), (a, b) >$ defined by:
$< (\varpi_\Delta, \xi_\Delta), (a, b) >= (< \varpi_\Delta, a >, < \xi_\Delta, b >)$ is called the extension of $(\varpi_\Delta, \xi_\Delta)$ by (a, b). If $a = b$, then it is denoted by $< (\varpi_\Delta, \xi_\Delta), a >$.

2.5 Intuitionistic fuzzy p-ideal

An IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω is called an "intuitionistic fuzzy p-ideal" (IF_pI) of Ω if it satisfies:

(IF-PI-1) $\varpi_\Delta(0) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0) \leq \xi_\Delta(\imath)$, for all $\imath \in \Omega$

$(IF-PI-2)$ $\varpi_\Delta(\imath) \geq \min\{\varpi_\Delta((\imath \cdot \ell) \cdot (\jmath \cdot \ell)), \varpi_\Delta(\jmath)\}$

$(IF-PI-3)\xi_\Delta(\imath) \leq \max\{\xi_\Delta((\imath \cdot \ell) \cdot (\jmath \cdot \ell)), \xi_\Delta(\jmath)\}$

for all $\imath, \jmath, \ell \in \Omega$.

Example 2.5.1. Let $\Omega = \{0, \imath, \jmath, \ell\}$ be a BCI-algebra defined by the following cayley table:

$\cdot$	0	$\imath$	$\jmath$	ℓ
0	0	$\imath$	$\jmath$	ℓ
$\imath$	$\imath$	0	ℓ	$\jmath$
$\jmath$	$\jmath$	ℓ	0	$\imath$
ℓ	ℓ	$\jmath$	$\imath$	0

Define an $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω as:

$\varpi_\Delta(0) = \varpi_\Delta(\ell) = 1$, $\varpi_\Delta(\imath) = \varpi_\Delta(\jmath) = t$

and $\xi_\Delta(0) = \xi_\Delta(\ell) = 0$, $\xi_\Delta(\imath) = \xi_\Delta(\jmath) = s$

where $s, t \in (0,1)$ and $s + t \leq 1$

By routine calculations it is easy to verify that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an IF_pI of Ω.

An $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω is called an "intuitionistic fuzzy closed p-ideal" of Ω if it satisfies $(IF-PI-2)$, $(IF-PI-3)$ and

$(IF-PI-4)$ $\varpi_\Delta(0 \cdot \imath) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0 \cdot \imath) \leq \xi_\Delta(\imath)$, for all $\imath \in \Omega$.

Theorem 2.5.2. *Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an IFI of Ω. Then the following conditions are equivalent:*

1. *$IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an IF_pI of Ω.*

2. *$\varpi_\Delta(\imath) \geq \varpi_\Delta(0 \cdot (0 \cdot \imath))$ and $\xi_\Delta(\imath) \leq \xi_\Delta(0 \cdot (0 \cdot \imath))$.*

3. *$\varpi_\Delta(\imath) = \varpi_\Delta(0 \cdot (0 \cdot \imath))$ and $\xi_\Delta(\imath) = \xi_\Delta(0 \cdot (0 \cdot \imath))$.*

Proof. $(1 \Rightarrow 2)$ Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an IF_pI of Ω. Then,

$\varpi_\Delta(\imath) \geq \min\{\varpi_\Delta((\imath \cdot \ell) \cdot (\jmath \cdot \ell)), \varpi_\Delta(\jmath)\}$

and $\xi_\Delta(\imath) \leq \max\{\xi_\Delta((\imath \cdot \ell) \cdot (\jmath \cdot \ell)), \xi_\Delta(\jmath)\}$,

for any $\imath, \jmath, \ell \in \Omega$.

Now putting $\ell = \imath$ and $\jmath = 0$ we get

$\varpi_\Delta(\imath) \geq \min\{\varpi_\Delta((\imath \cdot \imath) \cdot (0 \cdot \imath)), \varpi_\Delta(0)\}$

and $\xi_\Delta(\imath) \leq \max\{\xi_\Delta((\imath \cdot \imath) \cdot (0 \cdot \imath)), \xi_\Delta(0)\}$

$\Rightarrow \varpi_\Delta(\imath) \geq \varpi_\Delta(0 \cdot (0 \cdot \imath))$ and $\xi_\Delta(\imath) \leq \xi_\Delta(0 \cdot (0 \cdot \imath))$.

$(2 \Rightarrow 3)$ Let $\varpi_\Delta(\imath) \geq \varpi_\Delta(0 \cdot (0 \cdot \imath))$ and $\xi_\Delta(\imath) \leq \xi_\Delta(0 \cdot (0 \cdot \imath))$.

Since by 1.2.2, $0 \cdot (0 \cdot \imath) \leq \imath$.

Therefore by Lemma 2.4.3,

$\varpi_\Delta(0 \cdot (0 \cdot \imath)) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0 \cdot (0 \cdot \imath)) \leq \xi_\Delta(\imath)$.

Therefore we have

$\varpi_\Delta(\imath) = \varpi_\Delta(0 \cdot (0 \cdot \imath))$ and $\xi_\Delta(\imath) = \xi_\Delta(0 \cdot (0 \cdot \imath))$.

which is required condition.

$(3 \Rightarrow 1)$ Let $\varpi_\Delta(\imath) = \varpi_\Delta(0 \cdot (0 \cdot \imath))$ and $\xi_\Delta(\imath) = \xi_\Delta(0 \cdot (0 \cdot \imath))$.

Now $(0 \cdot (0 \cdot \imath)) \cdot ((\imath \cdot \ell) \cdot (\jmath \cdot \ell)) = (0 \cdot ((\imath \cdot \ell) \cdot (\jmath \cdot \ell))) \cdot (0 \cdot \imath)$

$= ((0 \cdot (\imath \cdot \ell)) \cdot (0 \cdot (\jmath \cdot \ell))) \cdot (0 \cdot \imath) = ((0 \cdot (\imath \cdot \ell)) \cdot (0 \cdot \imath)) \cdot (0 \cdot (\jmath \cdot \ell))$

$= (((0 \cdot \imath) \cdot (0 \cdot \ell)) \cdot (0 \cdot \imath)) \cdot ((0 \cdot \jmath) \cdot (0 \cdot \ell))$

$= (((0 \cdot \imath) \cdot (0 \cdot \imath)) \cdot (0 \cdot \ell)) \cdot ((0 \cdot \jmath) \cdot (0 \cdot \ell))$

$= (0 \cdot (0 \cdot \ell)) \cdot ((0 \cdot \jmath) \cdot (0 \cdot \ell)) \leq 0 \cdot (0 \cdot \jmath)$ (By 1.2.11)

$\Rightarrow (0 \cdot (0 \cdot \imath)) \cdot ((\imath \cdot \ell) \cdot (\jmath \cdot \ell)) \leq 0 \cdot (0 \cdot \jmath)$

Therefore by using Lemma 2.4.2 we get

$\varpi_\Delta(0 \cdot (0 \cdot \imath)) \geq \min\{\varpi_\Delta((\imath \cdot \ell) \cdot (\jmath \cdot \ell)), \varpi_\Delta(0 \cdot (0 \cdot \jmath))\}$

and $\xi_\Delta(0 \cdot (0 \cdot \imath)) \leq \max\{\xi_\Delta((\imath \cdot \ell) \cdot (\jmath \cdot \ell)), \xi_\Delta(0 \cdot (0 \cdot \jmath))\}$

Since $\varpi_\Delta(0 \cdot (0 \cdot \jmath)) \geq \varpi_\Delta(\jmath)$ and $\xi_\Delta(0 \cdot (0 \cdot \jmath)) \leq \xi_\Delta(\jmath)$

Thus we get

$\varpi_\Delta(0 \cdot (0 \cdot \imath)) \geq \min\{\varpi_\Delta((\imath \cdot \ell) \cdot (\jmath \cdot \ell)), \varpi_\Delta(\jmath)\}$

and $\xi_\Delta(0\cdot(0\cdot\imath))\leq\max\{\xi_\Delta((\imath\cdot\ell)\cdot(\jmath\cdot\ell)),\xi_\Delta(\jmath)\}$

that is

$\varpi_\Delta(\imath)\geq\min\{\varpi_\Delta((\imath\cdot\ell)\cdot(\jmath\cdot\ell)),\varpi_\Delta(\jmath)\}$

and $\xi_\Delta(\imath)\leq\max\{\xi_\Delta((\imath\cdot\ell)\cdot(\jmath\cdot\ell)),\xi_\Delta(\jmath)\}$

Hence $IFS\ \Delta=(\varpi_\Delta,\xi_\Delta)$ is an IF_pI of Ω. □

2.6 Intuitionistic fuzzy h-ideal

An $IFS\ \Delta=(\varpi_\Delta,\xi_\Delta)$ in Ω is called an "intuitionistic fuzzy h-ideal" (IF_hI) of Ω if it satisfies:

$(IF-HI-1)$ $\varpi_\Delta(0)\geq\varpi_\Delta(\imath)$ and $\xi_\Delta(0)\leq\xi_\Delta(\imath)$, for all $\imath\in\Omega$.

$(IF-HI-2)$ $\varpi_\Delta(\imath\cdot\ell)\geq\min\{\varpi_\Delta(\imath\cdot(\jmath\cdot\ell)),\varpi_\Delta(\jmath)\}$

$(IF-HI-3)$ $\xi_\Delta(\imath\cdot\ell)\leq\max\{\xi_\Delta(\imath\cdot(\jmath\cdot\ell)),\xi_\Delta(\jmath)\}$,

for all $\imath,\jmath,\ell\in\Omega$.

Example 2.6.1. Let $\Omega=\{0,\imath,\jmath\}$ be the BCI-algebra defined in Example 1.4.2. Define an $IFS\ \Delta=(\varpi_\Delta,\xi_\Delta)$ in Ω as:

$\varpi_\Delta(0)=1,\ \varpi_\Delta(\imath)=\varpi_\Delta(\jmath)=t$ and $\xi_\Delta(0)=0,\xi_\Delta(\imath)=\xi_\Delta(\jmath)=s.$

where $s,t\in(0,1)$ and $s+t\leq1$

By routine calculations it is easy to verify that $IFS\ \Delta=(\varpi_\Delta,\xi_\Delta)$ is an IF_hI of Ω.

Theorem 2.6.2. *Let $IFS\ \Delta=(\varpi_\Delta,\xi_\Delta)$ be an IFI of Ω. Then the following conditions are equivalent:*

1. *$IFS\ \Delta=(\varpi_\Delta,\xi_\Delta)$ is an IF_hI of Ω.*
2. *$\varpi_\Delta((\imath\cdot\jmath)\cdot\ell)\geq\varpi_\Delta(\imath\cdot(\jmath\cdot\ell))$ and $\xi_\Delta((\imath\cdot\jmath)\cdot\ell)\leq\xi_\Delta(\imath\cdot(\jmath\cdot\ell))$.*
3. *$\varpi_\Delta(\imath\cdot\jmath)\geq\varpi_\Delta(\imath\cdot(0\cdot\jmath))$ and $\xi_\Delta(\imath\cdot\jmath)\leq\xi_\Delta(\imath\cdot(0\cdot\jmath))$.*

Proof. (1 ⇒ 2) Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an IF_hI of Ω. Then:

$\varpi_\Delta((\imath \cdot \jmath) \cdot \ell) \geq \min\{\varpi_\Delta((\imath \cdot \jmath) \cdot (0 \cdot \ell)), \varpi_\Delta(0)\} = \varpi_\Delta((\imath \cdot \jmath) \cdot (0 \cdot \ell))$ and

$\xi_\Delta((\imath \cdot \jmath) \cdot \ell) \leq \max\{\xi_\Delta((\imath \cdot \jmath) \cdot (0 \cdot \ell)), \xi_\Delta(0)\} = \xi_\Delta((\imath \cdot \jmath) \cdot (0 \cdot \ell))$.

$\Rightarrow \varpi_\Delta((\imath \cdot \jmath) \cdot \ell) \geq \varpi_\Delta((\imath \cdot \jmath) \cdot (0 \cdot \ell))$

and $\xi_\Delta((\imath \cdot \jmath) \cdot \ell) \leq \xi_\Delta((\imath \cdot \jmath) \cdot (0 \cdot \ell))$

Since $(\imath \cdot \jmath) \cdot (0 \cdot \ell) = (\imath \cdot \jmath) \cdot ((\jmath \cdot \ell) \cdot \jmath) \leq \imath \cdot (\jmath \cdot \ell)$ (by 1.2.11)

Therefore we get $\varpi_\Delta((\imath \cdot \jmath) \cdot (0 \cdot \ell)) \geq \varpi_\Delta(\imath \cdot (\jmath \cdot \ell))$

and $\xi_\Delta((\imath \cdot \jmath) \cdot (0 \cdot \ell)) \leq \xi_\Delta(\imath \cdot (\jmath \cdot \ell))$. (by Lemma 2.4.3)

i.e., $\varpi_\Delta((\imath \cdot \jmath) \cdot \ell) \geq \varpi_\Delta((\imath \cdot \jmath) \cdot (0 \cdot \ell)) \geq \varpi_\Delta(\imath \cdot (\jmath \cdot \ell))$

and

$\xi_\Delta((\imath \cdot \jmath) \cdot \ell) \leq \xi_\Delta((\imath \cdot \jmath) \cdot (0 \cdot \ell)) \leq \xi_\Delta(\imath \cdot (\jmath \cdot \ell))$

Which is the required condition.

(2 ⇒ 3) Assume that $\varpi_\Delta((\imath \cdot \jmath) \cdot \ell) \geq \varpi_\Delta(\imath \cdot (\jmath \cdot \ell))$

and $\xi_\Delta((\imath \cdot \jmath) \cdot \ell) \leq \xi_\Delta(\imath \cdot (\jmath \cdot \ell))$.

Substituting $\jmath = 0$ and $\ell = \jmath$,

$\varpi_\Delta((\imath \cdot 0) \cdot \jmath) \geq \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$ and $\xi_\Delta((\imath \cdot 0) \cdot \jmath) \leq \xi_\Delta(\imath \cdot (0 \cdot \jmath))$.

$\Rightarrow \varpi_\Delta(\imath \cdot \jmath) \geq \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$ and $\xi_\Delta(\imath \cdot \jmath) \leq \xi_\Delta(\imath \cdot (0 \cdot \jmath))$.

which is the required condition.

(3 ⇒ 1) Assume that $\varpi_\Delta(\imath \cdot \jmath) \geq \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$ and $\xi_\Delta(\imath \cdot \jmath) \leq \xi_\Delta(\imath \cdot (0 \cdot \jmath))$.

Since $(\imath \cdot (0 \cdot \jmath))(\imath \cdot (\ell \cdot \jmath)) \leq (\ell \cdot \jmath) \cdot (0 \cdot \jmath) \leq \ell \cdot 0 = \ell$ (by 1.2.1 and 1.2.11)

Therefore by Lemma 2.4.2,

$\varpi_\Delta(\imath \cdot (0 \cdot \jmath)) \geq \min\{\varpi_\Delta(\imath \cdot (\ell \cdot \jmath)), \varpi_\Delta(\ell)\}$

and $\xi_\Delta(\imath \cdot (0 \cdot \jmath)) \leq \max\{\xi_\Delta(\imath \cdot (\ell \cdot \jmath)), \xi_\Delta(\ell)\}$.

i.e., $\varpi_\Delta(\imath \cdot \jmath) \geq \min\{\varpi_\Delta(\imath \cdot (\ell \cdot \jmath)), \varpi_\Delta(\ell)\}$

and $\xi_\Delta(\imath \cdot \jmath) \leq \max\{\xi_\Delta(\imath \cdot (\ell \cdot \jmath)), \xi_\Delta(\ell)\}$.

Hence $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an IF_hI of Ω. □

Theorem 2.6.3. *Let IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ be an IFI of Ω.*
If $\varpi_\Delta(\imath \cdot \jmath) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(\imath \cdot \jmath) \leq \xi_\Delta(\imath)$ for all $\imath, \jmath \in \Omega$, then $\Delta = (\varpi_\Delta, \xi_\Delta)$ is an IF_hI of Ω.

Proof. By given hypothesis

$\min\{\varpi_\Delta(\imath \cdot (\jmath \cdot \ell)), \varpi_\Delta(\jmath)\} \leq \min\{\varpi_\Delta((\imath \cdot \ell) \cdot (\jmath \cdot \ell)), \varpi_\Delta(\jmath \cdot \ell)\} \leq \varpi_\Delta(\imath \cdot \ell)$

and $\max\{\xi_\Delta(\imath \cdot (\jmath \cdot \ell)), \xi_\Delta(\jmath)\} \geq \max\{\xi_\Delta((\imath \cdot \ell) \cdot (\jmath \cdot \ell)), \xi_\Delta(\jmath \cdot \ell)\} \geq \xi_\Delta(\imath \cdot \ell)$.

Hence IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ is an IF_hI of Ω. □

Theorem 2.6.4. *Let IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ be an IF_hI of Ω.*
Then $\varpi_\Delta(0 \cdot \imath) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0 \cdot \imath) \leq \xi_\Delta(\imath)$ for all $\imath \in \Omega$.

Proof. Let IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ be an IF_hI of Ω. Then

$\varpi_\Delta(\imath \cdot \ell) \geq \min\{\varpi_\Delta(\imath \cdot (\jmath \cdot \ell)), \varpi_\Delta(\jmath)\}$

and $\xi_\Delta(\imath \cdot \ell) \leq \max\{\xi_\Delta(\imath \cdot (\jmath \cdot \ell)), \xi_\Delta(\jmath)\}$.

Substituting $\imath = 0$, $\ell = \imath$ and $\jmath = \imath$,

$\varpi_\Delta(0 \cdot \imath) \geq \min\{\varpi_\Delta(0 \cdot (\imath \cdot \imath)), \varpi_\Delta(\imath)\}$

and $\xi_\Delta(0 \cdot \imath) \leq \max\{\xi_\Delta(0 \cdot (\imath \cdot \imath)), \xi_\Delta(\imath)\}$

$\Rightarrow \varpi_\Delta(0 \cdot \imath) \geq \min\{\varpi_\Delta(0), \varpi_\Delta(\imath)\} = \varpi_\Delta(\imath)$

and $\xi_\Delta(0 \cdot \imath) \leq \max\{\xi_\Delta(0), \xi_\Delta(\imath)\} = \xi_\Delta(\imath)$. □

2.7 Intuitionistic fuzzy α-ideal

An IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω is called an "intuitionistic fuzzy α-ideal" ($IF_\alpha I$) of Ω if it satisfies:

$(IF - \alpha I - 1)$ $\varpi_\Delta(0) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0) \leq \xi_\Delta(\imath)$, for all $\imath \in \Omega$.

$(IF - \alpha I - 2)$ $\varpi_\Delta(\jmath \cdot \imath) \geq \min\{\varpi_\Delta((\imath \cdot \ell) \cdot (0 \cdot \jmath)), \varpi_\Delta(\ell)\}$

$(IF - \alpha I - 3)$ $\xi_\Delta(\jmath \cdot \imath) \leq \max\{\xi_\Delta((\imath \cdot \ell) \cdot (0 \cdot \jmath)), \xi_\Delta(\ell)\}$,

for all $\imath, \jmath, \ell \in \Omega$.

Example 2.7.1. Let $\Omega = \{0, \imath, \jmath, \ell\}$ be the BCI-algebra defined in Example 2.5.1. Define an IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω as:

$\varpi_\Delta(0) = \varpi_\Delta(\imath) = 1, \varpi_\Delta(\jmath) = \varpi_\Delta(\ell) = t$

$\xi_\Delta(0) = \xi_\Delta(\imath) = 0, \xi_\Delta(\jmath) = \xi_\Delta(\ell) = s$

where $s, t \in (0, 1)$ and $s + t \leq 1$

By routine calculations it is easy to verify that IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_\alpha I$ of Ω.

An IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω is an "intuitionistic fuzzy closed α-ideal" of Ω if it satisfies $(IF - \alpha I - 2)$, $(IF - \alpha I - 3)$ and

$(IF - \alpha I - 4)$ $\varpi_\Delta(0 \cdot \imath) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0 \cdot \imath) \leq \xi_\Delta(\imath)$, for all $\imath \in \Omega$.

Theorem 2.7.2. *Let IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ be an IFI of Ω. Then the following conditions are equivalent:*

1. *IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_\alpha I$ of Ω.*
2. *$\varpi_\Delta(\jmath \cdot (\imath \cdot \ell)) \geq \varpi_\Delta((\imath \cdot \ell) \cdot (0 \cdot \jmath))$ and $\xi_\Delta(\jmath \cdot (\imath \cdot \ell)) \leq \xi_\Delta((\imath \cdot \ell) \cdot (0 \cdot \jmath))$*
3. *$\varpi_\Delta(\jmath \cdot \imath) \geq \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$ and $\xi_\Delta(\jmath \cdot \imath) \leq \xi_\Delta(\imath \cdot (0 \cdot \jmath))$*

for all $\imath, \jmath, \ell \in \Omega$.

Proof. $(1 \Rightarrow 2)$ Let IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ be an $IF_\alpha I$ of Ω.

Then for any $\imath, \jmath, \ell \in \Omega$,

$\varpi_\Delta(\jmath \cdot (\imath \cdot \ell)) \geq \min\{\varpi_\Delta(((\imath \cdot \ell) \cdot 0) \cdot (0 \cdot \jmath)), \varpi_\Delta(0)\} = \varpi_\Delta((\imath \cdot \ell) \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\jmath \cdot (\imath \cdot \ell)) \leq \max\{\xi_\Delta(((\imath \cdot \ell) \cdot 0) \cdot (0 \cdot \jmath)), \xi_\Delta(0)\} = \xi_\Delta((\imath \cdot \ell) \cdot (0 \cdot \jmath))$.

$\Rightarrow \varpi_\Delta(\jmath \cdot (\imath \cdot \ell)) \geq \varpi_\Delta((\imath \cdot \ell) \cdot (0 \cdot \jmath))$ and $\xi_\Delta(\jmath \cdot (\imath \cdot \ell)) \leq \xi_\Delta((\imath \cdot \ell) \cdot (0 \cdot \jmath))$.

Which is the required condition.

$(2 \Rightarrow 3)$ Assume that $\varpi_\Delta(\jmath \cdot (\imath \cdot \ell)) \geq \varpi_\Delta((\imath \cdot \ell) \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\jmath \cdot (\imath \cdot \ell)) \leq \xi_\Delta((\imath \cdot \ell) \cdot (0 \cdot \jmath))$.

Substituting $\ell = 0$,

$\varpi_\Delta(\jmath \cdot (\imath \cdot 0)) \geq \varpi_\Delta((\imath \cdot 0) \cdot (0 \cdot \jmath))$ and $\xi_\Delta(\jmath \cdot (\imath \cdot 0)) \leq \xi_\Delta((\imath \cdot 0) \cdot (0 \cdot \jmath))$

$\Rightarrow \varpi_\Delta(\jmath \cdot \imath) \geq \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$ and $\xi_\Delta(\jmath \cdot \imath) \leq \xi_\Delta(\imath \cdot (0 \cdot \jmath))$.

which is the required condition.

$(3 \Rightarrow 1)$ Assume that $\varpi_\Delta(\jmath \cdot \imath) \geq \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\jmath \cdot \imath) \leq \xi_\Delta(\imath \cdot (0 \cdot \jmath))$.

Since $(\imath \cdot (0 \cdot \jmath))((\imath \cdot \ell) \cdot (0 \cdot \jmath)) \leq \imath \cdot (\imath \cdot \ell) \leq \ell$ (By 1.2.11 and 1.2.2)

Therefore by Lemma 2.4.2,

$\varpi_\Delta(\imath \cdot (0 \cdot \jmath)) \geq \min\{\varpi_\Delta((\imath \cdot \ell) \cdot (0 \cdot \jmath)), \varpi_\Delta(\ell)\}$

and $\xi_\Delta(\imath \cdot (0 \cdot \jmath)) \leq \max\{\xi_\Delta((\imath \cdot \ell) \cdot (0 \cdot \jmath)), \xi_\Delta(\ell)\}$

i.e., $\varpi_\Delta(\jmath \cdot \imath) \geq \min\{\varpi_\Delta((\imath \cdot \ell) \cdot (0 \cdot \jmath)), \varpi_\Delta(\ell)\}$

and $\xi_\Delta(\jmath \cdot \imath) \leq \max\{\xi_\Delta((\imath \cdot \ell) \cdot (0 \cdot \jmath)), \xi_\Delta(\ell)\}$.

Hence $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_\alpha I$ of Ω. □

2.8 Interrelationship between intuitionistic fuzzy (p, h, α)-ideals in BCI-algebras

Theorem 2.8.1. *Any $IF_\alpha I$ of Ω is an $IF_p I$ of Ω.*

Proof. Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an $IF_\alpha I$ of Ω. Then by Theorem 2.7.2,

$\varpi_\Delta(\jmath \cdot \imath) \geq \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$ and $\xi_\Delta(\jmath \cdot \imath) \leq \xi_\Delta(\imath \cdot (0 \cdot \jmath))$

for all $\imath, \jmath, \ell \in \Omega$.

Putting $\imath = 0$ we get

$\varpi_\Delta(\jmath) \geq \varpi_\Delta(0 \cdot (0 \cdot \jmath))$ and $\xi_\Delta(\jmath) \leq \xi_\Delta(0 \cdot (0 \cdot \jmath))$.

Therefore by Theorem 2.5.2, $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_p I$ of Ω. □

Whereas the converse of this theorem is not true in general and can be observed by the succeeding example.

Example 2.8.2. Consider the BCI-algebra $\Omega = \{0, \imath, \jmath\}$ defined in Example 1.4.1. Define an $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω as follows:

$\varpi_\Delta(0) = 1, \varpi_\Delta(\imath) = \varpi_\Delta(\jmath) = t,\ \xi_\Delta(0) = 0, \xi_\Delta(\imath) = \xi_\Delta(\jmath) = s$

where $s, t \in (0,1)$ and $s + t \leq 1$

By routine calculations it is easy to verify that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an IF_pI of X but it is not an $IF_\alpha I$ of Ω because

$\varpi_\Delta(\imath \cdot \jmath) = \varpi_\Delta(\jmath) = t < 1 = \varpi_\Delta(0) = min\{\varpi_\Delta((\jmath \cdot 0) \cdot (0 \cdot \imath)), \varpi_\Delta(0)\}$.

Theorem 2.8.3. *Any $IF_\alpha I$ of Ω is an IF_hI of Ω.*

Proof. Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an $IF_\alpha I$ of Ω.

Since by 1.2.9, 1.2.10 and 1.2.11,

$(0 \cdot (0 \cdot (\jmath \cdot (0 \cdot \imath)))) \cdot (\imath \cdot (0 \cdot \jmath)) \leq 0$

Therefore by using Lemma 2.4.2 we get

$\varpi_\Delta(0 \cdot (0 \cdot (\jmath \cdot (0 \cdot \imath)))) \geq \min\{\varpi_\Delta(\imath \cdot (0 \cdot \jmath)), \varpi_\Delta(0)\} = \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$

and $\xi_\Delta(0 \cdot (0 \cdot (\jmath \cdot (0 \cdot \imath)))) \leq \max\{\xi_\Delta(\imath \cdot (0 \cdot \jmath)), \xi_\Delta(0)\} = \xi_\Delta(\imath \cdot (0 \cdot \jmath))$

Now since by Theorem 2.8.1, $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is also an IF_pI of Ω. Therefore by Theorem 2.5.2,

$\varpi_\Delta(\jmath \cdot (0 \cdot \imath)) \geq \varpi_\Delta(0 \cdot (0 \cdot (\jmath \cdot (0 \cdot \imath)))) \geq \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$

$\Rightarrow \varpi_\Delta(\jmath \cdot (0 \cdot \imath)) \geq \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\jmath \cdot (0 \cdot \imath)) \leq \xi_\Delta(0 \cdot (0 \cdot (\jmath \cdot (0 \cdot \imath)))) \leq \xi_\Delta(\imath \cdot (0 \cdot \jmath))$

$\Rightarrow \xi_\Delta(\jmath \cdot (0 \cdot \imath)) \leq \xi_\Delta(\imath \cdot (0 \cdot \jmath))$

Now since $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_\alpha I$ of Ω, so by Theorem 2.7.2,

$\varpi_\Delta(\imath \cdot \jmath) \geq \varpi_\Delta(\jmath \cdot (0 \cdot \imath)) \geq \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\imath \cdot \jmath) \leq \xi_\Delta(\jmath \cdot (0 \cdot \imath)) \leq \xi_\Delta(\imath \cdot (0 \cdot \jmath))$

Therefore by Theorem 2.6.2, $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an IF_hI of Ω. □

The converse of above theorem is not generally true. To observe this, we consider Example 1.4.2. It is clear that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an IF_hI of Ω but it is not an $IF_\alpha I$ of Ω because:

$\varpi_\Delta(\imath \cdot 0) = \varpi_\Delta(\imath) = t < 1 = \min\{\varpi_\Delta((0 \cdot 0) \cdot (0 \cdot \imath)), \varpi_\Delta(0)\}$.

Theorem 2.8.4. *An $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω is an $IF_\alpha I$ of Ω if and only if $\Delta = (\varpi_\Delta, \xi_\Delta)$ is both an IF_pI and an IF_hI of Ω.*

Proof. Assume that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_\alpha I$ of Ω. Then by Theorem 2.8.1 and Theorem 2.8.3, $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an IF_pI and also an IF_hI of Ω.

Conversely suppose that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is both an IF_pI and an IF_hI of Ω. Then by Theorem 2.6.2,

$\varpi_\Delta(\imath \cdot \jmath) \geq \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$ and $\xi_\Delta(\imath \cdot \jmath) \leq \xi_\Delta(\imath \cdot (0 \cdot \jmath))$.

Since $0 \cdot (\jmath \cdot \imath) = (\imath \cdot \imath) \cdot (\jmath \cdot \imath) \leq \imath \cdot \jmath$ (by 1.2.11)

Therefore by Lemma 2.4.3,

$\varpi_\Delta(0 \cdot (\jmath \cdot \imath)) \geq \varpi_\Delta(\imath \cdot \jmath) \geq \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$

$\Rightarrow \varpi_\Delta(0 \cdot (\jmath \cdot \imath)) \geq \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$

and $\xi_\Delta(0 \cdot (\jmath \cdot \imath)) \leq \xi_\Delta(\imath \cdot \jmath) \leq \xi_\Delta(\imath \cdot (0 \cdot \jmath))$

$\Rightarrow \xi_\Delta(0 \cdot (\jmath \cdot \imath)) \leq \xi_\Delta(\imath \cdot (0 \cdot \jmath))$

Thus by Theorem 2.6.4,

$\varpi_\Delta(0 \cdot (0 \cdot (\jmath \cdot \imath))) \geq \varpi_\Delta(0 \cdot (\jmath \cdot \imath)) \geq \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$

and $\xi_\Delta(0 \cdot (0 \cdot (\jmath \cdot \imath))) \leq \xi_\Delta(0 \cdot (\jmath \cdot \imath)) \leq \xi_\Delta(\imath \cdot (0 \cdot \jmath))$.

Since $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is also an IF_pI of Ω, therefore by using Theorem 2.5.2, we get,

$\varpi_\Delta(\jmath \cdot \imath) \geq \varpi_\Delta(0 \cdot (0 \cdot (\jmath \cdot \imath))) \geq \varpi_\Delta(\imath \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\jmath \cdot \imath) \leq \xi_\Delta(0 \cdot (0 \cdot (\jmath \cdot \imath))) \leq \xi_\Delta(\imath \cdot (0 \cdot \jmath))$.

Therefore by Theorem 2.7.2, $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_\alpha I$ of Ω. □

2.9 Intuitionistic fuzzy BCI-implicative ideal

An IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω is called an "Intuitionistic fuzzy BCI-implicative ideal" ($IF_{BCI}II$) of Ω if it satisfies:

$(IFBCI - I - 1)$ $\varpi_\Delta(0) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0) \leq \xi_\Delta(\imath)$, for all $\imath \in \Omega$.

$(IFBCI - I - 2)$ $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq$
$\min\{\varpi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \varpi_\Delta(\ell)\}$

$(IFBCI - I - 3)$ $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq$
$\max\{\xi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \xi_\Delta(\ell)\}$,
for all $\imath, \jmath, \ell \in \Omega$.

Example 2.9.1. Consider the BCI-algebra $\Omega = \{0, \imath, \jmath, \ell\}$ with the following cayley table:

$\cdot$	0	$\imath$	$\jmath$	ℓ
0	0	0	ℓ	$\jmath$
$\imath$	$\imath$	0	ℓ	$\jmath$
$\jmath$	$\jmath$	$\jmath$	0	ℓ
ℓ	ℓ	ℓ	$\jmath$	0

Define an IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω as:
$\varpi_\Delta(0) = \varpi_\Delta(\ell) = 1, \varpi_\Delta(\imath) = \varpi_\Delta(\jmath) = \delta$
and $\xi_\Delta(0) = \xi_\Delta(\ell) = 0, \xi_\Delta(\imath) = \xi_\Delta(\jmath) = \eta$
where $\delta, \eta \in (0, 1)$ and $\delta + \eta \leq 1$
By routine calculations it is easy to verify that IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω.

An IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω is an "intuitionistic fuzzy closed BCI-implicative ideal" of Ω if it satisfies $(IFBCI - I - 2)$, $(IFBCI - I - 3)$

and

$(IFBCI-I-4)$ $\varpi_\Delta(0\cdot\imath)\geq\varpi_\Delta(\imath)$ and $\xi_\Delta(0\cdot\imath)\leq\xi_\Delta(\imath)$, for all $\imath\in\Omega$.

Theorem 2.9.2. *Any $IF_{BCI}II$ of Ω is an IFI of Ω.*

Proof. Assume that IFS $\Delta=(\varpi_\Delta,\xi_\Delta)$ is an $IF_{BCI}II$ of Ω. Then,
$\varpi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\geq\min\{\varpi_\Delta((((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))\cdot\ell),\varpi_\Delta(\ell)\}$
and $\xi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\leq\max\{\xi_\Delta((((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))\cdot\ell),\xi_\Delta(\ell)\}$
for all $\imath,\jmath,\ell\in\Omega$.
Substituting $\ell=\jmath$ and $\jmath=0$,
$\varpi_\Delta(\imath\cdot((0\cdot(0\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot 0)))))\geq\min\{\varpi_\Delta((((\imath\cdot 0)\cdot 0)\cdot(0\cdot 0))\cdot\jmath),\varpi_\Delta(\jmath)\}$
$\xi_\Delta(\imath\cdot((0\cdot(0\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot 0)))))\leq\max\{\xi_\Delta((((\imath\cdot 0)\cdot 0)\cdot(0\cdot 0))\cdot\jmath),\xi_\Delta(\jmath)\}$
i.e., $\varpi_\Delta(\imath)\geq\min\{\varpi_\Delta(\imath\cdot\jmath),\varpi_\Delta(\jmath)\}$ and $\xi_\Delta(\imath)\leq\max\{\xi_\Delta(\imath\cdot\jmath),\xi_\Delta(\jmath)\}$.
Hence IFS $\Delta=(\varpi_\Delta,\xi_\Delta)$ is an IFI of Ω. □

Whereas the converse of this Theorem isn't true. For this cogitate the succeeding example.

Example 2.9.3. Consider the BCI-algebra $\Omega=\{0,\imath,\jmath,\ell,\wp\}$ with the following caley table:

$\cdot$	0	$\imath$	$\jmath$	ℓ	$\wp$
0	0	0	0	0	0
$\imath$	$\imath$	0	$\imath$	0	0
$\jmath$	$\jmath$	$\jmath$	0	0	0
ℓ	ℓ	ℓ	ℓ	0	0
$\wp$	$\wp$	ℓ	$\wp$	$\imath$	0

Define an IFS $\Delta=(\varpi_\Delta,\xi_\Delta)$ in Ω by:

$\varpi_\Delta(0) = \varpi_\Delta(\jmath) = 1, \varpi_\Delta(\imath) = \varpi_\Delta(\ell) = \varpi_\Delta(\wp) = \delta$

$\xi_\Delta(0) = \xi_\Delta(\jmath) = 0, \xi_\Delta(\imath) = \xi_\Delta(\ell) = \xi_\Delta(\wp) = \eta$

where $\delta, \eta \in (0, 1)$ and $\delta + \eta \leq 1$.

By routine calculations it is easy to verify that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an IFI of Ω but it is not an $IF_{BCI}II$ of Ω because:

$\varpi_\Delta(\wp \cdot ((\ell \cdot (\ell \cdot \wp)) \cdot (0 \cdot (0 \cdot (\wp \cdot \ell))))) = \varpi_\Delta(\imath) = \delta < 1$

$= \varpi_\Delta(0) \min\{\varpi((((\wp \cdot \ell) \cdot \ell) \cdot (0 \cdot \ell)) \cdot 0), \varpi_\Delta(0)\}$.

Theorem 2.9.4. *Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an IFI of Ω. Then the following conditions are equivalent:*

1. *$IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω.*

2. $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$
and $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

3. $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) = \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$
and $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) = \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$,

for all $\imath, \jmath \in \Omega$.

Proof. $(1 \Rightarrow 2)$ Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an $IF_{BCI}II$ of Ω. Then for any $\imath, \jmath, \ell \in \Omega$,

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq$

$\min\{\varpi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \varpi_\Delta(\ell)\}$ and

$\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \max\{\xi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \xi_\Delta(\ell)\}$.

By putting $\ell = 0$ we get

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \min\{\varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)), \varpi_\Delta(0)\}$

$\xi_(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \max\{\xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)), \xi_(0)\}$,

i.e., $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$.

which are the required conditions.

$(2 \Rightarrow 3)$ Assume that

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$ (a)

and $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$ (b)

Since by 1.2.6, 1.2.8, 1.2.10 and 1.2.11,

$(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot (\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq 0$

Therefore by using Lemma 2.4.2 we have

$\varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \geq \min\{\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))), \varpi_\Delta(0)\}$

$= \varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$

and $\xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \leq \max\{\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))), \xi_\Delta(0)\}$

$= \xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))),$

i.e., $\varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \geq \varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$ (c)

and

$\xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \leq \xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$ (d)

From (a) and (c) and (b) and (d) we have

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) = \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) = \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$.

$(3 \Rightarrow 1)$ Assume that:

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) = \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$ (a1)

and

$\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) = \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$ (b1)

for all $\imath, \jmath \in \Omega$.

Since $(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell) \leq \ell$ (by 1.2)

Therefore by using Lemma 2.4.2 we get

$\varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \geq \min\{\varpi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \varpi_\Delta(\ell)\}$ $(c1)$

$\xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \leq \max\{\xi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \xi_\Delta(\ell)\}$ (d1)

combining (a1) and (c1) and (b1) and (d1) we get

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq$

$\min\{\varpi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \varpi_\Delta(\ell)\}$ and

$\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \max\{\xi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \xi_\Delta(\ell)\}$.

Hence $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω. □

Lemma 2.9.5. *An $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω if and only if ϖ_Δ and $\bar{\xi}_\Delta$ are $F_{BCI}IIs$ of Ω.*

Proof. Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an $IF_{BCI}II$ of Ω.Then for any $\imath \in \Omega$,

$\varpi_\Delta(0) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0) \leq \xi_\Delta(\imath)$.

Also $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq$

$\min\{\varpi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \varpi_\Delta(\ell)\}$ and

$\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \max\{\xi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \xi_\Delta(\ell)\}$

for any $\imath, \jmath, \ell \in \Omega$.

Then clearly ϖ_Δ is a $F_{BCI}II$ of Ω.

Now $\xi_\Delta(0) \leq \xi_\Delta(\imath) \Rightarrow 1 - \bar{\xi}_\Delta(0) \leq 1 - \bar{\xi}_\Delta(\imath) \Rightarrow \bar{\xi}_\Delta(0) \geq \bar{\xi}_\Delta(\imath)$.

Moreover, $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq$

$\max\{\xi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \xi_\Delta(\ell)\}$

$\Rightarrow 1 - \bar{\xi}_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq$

$\max\{1 - \bar{\xi}_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), 1 - \bar{\xi}_\Delta(\ell)\}$

$\Rightarrow \bar{\xi}_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq$

$1 - \max\{1 - \bar{\xi}_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), 1 - \bar{\xi}_\Delta(\ell)\}$

$\Rightarrow \bar{\xi}_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq$

$\min\{\bar{\xi}_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \bar{\xi}_\Delta(\ell)\}$

Hence $\bar{\xi}_\Delta$ is also a $F_{BCI}II$ of Ω.

Conversely suppose that ϖ_Δ and $\bar{\xi}_\Delta$ are $F_{BCI}IIs$ of Ω.

Then $\varpi_\Delta(0) \geq \varpi_\Delta(\imath)$ and $\bar{\xi}_\Delta(0) \geq \bar{\xi}_\Delta(\imath)$.

Also $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq$

$\min\{\varpi_\Delta((((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))\cdot\ell),\varpi_\Delta(\ell)\}$ and

$\bar{\xi_\Delta}(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\geq\min\{\bar{\xi_\Delta}((((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))\cdot\ell),\bar{\xi_\Delta}(\ell)\}$

Then $\bar{\xi_\Delta}(0)\geq\bar{\xi_\Delta}(\imath)\Rightarrow 1-\xi_\Delta(0)\geq 1-\xi_\Delta(\imath)\Rightarrow\xi_\Delta(0)\leq\xi_\Delta(\imath)$

Also $\bar{\xi_\Delta}(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\geq$

$\min\{\bar{\xi_\Delta}((((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))\cdot\ell),\bar{\xi_\Delta}(\ell)\}$

$\Rightarrow 1-\xi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\geq$

$\min\{1-\xi_\Delta((((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))\cdot\ell),1-\xi_\Delta(\ell)\}$

$\Rightarrow\xi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\leq$

$1-\min\{1-\xi_\Delta((((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))\cdot\ell),1-\xi_\Delta(\ell)\}$

$\Rightarrow\xi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\leq$

$\max\{\xi_\Delta((((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))\cdot\ell),\xi_\Delta(\ell)\}$.

Hence $IFS\ \Delta=(\varpi_\Delta,\xi_\Delta)$ an $IF_{BCI}II$ of Ω. □

Lemma 2.9.6. *An $IFS\ \Delta=(\varpi_\Delta,\xi_\Delta)$ is an "intuitionistic fuzzy closed BCI-implicative ideal" of Ω if and only if ϖ_Δ and $\bar{\xi_\Delta}$ are "fuzzy closed BCI-implicative ideals" of Ω.*

Proof. Suppose that $IFS\ \Delta=(\varpi_\Delta,\xi_\Delta)$ is an "intuitionistic fuzzy closed BCI-implicative ideal" of Ω. Then it satisfies $(IFBCI-I-2)$, $(IFBCI-I-3)$ and

$(IFBCI-I-4)$ $\varpi_\Delta(0\cdot\imath)\geq\varpi_\Delta(\imath)$ and $\xi_\Delta(0\cdot\imath)\leq\xi_\Delta(\imath)$, for all $\imath\in\Omega$.

Then it is clear that ϖ_Δ is fuzzy closed BCI-implicative ideal of Ω. For $\bar{\xi_\Delta}$ it can be easily verified as done earlier in Lemma 2.9.5 that,

$\bar{\xi_\Delta}(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\geq\min\{\bar{\xi_\Delta}((((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))\cdot\ell),\bar{\xi_\Delta}(\ell)\}$

for any $\imath,\jmath,\ell\in\Omega$. It is therefore required to elaborate that $\bar{\xi}(0\cdot\imath)\geq\bar{\xi}(\imath)$.

Since $\xi(0\cdot\imath)\leq\xi(\imath)\Rightarrow 1-\bar{\xi_\Delta}(0\cdot\imath)\leq 1-\bar{\xi_\Delta}(\imath)\Rightarrow\bar{\xi_\Delta}(0\cdot\imath)\geq\bar{\xi_\Delta}(\imath)$.

Thus $\bar{\xi_\Delta}$ is also a "fuzzy closed BCI-implicative ideal" of Ω.

Conversely suppose that ϖ_Δ and $\bar{\xi_\Delta}$ are "fuzzy closed BCI-implicative ideals" of Ω.

Then $\varpi_\Delta(0 \cdot \imath) \geq \varpi_\Delta(\imath)$ and $\bar{\xi_\Delta}(0 \cdot \imath) \geq \bar{\xi_\Delta}(\imath)$ for all $\imath \in \Omega$. Now $\bar{\xi_\Delta}(0 \cdot \imath) \geq \bar{\xi_\Delta}(\imath) \Rightarrow 1 - \xi_\Delta(0 \cdot \imath) \geq 1 - \xi_\Delta(\imath) \Rightarrow \xi_\Delta(0 \cdot \imath) \leq \xi_\Delta(\imath)$.

Also $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq$

$\min\{\varpi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \varpi_\Delta(\ell)\}$ and

$\bar{\xi_\Delta}(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \min\{\bar{\xi_\Delta}((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \bar{\xi_\Delta}(\ell)\}$,

for any $\imath, \jmath, \ell \in \Omega$.

$\bar{\xi_\Delta}(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \min\{\bar{\xi_\Delta}((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \bar{\xi_\Delta}(\ell)\}$

$\Rightarrow 1 - \xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq$

$\min\{1 - \xi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), 1 - \xi_\Delta(\ell)\}$

$\Rightarrow \xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq$

$1 - \min\{1 - \xi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), 1 - \xi_\Delta(\ell)\}$

$\Rightarrow \xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq$

$\max\{\xi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \xi_\Delta(\ell)\}$.

Hence $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an "intuitionistic fuzzy closed BCI-implicative ideal" of Ω. □

Theorem 2.9.7. *An $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω if and only if $\Box\Delta = (\varpi_\Delta, \bar{\varpi_\Delta})$ and $\diamond\Delta = (\bar{\xi_\Delta}, \xi_\Delta)$ are $IF_{BCI}IIs$ of Ω.*

Proof. Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an $IF_{BCI}II$ of Ω. Then by Lemma 2.9.5 ϖ_Δ and $\bar{\xi_\Delta}$ are $F_{BCI}IIs$ of Ω i.e $\bar{\bar{\varpi}} = \varpi_\Delta$ and $\bar{\xi_\Delta}$ are $F_{BCI}IIs$ of Ω. Therefore by Lemma 2.9.5, $\Box\Delta = (\varpi_\Delta, \bar{\varpi_\Delta})$ and $\diamond\Delta = (\bar{\xi_\Delta}, \xi_\Delta)$ are $IF_{BCI}IIs$ of Ω.

Conversely suppose that $\Box\Delta = (\varpi_\Delta, \bar{\varpi_\Delta})$ and $\diamond\Delta = (\bar{\xi_\Delta}, \xi_\Delta)$ are $IF_{BCI}IIs$ of Ω. Then by Lemma 2.9.5 ϖ_Δ and $\bar{\xi_\Delta}$ are $F_{BCI}IIs$ of Ω. Therefore by Lemma 2.9.5, $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω. □

Theorem 2.9.8. *An $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an "intuitionistic fuzzy closed*

BCI-implicative ideal" of Ω *if and only if* $\Box\Delta = (\varpi_\Delta, \bar{\varpi_\Delta})$ *and* $\diamond\Delta = (\bar{\xi_\Delta}, \xi_\Delta)$ *are "intuitionistic fuzzy closed BCI-implicative ideals" of* Ω.

Proof. Suppose that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an "intuitionistic fuzzy closed BCI-implicative ideal" of Ω. Then by Lemma 2.9.6, ϖ_Δ and $\bar{\xi_\Delta}$ are "fuzzy closed BCI-implicative ideals" of Ω i.e $\bar{\bar{\varpi}} = \varpi_\Delta$ and $\bar{\xi_\Delta}$ are "fuzzy closed BCI-implicative ideals" of Ω. Therefore by Lemma 2.9.6, $\Box\Delta = (\varpi_\Delta, \bar{\varpi_\Delta})$ and $\diamond\Delta = (\bar{\xi_\Delta}, \xi_\Delta)$ are "intuitionistic fuzzy closed BCI-implicative ideals" of Ω.

Conversely suppose that $\Box\Delta = (\varpi_\Delta, \bar{\varpi_\Delta})$ and $\diamond\Delta = (\bar{\xi_\Delta}, \xi_\Delta)$ are "intuitionistic fuzzy closed BCI-implicative ideals" of Ω. Then by Lemma 2.9.6, ϖ_Δ and $\bar{\xi_\Delta}$ are "fuzzy closed BCI-implicative ideals" of Ω. Therefore by Lemma 2.9.6, $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an "intuitionistic fuzzy closed BCI-implicative ideal" of Ω. $\Box$

Theorem 2.9.9. *An* $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ *is an* $IF_{BCI}II$ *of* Ω *if and only if the non-empty* δ*-level cut* $U(\varpi_\Delta; \delta)$ *and the non-empty lower* η*-level cut* $L(\xi_\Delta; \eta)$ *are BCI-implicative ideals of* Ω *for any* $\eta, \delta \in [0, 1]$.

Proof. Suppose that An $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω. Since $U(\varpi_\Delta; \delta) \neq \emptyset$, $L(\xi_\Delta; \eta) \neq \emptyset$. So for $\imath \in U(\varpi_\Delta; \delta)$,

$\varpi_\Delta(\imath) \geq \delta \Rightarrow \varpi_\Delta(0) \geq \varpi_\Delta(\imath) \geq \delta \Rightarrow 0 \in U(\varpi_\Delta; \delta)$.

Now let $(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell \in U(\varpi_\Delta; \delta)$ and $\ell \in U(\varpi_\Delta; \delta)$.

Then $\varpi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell) \geq \delta$ and $\varpi_\Delta(\ell) \geq \delta$.

Since $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq$

$\min\{\varpi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \varpi_\Delta(\ell)\} \geq \delta$.

Thus $\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))) \in U(\varpi_\Delta; \delta)$. Hence $U(\varpi_\Delta; \delta)$ is BCI-implicative ideal of Ω. Similarly we can prove that $L(\xi_\Delta; \eta)$ is a BCI-implicative ideal of Ω.

Conversely suppose that the non-empty upper δ-level cut $U(\varpi_\Delta;\delta)$ and the non-empty lower η-level cut $L(\xi_\Delta;\eta)$ are BCI-implicative ideals of Ω for any $\eta,\delta \in [0,1]$. If for some $\imath_0 \in \Omega$, $\varpi_\Delta(0) < \varpi_\Delta(\imath_0)$ and $\xi_\Delta(0) > \xi_\Delta(\imath_0)$.

Take $\delta_0 = 1/2\{\varpi_\Delta(0) + \varpi_\Delta(\imath_0)\}$ then $\varpi_\Delta(0) < \delta_0 < \varpi_\Delta(\imath_0)$.

This implies $\imath_0 \in U(\varpi_\Delta;\delta_0)$ and 0 does not belong to $U(\varpi_\Delta;\delta_0)$ which is a contradiction to the fact that $U(\varpi_\Delta;\delta_0)$ is a BCI-implicative ideal of Ω. Therefore we must have $\varpi_\Delta(0) \geq \varpi_\Delta(\imath)$ for all $\imath \in \Omega$. Similarly by taking $\eta_0 = 1/2\{\xi_\Delta(0) + \xi_\Delta(\imath_0)\}$ we can prove that $\xi_\Delta(0) \leq \xi_\Delta(\imath)$ for all $\imath \in \Omega$.

If possible assume that there exists some $\imath_0, \jmath_0, \ell_0 \in \Omega$ such that,

$p = \varpi_\Delta(\imath_0 \cdot ((\jmath_0 \cdot (\jmath_0 \cdot \imath_0)) \cdot (0 \cdot (0 \cdot (\imath_0 \cdot \jmath_0))))) <$

$\min\{\varpi_\Delta((((\imath_0 \cdot \jmath_0) \cdot \jmath_0) \cdot (0 \cdot \jmath_0)) \cdot \ell_0), \varpi_\Delta(\ell_0)\} = q$

Put $\delta_0 = 1/2\{p + q\}$ then $p < \delta_0 < q$

$\Rightarrow$ $(((\imath_0 \cdot \jmath_0) \cdot \jmath_0) \cdot (0 \cdot \jmath_0)) \cdot \ell_0 \in U(\varpi_\Delta;\delta_0)$ and $\ell_0 \in U(\varpi_\Delta;\delta_0)$ whereas $\imath_0 \cdot ((\jmath_0 \cdot (\jmath_0 \cdot \imath_0)) \cdot (0 \cdot (0 \cdot (\imath_0 \cdot \jmath_0))))$ does not belong to $U(\varpi_\Delta;\delta_0)$ which is a contradiction to the fact that $U(\varpi_\Delta;\delta_0)$ is a BCI-implicative ideal of Ω. Therefore for any $\imath, \jmath, \ell \in \Omega$,

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \min\{\varpi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \varpi_\Delta(\ell)\}$.

Similarly we can prove that,

$\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \max\{\xi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \xi_\Delta(\ell)\}$,

for all $\imath, \jmath, \ell \in \Omega$. Hence $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω. □

Theorem 2.9.10. *An $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an "intuitionistic fuzzy closed BCI-implicative ideal" of Ω if and only if the non-empty upper δ-level cut $U(\varpi_\Delta;\delta)$ and the non-empty lower η-level cut $L(\xi_\Delta;\eta)$ are closed BCI-implicative ideals of Ω for any $\eta,\delta \in [0,1]$.*

Proof. Suppose that An $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an "intuitionistic fuzzy closed BCI-implicative ideal" of Ω. Then $\varpi_\Delta(0 \cdot \imath) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0 \cdot \imath) \leq \xi_\Delta(\imath)$

for all $\imath \in \Omega$.

Since $U(\varpi_\Delta;\delta) \neq \emptyset$, $L(\xi_\Delta;\eta) \neq \emptyset$. So for $\imath \in U(\varpi_\Delta;\delta)$, $\varpi_\Delta(\imath) \geq \delta \Rightarrow \varpi_\Delta(0\cdot\imath) \geq \varpi_\Delta(\imath) \geq \delta \Rightarrow 0\cdot\imath \in U(\varpi_\Delta;\delta)$. Similarly, for every $\imath \in L(\xi_\Delta;\eta)$, $\xi_\Delta(\imath) \leq \eta \Rightarrow \xi_\Delta(0\cdot\imath) \leq \xi_\Delta(\imath) \leq \eta \Rightarrow 0\cdot\imath \in L(\xi_\Delta;\eta)$. Hence $U(\varpi_\Delta;\delta)$ and $L(\xi_\Delta;\eta)$ are closed BCI-implicative ideals of Ω.

Conversely suppose that the non-empty upper δ-level cut $U(\varpi_\Delta;\delta)$ and the non-empty lower η-level cut $L(\xi_\Delta;\eta)$ are closed BCI-implicative ideals of Ω for any $\eta,\delta \in [0,1]$. We want to show that $IFS\ \Delta = (\varpi_\Delta,\xi_\Delta)$ is an "intuitionistic fuzzy closed BCI-implicative ideal" of Ω. It is enough to show that $\varpi_\Delta(0\cdot\imath) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0\cdot\imath) \leq \xi_\Delta(\imath)$ for all $\imath \in \Omega$. If for some $\imath_0 \in \Omega$, $\varpi_\Delta(0\cdot\imath_0) < \varpi_\Delta(\imath_0)$. Take $\delta_0 = 1/2\{\varpi_\Delta(0\cdot\imath_0) + \varpi_\Delta(\imath_0)\}$ then $\varpi_\Delta(0\cdot\imath_0) < \delta_0 < \varpi_\Delta(\imath_0) \Rightarrow \imath_0 \in U(\varpi_\Delta;\delta_0)$ whereas $0\cdot\imath_0$ does not belong to $U(\varpi_\Delta;\delta_0)$ which is a contradiction to the fact that $U(\varpi_\Delta;\delta_0)$ is a closed BCI-implicative ideal of Ω. Therefore we must have $\varpi_\Delta(0\cdot\imath) \geq \varpi_\Delta(\imath)$ for all $\imath \in \Omega$. Similarly we can prove that $\xi_\Delta(0\cdot\imath) \leq \xi_\Delta(\imath)$ for all $\imath \in \Omega$. Hence IFSA $= (\varpi_\Delta,\xi_\Delta)$ is an "intuitionistic fuzzy closed BCI-implicative ideal" of Ω. □

Theorem 2.9.11. *Let $\{I_\delta \mid \delta \in \Lambda\}$ be a collection of BCI-implicative ideals of Ω such that*

1. $\Omega = \bigcup_{\delta\in\Lambda} I_\delta$.

2. $\eta > \delta$ if and only if $I_\eta \subset I_\delta$ for all $\eta,\delta \in \Lambda$.

Then an $IFS\ \Delta = (\varpi_\Delta,\xi_\Delta)$ defined by $\varpi_\Delta(\imath) = \sup\{\delta \in \Lambda \mid \imath \in I_\delta\}$ and $\xi_\Delta(\imath) = \inf\{\eta \in \Lambda \mid \imath \in I_\eta\}$ for all $\imath \in \Omega$ is an $IF_{BCI}II$ of Ω.

Proof. By Theorem 2.9.9, it is sufficient to prove that $U(\varpi_\Delta;\delta)$ and $L(\xi_\Delta;\eta)$ are BCI-implicative ideals of Ω. To prove that $U(\varpi_\Delta;\delta)$ is a BCI-implicative ideal of Ω, we divide the proof into the following two cases:

1. $\delta = \sup\{\varrho \in \Lambda \mid \varrho < \delta\}$

2. $\delta \neq \varrho \sup\{\varrho \in \Lambda \mid \varrho < \delta\}$

The case (1) implies that $\imath \in U(\varpi_\Delta;\delta) \Leftrightarrow \imath \in I_\varrho$, for all $\varrho < \delta \Leftrightarrow \imath \in \bigcap_{\varrho<\delta} I_\varrho$ so that $U(\varpi_\Delta;\delta) = \bigcap_{\varrho<\delta} I_\varrho$ which is a BCI-implicative ideal of Ω.

For the case (2) we claim that $U(\varpi_\Delta;\delta) = \bigcup_{\varrho\geq\delta} I_\varrho$. If $\imath \in \bigcup_{\varrho\geq\delta} I_\varrho$ then $\imath \in I_\varrho$ for some $\varrho \geq \delta$. It follows that $\varpi_\Delta(\imath) \geq \varrho \geq \delta$, so that $\imath \in U(\varpi_\Delta;\delta)$. This shows that $\bigcup_{\varrho\geq\delta} I_\varrho \subseteq U(\varpi_\Delta;\delta)$. Now assume that $\imath \notin \bigcup_{\varrho\geq\delta} I_\varrho$ then $\imath \notin I_\varrho$ for all $\varrho \geq \delta$. Since $t \neq \sup\{\varrho \in \Lambda \mid \varrho < \delta\}$, there exists some $\epsilon > 0$ such that $(\delta-\epsilon,\delta)\bigcap\Lambda = \emptyset$. Hence $\imath \notin I_\varrho$ for all $\varrho > \delta - \epsilon$ which means that $\imath \in I_\varrho$ if $\varrho \leq \delta-\epsilon < \delta$. Thus $\varpi_\Delta(\imath) \leq \delta - \epsilon < \delta$ and so $\imath \notin U(\varpi_\Delta;\delta)$. Therefore $U(\varpi_\Delta;\delta) \subseteq \bigcup_{\varrho\geq\delta} I_\varrho$ and that $U(\varpi_\Delta;\delta) = \bigcup_{\varrho\geq\delta} I_\varrho$ which is a BCI-implicative ideal of Ω.

Next we prove that $L(\xi_\Delta;\eta)$ is a BCI-implicative ideal of Ω. For this we divide the proof into the following two cases:

1. $\eta = \inf\{\varsigma \in \Lambda \mid \eta < \varsigma\}$

2. $\eta \neq \inf\{\varsigma \in \Lambda \mid \eta < \varsigma\}$

The case (1) implies that The case (1) implies that $\imath \in L(\xi_\Delta;\eta) \Leftrightarrow \imath \in I_\varsigma$, for all $\eta < \varsigma \Leftrightarrow \imath \in \bigcap_{\eta<\varsigma} I_\varsigma$ so that $L(\xi_\Delta;\eta) = \bigcap_{\eta<\varsigma} I_\varsigma$ which is a BCI-implicative ideal of Ω.

For the case (2), we state that $L(\xi_\Delta;\eta) = \bigcup_{\varsigma\leq\eta} I_\varsigma$. If $\imath \in \bigcup_{\varsigma\leq\eta} I_\varsigma$ then $\imath \in I_\varsigma$ for some $\varsigma \leq \eta$. Thus $\xi_\Delta(\imath) \leq \varsigma \leq \eta$, so that $\imath \in L(\xi_\Delta;\eta)$. This shows that $\bigcup_{\varsigma\leq\eta} I_\varsigma \subseteq L(\xi_\Delta;\eta)$. Now assume that $\imath \notin \bigcup_{\varsigma\leq\eta} I_\varsigma$ then $\imath \notin I_\varsigma$ for all $\varsigma \leq \eta$. Since $\eta \neq \inf\{\varsigma \in \Lambda \mid \eta < \varsigma\}$, $\exists\ \epsilon > 0$ such that $(\eta, \eta+\epsilon)\bigcap\Lambda = \emptyset$. Hence $\imath \notin I_\varsigma$ for all $\varsigma < \eta + \epsilon$ which means that $\imath \in I_\varsigma$ if $\varsigma \geq \eta + \epsilon > \eta$. Thus $\xi_\Delta(\imath) \geq \eta + \epsilon > \eta$ and so $\imath \notin L(\xi_\Delta;\eta)$. Therefore $L(\xi_\Delta;\eta) \subseteq \bigcup_{\varsigma\leq\eta} I_\varsigma$ and that $L(\xi_\Delta;\eta) = \bigcup_{\varsigma\leq\eta} I_\varsigma$ which is a BCI-implicative ideal of Ω. This completes the proof. $\square$

Theorem 2.9.12. *If IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ is an "intuitionistic fuzzy closed BCI-implicative ideal" of Ω, then the sets $J = \{\imath \in \Omega \mid \varpi_\Delta(\imath) = \varpi_\Delta(0)\}$ and $K = \{\imath \in \Omega \mid \xi_\Delta(\imath) = \xi_\Delta(0)\}$ are BCI-implicative ideals of Ω.*

Proof. Since $0 \in \Omega$, $\varpi_\Delta(0) = \varpi_\Delta(0)$ and $\xi_\Delta(0) = \xi_\Delta(0)$ implies $0 \in J$ and $0 \in K$, so $J \neq \Phi$ and $K \neq \Phi$. Now let $(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell \in J$ and $\ell \in J$. Then $\varpi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell) = \varpi_\Delta(0)$ and $\varpi_\Delta(\ell) = \varpi_\Delta(0)$.Since $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \min\{\varpi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \varpi_\Delta(\ell)\} = \varpi_\Delta(0)$. But $\varpi_\Delta(0) \geq \varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$. Therefore $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) = \varpi_\Delta(0)$. It follows that that $\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))) \in J$ for all $\imath, \jmath, \ell \in \Omega$. Hence J is a BCI-implicative ideal of Ω. Similarly we can prove that K is a BCI-implicative ideal of Ω. $\square$

Definition 2.9.13. Let θ be a mapping on a set Ω and $\Delta = (\varpi_\Delta, \xi_\Delta)$ be an IFS in Ω. Then the fuzzy sets ς and υ on $\theta(\Omega)$ defined by $\varsigma(\jmath) = \sup_{\imath \in \theta^{-1}(\jmath)} \varpi_\Delta(\imath)$ and $\upsilon(\jmath) = \inf_{\imath \in \theta^{-1}(\jmath)} \xi_\Delta(\imath)$, for all $\jmath \in \theta(\Omega)$, are called the images of Δ under θ. If ς, υ are fuzzy sets in $\theta(\Omega)$ then the fuzzy sets $\varpi_\Delta = \varsigma \circ \theta$ and $\xi_\Delta = \upsilon \circ \theta$ are called the pre-images of ς and υ under θ.

Theorem 2.9.14. *Let $\theta : \Omega \mapsto \Omega'$ be an onto homomorphism of BCI-algebras. If $A' = (\varsigma, \upsilon)$ is an $IF_{BCI}II$ of BCI-algebra Ω' then the pre-image of $A' = (\varsigma, \upsilon)$ under θ is an $IF_{BCI}II$ of Ω.*

Proof. Let an $\Delta = (\varpi_\Delta, \xi_\Delta)$ where $\varpi_\Delta = \varsigma \circ \theta$ and $\xi_\Delta = \upsilon \circ \theta$ be the pre-image of $A' = (\varsigma, \upsilon)$ under θ. Since $A' = (\varsigma, \upsilon)$ is an $IF_{BCI}II$ of Ω', we have $\varsigma(0') \geq \varsigma(\theta(\imath)) = \varpi_\Delta(\imath)$ and $\upsilon(0') \leq \upsilon(\theta(\imath)) = \xi_\Delta(\imath)$. On the other hand $\varsigma(0') = \varsigma(\theta(0)) = \varpi_\Delta(0)$ and $\upsilon(0') = \upsilon(\theta(0)) = \xi_\Delta(0)$. Therefore $\varpi_\Delta(0) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0) \leq \xi_\Delta(\imath)$, for all $\imath \in \Omega$. Now we show that

$\varpi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\geq\min\{\varpi_\Delta((((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))\cdot\ell),\varpi_\Delta(\ell)\}$
and $\xi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\leq\max\{\xi_\Delta((((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))\cdot\ell),\xi_\Delta(\ell)\}$
for all $\imath,\jmath,\ell\in\Omega$.

Now $\varpi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))$
$=\varsigma(\theta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath))))))$
$=\varsigma(\theta(\imath)\cdot((\theta(\jmath)\cdot(\theta(\jmath)\cdot\theta(\imath)))\cdot(\theta(0)\cdot(\theta(0)\cdot(\theta(\imath)\cdot\theta(\jmath))))))\geq$
$\min\{\varsigma((((\theta(\imath)\cdot\theta(\jmath))\cdot\theta(\jmath))\cdot(\theta(0)\cdot\theta(\jmath)))\cdot\ell'),\varsigma(\ell')\}$ for $\ell'\in\Omega'$.

Since θ is an onto homomorphism, so $\exists\ \ell\in\Omega$, s.t, $\theta(\ell)=\ell'$. Thus
$\varpi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\geq$
$\min\{\varsigma((((\theta(\imath)\cdot\theta(\jmath))\cdot\theta(\jmath))\cdot(\theta(0)\cdot\theta(\jmath)))\cdot\theta(\ell)),\varsigma(\theta(\ell))\}$
$=\min\{\varsigma(\theta((((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))\cdot\ell)),\varsigma(\theta(\ell))\}$
$=\min\{\varpi_\Delta((((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))\cdot\ell)),\varpi_\Delta(\ell)\}$.

Therefore the result,
$\varpi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\geq$
$\min\{\varpi_\Delta((((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))\cdot\ell),\varpi_\Delta(\ell)\}$,
is true for all $\imath,\jmath,\ell\in\Omega$ because ℓ' is an arbitrary element of Ω' and θ is an onto mapping. Similarly we can prove that,
$\xi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\leq\max\{\xi_\Delta((((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))\cdot\ell),\xi_\Delta(\ell)\}$
for all $\imath,\jmath,\ell\in\Omega$.

Hence the pre-image $\Delta=(\varpi_\Delta,\xi_\Delta)$ of $A'=(\varsigma,\upsilon)$ under θ is an $IF_{BCI}II$ of Ω. □

Definition 2.9.15. Let $\theta:\Omega\mapsto\Upsilon$ be a homomorphism of BCI-algebras. For any IFS $\Delta=(\varpi_\Delta,\xi_\Delta)$ in Υ we define a new IFS $\Delta^\theta=(\varpi_\Delta^\theta,\xi_\Delta^\theta)$ in Ω by $\varpi_\Delta^\theta(\imath)=\varpi_\Delta(\theta(\imath))$, $\xi_\Delta^\theta(\imath)=\xi_\Delta(\theta(\imath))$ for all $\imath\in\Omega$. If $\theta:\Omega\mapsto\Upsilon$ is a homomorphism of BCI-algebras then $\theta(0)=0$.

Theorem 2.9.16. *Let $\theta:\Omega\mapsto\Upsilon$ be a homomorphism of BCI-algebras.*

If an IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ *in* Υ *is an* $IF_{BCI}II$ *of* Υ*, then the IFS* $\Delta^\theta = (\varpi^\theta_\Delta, \xi^\theta_\Delta)$ *in* Ω *is an* $IF_{BCI}II$ *of* Ω*.*

Proof. We first have that

$\varpi^\theta_\Delta(\imath) = \varpi_\Delta(\theta(\imath)) \leq \varpi_\Delta(0) = \varpi_\Delta(\theta(0)) = \varpi^\theta_\Delta(0)$

$\Rightarrow \varpi^\theta_\Delta(\imath) \leq \varpi^\theta_\Delta(0)$

$\xi^\theta_\Delta(\imath) = \xi_\Delta(\theta(\imath)) \geq \xi_\Delta(0) = \xi_\Delta(\theta(0)) = \xi^\theta_\Delta(0)$

$\Rightarrow \xi^\theta_\Delta(\imath) \geq \xi^\theta_\Delta(0)$.

Let $\imath, \jmath, \ell \in \Omega$. Then

$\min\{\varpi^\theta_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \varpi^\theta_\Delta(\ell)\}$

$= \min\{\varpi_\Delta(\theta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell)), \varpi_\Delta(\theta(\ell))\}$

$= \min\{\varpi_\Delta((((\theta(\imath) \cdot \theta(\jmath)) \cdot \theta(\jmath)) \cdot (\theta(0) \cdot \theta(\jmath))) \cdot \theta(\ell)), \varpi_\Delta(\theta(\ell))\}$

$\leq \varpi_\Delta(\theta(\imath) \cdot ((\theta(\jmath) \cdot (\theta(\jmath) \cdot \theta(\imath))) \cdot (\theta(0) \cdot (\theta(0) \cdot (\theta(\imath) \cdot \theta(\jmath))))))$

$= \varpi_\Delta(\theta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))))$

$= \varpi^\theta_\Delta((\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))))$.

Similarly, $\max\{\xi^\theta_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \xi^\theta_\Delta(\ell)\}$

$= \max\{\xi_\Delta(\theta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell)), \xi_\Delta(\theta(\ell))\}$

$= \max\{\xi_\Delta((((\theta(\imath) \cdot \theta(\jmath)) \cdot \theta(\jmath)) \cdot (\theta(0) \cdot \theta(\jmath))) \cdot \theta(\ell)), \xi_\Delta(\theta(\ell))\}$

$\geq \xi_\Delta(\theta(\imath) \cdot ((\theta(\jmath) \cdot (\theta(\jmath) \cdot \theta(\imath))) \cdot (\theta(0) \cdot (\theta(0) \cdot (\theta(\imath) \cdot \theta(\jmath))))))$

$= \xi_\Delta(\theta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))))$

$= \xi^\theta_\Delta((\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))))$.

Hence IFS $\Delta^\theta = (\varpi^\theta_\Delta, \xi^\theta_\Delta)$ in Ω is an $IF_{BCI}II$ of Ω. □

Theorem 2.9.17. *Let* $\theta : \Omega \mapsto \Upsilon$ *be an epimorphism of BCI-algebras and IFS* $\Delta = (\varpi_\Delta, \xi_\Delta)$ *be in* Υ*. If IFS* $\Delta^\theta = (\varpi^\theta_\Delta, \xi^\theta_\Delta)$ *is an* $IF_{BCI}II$ *of* Ω*, then IFS* $\Delta = (\varpi_\Delta, \xi_\Delta)$ *is an* $IF_{BCI}II$ *of* Υ*.*

Proof. For any $\imath, \jmath, \ell \in \Upsilon$, $\exists\ \wp, \Im, \vartheta \in \Omega$, s.t, $\theta(\wp) = \imath$, $\theta(\Im) = \jmath$, $\theta(\vartheta) = \ell$.

Then for any $\imath \in \Upsilon$,

$\varpi_\Delta(\imath) = \varpi_\Delta(\theta(\wp)) = \varpi^\theta_\Delta(\wp) \leq \varpi^\theta_\Delta(0) = \varpi_\Delta(\theta(0)) = \varpi_\Delta(0)$

$\Rightarrow \varpi_\Delta(\imath) \leq \varpi_\Delta(0)$

$\xi_\Delta(\imath) = \xi_\Delta(\theta(\wp)) = \xi^\theta_\Delta(\wp) \geq \xi^\theta_\Delta(0) = \xi_\Delta(\theta(0)) = \xi_\Delta(0)$

$\Rightarrow \xi_\Delta(\imath) \geq \xi_\Delta(0)$

Now $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$

$= \varpi_\Delta(\theta(\wp) \cdot ((\theta(\Im) \cdot (\theta(\Im) \cdot \theta(\wp))) \cdot (\theta(0) \cdot (\theta(0) \cdot (\theta(\wp) \cdot \theta(\Im))))))$

$= \varpi_\Delta(\theta(\wp \cdot ((\Im \cdot (\Im \cdot \wp)) \cdot (0 \cdot (0 \cdot (\wp \cdot \Im))))))$

$= \varpi^\theta_\Delta(\wp \cdot ((\Im \cdot (\Im \cdot \wp)) \cdot (0 \cdot (0 \cdot (\wp \cdot \Im)))))$

$\geq \min\{\varpi^\theta_\Delta((((\wp \cdot \Im) \cdot \Im) \cdot (0 \cdot \Im)) \cdot \vartheta), \varpi^\theta_\Delta(\vartheta)\}$

$= \min\{\varpi_\Delta(\theta((((\wp \cdot \Im) \cdot \Im) \cdot (0 \cdot \Im)) \cdot \vartheta)), \varpi_\Delta(\theta(\vartheta))\}$

$= \min\{\varpi_\Delta((((\theta(\wp) \cdot \theta(\Im)) \cdot \theta(\Im)) \cdot ((\theta(0) \cdot \theta(\Im)))) \cdot \theta(\vartheta)), \varpi_\Delta(\theta(\vartheta))\}$

$= \min\{\varpi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \varpi_\Delta(\ell)\}.$

Similarly, $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$

$= \xi_\Delta(\theta(\wp) \cdot ((\theta(\Im) \cdot (\theta(\Im) \cdot \theta(\wp))) \cdot (\theta(0) \cdot (\theta(0) \cdot (\theta(\wp) \cdot \theta(\Im))))))$

$= \xi_\Delta(\theta(\wp \cdot ((\Im \cdot (\Im \cdot \wp)) \cdot (0 \cdot (0 \cdot (\wp \cdot \Im))))))$

$= \xi^\theta_\Delta(\wp \cdot ((\Im \cdot (\Im \cdot \wp)) \cdot (0 \cdot (0 \cdot (\wp \cdot \Im)))))$

$\leq \max\{\xi^\theta_\Delta((((\wp \cdot \Im) \cdot \Im) \cdot (0 \cdot \Im)) \cdot \vartheta), \xi^\theta_\Delta(\vartheta)\}$

$= \max\{\xi_\Delta(\theta((((\wp \cdot \Im) \cdot \Im) \cdot (0 \cdot \Im)) \cdot \vartheta)), \xi_\Delta(\theta(\vartheta))\}$

$= \max\{\xi_\Delta((((\theta(\wp) \cdot \theta(\Im)) \cdot \theta(\Im)) \cdot ((\theta(0) \cdot \theta(\Im)))) \cdot \theta(\vartheta)), \xi_\Delta(\theta(\vartheta))\}$

$= \max\{\xi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), \xi_\Delta(\ell)\}$

for all $\imath, \jmath, \ell \in \Upsilon$.

Hence $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Υ. □

Theorem 2.9.18. *Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an "intuitionistic fuzzy closed ideal" of Ω. Then $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω if and only if:*

(a) $\varpi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

(b) $\xi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \leq \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$,

for all $\imath, \jmath \in \Omega$.

Proof. Suppose that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω. Since $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an "intuitionistic fuzzy closed ideal" of Ω, so for any $\imath, \jmath \in \Omega$,

$\varpi_\Delta(0 \cdot (((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

and $\xi_\Delta(0 \cdot (((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))) \leq \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$.

By 1.2.1, 1.2.6 and 1.2.10,

$(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \cdot (\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq (0 \cdot (\imath \cdot \jmath))$

Moreover by 1.2.6 and 1.2.9,

$0 \cdot (((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) = 0 \cdot (\imath \cdot \jmath)$

Hence by Lemma 2.4.2,

$\varpi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath)))$

$\geq \min\{\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))), \varpi_\Delta(0 \cdot (((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)))\}$

and $\xi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \leq$

$\max\{\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))), \xi_\Delta(0 \cdot (((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)))\}$.

Now by Theorem 2.9.4,

$\varpi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \geq$

$\min\{\varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)), \varpi_\Delta(0 \cdot (((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)))\}$

$= \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \leq$

$\max\{\xi_\Delta((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath), \xi_\Delta(0 \cdot (((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)))\} = \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$.

Now conversely suppose that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is "intuitionistic fuzzy closed ideal" of Ω satisfying the conditions:

$\varpi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

$\xi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \leq \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

for all $\imath, \jmath \in \Omega$.

Consider $(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \cdot (\imath \cdot (\jmath \cdot (\jmath \cdot \imath)))$

$\leq (\jmath \cdot (\jmath \cdot \imath)) \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))) \leq 0 \cdot (0 \cdot (\imath \cdot \jmath))$ (by 1.2)

By using Lemma 2.4.2,

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$

$\geq \min\{\varpi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))), \varpi_\Delta(0 \cdot (0 \cdot (\imath \cdot \jmath))))\}$

$\geq \min\{\varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)), \varpi_\Delta((0 \cdot (\imath \cdot \jmath)))\}$ (By given conditions)

$= \min\{\varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)), \varpi_\Delta((0 \cdot (((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))))\}$

$= \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$.

Similarly by Lemma 2.4.2,

$\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$

$\leq \max\{\xi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))), \xi_\Delta(0 \cdot (0 \cdot (\imath \cdot \jmath))))\}$

$\leq \max\{\xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)), \xi_\Delta((0 \cdot (\imath \cdot \jmath)))\}$ (By given conditions)

$= \max\{\xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)), \xi_\Delta((0 \cdot (((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))))\}$

$= \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$.

Hence by Theorem 2.9.4, $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω. □

Theorem 2.9.19. *Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an $IF_{BCI}II$ of Ω, s.t, $\varpi_\Delta(\imath \wedge \jmath) \leq \varpi_\Delta(\jmath)$ and $\xi_\Delta(\imath \wedge \jmath) \geq \xi_\Delta(\jmath)$. Then $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an IF_hI of Ω.*

Proof. Suppose that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω. Then by Theorem 2.9.4,

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

$\geq \min\{\varpi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \imath), \varpi_\Delta(\imath)\}$

$= \min\{\varpi_\Delta(0 \cdot \jmath), \varpi_\Delta(\imath)\} \geq \min\{\varpi_\Delta(\imath \wedge (0 \cdot \jmath)), \varpi_\Delta(\imath)\} = \varpi_\Delta(\imath)$

$\Rightarrow \varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \varpi_\Delta(\imath)$.

Similarly by Theorem 2.9.4,

$\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

$\leq \max\{\xi_\Delta((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \imath), \xi_\Delta(\imath)\} = \max\{\xi_\Delta(0 \cdot \jmath), \xi_\Delta(\imath)\}$

$\leq \max\{\xi_\Delta(\imath \wedge (0 \cdot \jmath)), \xi_\Delta(\imath)\} = \xi_\Delta(\imath)$

$\Rightarrow \xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \xi_\Delta(\imath)$,

for all $\imath, \jmath \in \Omega$.

Therefore by Theorem 2.6.3, $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an IF_hI of Ω. □

Theorem 2.9.20. *Let an $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be $IF_{BCI}II$ of a "positive implicative BCK-algebra" Ξ and $\mathrm{u}, \mathrm{v} \in \Xi$. Then the extension $< (\varpi_\Delta, \xi_\Delta), (\mathrm{u}, \mathrm{v}) >$ of $(\varpi_\Delta, \xi_\Delta)$ by (u, v) is also an $IF_{BCI}II$ of Ξ.*

Proof. Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be $IF_{BCI}II$ of a "positive implicative BCK-algebra" Ξ and $\wp, \Im \in \Xi$. Let $\imath, \jmath \in \Xi$. Then we have $< \varpi_\Delta, \wp > (0) = \varpi_\Delta(0 \cdot \wp) = \varpi_\Delta(0) \geq \varpi_\Delta(\imath \cdot \wp) =< \varpi_\Delta, \wp > (\imath)$ and

$< \xi_\Delta, \Im > (0) = \xi_\Delta(0 \cdot \Im) = \xi_\Delta(0) \leq \xi_\Delta(\imath \cdot \Im) =< \xi_\Delta, \Im > (\imath)$.

Now, $< \varpi_\Delta, \wp > (\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$

$= \varpi_\Delta((\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \cdot \wp)$

$= \varpi_\Delta(((\imath \cdot \wp) \cdot (((\jmath \cdot \wp) \cdot ((\jmath \cdot \wp) \cdot (\imath \cdot \wp))) \cdot ((0 \cdot \wp) \cdot ((0 \cdot \wp) \cdot ((\imath \cdot \wp) \cdot (\jmath \cdot \wp)))))))$

$= \varpi_\Delta(((\imath \cdot \wp) \cdot (((\jmath \cdot \wp) \cdot ((\jmath \cdot \wp) \cdot (\imath \cdot \wp))) \cdot (0 \cdot (0 \cdot ((\imath \cdot \wp) \cdot (\jmath \cdot \wp)))))))$

$\geq \min\{\varpi_\Delta(((((\imath \cdot \wp) \cdot (\jmath \cdot \wp)) \cdot (\jmath \cdot \wp)) \cdot (0 \cdot (\jmath \cdot \wp))) \cdot (\ell \cdot \wp)), \varpi_\Delta(\ell \cdot \wp)\}$

$= \min\{\varpi_\Delta(((((\imath \cdot \wp) \cdot (\jmath \cdot \wp)) \cdot (\jmath \cdot \wp)) \cdot ((0 \cdot \wp) \cdot (\jmath \cdot \wp))) \cdot (\ell \cdot \wp)), \varpi_\Delta(\ell \cdot \wp)\}$

$= \min\{\varpi_\Delta(((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell) \cdot \wp), \varpi_\Delta(\ell \cdot \wp\})$

$= \min\{< \varpi_\Delta, \wp > ((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), < \varpi_\Delta, \wp > (\ell)\}$.

Similarly, $< \xi_\Delta, \Im > (\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$

$= \xi_\Delta((\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \cdot \Im)$

$= \xi_\Delta(((\imath \cdot \Im) \cdot (((\jmath \cdot \Im) \cdot ((\jmath \cdot \Im) \cdot (\imath \cdot \Im))) \cdot ((0 \cdot \Im) \cdot ((0 \cdot \Im) \cdot ((\imath \cdot \Im) \cdot (\jmath \cdot \Im)))))))$

$= \xi_\Delta(((\imath \cdot \Im) \cdot (((\jmath \cdot \Im) \cdot ((\jmath \cdot \Im) \cdot (\imath \cdot \Im))) \cdot (0 \cdot (0 \cdot ((\imath \cdot \Im) \cdot (\jmath \cdot \Im)))))))$

$\leq \max\{\xi_\Delta(((((\imath \cdot \Im) \cdot (\jmath \cdot \Im)) \cdot (\jmath \cdot \Im)) \cdot (0 \cdot (\jmath \cdot \Im))) \cdot (\ell \cdot \Im)), \xi_\Delta(\ell \cdot \Im)\}$

$= \max\{\xi_\Delta(((((\imath \cdot \Im) \cdot (\jmath \cdot \Im)) \cdot (\jmath \cdot \Im)) \cdot ((0 \cdot \Im) \cdot (\jmath \cdot \Im))) \cdot (\ell \cdot \Im)), \xi_\Delta(\ell \cdot \Im)\}$

$= \max\{\xi_\Delta(((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell) \cdot \Im), \xi_\Delta(\ell \cdot \Im)\}$

$= \max\{< \xi_\Delta, \Im > ((((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell), < \xi_\Delta, \Im > (\ell)\}$.

Hence the extension $< (\varpi_\Delta, \xi_\Delta), (\wp, \Im) >$ of $(\varpi_\Delta, \xi_\Delta)$ by $(\wp, \Im)$ is an $IF_{BCI}II$ of Ξ. $\square$

Corollary 2.9.21. *Let an IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ be $IF_{BCI}II$ of a "positive implicative BCK-algebra" Ξ and $\wp \in \Xi$. Then the extension $< (\varpi_\Delta, \xi_\Delta), \wp >$ of $(\varpi_\Delta, \xi_\Delta)$ by $\wp$ is also an $IF_{BCI}II$ of Ξ.*

2.10 Intuitionistic fuzzy BCI-positive implicative ideal

An IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω is called an "intuitionistic fuzzy BCI-positive implicative ideal" $(IF_{BCI}PII)$ of Ω if it satisfies:

$(IFBCI-PI-1)$ $\varpi_\Delta(0) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0) \leq \xi_\Delta(\imath)$

$(IFBCI-PI-2)$ $\varpi_\Delta(\imath \cdot \ell) \geq \min\{\varpi_\Delta(((\imath \cdot \ell) \cdot \ell) \cdot (\jmath \cdot \ell)), \varpi_\Delta(\jmath)\}$

$(IFBCI-PI-3)$ $\xi_\Delta(\imath \cdot \ell) \leq \max\{\xi_\Delta(((\imath \cdot \ell) \cdot \ell) \cdot (\jmath \cdot \ell)), \xi_\Delta(\jmath)\}$,

for all $\imath, \jmath, \ell \in \Omega$.

Example 2.10.1. Let $\Omega = \{0, \imath, \jmath, \ell\}$ be a BCI-algebra defined by the following cayley table:

$\cdot$	0	$\imath$	$\jmath$	ℓ
0	0	0	0	ℓ
$\imath$	$\imath$	0	0	ℓ
$\jmath$	$\jmath$	$\jmath$	0	ℓ
ℓ	ℓ	ℓ	ℓ	0

Define an $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω as follows:

$\varpi_\Delta(0) = \varpi_\Delta(\ell) = 1, \varpi_\Delta(\imath) = \varpi_\Delta(\jmath) = \delta$

and $\xi_\Delta(0) = \xi_\Delta(\ell) = 0, \xi_\Delta(\imath) = \xi_\Delta(\jmath) = \eta$

where $\delta, \eta \in (0,1)$ and $\delta + \eta \leq 1$

By routine calculations it is easy to verify that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}PII$ of Ω.

An $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω is called an "intuitionistic fuzzy closed BCI-positive implicative ideal" of Ω if it satisfies $(IFBCI - PI - 2)$, $(IFBCI - PI - 3)$ and

$(IFBCI - PI - 4)$ $\varpi_\Delta(0 \cdot \imath) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0 \cdot \imath) \leq \xi_\Delta(\imath)$, for all $\imath \in \Omega$.

Theorem 2.10.2. *Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an IFI of Ω. Then the following conditions are equivalent:*

1. *$IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}PII$ of Ω.*

2. $\varpi_\Delta((\imath \cdot \jmath) \cdot \ell) \geq \varpi_\Delta(((\imath \cdot \ell) \cdot \ell) \cdot (\jmath \cdot \ell))$

and $\xi_\Delta((\imath \cdot \jmath) \cdot \ell)) \leq \xi_\Delta(((\imath \cdot \ell) \cdot \ell) \cdot (\jmath \cdot \ell))$

3. $\varpi_\Delta((\imath \cdot \jmath) \cdot \ell)) = \varpi_\Delta(((\imath \cdot \ell) \cdot \ell) \cdot (\jmath \cdot \ell))$

and $\xi_\Delta((\imath \cdot \jmath) \cdot \ell)) = \xi_\Delta(((\imath \cdot \ell) \cdot \ell) \cdot (\jmath \cdot \ell))$

4. $\varpi_\Delta(\imath \cdot \jmath) = \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\imath \cdot \jmath) = \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

5. $\varpi_\Delta(\imath \cdot \jmath) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\imath \cdot \jmath) \leq \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$,

for any $\imath, \jmath, \ell \in \Omega$.

Proof. $(1 \Rightarrow 2)$ Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an $IF_{BCI}PII$ of Ω.

Then for any $\imath, \jmath, \ell \in \Omega$,

$\varpi_\Delta((\imath\cdot\jmath)\cdot\ell)\geq\min\{\varpi_\Delta((((\imath\cdot\jmath)\cdot\ell)\cdot\ell)\cdot(0\cdot\ell)),\varpi_\Delta(0)\}$

$=\varpi_\Delta(((\imath\cdot\ell)\cdot\ell)\cdot(0\cdot\ell))$

and $\xi_\Delta((\imath\cdot\jmath)\cdot\ell)\leq\max\{\xi_\Delta((((\imath\cdot\jmath)\cdot\ell)\cdot\ell)\cdot(0\cdot\ell)),\xi_\Delta(0)\}$

$=\xi_\Delta((((\imath\cdot\jmath)\cdot\ell)\cdot\ell)\cdot(0\cdot\ell))$.

Now $(((\imath\cdot\jmath)\cdot\ell)\cdot\ell)\cdot(0\cdot\ell)=(((\imath\cdot\ell)\cdot\ell)\cdot\jmath)\cdot((\jmath\cdot\ell)\cdot\jmath)$

$\leq((\imath\cdot\ell)\cdot\ell)\cdot(\jmath\cdot\ell)$ (by 1.2.11)

Therefore by Lemma 2.4.3,

$\varpi_\Delta((((\imath\cdot\jmath)\cdot\ell)\cdot\ell)\cdot(0\cdot\ell))\geq\varpi_\Delta(((\imath\cdot\ell)\cdot\ell)\cdot(\jmath\cdot\ell))$

and $\xi_\Delta((((\imath\cdot\jmath)\cdot\ell)\cdot\ell)\cdot(0\cdot\ell))\leq\xi_\Delta(((\imath\cdot\ell)\cdot\ell)\cdot(\jmath\cdot\ell))$

Thus $\varpi_\Delta((\imath\cdot\jmath)\cdot\ell)\geq\varpi_\Delta(((\imath\cdot\ell)\cdot\ell)\cdot(\jmath\cdot\ell))$

and $\xi_\Delta((\imath\cdot\jmath)\cdot\ell)\leq\xi_\Delta(((\imath\cdot\ell)\cdot\ell)\cdot(\jmath\cdot\ell))$.

$(2\Rightarrow3)$ Assume that

$\varpi_\Delta((\imath\cdot\jmath)\cdot\ell)\geq\varpi_\Delta(((\imath\cdot\ell)\cdot\ell)\cdot(\jmath\cdot\ell))$

and $\xi_\Delta((\imath\cdot\jmath)\cdot\ell)\leq\xi_\Delta(((\imath\cdot\ell)\cdot\ell)\cdot(\jmath\cdot\ell))$

Now $((\imath\cdot\ell)\cdot\ell)\cdot(\jmath\cdot\ell)\leq(\imath\cdot\ell)\cdot\jmath=(\imath\cdot\jmath)\cdot\ell$

Therefore by using Lemma 2.4.3 we have

$\varpi_\Delta(((\imath\cdot\ell)\cdot\ell)\cdot(\jmath\cdot\ell))\geq\varpi_\Delta((\imath\cdot\jmath)\cdot\ell)$

and $\xi_\Delta(((\imath\cdot\ell)\cdot\ell)\cdot(\jmath\cdot\ell))\leq\xi_\Delta((\imath\cdot\jmath)\cdot\ell)$

Thus, $\varpi_\Delta((\imath\cdot\jmath)\cdot\ell)=\varpi_\Delta(((\imath\cdot\ell)\cdot\ell)\cdot(\jmath\cdot\ell))$

and $\xi_\Delta((\imath\cdot\jmath)\cdot\ell)=\xi_\Delta(((\imath\cdot\ell)\cdot\ell)\cdot(\jmath\cdot\ell))$

$(3\Rightarrow4)$ Assume that

$\varpi_\Delta((\imath\cdot\jmath)\cdot\ell)=\varpi_\Delta(((\imath\cdot\ell)\cdot\ell)\cdot(\jmath\cdot\ell))$

and $\xi_\Delta((\imath\cdot\jmath)\cdot\ell)=\xi_\Delta(((\imath\cdot\ell)\cdot\ell)\cdot(\jmath\cdot\ell))$

Putting $\ell=\jmath$ and $\jmath=0$ we get

$\varpi_\Delta((\imath\cdot0)\cdot\jmath)=\varpi_\Delta(((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))$

and $\xi_\Delta((\imath\cdot0)\cdot\jmath)=\xi_\Delta(((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))$

i.e., $\varpi_\Delta(\imath \cdot \jmath) = \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\imath \cdot \jmath) = \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$.

$(4 \Rightarrow 5)$ Obviously.

$(5 \Rightarrow 1)$ Assume that

$\varpi_\Delta(\imath \cdot \jmath) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\imath \cdot \jmath) \leq \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$.

Now $(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot (((\imath \cdot \jmath) \cdot \jmath) \cdot (\ell \cdot \jmath)) \leq (\ell \cdot \jmath) \cdot (0 \cdot \jmath) \leq \ell \cdot 0 = \ell$.

Therefore by Lemma 2.4.2,

$\varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \geq \min\{\varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (\ell \cdot \jmath)), \varpi_\Delta(\ell)\}$

and $\xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \leq \max\{\xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (\ell \cdot \jmath)), \xi_\Delta(\ell)\}$,

i.e., $\varpi_\Delta(\imath \cdot \jmath) \geq \min\{\varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (\ell \cdot \jmath)), \varpi_\Delta(\ell)\}$

and $\xi_\Delta(\imath \cdot \jmath) \leq \max\{\xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (\ell \cdot \jmath)), \xi_\Delta(\ell)\}$.

Hence $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}PII$ of Ω. □

Theorem 2.10.3. *Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an IFI of Ω. If for any $\imath, \jmath \in \Omega$,*

$\varpi_\Delta(\imath \cdot (\imath \cdot (\jmath \cdot (\jmath \cdot \imath)))) \geq \varpi_\Delta((\imath \cdot (\imath \cdot \jmath)) \cdot (\jmath \cdot \imath))$

and $\xi_\Delta(\imath \cdot (\imath \cdot (\jmath \cdot (\jmath \cdot \imath)))) \leq \xi_\Delta((\imath \cdot (\imath \cdot \jmath)) \cdot (\jmath \cdot \imath))$.

Then $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}PII$ of Ω.

Proof. Substituting $\wp = \imath \cdot \jmath$ gives,

$(\wp \cdot (\wp \cdot \imath)) \cdot (\imath \cdot \wp) = ((\imath \cdot \jmath) \cdot ((\imath \cdot \jmath) \cdot \imath)) \cdot (\imath \cdot (\imath \cdot \jmath))$

$= ((\imath \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot (\imath \cdot (\imath \cdot \jmath)) = ((\imath \cdot \jmath) \cdot (\imath \cdot (\imath \cdot \jmath))) \cdot (0 \cdot \jmath)$

$= ((\imath \cdot (\imath \cdot (\imath \cdot \jmath))) \cdot \jmath) \cdot (0 \cdot \jmath) = ((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)$ (Since by using 1.2.1 $\imath \cdot (\imath \cdot (\imath \cdot \jmath)) = \imath \cdot \jmath$)

Thus, $\varpi_\Delta((\wp \cdot (\wp \cdot \imath)) \cdot (\imath \cdot \wp)) = \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

and $\xi_\Delta((\wp \cdot (\wp \cdot \imath)) \cdot (\imath \cdot \wp)) = \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$.

Now $\wp \cdot (\wp \cdot (\imath \cdot (\imath \cdot \wp))) = (\imath \cdot \jmath) \cdot ((\imath \cdot \jmath) \cdot (\imath \cdot (\imath \cdot (\imath \cdot \jmath)))) = (\imath \cdot \jmath) \cdot ((\imath \cdot \jmath) \cdot (\imath \cdot \jmath))$

$= (\imath \cdot \jmath) \cdot 0 = \imath \cdot \jmath$

Thus, $\varpi_\Delta(\wp \cdot (\wp \cdot (\imath \cdot (\imath \cdot \wp)))) = \varpi_\Delta(\imath \cdot \jmath)$ and $\xi_\Delta(\wp \cdot (\wp \cdot (\imath \cdot (\imath \cdot \wp)))) = \xi_\Delta(\imath \cdot \jmath)$

By given hypothesis,

$\varpi_\Delta(\wp \cdot (\wp \cdot (\imath \cdot (\imath \cdot \wp)))) \geq \varpi_\Delta((\wp \cdot (\wp \cdot \imath)) \cdot (\imath \cdot \wp))$

and $\xi_\Delta(\wp \cdot (\wp \cdot (\imath \cdot (\imath \cdot \wp)))) \leq \xi_\Delta((\wp \cdot (\wp \cdot \imath)) \cdot (\imath \cdot \wp))$.

Thus, $\varpi_\Delta(\imath \cdot \jmath) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$ and $\xi_\Delta(\imath \cdot \jmath) \leq \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$.

Hence by Theorem 2.10.2, $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}PII$ of Ω. □

2.11 Intuitionistic fuzzy BCI-commutative ideal

An $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω is called an "intuitionistic fuzzy BCI-commutative ideal" ($IF_{BCI}CI$) of Ω if it satisfies:

$(IFBCI-C-1)$ $\varpi_\Delta(0) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0) \leq \xi_\Delta(\imath)$, for all $\imath \in \Omega$.

$(IFBCI-C-2)$ $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$

$\geq \min\{\varpi_\Delta((\imath \cdot \jmath) \cdot \ell), \varpi_\Delta(\ell)\}$

$(IFBCI-C-3)$ $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$

$\leq \max\{\xi_\Delta((\imath \cdot \jmath) \cdot \ell), \xi_\Delta(\ell)\}$,

for all $\imath, \jmath, \ell \in \Omega$.

Example 2.11.1. Let $\Omega = \{0, \imath, \jmath, \ell\}$ be a BCI-algebra defined by the following cayley table:

$\cdot$	0	$\imath$	$\jmath$	ℓ
0	0	0	0	0
$\imath$	$\imath$	0	0	$\imath$
$\jmath$	$\jmath$	$\imath$	0	$\jmath$
ℓ	ℓ	ℓ	ℓ	0

Define an IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω as follows:

$\varpi_\Delta(0) = \varpi_\Delta(\ell) = 1, \varpi_\Delta(\imath) = \varpi_\Delta(\jmath) = \delta$

and $\xi_\Delta(0) = \xi_\Delta(\ell) = 0, \xi_\Delta(\imath) = \xi_\Delta(\jmath) = \eta$

where $\delta, \eta \in (0, 1)$ and $\delta + \eta \leq 1$

By routine calculations it is easy to verify that IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}CI$ of Ω.

An IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω is an "intuitionistic fuzzy closed BCI-commutative ideal" of Ω if it satisfies $(IFBCI - C - 2)$, $(IFBCI - C - 3)$ and

$(IFBCI - C - 4)$ $\varpi_\Delta(0 \cdot \imath) \geq \varpi_\Delta(\imath)$ and $\xi_\Delta(0 \cdot \imath) \leq \xi_\Delta(\imath)$, for all $\imath \in \Omega$.

Theorem 2.11.2. *Let IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ be an IFI of Ω. Then the following conditions are equivalent:*

1. IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}CI$ of Ω.

2. $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \varpi_\Delta(\imath \cdot \jmath)$

and $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \xi_\Delta(\imath \cdot \jmath)$

3. $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) = \varpi_\Delta(\imath \cdot \jmath)$

and $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) = \xi_\Delta(\imath \cdot \jmath)$,

for any $\imath, \jmath, \ell \in \Omega$.

Proof. $(1 \Rightarrow 2)$ Let IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ be an $IF_{BCI}CI$ of Ω.

Then for any $\imath, \jmath, \ell \in \Omega$,

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \min\{\varpi_\Delta((\imath \cdot \jmath) \cdot \ell), \varpi_\Delta(\ell)\}$

$\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \max\{\xi_\Delta((\imath \cdot \jmath) \cdot \ell), \xi_\Delta(\ell)\}$.

By substituting $\ell = 0$,

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \min\{\varpi_\Delta((\imath \cdot \jmath) \cdot 0), \varpi_\Delta(0)\}$

$\xi_(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \max\{\xi_\Delta((\imath \cdot \jmath) \cdot 0), \xi_(0)\}$, i.e., $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \varpi_\Delta(\imath \cdot \jmath)$

and $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \xi_\Delta(\imath \cdot \jmath)$.

which are the required conditions.

$(2 \Rightarrow 3)$ Assume that

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \varpi_\Delta(\imath \cdot \jmath)$ (a)

and $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \xi_\Delta(\imath \cdot \jmath)$ (b)

Since $(\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))) = (\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (\jmath \cdot \imath)) \leq \jmath$ (by 1.2.10 and 1.2.11)

$\Rightarrow \imath \cdot \jmath \leq \imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))$.

Then by Lemma 2.4.3,

$\varpi_\Delta(\imath \cdot \jmath) \geq \varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$ (c)

and

$\xi_\Delta(\imath \cdot \jmath) \leq \xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$ (d)

From (a) and (c) and (b) and (d),

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) = \varpi_\Delta(\imath \cdot \jmath)$

and

$\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) = \xi_\Delta(\imath \cdot \jmath)$

which are the required conditions.

$(3 \Rightarrow 1)$ Assume that:

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) = \varpi_\Delta(\imath \cdot \jmath)$ (a)

and

$\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) = \xi_\Delta(\imath \cdot \jmath)$ (b)

for all $\imath, \jmath \in \Omega$.

Since $(\imath \cdot \jmath)((\imath \cdot \jmath) \cdot \ell) \leq \ell$ (by 1.2)

Therefore by using Lemma 2.4.2,

$\varpi_\Delta(\imath \cdot \jmath) \geq \min\{\varpi_\Delta((\imath \cdot \jmath) \cdot \ell), \varpi_\Delta(\ell)\}$ (c)

$\xi_\Delta(\imath \cdot \jmath) \leq \max\{\xi_\Delta((\imath \cdot \jmath) \cdot \ell), \xi_\Delta(\ell)\}$ (d)

From (a) and (c) and (b) and (d),

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \min\{\varpi_\Delta((\imath \cdot \jmath) \cdot \ell), \varpi_\Delta(\ell)\}$

$\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \max\{\xi_\Delta((\imath \cdot \jmath) \cdot \ell), \xi_\Delta(\ell)\}$.

Hence $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}CI$ of Ω. □

Theorem 2.11.3. *Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an "intuitionistic fuzzy closed ideal" of Ω. Then $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}CI$ of Ω if and only if*

(a) $\varpi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \geq \varpi_\Delta(\imath \cdot \jmath)$

(b) $\xi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \leq \xi_\Delta(\imath \cdot \jmath)$,

for all $\imath, \jmath \in \Omega$.

Proof. Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an $IF_{BCI}CI$ of Ω. Since $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an "intuitionistic fuzzy closed ideal" of Ω, so for any $\imath, \jmath \in \Omega$,

$\varpi_\Delta(0 \cdot (\imath \cdot \jmath)) \geq \varpi_\Delta(\imath \cdot \jmath)$ and $\xi_\Delta(0 \cdot (\imath \cdot \jmath)) \leq \xi_\Delta(\imath \cdot \jmath)$.

Since by 1.2.1, 1.2.6 and 1.2.10,

$(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \cdot (\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq (0 \cdot (\imath \cdot \jmath))$

Hence by Lemma 2.4.2,

$\varpi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \geq \min\{\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))), \varpi_\Delta(0 \cdot (\imath \cdot \jmath))\}$

and

$\xi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \leq \max\{\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))), \xi_\Delta(0 \cdot (\imath \cdot \jmath))\}$.

Now by using Theorem 2.11.2,

$\varpi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \geq \min\{\varpi_\Delta(\imath \cdot \jmath), \varpi_\Delta(0 \cdot (\imath \cdot \jmath))\} = \varpi_\Delta(\imath \cdot \jmath)\}$ and $\xi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \leq \max\{\xi_\Delta(\imath \cdot \jmath), \xi_\Delta(0 \cdot (\imath \cdot \jmath))\} = \xi_\Delta(\imath \cdot \jmath)$.

Which are the required conditions.

Now, conversely, let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an "intuitionistic fuzzy closed ideal" of Ω satisfying the conditions:

$\varpi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \geq \varpi_\Delta(\imath \cdot \jmath)$

$\xi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \leq \xi_\Delta(\imath \cdot \jmath)$

for all $\imath, \jmath, \ell \in \Omega$.

Since by 1.2.1 and 1.2.2,

$(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \cdot (\imath \cdot (\jmath \cdot (\jmath \cdot \imath))) \leq 0 \cdot (0 \cdot (\imath \cdot \jmath))$

Therefore by using Lemma 2.4.2,

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$

$\geq \min\{\varpi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))), \varpi_\Delta(0 \cdot (0 \cdot (\imath \cdot \jmath))))$

$\geq \min\{\varpi_\Delta(\imath \cdot \jmath), \varpi_\Delta((0 \cdot (\imath \cdot \jmath))))$ (By given conditions)

$= \varpi_\Delta(\imath \cdot \jmath)$

Similarly by Lemma 2.4.2 we have

$\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$

$\leq \max\{\xi_\Delta(\imath \cdot (\jmath \cdot (\jmath \cdot \imath))), \xi_\Delta(0 \cdot (0 \cdot (\imath \cdot \jmath))))$

$\leq \max\{\xi_\Delta(\imath \cdot \jmath), \xi_\Delta((0 \cdot (\imath \cdot \jmath))))$ (By given conditions)

$= \xi_\Delta(\imath \cdot \jmath)$.

Hence by Theorem 2.11.2, $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}CI$ of Ω. □

Now, firstly we interrelate "intuitionistic fuzzy BCI-(positive implicative, implicative, commutative) ideals" in BCI-algebras with each other and show that an IFS in Ω is an $IF_{BCI}II$ of Ω if and only if it is both an $IF_{BCI}PI$ and an $IF_{BCI}CI$ of Ω. Then secondly we interrelate these three $IFIs$ with IF_pIs and $IF_\alpha Is$ and show that every IF_pI of Ω is an $IF_{BCI}II$ of Ω and also every $IF_\alpha I$ of Ω is an $IF_{BCI}II$ of Ω.

2.12 Interrelationship between intuitionistic fuzzy BCI-(implicative, positive implicative, commutative) ideals

Theorem 2.12.1. *Any $IF_{BCI}II$ of Ω is an $IF_{BCI}CI$ of Ω.*

Proof. Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an $IF_{BCI}II$ of Ω.
Then by Theorem 2.9.4,
$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$
and $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$
for all $\imath, \jmath \in \Omega$.
Consider $((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath) = ((\imath \cdot \jmath) \cdot \jmath) \cdot ((\imath \cdot \jmath) \cdot \imath) \leq \imath \cdot \jmath$. (By 1.2.1)
Therefore by using Lemma 2.4.3,
$\varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \geq \varpi_\Delta(\imath \cdot \jmath)$ and $\xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \leq \xi_\Delta(\imath \cdot \jmath)$
i.e., $\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \varpi_\Delta(\imath \cdot \jmath)$
and $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \xi_\Delta(\imath \cdot \jmath)$.
Therefore by Theorem 2.11.2, $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}CI$ of Ω. □

Whereas the converse of this theorem is not true in general. To examine this we cogitate the succeeding example.

Example 2.12.2. Cogitate the Example 2.11.1. By routine calculations it can be easily verified that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}CI$ of Ω but it is not an $IF_{BCI}II$ of Ω since,
$\varpi_\Delta(\jmath \cdot ((\imath \cdot (\imath \cdot \jmath)) \cdot (0 \cdot (0 \cdot (\jmath \cdot \imath))))) = \varpi_\Delta(\imath) = \delta < 1 = \varpi_\Delta(0) =$
$\min\{\varpi_\Delta((((\jmath \cdot \imath) \cdot \imath) \cdot (0 \cdot \imath)) \cdot 0), \varpi_\Delta(0)\}$.

Theorem 2.12.3. *Any $IF_{BCI}II$ of Ω is an $IF_{BCI}PII$ of Ω.*

Proof. Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an $IF_{BCI}II$ of Ω.

Then by Theorem 2.9.4,

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$,

for all $\imath, \jmath \in \Omega$.

Consider $(\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))) = (\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (\jmath \cdot \imath)) \leq \jmath \cdot 0 = \jmath$. (By 1.2.10 and 1.2.11)

$\Rightarrow (\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))) \leq \jmath$

$\Rightarrow \imath \cdot \jmath \leq \imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))$ (By 1.2.8)

Therefore by Lemma 2.4.3,

$\varpi_\Delta(\imath \cdot \jmath) \geq \varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$

and $\xi_\Delta(\imath \cdot \jmath) \leq \xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))))$

Thus, $\varpi_\Delta(\imath \cdot \jmath) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$ and $\xi_\Delta(\imath \cdot \jmath) \leq \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$.

Therefore by Theorem 2.10.2, $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}PII$ of Ω. □

Whereas the converse of this Theorem is not true in general and can be examined by the succeeding example.

Example 2.12.4. Consider the Example 2.10.1. Here it can be easily verified that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}PII$ of Ω but it is not an $IF_{BCI}II$ of Ω since,

$\varpi_\Delta(\imath * ((\jmath * (\jmath * \imath)) * (0 * (0 * (\imath * \jmath))))) = \varpi_\Delta(\imath) = \delta < 1 = \varpi_\Delta(0) = \min\{\varpi_\Delta((((\imath * \jmath) * \jmath) * (0 * \jmath)) * 0), \varpi_\Delta(0)\}$.

Theorem 2.12.5. *An $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ in Ω is an $IF_{BCI}II$ of Ω if and only if $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is both an $IF_{BCI}PII$ and an $IF_{BCI}CI$ of Ω.*

Proof. Suppose that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω. Then by Theorem 2.12.1 and Theorem 2.12.3, $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is both an $IF_{BCI}CI$

and an $IF_{BCI}PII$ of Ω.

Conversely suppose that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is both an $IF_{BCI}CI$ and an $IF_{BCI}PII$ of Ω. Then by Theorem 2.11.2, for any $\imath, \jmath \in \Omega$,

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \varpi_\Delta(\imath \cdot \jmath)$ (a)

and

$\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \xi_\Delta(\imath \cdot \jmath)$ (b)

Since is also an $IF_{BCI}PII$ of Ω. Therefore by Theorem 2.10.2,

$\varpi_\Delta(\imath \cdot \jmath) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$ (c)

and

$\xi_\Delta(\imath \cdot \jmath) \leq \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$ (d)

From (a) and (c) and (b) and (d) we have

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \leq \xi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

Hence by Theorem 2.9.4, $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω. □

2.13 Relationship of intuitionistic fuzzy BCI-(implicative, positive implicative, commutative) ideals with intuitionistic fuzzy p-ideals

Theorem 2.13.1. *Any IF_pI of Ω is an $IF_{BCI}II$ of Ω.*

Proof. Let $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ be an IF_pI of Ω. Then we know that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is also an IFI of Ω. In order to prove that $IFS\ \Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω, we need to corroborate that,

$\varpi_\Delta(\imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath))))) \geq \varpi_\Delta(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath))$

and $\xi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\leq\xi_\Delta(((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))$,

for any $\imath,\jmath\in\Omega$. (From Theorem 2.9.4)

Since $(0\cdot(0\cdot(\imath\cdot\jmath)))\cdot(((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))=0$

$\Rightarrow(0\cdot(0\cdot(\imath\cdot\jmath)))\leq(((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))$

Therefore by Lemma 2.4.3,

$\varpi_\Delta(0\cdot(0\cdot(\imath\cdot\jmath)))\geq\varpi_\Delta(((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))$

and $\xi_\Delta(0\cdot(0\cdot(\imath\cdot\jmath)))\leq\xi_\Delta(((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))$

By using 1.2.1, 1.2.10, 1.2.6 and 1.2.9 respectively we get,

$0\cdot(0\cdot(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath))))))\leq0\cdot(0\cdot(\imath\cdot\jmath))$.

Therefore by Lemma 2.4.3,

$\varpi_\Delta(0\cdot(0\cdot(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))))\geq\varpi_\Delta(0\cdot(0\cdot(\imath\cdot\jmath)))$

and $\xi_\Delta(0\cdot(0\cdot(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))))\leq\xi_\Delta(0\cdot(0\cdot(\imath\cdot\jmath)))$.

Now, since $IFS\ \Delta=(\varpi_\Delta,\xi_\Delta)$ is an IF_pI of Ω, therefore by Theorem 2.5.2,

$\varpi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))$

$\geq\varpi_\Delta(0\cdot(0\cdot(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))))$

$\geq\varpi_\Delta(0\cdot(0\cdot(\imath\cdot\jmath)))\geq\varpi_\Delta(((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))$

$\Rightarrow\varpi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\geq\varpi_\Delta(((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))$.

and $\xi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))$

$\leq\xi_\Delta(0\cdot(0\cdot(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))))$

$\leq\xi_\Delta(0\cdot(0\cdot(\imath\cdot\jmath)))\leq\xi_\Delta(((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))$

$\Rightarrow\xi_\Delta(\imath\cdot((\jmath\cdot(\jmath\cdot\imath))\cdot(0\cdot(0\cdot(\imath\cdot\jmath)))))\leq\xi_\Delta(((\imath\cdot\jmath)\cdot\jmath)\cdot(0\cdot\jmath))$

Therefore by Theorem 2.9.4, $IFS\ \Delta=(\varpi_\Delta,\xi_\Delta)$ is an $IF_{BCI}II$ of Ω. □

Whereas converse of this theorem isn't valid. For this we cogitate the succeeding example.

Example 2.13.2. Consider the Example 2.9.1. Here by routine calculations it can be easily verified that $IFS\ \Delta=(\varpi_\Delta,\xi_\Delta)$ is an $IF_{BCI}II$ of Ω

but it is not an IF_pI of Ω because:

$\varpi_\Delta(\imath) = \delta < 1 = \varpi_\Delta(0) = \min\{\varpi_\Delta((\imath \cdot \ell) \cdot (0 \cdot \ell), \varpi_\Delta(0)\}$.

Corollary 2.13.3. *Any IF_pI of Ω is an $IF_{BCI}PII$ of Ω.*

Proof. It is evident since every $IF_{BCI}II$ is an $IF_{BCI}PII$. □

Corollary 2.13.4. *Any IF_pI of Ω is an $IF_{BCI}CI$ of Ω.*

Proof. It is evident since every $IF_{BCI}II$ is an $IF_{BCI}CI$. □

2.14 Relationship of intuitionistic fuzzy BCI-(implicative, positive implicative, commutative) ideals with intuitionistic fuzzy α-ideals

Theorem 2.14.1. *Any $IF_\alpha I$ of Ω is an $IF_{BCI}II$ of Ω.*

Proof. By Theorem 2.8.1, since every $IF_\alpha I$ of Ω is an IF_pI of Ω and by Theorem 2.13.1, every IF_pI of Ω is an $IF_{BCI}II$ of Ω. Therefore every $IF_\alpha I$ of Ω is an $IF_{BCI}II$ of Ω. □

Whereas converse of this theorem does not hold in general and can be observed by the following example.

Example 2.14.2. Consider the Example 2.9.1. Here by routine calculations it can be easily verified that IFS $\Delta = (\varpi_\Delta, \xi_\Delta)$ is an $IF_{BCI}II$ of Ω but it is not an $IF_\alpha I$ of Ω since,

$\varpi_\Delta(\imath \cdot \ell) = \varpi_\Delta(\jmath) = \delta < 1 = \varpi_\Delta(\ell) = \min\{\varpi_\Delta((\ell \cdot 0) \cdot (0 \cdot \imath)), \varpi_\Delta(0)\}$.

Corollary 2.14.3. *Any* $IF_{\alpha}I$ *of* Ω *is an* $IF_{BCI}PII$ *of* Ω.

Proof. It is evident since every $IF_{BCI}II$ is an $IF_{BCI}PII$. □

Corollary 2.14.4. *Any* $IF_{\alpha}I$ *of* Ω *is an* $IF_{BCI}CI$ *of* Ω.

Proof. It is evident since every $IF_{BCI}II$ is an $IF_{BCI}CI$. □

Chapter 3

Hyperstructure theoretic approaches to ideals in BCK-algebras

In this chapter, basic concepts about hyper BCK-algebras are discussed and notions of (weak, strong, reflexive) hyper (h, p)-ideals, also denoted as $wH_{(h,p)}Is$, $sH_{(h,p)}Is$ and $rH_{(h,p)}Is$ respectively are presented and their relations are conferred. Also notions of fuzzy (weak, strong, reflexive) hyper (h, p)-ideals, denoted as $FwH_{(h,p)}Is$, $FsH_{(h,p)}Is$ and $FrH_{(h,p)}Is$ are familiarized and their relations are discussed. Similarly, the notions of (weak, strong, reflexive) hyper (BCK-commutative, BCK-implicative, BCK-positive implicative) ideals, also denoted as $wH_{BCK}(C, I, PI)Is$, $sH_{BCK}(C, I, PI)Is$ and $rH_{BCK}(C, I, PI)Is$ respectively are presented and detail properties are conferred. Moreover, fuzzy (weak, strong, reflexive) hyper (BCK-commutative, BCK-implicative, BCK-positive implicative) ideals, also written as $FwH_{BCK}(C, I, PI)Is$, $FsH_{BCK}(C, I, PI)Is$ and $FrH_{BCK}(C, I, PI)Is$ are introduced and their relations are discussed. Also their characterizations are conferred using the idea of level subsets. Lastly, relations between $FwH_{BCK}(C, I, PI)Is$, $FsH_{BCK}(C, I, PI)Is$ and

$FrH_{BCK}(C, I, PI)Is$ have been established with the help of different examples.

3.1 Preliminaries

Consider a non-empty set $\digamma$ equipped with a hyper operation "$\circ$", i.e., $\circ$ is a mapping from $\digamma \times \digamma$ to the family $P(\digamma)$ of all non-empty subsets of $\digamma$. For two subset U and V of $\digamma$, the set $\bigcup\{u \circ v \mid u \in U, v \in V\}$ is represented as $U \circ V$. We shall use $\imath \circ \jmath$ instead of $\imath \circ \{\jmath\}$, $\{\imath\} \circ \jmath$ or $\{\imath\} \circ \{\jmath\}$.

Definition 3.1.1. [34] A *"hyper BCK-algebra"* refers to a a non-empty set $\digamma$ with a hyperoperation "$\circ$" and a constant 0 satisfying the axioms:

$(HK1)$ $(\imath \circ \ell) \circ (\jmath \circ \ell) \ll \imath \circ \jmath$

$(HK2)$ $(\imath \circ \jmath) \circ \ell = (\imath \circ \ell) \circ \jmath$

$(HK3)$ $\imath \circ H \ll \{\imath\}$

$(HK4)$ $\imath \ll \jmath$ and $\jmath \ll \imath$ imply $\imath = \jmath$

$\forall\ \imath, \jmath, \ell \in \digamma$, where $\imath \ll \jmath$ is defined by $0 \in \imath \circ \jmath$ and for every $U,\ V \subseteq \digamma$, $U \ll V$ is defined by $\forall\ u \in U$, $\exists\ v \in V$ such that $u \ll v$. The operator "$\ll$" is termed as the "hyper order" in $\digamma$.

Obviously, any BCK-algebra $(G, *, 0)$ becomes a hyper BCK-algebra w.r.t the operation $\imath \circ \jmath = \{\imath * \jmath\}$. Another interesting example of a "hyper BCK-algebra" is the set of all non-negative reals, i.e., $\digamma = [0, \infty)$ with the operation

$$\imath \circ \jmath = \begin{cases} [0, \imath] & if \quad \imath < \jmath, \\ [0, \jmath] & if \quad \imath > \jmath \neq 0, \\ \{\imath\} & if \quad \jmath = 0. \end{cases}$$

Proposition 3.1.2. *[34]* In any "hyper BCK-algebra" $\digamma$, the following axioms are valid:

(i) $\imath \circ 0 = \{\imath\}$

(ii) $\imath \circ \jmath \ll \imath$

(iii) $0 \circ U = \{0\}$

(iv) $U \ll U$

(v) $U \subseteq V$ implies $U \ll V$

(vi) $U \circ \{0\} = \{0\}$ implies $U = \{0\}$

(vii) $0 \ll \imath$

$(viii)$ $0 \circ \imath = \{0\}$

(ix) $0 \circ 0 = \{0\}$

(x) $\jmath \ll \ell$ implies $\imath \circ \ell \ll \imath \circ \jmath$

$\forall\ \imath, \jmath, \ell \in F$ and for all non-empty subsets U and V of F.

Let Υ be a non-empty subset of "hyper BCK-algebra" F and $0 \in \Upsilon$. Then Υ is a *"hyper BCK-subalgebra" (or $H_{BCK}S$)* of F if $\imath \circ \jmath \subseteq \Upsilon$, for any $\imath, \jmath \in \Upsilon$, a *"weak hyper BCK-ideal" (or $wH_{BCK}I$)* of F if $\imath \circ \jmath \subseteq \Upsilon$ and $\jmath \in \Upsilon$ imply $\imath \in \Upsilon$, for all $\imath, \jmath \in F$, a *"hyper BCK-ideal" (or $H_{BCK}I$)* of F if $\imath \circ \jmath \ll \Upsilon$ and $\jmath \in \Upsilon$ imply $\imath \in \Upsilon$, for any $\imath, \jmath \in F$, a *"strong hyper BCK-ideal" (or $sH_{BCK}I$)* of F if $(\imath \circ \jmath) \cap \Upsilon \neq \emptyset$ and $\jmath \in \Upsilon$ imply $\imath \in \Upsilon$, for all $\imath, \jmath \in F$. Υ is termed as *reflexive* if $\imath \circ \imath \subseteq \Upsilon$ for any $\imath \in F$. In the sequel, F will be a "hyper BCK-algebra".

Lemma 3.1.3. *[33, 34]* For any F,

- any "reflexive hyper BCK-ideal" (or $rH_{BCK}I$) of F is a $sH_{BCK}I$ of F.
- any $sH_{BCK}I$ of F is a $H_{BCK}I$ of F.
- any $H_{BCK}I$ of F is a $wH_{BCK}I$ of F.

Lemma 3.1.4. *[33]* Let Υ be a $rH_{BCK}I$ of F. Then $(\imath \circ \jmath) \cap I \neq \emptyset$ implies $\imath \circ \jmath \ll I$, $\forall\ \imath, \jmath \in F$.

Proposition 3.1.5. *[32]* Let U be a subset of F. If Υ is a $H_{BCK}I$ of H such that $U \ll \Upsilon$ then $U \subseteq \Upsilon$.

Definition 3.1.6. A non-empty subset $\Upsilon \subseteq F$ containing 0 is

- a *"weak hyper h-ideal" (or wH_hI)* of F if

$\imath \circ (\jmath \circ \ell) \subseteq \Upsilon$ and $\jmath \in \Upsilon$ imply $(\imath \circ \ell) \subseteq \Upsilon$.

• a *"hyper h-ideal" (or H_hI)* of $\digamma$ if

$\imath \circ (\jmath \circ \ell) \ll \Upsilon$ and $\jmath \in \Upsilon$ imply $(\imath \circ \ell) \subseteq \Upsilon$.

• a *"strong hyper h-ideal" (or sH_hI)* of $\digamma$ if

$(\imath \circ (\jmath \circ \ell)) \bigcap \Upsilon \neq \emptyset$ and $\jmath \in \Upsilon$ imply $(\imath \circ \ell) \subseteq \Upsilon$.

for any $\imath, \jmath, \ell \in \digamma$.

Theorem 3.1.7. Any H_hI (resp. wH_hI, sH_hI, rH_hI) of $\digamma$ is a $H_{BCK}I$ (resp. $wH_{BCK}I$, $sH_{BCK}I$, $rH_{BCK}I$) of $\digamma$.

Proof. Straightforward. □

Theorem 3.1.8. For any $\digamma$,

(i) Any H_hI of $\digamma$ is a wH_hI of $\digamma$.

(ii) Any sH_hI of $\digamma$ is a H_hI of $\digamma$.

(iii) Any "reflexive hyper h-ideal" (or rH_hI) of $\digamma$ is a sH_hI of $\digamma$.

Proof. (i). Let Υ be a H_hI of $\digamma$. For any $\imath, \jmath, \ell \in \digamma$, let $\imath \circ (\jmath \circ \ell) \subseteq \Upsilon$ and $\jmath \in \Upsilon$. Then $\imath \circ (\jmath \circ \ell) \subseteq \Upsilon$ implies $(\imath \circ \ell) \circ (\jmath \circ \ell) \ll \Upsilon$ (by Proposition 3.1.2(v)), which along with $\jmath \in \Upsilon$ implies $\imath \circ \ell \subseteq \Upsilon$. Hence Υ is a wH_hI of $\digamma$.

(ii). Let Υ be a sH_hI of $\digamma$. Let $\imath \circ (\jmath \circ \ell) \ll \Upsilon$ and $\jmath \in \Upsilon$. Then for all $a \in \imath \circ (\jmath \circ \ell)$, $\exists\ b \in \Upsilon$, s.t, $a \ll b$. This implies $0 \in a \circ b$ and thus $a \circ b \cap \Upsilon \neq \emptyset$. By Theorem 3.1.7, Υ is also a $sH_{BCK}I$ of $\digamma$. Therefore $a \circ b \cap \Upsilon \neq \emptyset$ along with $b \in \Upsilon$ implies $a \in \Upsilon$, that is $\imath \circ (\jmath \circ \ell) \subseteq \Upsilon$. Therefore $(\imath \circ (\jmath \circ \ell)) \cap \Upsilon \neq \emptyset$, which along with $\jmath \in \Upsilon$ implies $\imath \circ \ell \subseteq \Upsilon$. Hence Υ is a H_hI of $\digamma$.

(iii). Let Υ be a rH_hI of $\digamma$. For any $\imath, \jmath, \ell \in \digamma$, let $\imath \circ (\jmath \circ \ell) \bigcap \Upsilon \neq \emptyset$ and $\jmath \in \Upsilon$. Being a rH_hI, Υ is also a $rH_{BCK}I$ of $\digamma$ (by Theorem 3.1.7), therefore by Lemma 3.1.4, $\imath \circ (\jmath \circ \ell) \bigcap \Upsilon \neq \emptyset \Rightarrow \imath \circ (\jmath \circ \ell) \ll \Upsilon$, which along with $\jmath \in \Upsilon$ implies $\imath \circ \ell \subseteq \Upsilon$. Hence Υ is a sH_hI of $\digamma$. $\square$

The converse of above theorem isn't valid. It can be observed by the succeeding examples.

Example 3.1.9. Let $\digamma = \{0, \imath, \jmath\}$ be a "hyper BCK-algebra" defined by the succeeding table:

$\circ$	$\{0\}$	$\{\imath\}$	$\{\jmath\}$
0	$\{0\}$	$\{0\}$	$\{0\}$
$\imath$	$\{\imath\}$	$\{0, \imath\}$	$\{0, \imath\}$
$\jmath$	$\{\jmath\}$	$\{\jmath\}$	$\{0, \jmath\}$

Take $\Upsilon = \{0, \jmath\}$. Then Υ is a wH_hI of $\digamma$ but it is not a H_hI of $\digamma$ because $\imath \circ (\jmath \circ \imath) = \{0, \imath\} \ll \Upsilon$ and $\jmath \in \Upsilon$ but $\imath \circ \imath = \{0, \imath\} \nsubseteq \Upsilon$.

Now take $\Upsilon = \{0, \imath\}$. Then Υ is a sH_hI of $\digamma$ but it is not a sH_hI of $\digamma$ because
$\jmath \circ \jmath = \{0, \jmath\} \nsubseteq \Upsilon$.

Example 3.1.10. Let $\digamma = \{0, \imath, \jmath\}$ be a "hyper BCK-algebra" defined by the succeeding table:

$\circ$	$\{0\}$	$\{\imath\}$	$\{\jmath\}$
0	$\{0\}$	$\{0\}$	$\{0\}$
$\imath$	$\{\imath\}$	$\{0\}$	$\{0\}$
$\jmath$	$\{\jmath\}$	$\{\imath, \jmath\}$	$\{0, \imath, \jmath\}$

Take $\Upsilon = \{0, \imath\}$. Then Υ is a H_hI of $\digamma$ but it is not a sH_hI of $\digamma$ because $(\jmath \circ (\imath \circ 0)) \cap \Upsilon = \{\imath, \jmath\} \cap \Upsilon \neq \emptyset$ and $\imath \in \Upsilon$ but $\jmath \circ 0 = \{\jmath\} \nsubseteq \Upsilon$.

Definition 3.1.11. [31] A fuzzy set ϖ in $\digamma$ is

• a *"fuzzy weak hyper BCK-ideal" (or $FwH_{BCK}I$)* of $\digamma$ if

$\varpi(0) \geq \varpi(\imath) \geq \min\{\inf_{u\in\imath\circ\jmath}\ \varpi(u),\ \varpi(\jmath)\}$.

• a *"fuzzy hyper BCK-ideal" (or $FH_{BCK}I$)* of $\digamma$ if

$\imath \ll \jmath$ implies $\varpi(\imath) \geq \varpi(\jmath)$ and $\varpi(\imath) \geq \min\ \{\inf_{u\in\imath\circ\jmath}\ \varpi(u),\ \varpi(\jmath)\}$.

• a *"fuzzy strong hyper BCK-ideal" (or $FsH_{BCK}I$)* of $\digamma$ if

$\inf_{u\in\imath\circ\imath}\ \varpi(u) \geq \varpi(\imath) \geq \min\{\sup_{v\in\imath\circ\jmath}\ \varpi(v),\ \varpi(\jmath)\}$.

• a *"fuzzy reflexive hyper BCK-ideal" (or $FrH_{BCK}I$)* of $\digamma$ if

$\inf_{u\in\imath\circ\imath}\ \varpi(u) \geq \varpi(\jmath)$ and $\varpi(\imath) \geq \min\{\sup_{v\in\imath\circ\jmath}\ \varpi(v),\ \varpi(\jmath)\}$.

for all $\imath, \jmath \in \digamma$.

Theorem 3.1.12. *[31] For any $\digamma$,*

• *Any $FH_{BCK}I$ of $\digamma$ is a $FwH_{BCK}I$ of $\digamma$.*

• *Any $FsH_{BCK}I$ of $\digamma$ is a $FH_{BCK}I$ of $\digamma$.*

• *Any $FrH_{BCK}I$ of $\digamma$ is a $FsH_{BCK}I$ of $\digamma$.*

3.2 Fuzzy hyper h-ideals

Now, we present the notions of "fuzzy weak hyper h-ideals" (or FwH_hIs), "fuzzy hyper h-ideals" (or FH_hIs), "fuzzy strong hyper h-ideals" (or FsH_hIs) and "fuzzy reflexive hyper h-ideals" FrH_hIs and confer some of their properties.

Definition 3.2.1. A fuzzy set ϖ in $\digamma$ is

• a *"fuzzy weak hyper h-ideal" (or FwH_hI)* of $\digamma$ if

$\varpi(0) \geq \varpi(\imath)$ and for all $t \in \imath \circ \ell$,

$\varpi(t) \geq \min\{\inf_{u\in\imath\circ(\jmath\circ\ell)}\ \varpi(u),\ \varpi(\jmath)\}$.

• a *"fuzzy hyper h-ideal" (or FH_hI)* of $\digamma$ if

$\imath \ll \jmath$ implies $\varpi(\imath) \geq \varpi(\jmath)$ and for all $t \in \imath \circ \ell$,

$\varpi(t) \geq \min\{\inf_{u \in \imath \circ (\jmath \circ \ell)} \ \varpi(u), \ \varpi(\jmath)\}$.

• a *"fuzzy strong hyper h-ideal" (or FsH_hI)* of $\digamma$ if

$\inf_{u \in \imath \circ \imath} \ \varpi(u) \geq \varpi(\imath)$ and for all $t \in \imath \circ \ell$,

$\varpi(t) \geq \min\{\sup_{v \in \imath \circ (\jmath \circ \ell)} \ \varpi(v), \ \varpi(\jmath)\}$.

• a *"fuzzy reflexive hyper h-ideal" (or FrH_hI)* of $\digamma$ if

$\inf_{u \in \imath \circ \imath} \ \varpi(u) \geq \varpi(\jmath)$ and for all $t \in \imath \circ \ell$,

$\varpi(t) \geq \min\{\sup_{v \in \imath \circ (\jmath \circ \ell)} \ \varpi(v), \ \varpi(\jmath)\}$.

for any $\imath, \jmath, \ell \in \digamma$

Theorem 3.2.2. Any FH_hI (resp. $FwH_hI, \ FsH_hI, \ FrH_hI$) of $\digamma$ is a $FH_{BCK}I$ (resp. $FwH_{BCK}I, \ FsH_{BCK}I, \ FrH_{BCK}I$) of $\digamma$.

Proof. Straightforward. □

Theorem 3.2.3. For any $\digamma$,

(i). Any FH_hI of $\digamma$ is a FwH_hI of $\digamma$.

(ii). Any FsH_hI of H is a FH_hI of $\digamma$.

(iii). Any FrH_hI of H is a FsH_hI of $\digamma$.

Proof. (i). Let ϖ be a FH_hI of $\digamma$. Since any FH_hI is a $FH_{BCK}I$ (by Theorem 3.2.2) and every $FH_{BCK}I$ is a $FwH_{BCK}I$ (By Theorem 3.1.12), therefore ϖ is also a $FwH_{BCK}I$ of $\digamma$. Hence ϖ satisfies $\varpi(0) \geq \varpi(\imath)$ for all $\imath \in \digamma$. Also being a FH_hI, $\forall \ \imath, \jmath, \ell \in \digamma$ and $t \in \imath \circ \ell$, ϖ satisfies:

$$\varpi(t) \geq \min\{\inf_{u \in \imath \circ (\jmath \circ \ell)} \ \varpi(u), \ \varpi(\jmath)\}.$$

Hence ϖ is a FwH_hI of $\digamma$.

(ii). Let ϖ be a FsH_hI of $\digamma$. Since FsH_hI is a $FsH_{BCK}I$ (by Theorem 3.2.2) and every $FsH_{BCK}I$ is a $FH_{BCK}I$ (by Theorem 3.1.12), therefore ϖ

is also a $FH_{BCK}I$ of $\digamma$. Hence for any $\imath, \jmath \in \digamma$, if $\imath \ll \jmath$ then $\varpi(\imath) \geq \varpi(\jmath)$. Also being a FsH_hI, $\forall\ \imath, \jmath, \ell \in \digamma$ and $t \in \imath \circ \ell$, ϖ satisfies

$$\varpi(t) \geq \min\{\sup_{a \in (\imath \circ (\jmath \circ \ell)} \varpi(a),\ \varpi(\jmath)\}.$$

Since $\sup_{a \in \imath \circ (\jmath \circ \ell)} \varpi(a) \geq \varpi(b)$, for all $b \in \imath \circ (\jmath \circ \ell)$,

Thus, $\varpi(t) \geq \min\{\sup_{a \in \imath \circ (\jmath \circ \ell)} \varpi(a),\ \varpi(\jmath)\} \geq \min\{\varpi(b),\ \varpi(\jmath)\}$, for all $b \in \imath \circ (\jmath \circ \ell)$

Since $\varpi(b) \geq \inf_{c \in \imath \circ (\jmath \circ \ell)} \varpi(c)$ for all $b \in \imath \circ (\jmath \circ \ell)$,

Therefore, $\varpi(t) \geq \min\{\varpi(b),\ \varpi(\jmath)\} \geq \min\{\inf_{c \in \imath \circ (\jmath \circ \ell)} \varpi(c),\ \varpi(\jmath)\}$,

i.e., $\varpi(t) \geq \min\{\inf_{c \in \imath \circ (\jmath \circ \ell)} \varpi(c),\ \varpi(\jmath)\}$.

Hence ϖ is a FH_hI of $\digamma$.

(iii). Let ϖ be a FrH_hI of $\digamma$. Then ϖ satisfies

$\inf_{u \in \imath \circ \imath} \varpi(u) \geq \varpi(\jmath)$, $\forall\ \imath, \jmath \in \digamma$

$\Rightarrow \inf_{u \in \imath \circ \imath} \varpi(u) \geq \varpi(\imath)$, for all $\imath \in \digamma$.

Hence the first condition for ϖ to be a FsH_hI of $\digamma$ is satisfied. Also being a FrH_hI, for all $\imath, \jmath, \ell \in \digamma$ and $t \in \imath \circ \ell$, ϖ satisfies

$\varpi(t) \geq \min\{\sup_{v \in (\imath \circ (\jmath \circ \ell)} \varpi(v),\ \varpi(\jmath)\}$.

Hence ϖ is a FsH_hI of $\digamma$. $\square$

The converse of the above theorem isn't valid. Consider the "hyper BCK-algebra" $\digamma = \{0, \imath, \jmath\}$ delineated by the table given in Example 3.1.9. Define a fuzzy set ϖ in $\digamma$ by:

$$\varpi(0) = \varpi(\jmath) = 1,\ \varpi(\imath) = 0.$$

Then ϖ is a FwH_hI of $\digamma$ but it is not a FH_hI of $\digamma$ because:

$$\imath \leq \jmath \Rightarrow \varpi(\imath) = 0 < 1 = \varpi(\jmath).$$

Example 3.2.4. Let $\digamma = \{0, \imath, \jmath\}$ be a hyper BCK-algebra defined by the succeeding table:

$\circ$	$\{0\}$	$\{\imath\}$	$\{\jmath\}$
0	$\{0\}$	$\{0\}$	$\{0\}$
$\imath$	$\{\imath\}$	$\{0, \imath\}$	$\{0, \imath\}$
$\jmath$	$\{\jmath\}$	$\{\imath, \jmath\}$	$\{0, \imath, \jmath\}$

Define a fuzzy set ϖ in $\digamma$ by:

$$\varpi(0) = \varpi(\imath) = 0.8,\ \varpi(\jmath) = 0.2.$$

Then ϖ is a FH_hI of $\digamma$ but it is not a FsH_hI of $\digamma$ because:

$$\varpi(\jmath) = 0.2 < 0.8 = \min\{\sup_{a \in \jmath \circ (\imath \circ \jmath)} \varpi(a),\ \varpi(\imath)\}.$$

Again consider the hyper BCK-algebra $\digamma = \{0, \imath, \jmath\}$ defined by the table given in Example 3.2.4. Define a fuzzy set ϖ in $\digamma$ by:

$$\varpi(0) = 1,\ \varpi(\imath) = \varpi(\jmath) = \frac{1}{2}$$

Then ϖ is a FsH_hI of $\digamma$ but it is not a FrH_hI of $\digamma$ because for $\imath, \jmath \in \jmath \circ \jmath$,

$$\varpi(\imath) = \varpi(\jmath) = \frac{1}{2} < 1 = \varpi(0).$$

3.3 Product of fuzzy hyper h-ideals

Definition 3.3.1. [9] Let $(\digamma_1, \circ_1, 0_1)$ and $(\digamma_2, \circ_2, 0_2)$ are hyper BCK-algebras and $\digamma = \digamma_1 \times \digamma_2$. We define a hyperoperation "$\circ$" on $\digamma$ by

$$(u_1, v_1) \circ (u_2, v_2) = (u_1 \circ u_2,\ v_1 \circ v_2),$$

for all $(u_1, v_1), (u_2, v_2) \in \digamma$, where for $U \subseteq \digamma_1$ and $V \subseteq \digamma_2$ by (U, V) we mean

$(U, V) = \{(u, v) : u \in U, v \in V\}$ and $0 = (0_1, 0_2)$ and a hyperorder " $\ll$ " on $\digamma$ by

$(u_1, v_1) \ll (u_2, v_2) \Leftrightarrow u_1 \ll u_2$ and $v_1 \ll v_2$.

Thus $(\digamma, \circ, 0)$ is a "hyper BCK-algebra".

Let ϖ and ζ be fuzzy sets in hyper BCK-algebras $\digamma_1$ and $\digamma_2$ respectively. Then $\varpi \times \zeta$, the product of ϖ and ζ of $\digamma = \digamma_1 \times \digamma_2$ is delineated as

$$(\varpi \times \zeta)((\imath, \jmath)) = \min\{\varpi(\imath), \zeta(\jmath)\}.$$

In the sequel, $\digamma_1$ and $\digamma_2$ will bee hyper BCK-algebras and $\digamma = \digamma_1 \times \digamma_2$.

Definition 3.3.2. Let ϖ be a fuzzy set in $\digamma$. Then fuzzy sets ϖ_1 and ϖ_2 on $\digamma_1$ and $\digamma_2$ respectively, are delineated as

$\varpi_1(\imath) = \varpi((\imath, 0)), \quad \varpi_2(\jmath) = \varpi((0, \jmath))$.

Theorem 3.3.3. Let ϖ be a fuzzy set in $\digamma$. Then ϖ is a FH_hI (resp. FwH_hI, FsH_hI, FrH_hI) of $\digamma$ if and only if ϖ_1 and ϖ_2 are FH_hIs (resp. FwH_hIs, FsH_hIs, FrH_hIs) of $\digamma_1$ and $\digamma_2$ respectively.

Proof. Let ϖ be a FH_hI of $\digamma$ and let $\imath_1 \ll \imath_2$ for some $\imath_1, \imath_2 \in \digamma_1$. Then $(\imath_1, 0) \ll (\imath_2, 0)$ which implies $\varpi((\imath_1, 0)) = \varpi_1(\imath_1) \geq \varpi((\imath_2, 0)) = \varpi_1(\imath_2)$, that is, $\varpi_1(\imath_1) \geq \varpi_1(\imath_2)$

Moreover for any $\imath_1, \jmath_1, \ell_1 \in \digamma_1$, let $t = \min\{\inf_{a \in \imath_1 \circ (\jmath_1 \circ \ell_1)} \ \varpi_1(a), \ \varpi_1(\jmath_1)\}$

Then for all $b \in \imath_1 \circ (\jmath_1 \circ \ell_1)$, $\varpi_1(b) \geq \inf_{a \in \imath_1 \circ (\jmath_1 \circ \ell_1)} \ \varpi_1(a) \geq t$ and $\varpi_1(\jmath_1) \geq t$

$\Rightarrow \varpi((b, 0)) \geq t$ and $\varpi((\jmath_1, 0)) \geq t$, for all $(b, 0) \in (\imath_1, 0) \circ ((\jmath_1, 0) \circ (\ell_1, 0))$

$\Rightarrow (b, 0) \in \varpi_t$ and $(\jmath_1, 0) \in \varpi_t$, for all $(b, 0) \in (\imath_1, 0) \circ ((\jmath_1, 0) \circ (\ell_1, 0))$

$\Rightarrow (\imath_1, 0) \circ ((\jmath_1, 0) \circ (\ell_1, 0)) \subseteq \varpi_t$ and $(\jmath_1, 0) \in \varpi_t$.

By Transfer principle for fuzzy sets, $\varpi_t \neq \emptyset$ is a H_hI of $\digamma$ and so is a wH_hI

of $\digamma$ (by Theorem 3.1.8). Thus $(\imath_1, 0)\circ((\jmath_1, 0)\circ(\ell_1, 0)) \subseteq \varpi_t$ and $(\jmath_1, 0) \in \varpi_t$ imply $(\imath_1, 0) \circ (\ell_1, 0) \subseteq \varpi_t$

Therefore, $\varpi((s, 0)) \geq t$, for all $(s, 0) \in (\imath_1, 0) \circ (\ell_1, 0) = (\imath_1 \circ \ell_1,\ 0)$

$\Rightarrow \varpi_1(s) \geq t = \min\{\inf_{a\in \imath_1\circ(\jmath_1\circ\ell_1)}\ \varpi_1(a),\ \varpi_1(\jmath_1)\}$, for all $s \in \imath_1 \circ \ell_1$

Hence ϖ_1 is a FH_hI of $\digamma_1$.

Similarly, we can prove that ϖ_2 is a FH_hI of $\digamma_2$.

Conversely, let ϖ_1 and ϖ_2 be FH_hIs of $\digamma_1$ and $\digamma_2$ respectively.

For any $(\imath, u), (\jmath, v) \in \digamma$, where $\imath, \jmath \in \digamma_1$ and $u, v \in \digamma_2$, let $(\imath, u) \ll (\jmath, v)$

Since $(\imath, u) \ll (\jmath, v) \Leftrightarrow \imath \ll \jmath$ and $u \ll v$

$\Rightarrow \varpi_1(\imath) \geq \varpi_1(\jmath)$ and $\varpi_2(u) \geq \varpi_2(v)$

$\Rightarrow \min\ \{\varpi_1(\imath),\ \varpi_2(u)\} \geq \min\{\varpi_1(\jmath),\ \varpi_2(v)\}$

$\Rightarrow (\varpi_1 \times \varpi_2)((\imath, u)) \geq (\varpi_1 \times \varpi_2)((\jmath, v))$

$\Rightarrow \varpi((\imath, u)) \geq \varpi((\jmath, v))$

Thus $(\imath, u) \ll (\jmath, v) \Rightarrow \varpi((\imath, u)) \geq \varpi((\jmath, v))$.

Moreover for any $(\imath, u),\ (\jmath, v),\ (\ell, w) \in \digamma$, where $\imath, \jmath, \ell \in \digamma_1$ and $u, v, w \in \digamma_2$ and for all $(a, b) \in (\imath, u) \circ (\ell, w) = (\imath \circ \ell,\ u \circ w)$,

$\varpi((a, b)) = (\varpi_1 \times \varpi_2)((a, b)) = \min\{\varpi_1(a),\ \varpi_2(b)\}$

$\geq \min[\min\{\inf_{c\in\imath\circ(\jmath\circ\ell)}\ \varpi_1(c),\ \varpi_1(\jmath)\},\ \min\{\inf_{d\in u\circ(v\circ w)}\ \varpi_2(d),\ \varpi_2(v)\}]$

$= \min[\min\{\inf_{c\in\imath\circ(\jmath\circ\ell)}\ \varpi_1(c),\ \inf_{d\in u\circ(v\circ w)}\ \varpi_2(d)\},\ \min\{\ \varpi_1(\jmath),\ \varpi_2(v)\}]$

$= \min[\inf_{c\in\imath\circ(\jmath\circ\ell),\ d\in u\circ(v\circ w)}\ \{\min\{\varpi_1(c),\ \varpi_2(d)\}\},\ \min\{\ \varpi_1(\jmath),\ \varpi_2(v)\}]$

$= \min\{\inf_{(c,d)\in(\imath\circ(\jmath\circ\ell),\ u\circ(v\circ w))}\ (\varpi_1 \times \varpi_2) \min d)),\ (\varpi_1 \times \varpi_2)((\jmath, v))\}$

$= \min\{\inf_{(c,d)\in(\imath\circ(\jmath\circ\ell),\ u\circ(v\circ w))}\ \varpi((c, d)),\ \varpi((\jmath, v))\}$

$\Rightarrow \varpi((a, b)) \geq \min\{\inf_{(c,d)\in(\imath,u)\circ((\jmath,v)\circ(\ell,w))}\ \varpi((c, d)),\ \varpi((\jmath, v))\}$

Hence, ϖ is a FH_hI of H. $\square$

3.4 Fuzzy hyper p-ideals

Now we present the idea of (fuzzy) (weak, strong) hyper p-ideals and confer associated properties.

Definition 3.4.1. A non-empty subset $\Upsilon \subseteq \digamma$ containing 0 is

• a "weak hyper p-ideal" (or wH_pI) of $\digamma$ if

$(\imath \circ \ell) \circ (\jmath \circ \ell) \subseteq \Upsilon$ and $\jmath \in \Upsilon$ imply $\imath \in \Upsilon$.

• a "hyper p-ideal" (or H_pI) of $\digamma$ if

$(\imath \circ \ell) \circ (\jmath \circ \ell) \ll \Upsilon$ and $\jmath \in \Upsilon$ imply $\imath \in \Upsilon$.

• a "strong hyper p-ideal" (or sH_pI) of $\digamma$ if

$((\imath \circ \ell) \circ (\jmath \circ \ell)) \bigcap \Upsilon \neq \emptyset$ and $\jmath \in \Upsilon$ imply $\imath \in \Upsilon$.

for any $\imath, \jmath, \ell \in \digamma$.

Theorem 3.4.2. Any H_pI (resp. wH_pI, sH_pI) of $\digamma$ is a $H_{BCK}I$ (resp. $wH_{BCK}I$, $sH_{BCK}I$) of $\digamma$.

Proof. Straightforward. □

Theorem 3.4.3. For any $\digamma$,

(i) Any H_pI of $\digamma$ is a wH_pI of $\digamma$.

(ii) Any sH_pI of $\digamma$ is a H_pI of $\digamma$.

Proof. (i) Let Υ be a H_pI of $\digamma$.

Let, $(\imath \circ \ell) \circ (\jmath \circ \ell) \subseteq \Upsilon$ and $\jmath \in \Upsilon$. Then, $(\imath \circ \ell) \circ (\jmath \circ \ell) \subseteq \Upsilon$ implies $(\imath \circ \ell) \circ (\jmath \circ \ell) \ll \Upsilon$ (by Proposition 3.1.2(v)), which along with $\jmath \in \Upsilon$ implies $\imath \in \Upsilon$. Hence Υ is a wH_pI of $\digamma$.

(ii) Let Υ be a sH_pI of $\digamma$. Let, $(\imath \circ \ell) \circ (\jmath \circ \ell) \ll \Upsilon$ and $\jmath \in \Upsilon$. Then, $\forall\ \alpha \in (\imath \circ \ell) \circ (\jmath \circ \ell)$, $\exists\ \beta \in \Upsilon$ such that $\alpha \ll \beta$. Thus $0 \in \alpha \circ \beta$ and $(\alpha \circ \beta) \cap \Upsilon \neq \emptyset$, which along with $\beta \in \Upsilon$ implies $\alpha \in \Upsilon$, i.e., $(\imath \circ \ell) \circ (\jmath \circ \ell) \subseteq \Upsilon$.

Thus $(\imath \circ \ell) \circ (\jmath \circ \ell) \cap \Upsilon \neq \emptyset$, which along with $\jmath \in \Upsilon$ implies $\imath \in \Upsilon$. Hence Υ is a H_pI of $\digamma$.

□

Generally, the converse of above theorem doesn't hold. It can be observed by the succeeding examples:

Example 3.4.4. Let $\digamma = \{0, \imath, \jmath\}$. We Contemplate the succeeding table:

$\circ$	0	$\imath$	$\jmath$
0	$\{0\}$	$\{0\}$	$\{0\}$
$\imath$	$\{\imath\}$	$\{0, \imath\}$	$\{0, \imath\}$
$\jmath$	$\{\jmath\}$	$\{\jmath\}$	$\{0, \imath, \jmath\}$

Then $\digamma$ is a "hyper BCK-algebra". Take $\Upsilon = \{0, \jmath\}$. Clearly, Υ is a wH_pI of $\digamma$. But for $(\imath \circ \imath) \circ (0 \circ \imath) = \{0, \imath\} \ll \Upsilon$ and $0 \in \Upsilon$, $\imath \notin \Upsilon$, so Υ isn't a H_pI of $\digamma$.

Example 3.4.5. Cogitate the "hyper BCK-algebra" $\digamma = \{0, \imath, \jmath\}$ delineated in Example 3.2.4. Take $\Upsilon = \{0, \imath\}$. Clearly, Υ is a H_pI but not a H_pI of $\digamma$ as, $(\jmath \circ 0) \circ (\imath \circ 0) \cap \Upsilon = \{\imath, \jmath\} \cap \Upsilon \neq \emptyset$ and $\imath \in \Upsilon$ but $\jmath \notin \Upsilon$.

Definition 3.4.6. A fuzzy set ϖ in $\digamma$ is

• a "fuzzy weak hyper p-ideal" (or FwH_pI) of $\digamma$ if

$\varpi(0) \geq \varpi(\imath) \geq \min\{\inf_{x \in (\imath \circ \ell) \circ (\jmath \circ \ell)} \ \varpi(x),\ \varpi(\jmath)\}$

• a "fuzzy hyper p-ideal" (or FH_pI) of $\digamma$ if,

$\imath \ll \jmath$ implies $\varpi(\imath) \geq \varpi(\jmath)$ and

$\varpi(\imath) \geq \min\{\inf_{x \in (\imath \circ \ell) \circ (\jmath \circ \ell)} \ \varpi(x),\ \varpi(\jmath)\}$

• a "fuzzy strong hyper p-ideal" (or FsH_pI) of $\digamma$ if

$\inf_{x \in \imath \circ \imath} \ \varpi(x) \geq \varpi(\imath) \geq \min\{\sup_{y \in (\imath \circ \ell) \circ (\jmath \circ \ell)} \ \varpi(y),\ \varpi(\jmath)\}$

for any $\imath, \jmath, \ell \in \digamma$.

Theorem 3.4.7. Any FH_pI (resp. FwH_pI, FsH_pI) of $\digamma$ is a $FH_{BCK}I$ (resp. $FwH_{BCK}I$, $FsH_{BCK}I$) of $\digamma$.

Proof. Straightforward. □

Theorem 3.4.8. For any $\digamma$,

(i). Any FH_pI of $\digamma$ is a FwH_pI of $\digamma$.

(ii). Any FsH_pI of H is a FH_pI of $\digamma$.

Proof. (i). Let ϖ be a FH_pI of $\digamma$. Since, any FH_pI is a $FH_{BCK}I$ (by Theorem 3.4.7) and every $FH_{BCK}I$ is a $FwH_{BCK}I$ (by Theorem 3.1.12), therefore ϖ is also a $FwH_{BCK}I$ of $\digamma$. Hence ϖ satisfies $\varpi(0) \geq \varpi(\imath)$, for all $\imath \in \digamma$. Also being a FH_pI ϖ satisfies:

$\varpi(\imath) \geq \min\{\inf_{x\in(\imath\circ\ell)\circ(\jmath\circ\ell)} \ \varpi(x),\ \varpi(\jmath)\}, \forall\ \imath, \jmath, \ell \in \digamma$.

Hence ϖ is a FwH_pI of $\digamma$.

(ii). Let ϖ be a FsH_pI of $\digamma$. Since, any FsH_pI is a $FsH_{BCK}I$ (by Theorem 3.4.7) and every $FsH_{BCK}I$ is a $FH_{BCK}I$ (by Theorem 3.1.12), therefore ϖ is also a $FH_{BCK}I$ of $\digamma$. Hence for any $\imath, \jmath \in \digamma$, if $\imath \ll \jmath$ then $\varpi(\imath) \geq \varpi(\jmath)$.

Also being a FsH_pI, ϖ satisfies for any $\imath, \jmath, \ell \in \digamma$

$\varpi(\imath) \geq \min\{\sup_{x\in(\imath\circ\ell)\circ(\jmath\circ\ell)} \ \varpi(x),\ \varpi(\jmath)\}$

Since $\sup_{x\in(\imath\circ\ell)\circ(\jmath\circ\ell)} \ \varpi(x) \geq \varpi(y), \forall\ y \in (\imath \circ \ell) \circ (\jmath \circ \ell)$,

Thus, $\varpi(\imath) \geq \min\{\sup_{x\in(\imath\circ\ell)\circ(\jmath\circ\ell)} \ \varpi(x),\ \varpi(\jmath)\} \geq \min\{\varpi(y),\ \varpi(\jmath)\}$,

for all $y \in (\imath \circ \ell) \circ (\jmath \circ \ell)$.

Since $\varpi(y) \geq \inf_{z\in(\imath\circ\ell)\circ(\jmath\circ\ell)} \ \varpi(\ell), \forall\ y \in (\imath \circ \ell) \circ (\jmath \circ \ell)$,

Therefore, $\varpi(\imath) \geq \min\{\varpi(y),\ \varpi(\jmath)\} \geq \min\{\inf_{z\in(\imath\circ\ell)\circ(\jmath\circ\ell)} \ \varpi(z),\ \varpi(\jmath)\}$,

i.e., $\varpi(\imath) \geq \min\{\inf_{z\in(\imath\circ\ell)\circ(\jmath\circ\ell)} \ \varpi(z),\ \varpi(\jmath)\}$.

Hence proved. □

Generally, the converse of above theorem doesn't hold. Consider the "hyper BCK-algebra" $\digamma = \{0, \imath, \jmath\}$ defined by the table given in Example 3.4.4. Define a fuzzy set ϖ in $\digamma$ by:

$$\varpi(0) = 0.9,\ \varpi(\imath) = 0.6,\ \varpi(\jmath) = 0.8.$$

Then ϖ is a FwH_pI but not a FH_pI of $\digamma$ as:

$$\imath \leq \jmath \Rightarrow \varpi(\imath) = 0.6 < 0.8 = \varpi(\jmath).$$

Example 3.4.9. Consider a "hyper BCK-algebra" $\digamma = \{0, \imath, \jmath\}$ delineated by the succeeding table:

$\circ$	$\{0\}$	$\{\imath\}$	$\{\jmath\}$
0	$\{0\}$	$\{0\}$	$\{0\}$
$\imath$	$\{\imath\}$	$\{0, \imath\}$	$\{\imath\}$
$\jmath$	$\{\jmath\}$	$\{\jmath\}$	$\{0, \jmath\}$

Define a fuzzy set ϖ in $\digamma$ by:

$$\varpi(0) = \varpi(\imath) = 0.7,\ \varpi(\jmath) = \frac{1}{2}$$

Then ϖ is a FH_pI but it is not a FsH_pI of $\digamma$ as:

$$\varpi(\jmath) = \frac{1}{2} < 0.7 = \min\{\sup_{x \in (\jmath \circ \jmath) \circ (\imath \circ \jmath)} \varpi(x),\ \varpi(\imath)\}.$$

If ϖ is a fuzzy set in $\digamma$ then the "strongest fuzzy relation" on $\digamma$ is a fuzzy relation on ϖ, denoted by "χ_ϖ" and defined as:

$$\chi_\varpi(\imath, \jmath) = \min\{\varpi(\imath), \varpi(\jmath)\},\ \forall\ \imath, \jmath \in \digamma.$$

Theorem 3.4.10. *Let ϖ be a fuzzy set and let χ_ϖ be the "strongest fuzzy relation" on $\digamma$. Then ϖ is a FsH_pI (resp. $FH_pI,\ FwH_pI$) of $\digamma \iff \chi_\varpi$ is a FsH_pI (resp. $FH_pI,\ FwH_pI$) of $\digamma \times \digamma$.*

Proof. Let ϖ be a FsH_pI of F. Consider

$\inf_{(x,y)\in(\imath_1,\imath_2)\circ(\imath_1,\imath_2)}\ \chi_\varpi(x,y) = \inf_{(x,y)\in(\imath_1\circ\imath_1,\imath_2\circ\imath_2)}\ [\min\{\varpi(x),\ \varpi(y)\}]$

$= \min\{\inf_{x\in\imath_1\circ\imath_1}\ \varpi(x),\ \inf_{y\in\imath_2\circ\imath_2}\ \varpi(y)\} \geq \min\{\varpi(\imath_1),\ \varpi(\imath_2)\} = \chi_\varpi(\imath_1,\imath_2)$

$\Rightarrow \inf_{(x,y)\in(\imath_1,\imath_2)\circ(\imath_1,\imath_2)}\ \chi_\varpi(x,y) \geq \chi_\varpi(\imath_1,\imath_2),\ \forall\ (\imath_1,\imath_2)\in F\times F$

Now, for any $(\imath_1,\imath_2),\ (\jmath_1,\jmath_2),\ (\ell_1,\ell_2)$ in $F\times F$,

$\chi_\varpi(\imath_1,\imath_2) = \min\{\varpi(\imath_1),\ \varpi(\imath_2)\} \geq$

$\min[\min\{\sup_{z\in(\imath_1\circ\ell_1)\circ(\jmath_1\circ\ell_1)}\ \varpi(z),\ \varpi(\jmath_1)\},\ \min\{\sup_{d\in(\imath_2\circ\ell_2)\circ(\jmath_2\circ\ell_2)}\ \varpi(d),\varpi(\jmath_2)\}]$

$= \min[\min\{\sup_{z\in(\imath_1\circ\ell_1)\circ(\jmath_1\circ\ell_1)}\ \varpi(z),\ \sup_{d\in(\imath_2\circ\ell_2)\circ(\jmath_2\circ\ell_2)}\ \varpi(d)\},\ \min\ \{\varpi(\jmath_1),\ \varpi(\jmath_2)\}]$

$= \min\ [\min\ \{\sup\ (\varpi(z),\ \varpi(d))\},\ \chi_\varpi(\jmath_1,\jmath_2)],$

where $z\in(\imath_1\circ\ell_1)\circ(\jmath_1\circ\ell_1)$ and $d\in(\imath_2\circ\ell_2)\circ(\jmath_2\circ\ell_2)$

$\Rightarrow \chi_\varpi(\imath_1,\imath_2) \geq \min\ [\sup\{\min(\varpi(z),\varpi(d))\},\ \chi_\varpi(\jmath_1,\jmath_2)],$

where $z\in(\imath_1\circ\ell_1)\circ(\jmath_1\circ\ell_1)$, $d\in(\imath_2\circ\ell_2)\circ(\jmath_2\circ\ell_2)$

$\Rightarrow \chi_\varpi(\imath_1,\imath_2) \geq \min\ \{\sup\ \chi_\varpi(z,d),\ \chi_\varpi(\jmath_1,\jmath_2)\},$

where $(z,d)\in((\imath_1\circ\ell_1)\circ(\jmath_1\circ\ell_1),\ (\imath_2\circ\ell_2)\circ(\jmath_2\circ\ell_2))$

$= ((\imath_1,\imath_2)\circ(\ell_1,\ell_2))\circ((\jmath_1,\jmath_2)\circ(\ell_1,\ell_2))$

Hence, χ_ϖ is a FsH_pI of $F\times F$.

Conversely, let χ_ϖ is a FsH_pI of $F\times F$.

Then $\inf_{(x,y)\in(\imath_1,\imath_2)\circ(\imath_1,\imath_2)}\ \chi_\varpi(x,y) \geq \chi_\varpi(\imath_1,\imath_2),\ \forall\ (\imath_1,\imath_2)\in F\times F$

$\Rightarrow \inf_{(x,y)\in(\imath_1\circ\imath_1,\imath_2\circ\imath_2)}\ [\min\{\varpi(x),\ \varpi(y)\}] \geq \min\{\varpi(\imath_1),\ \varpi(\imath_2)\}$

$\Rightarrow \min\{\inf_{x\in\imath_1\circ\imath_1}\ \varpi(x),\ \inf_{y\in\imath_2\circ\imath_2}\ \varpi(y)\} \geq \min\{\varpi(\imath_1),\ \varpi(\imath_2)\}$

$\Rightarrow \{\inf_{x\in\imath_1\circ\imath_1}\ \varpi(x),\ \inf_{y\in\imath_2\circ\imath_2}\ \varpi(y)\} \geq \{\varpi(\imath_1),\ \varpi(\imath_2)\}$

$\Leftrightarrow \inf_{x\in\imath_1\circ\imath_1}\ \varpi(x) \geq \varpi(\imath_1)$ and $\inf_{y\in\imath_2\circ\imath_2}\ \varpi(y) \geq \varpi(\imath_2),\ \forall\ \imath_1,\imath_2\in F$.

Hence the first condition for ϖ to be a FsH_pI is satisfied.

Note that being a FsH_pI of $F\times F$, χ_ϖ is also a FwH_pI of $F\times F$ (by Theorem 3.4.8), thus χ_ϖ satisfies:

$\chi_{\varpi}(0,0) \geq \chi_{\varpi}(\imath,\imath)$, $\forall$ $(0,0)$, $(\imath,\imath) \in \digamma \times \digamma$

$\Rightarrow \min\{\varpi(0),\ \varpi(0)\} \geq \min\{\varpi(\imath),\ \varpi(\imath)\}$

$\Rightarrow \varpi(0) \geq \varpi(\imath)$, $\forall\ \imath \in \digamma$

Now, for any, $(\imath_1, \imath_2)$, $(\jmath_1, \jmath_2)$, (ℓ_1, ℓ_2) in $\digamma \times \digamma$, χ_{ϖ} satisfies:

$\Rightarrow \chi_{\varpi}(\imath_1, \imath_2) \geq \min\{\sup\ \chi_{\varpi}(e,f),\ \chi_{\varpi}(\jmath_1, \jmath_2)\}$

where, $(e,f) \in ((\imath_1, \imath_2) \circ (\ell_1, \ell_2)) \circ ((\jmath_1, \jmath_2) \circ (\ell_1, \ell_2))$

$= ((\imath_1 \circ \ell_1) \circ (\jmath_1 \circ \ell_1),\ (\imath_2 \circ \ell_2) \circ (\jmath_2 \circ \ell_2))$

$\Rightarrow \min\{\varpi(\imath_1),\ \varpi(\imath_2)\} \geq \min[\sup\{\min\{\varpi(e),\ \varpi(f)\}\},\ \min\{\varpi(\jmath_1),\ \varpi(\jmath_2)\}]$,

where, $(e,f) \in ((\imath_1 \circ \ell_1) \circ (\jmath_1 \circ \ell_1),\ (\imath_2 \circ \ell_2) \circ (\jmath_2 \circ \ell_2))$

Substituting $\imath_1 = \jmath_1 = \ell_1 = 0$,

$\Rightarrow \min\{\varpi(0),\ \varpi(\imath_2)\} \geq \min[\sup\{\min\{\varpi(0),\ \varpi(f)\}\}, \min\{\varpi(0),\ \varpi(\jmath_2)\}]$

Where, $(e,f) \in (0, (\imath_2 \circ \ell_2) \circ (\jmath_2 \circ \ell_2))$

$\Rightarrow \varpi(\imath_2) \geq \min\{\sup_{f \in (\imath_2 \circ \ell_2) \circ (\jmath_2 \circ \ell_2))}\ \varpi(f),\ \varpi(\jmath_2)\}$,

since $\varpi(0) \geq \varpi(\imath)$, $\forall\ \imath \in \digamma$.

Similarly by putting $\imath_2 = \jmath_2 = \ell_2 = 0$, we get,

$\Rightarrow \varpi(\imath_1) \geq \min\{\sup_{e \in (\imath_1 \circ \ell_1) \circ (\jmath_1 \circ \ell_1))}\ \varpi(e),\ \varpi(\jmath_1)\}$

Hence ϖ is a FsH_pI of $\digamma$. □

Identically, as done above, we can corroborate the statement for the other two cases.

Theorem 3.4.11. *A fuzzy set $\varpi = \varpi_1 \times \varpi_2$ is a FH_pI (resp. FwH_pI, FsH_pI) of $\digamma = \digamma_1 \times \digamma_2$ if and only if ϖ_1 and ϖ_2 are FH_pIs (resp. FwH_pIs, FsH_pIs) of $\digamma_1$ and $\digamma_2$ respectively.*

Proof. Let $\varpi = \varpi_1 \times \varpi_2$ be a FH_pI of $\digamma = \digamma_1 \times \digamma_2$ and let $\imath_1 \ll \imath_2$ for some $\imath_1, \imath_2 \in \digamma_1$. Then $(\imath_1, 0) \ll (\imath_2, 0)$ which implies

$\varpi((\imath_1, 0)) = \varpi_1(\imath_1) \geq \varpi((\imath_2, 0)) = \varpi_1(\imath_2)$, that is, $\varpi_1(\imath_1) \geq \varpi_1(\imath_2)$

Moreover, for any $\imath_1, \jmath_1, \ell_1 \in \digamma_1$, let $\delta = \min\{\inf_{a\in(\imath_1\circ\ell_1)\circ(\jmath_1\circ\ell_1)} \varpi_1(a),\ \varpi_1(\jmath_1)\}$

Then, $\forall\ b \in (\imath_1 \circ \ell_1) \circ (\jmath_1 \circ \ell_1),\ \varpi_1(b) \geq \inf_{a\in(\imath_1\circ\ell_1)\circ(\jmath_1\circ\ell_1)} \varpi_1(a) \geq \delta$ and $\varpi_1(\jmath_1) \geq \delta$

$\Rightarrow \varpi((b,0)) \geq \delta$ and $\varpi((\jmath_1,0)) \geq \delta$,

$\forall\ (b,0) \in ((\imath_1,0) \circ (\ell_1,0)) \circ ((\jmath_1,0) \circ (\ell_1,0))$

$\Rightarrow (b,0) \in \varpi_\delta$ and $(\jmath_1,0) \in \varpi_\delta$,

$\Rightarrow ((\imath_1,0) \circ (\ell_1,0)) \circ ((\jmath_1,0) \circ (\ell_1,0)) \subseteq \varpi_\delta$ and $(\jmath_1,0) \in \varpi_\delta$

$\Rightarrow ((\imath_1,0) \circ (\ell_1,0)) \circ ((\jmath_1,0) \circ (\ell_1,0)) \ll \varpi_\delta$ and $(\jmath_1,0) \in \varpi_\delta$

$\Rightarrow (\imath_1,0) \in \varpi_\delta$, since ϖ_δ is a H_pI (by transfer principle for fuzzy sets).

Therefore, $\varpi((\imath_1,0)) \geq \delta$.

Thus, $\varpi_1(\imath_1) \geq \delta = \min\{\inf_{a\in(\imath_1\circ\ell_1)\circ(\jmath_1\circ\ell_1)} \varpi_1(a),\ \varpi_1(\jmath_1)\}$, which is our required condition.

Likewise, it can be proved that, ϖ_2 is a FH_pI of $\digamma_2$.

Conversely, let ϖ_1 and ϖ_2 be two FH_pIs of $\digamma_1$ and $\digamma_2$" respectively.

For any $(\imath,l),(\jmath,m) \in \digamma = \digamma_1 \times \digamma_2$, where $i,\jmath \in \digamma_1$ and $l,m \in \digamma_2$, let $(\imath,l) \ll (\jmath,m)$.

Since $(\imath,l) \ll (\jmath,m)$ imply $\imath \ll \jmath$ and $l \ll m$

$\Rightarrow \varpi_1(\imath) \geq \varpi_1(\jmath)$ and $\varpi_2(l) \geq \varpi_2(m)$

$\Rightarrow \min\ \{\varpi_1(\imath),\ \varpi_2(l)\} \geq \min\{\varpi_1(\jmath),\ \varpi_2(m)\}$

$\Rightarrow (\varpi_1 \times \varpi_2)((\imath,l)) \geq (\varpi_1 \times \varpi_2)((\jmath,m))$

$\Rightarrow \varpi((\imath,l)) \geq \varpi((\jmath,m))$

Thus, $(\imath,l) \ll (\jmath,m) \Rightarrow \varpi((\imath,l)) \geq \varpi((\jmath,m))$

Moreover, for any $(\imath,l),\ (\jmath,m),\ (\ell,n) \in \digamma$, s.t, $\imath,\jmath,\ell \in \digamma_1$ and $l,m,n \in \digamma_2$,

$\varpi((\imath,l)) = (\varpi_1 \times \varpi_2)((\imath,l)) = \min\{\varpi_1(\imath),\ \varpi_2(l)\}$

$\geq \min[\min\{\inf_{c\in(\imath\circ\ell)\circ(\jmath\circ\ell)} \varpi_1(c),\ \varpi_1(\jmath)\},\ \min\{\inf_{d\in(l\circ n)\circ(m\circ n)} \varpi_2(d),\ \varpi_2(m)\}]$

$= \min[\min\{\inf_{c\in(\imath\circ\ell)\circ(\jmath\circ\ell)} \varpi_1(c),\ \inf_{d\in(l\circ n)\circ(m\circ n)} \varpi_2(d)\},\ \min\{\ \varpi_1(\jmath),\ \varpi_2(m)\}]$

$= \min[\inf_{c\in(\imath\circ\ell)\circ(\jmath\circ\ell),\ d\in(l\circ n)\circ(m\circ n)}\ \{\min\{\varpi_1(c),\ \varpi_2(d)\}\},\ \min\{\ \varpi_1(\jmath),\ \varpi_2(m)\}]$

$= \min\{\inf_{(c,d)\in((\imath\circ\ell)\circ(\jmath\circ\ell),\ (l\circ n)\circ(m\circ n))}\ (\varpi_1 \times \varpi_2)(c,d),\ (\varpi_1 \times \varpi_2)((\jmath,m))\}$

$= \min\{\inf_{(c,d)\in((\imath\circ\ell)\circ(\jmath\circ\ell),\ (l\circ n)\circ(m\circ n))}\ \varpi((c,d)),\ \varpi((\jmath,m))\}$

$\Rightarrow \varpi((\imath,l)) \geq \min\{\inf_{(c,d)\in(((\imath,l)\circ(\ell,n))\circ((\jmath,m)\circ(\ell,n)))}\ \varpi((c,d)),\ \varpi((\jmath,m))\}$

Hence proved. □

3.5 Fuzzy hyper BCK-commutative ideals.

Here, we present the idea of (fuzzy) (weak, strong, reflexive) "hyper BCK-commutative ideals" and elaborate apposite properties.

Definition 3.5.1. A non-empty subset $\Upsilon \subseteq \digamma$ containing 0 is

• a "weak hyper BCK-commutative ideal" (or $wH_{BCK}CI$) of $\digamma$ if

$(\imath \circ \jmath) \circ \ell \subseteq \Upsilon$ and $\ell \in \Upsilon$ imply $\imath \circ (\jmath \circ (\jmath \circ \imath)) \subseteq \Upsilon$.

• a "hyper BCK-commutative ideal" (or $H_{BCK}CI$) of $\digamma$ if

$(\imath \circ \jmath) \circ \ell \ll \Upsilon$ and $\ell \in \Upsilon$ imply $\imath \circ (\jmath \circ (\jmath \circ \imath)) \subseteq \Upsilon$.

• a "strong hyper BCK-commutative ideal" (or $sH_{BCK}CI$) of $\digamma$ if

$((\imath \circ \jmath) \circ \ell) \cap \Upsilon \neq \emptyset$ and $\ell \in \Upsilon$ imply $\imath \circ (\jmath \circ (\jmath \circ \imath)) \subseteq \Upsilon$.

for any $\imath, \jmath, \ell \in \digamma$.

Theorem 3.5.2. Any $H_{BCK}CI$ (resp. $wH_{BCK}CI,\ sH_hCI,\ rH_{BCK}CI$) of $\digamma$ is a $H_{BCK}I$ (resp. $wH_{BCK}I,\ sH_{BCK}I,\ rH_{BCK}I$) of $\digamma$.

Proof. Straightforward. □

Theorem 3.5.3. For any $\digamma$,

(i). Any $H_{BCK}CI$ of $\digamma$ is a $wH_{BCK}CI$ of $\digamma$.

(ii). Any $sH_{BCK}CI$ of $\digamma$ is a $H_{BCK}CI$ of $\digamma$.

(iii). Any $rH_{BCK}CI$ of $\digamma$ is a $sH_{BCK}CI$ of $\digamma$.

Proof. (i). Let I be a $H_{BCK}CI$ of $\digamma$.

For any $\imath, \jmath, \ell \in \digamma$, let $(\imath \circ \jmath) \circ \ell \subseteq \Upsilon$ and $\ell \in \Upsilon$. Then $(\imath \circ \jmath) \circ \ell \subseteq \Upsilon$ implies $(\imath \circ \jmath) \circ \ell \ll \Upsilon$ (by Proposition 3.1.2(v)), which along with $\ell \in \Upsilon$ implies $\imath \circ (\jmath \circ (\jmath \circ \imath)) \subseteq \Upsilon$. Hence Υ is a $wH_{BCK}CI$ of $\digamma$.

(ii). Let Υ be a $sH_{BCK}CI$ of $\digamma$. Let $(\imath \circ \jmath) \circ \ell \ll \Upsilon$ and $\ell \in \Upsilon$. Then for all $a \in (\imath \circ \jmath) \circ \ell$, $\exists\ b \in \Upsilon$ such that $a \ll b$. This implies $0 \in a \circ b$ and thus $(a \circ b) \cap \Upsilon \neq \emptyset$. By Theorem 3.5.2, Υ is also a $sH_{BCK}I$ of $\digamma$, therefore $(a \circ b) \cap \Upsilon \neq \emptyset$ along with $b \in \Upsilon$ implies $a \in \Upsilon$, i.e., $(\imath \circ \jmath) \circ \ell \subseteq \Upsilon$. Therefore $((\imath \circ \jmath) \circ \ell) \cap \Upsilon \neq \emptyset$, which along with $\ell \in \Upsilon$ implies $\imath \circ (\jmath \circ (\jmath \circ \imath)) \subseteq \Upsilon$. Hence Υ is a $H_{BCK}CI$ of $\digamma$.

(iii). Let Υ be a $rH_{BCK}CI$ of $\digamma$. For any $\imath, \jmath, \ell \in \digamma$, let $((\imath \circ \jmath) \circ \ell) \cap \Upsilon \neq \emptyset$ and $\ell \in \Upsilon$. Being a $rH_{BCK}CI$, Υ is also a $rH_{BCK}I$ of $\digamma$ (by Theorem 3.5.2), therefore by Lemma 3.1.4, $((\imath \circ \jmath) \circ \ell)\ \cap\ \Upsilon \neq \emptyset \Rightarrow (\imath \circ \jmath) \circ \ell \ll \Upsilon$, which along with $\ell \in \Upsilon$ implies $\imath \circ (\jmath \circ (\jmath \circ \imath)) \subseteq \Upsilon$. Hence Υ is a $sH_{BCK}CI$ of $\digamma$. □

The converse of above theorem isn't valid. It can be observed by the succeeding examples:

Example 3.5.4. Consider the "hyper BCK-algebra" $\digamma = \{0, \imath, \jmath\}$ defined by the table given in Example 3.1.9. Take $\Upsilon = \{0, \jmath\}$. Then Υ is a $wH_{BCK}CI$ of $\digamma$ but it is not a $H_{BCK}CI$ of $\digamma$ because

$(\imath \circ \jmath) \circ 0 = \{0, \imath\} \ll \Upsilon$ and $0 \in \Upsilon$ but $\imath \circ (\jmath \circ (\jmath \circ \imath)) = \{0, \imath\} \nsubseteq \Upsilon$.

Example 3.5.5. Consider the "hyper BCK-algebra" $\digamma = \{0, \imath, \jmath\}$ defined by the table given in Example 3.2.4. Take $\Upsilon = \{0, \imath\}$. Then Υ is a $H_{BCK}CI$ of $\digamma$ but it is not a $sH_{BCK}CI$ of $\digamma$ because

$((\jmath \circ \imath) \circ \imath) \cap \Upsilon = \{0, \imath, \jmath\} \cap \Upsilon \neq \emptyset$ and $\imath \in I$ but $\jmath \circ (\imath \circ (\imath \circ \jmath)) = \{\imath, \jmath\} \nsubseteq \Upsilon$.

Definition 3.5.6. A fuzzy set ϖ in $\digamma$ is

• a "fuzzy weak hyper BCK-commutative ideal" (or $FwH_{BCK}CI$) of $\digamma$ if $\varpi(0) \geq \varpi(\imath)$ and for all $t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$,
$\varpi(t) \geq \min\{\inf_{a \in (\imath \circ \jmath) \circ \ell} \ \varpi(a), \ \varpi(\ell)\}$.

• a "fuzzy hyper BCK-commutative ideal" (or $FH_{BCK}CI$) of $\digamma$ if $\imath \ll \jmath$ implies $\varpi(\imath) \geq \varpi(\jmath)$ and for all $t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$,
$\varpi(t) \geq \min\{\inf_{a \in (\imath \circ \jmath) \circ \ell} \ \varpi(a), \ \varpi(\ell)\}$.

• a "fuzzy strong hyper BCK-commutative ideal" (or $FsH_{BCK}CI$) of $\digamma$ if $\inf_{a \in \imath \circ \imath} \ \varpi(a) \geq \varpi(\imath)$ and for all $t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$,
$\varpi(t) \geq \min\{\sup_{b \in (\imath \circ \jmath) \circ \ell} \ \varpi(b), \ \varpi(\ell)\}$.

• a "fuzzy reflexive hyper BCK-commutative ideal" (or $FrH_{BCK}CI$) of $\digamma$ if $\inf_{a \in \imath \circ \imath} \ \varpi(a) \geq \varpi(\jmath)$ and for all $t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$,
$\varpi(t) \geq \min\{\sup_{b \in (\imath \circ \jmath) \circ \ell} \ \varpi(b), \ \varpi(\ell)\}$.

for any $\imath, \jmath, \ell \in \digamma$.

Theorem 3.5.7. Any $FH_{BCK}CI$ (resp. $FwH_{BCK}CI$, $FsH_{BCK}CI$, $FrH_{BCK}CI$) of $\digamma$ is a $FH_{BCK}I$ (resp. $FwH_{BCK}I$, $FsH_{BCK}I$, $FrH_{BCK}I$) of $\digamma$.

Proof. Straightforward. □

Theorem 3.5.8. For any $\digamma$,

(i). Any $FH_{BCK}CI$ of $\digamma$ is a $FwH_{BCK}CI$ of $\digamma$.

(ii). Any $FsH_{BCK}CI$ of H is a $FH_{BCK}CI$ of $\digamma$.

(iii). Any $FrH_{BCK}CI$ of H is a $FsH_{BCK}CI$ of $\digamma$.

Proof. (i). Let ϖ be a $FH_{BCK}CI$ of $\digamma$. Since, any $FH_{BCK}CI$ is a $FH_{BCK}I$ (By Theorem 3.5.7) and every $FH_{BCK}I$ is a $FwH_{BCK}I$ (By Theorem 3.1.12), therefore ϖ is a $FwH_{BCK}I$ of $\digamma$. Hence ϖ satisfies $\varpi(0) \geq \varpi(\imath)$ for all $\imath \in \digamma$. Also being a $FH_{BCK}CI$, for any $\imath, \jmath, \ell \in \digamma$ and for all

$t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$, ϖ satisfies:

$\varpi(t) \geq \min\{\inf_{a \in (\imath \circ \jmath) \circ \ell}\ \varpi(a),\ \varpi(\ell)\}$.

Hence ϖ is a $FwH_{BCK}CI$ of $\digamma$.

(ii). Suppose that ϖ is a $FsH_{BCK}CI$ of H. Since any $FsH_{BCK}CI$ is a $FsH_{BCK}I$ (by Theorem 3.5.7) and every $FsH_{BCK}I$ is a $FH_{BCK}I$ (by Theorem 3.1.12), therefore ϖ is a $FH_{BCK}I$ of $\digamma$. Hence for any $\imath, \jmath \in \digamma$, if $\imath \ll \jmath$ then $\varpi(\imath) \geq \varpi(\jmath)$.

Also being a $FsH_{BCK}CI$, for any $\imath, \jmath, \ell \in \digamma$ and for all $t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$, ϖ satisfies

$\varpi(t) \geq \min\{\sup_{a \in (\imath \circ \jmath) \circ \ell}\ \varpi(a),\ \varpi(\ell)\}$

Since $\sup_{a \in (\imath \circ \jmath) \circ \ell}\ \varpi(a) \geq \varpi(b)$, for all $b \in (\imath \circ \jmath) \circ \ell$, therefore we get,

$\varpi(t) \geq \min\{\varpi(b),\ \varpi(\ell)\}$, for all $b \in (\imath \circ \jmath) \circ \ell$

Since $\varpi(b) \geq \inf_{c \in (\imath \circ \jmath) \circ \ell}\ \varpi(c)$ for all $b \in (\imath \circ \jmath) \circ \ell$,

Therefore, $\varpi(t) \geq \min\{\varpi(b),\ \varpi(\ell)\} \geq \min\{\inf_{c \in (\imath \circ \jmath) \circ \ell}\ \varpi(c),\ \varpi(\ell)\}$,

i.e., $\varpi(t) \geq \min\{\inf_{c \in (\imath \circ \jmath) \circ \ell}\ \varpi(c),\ \varpi(\ell)\}$.

Hence ϖ is a $FH_{BCK}CI$ of $\digamma$.

(iii). Let ϖ be a $FrH_{BCK}CI$ of $\digamma$. Then ϖ satisfies

$\inf_{a \in \imath \circ \imath}\ \varpi(a) \geq \varpi(\jmath)$, $\forall\ \imath, \jmath \in \digamma$

$\Rightarrow \inf_{a \in \imath \circ \imath}\ \varpi(a) \geq \varpi(\imath)$, for all $\imath \in \digamma$

Hence, the first condition for ϖ to be a $FsH_{BCK}CI$ of $\digamma$ is satisfied. Also being a $FrH_{BCK}CI$, for any $\imath, \jmath, \ell \in \digamma$ and for all $t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$, ϖ satisfies

$\varpi(t) \geq \min\{\sup_{b \in (\imath \circ \jmath) \circ \ell}\ \varpi(b),\ \varpi(\ell)\}$.

Hence ϖ is a $FsH_{BCK}CI$ of $\digamma$. $\square$

The converse of above doesn't hold. Consider the "hyper BCK-algebra" $\digamma = \{0, \imath, \jmath\}$ defined by the table given in Example 3.1.9. Define a fuzzy set ϖ in $\digamma$ by:

$$\varpi(0) = \varpi(\jmath) = 0.6,\ \varpi(\imath) = 0.4.$$

Then ϖ is a $FwH_{BCK}CI$ of $\digamma$ but it is not a $FH_{BCK}CI$ of $\digamma$ because:

$$\imath \ll \jmath \Rightarrow \varpi(\imath) = 0.4 < 0.6 = \varpi(\jmath).$$

Now consider the "hyper BCK-algebra" $\digamma = \{0, \imath, \jmath\}$ defined by the table given in Example 3.2.4. Define a fuzzy set ϖ in $\digamma$ by:

$$\varpi(0) = \varpi(\imath) = 0.7,\ \varpi(\jmath) = 0.3.$$

Then ϖ is a $FH_{BCK}CI$ of $\digamma$ but it is not a $FsH_{BCK}CI$ of $\digamma$ because for $\jmath \in (\jmath \circ (\jmath \circ (\jmath \circ \jmath)))$,

$$\varpi(\jmath) = 0.3 < 0.7 = \min\{\sup_{a\in(\jmath\circ\jmath)\circ 0} \varpi(a),\ \varpi(0)\}.$$

Theorem 3.5.9. *A fuzzy set $\varpi = \varpi_1 \times \varpi_2$ is a $FH_{BCK}CI$ (resp. $FwH_{BCK}CI$, $FsH_{BCK}CI$, $FrH_{BCK}CI$) of $\digamma$ if and only if ϖ_1 and ϖ_2 are $FH_{BCK}CIs$ (resp. $FwH_{BCK}CIs$, $FsH_{BCK}CIs$, $FrH_{BCK}CIs$) of $\digamma_1$ and $\digamma_2$ respectively.*

Proof. Let ϖ be a $FH_{BCK}CI$ of $\digamma$ and let $\imath_1 \ll \imath_2$ for some $\imath_1, \imath_2 \in \digamma_1$. Then $(\imath_1, 0) \ll (\imath_2, 0)$ which implies $\varpi((\imath_1, 0)) = \varpi_1(\imath_1) \geq \varpi((\imath_2, 0)) = \varpi_1(\imath_2)$, i.e., $\varpi_1(\imath_1) \geq \varpi_1(\imath_2)$

Moreover, for any $\imath_1, \jmath_1, \ell_1 \in \digamma_1$, let $\delta = \min\ \{\inf_{a\in(\imath_1\circ\jmath_1)\circ\ell_1}\ \varpi_1(a),\ \varpi_1(\ell_1)\}$

Then for all $b \in (\imath_1 \circ \jmath_1) \circ \ell_1$, $\varpi_1(b) \geq \inf_{a\in(\imath_1\circ\jmath_1)\circ\ell_1}\ \varpi_1(a) \geq \delta$ and $\varpi_1(\ell_1) \geq \delta$

$\Rightarrow \varpi((b, 0)) \geq \delta$ and $\varpi((\ell_1, 0)) \geq \delta$, for all $(b, 0) \in ((\imath_1, 0) \circ (\jmath_1, 0)) \circ (\ell_1, 0)$

$\Rightarrow (b, 0) \in \varpi_\delta$ and $(\ell_1, 0) \in \varpi_\delta$, for all $(b, 0) \in ((\imath_1, 0) \circ (\jmath_1, 0)) \circ (\ell_1, 0)$

$\Rightarrow ((\imath_1, 0) \circ (\jmath_1, 0)) \circ (\ell_1, 0) \subseteq \varpi_\delta$ and $(\ell_1, 0) \in \varpi_\delta$

By transfer principle for fuzzy sets, $\varpi_\delta \neq \emptyset$ is a $H_{BCK}CI$ of $\digamma$ and so is a

$wH_{BCK}CI$ of $\digamma$ (by Theorem 3.5.3).

Thus $((\imath_1,0)\circ(\jmath_1,0))\circ(\ell_1,0)\subseteq\varpi_\delta$ and $(\ell_1,0)\in\varpi_\delta$ imply

$(\imath_1,0)\circ((\jmath_1,0)\circ((\jmath_1,0)\circ(\imath_1,0)))\subseteq\varpi_\delta$

Therefore $\varpi((s,0))\geq\delta$, for all $(s,0)\in(\imath_1,0)\circ((\jmath_1,0)\circ((\jmath_1,0)\circ(\imath_1,0)))=(\imath_1\circ(\jmath_1\circ(\jmath_1\circ\imath_1)),\ 0)$

$\Rightarrow\varpi_1(s)\geq\delta=\min\ \{\inf_{a\in(\imath_1\circ\jmath_1)\circ\ell_1}\ \varpi_1(a),\ \varpi_1(\ell_1)\}$,

for all $s\in\imath_1\circ(\jmath_1\circ(\jmath_1\circ\imath_1))$.

Hence ϖ_1 is a $FH_{BCK}CI$ of $\digamma_1$.

Similarly, we can demonstrate that ϖ_2 is a $FH_{BCK}CI$ of $\digamma_2$.

Conversely, let ϖ_1 and ϖ_2 be $FH_{BCK}CIs$ of $\digamma_1$ and $\digamma_2$ respectively.

For any $(\imath,u),(\jmath,v)\in\digamma$, where $\imath,\jmath\in\digamma_1$ and $u,v\in\digamma_2$, let $(\imath,u)\ll(\jmath,v)$

Since $(\imath,u)\ll(\jmath,v)\Leftrightarrow\imath\ll\jmath$ and $u\ll v$

$\Rightarrow\varpi_1(\imath)\geq\varpi_1(\jmath)$ and $\varpi_2(u)\geq\varpi_2(v)$

$\Rightarrow\min\ \{\varpi_1(\imath),\ \varpi_2(u)\}\geq\min\ \{\varpi_1(\jmath),\ \varpi_2(v)\}$

$\Rightarrow(\varpi_1\times\varpi_2)((\imath,u))\geq(\varpi_1\times\varpi_2)((\jmath,v))$

$\Rightarrow\varpi((\imath,u))\geq\varpi((\jmath,v))$

Thus $(\imath,u)\ll(\jmath,v)\Rightarrow\varpi((\imath,u))\geq\varpi((\jmath,v))$

Moreover, for any $(\imath,u),\ (\jmath,v),\ (\ell,w)\in\digamma$, where $\imath,\jmath,\ell\in\digamma_1$ and $u,v,w\in\digamma_2$ and for all $(a,b)\in(\imath,u)\circ((\jmath,v)\circ((\jmath,v)\circ(\imath,u)))$

$=(\imath\circ(\jmath\circ(\jmath\circ\imath)),\ u\circ(v\circ(v\circ u)))$,

$\varpi((a,b))=(\varpi_1\times\varpi_2)((a,b))=\min\ \{\varpi_1(a),\ \varpi_2(b)\}$

$\geq\min\ [\min\ \{\inf_{c\in(\imath\circ\jmath)\circ\ell}\ \varpi_1(c),\ \varpi_1(\ell)\},\ \min\ \{\inf_{d\in(u\circ v)\circ w}\ \varpi_2(d),\ \varpi_2(w)\}]$

$=\min\ [\min\ \{\inf_{c\in(\imath\circ\jmath)\circ\ell}\ \varpi_1(c),\ \inf_{d\in(u\circ v)\circ w}\ \varpi_2(d)\},\ \min\ \{\ \varpi_1(\ell),\ \varpi_2(w)\}]$

$=\min\ [\inf_{c\in(\imath\circ\jmath)\circ\ell,\ d\in(u\circ v)\circ w}\ \{\min\ \{\varpi_1(c),\ \varpi_2(d)\}\},\ \min\ \{\ \varpi_1(\ell),\ \varpi_2(w)\}]$

$=\min\ \{\inf_{(c,d)\in((\imath\circ\jmath)\circ\ell,\ (u\circ v)\circ w)}\ (\varpi_1\times\varpi_2)((c,d)),\ (\varpi_1\times\varpi_2)((\ell,w))\}$

$=\min\ \{\inf_{(c,d)\in((\imath\circ\jmath)\circ\ell,\ (u\circ v)\circ w)}\ \varpi((c,d)),\ \varpi((\ell,w))\}$

$\Rightarrow \varpi((a,b)) \geq \min\ \{\inf_{(c,d)\in((\imath,u)\circ(\jmath,v))\circ(\ell,w)}\ \varpi((c,d)),\ \varpi((\ell,w))\}$

Hence ϖ is a $FH_{BCK}CI$ of $\digamma$. □

3.6 Fuzzy hyper BCK-implicative ideals

Here, we present the idea of (fuzzy) (weak, strong, reflexive) "hyper BCK-implicative ideals" and elaborate apposite properties.

Definition 3.6.1. A non-empty subset $\Upsilon \subseteq \digamma$ containing 0 is

• a "weak hyper BCK-implicative ideal" (or $wH_{BCK}II$) of $\digamma$ if

$((\imath \circ \jmath) \circ \jmath) \circ \ell \subseteq \Upsilon$ and $\ell \in \Upsilon$ imply $\imath \circ (\jmath \circ (\jmath \circ \imath)) \subseteq \Upsilon$.

• a "hyper BCK-implicative ideal" (or $H_{BCK}II$) of $\digamma$ if

$((\imath \circ \jmath) \circ \jmath) \circ \ell \ll \Upsilon$ and $\ell \in \Upsilon$ imply $\imath \circ (\jmath \circ (\jmath \circ \imath)) \subseteq \Upsilon$.

• a "strong hyper BCK-implicative ideal" (or $sH_{BCK}II$) of $\digamma$ if

$(((\imath \circ \jmath) \circ \jmath) \circ \ell) \cap \Upsilon \neq \emptyset$ and $\ell \in \Upsilon$ imply $\imath \circ (\jmath \circ (\jmath \circ \imath)) \subseteq \Upsilon$.

for any $\imath, \jmath, \ell \in \digamma$.

Theorem 3.6.2. Any $H_{BCK}II$ (resp. $wH_{BCK}II$, sH_hII, $rH_{BCK}II$) of $\digamma$ is a $H_{BCK}I$ (resp. $wH_{BCK}I$, $sH_{BCK}I$, $rH_{BCK}I$) of $\digamma$.

Proof. Straightforward. □

Theorem 3.6.3. For any $\digamma$,

(i). Any $H_{BCK}II$ of $\digamma$ is a $wH_{BCK}II$ of $\digamma$.

(ii). Any $sH_{BCK}II$ of $\digamma$ is a $H_{BCK}II$ of $\digamma$.

(iii). Any $rH_{BCK}II$ of $\digamma$ is a $sH_{BCK}II$ of $\digamma$.

Proof. (i). Let I be a $H_{BCK}II$ of $\digamma$.

For any $\imath, \jmath, \ell \in \digamma$, let $((\imath \circ \jmath) \circ \jmath) \circ \ell \subseteq \Upsilon$ and $\ell \in \Upsilon$. Then $((\imath \circ \jmath) \circ \jmath) \circ \ell \subseteq \Upsilon$ implies $((\imath \circ \jmath) \circ \jmath) \circ \ell \ll \Upsilon$ (by Proposition 3.1.2(v)), which along with $\ell \in \Upsilon$

implies $\imath \circ (\jmath \circ (\jmath \circ \imath)) \subseteq \Upsilon$. Hence Υ is a $wH_{BCK}II$ of F.

(ii). Let I be a $sH_{BCK}II$ of F. Let $((\imath \circ \jmath) \circ \jmath) \circ \ell \ll \Upsilon$ and $\ell \in \Upsilon$. Then for all $a \in ((\imath \circ \jmath) \circ \jmath) \circ \ell$, $\exists\ b \in \Upsilon$ such that $a \ll b$. This implies $0 \in a \circ b$ and thus $(a \circ b) \cap \Upsilon \neq \emptyset$. By Theorem 3.6.2, Υ is also a $sH_{BCK}I$ of F, therefore $(a \circ b) \cap \Upsilon \neq \emptyset$ along with $b \in \Upsilon$ implies $a \in \Upsilon$, that is $((\imath \circ \jmath) \circ \jmath) \circ \ell \subseteq \Upsilon$. Therefore $(((\imath \circ \jmath) \circ \jmath) \circ \ell) \cap \Upsilon \neq \emptyset$, which along with $\ell \in \Upsilon$ implies $\imath \circ (\jmath \circ (\jmath \circ \imath)) \subseteq \Upsilon$. Hence Υ is a $H_{BCK}II$ of F.

(iii). Let I be a $rH_{BCK}II$ of F. For any $\imath, \jmath, \ell \in F$, let $(((\imath \circ \jmath) \circ \jmath) \circ \ell) \cap \Upsilon \neq \emptyset$ and $\ell \in \Upsilon$. Being a $rH_{BCK}II$, Υ is also a $rH_{BCK}I$ of F (by Theorem 3.6.2), therefore by Lemma 3.1.4, $(((\imath \circ \jmath) \circ \jmath) \circ \ell) \cap \Upsilon \neq \emptyset \Rightarrow ((\imath \circ \jmath) \circ \jmath) \circ \ell \ll \Upsilon$, which along with $\ell \in \Upsilon$ implies $\imath \circ (\jmath \circ (\jmath \circ \imath)) \subseteq \Upsilon$. Hence Υ is a $sH_{BCK}II$ of H. □

The converse of the above theorem isn't valid. It can be observed by the succeeding examples:

Example 3.6.4. Consider the "hyper BCK-algebra" $F = \{0, \imath, \jmath\}$ defined by the table given in Example 3.1.9. Take $\Upsilon = \{0, \jmath\}$. Then Υ is a $wH_{BCK}II$ of F but it is not a $H_{BCK}II$ of F because
$((\imath \circ 0) \circ 0) \circ \jmath = \{0, \imath\} \ll \Upsilon$ and $\jmath \in \Upsilon$ but $\imath \circ (0 \circ (0 \circ \imath)) = \{\imath\} \nsubseteq \Upsilon$.

Example 3.6.5. Consider the "hyper BCK-algebra" $F = \{0, \imath, \jmath\}$ defined by the table given in Example 3.1.10. Take $\Upsilon = \{0, \imath\}$. Then Υ is a $H_{BCK}II$ of F but it is not a $sH_{BCK}II$ of F because
$(((\jmath \circ 0) \circ 0) \circ \imath) \cap \Upsilon = \{\imath, \jmath\} \cap \Upsilon \neq \emptyset$ and $\imath \in I$ but $\jmath \circ (0 \circ (0 \circ \jmath)) = \{\jmath\} \nsubseteq \Upsilon$.

Definition 3.6.6. A fuzzy set ϖ in F is

• a "fuzzy weak hyper BCK-implicative ideal" (or $FwH_{BCK}II$) of F if $\varpi(0) \geq \varpi(\imath)$ and for all $t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$,

$\varpi(t) \geq \min\{\inf_{a\in((\imath\circ\jmath)\circ\jmath)\circ\ell}\ \varpi(a),\ \varpi(\ell)\}$.

• a "fuzzy hyper BCK-implicative ideal" (or $FH_{BCK}II$) of $\digamma$ if

$\imath \ll \jmath$ implies $\varpi(\imath) \geq \varpi(\jmath)$ and for all $t \in \imath\circ(\jmath\circ(\jmath\circ\imath))$,

$\varpi(t) \geq \min\{\inf_{a\in((\imath\circ\jmath)\circ\jmath)\circ\ell}\ \varpi(a),\ \varpi(\ell)\}$.

• a "fuzzy strong hyper BCK-implicative ideal" (or $FsH_{BCK}II$) of $\digamma$ if

$\inf_{a\in\imath\circ\imath}\ \varpi(a) \geq \varpi(\imath)$ and for all $t \in \imath\circ(\jmath\circ(\jmath\circ\imath))$,

$\varpi(t) \geq \min\{\sup_{b\in((\imath\circ\jmath)\circ\jmath)\circ\ell}\ \varpi(b),\ \varpi(\ell)\}$.

• a "fuzzy reflexive hyper BCK-implicative ideal" (or $FrH_{BCK}II$) of $\digamma$ if

$\inf_{a\in\imath\circ\imath}\ \varpi(a) \geq \varpi(\jmath)$ and for all $t \in \imath\circ(\jmath\circ(\jmath\circ\imath))$,

$\varpi(t) \geq \min\{\sup_{b\in((\imath\circ\jmath)\circ\jmath)\circ\ell}\ \varpi(b),\ \varpi(\ell)\}$.

for all $\imath,\jmath,\ell \in \digamma$

Theorem 3.6.7. Any $FH_{BCK}II$ (resp. $FwH_{BCK}II$, $FsH_{BCK}II$, $FrH_{BCK}II$) of $\digamma$ is a $FH_{BCK}I$ (resp. $FwH_{BCK}I$, $FsH_{BCK}I$, $FrH_{BCK}I$) of $\digamma$.

Proof. Straightforward. □

Theorem 3.6.8. For any $\digamma$,

(i). Any $FH_{BCK}II$ of $\digamma$ is a $FwH_{BCK}II$ of $\digamma$.

(ii). Any $FsH_{BCK}II$ of H is a $FH_{BCK}II$ of $\digamma$.

(iii). Any $FrH_{BCK}II$ of H is a $FsH_{BCK}II$ of $\digamma$.

Proof. (i). Let ϖ be a $FH_{BCK}II$ of $\digamma$. Since any $FH_{BCK}II$ is a $FH_{BCK}I$ (By Theorem 3.6.7) and every $FH_{BCK}I$ is a $FwH_{BCK}I$ (By Theorem 3.1.12), therefore ϖ is a $FwH_{BCK}I$ of $\digamma$. Hence ϖ satisfies $\varpi(0) \geq \varpi(\imath)$ for all $\imath \in \digamma$. Also being a $FH_{BCK}II$, for any $\imath,\jmath,\ell \in \digamma$ and for all $t \in \imath\circ(\jmath\circ(\jmath\circ\imath))$, ϖ satisfies:

$\varpi(t) \geq \min\{\inf_{a\in((\imath\circ\jmath)\circ\jmath)\circ\ell}\ \varpi(a),\ \varpi(\ell)\}$.

Hence ϖ is a $FwH_{BCK}II$ of $\digamma$.

(ii). Suppose that ϖ is a $FsH_{BCK}II$ of $\digamma$. Since any $FsH_{BCK}II$ is a $FsH_{BCK}I$ (by Theorem 3.6.7) and every $FsH_{BCK}I$ is a $FH_{BCK}I$ (by Theorem 3.1.12), therefore ϖ is a $FH_{BCK}I$ of $\digamma$. Hence for any $\imath, \jmath \in \digamma$, if $\imath \ll \jmath$ then $\varpi(\imath) \geq \varpi(\jmath)$.

Also being a $FsH_{BCK}II$, for any $\imath, \jmath, \ell \in \digamma$ and for all $t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$, ϖ satisfies

$\varpi(t) \geq \min\{\sup_{a \in ((\imath \circ \jmath) \circ \jmath) \circ \ell} \ \varpi(a), \ \varpi(\ell)\}$.

Since $\sup_{a \in ((\imath \circ \jmath) \circ \jmath) \circ \ell} \ \varpi(a) \geq \varpi(b)$, for all $b \in ((\imath \circ \jmath) \circ \jmath) \circ \ell$, therefore we get,

$\varpi(t) \geq \min\{\varpi(b), \ \varpi(\ell)\}$, for all $b \in ((\imath \circ \jmath) \circ \jmath) \circ \ell$.

Since $\varpi(b) \geq \inf_{c \in ((\imath \circ \jmath) \circ \jmath) \circ \ell} \ \varpi(c)$ for all $b \in ((\imath \circ \jmath) \circ \jmath) \circ \ell$, therefore we have,

$\varpi(t) \geq \min\{\varpi(b), \ \varpi(\ell)\} \geq \min\{\inf_{c \in ((\imath \circ \jmath) \circ \jmath) \circ \ell} \ \varpi(c), \ \varpi(\ell)\}$, that is

$\varpi(t) \geq \min\{\inf_{c \in ((\imath \circ \jmath) \circ \jmath) \circ \ell} \ \varpi(c), \ \varpi(\ell)\}$.

Hence ϖ is a $FH_{BCK}II$ of $\digamma$.

(iii). Let ϖ be a $FrH_{BCK}II$ of $\digamma$. Then ϖ satisfies

$\inf_{a \in \imath \circ \imath} \ \varpi(a) \geq \varpi(\jmath)$, for all $\imath, \jmath \in \digamma$.

$\Rightarrow \inf_{a \in \imath \circ \imath} \ \varpi(a) \geq \varpi(\imath)$, for all $\imath \in \digamma$.

Hence the first condition for ϖ to be a $FsH_{BCK}II$ of $\digamma$ is satisfied. Also being a $FrH_{BCK}II$, for any $\imath, \jmath, \ell \in \digamma$ and for all $t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$, ϖ satisfies

$\varpi(t) \geq \min\{\sup_{b \in ((\imath \circ \jmath) \circ \jmath) \circ \ell} \ \varpi(b), \ \varpi(\ell)\}$.

Hence ϖ is a $FsH_{BCK}II$ of $\digamma$. □

The converse of the above theorem isn't valid. Consider the "hyper BCK-algebra" $\digamma = \{0, \imath, \jmath\}$ defined by the table given in Example 3.1.9. Delineate a fuzzy set ϖ in $\digamma$ by:

$$\varpi(0) = \varpi(\jmath) = 0.8, \ \varpi(\imath) = 0.2.$$

Then ϖ is a $FwH_{BCK}II$ of $\digamma$ but it is not a $FH_{BCK}II$ of $\digamma$ because:

$$\imath \ll \jmath \Rightarrow \varpi(\imath) = 0.2 < 0.8 = \varpi(\jmath).$$

Example 3.6.9. Consider the "hyper BCK-algebra" $\digamma = \{0, \imath, \jmath\}$ defined by the table given in Example 3.2.4. Define a fuzzy set ϖ in $\digamma$ by:

$$\varpi(0) = \varpi(\imath) = 0.9,\ \varpi(\jmath) = 0.1.$$

Then ϖ is a $FH_{BCK}II$ of $\digamma$ but it is not a $FsH_{BCK}II$ of $\digamma$ because for $\jmath \in (\jmath \circ (\jmath \circ (\jmath \circ \jmath)))$,

$$\varpi(\jmath) = 0.1 < 0.9 = \min\{\sup_{a \in (((\jmath \circ \jmath) \circ \jmath) \circ 0)} \varpi(a),\ \varpi(0)\}.$$

Theorem 3.6.10. *A fuzzy set $\varpi = \varpi_1 \times \varpi_2$ is a $FH_{BCK}II$ (resp. $FwH_{BCK}II$, $FsH_{BCK}II$, $FrH_{BCK}II$) of $\digamma$ if and only if ϖ_1 and ϖ_2 are $FH_{BCK}IIs$ (resp. $FwH_{BCK}IIs$, $FsH_{BCK}IIs$, $FrH_{BCK}IIs$) of $\digamma_1$ and $\digamma_2$ respectively.*

Proof. Let ϖ be a $FH_{BCK}II$ of $\digamma$ and let $\imath_1 \ll \imath_2$ for some $\imath_1, \imath_2 \in \digamma_1$. Then $(\imath_1, 0) \ll (\imath_2, 0)$ which implies $\varpi((\imath_1, 0)) = \varpi_1(\imath_1) \geq \varpi((\imath_2, 0)) = \varpi_1(\imath_2)$, i.e, $\varpi_1(\imath_1) \geq \varpi_1(\imath_2)$

Moreover, for any $\imath_1, \jmath_1, \ell_1 \in \digamma_1$, let $\delta = \min\ \{\inf_{a \in ((\imath_1 \circ \jmath_1) \circ \jmath_1) \circ \ell_1} \varpi_1(a),\ \varpi_1(\ell_1)\}$

Then for all $b \in ((\imath_1 \circ \jmath_1) \circ \jmath_1) \circ \ell_1$, $\varpi_1(b) \geq \inf_{a \in ((\imath_1 \circ \jmath_1) \circ \jmath_1) \circ \ell_1} \varpi_1(a) \geq \delta$ and $\varpi_1(\ell_1) \geq \delta$

$\Rightarrow \varpi((b, 0)) \geq \delta$ and $\varpi((\ell_1, 0)) \geq \delta$,

for all $(b, 0) \in (((\imath_1, 0) \circ (\jmath_1, 0)) \circ (\jmath_1, 0)) \circ (\ell_1, 0)$

$\Rightarrow (b, 0) \in \varpi_\delta$ and $(\ell_1, 0) \in \varpi_\delta$,

$\Rightarrow (((\imath_1, 0) \circ (\jmath_1, 0)) \circ (\jmath_1, 0)) \circ (\ell_1, 0) \subseteq \varpi_\delta$ and $(\ell_1, 0) \in \varpi_\delta$

Since by transfer principle for fuzzy sets, $\varpi_\delta \neq \emptyset$ is a $H_{BCK}II$ of $\digamma$ and so is a $wH_{BCK}II$ of $\digamma$ (by Theorem 3.6.3).

Thus, $((((\imath_1,0)\circ(\jmath_1,0))\circ(\jmath_1,0))\circ(\ell_1,0))\subseteq\varpi_\delta$ and $(\ell_1,0)\in\varpi_\delta$ imply

$(\imath_1,0)\circ((\jmath_1,0)\circ((\jmath_1,0)\circ(\imath_1,0)))\subseteq\varpi_\delta$

Therefore $\varpi((s,0))\geq\delta$, for all $(s,0)\in(\imath_1,0)\circ((\jmath_1,0)\circ((\jmath_1,0)\circ(\imath_1,0)))=$

$(\imath_1\circ(\jmath_1\circ(\jmath_1\circ\imath_1)),\ 0)$

$\Rightarrow\varpi_1(s)\geq\delta=\min\ \{\inf_{a\in((\imath_1\circ\jmath_1)\circ\jmath_1)\circ\ell_1}\ \varpi_1(a),\ \varpi_1(\ell_1)\}$,

for all $s\in\imath_1\circ(\jmath_1\circ(\jmath_1\circ\imath_1))$.

Hence ϖ_1 is a $FH_{BCK}II$ of $\digamma_1$.

Similarly, we can demonstrate that ϖ_2 is a $FH_{BCK}II$ of $\digamma_2$.

Conversely, let ϖ_1 and ϖ_2 are $FH_{BCK}IIs$ of $\digamma_1$ and $\digamma_2$ respectively.

For any $(\imath,u),(\jmath,v)\in\digamma$, where $\imath,\jmath\in\digamma_1$ and $u,v\in\digamma_2$, let $(\imath,u)\ll(\jmath,v)$

Since $(\imath,u)\ll(\jmath,v)\Leftrightarrow\imath\ll\jmath$ and $u\ll v$

$\Rightarrow\varpi_1(\imath)\geq\varpi_1(\jmath)$ and $\varpi_2(u)\geq\varpi_2(v)$

$\Rightarrow\min\ \{\varpi_1(\imath),\ \varpi_2(u)\}\geq\min\ \{\varpi_1(\jmath),\ \varpi_2(v)\}$

$\Rightarrow(\varpi_1\times\varpi_2)((\imath,u))\geq(\varpi_1\times\varpi_2)((\jmath,v))$

$\Rightarrow\varpi((\imath,u))\geq\varpi((\jmath,v))$

Thus $(\imath,u)\ll(\jmath,v)\Rightarrow\varpi((\imath,u))\geq\varpi((\jmath,v))$

Moreover, for any $(\imath,u),\ (\jmath,v),\ (\ell,w)\in\digamma$, where $\imath,\jmath,\ell\in\digamma_1$ and

$u,v,w\in\digamma_2$ and for all $(a,b)\in(\imath,u)\circ((\jmath,v)\circ((\jmath,v)\circ(\imath,u)))$

$=(\imath\circ(\jmath\circ(\jmath\circ\imath)),\ u\circ(v\circ(v\circ u)))$,

$\varpi((a,b))=(\varpi_1\times\varpi_2)((a,b))=\min\ \{\varpi_1(a),\ \varpi_2(b)\}$

$\geq\min\ [\min\ \{\inf_{c\in((\imath\circ\jmath)\circ\jmath)\circ\ell}\ \varpi_1(c),\ \varpi_1(\ell)\},\ \min\ \{\inf_{d\in((u\circ v)\circ v)\circ w}\ \varpi_2(d),\ \varpi_2(w)\}]$

$=\min\ [\min\ \{\inf_{c\in((\imath\circ\jmath)\circ\jmath)\circ\ell}\ \varpi_1(c),\ \inf_{d\in((u\circ v)\circ v)\circ w}\ \varpi_2(d)\},\ \min\ \{\ \varpi_1(\ell),\ \varpi_2(w)\}]$

$=\min\ [\inf_{c\in((\imath\circ\jmath)\circ\jmath)\circ\ell,\ d\in((u\circ v)\circ v)\circ w}\ \{\min\ \{\varpi_1(c),\ \varpi_2(d)\}\},\ \min\ \{\ \varpi_1(\ell),\ \varpi_2(w)\}]$

$=\min\ \{\inf_{(c,d)\in(((\imath\circ\jmath)\circ\jmath)\circ\ell,\ ((u\circ v)\circ v)\circ w)}\ (\varpi_1\times\varpi_2)((c,d)),\ (\varpi_1\times\varpi_2)((\ell,w))\}$

$=\min\ \{\inf_{(c,d)\in(((\imath\circ\jmath)\circ\jmath)\circ\ell,\ ((u\circ v)\circ v)\circ w)}\ \varpi((c,d)),\ \varpi((\ell,w))\}$

$\Rightarrow\varpi((a,b))\geq\min\ \{\inf_{(c,d)\in(((\imath,u)\circ(\jmath,v))\circ(\jmath,v))\circ(\ell,w)}\ \varpi((c,d)),\ \varpi((\ell,w))\}$

Hence ϖ is a $FH_{BCK}II$ of $\digamma$. □

3.7 Fuzzy hyper BCK-positive implicative ideals.

We now present the idea of (fuzzy) (weak, strong, reflexive) "hyper BCK-positive implicative ideals" and discuss apposite properties.

Definition 3.7.1. A non-empty subset $\Upsilon \subseteq \digamma$ containing 0 is

• a "weak hyper BCK-positive implicative ideal" (or $wH_{BCK}PII$) of $\digamma$ if $((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \subseteq \Upsilon$ and $\jmath \in \Upsilon$ imply $(\imath \circ \ell) \subseteq \Upsilon$.

• a "hyper BCK-positive implicative ideal" (or $H_{BCK}PII$) of $\digamma$ if $((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \ll \Upsilon$ and $\jmath \in \Upsilon$ imply $(\imath \circ \ell) \subseteq \Upsilon$.

• a "strong hyper BCK-positive implicative ideal" (or $sH_{BCK}PII$) of $\digamma$ if $((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \cap \Upsilon \neq \emptyset$ and $\jmath \in \Upsilon$ imply $(\imath \circ \ell) \subseteq \Upsilon$.

for any $\imath, \jmath, \ell \in \digamma$

Theorem 3.7.2. Any $H_{BCK}PII$ (resp. $wH_{BCK}PII$, sH_hPII, $rH_{BCK}PII$) of $\digamma$ is a $H_{BCK}I$ (resp. $wH_{BCK}I$, $sH_{BCK}I$, $rH_{BCK}I$) of $\digamma$.

Proof. Straightforward. □

Theorem 3.7.3. For any $\digamma$,

(i). Any $H_{BCK}PII$ of $\digamma$ is a $wH_{BCK}PII$ of $\digamma$.

(ii). Any $sH_{BCK}PII$ of $\digamma$ is a $H_{BCK}PII$ of $\digamma$.

(iii). Any $rH_{BCK}PII$ of $\digamma$ is a $sH_{BCK}PII$ of $\digamma$.

Proof. (i). Let Υ be a $H_{BCK}PII$ of $\digamma$.

For any $\imath, \jmath, \ell \in \digamma$, let $((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \subseteq \Upsilon$ and $\jmath \in \Upsilon$.

Then $((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \subseteq \Upsilon$ implies $((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \ll \Upsilon$ (by Proposition 3.1.2(v)), which along with $\jmath \in \Upsilon$ implies $\imath \circ \ell \subseteq \Upsilon$. Hence Υ is a $wH_{BCK}PII$

of $\digamma$.

(ii). Let Υ be a $sH_{BCK}PII$ of $\digamma$. Let $((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \ll \Upsilon$ and $\jmath \in \Upsilon$. Then for any $a \in ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)$, $\exists\, b \in \Upsilon$, s.t, $a \ll b$. This implies $0 \in a\circ b$ and thus $a \circ b \cap \Upsilon \neq \emptyset$. By Theorem 3.7.2, Υ is a $sH_{BCK}I$ of $\digamma$. Therefore $a \circ b \cap \Upsilon \neq \emptyset$ along with $b \in \Upsilon$ implies $a \in \Upsilon$, that is $((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \subseteq \Upsilon$. Therefore $(((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell)) \cap \Upsilon \neq \emptyset$, which along with $\jmath \in \Upsilon$ implies $\imath \circ \ell \subseteq \Upsilon$. Hence Υ is a $H_{BCK}PII$ of $\digamma$.

(iii). Let Υ be a $rH_{BCK}PII$ of $\digamma$.

For any $\imath, \jmath, \ell \in \digamma$, let $(((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell)) \cap \Upsilon \neq \emptyset$ and $\jmath \in \Upsilon$. Being a $rH_{BCK}PII$, Υ is also a $rH_{BCK}I$ of $\digamma$ (by Theorem 3.7.2), therefore by Lemma 3.1.4, $(((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell)) \cap \Upsilon \neq \emptyset \Rightarrow ((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \ll \Upsilon$, which along with $\jmath \in \Upsilon$ implies $\imath \circ \ell \subseteq \Upsilon$. Hence Υ is a $sH_{BCK}PII$ of $\digamma$. □

The converse of the above theorem doesn't hold. It can be observed by the succeeding examples.

Example 3.7.4. Consider the "hyper BCK-algebra" $\digamma = \{0, \imath, \jmath\}$ defined by the table given in Example 3.1.9. Take $\Upsilon = \{0, \jmath\}$. Then Υ is a $wH_{BCK}PII$ of $\digamma$ but it isn't a $H_{BCK}PII$ of $\digamma$ because,
$((\imath \circ 0) \circ 0) \circ (0 \circ 0) = \{\imath\} \ll \Upsilon$ and $0 \in \Upsilon$ but $\imath \circ 0 = \{\imath\} \nsubseteq \Upsilon$.

Example 3.7.5. Consider the "hyper BCK-algebra" $\digamma = \{0, \imath, \jmath\}$ defined by the table given in Example 3.1.10. Take $\Upsilon = \{0, \imath\}$. Then Υ is a $H_{BCK}PII$ of $\digamma$ but it isn't a $sH_{BCK}PII$ of $\digamma$ because,
$(((\jmath \circ 0) \circ 0) \circ (\imath \circ 0)) \cap \Upsilon = \{\imath, \jmath\} \cap \Upsilon \neq \emptyset$ and $\imath \in \Upsilon$ but $\jmath \circ 0 = \{\jmath\} \nsubseteq \Upsilon$.

Definition 3.7.6. A fuzzy set ϖ in $\digamma$ is

• a "fuzzy weak hyper BCK-positive implicative ideal" (or $FwH_{BCK}PII$) of $\digamma$ if $\varpi(0) \geq \varpi(\imath)$ and for all $t \in \imath \circ \ell$,

$\varpi(t) \geq \min\{\inf_{a\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi(a),\ \varpi(\jmath)\}$.

• a "fuzzy hyper BCK-positive implicative ideal" (or $FH_{BCK}PII$) of $\digamma$ if $\imath \ll \jmath$ implies $\varpi(\imath) \geq \varpi(\jmath)$ and for all $t \in \imath \circ \ell$,

$\varpi(t) \geq \min\{\inf_{a\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi(a),\ \varpi(\jmath)\}$.

• a "fuzzy strong hyper BCK-positive implicative ideal" (or $FsH_{BCK}PII$) of $\digamma$ if $\inf_{a\in\imath\circ\imath}\ \varpi(a) \geq \varpi(\imath)$ and for all $t \in \imath \circ \ell$,

$\varpi(t) \geq \min\{\sup_{b\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi(b),\ \varpi(\jmath)\}$.

• a "fuzzy reflexive hyper BCK-positive implicative ideal" (or $FrH_{BCK}PII$) of $\digamma$ if $\inf_{a\in\imath\circ\imath}\ \varpi(a) \geq \varpi(\jmath)$ and for all $t \in \imath \circ \ell$,

$\varpi(t) \geq \min\{\sup_{b\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi(b),\ \varpi(\jmath)\}$.

for any $\imath, \jmath, \ell \in \digamma$.

Theorem 3.7.7. Any $FH_{BCK}PII$ (resp. $FwH_{BCK}PII$, $FsH_{BCK}PII$, $FrH_{BCK}PII$) of $\digamma$ is a $FH_{BCK}I$ (resp. $FwH_{BCK}I$, $FsH_{BCK}I$, $FrH_{BCK}I$) of $\digamma$.

Proof. Let ϖ be a $FH_{BCK}PII$ of $\digamma$.

Then for any $\imath, \jmath, \ell \in \digamma$ and for all $t \in \imath \circ \ell$,

$\varpi(t) \geq \min\{\inf_{a\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi(a),\ \varpi(\jmath)\}$.

By substituting $\ell = 0$,

$\varpi(t) \geq \min\{\inf_{a\ \in\ ((\imath\circ 0)\circ 0)\circ(\jmath\circ 0)}\ \varpi(a),\ \varpi(\jmath)\}$.

i.e., $\varpi(\imath) \geq \min\{\inf_{a\ \in\ \imath\circ\jmath}\ \varpi(a),\ \varpi(\jmath)\}$

Thus ϖ is a $FH_{BCK}I$ of $\digamma$. □

The converse of above theorem does't hold. It can be observed by the succeeding example.

Example 3.7.8. Let the "hyper BCK-algebra" $\digamma = \{0, \imath, \jmath, \ell\}$ be defined by the succeeding table.

$\circ$	0	$\imath$	$\jmath$	ℓ
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\imath$	$\{\imath\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\jmath$	$\{\jmath\}$	$\{\jmath\}$	$\{0\}$	$\{0\}$
ℓ	$\{\ell\}$	$\{\ell\}$	$\{\jmath\}$	$\{0, \imath\}$

Define a fuzzy set ϖ in $\digamma$ by:

$$\varpi(0) = \varpi(\imath) = 0.8, \ \varpi(\jmath) = \varpi(\ell) = 0.5.$$

Then ϖ is a $FH_{BCK}I$ (resp. $FwH_{BCK}I, \ FsH_{BCK}I, \ FrH_{BCK}I$) of $\digamma$ but it isn't a $FH_{BCK}PII$ (resp. $FwH_{BCK}PII, \ FsH_{BCK}PII, \ FrH_{BCK}PII$) of $\digamma$ because for $\jmath \in \ell \circ \jmath$,

$\varpi(\jmath) = 0.5 < 0.8 = \varpi(0) = \min\{\inf_{a \ \in \ ((\ell\circ\jmath)\circ\jmath)\circ(0\circ\jmath)} \ \varpi(a), \ \varpi(0)\}$.

and $\varpi(\jmath) = 0.5 < 0.8 = \varpi(0) = \min\{\sup_{a \ \in \ ((\ell\circ\jmath)\circ\jmath)\circ(0\circ\jmath)} \ \varpi(a), \ \varpi(0)\}$.

Theorem 3.7.9. For any $\digamma$,

(i). Any $FH_{BCK}PII$ of $\digamma$ is a $FwH_{BCK}PII$ of $\digamma$.

(ii). Any $FsH_{BCK}PII$ of H is a $FH_{BCK}PII$ of $\digamma$.

(iii). Any $FrH_{BCK}PII$ of H is a $FsH_{BCK}PII$ of $\digamma$.

Proof. (i). Let ϖ be a $FH_{BCK}PII$ of $\digamma$. Since any $FH_{BCK}PII$ is a $FH_{BCK}I$ (By Theorem 3.7.7) and every $FH_{BCK}I$ is a $FwH_{BCK}I$ (By Theorem 3.1.12), therefore ϖ is also a $FwH_{BCK}I$ of $\digamma$. Hence ϖ satisfies $\varpi(0) \geq \varpi(\imath)$ for all $\imath \in \digamma$. Also being a $FH_{BCK}PII$, for any $\imath, \jmath, \ell \in \digamma$ and for all $t \in \imath \circ \ell$, ϖ satisfies:

$\varpi(t) \geq \min\{\inf_{a \ \in \ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)} \ \varpi(a), \ \varpi(\jmath)\}$.

Hence ϖ is a $FwH_{BCK}PII$ of $\digamma$.

(ii). Let ϖ be a $FsH_{BCK}PII$ of $\digamma$. Since any $FsH_{BCK}PII$ is a $FsH_{BCK}I$ (by Theorem 3.7.7) and every $FsH_{BCK}I$ is a $FH_{BCK}I$ (by Theorem 3.1.12), therefore ϖ is also a $FH_{BCK}I$ of $\digamma$. Hence for any $\imath, \jmath \in \digamma$, if $\imath \ll \jmath$ then $\varpi(\imath) \geq \varpi(\jmath)$. Also being a $FsH_{BCK}PII$, for any $\imath, \jmath, \ell \in \digamma$ and for all $t \in \imath \circ \ell$, ϖ satisfies,

$\varpi(t) \geq \min\{\sup_{a\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi(a),\ \varpi(\jmath)\}$.

Since $\sup_{a\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi(a) \geq \varpi(b)$, for all $b \in ((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell)$,

Thus, $\varpi(t) \geq \min\{\sup_{a\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi(a),\ \varpi(\jmath)\} \geq \min\{\varpi(b),\ \varpi(\jmath)\}$, for all $b \in ((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell)$.

Since $\varpi(b) \geq \inf_{c\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi(c)$ for all $b \in ((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell)$,

Therefore, $\varpi(t) \geq \min\{\varpi(b),\ \varpi(\jmath)\} \geq \min\{\inf_{c\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi(c),\ \varpi(\jmath)\}$

$\Rightarrow \varpi(t) \geq \min\{\inf_{c\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi(c),\ \varpi(\jmath)\}$.

Hence ϖ is a $FH_{BCK}PII$ of $\digamma$.

(iii). Let ϖ be a $FrH_{BCK}PII$ of $\digamma$. Then ϖ satisfies,

$\inf_{a\in\imath\circ\imath}\ \varpi(a) \geq \varpi(\jmath)$, for all $\imath, \jmath \in \digamma$.

$\Rightarrow \inf_{a\in\imath\circ\imath}\ \varpi(a) \geq \varpi(\imath)$, for all $\imath \in \digamma$.

Hence the first condition for ϖ to be a $FsH_{BCK}PII$ of $\digamma$ is satisfied. Also being a $FrH_{BCK}PII$, for any $\imath, \jmath, \ell \in \digamma$ and for all $t \in \imath \circ \ell$, ϖ satisfies,

$\varpi(t) \geq \min\{\sup_{b\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi(b),\ \varpi(\jmath)\}$.

Hence ϖ is a $FsH_{BCK}PII$ of $\digamma$. □

The converse of the above theorem isn't valid. Consider the "hyper BCK-algebra" $\digamma = \{0, \imath, \jmath\}$ defined by the table given in Example 3.1.9. Define a fuzzy set ϖ in $\digamma$ by:

$$\varpi(0) = \varpi(\jmath) = 0.7,\ \varpi(\imath) = 0.4.$$

Then ϖ is a $FwH_{BCK}PII$ of $\digamma$ but it is not a $FH_{BCK}PII$ of $\digamma$ because:

$\imath \leq \jmath$ but $\varpi(\imath) = 0.4 < 0.7 = \varpi(\jmath)$.

Now consider the "hyper BCK-algebra" $\digamma = \{0, \imath, \jmath\}$ defined by the table given in Example 3.1.10. Define a fuzzy set ϖ in $\digamma$ by:

$$\varpi(0) = \varpi(\imath) = 0.6, \ \varpi(\jmath) = 0.3.$$

Then ϖ is a $FH_{BCK}PII$ of $\digamma$ but it is not a $FsH_{BCK}PII$ of $\digamma$ because for $\jmath \in \jmath \circ \jmath$,

$$\varpi(\jmath) = 0.3 < 0.6 = \min \ \{\sup_{a \in ((\jmath \circ \jmath) \circ \jmath) \circ (0 \circ \jmath)} \varpi(a), \ \varpi(0)\}.$$

Theorem 3.7.10. *A fuzzy set* ϖ *is a* $FH_{BCK}PII$ *(resp.* $FwH_{BCK}PII$, $FsH_{BCK}PII$, $FrH_{BCK}PII$*) of* $\digamma$ *if and only if for all* $\delta \in [0, 1]$, $\varpi_\delta \neq \emptyset$ *is a* $H_{BCK}PII$ *(resp.* $wH_{BCK}PII$, $sH_{BCK}PII$, $rH_{BCK}PII$*) of* $\digamma$*.*

Proof. Let ϖ be a $FH_{BCK}PII$ of $\digamma$. Since $\varpi_\delta \neq \emptyset$, so for any $\imath \in \varpi_\delta$, $\varpi(\imath) \geq \delta$. Since any $FH_{BCK}PII$ is also a $FwH_{BCK}PII$ (by Theorem 3.7.9), so ϖ is also a $FwH_{BCK}PII$ of $\digamma$. Thus $\varpi(0) \geq \varpi(\imath) \geq \delta$, for all $\imath \in \digamma$, which implies $0 \in \varpi_\delta$.

Let $((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \ll \varpi_\delta$ and $\jmath \in \varpi_\delta$, for some $\imath, \jmath, \ell \in \digamma$. Then for all $a \in ((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell)$, $\exists \ b \in \varpi_\delta$, s.t, $a \ll b$. So $\varpi(a) \geq \varpi(b) \geq \delta$, for all $a \in ((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell)$. Thus $\inf_{a \ \in \ ((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell)} \ \varpi(a) \geq \delta$. Also $\varpi(\jmath) \geq \delta$, as $\jmath \in \varpi_\delta$. Therefore for all $v \in \imath \circ \ell$, ϖ satisfies,

$\varpi(v) \geq \min\{\inf_{a \ \in \ ((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell)} \ \varpi(a), \ \varpi(\jmath)\} \geq \min\{\delta, \delta\} = \delta$

$\Rightarrow v \in \varpi_\delta \Rightarrow \imath \circ \ell \subseteq \varpi_\delta$.

Hence ϖ_δ is $H_{BCK}PII$ of $\digamma$.

Conversely, let $\varpi_\delta \neq \emptyset$ be a $H_{BCK}PII$ of $\digamma$ for all $\delta \in [0, 1]$. Let $\imath \ll \jmath$ for some $\imath, \jmath \in \digamma$ and put $\varpi(\jmath) = \delta$. Then $\jmath \in \varpi_\delta$. So $\imath \ll \jmath \in \varpi_\delta \Rightarrow \imath \ll \varpi_\delta$. Being a $H_{BCK}PII$, ϖ_δ is also a $H_{BCK}I$ of $\digamma$ (by Theorem 3.7.7) therefore by Proposition 3.1.5, $\imath \in \varpi_\delta$. Hence $\varpi(\imath) \geq \delta = \varpi(\jmath)$. That is $\imath \ll \jmath \Rightarrow \varpi(\imath) \geq \varpi(\jmath)$, for all $\imath, \jmath \in \digamma$.

Moreover, for any $\imath, \jmath, \ell \in \digamma$, let $d = \min\{\inf_{c \,\in\, ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)} \varpi(c),\ \varpi(\jmath)\}$. Then $\varpi(\jmath) \geq d \Rightarrow \jmath \in \varpi_d$ and for all $e \in ((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell)$, $\varpi(e) \geq \inf_{c \,\in\, ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)} \varpi(c) \geq d$, which implies that $e \in \varpi_d$.

Thus $((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \subseteq \varpi_d$.

By Proposition 3.1.2(v), $((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell) \subseteq \varpi_d \Rightarrow ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell) \ll \varpi_d$, which along with $\jmath \in \varpi_d$ implies $\imath \circ \ell \subseteq \varpi_d$. Hence for all $u \in \imath \circ \ell$,

$\varpi(u) \geq d = \min\{\inf_{c \,\in\, ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)} \varpi(c),\ \varpi(\jmath)\}$.

Thus ϖ is a $FH_{BCK}PII$ of $\digamma$. □

Theorem 3.7.11. *If a fuzzy set ϖ is a $FH_{BCK}PII$ (resp. $FwH_{BCK}PII$, $FsH_{BCK}PII$, $FrH_{BCK}PII$) of $\digamma$ then the set $\Xi = \{\imath \in \digamma \mid \varpi(\imath) = \varpi(0)\}$ is a $H_{BCK}PII$ (resp. $wH_{BCK}PII$, $sH_{BCK}PII$, $rH_{BCK}PII$) of $\digamma$.*

Proof. Let ϖ be a $FH_{BCK}PII$ of $\digamma$. Clearly $0 \in \Xi$.

Let $((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \ll \Xi$ and $\jmath \in \Xi$ for some $\imath, \jmath, \ell \in \digamma$. Then for all $a \in ((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell)$, $\exists\ b \in \Xi$ such that $a \ll b$. Therefore $\varpi(a) \geq \varpi(b) = \varpi(0)$. But being a $FH_{BCK}PII$, ϖ is also a $FwH_{BCK}PII$ of $\digamma$ (by Theorem 3.7.9), so ϖ satisfies $\varpi(0) \geq \varpi(v)$, for all $v \in \digamma$. This implies $\varpi(0) \geq \varpi(a)$, for all $a \in ((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell)$. Therefore $\varpi(a) = \varpi(0)$, for all $a \in ((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell)$, i.e., $\inf_{a \,\in\, ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)} \varpi(a) = \varpi(0)$. Also $\varpi(\jmath) = \varpi(0)$. Being a $FH_{BCK}PII$, for all $t \in \imath \circ \ell$, ϖ satisfies,

$\varpi(t) \geq \min\{\inf_{a \,\in\, ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)} \varpi(a),\ \varpi(\jmath)\} = \min\{\varpi(0),\ \varpi(0)\} = \varpi(0)$.

Since $\varpi(0) \geq \varpi(v)$, for all $v \in \digamma$, therefore $\varpi(t) = \varpi(0)$, for all $t \in \imath \circ \ell$.

Thus $\imath \circ \ell \subseteq \Xi$. Hence Ξ is a $H_{BCK}PII$ of $\digamma$. □

The transfer principle for fuzzy sets described in [38] suggest the following theorem.

Theorem 3.7.12. *For any subset* Ξ *of* $\digamma$*, let* ϖ *be a fuzzy set in* $\digamma$ *defined by:*

$$\varpi(\imath) = \begin{cases} \delta & if\ \imath \in \Xi \\ 0 & if\ \imath \notin \Xi \end{cases}$$

for all $\imath \in \digamma$*, where* $\delta \in (0,1]$*. Then* Ξ *is a* $H_{BCK}PII$ *(resp.* $wH_{BCK}PII$*,* $sH_{BCK}PII$*,* $rH_{BCK}PII$*) of* $\digamma \iff \varpi$ *is a* $FH_{BCK}PII$ *(resp.* $FwH_{BCK}PII$*,* $FsH_{BCK}PII$*,* $FrH_{BCK}PII$*) of* $\digamma$*.*

Proof. Let Ξ be a $H_{BCK}PII$ of $\digamma$. Let $\imath \ll \jmath$ for some $\imath, \jmath \in \digamma$ and put $\varpi(\jmath) = \delta$. Then $\jmath \in \varpi_\delta$. So $\imath \ll \jmath \in \varpi_\delta \Rightarrow \imath \ll \varpi_\delta$. Being a $H_{BCK}PII$, ϖ_δ is also a $H_{BCK}I$ of $\digamma$ (by Theorem 3.7.2) therefore by Proposition 3.1.5, $\imath \in \varpi_\delta$. Hence $\varpi(\imath) \geq \delta = \varpi(\jmath)$. That is $\imath \ll \jmath \Rightarrow \varpi(\imath) \geq \varpi(\jmath)$, for all $\imath, \jmath \in \digamma$.

Moreover, for any $\imath, \jmath, \ell \in \digamma$,

If $((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \ll \Xi$ and $\jmath \in \Xi$ then $\imath \circ \ell \subseteq \Xi$. Since Ξ is a $H_{BCK}PII$ of $\digamma$, so it is also a $H_{BCK}I$ of $\digamma$ (by Theorem 3.7.2). Therefore by Proposition 3.1.5,

$((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \subseteq \Xi$. Thus $\varpi(a) = \delta$, for all $a \in ((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell)$ which implies $\inf_{a\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi(a) = \delta$. Also $\varpi(\jmath) = \delta$. Since $\imath \circ \ell \subseteq \Xi$, for all $u \in \imath \circ \ell$,

$\varpi(u) = \delta = \min\{\inf_{a\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi(a),\ \varpi(\jmath)\}$.

If $((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \not\ll \Xi$ and $\jmath \notin \Xi$ then,

$\min\{\inf_{a\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi(a),\ \varpi(\jmath)\} = 0 \leq \varpi(u)$, for all $u \in \imath \circ \ell$.

If $((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \not\ll \Xi$ and $\jmath \in \Xi$ (OR) If $((\imath \circ \ell) \circ \ell) \circ (\jmath \circ \ell) \ll \Xi$ and $\jmath \notin \Xi$. Then in both of these cases,

$\min\{\inf_{a \ \in \ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)} \ \varpi(a), \ \varpi(\jmath)\} = 0 \leq \varpi(u)$, for all $u \in \imath \circ \ell$.

Hence ϖ is a $FH_{BCK}PII$ of $\digamma$.

Conversely, let ϖ be a $FH_{BCK}PII$ of $\digamma$. Then by Theorem 3.7.10, for all $\delta \in (0, 1]$, $\varpi_\delta = \Xi$ is a $H_{BCK}PII$ of $\digamma$. □

For a family $\{\varpi_\varepsilon \mid \varepsilon \in \Upsilon\}$ of fuzzy sets in a non-empty set Ξ, define the join $\vee_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon$ and meet $\wedge_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon$ as follows:

$$(\vee_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon)(\imath) = \sup_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon(\imath).$$

$$(\wedge_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon)(\imath) = \inf_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon(\imath).$$

for all $\imath \in \Xi$, where Υ is any indexing set.

Theorem 3.7.13. *The family of $FH_{BCK}PIIs$ (resp. $FwH_{BCK}PIIs$, $FsH_{BCK}PIIs$, $FrH_{BCK}PIIs$) of $\digamma$ is a completely distributive lattice with respect to join and meet.*

Proof. Let $\{\varpi_\varepsilon \mid \varepsilon \in \Upsilon\}$ be a family of $FH_{BCK}PIIs$ of $\digamma$. Since $[0, 1]$ is a "completely distributive lattice" w.r.t the normal ordering in $[0, 1]$, it is enough to demonstrate that $\vee_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon$ and $\wedge_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon$ are $FH_{BCK}PIIs$ of $\digamma$.

For any $\imath, \jmath \in \digamma$, if $\imath \ll \jmath$ then,

$(\vee_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon)(\imath) = \sup_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon(\imath) \geq \sup_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon(\jmath) = (\vee_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon)(\jmath)$.

$\Rightarrow (\vee_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon)(\imath) \geq (\vee_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon)(\jmath)$.

Moreover, for any $\imath, \jmath, \ell \in \digamma$ and for all $t \in \imath \circ \ell$,

$(\vee_{\varepsilon\in\Upsilon} \varpi_\varepsilon)(t) = \sup_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon(t) \geq \sup_{\varepsilon\in\Upsilon}[\min\{\inf_{a \ \in \ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)} \ \varpi_\varepsilon(a), \ \varpi_\varepsilon(\jmath)\}]$

$= \min\{\sup_{\varepsilon\in\Upsilon} \ (\inf_{a \ \in \ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)} \ \varpi_\varepsilon(a)), \ \sup_{\varepsilon\in\Upsilon} \ (\varpi_\varepsilon(\jmath))\}$

$= \min\{\inf_{a \ \in \ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)} \ (\sup_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon(a)), \ \sup_{\varepsilon\in\Upsilon} \ (\varpi_\varepsilon(\jmath))\}$

$= \min\{\inf_{a \ \in \ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)} \ ((\vee_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon)(a)), \ (\vee_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon)(\jmath)\}$

$\Rightarrow (\vee_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon)(t) \geq \min\{\inf_{a \ \in \ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)} \ ((\vee_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon)(a)), \ (\vee_{\varepsilon\in\Upsilon} \ \varpi_\varepsilon)(\jmath)\}$.

Hence $\vee_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon$ is a $FH_{BCK}PII$ of $\digamma$.

Now, we demonstrate that $\wedge_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon$ is a $FH_{BCK}PII$ of $\digamma$.

For any $\imath, \jmath \in \digamma$, if $\imath \ll \jmath$ then

$(\wedge_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon)(\imath) = \inf_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon(\imath) \geq \inf_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon(\jmath) = (\wedge_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon)(\jmath)$

$\Rightarrow (\wedge_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon)(\imath) \geq (\wedge_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon)(\jmath)$.

Moreover, for any $\imath, \jmath, \ell \in \digamma$ and for all $t \in \imath \circ \ell$,

$(\wedge_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon)(t) = \inf_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon(t) \geq \inf_{\varepsilon\in\Upsilon}\ [\min\{\inf_{a\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi_\varepsilon(a),\ \varpi_\varepsilon(\jmath)\}]$

$= \min\{\inf_{\varepsilon\in\Upsilon}\ (\inf_{a\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi_\varepsilon(a)),\ \inf_{\varepsilon\in\Upsilon}\ (\varpi_\varepsilon(\jmath))\}$

$= \min\{\inf_{a\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ (\inf_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon(a)),\ \inf_{\varepsilon\in\Upsilon}\ (\varpi_\varepsilon(\jmath))\}$

$= \min\{\inf_{a\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ ((\wedge_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon)(a)),\ (\wedge_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon)(\jmath)\}$

$\Rightarrow (\wedge_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon)(t) \geq \min\ \{\inf_{a\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ ((\wedge_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon)(a)),\ (\wedge_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon)(\jmath)\}$.

Hence $\wedge_{\varepsilon\in\Upsilon}\ \varpi_\varepsilon$ is a $FH_{BCK}PII$ of $\digamma$. $\square$

Let $\digamma$ and $\daleth$ be hyper BCK-algebras. A mapping $\Gamma : \digamma \to \daleth$ is called a "hyper homomorphism" if

(i) $\Gamma(0) = 0$

(ii) $\Gamma(\imath \circ \jmath) = \Gamma(\imath) \circ \Gamma(\jmath),\ \forall\ \imath, \jmath \in \digamma$.

Theorem 3.7.14. *Let $\Gamma : \digamma \to \daleth$ be an onto hyper homomorphism from a hyper BCK-algebra $\digamma$ to a hyper BCK-algebra $\daleth$. If ξ is a $FH_{BCK}PII$ (resp. $FwH_{BCK}PII$, $FsH_{BCK}PII$, $FrH_{BCK}PII$) of $\daleth$ then the hyper homomorphic pre-image ϖ of ξ under Γ is a $FH_{BCK}PII$ (resp. $FwH_{BCK}PII$, $FsH_{BCK}PII$, $FrH_{BCK}PII$) of $\digamma$.*

Proof. Let ξ be a $FH_{BCK}PII$ of $\daleth$. Since ϖ is a hyper homomorphic pre-image of ξ under Γ then ϖ is defined by $\varpi = \xi \circ \Gamma$ that is $\varpi(\imath) = \xi(\Gamma(\imath))$ for all $\imath \in \digamma$.

For any $\Gamma(\imath), \Gamma(\imath) \in \daleth$, where $\imath, \jmath \in \digamma$, If $\imath \ll \jmath$ then $0 \in \imath \circ \jmath$, which implies $\Gamma(0) \in \Gamma(\imath \circ \jmath)$

$\Rightarrow 0 \in \Gamma(\imath) \circ \Gamma(\jmath) \Rightarrow \Gamma(\imath) \ll \Gamma(\jmath)$

$\Rightarrow \xi(\Gamma(\imath)) \geq \xi(\Gamma(\jmath)) \Rightarrow \varpi(\imath) \geq \varpi(\jmath)$

i.e., $\imath \ll \jmath \Rightarrow \varpi(\imath) \geq \varpi(\jmath)$, for all $\imath, \jmath \in F$.

Moreover, for all $t \in \imath \circ \ell$, $\Gamma(t) \in \Gamma(\imath \circ \ell) = \Gamma(\imath) \circ (\ell)$, where $\imath, \ell \in F$ and $\Gamma(\imath), \Gamma(\ell) \in \daleth$,

$\varpi(t) = \xi(\Gamma(t)) \geq \min\{\inf_{\Gamma(a) \in ((\Gamma(\imath)\circ\Gamma(\ell))\circ\Gamma(\ell))\circ(\jmath'\circ\Gamma(\ell))} \xi(\Gamma(a)),\ \xi(\jmath')\}$,

where $\jmath' \in \daleth$. Since $\Gamma : F \to \daleth$ is an onto hyper homomorphism, so for $\jmath' \in \daleth$, $\exists\ \jmath \in F$ such that $\Gamma(\jmath) = \jmath'$. Hence we get,

$\varpi(t) \geq \min\{\inf_{\Gamma(a) \in ((\Gamma(\imath)\circ\Gamma(\ell))\circ\Gamma(\ell))\circ(\Gamma(\jmath)\circ\Gamma(\ell))=\Gamma(((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell))} \xi(\Gamma(a)),\ \xi(\Gamma(\jmath))\}$

$\Rightarrow \varpi(t) \geq \min\{\inf_{a \in ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)} \varpi(a),\ \varpi(\jmath)\}\ \forall\ \imath, \jmath, \ell \in F$.

Hence ϖ is a $FH_{BCK}PII$ of F. □

Theorem 3.7.15. *If a fuzzy set ϖ is a $FH_{BCK}PII$ (resp. $FwH_{BCK}PII$, $FsH_{BCK}PII$, $FrH_{BCK}PII$) of F, then $\varpi = \varpi_1 \times \varpi_2$, where ϖ_1 and ϖ_2 are fuzzy sets on F_1 and F_2 respectively.*

Proof. Let ϖ be a $FH_{BCK}PII$ of F. Then for any $(\imath, u),\ (\jmath, v),\ (\ell, w) \in F$, where $\imath, \jmath, \ell \in F_1$ and $u, v, w \in F_2$ and for all $(a, b) \in (\imath, u) \circ (\ell, w) = (\imath \circ \ell,\ u \circ w)$,

$\varpi((a, b)) \geq \min\{\inf_{(c,d) \in (((\imath,u)\circ(\ell,w))\circ(\ell,w))\circ((\jmath,v)\circ(\ell,w))} \varpi((c, d)),\ \varpi((\jmath, v))\}$.

Substituting $d = \jmath = \ell = w = 0$ and $v = u$,

$\varpi((\imath, u)) \geq \min\{inf_{(c,0) \in (((\imath,u)\circ(0,0))\circ(0,0))\circ((0,u)\circ(0,0))} \varpi((c, 0)),\ \varpi((0, u))\}$

$\Rightarrow \varpi((\imath, u)) \geq \min\{inf_{(c,0) \in (\imath, u\circ u)} \varpi((c, 0)),\ \varpi((0, u))\}$

$\Rightarrow \varpi((\imath, u)) \geq \min\{\varpi_1(\imath),\ \varpi_2(u)\}$

$\Rightarrow \varpi((\imath, u)) \geq (\varpi_1 \times \varpi_2)((\imath, u))$

$\Rightarrow \varpi_1 \times \varpi_2 \subseteq \varpi \qquad (1)$

Conversely, since $(\imath, 0) \ll (\imath, u)$ and $(0, u) \ll (\imath, u)$

$\Rightarrow \varpi((\imath, 0)) \geq \varpi((\imath, u))$ and $\varpi((0, u)) \geq \varpi((\imath, u))$

Therefore $(\varpi_1 \times \varpi_2)((\imath, u)) = \min\{\varpi_1(\imath),\ \varpi_2(u)\} = \min\{\varpi(\imath, 0),\ \varpi(0, u)\}$

$\geq \min\{\varpi(\imath, u),\ \varpi(\imath, u)\} =\ \varpi(\imath, u)$

$\Rightarrow (\varpi_1 \times \varpi_2)((\imath, u)) \geq \varpi(\imath, u)$

$\Rightarrow \varpi \subseteq \varpi_1 \times \varpi_2$ (2)

Hence from (1) and (2) we have, $\varpi_1 \times \varpi_2 = \varpi$ □

Theorem 3.7.16. *A fuzzy set ϖ is a $FH_{BCK}PII$ (resp. $FwH_{BCK}PII$, $FsH_{BCK}PII$, $FrH_{BCK}PII$) of $\digamma$ if and only if ϖ_1 and ϖ_2 are $FH_{BCK}PIIs$ (resp. $FwH_{BCK}PIIs$, $FsH_{BCK}PIIs$, $FrH_{BCK}PIIs$) of $\digamma_1$ and $\digamma_2$ respectively.*

Proof. Let ϖ be a $FH_{BCK}PII$ of $\digamma$ and let $\imath_1 \ll \imath_2$ for some $\imath_1, \imath_2 \in \digamma_1$. Then $(\imath_1, 0) \ll (\imath_2, 0)$ which implies $\varpi((\imath_1, 0)) = \varpi_1(\imath_1) \geq \varpi((\imath_2, 0)) = \varpi_1(\imath_2)$, that is, $\varpi_1(\imath_1) \geq \varpi_1(\imath_2)$.

Moreover for any $\imath_1, \jmath_1, \ell_1 \in \digamma_1$,

Let $\delta = \min\{\inf_{a\ \in\ ((\imath_1 \circ \ell_1) \circ \ell_1) \circ (\jmath_1 \circ \ell_1)}\ \varpi_1(a),\ \varpi_1(\jmath_1)\}$

Then for all $b \in ((\imath_1 \circ \ell_1) \circ \ell_1) \circ (\jmath_1 \circ \ell_1)$,

$\varpi_1(b) \geq inf_{a\ \in\ ((\imath_1 \circ \ell_1) \circ \ell_1) \circ (\jmath_1 \circ \ell_1)}\ \varpi_1(a) \geq \delta$ and $\varpi_1(\jmath_1) \geq \delta$

$\Rightarrow \varpi((b, 0)) \geq \delta$ and $\varpi((\jmath_1, 0)) \geq \delta$,

for all $(b, 0) \in (((\imath_1, 0) \circ (\ell_1, 0)) \circ (\ell_1, 0)) \circ ((\jmath_1, 0) \circ (\ell_1, 0))$

$\Rightarrow (b, 0) \in \varpi_\delta$ and $(\jmath_1, 0) \in \varpi_\delta$

$\Rightarrow (((\imath_1, 0) \circ (\ell_1, 0)) \circ (\ell_1, 0)) \circ ((\jmath_1, 0) \circ (\ell_1, 0)) \subseteq \varpi_\delta$ and $(\jmath_1, 0) \in \varpi_\delta$

Since by Theorem 3.7.10, $\varpi_\delta \neq \emptyset$ is a $H_{BCK}PII$ of $\digamma$ and so is a $wH_{BCK}PII$ of $\digamma$ (by Theorem 3.7.3), for any $\delta \in [0, 1]$.

Thus $(((\imath_1, 0) \circ (\ell_1, 0)) \circ (\ell_1, 0)) \circ ((\jmath_1, 0) \circ (\ell_1, 0)) \subseteq \varpi_\delta$ and $(\jmath_1, 0) \in \varpi_\delta$ imply $(\imath_1, 0) \circ (\ell_1, 0) \subseteq \varpi_\delta$

Therefore $\varpi((s, 0)) \geq \delta$, for all $(s, 0) \in (\imath_1, 0) \circ (\ell_1, 0) = (\imath_1 \circ \ell_1,\ 0)$,

$\Rightarrow \varpi_1(s) \geq \delta = \min\{inf_{a\ \in\ ((\imath_1 \circ \ell_1)\circ \ell_1)\circ(\jmath_1 \circ \ell_1)}\ \varpi_1(a),\ \varpi_1(\jmath_1)\}$, for all $s \in \imath_1 \circ \ell_1$.

Hence ϖ_1 is a $FH_{BCK}PII$ of F_1.

Similarly, we can demonstrate that ϖ_2 is a $FH_{BCK}PII$ of F_2.

Conversely, let ϖ_1 and ϖ_2 be two $FH_{BCK}PIIs$ of F_1 and F_2 respectively.

For any $(\imath, u), (\jmath, v) \in F$, where $\imath, \jmath \in F_1$ and $u, v \in F_2$, let $(\imath, u) \ll (\jmath, v)$

Since $(\imath, u) \ll (\jmath, v)$ if $\imath \ll \jmath$ and $u \ll v$

$\Rightarrow \varpi_1(\imath) \geq \varpi_1(\jmath)$ and $\varpi_2(u) \geq \varpi_2(v)$

$\Rightarrow \min\{\varpi_1(\imath),\ \varpi_2(u)\} \geq \min\{\varpi_1(\jmath),\ \varpi_2(v)\}$

$\Rightarrow (\varpi_1 \times \varpi_2)((\imath, u)) \geq (\varpi_1 \times \varpi_2)((\jmath, v))$

$\Rightarrow \varpi((\imath, u)) \geq \varpi((\jmath, v))$

Therefore $(\imath, u) \ll (\jmath, v) \Rightarrow \varpi((\imath, u)) \geq \varpi((\jmath, v))$.

Moreover, for any $(\imath, u),\ (\jmath, v),\ (\ell, w) \in F$, where $\imath, \jmath, \ell \in F_1$ and $u, v, w \in F_2$ and for all $(a, b) \in (\imath, u) \circ (\ell, w) = (\imath \circ \ell,\ u \circ w)$,

$\varpi((a, b)) = (\varpi_1 \times \varpi_2)((a, b)) = \min\{\varpi_1(a),\ \varpi_2(b)\}$

$\geq \min\ [\min\{\inf_{c\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi_1(c),\ \varpi_1(\jmath)\},\ \min\{\inf_{d\ \in\ ((u\circ w)\circ w)\circ(v\circ w)}\ \varpi_2(d),\ \varpi_2(v)\}]$

$= \min\ [\min\{\inf_{c\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell)}\ \varpi_1(c),\ \inf_{d\ \in\ ((u\circ w)\circ w)\circ(v\circ w)}\ \varpi_2(d)\},\ \min\{\ \varpi_1(\jmath),\ \varpi_2(v)\}]$

$= \min\ [\inf_{c\ \in\ ((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell),\ d\in((u\circ w)\circ w)\circ(v\circ w)}\ \{\min\{\varpi_1(c),\ \varpi_2(d)\}\},\ \min\{\ \varpi_1(\jmath),\ \varpi_2(v)\}]$

$= \min\{\inf_{(c,d)\ \in\ (((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell),\ ((u\circ w)\circ w)\circ(v\circ w)}\ (\varpi_1 \times \varpi_2)((c, d)),\ (\varpi_1 \times \varpi_2)((\jmath, v))\}$

$= \min\{\inf_{(c,d)\ \in\ (((\imath\circ\ell)\circ\ell)\circ(\jmath\circ\ell),\ ((u\circ w)\circ w)\circ(v\circ w)}\ \varpi((c, d)),\ \varpi((\jmath, v))\}$

$\Rightarrow \varpi((a, b)) \geq \min\{\inf_{(c,d)\ \in\ (((\imath,u)\circ(\ell,w))\circ(\ell,w))\circ((\jmath,v)\circ(\ell,w))}\ \varpi((c, d)),\ \varpi((\jmath, v))\}$.

Hence ϖ is a $FH_{BCK}PII$ of F. $\square$

3.8 Relationship between fuzzy (weak, strong, reflexive) hyper BCK-(implicative, positive implicative, commutative) ideals

Theorem 3.8.1. *Any $FH_{BCK}II$ (resp. $FwH_{BCK}II$, $FsH_{BCK}II$, $FrH_{BCK}II$) of $\digamma$ is a $FH_{BCK}CI$ (resp. $FwH_{BCK}CI$, $FsH_{BCK}CI$, $FrH_{BCK}PII$) of $\digamma$.*

Proof. Let ϖ be a $FH_{BCK}II$ of $\digamma$. Then for any $\imath, \jmath, \ell \in \digamma$ and for all $t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$,

$\varpi(t) \geq \min\{\inf_{a\ \in\ ((\imath\circ\jmath)\circ\jmath)\circ\ell}\ \varpi(a),\ \varpi(\ell)\}$.

Since, $((\imath \circ \jmath) \circ \jmath) \circ \ell = ((\imath \circ \jmath) \circ \ell) \circ \jmath \ll (\imath \circ \jmath) \circ \ell$ (by $(HK2)$ and Proposition 3.1.2(ii))

Then for all $a \in ((\imath \circ \jmath) \circ \jmath) \circ \ell$, $\exists\ b \in (\imath \circ \jmath) \circ \ell$, s.t, $a \ll b$

$\Rightarrow \varpi(a) \geq \varpi(b)$, for all $a \in ((\imath \circ \jmath) \circ \jmath) \circ \ell$ and for some $b \in (\imath \circ \jmath) \circ \ell$

$\Rightarrow \inf_{a\ \in\ ((\imath\circ\jmath)\circ\jmath)\circ\ell}\ \varpi(a) \geq \varpi(b) \geq \inf_{c\ \in\ (\imath\circ\jmath)\circ\ell}\ \varpi(c)$

Thus for all $t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$,

$\varpi(t) \geq \min\{\inf_{a\ \in\ ((\imath\circ\jmath)\circ\jmath)\circ\ell}\ \varpi(a),\ \varpi(\ell)\} \geq \min\{\inf_{c\ \in\ (\imath\circ\jmath)\circ\ell}\ \varpi(c),\ \varpi(\ell)\}$.

Hence ϖ is a $FH_{BCK}CI$ of $\digamma$. □

The converse of the above theorem isn't valid. It can be observed by the succeeding example.

Example 3.8.2. Let $\digamma = \{0, \imath, \jmath, \ell\}$ be a "hyper BCK-algebra" defined by the succeeding table:

$\circ$	0	$\imath$	$\jmath$	ℓ
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\imath$	$\{\imath\}$	$\{0\}$	$\{0\}$	$\{\imath\}$
$\jmath$	$\{\jmath\}$	$\{\imath\}$	$\{0\}$	$\{\jmath\}$
ℓ	$\{\ell\}$	$\{\ell\}$	$\{\ell\}$	$\{0\}$

Define a fuzzy set ϖ in $\digamma$ by:

$\varpi(0) = \varpi(\ell) = 0.9,\ \varpi(\imath) = \varpi(\jmath) = 0.3.$

Then ϖ is a $FH_{BCK}CI$ (resp. $FwH_{BCK}CI,\ FsH_{BCK}CI,\ FrH_{BCK}PII$) of $\digamma$ but it is not a $FH_{BCK}II$ (resp. $FwH_{BCK}II,\ FsH_{BCK}II,\ FrH_{BCK}II$) of $\digamma$ because for $\imath \in \jmath \circ (\imath \circ (\imath \circ \jmath))$,

$\varpi(\imath) = 0.3 < 0.9 = \varpi(0) = \min\{\inf_{a\ \in\ ((\jmath\circ\imath)\circ\imath)\circ 0}\ \varpi(a),\ \varpi(0)\}$

and $\varpi(\imath) = 0.3 < 0.9 = \varpi(0) = \min\{\sup_{a\ \in\ ((\jmath\circ\imath)\circ\imath)\circ 0}\ \varpi(a),\ \varpi(0)\}$.

Theorem 3.8.3. *Any $FH_{BCK}II$ (resp. $FwH_{BCK}II,\ FsH_{BCK}II,\ FrH_{BCK}II$) of $\digamma$ is a $FH_{BCK}PII$ (resp. $FwH_{BCK}PII,\ FsH_{BCK}PII,\ FrH_{BCK}PII$) of $\digamma$.*

Proof. Let ϖ be a $FH_{BCK}II$ of $\digamma$. Then for any $\imath, \jmath, \ell \in \digamma$ and for all $t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$,

$\varpi(t) \geq \min\{inf_{a\ \in\ ((\imath\circ\jmath)\circ\jmath)\circ\ell}\ \varpi(a),\ \varpi(\ell)\}$.

Since $\jmath \circ (\jmath \circ \imath) \ll \jmath \Rightarrow \imath \circ \jmath \ll \imath \circ (\jmath \circ (\jmath \circ \imath))$ (by Proposition 3.1.2 (ii, x))

Then for all $s \in \imath \circ \jmath,\ \exists\ t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$ such that $s \ll t$

$\Rightarrow \varpi(s) \geq \varpi(t)$ for all $s \in \imath \circ \jmath$ and for some $t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$

Moreover, $\ell \circ \jmath \ll \ell \Rightarrow ((\imath \circ \jmath) \circ \jmath) \circ \ell \ll ((\imath \circ \jmath) \circ \jmath) \circ (\ell \circ \jmath)$.

Then for all $a \in ((\imath \circ \jmath) \circ \jmath) \circ \ell,\ \exists\ b \in ((\imath \circ \jmath) \circ \jmath) \circ (\ell \circ \jmath)$, s.t, $a \ll b$

$\Rightarrow \varpi(a) \geq \varpi(b)$ for all $a \in ((\imath \circ \jmath) \circ \jmath) \circ \ell$ and some $b \in ((\imath \circ \jmath) \circ \jmath) \circ (\ell \circ \jmath)$

$\Rightarrow \inf_{a \in ((\imath \circ \jmath) \circ \jmath) \circ \ell} \varpi(a) \geq \varpi(b) \geq \inf_{c \in ((\imath \circ \jmath) \circ \jmath) \circ (\ell \circ \jmath)} \varpi(c)$

Thus for all $s \in \imath \circ \jmath$,

$\varpi(s) \geq \varpi(t) \geq \min\{\inf_{a \in ((\imath \circ \jmath) \circ \jmath) \circ \ell} \varpi(a),\ \varpi(\ell)\} \geq \min\{\inf_{c \in ((\imath \circ \jmath) \circ \jmath) \circ (\ell \circ \jmath)} \varpi(c),\ \varpi(\ell)\}$

$\Rightarrow \varpi(s) \geq \min\{\inf_{c \in ((\imath \circ \jmath) \circ \jmath) \circ (\ell \circ \jmath)} \varpi(c),\ \varpi(\ell)\}$.

Hence ϖ is a $FH_{BCK}PII$ of $\digamma$. □

The converse of the above theorem isn't valid. It can be observed by the succeeding example.

Example 3.8.4. Let $\digamma = \{0, \imath, \jmath, \ell\}$ be a "hyper BCK-algebra" defined by the succeeding table:

$\circ$	0	$\imath$	$\jmath$	ℓ
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\imath$	$\{\imath\}$	$\{0\}$	$\{0\}$	$\{\imath\}$
$\jmath$	$\{\jmath\}$	$\{\jmath\}$	$\{0\}$	$\{\jmath\}$
ℓ	$\{\ell\}$	$\{\ell\}$	$\{\ell\}$	$\{0\}$

Define a fuzzy set ϖ in $\digamma$ by:

$\varpi(0) = \varpi(\ell) = 0.6,\ \varpi(\imath) = \varpi(\jmath) = 0.2.$

Then ϖ is a $FH_{BCK}PII$ (resp. $FwH_{BCK}PII,\ FsH_{BCK}PII,\ FrH_{BCK}PII$) of $\digamma$ but it isn't a $FH_{BCK}II$ (resp. $FwH_{BCK}II,\ FsH_{BCK}II,\ FrH_{BCK}II$) of $\digamma$ because for $\imath \in \imath \circ (\jmath \circ (\jmath \circ \imath))$,

$\varpi(\imath) = 0.2 < 0.6 = \varpi(0) = \min\{\inf_{a \in ((\imath \circ \jmath) \circ \jmath) \circ 0} \varpi(a),\ \varpi(0)\}$

and $\varpi(\imath) = 0.2 < 0.6 = \varpi(0) = \min\{\sup_{a \in ((\imath \circ \jmath) \circ \jmath) \circ 0} \varpi(a),\ \varpi(0)\}$.

Theorem 3.8.5. *A fuzzy set ϖ in $\digamma$ is a $FH_{BCK}II$ (resp. $FwH_{BCK}II,\ FsH_{BCK}II,\ FrH_{BCK}II$) of $\digamma$ if and only if ϖ is both a $FH_{BCK}PII$ (resp. $FwH_{BCK}PII,\ FsH_{BCK}PII,\ FrH_{BCK}PII$) and a $FH_{BCK}CI$ (resp. $FwH_{BCK}CI,\ FsH_{BCK}CI,\ FrH_{BCK}PII$) of $\digamma$.*

Proof. Let ϖ be a $FH_{BCK}II$ of $\digamma$. Then by Theorem 3.8.1 and Theorem 3.8.3, ϖ is both a $FH_{BCK}CI$ and a $FH_{BCK}PII$ of $\digamma$.

Conversely, let ϖ be both a $FH_{BCK}CI$ and a $FH_{BCK}PII$ of $\digamma$.

Then for any $\imath, \jmath, \ell \in \digamma$ and for all $t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$,

$\varpi(t) \geq \min\{\inf_{a\ \in\ (\imath\circ\jmath)\circ\ell}\ \varpi(a),\ \varpi(\ell)\}$.

Since ϖ is also a $FH_{BCK}PII$ of $\digamma$, thus for any $\imath, \jmath, \ell \in \digamma$ and for all $a \in (\imath \circ \jmath) \circ \ell = (\imath \circ \ell) \circ \jmath$,

$\varpi(a) \geq \min\{\inf_{b\ \in\ (((\imath\circ\ell)\circ\jmath)\circ\jmath)\circ(0\circ\jmath)}\ \varpi(b),\ \varpi(0)\}$

$\Rightarrow \varpi(a) \geq \inf_{b\ \in\ ((\imath\circ\ell)\circ\jmath)\circ\jmath}\ \varpi(b)$, for all $a \in (\imath \circ \jmath) \circ \ell$

$\Rightarrow \inf_{a\ \in\ (\imath\circ\jmath)\circ\ell} \geq \inf_{b\ \in\ ((\imath\circ\jmath)\circ\jmath)\circ\ell}\ \varpi(b)$ (since $((\imath \circ \ell) \circ \jmath) \circ \jmath = ((\imath \circ \jmath) \circ \jmath) \circ \ell)$

Therefore, for any $\imath, \jmath, \ell \in \digamma$ and for all $t \in \imath \circ (\jmath \circ (\jmath \circ \imath))$,

$\varpi(t) \geq \min\{\inf_{a\ \in\ (\imath\circ\jmath)\circ\ell}\ \varpi(a),\ \varpi(\ell)\} \geq \min\{\inf_{b\ \in\ ((\imath\circ\jmath)\circ\jmath)\circ\ell}\ \varpi(b),\ \varpi(\ell)\}$

$\Rightarrow \varpi(t) \geq \min\{\inf_{b\ \in\ ((\imath\circ\jmath)\circ\jmath)\circ\ell}\ \varpi(b),\ \varpi(\ell)\}$.

Hence ϖ is a $FH_{BCK}II$ of $\digamma$. □

Chapter 4

Fuzzy soft set theoretic approaches to α-ideals in BCI-algebras

To overcome the difficulties which usually occur while using classical methods, out of which the inability of the parametrization tool for these methods is one of he main reasons, Molodtsov [57] presented a new technique of soft set theory to cope with such type of problems. Now a days, soft set theory is considered to be the one of the most reliable method for dealing with uncertainties. In 2008, Jun [26] applied soft set theory to the theory of BCK/BCI-algebras. In [35, 36] soft subalgebras and soft ideals of BCK/BCI-algebras are characterized by using the idea of fuzzy sets. Continuing these studies, below we introduce the concept of fuzzy soft α-ideals of BCI-algebras and prove their basic properties. We also describe connections between various types of fuzzy soft α-ideals and fuzzy soft ideals, and characterize some types of fuzzy α-ideals by $\in$-soft sets. Presented examples give applications of our results. Here, first we will discuss the basics of soft set theory and its applications in ideal theory of BCI-algebras.

4.1 Preliminaries

Molodtsov in [57] defined the "soft sets" (or SSs) as under: Let Ω be an universal set and ζ be a set of parameters. Let $\mathfrak{P}(\Omega)$ denotes the power set of Ω and $\varsigma \subset \zeta$.

Definition 4.1.1. [57] A pair (Γ, ς) is is termed as a "soft set" (or SS) over Ω, where Γ is a function $\Gamma : \varsigma \rightarrow \mathfrak{P}(\Omega)$
i.e., a SS over Ω is a family of parameters of subsets of universal set Ω.

Definition 4.1.2. [41] For two SSs (Γ, ς) and (Υ, τ) over a common universe Ω, the "union" of (Γ, ς) and (Υ, τ), denoted by
$(\Gamma, \varsigma)\widetilde{\cup}(\Upsilon, \tau)$, is defined to be the SS (Π, ϱ), where $\varrho = \varsigma \cup \tau$ and for any $\wp \in \varrho$,

$$\Pi[\wp] = \begin{cases} \Gamma[\wp] & if\ \wp \in \varsigma - \tau \\ \Upsilon[\wp] & if\ \wp \in \tau - \varsigma \\ \Gamma[\wp] \cup \Upsilon[\wp] & if\ \wp \in \varsigma \cap \tau \end{cases}$$

Definition 4.1.3. [2] For two SSs (Γ, ς) and (Υ, τ) over a common universe Ω, the "extended intersection" of (Γ, ς) and (Υ, τ), denoted by $(\Gamma, \varsigma) \sqcap_E (\Upsilon, \tau)$, is defined to be the SS (Π, ϱ), where $\varrho = \varsigma \cup \tau$ and for any $\wp \in \varrho$,

$$\Pi[\wp] = \begin{cases} \Gamma[\wp] & if\ \wp \in \varsigma - \tau \\ \Upsilon[\wp] & if\ \wp \in \tau - \varsigma \\ \Gamma[\wp] \cap \Upsilon[\wp] & if\ \wp \in \varsigma \cap \tau \end{cases}$$

Definition 4.1.4. [2] For two SSs (Γ, ς) and (Υ, τ) over a common universe Ω such that $\varsigma \cap \tau \neq \emptyset$, the "restricted intersection" of (Γ, ς) and (Υ, τ), denoted by $(\Gamma, \varsigma) \sqcap_{\Re} (\Upsilon, \tau)$, is defined to be the SS (Π, ϱ), where $\varrho = \varsigma \cap \tau$ and for any $\wp \in \varrho$, $\Pi[\wp] = \Gamma[\wp] \cap \Upsilon[\wp]$.

Definition 4.1.5. [2] For two SSs (Γ, ς) and (Υ, τ) over a common universe Ω such that $\varsigma \cap \tau \neq \emptyset$, the "restricted union" of (Γ, ς) and (Υ, τ), denoted by $(\Gamma,\ \varsigma) \sqcup_{\Re} (\Upsilon, \tau)$, is defined to be the SS (Π, ϱ), where $\varrho = \varsigma \cap \tau$ and for any $\wp \in \varrho$, $\Pi[\wp] = \Gamma[\wp] \cup \Upsilon[\wp]$.

Definition 4.1.6. [2] For two SSs (Γ, ς) and (Υ, τ) over a common universe Ω such that $\varsigma \cap \tau \neq \emptyset$, the "restricted difference" of (Γ, ς) and (Υ, τ), denoted by $(\Gamma, \varsigma) \smile_{\Re} (\Upsilon, \tau)$, is defined to be the SS (Π, ϱ), where $\varrho = \varsigma \cap \tau$ and for any $\wp \in \varrho$, $\Pi[\wp] = \Gamma[\wp] \setminus \Upsilon[\wp]$.

Definition 4.1.7. [41] Let (Γ, ς) and (Υ, τ) be two SSs over a common universe Ω. Then "(Γ, ς) AND (Υ, τ)" denoted by $(\Gamma, \varsigma)\widetilde{\wedge}(\Upsilon, \tau)$ is defined as $(\Gamma, \varsigma)\widetilde{\wedge}(\Upsilon, \tau) = (\Pi, \varsigma \times \tau)$,
where $\Pi(u, v) = \Gamma(u) \cap \Upsilon(v)$, for all $(u, v) \in \varsigma \times \tau$.

Definition 4.1.8. [41] Let (Γ, ς) and (Υ, τ) be two SSs over a common universe Ω. Then "(Γ, ς) OR (Υ, τ)" denoted by $(\Gamma, \varsigma)\widetilde{\vee}(\Upsilon, \tau)$ is defined as $(\Gamma, \varsigma)\widetilde{\vee}(\Upsilon, \tau) = (\Pi, \varsigma \times \tau)$,
where $\Pi(u, v) = \Gamma(u) \cup \Upsilon(v)$, for all $(u, v) \in \varsigma \times \tau$.

In the sequel, Ω will be a BCI-algebra.

4.2 Soft α-ideals

Let R be an arbitrary binary relation between an element of a non-empty set Ξ and an element of Ω. A set valued function $\Gamma : \Xi \to \mathfrak{P}(\Omega)$ can be delineated as $\Gamma(\imath) = \{\jmath \in \Omega \mid \imath R \jmath\}$ for all $\imath \in \Xi$. The pair $(\Gamma,\ \Xi)$ is then a SS over Ω".

Definition 4.2.1. [30] Let Ξ be a subalgebra of Ω. A subset Υ of Ω is an "ideal of Ω related to Ξ" (briefly, "Ξ-ideal" of Ω), symbolized as $\Upsilon \lhd \Xi$, if it satiates:

(i). $0 \in \Upsilon$

(ii). $\imath \cdot \jmath \in \Upsilon$ and $\jmath \in \Upsilon \Rightarrow \imath \in \Upsilon$ for all $\imath \in \Xi$.

Definition 4.2.2. Let Ξ be a subalgebra of Ω. A subset Υ of Ω is an "α-ideal (or αI) of Ω related to Ξ" (briefly, "Ξ-α-ideal" of Ω), symbolized as $\Upsilon \lhd_\alpha \Xi$, if it satiates:

(i). $0 \in \Upsilon$

(ii). $(\imath \cdot \ell) \cdot (0 \cdot \jmath)) \in \Upsilon$ and $\ell \in \Upsilon \Rightarrow \jmath \cdot \imath \in \Upsilon$ for all $\imath, \jmath \in \Xi$.

Example 4.2.3. Consider the BCI-algebra $\Omega = \{0, \imath, \jmath, \ell\}$ defined in Example 2.5.1. Then $\Xi = \{0, \imath\}$ is a subalgebra of Ω and $\Upsilon = \{0, \imath, \jmath\}$ is an "Ξ-α-ideal" of Ω.

It is evident that any "Ξ-α-ideal" of Ω is an "Ξ-ideal" of Ω.

Definition 4.2.4. [26] Let (Γ, ς) be a SS over Ω. Then (Γ, ς) is is termed as a "soft BCI-algebra" (or $S_{BCI}A$) over Ω if $\Gamma(\wp)$ is a subalgebra of Ω for any $\wp \in \varsigma$.

Definition 4.2.5. [30] Let (Γ, ς) be a $S_{BCI}A$ over Ω. A SS (Υ, τ) over Ω is termed as a "soft ideal" (or SI) of (Γ, ς), symbolized as $(\Upsilon, \tau) \,\tilde{\lhd}\, (\Gamma, \varsigma)$, if $\tau \subset \varsigma$ and for any $\wp \in \tau$, $\Upsilon(\wp) \lhd \Gamma(\wp)$.

Definition 4.2.6. Let (Γ, ς) be a $S_{BCI}A$ over Ω. A SS (Υ, τ) over Ω is termed as a "soft α-ideal" (or $S_\alpha I$) of (Γ, ς), symbolized as $(\Upsilon, \tau) \,\tilde{\lhd}_\alpha\, (\Gamma, \varsigma)$, if $\tau \subset \varsigma$ and for all $\wp \in \Upsilon$, $\Upsilon(\wp) \lhd_\alpha \Gamma(\wp)$.

Let us elaborate the above definition by using the succeeding example.

Example 4.2.7. Let $\Omega = \{0, \imath, \jmath, \ell\}$ be the BCI-algebra defined in Example 2.5.1. Let (Γ, ς) be a SS over Ω, where $\varsigma = \Omega$ and $\Gamma : \varsigma \rightarrow \mathfrak{P}(\Omega)$ is a set-valued function delineated as:

$\Gamma(\wp) = \{0\} \cup \{\kappa \in \Omega \mid \kappa \cdot (\kappa \cdot \wp) \in \{0, \imath\}\}$, for all $\wp \in \varsigma$. Then $\Gamma(0) = \Gamma(\imath) = \Omega$, $\Gamma(\jmath) = \Gamma(\ell) = \{0\}$, which are subalgebras of Ω. Hence (Γ, ς) is a $S_{BCI}A$ over Ω. Let $\tau = \{0, \imath, \jmath\} \subset \varsigma$ and $\Upsilon : \tau \rightarrow \mathfrak{P}(\Omega)$ be a set-valued function delineated as:

$$\Upsilon(\wp) = \begin{cases} \vartheta(\{0, \imath\}) & if \ \wp = \jmath \\ \{0\} & if \ \wp \in \{0, \imath\} \end{cases}$$

where $\vartheta(\{0, \imath\}) = \{\wp \in \Omega \mid 0 \cdot (0 \cdot \wp) \in \{0, \imath\}\}$. Then $\Upsilon(0) = \{0\} \lhd_\alpha \Omega = \Gamma(0)$, $\Upsilon(\imath) = \{0\} \lhd_\alpha \Omega = \Gamma(\imath)$, $\Upsilon(\jmath) = \{0, \imath\} \lhd_\alpha \{0\} = \Gamma(\jmath)$. Hence (Υ, τ) is a $S_\alpha I$ of (Γ, ς).

Any $S_\alpha I$ is a SI but the converse isn't valid as can be observed by the succeeding example.

Example 4.2.8. Consider a BCI-algebra $\Omega = \{0, \imath, \jmath, \ell, \kappa\}$ defined by the succeeding Cayley table:

$\cdot$	0	$\imath$	$\jmath$	ℓ	κ
0	0	0	0	0	0
$\imath$	$\imath$	0	0	0	0
$\jmath$	$\jmath$	$\jmath$	0	$\jmath$	0
ℓ	ℓ	ℓ	ℓ	0	0
κ	κ	κ	ℓ	$\jmath$	0

Let (Γ, ς) be a SS over Ω, where $\varsigma = \Omega$ and $\Gamma : \varsigma \rightarrow \mathfrak{P}(\Omega)$ is a set-valued function delineated as:

$$\Gamma(u) = \{y \in \Omega \mid y \cdot (y \cdot u) \in \{0, \jmath\}\},$$

for any $u \in \varsigma$.

Then $\Gamma(0) = \Omega,\ \Gamma(\imath) = \Gamma(\jmath) = \{0, \jmath, \ell, \kappa\},\ \Gamma(\ell) = \Gamma(\kappa) = \{0, \jmath\}$, which are subalgebras of Ω. Hence (Γ, ς) is a $S_{BCI}A$ over Ω.

Let (Υ, τ) be a SS over Ω, where $\tau = \{\jmath, \ell, \kappa\} \subset \varsigma$ and $\Upsilon : \tau \to \mathfrak{P}(\Omega)$ be a set-valued function delineated as:

$\Upsilon(u) = \{y \in \Omega \mid y \cdot u = 0\}$, for any $u \in \tau$.

Then $\Upsilon(\jmath) = \{0, \imath, \jmath\} \lhd \{0, \jmath, \ell, \kappa\} = \Gamma(\jmath),\ \Upsilon(\ell) = \{0, \imath, \ell\} \lhd \{0, \jmath\} = \Gamma(\ell),\ \Upsilon(\kappa) = \Omega \lhd \{0, \jmath\} = \Gamma(\kappa)$. Hence $(\Upsilon,\ \tau)$ is a SI of (Γ, ς) but it is not a $S_\alpha I$ of (Γ, ς) because $\Upsilon(\jmath)$ isn't an $\Gamma(\jmath)$-α-ideal of Ω since $(\jmath \cdot \jmath) \cdot (0 \cdot \kappa) = 0 \in \Upsilon(\jmath)$ and $\jmath \in \Upsilon(\jmath)$ but $\kappa \cdot \jmath = \ell \notin \Upsilon(\jmath)$.

4.3 α-idealistic soft BCI-algebras

Definition 4.3.1. [30] A SS (Γ, ς) over Ω is termed as an "idealistic soft BCI-algebra" (or $IS_{BCI}A$) over Ω if $\Gamma(\wp)$ is an ideal of Ω, for any $\wp \in \varsigma$.

Definition 4.3.2. A SS (Γ, ς) over Ω is termed as an "α-idealistic soft BCI-algebra" (or $\alpha IS_{BCI}A$) over Ω if $\Gamma(\wp)$ is an αI of Ω, for any $\wp \in \varsigma$.

Example 4.3.3. Let $\Omega = \{0, \imath, \jmath, \ell\}$ be the BCI-algebra which is defined by the Cayley table given in Example 2.5.1. Let (Γ, ς) be a SS over Ω, where $\varsigma = \Omega$ and define a set valued function, $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$ as:

$$\Gamma(\wp) = \begin{cases} \vartheta(\{0, \imath\}) & if\ \wp \in \{\jmath, \ell\} \\ \Omega & if\ \wp \in \{0, \imath\} \end{cases}$$

where $\vartheta(\{0, \imath\}) = \{\upsilon \in \Omega \mid 0 \cdot (0 \cdot \upsilon) \in \{0, \imath\}\}$. Then (Γ, ς) is an $\alpha IS_{BCI}A$ over Ω.

The order of any member $\wp$ of Ω, denoted as $o(\wp)$, is delineated as:

$$o(\wp) = \min\{n \in N \mid 0 \cdot \wp^n = 0\},$$

where $0 \cdot \wp^n = (...((0 \cdot \wp) \cdot \wp)...) \cdot \wp$, i.e., $\wp$ occurs n-times.

Example 4.3.4. Let $\Omega = \{0, \imath, \jmath, \ell, \varepsilon, \eta, \kappa, \lambda\}$ be a BCI-algebra defined by the succeeding Cayley table:

$\cdot$	0	$\imath$	$\jmath$	ℓ	ε	η	κ	λ
0	0	0	0	0	ε	ε	ε	ε
$\imath$	$\imath$	0	0	0	η	ε	ε	ε
$\jmath$	$\jmath$	$\jmath$	0	0	κ	κ	ε	ε
ℓ	ℓ	$\jmath$	$\imath$	0	λ	κ	η	ε
ε	ε	ε	ε	ε	0	0	0	0
η	η	ε	ε	ε	$\imath$	0	0	0
κ	κ	κ	ε	ε	$\jmath$	$\jmath$	0	0
λ	λ	κ	η	ε	ℓ	$\jmath$	$\imath$	0

Let (Γ, ς) be a SS over Ω, where $\varsigma = \{\imath, \jmath, \ell\} \subset \Omega$ and $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$ is a set-valued function defined by:

$$\Gamma(\wp) = \{\iota \in \Omega \mid o(\wp) = o(\iota)\},$$

for all $\wp \in \varsigma$. Then $\Gamma(\imath) = \Gamma(\jmath) = \Gamma(\ell) = \{0, \imath, \jmath, \ell\}$ is an α-ideal of Ω. Hence (Γ, ς) is an $\alpha IS_{BCI}A$ over Ω. But if we take $\tau = \{\imath, \jmath, \kappa, \lambda\} \subset \Omega$ and defined a set-valued function $\Upsilon : \tau \to \mathfrak{P}(\Omega)$ by:

$$\Upsilon(\wp) = \{0\} \cup \{\iota \in \Omega \mid o(\wp) = o(\iota)\},$$

for all $\wp \in \tau$, then (Υ, τ) is not an $\alpha IS_{BCI}A$ over Ω, since $\Upsilon(\kappa) = \{0, \varepsilon, \eta, \kappa, \lambda\}$ is not an αI of Ω because $(\varepsilon \cdot 0) \cdot (0 \cdot \kappa) = 0 \in \Upsilon(\kappa)$ and $0 \in \Upsilon(\kappa)$ but $\kappa \cdot \varepsilon = \jmath \notin \Upsilon(\kappa)$.

Example 4.3.5. Consider the BCI-algebra $\Omega = \{0, \imath, \jmath, \ell\}$ which is given in Example 2.5.1. Let (Γ, ς) be a SS over Ω, where $\varsigma = \Omega$ and $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$

is a set-valued function defined by:

$$\Gamma(\wp) = \{\iota \in \Omega \mid \iota = \wp^n,\ n \in N\},$$

for all $\wp \in \varsigma$. Then $\Gamma(0) = \{0\},\ \Gamma(\imath) = \{0, \imath\},\ \Gamma(\jmath) = \{0, \jmath\},\ \Gamma(\ell) = \{0, \ell\}$, which are αIs of Ω. Hence (Γ, ς) is an $\alpha IS_{BCI}A$ over Ω.

Any $\alpha IS_{BCI}A$ over Ω is an $IS_{BCI}A$ over Ω. But the converse doesn't hold as can be observed through the succeeding example.

Example 4.3.6. Let $\Omega = \{0, \imath, \jmath, \ell, \kappa\}$ be the BCI-algebra defined by the Cayley table given in Example 4.2.8. Let (Γ, ζ) be a SS over Ω, where $\zeta = \{\jmath, \ell, \kappa\}$ and the set valued function $\Gamma : \zeta \to \mathfrak{P}(\Omega)$ is delineated as:

$$\Gamma(\wp) = \{\iota \in \Omega \mid \iota \cdot \wp = 0\}$$

for all $\wp \in \zeta$. Then $\Gamma(\jmath) = \{0, a, \jmath\},\ \Gamma(\ell) = \{0, \imath, \ell\},\ \Gamma(\kappa) = \Omega$, which are ideals of Ω. Hence $(\Gamma,\ \zeta)$ is an $IS_{BCI}A$ over Ω but it is not an $\alpha IS_{BCI}A$ over Ω because $\Gamma(\jmath)$ is not an αI of Ω since, $(\jmath \cdot \jmath) \cdot (0 \cdot \kappa) = 0 \in \Gamma(\jmath)$ and $\jmath \in \Gamma(\jmath)$ but $\kappa \cdot \jmath = \ell \notin \Gamma(\jmath)$.

Proposition 4.3.7. *Let (Γ, ς) and (Γ, τ) be SSs over Ω where $\tau \subseteq \varsigma \subseteq \Omega$. If (Γ, ς) is an $\alpha IS_{BCI}A$ over Ω, then (Γ, τ) is also an $\alpha IS_{BCI}A$ over Ω.*

Proof. Straightforward. □

The converse of the above mentioned statement isn't generally true as is evident from the succeeding example.

Example 4.3.8. Consider an $\alpha IS_{BCI}A$ (Γ, ς) over Ω which is described in Example 4.3.4. Take $\tau = \{\imath, \jmath, \ell, \varepsilon\} \supseteq \varsigma$, then (Γ, τ) is not an $\alpha IS_{BCI}A$ over Ω since $\Gamma(\varepsilon) = \{\varepsilon, \eta, \kappa, \lambda\}$ is not an αI of Ω.

Definition 4.3.9. An $\alpha IS_{BCI}A$ (Γ, ς) over Ω is termed as trivial (resp., whole) if $\Gamma(\wp) = 0$ (resp., $\Gamma(\wp) = \Omega$), for any $\wp \in \varsigma$.

Example 4.3.10. Let $\Omega = \{0, \imath, \jmath, \ell\}$ be a BCI-algebra defined in Example 2.5.1 and let $\Gamma : \Omega \to \mathfrak{P}(\Omega)$ be a set-valued function delineated as

$$\Gamma(\wp) = \{0\} \cup \{\iota \in \Omega \mid o(\wp) = o(\iota)\},$$

for any $\wp \in \Omega$. Then $\Gamma(0) = \{0\}$ and $\Gamma(\imath) = \Gamma(\jmath) = \Gamma(\ell) = \Omega$, that are αIs of Ω. Hence $(\Gamma, \{0\})$ is a trivial $\alpha IS_{BCI}A$ over Ω and $(\Gamma, \Omega \setminus \{0\})$ is a whole $\alpha IS_{BCI}A$ over Ω.

Let $\vartheta : \Omega \to \Delta$ be a mapping of BCI-algebras. For a SS (Γ, ς) over Ω, $(\vartheta(\Gamma), \varsigma)$ is SS over Δ, where $\vartheta(\Gamma) : \varsigma \to \mathfrak{P}(\Delta)$ is defined by $\vartheta(\Gamma)(\wp) = \vartheta(\Gamma(\wp))$ for all $\wp \in \varsigma$.

Lemma 4.3.11. *Let $\vartheta : \Omega \to \Delta$ be an isomorphism of BCI-algebras. If (Γ, ς) is an $\alpha IS_{BCI}A$ over Ω, then $(\vartheta(\Gamma), \varsigma)$ is an $\alpha IS_{BCI}A$ over Δ.*

Theorem 4.3.12. *Let $\vartheta : \Omega \to \Delta$ be an isomorphism of BCI-algebras and let (Γ, ς) be an $\alpha IS_{BCI}A$ over Ω.*
(1). *If $\Gamma(\wp) = ker(\vartheta)$, for any $\wp \in \varsigma$, then $(\vartheta(\Gamma), \varsigma)$ is a trivial $\alpha IS_{BCI}A$ over Δ.*
(2). *If (Γ, ς) is whole, then $(\vartheta(\Gamma), \varsigma)$ is a whole $\alpha IS_{BCI}A$ over Δ.*

Proof. (1). Let $\Gamma(\wp) = ker(\vartheta)$, for any $\wp \in \varsigma$. Then $\vartheta(\Gamma)(\wp) = \vartheta(\Gamma(\wp)) = \{0_\Delta\}$, for any $\wp \in \varsigma$. Hence (Γ, ς) is a trivial $\alpha IS_{BCI}A$ over Δ (by Lemma 4.3.11 and Definition 4.3.9).
(2). Let (Γ, ς) be whole. Then $\Gamma(\wp) = \Omega$, for any $\wp \in \varsigma$ and so $\vartheta(\Gamma)(\wp) = \vartheta(\Gamma(\wp)) = \vartheta(\Omega) = \Delta$, for any $\wp \in \varsigma$. Then from Lemma 4.3.11 and Definition 4.3.9, it is evident that $(\vartheta(\Gamma), \varsigma)$ is a whole $\alpha IS_{BCI}A$ over Δ. □

Theorem 4.3.13. *For every $F_\alpha I$ ϖ of Ω, $\exists$ an $\alpha IS_{BCI}A$ (Γ, ς) over Ω.*

Proof. Let ϖ be a $F_\alpha I$ of Ω. Then $U(\varpi;\delta) := \{\wp \in \Omega \mid \varpi(\wp) \geq \delta\}$ is an αI of Ω, for any $\delta \in Im(\varpi)$. Take $\varsigma = Im(\varpi)$ and consider a set valued function $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$ delineated as $\Gamma(\delta) = U(\varpi;\delta)$, for any $\delta \in \varsigma$, then (Γ, ς) is an $\alpha IS_{BCI}A$ over Ω. □

Conversely, the succeeding proposition is evident.

Proposition 4.3.14. *For any FS ϖ in Ω, if an $\alpha IS_{BCI}A$ (Γ, ς) over Ω is defined as $\varsigma = Im(\varpi)$ and $\Gamma(\delta) = U(\varpi;\delta)$, for any $\delta \in \varsigma$, then ϖ is a $F_\alpha I$ of Ω.*

Let ϖ be a FS in Ω and let (Γ, ς) be a SS over Ω in which $\varsigma = Im(\varpi)$ and $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$ is a set-valued function delineated as

$$\Gamma(\delta) = \{\wp \in \Omega \mid \varpi(\wp) + \delta > 1\} \qquad (4.3.14(a))$$

for any $\delta \in \varsigma$. Then there exists $\delta \in \varsigma$, s.t, $\Gamma(\delta)$ is not an αI of Ω as seen in the example stated below.

Example 4.3.15. Delineate a FS ϖ in Ω as, $\varpi(0) = \delta_\circ < 0.5$ and $\varpi(\wp) = 1 - \delta_\circ$ for any $\wp \neq 0$. Let $\varsigma = Im(\varpi)$ and $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$ be a set-valued function defined by (4.3.14(a)). Then $\Gamma(1 - \delta_\circ) = \Omega \setminus \{0\}$, which isn't an αI of Ω.

4.4 Soft h-ideals

Definition 4.4.1. Let Ξ be a subalgebra of Ω. A subset Υ of Ω is an "h-ideal (or hI) of Ω related to Ξ" (briefly, "Ξ-h-ideal" of Ω), symbolized as $\Upsilon \lhd_h \Xi$, if it satiates:

(i). $0 \in \Upsilon$

(ii). $\imath \cdot (\jmath \cdot \ell) \in \Upsilon$ and $\jmath \in \Upsilon \Rightarrow \imath \cdot \ell \in \Upsilon$, for any $\imath, \ell \in \Xi$.

Example 4.4.2. Let $\Omega = \{0, \imath, \jmath, \ell, \kappa\}$ be a BCK-algebra and hence a BCI-algebra, with the succeeding Cayley table:

$\cdot$	0	$\imath$	$\jmath$	ℓ	κ
0	0	0	0	0	0
$\imath$	$\imath$	0	$\imath$	0	0
$\jmath$	$\jmath$	$\jmath$	0	0	$\jmath$
ℓ	ℓ	$\jmath$	$\imath$	0	$\jmath$
κ	κ	$\imath$	κ	$\imath$	0

Then $\Xi = \{0, \imath, \jmath\}$ is a subalgebra of Ω and $\Upsilon = \{0, \imath, \jmath, \kappa\}$ is an "Ξ-h-ideal" of Ω.

Definition 4.4.3. Let (Γ, ς) be a $S_{BCI}A$ over Ω. A SS (Υ, τ) over Ω is termed as a "soft h-ideal" (or S_hI) of (Γ, ς), symbolized as $(\Upsilon, \tau) \tilde{\lhd}_h (\Gamma, \varsigma)$, if $\tau \subset \varsigma$ and for all $\wp \in \Upsilon$, $\Upsilon(\wp) \lhd_h \Gamma(\wp)$.

Let us elaborate the above definition by using the succeeding example.

Example 4.4.4. Consider a BCI-algebra $\Omega = \{0, \imath, \jmath, \ell, \kappa\}$ which is given in Example 4.4.2. Let (Γ, ς) be a SS over Ω, where $\varsigma = \Omega$ and $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$ is a set-valued function delineated as:

$$\Gamma(\wp) = \{\iota \in \Omega \mid \iota \cdot (\iota \cdot \wp) \in \{0, a\}\},$$

for any $\wp \in \varsigma$. Then $\Gamma(0) = \Gamma(\imath) = \Omega$, $\Gamma(\jmath) = \Gamma(\ell) = \{0, \imath, \kappa\}$, $\Gamma(\kappa) = \{0, \imath, \jmath, \ell\}$, which are subalgebras of Ω. Hence (Γ, ς) is a $S_{BCI}A$ over Ω. Let $\tau = \{0, \imath, \jmath\} \subset \varsigma$ and $\Upsilon : \tau \to \mathfrak{P}(\Omega)$ be a set-valued function delineated as:

$$\Upsilon(\wp) = \{\iota \in \Omega \mid \iota \cdot (\iota \cdot \wp) \in \{0, \ell\}\},$$

for any $\wp \in \tau$. Then $\Upsilon(0) = \Omega \lhd_h \Omega = \Gamma(0)$, $\Upsilon(\imath) = \{0, \jmath\} \lhd_h \Omega = \Gamma(\imath)$, $\Upsilon(\jmath) = \{0, \imath, \kappa\} \lhd_h \{0, \imath, \kappa\} = \Gamma(\jmath)$. Hence (Υ, τ) is a S_hI of (Γ, ς).

Any S_hI is a SI but the converse isn't valid as can be observed by the succeeding example.

Example 4.4.5. Let $\Omega = \{0, \imath, \jmath, \ell\}$ be the BCI-algebra defined in Example 2.9.1. Let (Γ, ς) be a SS over Ω, where $\varsigma = \Omega$ and $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$ is a set-valued function delineated as:

$$\Gamma(\wp) = \{0\} \cup \{\iota \in \Omega \mid \iota \cdot (\iota \cdot \wp) \in \{0, \imath\}\},$$

for any $\wp \in \varsigma$. Then $\Gamma(0) = \Gamma(\imath) = \Omega$ and $\Gamma(\jmath) = \Gamma(\ell) = \{0\}$, which are subalgebras of Ω. Hence (Γ, ς) is a $S_{BCI}A$ over Ω.
Let (Υ, τ) be a SS over Ω, where $\tau = \{0, \imath\} \subset \varsigma$ and $\Upsilon : \tau \to \mathfrak{P}(\Omega)$ be a set-valued function delineated as:

$$\Upsilon(\wp) = \{0\} \cup \{\iota \in \Omega \mid \wp \leq \iota\},$$

for any $\wp \in \tau$. Then $\Upsilon(0) = \{0, \imath\} \lhd \Omega = \Gamma(0)$ and $\Upsilon(\imath) = \{0, \imath\} \lhd \Omega = \Gamma(\imath)$. Hence $(\Upsilon,\ I)$ is a SI of (Γ, ς) but it isn't a S_hI of (Γ, ς) because $\Upsilon(\imath)$ is not an $\Gamma(\imath)$-h-ideal of Ω since $\jmath \cdot (0 \cdot \ell) = \jmath \cdot \jmath = 0 \in \Upsilon(\imath)$ and $0 \in \Upsilon(\imath)$ but $\jmath \cdot \ell = \ell \notin \Upsilon(\imath)$.

4.5 h-idealistic soft BCI-algebras

Definition 4.5.1. A SS (Γ, ς) over Ω is termed as an "h-idealistic soft BCI-algebra" (or $hIS_{BCI}A$) over Ω if $\Gamma(\wp)$ is an "h-ideal" of Ω, for any $\wp \in \varsigma$.

Example 4.5.2. Let $\Omega = \{0, \imath, \jmath, \ell, \varepsilon, \eta, \kappa, \lambda\}$ be the BCI-algebra defined in Example 4.3.4. Let $(\Gamma,\ \varsigma)$ be a SS over Ω, where $\varsigma = \{\imath, \jmath, \ell\} \subset \Omega$ and $\Gamma : \varsigma \rightarrow \mathfrak{P}(\Omega)$ is a set-valued function delineated as:

$$\Gamma(\wp) = \{\iota \in \Omega \mid o(\wp) = o(\iota)\},$$

for any $\wp \in \varsigma$. Then $\Gamma(\imath) = \Gamma(\jmath) = \Gamma(\ell) = \{0, \imath, \jmath, \ell\}$ is an h-ideal of Ω. Hence $(\Gamma,\ \varsigma)$ is an $hIS_{BCI}A$ over Ω.

4.6 Soft BCI-positive implicative ideals

Definition 4.6.1. Let Ξ be a subalgebra of Ω. A subset Υ of Ω is termed as a "BCI-positive implicative ideal of Ω related to Ξ"
(briefly, "$\Xi - (BCI - PI)$-ideal" of Ω), symbolized as $\Upsilon \lhd_{bci-pi} \Xi$, if it satiates :
(i). $0 \in \Upsilon$
(ii). $((\imath \cdot \ell) \cdot \ell) \cdot (\jmath \cdot \ell) \in \Upsilon$ and $\jmath \in \Upsilon \Rightarrow \imath \cdot \ell \in \Upsilon$ for all $\imath, \ell \in \Xi$.

Example 4.6.2. Let $\Omega = \{0, \imath, \jmath, \ell\}$ be the BCI-algebra defined in Example 2.10.1. Then $\Xi = \{0, \jmath\}$ is a subalgebra of Ω and $\Upsilon = \{0, \imath, \jmath\}$ is an $\Xi - (BCI - PI)$-ideal of Ω.

Definition 4.6.3. Let (Γ, ς) be a $S_{BCI}A$ over Ω. A SS (Υ, τ) over Ω is termed as a "soft BCI-positive implicative ideal" (or $S_{BCI}PII$) of (Γ, ς), symbolized as $(\Upsilon, \tau) \tilde{\lhd}_{bci-pi} (\Gamma, \varsigma)$, if $\tau \subset \varsigma$ and for any $\wp \in \Upsilon$, $\Upsilon(\wp) \lhd_{bci-pi} \Gamma(\wp)$.

Let us elaborate the above definition by using the succeeding example.

Example 4.6.4. Let $\Omega = \{0, \imath, \jmath, \ell\}$ be the BCI-algebra defined in Example

2.10.1. Let (Γ, ς) be a SS over Ω, where $\varsigma = \Omega$ and $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$ is a set-valued function delineated as:

$$\Gamma(\wp) = \{0\} \cup \{\iota \in \Omega \mid \iota \cdot (\iota \cdot \wp) \in \{0, \imath\}\},$$

for any $\wp \in \varsigma$. Then $\Gamma(0) = \Gamma(\imath) = \Omega$, $\Gamma(\jmath) = \{0, \imath, \ell\}$, $\Gamma(\ell) = \{0\}$, which are subalgebras of Ω. Hence (Γ, ς) is a $S_{BCI}A$ over Ω. Let $\tau = \{0, \imath, \jmath\} \subset \varsigma$ and $\Upsilon : \tau \to \mathfrak{P}(\Omega)$ be a set-valued function delineated as:

$$\Upsilon(\wp) = \begin{cases} Z(\{0, \imath\}) & if\ \wp = \jmath \\ \{0\} & if\ \wp \in \{0, \imath\} \end{cases}$$

where $Z(\{0, \imath\}) = \{\wp \in \Omega \mid 0 \cdot (0 \cdot \wp) \in \{0, \imath\}\}$. Then $\Upsilon(0) = \{0\} \lhd_{bci-pi} \Omega = \Gamma(0)$, $\Upsilon(\imath) = \{0\} \lhd_{bci-pi} \Omega = \Gamma(\imath)$, $\Upsilon(\jmath) = \{0, \imath, \jmath\} \lhd_{bci-pi} \{0, \imath, \ell\} = \Gamma(\jmath)$. Hence (Υ, τ) is a $S_{BCI}PII$ of (Γ, ς).

Any $S_{BCI}PII$ is a SI but the converse isn't valid as can be observed by the succeeding example.

Example 4.6.5. Let $\Omega = \{0, \imath, \jmath, \ell, \kappa\}$ be a BCK-algebra and hence a BCI-algebra, with the succeeding Cayley table:

$\cdot$	0	$\imath$	$\jmath$	ℓ	κ
0	0	0	0	0	0
$\imath$	$\imath$	0	0	0	0
$\jmath$	$\jmath$	$\jmath$	0	0	0
ℓ	ℓ	ℓ	ℓ	0	0
κ	κ	κ	κ	ℓ	0

Let (Γ, ς) be a SS over Ω, where $\varsigma = \Omega$ and $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$ is a set-valued function delineated as:

$$\Gamma(\wp) = \{\iota \in \Omega \mid \iota \cdot (\iota \cdot \wp) \in \{0, \imath\}\},$$

for any $\wp \in \varsigma$. Then $\Gamma(0) = \Gamma(\imath) = \Omega,\ \Gamma(\jmath) = \{0, \imath, \ell, \kappa\}$ and $\Gamma(\ell) = \Gamma(\kappa) = \{0, \imath\}$, which are subalgebras of Ω. Hence (Γ, ς) is a $S_{BCI}A$ over Ω.

Let (Υ, τ) be a SS over Ω, where $\tau = \{\imath, \jmath\} \subset \varsigma$ and $\Upsilon : \tau \to \mathfrak{P}(\Omega)$ be a set-valued function delineated as:

$$\Upsilon(\wp) = \{\iota \in \Omega \mid \iota \cdot \wp = 0\},$$

for any $\wp \in \tau$. Then $\Upsilon(\imath) = \{0, \imath\} \lhd \Omega = \Gamma(\imath),\ \Upsilon(\jmath) = \{0, \imath, \jmath\} \lhd \{0, \imath, \ell, \kappa\} = \Gamma(\jmath)$. Hence (Υ, τ) is a SI of (Γ, ς) but it isn't a $S_{BCI}PII$ of (Γ, ς) because $\Upsilon(\imath)$ is not a "BCI-positive implicative ideal" of Ω related to $\Gamma(\imath)$ since $((\kappa \cdot \ell) \cdot \ell) \cdot (0 \cdot \ell) = 0 \in \Upsilon(\imath)$ and $0 \in \Upsilon(\imath)$ but $\kappa \cdot \ell = \ell \notin \Upsilon(\imath)$.

4.7 BCI-positive implicative idealistic soft BCI-algebras

Definition 4.7.1. A SS (Γ, ς) over Ω is termed as a "BCI-positive implicative idealistic soft BCI-algebra" (or $BCI - PIIS_{BCI}A$) over Ω if $\Gamma(\wp)$ is a "BCI-positive implicative ideal" of Ω, for any $\wp \in \varsigma$.

Example 4.7.2. Consider a BCI-algebra $\Omega = \{0, \imath, \jmath, \ell\}$ which is given in Example 2.10.1. Let (Γ, ς) be a SS over Ω, where $\varsigma = \Omega$ and $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$ is a set-valued function delineated as:

$$\Gamma(\wp) = \begin{cases} \theta(\{0, \imath\}) & if\ \wp \in \{\jmath, \ell\} \\ \Omega & if\ \wp \in \{0, \imath\} \end{cases}$$

where $\theta(\{0, \imath\}) = \{\wp \in \Omega \mid 0 \cdot (0 \cdot \wp) \in \{0, \imath\}\}$. Then (Γ, ς) is a $BCI - PIIS_{BCI}A$ over Ω.

4.8 Soft BCI-implicative ideals

Definition 4.8.1. Let Ξ be a subalgebra of Ω. A subset Υ of Ω is termed as a "BCI-implicative ideal of Ω related to Ξ" (briefly, "$\Xi-(BCI-I)$-ideal" of Ω), symbolized as $\Upsilon \lhd_{bci-i} \Xi$, if it satiates :

(i). $0 \in \Upsilon$

(ii). $(((\imath \cdot \jmath) \cdot \jmath) \cdot (0 \cdot \jmath)) \cdot \ell \in \Upsilon$ and $\ell \in \Upsilon \Rightarrow \imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))) \in \Upsilon$,

for any $\imath, \jmath \in \Xi$.

Example 4.8.2. Let $\Omega = \{0, \imath, \jmath, \ell\}$ be the BCI-algebra defined in Example 2.9.1. Then $\Xi = \{0, \imath\}$ is a subalgebra of Ω and $\Upsilon = \{0, \imath, \jmath\}$ is a $\Xi-(BCI-I)$-ideal of Ω.

Definition 4.8.3. Let (Γ, ς) be a $S_{BCI}A$ over Ω. A SS (Υ, τ) over Ω is termed as a "soft BCI-implicative ideal" (or $S_{BCI}II$) of (Γ, ς), symbolized as $(\Upsilon, \tau)\ \tilde{\lhd}_{bci-i}\ (\Gamma, \varsigma)$, if $\tau \subset \varsigma$ and for any $\wp \in \Upsilon$, $\Upsilon(\wp) \lhd_{bci-i} \Gamma(\wp)$.

Let us elaborate the above definition by using the succeeding example.

Example 4.8.4. Consider a BCI-algebra $\Omega = \{0, \imath, \jmath, \ell\}$ which is defined in Example 2.9.1. Let (Γ, ς) be a SS over Ω, where $\varsigma = \Omega$ and $\Gamma : \varsigma \rightarrow \mathfrak{P}(\Omega)$ is a set-valued function delineated as:

$$\Gamma(\wp) = \{0\} \cup \{\iota \in \Omega \mid \iota \cdot (\iota \cdot \wp) \in \{0, \imath\}\},$$

for any $\wp \in \varsigma$. Then $\Gamma(0) = \Gamma(\imath) = \Omega$, $\Gamma(\jmath) = \Gamma(\ell) = \{0\}$, which are subalgebras of Ω. Hence (Γ, ς) is a $S_{BCI}A$ over Ω. Let $\tau = \{0, \imath, \jmath\} \subset \varsigma$ and $\Upsilon : \tau \rightarrow \mathfrak{P}(\Omega)$ be a set-valued function delineated as:

$$\Upsilon(\wp) = \begin{cases} Z(\{0, \imath\}) & if\ \wp = \jmath \\ \{0\} & if\ \wp \in \{0, \imath\} \end{cases}$$

where $Z(\{0,\imath\}) = \{\wp \in \Omega \mid 0 \cdot (0 \cdot \wp) \in \{0,\imath\}\}$. Then $\Upsilon(0) = \{0\} \lhd_{bci-i} \Omega = \Gamma(0)$, $\Upsilon(\imath) = \{0\} \lhd_{bci-i} \Omega = \Gamma(\imath)$, $\Upsilon(\jmath) = \{0,\imath\} \lhd_{bci-i} \{0\} = \Gamma(\jmath)$. Hence $(\Upsilon,\ \tau)$ is a $S_{BCI}II$ of (Γ, ς).

Any $S_{BCI}II$ is a SI but the converse isn't valid as can be observed by the succeeding example.

Example 4.8.5. Let $\Omega = \{0, \imath, \jmath, \ell, \kappa\}$ be the BCI-algebra defined in Example 4.6.5. Let (Γ, ς) be a SS over Ω, where $\varsigma = \Omega$ and $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$ is a set-valued function delineated as:

$$\Gamma(\wp) = \{\iota \in \Omega \mid \iota \cdot (\iota \cdot \wp) \in \{0, \imath\}\},$$

for any $\wp \in \varsigma$. Then $\Gamma(0) = \Gamma(\imath) = \Omega$, $\Gamma(\jmath) = \{0, \imath, \ell, \kappa\}$ and $\Gamma(\ell) = \Gamma(\kappa) = \{0, \imath\}$, which are subalgebras of Ω. Hence (Γ, ς) is a $S_{BCI}A$ over Ω.
Let (Υ, τ) be a SS over Ω, where $\tau = \{\imath, \jmath\} \subset \varsigma$ and $\Upsilon : \tau \to \mathfrak{P}(\Omega)$ be a set-valued function delineated as:

$$\Upsilon(\wp) = \{\iota \in \Omega \mid \iota \cdot \wp = 0\},$$

for any $\wp \in \tau$. Then $\Upsilon(\imath) = \{0, \imath\} \lhd \Omega = \Gamma(\imath)$, $\Upsilon(\jmath) = \{0, \imath, \jmath\} \lhd \{0, \imath, \ell, \kappa\} = \Gamma(\jmath)$. Hence (Υ, τ) is a SI of (Γ, ς) but it isn't a $S_{BCI}II$ of (Γ, ς) because $\Upsilon(\imath)$ isn't a "BCI-implicative ideal" of Ω related to $\Gamma(a)$ since $(((\jmath \cdot \ell) \cdot \ell) \cdot (0 \cdot \ell)) \cdot \imath = 0 \in \Upsilon(\imath)$ and $\imath \in \Upsilon(\imath)$ but $\jmath \cdot ((\ell \cdot (\ell \cdot \jmath)) \cdot (0 \cdot (0 \cdot (\jmath \cdot \ell)))) = \jmath \cdot 0 = \jmath \notin \Upsilon(\imath)$.

4.9 BCI-implicative idealistic soft BCI-algebras

Definition 4.9.1. A SS (Γ, ς) over Ω is termed as a "BCI-implicative idealistic soft BCI-algebra" (or $BCI - IIS_{BCI}A$) over Ω if $\Gamma(\wp)$ is a "BCI-implicative ideal" of Ω, for any $\wp \in \varsigma$.

Example 4.9.2. Let $\Omega = \{0, \imath, \jmath, \ell\}$ be the BCI-algebra defined in Example 2.9.1. Let (Γ, ς) be a SS over Ω, where $\varsigma = \Omega$ and $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$ is a set-valued function delineated as:

$$\Gamma(\wp) = \begin{cases} \theta(\{0, \imath\}) & if \ \wp \in \{\jmath, \ell\} \\ \Omega & if \ \wp \in \{0, \imath\} \end{cases}$$

where $\theta(\{0, \imath\}) = \{\wp \in \Omega \mid 0 \cdot (0 \cdot \wp) \in \{0, \imath\}\}$.
Then (Γ, ς) is a $BCI - IIS_{BCI}A$ over Ω.

4.10 Soft BCI-commutative ideals

Definition 4.10.1. Let Ξ be a subalgebra of Ω. A subset Υ of Ω is termed as a "BCI-commutative ideal of Ω related to Ξ" (briefly, "$\Xi - (BCI - C)$-ideal" of Ω), symbolized as $\Upsilon \lhd_{bci-c} \Xi$, if it satiates :
(i). $0 \in \Upsilon$
(ii). $(\imath \cdot \jmath) \cdot \ell \in \Upsilon$ and $\ell \in \Upsilon \Rightarrow \imath \cdot ((\jmath \cdot (\jmath \cdot \imath)) \cdot (0 \cdot (0 \cdot (\imath \cdot \jmath)))) \in \Upsilon$,
for any $\imath, \jmath \in \Xi$.

Example 4.10.2. Let $\Omega = \{0, \imath, \jmath, \ell\}$ be the BCI-algebra defined in Example 2.5.1. Then $\Xi = \{0, \jmath\}$ is a subalgebra of Ω and $\Upsilon = \{0, \imath, \jmath\}$ is an $\Xi - (BCI - C)$-ideal of Ω.

Definition 4.10.3. Let (Γ, ς) be a $S_{BCI}A$ over Ω. A SS (Υ, τ) over Ω is termed as a "soft BCI-commutative ideal" (or $S_{BCI}CI$) of (Γ, ς), symbolized as $(\Upsilon, \tau) \,\tilde{\lhd}_{bci-c}\, (\Gamma, \varsigma)$, if $\tau \subset \varsigma$ and for any $\wp \in \Upsilon$, $\Upsilon(\wp) \lhd_{bci-c} \Gamma(\wp)$.

Let us elaborate the above definition by using the succeeding example.

Example 4.10.4. Let $\Omega = \{0, \imath, \jmath, \ell\}$ be the BCI-algebra defined in Example 2.5.1. Let (Γ, ς) be a SS over Ω, where $\varsigma = \Omega$ and $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$ is a set-valued function delineated as:

$$\Gamma(\wp) = \{0\} \cup \{\iota \in \Omega \mid \iota \cdot (\iota \cdot \wp) \in \{0, \imath\}\},$$

for any $\wp \in \varsigma$. Then $\Gamma(0) = \Gamma(\imath) = \Omega$, $\Gamma(\jmath) = \Gamma(\ell) = \{0\}$, which are subalgebras of Ω. Hence (Γ, ς) is a $S_{BCI}A$ over Ω. Let $\tau = \{0, \imath, \jmath\} \subset \varsigma$ and $\Upsilon : \tau \to \mathfrak{P}(\Omega)$ be a set-valued function delineated as:

$$\Upsilon(\wp) = \begin{cases} \theta(\{0, \imath\}) & if \ \wp = \jmath \\ \{0\} & if \ \wp \in \{0, \imath\} \end{cases}$$

where $\theta(\{0, \imath\}) = \{\wp \in \Omega \mid 0 \cdot (0 \cdot \wp) \in \{0, \imath\}\}$. Then $\Upsilon(0) = \{0\} \lhd_{bci-c} \Omega = \Gamma(0)$, $\Upsilon(\imath) = \{0\} \lhd_{bci-c} \Omega = \Gamma(\imath)$, $\Upsilon(\jmath) = \{0, \imath\} \lhd_{bci-c} \{0\} = \Gamma(\jmath)$. Hence $(\Upsilon, \ \tau)$ is a $S_{BCI}CI$ of (Γ, ς).

Any $S_{BCI}CI$ is a SI but the converse isn't valid as can be observed by the succeeding example.

Example 4.10.5. Let $\Omega = \{0, \imath, \jmath, \ell, \kappa\}$ be a BCI-algebra with the succeeding Cayley table:

$\cdot$	0	$\imath$	$\jmath$	ℓ	κ
0	0	0	0	0	κ
$\imath$	$\imath$	0	0	0	κ
$\jmath$	$\jmath$	$\jmath$	0	0	κ
ℓ	ℓ	ℓ	ℓ	0	κ
κ	κ	κ	κ	κ	0

Let (Γ, ς) be a SS over Ω, where $\varsigma = \Omega$ and $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$ is a set-valued function delineated as:

$$\Gamma(\wp) = \{0\} \cup \{\iota \in \Omega \mid \iota \cdot (\iota \cdot \wp) \in \{0, \imath\}\},$$

for any $\wp \in \varsigma$. Then $\Gamma(0) = \Gamma(\imath) = \Omega$, $\Gamma(\jmath) = \{0, \imath, \ell, \kappa\}$ and $\Gamma(\ell) = \{0, \imath, \kappa\}$, $\Gamma(\kappa) = \{0\}$, which are subalgebras of Ω. Hence (Γ, ς) is a $S_{BCI}A$ over Ω.

Let (Υ, τ) be a SS over Ω, where $\tau = \{\imath, \jmath\} \subset \varsigma$ and $\Upsilon : \tau \to \mathfrak{P}(\Omega)$ be a set-valued function delineated as:

$$\Upsilon(\wp) = \{\iota \in \Omega \mid \iota \cdot \wp = 0\},$$

for any $\wp \in \tau$. Then $\Upsilon(\imath) = \{0, \imath\} \lhd \Omega = \Gamma(\imath)$, $\Upsilon(\jmath) = \{0, \imath, \jmath\} \lhd \{0, \imath, \ell, \kappa\} = \Gamma(\jmath)$. Hence (Υ, τ) is a SI of (Γ, ς) but it isn't a $S_{BCI}CI$ of (Γ, ς) because $\Upsilon(\imath)$ isn't a "BCI-commutative ideal" of Ω related to $\Gamma(\imath)$ since $(\jmath \cdot \ell) \cdot \imath = 0 \in \Upsilon(\imath)$ and $\imath \in \Upsilon(\imath)$ but $\jmath \cdot ((\ell \cdot (\ell \cdot \jmath)) \cdot (0 \cdot (0 \cdot (\jmath \cdot \ell)))) = \jmath \cdot 0 = \jmath \notin \Upsilon(\imath)$.

4.11 BCI-commutative idealistic soft BCI-algebras

Definition 4.11.1. A SS (Γ, ς) over Ω is termed as a "BCI-commutative idealistic soft BCI-algebra" (or $BCI-CIS_{BCI}A$) over Ω if $\Gamma(\wp)$ is a "BCI-commutative ideal" of Ω, for any $\wp \in \varsigma$.

Example 4.11.2. Let $\Omega = \{0, \imath, \jmath, \ell, \kappa\}$ be the BCI-algebra defined in Example 4.10.5. Let (Γ, ς) be a SS over Ω, where $\varsigma = \Omega$ and $\Gamma : \varsigma \to \mathfrak{P}(\Omega)$ is a set-valued function delineated as:

$$\Gamma(\wp) = \begin{cases} \theta(\{0, \imath\}) & if\ \wp \in \{\jmath, \ell, \kappa\} \\ \Omega & if\ \wp \in \{0, \imath\} \end{cases}$$

where $\theta(\{0, \imath\}) = \{\wp \in \Omega \mid 0 \cdot (0 \cdot \wp) \in \{0, \imath\}\}$.
Then (Γ, ς) is a $BCI-CIS_{BCI}A$ over Ω.

For simplicity, now a multiplication will be denoted by juxtaposition. Dots we use only to avoid repetitions of brackets. For example, the formula $((\imath \cdot \jmath) \cdot (\ell \cdot \jmath)) \cdot (\imath \cdot \ell) = 0$ will be written as $(\imath\jmath \cdot \ell\jmath) \cdot \imath\ell = 0$. A fuzzy set ϖ in Ω is a fuzzy α-ideal of Ω if

(α_1) $\varpi(0) \geq \varpi(\imath)$,

(α_2) $\varpi(\jmath\imath) \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\}$,

valid for any $\imath, \jmath, \ell \in \Omega$

Useful facts on various operations on soft sets are given in [2]. A basic literature relevant to the fuzzy soft theory one can find in [29].

4.12 Fuzzy soft set theoretic approach to α-ideals

Let (F, ς) be a fuzzy soft set over a BCI-algebra Ω. If for some $u \in \varsigma$, the fuzzy set $F[u] : \Omega \to [0, 1]$ is a fuzzy α-ideal of Ω, then (F, ς) is referred as a *fuzzy soft α-ideal over Ω with respect to the parameter u.* If (F, ς) is a fuzzy soft α-ideal over Ω with respect to all the members of ς (i.e., for all the parameters in ς), then (F, ς) is referred as a *fuzzy soft α-ideal* over Ω.

Example 4.12.1. Consider the following four different teaching styles: authority, demonstrator, facilitator, delegator and compose the universe

$$\Omega = \{a = authority,\ d = demonstrator,\ f = facilitator,\ g = delegator\}$$

and the multiplication

$\cdot$	a	d	f	g
a	a	d	f	g
d	d	a	g	f
f	f	g	a	d
g	g	f	d	a

Then, $(X, \cdot, a)$ is a BCI-algebra. Let

$$\varsigma = \{m = motivational,\ i = influential,\ c = creative\}$$

be the set of characteristics of the teaching styles given in Ω and let (F, ς) be a fuzzy soft set over Ω with $F[m]$, $F[i]$ and $F[c]$ defined defined by the table:

F	a	d	f	g
m	0.9	0.9	0.4	0.4
i	0.8	0.8	0.5	0.5
c	0.6	0.6	0.2	0.2

Then $F[m]$, $F[i]$ and $F[c]$ are fuzzy soft α-ideals over Ω with respect to the parameters m, i and c respectively. Hence (F, ς) is a fuzzy soft α-ideal over Ω.

In the sequel, Ω will be a BCI-algebra.

Proposition 4.12.2. *If* (F, ς) *is a fuzzy soft* α*-ideal over* Ω*, then*

$$F[\mathrm{u}](\jmath\imath) \geq F[\mathrm{u}](\imath \cdot 0\jmath)$$

for all $\mathrm{u} \in \varsigma$ *and* $\imath, \jmath \in \Omega$.

Proof. It is a consequence of (α_2) and (1.2.7). □

Theorem 4.12.3. *A fuzzy soft* α*-ideal over* Ω *is a fuzzy soft ideal over* Ω.

Proof. Indeed, since (F, ς) is a fuzzy soft α-ideal over Ω, by (α_2), for any $u \in \varsigma$ and $\jmath, \ell \in \Omega$ we have $F[u](\jmath 0) \geq \min\{F[u](0\ell \cdot 0\jmath), F[u](\ell)\}$, whence, applying (1.2.7) and (1.2.11) we conclude $F[u](\jmath) \geq \min\{F[u](\jmath\ell), F[u](\ell)\}$. Hence (F, ς) is a fuzzy soft ideal over Ω. □

The following example shows that a fuzzy soft ideal may not be fuzzy soft α-ideal.

Example 4.12.4. Consider five different teaching styles: authority, demonstrator, facilitator, delegator, hybrid and compose the universe Ω by putting

$$\Omega = \{a = authority,\ d = demonstrator,\ f = facilitator,\ g = delegator,\ h = hybrid\}$$

Such defined set Ω with the multiplication determined by the table

$\cdot$	a	d	f	g	h
a	a	a	a	a	a
d	d	a	d	a	a
f	f	f	a	a	a
g	g	g	g	a	a
h	h	h	h	h	a

and $0 = a$ is a BCI-algebra. (In fact it is a BCK-algebra.)

Let $\varsigma = \{m = motivational,\ i = influential,\ c = creative\}$ be the set of characteristics of the teaching styles given in Ω and let $F[m]$, $F[i]$, $F[c]$ be fuzzy sets in Ω defined as follows:

F	a	d	f	g	h
m	0.8	0.6	0.4	0.4	0.4
i	0.7	0.3	0.3	0.6	0.3
c	0.9	0.8	0.7	0.5	0.5

Then (F, ς) is a fuzzy soft ideal over Ω but it is not a fuzzy α-ideal because

$$F[i](hd) = F[i](h) = 0.3 < 0.6 = \min\{F[i](dg \cdot ah), F[i](g)\}.$$

Proposition 4.12.5. *Let (F, ς) be a fuzzy soft α-ideal over Ω. Then*

$$F[\mathrm{u}](\imath\ell \cdot 0\jmath) \geq F[\mathrm{u}](\imath \cdot \ell\jmath)$$

for any parameter $\mathrm{u} \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$.

Proof. Let (F, ς) be a fuzzy soft α-ideal over Ω. Since $\imath\ell \cdot 0\jmath = \imath\ell \cdot (\ell\jmath \cdot \ell) \leq \imath \cdot \ell\jmath$. Therefore, $(\imath\ell \cdot 0\jmath)(\imath \cdot \ell\jmath) = 0$. By Theorem 4.12.3, (F, ς) is a fuzzy soft ideal over Ω. Thus for any $u \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$ we have

$$F[u](\imath\ell \cdot 0\jmath) \geq \min\{F[u]((\imath\ell \cdot 0\jmath) \cdot (\imath \cdot \ell\jmath)), F[u](\imath \cdot \ell\jmath)\}$$
$$= \min\{F[u](0), F[u](\imath \cdot \ell\jmath)\} = F[u](\imath \cdot \ell\jmath),$$

which completes the proof. □

Theorem 4.12.6. *Let* (F, ς) *be a fuzzy soft ideal over* Ω*. If for any parameter* u $\in \varsigma$ *and* $\imath, \jmath \in \Omega$ *we have* $F[\mathrm{u}](\jmath\imath) \geq F[\mathrm{u}](\imath \cdot 0\jmath)$*, then* (F, ς) *is a fuzzy soft* α*-ideal over* Ω*.*

Proof. Indeed, in this case

$$\begin{aligned} F[u](\jmath\imath) &\geq F[u](\imath \cdot 0\jmath) \geq \min\{F[u]((\imath \cdot 0\jmath)\ell), F[u](\ell)\} \\ &= \min\{F[u](\imath\ell \cdot 0\jmath), F[u](\ell)\} \end{aligned}$$

for any arbitrary parameter $u \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$. Hence (F, ς) is a fuzzy soft α-ideal over Ω. □

Theorem 4.12.7. *If* (F, ς) *and* (G, τ) *are fuzzy soft* α*-ideals over* Ω*, then the extended intersection of* (F, ς) *and* (G, τ) *is a fuzzy soft* α*-ideal over* Ω*.*

Proof. The extended intersection of fuzzy soft sets (F, ς) and (G, τ) is defined as a fuzzy soft set $(F, \varsigma) \sqcap_E (G, \tau) = (H, \varrho)$ such that $\varrho = \varsigma \cup \tau$ and

$$H[u] = \begin{cases} F[u] & if\ u \in \varsigma - \tau, \\ G[u] & if\ u \in \tau - \varsigma, \\ F[u] \cap G[u] & if\ u \in \varsigma \cap \tau \end{cases}$$

for any $u \in \varrho$.

Simple computations show that $H[u]$ is a fuzzy α-ideal for any $u \in \varrho$. Hence (H, ϱ) is a fuzzy α-ideal of Ω. □

Corollary 4.12.8. *Extended and restricted intersections of any two fuzzy soft* α*-ideals are fuzzy soft* α*-ideals.*

Theorem 4.12.9. *Let* (F, ς) *and* (G, τ) *be two fuzzy soft* α*-ideals over* Ω*. If* $\varsigma \cap \tau = \phi$*, then the union* $(F, \varsigma)\widetilde{\cup}(G, \tau)$ *is a fuzzy soft* α*-ideal over* Ω*.*

Proof. The union $(F, \varsigma)\widetilde{\cup}(G, \tau)$ of fuzzy soft sets (F, ς) and (G, τ) is defined as a fuzzy soft set (H, ϱ) such that $\varrho = \varsigma \cup \tau$ and

$$H[u] = \begin{cases} F[u] & if\ u \in \varsigma - \tau, \\ G[u] & if\ u \in \tau - \varsigma, \\ F[u] \cup G[u] & if\ u \in \varsigma \cap \tau \end{cases}$$

for any $u \in \varrho$.

If $\varsigma \cap \tau = \phi$, then obviously either $H[u] = F[u]$ or $H[u] = G[u]$, which means that if $F[u]$ and $G[u]$ are fuzzy α-ideal of Ω, then also Hu is a fuzzy soft α-ideal over Ω. Hence (H, ϱ) is a fuzzy soft α-ideal over Ω. □

In the case $\varsigma \cap \tau \neq \phi$ the union $(F, \varsigma)\widetilde{\cup}(G, \tau)$ of two fuzzy soft α-ideals may not be a fuzzy soft α-ideal.

Example 4.12.10. Let Ω be as in Example 4.12.4. Defining on Ω a new multiplication

$\cdot$	a	d	f	g	h
a	a	a	f	g	h
d	d	a	f	g	h
f	f	f	a	h	g
g	g	g	h	a	f
h	h	h	g	f	a

we obtain a BCI-algebra $(\Omega, \cdot, a)$ which is not a BCK-algebra.

Consider on Ω be two sets of characteristics of the teaching styles:

$\varsigma = \{m = motivational,\ i = influential,\ c = creative,\ p = comprehensive\}$

and

$\tau = \{c = creative,\ p = comprehensive,\ v = perceived\}$.

Then fuzzy sets $F[m]$, $F[i]$, $F[c]$, $F[p]$ defined by

F	a	d	f	g	h
m	0.8	0.8	0.4	0.4	0.4
i	0.7	0.7	0.5	0.3	0.3
c	0.9	0.9	0.2	0.4	0.2
p	0.6	0.6	0.3	0.3	0.5

and fuzzy sets $G[c]$, $G[p]$, $G[v]$ defined by

G	a	d	f	g	h
c	0.9	0.9	0.4	0.2	0.2
p	0.8	0.8	0.4	0.4	0.6
v	1	1	0.3	0.5	0.3

are fuzzy soft α-ideals over Ω. Hence, also (F, ς) and (G, τ) are fuzzy soft α-ideals over Ω.

Now, we consider the union $(F, \varsigma)\widetilde{\cup}(G, \tau) = (H, \varrho)$. For any parameter $c \in \varsigma \cap \tau$ we have $H[c](\imath) = (F[c] \cup G[c])(\imath) = \max\{F[c](\imath), G[c](\imath)\}$ for any $\imath \in \Omega$. Therefore $H[c](gf) = H[c](h) = \max\{F[c](h), G[c](h)\} = 0.2$ and $\min\{H[c](ff \cdot ag), H[c](f)\} = \min\{H[c](g), H[c](f)\} = 0.4$, which shows that $H[c]$, and consequently (H, ϱ), is not a fuzzy soft α-ideal over Ω.

Theorem 4.12.11. *If (F, ς) and (G, τ) are two fuzzy soft α-ideals over Ω, then $(F, \varsigma)\widetilde{\wedge}(G, \tau)$ also is a fuzzy soft α-ideal over Ω.*

Proof. By definition, $(F, \varsigma)\widetilde{\wedge}(G, \tau) = (H, \varsigma \times \tau)$, where $H[u, v] = F[u] \cap G[v]$

for all $(u, v) \in \varsigma \times \tau$. Thus for any $(u, v) \in \varsigma \times \tau$ and $\imath, \jmath, \ell \in \Omega$ we have

$$\begin{aligned} H[u, v](\jmath\imath) &= (F[u] \cap G[v])(\jmath\imath) = \min\{F[u](\jmath\imath), G[v](\jmath\imath)\} \\ &\geq \min\{\min\{F[u](\imath\ell \cdot 0\jmath), F[u](\ell)\}\}, \min\{G[v](\imath\ell \cdot 0\jmath), G[v](\ell)\}\} \\ &= \min\{\min\{F[u](\imath\ell \cdot 0\jmath), G[v](\imath\ell \cdot 0\jmath)\}\}, \min\{F[u](\ell), G[v](\ell)\}\} \\ &= \min\{(F[u] \cap G[v])(\imath\ell \cdot 0\jmath), (F[u] \cap G[v])(\ell)\} \\ &= \min\{H[u, v](\imath\ell \cdot 0\jmath), H[u, v](\ell)\}. \end{aligned}$$

Hence $(H, \varsigma \times \tau)$ is a fuzzy soft α-ideal over Ω. □

Definition 4.12.12. Let K be a subalgebra of Ω. A fuzzy set ϖ in Ω such that $\varpi(0) \geq \varpi(\imath)$ for every $\imath \in \Omega$ is called

- a *fuzzy ideal of* Ω *related to* K (or a K*-fuzzy ideal of* Ω) if

$$\varpi(\imath) \geq \min\{\varpi(\imath\jmath), \varpi(\jmath)\} \quad \text{holds for all } \imath, \jmath \in K,$$

- a *fuzzy* α*-ideal of* Ω *related to* K (or a K*-fuzzy* α*-ideal of* Ω) if

$$\varpi(\jmath\imath) \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\} \quad \text{holds for all } \imath, \jmath, \ell \in K.$$

A K-fuzzy ideal is denoted by $\varpi \blacktriangleleft K$, a K-fuzzy α-ideal by $\varpi \blacktriangleleft_\alpha K$.

Example 4.12.13. Consider the BCI-algebra $(\Omega, \cdot, a)$ defined in Example 4.12.10 and its subalgebra $K = \{a, f, g, h\}$. It is easily to see that a fuzzy set ϖ defined by $\varpi(a) = \varpi(d) = \varpi(f) = 0.9$ and $\varpi(g) = \varpi(h) = 0.3$ is a K-fuzzy α-ideal of Ω.

As a consequence of Theorem 4.12.3 we obtain

Proposition 4.12.14. *Each* K*-fuzzy* α*-ideal of* Ω *is a* K*-fuzzy ideal of* Ω.

Definition 4.12.15. Let (F, ς) be a soft BCI-algebra over Ω. A fuzzy soft set (G, τ) over Ω such that $\tau \subset \varsigma$ is called

- a *fuzzy soft ideal* of (F,ς) if $G[u] \blacktriangleleft F[u]$ for all $u \in \tau$,

- a *fuzzy soft α-ideal* of (F,ς) if $G[u] \blacktriangleleft_\alpha F[u]$ for all $u \in \tau$.

In the first case we write $(G,\tau)\widetilde{\blacktriangleleft}(F,\varsigma)$, in the second $(G,\tau)\widetilde{\blacktriangleleft}_\alpha(F,\varsigma)$.

We clarify the above definition by the example.

Example 4.12.16. Let $(\Omega,\cdot,a)$ and (F,ς) be as in Example 4.12.10. Then $F[m] = F[i] = \{a,f,g,h\}$, $F[c] = \{a,f\}$ and $F[p] = \{a,h\}$ are subalgebras of Ω. Hence (F,ς) is a soft BCI-algebra over Ω.

Consider a fuzzy soft set (G,τ) over Ω such that $\tau = \{m,i\} \subset \varsigma$ and

G	a	d	f	g	h
m	0.9	0.9	0.9	0.3	0.3
i	0.7	0.7	0.7	0.2	0.2

Then $G[m]$ and $G[i]$ are fuzzy α-ideals of Ω related to $F[m]$ and $F[i]$ respectively. Hence $(G,\tau)\widetilde{\blacktriangleleft}_\alpha(F,\varsigma)$.

From our results and examples presented in this note we can deduce that a fuzzy soft α-ideal is a fuzzy soft ideal but the converse statement is not true.

As a consequence of Theorem 4.12.7 and Corollary 4.12.8 we obtain

Theorem 4.12.17. *The extended intersection of two fuzzy soft α-ideals of a BCI-algebra (F,ς) also is a fuzzy soft α-ideal of (F,ς).*

Corollary 4.12.18. *Extended and restricted intersections of any two fuzzy soft α-ideals of a soft BCI-algebra (F,ς) also are fuzzy soft α-ideal of (F,ς).*

Similarly, from Theorem 4.12.9 we can deduce

Theorem 4.12.19. *Let (G, τ) and (H, ϱ) be two fuzzy soft α-ideals of a soft BCI-algebra (F, ς). If $\tau \cap \varrho = \phi$, then the union, $(G, \tau)\widetilde{\cup}(H, \varrho)$ is a fuzzy soft α-ideal of (F, ς).*

Example 4.12.10 shows that in the case $\tau \cap \varrho \neq \phi$ the union $(G, \tau)\widetilde{\cup}(H, \varrho)$ of two fuzzy soft α-ideals may not be a fuzzy soft α-ideal of (F, ς). Moreover, the transfer principle for fuzzy sets described in [38] suggest that any fuzzy soft α-ideal (F, ς) over Ω can be characterized by its *level subsets*, i.e., by the sets

$$L(F[u]; \delta) = \{\imath \in \Omega \mid F[u](\imath) \geq \delta\},$$

where $u \in \varsigma$ and $\delta \in [0, 1]$. Below we present such characterization.

Theorem 4.12.20. *A fuzzy soft set (F, ς) over Ω is a fuzzy soft α-ideal over Ω if and only if each its nonempty level subset $L(F[\mathrm{u}]; \delta)$ is an α-ideal of Ω.*

Proof. Let (F, ς) be a fuzzy soft α-ideal over Ω. Then $F[u]$ is fuzzy α-ideal of Ω for any parameter $u \in \varsigma$. If $L(F[u]; \delta)$ is an arbitrary nonempty level subset of (F, ς), then $F[u](0) \geq F[u](\imath) \geq \delta$ for any $\imath \in L(F[u]; \delta)$. So, $0 \in L(F[u]; \delta)$. Moreover, if $\imath\ell \cdot 0\jmath \in L(F[u]; \delta)$ and $\ell \in L(F[u]; \delta)$, then $F[u](\imath\ell \cdot 0\jmath) \geq \delta$ and $F[u](\ell) \geq \delta$. Thus $F[u](\jmath\imath) \geq \min\{F[u](\imath\ell \cdot 0\jmath), F[u](\ell)\} \geq \delta$, i.e., $\jmath\imath \in L(F[u]; \delta)$. Hence $L(F[u]; \delta)$ is an α-ideal of Ω.

Conversely assume that each nonempty level subset $L(F[u]; \delta)$ is an α-ideal of Ω. If for some $\imath_\circ \in \Omega$ and $u_\circ \in \varsigma$, $F[u_\circ](0) < F[u_\circ](\imath_\circ)$, then $F[u_\circ](0) < \delta_\circ \leq F[u_\circ](\imath_\circ)$ for some $\delta_\circ \in [0, 1]$. This implies that $\imath_\circ \in L(F[u_\circ]; \delta_\circ)$ and $0 \notin L(F[u_\circ]; \delta_\circ)$, which is a contradiction. Thus $F[u](0) \geq F[u](\imath)$ for any $u \in \varsigma$ and $\imath \in \Omega$.

Moreover, if $F[u_\circ](\jmath_\circ\imath_\circ) < \min\{F[u_\circ](\imath_\circ\ell_\circ \cdot 0\jmath_\circ), F[u_\circ](\ell_\circ)\}$ for some $u_\circ \in$

ς and $\imath_\circ, \jmath_\circ, \ell_\circ \in \Omega$, then $F[u_\circ](\jmath_\circ\imath_\circ) < s_\circ \leq \min\{F[u_\circ](\imath_\circ\ell_\circ \cdot 0\jmath_\circ), F[u_\circ](\ell_\circ)\}$ for some $s_\circ \in [0,1]$. So, $\imath_\circ\ell_\circ \cdot 0\jmath_\circ \in L(F[u_\circ]; s_\circ)$ and $\ell_\circ \in L(F[u_\circ]; s_\circ)$, but $\jmath_\circ\imath_\circ \notin L(F[u_\circ]; s_\circ)$, again a contradiction to the fact that $L(F[u_\circ]; s_\circ)$ is an α-ideal of Ω. Thus $F[u](\jmath\imath) \geq \min\{F[u](\imath\ell \cdot 0\jmath), F[u](\ell)\}$ for any $\imath, \jmath, \ell \in \Omega$ and $u \in \varsigma$. This proves that $F[u]$ is a fuzzy α-ideal of Ω for any $u \in \varsigma$. Consequently, (F, ς) is a fuzzy soft α-ideal over Ω. □

4.13 Fuzzy points approach to α-ideals

A fuzzy subset ϖ of Ω such that $\varpi(\jmath) = t \in (0,1]$ for $\jmath = \imath$ and $\varpi(\jmath) = 0$ in other cases is called a *fuzzy point* and is denoted by $\imath_t$. The symbol $\imath_t \in \varpi$ means that $\varpi(\imath) \geq t$. In the case $\varpi(\imath) + t > 1$ we write $\imath_t q\, \varpi$. The symbol $\imath_t \in\vee q\, \varpi$ means that $\imath_t \in \varpi$ or $\imath_t q\, \varpi$. If $\imath_t \in \varpi$ or $\imath_t \in\vee q\, \varpi$ do not hold, then we write $\imath_t \tilde{\in} \varpi$ and $\imath_t \tilde{\in}\vee\, \tilde{q}\, \varpi$, respectively. For other terminology used in this section one can consult [35] and [36]. In the sequel, $\alpha IS_{BCI}A$ will be an *α-idealistic soft BCI-algebra.*

Proposition 4.13.1. *For a fuzzy set ϖ in Ω, the conditions*

(α_1) $\varpi(0) \geq \varpi(\imath)$,

(α_2) $\varpi(\jmath\imath) \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\}$,

are equivalent to the conditions

(α_4) $\imath_\eta \in \varpi \Rightarrow 0_\eta \in \varpi$,

(α_5) $(\imath\ell \cdot 0\jmath)_{\xi_1} \in \varpi,\ \ell_{\xi_2} \in \varpi \Rightarrow (\jmath\imath)_{\min\{\xi_1,\xi_2\}} \in \varpi$

valid for any $\imath, \jmath, \ell \in \Omega$ and $\eta, \xi_1, \xi_2 \in (0,1]$.

Proof. Suppose that the conditions (α_1) and (α_2) are valid. Let $\imath_\eta \in \varpi$. Then $\varpi(0) \geq \varpi(\imath) \geq \eta$. Hence $0_\eta \in \varpi$. This proves (α_4).

Now, if $(\imath\ell\cdot 0\jmath)_{\xi_1}\in\varpi$ and $\ell_{\xi_2}\in\varpi$ for some $\imath,\jmath,\ell\in\Omega$ and $\xi_1,\xi_2\in(0,1]$, then $\varpi(\imath\ell\cdot 0\jmath)\geq\xi_1$ and $\varpi(\ell)\geq\xi_2$, which, by (α_2), implies $\varpi(\jmath\imath)\geq\min\{\varpi(\imath\ell\cdot 0\jmath),\varpi(\ell)\}\geq\min\{\xi_1,\xi_2\}$. So, $(\jmath\imath)_{\min\{\xi_1,\xi_2\}}\in\varpi$. This proves (α_5).

Conversely, suppose that (α_4) and (α_5) are valid. Since, $\imath_{\varpi(\imath)}\in\varpi$, for any $\imath\in\Omega$, therefore, by (α_4), $0_{\varpi(\imath)}\in\varpi$, i.e., $\varpi(0)\geq\varpi(\imath)$ for any $\imath\in\Omega$. This proves (α_1). Moreover, since $(\imath\ell\cdot 0\jmath)_{\varpi(\imath\ell\cdot 0\jmath)}\in\varpi$, $\ell_{\varpi(\ell)}\in\varpi$, from (α_5), we conclude $(\jmath\imath)_{\min\{\varpi(\imath\ell\cdot 0\jmath),\varpi(\ell)\}}\in\varpi$. Thus, $\varpi(\jmath\imath)\geq\min\{\varpi(\imath\ell\cdot 0\jmath),\varpi(\ell)\}$. So, (α_2) is valid too. □

Theorem 4.13.2. *An $\in$-soft set $(F[\varpi],\varsigma)$, where $\varsigma=(0,1]$, is an $\alpha IS_{BCI}A$ over Ω if and only if ϖ is a fuzzy α-ideal of Ω.*

Proof. Let ϖ be a fuzzy α-ideal of Ω. To prove that $(F[\varpi],\varsigma)$ is an $\alpha IS_{BCI}A$ over Ω, it is sufficient to show that $F[\varpi](\delta)$ is an α-ideal of Ω for any $\delta\in\varsigma$.

First observe that $\imath_\delta\in\varpi$ for any $\imath\in F[\varpi](\delta)$. This implies $0_\delta\in\varpi$. Hence, $0\in F[\varpi](\delta)$. Now, if $\imath\ell\cdot 0\jmath\in F[\varpi](\delta)$ and $\ell\in F[\varpi](\delta)$, then $(\imath\ell\cdot 0\jmath)_\delta\in\varpi$ and $\ell_\delta\in\varpi$, which gives $(\jmath\imath)_\delta\in\varpi$. So, $\jmath\imath\in F[\varpi](\delta)$. Hence $F[\varpi](\delta)$ is an α-ideal of Ω for any $\delta\in\varsigma$, and consequently, $(F[\varpi],\varsigma)$ is an $\alpha IS_{BCI}A$ over Ω.

Conversely, let $(F[\varpi],\varsigma)$ is an $\alpha IS_{BCI}A$ over Ω. If for some $\imath\in\Omega$, $\varpi(0)<\varpi(\imath)$, then $\varpi(0)<\delta_\circ\leq\varpi(\imath)$ for some $\delta_\circ\in\varsigma$. This implies $\imath_{\delta_\circ}\in\varpi$, i.e., $\imath\in F[\varpi](\delta_\circ)$. But $0_{\delta_\circ}\notin\varpi$, so, $0\notin F[\varpi](\delta_\circ)$, a contradiction. Therefore $\varpi(0)\geq\varpi(\imath)$, for any $\imath\in\Omega$.

Moreover, if $\varpi(\jmath\imath)<\min\{\varpi(\imath\ell\cdot 0\jmath),\varpi(\ell)\}$ for some $\imath,\jmath,\ell\in\Omega$, then $\varpi(\jmath\imath)<\delta\leq\min\{\varpi(\imath\ell\cdot 0\jmath),\varpi(\ell)\}$ for some $\delta\in\varsigma$. This implies $(\imath\ell\cdot 0\jmath)_\delta\in\varpi$ and $\ell_\delta\in\varpi$. Thus $\imath\ell\cdot 0\jmath\in F[\varpi](\delta)$ and $\ell\in F[\varpi](\delta)$ but $\jmath\imath\notin F[\varpi](\delta)$, which contradicts to the hypothesis. Hence $\varpi(\jmath\imath)\geq\min\{\varpi(\imath\ell\cdot 0\jmath),\varpi(\ell)\}$ for all $\imath\jmath,\ell\in\Omega$. This means that ϖ is a fuzzy α-ideal of Ω. □

Theorem 4.13.3. *A q-soft set $(F[\varpi]_q, \varsigma)$, where $\varsigma = (0,1]$, is an $\alpha IS_{BCI}A$ over Ω if and only if ϖ is a fuzzy α-ideal of Ω.*

Proof. Let ϖ be a fuzzy α-ideal of Ω. Since $\imath_\delta q\varpi$ for any $\imath \in F[\varpi]_q(\delta)$, we have $\varpi(0) + \delta \geq \varpi(\imath) + \delta > 1$. Thus $0_\delta \in \varpi$, i.e., $0 \in F[\varpi]_q(\delta)$.

Let $\imath\ell \cdot 0\jmath \in F[\varpi]_q(\delta)$ and $\ell \in F[\varpi]_q(\delta)$. Then

$$\varpi(\jmath\imath) + \delta \geq \min\{\varpi(\imath\ell \cdot 0\jmath) + \delta, \varpi(\ell) + \delta\} > 1,$$

which implies $(\jmath\imath)_\delta q\varpi$, i.e., $\jmath\imath \in F[\varpi]_q(\delta)$. Hence $F[\varpi]_q(\delta)$ is an α-ideal of Ω.

Conversely assume that $(F[\varpi]_q, \varsigma)$ is an $\alpha IS_{BCI}A$ over Ω. If $\varpi(0) < \varpi(\imath)$ for some $\imath \in \Omega$, then $\varpi(0) + \delta_\circ \leq 1 < \varpi(\imath) + \delta_\circ$ for some $\delta_\circ \in \varsigma$. This implies $\imath_{\delta_\circ} q\varpi$ and $0_{\delta_\circ} \tilde{q}\varpi$. Thus, $\imath \in F[\varpi]_q(\delta_\circ)$ and $0 \notin F[\varpi]_q(\delta_\circ)$, which is a contradiction. Therefore $\varpi(0) \geq \varpi(\imath)$ for every $\imath \in \Omega$.

Moreover, if $\varpi(\jmath\imath) < \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\}$ for some $\imath, \jmath, \ell \in \Omega$, then $\varpi(\jmath\imath) + \delta \leq 1 < \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\} + \delta$ for some $\delta \in \varsigma$. Therefore, $\imath\ell \cdot 0\jmath \in F[\varpi]_q(\delta)$ and $\ell \in F[\varpi]_q(\delta)$ but $\jmath\imath \notin F[\varpi]_q(\delta)$, which contradicts to the hypothesis. Thus $\varpi(\jmath\imath) \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\}$ for any $\imath, \jmath, \ell \in \Omega$. Hence ϖ is a fuzzy α-ideal of Ω. □

Definition 4.13.4. A fuzzy set ϖ in Ω is called a $(\in, \in\vee q)$-*fuzzy α-ideal* of Ω if

(a_1) ϖ is a $(\in, \in\vee q)$-fuzzy ideal of Ω,

(a_2) $\varpi(\jmath\imath) \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell), 0.5\}$, for any $\imath, \jmath, \ell \in \Omega$.

Lemma 4.13.5. *The condition (a_2) is equivalent to the implication*

(a_3) $(\imath\ell \cdot 0\jmath)_{\delta_1} \in \varpi$ *and* $\ell_{\delta_2} \in \varpi \Rightarrow (\jmath\imath)_{\min\{\delta_1,\delta_2\}} \in \vee q\,\varpi$,

for any $\imath, \jmath, \ell \in \Omega$ and $\delta_1, \delta_2 \in (0,1]$.

Proof. Suppose that (a_2) holds and for some $\imath, \jmath, \ell \in \Omega$ and $\delta_1, \delta_2 \in (0,1]$ we have $(\imath\ell \cdot 0\jmath)_{\delta_1} \in \varpi$, $\ell_{\delta_2} \in \varpi$ and $(\jmath\imath)_{\min\{\delta_1,\delta_2\}} \notin \varpi$. Then $\varpi(\imath\ell \cdot 0\jmath) \geq \delta_1$, $\varpi(\ell) \geq \delta_2$ and $\varpi(\jmath\imath) < \min\{\delta_1, \delta_2\}$. If $\min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\} < 0.5$, then

$$\varpi(\jmath\imath) \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell), 0.5\} = \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\} \geq \min\{\delta_1, \delta_2\},$$

a contradiction. Thus, $\min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\} \geq 0.5$. Therefore,

$$\varpi(\jmath\imath) + \min\{\delta_1, \delta_2\} > 2\varpi(\jmath\imath) \geq 2\min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell), 0.5\} = 1.$$

Hence $(\jmath\imath)_{\min\{\delta_1,\delta_2\}} q\, \varpi$. Consequently, $(\jmath\imath)_{\min\{\delta_1,\delta_2\}} \in \vee q\, \varpi$. So, (a_2) implies (a_3).

To prove the converse implication observe that $\min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\} < 0.5$ gives $\varpi(\jmath\imath) \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\}$. If not then there exist $\delta \in (0, 0.5)$ such that $\varpi(\jmath\imath) < \delta \leq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\}$. Hence $(\imath\ell \cdot 0\jmath)_\delta \in \varpi$ and $(\ell)_\delta \in \varpi$ but $(\jmath\imath)_\delta \notin \varpi$. Thus $\varpi(\jmath\imath) + \delta < 1$, i.e., $(\jmath\imath)_\delta \tilde{q} \varpi$, a contradiction. Hence $\varpi(\jmath\imath) \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\}$, whenever $\min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\} < 0.5$. If $\min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\} \geq 0.5$. Then $(\imath\ell \cdot 0\jmath)_{0.5} \in \varpi$ and $(\ell)_{0.5} \in \varpi$, which imply $(\jmath\imath)_{0.5} \in \varpi$. Therefore, $\varpi(\jmath\imath) \geq 0.5$ because in the case $\varpi(\jmath\imath) < 0.5$ we obtain $\varpi(\jmath\imath) + 0.5 < 1$, which is impossible. Hence $\varpi(\jmath\imath) \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell), 0.5\}$ for all $\imath, \jmath, \ell \in \Omega$. □

Definition 4.13.6. A fuzzy set ϖ in Ω is called a $(\in, \in\vee q)$-*fuzzy* α-*ideal* of Ω if

(*i*) $\imath_\delta \in \varpi \Rightarrow 0_\delta \in \vee q\, \varpi$,

(*ii*) $(\imath\ell \cdot 0\jmath)_{\delta_1} \in \varpi$ and $\ell_{\delta_2} \in \varpi \Rightarrow (\jmath\imath)_{\min\{\delta_1,\delta_2\}} \in \vee q\, \varpi$

for any $\imath, \jmath, \ell \in \Omega$ and $\delta, \delta_1, \delta_2 \in (0,1]$.

Proposition 4.13.7. *For any* $(\in, \in\vee q)$-*fuzzy* α-*ideal* ϖ *of* Ω *and any* $\imath, \jmath \in \Omega$, $\delta \in (0,1]$

(a) $(\imath \cdot 0\jmath)_\delta \in \varpi \Rightarrow (\jmath\imath)_\delta \in \vee q\, \varpi$,

(b) $(\imath \cdot \ell\jmath)_\delta \in \varpi \Rightarrow (\imath\ell \cdot 0\jmath)_\delta \in \vee q\, \varpi$

Proof. By the assumption $\varpi(\jmath\imath) \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell), 0.5\}$ for all $\imath, \jmath, \ell \in \Omega$, which for $\ell = 0$ gives $\varpi(\jmath\imath) \geq \min\{\varpi(\imath \cdot 0\jmath), \varpi(0), 0.5\} \geq \min\{\varpi(\imath \cdot 0\jmath), 0.5\}$.

Let $(\imath \cdot 0\jmath)_\delta \in \varpi$ for some $\imath, \jmath \in \Omega$ and $\delta \in (0, 1]$. Then $\varpi(\imath \cdot 0\jmath) \geq \delta$. For $(\jmath\imath)_\delta \in \varpi$ the implication (a) is true. If $(\jmath\imath)_\delta \notin \varpi$, then $\varpi(\jmath\imath) < \delta$ and $\varpi(\imath \cdot 0\jmath) \geq 0.5$ since for $\varpi(\imath \cdot 0\jmath) < 0.5$ we obtain $\varpi(\jmath\imath) \geq \min\{\varpi(\imath \cdot 0\jmath), 0.5\} = \varpi(\imath \cdot 0\jmath) \geq \delta$, which gives a contradiction. Therefore, $\varpi(\imath \cdot 0\jmath) \geq 0.5$ and hence, $\varpi(\jmath\imath) + \delta > 2\varpi(\jmath\imath) \geq 2\min\{\varpi(\imath \cdot 0\jmath), 0.5\} = 1$, i.e., $(\jmath\imath)_\delta\, q\, \varpi$. Hence $(\jmath\imath)_\delta \in \vee q\, \varpi$. This proves (a).

To prove (b) observe first that from (1.2.2) – (1.2.11) for any $\imath, \jmath, \ell \in \Omega$ we obtain $(\imath\ell \cdot 0\jmath) \cdot (\imath \cdot \ell\jmath) = (\imath\ell \cdot (\ell\jmath \cdot \ell)) \cdot (\imath \cdot \ell\jmath) \leq (\imath \cdot \ell\jmath) \cdot (\imath \cdot \ell\jmath) = 0$. Therefore, $(\imath\ell \cdot 0\jmath) \cdot (\imath \cdot \ell\jmath) = 0$. Since any $(\in, \in \vee q)$-fuzzy α-ideal is an $(\in, \in \vee q)$-fuzzy ideal, we have $\varpi(\imath\ell \cdot 0\jmath) \geq \min\{\varpi((\imath\ell \cdot 0\jmath) \cdot (\imath \cdot \ell\jmath)), \varpi(\imath \cdot \ell\jmath), 0.5\} = \min\{\varpi(0), \varpi(\imath \cdot \ell\jmath), 0.5\} \geq \min\{\varpi(\imath \cdot \ell\jmath), 0.5\}$.

Let $(\imath \cdot \ell\jmath)_\delta \in \varpi$. For $(\imath\ell \cdot 0\jmath)_\delta \in \varpi$ the implication (b) is true. For $(\imath\ell \cdot 0\jmath)_\delta \notin \varpi$ we have $\varpi(\imath\ell \cdot 0\jmath) < \delta$. If $\varpi(\imath \cdot \ell\jmath) < 0.5$, then $\varpi(\imath\ell \cdot 0\jmath) \geq \min\{\varpi(\imath \cdot \ell\jmath), 0.5\} = \varpi(\imath \cdot \ell\jmath) \geq \delta$, a contradiction. Therefore $\varpi(\imath \cdot \ell\jmath) \geq 0.5$, and consequently, $\varpi(\imath\ell \cdot 0\jmath) + \delta > 2\varpi(\imath\ell \cdot 0\jmath) \geq 2\min\{\varpi(\imath \cdot \ell\jmath), 0.5\} = 1$, i.e., $(\imath\ell \cdot 0\jmath)_\delta\, q\, \varpi$. Hence $(\imath\ell \cdot 0\jmath)_\delta \in \vee q\, \varpi$. This completes the proof of (b). □

Theorem 4.13.8. *An* $\in$*-soft set* $(F[\varpi], \varsigma)$*, where* $\varsigma = (0, 0.5]$*, is an* $\alpha IS_{BCI}A$ *over* Ω *if and only if* ϖ *is a* $(\in, \in \vee q)$*-fuzzy* α*-ideal of* Ω*.*

Proof. Let ϖ be an $(\in, \in \vee q)$-fuzzy α-ideal of Ω. To prove that $(F[\varpi], \varsigma)$, where $\varsigma = (0, 0.5]$, is an $\alpha IS_{BCI}A$ over Ω, it is sufficient to show that $F[\varpi](\delta)$ is an α-ideal of Ω for any $\delta \in \varsigma$.

$0 \in F[\varpi](\delta)$ since $\varpi(0) \geq \min\{\varpi(\imath), 0.5\} \geq \min\{\delta, 0.5\} = \delta$ for any $\imath \in F[\varpi](\delta)$. If $\imath\ell \cdot 0\jmath \in F[\varpi](\delta)$ and $\ell \in F[\varpi](\delta)$, then $\varpi(\jmath\imath) \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell), 0.5\} \geq \min\{\delta, 0.5\} = \delta$, i.e., $\jmath\imath \in F[\varpi](\delta)$. Thus $F[\varpi](\delta)$ is an α-ideal of Ω for any $\delta \in \varsigma$. Hence $(F[\varpi], \varsigma)$ is an $\alpha IS_{BCI}A$ over Ω.

Conversely, let $(F[\varpi], \varsigma)$ is an $\alpha IS_{BCI}A$ over Ω. If for some $\imath \in \Omega$, $\varpi(0) < \min\{\varpi(\imath), 0.5\}$, then $\varpi(0) < \delta_\circ \leq \min\{\varpi(\imath), 0.5\}$ for some $\delta_\circ \in \varsigma$. This implies $\imath \in F[\varpi](\delta_\circ)$ but $0 \notin F[\varpi](\delta_\circ)$, a contradiction. Therefore $\varpi(0) \geq \min\{\varpi(\imath), 0.5\}$, for any $\imath \in \Omega$. Moreover, if for some $\imath, \jmath, \ell \in \Omega$, $\varpi(\jmath\imath) < \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell), 0.5\}$. Then $\varpi(\jmath\imath) < \delta \leq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell), 0.5\}$ for some $\delta \in \varsigma$. This implies $\imath\ell \cdot 0\jmath \in F[\varpi](\delta)$ and $\ell \in F[\varpi](\delta)$ but $\jmath\imath \notin F[\varpi](\delta)$, which contradicts to the assumption that $F[\varpi](\delta)$ is an α-ideal of Ω. So, $\varpi(\jmath\imath) \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell), 0.5\}$ for all $\imath, \jmath, \ell \in \Omega$. Hence ϖ is a $(\in, \in\vee q)$-fuzzy α-ideal of Ω. □

Theorem 4.13.9. *An $\in$-soft set $(F[\varpi], \varsigma)$, where $\varsigma = (0.5, 1]$, is an $\alpha IS_{BCI}A$ over Ω if only if*

(i) $\max\{\varpi(0), 0.5\} \geq \varpi(\imath)$,

(ii) $\max\{\varpi(\jmath\imath), 0.5\} \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\}$

for any $\imath, \jmath, \ell \in \Omega$.

Proof. Let $(F[\varpi], \varsigma)$, where $\varsigma = (0.5, 1]$, be an $\alpha IS_{BCI}A$ over Ω. If for some $\imath \in \Omega$, $\max\{\varpi(0), 0.5\} < \varpi(\imath)$, then $\max\{\varpi(0), 0.5\} < \delta_\circ \leq \varpi(\imath)$ for some $\delta_\circ \in \varsigma$. So, $\imath \in F[\varpi](\delta_\circ)$ and $0 \notin F[\varpi](\delta_\circ)$, which is a contradiction. Thus (i) is true for all $\imath \in \Omega$.

If $\max\{\varpi(\jmath\imath), 0.5\} < \delta \leq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\}$ for some $\imath, \jmath, \ell \in \Omega$, then $\max\{\varpi(\jmath\imath), 0.5\} < \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\}$ for some $\delta \in \varsigma$. This means that $\imath\ell \cdot 0\jmath \in F[\varpi](\delta)$ and $\ell \in F[\varpi](\delta)$ but $\jmath\imath \notin F[\varpi](\delta)$. Obtained contradiction proves (ii).

Conversely, assume that (i) and (ii) are valid. Now here we need to show that $F[\varpi](\delta)$ is an α-ideal of Ω for any $\delta \in (0.5, 1]$. If $\imath \in F[\varpi](\delta)$, then, by (i), $\max\{\varpi(0), 0.5\} \geq \varpi(\imath) \geq \delta > 0.5$. Hence $\varpi(0) \geq \delta$, i.e., $0 \in F[\varpi](\delta)$. Now let $\imath\ell \cdot 0\ell \in F[\varpi](\delta)$ and $\ell \in F[\varpi](\delta)$ for some $\delta \in (0.5, 1]$. Then $\varpi(\imath\ell \cdot 0\ell) \geq \delta$ and $\varpi(\ell) \geq \delta$, which, by (ii), gives $\varpi(\jmath\imath) \geq \delta$. Thus, $\jmath\imath \in F[\varpi](\delta)$. So, $F[\varpi](\delta)$ is an α-ideal of Ω for any $\delta \in (0.5, 1]$. □

Definition 4.13.10. A fuzzy set ϖ in Ω is a $(\tilde{\in}, \tilde{\in}\vee\tilde{q})$-*fuzzy* α-*ideal* of Ω if

(d_1) $0_\delta \tilde{\in} \varpi \Rightarrow \imath_\delta \tilde{\in}\vee\tilde{q}\,\varpi$,

(d_2) $(\jmath\imath)_{\min\{\delta_1,\delta_2\}} \tilde{\in} \varpi \Rightarrow (\imath\ell \cdot 0\jmath)_{\delta_1} \tilde{\in}\vee\tilde{q}\,\varpi$ or $\ell_{\delta_2} \tilde{\in}\vee\tilde{q}\,\varpi$

for any $\imath, \jmath, \ell \in \Omega$ and $\delta, \delta_1, \delta_2 \in [0, 1]$.

Example 4.13.11. Let Ω be a BCI-algebra defined in Example 4.12.1. By a simple calculation we can verify that a fuzzy set ϖ such that: $\varpi(a) = 0.8$, $\varpi(d) = 0.7$, $\varpi(f) = 0.2$, $\varpi(g) = 0.5$ is an $(\tilde{\in}, \tilde{\in}\vee\tilde{q})$-fuzzy α-ideal of Ω.

Theorem 4.13.12. *An $\in$-soft set $(F[\varpi], \varsigma)$, where $\varsigma = (0.5, 1]$, is an $\alpha IS_{BCI}A$ over Ω if and only if ϖ is an $(\tilde{\in}, \tilde{\in}\vee\tilde{q})$-fuzzy α-ideal of Ω.*

Proof. If $(F[\varpi], \varsigma)$, where $\varsigma = (0.5, 1]$, is an $\alpha IS_{BCI}A$ over Ω, then for all $\imath, \jmath, \ell \in \Omega$ the conditions (i) and (ii) of Theorem 4.13.9 are satisfied. We will show that ϖ satisfies (d_1) and (d_2).

Let $(\jmath\imath)_{\min\{\delta_1,\delta_2\}} \tilde{\in} \varpi$ for some $\imath, \jmath \in \Omega$ and $\delta_1, \delta_2 \in [0, 1]$. Then $\varpi(\jmath\imath) < \min\{\delta_1, \delta_2\}$. If $\varpi(\jmath\imath) \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\}$, then $\min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\} < \min\{\delta_1, \delta_2\}$, i.e., $\varpi(\imath\ell \cdot 0\jmath) < \delta_1$ or $\varpi(\ell) < \delta_2$. Thus $(\imath\ell \cdot 0\jmath)_{\delta_1} \tilde{\in} \varpi$ or $\ell_{\delta_2} \tilde{\in} \varpi$, so, $(\imath\ell \cdot 0\jmath)_{\delta_1} \tilde{\in}\vee\tilde{q}\,\varpi$ or $\ell_{\delta_2} \tilde{\in}\vee\tilde{q}\,\varpi$.

In the case $\varpi(\jmath\imath) < \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell)\}$, by (ii), we get $\imath\{\varpi(\jmath\imath), 0.5\} = 0.5$. Therefore $\varpi(\imath\ell \cdot 0\jmath) \leq 0.5$ or $\varpi(\ell) \leq 0.5$. If $(\imath\ell \cdot 0\jmath)_{\delta_1} \in \varpi$ or $\ell_{\delta_2} \in \varpi$, then $\delta_1 \leq \varpi(\imath\ell \cdot 0\jmath) \leq 0.5$ or $\delta_2 \leq \varpi(\ell) \leq 0.5$. Thus $\varpi(\imath\ell \cdot 0\jmath) + \delta_1 \leq 1$ or

$\varpi(\ell)+\delta_2 \le 1$. Hence $(\imath\ell\cdot 0\jmath)_{\delta_1}\tilde{q}\,\varpi$ or $\ell_{\delta_2}\tilde{q}\,\varpi$. Consequently, $(\imath\ell\cdot 0\jmath)_{\delta_1}\tilde{\in}\vee\tilde{q}\,\varpi$ or $\ell_{\delta_2}\tilde{\in}\vee\tilde{q}\,\varpi$. This means that (d_2) is satisfied in any case.

To prove (d_1), let $0_\delta\tilde{\in}\varpi$ for some $\delta\in[0,1]$. If $\varpi(0)\ge\varpi(\imath)$, then $\varpi(\imath)<\delta$, i.e., $\imath_\delta\tilde{\in}\varpi$. Thus $\imath_\delta\tilde{\in}\vee\tilde{q}\,\varpi$. In the case $\varpi(0)<\varpi(\imath)$, according to (i), we have $\max\{\varpi(0),0.5\}=0.5$. So, $\varpi(\imath)\le 0.5$. If $\imath_\delta\tilde{\in}\varpi$, then (d_1) is valid. If $\imath_\delta\in\varpi$, then $\delta\le\varpi(\imath)\le 0.5$, and consequently, $\varpi(\imath)+\delta\le 1$. So, in this case (d_1) also is valid. This proves that ϖ is an $(\tilde{\in},\tilde{\in}\vee\tilde{q})$-fuzzy α-ideal of Ω.

Conversely, let ϖ be an $(\tilde{\in},\tilde{\in}\vee\tilde{q})$-fuzzy α-ideal of Ω. If $\max\{\varpi(0),0.5\}<\varpi(\imath)$ for some $\imath\in\Omega$, then $\max\{\varpi(0),0.5\}<\delta_\circ\le\varpi(\imath)$ for some $\delta_\circ\in(0.5,1]$. Thus, $\imath_{\delta_\circ}\in\varpi$ and $0_{\delta_\circ}\tilde{\in}\varpi$. This, by (d_1), implies $\imath_{\delta_\circ}\tilde{\in}\vee\tilde{q}\,\varpi$. Hence, $2\delta_\circ\le\varpi(\imath)+\delta_\circ\le 1$. Therefore, $\delta_\circ\le 0.5$, which contradicts that $\delta_\circ\in(0.5,1]$. Hence (i) is valid.

If for some $\imath,\jmath,\ell\in\Omega$, $\max\{\varpi(\jmath\imath),0.5\}<\min\{\varpi(\imath\ell\cdot 0\jmath),\varpi(\ell)\}$. Then, $\max\{\varpi(\jmath\imath),0.5\}<\delta\le\min\{\varpi(\imath\ell\cdot 0\jmath),\varpi(\ell)\}$ for some $\delta\in(0.5,1]$. This means that $(\imath\ell\cdot 0\jmath)_\delta\in\varpi,\ell_\delta\in\varpi$ and $(\jmath\imath)_\delta\tilde{\in}\varpi$. From this, by (d_2), we obtain $(\imath\ell\cdot 0\jmath)_\delta\tilde{\in}\vee\tilde{q}\,\varpi$ or $\ell_\delta\tilde{\in}\vee\tilde{q}\,\varpi$, i.e., $2\delta\le\varpi(\imath\ell\cdot 0\jmath)+\delta\le 1$ or $2\delta\le\varpi(\ell)+\delta\le 1$. Thus $\delta\le 0.5$, which is impossible because $\delta\in(0.5,1]$. Hence $\max\{\varpi(\jmath\imath),0.5\}\ge\min\{\varpi(\imath\ell\cdot 0\jmath),\varpi(\ell)\}$. This proves (ii). Theorem 4.13.9 completes the proof. □

Definition 4.13.13. Let $0\le\delta_1<\delta_2\le 1$. A fuzzy set ϖ of Ω such that

(e_1) $\max\{\varpi(0),\delta_1\}\ge\min\{\varpi(\imath),\delta_2\}$,

(e_2) $\max\{\varpi(\jmath\imath),\delta_1\}\ge\min\{\varpi(\imath\ell\cdot 0\jmath),\varpi(\ell),\delta_2\}$

for all $\imath,\jmath,\ell\in\Omega$, is called a (δ_1,δ_2)-*fuzzy* α-*ideal* of Ω.

Theorem 4.13.14. *Let* $0\le\delta_1<\delta_2\le 1$. *A fuzzy set* ϖ *of a BCI-algebra* Ω *is its* (δ_1,δ_2)-*fuzzy* α-*ideal if and only if an* $\in$-*soft set* $(F[\varpi],\varsigma)$, *where*

$\varsigma = (\delta_1, \delta_2]$, *is an* $\alpha IS_{BCI}A$ *over* Ω.

Proof. Let ϖ be a (δ_1, δ_2)-fuzzy α-ideal of Ω. We have to prove that $F[\varpi](\eta)$ is an α-ideal of Ω for any $\eta \in \varsigma = (\delta_1, \delta_2]$.

For any $\imath \in F[\varpi](\eta)$, $\max\{\varpi(0), \delta_1\} \geq \min\{\varpi(\imath), \delta_2\} \geq \min\{\eta, \delta_2\} = \eta > \delta_1$. This implies, $\varpi(0) \geq \eta$, i.e., $0 \in F[\varpi](\eta)$. Moreover, if $\imath\ell \cdot 0\jmath \in F[\varpi](\eta)$ and $\ell \in F[\varpi](\eta)$, then, by (e_2), we have $\max\{\varpi(\jmath\imath), \delta_1\} \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell), \delta_2\} \geq \min\{\eta, \delta_2\} = \eta > \delta_1$. This implies $\varpi(\jmath\imath) \geq \eta$, i.e., $\jmath\imath \in F[\varpi](\eta)$. So, $F[\varpi](\eta)$ is an α-ideal of Ω for each $\eta \in \varsigma = (\delta_1, \delta_2]$. Hence $(F[\varpi], \varsigma)$ is an $\alpha IS_{BCI}A$ over Ω.

Conversely, let $(F[\varpi], \varsigma)$ be an $\alpha IS_{BCI}A$ over Ω. If $\max\{\varpi(0), \delta_1\} < \min\{\varpi(\imath), \delta_2\}$ for some $\imath \in \Omega$, then $\max\{\varpi(0), \delta_1\} < \eta \leq \min\{\varpi(\imath), \delta_2\}$ for some $\eta \in \varsigma = (\delta_1, \delta_2]$. Hence $\imath \in F[\varpi](\eta)$ and $0 \tilde{\in} F[\varpi](\eta)$, which is a contradiction. Thus, $\max\{\varpi(0), \delta_1\} \geq \min\{\varpi(\imath), \delta_2\}$ for all $\imath \in \Omega$. This proves (e_1). To prove (e_2) suppose that $\max\{\varpi(\jmath\imath), \delta_1\} < \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell), \delta_2\}$ for some $\imath, \jmath, \ell \in \Omega$. Then $(\imath\ell \cdot 0\jmath) \in F[\varpi](\eta)$, $\ell \in F[\varpi](\eta)$ and $\varpi(\jmath\imath) < \eta$ for some $\eta \in \varsigma = (\delta_1, \delta_2]$. So, $\jmath\imath \tilde{\in} F[\varpi](\eta)$. The last is impossible because $F[\varpi](\eta)$ is an α-ideal of Ω for any $\eta \in \varsigma$. Hence $\max\{\varpi(\jmath\imath), \delta_1\} \geq \min\{\varpi(\imath\ell \cdot 0\jmath), \varpi(\ell), \delta_2\}$. This proves (e_2). $\square$

Chapter 5

Intuitionistic Fuzzy soft set theoretic approaches to α-ideals in BCI-algebras

By considering the idea of intuitionistic fuzzy sets by Atanassov [4], below we extend the study of applications of soft sets in α-ideals of BCI-algebras and introduce the concept of intuitionistic fuzzy soft α-ideals and prove their basic properties. We also describe connections between various types of intuitionistic fuzzy soft α-ideals and intuitionistic fuzzy soft ideals. We explore useful facts on various operations given in [40] and [2] on intuitionistic fuzzy soft α-ideals and characterize intuitionistic fuzzy α-ideals by soft (δ, η)-level sets. Presented examples give applications of our results. A basic literature relevant to the intuitionistic fuzzy soft theory one can find in [25, 40].

5.1 Intuitionistic fuzzy soft set theoretic approach to subalgebras and ideals in BCI-algebras

In this section, we present the notions of "intuitionistic fuzzy soft BCI-algebras" and "intuitionistic fuzzy soft ideals" and elaborate apposite properties. We will also establish relation between them with the help of different examples. In the sequel, IFS ($resp.$ $IFSs$), $IFSS$ ($resp.$ $IFSSs$) will be "intuitionistic fuzz set" (resp. "intuitionistic fuzz sets"), "intuitionistic fuzzy soft set" (resp. "intuitionistic fuzzy soft sets") and Ω will be a BCI-algebra.

5.2 Intuitionistic fuzzy soft BCI-algebras

Definition 5.2.1. Let (Γ, ς) be an $IFSS$ over Ω. If for some $\delta \in \varsigma$, $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$ is an "intuitionistic fuzzy BCI-algebra" (or $IF_{BCI}A$) of Ω, then (Γ, ς) is referred as an "intuitionistic fuzzy soft BCI-algebra" (or $IFS_{BCI}A$) over Ω with respect to the parameter δ. If (Γ, ς) is an $IFS_{BCI}A$ over Ω with respect to all the members of ς (i.e., for all the parameters in ς), then (Γ, ς) is referred as an $IFS_{BCI}A$ over Ω.

We clarify the above definition by the succeeding example.

Example 5.2.2. Let the countries; Denmark, Finland, France, Georgia and Hungary constitute a universe set Ω, i.e.,
$\Omega = \{Denmark, Finland, France, Georgia, Hungary\}$.
Suppose that Ⓢ is an operator which acts upon the members of Ω accordingly as:

Denmark Ⓢ $\imath$ = Denmark, for any $\imath \in \Omega$.

$$Finland Ⓢ \imath = \begin{cases} Denmark & if\ \imath \in \{Finland, Georgia, Hungary\} \\ Finland & if\ \imath \in \{Denmark, France\} \end{cases}$$

$$France Ⓢ \imath = \begin{cases} Denmark & if\ \imath \in \{France, Hungary\} \\ France & if\ \imath \in \{Denmark, Finland, Georgia\} \end{cases}$$

$$Georgia Ⓢ \imath = \begin{cases} Georgia & if\ \imath \in \{Denmark, Finland, France\} \\ Denmark & if\ \imath \in \{Georgia, Hungary\} \end{cases}$$

$$Hungary Ⓢ \imath = \begin{cases} Hungary & if\ \imath \in \{Denmark, Finland\} \\ Georgia & if\ \imath = France \\ France & if\ \imath = Georgia \\ Denmark & if\ \imath = Hungary \end{cases}$$

Then $(\Omega, Ⓢ, Denmark)$ is a BCK-algebra and hence a BCI-algebra.

Let $\varsigma = \{Tourist, Investor, Student\}$ be a set of types of visas offered by the countries in Ω to the under-developing countries.

Let (Γ, ς) be an $IFSS$ over Ω. Then $\Gamma[Tourist]$, $\Gamma[Investor]$ and $\Gamma[Student]$ are $IFSs$ in Ω delineated as:

Γ	$Denmark$	$Finland$	$France$	$Georgia$	$Hungary$
$Tourist$	$(0.8, 0.1)$	$(0.8, 0.2)$	$(0.8, 0.2)$	$(0.7, 0.3)$	$(0.7, 0.3)$
$Investor$	$(0.9, 0.1)$	$(0.8, 0.2)$	$(0.4, 0.3)$	$(0.6, 0.4)$	$(0.4, 0.4)$
$Student$	$(0.7, 0.3)$	$(0.2, 0.4)$	$(0.6, 0.4)$	$(0.2, 0.5)$	$(0.2, 0.5)$

Then $\Gamma[Tourist]$, $\Gamma[Investor]$ and $\Gamma[Student]$ are "intuitionistic fuzzy soft BCI-algebras" ($IFS_{BCI}As$) over Ω based on parameters Tourist, Investor and Student respectively. Hence (Γ, ς) is an $IFS_{BCI}A$ over Ω.

Proposition 5.2.3. *If (Γ, ς) is an $IFS_{BCI}A$ over Ω, then for any parameter $\delta \in \varsigma$ and $\imath \in \Omega$, $\varpi_{\Gamma[\delta]}(0) \geq \varpi_{\Gamma[\delta]}(\imath)$ and $\xi_{\Gamma[\delta]}(0) \leq \xi_{\Gamma[\delta]}(\imath)$.*

Proof. By given hypothesis, $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$ is an $IF_{BCI}A$ of Ω for any $\delta \in \varsigma$. Thus for any parameter $\delta \in \varsigma$ and $\imath \in \Omega$,
$\varpi_{\Gamma[\delta]}(0) = \varpi_{\Gamma[\delta]}(\imath * \imath) \geq \min\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Gamma[\delta]}(\imath)\} = \varpi_{\Gamma[\delta]}(\imath)$ and
$\xi_{\Gamma[\delta]}(0) = \xi_{\Gamma[\delta]}(\imath * \imath) \leq \min\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)\} = \xi_{\Gamma[\delta]}(\imath)$.
Hence proved. □

The succeeding statement is evident.

Theorem 5.2.4. *Let* (Γ, ς) *be an* $IFS_{BCI}A$ *over* Ω*. If* $\tau \subset \varsigma$*, then* $(\Gamma|_\tau, \tau)$ *is an* $IFS_{BCI}A$ *over* Ω*.*

Now, we demonstrate that there exists an $IFSS$ (Γ, ς) over Ω which is not an $IFS_{BCI}A$ over Ω but there exists $\tau \subset \varsigma$ such that $(\Gamma|_\tau, \tau)$ is an $IFS_{BCI}A$ over Ω.

Example 5.2.5. Let $(\Omega, Ⓢ, Denmark)$ be the BCI-algebra established in Example 5.2.2 and $\varsigma = \{Tourist, Investor, Student, Worker, Athlete\}$ be a set of characteristics of members of Ω.
Let (Γ, ς) be an $IFSS$ over Ω. Then $\Gamma[Tourist]$, $\Gamma[Investor]$, $\Gamma[Student]$, $\Gamma[Worker]$ and $\Gamma[Athlete]$ are $IFSs$ in Ω delineated as:

Γ	$Denmark$	$Finland$	$France$	$Georgia$	$Hungary$
$Tourist$	$(0.7, 0.1)$	$(0.7, 0.2)$	$(0.7, 0.2)$	$(0.2, 0.3)$	$(0.2, 0.4)$
$Investor$	$(0.8, 0.2)$	$(0.7, 0.3)$	$(0.2, 0.3)$	$(0.5, 0.4)$	$(0.2, 0.5)$
$Student$	$(0.6, 0.1)$	$(0.2, 0.2)$	$(0.4, 0.2)$	$(0.2, 0.3)$	$(0.2, 0.3)$
$Worker$	$(0.1, 0.1)$	$(0.2, 0.2)$	$(0.3, 0.2)$	$(0.5, 0.3)$	$(0.6, 0.4)$
$Athlete$	$(0.3, 0.2)$	$(0.2, 0.3)$	$(0.5, 0.3)$	$(0.6, 0.4)$	$(0.2, 0.5)$

Then it can be perceived that (Γ, ς) is not an $IFS_{BCI}A$ over Ω, since, $\Gamma[Worker]$ and $\Gamma[Athlete]$ aren't "intuitionistic fuzzy BCI-algebras" (or

$IF_{BCI}As$) in Ω. Whereas, if we contemplate $\tau = \{Tourist, Investor, Student\} \subset \varsigma$, then $(\Gamma|_{\tau}, \tau)$ is an $IFS_{BCI}A$ over Ω.

5.3 Intuitionistic fuzzy soft ideals

Definition 5.3.1. Let (Γ, ς) be an $IFSS$ over Ω. If for some $\delta \in \varsigma$, $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$ is an "intuitionistic fuzzy ideal" (or IFI) of Ω, then (Γ, ς) is referred as an "intuitionistic fuzzy soft ideal" (or $IFSI$) over Ω with respect to the parameter δ. If (Γ, ς) is an $IFSI$ over Ω with respect to all the members of ς (i.e., for all the parameters in ς), then (Γ, ς) is referred as an $IFSI$ over Ω.

We clarify the above definition by the succeeding example.

Example 5.3.2. Let

$$\Omega = \{Denmark, Finland, France, Georgia, Hungary\}$$

be a universe set.

Suppose that $\circledcirc$ is an operator which acts upon the members of Ω accordingly as:

Denmark $\circledcirc \imath =$ Denmark, for any $\imath \in \Omega$.

$$Finland \circledcirc \imath = \begin{cases} Denmark & if\ \imath \in \{Finland, Georgia, Hungary\} \\ Finland & if\ \imath \in \{Denmark, France\} \end{cases}$$

$$France \circledcirc \imath = \begin{cases} Denmark & if\ \imath \in \{France, Georgia\} \\ France & if\ \imath \in \{Denmark, Finland, Hungary\} \end{cases}$$

$$Georgia \circledcirc \imath = \begin{cases} Georgia & if\ \imath = Denmark \\ Denmark & if\ \imath = Georgia \\ France & if\ \imath \in \{Finland, Hungary\} \\ Finland & if\ \imath = France \end{cases}$$

$$Hungary \circledcirc \imath = \begin{cases} Denmark & if\ \imath = Hungary \\ Finland & if\ \imath \in \{Finland, Georgia\} \\ Hungary & if\ \imath \in \{Denmark, France\} \end{cases}$$

Then $(\Omega, \circledcirc, Denmark)$ is a BCK-algebra and hence a BCI-algebra.

Let $\varsigma = \{Tourist, Investor, Student, Worker, Athlete, Artist\}$ be a set of characteristics of members of Ω.

Let (Γ, ς) be an $IFSS$ over Ω. Then $\Gamma[Tourist]$, $\Gamma[Investor]$, $\Gamma[Student]$, $\Gamma[Worker]$, $\Gamma[Athlete]$ and $\Gamma[Artist]$ are $IFSs$ in Ω delineated as:

Γ	$Denmark$	$Finland$	$France$	$Georgia$	$Hungary$
$Tourist$	$(0.6, 0.1)$	$(0.4, 0.2)$	$(0.4, 0.2)$	$(0.4, 0.3)$	$(0.4, 0.4)$
$Investor$	$(0.7, 0.2)$	$(0.5, 0.3)$	$(0.7, 0.3)$	$(0.5, 0.4)$	$(0.5, 0.5)$
$Student$	$(0.8, 0.1)$	$(0.8, 0.2)$	$(0.3, 0.2)$	$(0.3, 0.4)$	$(0.8, 0.2)$
$Worker$	$(0.5, 0.3)$	$(0.3, 0.4)$	$(0.4, 0.4)$	$(0.2, 0.5)$	$(0.2, 0.6)$
$Athlete$	$(0.5, 0.1)$	$(0.5, 0.2)$	$(0.8, 0.2)$	$(0.7, 0.3)$	$(0.4, 0.5)$
$Artist$	$(0.6, 0.3)$	$(0.5, 0.4)$	$(0.2, 0.4)$	$(0.2, 0.5)$	$(0.5, 0.6)$

Then (Γ, ς) is an $IFSI$ over Ω based on the parameters Tourist, Investor, Student, Worker and Artist. But since,

$\varpi_{\Gamma[Athelet]}(Hungary) = 0.4 < 0.5$

$= \min\{\varpi_{\Gamma[Athelet]}(Hungary \circledcirc Georgia), \varpi_{\Gamma[Athelet]}(Georgia)\}$

and $\xi_{\Gamma[Athelet]}(Hungary) = 0.5 > 0.3$

$= \max\{\xi_{\Gamma[Athelet]}(Hungary \circledcirc Georgia), \xi_{\Gamma[Athelet]}(Georgia)\}$,

i.e., the IFS $\Gamma[Athelet] = \{(\varpi_{\Gamma[Athelet]}(\imath), (\xi_{\Gamma[Athelet]}(\imath)) \mid \imath \in \Omega\}$ is not an IFI of Ω. Thus (Γ, ς) isn't an $IFSI$ over Ω based on the parameter "Athlete". Hence (Γ, ς) isn't an $IFSI$ over Ω.

Example 5.3.3. Let

$$\Omega = \{Denmark, Finland, France, Georgia, Hungary\}$$

be a universe set.

Suppose that $\circledast$ is an operator which acts upon the members of Ω accordingly as:

$$Denmark \circledast \imath = \begin{cases} Denmark & if\ \imath \in \{Denmark, Finland, France\} \\ Hungary & if\ \imath = Georgia \\ Georgia & if\ \imath = Hungary \end{cases}$$

$$Finland \circledast \imath = \begin{cases} Denmark & if\ \imath = Finland \\ Finland & if\ \imath \in \{Denmark, France\} \\ Hungary & if\ \imath = Georgia \\ Georgia & if\ \imath = Hungary \end{cases}$$

$$France \circledast \imath = \begin{cases} Denmark & if\ \imath = France \\ France & if\ \imath \in \{Denmark, Finland\} \\ Hungary & if\ \imath = Georgia \\ Georgia & if\ \imath = Hungary \end{cases}$$

$$Georgia \circledast \imath = \begin{cases} Denmark & if\ \imath = Georgia \\ Georgia & if\ \imath \in \{Denmark, Finland, France\} \\ Hungary & if\ \imath = Hungary \end{cases}$$

$$Hungary \circledast \imath = \begin{cases} Denmark & if\ \imath = Hungary \\ Hungary & if\ \imath \in \{Denmark, Finland, France\} \\ Georgia & if\ \imath = Georgia \end{cases}$$

Then $(\Omega, \circledast, Denmark)$ is BCI-algebra.

Let $\varsigma = \{Tourist, Investor, Student\}$ be a set of characteristics of members of Ω.

Let (Γ, ς) be an $IFSS$ over Ω. Then $\Gamma[Tourist]$, $\Gamma[Investor]$ and $\Gamma[Student]$ are $IFSs$ in Ω delineated as:

Γ	$Denmark$	$Finland$	$France$	$Georgia$	$Hungary$
$Tourist$	$(0.6, 0.1)$	$(0.4, 0.2)$	$(0.4, 0.3)$	$(0.4, 0.3)$	$(0.6, 0.1)$
$Investor$	$(0.3, 0.2)$	$(0.5, 0.3)$	$(0.3, 0.4)$	$(0.3, 0.2)$	$(0.5, 0.5)$
$Student$	$(0.5, 0.3)$	$(0.6, 0.3)$	$(0.4, 0.5)$	$(0.4, 0.4)$	$(0.6, 0.4)$

Then (Γ, ς) is an $IFSI$ over Ω.

Any $IFSI$ of a BCK-algebra is an "intuitionistic fuzzy soft BCK-algebra" (or $IFS_{BCK}A$) but the converse isn't valid. To comprehend this we contemplate the succeeding example.

Example 5.3.4. Let the flowers, Rose, Tulip, Sunflower, Camation and Gerbera constitute a universe set Ω, i.e.,

$$\Omega = \{Rose, Tulip, Sunflower, Camation, Gerbera\}.$$

Suppose that $\boxminus$ is an operator which acts upon the members of Ω accordingly as:

Rose $\boxminus\ \imath =$ Rose, for any $\imath \in \Omega$.

$$Tulip \boxminus \imath = \begin{cases} Rose & if\ \imath \in \{Tulip, Camation, Gerbera\} \\ Tulip & if\ \imath \in \{Rose, Sunflower\} \end{cases}$$

$$Sunflower \boxminus \imath = \begin{cases} Rose & if\ \imath \in \{Sunflower, Camation, Gerbera\} \\ Sunflower & if\ \imath \in \{Rose, Tulip\} \end{cases}$$

$$Camation \boxminus \imath = \begin{cases} Rose & if\ \imath \in \{Camation, Gerbera\} \\ Camation & if\ \imath \in \{Rose, Tulip, Sunflower\} \end{cases}$$

$$Gerbera \boxminus \imath = \begin{cases} Rose & if\ \imath = Gerbera \\ Gerbera & if\ \imath \in \{Rose, Tulip, Sunflower, Camation\} \end{cases}$$

Then $(\Omega, \boxminus, Rose)$ is a BCK-algebra.

Let $\varsigma = \{white, gold, pink\}$ be a set of different colors in which the flowers in Ω exist in nature.

Let (Γ, ς) be an $IFSS$ over Ω. Then $\Gamma[white]$, $\Gamma[gold]$ and $\Gamma[pink]$ are $IFSs$ in Ω delineated as:

Γ	$Rose$	$Tulip$	$Sunflower$	$Camation$	$Gerbera$
$white$	$(0.8, 0.1)$	$(0.6, 0.2)$	$(0.4, 0.2)$	$(0.4, 0.3)$	$(0.4, 0.4)$
$gold$	$(0.7, 0.2)$	$(0.3, 0.3)$	$(0.3, 0.3)$	$(0.6, 0.2)$	$(0.3, 0.5)$
$pink$	$(0.9, 0.1)$	$(0.8, 0.2)$	$(0.7, 0.2)$	$(0.5, 0.3)$	$(0.5, 0.4)$

Then (Γ, ς) is an $IFS_{BCK}A$ over Ω. But since,

$\varpi_{\Gamma[gold]}(Tulip) = 0.3 < 0.6$

$= \min\{\varpi_{\Gamma[gold]}(Tulip \boxminus Camation), \varpi_{\Gamma[gold]}(Camation)\}$

and $\xi_{\Gamma[gold]}(Tulip) = 0.3 > 0.2$

$= \max\{\xi_{\Gamma[gold]}(Tulip \boxminus Camation), \xi_{\Gamma[gold]}(Camation)\}$,

i.e., the IFS $\Gamma[gold] = \{(\varpi_{\Gamma[gold]}(\imath), (\xi_{\Gamma[gold]}(\imath)) \mid \imath \in \Omega\}$ is not an IFI of Ω. Thus (Γ, ς) isn't an $IFSI$ over Ω based on the parameter gold. Hence (Γ, ς) isn't an $IFSI$ over Ω.

Theorem 5.3.5. *Let (Γ, ς) be an $IFSS$ over Ω. If (Γ, ς) satisfies Proposition 5.2.3 and the implication,*

$\imath * \jmath \leq \ell \Rightarrow \varpi_{\Gamma[\delta]}(\imath) \geq \min\{\varpi_{\Gamma[\delta]}(\jmath), \varpi_{\Gamma[\delta]}(\ell)\}$

and $\xi_{\Gamma[\delta]}(\imath) \leq \max\{\xi_{\Gamma[\delta]}(\jmath), \xi_{\Gamma[\delta]}(\ell)\}$, (10.1)

for any $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$. Then (Γ, ς) is an $IFSI$ over Ω.

Proof. Since by axiom (II), $\imath * (\imath * \jmath) \leq \jmath$, for any $\imath, \jmath, \ell \in \Omega$, it follows by the given implication,

$\varpi_{\Gamma[\delta]}(\imath) \geq \min\{\varpi_{\Gamma[\delta]}(\imath * \jmath), \varpi_{\Gamma[\delta]}(\jmath)\}$ and $\xi_{\Gamma[\delta]}(\imath) \leq \max\{\xi_{\Gamma[\delta]}(\imath * \jmath), \xi_{\Gamma[\delta]}(\jmath)\}$.

This along with Proposition 5.2.3 implies that (Γ, ς) is an $IFSI$ over Ω. □

Theorem 5.3.6. *Let* (Γ, ς) *be an* $IFSI$ *over* Ω. *If for any* $\imath, \jmath, \ell \in \Omega$, $\imath * \jmath \leq \ell$, *then,*

$\varpi_{\Gamma[\delta]}(\imath) \geq \min\{\varpi_{\Gamma[\delta]}(\jmath), \varpi_{\Gamma[\delta]}(\ell)\}$ *and* $\xi_{\Gamma[\delta]}(\imath) \leq \max\{\xi_{\Gamma[\delta]}(\jmath), \xi_{\Gamma[\delta]}(\ell)\}$,

for any parameter $\delta \in \varsigma$.

Proof. Let $\imath * \jmath \leq \ell$, for any $\imath, \jmath, \ell \in \Omega$. Then $(\imath * \jmath) * \ell = 0$. By given hypothesis, for any $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$,

$\varpi_{\Gamma[\delta]}(\imath * \jmath) \geq \min\{\varpi_{\Gamma[\delta]}((\imath * \jmath) * \ell), \varpi_{\Gamma[\delta]}(\ell)\} = \min\{\varpi_{\Gamma[\delta]}(0), \varpi_{\Gamma[\delta]}(\ell)\} = \varpi_{\Gamma[\delta]}(\ell)$

and

$\xi_{\Gamma[\delta]}(\imath * \jmath) \leq \max\{\xi_{\Gamma[\delta]}((\imath * \jmath) * \ell), \xi_{\Gamma[\delta]}(\ell)\} = \max\{\xi_{\Gamma[\delta]}(0), \xi_{\Gamma[\delta]}(\ell)\} = \xi_{\Gamma[\delta]}(\ell)$,

Thus for any $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$,

$\varpi_{\Gamma[\delta]}(\imath) \geq \min\{\varpi_{\Gamma[\delta]}(\imath * \jmath), \varpi_{\Gamma[\delta]}(\jmath)\} \geq \min\{\varpi_{\Gamma[\delta]}(\jmath), \varpi_{\Gamma[\delta]}(\ell)\}$

and

$\xi_{\Gamma[\delta]}(\imath) \leq \max\{\xi_{\Gamma[\delta]}(\imath * \jmath), \xi_{\Gamma[\delta]}(\jmath)\} \leq \max\{\xi_{\Gamma[\delta]}(\jmath), \xi_{\Gamma[\delta]}(\ell)\}$.

Hence proved. □

Theorem 5.3.7. *Let* (Γ, ς) *be an* $IFS_{BCI}A$ *over* Ω. *Then* (Γ, ς) *is an* $IFSI$ *of* $\Omega \iff$ *the implication* (10.1) *is valid.*

Proof. The necessity is evident by Theorem 5.3.6.

Conversely, let the implication (10.1) is valid.

Since $\imath * (\imath * \jmath) \leq \jmath$, for any $\imath, \jmath, \ell \in \Omega$, thus by (10.1),

$\varpi_{\Gamma[\delta]}(\imath) \geq \min\{\varpi_{\Gamma[\delta]}(\imath * \jmath), \varpi_{\Gamma[\delta]}(\jmath)\}$ and $\xi_{\Gamma[\delta]}(\imath) \leq \max\{\xi_{\Gamma[\delta]}(\imath * \jmath), \xi_{\Gamma[\delta]}(\jmath)\}$, for any $\delta \in \varsigma$. This along with Proposition 5.2.3 implies that (Γ, ς) is an $IFSI$ over Ω. $\square$

Theorem 5.3.8. *Any $IFSI$ (Γ, ς) over Ω satisfies,*
$\varpi_{\Gamma[\delta]}(0 * (0 * \imath)) \geq \varpi(\imath)$ *and* $\xi_{\Gamma[\delta]}(0 * (0 * \imath)) \leq \xi(\imath)$,
for any $\delta \in \varsigma$ and $\imath \in \Omega$.

Proof. By given hypothesis, for any $\delta \in \varsigma$ and $\imath \in \Omega$,
$\varpi_{\Gamma[\delta]}(0 * (0 * \imath)) \geq \min\{\varpi_{\Gamma[\delta]}((0 * (0 * \imath)) * \imath), \varpi_{\Gamma[\delta]}(\imath)\}$
$= \min\{\varpi_{\Gamma[\delta]}((0 * \imath) * (0 * \imath)), \varpi_{\Gamma[\delta]}(\imath)\} = \min\{\varpi_{\Gamma[\delta]}(0), \varpi_{\Gamma[\delta]}(\imath)\} = \varpi_{\Gamma[\delta]}(\imath)$
and
$\xi_{\Gamma[\delta]}(0 * (0 * \imath)) \leq \max\{\xi_{\Gamma[\delta]}((0 * (0 * \imath)) * \imath), \xi_{\Gamma[\delta]}(\imath)\}$
$= \max\{\xi_{\Gamma[\delta]}((0 * \imath) * (0 * \imath)), \xi_{\Gamma[\delta]}(\imath)\} = \max\{\xi_{\Gamma[\delta]}(0), \xi_{\Gamma[\delta]}(\imath)\} = \xi_{\Gamma[\delta]}(\imath)$
Hence proved. $\square$

Theorem 5.3.9. *The AND operation of two intuitionistic fuzzy soft ideals (Γ, ς) and (Υ, τ) over Ω is an $IFSI$ over Ω.*

Proof. The AND operation of (Γ, ς) and (Υ, τ) denoted by, $(\Gamma, \varsigma)\tilde{\wedge}(\Upsilon, \tau)$, is defined as, $(\Gamma, \varsigma)\tilde{\wedge}(\Upsilon, \tau) = (\Delta, \varsigma \times \tau)$, where
$\Delta[\delta, \eta] = \Gamma[\delta] \cap \Upsilon[\eta] = \{(\varpi_{\Delta[\delta,\eta]}(\imath), \xi_{\Delta[\delta,\eta]}(\imath)) \mid \imath \in \Omega\}$
$= \{(\varpi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath), \xi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath)) \mid \imath \in \Omega\}$
$= \{(\min\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Upsilon[\eta]}(\imath)\}, \max\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Upsilon[\eta]}(\imath)\}) \mid \imath \in \Omega\}$,
for any $(\delta, \eta) \in \varsigma \times \tau$ and obviously $\delta \in \varsigma$, $\eta \in \tau$.
Thus for any $\imath, \jmath \in \Omega$ and $(\delta, \eta) \in \varsigma \times \tau$,
$\varpi_{\Delta[\delta,\eta]}(\imath) = \varpi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath) = \min\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Upsilon[\eta]}(\imath)\} \geq$
$\min\{\min\{\varpi_{\Gamma[\delta]}(\imath * \jmath), \varpi_{\Gamma[\delta]}(\jmath)\}, \min\{\varpi_{\Upsilon[\eta]}(\imath * \jmath), \varpi_{\Upsilon[\eta]}(\jmath)\}\}$
$= \min\{\min\{\varpi_{\Gamma[\delta]}(\imath * \jmath), \varpi_{\Upsilon[\eta]}(\imath * \jmath)\}, \min\{\varpi_{\Gamma[\delta]}(\jmath), \varpi_{\Upsilon[\eta]}(\jmath)\}\}$
$= \min\{\varpi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath * \jmath), \varpi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\jmath)\}$

$= \min\{\varpi_{\Delta[\delta,\eta]}(\imath * \jmath), \varpi_{\Delta[\delta,\eta]}(\jmath)\}$.

and $\xi_{\Delta[\delta,\eta]}(\imath) = \xi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath) = \max\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Upsilon[\eta]}(\imath)\} \leq$

$\max\{\max\{\xi_{\Gamma[\delta]}(\imath * \jmath), \xi_{\Gamma[\delta]}(\jmath)\}, \max\{\xi_{\Upsilon[\eta]}(\imath * \jmath), \xi_{\Upsilon[\eta]}(\jmath)\}\}$

$= \max\{\max\{\xi_{\Gamma[\delta]}(\imath * \jmath), \xi_{\Upsilon[\eta]}(\imath * \jmath)\}, \max\{\xi_{\Gamma[\delta]}(\jmath), \xi_{\Upsilon[\eta]}(\jmath)\}\}$

$= \max\{\xi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath * \jmath), \xi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\jmath)\}$

$= \max\{\xi_{\Delta[\delta,\eta]}(\imath * \jmath), \xi_{\Delta[\delta,\eta]}(\jmath)\}$.

Hence $(\Gamma, \varsigma)\tilde{\wedge}(\Upsilon, \tau) = (\Delta, \varsigma \times \tau)$ is an $IFSI$ over Ω. □

For a BCI-algebra Ω, an $IFSI$ over Ω isn't necessarily an $IFS_{BCI}A$ as can be perceived by the succeeding example.

Example 5.3.10. Let R be the set of all non-zero rational numbers. Then it is easy to substantiate that $(R, \div, 1)$ is a BCI-algebra. Delineate an IFS $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$, for any $\delta \in \varsigma$ and $\imath \in R$ as:

$$\varpi_{\Gamma[\delta]}(\imath) = \begin{cases} 0.9 & if\ \imath \in Z' \\ 0.09 & otherwise \end{cases}$$

and

$$\xi_{\Gamma[\delta]}(\imath) = \begin{cases} 0.08 & if\ \imath \in Z' \\ 0.8 & otherwise \end{cases}$$

Here Z' = set of all non-zero integers. Then (Γ, ς) is an $IFSI$ over R but since for any $\delta \in \varsigma$,

$\varpi_{\Gamma[\delta]}(5 \div 4) = 0.09 < 0.9 = \min\{\varpi_{\Gamma[\delta]}(5), \varpi_{\Gamma[\delta]}(4)\}$.

and

$\xi_{\Gamma[\delta]}(5 \div 4) = 0.8 > 0.08 = \min\{\xi_{\Gamma[\delta]}(5), \xi_{\Gamma[\delta]}(4)\}$.

i.e, $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$ is not an $IF_{BCI}A$ of R for $\delta \in \varsigma$. Thus (Γ, ς) is not an $IFS_{BCI}A$ over R.

An $IFSI$ (Γ, ς) over Ω is termed as closed if $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$ is a "closed intuitionistic fuzzy ideal" (or $CIFI$) of Ω for any $\delta \in \varsigma$.

Theorem 5.3.11. *Any closed $IFSI$ over Ω is an $IFS_{BCI}A$ over Ω.*

Proof. By given hypothesis, $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$ is a $CIFI$ of Ω for any $\delta \in \varsigma$, i.e., $\varpi_{\Gamma[\delta]}(0 * \imath) \geq \varpi_{\Gamma[\delta]}(\imath)$ and $\xi_{\Gamma[\delta]}(0 * \imath) \leq \xi_{\Gamma[\delta]}(\imath)$, for any $\imath \in \Omega$ and $\delta \in \varsigma$.

Thus for any $\imath, \jmath \in \Omega$ and $\delta \in \varsigma$,

$\varpi_{\Gamma[\delta]}(\imath * \jmath) \geq \min\{\varpi_{\Gamma[\delta]}((\imath * \jmath) * \imath), \varpi_{\Gamma[\delta]}(\imath)\}$

$= \min\{\varpi_{\Gamma[\delta]}((\imath * \imath) * \jmath), \varpi_{\Gamma[\delta]}(\imath)\}$

$= \min\{\varpi_{\Gamma[\delta]}(0 * \jmath), \varpi_{\Gamma[\delta]}(\imath)\} \geq \min\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Gamma[\delta]}(\jmath)\}$.

and $\xi_{\Gamma[\delta]}(\imath * \jmath) \leq \max\{\xi_{\Gamma[\delta]}((\imath * \jmath) * \imath), \xi_{\Gamma[\delta]}(\imath)\}$

$= \max\{\xi_{\Gamma[\delta]}((\imath * \imath) * \jmath), \xi_{\Gamma[\delta]}(\imath)\}$

$= \max\{\xi_{\Gamma[\delta]}(0 * \jmath), \xi_{\Gamma[\delta]}(\imath)\} \leq \max\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\jmath)\}$.

Hence proved. □

Theorem 5.3.12. *An $IFSI$ (Γ, ς) over Ω is closed $\iff$ for any $\delta \in \varsigma$ and $\imath, \jmath \in \Omega$ it placates,*

$$\varpi_{\Gamma[\delta]}(\imath * \jmath) \geq \min\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Gamma[\delta]}(\jmath)\}$$
$$\text{and } \xi_{\Gamma[\delta]}(\imath * \jmath) \leq \max\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\jmath)\} \qquad (17.1)$$

Proof. The necessity is evident by Theorem 5.3.11.

Conversely, let (17.1) is valid. Then for any $\delta \in \varsigma$ and $\wp \in \Omega$,

$\varpi_{\Gamma[\delta]}(0 * \wp) \geq \min\{\varpi_{\Gamma[\delta]}(0), \varpi_{\Gamma[\delta]}(\wp)\} = \varpi_{\Gamma[\delta]}(\wp)$

and $\xi_{\Gamma[\delta]}(0 * \wp) \leq \max\{\xi_{\Gamma[\delta]}(0), \xi_{\Gamma[\delta]}(\wp)\} = \xi_{\Gamma[\delta]}(\wp)$,

i.e., $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$ is a $CIFI$ of Ω for any $\delta \in \varsigma$.

Hence proved. □

5.4 Intuitionistic fuzzy soft set theoretic approach to α-ideals in BCI-algebras

In this section, we present the notions of "intuitionistic fuzzy soft α-ideals" and elaborate apposite properties. We will also establish their relation with "intuitionistic fuzzy soft ideals" with the help of different examples. The "AND" operation, "extended intersection", "restricted intersection" and "union" of "intuitionistic fuzzy soft α-ideals" will also be conferred. In the sequel, $IFS_\alpha I$ ($resp.$ $IFS_\alpha Is$) will be "intuitionistic fuzzy soft α-ideal" (resp. "intuitionistic fuzzy soft α-ideals") and Ω will be a BCI-algebra.

5.5 Intuitionistic fuzzy soft α-ideals

Definition 5.5.1. Let (Γ, ς) be an $IFSS$ over Ω. If for some $\delta \in \varsigma$, $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$ is an "intuitionistic fuzzy α-ideal" (or $IF_\alpha I$) of Ω, then (Γ, ς) is referred as an "intuitionistic fuzzy soft α-ideal" (or $IFS_\alpha I$) over Ω with respect to the parameter δ. If (Γ, ς) is an $IFS_\alpha I$ over Ω with respect to all the members of ς (i.e., for all the parameters in ς), then (Γ, ς) is referred as an $IFS_\alpha I$ over Ω.

We clarify the above definition by the succeeding example.

Example 5.5.2. Let the four different types of flowers; Rose, Tulip, Sunflower and Camation compose the universe Ω, i.e.,

$\Omega = \{Rose, Tulip, Sunflower, Camation\}$.

Suppose that $\Diamond$ is an operator which acts upon the members of Ω accordingly as:

Rose $\Diamond\ \imath = \imath$, for any $\imath \in \Omega$.

$$Tulip\Diamond\imath = \begin{cases} Tulip & if\ \imath = Rose \\ Rose & if\ \imath = Tulip \\ Camation & if\ \imath = Sunflower \\ Sunflower & if\ \imath = Camation \end{cases}$$

$$Sunflower\Diamond\imath = \begin{cases} Sunflower & if\ \imath = Rose \\ Camation & if\ \imath = Tulip \\ Rose & if\ \imath = Sunflower \\ Tulip & if\ \imath = Camation \end{cases}$$

$$Camation\Diamond\imath = \begin{cases} Camation & if\ \imath = Rose \\ Sunflower & if\ \imath = Tulip \\ Tulip & if\ \imath = Sunflower \\ Rose & if\ \imath = Camation \end{cases}$$

Then $(\Omega, \Diamond, Rose)$ is a BCI-algebra.

Let $\varsigma = \{lavender, red, orange\}$ be a set of different colors in which the flowers in Ω exist in nature.

Let (Γ, ς) be an $IFSS\ \Omega$. Then $\Gamma[lavender]$, $\Gamma[red]$ and $\Gamma[orange]$ are $IFSs$ in Ω delineated as:

Γ	$Rose$	$Tulip$	$Sunflower$	$Camation$
$lavender$	$(0.8, 0.1)$	$(0.8, 0.1)$	$(0.6, 0.2)$	$(0.6, 0.2)$
red	$(0.9, 0)$	$(0.9, 0)$	$(0.7, 0.3)$	$(0.7, 0.3)$
$orange$	$(0.7, 0.2)$	$(0.7, 0.2)$	$(0.6, 0.4)$	$(0.6, 0.4)$

Then (Γ, ς) is an $IFS_\alpha I$ over Ω with respect to the parameters lavender, red and orange respectively. Hence (Γ, ς) is an $IFS_\alpha I$ over Ω.

Proposition 5.5.3. *For any $IFS_\alpha I$ (Γ, ς) over Ω, the succeeding inequalities hold:*

$\varpi_{\Gamma[\delta]}(\jmath * \imath) \geq \varpi_{\Gamma[\delta]}(\imath * (0 * \jmath))$ *and* $\xi_{\Gamma[\delta]}(\jmath * \imath) \leq \xi_{\Gamma[\delta]}(\imath * (0 * \jmath))$,

for any $\delta \in \varsigma$ and $\imath, \jmath \in \Omega$.

Proof. Let (Γ, ς) be an $IFS_\alpha I$ over Ω.

Then $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$ is an $IF_\alpha I$ of Ω for any $\delta \in \varsigma$. Thus for any $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$,

$\varpi_{\Gamma[\delta]}(\jmath * \imath) \geq \min\{\varpi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Gamma[\delta]}(\ell)\}$

and $\xi_{\Gamma[\delta]}(\jmath * \imath) \leq \max\{\xi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \xi_{\Gamma[\delta]}(\ell)\}$.

By substituting $\ell = 0$ we get,

$\varpi_{\Gamma[\delta]}(\jmath * \imath) \geq \min\{\varpi_{\Gamma[\delta]}((\imath * 0) * (0 * \jmath)), \varpi_{\Gamma[\delta]}(0)\} = \varpi_{\Gamma[\delta]}(\imath * (0 * \jmath))$

and $\xi_{\Gamma[\delta]}(\jmath * \imath) \leq \max\{\xi_{\Gamma[\delta]}((\imath * 0) * (0 * \jmath)), \xi_{\Gamma[\delta]}(0)\} = \xi_{\Gamma[\delta]}(\imath * (0 * \jmath))$.

Hence proved □

Theorem 5.5.4. *Any $IFS_\alpha I$ over Ω is an $IFSI$ over Ω.*

Proof. Let (Γ, ς) be an $IFS_\alpha I$ over Ω.

Then $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$ is an $IF_\alpha I$ of Ω for any $\delta \in \varsigma$.

Thus for any $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$,

$\varpi_{\Gamma[\delta]}(\jmath * \imath) \geq \min\{\varpi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Gamma[\delta]}(\ell)\}$

and $\xi_{\Gamma[\delta]}(\jmath * \imath) \leq \max\{\xi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \xi_{\Gamma[\delta]}(\ell)\}$.

By substituting $\imath = 0$ we get,

$\varpi_{\Gamma[\delta]}(\jmath * 0) \geq \min\{\varpi_{\Gamma[\delta]}((0 * \ell) * (0 * \jmath)), \varpi_{\Gamma[\delta]}(\ell)\}$

and $\xi_{\Gamma[\delta]}(\jmath * 0) \leq \max\{\xi_{\Gamma[\delta]}((0 * \ell) * (0 * \jmath)), \xi_{\Gamma[\delta]}(\ell)\}$.

Since we know that $(0 * \ell) * (0 * \jmath) \leq \jmath * \ell$, therefore,

$\varpi_{\Gamma[\delta]}((0 * \ell) * (0 * \jmath)) \geq \varpi_{\Gamma[\delta]}(\jmath * \ell)$

and $\xi_{\Gamma[\delta]}((0 * \ell) * (0 * \jmath)) \leq \xi_{\Gamma[\delta]}(\jmath * \ell)$.

Thus we acquire,

$\varpi_{\Gamma[\delta]}(\jmath) \geq \min\{\varpi_{\Gamma[\delta]}((0*\ell)*(0*\jmath)), \varpi_{\Gamma[\delta]}(\ell)\}$

$\geq \min\{\varpi_{\Gamma[\delta]}(\jmath*\ell), \varpi_{\Gamma[\delta]}(\ell)\}$

and $\xi_{\Gamma[\delta]}(\jmath) \leq \max\{\xi_{\Gamma[\delta]}((0*\ell)*(0*\jmath)), \xi_{\Gamma[\delta]}(\ell)\}$

$\leq \max\{\xi_{\Gamma[\delta]}(\jmath*\ell), \xi_{\Gamma[\delta]}(\ell)\}$,

i.e., $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$ is an IFI of Ω for any $\delta \in \varsigma$. Hence (Γ, ς) is an $IFSI$ over Ω. □

The succeeding example proves that an $IFSI$ may not be an $IFS_\alpha I$.

Example 5.5.5. Let the five different kinds of flowers; Rose, Tulip, Sunflower , Camation and Lily compose the universe Ω, i.e.,

$\Omega = \{Rose, Tulip, Sunflower, Camation, Lily\}$.

Suppose that $\oplus$ is an operator which acts upon the members of Ω accordingly as:

$\imath\oplus$ Rose $= \imath$, for any $\imath \in \Omega$.

$$\imath \oplus Tulip = \begin{cases} Rose & if\ \imath \in \{Rose, Tulip\} \\ \imath & if\ \imath \in \{Sunflower, Camation, Lily\} \end{cases}$$

$$\imath \oplus Sunflower = \begin{cases} Lily & if\ \imath \in \{Rose, Tulip\} \\ Rose & if\ \imath = Sunflower \\ Sunflower & if\ \imath = Camation \\ Camation & if\ \imath = Lily \end{cases}$$

$$\imath \oplus Camation = \begin{cases} Camation & if\ \imath \in \{Rose, Tulip\} \\ Lily & if\ \imath = Sunflower \\ Rose & if\ \imath = Camation \\ Sunflower & if\ \imath = Lily \end{cases}$$

$$\imath \oplus Lily = \begin{cases} Sunflower & if \ \imath \in \{Rose, Tulip\} \\ Camation & if \ \imath = Sunflower \\ Lily & if \ \imath = Camation \\ Rose & if \ \imath = Lily \end{cases}$$

Then $(\Omega, \oplus, Rose)$ is a BCI-algebra.

Let $\varsigma = \{lavender, red, green\}$ be a set of characteristics of the flowers given in Ω.

Let (Γ, ς) be an $IFSS$ over Ω. Then $\Gamma[lavender]$, $\Gamma[red]$ and $\Gamma[green]$ are $IFSs$ in Ω delineated as:

Γ	$Rose$	$Tulip$	$Sunflower$	$Camation$	$Lily$
$lavender$	$(0.9, 0.1)$	$(0.4, 0.4)$	$(0.6, 0.3)$	$(0.8, 0.2)$	$(0.1, 0.5)$
red	$(0.9, 0)$	$(0.7, 1)$	$(0.4, 0.4)$	$(0.5, 0.3)$	$(0.6, 0.2)$
$green$	$(1, 0)$	$(0.6, 0.2)$	$(0.5, 0.3)$	$(0.3, 0.4)$	$(0.7, 0.1)$

Then (Γ, ς) is an $IFSI$ over Ω but since

$\varpi_{\Gamma[red]}(Tulip \oplus Lily) = \varpi_{\Gamma[red]}(Sunflower) = 0.4 < 0.6 =$

$\min\{\varpi_{\Gamma[red]}((Lily \oplus Rose) \oplus (Rose \oplus Tulip)), \varpi_{\Gamma[red]}(Rose)\}$

and $\xi_{\Gamma[red]}(Tulip \oplus Lily) = \xi_{\Gamma[red]}(Sunflower) = 0.4 > 0.2 =$

$\max\{\xi_{\Gamma[red]}((Lily \oplus Rose) \oplus (Rose \oplus Tulip)), \xi_{\Gamma[red]}(Rose)\}$,

i.e., $\Gamma[red] = \{(\varpi_{\Gamma[red]}(\imath), \xi_{\Gamma[red]}(\imath)) \mid \imath \in \Omega\}$ isn't an $IF_\alpha I$ of Ω. Therefore (Γ, ς) is not an $IFS_\alpha I$ over Ω with respect to the parameter "red". Hence (Γ, ς) is not an $IFS_\alpha I$ over Ω.

Proposition 5.5.6. *Let (Γ, ς) be an $IFS_\alpha I$ over Ω. Then for any parameter $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$,*

$\varpi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)) \geq \varpi_{\Gamma[\delta]}(\imath * (\ell * \jmath))$ *and* $\xi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)) \leq \xi_{\Gamma[\delta]}(\imath * (\ell * \jmath))$.

Proof. Let (Γ, ς) be an $IFS_\alpha I$ over Ω.

Since $(\imath * \ell) * (0 * \jmath) = (\imath * \ell) * ((\ell * \jmath) * \ell) \leq \imath * (\ell * \jmath)$.

Therefore, $(\imath * \ell) * (0 * \jmath) * (\imath * (\ell * \jmath)) = 0$.

By Theorem 5.5.4, (Γ, ς) is an $IFSI$ over Ω.

Thus, $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$ is an IFI of Ω for any $\delta \in \Omega$.

Thus for any $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$,

$\varpi_{\Gamma[\delta]}((\imath*\ell)*(0*\jmath)) \geq \min\{\varpi_{\Gamma[\delta]}(((\imath*\ell)*(0*\jmath))*(\imath*(\ell*\jmath))), \varpi_{\Gamma[\delta]}((\imath*(\ell*\jmath)))\}$

$= \min\{\varpi_{\Gamma[\delta]}(0), \varpi_{\Gamma[\delta]}((\imath * (\ell * \jmath)))\} = \varpi_{\Gamma[\delta]}((\imath * (\ell * \jmath)))$

and $\xi_{\Gamma[\delta]}((\imath*\ell)*(0*\jmath)) \leq \max\{\xi_{\Gamma[\delta]}(((\imath*\ell)*(0*\jmath))*(\imath*(\ell*\jmath))), \xi_{\Gamma[\delta]}((\imath*(\ell*\jmath)))\}$

$= \max\{\xi_{\Gamma[\delta]}(0), \xi_{\Gamma[\delta]}((\imath * (\ell * \jmath)))\} = \xi_{\Gamma[\delta]}((\imath * (\ell * \jmath)))$.

Hence proved. □

Theorem 5.5.7. *Let* (Γ, ς) *be an* $IFSI$ *over* Ω. *If for any parameter* $\delta \in \varsigma$ *and* $\imath, \jmath \in \Omega$,

$\varpi_{\Gamma[\delta]}(\jmath * \imath) \geq \varpi_{\Gamma[\delta]}(\imath * (0 * \jmath))$ *and* $\xi_{\Gamma[\delta]}(\jmath * \imath) \leq \xi_{\Gamma[\delta]}(\imath * (0 * \jmath))$.

Then (Γ, ς) *is an* $IFS_\alpha I$ *over* Ω.

Proof. Since (Γ, ς) is an $IFSI$ over Ω, therefore

$\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$ is an IFI of Ω for any $\delta \in \Omega$.

Thus for any parameter $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$,

$\varpi_{\Gamma[\delta]}(\jmath * \imath) \geq \varpi_{\Gamma[\delta]}(\imath * (0 * \jmath)) \geq \min\{\varpi_{\Gamma[\delta]}((\imath * (0 * \jmath)) * \ell), \varpi_{\Gamma[\delta]}\ell)\}$

$= \min\{\varpi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \Gamma[\delta](\ell)\}$.

and $\xi_{\Gamma[\delta]}(\jmath * \imath) \leq \xi_{\Gamma[\delta]}(\imath * (0 * \jmath)) \leq \min\{\xi_{\Gamma[\delta]}((\imath * (0 * \jmath)) * \ell), \xi_{\Gamma[\delta]}\ell)\}$

$= \min\{\xi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \Gamma[\delta](\ell)\}$,

i.e., $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$ is an $IF_\alpha I$ of Ω for any $\delta \in \Omega$. Hence (Γ, ς) is an $IFS_\alpha I$ over Ω. □

Theorem 5.5.8. *If* (Γ, ς) *and* (Υ, τ) *are* $IFS_\alpha Is$ *over* Ω, *then the "extended intersection" of* (Γ, ς) *and* (Υ, τ) *is an* $IFS_\alpha I$ *over* Ω.

Proof. We know that the "extended intersection" of (Γ, ς) and (Υ, τ), denoted by $(\Gamma, \varsigma) \sqcap_E (\Upsilon, \tau)$, can be defined as, $(\Gamma, \varsigma) \sqcap_E (\Upsilon, \tau) = (\Pi, \varrho)$, where $\varrho = \varsigma \cup \tau$ and for any $\wp \in \varrho$,

$$\Pi[\wp] = \begin{cases} \Gamma[\wp] = \{(\varpi_{\Gamma[\wp]}(\imath), \xi_{\Gamma[\wp]}(\imath)) \mid \imath \in \Omega\} & if\ \wp \in \varsigma - \tau \\ \Upsilon[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)) \mid \imath \in \Omega\} & if\ \wp \in \tau - \varsigma \\ \Gamma[\wp] \cap \Upsilon[\wp] = \{(\min\{\varpi_{\Gamma[\wp]}(\imath), \varpi_{\Upsilon[\wp]}(\imath)\}, & \\ \max\{\xi_{\Gamma[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)\}) \mid \imath \in \Omega\} & if\ \wp \in \varsigma \cap \tau \end{cases}$$

For any $\wp \in \varrho$ if $\wp \in \varsigma - \tau$, then $\Pi[\wp] = \Gamma[\wp] = \{(\varpi_{\Gamma[\wp]}(\imath), \xi_{\Gamma[\wp]}(\imath)) \mid \imath \in \Omega\}$, which is an $IF_\alpha I$ of Ω.

Similarly, if $\wp \in \tau - \varsigma$, then $\Pi[\wp] = \Upsilon[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)) \mid \imath \in \Omega\}$, which is an $IF_\alpha I$ of Ω.

Moreover if $\wp \in \varrho$ such that $\wp \in \varsigma \cap \tau$, then

$\Pi[\wp] = \Gamma[\wp] \cap \Upsilon[\wp]$

$= \{(\min\{\varpi_{\Gamma[\wp]}(\imath), \varpi_{\Upsilon[\wp]}(\imath)\}, \max\{\xi_{\Gamma[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)\}) \mid \imath \in \Omega\}$,

which is also an $IF_\alpha I$ of Ω since, the intersection of two $IF_\alpha I$ is an $IF_\alpha I$. Hence $\Pi[\wp]$ is an $IF_\alpha I$ of Ω for any $\wp \in \varrho$. Hence $(\Pi, \varrho) = (\Gamma, \varsigma) \sqcap_E (\Upsilon, \tau)$ is an $IFS_\alpha I$ over Ω. □

The corollaries stated below can be deduced from the above theorem.

Corollary 5.5.9. *If* (Γ, ς) *and* (Υ, ς) *are* $IFS_\alpha Is$ *over* Ω*, then the "extended intersection" of* (Γ, ς) *and* (Υ, ς) *is an* $IFS_\alpha I$ *over* Ω.

Corollary 5.5.10. *The "restricted intersection" of two* $IFS_\alpha Is$ *is an* $IFS_\alpha I$.

Theorem 5.5.11. *Let* (Γ, ς) *and* (Υ, τ) *be two* $IFS_\alpha Is$ *over* Ω*. If* $\varsigma \cap \tau = \phi$ *then the "union",* $(\Gamma, \varsigma) \widetilde{\cup} (\Upsilon, \tau)$ *is an* $IFS_\alpha I$ *over* Ω.

Proof. We know that the "union" of (Γ, ς) and (Υ, τ), denoted by $(\Gamma, \varsigma)\widetilde{\cup}(\Upsilon, \tau)$, can be defined as, $(\Gamma, \varsigma)\widetilde{\cup}(\Upsilon, \tau) = (\Pi, \varrho)$, where $\varrho = \varsigma \cup \tau$ and for any $\wp \in \varrho$,

$$\Pi[\wp] = \begin{cases} \Gamma[\wp] = \{(\varpi_{\Gamma[\wp]}(\imath), \xi_{\Gamma[\wp]}(\imath)) \mid \imath \in \Omega\} & if\ \wp \in \varsigma - \tau \\ \Upsilon[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)) \mid \imath \in \Omega\} & if\ \wp \in \tau - \varsigma \\ \Gamma[\wp] \cup \Upsilon[\wp] = \{(\max\{\varpi_{\Gamma[\wp]}(\imath), \varpi_{\Upsilon[\wp]}(\imath)\}, & \\ \min\{\xi_{\Gamma[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)\}) \mid \imath \in \Omega\} & if\ \wp \in \varsigma \cap \tau \end{cases}$$

Since $\varsigma \cap \tau = \phi$, either $\wp \in \varsigma - \tau$ or $\wp \in \tau - \varsigma$ for all $\wp \in \varrho$.
If $\wp \in \varsigma - \tau$ then $\Pi[\wp] = \Gamma[\wp] = \{(\varpi_{\Gamma[\wp]}(\imath), \xi_{\Gamma[\wp]}(\imath)) \mid \imath \in \Omega\}$, which is an $IF_\alpha I$ of Ω as (Γ, ς) is an $IFS_\alpha I$ over Ω.
If $\wp \in \tau - \varsigma$ then $\Pi[\wp] = \Upsilon[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)) \mid \imath \in \Omega\}$, which is an $IF_\alpha I$ of Ω as (Υ, τ) is an $IFS_\alpha I$ over Ω.
Hence $(\Pi, \varrho) = (\Gamma, \varsigma)\widetilde{\cup}(\Upsilon, \tau)$ is an $IFS_\alpha I$ over Ω. □

Below, we consider an example in which we have a non-empty intersection of the sets of parameters (i.e., $\varsigma \cap \tau \neq \phi$).

Example 5.5.12. Let $\Omega = \{Rose, Tulip, Sunflower, Camation, Lily\}$ be a universe set.
Suppose that $\otimes$ is an operator which acts upon the members of Ω accordingly as:

$$Rose \otimes\ \imath = \begin{cases} Rose & if\ \imath \in \{Rose, Tulip\} \\ \imath & if\ \imath \in \{Sunflower, Camation, Lily\} \end{cases}$$

$$Tulip \otimes\ \imath = \begin{cases} Rose & if\ \imath = Tulip \\ Tulip & if\ \imath = Rose \\ \imath & if\ \imath \in \{Sunflower, Camation, Lily\} \end{cases}$$

$$Sunflower \otimes \ \imath = \begin{cases} Sunflower & if \ \imath \in \{Rose, Tulip\} \\ Camation & if \ \imath = Lily \\ Rose & if \ \imath = Sunflower \\ Lily & if \ \imath = Camation \end{cases}$$

$$Camation \otimes \ \imath = \begin{cases} Camation & if \ \imath \in \{Rose, Tulip\} \\ Sunflower & if \ \imath = Lily \\ Lily & if \ \imath = Sunflower \\ Rose & if \ \imath = Camation \end{cases}$$

$$Lily \otimes \ \imath = \begin{cases} Lily & if \ \imath \in \{Rose, Tulip\} \\ Camation & if \ \imath = Sunflower \\ Sunflower & if \ \imath = Camation \\ Rose & if \ \imath = Lily \end{cases}$$

Then $(\Omega, \otimes, Rose)$ is a BCI-algebra.

Let $\varsigma = \{lavender, red, green, purple\}$ and $\tau = \{green, purple, blue\}$ be two sets of characteristics of the flowers given in Ω.

Let (Γ, ς) be an $IFSS$ over Ω. Then $\Gamma[lavender]$, $\Gamma[red]$, $\Gamma[green]$ and $\Gamma[purple]$ are $IFSs$ in Ω delineated as:

Γ	$Rose$	$Tulip$	$Sunflower$	$Camation$	$Lily$
$lavender$	$(0.9, 0)$	$(0.9, 0)$	$(0.4, 0.3)$	$(0.4, 0.1)$	$(0.4, 0.3)$
red	$(0.6, 0.2)$	$(0.6, 0.2)$	$(0.3, 0.4)$	$(0.3, 0.4)$	$(0.5, 0.3)$
$green$	$(0.8, 0.1)$	$(0.8, 0.1)$	$(0.2, 0.5)$	$(0.5, 0.3)$	$(0.2, 0.5)$
$purple$	$(0.7, 0.2)$	$(0.7, 0.2)$	$(0.5, 0.3)$	$(0.3, 0.5)$	$(0.3, 0.5)$

Then (Γ, ς) is an $IFS_\alpha I$ over Ω with respect to the parameters lavender, red, green and purple respectively. Hence (Γ, ς) is an $IFS_\alpha I$ over Ω.

Let (Υ, τ) be an $IFSS$ over Ω. Then $\Upsilon[green]$, $\Upsilon[purple]$ and $\Upsilon[blue]$ are

$IFSs$ in Ω defined as follows:

Υ	$Rose$	$Tulip$	$Sunflower$	$Camation$	$Lily$
$green$	$(0.7, 0)$	$(0.7, 0)$	$(0.5, 0.3)$	$(0.2, 0.5)$	$(0.2, 0.5)$
$purple$	$(0.6, 0.2)$	$(0.6, 0.2)$	$(0.2, 0.5)$	$(0.2, 0.5)$	$(0.4, 0.3)$
$blue$	$(0.9, 0)$	$(0.9, 0)$	$(0.4, 0.3)$	$(0.6, 0.1)$	$(0.4, 0.3)$

Then (Υ, τ) is an $IFS_\alpha I$ over Ω with respect to the parameters creative, comprehensive and perceived respectively. Then (Υ, τ) is an $IFS_\alpha I$ over Ω. Now, we cogitate the union of (Γ, ς) and (Υ, τ), i.e., $(\Gamma, \varsigma)\widetilde{\cup}(\Upsilon, \tau) = (\Pi, \varrho)$, where $\varrho = \varsigma \cup \tau$.

Note that for any parameter $\delta \in \varsigma \cap \tau$,

$\Pi[\delta] = \Gamma[\delta] \cup \Upsilon[\delta] = \{(\varpi_{\Gamma[\delta]\cup\Upsilon[\delta]}, \xi_{\Gamma[\delta]\cup\Upsilon[\delta]}) \mid \imath \in \Omega\}$

$= \{(\max\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Upsilon[\delta]}(\imath)\}, \min\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Upsilon[\delta]}(\imath)\}) \mid \imath \in \Omega\}$.

Then, $\varpi_{\Pi[green]}(Camation \otimes Sunflower) = \varpi_{\Pi[green]}(Lily)$

$= \varpi_{(\Gamma[green]\cup\Upsilon[green])}(Lily)$

$= \max\{\varpi_{\Gamma[green]}(Lily), \varpi_{\Upsilon}[green](Lily)\} = \max\{0.2, 0.2\} = 0.2 < 0.5$

$= \min\{\varpi_{\Pi[green]}((Sunflower \otimes Sunflower) \otimes (Rose \otimes Camation)),$

$\varpi_{\Pi[green]}(Sunflower)\}$

$= \min\{\varpi_{\Pi[green]}(Camation), \varpi_{\Pi[green]}(Sunflower)\}$

$= \min\{\varpi_{(\Gamma[green]\cup\Upsilon[green])}(Camation), \varpi_{(\Gamma[green]\cup\Upsilon[green])}(Sunflower)\}$

$= \min\{\max\{\varpi_{\Gamma[green]}(Camation), \varpi_{\Upsilon[green]}(Camation)\},$

$\max\{\varpi_{\Gamma[green]}(Sunflower), \varpi_{\Upsilon[green]}(Sunflower)\}\}$

$= \min\{\max\{0.5, 0.2\}, \max\{0.2, 0.5\}\} = \min\{0.5, 0.5\}$.

and $\xi_{\Pi[green]}(Camation \otimes Sunflower) = \xi_{\Pi[green]}(Lily)$

$= \xi_{(\Gamma[green]\cup\Upsilon[green])}(Lily) = \min\{\xi_{\Gamma[green]}(Lily), \xi_{\Upsilon}[green](Lily)\}$

$= \min\{0.5, 0.5\} = 0.5 > 0.3$

$= \max\{\xi_{\Pi[green]}((Sunflower \otimes Sunflower) \otimes (Rose \otimes Camation)),$

$\xi_{\Pi[green]}(Sunflower)\}$

$= \max\{\xi_{\Pi[green]}(Camation), \xi_{\Pi[green]}(Sunflower)\}$

$= \max\{\xi_{(\Gamma[green]\cup\Upsilon[green])}(Camation), \xi_{(\Gamma[green]\cup\Upsilon[green])}(Sunflower)\}$

$= \max\{\min\{\xi_{\Gamma[green]}(Camation), \xi_{\Upsilon[green]}(Camation)\},$

$\min\{\xi_{\Gamma[green]}(Sunflower), \xi_{\Upsilon[green]}(Sunflower)\}\}$

$= \max\{\min\{0.3, 0.5\}, \min\{0.5, 0.3\}\} = \max\{0.3, 0.3\}.$

Therefore $\Pi[green] = \Gamma[green] \cup \Upsilon[green]$

$= \{(\varpi_{\Gamma[green]\cup\Upsilon[green]}, \xi_{\Gamma[green]\cup\Upsilon[green]}) \mid \imath \in \Omega\}$ isn't an $IF_\alpha I$ of Ω. Thus $(\Pi, \varrho) = (\Gamma, \varsigma)\widetilde{\cup}(\Upsilon, \tau)$ isn't an $IFS_\alpha I$ over Ω based on the parameter "green". Hence $(\Pi, \varrho) = (\Gamma, \varsigma)\widetilde{\cup}(\Upsilon, \tau)$ is not an $IFS_\alpha I$ over Ω.

Theorem 5.5.13. *If (Γ, ς) and (Υ, τ) are two $IFS_\alpha Is$ over Ω, then the "AND" operation, $(\Gamma, \varsigma)\widetilde{\wedge}(\Upsilon, \tau)$ is also an $IFS_\alpha I$ over Ω.*

Proof. By definition, $(\Gamma, \varsigma)\widetilde{\wedge}(\Upsilon, \tau) = (\Pi, \varsigma \times \tau)$, where

$\Pi[\delta, \eta] = \Gamma[\delta] \cap \Upsilon[\eta] = \{(\varpi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath), \xi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath)) \mid \imath \in \Omega\}$

$= \{(\min\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Upsilon[\eta]}(\imath)\}, \max\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Upsilon[\eta]}(\imath)\}) \mid \imath \in \Omega\},$

for all $(\delta, \eta) \in \varsigma \times \tau$.

For any $(\delta, \eta) \in \varsigma \times \tau$ (i.e., $\delta \in \varsigma$ and $\eta \in \tau$) and $\imath \in \Omega$,

$\varpi_{\Pi[\delta,\eta]}(0) = \varpi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(0) = \min\{\varpi_{\Gamma[\delta]}(0), \varpi_{\Upsilon[\eta]}(0)\}$

$\geq \min\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Upsilon[\eta]}(\imath)\} = \varpi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(\imath) = \varpi_{\Pi[\delta,\eta]}(\imath)$

and $\xi_{\Pi[\delta,\eta]}(0) = \xi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(0) = \max\{\xi_{\Gamma[\delta]}(0), \xi_{\Upsilon[\eta]}(0)\}$

$\leq \max\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Upsilon[\eta]}(\imath)\} = \xi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(\imath) = \xi_{\Pi[\delta,\eta]}(\imath).$

For any $(\delta, \eta) \in \varsigma \times \tau$ (i.e., $\delta \in \varsigma$ and $\eta \in \tau$) and $\imath, \jmath, \ell \in \Omega$,

$\varpi_{\Pi[\delta,\eta]}(\jmath * \imath) = \varpi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(\jmath * \imath) = \min\{\varpi_{\Gamma[\delta]}(\jmath * \imath), \varpi_{\Upsilon[\eta]}(\jmath * \imath)\}$

$\geq \min\{\min\{\varpi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Gamma[\delta]}(\ell)\}\},$

$\min\{\varpi_{\Upsilon[\eta]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Upsilon[\eta]}(\ell)\}\}$

$= \min\{\min\{\varpi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)),$

$\varpi_{\Upsilon[\eta]}((\imath * \ell) * (0 * \jmath))\}\}, \min\{\varpi_{\Gamma[\delta]}(\ell), \varpi_{\Upsilon[\eta]}(\ell)\}\}$

$= \min\{\varpi_{(\Gamma[\delta]\cap\Upsilon[\eta])}((\imath * \ell) * (0 * \jmath)), \varpi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(\ell)\}$

$= \min\{\varpi_{\Pi[\delta,\eta]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Pi[\delta,\eta]}(\ell)\}$.

and $\xi_{\Pi[\delta,\eta]}(\jmath * \imath) = \xi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(\jmath * \imath) = \max\{\xi_{\Gamma[\delta]}(\jmath * \imath), \xi_{\Upsilon[\eta]}(\jmath * \imath)\}$

$\leq \max\{\max\{\xi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \xi_{\Gamma[\delta]}(\ell)\}\},$

$\max\{\xi_{\Upsilon[\eta]}((\imath * \ell) * (0 * \jmath)), \xi_{\Upsilon[\eta]}(\ell)\}\}$

$= \max\{\max\{\xi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)),$

$\xi_{\Upsilon[\eta]}((\imath * \ell) * (0 * \jmath))\}\}, \max\{\xi_{\Gamma[\delta]}(\ell), \xi_{\Upsilon[\eta]}(\ell)\}\}$

$= \max\{\xi_{(\Gamma[\delta]\cap\Upsilon[\eta])}((\imath * \ell) * (0 * \jmath)), \xi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(\ell)\}$

$= \max\{\xi_{\Pi[\delta,\eta]}((\imath * \ell) * (0 * \jmath)), \xi_{\Pi[\delta,\eta]}(\ell)\}$.

Thus $\Pi[\delta,\eta] = \Gamma[\delta] \cap \Upsilon[\eta] = \{(\varpi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath), \xi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath)) \mid \imath \in \Omega\}$ is an $IF_\alpha I$ of Ω for any $(\delta,\eta) \in \varsigma \times \tau$.

Hence $(\Gamma,\varsigma)\widetilde{\wedge}(\Upsilon,\tau) = (\Pi,\varsigma\times\tau)$ is an $IFS_\alpha I$ over Ω with respect to the parameter (δ,η). Since (δ,η) is an arbitrary parameter, therefore $(\Gamma,\varsigma)\widetilde{\wedge}(\Upsilon,\tau) = (\Pi,\varsigma\times\tau)$ is an $IFS_\alpha I$ over Ω. □

Any $IFS_\alpha I$ over a BCK-algebra Ω is an "intuitionistic fuzzy soft BCK-algebra" (or $IFS_{BCK}A$). We perceive by the succeeding example that the converse isn't valid.

Example 5.5.14. Let $\Omega = \{Rose, Tulip, Sunflower, Camation, Lily\}$ be a universe set.

Suppose that '$\boxplus$' is an operator which acts upon the members of Ω accordingly as:

Rose $\boxplus \imath =$ Rose, for all $\imath \in \Omega$.

$$Tulip \boxplus \imath = \begin{cases} Rose & if\ \imath \in \{Tulip, Camation, Lily\} \\ Tulip & if\ \imath \in \{Rose, Sunflower\} \end{cases}$$

$$Sunflower \boxplus \imath = \begin{cases} Sunflower & if\ \imath \in \{Rose, Tulip\} \\ Rose & if\ \imath \in \{Sunflower, Camation, Lily\} \end{cases}$$

$$Camation \boxplus \imath = \begin{cases} Camation & if\ \imath \in \{Rose, Tulip, Sunflower\} \\ Rose & if\ \imath \in \{Camation, Lily\} \end{cases}$$

$$Lily \boxplus \imath = \begin{cases} Lily & if\ \imath \in \{Rose, Tulip, Sunflower, Camation\} \\ Rose & if\ \imath = Lily \end{cases}$$

Then $(\Omega, \boxplus, Rose)$ is a BCK-algebra.

Let $\varsigma = \{lavender, red, green\}$ be a set of parameters of Ω.

Let (Γ, ς) be an $IFSS$ over Ω. Then $\Gamma[lavender]$, $\Gamma[red]$ and $\Gamma[green]$ are $IFSs$ in Ω delineated as:

Γ	$Rose$	$Tulip$	$Sunflower$	$Camation$	$Lily$
$lavender$	$(0.7, 0.1)$	$(0.5, 0.2)$	$(0.3, 0.4)$	$(0.3, 0.4)$	$(0.3, 0.4)$
red	$(0.8, 0.2)$	$(0.4, 0.5)$	$(0.4, 0.5)$	$(0.7, 0.3)$	$(0.4, 0.5)$
$green$	$(0.8, 0)$	$(0.7, 0.1)$	$(0.6, 0.3)$	$(0.4, 0.4)$	$(0.4, 0.4)$

Then (Γ, ς) is an $IFS_{BCK}A$ over Ω but since,

$\varpi_{\Gamma[red]}(Lily \boxplus Tulip) = \varpi_{\Gamma[red]}(Lily) = 0.4 < 0.7$

$= \min\{\varpi_{\Gamma[red]}((Tulip \boxplus Camation) \boxplus (Rose \boxplus Lily)), \varpi_{\Gamma[red]}(Camation)\}$

and $\xi_{\Gamma[red]}(Lily \boxplus Tulip) = \xi_{\Gamma[red]}(Lily) = 0.5 > 0.3$

$= \max\{\xi_{\Gamma[red]}((Tulip \boxplus Camation) \boxplus (Rose \boxplus Lily)), \xi_{\Gamma[red]}(Camation)\}$,

i.e., $\Gamma[red] = \{(\varpi_{\Gamma[red]}(\imath), \xi_{\Gamma[red]}(\imath)) \mid \imath \in \Omega\}$ is not an $IF_\alpha I$ of Ω. Therefore (Γ, ς) isn't an $IFS_\alpha I$ over Ω based on the parameter "red". Hence (Γ, ς) isn't an $IFS_\alpha I$ over Ω.

5.6 Intuitionistic fuzzy soft set theoretic approach to α-ideals based on soft set theoretic approach to BCI-algebras

Now, we will confer "intuitionistic fuzzy soft α-ideal" of a "soft BCI-algebra" and discuss its properties. We will elaborate the "AND" operation, "extended intersection", "restricted intersection" and "union" of "intuitionistic fuzzy soft α-ideals" of a "soft BCI-algebra". Here first we familiarize with IFI and $IF_\alpha I$ related to a subalgebra. In the sequel, Ω, as usual will be a BCI-algebra.

5.7 Intuitionistic fuzzy α-ideal related to a subalgebra

Definition 5.7.1. Let Ξ be a subalgebra of Ω.

An IFS $\Theta = \{(\varpi_\Theta(\imath), \xi_\Theta(\imath)) \mid \imath \in \Omega\}$ in Ω is an IFI of Ω related to Ξ (or briefly, Ξ-IFI of Ω), symbolized as $\Theta\blacktriangle\Xi$ if,

(i) $\varpi_\Theta(0) \geq \varpi_\Theta(\imath)$ and $\xi_\Theta(0) \leq \xi_\Theta(\imath)$, for any $\imath \in \Xi$.

(ii) $\varpi_\Theta(\imath) \geq \min\{\varpi_\Theta(\imath * \jmath), \varpi_\Theta(\jmath)\}$ and $\xi_\Theta(\imath) \leq \max\{\xi_\Theta(\imath * \jmath), \xi_\Theta(\jmath)\}$, for any $\imath, \jmath \in \Xi$.

Definition 5.7.2. Let Ξ be a subalgebra of Ω.

An IFS $\Theta = \{(\varpi_\Theta(\imath), \xi_\Theta(\imath)) \mid \imath \in \Omega\}$ in Ω is an $IF_\alpha I$ of Ω related to Ξ (or briefly, Ξ-$IF_\alpha I$ of Ω), symbolized as $\Theta\blacktriangle_\alpha\Xi$, if,

(i) $\varpi_\Theta(0) \geq \varpi_\Theta(\imath)$ and $\xi_\Theta(0) \leq \xi_\Theta(\imath)$, for any $\imath \in \Xi$.

(ii) $\varpi_\Theta(\jmath * \imath) \geq \min\{\varpi_\Theta((\imath * \ell) * (0 * \jmath)), \varpi_\Theta(\ell)\}$

and $\xi_{\Theta}(\jmath * \imath) \leq \max\{\xi_{\Theta}((\imath * \ell) * (0 * \jmath)), \xi_{\Theta}(\ell)\}$, for any $\imath, \jmath, \ell \in \Xi$.

Example 5.7.3. Cogitate the BCI-algebra $(\Omega, \otimes, Rose)$ defined in Example 5.5.12. Let $\Xi = \{Rose, Sunflower, Camation, Lily\}$ be a subset of Ω. Then $(\Xi, \otimes, Rose)$ is also a BCI-algebra, i.e., Ξ is a subalgebra of Ω.

Delineate an IFS $\Theta = \{(\varpi_{\Theta}(\imath), \xi_{\Theta}(\imath)) \mid \imath \in \Omega\}$ in Ω as:

$\varpi_{\Theta}(Rose) = \varpi_{\Theta}(Tulip) = \varpi_{\Theta}(Sunflower) = 0.8$,

$\varpi_{\Theta}(Camation) = \varpi_{\Theta}(Lily) = 0.2$.

and $\xi_{\Theta}(Rose) = \xi_{\Theta}(Tulip) = \xi_{\Theta}(Sunflower) = 0.1$,

$\xi_{\Theta}(Camation) = \xi_{\Theta}(Lily) = 0.6$.

Then it can be observed that,

(i) $\varpi_{\Theta}(0) \geq \varpi_{\Theta}(\imath)$ and $\xi_{\Theta}(0) \leq \xi_{\Theta}(\imath)$, for any $\imath \in \Xi$.

(ii) $\varpi_{\Theta}(\jmath * \imath) \geq \min\{\varpi_{\Theta}((\imath * \ell) * (0 * \jmath)), \varpi_{\Theta}(\ell)\}$

and $\xi_{\Theta}(\jmath * \imath) \leq \max\{\xi_{\Theta}((\imath * \ell) * (0 * \jmath)), \xi_{\Theta}(\ell)\}$, for any $\imath, \jmath, \ell \in \Xi$.

Hence $\Theta = \{(\varpi_{\Theta}(\imath), \xi_{\Theta}(\imath)) \mid \imath \in \Omega\}$ is an Ξ-$IF_{\alpha}I$ of Ω.

It is eminent that any Ξ-$IF_{\alpha}I$ of Ω is an Ξ-IFI of Ω.

5.8 Intuitionistic fuzzy soft α-ideal of a soft BCI-algebra

In the sequel, $S_{BCI}A$ will be a "soft BCI-algebra" and Ω will be a "BCI-algebra".

Definition 5.8.1. Let (Γ, ς) be a $S_{BCI}A$ over Ω. An $IFSS$ (Υ, τ) over Ω is an $IFSI$ of (Γ, ς), symbolized as $(\Upsilon, \tau)\widetilde{\blacktriangle}(\Gamma, \varsigma)$, if $\tau \subset \varsigma$ and for any $\delta \in \tau$,

$$\Upsilon[\delta] = \{(\varpi_{\Upsilon[\delta]}(\imath), \xi_{\Upsilon[\delta]}(\imath)) \mid \imath \in \Omega\}\blacktriangle\Gamma[\delta].$$

Definition 5.8.2. Let (Γ, ς) be a $S_{BCI}A$ over Ω. An $IFSS$ (Υ, τ) over Ω is an $IFS_\alpha I$ of (Γ, ς), symbolized as $(\Upsilon, \tau)\widetilde{\blacktriangle}_\alpha(\Gamma, \varsigma)$, if $\tau \subset \varsigma$ and for any $\delta \in \tau$,

$$\Upsilon[\delta] = \{(\varpi_{\Upsilon[\delta]}(\imath), \xi_{\Upsilon[\delta]}(\imath)) \mid \imath \in \Omega\}\blacktriangle_\alpha\Gamma[\delta].$$

Below, we discuss an example to explore the above definition.

Example 5.8.3. Cogitate the BCI-algebra $(\Omega, \otimes, Rose)$ defined in Example 5.5.12, where $\Omega = \{Rose, Tulip, Sunflower, Camation, Lily\}$.
Let $\varsigma = \{lavender, pink, golden, purple\}$ be a set of characteristics of members of Ω.
Let (Γ, ς) be a soft set over Ω.
Then $\Gamma[lavender] = \Gamma[pink] = \{Rose, Sunflower, Camation, Lily\}$, $\Gamma[golden] = \{Rose, Sunflower\}$ and $\Gamma[purple] = \{Rose, Lily\}$, that are all subalgebras of Ω. Hence (Γ, ς) is a "soft BCI-algebra" over Ω.
Let (Υ, τ) be an $IFSS$ over Ω,
where $\tau = \{lavender, pink\} \subset \varsigma$. Then $\Upsilon[lavender]$ and $\Upsilon[inflential]$ are $IFSs$ in Ω delineated as:

Υ	$Rose$	$Tulip$	$Sunflower$	$Camation$	$Lily$
$lavender$	$(0.8, 0.1)$	$(0.8, 0.1)$	$(0.8, 0.1)$	$(0.2, 0.3)$	$(0.2, 0.3)$
$pink$	$(0.6, 0.2)$	$(0.6, 0.2)$	$(0.6, 0.2)$	$(0.3, 0.4)$	$(0.3, 0.4)$

Then $\Upsilon[lavender] = \{(\varpi_{\Upsilon[lavender]}(\imath), \xi_{\Upsilon[lavender]}(\imath)) \mid \imath \in \Omega\}$ and $\Upsilon[pink] = \{(\varpi_{\Upsilon[pink]}(\imath), \xi_{\Upsilon[pink]}(\imath)) \mid \imath \in \Omega\}$ are $IFS_\alpha Is$ of Ω related to $\Gamma[lavender]$ and $\Gamma[pink]$ respectively.
Hence $(\Upsilon, \tau)\widetilde{\blacktriangle}_\alpha(\Gamma, \varsigma)$.

Any $IFS_\alpha I$ (Υ, τ) of a $S_{BCI}A$ (Γ, ς) is an $IFSI$ of (Γ, ς) but the converse isn't true, as can be perceived by the succeeding example.

Example 5.8.4. Assume $\Omega = \{Rose, Tulip, Sunflower, Camation, Lily\}$ is a universe set.

Let '$\circledast$' be an operator which operates on the elements of Ω accordingly as Rose $\circledast\ \imath$ = Rose, for all $\imath \in \Omega$.

$$Tulip \circledast \imath = \begin{cases} Tulip & if\ \imath = Rose \\ Rose & if\ \imath \in \{Tulip, Sunflower, Camation, Lily\} \end{cases}$$

$$Sunflower \circledast \imath = \begin{cases} Sunflower & if\ \imath \in \{Rose, Tulip, Camation\} \\ Rose & if\ \imath \in \{Sunflower, Lily\} \end{cases}$$

$$Camation \circledast \imath = \begin{cases} Camation & if\ \imath \in \{Rose, Tulip, Sunflower\} \\ Rose & if\ \imath \in \{Camation, Lily\} \end{cases}$$

$$Lily \circledast \imath = \begin{cases} Lily & if\ \imath \in \{Rose, Tulip\} \\ Rose & if\ \imath = Lily \\ Camation & if\ \imath = Sunflower \\ Sunflower & if\ \imath = Camation \end{cases}$$

Then $(\Omega, \circledast, Rose)$ is a "BCK-algebra" and thus a "BCI-algebra".

Let $\varsigma = \{lavender, pink, gold, purple, yellow\}$ be a set of different types of colors in which the flowers in Ω exist in nature.

Let (Γ, ς) be a soft set over Ω.

Then $\Gamma[lavender] = \Omega,\ \Gamma[pink] = \Gamma[gold]$

$= \{Rose, Sunflower, Camation, Lily\}$

and $\Gamma[purple] = \Gamma[yellow] = \{Rose, Sunflower\}$, that are all subalgebras of Ω. Hence (Γ, ς) is a $S_{BCI}A$ over Ω.

Suppose that (Υ, τ), where $\tau = \{gold, purple, yellow\} \subset \varsigma$ is an $IFSS$ over

Ω. Then $\Upsilon[gold]$, $\Gamma[purple]$ and $\Upsilon[yellow]$ are $IFSs$ in Ω delineated as:

Υ	*Rose*	*Tulip*	*Sunflower*	*Camation*	*Lily*
Gold	$(0.8, 0.1)$	$(0.7, 0.2)$	$(0.6, 0.3)$	$(0.4, 0.4)$	$(0.4, 0.4)$
Purple	$(0.7, 0)$	$(0.6, 0.1)$	$(0.5, 0.2)$	$(0.3, 0.3)$	$(0.3, 0.3)$
Yellow	$(0.6, 0.2)$	$(0.5, 0.3)$	$(0.4, 0.4)$	$(0.2, 0.5)$	$(0.2, 0.5)$

Then (Υ, τ) is an $IFSI$ of (Γ, ς) but since

$\varpi_{\Upsilon[gold]}(Camation \divideontimes Sunflower) = \varpi_{\Upsilon[gold]}(Camation) = 0.4 < 0.6 =$

$\min\{\varpi_{\Upsilon[gold]}((Sunflower \divideontimes Rose) \divideontimes (Rose \divideontimes Camation)), \varpi_{\Upsilon[gold]}(Rose)\}$

$\xi_{\Upsilon[gold]}(Camation \divideontimes Sunflower) = \xi_{\Upsilon[gold]}(Camation) = 0.4 > 0.3 =$

$\max\{\xi_{\Upsilon[gold]}((Sunflower \divideontimes Rose) \divideontimes (Rose \divideontimes Camation)), \xi_{\Upsilon[gold]}(Rose)\}$.

i.e., $\Upsilon[gold] = \{(\varpi_{\Upsilon[gold]}(\imath), \xi_{\Upsilon[gold]}(\imath)) \mid \imath \in \Omega\}$ is not an $IF_\alpha I$ of Ω related to $\Gamma[gold]$. Therefore (Υ, τ) is not an $IFS_\alpha I$ of $S_{BCI}A$ (Γ, ς).

Theorem 5.8.5. *Let* (Γ, ς) *be a* $S_{BCI}A$ *over* Ω. *If* (Υ, τ) *and* (Π, ϱ) *are* $IFS_\alpha Is$ *of* (Γ, ς)*, then, the "extended intersection" of* (Υ, τ) *and* (Π, ϱ) *is an* $IFS_\alpha I$ *of* (Γ, ς).

Proof. We know that the "extended intersection" of (Υ, τ) and (Π, ϱ), denoted by $(\Upsilon, \tau) \sqcap_E (\Pi, \varrho)$, can be defined as, $(\Upsilon, \tau) \sqcap_E (\Pi, \varrho) = (\Xi, \zeta)$, where $\zeta = \tau \cup \varrho \subset \varsigma$ and for any $\wp \in \zeta$,

$$\Xi[\wp] = \begin{cases} \Upsilon[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)) \mid \imath \in \Omega\} & if\ \wp \in \tau - \varrho \\ \Pi[\wp] = \{(\varpi_{\Pi[\wp]}(\imath), \xi_{\Pi[\wp]}(\imath)) \mid \imath \in \Omega\} & if\ \wp \in \varrho - \tau \\ \Upsilon[\wp] \cap \Pi[\wp] = & \\ \{(\min\{\varpi_{\Upsilon[\wp]}(\imath), \varpi_{\Pi[\wp]}(\imath)\}, \max\{\xi_{\Upsilon[\wp]}(\imath), & \\ \xi_{\Pi[\wp]}(\imath)\}) \mid \imath \in \Omega\} & if\ \wp \in \tau \cap \varrho \end{cases}$$

For any $\wp \in \zeta$ if $\wp \in \tau - \varrho$, then

$\Xi[\wp] = \Upsilon[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)) \mid \imath \in \Omega\} \blacktriangle_\alpha \Gamma[\wp]$, since $(\Upsilon, \tau) \widetilde{\blacktriangle}_\alpha (\Gamma, \varsigma)$.

Similarly, if $\wp \in \varrho - \tau$, then

$\Xi[\wp] = \Pi[\wp] = \{(\varpi_{\Pi[\wp]}(\imath), \xi_{\Pi[\wp]}(\imath)) \mid \imath \in \Omega\} \blacktriangle_\alpha \Gamma[\wp]$, since $(\Pi, \varrho) \widetilde{\blacktriangle}_\alpha (\Gamma, \varsigma)$.

Moreover if $\wp \in \zeta$ such that $\wp \in \tau \cap \varrho$, then,

$\Xi[\wp] = \Upsilon[\wp] \cap \Pi[\wp] =$

$\{(\min\{\varpi_{\Upsilon[\wp]}(\imath), \varpi_{\Pi[\wp]}(\imath)\}, \max\{\xi_{\Upsilon[\wp]}(\imath), \xi_{\Pi[\wp]}(\imath)\}) \mid \imath \in \Omega\} \blacktriangle_\alpha \Gamma[\wp]$

(since the intersection of two $IF_\alpha Is$ is an $IF_\alpha I$).

Hence $\Xi[\wp] \blacktriangle_\alpha \Gamma[\wp]$ for any $\wp \in \zeta$. Hence $(\Xi, \zeta) = (\Upsilon, \tau) \sqcap_E (\Pi, \varrho) \widetilde{\blacktriangle}_\alpha (\Gamma, \varsigma)$.

□

It is easy to extract the following corollaries from the above result.

Corollary 5.8.6. *If* (Υ, τ) *and* (Π, τ) *are* $IFS_\alpha Is$ *of a* $S_{BCI}A$ (Γ, ς)*, then the "extended intersection" of* (Υ, τ) *and* (Π, τ) *is an* $IFS_\alpha I$ *of* (Γ, ς).

Corollary 5.8.7. *The "restricted intersection" of two* $IFS_\alpha Is$ (Υ, τ) *and* (Π, ϱ) *of a* $S_{BCI}A$ (Γ, ς) *is an* $IFS_\alpha I$ *of* (Γ, ς).

Theorem 5.8.8. *Let* (Υ, τ) *and* (Π, ϱ) *be two* $IFS_\alpha Is$ *of a* $S_{BCI}A$ (Γ, ς). *If* $\tau \cap \varrho = \phi$ *then the "union",* $(\Upsilon, \tau) \widetilde{\cup} (\Pi, \varrho)$ *is an* $IFS_\alpha I$ *of* (Γ, ς).

Proof. We know that the "union" of (Υ, τ) and (Π, ϱ), denoted by $(\Upsilon, \tau) \widetilde{\cup} (\Pi, \varrho)$, can be defined as, $(\Upsilon, \tau) \widetilde{\cup} (\Pi, \varrho) = (\Xi, \zeta)$, where $\zeta = \tau \cup \rho \subset \varsigma$ and for any $\wp \in \zeta$,

$$\Xi[\wp] = \begin{cases} \Upsilon[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)) \mid \imath \in \Omega\} & if\ \wp \in \tau - \varrho \\ \Pi[\wp] = \{(\varpi_{\Pi[\wp]}(\imath), \xi_{\Pi[\wp]}(\imath)) \mid \imath \in \Omega\} & if\ \wp \in \varrho - \tau \\ \Upsilon[\wp] \cup \Pi[\wp] = & \\ \{(\max\{\varpi_{\Upsilon[\wp]}(\imath), \varpi_{\Pi[\wp]}(\imath)\}, \min\{\xi_{\Upsilon[\wp]}(\imath), & \\ \xi_{\Pi[\wp]}(\imath)\}) \mid \imath \in \Omega\} & if\ \wp \in \tau \cap \varrho \end{cases}$$

Since $\tau \cup \rho = \phi$, either $\wp \in \tau - \varrho$ or $\wp \in \varrho - \tau$ for all $\wp \in \zeta$.

If $\wp \in \tau - \varrho$ then

$\Xi[\wp] = \Upsilon[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)) \mid \imath \in \Omega\}\blacktriangle\Gamma[\wp]$, since $(\Upsilon, \tau)\widetilde{\blacktriangle}_\alpha(\Gamma, \varsigma)$.

If $\wp \in \varrho - \tau$ then

$\Xi[\wp] = \Pi[\wp] = \{(\varpi_{\Pi[\wp]}(\imath), \xi_{\Pi[\wp]}(\imath)) \mid \imath \in \Omega\}\blacktriangle\Gamma[\wp]$, since $(\Pi, \varrho)\widetilde{\blacktriangle}_\alpha(\Gamma, \varsigma)$.

Hence $(\Xi, \zeta) = ((\Upsilon, \tau)\widetilde{\cup}(\Pi, \varrho))\widetilde{\blacktriangle}_\alpha(\Gamma, \varsigma)$. □

Below, we discuss the case when we have a non-empty intersection of the sets of parameters (i.e., $\tau \cap \varrho \neq \phi$).

Example 5.8.9. Cogitate the BCI-algebra $(\Omega, \otimes, Rose)$ defined in Example 5.5.12.

Suppose that $\varsigma = \{lavender, pink, purple, yellow\}$ is a set of parameters relevant to the universe set Ω.

Let (Γ, ς) be a soft set over Ω. Then $\Gamma[lavender] = \Omega$,

$\Gamma[pink] = \Gamma[purple] = \{Rose, Sunflower, Camation, Lily\}$ and

$\Gamma[yellow] = \{Rose, Sunflower\}$, that are all subalgebras of Ω. Hence (Γ, ς) is a "soft BCI-algebra" over Ω.

Let $\tau = \{lavender, pink, purple\} \subset \varsigma$ and

$\varrho = \{purple, yellow\} \subset \varsigma$ be two sets of characteristics of the flowers given in Ω.

Let (Υ, τ) be an $IFSS$ over Ω. Then $\Upsilon[lavender]$, $\Upsilon[pink]$ and $\Upsilon[purple]$ are $IFSs$ in Ω delineated as:

Υ	*Rose*	*Tulip*	*Sunflower*	*Camation*	*Lily*
lavender	(0.7, 0.1)	(0.7, 0.1)	(0.4, 0.2)	(0.2, 0.4)	(0.2, 0.4)
pink	(0.8, 0)	(0.8, 0)	(0.2, 0.5)	(0.3, 0.2)	(0.2, 0.5)
purple	(0.5, 0.1)	(0.5, 0.1)	(0.2, 0.3)	(0.2, 0.3)	(0.4, 0.2)

Then $\Upsilon[lavender] = \{(\varpi_{\Upsilon[lavender]}(\imath), \xi_{\Upsilon[lavender]}(\imath)) \mid \imath \in \Omega\}\blacktriangle_\alpha\Gamma[lavender]$,

$\Upsilon[pink] = \{(\varpi_{\Upsilon[pink]}(\imath), \xi_{\Upsilon[pink]}(\imath)) \mid \imath \in \Omega\}\blacktriangle_\alpha\Gamma[pink]$ and

$\Upsilon[purple] = \{(\varpi_{\Upsilon[purple]}(\imath), \xi_{\Upsilon[purple]}(\imath)) \mid \imath \in \Omega\}\blacktriangle_\alpha\Gamma[purple]$.

Hence $(\Upsilon, \tau)\widetilde{\blacktriangle}_\alpha(\Gamma, \varsigma)$.

Let (Π, ϱ) be an $IFSS$ over Ω. Then $\Pi[purple]$ and $\Pi[yellow]$ are $IFSs$ in Ω delineated as:

Π	$Rose$	$Tulip$	$Sunflower$	$Camation$	$Lily$
$purple$	$(0.8, 0.1)$	$(0.8, 0.1)$	$(0.5, 0.2)$	$(0.3, 0.4)$	$(0.3, 0.4)$
$yellow$	$(0.7, 0.2)$	$(0.7, 0.2)$	$(0.4, 0.5)$	$(0.4, 0.5)$	$(0.6, 0.3)$

Then $\Pi[purple] = \{(\varpi_{\Pi[purple]}(\imath), \xi_{\Pi[purple]}(\imath)) \mid \imath \in \Omega\}\blacktriangle_\alpha\Gamma[purple]$ and

$\Pi[yellow] = \{(\varpi_{\Pi[yellow]}(\imath), \xi_{\Pi[yellow]}(\imath)) \mid \imath \in \Omega\}\blacktriangle_\alpha\Gamma[yellow]$.

Thus $(\Pi, \varrho)\widetilde{\blacktriangle}_\alpha(\Gamma, \varsigma)$.

Now we consider the union of (Υ, τ) and (Π, ϱ), i.e.,

$(\Upsilon, \tau)\widetilde{\cup}(\Pi, \varrho) = (\Xi, \zeta)$, where $\zeta = \tau \cup \varrho$.

Note that for any parameter like, purple $\in \tau \cap \varrho$,

$\Xi[purple] = \Upsilon[purple] \cup \Pi[purple]$

$= \{(\varpi_{\Upsilon[purple]\cup\Pi[purple]}(\imath), \xi_{\Upsilon[purple]\cup\Pi[purple]}(\imath)) \mid \imath \in \Omega\}$

$= \{(\max\{\varpi_{\Upsilon[purple]}(\imath), \varpi_{\Pi[purple]}(\imath)\}, \min\{\xi_{\Upsilon[purple]}(\imath),$

$\xi_{\Pi[purple]}(\imath)\}) \mid \imath \in \Omega\}$.

Since, $\varpi_{\Xi[purple]}(Lily \otimes Sunflower) = \varpi_{\Xi[purple]}(Camation)$

$= \varpi_{(\Upsilon[purple]\cup\Pi[purple])}(Camation)$

$= \max\{\varpi_{\Upsilon[purple]}(Camation), \varpi_{\Pi[purple]}(Camation)\}$

$= \max\{0.2, 0.3\} = 0.3 < 0.4$

$= \min\{\varpi_{\Xi[purple]}((Sunflower \otimes Sunflower) \otimes (Rose \otimes Lily)),$

$\varpi_{\Xi[purple]}(Sunflower)\}$

$= \min\{\varpi_{\Xi[purple]}(Lily), \varpi_{\Xi[purple]}(Sunflower)\}$

$= \min\{\varpi_{(\Upsilon[purple]\cup\Pi[purple])}(Lily), \varpi_{(\Upsilon[purple]\cup\Pi[purple])}(Sunflower)\}$

$= \min\{\max\{\varpi_{\Upsilon[purple]}(Lily), \varpi_{\Pi[purple]}(Lily)\},$

$\max\{\varpi_{\Upsilon[purple]}(Sunflower), \varpi_{\Pi[purple]}(Sunflower)\}\}$

$= \min\{\max\{0.4, 0.3\}, \max\{0.2, 0.5\}\} = \min\{0.4, 0.5\}.$

and $\xi_{\Xi[purple]}(Lily \otimes Sunflower) = \xi_{\Xi[purple]}(Camation)$

$= \xi_{(\Upsilon[purple]\cup\Pi[purple])}(Camation)$

$= \min\{\xi_{\Upsilon[purple]}(Camation), \xi_{\Pi[purple]}(Camation)\}$

$= \min\{0.3, 0.4\} = 0.3 > 0.2$

$= \max\{\xi_{\Xi[purple]}((Sunflower \otimes Sunflower) \otimes (Rose \otimes Lily)),$

$\xi_{\Xi[purple]}(Sunflower)\}$

$= \max\{\xi_{\Xi[purple]}(Lily), \xi_{\Xi[purple]}(Sunflower)\}$

$= \max\{\xi_{(\Upsilon[purple]\cup\Pi[purple])}(Lily), \xi_{(\Upsilon[purple]\cup\Pi[purple])}(Sunflower)\}$

$= \max\{\min\{\xi_{\Upsilon[purple]}(Lily), \xi_{\Pi[purple]}(Lily)\},$

$\min\{\xi_{\Upsilon[purple]}(Sunflower), \xi_{\Pi[purple]}(Sunflower)\}\}$

$= \max\{\min\{0.2, 0.4\}, \min\{0.3, 0.2\}\} = \max\{0.2, 0.2\}.$

i.e., $\Xi[purple] = \Upsilon[purple] \cup \Pi[purple]$ is not an $IF_\alpha I$ of Ω related to $\Gamma[purple]$. Therefore, $(\Xi, \zeta) = (\Upsilon, \tau)\widetilde{\cup}(\Pi, \varrho)$ is not an $IFS_\alpha I$ of $S_{BCI}A$ (Γ, ς).

5.9 Characterization of Intuitionistic fuzzy soft α-ideals by soft (δ, η)-level sets

Now, we confer the characterization of an $IFS_\alpha I$ (Γ, ς) over Ω using the idea of a soft (δ, η)-level set, $L(\Gamma[\wp]; \delta; \eta) = \{\imath \in \Omega \mid \varpi_{\Gamma[\wp]}(\imath) \geq \delta$ and $\xi_{\Gamma[\wp]}(\imath) \leq \eta\}$, for any $\wp \in \varsigma$ and $\delta, \eta \in [0, 1]$.

Theorem 5.9.1. *An $IFSS$ (Γ, ς) over Ω is an $IFS_\alpha I$ over Ω $\iff$ soft*

(δ,η)-level set, $L(\Gamma[\wp];\delta;\eta)=\{\imath\in\Omega\mid\varpi_{\Gamma[\wp]}(\imath)\geq\delta$ and $\xi_{\Gamma[\wp]}(\imath)\leq\eta\}\neq\emptyset$, is an α-ideal of Ω, for any $\wp\in\varsigma$ and $\delta,\eta\in[0,1]$.

Proof. Let (Γ,ς) be an $IFS_\alpha I$ over Ω.

Then $\Gamma[\wp]=\{(\varpi_{\Gamma[\wp]}(\Im),\xi_{\Gamma[\wp]}(\Im))\mid\Im\in\Omega\}$ is an $IF_\alpha I$ of Ω, for any parameter $\wp\in\varsigma$.

Let $L(\Gamma[\wp];\delta;\eta)=\{\imath\in\Omega\mid\varpi_{\Gamma[\wp]}(\imath)\geq\delta$ and $\xi_{\Gamma[\wp]}(\imath)\leq\eta\}\neq\emptyset$, for any $\wp\in\varsigma$ and $\delta,\eta\in[0,1]$. Then for any $\imath\in L(\Gamma[\wp];\delta;\eta)$,

$\varpi_{\Gamma[\wp]}(0)\geq\varpi_{\Gamma[\wp]}(\imath)\geq\delta$ and $\xi_{\Gamma[\wp]}(0)\leq\xi_{\Gamma[\wp]}(\imath)\leq\eta$,

i.e., $0\in L(\Gamma[\wp];\delta;\eta)$.

Let $(\imath*\ell)*(0*\jmath)\in L(\Gamma[\wp];\delta;\eta)$ and $\ell\in L(\Gamma[\wp];\delta;\eta)$, for any $\imath,\jmath,\ell\in\Omega$.

Then $\varpi_{\Gamma[\wp]}((\imath*\ell)*(0*\jmath))\geq\delta,\ \varpi_{\Gamma[\wp]}(\ell)\geq\delta$

and $\xi_{\Gamma[\wp]}((\imath*\ell)*(0*\jmath))\leq\eta,\ \xi_{\Gamma[\wp]}(\ell)\leq\eta$.

Thus for any $\imath,\jmath,\ell\in\Omega$,

$\varpi_{\Gamma[\wp]}(\jmath*\imath)\geq\min\{\varpi_{\Gamma[\wp]}((\imath*\ell)*(0*\jmath)),\varpi_{\Gamma[\wp]}(\ell)\}\geq\delta$.

$\xi_{\Gamma[\wp]}(\jmath*\imath)\leq\max\{\xi_{\Gamma[\wp]}((\imath*\ell)*(0*\jmath)),\xi_{\Gamma[\wp]}(\ell)\}\leq\eta$.

i.e., $\jmath*\imath\in L(\Gamma[\wp];\delta;\eta)$. Hence $L(\Gamma[\wp];\delta;\eta)\neq\emptyset$ is an α-ideal of Ω for any $\wp\in\varsigma$ and $\delta,\eta\in[0,1]$.

Conversely assume that $L(\Gamma[\wp];\delta;\eta)$ is an α-ideal of Ω, for any $\wp\in\varsigma$ and $\delta,\eta\in[0,1]$.

If for some $\imath_\circ\in\Omega$ and

$\wp_\circ\in\varsigma$, $\varpi_{\Gamma[\wp_\circ]}(0)<\varpi_{\Gamma[\wp_\circ]}(\imath_\circ)$ and $\varpi_{\Gamma[\wp_\circ]}(0)>\varpi_{\Gamma[\wp_\circ]}(\imath_\circ)$, then

$\Gamma[\wp_\circ](0)<\delta_\circ\leq\Gamma[\wp_\circ](\imath_\circ)$ and $\Gamma[\wp_\circ](0)>\eta_\circ\geq\Gamma[\wp_\circ](\imath_\circ)$,

for some $\delta_\circ,\eta_\circ\in[0,1]$.

This implies that $\imath_\circ\in L(\Gamma[\wp_\circ];\delta_\circ;\eta_\circ)$ but $0\notin L(\Gamma[\wp_\circ];\delta_\circ;\eta_\circ)$, a contradiction.

Thus $\varpi_{\Gamma[\wp]}(0)\geq\varpi_{\Gamma[\wp]}(\imath)$ and $\xi_{\Gamma[\wp]}(0)\leq\xi_{\Gamma[\wp]}(\imath)$, for any $\wp\in\varsigma$ and $\imath\in\Omega$.

Moreover if there are elements $\imath_\circ, \jmath_\circ, \ell_\circ \in \Omega$ and $\wp_\circ \in \varsigma$ such that,

$\varpi_{\Gamma[\wp_\circ]}(\jmath_\circ * \imath_\circ) < \min\{\varpi_{\Gamma[\wp_\circ]}((\imath_\circ * \ell_\circ) * (0 * \jmath_\circ)), \varpi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$

and $\xi_{\Gamma[\wp_\circ]}(\jmath_\circ * \imath_\circ) > \max\{\xi_{\Gamma[\wp_\circ]}((\imath_\circ * \ell_\circ) * (0 * \jmath_\circ)), \xi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$.

Then for some $\delta_\circ, \eta_\circ \in [0, 1]$,

$\varpi_{\Gamma[\wp_\circ]}(\jmath_\circ * \imath_\circ) < \delta_\circ \leq \min\{\varpi_{\Gamma[\wp_\circ]}((\imath_\circ * \ell_\circ) * (0 * \jmath_\circ)), \varpi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$

and $\xi_{\Gamma[\wp_\circ]}(\jmath_\circ * \imath_\circ) > \eta_\circ \geq \max\{\xi_{\Gamma[\wp_\circ]}((\imath_\circ * \ell_\circ) * (0 * \jmath_\circ)), \xi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$.

i.e., $(\imath_\circ * \ell_\circ) * (0 * \jmath_\circ) \in L(\Gamma[\wp_\circ]; \delta_\circ, \eta_\circ)$ and $\ell_\circ \in L(\Gamma[\wp_\circ]; \delta_\circ, \eta_\circ)$ but $\jmath_\circ * \imath_\circ \notin L(\Gamma[\wp_\circ]; \delta_\circ, \eta_\circ)$, again contradicts the hypothesis that $L(\Gamma[\wp_\circ]; \delta_\circ, \eta_\circ) \neq \emptyset$ is an α-ideal of Ω. Thus for any $\imath, \jmath, \ell \in \Omega$ and any $\wp \in \varsigma$,

$\varpi_{\Gamma[\wp]}(\jmath * \imath) \geq \min\{\varpi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Gamma[\wp]}(\ell)\}$

and $\xi_{\Gamma[\wp]}(\jmath * \imath) \leq \max\{\xi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \xi_{\Gamma[\wp]}(\ell)\}$

i.e., $\Gamma[\wp] = \{(\varpi_{\Gamma[\wp]}(\Im), \xi_{\Gamma[\wp]}(\Im)) \mid \Im \in \Omega\}$ is an $IF_\alpha I$ of Ω for any $\wp \in \varsigma$. Hence (Γ, ς) is an $IFS_\alpha I$ over Ω. □

From the above statement the following corollary is evident.

Corollary 5.9.2. *An $IFSS$ (Γ, ς) over Ω is an $IFS_\alpha I$ over Ω $\iff$ the soft (δ, η)-level set $L(\Gamma[\wp]; \delta; \eta) = \{\imath \in \Omega \mid \varpi_{\Gamma[\wp]}(\imath) \geq \delta$ and $\xi_{\Gamma[\wp]}(\imath) \leq \eta\} \neq \emptyset$, is an α-ideal of Ω, for any $\wp \in \varsigma$ and $\delta, \eta \in (0.5, 1]$.*

Theorem 5.9.3. *A soft (δ, η)-level set $L(\Gamma[\wp]; \delta; \eta) = \{\imath \in \Omega \mid \varpi_{\Gamma[\wp]}(\imath) \geq \delta$ and $\xi_{\Gamma[\wp]}(\imath) \leq \eta\} \neq \emptyset$, is an α-ideal of Ω, for any $\wp \in \varsigma$ and $\delta, \eta \in (0.5, 1]$ $\iff$ the following conditions are valid:*

(i) $\max\{\varpi_{\Gamma[\wp]}(0), 0.5\} \geq \varpi_{\Gamma[\wp]}(\imath)$ *and* $\max\{\xi_{\Gamma[\wp]}(0), 0.5\} \leq \xi_{\Gamma[\wp]}(\imath)$.

(ii) $\max\{\varpi_{\Gamma[\wp]}(\jmath * \imath), 0.5\} \geq \min\{\varpi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Gamma[\wp]}(\ell)\}$

and $\max\{\xi_{\Gamma[\wp]}(\jmath * \imath), 0.5\} \leq \max\{\xi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \xi_{\Gamma[\wp]}(\ell)\}$

for any $\wp \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$.

Proof. Let the soft (δ,η)-level set $L(\Gamma[\wp];\delta;\eta)=\{\imath\in\Omega\mid\varpi_{\Gamma[\wp]}(\imath)\geq\delta$ and $\xi_{\Gamma[\wp]}(\imath)\leq\eta\}\neq\emptyset$, is an α-ideal of Ω, for any $\wp\in\varsigma$ and $\delta,\eta\in(0.5,1]$.

If for some $\imath_\circ\in\Omega$ and $\wp_\circ\in\varsigma$.

$\max\{\varpi_{\Gamma[\wp_\circ]}(0),0.5\}<\varpi_{\Gamma[\wp_\circ]}(\imath_\circ)$ and $\max\{\xi_{\Gamma[\wp_\circ]}(0),0.5\}>\xi_{\Gamma[\wp_\circ]}(\imath_\circ)$.

Then there are $\delta_\circ,\eta_\circ\in(0.5,1]$ such that,

$\max\{\varpi_{\Gamma[\wp_\circ]}(0),0.5\}<\delta_\circ\leq\varpi_{\Gamma[\wp_\circ]}(\imath_\circ)$

and $\max\{\xi_{\Gamma[\wp_\circ]}(0),0.5\}>\eta_\circ\geq\xi_{\Gamma[\wp_\circ]}(\imath_\circ)$.

This implies, $\varpi_{\Gamma[\wp_\circ]}(0)<\delta_\circ\leq\varpi_{\Gamma[\wp_\circ]}(\imath_\circ)$ and $\xi_{\Gamma[\wp_\circ]}(0)>\eta_\circ\geq\xi_{\Gamma[\wp_\circ]}(\imath_\circ)$,

i.e., $\imath_\circ\in L(\Gamma[\wp_\circ];\delta_\circ;\eta_\circ)$ but $0\notin L(\Gamma[\wp_\circ];\delta_\circ;\eta_\circ)$, a contradiction. Thus (i) is valid.

Moreover if for some $\imath_\circ,\jmath_\circ,\ell_\circ\in\Omega$ and $\wp_\circ\in\varsigma$,

$\max\{\varpi_{\Gamma[\wp_\circ]}(\jmath_\circ*\imath_\circ),0.5\}<\min\{\varpi_{\Gamma[\wp_\circ]}((\imath_\circ*\ell_\circ)*(0*\jmath_\circ)),\varpi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$

and $\max\{\xi_{\Gamma[\wp_\circ]}(\jmath_\circ*\imath_\circ),0.5\}>\max\{\xi_{\Gamma[\wp_\circ]}((\imath_\circ*\ell_\circ)*(0*\jmath_\circ)),\xi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$

Then for some $\delta_\circ,\eta_\circ\in(0.5,1]$,

$\max\{\varpi_{\Gamma[\wp_\circ]}(\jmath_\circ*\imath_\circ),0.5\}<\delta_\circ\leq\min\{\varpi_{\Gamma[\wp_\circ]}((\imath_\circ*\ell_\circ)*(0*\jmath_\circ)),$

$\varpi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$

and

$\max\{\xi_{\Gamma[\wp_\circ]}(\jmath_\circ*\imath_\circ),0.5\}>\eta_\circ\geq\max\{\xi_{\Gamma[\wp_\circ]}((\imath_\circ*\ell_\circ)*(0*\jmath_\circ)),\xi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$.

i.e., $\varpi_{\Gamma[\wp_\circ]}(\jmath_\circ*\imath_\circ)<\delta_\circ\leq\min\{\varpi_{\Gamma[\wp_\circ]}((\imath_\circ*\ell_\circ)*(0*\jmath_\circ)),\varpi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$

and $\xi_{\Gamma[\wp_\circ]}(\jmath_\circ*\imath_\circ)>\eta_\circ\geq\max\{\xi_{\Gamma[\wp_\circ]}((\imath_\circ*\ell_\circ)*(0*\jmath_\circ)),\xi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$.

i.e., $(\imath_\circ*\ell_\circ)*(0*\jmath_\circ)\in L(\Gamma[\wp_\circ];\delta_\circ;\eta_\circ)$ and $\ell_\circ\in L(\Gamma[\wp_\circ];\delta_\circ;\eta_\circ)$ but $\jmath_\circ*\imath_\circ\notin L(\Gamma[\wp_\circ];\delta_\circ;\eta_\circ)$, which contradicts the hypothesis that $L(\Gamma[\wp_\circ];\delta_\circ;\eta_\circ)\neq\emptyset$ is an α-ideal of Ω, for any $\wp_\circ\in\varsigma$ and $\delta_\circ,\eta_\circ\in(0.5,1]$. Hence (ii) is also valid.

Conversely, suppose that (i) and (ii) are valid. Let $L(\Gamma[\wp];\delta;\eta)\neq\emptyset$, for any $\wp\in\varsigma$ and $\delta,\eta\in(0.5,1]$. Then for any $\imath\in L(\Gamma[\wp];\delta;\eta)$,

$\max\{\varpi_{\Gamma[\wp]}(0),0.5\}\geq\varpi_{\Gamma[\wp]}(\imath)\geq\delta>0.5$

and $\max\{\xi_{\Gamma[\wp]}(0), 0.5\} \leq \xi_{\Gamma[\wp]}(\imath) \leq \eta$

which implies $\varpi_{\Gamma[\wp]}(0) \geq \delta$ and $\xi_{\Gamma[\wp]}(0) \leq \eta$ and thus $0 \in L(\Gamma[\wp]; \delta; \eta)$.

Let $(\imath * \ell) * (0 * \jmath) \in L(\Gamma[\wp]; \delta; \eta))$ and $\ell \in L(\Gamma[\wp]; \delta; \eta)$, for any $\imath, \jmath, \ell \in \Omega$.

Then $\varpi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)) \geq \delta,\ \varpi_{\Gamma[\wp]}(\ell) \geq \delta$

and $\xi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)) \leq \eta,\ \xi_{\Gamma[\wp]}(\ell) \leq \eta$.

Thus from (ii) we get,

$\max\{\varpi_{\Gamma[\wp]}(\jmath * \imath), 0.5\} \geq \min\{\varpi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Gamma[\wp]}(\ell)\} \geq \delta > 0.5$

and $\max\{\xi_{\Gamma[\wp]}(\jmath * \imath), 0.5\} \leq \max\{\xi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \xi_{\Gamma[\wp]}(\ell)\} \leq \eta$

This implies, $\varpi_{\Gamma[\wp]}(\jmath * \imath) \geq \delta$ and $\xi_{\Gamma[\wp]}(\jmath * \imath) \leq \eta$.

Thus $\jmath * \imath \in L(\Gamma[\wp]; \delta; \eta)$. Therefore $L(\Gamma[\wp]; \delta; \eta) \neq \emptyset$ is an α-ideal of Ω, for any $\wp \in \varsigma$ and $\delta, \eta \in (0.5, 1]$. $\square$

Chapter 6

Summary and Discussion

Now, we summarize our results presented in this thesis and conclude our discussion. Our research is devoted to the study of applications of hyperstructures and soft sets in different ideals of BCK/BCI-algebras. We started by considering the basics of BCK/BCI-algebras and different ideals in these algebras. The intuitionistic fuzzy sets have been applied to these ideals. Characterizations of these ideals using level subsets and Transfer principle and with respect to various aspects were described. Connections between these ideals with the help of different examples have been considered. Then hyperstructures have been applied to these ideals in BCK-algebras and apposite properties were coffered. Different types of fuzzy hyper ideals were presented and their relations were discussed. These fuzzy hyper ideals were characterized using the idea of strongest fuzzy relations, level sets and hyper homomorphism and the products of these fuzzy hyper ideals were discussed.

Moreover, the applications of soft sets in the ideal theory of BCI-algebras have been discussed. The concept of fuzzy soft α-ideals of BCI-algebras has been presented and their basic properties were proved. Fuzzy soft α-ideals have been related with fuzzy soft ideals and different types of fuzzy α-ideals were characterized by $\in$-soft sets. Lastly, concept of intuitionistic fuzzy soft

α-ideals was considered and relevant properties were explored. Intuitionistic fuzzy soft α-ideals have been related with intuitionistic fuzzy soft ideals. All of our above discussion was supported by concrete examples.

6.1 Background

The theory of Hyperstructures presented by Marty [53] is used in a variety of mathematical disciplines. As some branches of Mathematics are expressed and developed in terms of other branches, at the very beginning, Marty applied the theory of Hyperstructures to groups, rational fractions and algebraic functions. Afterwards many researchers focussed their study on new applications of Hyperstructure theory in groups. Rosenberg, in 1966, [64] for the first time considered, in the most general meanings relations between Binary Relations and Hyperstructures. Moreover, hyperstructures have applications to probability, cryptography, combinatorics, hypergraphs, geometry etc.

Now hyper structures are studied by many authors in various directions. For example, Borzooei et al. [8] studied hyper BCC-algebras which are a generalization of BCC-algebras. Long [39] presented the idea of a hyper BCI-algebras as a generalization of BCI-algebra. In [34], Jun et al. applied the hyper structures to BCK-algebras and defined the idea of a hyper BCK-algebra and discussed their characteristics. Continuing these studies, we have applied the hyperstructures to different type of ideals in BCK-algebras. We have been able to explore a number of new results by characterizing these hyper BCK-ideals with respect to various aspects and by describing their connections. We extended the study of these hyper BCK-ideals by involving the concepts of weak, strong and reflexive hyper BCK-ideals and also by considering strongest fuzzy relation, hyper homomorphism and the product of these ideals.

Now a days, soft set theory is considered to be the one of the most reliable method for dealing with uncertainties. Research involving soft sets and its application in various fields of science and technology are currently going on in a rapid pace. In 2008, Jun [26] applied soft set theory to BCK/BCI-algebras. In [35, 36] soft subalgebras and soft ideals of BCK/BCI-algebras are characterized by using the idea of fuzzy sets. Continuing these studies, we have presented the idea of fuzzy soft α-ideals of BCI-algebras and proved their basic properties. We have also described connections between various types of fuzzy soft α-ideals and fuzzy soft ideals, and characterized some types of fuzzy α-ideals by $\in$-soft sets.

Maji [40] in 2009, considered some new operations on intuitionistic fuzzy soft sets and also discussed an application of these newly defined operations. Jiang et al. [25], used intuitionistic fuzzy soft set theoretic approach to solve different decision making problems. By considering the idea of intuitionistic fuzzy sets by Atanassov [4], we have extended the study of applications of soft sets in α-ideals of BCI-algebras and presented the idea of intuitionistic fuzzy soft α-ideals and proved their basic properties. Useful facts on various operations given in [40] and [2] have been explored on intuitionistic fuzzy soft α-ideals and intuitionistic fuzzy α-ideals have been characterized by soft (δ, η)-level sets.

6.2 Main Results

Intuitionistic fuzzy sets have a variety of applications in the ideal theory of BCK/BCI-algebras. We have discussed the intuitionistic fuzzification of about six different types of ideals in BCK/BCI-algebras, namely "intuitionistic fuzzy p-ideals" (IF_pIs), "intuitionistic fuzzy h-ideals" (IF_hIs), "intuitionistic fuzzy α-ideals" ($IF_\alpha Is$), "intuitionistic fuzzy BCI-implicative ideals" ($IF_{BCI}IIs$), "intuitionistic fuzzy BCI-positive implicative ideals" (IF_pPIIs) and "intuitionistic fuzzy BCI-commutative ideals" IF_pCIs. We have considered the notions and characterization of each of these ideals and discussed their properties. An effective development which is achieved here is that we have tried to relate all of these six intuitionistic fuzzy ideals ($IFIs$). First we have discussed the relations among IF_pIs, IF_hIs and $IF_\alpha Is$ and observed that an IFS is an $IF_\alpha I$ if and only if it is both an IF_pI and an IF_hI.

A detail study of $IF_{BCI}IIs$ has been conducted. $IF_{BCI}IIs$ have been related with $IFIs$ and some types of characterizations are defined in Theorem 2.9.4. $IF_{BCI}IIs$ are also characterized using the Transfer principle for fuzzy sets and using this characterization we have proved an interesting result in Theorem 2.9.11. Relations among $IF_{BCI}IIs$, IF_pPIIs and IF_pCIs have been explored and it is observed that an IFS is an $IF_{BCI}II$ if and only if it is both an $IF_{BCI}PII$ and an $IF_{BCI}CI$. Also $IF_{BCI}IIs$, IF_pPIIs and IF_pCIs are related with IF_pIs and $IF_\alpha Is$. It is demonstrated that any IF_pI is an $IF_{BCI}II$ and thus also an $IF_{BCI}PII$ and an $IF_{BCI}CI$. Since in Theorem 2.8.1, it is demonstrated that any IF_α is an IF_pI, thus every IF_α is also an $IF_{BCI}II$ and therefore is an $IF_{BCI}PII$ and an $IF_{BCI}CI$.

By extending the approach of Jun et al. [34] on hyperstructures, we have applied the hyperstructures to five types of ideals in BCK-algebras. The concepts of "weak hyper h-ideals" (wH_hIs), "hyper h-ideals" (H_hIs), "strong hyper h-ideals" (sH_hIs) and "reflexive hyper h-ideals" (rH_hIs) are defined and their connections are discussed. Similarly, the ideas of "fuzzy weak hyper h-ideals" (FwH_hIs), "fuzzy hyper h-ideals" (FH_hIs), "fuzzy strong hyper h-ideals" (FsH_hIs) and "fuzzy reflexive hyper h-ideals" (FrH_hIs) are defined and their connections are discussed. We have also considered their products and proved that a fuzzy set $\varpi = \varpi_1 \times \varpi_2$ is a FH_hI (resp. FwH_hI, FsH_hI, FrH_hI) of a hyper BCK-algebra $\digamma = \digamma_1 \times \digamma_2$ if and only if ϖ_1 and ϖ_2 are FH_hIs (resp. FwH_hIs, FsH_hIs, FrH_hIs) of hyper BCK-algebras $\digamma_1$ and $\digamma_2$ respectively. Similarly, we have applied hyperstructures to the other four ideals of BCI-algebras.

For a "fuzzy strong hyper p-ideal" (FsH_pI), using the idea of a strongest fuzzy relation (χ_ϖ) on a hyper BCK-algebra $\digamma$, it has been proved that a fuzzy set ϖ is a FsH_pI of $\digamma$ if and only if χ_ϖ is a FsH_pI of $\digamma \times \digamma$. We have discussed the properties of "fuzzy hyper BCK-positive implicative ideals" ($FH_{BCK}PIIS$) in detail. A fuzzy set ϖ is a $FH_{BCK}PII$ if and only if for any $\delta \in [0, 1]$, $\varpi_\delta \neq \emptyset$ is a hyper BCK-positive implicative ideal ($H_{BCK}PII$). The family of $FH_{BCK}PIIs$ is a completely distributive lattice with respect to join and meet. For an onto hyper homomorphism $\Gamma : \digamma \to \daleth$ from a hyper BCK-algebra $\digamma$ to a hyper BCK-algebra $\daleth$, if ξ is a $FH_{BCK}PII$ of $\daleth$ then the hyper homomorphic pre-image ϖ of ξ under Γ is a $FH_{BCK}PII$ of $\digamma$. Moreover, we have considered the connections between $FH_{BCK}PIIs$, "fuzzy hyper BCK-implicative ideals" ($FH_{BCK}IIs$) and "fuzzy hyper BCK-commutative ideals" ($FH_{BCK}CIs$) and proved that a fuzzy set is a $FH_{BCK}II$ if and only if it is both a $FH_{BCK}PII$ and a $FH_{BCK}CI$.

Soft set theory has proved to be an effective tool in overcoming the difficulties which one has to face while using classical methods. Jun [26], in 2008, applied soft sets to BCK/BCI-algebras. He characterized soft subalgebras and soft ideals in BCK/BCI-algebras. Continuing these studies, we have applied soft sets to five types of ideals in BCI-algebras and discussed apposite properties. Fuzzy soft α-ideals are introduced and their basic properties are conferred. Fuzzy soft α-ideals are related with fuzzy soft ideals. Different operations have been described on fuzzy soft α-ideals and it is observed that extended intersection of two fuzzy soft α-ideals is a fuzzy soft α-ideal. For the union of two fuzzy soft α-ideals to be a fuzzy soft α-ideal, the intersection of the sets of parameters should be empty. In the case when we have a non-empty intersection of the sets of parameters, then it can be observed from Example 4.12.10 that the union of two fuzzy soft α-ideal is not a fuzzy soft α-ideal. Moreover, fuzzy soft α-ideal of a soft BCI-algebra is introduced and its characteristics are explored. A fuzzy soft set (F, ς) over a BCI-algebra Ω is a fuzzy soft α-ideal over Ω if and only if each its nonempty level subset $L(F[u]; \delta)$ is an α-ideal of Ω.

Fuzzy points approach to α-ideals is considered and some types of fuzzy α-ideals are characterized by $\in$-soft sets. An $\in$-soft set $(F[\varpi], \varsigma)$, where $\varsigma = (0, 1]$, is an α-idealistic soft BCI-algebra over a BCI-algebra Ω if and only if ϖ is a fuzzy α-ideal of Ω. Similarly, a q-soft set $(F[\varpi]_q, \varsigma)$, where $\varsigma = (0, 1]$, is an α-idealistic soft BCI-algebra over a BCI-algebra Ω if and only if ϖ is a fuzzy α-ideal of Ω. An $\in$-soft set $(F[\varpi], \varsigma)$, where $\varsigma = (0, 0.5]$, is an α-idealistic soft BCI-algebra over a BCI-algebra Ω if and only if ϖ

is a $(\in, \in\vee q)$-fuzzy α-ideal of Ω. The concept of $(\tilde{\in}, \tilde{\in}\vee\tilde{q})$-*fuzzy* α-*ideal* is defined and it is proved that an $\in$-soft set $(F[\varpi], \varsigma)$, where $\varsigma = (0.5, 1]$, is an α-idealistic soft BCI-algebra over a BCI-algebra Ω if and only if ϖ is an $(\tilde{\in}, \tilde{\in}\vee\tilde{q})$-fuzzy α-ideal of Ω. Also (δ_1, δ_2)-*fuzzy* α-*ideal* is defined and it is demonstrated that a fuzzy set ϖ of a BCI-algebra Ω is its (δ_1, δ_2)-fuzzy α-ideal, where $0 \leq \delta_1 < \delta_2 \leq 1$, if and only if an $\in$-soft set $(F[\varpi], \varsigma)$, where $\varsigma = (\delta_1, \delta_2]$, is an α-idealistic soft BCI-algebra over Ω.

By extending the study of applications of soft sets in α-ideals of BCI-algebras, we have defined the idea of "intuitionistic fuzzy soft α-ideals" ($IFS_\alpha Is$) and proved their basic properties. In chapter 5, initially the properties of "intuitionistic fuzzy soft BCI-algebras" ($IFS_{BCI}As$) and "intuitionitic fuzzy soft ideals" ($IFSIs$) have been described with the help of concrete examples. We proved that any $IFSI$ of a BCK-algebra is an "intuitionistic fuzzy soft BCK-algebra" ($IFS_{BCK}A$). Afterwards we have proceeded towards the detail discussion of $IFS_\alpha Is$. $IFS_\alpha Is$ are related with $IFSIs$ and various characterizations are discussed. Useful facts have been explored on various operations on intuitionistic fuzzy soft α-ideals. For instance, it has been proved that the "AND" operation, extended intersection and restricted intersection of two $IFS_\alpha Is$ is an $IFS_\alpha I$. The union of two $IFS_\alpha Is$ is an $IFS_\alpha I$ if the intersection of the sets of parameters is empty.

$IFS_\alpha I$ of a "soft BCI-algebra" ($S_{BCI}A$ has been defined and apposite properties have been explored. We have proved that any $IFS_\alpha I$ (Υ, τ) of a $S_{BCI}A$ (Γ, ς) is an $IFSI$ of (Γ, ς). If (Γ, ς) is a $S_{BCI}A$ over Ω and (Υ, τ) and (Π, ϱ) are $IFS_\alpha Is$ of (Γ, ς), then, the extended intersection of (Υ, τ) and (Π, ϱ) is an $IFS_\alpha I$ of (Γ, ς). Also, if (Υ, τ) and (Π, ϱ) be two

$IFS_\alpha Is$ of a $S_{BCI}A$ (Γ, ς) and $\tau \cap \varrho = \phi$, then the union, $(\Upsilon, \tau)\widetilde{\cup}(\Pi, \varrho)$ is an $IFS_\alpha I$ of (Γ, ς). Lastly we have characterized $IFS_\alpha Is$ by a soft (δ, η)-level set. It has been proved that an intuitionistic fuzzy soft set (Γ, ς) over a BCI-algebra Ω is an $IFS_\alpha I$ over Ω if and only if the soft (δ, η)-level set, $L(\Gamma[\wp]; \delta; \eta) = \{\imath \in \Omega \mid \varpi_{\Gamma[\wp]}(\imath) \geq \delta$ and $\xi_{\Gamma[\wp]}(\imath) \leq \eta\} \neq \emptyset$, is an α-ideal of Ω, for any $\wp \in \varsigma$ and $\delta, \eta \in [0, 1]$. Moreover, a soft (δ, η)-level set $L(\Gamma[\wp]; \delta; \eta) = \{\imath \in \Omega \mid \varpi_{\Gamma[\wp]}(\imath) \geq \delta$ and $\xi_{\Gamma[\wp]}(\imath) \leq \eta\} \neq \emptyset$, is an α-ideal of Ω, for any $\wp \in \varsigma$ and $\delta, \eta \in (0.5, 1]$ if and only if the following conditions are valid:

(i) $\max\{\varpi_{\Gamma[\wp]}(0), 0.5\} \geq \varpi_{\Gamma[\wp]}(\imath)$ and $\max\{\xi_{\Gamma[\wp]}(0), 0.5\} \leq \xi_{\Gamma[\wp]}(\imath)$.

(ii) $\max\{\varpi_{\Gamma[\wp]}(\jmath * \imath), 0.5\} \geq \min\{\varpi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Gamma[\wp]}(\ell)\}$
and $\max\{\xi_{\Gamma[\wp]}(\jmath * \imath), 0.5\} \leq \max\{\xi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \xi_{\Gamma[\wp]}(\ell)\}$
for any $\wp \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$.

6.3 Recommendations for further study

From our discussion, it appears that the study of hyperstructures in the ideal theory of BCK-algebras can be extended. We should focus on the applications of soft sets in hyper BCK-ideals introduced in chapter 3. The introduction of concepts like soft hyper BCK-ideals, soft hyper p-ideals, soft hyper h-ideals etc. will lay a foundation towards the connection of soft sets and hyperstructures in terms of BCK/BCI-algebras. This study may further pave the way for applying fuzzy sets and intuitionistic fuzzy sets to soft hyper BCK-ideals, soft hyper p-ideals, soft hyper h-ideals etc.

Also some types of intuitionistic fuzzy α-ideals may also be characterized by $\in$-soft sets. Moreover, fuzzy sets and intuitionistic fuzzy sets may be applied to other soft ideals introduced in chapter 4. After this application, the connections between different fuzzy soft ideals and intuitionistic fuzzy soft ideals may be considered.

Bibliography

[1] H. Aktaş, N. çağman: Soft sets and soft groups, Inform. Sci. 177 (2007) 2726-2735.

[2] M.A. Ali, F. Feng, X. Liu, W.K. Min, M. Shabir: On some new operations in soft set theory, Comp. Math. Appl. 57 (2009) 1547-1553.

[3] R. Ameri, M.M. Zahedi: Hypergroup and Join Space induced by a Fuzzy Subset, PU. M. A. 8 (1997).

[4] K.T. Atanassov: Intuitionistic fuzzy sets, Fuzzy Set Syst. 20 (1986) 87-96.

[5] K.V. Babitha, J.J. Sunil: Soft sets relations and functions, Comp. Math. Appl. 60 (2010) 1840-1849.

[6] L. Berardi, F. Eugeni, S. Innamorati: Remarks on Hypergroupoids and Criptography, Journal of Combinatorics, Information and System Sciences 17(3-4) (1992) 217-231.

[7] G. Birkhoff, S.A. Kiss: A ternary operation in distributive lattices, Bull. Amer. Math. Soc. 53 (1974) 749-752.

[8] R.A. Borzooei, W.A. Dudek, N. Koohestani: On hyper BCC-algebras, International J. Math. and Math. Sciences (2006), Article ID 49703.

[9] R.A. Borzooei, A. Hasankhani, M.M. Zahedi, Y.B. Jun: On hyper K-algebras, Math. Japonica 52 (1) (2000), 113-121.

[10] J. Chvalina: Commutative hypergroups in the sense of Marty and ordered sets, Summer School on General Algebra and Ordered Sets, Proc. Internat. Conf. Olomouc (Czech Rep.) (1994) 19-30.

[11] P. Corsini: Hypergraphs and hypergroups, Algebra Universalis 36 (1996) 548-555.

[12] P. Corsini: Properties of hyperoperations associated with fuzzy sets and with factor spaced, Int. Journal of Science and Technology, Kashan University 1 (2000).

[13] P. Corsini, V. Leoreanu: Applications of hyperstructure theory, Advances in Mathematics, Vol. 5, Kluwer Academic Publishers, 2003.

[14] P. Corsini, I. Tofan: On fuzzy hypergroups, PU. M. A. 8 (1) (1997).

[15] K. Drbohlav: Gruppenartige Multigruppen, Czech. Math. J. 7 (82) (1957).

[16] W.A. Dudek, M. Touqeer, M.A. Malik: Fuzzy soft set theoretic approaches to α-ideals in BCI-algebras (submitted).

[17] J. E. Eaton: Theory of cogroups, Duke Math. J. 6 (1) (1940).

[18] M. Golmohammadian, M.M. Zahedi: BCK-algebras and hyper BCK-algebras induced by a deterministic finite automaton, Iranian J. Math. Sci. and Informatics 4 (2009), 79-98.

[19] K. Gong, Z. Xiao, X.Zhang: The bijective soft set with its operations, Comput. Math. Appl. 60 (2010) 2270-2278.

[20] L. Haddad, Y. Sureau: Les cogroups et les D-hypergroupes, Algebra 118 (2) (1988).

[21] D.K. Harrison: Double coset and orbit spaces, Pacific J. Math. 80 (1979).

[22] T. Herawan, R. Ghazali, M.M. Deris: Soft set theoretic approach for dimensionality reduction, Intern. J. Database Theory Appl. 3 (2) (2010) 47-60.

[23] Y.S. Huang: BCI-algebra, Science Press, Beijing 2006.

[24] Y. Imai, K. Iséki: On axioms of propositional calculi, XIV Proc., Jpn. Acad. 42 (1966) 19-22.

[25] Y. Jiang, Y. Tang, Q. Chen: An adjustable approach to intuitionistic fuzzy soft sets based decision making, Appl. Math. Model. 35 (2011) 824836.

[26] Y.B. Jun: Soft BCK/BCI-algebras, Comput. Math. Appl. 56 (2008) 1408-1413.

[27] Y.B. Jun: Union-soft sets with applications in BCK/BCI-algebras, Bull. Korean Math. Soc 50 (6) (2013) 1937-1956.

[28] Y.B. Jun, K.H. Kim, Intuitionistic fuzzy ideals of BCK-algebras, Internat. J. Math. Math. Sci. 24 (12) (2000) 839-849.

[29] Y.B. Jun, K.J. Lee, C.H. Park: Fuzzy soft set theory applied to BCK/BCI-algebras, Comput. Math. Appl. 59 (2010) 3180-3192.

[30] Y.B. Jun, C.H. Park: Applications of soft sets in ideal theory of BCK/BCI-algerbas, Inform. Sci. 178 (2008) 2466-2475.

[31] Y.B. Jun, X.L. Xin: Fuzzy hyper BCK-ideals of hyper BCK-algebras, Sci. Math. Jpn. 53 (2) (2001), 415-422.

[32] Y.B. Jun, X.L. Xin: Scalar elements and hyperatoms of hyper BCK-algebras, Sci. Math. Jpn. 2 (3)(1999), 303-309.

[33] Y.B. Jun, X.L. Xin, M.M. Zahedi, E.H. Roh: Strong hyper BCK-ideals of hyper BCK-algebras, Math. Jpn. 51 (3) (2000), 493-498.

[34] Y.B. Jun, M.M. Zahedi, X.L. Xin, R. A. Borzooei: On hyper BCK-algebras, Italian J. Pure Appl. Math. 8 (2000), 127-136.

[35] Y.B. Jun, S.K. Song: Soft subalgebras and soft ideals of BCK/BCI-algebras related to fuzzy set theory, Math. Commun. 14 (2009) 271-282.

[36] Y.B. Jun, J. Zhang: Soft ideals of BCK/BCI-algebras based on fuzzy set theory, J. Comput. Math. 88 (12) (2011) 2502-2515.

[37] A. Kharal, B. Ahmad: Mappings on soft classes, Inf. Sci., INS-D-08-1231 by ESS (2010) 1-11.

[38] M. Kondo, W.A. Dudek: On the transfer principle in fuzzy theory, Mathware Soft Comput. 12 (2005) 41-55.

[39] X.X. Long: Hyper BCI-algebras, Discuss. Math. 26 (2006), 5-19.

[40] P.K. Maji: More on intuitionistic fuzzy soft sets, in: Proceedings of the 12th International Conference on Rough Sets, Fuzzy Sets, Data Mining and Granular Computing (RSFDGrC 2009), in: H. Sakai, M.K. Chakraborty, A.E. Hassanien, D. Slezak, W. Zhu (Eds.), Lecture Notes in Computer Science, 5908, Springer, 2009, pp. 231240.

[41] P.K. Maji, A.R. Biswas, A.R. Roy: Soft set theory, Comput. Math. Appl. 45 (2003) 555-562.

[42] M.A. Malik, M. Touqeer: Some Results on fuzzy hyper BCK-ideals of hyper BCK-algebras, Indian Journal of Science and Technology 8(S3) (2015) 10-15.

[43] M.A. Malik, M. Touqeer: Soft h-ideals of soft BCI-algebras, Indian Journal of Science and Technology 8(S3) (2015) 16-23.

[44] M.A. Malik, M. Touqeer: Soft BCI-positive implicative ideals of soft BCI-algebras, Sci. Math. Jpn. e-2015, Whole Number 28 (2015) 57-78.

[45] M.A. Malik, M. Touqeer: Fuzzy hyper BCK-implicative ideals of hyper BCK-algebras, Sci. Math. Jpn. e-2015, Whole Number 28 (2015) (online).

[46] M.A. Malik, M. Touqeer: Intuitionistic fuzzy BCI-commutative ideals in BCI-algebras, Pakistan Journal of Science 64 (4) (2012) 353-358.

[47] M.A. Malik, M. Touqeer: Intuitionistic fuzzy BCI-positive implicative ideals in BCI-algebras, Int. Math. Forum 6 (47) (2011) 2317 - 2334.

[48] M.A. Malik, M. Touqeer: Some results on intuitionistic fuzzy BCI-(positive implicative, implicative, commutative) ideals in BCI-algebras, Int. Math. Forum 6 (47) (2011) 2335 - 2347.

[49] M.A. Malik, M. Touqeer: Fuzzy hyper p-ideals of hyper BCK-algebras, Filomat (in press, 2015).

[50] M.A. Malik, M. Touqeer: Soft BCI-implicative ideals of soft BCI-algebras, Novi Sad J. Math. (in press, 2015).

[51] M.A. Malik, M. Touqeer: Some results on soft α-ideals, Journal of Applied Mathematics and Informatics (JAMI) (in press, 2015).

[52] M.A. Malik, M. Touqeer: Soft BCI-commutative ideals of soft BCI-algebras, World Appl. Sci. J. (in press, 2015).

[53] F. Marty: Sur une generalization de la notion de groupe, 8th Congress Math. Scandinaves, Stockholm 1934, 45-49.

[54] A. Maturo: On a non-standard algebraic hyperstructue and its application to the coherent probability assessments, Italian J. of Pure and Appl. Math. 7 (200) 33-50.

[55] R. Migliorato, G. Gentile: Hypergroupoids and cryptosystems, J. of Discrete Math. and Cryptography 2002.

[56] J. Mockor: A realization of d-groups, Czech. Math. J. 27 (102) (1977).

[57] D. Molodtsov: Soft set theory, First results, Comput. Math. Appl. 37 (1999) 19-31.

[58] D. Ore: Structure and group theory, Duke Math. J. 3 (1937).

[59] W. Prenowitz: Descriptive Geometries as Multigroups, Trans. Amer. Math. Soc. 59 (1946) 333-380.

[60] W. Prenowitz: Projective Geometries as Multigroups, Amer. J. Math. 65 (1943) 235-256.

[61] W. Prenowitz: Spherical Geometries and Multigroups, Canad. J. Math. 2 (1950) 100-119.

[62] W. Prenowitz, J. Jantosiak: Geometries and Join Spaces, J. reine and angewandte Math. 257 (1972) 100-128.

[63] K. Qin, Z. Hong: On soft equality, Computers Math. Appl. 234 (2010) 1347-1355.

[64] I.G., Rosenberg: An algebraic approach to hyperalgebras, Proc. 26th Int. Symp. Multiple-Valued Logic, Santiago de Compostela, IEEE (1996).

[65] R. Roth: On derived canonical hypergroups, Riv. Mat. Pura e Appl. 3 (1988).

[66] A. Sezgin, A.O. Atagun: On operation of soft sets, Comp. Math. Appl. 61 (2011) 1457-1467.

[67] G. Tallini: Ipergruppoidi di steiner e geometrie combinatorie, Convegno "Sistemi binary e loro applicazioni", Taormina, 1978.

[68] M. Touqeer, M.A. Malik: Fuzzy hyper h-ideals of hyper BCK-algebras (submitted).

[69] M. Touqeer, M.A. Malik: Fuzzy hyper BCK-commutative ideals of hyper BCK-algebras (submitted).

[70] M. Touqeer, M.A. Malik: Intuitionistic fuzzy BCI-implicative ideals in BCI-algebras (submitted).

[71] M. Touqeer, M.A. Malik: Intuitionistic fuzzy p-ideals in BCI-algebras (submitted).

[72] M. Touqeer, M.A. Malik: Correlation of intuitionistic fuzzy (p, H, α)-ideals in BCI-algebras (submitted).

[73] M. Touqeer, M.A. Malik: Intuitionistic fuzzy soft set theoretic approaches to α-ideals in BCI-algebras (submitted).

[74] Y. Utumi: On hypergroups of group right cosets, Osaka Math. J. 1 (1949) 73-80.

[75] L. A Zadeh: Fuzzy sets, Inf. Control 8 (1965), 338-353.

Appendix

List of Research Work

The following is a complete list of published, in press and submitted research articles of our dissertation.

Published

1. M.A. Malik, M. Touqeer: Some Results on fuzzy hyper BCK-ideals of hyper BCK-algebras, Indian Journal of Science and Technology 8(S3) (2015) 10-15.

2. M.A. Malik, M. Touqeer: Soft h-ideals of soft BCI-algebras, Indian Journal of Science and Technology 8(S3) (2015) 16-23.

3. M.A. Malik, M. Touqeer: Soft BCI-positive implicative ideals of soft BCI-algebras, Sci. Math. Jpn. e-2015, Whole Number 28 (2015) 57-78.

4. M.A. Malik, M. Touqeer: Fuzzy hyper BCK-implicative ideals of hyper BCK-algebras, Sci. Math. Jpn. e-2015, Whole Number 28 (2015) (online).

5. M.A. Malik, M. Touqeer: Intuitionistic fuzzy BCI-commutative ideals in BCI-algebras, Pakistan Journal of Science 64 (4) (2012) 353-358.

6. M.A. Malik, M. Touqeer: Intuitionistic fuzzy BCI-positive implicative ideals in BCI-algebras, Int. Math. Forum 6 (47) (2011) 2317 - 2334.

7. M.A. Malik, M. Touqeer: Some results on intuitionistic fuzzy BCI-(positive implicative, implicative, commutative) ideals in BCI-algebras, Int. Math. Forum 6 (47) (2011) 2335 - 2347.

In Press

8. M.A. Malik, M. Touqeer: Fuzzy hyper p-ideals of hyper BCK-algebras, Filomat (in press, 2015).

9. M.A. Malik, M. Touqeer: Soft BCI-implicative ideals of soft BCI-algebras, Novi Sad J. Math. (in press, 2015).

10. M.A. Malik, M. Touqeer: Some results on soft α-ideals, Journal of Applied Mathematics and Informatics (JAMI) (in press, 2015).

11. M.A. Malik, M. Touqeer: Soft BCI-commutative ideals of soft BCI-algebras, World Appl. Sci. J. (in press, 2015).

Submitted

12. M. Touqeer, M.A. Malik: Fuzzy hyper h-ideals of hyper BCK-algebras (submitted).

13. M. Touqeer, M.A. Malik: Fuzzy hyper BCK-commutative ideals of hyper BCK-algebras (submitted).

14. M. Touqeer, M.A. Malik: Intuitionistic fuzzy BCI-implicative ideals in BCI-algebras (submitted).

15. M. Touqeer, M.A. Malik: Intuitionistic fuzzy p-ideals in BCI-algebras (submitted).

16. M. Touqeer, M.A. Malik: Correlation of intuitionistic fuzzy (p, H, α)-ideals in BCI-algebras (submitted).

17. W.A. Dudek, M. Touqeer, M.A. Malik: Fuzzy soft set theoretic approaches to α-ideals in BCI-algebras (submitted).

18. M. Touqeer, M.A. Malik: Intuitionistic fuzzy soft set theoretic approaches to α-ideals in BCI-algebras (submitted).