
بسم الله الرحمن الرحيم

Certificate of Approval

It is certified that the research work presented in this thesis, entitled "Motion of Oldroyd-B Fluids with Fractional Derivatives in Cylinderical Domain" was conducted by Ms. Qammar Rubbab under the supervision of Dr. S.M. Husnine.

No part of this thesis has been submitted anywhere else for any other degree.
This thesis is submitted to the Department of Sciences & Humanities
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in Mathematics
at the
National University of Computer and Emerging Sciences
Lahore, PAKISTAN
May 2012

Signature of candidate:

Ms. Qammar Rubbab ______ Qammar Rubbab

May 26, 2012

Signatures, Examination Committee Members:

Dr. Shahid S. Siddiqi ______
Professor, PU, Lahore

Dr. Muhammad Ozair Ahmad ______ M. O. Ahmed
Professor, UET, Lahore

Dr. Muhammad Asif Gondal ______
Associate Professor, NUCES, Islamabad Campus

Dr. Arshad Hussain ______ Arshad Hussain

Director, National University of Computer and Emerging Sciences, Lahore Campus

Dr. M. Ayub Alvi ______

Dean, National University of Computer and Emerging Sciences

To my beloved Parents and Family

Table of Contents

Abstract

In this thesis, we determine the exact solutions for certain flows of non-Newtonian fluids of differential and rate type including second grade fluids, ordinary Maxwell fluids and Oldroyd-B fluids with fractional derivatives model. The research investigates longitudinal oscillatory and the unsteady rotational flows between two coaxial cylinders for second grade fluids, the Stokes' second problem for the Maxwell fluids by considering both sine and cosine oscillations of the plate, as well as the unsteady flow of Oldroyd-B fluids with fractional derivatives model in a circular cylinder. Ultimately we are lead to partial differential equations which are solved by using several transforms. We have determined the time required to attain the steady state for oscillating flows of second grade and Maxwell fluids by exploiting numerical methods and graphical illustrations. All the equations and boundary conditions governing the flow are satisfied by the solutions achieved in this thesis. As special cases, solutions are derived for some Newtonian fluids as well.

Publications from the thesis

1. Amir Mahmood, Najeeb Alam Khan, Corina Fetecau, Muhammad Jamil and Qammar Rubbab, Exact analytic solutions for the flow of second grade fluid between two longitudinally oscillating cylinders Published in " Journal of Prime Research in Mathematics 5 (2009), 192-204."

2. Qammar Rubbab, M.Imran and Syed Muhammad Husnine, A note On rotational flow of a second grade fluid in an annular pipe due to a time dependent shear stress presented in " 5th World Conference on 21st Century Mathematics 2011 held in Abdus Salam School of Mathematical Sciences, GC University Lahore-Pakistan February (2011) 9-13."

3. Imran Siddique and Qammar Rubbab, New exact solutions corresponding to Stokes'second problem for Maxwell fluids Published in " Buletinul Institutului Politehnic Din IASI Tomul LV (LIX), Fasc. 1, 2009."

4. A.U.Awan, Corina Fetecau and Qammar Rubbab, Axial Couette flow of an Oldroyd-B fluid with fractional derivatives due to a longitudinal time-dependent shear stress Published in " Journal of Quaestiones Mathematicae 33 (2010), 429-441."

5. Qammar Rubbab, Syed Muhammad Husnine and Amir Mahmood, Exact solutions of generalized Oldroyd-B fluid subject to a time-dependent shear stress in a pipe Published in "Journal of Prime Research in Mathematics 5 (2009), 139-148."

Other publications

6. Amir Mahmood, Saifullah and Qammar Rubbab, Exact solutions for a rotational flow of generalized second grade fluids through a circular cylinder Published in " BULETINUL ACADEMIEI DE STIINTE A REPUBLICII MOLDOVA MATEMATICA 3 (58), 2008, Pages 9-17 ISSN 1024-7696."

7. Q.Rubbab, S.M.Husnine, M.Imran, M.Athar and M.Kamran, Unsteady axial flow of an Oldroyd-B fluid with fractional derivatives induced by a circular cylinder Sent into "International Journal of Nonlinear Sciences and Numerical Simulation"

8. Muhammad Kamran, Muhammad Athar, Qammar Rubbab and Constantin Fetecau, Exact solutions for the flow of generalized second grade fluid in an annular pipe Published in " Buletinul Institutului Politehnic Din IASI Publicate de Universitatea Tehnica,, Gheorghe Asachi" din Iasi Tomul LV (LIX), Fasc. 3, 2010 Sectia MATEMATICA. MECANICA TEORETICA. FIZICA.

Acknowledgements

In the name of **Allah**, the most Gracious and the most Merciful. All praise and glory to **Allah** for His blessing upon me to finish my thesis. My sincerest thanks are due to my supervisor **Dr. S. M. Husnine** and co-supervisor **Dr. Constantin Fetecau**, for their valuable supervision. It would not have been possible for me to complete this work without their help and keen interest. I am deeply grateful to all my teachers who taught me graduate courses and enabled me to start research. I would like to thank our director Professor **Dr. Arshad Hussain** and the Head of Department Miss **Summaira Sarfraz** for their sympathetic attitude in solving all the administrative problems.

My gratitude is also due to **Prof. A. D. R. Chaudhary**, the Director General, Abdus Salam School of Mathematical Sciences (ASSMS) GCU Lahore for allowing me to avail all the facilities at his disposal so that I could do of my research work there.

I am especially thankful to the **HEC (Higher Education Commission of Pakistan)** for the financial support which provided me with an excellent opportunity to complete my PhD. I must also thank all the colleagues and official staff of **FAST-NU** and **ASSMS** for their moral support and cooperation. Of course, I am grateful to my parents, husband, children, **Waqas**, **Hassan** and **Hussain** and my younger sister, **Rissalat**, for their love and support.

FAST, NU, Lahore — Qammar Rubbab
November, 2011

Introduction

The world faces a lot of flow problems in natural disasters like mud slides, snowstorms, in volcanic eruptions etc. as well as many industrial, biological and agricultural processes. Scientists have to study these complex flows to find preventive measures and to use such phenomenons in devising many useful equipments. Most of these flows are viscous. However, the change of viscosity is non-linear and elasticity depends upon time and elongational effects. So they are treated like viscoelastic fluids.

Low molecular fluids are treated by the simplest model like Navier-Stokes models but non-Newtonian fluids like plastic, polymers, oils, detergents, greases, soaps and other biological fluids are dealt by different models. Several non-Newtonian models have been developed during forgoing few decades. One of them, the most famous, is fluids of differential type by Dunn *et al* [22] and Rivlin *et al* [63]. These fluids describe stress differences in shear thinning, shear flow, shear thickening and other non-linear creep properties but they fail to explain the stress relaxation. By Rajagopal [60] and Rajagopal *et al* [62], one finds the rate-type fluids which can better explain the above mentioned characteristics in shear flow. It is difficult to formulate shear stress for non-Newtonian fluids because little theoretical work is available in this area.

Nevertheless, non-Newtonian fluids have become very important because of their use in modern technology including geophysics, chemical engineering, bio-engineering and oil-reservoir engineering. Non-Newtonian fluids are divided into various classes according to their different shear behavior. Newtonian fluids have the characteristic of linear relationship between the shear rate and the shear stress which gives a constant viscosity. However, some non-Newtonian fluids like second grade fluid, Maxwell fluid and Oldroyd-B fluid have also constant viscosity. Second grade fluid is a subgroup of differential type fluids which has been investigated by Hayat *et al* [39, 41]. The analytical solutions of these fluids are well within reach as by Fetecau [29].

In order to describe stress relaxation, the first rate type model was developed by Maxwell [51]. A one-dimensional model was developed due to Burgers [12] which was very successful in dealing with biomaterial class. Burgers rate type-model in one dimension includes Oldroyd-B model as a special case. Oldroyd [53] developed rate-type model in three dimensions. This fulfilled the needs of frame indifference. But this model known as Oldroyd-B model failed to describe the shear thickening and shear thinning although it successfully explained the stress relaxation and normal stress differences in shear flow.

Flow around oscillating bodies is one of the most interesting phenomenon to industrialists and academicians. An exact solution was found by Stokes [71] for rotational oscillations of an infinite rod immersed in viscous fluids. Casarella and Laura [14] also found an exact solution for the same fluid motion due to longitudinal and torsional oscillation of the rod. Later on, Rajagopal [57] gave two fantastic solutions about the flow of second grade fluids for the above mentioned motion of an immersed rod. Rajagopal and Bhatnagar [61] extended these solutions to Oldroyd-B model. Further interesting results were obtained by Hayat *et al* [39, 45].

So far Newtonian fluids are concernd, the transient velocity distribution in a circular cylinder is discussed by Batchelor [8]. The exact solutions of second grade fluids within a circular cylinder was given by Ting [72] while the exact solutions of Maxwell fluids were given by Srivastava [69] and Rajagopal [56]. Subsequently, the Poiseuille flow of an Oldroyd-B fluid in a circular tube was investigated by Waters and King [78]. They also studied the loss of steady-state condition after removal of pressure gradient. The energy storage in Oldroyd-B fluids indicates that on one hand these fluids behave like elastic solids and on the other they appear to be mixture of two viscous fluids. This led to the study of viscoelastic fluids. Oldroyd [53] gave a systematic method for developing rate type viscoelastic fluids. Although his work met the invariance requirement of framework yet it left out the thermodynamical issues. Recently, a thermodynamic framework for rate type viscoelastic fluids has been described by Rajagopal and Srinivasa [62]. This framework basically deals with the saving and dissipating of energy.

The Oldroyd-B model involves the time for relaxation and retardation as well as viscosity. This model gave successful response to polymeric fluids as described by Bird *et al* [9] and Böhme [11] due to some analytic and experimental considerations. Oldroyd-B fluids describe adequately creep, stress-relaxation and normal stress differences which are the result of shear flows although it cannot describe shear thickening or thinning which is a characteristic of polymeric materials. The success of this model lies in its ability to describe the behaviors of a sub-class of polymeric fluids. As a particular case, the Maxwell model was deduced from this model. However, Choi *et al.* [15] found some inadequacy in the deduction.

Shear stress and shear rate have linear relation in a shear flow of real fluids. It was not possible for the Maxwell model to get the range of frequencies in experiments as mentioned by Bandelli *et al* [7]. The Maxwell model with fractional calculus by Hilfer [43] and Lorenzo *et al* [47] replaced the ordinary Maxwell model. This achieved an excellent fit of experimental data. Recently, the fractional calculus has achieved great success in describing viscoelasticity. The behavior of polymeric solutions and melts is better described by the rheological constitutive equations with fractional derivatives. The viscoelastic response in the α-relaxation regions as given by Heibig *et al.* [42] and in the glass transition has been properly described by one-dimensional fractional derivative. Maxwell model resulted in more work on generalization of nonlinear viscoelastic fractional constitutive equations. Palade *et al.*[54] gave first fully objective constitutive equations with fractional derivatives. Small deformations hypothesis are used to reduce them into the linear fractional model. In the ordinary governing equations of non-Newtonian fluid, one replaces the integer order derivatives by the Riemann-Liouville fractional operators to obtain the fractional derivatives model. Fractional derivatives of non-integer order as given by Podlubny [55] replace the ordinary derivatives of first, second or higher orders.

The main object of this thesis is to find some exact solutions for flows of second grade and Oldroyd-B fluids with fractional derivatives model in cylindrical domains. We also investigate Stokes' second problem for Maxwell fluids. Laplace and Hankel transforms are extensively used in our work to determine the exact solutions. The whole work comprises 5 chapters.

Chapter 1 includes the preliminaries relating to the fluids of rate type, constitutive equations, the continuity equation, equations of motion and some integral transforms which are used to solve the partial differential equations resulting from the mathematical formulation of different types of fluid flows. The exact solutions of different rate type fluids are obtained by large number of researchers. In **chapter 2**, we exploit the Laplace and Hankel transforms to obtain the exact solutions for the velocity field and corresponding shear stress related to the longitudinal oscillatory flow of a second grade fluid in the annular region produced by two infinite coaxial circular cylinders. As a special case we have recovered the solutions for Newtonian fluid. To compare the flows of second grade and Newtonian fluids, we graph their velocity profiles. The content of this chapter has been published in **Journal of Prime Research in Mathematics 5 (2009) 192-204**.

Chapter 3 studies the unsteady rotational flow of second grade fluid in the annular region produced by two infinite coaxial cylinders, one of them being set in rotation about its axis by a time dependent azimuthal shear stress. We present in series the solutions obtained by means of the finite Hankel transform, thereby obtaining them in series form in terms of Bessel functions such that the later satisfy all given boundary conditions. The results for Newtonian fluids, are derived as particular case. The effect of the specific parameters for given material on the velocity and shear stress is illustrated by graphs. The content of this chapter has been presented in **5th World Conference on 21st Century Mathematics 2011** held in Abdus Salam School of Mathematical Sciences, GC University Lahore-Pakistan February (2011) 9-13.

Moreover the exact solutions relating to the flow of non-Newtonian fluids, over an extended flat plate, are described in [6, 18, 23, 32, 48, 49, 56, 58, 65, 77]. In **chapter 4**, we consider both sine and cosine oscillations of the plate to recover the exact solutions for Stokes' second problem for Maxwell fluids. It seems that the solutions corresponding to this problem for non-Newtonian fluids achieved by Rajagopal [56], Hayat et al. [36, 37] and Siddique et al. [66] are the earliest efforts. These authors determined the oscillatory pressure gradient in second grade fluids and some other steady-state solutions relating to various oscillations of a rigid plate. Hayat et al. [38] also studied the steady case for Oldroyd-B fluid. Erdogan [23] gave starting solutions about the motion of Newtonian fluid by using the Laplace transform, which is the result of sine and cosine oscillations of a flat plate. In this chapter, we find new starting solutions by using Laplace and Fourier sine transforms and determine the flow for low and high values of time t. These solutions have two components, namely transient and steady-state. The transient components are simpler than the previous ones and the steady-state components are identical with those obtained in previous works. We determine the time values for attaining the steady-flow numerically and also illustrate them graphically. This work has been published in **BULETINUL INSTITUTULUI POLITEHNIC DIN IASI Tomul LV (LIX), Fasc. 1 (2009) 19-31**.

Several authors like Mainardi [50], Slonimsky [67], Stiassnie [70] argued that integer-order models for viscoelastic models are not suitable for both quantitative and qualitative aspects. At that time, they modeled the viscoelastic behavior of real materials by using fractional order laws of deformation. Zener's integer-order model as discussed by Zener [79] was extended to fractional-order model by Caputo and Mainardi [13]. Experiments have proved that many viscoelastic materials can be modeled very successfully by this model (see Bagley and Torvik [3, 4] and Rogers [64]). Bagley and Torvik's analysis [5] have shown that the previously adopted classical models of viscoelasticity were less reliable and satisfactory than the fractional calculus models of viscoelastic behavior.