Exclusive Radiative Decays of B Mesons

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Contents

1 Introduction ................................................. 1

I Fundamentals ....................................................... 6

2 Basis Concepts .................................................. 7
  2.1 Standard Model .............................................. 7
  2.2 Renormalization and Renormalization Group ................. 11
    2.2.1 General Remarks ....................................... 11
    2.2.2 QCD Lagrangian ........................................ 12
    2.2.3 Renormalization ........................................ 13
    2.2.4 Renormalization Group Equation ....................... 15
  2.3 Operator Product Expansion ................................ 17
  2.4 Effective Hamiltonian ...................................... 18
    2.4.1 Operator Product Expansion and Short Distance QCD Effects ................. 19

3 Approaches to Non-Leptonic Decays ......................... 25
  3.1 The Effective $b \rightarrow s \gamma$ Hamiltonian .............. 25
  3.2 Heavy Quark Effective Theory ............................. 28
  3.3 QCD Factorization ......................................... 30
    3.3.1 The factorization formula ............................ 30
  3.4 Large Energy Effective Theory (LEET) ..................... 34
    3.4.1 LEET symmetries, their breaking in perturbative QCD and form factors .... 38
Abstract

This PhD thesis presents the exclusive radiative decays of $B$ meson within the Standard Model (SM). In particular, we consider the decays $B \rightarrow \gamma \ell \nu_{\ell}$ and $B \rightarrow (K_1, b_1, h_1) \gamma$, where $K_1$, $b_1$ and $h_1$ are orbitally excited axial vector mesons of $K^*$, $\rho$ and $\omega$ respectively. At quark level, all these decays are governed by the flavor changing neutral currents $b \rightarrow (s, d) \gamma$ transitions, which are not allowed at tree level in the SM. These processes will provide quantitative information on the Standard Model parameters especially the CKM matrix elements and are also sensitive to the presence of physics beyond the SM. The exclusive decays are experimentally better accessible but pose more problem for the theoretical analysis. It is therefore imperative to firm up theoretical predictions in exclusive decays for precision tests of SM and to interpret data for possible new physics effects in these decays.

The main results of our work are the following:

- Form factors parameterizing radiative leptonic decays of heavy mesons ($B^+ \rightarrow \gamma l^+ \nu_{l}$) for photon energy are computed in the language of dispersion relations. The contributing states to the absorptive part in the dispersion relation are the multiparticle continuum, estimated by quark triangle graph and resonances with quantum numbers $1^-$ and $1^+$ which includes $B^*$ and $B_{\ast}\gamma$ and their radial excitations, which model the higher state contributions. Constraints provided by the asymptotic behavior of the structure dependent amplitude, Ward Identities and gauge invariance are used to provide useful information for parameters needed. The couplings $g_{BB^*\gamma}$ and $f_{BB_{\ast}\gamma}$ are predicted if we restrict to first radial excitation; otherwise using these as an input the radiative decay coupling constants for radial excitations are predicted. The value of the branching ratio for the process $B^+ \rightarrow \gamma \mu^+ \nu_{\mu}$ is found to be in the range $0.5 \times 10^{-6}$. A detailed comparison is given with other approaches.

- For $B \rightarrow (K_1, b_1, h_1) \gamma$ decays we calculate the hard spectator corrections in $O(\alpha_s)$ in the leading-twist approximation using Large Energy Effective Theory (LEET) techniques. Combining with the hard vertex, already calculated in the literature, and annihilation contributions, they are used to compute the branching ratios for these decays in the next-to-leading order (NLO) in the strong coupling $\alpha_s$ and in leading power in $\Lambda_{QCD}/M_B$. It is found that the theoretical branching ratios for the decays $B \rightarrow K_1 \gamma$ in the LEET approach can be compared with the data only for significantly large values of the form factors than their estimates in the Light Cone Sum Rules (LCSR). Using the $SU(3)$ symmetry for the form factors, the branching ratio for $B \rightarrow (b_1, h_1) \gamma$ is expressed in terms of the branching ratio of the $B \rightarrow K_1 \gamma$ and they are found to be $B(B \rightarrow b_1 \gamma) = 0.53 \times 10^{-6}$ and $B(B \rightarrow h_1 \gamma) = 0.51 \times 10^{-6}$.

- We also calculate the direct $CP$ asymmetry for the decays $B \rightarrow (K_1, b_1, h_1) \gamma$ and find, in conformity with the observations made in the literature, that the hard spectator contributions significantly reduce the asymmetry arising from the vertex corrections. The sensitivity of the $CP$ asymmetry on the underlying parameters is found to be significantly large.
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I am filled with the praise and glory to All Mighty Allah, the most merciful and benevolent, who created the universe, with ideas of beauty, symmetry and harmony, with regularity and without any chaos, and gave us the abilities to discover what He thought.

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TO MY LOVING MAA’N Jee AND BABA Jee
Certificate

Certified that the work contained in this dissertation was carried out by Mr. Muhammad Jamil Aslam under my supervision.

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Chapter 1

Introduction

The most fundamental element of physics is the reduction principle. The large variety of macroscopic forms of matter can be traced back, according to this principle, to a few microscopic constituents which interact by a small number of forces. The reduction principle has provided a guide to unraveling of the structure of physics from the macroscopic world through atomic and nuclear physics to particle physics. Elementary particle Physics is the best representation of man's effort to answer the basic question: "What is the World made of?" Starting from the Aristotelian idea of water, fire, air and soil till the finding of neutrino by Reines and Cowan reflects his curiosity to explore the nature. What underlies our current theoretical understanding of nature is quantum field theory in combination with a gauge principle. The laws of nature governing the subatomic world are summarized in the Standard Model of the particle physics.

Standard Model, proposed by the Glashow, Weinberg and Salam [1, 2] is made up of electromagnetism, weak and strong nuclear forces. The interactions of these forces with quarks and leptons are also described by this model. The electroweak sector of the Standard Model which unifies electromagnetic and weak forces is one of the most successful model of 20th century and this has provided the plenty of successful predictions with an impressive level of precision.

Despite its many successes, no one can say that it is the end of Physics. It is not complete and has some deficiencies:

- Gravity is not included in the Standard Model.

- Why is the electroweak scale so small (hierarchy problem)?

- The Standard Model incorporates but does not explain electric charge quantization. Sim-
ilarly, there is no explanation why only the electroweak part is chiral.

- Strong CP violation problem.

- Why there are three particle generations? What is the origin of the mass spectrum?

- In Standard Model the neutrinos are massless but now the experiments have shown that the neutrinos do in fact have non-zero mass.

These problems indicate that there must be some New Physics beyond the Standard Model. The ideas of grand unification, extra dimensions, or supersymmetry were put forward to find a more complete theory. But these theories are not very simple and have many arbitrary parameters like the Standard Model. To date, string theory, the relativistic theory of one-dimensional objects, is a promising, and so far the only candidate for such a “Theory of Everything”.

In this thesis we work exclusively in the Standard Model, and more specifically in the flavor sector. We deal with the bottom quark system, which is an ideal laboratory for studying the flavor physics. Historically $B$ physics started in 1977 with the observation of a dimuon resonance at 9.5 GeV in 400 GeV proton-nucleon collision at Fermilab [3]. It was named as “$Y$ resonances” and its quark contents is $b\bar{b}$. Babar [4] and Belle [5] started working in 1999 and these dedicated $B$ factories add a wealth of data to the results of CLEO [6], CERN [7] and Fermilab [8] Experiments. The upcoming $B$ physics experiments at the Tevatron Run II [9] and LHC [10] will bring us ever closer to the main goal of $B$ physics, which is a precision study of the flavor sector with its phenomenon of CP violation to pass the buck of being the experimentally least constrained part of the Standard Model. This will not only give us the deep understanding of the parameters of the Standard Model, but also help us to study the New Physics effects via deviations of measured observables from the Standard Model expectations. Such an indirect search of New Physics is complementary to the direct search at particle accelerator and invites both theoreticians and experimentalists to work with precision. We need accurate and reliable measurements and calculations. The calculation challenge we will meet for this thesis are exclusive radiative decays of $B$ mesons.

Let us start with a very natural and common question, why we study meson decays? Due to confinement quarks appear in nature not separately, but have to be bound into colorless hadrons and quark-antiquark bound state is known as a meson. The bound states with a $b$ quark and a $\bar{d}$ or $\bar{u}$ antiquark are referred to as the $B^0$ and $B^-$ mesons, respectively. Therefore choice to study the “meson” decays is the fact that mesons are the simplest hadrons. Now if
we study the meson decays then why "B" mesons? The most obvious reason is that the B mesons are heaviest mesons, as the top quark decays before it can hadronize. The fact that B meson is heavy has two weighty consequences: B decays show an extremely rich phenomenology and theoretical techniques using an expansion in the heavy mass allow for model-independent predictions. The large available phase space and the the possibility for large CP-violating asymmetries in the B decays make it a topic of rich phenomenological study. The large CP-violating feature of B meson is in contrast to the Standard Model expectations for the decays of K and D mesons. The pattern of CP violation in K and B system just represents the hierarchy of the CKM matrix. The B meson system offers an excellent laboratory to quantitatively test the CP-violating sector of the Standard Model, determine fundamental parameters, study the interplay of strong and electroweak interactions, or search for New Physics.

The decay of the heavy quark is governed by weak interactions but it is the strong force that is responsible for the formation of the hadrons that are observed in the detectors. The easily accessible decays for the experimentalists are the exclusive ones, i.e. those where all decay products are detected. But on the other hand, theoretically it is much easier to calculate the inclusive decays (e.g. b → sγ) instead of the exclusive B mesons where we have to dress the b quark with the light degrees of freedom inside the B meson, and have to keep struggling with the hadronization. Bearing in mind all these theoretical complexities for exclusive decays, it is still worthwhile to better understand them. Especially in the difficult environment of the hadron machines, like the Fermilab Tevatron, LHC at CERN they are easier to investigate experimentally. The systematical uncertainties both theoretically and experimentally are very different for both type of modes, i.e. inclusive and exclusive decay modes. A careful study of the exclusive modes can therefore yield valuable complementary information in testing the Standard Model. In order to study these decay modes, the field theoretical kit includes operator product expansion (OPE) and the renormalization group equations in the framework of an effective theory. In order to calculate the transition amplitudes one can separate it into the perturbatively calculable short distance Wilson coefficients and the long distance operator matrix elements. These long distance contributions have to be calculated by means of a non-perturbative methods like lattice QCD or QCD sum rules. For the exclusive decays of B mesons, however, one can use additionally the fact that the b quark mass is large compared to the typical QCD scale λQCD. Therefore one can use the factorization formula to calculate the relevant hadronic matrix elements of local operators in the weak Hamiltonian. With this further separation of
the long distance contributions to the process from a perturbatively calculable short-distance part, that depends only on the large scale $m_b$, is achieved. The long distance contributions are much simpler in structure then the original matrix element and have to be calculated either non-perturbatively or from the experiments. The QCD factorization technique \cite{11} and the Large Energy Effective Theory (LEET) \cite{12-14} have been developed to study these non-leptonic decays, and these will be covered to some length in this thesis.

This thesis is organized in three parts. In the first part we fill our toolbox with the necessary ingredients. After giving a bird eye view of the Standard Model we present some basic tools: operator product expansion, effective theories and renormalization group improved perturbation theory. A discussion of the effective $b \to s\gamma$ Hamiltonian, Heavy Quark Effective Theory (HQET), QCD factorization, LEET and its symmetry breaking are also the subjects of this part.

Part II and part III deal with the main subject of the work: the study of exclusive semileptonic and hadronic $B$ meson decays. In part II we have studied the exclusive semileptonic $B$ meson decay $B^+ \to \gamma l^+ \nu_l$ with in the Standard Model. After presenting the decay kinematics and current matrix elements for $B^+ \to l^+ \nu_l\gamma$, we discuss the various contributions to the absorptive part of the SD amplitude $iH_{\mu\nu}$, needed in the dispersion relation. This include multiparticle continuum and resonances with quantum numbers $1^-$ and $1^+$. The resonances include $B^*$ and $B^*_A$ mesons and their radial excitations, which model the higher states. The continuum is estimated by quark triangle graphs. The asymptotic behavior of the SD amplitude is also studied which provide a usual constraint on the residues of the resonance contribution, in terms of the continuum contribution. Then we discuss Ward Identities which together with gauge invariance relates various form factors. These identities which are expected to hold below the resonance regime, fix the normalization of the forms at $q^2 = 0$ in terms of a universal function $g_+(0)$ as well as another constraint on the residues. Thus in our approach, a parametrization of $q^2$ dependence of form factors is not approximated by single pole contribution. But this parametrization is dictated by considerations mentioned above and also predict the coupling constants of $1^-$ and $1^+$ resonances with photon if we restrict to one radial excitation; otherwise using these as input, the radiative coupling constants of radial excitations are predicted. We end this part by comparing our numerical value of branching ratios with some other approaches already existing in the literature.

Part III presents the next-to-leading order (NLO) calculations of $B \to (K_1, \, ^1P_1)\gamma \left(^1P_1 = b_1, \right.$
in LEET. These \( K_1, b_1 \) and \( h_1 \) are the corresponding axial vector states of \( K^*, \rho \) and \( \omega \) respectively. The distribution amplitude (DA) of these axial vector meson states contain an extra factor of \( \gamma_5 \) in comparison to the DA of vector meson state. But we will see this extra \( \gamma_5 \) does not alter the calculation and give the same result of the perturbative part.

We first derive the decay amplitude for \( B \to K_1 \gamma \) complete at next-to-leading order in LEET. Hard vertex and Hard spectator corrections are discussed separately. The most important phenomenological quantity is the branching ratio which has only one unknown and that is the LEET form factor. We will extract the value of this unknown from the data and will show that its value remains unchanged even if we include the higher twist effects (non-asymptotic effects) in DA or include the annihilation contributions. Our calculation also allows us to estimate CP asymmetries for \( B \to K_1 \gamma \) decays which is explicitly studied in this thesis.

Then we apply the same expressions, with an obvious replacement, calculated for \( B \to K_1 \gamma \) to \( B \to (b_1, h_1) \gamma \) decays which are \( b \to d \gamma \) transition instead of the \( b \to s \gamma \). As we have not any experimental bounds on these decays at present, we have to relate these unknown quantities to some known results in the literature. To meet the task we use the \( SU(3) \) symmetry to relate the form factors of \( B \to (b_1, h_1) \gamma \) decay to that of the \( B \to K_1 \gamma \) decay. After this the numerical results for branching ratio and CP asymmetries are presented.

We give our conclusions and outlook in the chapter 7.
Part I

Fundamentals
Chapter 2

Basis Concepts

This chapter introduces the basic concepts and tools for doing calculations in elementary particle physics. We give a short introduction of the Standard Model, renormalization, renormalization groups, operator product expansion (OPE) and the short distance (SD) effects. The prerequisites to read this dissertation are quantum field and gauge theories and we refer to some pertinent textbooks [15]. The detailed discussion on some of the topics presented here is given in references [16–20].

2.1 Standard Model

The Standard Model of Electroweak and Strong interactions is one of the successful models of the 20th century giving a complete and correct description of all non-gravitational physics tested so far. Here we will introduce the Standard Model, with massless neutrinos. The recent evidence of neutrino masses, coming from the observation of neutrino oscillations, has no direct effect on our work.

The Standard Model is made up of the Glashow-Salam-Weinberg Model [1] of electroweak interaction and Quantum Chromodynamics (QCD) [2]. It is a model whose foundation is symmetry and the basic one is gauge symmetry. The Lagrangian of a gauge theory is invariant under local gauge transformations of a symmetry group. Such a symmetry can be used to generate dynamics - the gauge interactions. The prototype gauge theory is quantum electrodynamics (QED) with its Abelian U(1) local symmetry. It is believed that all fundamental interactions are described by some form of gauge theory.

For strong interactions the gauge group is the non-Abelian SU(3)C which has eight genera-
tors which corresponds to the eight gluons. These gluons communicate the strong force between objects carrying color charge - subscript $C$ denotes this fact. Since the gluons themselves are colored, they can directly interact with each other, which leads to the phenomena of "asymptotic freedom" and "confinement". At short distances, the coupling constant $\alpha_s$ becomes small. This allows us to compute color interactions using perturbative techniques and turns QCD into a quantitative calculational scheme. For long distances, the coupling gets large, which causes the quarks to be confined into colorless hadrons.

Electroweak interaction is based on the gauge group $SU(2)_L\cdot U(1)_Y$. $L$ stands for left and $Y$ denotes the hypercharge. This gauge group is spontaneously broken to $U(1)_{\text{QED}}$ through the non-vanishing vacuum expectation value of a scalar isospin doublet $\Phi$ field [21]

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}.$$ (2.1)

This Higgs doublet has four scalar degrees of freedom. Out of these four, the three give masses to the $W$ and $Z$ bosons. The remaining one manifests itself in a massive neutral spin zero boson, the physical Higgs boson. It is the only unknown parameter of the Standard Model which lacks direct experimental detection. The current lower limit on its mass is 114.1 GeV at the 95% confidence level [22]. From electroweak precision data there is much evidence for a light Higgs. But as soon as such a light Higgs is found, this gives birth to the hierarchy problem. A scalar (Higgs) mass is not protected by gauge or chiral symmetries so we expect $m_H \approx \Lambda \approx 10^{16}$ GeV if we do not want to fine-tune the bare Higgs mass against the mass acquired from quantum effects. Why should $m_H$ be much smaller than $\Lambda$?

In the Standard Model, the fermions which are building blocks of matter appear in three generations which differ only in their masses. The two fundamental species of fermions are leptons and quarks that can be classified in left-handed doublets and right-handed singlets.

Quarks :

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix}_{\text{L}} \begin{pmatrix} c_L \\ s'_L \end{pmatrix}_{\text{L}} \begin{pmatrix} t_L \\ b'_L \end{pmatrix}_{\text{L}}$$

$$u_R \quad c_R \quad t_R$$

$$d_R \quad s_R \quad b_R$$
Leptons: 
\[
\begin{pmatrix}
\nu_e \\
e^- \\
e_R \end{pmatrix}_L, \quad \begin{pmatrix}
\nu_\mu \\
\mu^- \\
\mu_R \end{pmatrix}_L, \quad \begin{pmatrix}
\nu_\tau \\
\tau^- \\
\tau_R \end{pmatrix}_L
\]

Obviously there is no right handed neutrino in the Standard Model. The quarks transform as triplet under \( SU(3)_C \) transformation due to their color charge whereas the colorless leptons are \( SU(3)_C \) singlets.

Fermions get their masses due to the Yukawa interactions, viz. \( \bar{\psi}(x)\phi(x)\psi(x) \), with the Higgs field (2.1). The flavor and mass eigenstates of the quarks are related through the global unitary transformations. Using these transformations in flavor space, the Yukawa interactions can be diagonalized to obtain the physical mass eigenstates

\[
\begin{pmatrix}
d' \\
s' \\
b' \end{pmatrix} = V_{\text{CKM}} \cdot \begin{pmatrix}
d \\
s \\
b \end{pmatrix},
\]

where \( V_{\text{CKM}} \) is called the Cabbibo-Kobayashi-Maskawa matrix and is symbolically written as

\[
V_{\text{CKM}} = \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb} \\
\end{pmatrix},
\]

The off-diagonal elements of the Cabbibo-Kobayashi-Maskawa matrix [23] allow for transitions between the quark generations in the charged quark current

\[
J^\mu = (\bar{u}, \bar{c}, \bar{t})_L \gamma^\mu V_{\text{CKM}} \begin{pmatrix}
d \\
s \\
b \end{pmatrix}_L.
\]

Due to the unitarity of \( V_{\text{CKM}} \) the flavor-changing-neutral-currents (FCNC) are absent at tree level in the Standard Model. This Glashow-Iliopoulos-Maiani (GIM) mechanism [24] would forbid FCNC transitions even beyond the tree level if we had exact horizontal flavour symmetry which assures the equality of quark masses of a given charge. Such a symmetry is in nature obviously broken by the different quark masses so that at the one-loop level effective \( b \to s \), \( b \to d \), processes like \( B \to X_s \gamma \) can appear.
Because of the unitarity of CKM matrix, it is characterized by three independent rotation angles and a complex phase. The KM theory was formulated to incorporate the CP violation observed in Kaon decays in 1964 by Christenson et al. [25]. In this theory the CP symmetry is broken at the Lagrangian level in the charged current weak interactions and nowhere else. In principle, all the elements of the matrix $V_{\text{CKM}}$ are complex. In practice, only two of the matrix elements have measurable phases. But, this is sufficient to anticipate CP violation in a large number of processes, some of which are now being measured with ever-increasing precision in the $K$ and $B$ decays.

In practice there are many different ways of parameterizing the CKM matrix. But, it has become customary to discuss the CKM phenomenology by using the Wolfenstein parametrization [26]:

$$V_{\text{CKM}} = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3 (\rho - i\eta) \\
-\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\
A\lambda^3 (1 - \rho - i\eta) & -A\lambda^2 & 1
\end{pmatrix}, \quad (2.5)$$

where the four independent parameters are: $A$, $\lambda = \sin \theta_C$, $\rho$ and $\eta$, of which $\eta$ is what makes the matrix complex and cause CP violation. Anticipating precise data, a perturbatively improved Wolfenstein parameterization [27] with $\bar{\rho} = \rho (1 - \lambda^2 / 2)$ and $\bar{\eta} = \eta (1 - \lambda^2 / 2)$ will be used. This recasting effects mainly the matrix element $V_{ud}$ and $V_{ub}$ and the other matrix elements remain essentially unchanged.

The unitarity of the CKM-matrix implies various relations between its elements. In particular, we have

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0. \quad (2.6)$$

Phenomenologically this relation is very interesting as it involves simultaneously the elements $V_{ub}$, $V_{cb}$ and $V_{td}$ which are under extensive discussion at present. Other relevant unitarity relations will be presented as we proceed.

The relation (2.6) can be represented as a unitarity triangle in the complex ($\bar{\rho}$, $\bar{\eta}$) plane. The invariance of Eq. (2.6) under any phase-transformations implies that the corresponding triangle is rotated in the ($\bar{\rho}$, $\bar{\eta}$) plane under such transformations. Since the angles and the sides (given by the moduli of the elements of the mixing matrix) in this triangle remain unchanged, they are phase convention independent and are physical observables. Consequently they can be measured directly in suitable experiments. One can construct five additional unitarity triangles [28] corresponding to other orthogonality relations, like the one in Eq. (2.6) but all triangles
have the same area. Some of them should be useful when the data on rare and CP violating
decays improve. The areas ($A_{\Delta}$) of all unitarity triangles is related to the measure of CP
violation $J_{CP} : |J_{CP}| = 2 \cdot A_{\Delta}$ [29].

Noting that to an excellent accuracy $V_{cd}V_{cd}^*$ in the parametrization (2.5) is real with $|V_{cd}V_{cd}^*| =
A \lambda^3 + O (\lambda^7)$ and rescaling all terms in Eq. (2.6) by $A \lambda^3$ we indeed find that the relation (2.6)
can be represented as the triangle in the complex ($\bar{\rho}, \eta$) plane as shown in Fig. 2-1. The state-
of-the-art results of unitarity triangle from the 2004 International Conference on High Energy
Physics (ICHEP) is displayed in Fig. 2-2 [30]. Actually the good agreement of measurements
with the Kobayashi-Maskawa mechanism give rise to some theoretical puzzles: the KM mecha-
nism for example does explain neither the cosmic baryon asymmetry nor the smallness of $\theta_{QCD}$
and basically all extensions of Standard Model introduce a large number of new CP-violating
phases.

2.2 Renormalization and Renormalization Group

2.2.1 General Remarks

In this section we give a brief review of some facts of QCD, its Lagrangian, renormalization
and the renormalization group, which are indispensable for our climb. In particular we discuss
the dimensional regularization, the MS and $\overline{\text{MS}}$ renormalization schemes and renormalization
group equations for the running QCD coupling and running quark masses.
2.2.2 QCD Lagrangian

The Lagrangian density of QCD, omitting the ghosts and setting the gauge parameter to $\xi = 1$, reads

$$\mathcal{L}_{\text{QCD}} = -\frac{i}{2} \left( \partial_\mu A_\mu^a - \partial_\nu A_\nu^a \right) \left( \partial_\rho A_\rho^{\mu a} - \partial_\nu A_\nu^{\rho a} \right) - \frac{i}{2} \left( \partial_\mu A_\mu^a \right)^2 + \bar{q}_a (i \gamma^\mu - m_q) q_a$$

$$- g q_\alpha T_\alpha^a q_\beta A_\mu^a + \frac{g_\alpha}{4} f^{abc} \big( \partial_\mu A_\nu^a - \partial_\nu A_\mu^a \big) A^{b \mu} A^{c \nu} - \frac{g^2}{4} f^{abc} f^{d e c} A_\mu^a A_\nu^b A^{a e d} .$$

(2.7)

Here $A_\mu^a$ are the gluon fields with $(a, b, c = 1, \ldots, 8)$ and $q$ is the color triplet of the quark flavor $q$, $q = u, d, s, c, b, t$. $g$ is the QCD coupling so that

$$\alpha_s = \frac{g^2}{4\pi} ,$$

and $T^a$ and $f^{abc}$ are the generator and structure constants of $SU(3)_C$, respectively.

With this Lagrangian one can deduce the Feynman rules by means of which amplitude of the processes occurring in QCD can be calculated in perturbative QCD. If one goes beyond the tree level, Feynman diagrams with internal loops, one often encounter the ultraviolet divergences.
due to the momentum variable of the virtual particle in the loop integration which ranges from zero to infinity. The theory of renormalization is a prescription which allows us to consistently isolate and remove all these infinities from the physically measurable quantities.

2.2.3 Renormalization

To get rid of the difficulty of infinities in the physically measurable quantity, a two step procedure is needed. Firstly, one regulates the theory, that is, one modifies it in a way that observable quantities are finite and well defined to all orders of perturbation theory. We are then free to manipulate formally these quantities, which are divergent only once regularization is removed. The momentum cutoff method is the most straightforward way to make these integrals finite. The cost we have to pay for this is the violation of Lorentz invariance and Ward Identities. Dimensional regularization method is one that preserves all symmetries of gauge theory [31, 32]. In this regularization scheme the Feynman diagrams are evaluated in $D = 4 - 2\varepsilon$ space-time dimensions and singularities are extracted as poles for $\varepsilon \to 0$.

Although the dimensional regularization is the favorite regularization in gauge theories, but potential problems are connected with the treatment of $\gamma_5$ in $D \neq 4$ dimensions, which clearly is of deep concern with the theory of weak interactions. The definition

$$\gamma_5 = \frac{i}{4! \varepsilon_{\mu\nu\alpha\beta}} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta,$$

with $\varepsilon_{\mu\nu\alpha\beta}$ the completely antisymmetric tensor in four dimensions, cannot straightforwardly be translated to $D \neq 4$ dimensions. In naive dimensional regularization (NDR) scheme [33] the metric tensor is generalized in $D$ dimensions and the $\gamma$ matrices obey the same anticommutation rules as the four dimensions. Even if these rules are algebraically inconsistent [34], the NDR scheme give correct results provided one can avoid the calculation of traces like $\text{Tr}(\gamma^3 \gamma_4 \gamma_5 \gamma^\alpha \gamma^\beta)$ [35].

The other scheme which is originally proposed by 't Hooft and Veltman (HV scheme) [31] and by Akyeampong and Delbourgo [36] and systematized by Breitenlohner and Maison [34] allows a consistent formulation of dimensional regularization even when $\gamma_5$ couplings are present. This scheme introduces in addition to the $D$- and 4- dimensional metric tensors $g$ and $\tilde{g}$, the $-2\varepsilon$-dimensional tensor $\hat{g}$. One can split $D$-dimensional Dirac matrix $\gamma_\mu$ into a 4- and a $-2\varepsilon$-dimensional part $\hat{\gamma}_\mu$ and $\hat{\gamma}_\mu$ which separately obey commutation relations with the appropriate
metric tensors. In Ref. [34] it is shown that a $\gamma_5$ can be introduced which anticommutes with $\hat{\gamma}$ but commutes with $\hat{\gamma}$. The price we have to pay for a consistent dimensional regularization scheme is a substantial increase in the complexity of calculations.

After regularization, the second step, called \textit{renormalization}, consists of relating the properties of the unphysical (\textit{bare}) and physical (\textit{renormalized}) parameters like couplings $g$ and masses $m$ and rewrite observables as functions of the physical quantities. In this procedure all the divergences are hidden in a redefinition of the fields and parameters in the Lagrangian, i.e.

\begin{equation}
\begin{align*}
g^{(0)} &= Z_g g \mu^\varepsilon, \quad m^{(0)} = Z_m m \\
q^{(0)} &= Z_q^{1/2} q, \quad A_\mu^{(0)} = Z_3^{1/2} A_\mu.
\end{align*}
\end{equation}

The bare quantities are indicated by superscript $(0)$. $A_\mu$ and $q$ are the renormalized fields, $g$ is the renormalized QCD coupling and $m$ is the renormalized quark mass. Introducing a mass parameter $\mu$ is necessary to keep the couplings dimensionless. The $Z$'s are the renormalization constants. The renormalization is the recursive process in the powers of the coupling constant $g$ and if to every order of perturbation theory all divergences are absorbed in $Z$'s, the theory is called \textit{renormalizable}. Standard Model is one of the best example of the renormalizable theory. It remains renormalizable even if the gauge symmetry is spontaneously broken via the Higgs mechanism because gauge invariance of the Lagrangian is conserved [37].

Counter-term method is a straightforward way to implement the renormalization. In it the parameters and fields in the original Lagrangian, considered as unrenormalized quantities, are reexpressed through renormalized ones by means of Eq. (2.8). Thus

\begin{equation}
L_{\text{QCD}}^{(0)} = L_{\text{QCD}} + L_{\text{counter}}
\end{equation}

where $L_{\text{QCD}}$ is given in Eq. (2.7) and $L_{\text{QCD}}^{(0)}$ is the QCD Lagrangian expressed in terms of the bare quantities. $L_{\text{counter}}$ is proportional to $(Z - 1)$ and can be formally treated as new interaction term that contribute to Green functions calculated in perturbation theory. Feynman rules for these new quantities can be derived and the renormalization constants $Z_i$ are determined such that the contribution from these new interactions cancel the divergences in the Greens function. There is some arbitrariness how this can be done because a given renormalization prescription can in general subtract not only the divergences but also finite parts. Different finite parts define renormalization schemes. The simplest one is Minimal Subtraction (MS)
scheme in which only divergences are subtracted [38]. Of particular interest is the modified MS scheme \((\overline{\text{MS}})\) [39], with which we will exclusively work here, defines the finite parts such that terms \(\ln 4\pi - \gamma_E\), the artifacts of the dimensional regularization, are absent. This can be achieved if one calculates with

\[
\mu_{\overline{\text{MS}}} = \frac{\mu e^{\gamma_E/2}}{\sqrt{4\pi}},
\]

instead of \(\mu\) and performs minimal subtraction afterwards. The renormalization constants \(Z\) defined in Eq. (2.8) in \(\overline{\text{MS}}\) scheme are:

\[
\begin{align*}
Z_g &= 1 - \frac{1}{\varepsilon} \left( \frac{1}{6} N_c - \frac{1}{3} N_f \right) \alpha_s, \\
Z_m &= 1 - 3 \alpha_s \frac{1}{\varepsilon} N_f, \\
Z_3 &= 1 - \frac{1}{\varepsilon} \left( \frac{2}{3} N_c - \frac{5}{3} N_f \right) \alpha_s, \\
Z_q &= 1 - \frac{1}{\varepsilon} \alpha_s. 
\end{align*}
\]  

(2.9)

\(N_c\) and \(N_f\) denotes the number of quark colors and flavors respectively. In general this can be written as

\[
Z_i \approx 1 + \sum_{k=1}^{\infty} \frac{Z_{i,k} (g)}{\varepsilon^k}.
\]

2.2.4 Renormalization Group Equation

In the process of renormalization we have introduced an arbitrary mass parameter \(\mu\). Even after renormalization the theoretical predictions depend on this renormalization scale \(\mu\). To determine the renormalized parameters from experiment, a specific choice of \(\mu\) is necessary: \(g \equiv g (\mu), m \equiv m (\mu), q \equiv q (\mu)\). Different values of \(\mu\) defines different parameter sets \(g (\mu), m (\mu), q (\mu)\) and set of equations that relates parameter sets with different \(\mu\) is called renormalization group equations (RGE). The equations for running coupling and running mass are derived from the definitions (2.8) using the fact that the bare quantities are independent of \(\mu\), one finds:

\[
\begin{align*}
\frac{dg (\mu)}{d \ln (\mu)} &= \beta (g (\mu), \varepsilon), \\
\frac{dm (\mu)}{d \ln (\mu)} &= -\gamma_m (g (\mu)) m (\mu),
\end{align*}
\]  

(2.10)

with the \(\beta\)-function

\[
\begin{align*}
\beta (g (\mu), \varepsilon) &= -\varepsilon g + \beta (g), \\
\beta (g) &= -\frac{1}{Z_g} \frac{d Z_g}{d \ln (\mu)},
\end{align*}
\]  

(2.11)

15
and the anomalous dimensions of mass operator

$$\gamma_m (g (\mu)) = \frac{1}{Z_m} \frac{dZ_m}{d \ln (\mu)}. \quad (2.12)$$

Calculating to two loop accuracy we get

$$\beta (g) = - \frac{g^3}{16 \pi^2} \beta_0 - \frac{g^5}{(16 \pi^2)^2} \beta_1,$$

$$\gamma_m (\alpha_s) = \frac{\alpha_s}{4 \pi} \gamma_m^{(0)} + \left( \frac{\alpha_s}{4 \pi} \right)^2 \gamma_m^{(1)}, \quad (2.13)$$

where

$$\beta_0 = \frac{11 N_c - 2 N_f}{3}, \quad \beta_1 = \frac{34}{3} N_c^2 - \frac{10}{3} N_c N_f - 2 C_F N_f,$$

$$\gamma_m^{(0)} = 6 C_F, \quad \gamma_m^{(1)} = C_F \left( 3 C_F + \frac{97}{3} N_c - \frac{10}{3} N_f \right),$$

$$\alpha_s (\mu) = \frac{g^2 (\mu)}{4 \pi}, \quad C_F = \frac{N_c^2 - 1}{2 N_c}. \quad (2.14)$$

The solutions for $\alpha_s (\mu)$ and $m (\mu)$ are [39]

$$\alpha_s (\mu) = \frac{4 \pi}{\beta_0 \ln \left( \frac{\mu^2 / \Lambda_{\overline{\text{MS}}}^2}{\Lambda_{\overline{\text{MS}}}^2} \right)} \left[ 1 - \frac{\beta_1}{\beta_0^2} \ln \left( \frac{\ln \left( \frac{\mu^2 / \Lambda_{\overline{\text{MS}}}^2}{\Lambda_{\overline{\text{MS}}}^2} \right)}{\beta_0^2} \right) \right], \quad (2.15)$$

$$m (\mu) = m (\mu_0) \left[ \frac{\alpha_s (\mu)}{\alpha_s (\mu_0)} \right]^{\frac{\gamma_m^{(0)}}{2 \beta_0}} \left[ 1 + \left( \frac{\gamma_m^{(1)}}{2 \beta_0} - \frac{\beta_1 \gamma_m^{(0)}}{2 \beta_0^2} \right) \frac{\alpha_s (\mu) - \alpha_s (\mu_0)}{4 \pi} \right]. \quad (2.16)$$

Here, $\Lambda_{\overline{\text{MS}}}$ is a characteristic scale both for QCD and the used $\overline{\text{MS}}$ scheme and depends also on the number of effective flavors present in the $\beta_0$ and $\beta_1$. An $\alpha_s^{(5)} (M_Z) = 0.118 \pm 0.005$ corresponds to $\Lambda_{\overline{\text{MS}}}^{(5)} = 225^{+70}_{-57}$ MeV at NLO. The interesting thing about this is that it emerges without making reference to any dimensional quantity and would be present also in a theory with completely massless particles. Both $\beta_0$ and $\gamma_m^{(0)}/2\beta_0$ are positive even if we have three colors and six active flavors in QCD. This leads to asymptotic freedom as the coupling tends to zero with increasing $\mu$. The pole at $\Lambda_{\overline{\text{MS}}}$ signals the breakdown of perturbation theory but gives a plausible argument for confinement. Similarly, the mass $m (\mu)$ decreases with increase of $\mu$.

One of the useful application or renormalization group is the summation of large logarithms.
To see this we re-express $\alpha_s$ given in Eq. (2.15) as

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{v(\mu)} \left[ 1 - \frac{\beta_1}{\beta_0} \frac{\alpha_s(\mu_0)}{4\pi} \ln \frac{\mu^2}{\mu_0^2} \ln \frac{\mu}{\mu_0} \right],$$

(2.17)

with

$$v(\mu) = 1 - \frac{\alpha_s(\mu_0)}{4\pi} \ln \frac{\mu_0^2}{\mu^2}. \quad (2.18)$$

If one expand the leading order term of Eq. (2.17) in $\alpha_s(\mu_0)$ we get

$$\alpha_s(\mu) = \alpha_s(\mu_0) \sum_{m=0}^{\infty} \left( \beta_0 \frac{\alpha_s(\mu_0)}{4\pi} \ln \frac{\mu_0^2}{\mu^2} \right)^m. \quad (2.19)$$

Thus the solution of RGE automatically sums the logarithms $\ln \left( \frac{\mu_0^2}{\mu^2} \right)$ which get large for $\mu \ll \mu_0$. Generally, solving RGE to order $n$ sums in $\alpha_s(\mu)$ all terms of the form

$$\alpha_s(\mu_0)^{m+1} \left( \alpha_s(\mu_0) \ln \frac{\mu_0^2}{\mu^2} \right)^k, \quad 0 \leq m \leq n, \ n \in \mathbb{N}_0. \quad (2.20)$$

This is particularly useful if, though $\alpha_s(\mu_0)$ is smaller than one, the combination $\alpha_s(\mu_0) \ln \left( \frac{\mu_0^2}{\mu^2} \right)$ is closer or even larger than one. Then the large logarithms would spoil the convergence of the perturbation series.

### 2.3 Operator Product Expansion

Weak decays of hadrons are mediated through weak interactions of quarks, whose strong interactions, binding quarks into hadrons, are characterized by typical hadronic energy scale $O(1\text{GeV})$ which is much lower than the scale of weak interactions: $O(M_{Z,W})$. In dimensional regularization for example we encounter logarithms of ratios of any scale with the renormalization scale $\mu$. As we have seen in the last section that these large logarithms can be summed systematically using RGE. It is alarming when we have the energy scale which is $O(1\text{GeV})$. Here, even without large logarithms the strong coupling $\alpha_s$ is too large for perturbation theory to make sense. Our goal is therefore to derive an effective low energy theory describing the weak interactions of quarks. The theoretical tool for this purpose is Operator Product Expansion (OPE) [40–42].

Consider the quark level transition $b \to cs\bar{u}$. Disregarding QCD effects for the moment, the
Figure 2.3: Replacing a $W$ propagator with an effective four-fermion vertex.

The corresponding tree-level $W$-exchange amplitude is given by

$$ A(b \rightarrow cs\bar{u}) = -\frac{G_F}{\sqrt{2}} V_{cb} V_{us}^{*} \left( \frac{m_W^2}{k^2 - M_W^2} \right) (\bar{s}u)_{V-A} (\bar{c}u)_{V-A} $$

$$ = \frac{G_F}{\sqrt{2}} V_{cb} V_{us}^{*} \left( \bar{s}u \right)_{V-A} (\bar{c}u)_{V-A} + \mathcal{O} \left( \frac{k^2}{M_W^2} \right), \quad (2.21) $$

where

$$ (\bar{q}_1 q_2)_{V-A} \equiv q_1 \gamma_{\mu} (1 - \gamma_5) q_2. $$

Since $k$, the momentum transfer through the $W$ propagator, is very small as compared to $M_W$, we can safely neglect the terms $\mathcal{O} \left( k^2 / M_W^2 \right)$. The $W$ propagator then quasi shrinks to a point (see Fig. 2.3) and we obtain an effective four fermion interaction. This simple example illustrates the basic idea of OPE: the product of two charged current operators is expanded into a series of local operators, whose contributions are weighted by effective coupling constants, the Wilson coefficients. The Wilson coefficients in this example is simply one.

### 2.4 Effective Hamiltonian

The result (2.21) can also be derived from effective Hamiltonian

$$ \mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{cb} V_{us}^{*} (\bar{s}u)_{V-A} (\bar{c}u)_{V-A} + \text{operator of higher dimension}, \quad (2.22) $$

where the operators of higher dimensions correspond to the terms $\mathcal{O} \left( k^2 / M_W^2 \right)$ in Eq. (2.22) and can likewise be neglected. In the effective theory the $W$ boson is removed as an explicit.
dynamical degree of freedom. It is “integrated out” or “contracted out” using the language of
the path integral or canonical operator formalism, respectively. One can proceed in a completely
analogous way with the heavy quarks. This leads to effective $f$ quark theories where $f$ denotes
the “active” quarks, i.e. those that have not been integrated out.

### 2.4.1 Operator Product Expansion and Short Distance QCD Effects

If we include also short distance QCD or electroweak corrections more operators have to be
added to the effective Hamiltonian which we generalize to

$$
\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_i V_{\text{CKM}}^i C_i(\mu) Q_i(\mu),
$$

(2.23)

$V_{\text{CKM}}^i$ denotes the CKM structure of the particular operator. Now the amplitude for the decay
of any meson $M$ to a final state $F$ can be written as

$$
A(M \to F) = \langle F | \mathcal{H}_{\text{eff}} | M \rangle = \frac{G_F}{\sqrt{2}} \sum_i V_{\text{CKM}}^i C_i(\mu) \langle F | Q_i(\mu) | M \rangle.
$$

(2.24)

The Wilson coefficients $C_i(\mu)$ can be interpreted as the coupling constants for the effective
interaction terms $Q_i(\mu)$. They are calculable functions of $\alpha_s$, $M_W$, and the renormalization
scale $\mu$. To any order in perturbation theory the Wilson coefficients can be obtained by matching
the full theory with the effective one. This simply is the requirement that the amplitude in the
effective theory should reproduce the corresponding amplitude in the full theory. Hence, we
first have to calculate the amplitude in the full theory and then the matrix elements $\langle Q_i \rangle$. In
this second step the resulting expressions may, even after quark field renormalization, be still
divergent. Consequently we have to perform an operator renormalization

$$
Q_i^{(0)} = Z_{ij} Q_j,
$$

where $\langle Q_i \rangle^{(0)}$ denotes the unrenormalized operator. This notation is somewhat misleading.
What actually is renormalized is not the operator but the operator matrix elements, or, even
more exactly, the amputated Green functions $\langle Q_i \rangle$. Then we have to include the renormalization

19
constant $Z_q^{1/2}$ for each of the four external fields:

$\langle Q_i \rangle^{(0)} = Z_q^{-2} Z_{ij} \langle Q_j \rangle$.

Generally, $Z_{ij}$ is a matrix so that operators carrying the same quantum numbers can mix under renormalization. The operators of given dimensions mix only into operators of the same or of lower dimensions. Again, the divergent parts of the renormalization constant are determined from the requirement that the amplitude in the effective theory is finite. The finite part in $Z_{ij}$ on the other hand defines a specific renormalization scheme. In a third step we extract the Wilson coefficients by comparing the full and the effective theory amplitude. These are the Wilson coefficients at some fixed scale $\mu_0$. A condition here is that the external states in the full and the effective theory have to be treated in the same manner. Especially the same regularization and renormalization schemes have to be used on both sides.

The effective Hamiltonian $H_{\text{eff}}$ is projected onto some external states to obtain the complete amplitude. But the Wilson coefficients appear already at the level of the effective Hamiltonian and so they are independent of these external states. When determining the Wilson coefficients, any external, even unphysical, state can be used and the coefficient functions represent the short-distance structure of the theory. Because they depend for example on the masses of the particles that were integrated out, they contain all information about the physics at the high energy scale. The long-distance contribution, on the other hand, is parametrized by the process-dependent matrix elements of the local operators. This factorization of SD and Long distance (LD) dynamics is one of the salient features of OPE. One can calculate the Wilson coefficients in perturbation theory and the hadronic matrix elements by means of some non-perturbative technique like $1/N$ expansion, sum rules, or lattice gauge theory. Especially to use the latter one, a separation of the SD part is essential for todays lattice sizes. The factorization can be visualized with large logarithms $\ln (M_{12}^2/m_q^2)$ being split into $\ln (M_{12}^2/\mu^2) + \ln (\mu^2/m_q^2)$. In doing so, the first logarithm will be retrieved in the Wilson coefficients and the second one in the matrix elements. From this point of view the renormalization scale $\mu$ can be interpreted as the factorization scale at which the full contribution is separated into a low energy and a high energy part [16, 20].
OPE and Renormalization Group

A typical scale used to calculate the hadronic matrix elements of local operators is low compared to $M_W$. For $B$ decays we would choose $\mu = \mathcal{O}(m_B)$. Therefore, the logarithm in $(M_W^2/\mu^2)$ contained in the Wilson coefficient is large. So why not use the powerful technique of summing large logarithms developed in section 2.2? In order to do so we have to find the renormalization group equations for the Wilson coefficients and solve them. But so far the Wilson coefficients were not renormalized at all. If we remember, however, that in the effective Hamiltonian the operators, which have to be renormalized, are accompanied always by the appropriate Wilson coefficient we can shuffle the renormalization as well to the Wilson coefficients. Let us start with the Hamiltonian of the effective theory with fields and coupling constants as bare quantities, which are renormalized according to

\begin{align*}
q^{(0)} &= Z_q^{1/2} q, \\
C_i^{(0)} &= Z_{ij}C_j.
\end{align*}

So the Hamiltonian (2.23) becomes

\begin{align*}
\mathcal{H}_{\text{eff}} \propto C_i^{(0)} Q_i^{(0)} \left(q^{(0)}\right) \\
&= Z_{ij}^c C_j Z_q^2 Q_i \\
&= C_i Q_i + (Z_q^2 Z_{ij}^c - \delta_{ij}) C_j Q_i,
\end{align*}

i.e. it can be written in terms of the renormalized couplings $C_i$ and fields $Q_i$ plus counter terms. By calculating the amplitude with the Hamiltonian (2.26) including the counter terms, we get the finite renormalized results

\begin{align*}
Z_q^2 Z_{ij}^c C_j \langle Q_i \rangle^{(0)} = C_j \langle Q_j \rangle.
\end{align*}

Comparing the last two equations we finally find the relation

\begin{align*}
Z_{ij}^c = Z_{ji}^{-1}.
\end{align*}
This result is very useful in deriving the renormalization group equations for the couplings $C_i$, which are

$$\frac{dC_i(\mu)}{d\ln \mu} = \gamma_{ji}(\mu) C_j(\mu),$$  \hspace{1cm} (2.29)

with the anomalous dimension matrix for the operators

$$\gamma_{ji}(\mu) = Z_{ik}^{-1} \frac{dZ_{kj}}{d\ln \mu}. \hspace{1cm} (2.30)$$

Now, to calculate the numerical values of these anomalous dimension matrix we will use the technique we have developed for the running mass in the previous section. To leading order this is in fact possible. But if we want to go to next-to-leading-order accuracy we run into problems, because the matrices $\gamma_{ij}^{(0)}$ and $\gamma_{ij}^{(1)}$ in the perturbation expansion

$$\gamma_{ij} = \gamma_{ij}^{(0)} \frac{\alpha_s}{4\pi} + \gamma_{ij}^{(1)} \left( \frac{\alpha_s}{4\pi} \right)^2 + \mathcal{O}(\alpha_s^3),$$  \hspace{1cm} (2.31)

do not commute with each other. Let us instead formally write the solution for the Wilson coefficients with an evolution matrix $U(\mu, \mu_0)$

$$C_i(\mu) = U_{ij}(\mu, \mu_0) C_j(\mu_0).$$  \hspace{1cm} (2.32)

The leading order evolution matrix is

$$U^{(0)}(\mu, \mu_0) = \begin{bmatrix} \alpha(\mu) & \gamma_{ij}^{(0)} \frac{\alpha_s}{\alpha(\mu_0)} \end{bmatrix}$$

$$= V \begin{bmatrix} \alpha(\mu_0) & \gamma_{ij}^{(0)} \frac{\alpha_s}{\alpha(\mu_0)} \end{bmatrix}_D V^{-1}, \hspace{1cm} (2.33)$$

where $V$ is the matrix that diagonalizes $\gamma^{(0)\gamma}$

$$\gamma_{ij}^{(0)} = V^{-1} \gamma^{(0)\gamma} V, \hspace{1cm} (2.34)$$
and $\gamma^{(0)T}$ is the vector containing the eigenvalues of $\gamma^{(0)}$. For the next-to-leading order solution we make the clever ansatz

$$U(\mu, \mu_0) = \left[ 1 + \frac{\alpha_s(\mu)}{4\pi} J \right] U^{(0)}(\mu, \mu_0) \left[ 1 - \frac{\alpha_s(\mu_0)}{4\pi} J \right], \quad (2.35)$$

which proves to solve (2.30) if [43]

$$J = VH\nu^{-1}, \quad (2.36)$$

where the elements of $H$ are

$$H_{ij} = \delta_{ij} \gamma_i^{(0)} \frac{\beta_1}{2\beta_0} - \frac{G_{ij}}{2\beta_0 + \gamma_i^{(0)} - \gamma_j^{(0)}}, \quad (2.37)$$

with

$$G = V^{-1} \gamma^{(1)T} V. \quad (2.38)$$

A point to note is that physical quantities must clearly be independent of the renormalization scheme chosen, at NLO unphysical quantities, like the Wilson coefficients and the anomalous dimensions, depend on the choice of the renormalization scheme. To ensure a proper cancellation of this scheme dependence in the product of Wilson coefficients and matrix elements the same scheme has to be used for both. In order to uniquely define a renormalization scheme it is not sufficient to quote only the regularization and renormalization procedure but one also has to choose a specific form for the so-called evanescent operators. These are operators which exist in $D \neq 4$ dimensions but vanish in $D = 4$ [35, 44, 45].

So what we have in our hands so far? We have determined the Wilson coefficients at a scale $\mu_0$ via a matching procedure. These are the initial conditions for the evolution from $\mu_0$ down to an appropriate low energy scale $\mu$ via $U(\mu, \mu_0)$ which sums large logarithms. Herefore, we had to determine the anomalous dimensions of the operators and solve the renormalization group equation for the Wilson coefficients. We thus arrive at a RG improved perturbation theory and officially do not speak any more of leading (LO) and next-to-leading order (NLO) but rather of leading (LL) and next-to-leading logarithmic order (NLL). Yet, we might carelessly use the terms synonymously. In our task to evaluate weak decay amplitudes involving hadrons in the framework of a low energy effective theory we then only lack the calculation of the hadronic matrix elements $\langle Q_j(\mu) \rangle$. This, however, is a highly non-trivial problem and lot of work has been done on this issue in Refs. [17, 18] and the references therein.
The NLO formulas are very complicated and this give birth to a question: why at all going to next-to-leading-order accuracy? To argue about this question one can say, though all these calculations imply the evaluation of two or even more loop diagrams which are technically very challenging but they are very important. First of all we can test the validity of the renormalization group improved perturbation theory. Then, of course, we hope that the theoretical uncertainties get reduced. One particular issue is the residual renormalization scale dependence of the result. The scale $\mu$ enters for example in $\alpha_s(\mu)$ or the running quark masses, in particular $m_t(\mu)$, $m_\ell(\mu)$, and $m_c(\mu)$. In principle, a physical quantity cannot depend on the renormalization scale. But this symmetry is broken because we have to truncate the perturbative series at some fixed order. The renormalization scale dependence of Wilson coefficients and operator matrix elements cancels only to the order of perturbation theory included in the calculation. Therefore, one can use the remaining scale ambiguity as an estimate for the neglected higher order corrections. Usually one varies $\mu$ between half and twice the typical scale of the problem, i.e. $m_b/2 < \mu < 2m_b$ for $B$ decays and we will also use it in the work we are going to present in chapters 5 and 6. Going to NLO not only reduces these scale ambiguities but also the renormalization scheme dependence of the Wilson coefficients appears at NLO for the first time. Only if we properly match the long distance matrix elements, obtained for example from lattice calculations, to the short distance contributions, these unphysical scheme dependences will cancel. Another issue is that the QCD scale $\Lambda_{\overline{MS}}$, which can be extracted from various high energy processes, cannot be used meaningfully in weak decays without going to NLO [16].
Chapter 3

Approaches to Non-Leptonic Decays

In the last chapter we developed some basic concepts for doing calculations in elementary particle physics both at leading order (LO) as well as at NLO. Here we will talk about some tools to deal with the calculation of non-leptonic decays. First, we will discuss the effective $b \to s\gamma$ Hamiltonian and the nomenclature of different operators appearing in this Hamiltonian. Then we talk about the effective theories like Heavy Quark Effective Theory (HQET), QCD factorization as well as the Large Energy Effective Theory (LEET) which is used in the calculation done in Part III of this dissertation. We will end this chapter by giving brief remarks about the LEET symmetries, which are useful to relate different form factors, as well as their breaking.

3.1 The Effective $b \to s\gamma$ Hamiltonian

Starting with a usual framework of an effective theory with five quarks, obtained by integrating out the heavier degrees of freedom which in the Standard Model are top quark and the $W$ boson. The effective Hamiltonian includes a complete set of dimension-6 operators relevant for the process $b \to s\gamma$:

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_{p=u,c} \lambda_p^{(s)} \left[ C_1 O_1^p + C_2 O_2^p + \sum_{i=3,\ldots,8} C_i O_i \right]$$

(3.1)

where

$$\lambda_p^{(s)} = V_{ps}^* V_{pb}$$

(3.2)
Figure 3-1: The diagrams in full theory from which the operator basis for $b \rightarrow s \gamma$ originates. The cross in diagram (f) means a mass-insertion. It indicates that magnetic penguins originate from the mass term on the external line in the usual QCD or QED penguin diagrams.
The operators originate from the diagrams in Fig. 3-1 and are given by

\begin{align}
O_1^p &= (\bar{s}_i p_j) v_{-A} (\bar{q}_j b_i) v_{-A}, \\
O_2 &= (\bar{s}_i c_j) v_{-A} (\bar{q}_j b_j) v_{-A}, \\
O_3 &= (\bar{s}_i b_i) v_{-A} \sum_q (\bar{q}_j q_j) v_{-A}, \\
O_4 &= (\bar{s}_i b_j) v_{-A} \sum_q (\bar{q}_j q_j) v_{-A}, \\
O_5 &= (\bar{s}_i b_i) v_{-A} \sum_q (\bar{q}_j q_j) v_{+A}, \\
O_6 &= (\bar{s}_i b_j) v_{-A} \sum_q (\bar{q}_j q_j) v_{+A}, \\
O_7 &= \frac{e m_s^2}{8 \pi^2} \bar{s}_i \sigma_{\mu \nu} (1 + \gamma_5) b_i F_{\mu \nu}, \\
O_8 &= \frac{g_s m_b}{8 \pi^2} \bar{s}_i \sigma_{\mu \nu} (1 + \gamma_5) T^a_{ij} b_j G_{\mu \nu}^a. 
\end{align}

(3.3)

with \(i, j\) the color indices, \(e\) and \(g_s\) the coupling constants of electromagnetic and strong interactions and \(F_{\mu \nu}\) and \(G_{\mu \nu}\) the photonic and gluonic fields tensors, respectively. As \(m_b\) is much larger than \(m_s\) therefore the contribution from \(m_s (1 - \gamma_5)\) is ignored in operators \(O_7\) and \(O_8\). The operators given in Eq. (3.3) are all possible gauge invariant structures with the following properties:

- They have the correct quantum numbers to contribute to \(b \to s \gamma\),

- they are compatible with the symmetries of electroweak interactions,

- they cannot be transformed into each other by applying equations of motion.

These all operators have special names due to their nomenclature. \(O_1\) and \(O_2\) are called \textbf{Current-Current} operators, \(O_3, \ldots, O_6\) are \textbf{penguin} operators and \(O_7\) and \(O_8\) \textbf{electromagnetic} and \textbf{chromomagnetic} penguin operator, respectively. The coefficients \(C_7\) and \(C_8\) are negative in the Standard Model, which is a choice generally adopted in the literature.

For the construction of effective Hamiltonian defined in Eq. (3.3) the fact that hadrons are bound states of quarks is not relevant. However, once we want to calculate a physical process involving hadrons, we have to deal with non-perturbative matrix elements for which there are different methods, depending on the chosen energy scale. In next few sections we will briefly discuss these methods one by one.
3.2 Heavy Quark Effective Theory

The QCD Lagrangian describing a quark $Q$ of mass $m_Q$ and its interactions with gluon is given by

$$ L = \bar{\Psi}_Q i D_\mu \gamma^\mu \Psi_Q - m_Q \bar{\Psi}_Q \Psi_Q, $$

(3.4)

with

$$ D_\mu = \partial_\mu - ig_s T^a \Lambda_\mu. $$

The Lagrangian (3.4) does not have manifest heavy quark spin-flavor symmetry as $m_Q \to \infty$. It is convenient to use an effective field theory for QCD in which heavy quark symmetry is manifest in the above limit. This effective field theory is known as heavy quark effective theory (HQET) and it describes the dynamics of hadrons containing single heavy quark.

In the heavy-quark limit ($m_Q \to \infty$), the velocity $v_\mu$ of the quark is conserved and its four momentum may be decomposed as:

$$ p_\mu = m_Q v_\mu + k_\mu, \quad \text{with } v^2 = 1. $$

(3.5)

where $m_Q v_\mu$ and $k_\mu$ are on-shell and off-shell parts respectively.

The components of residual momentum $k$ are much smaller than $m_Q$ and are changed by interactions of the heavy quark with light degrees of freedom by $\Delta k \sim \Lambda_{\text{QCD}}$. The large and small component fields

$$ h_v (x) \equiv e^{i m_Q x} \frac{1 + \gamma^\mu}{2} \Psi_Q (x) $$

(3.6)

and

$$ H_v (x) \equiv e^{i m_Q x} \frac{1 - \gamma^\mu}{2} \Psi_Q (x), $$

(3.7)

with the properties $\gamma h_v = h_v$ and $\gamma H_v = H_v$, respectively. The quark field in terms of the new fields can be expressed as

$$ \Psi_Q (x) = e^{-i m_Q x} \left( h_v (x) + H_v (x) \right). $$

(3.8)

One may split the covariant derivative $D$ into "longitudinal" and "transverse" parts:

$$ D_\perp = D^\mu - v^\mu v \cdot D, \quad \text{with } v \cdot D_\perp = 0, \quad \{ \mathcal{P}_\perp, \gamma \} = 0, $$

28
Using relations as $\bar{h}_v H_v = 0$ and $\bar{h}_v \not{\partial} H_v = 0$, the Lagrangian takes the form

$$\mathcal{L} = \bar{h}_v i (v \cdot D) h_v - \bar{H}_v i (i v \cdot D + 2m_Q) H_v + \bar{h}_v i \not{\partial} H_v + \bar{H}_v i \not{\partial} h_v.$$  \hspace{1cm} (3.9)

Thus equation of motion for $H_v$ becomes

$$H_v(x) = \frac{1}{2m_Q + i v \cdot D} i \not{\partial} h_v.$$  

This allows us, on a classical level, to eliminate the heavy degree of freedom $H_v$ from the Lagrangian:

$$\mathcal{L}_{\text{eff}} = \bar{h}_v i (v \cdot D) h_v + \bar{h}_v i \not{\partial} \left( \frac{1}{2m_Q + i v \cdot D} i \not{\partial} h_v \right)$$

$$= \bar{h}_v i (v \cdot D) h_v + \frac{1}{2m_Q} \sum_{n=0}^{\infty} \bar{h}_v i \not{\partial} \left( \frac{-i v \cdot D}{2m_Q} \right)^n i \not{\partial} h_v.$$  \hspace{1cm} (3.10)

The above equation can also be written as

$$\mathcal{L}_{\text{eff}} = \bar{h}_v i (v \cdot D) h_v + \frac{1}{2m_Q} \bar{h}_v (i \not{\partial})^2 h_v + \frac{g_s}{4m_Q} \bar{h}_v \sigma_{\mu \nu} G^{\mu \nu} h_v + \mathcal{O}(1/m_Q^2).$$  \hspace{1cm} (3.11)

In the limit $m_Q \rightarrow \infty$, only the term

$$\mathcal{L}_{\infty} = \bar{h}_v i (v \cdot D) h_v$$  \hspace{1cm} (3.12)

survives. There appear neither Dirac matrices nor quark masses in this equation. For $m_Q \rightarrow \infty$, the interactions of heavy quarks and gluons become independent of the spin of the quark. Furthermore, when extending the theory to more than one heavy quark moving at the same velocity, the Lagrangian $\mathcal{L}_{\infty}$ is symmetric under rotations in the flavor space. This is heavy quark flavor symmetry [46]. The spin-flavor symmetry leads to many interesting relations between the properties, especially the spectroscopy, of hadrons containing a heavy quark but we do not want to go into the detail of that because, these are out of the scope of this thesis.
3.3 QCD Factorization

QCD factorization, introduced by Beneke, Buchalla, Neubert, and Sachradja [11] uses the fact that the typical scale of $B$ meson decay is of order $m_B$ and therefore much larger than $\Lambda_{\text{QCD}}$, the long distance scale where non-perturbative QCD takes over. This time, something is factorized in the general sense of QCD applications: Namely the long-distance dynamics in the matrix elements and the short-distance interactions that depend only on the large scale $m_B$. And again, the short-distance contributions can be computed in a perturbative expansion in the strong coupling $\alpha_s (O (m_b))$. The long-distance part has still to be computed non-perturbatively or determined experimentally. However, these non-perturbative parameters are mostly simpler in structure than the original matrix element and they are process independent.

3.3.1 The factorization formula

We consider $B \rightarrow M_1 M_2$ in the heavyquark limit and differentiate between decays into final states containing a heavy and a light meson $H_1$ and $L_2$ or two light mesons $L_1$ and $L_2$. Up to the power corrections of order $\Lambda_{\text{QCD}}/m_b$ the transition matrix elements of an operator $O_i$ in the weak effective Hamiltonian is given by

$$
\langle L_1 L_2 | O_i | \bar{B} \rangle = \sum_j F_j^{B \rightarrow L_1} (m_B^2) \int_0^1 du T_{ij} (u) \Phi_{L_2} (u)
$$

$$
+ \sum_k F_k^{B \rightarrow L_2} (m_B^2) \int_0^1 dv T_{ik} (v) \Phi_{L_1} (v)
$$

$$
+ \int_0^1 d\xi du dv T^{III} (\xi, u, v) \Phi_B (\xi) \Phi_{L_1} (u) \Phi_{L_2} (v),
$$

(3.13)

$$
\langle H_1 L_2 | O_i | \bar{B} \rangle = \sum_j F_j^{B \rightarrow H_1} (m_B^2) \int_0^1 du T_{ij} (u) \Phi_{L_2} (u).
$$

(3.14)

Here $F_j^{B \rightarrow M}$ denotes a $B \rightarrow M$ form factor and $\Phi_M$ is the light cone distribution amplitude (LCDA) for the quark-antiquark Fock state of meson $M$. These non-perturbative quantities are much simpler than the original non-leptonic matrix element. The form factors refer only to a relatively simple $B \rightarrow M$ transition matrix element of a local current whereas, LCDA $\Phi_M$ reflects universal properties of single meson state. Both can be obtained from experiments or can be calculated using some non-perturbative technique, like lattice QCD or QCD sum rules. $T_{ij} (u)$, $T_{ik} (v)$ and $T^{III} (\xi, u, v)$ are the hard-scattering functions. These are perturbatively
Figure 3-2: Graphical representation of the factorization formula for the $B$ meson decaying into two light mesons, e.g. $B^- \rightarrow \pi^0 K^-$. At leading order in $\Lambda_{QCD}/m_b$ there are no long-distance interactions between the system of $B$ meson and the meson that pick up the spectator quark, and the other final state meson.

calculable functions of light-cone momentum fractions $u$, $v$, and $\xi$ of the quark inside the final state mesons and the $B$ meson, respectively. We distinguish "type I" or "hard vertex" and "type II" or "hard spectator contributions. Eq. (3.13) is represented graphically in Fig. 3-2.

The factorization formula (3.13) simplifies when the spectator quark goes to a heavy meson. In this case the third term on the right-hand side of Eq. (3.13), which accounts for the hard interactions with the spectator quark can be dropped because it is power suppressed in the heavy-quark limit. In the opposite case, the factorization does not hold because the heavy meson is neither fast nor small and cannot be factorized from the $B \rightarrow M_1$ transition. At leading order in the heavy-quark expansion, the annihilation topologies do not contribute.

Now let us discuss the non-perturbative parameters, form-factors and light-cone distribution amplitude, involved in the factorization formula (3.13).

Form factors

A form factor is a function of scalar variables accompanying the independent terms in the most general decomposition of the matrix element of a current consistent with Lorentz and gauge invariance. In the context of QCD factorization we often need the matrix element of the vector current which is conventionally parameterized by two scalar form factors

$$\langle P (k) | \bar{q} \gamma^\mu q | B (p) \rangle = F_+^{B \rightarrow P} (q^2) (p^\mu + k^\mu) + \left[ F_0^{B \rightarrow P} (q^2) - F_+^{B \rightarrow P} (q^2) \right] \frac{m_B^2 - m_P^2}{q^2} q^\mu, \quad (3.15)$$
where $q = p - k$. For $q^2 = 0$ the two form factors coincide, $F^{B \to P}_{+} (0) = F^{B \to P}_{0} (0)$. The two form factors describes the overlap of the $B$ meson and pseudoscalar meson $P$ during the weak decay. These form factors receives the leading contribution from soft gluon exchange and that is why they enter to the factorization formula as a non-perturbative input.

**Light-cone distribution amplitudes for light mesons**

Let us define the light cone components

$$k_\pm = \frac{k^0 \pm k^3}{\sqrt{2}},$$  \hspace{1cm} (3.16)

for any four vector $k^\mu = \begin{pmatrix} k_+, k_-, k_\perp \end{pmatrix}$. Let us construct a light pseudoscalar meson out of the on-shell constituent quarks in a spin singlet state and with no net transverse momentum:

$$|P (k) \rangle = \int \frac{du \, d^2 l_\perp}{\sqrt{u a} \, 16 \pi^3} \frac{1}{\sqrt{2}} \left( a_{i_1,1}^\dagger b_{i_2,1}^\dagger - a_{i_1,1}^\dagger b_{i_2,1}^\dagger \right) |0 \rangle \psi (u, l_\perp),$$  \hspace{1cm} (3.17)

with $u \equiv 1 - a$. Here, $a_{i_1}^\dagger (b_{i_1}^\dagger)$ creates a (anti)quark with momentum $l$ and spin up. The momenta $l_1$ and $l_2$ are fractions $u$ and $\bar{u}$ of the meson momentum $k$ plus a momentum $l_\perp$ perpendicular to meson momentum direction, which adds up to zero for both quarks. At leading twist, the light-cone wave function $\Phi_P$ can be obtained from

$$\int \frac{d^2 l_\perp}{16 \pi^3} \frac{1}{\sqrt{2 N}} \psi (u, l_\perp) \equiv - \frac{i f_P}{4 N} \Phi_P (u),$$  \hspace{1cm} (3.18)

with $f_P$ being the meson decay constant. The latter is normalized as $\int_0^1 du \Phi_P (u) = 1$ and has the asymptotic form $\Phi_P (u) = 6 u u$. One can write

$$\left< P (k) \left| q_i (z) \bar{q}_j (0) \right| 0 \right> = \frac{i f_P}{4 N} \bar{c}_{i_2} \left( \gamma_5 \gamma^\mu \right)_{\alpha \beta} \int_0^1 du e^{i k_\perp z} \Phi_P (u),$$  \hspace{1cm} (3.19)
with $i, j$ color and $\alpha$ and $\beta$ spinor indices. The leading twist LCDA for a vector meson with polarization $\varepsilon$ [47]:

$$
\left\langle V(k, \varepsilon) \mid q_i'(z) \bar{q}_j(0) \right\rangle = -\frac{f_{V1}^+}{4N_c} \delta_{ij} \int_0^1 du e^{i t \bar{u} \cdot z} \Phi_V^+(u) \\
- \frac{f_{V0}}{4N_c} \varepsilon_{ij} \int_0^1 du e^{i t \bar{u} \cdot z} \Phi_V^0(u) \\
\delta_{\alpha \beta} \int_0^1 du e^{i t \bar{u} \cdot z} g_{\perp}^\perp(u) \\
- \frac{1}{4} \left( \varepsilon_{\alpha} \lambda_{\mu' \nu'} \epsilon_{\sigma} k_{\lambda} z_{\mu} \gamma_{\nu} \gamma_5 \right)^\alpha_{\beta} \int_0^1 du e^{i t \bar{u} \cdot z} g_{\perp}^{(a)}(u).
$$

(3.20)

The light-cone wavefunction $\Phi_V^+(u)$ has an expansion in terms of Gegenbauer polynomials $C_n^{3/2}(2u - 1)$

$$
\Phi_V^+(u) = 6u \bar{u} - \sum_{n=1}^{\infty} a_n^V(\mu) C_n^{3/2}(2u - 1),
$$

(3.21)

where $C_n^{3/2}(x) = 3x, C_n^{3/2}(x) = \frac{3}{2}(5x^2 - 1)$, etc. The Gegenbauer moments $a_n^V(\mu)$ are multiplicatively normalized and they vanish logarithmically as the scale $\mu \to \infty$. This limit reduces $\Phi_V^+(u)$ to its asymptotic form $\Phi_V^+(u) = 6u \bar{u}$, which often is a reasonable first approximation. $\Phi_V^0(u)$, $g_{\perp}^\perp(u)$, and $g_{\perp}^{(a)}$ do not contribute to leading power if the mesons are transversely polarized.

In short we end the discussion of light meson LCDAs with the counting rules for the wave functions. Using the asymptotic form $\Phi_X(u) = 6u \bar{u}$ for $X = P$ and $X = L$ we count the endpoint region, where $u$ and $\bar{u}$ is of order $\Lambda_{QCD}/m_b$. Away from end point region the wave function is $O(1)$.

Light-cone distribution amplitude for $B$ meson

It is intuitive that light-cone distribution amplitudes for light mesons appear in non-leptonic decays. The relevance of light-cone distribution amplitudes for $B$ meson is less clear, because the spectator quark in $B$ meson is not energetic in the $B-$meson rest frame. So one can say that the $b$ quark carries largest part of $B$ mesons momentum $p$: $\not{p}_b^+ = \not{\xi} p^+ \approx p^+$, whereas for the spectator quark we have $l^+ = \xi p^+$ with $\xi (\Lambda_{QCD}/m_b)$.

Now, for most general decomposition of the light-cone distribution amplitude at leading order in $1/m_b$, we make use of the fact that in the $B-$meson rest frame only the upper two
components of the $b$-quark spinor are large. However, since the spectator quark is neither energetic nor heavy, no further restriction on the components of the spectator quark spinors exists. We then find that the $B$ meson is described by two scalar wave functions at leading power, which we can choose as [11, 48, 49]

$$\langle 0 \left| b_i(0)_{\alpha} \bar{q}_j(z)_{\beta} \right| B(p) \rangle = \frac{ig}{4N_c} \delta_{ij} [\gamma^\mu \gamma_5]_{\alpha\beta} \int_0^1 d\xi e^{i\xi p \cdot z} \left[ \Phi_{B_1}(\xi) + i \Phi_{B_2}(\xi) \right]_{\lambda\beta}, \quad (3.22)$$

where $n_-= (1, 0, 0, -1)$, and the normalization conditions are

$$\int_0^1 d\xi \Phi_{B_1}(\xi) = 1 \quad \int_0^1 d\xi \Phi_{B_2}(\xi) = 0. \quad (3.23)$$

Contrary to the distribution amplitudes of light mesons, the $B$-meson distribution amplitudes are poorly known, even theoretically. At scales much larger than $m_b$, the $B$ meson is like a light meson and the distribution amplitude should approach a symmetric form. At scales of $m_b$ and smaller, one expects the distribution amplitude to be highly asymmetric with $\xi = O(A_{QCD}/m_b)$.

Thus with this introduction of the QCD factorization we can say that we really have a method based on solid theoretical grounds namely the heavy quark limit. It is widely used to study the hadronic and Exclusive radiative $B$ meson decays and detail discussion about this issue can be found in Ref. [16].

### 3.4 Large Energy Effective Theory (LEET)

LEET, originally proposed by Dugan and Grinstein [12], is applicable to exclusive semileptonic, radiative and rare heavy-to-light transitions in the region where the energy release $E$ is large compared to the strong interaction scale and to the mass of the final hadron. They introduced it to study factorization of non-leptonic matrix elements in decays like $B \rightarrow D^{(*)}\pi, D^{(*)}\rho...$, where the light meson is emitted by the $W$-boson. In this case, both quarks constituting the light energetic meson are fast. However, Aglietti et al. [13] then argued that such a situation could not be described by LEET, as the relative transverse momentum of the fast quarks may be hard. They proposed to use instead the LEET effective theory, a variant of LEET which takes into account hard transverse degrees of freedom. This seems to be similar to the description of the heavy quark systems: HQET is the appropriate theory for the heavy-light hadrons, while NRQCD (Non-Relativistic QCD) should be used for the quarkonia. Conversely, Aglietti et al.
found that LEET could be used in semi-inclusive non-leptonic decays such as \( B \rightarrow DX_\nu \), where factorization should hold at the leading order [13].

In this section we summarize the LEET in a systematic way and for its detail we will refer to [14]. From now on, we will refer to high energy exclusive heavy-to-light decays, and consider only the ground state mesons. The appropriate kinematical variables for such decays are:

- The four-momentum \( p \), mass \( M \) and four-velocity \( v \) of the heavy hadron

\[
p = Mv
\]  
(3.24)

It is clear from the above equation that the heavy quark is treated classically as in HQET.

- The four-vector \( n \) and the scalar \( E \) defined by

\[
p' \equiv En, \quad v \cdot n \equiv 1
\]  
(3.25)

where \( p' \) is the four-momentum of the light hadron, \( p'^2 = m'^2 \). Thus

\[
E = v \cdot p'
\]  
(3.26)

is just the energy of the light hadron in the rest frame of the heavy hadron. This shows that in case of the heavy hadrons the momentum \( p \) of the light quarks also scale with the large mass.

The approximation we will consider are the Heavy Mass to Large Energy limit:

\[
(\Lambda_{QCD}, m') \ll (M, E), \quad \text{with } v \text{ and } n \text{ fixed.}
\]  
(3.27)

Note that we do not assume anything for the ratios \( E/M \) and \( \Lambda_{QCD}/m' \). As \( n^2 = m'^2/E^2 \rightarrow 0 \), \( n \) becomes light-like in the above limit. In the rest frame of \( v \), with the \( z \) direction along \( p' \), one has simply

\[
v = (1, 0, 0, 0), \quad n \simeq (1, 0, 0, 1).
\]  
(3.28)

In a general frame one has the normalization conditions

\[
v^2 = 1, \quad v \cdot n = 1, \quad n^2 \simeq 0.
\]  
(3.29)
In a decay like \( B \to \pi \), not too close from \( q^2 = q_{\text{max}}^2 = (m_B - m_\pi)^2 \), the final active quark gets a very large energy and should form with the spectator a hadron of finite mass. Thus, neglecting as said above hard spectator effects, the momentum \( l \) of the active quark is close to the momentum of the hadron:

\[
l = En + k, \quad \text{with} \quad k \sim \ll E \quad \text{is the residual momentum.} \tag{3.30}
\]

As the residual momentum is very small therefore the particle has the small fluctuations about its worldline. Our goal is now to derive the LEET Lagrangian from the QCD one in the limit (3.30). We would like to separate the large components of the quark field from the small ones which, corresponding to the negative energy solutions, should be suppressed by \( 1/E \). This is in analogy with the derivation of HQET Lagrangian [50]. Define the projectors

\[
P_+ = \frac{\gamma^\mu \gamma_5}{2}, \quad P_- = \frac{\gamma^\mu}{2}, \tag{3.31}
\]

which indeed verify from Eq. (3.29)

\[
P_\pm^2 = P_\pm, \quad P_\pm P_\mp = 0, \quad P_+ + P_- = \frac{\{\gamma^\mu, \gamma_5\}}{2} = 1. \tag{3.32}
\]

One may define two two-component projected fields \( q_\pm(x) \) from the full, four component quark field \( q(x) \)

\[
q_\pm(x) \equiv e^{iE_0 x} P_\pm q(x). \tag{3.33}
\]

Thus from the projector properties one has

\[
q(x) = e^{-iE_0 x} [q_+(x) + q_-(x)] \tag{3.34}
\]

with

\[
P_\pm q_\pm = q_\pm, \quad P_\mp q_\pm = 0 \tag{3.35}
\]

and

\[
\bar{q}_\pm P_\mp = \bar{q}_\pm, \quad \bar{q}_\mp P_\pm = 0. \tag{3.36}
\]

Thus the QCD Lagrangian for the quark \( q \), \( \mathcal{L}_{\text{QCD}} = \bar{q}(i\gamma^\mu - m_q)q \) can be expressed in terms of
the \( q_{\pm} \) fields:

\[
\mathcal{L}_{\text{QCD}} = \bar{q}_+ i \not{n} \cdot \not{D} q_+ + \bar{q}_- (i \not{\partial} - m_q) q_- - \bar{q}_- (i \not{\partial} - m_q) q_+ + \bar{q}_- \not{\gamma} (2E + 2iv \cdot D - in \cdot D) q_- \]  \tag{3.37}

The equation of motion \((i \not{\partial} - m_q) q(x) = 0\), projected by \( P_{\pm} \), reads

\[
\not{\gamma} \not{\gamma} \not{n} \cdot D q_+ + [i \not{\partial} - m_q - \not{\gamma} (2iv \cdot D - in \cdot D)] q_- = 0, \]  \tag{3.38}

\[
(i \not{\partial} - m_q - \not{\gamma} \not{n} \cdot D) q_+ - \not{\gamma} (2E + 2iv \cdot D - in \cdot D) q_- = 0. \]  \tag{3.39}

The latter equation can formally be solved to express \( q_- \) in terms of \( q_+ \):

\[
q_-(x) = (2E + 2iv \cdot D - in \cdot D + i\epsilon)^{-1} \not{\gamma} (i \not{\partial} - m_q - \not{\gamma} \not{n} \cdot D) q_+(x). \]  \tag{3.40}

Thus the field \( q_-(x) \), corresponding to the negative energy solutions is of order \( 1/E \) with respect to \( q_+(x) \). Physically this means that the pair creation is suppressed in the effective theory.

To summarize we have obtained the result:

\[
\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{LEET}} + \mathcal{O}(1/E) \]  \tag{3.41}

with

\[
\mathcal{L}_{\text{LEET}} = \bar{q}_n \not{n} \cdot D q_n, \]  \tag{3.42}

where we have defined \( q_n(x) \equiv q_+(x) \) to recall the usual notation \( h_n(x) \) for the effective field of HQET. In addition the two-component field \( q_n(x) \) verifies the projection condition

\[
q_n(x) = \frac{i \not{\gamma}}{2} q_n(x) \]  \tag{3.43}

which implies in particular \( \not{n} q_n = 0 \). The LEET equation of motion is just

\[
n \cdot D q_n(x) = 0. \]  \tag{3.44}

In short the Feynman rules of the LEET effective theory are

\[
\text{LEET quark propagator:} \quad \frac{i \not{\gamma}}{n \cdot k + i\epsilon} \frac{i \not{\gamma}}{2}, \]  \tag{3.45}

\[
\text{LEET quark-gluon vertex:} \quad -ig \not{\gamma} T_a n^a. \]  \tag{3.46}
which look like the HQET ones [50]. The only difference is that in HQET one writes the Lagrangian for heavy quark involved in the initial state meson where as in LEET the Lagrangian is for light energetic quark of final meson.

One can see that the assumption of a massless quark, or even a light quark (compared to $\Lambda_{\text{QCD}}$), is not needed to write Eq. (3.41). The mass term $\bar{q}_n m_q q_n$ just vanishes because of the projector (3.43). As far as masses are concerned, we only need $m_q \ll E$ for the quark, and $m' \ll E$ for the hadron, in order for $n$ to become a light-like vector. Of course in phenomenological applications we will use LEET mainly for the light $u$, $d$ and $s$ quarks. At leading order the LEET Lagrangian (3.42) has some global symmetries which are used to relate different form factors for $B \to V$ transition. The next section is devoted to the issue of these symmetries and their breaking in perturbative QCD.

### 3.4.1 LEET symmetries, their breaking in perturbative QCD and form factors

Among the global symmetries of LEET the simplest one is the flavor symmetry, as there is no mass term in the Lagrangian, meaning that mass effects should be small for energetic quarks.

One can immediately check that the LEET Lagrangian (3.42) as well as the projection condition (3.43) are invariant under chiral transformation

$$q_n(x) \to e^{i\gamma_5/2} q_n(x). \quad (3.47)$$

The helicity operator of LEET quark, which is massless here is just $\gamma_5/2$. The fact is that there is no Dirac matrix in the LEET Lagrangian (coupled to the covariant derivative $D_\mu$), should indicate that the $U(1)$ chiral symmetry can be embedded in the large symmetry group. One defines in the rest frame of $v$

$$S^1 = \frac{1}{2} \gamma^0 \Sigma^1 = \frac{1}{2} \gamma^1 \gamma^5, \quad S^2 = \frac{1}{2} \gamma^0 \Sigma^2 = \frac{1}{2} \gamma^2 \gamma^5, \quad S^3 = \frac{1}{2} \Sigma^3 = \frac{1}{2} \gamma^5 \gamma^0 \gamma^3. \quad (3.48)$$

Generally these can be defined as

$$S^1 \equiv \frac{\gamma^5 d^1}{2}, \quad S^2 \equiv \frac{\gamma^5 d^2}{2}, \quad S^3 \equiv \frac{\gamma^5}{2} (1 - \psi \psi). \quad (3.49)$$

Here, $e^1$ and $e^2$ are the two four vectors transverse to both $v$ and $n$. As the $\Sigma^i$ generate the
SU (2) group, and from $[\gamma^0, \Sigma^i] = 0$ and $(\gamma^0)^2 = 1$, the $S^i$ operators also verify the SU (2) algebra:

$$[S^i, S^j] = i\epsilon^{ijk} S_k. \quad (3.50)$$

Finally, it is simple to check that both the Lagrangian (3.42) and the projection condition (3.43) remains invariant under the transformation generated by $\vec{S}$

$$q_n(x) \rightarrow e^{it\vec{S}} q_n(x). \quad (3.51)$$

Thus the LEET has as much global symmetry as HQET, i.e. a flavor and SU (2) symmetry [14]. We now try to reduce the number of form factors involved in the calculation of $B$ meson decays by using some of these symmetries.

It is well known that the form factors for $B$ decays into a vector meson are defined by the following Lorentz decompositions of bilinear quark current matrix elements:

$$\langle V(p', \epsilon^*)|\bar{q}\gamma^\mu b|\bar{B}(p)\rangle = \frac{2iV(q^2)}{M + m_V} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\nu}^* p_{\rho} p_{\sigma}, \quad (3.52)$$

$$\langle V(p', \epsilon^*)|\bar{q}\gamma^\mu \gamma^5 b|\bar{B}(p)\rangle = 2m_V A_0(q^2) \frac{\epsilon^* \cdot q}{q^2} q^\mu + (M + m_V) A_1(q^2) \left[ \epsilon^\mu - \frac{\epsilon^* \cdot q}{q^2} q^\mu \right]$$

$$- A_2(q^2) \frac{\epsilon^* \cdot q}{M + m_V} \left[ p^\mu + p'^\mu - \frac{M^2 - m_V^2}{q^2} q^\mu \right], \quad (3.53)$$

$$\langle V(p', \epsilon^*)|\bar{q}\sigma^{\mu\nu} q_{\nu} b|\bar{B}(p)\rangle = 2 T_1(q^2) \epsilon^{\mu\nu\rho\sigma} \epsilon_{\nu}^* p_{\rho} p_{\sigma}', \quad (3.54)$$

$$\langle V(p', \epsilon^*)|\bar{q}\sigma^{\mu\nu} \gamma_5 q_{\nu} b|\bar{B}(p)\rangle = (-i) T_2(q^2) \left[ (M^2 - m_V^2) \epsilon^{*\mu} - (\epsilon^* \cdot q) (p^\mu + p'^\mu) \right]$$

$$+ (-i) T_3(q^2) (\epsilon^* \cdot q) \left[ q^\mu - \frac{q^2}{M^2 - m_V^2} (p^\mu + p'^\mu) \right], \quad (3.55)$$

where $m_V$ (e) is the mass (polarization vector) of the vector meson and we use the sign convention $\epsilon^{0123} = -1$. Thus we have seven scalar functions (form factors): $V$, $A_i$, and $T_i \ (i = 1, 2, 3)$ of the momentum squared $q^2$ transferred from the heavy to the light vector meson. Now, when the energy of the final light meson $E$ is large (the large recoil limit), one can expand the interaction of the energetic quark in the meson with the soft gluons in terms of $\Lambda_{QCD}/E$. Using
then the effective heavy quark theory for the interaction of the heavy $b$-quark with the gluons, one can derive non-trivial relations between the soft contributions to the form factors [14]. The resulting theory (LEET), discussed in detail in the previous section, reduces the number of independent form factors from seven in the $B \to V$ transitions to two in this limit. We will try to summarize them here.

Let us recall the relation
\[ q^2 = M^2 - 2ME + m_b^2 \implies E = \frac{M}{2} \left( 1 - \frac{q^2}{M^2} + \frac{m_b^2}{M^2} \right). \]

Let us first write the matrix elements given in Eqs. (3.52-3.55) in the parametrization that is convenient to study the $M \to \infty$ and $E \to \infty$ limit, i.e. we use the variables $(q^2, \nu, \mu, E)$ rather than $(p^\mu, p'^\mu, q^2)$. Now the polarization vector for the vector meson $\varepsilon^\mu$ has $\varepsilon^\mu = \mathcal{O}(1)$ for transverse vector meson while $\varepsilon^\mu = \mathcal{O}(E/m_B)$ for the longitudinal meson in the limit $E \to \infty$. Thus we decompose the matrix elements in terms of Lorentz structures which remain finite in the asymptotic limit $M \to \infty$ and $E \to \infty$:

\[ \langle V(p', \varepsilon^*) | q \gamma^\mu b | \bar{B}(p) \rangle = i2E \varepsilon^{(u)}_1 (M, E) \varepsilon^{\mu\nu} \varepsilon_{\nu} n_{\gamma} n_{\sigma}, \quad (3.56) \]
\[ \langle V(p', \varepsilon^*) | q \gamma^\mu \gamma^5 b | \bar{B}(p) \rangle = 2E \varepsilon^{(a)}_1 (M, E) [\varepsilon^\mu - (\varepsilon^\nu \cdot \nu) n^\mu] + 2E \varepsilon^{(t)}_1 (\varepsilon^\mu \cdot \nu) [\varepsilon^{\nu} (M, E) n^\mu + \varepsilon^{(a)}_1 (M, E) n^\mu] \]
\[ -i2E \varepsilon^{(t)}_1 (M, E) \{[\varepsilon^\mu - (\varepsilon^\nu \cdot \nu) n^\mu] n^\nu - [\varepsilon^\nu - (\varepsilon^\mu \cdot \nu) n^\nu] n^\mu \} \quad (3.57) \]
\[ \langle V(p', \varepsilon^*) | q \sigma^{\mu\nu} \gamma_5 q_\nu b | \bar{B}(p) \rangle = -i2E \varepsilon^{(t)}_1 (M, E) (\varepsilon^{\mu\nu} n^\nu - \varepsilon^\nu n^\mu) - 2E \varepsilon^{(t)}_1 (M, E) \frac{\varepsilon^\mu \cdot \nu}{E} (n^\mu v^\nu - n^\nu v^\mu) \quad (3.58) \]

Here, $2E$ is the overall normalization factor and the superscripts $(u), (a) \ldots$ refer to the Dirac structure of the current operators. It is also clear that the matrix element to a longitudinal (respectively transverse) vector meson, only the form factor with $/ /$ (respectively $\perp$) subscripts contribute in the $M \to \infty$ and $E \to \infty$ limit.

Just to expose our argument in the above mentioned asymptotic limits, we use LEET to describe the final active quark and replace the quark field $q(0)$ in the current operators by the effective LEET field $\bar{q}_n(0)$, with $\bar{q}_n \gamma_5 / 2 = \bar{q}_n$. Two very useful Dirac identities are:

\[ \frac{\gamma^\mu}{2} = \frac{\gamma^\mu}{2} [n^\mu \gamma^\nu + \varepsilon^{\mu\nu\rho\sigma} v^\rho n_\sigma \gamma_5], \quad (3.59) \]
\[
\frac{\psi_f}{2} \sigma^{\mu \nu} = \frac{\psi_f}{2} \left[ i \left( n^\mu \nu - n^\nu \mu \right) - i \left( n^\mu \gamma^\nu - n^\nu \gamma^\mu \right) \gamma^5 + \epsilon^{\mu \nu \rho \sigma} v_\rho n_\sigma \gamma_5 \right].
\] (3.60)

But what about the initial quark: should it be treated by HQET? As it has already been noticed that to leading order LEET neglects hard spectator effects, and the hard momenta are integrated out, leaving only soft, non perturbative degrees of freedom. Thus for consistency one should use HQET to describe the initial quark, and replace the quark field \( b(0) \) by \( b_v(0) \) with \( \psi b_v = b_v \). Hence, HQET/LEET formalism is not more than a soft contribution dominance assumption, that will have short-distance \( \alpha_s \) corrections, and non-perturbative \( 1/M \) and \( 1/E \) corrections. By replacing the quark current operator \( \bar{q} \Gamma b \) by the effective one \( \bar{q}_n \Gamma b \) and using the identities given in Eqs. (3.59)-(3.60) we find the following relations between the currents, to leading order in \( 1/M \) and \( 1/E \):

\[
\bar{q}_n b_v = v_n \bar{q}_n \gamma^\mu b_v, \quad \tag{3.61}
\]
\[
\bar{q}_n \gamma^\mu b_v = n^\mu \bar{q}_n b_v + i \epsilon^{\mu \nu \rho \sigma} v_\nu n_\rho \bar{q}_n \gamma_\sigma \gamma_5 b_v, \quad \tag{3.62}
\]
\[
\bar{q}_n \gamma^\mu \gamma_5 b_v = -n^\mu \bar{q}_n \gamma_5 b_v + i \epsilon^{\mu \nu \rho \sigma} v_\nu n_\rho \bar{q}_n \gamma_\sigma b_v, \quad \tag{3.63}
\]
\[
\bar{q}_n \sigma^{\mu \nu} b_v = i \left[ n^\mu \nu \bar{q}_n b_v - n^\mu \bar{q}_n \gamma^\nu b_v - (\mu \leftrightarrow \nu) \right] + \epsilon^{\mu \nu \rho \sigma} v_\rho n_\sigma \bar{q}_n \gamma_5 b_v, \quad \tag{3.64}
\]
\[
\bar{q}_n \sigma^{\mu \nu} \gamma_5 b_v = i \left[ n^\mu \nu \bar{q}_n \gamma_5 b_v \right] + n^\mu \bar{q}_n \gamma^\nu \gamma_5 b_v - (\mu \leftrightarrow \nu) \right] + \epsilon^{\mu \nu \rho \sigma} v_\rho n_\sigma \bar{q}_n b_v. \quad \tag{3.65}
\]

In order to get these results we have used the addition constraints \( \eta b_v = b_v \) and \( \eta q_n = 0 \). Now reporting Eqs. (3.61)-(3.65) in Eqs. (3.56)-(3.58) we find

\[
\xi^{(u)}_\perp = \xi^{(a)}_\perp = \xi^{(t)}_\perp = \xi_\perp, \quad \tag{3.66}
\]
\[
\xi^{(a)}_\parallel = \xi^{(t)}_\parallel = \xi_\parallel. \quad \tag{3.67}
\]

Thus to leading order in \( 1/M, 1/E \), and \( \alpha_s \), there are only two independent form factors involved in heavy-to-light \( B \rightarrow V \) transitions, which from now we will denote by \( \xi_\perp \) and \( \xi_\parallel \). The form factors given in Eqs. (3.52)-(3.55) can be written in terms of these \( \xi_\perp \) and \( \xi_\parallel \) as
follows [14]:

\[
A_0(q^2) = \left(1 - \frac{m_V^2}{M^2E}\right) \xi_{\perp}(M,E) + \frac{m_V}{M} \xi_{\parallel}(M,E),
\]

(3.68)

\[
A_1(q^2) = \frac{2E}{M + m_V} \xi_{\perp}(M,E),
\]

(3.69)

\[
A_2(q^2) = \left(1 + \frac{m_V}{M}\right) \left[\xi_{\perp}(M,E) - \frac{m_V}{M} \xi_{\parallel}(M,E)\right],
\]

(3.70)

\[
V(q^2) = \left(1 + \frac{m_V}{M}\right) \xi_{\perp}(M,E),
\]

(3.71)

\[
T_1(q^2) = \xi_{\perp}(M,E),
\]

(3.72)

\[
T_2(q^2) = \left(1 - \frac{q^2}{M^2 - m_V^2}\right) \xi_{\perp}(M,E),
\]

(3.73)

\[
T_3(q^2) = \xi_{\perp}(M,E) - \frac{m_V}{M} \left(1 - \frac{m_V^2}{M^2}\right) \xi_{\parallel}(M,E).
\]

(3.74)

The symmetry argument given above to relate the form factors is valid only when active degrees of freedom interact softly with the spectator ones. These relations among the form factors in the symmetry limit are broken by perturbative QCD radiative corrections arising from the vertex renormalization and the hard spectator interactions. To incorporate both types of QCD corrections, a tentative factorization formula for the heavy-light form factors at large recoil and at leading order in the inverse heavy meson mass was introduced in Ref. [49]:

\[
f_k(q^2) = C_{\perp k} \xi_{\perp} + C_{\parallel k} \xi_{\parallel} + \Phi_B \otimes T_k \otimes \Phi_V,
\]

(3.75)

where \(f_k(q^2)\) is any of the seven independent form factors. The \(\xi_{\perp}\) and \(\xi_{\parallel}\) are the two independent form factors remaining in the LEET-symmetry limit; \(T_k\) is a hard-scattering kernel, convoluted with light-cone distribution amplitudes of the \(B\) and \(V\) meson; \(C_k = 1 + O(\alpha_s)\) are the hard vertex renormalization coefficients. The hard-scattering kernel has logarithmic endpoint divergences. To remove these divergencies one would have to introduce a factorization scale and factorize the end point divergencies into the soft form factor. Once these divergencies are removed the hard spectator corrections contribute to the convolution term in Eq. (3.75) break the factorization, implying that their contribution can not be absorbed in the redefinition of the first two terms, and they are suppressed by one power of the strong coupling \(\alpha_s\) relative to the soft contributions defined by \(\xi_{\perp}\) and \(\xi_{\parallel}\). To compute the hard spectator contribution to the \(B \to V\gamma\) decay amplitude, one has to assume distribution amplitudes for the initial and final mesons. To leading order in the inverse \(B\) meson mass, the dominant contribution is from
the leading-twist (twist-two) light-cone distribution amplitudes of the mesons. In this approach both the $B$ and $V$ mesons can be described by two constituents only, for example, $B^- = (b\bar{u})$ and $\rho^- = (d\bar{u})$, and the higher Fock states involving in addition gluons are ignored. It has been shown in the literature that the tentative factorization Ansatz given in Eq. (3.75) holds and derive the explicit corrections to the amplitudes $B \rightarrow V\gamma$, where $V = \rho, K^*$ in the LEET approach [52] and also an $O(\alpha_s)$ proof of the validity of Eq. (3.75) has also been provided by Beneke, Feldmann and Seidel [53], and by Bosch and Buchalla [54].
Part II

Radiative Leptonic Decays of $B$ meson
Chapter 4

Study of the Leptonic Decay modes of $B$ meson

It has been already mentioned that the rare $B$ meson decays, as being flavor changing neutral currents (FCNC) processes, are sensitive to the structure of the standard model (SM) and its possible extensions. Therefore, these decays may serve as an important tool to investigate the new physics prior to any possible experimental clue about it. Among the rare $B$ meson decays, the radiative decay $B \to l\nu\gamma$ is of viable interest, because it contains important information about the weak and hadronic interactions of the $B$ meson. Furthermore, with the introduction of the $B$-factories LHCb, BaBar, Belle and CLEOc, the radiative $B$ meson decay can be studied with enough statistics. Preliminary data from the CLEO Collaboration indicate the limit on the branching ratio $\mathcal{B}(B \to l\nu\gamma)$ which is:

$$\mathcal{B}(B \to e\nu_e\gamma) < 2.0 \times 10^{-4}$$
$$\mathcal{B}(B \to \mu\nu_\mu\gamma) < 5.2 \times 10^{-5}$$

at 90% confidence level [55]. With the better statistics expected from the upcoming $B$-factories, the observation and experimental study of this decay could soon become feasible. It is therefore of some interest to have a good theoretical control over the theoretical uncertainties affecting the relevant matrix elements.

The radiative leptonic decay $B^+ \to l^+\nu_l\gamma$ has received a great deal of attention in the literature [56–67] as a means of probing the aspects of the strong and weak interactions of a
heavy quark system. The presence of the additional photon in the final state can compensate for the helicity suppression of the decay rate present in the purely leptonic mode. As a result, the branching ratio for the radiative leptonic mode can be as large as $10^{-6}$ for the $\mu^+$ case [64], which would open up the possibility for directly measuring the decay constant $f_B$ [61]. A study of this decay can offer also useful information about the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{ub}|$ [68, 69].

In the radiative $B$-decay process, there are two contributions to the amplitude:

1. inner bremsstrahlung (IB) and

2. the structure dependent (SD) contribution which depends on the vector and axial vector form factor $F_V$ and $F_A$ respectively.

The IB contribution to the decay amplitude is associated with the tree diagrams shown in Figs. (4-1a) and (4-1b), and the SD contribution is associated with Fig. 4-1c.

In this chapter, we will study the radiative leptonic $B$-decays of $B^+ \to l^+ \nu_l \gamma$. The IB part is still helicity suppressed [56], while the SD one is free from the suppression [70]. Therefore, the radiative decay rates of $B^+ \to l^+ \nu_l \gamma$ ($l = e, \mu$) could have an enhancement with respect to the purely leptonic modes of $B^+ \to l^+ \nu_l$ due to the SD contributions in spite of the electromagnetic coupling constant $\alpha$. With the possible large branching ratios, the radiative leptonic $B$-decays could be measured in future experiments at the hadronic colliders, such as BTeV and the CERN Large Hadron Collider (LHC-B) experiments [71].

The chapter is organized as follows. In Sec. 4.1, we give the detailed calculation for the SM pure leptonic decay $B(p) \to l(p_l) \nu_l(p_\nu)$ as well as some comment about its helicity suppression, where as in Sec. 4.2 we present the decay kinematics and current matrix elements for $B^+ \to l^+ \nu_l \gamma$. In Sec. 4.3, we discuss the various contributions to the absorptive part of the SD amplitude $iH_{\mu\nu}$, needed in the dispersion relation. This includes the multiparticle continuum and resonances with the quantum numbers $1^-$ and $1^+$. The resonances include $B^{*-}$ and $B^{*+}$ mesons and their radial excitations, which model the higher states. The continuum is estimated by quark triangle graphs. In Sec. 4.4, the asymptotic behavior of the SD amplitude is studied. This provides a usual constraint on the residues of the resonance contribution, in terms of the continuum contribution. Ward Identities which together with gauge invariance relate various form factors are discussed in Sec. 4.5. These identities, which are expected to hold below the resonance regime, fix the normalization of the forms at $q^2 = 0$ in terms of the universal function.
Figure 4.1: $B \rightarrow l\nu\gamma$ radiative leptonic decay diagrams.
$g_+(0)$ as well as another constraint on the residues. Thus, in our approach, a parametrization of the $q^2$ dependence of the form factors is not approximated by a single pole contribution. But this parametrization is dictated by considerations mentioned above and also predicts the coupling constants of $1^-$ and $1^+$ resonances with the photon, if we restrict ourselves to one radial excitation; otherwise, using these as input, the radiative coupling constants of the radial excitations are predicted. In this and other respects our approach is different from the others mentioned previously. Our approach is closest to the one used in [72] for $B \to \pi l\nu_l$. We calculate the decay branching ratios in Sec. 4.6. What we have concluded in this chapter is described in Sec. 4.7.

4.1 The Standard Model pure leptonic decay $B(p) \to l(p_l)\nu_l(p_\nu)$

The pure leptonic decays of heavy mesons are very interesting from the point of not only theoretical but also experimental view. In principle, the pure leptonic decays $B \to l\nu$ is the most natural process to determine the decay constant $f_B$ and may also be sensitive to new physics beyond the Standard Model at tree level. With a great number of $B$-meson production at LHC, we may expect that careful experimental studies on the $B$-meson will be able to be accessible in the foreseeable future.

These decays proceed through the annihilation diagram in complete analogy to pion decays. The Feynman diagram for this decay is given in Fig. 4-2. The constituent quarks of the $B$-meson annihilate via the weak interaction to form a virtual $W$ boson. After propagating a short distance, the $W$ produces one of the three lepton pairs; $e\bar{\nu}$, $\mu\bar{\nu}_\mu$, or $\tau\bar{\nu}_\tau$.

For this pure leptonic decay, the matrix element in the $(V - A)$ theory is,

$$M = iG_F V_{ub}(0)\{V_\mu(0) - A_\mu(0)\}|B\rangle \bar{u}(p_l)\gamma^\mu(1 - \gamma^5)\nu(p_\nu),$$

(4.1)

where $p$, $p_l$, $p_\nu$ are four momenta of the $B$-meson, charged lepton ($l$), and the associated lepton neutrino($\nu_l$) respectively. The Fermi constant $G_F = 1.16 \times 10^{-5}$ GeV$^{-2}$, $V_{ub}$ is the $CKM$ mixing matrix element for $b \to u$ transitions. While the vector and axial-vector currents are defined as

$$V_\mu = \bar{u}\gamma_\mu b, ~ A_\mu = \bar{u}\gamma_\mu\gamma_5 b.$$  

(4.2)

Since, $B$-meson is a pseudoscalar particle, we can determine on grounds of Lorentz invariance
that:

$$\langle 0 | V_\mu(0) | B \rangle = 0,$$

due to the absence of available axial-vectors. On the other hand,

$$\langle 0 | A_\mu(0) | B \rangle = i f_B p^\mu,$$

where $f_B(p^2) = f_B(m_B^2) \equiv f_B$ is the $B$-meson decay constant.

Therefore,

$$M = G_F V_{ub} f_B \bar{u}(p_t)(1 - \gamma_5)v(p_\nu). \quad (4.3)$$

By the law of conservation of momentum, we have

$$p = p_t + p_\nu.$$

Using Dirac equation and also substituting $m_\nu = 0$, Eq. (4.3) becomes

$$M = G_F V_{ub} f_B m_t \bar{u}(p_t)(1 - \gamma_5)v(p_\nu),$$

49
and its complex conjugate is

$$M^* = G_F V_{ub}^* f_B m_1 \bar{\nu}(p_\nu)(1 + \gamma_5) u(p_l),$$

$$|M|^2 = M^* M$$

$$= G_F^2 |V_{ub}|^2 f_B^2 m_1^2 \bar{\nu}(p_\nu)(1 + \gamma_5) u(p_l) \times \bar{u}(p_\nu)(1 - \gamma_5) v(p_\nu)$$

$$= G_F^2 |V_{ub}|^2 f_B^2 m_1^2 T_\tau \left( m_\nu - m_\nu \right) \left( 1 + \gamma_5 \right) \left( m_l + m_l \right) \left( 1 - \gamma_5 \right)$$

$$= G_F^2 |V_{ub}|^2 f_B^2 m_1^2 \delta p_\nu \cdot p_l.$$

The partial decay width is defined as

$$d\Gamma = \frac{1}{2M_B} \sum_{s p m s} |M|^2 d_{LIPS}(p; p_l, p_\nu), \quad (4.4)$$

where $M_B$ is the mass of $B$-meson and

$$d_{LIPS}(p; p_l, p_\nu) = (2\pi)^4 \delta^4(p - p_l - p_\nu) \frac{d^3 p_l}{(2\pi)^3 2E_l} \times \frac{d^3 p_\nu}{(2\pi)^3 2E_\nu}, \quad (4.5)$$

is the phase space. Using Eq. (4.5) in Eq. (4.4), we can write

$$d\Gamma = \frac{1}{2M_B} G_F^2 |V_{ub}|^2 f_B^2 m_1^2 \delta p_\nu \cdot p_l (2\pi)^4 \delta^4(p - p_l - p_\nu) \frac{d^3 p_l}{(2\pi)^3 2E_l} \times \frac{d^3 p_\nu}{(2\pi)^3 2E_\nu}. \quad (4.6)$$

Hence, integrating over lepton momenta, we get

$$\Gamma(B \rightarrow l\bar{\nu}_l) = \frac{G_F^2}{8\pi} |V_{ub}|^2 f_B^2 m_1^2 M_B^3 \left( 1 - \frac{m_l^2}{M_B^2} \right)^2. \quad (4.7)$$

All quantities in this decay rate are well established except for $f_B$ and $V_{ub}$. The experimental difficulty in measuring $B \rightarrow l\bar{\nu}_l$ is primarily due to the well-known fact of helicity suppression factor $m_l^2/M_B^2$ (because of the lightness of the leptons, $e$ and $\mu$) which lowers its rate\(^1\). According to CPT invariance, we expect the decay rate for $B^- \rightarrow l^-\bar{\nu}_l$ and $B^+ \rightarrow l^+\nu_l$ to be identical.

---

\(^1\)This can be interpreted as a physical consequence of the term $(1 - \gamma_5)$ in Eq. (3.3) which is a helicity projection operator for massless leptons, allowing only left-handed massless particles and right-handed massless antiparticles.
Figure 4-3: Lepton helicity states in $B \rightarrow l\nu_l$ decay.

**Helicity Suppression:**

To understand the helicity suppression effect, consider the decay $B^- \rightarrow l\bar{\nu}_l$ in the $B$ rest frame (Fig. 4-3). Since the $B$ has zero spin, the two leptons, which are emitted in opposite directions, must have the same helicity, so that their spins add to zero, by angular momentum conservation. Since the weak interaction produces exclusively right-handed (positive helicity $H = +1$) massless $\nu_l$'s, the lepton must be emitted with positive helicity, as well. Weak interactions, however, couple only to the left-handed chiral component $(1 - \gamma_5)u_l$, the leptons produced are preferentially left-handed, so that this decay would be completely forbidden in the limit $m_l = 0$. Since $m_e \neq 0$, both positive and negative helicity states are mixed by an amount proportional to the mass, $(1 - \gamma_5)u_l$ contains a small part of positive helicity, resulting in non-zero decay rates. But the amplitude for this process is suppressed by a factor $m_l^2/M_B^2$.

The dependence of helicity suppression on the lepton mass is given by

$$\text{Helicity Suppression} \sim 1 - \beta_l,$$

$$= \frac{2m_l^2}{m_B^2 + m_l^2}. \quad (4.8)$$

$$= \frac{2m_l^2}{m_B^2 + m_l^2}. \quad (4.9)$$
where $\beta_1$ is the velocity of the lepton. For $l = \mu$, the same argument holds, but here, the corresponding factor $m_\mu^2/M_B^2$ is comparatively larger. The helicity suppression factor for $\tau, \mu$ and $e$ is approximately $1/5$, $1/1000$, and $1/50,000,000$ respectively. The experimental confirmation of helicity suppression in $\pi^- \rightarrow l^- \bar{\nu}_l$ decays [73] is one of the great achievements of the Standard Model.

### 4.2 Decay kinematics and current matrix elements for $B^+ \rightarrow l^+ \nu_l \gamma$ decay

In the last section, we have seen that SM decay $B \rightarrow l \nu_l$ suffers from helicity suppression $m_l^2/M_B^2$ for both the light leptons $e$ and $\mu$. In order to compensate this factor, the BGW model suggested an alternative way [56]. It was suggested, therefore, that the radiative effects in purely leptonic decays should also be investigated by studying the process in which an additional photon is emitted i.e. $B \rightarrow \gamma l \nu_l$. This partner process helps to turn the helicity suppression into an $\alpha$ (the c.m. fine-structure constant) suppression by taking into account the structure dependence, as we shall find it below.

The amplitude for the decay $B^+(p) \rightarrow l^+(p_l)\nu_l(p_{\nu})\gamma(k)$, where $l$ stands for $e$ or $\mu$, and $\gamma$ is a real photon with $k^2 = 0$, can be written in two parts, $M_{IB}$ and $M_{SD}$, as follows:

$$M(B^+ \rightarrow l^+ \nu_l \gamma) = M_{IB} + M_{SD}$$  \hspace{1cm} (4.10)

in terms of two emission types of the real photon from $B^+ \rightarrow l^+ \nu_l$. They are given by [74–77]

$$M_{IB} = iG_F \frac{V_{ub} f_B m_l e^*_\mu l^\mu}{\sqrt{2}}$$  \hspace{1cm} (4.11)

$$M_{SD} = -iG_F \frac{V_{ub} f_B m_l e^*_\mu \tilde{H}^{\mu\nu} l_\nu}{\sqrt{2}}$$  \hspace{1cm} (4.12)

with

$$L^\mu = m_l \bar{u}(p_{\nu}) (1 + \gamma_5) \left( \frac{2p^\mu}{2p \cdot k} - \frac{2p^\mu + k^\mu}{2p \cdot k} \right) v(p_l, s_l),$$  \hspace{1cm} (4.13)

$$l^\mu = \bar{u}(p_{\nu}) \gamma^\mu (1 + \gamma_5) v(p_l, s_l),$$  \hspace{1cm} (4.14)

$$\tilde{H}^{\mu\nu} = iF_V(q^2) \epsilon^{\mu\nu\alpha\beta} k_\alpha p_\beta - F_A(q^2) (p \cdot k) g^{\mu\nu} + p^\mu k^\nu,$$  \hspace{1cm} (4.15)

$$q^\mu = (p - k)^\mu = (p_l + p_{\nu})^\mu.$$  \hspace{1cm} (4.16)
Here $\epsilon^*_\mu$ denotes the polarization vector of the photon with $k^\mu \epsilon^*_\mu (k) = 0$. $p, p_l, p_\nu$, and $k$ are the four-momenta of $B^+, \ell^+, \nu$, and $\gamma$, respectively; $s_\ell$ is the polarization vector of the $\ell^+$, $f_B$ is the $B$ meson decay constant, and $F_A, F_V$ stand for two Lorentz invariant amplitudes (form factors).

The term proportional to $L^\mu$ in Eq. (4.13) does not contain unknown quantities—it is determined by the amplitude of the non-radiative decay $B^+ \to \ell^+\nu$. This part of the amplitude is usually referred as “inner bremsstrahlung contribution”, whereas the term proportional to $H^{\mu\nu}$ is called “structure dependent contribution”.

The form factor $F_A (F_V)$ is related to the matrix element of the axial (vector) current. The factors $f_B$ and $F_{V,A}$ are defined by

$$
\langle 0 | \bar{u} \gamma^\mu \gamma_5 b | B(p) \rangle = -if_B p^\mu \\
\langle \gamma (k) | \bar{u} \gamma^\mu \gamma_5 b | B(p) \rangle = -i(\epsilon^* \cdot p) k^\mu - \epsilon^* \mu (p \cdot k) f_A (q^2) \\
\langle \gamma (k) | \bar{u} \gamma^\mu b | B(p) \rangle = -i\epsilon^{\mu\nu\alpha\beta} \epsilon_\nu p_\alpha k_\beta F_V (q^2)
$$

In our phase convention, the form factors $F_A$ and $F_V$ are real in the physical region

$$m_t^2 \ll q^2 \ll M_B^2$$

where $q$ is the momentum transfer. The kinematics of the decay needs two variables, for which we choose the conventional quantities, and in the rest frame of $B$,

$$x = \frac{2 p \cdot k}{M_B^2} = \frac{2E_\gamma}{M_B}, \quad y = \frac{2 p \cdot p_l}{M_B^2} = \frac{2E_\ell}{M_B}$$

and the angle $\theta_{\ell\gamma}$ between the photon and the charged lepton is related to $x$ and $y$ by

$$x = \frac{1}{2} \left( \frac{2 - y + \sqrt{y^2 - 4r_1}}{2 - y + \sqrt{y^2 - 4r_1} \cos \theta_{\ell\gamma}} \right) \left( 2 - y - \sqrt{y^2 - 4r_1} \right).$$

In terms of these quantities, one can write the momentum transfer as

$$q^2 = M_B^2 (1 - x), \quad (k^2 = 0).$$
We write the physical region of $x$ and $y$ as

$$0 \leq x \leq 1 - r_l, \quad 1 - x + \frac{r_l}{1 - x} \leq y \leq 1 + r_l,$$

where

$$r_l = \frac{n_l^2}{M_B^2} = \begin{cases} 9.329 \times 10^{-9} & (l = e), \\ 4.005 \times 10^{-4} & (l = \mu). \end{cases}$$

(4.24)

(4.25)

4.2.1 Dispersion Relations

The structure dependent part, $H^{\mu\nu}$ is given by

$$iH^{\mu\nu} = i \int d^4x e^{ik\cdot x} \langle 0 | T(j^\mu_{em}(x), J_2^\nu(0)) | B(p) \rangle$$

(4.26)

We note that [78]

$$ik_{\mu} H^{\mu\nu} = if_B p_{\nu},$$

(4.27)

so that for the real photon we can write

$$H^{\mu\nu} = \delta^{\mu\nu} + f_R \frac{p^\mu p^\nu}{p \cdot k}$$

(4.28)

where $k_{\mu} \delta^{\mu\nu} = 0$ and $\tilde{H}^{\mu\nu}$ is parametrized as in Eq. (4.15). The second term in Eq. (4.28) is absorbed in $M_{1B}$. The absorptive part is

$$\text{Abs} [iH^{\mu\nu}] = \frac{1}{2} \int d^4x e^{ik\cdot x} \langle 0 | j^\mu_{em}(x), J_2^\nu(0) | B(p) \rangle$$

$$= \frac{1}{2} (2\pi)^4 \sum_n \langle 0 | j^\mu_{em}(0) | n \rangle \langle n | J_2^\nu(0) | B(p) \rangle \delta^4 (k - p_n)$$

$$- \sum_n \langle 0 | J_2^\nu(0) | n \rangle \langle n | j^\mu_{em}(0) | B(p) \rangle \delta^4 (k + p_n - p)$$

(4.29)

The $\delta$-function in the first term implies that only values with $p_n^2 = k^2 = 0$ are relevant and since there is no real particle with zero mass, the first term does not contribute. Thus contributing to the absorptive part are all possible intermediate states that couple to $B\gamma$ and annihilated by the weak vertex $\langle 0 | J_2^\nu(0) | n \rangle$. These include the multiparticle continuum as well as resonances.
with quantum numbers $I^-$ and $I^+$. Thus $(t = q^2)$

$$F_V(t) = \frac{gg_{B^*\gamma}}{M_{B^*}^2 - t} f_{B^*} + \cdots$$

(4.30)

$$F_A(t) = \frac{f_{B^*\gamma}}{M_{B^*}^2 - t} f_{B^*} + \cdots$$

The ellipses stand for contributions from higher states with the same quantum numbers. The couplings $g_{B^*\gamma}$ and $f_{B^*\gamma}$ are defined as

$$\langle B^{*\gamma}(q, \eta) | B^-(P) \rangle = ig_{B^*\gamma} \epsilon_{\alpha_\mu \rho \sigma} \epsilon^{\mu \eta \rho \sigma}$$

$$\langle B^*_A(q, \eta) | B^-(P) \rangle = ig_{B^*_A \gamma} (\epsilon^{\mu \eta \rho \sigma} - if_{B^*_A \gamma} (q, \eta))$$

(4.31)

We assume that the contributions from the radial excitations of $B^*$ and $B^*_A$ dominate the higher state contribution. Thus we write

$$F_V(t) = \frac{R_V}{1 - t/M_{B^*}^2} + \sum_i \frac{R_{V_i}}{1 - t/M_{B^*_i}^2} + \frac{1}{\pi} \int_{S_0}^{M^2} \frac{\text{Im} F^\text{Cont}_V(s)}{s - t - i\epsilon} ds + \cdots$$

(4.32)

$$F_A(t) = \frac{R_A}{1 - t/M_{B^*_A}^2} + \sum_i \frac{R_{A_i}}{1 - t/M_{B^*_A_i}^2} + \frac{1}{\pi} \int_{S_0}^{M^2} \frac{\text{Im} F^\text{Cont}_A(s)}{s - t - i\epsilon} ds + \cdots$$

where the ellipses stand for the contributions from the region with much larger mass than the physical mass of the heavy resonances up to $\infty$. Here, $M$ is a cut off near the first radial excitation of $M_{B^*}$ or $M_{B^*_A}$ and $S_0 = M_B + m_\pi$, and

$$R_V = \frac{gg_{B^*\gamma}}{M_{B^*}^2} f_{B^*}$$

(4.33)

$$R_A = \frac{f_{B^*_A \gamma}}{M_{B^*_A}^2} f_{B^*_A}$$

$R_V$ and $R_A$, are the corresponding quantities for the radial excitations with masses $M_{B^*}$ and $M_{B^*_A}$. In the next section we develop the constraints on some of the parameters appearing in
the above equations.

If we model the continuum contribution by a quark triangular graph (similar calculations exist in the literature [79]), we obtain

$$ F^\text{Cont}_V = F^\text{Cont}_A = \frac{f_B}{M_B} \left\{ \frac{Q_u}{\bar{\Lambda}} - \frac{Q_b}{M_B} \left(1 + \frac{\bar{\Lambda}}{M_B} \right) \right\} \frac{1}{1 - q^2/M_B^2} \tag{4.34} $$

where

$$ \bar{\Lambda} = M_B - m_b, \tag{4.35} $$

together with the term

$$ (Q_u - Q_b) f_B \frac{p^\mu p^\nu}{k \cdot p} = f_B \frac{p^\mu p^\nu}{k \cdot p} $$

which appears in Eq. (4.28). As is well known (see for example Ref. [80]), the pole at $q^2 = M_B^2$ in Eq. (4.34) arises due to a $(b)\text{quark propagator}$ which form one leg of quark $\Delta$, the other legs are the part of $B$ meson wave function.

4.3 Asymptotic Behavior

To get constraints on the residues $R_i$, it is useful to study the asymptotic behavior of the form factors $F_V$ and $F_A$. It has been argued that the behavior of the form factor for very large values of $|t|$ can be estimated reliably in perturbative QCD processes $[pQCD]$ [72, 81, 82]. For $t \ll 0$ and for $|t|$ much larger than the physical mass of the heavy resonances, $pQCD$ should yield a very good approximation to the form factors. First we note that by vector meson dominance

$$ \langle \gamma(k, \varepsilon^*(k)) | \bar{u} \gamma^\mu (1 - \gamma_5) b | B(p) \rangle \approx Q_u \frac{f_{\rho}}{m_{\rho}} \langle \rho(k, \varepsilon^*(k)) | \bar{u} \gamma^\mu (1 - \gamma_5) b | B(p) \rangle, \tag{4.36} $$

where $f_{\rho}$, having the dimension of mass, is defined as

$$ \langle 0 | \bar{u} \gamma^\mu u | \rho(k, \varepsilon(k)) \rangle = \frac{f_{\rho}}{m_{\rho}} \gamma^\mu \tag{4.37} $$

Then using the methods employed in [82], the corresponding hard scattering amplitude for Fig. (4-4a) is

$$ \langle \rho(k, \varepsilon^*(k)) | \bar{u} \gamma^\mu (1 - \gamma_5) b | B(p) \rangle = \frac{z_{\rho a}}{3} \int_0^1 dx \int_0^{1-x} dy \frac{\gamma_{\mu\nu\gamma_5} g_{\gamma_\mu(1-\gamma_5) \gamma_\nu}}{2k^2} \phi_B(x) \left\{ \frac{y_{\rho(x)+m_B} y_{\gamma_\gamma}}{y_{\rho(x)+m_B} + y_{\gamma_\gamma}} \right\} \phi_{\rho}(y) \tag{4.38} $$

56
Figure 4-4: Perturbative Diagrams for \( B \to \rho \gamma \)

and similarly for Fig. (4-4b) is

\[
\langle \rho (k, \varepsilon^* (k)) | \bar{u} \gamma^\mu (1 - \gamma_5) b | B (p) \rangle = \frac{8 \pi g_4}{3} \int_0^1 dx \int_0^{1-x} dy \, T_1 \frac{\not{p} \gamma^\nu \gamma^\mu (1 - \gamma_5) (p_2 + m_B) \gamma^\nu (p + g(z) m_B) \gamma_5 \gamma_\nu}{(k_2^2 - m_B^2) Q^2} \phi_\rho (y)
\]

(4.39)

Here \( \phi_B (x) \) and \( \phi_\rho (y) \) are the wave functions for \( B \) and \( \rho \) mesons respectively and

\[
\varepsilon \sim \mathcal{O} \left( \frac{\Lambda_{QCD}}{m_B} \right).
\]

The explicit form of these wave functions is

\[
\phi_\rho (y) \sim y (1 - y) \\
\phi_B (x) \sim \frac{1}{|z^2 / (1 - x) + 1 / x - 1|^2}
\]
These wavefunctions are normalized to
\[ \int_0^1 dx \phi_i(x) = \frac{1}{2\sqrt{3}} F_i \]
where \( F_i \) corresponds to \( F_B \) or \( F_\rho \).

The kinematic features which appear in the denominator of hard scattering amplitudes (Eq. (4.38) and (4.39) ) in the asymptotic limit \( q^2 \to \infty \) become
\[ k_i^2 Q^2 = q^4 (1 - x)^2 (1 - y) \]
\[ (k_i^2 - m_B^2) Q^2 = q^4 (1 - x) (1 - y)^2 \] (4.40)

The only contribution comes from the Fig. (4.1b) which by itself is gauge invariant. Using Eq. (4.40) in Eqs. (4.38 and 4.39) along with the static quark approximation to the \( B \) meson wave function i.e. \( g = 1 \) the value of the perturbative form factors become:
\[ F^{pQCD}_V = F^{pQCD}_A = \left( Q_u \frac{f_\rho}{m_\rho} \right) \frac{32\pi \alpha_s(t)}{3} (f_B f_\rho) m_B \left( \frac{1}{\varepsilon} \ln \varepsilon \right) \frac{1}{t^2} \] (4.41)

One can see that it is governed by the tail end of the \( B \) meson wave function characterized by \( \varepsilon \).

Now the asymptotic behavior of Eq. (4.32) is given by
\[ F(q^2) \to -\frac{1}{q^2} \left[ RM^2 + \sum_i R_i M_i^2 + \frac{1}{\pi} \int_{S_0}^{M^2} \text{Im} F^{\text{Cont}}(s) ds \right]. \] (4.42)

Since \( F^{pQCD}(t) \) is a reliable approximation to the form factor for \( t \to -\infty \), and \( (t F^{pQCD}) \to 0 \) in this limit, it follows that
\[ RM^2 + \sum_i R_i M_i^2 + c \approx 0, \] (4.43)

where we have defined
\[ c = \frac{1}{\pi} \int_{S_0}^{M^2} \text{Im} F^{\text{Cont}}(s) ds. \] (4.44)

The convergence relation (4.43) is a model independent result and constitutes a very binding constraint for model building. In other words, the various contributions in Eq. (4.42) may be individually much larger than the \( (t F^{pQCD}(t)) \) due to \( \alpha_s(t)/t \) suppression, but there must be
large cancellations among the non-perturbative contributions in Eq. (4.42). This is in the spirit of ref. [72]. We will explore the resonant contribution (in our model) in order to understand the effect of Eq. (4.43) on the behavior of form factors in the physical region. The imposition of this constraint will lead to a very distinct behavior of the photon momentum distribution, independently of how many resonances we choose to keep. As the radial excitations of $B^*$ become heavier, they are less relevant to the form factors since the spacing between the consecutive radial excitations are expected to become narrower and narrower [83]. Thus, heavier resonances contribute with a smaller value even in the narrow width approximation. Furthermore, as finite widths are considered, the contribution of heavier and thus broader excitations are additionally suppressed. This shows that the truncation of the sum over resonances is a reasonable approximation.

For the reasons stated above we will study a constrained dispersive model where only the first two radial excitations are kept. This is mainly for the reason mentioned above. On the other hand, the “minimal” choice of keeping only one radial excitation will determine $R_1$ in terms of $R$. The other necessary ingredient to specify the model is knowledge of the spectrum of radial excitations. These resonances [(2S) and (3S) excitations of $B^*$] have not yet been observed in the $B$ systems. We will then rely on potential model calculations for their masses [83]. These models have been very successful in predicting the masses of orbitally excited states, and therefore we are confident that the position of the radial excitations does not introduce a sizeable uncertainty. The resultant spectrum explicitly shows that the spacing among the 1S, 2S, 3S states are, to leading order, independent of the heavy quark mass and, therefore, constitute the property of the light degrees of freedom. The spectrum of radial excitations is given in Table 1, where the subindices 1 and 2 correspond to the 2S and 3S excitation of the $B^*$, etc. Thus the convergence condition (4.43) now reads

$$\bar{r}M^2 + R_1 M_1^2 + R_2 M_2^2 + c = 0,$$

(4.45)

This condition leaves two free parameters $R_1$ and $R_2$ in the model. This results in the correct scaling of form factors with the heavy meson mass. Solving Eq. (4.45) for $R_2$ and using
this in Eq. (4.42), we obtain

$$F(q^2) = \frac{RM^2 (M_B^2 - M^2)}{(M^2 - q^2) (M_B^2 - q^2)} + \frac{R_1 M^2 (M_B^2 - M^2)}{(M_1^2 - q^2) (M_2^2 - q^2)} + \frac{1}{M_2 - q^2} \frac{1}{\pi} \int_{S_0} \frac{M_B^2 - s}{s - q^2} \text{Im} F_{\gamma}^{\text{cont}}$$ (4.46)

If we model the continuum contribution by quark triangle graph as given in Eq. (4.34), we obtain

$$F(q^2) = \frac{RM^2 (M_B^2 - M^2)}{(M^2 - q^2) (M_B^2 - q^2)} + \frac{R_1 M^2 (M_B^2 - M^2)}{(M_1^2 - q^2) (M_2^2 - q^2)} + \frac{M_B^2 - M^2}{(M_2^2 - q^2) (M^2 - q^2)} c$$ (4.47)

where in the heavy quark limit $M_B = M_B^* = M$ and

$$c = f_B M_B \left[ \frac{Q_u}{k} + O \left( \frac{1}{M_B} \right) \right]$$ (4.48)

4.4 Ward Identity Constraints

It is useful to define

$$\langle \gamma (k, \epsilon) | i \bar{u} \sigma^{\mu \nu} q_{\nu} b | B(p) \rangle = -i \varepsilon_{\mu \nu \alpha \beta} \epsilon^*_{\alpha} k_{\beta} p_{\alpha} F_1(q^2)$$ (4.49)

$$\langle \gamma (k, \epsilon) | i \bar{u} \sigma^{\mu \nu} \gamma_5 q_{\nu} b | B(p) \rangle = [(q \cdot k) \epsilon^* \mu - (\epsilon^* \cdot q) k^\mu] F_3(q^2)$$ (4.50)

Now we will make use of the Ward identities and gauge invariance principle to relate different form factors.

Usually, gauge invariance is implemented by means of the Ward identities; another way, essentially the same, is to consider what happens if the polarization vector of an external (real) photon is replaced by its four-momentum. The result is zero, provided that one considers all diagrams where this particular photon is connected in all possible ways to a charge carrying line. In this way one understands the connection between gauge invariance and charge conservation.
The Ward identities\(^2\) used to relate different form factors appearing in our process are:

\[
\langle \gamma (k, \epsilon) | \bar{u} \sigma^{\mu \nu} q_{\nu} b | B(p) \rangle = -(m_b + m_q) \langle \gamma (k, \epsilon) | \bar{u} \gamma^\mu b | B(p) \rangle + (p^\mu + k^\mu) \langle \gamma (k, \epsilon) | \bar{u} b | B(p) \rangle = -(m_b + m_q) \langle \gamma (k, \epsilon) | \bar{u} \gamma^\mu b | B(p) \rangle \tag{4.51}
\]

\[
\langle \gamma (k, \epsilon) | \bar{u} i \sigma^{\mu \nu} \gamma_5 q_{\nu} b | B(p) \rangle = m_b - m_q \langle \gamma (k, \epsilon) | \bar{u} \gamma^\mu \gamma_5 b | B(p) \rangle + (p^\mu + k^\mu) \langle \gamma (k, \epsilon) | \bar{u} \gamma_5 b | B(p) \rangle = m_b - m_q \langle \gamma (k, \epsilon) | \bar{u} \gamma^\mu \gamma_5 b | B(p) \rangle \tag{4.52}
\]

where the matrix elements \(\langle \gamma (k, \epsilon) | \bar{u} b | B(p) \rangle\) and \(\langle \gamma (k, \epsilon) | \bar{u} \gamma_5 b | B(p) \rangle\) vanish for a real photon due to gauge invariance.

Using the Ward identities in Eqs. (4.49) and (4.50), and comparing the coefficients, we obtain \([p \cdot k = q \cdot k, \epsilon^* \cdot p = \epsilon^* \cdot q]\)

\[
F_\nu (q^2) = \frac{1}{m_b + m_q} F_1 (q^2) \tag{4.53}
\]

\[
F_A (q^2) = \frac{1}{m_b - m_q} F_3 (q^2) \tag{4.54}
\]

The results given in Eqs. (4.53) and (4.54) are model independent because these are derived by using Ward Identities.

In order to make use of the Ward identities to relate different form factors, we define

\[
\langle \gamma (k, \epsilon) | i \bar{u} \sigma_{\alpha \beta} b | B(p) \rangle = -i \varepsilon_{\alpha \beta \rho \sigma} \epsilon^\rho (k) [(p + k)^\sigma g_+ + q^\sigma g_-]
- i q \cdot \epsilon^* (k) \varepsilon_{\alpha \beta \rho \sigma} (p + k)^\rho q^\sigma h
- i \varepsilon_\alpha \varepsilon_{\beta \rho \sigma} \epsilon^\rho (k) (p + k)^\sigma q^\tau - \alpha \leftrightarrow \beta] h_1
- \varepsilon_\alpha \varepsilon_{\beta \rho \sigma} \epsilon^\rho (k) (p + k)^\sigma q^\tau - \alpha \leftrightarrow \beta] h_2. \tag{4.55}
\]

Since we have a real photon, gauge invariance requires that if we replace \(\epsilon^\mu (k)\) by \(k^\mu\), the matrix

\(^2\)See ref.\([84]\) for a detailed derivation of these Ward identities.
element should vanish. This requires

\[ g_+ + g_- + 2 (g \cdot k) h = 0 \quad (4.56) \]

From the Dirac algebra

\[ \sigma^{\mu\nu} \gamma_5 = -\frac{i}{2} \epsilon^{\mu\nu\alpha\beta} \sigma_{\alpha\beta}, \quad (4.57) \]

we can write

\[
\begin{align*}
\langle \gamma (k, \epsilon) | i \bar{u} \sigma^{\mu\nu} \gamma_5 b | B(p) \rangle \\
= -\frac{i}{2} \epsilon^{\mu\nu\alpha\beta} (\gamma (k, \epsilon) | i \bar{u} \sigma_{\alpha\beta} | B(p)) \\
= (\epsilon^{\nu} k^\nu - \epsilon^{\nu} k^\mu) [g_+ + g_- - (M_B^2 + q^2) h_1 - (3M_B^2 - q^2) h_2] \\
+ (\epsilon^\mu p^\nu - \epsilon^\nu p^\mu) \left[ g_+ + g_- + (M_B^2 - q^2) (h_1 + h_2) \right] \\
- 2q \cdot \epsilon^* (h - h_1 - h_2) (p^\mu k^\nu - p^\nu k^\mu) \quad (4.58)
\end{align*}
\]

By gauge invariance, namely, replacing $\epsilon^\mu$ by $t^\mu$, the matrix element should be zero, and this does not give any new relation other than Eq. (4.56). Using this relation and $2k \cdot q = M_B^2 - q^2$, we get

\[
\begin{align*}
\langle \gamma (k, \epsilon) | i \bar{u} \sigma^{\mu\nu} \gamma_5 b | B(p) \rangle \\
= (\epsilon^{\nu} k^\nu - \epsilon^{\nu} k^\mu) [2g_+ + (M_B^2 - q^2) (h - h_1 - h_2) - 2q^2 h_1 - 2M_B^2 h_2] \\
- [2k \cdot q (\epsilon^\mu p^\nu - \epsilon^\nu p^\mu) + 2q \cdot \epsilon^* (p^\mu k^\nu - p^\nu k^\mu)] (h - h_1 - h_2) \quad (4.59)
\end{align*}
\]

Contrary to what is stated in the literature, the gauge invariance does allow a second tensor structure in addition to $(\epsilon^{\nu} k^\nu - \epsilon^{\nu} k^\mu)$.

This gives

\[
\begin{align*}
\langle \gamma (k, \epsilon) | i \bar{u} \sigma^{\mu\nu} q_\nu \gamma_5 b | B(p) \rangle = 2 (g_+ - q^2 k - (M_B^2 - q^2) h_2) \\
\times \langle q \cdot k \epsilon^{\mu} (k) - q \cdot \epsilon^* (k) k^\mu \rangle. \quad (4.60)
\end{align*}
\]

This, in turn, gives [from Eq. (4.50)]

\[ F_3(q^2) = 2 \left[ -g_+ - q^2 k - (M_B^2 - q^2) h_2 \right] \quad (4.61) \]
Similarly, from Eq. (4.55), we get the relation

\[ \langle \gamma(k, \epsilon) | \bar{u}i\sigma_{\alpha\beta}q^\beta | B(p) \rangle = -i\epsilon_{\epsilon_\rho} \epsilon_\rho q^\alpha p^\alpha 2 \left[ g_+ - q^2 h_1 - M_B^2 h_2 \right] \]

Comparison of this equation with Eq. (4.49) gives

\[ F_1(q^2) = 2 \left[ g_+ (q^2) - q^2 h_1 (q^2) - M_B^2 h_2 (q^2) \right] \]

Thus, finally we obtain

\[ F_V(q^2) = \frac{2}{m_b + m_q} \left\{ g_+ (q^2) - q^2 h_1 (q^2) - M_B^2 h_2 (q^2) \right\} \], \hspace{1cm} (4.63)

\[ F_A(q^2) = \frac{2}{m_b - m_q} \left\{ g_+ (q^2) - q^2 h_1 (q^2) - (M_B^2 - q^2) h_2 (q^2) \right\} \]. \hspace{1cm} (4.64)

Therefore, the normalization of \( F_V \) and \( F_A \) at \( q^2 = 0 \) is determined by a universal form factor \( (g_+ (0) - M_B^2 h_2) \). Now the form factor \( h_2 \) does not get any contribution from quark triangle graph nor from the pole, and therefore we shall put it equal to zero. On the other hand, only \( g_+ (q^2) \) gets a contribution from quark \( \Delta \) graph. [79],

\[ g_+ (q^2) = f_B \left\{ \frac{Q_u}{2\tilde{\Lambda}} - \frac{Q_b}{2M_P} \left( 1 - \frac{m_q}{M_B} \right) \right\} \frac{1}{1 - q^2/M_B^2}. \]

(4.65)

We expect the Ward identities to hold at low \( q^2 \) below the resonance regime and as such we use the results obtained from them at \( q^2 = 0 \). Thus from Eqs. (4.63 and 4.64), we obtain

\[ (m_b + m_q) F_V (0) = 2g_+ (0) = (m_b - m_q) F_A (0). \]

(4.66)

Further, using Eq. (4.35) above, Eq. (4.66), and neglecting terms of the order of \( (\tilde{\Lambda} \pm m_q) /M_B \), we obtain another constraint using Eqs. (4.47, 4.48) at \( q^2 = 0 \):

\[ R \left( 1 - \frac{M_B^2}{M_\tilde{\Lambda}^2} \right) + R \left( 1 - \frac{M_B^2}{M_\tilde{\Lambda}^2} \right) = \left( \frac{2g_+ (0)}{M} \right) \frac{M^2}{M_B^2}. \]

(4.67)
Now if we restrict ourselves to one radial excitation \((M_2 = M_1)\) we obtain from Eq. (4.67)

\[
R = \frac{2g_+(0)}{(M^2_2/M^2 - 1)} M
\]

\[
F(q^2) = \frac{2}{M} \frac{g_+(0)}{(1 - q^2/M^2)(1 - q^2/M^2_1)}
\]

(4.68)

(4.69)

Restoring the subscripts and using the definitions (4.33)

\[
g_{B \cdot B \gamma} = \frac{2g_+(0)}{f_B} \frac{M^2_{B\gamma}}{M^2_2} \frac{M^2_2}{M^2_2 / M^2_{B\gamma} - 1}
\]

\[
g_{B \cdot B \gamma} = \frac{2g_+(0)}{f_B} \left( M^2_{B\gamma} / M^2_2 - 1 \right)
\]

(4.70)

while

\[
f_{B \cdot B \gamma} = \frac{M^2_{B\gamma}}{f_B} \frac{2g_+(0)}{f_B} \frac{2g_+(0)}{M^2_{B\gamma} / M^2_2 - 1}
\]

(4.71)

Using \(g_+(0)\) given in Eq. (4.65) with \(Q_q = 2/1\), namely

\[
g_+(0) = \frac{2f_B}{32\Lambda}
\]

(4.72)

we have the prediction

\[
g_{B \cdot B \gamma} = \frac{2}{32\Lambda} \left( M^2_{B\gamma} / M^2_2 - 1 \right)
\]

(4.73)

Further

\[
F_V(q^2) = \frac{2}{M_B} \frac{g_+(0)}{\left(1 - q^2/M^2_{B\gamma} \right) \left(1 - q^2/M^2_{B\gamma} \right)}
\]

\[
F_A(q^2) = \frac{2}{M_B} \frac{g_+(0)}{\left(1 - q^2/M^2_{B\gamma} \right) \left(1 - q^2/M^2_{B\gamma} \right)}
\]

(4.74)

(4.75)

This is the final expression for the form factors of our process \(B \rightarrow \gamma l \nu_l\), if we restrict ourselves to the one radial excitation. We also observe the approximate equality \(F_V(q^2) = F_A(q^2)\) of the form factors which also occur in some other models [66, 67]. For numerical work, we shall use the \(B\)-meson masses given in Table 4.1 and \(f_{3/2} = 0.180\) GeV.
Table 4.1: $B$-mesons masses in GeV [85]

<table>
<thead>
<tr>
<th>$J^P$</th>
<th>$M$</th>
<th>$M_1/M$</th>
<th>$M_2/M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_B$</td>
<td>0−</td>
<td>5.28</td>
<td>1.14</td>
</tr>
<tr>
<td>$M_{B^*}$</td>
<td>1−</td>
<td>5.33</td>
<td>1.14</td>
</tr>
<tr>
<td>$M_{B^{*+}}$</td>
<td>1+</td>
<td>5.71</td>
<td>1.12</td>
</tr>
</tbody>
</table>

This gives the prediction from Eq. (4.73)

$$g_{B\cdot B\gamma} = \frac{2.2}{\bar{\Lambda}} = 5.6 \text{ GeV}^{-1},$$

(4.76)

for $\bar{\Lambda} = 5.28 - 4.8 = 0.4 \text{ GeV}^{-1}$ [see Eq. (4.35) and Table 1]. Also, we obtain from Eq. (4.72)

$$g_+(0) = \frac{3}{20} = 0.15.$$  

(4.77)

Further from Eq. (4.71)

$$f_{B^{*+} B\gamma} = \frac{f_E}{f_{B^*}} \frac{M_{B^{*+}}}{\Lambda} = \frac{2.6}{f_{B^*}} \text{ GeV}^{-1}$$

(4.78)

We now study the effect of the second radial excitation. We go back to Eq. (4.47) and use the constraint (4.67) to obtain

$$F(q^2) = \frac{R \left( \frac{M_2^2}{M_2^2} - 1 \right) \left( \frac{M_2^2}{M_1^2} - 1 \right) \frac{M_2^2}{M_1^2} + \frac{2g_+(0)}{M} \left( 1 - q^2 \left( \frac{1}{M_2^2} + \frac{1}{M_1^2} - \frac{M_1^2 M_2^2}{M_1^2 M_2^2} \right) \right)}{(1 - q^2/M_2^2) \left( 1 - q^2/M_1^2 \right) (1 - q^2/M^2)}$$

If we parametrize $R$ as

$$R = \frac{2g_+(0) \left( 1 - \frac{M_2^2}{M_1^2} \right)}{M} \frac{A}{(M_1^2/M_2^2) - 1},$$

where $A$ is a parameter which in principle can be obtained when $g_{B\cdot B\gamma}$ and $f_{B^{*+} B\gamma}$ become known. Then

$$F(q^2) = \frac{2g_+(0) \left( 1 - \frac{q^2}{M_1^2} \left( 1 + \left( 1 - \frac{M_2^2}{M_1^2} \right) \left( 1 - \frac{M_1^2}{M_2^2} \right) A \right) \right)}{M} \frac{(1 - q^2/M_2^2) \left( 1 - q^2/M_1^2 \right) (1 - q^2/M^2)}$$

(4.79)

For $M_1 = M_2$ the above equation, (4.79), reduces to Eq. (4.69). So the couplings of $B$ with
$B^*\gamma$ and $B_A^*\gamma$ become

$$
g_{B^*\gamma} = \frac{2g_+(0)M_{B^*}^2}{M_B f_{B^*} (M_{B^*}^2/M_{B^*_1}^2 - 1) \left[ 1 - (1 - M_{B^*_1}^2/M_{B^*}^2) A \right]}
\quad \frac{1}{1 - (1 - M_{B^*_1}^2/M_{B^*}^2) A} \right] 5.6 \text{ GeV}^{-1} \tag{4.80}
$$

$$
f_{B_A^*\gamma} = \frac{f_{B^*} M_{B_A^*}^2}{f_{B_A^*}} \left[ 1 - \left( 1 - M_{B_A^*}^2/M_{B_A^*}^2 \right) A \right] 6.5 \text{ GeV}^{-1} \tag{4.81}
$$

and the corresponding form factors become

$$
F_V(q^2) = \frac{2g_+(0)}{M_B} \left( 1 - \frac{q^2}{M_{B^*}^2} \right) \left( 1 + \left( 1 - \frac{M_{B^*}^2}{M_{B_A^*}^2} \right) A \right) \left( 1 - \frac{M_{B_A^*}^2}{M_{B^*}^2} \right) \tag{4.82}
$$

$$
F_A(q^2) = \frac{2g_+(0)}{M_B} \left( 1 - \frac{q^2}{M_{B_A^*}^2} \right) \left( 1 + \left( 1 - \frac{M_{B_A^*}^2}{M_{B^*}^2} \right) A \right) \left( 1 - \frac{M_{B^*}^2}{M_{B_A^*}^2} \right) \tag{4.83}
$$

For the numerical values we shall use $A = 0$ [i.e., $M_1 = M_2$] and $A \approx 3$ and $A = 4.8$. The second value of $A (= 3)$ corresponds to the estimate of $g_{B^*\gamma}$ from vector meson dominance

$$
g_{B^*\gamma} = \frac{2}{3} g_{B^*\rho} \frac{f_{\rho^-}}{m_\rho^2} = 2.76 \text{ GeV}^{-1}
$$

where $g_{B^*\rho^-} = \sqrt{2}(11) \text{ GeV}^{-1}$ obtained in [36] and $f_{\rho^-}/m_\rho = 205 \text{ MeV}$. The third value of $A (= 4.8)$ gives more or less the width for $B^* \rightarrow B\gamma$ as obtained from the MI transition in non-relativistic quark model (NRQM). These values give the decay width for the $B^* \rightarrow B\gamma$ transition as 23 keV, 5.5 keV and 0.8 keV, respectively, while the MI transition in NRQM predicts it to be 0.9 keV. These predictions are testable when the above decay width is experimentally measured.
4.5 Decay distribution

The Dalitz plot density

\[
\rho(x,y) = \frac{d^2\Gamma}{dx\,dy} = \frac{d^2\gamma_{IB}}{dx\,dy} + \frac{d^2\Gamma_{SD}}{dx\,dy} + \frac{d^2\Gamma_{INT}}{dx\,dy}
\]

\[
= \rho_{IB}(x,y) + \rho_{SD}(x,y) + \rho_{INT}(x,y)
\]

(4.84)

is a Lorentz invariant which contains the form factors \(F_V\) and \(F_A\) in the following form [74, 75, 77]

\[
\rho_{IB}(x,y) = A_{IB} f_{IB}(x,y)
\]

\[
\rho_{SD}(x,y) = A_{SD} M_B^2 \left[(F_V + F_A)^2 f_{SD+}(x,y) + (F_V - F_A)^2 f_{SD-}(x,y)\right]
\]

\[
\rho_{INT}(x,y) = A_{INT} M_B \left[(F_V + F_A) f_{INT+}(x,y) + (F_V - F_A) f_{INT-}(x,y)\right]
\]

where

\[
f_{IB}(x,y) = \left(\frac{1 - y + r_l}{x^2(x + y - 1 - r_l)}\right) \times \left(x^2 + 2(1 - x)(1 - r_l) - \frac{2x r_l (1 - r_l)}{x + y - 1 - r_l}\right)
\]

\[
f_{SD+}(x,y) = (x + y - 1 - r_l) ((x + y - 1)(1 - x) - r_l)
\]

\[
f_{SD-}(x,y) = (1 - y + r_l) ((1 - x)(1 - y) + r_l)
\]

\[
f_{INT+}(x,y) = \left(\frac{1 - y + r_l}{x(x + y - 1 - r_l)}\right) ((1 - x)(1 - x - y) + r_l)
\]

\[
f_{INT-}(x,y) = \left(\frac{1 - y + r_l}{x(x + y - 1 - r_l)}\right) (x^2 - (1 - x)(1 - x - y) - r_l)
\]

and

\[
A_{IB} = 4r_l \left(\frac{f_B}{M_B}\right)^2 A_{SD}
\]

\[
A_{SD} = \frac{\alpha}{2} \frac{\gamma^2}{2} \frac{|V_{ub}|^2}{32\pi^2} M_B^5
\]

\[
A_{INT} = \epsilon r_l \left(\frac{f_B}{M_B}\right) A_{SD}
\]
The $SD^+$ term reaches its maximum at $x = 2/3$, $y = 1$, which corresponds to $\theta_{1\gamma} = \pi$. The $SD^-$ term reaches its maximum at $x = 2/3$, $y = 1/3$, corresponding to $\theta_{1\gamma} = 0$. Indeed, for a lepton of maximal energy ($y = 1$), only “right-handed” photons contribute. In this situation, the photon and the neutrino must be emitted in the direction opposite to that of the lepton. Angular momentum conservation forces the photon spin to be opposite to the total lepton spin, and the photon helicity has the same sign as that of the lepton. Then the photon and the neutrino are emitted parallel. This configuration corresponds to a neutrino of maximal energy ($E_\nu = E_\nu^{\text{max}}$ when $x + y = 1$). In this case, only the “left-handed” photon contributes. When $x + y = 1$, the $IB$ contribution becomes very large: this corresponds to $\theta_{1\gamma} = 0$. Consequently, it is very difficult to distinguish experimentally between the $IB$ and the $SD^-$ contribution. To summarize, an experiment performed in the region $\theta_{1\gamma} \approx \pi$ is essentially sensitive to $(F_V + F_A)^2$.

The form factors calculated in Eq. (4.69) can be expressed in terms of the dimensionless variable $x$,

$$F(x) = \frac{F(0)}{x \left[ 1 - (1 - x) / (M_1/M)^2 \right]}, \quad (4.85)$$

where $x$ is defined in Eq. (4.20) and $q^2$ in Eq. (4.23). After restoring subscripts, the form factors $F_V(q^2)$ in [Eq. (4.74)] and $F_A(q^2)$ in Eq. (4.75) can be written as

$$F_V(x) = \frac{F_V(0)}{x \left[ 1 - (1 - x) / (M_{B^*} / M_{B^*})^2 \right]}, \quad (4.86)$$

$$F_A(x) = \frac{F_V(0)}{x \left[ 1 - (1 - x) / (M_{B^*} / M_{B^*})^2 \right]}, \quad (4.87)$$

where

$$F_{V,A}(0) = \frac{2g_{1}(0)}{M_B}$$

We use these in Eq. (4.84) and integrate over $x$ and $y$ in the limit as mentioned in Eq. (4.24). The IB contribution diverges for the minimum value of $x$; we take an arbitrary lower limit for $x$ i.e. $x_{\text{min}} \approx y_1$ for which the divergence problem is cured and the IB part gives some definite value $O(10^{-26})$. But as the energy of the photon is increased, it approaches zero at $x_{\text{max}}$. Therefore in the total decay width, this does not contribute much. The $SD$ part is the most dominant part of the decay width which provides almost the whole contribution. This part increases initially with increasing $x$, reaches its peak value and then starts decreasing. The
Figure 4-5: The differential decay rate versus photon energy $x$ is plotted and a comparison is given with various approaches. The solid line (for $A = 0$), the dashed-tripple-dotted line (for $A = 3.0$) and the dotted line (for $A = 4.8$) are our calculation, the dash-dot-dot line is for [64], the dashed line for [66] and the dash-dotted line [67]. The thin-solid line is the Sudakov resummation calculation result from Ref. [66].

$INT$ part of the decay width is an increasingly vanishing contribution and can be neglected in comparison to the $SD$ part, because it is suppressed by $\mathcal{O}(10^{-21})$ and becomes flat (approaches zero) as $x$ (the photon energy) approaches 1 (its maxima). Therefore, this does not contribute fairly to the total decay width of the process.

In Fig. 4-5, the differential decay width of the process is plotted against $x$, and we see that for our calculations, the peak is shifted to a lower value of $x$ as compared to those for Eilam et al., [64], Korchemsky et al. [66] and Chekov et al. [67] and it is due to appearance of double pole in our form factors. So, for the process $B \to \gamma\nu_l$ the branching ratio obtained is

$$ B(B \to \gamma\nu_l) = 0.5 \times 10^{-6} \quad (l = \mu) $$

(4.88)

This value is for the form factors given in Eq. (4.86) and Eq. (4.87) which are obtained by
restricting to the first radial excitation only. Now if we consider the effect of the second radial excitation the expressions for the form factors are given in Eqs. (4.82, 4.83). The branching ratios thus obtained are

\[
B(B \rightarrow \gamma l \nu_l) \approx 0.38 \times 10^{-6} \quad (l = \mu, A = 3.0)
\]

\[
B(B \rightarrow \gamma l \nu_l) \approx 0.32 \times 10^{-6} \quad (l = \mu, A = 4.8)
\]

for the two representative cases of \( A = 3 \) and \( A = 4.8 \) respectively. These are not sensitive to the values of \( A \) in contrast to the decay width of \( B^* \rightarrow B \gamma \). The CLEO Collaboration indicates an upper limit on the branching ratio \( B(B^+ \rightarrow \gamma \nu_\ell e^+) \) of \( 2.0 \times 10^{-4} \) at the 90% confidence level [55]. The predicted values are within the upper limit provided by the CLEO Collaboration but differ from those predicted in [66, 67], namely \((2 - 5) \times 10^{-6}\) and \(0.9 \times 10^{-6}\), respectively. The Monte-Carlo simulation results are given in [87] where the upper limit on the branching ratio for this process is predicted to be \(5.2 \times 10^{-5}\).
Part III

The Radiative Decays

$B \to (K_1, b_1, h_1) \gamma$ at NLO in LEET
Chapter 5

Basic Formulas and Phenomenology

of $B \rightarrow K_1 \gamma$

The third part is devoted to the analysis of the exclusive radiative $B$ meson decays with an axial vector in the final state i.e. in particular the decays $B \rightarrow (K_1, b_1, h_1) \gamma$. We use the factorization formula (3.75) which allows us to factorize perturbatively calculable contributions from nonperturbative form factors and universal light cone distribution amplitudes. We present the complete next-to-leading order results in LEET and to leading power in heavy quark mass limit and include the branching ratios and estimate of direct $CP$ violation. The chapter 5 is about the analysis of the exclusive radiative $B$-meson decay, $B \rightarrow K_1 \gamma$, where $K_1$ is an axial vector meson using LEET. We follow the same framework as done by Ali et. al. [52] for $B \rightarrow K^* \gamma$, because $B \rightarrow K_1 \gamma$ shares many things with it. The only difference is the distribution amplitude (DA) for the daughter meson. As $K_1$ is an axial vector and is distinguished by vector by the $\gamma_5$ in the gamma structure of DA and some non perturbative parameters. But the presence of $\gamma_5$ does not alter the calculation, give the same result for the perturbative part. The higher twist terms are also included through the Gegenbauer moments in the Gegenbauer expansion. The calculation with out Gegenbauer has already been done in QCD factorization framework and using the LCSR results for form factors and decay constant [89, 90]. The NLO calculations for the $B \rightarrow K_1 \gamma$ decay involves an explicit calculation $O(\alpha_s)$ calculation of the matrix elements involving the hard vertex corrections [91–93], the so called hard-spectator contributions involving (virtual) hard gluons radiative corrections off the spectator quarks in the $B$ and $K_1$ mesons [11, 49] and the annihilation topologies [94–96]. These corrections will
shift the theoretical branching ratio and induce the \( CP \) asymmetry in these decays which is highly sensitive to the scale \( \mu \). The calculation performed in for the decay \( B \to K_1 \gamma \) can also be applied to calculate the branching ratio and \( CP \) asymmetry for \( B \to (b_1, h_1) \gamma \) decays, where \( b_1 \) and \( h_1 \) are the corresponding axial vector states for \( \rho \) and \( \omega \) mesons. This is the topic discussed in chapter 6. The results given here have to a great extent already been presented in [97] but here we try to explain them in some more detail.

### 5.1 Introduction and Developments

Rare \( B \) decays involving flavor-changing-neutral-current (FCNC) transitions, such as \( b \to s \gamma \), have received a lot of theoretical interest [98]. First measurements of the decay \( B \to X_s \gamma \) were reported by the CLEO collaboration [100]. These decays are now being investigated more precisely in experiments at the B factories. The current world average based on the improved measurements by the CLEO [101], ALEPH [132] and BELLE collaborations, \( B(B \to X_s \gamma) = (3.22 \pm 0.40) \times 10^{-4} \), is in good agreement with the estimates of the standard model (SM) [103–105], which we shall take as \( B(B \to X_s \gamma) = (3.50 \pm 0.50) \times 10^{-4} \), reflecting the parametric uncertainties dominated by the scheme-dependence of the quark masses. The decay \( B \to X_s \gamma \) also provides useful constraints on the parameters of the supersymmetric theories, which in the context of the minimal supersymmetric standard model (MSSM) have been recently updated [106].

Exclusive decays involving the \( b \to s \gamma \) transition are best exemplified by the decay \( B \to K^* \gamma \), which provide abundant issues for both theorists and experimentalists. After the first measurement at CLEO, \( B \to K^* \gamma \) is now also measured in Belle and Babar:

\[
B(B^0 \to K^{*0}\gamma) = \begin{cases} (4.09 \pm 0.21 \pm 0.19) \times 10^{-5} & \text{Belle}[107] \\ (4.23 \pm 0.40 \pm 0.22) \times 10^{-5} & \text{BaBar}[108] \\ (4.55 \pm 0.70 \pm 0.34) \times 10^{-5} & \text{CLEO}[109] \end{cases}
\]

\[
B(B^+ \to K^{*+}\gamma) = \begin{cases} (4.40 \pm 0.33 \pm 0.24) \times 10^{-5} & \text{Belle}[107] \\ (3.83 \pm 0.62 \pm 0.22) \times 10^{-5} & \text{BaBar}[108] \\ (3.76 \pm 0.86 \pm 0.28) \times 10^{-5} & \text{CLEO}[109] \end{cases}
\]

On theoretical side there have been noticeable advances in \( B \to K^* \gamma \) for a decade. QCD corrections at next-to-leading order (NLO) of \( O(\alpha_s) \) have already been considered [91–93].
Relevant Wilson coefficients have been improved up to three loop level calculations [110, 111]. Recent developments of the QCD factorizations helped one to calculate the hard spectator contributions systematically in the factorized form through the convolution at the heavy quark limit [112]. The detailed analysis of $B \to K^* \gamma$ has also been done at next to leading order in effective theories, such as large energy effective theory (LEET) which has covered some length in the Chapter 1, and in soft-collinear effective theory (SCET) [113].

In addition to $K^*$, higher resonances of kaon also deserve much attention. Recently, Belle has announced the first measurement of $B \to K_1^{+}(1270)\gamma$ [114]

$$B(B^+ \to K_1^{++}\gamma) = (4.23 \pm 0.94 \pm 0.43) \times 10^{-5}$$

Among many reasons to focus on the higher resonances, the first one is that they have much in common with $B \to K^* \gamma$, in particular at quark level both of them are governed by $b \to s \gamma$. Therefore all the achievements of $b \to s \gamma$ can be used in these decays, e.g. the same operators in the operator product expansion and the same Wilson coefficients that are available. The light cone distribution amplitudes (DA) are the same except for the overall factor of $\gamma_5$ and this gives few differences in many calculations [89]. Secondly, it was suggested that $B \to K_{res} (\to K\pi\pi)\gamma$ can provide a direct measurement of the photon polarization and it was shown that large polarization asymmetry $\approx 33\%$ has been produced due to decay of $B$ meson through the kaon resonances [115]. In the presences of anomalous right-handed couplings, the polarization can be severely reduced in the parameter space allowed by current experimental bounds of $B \to X_s \gamma$. It was also argued that the $B$ factories can now make a lot of $BB$ pairs, enough to check the anomalous couplings through the measurement of the photon polarization.

Theorists are also facing challenges from the discrepancy between their predictions and experiments. It was pointed out that the form factor, obtained using the LEET approach for $B \to K^* \gamma$, is found to be smaller compared to the values obtained by QCD sum rules or light-cone sum rules (LCSR) [52]. At this stage, the source of this mismatch is not well understood.

The situation is more complicated on $B \to K_1 \gamma$ decay. Based on the QCD factorization framework combined with the LCSR results, it is predicted that $B(B^0 \to K_1^0(1270)\gamma) \approx (0.825 \pm 0.335) \times 10^{-5}$ at the NLO of $\alpha_s$ which is very small as compared to the experimental value [cf. Eq. (5.3)] [89]. The value of the relevant form factor has been extracted from the experimental data and it is found to be $F_+^{K_1(1270)}(0) = 0.32 \pm 0.03$ which is very large as compared to
Figure 5-1: Leading order contribution by operator $O_7$

$F_{+}^{K_1(1270)}(0)_{\text{LCSR}} = 0.14 \pm 0.03$ obtained by the LCSR. These are contrary to the case of $B \to K^* \gamma$ where the form factor obtained from LCSR is larger than the LEET one and the source of discrepancy is not yet known. For the $B \to K_1 \gamma$ case the possible candidates to explain this discrepancy have also been discussed in detail in the literature [90].

5.2 Leading order contributions

The effective Hamiltonian for $b \to s \gamma$ is given in Sec. 2.1. The leading contribution to $B \to K_1 \gamma$ comes from the electromagnetic operator $O_7$ as shown in Fig. 5-1.

As in the case of the real photon emission ($q^2 = 0$), the only form factor appears in the calculation is $\xi_{\perp}^{(K_1)}$. Therefore one can write

$$\langle O_7 \rangle_A \equiv \langle K_1(p', \epsilon) \gamma(q, e)|O_7|B(p)\rangle = \frac{em_b}{4\pi^2} \xi_{\perp}^{(K_1)} \left[ \epsilon^\ast \cdot q(p + p') \cdot e^\ast - \epsilon^\ast \cdot e^\ast (p^2 - p'^2) + i\epsilon_{\mu\nu} \epsilon^{\mu} q^\ast (p + p')^\nu \right], \quad (5.4)$$

with $\epsilon^\ast$ and $e^\mu$ being the polarization vector for axial kaon and the photon respectively. The
The decay rate is straightforwardly obtained to be \cite{89}

\[
\Gamma(B \rightarrow K_1\gamma) = \frac{G_F^2\alpha m_{B}^2 m_{K}^2}{32\pi^4} |V_{ts}V_{cb}^*|^2 \left(1 - \frac{m_{K}^2}{m_{B}^2}\right)^3 \left|\xi_{\perp}^{(K_1)}\right|^2 |C_{7}^{\text{eff}(0)}|^2 ,
\]

(5.5)

where \(\alpha\) is the fine-structure constant and \(C_{7}^{\text{eff}(0)}\) is the effective Wilson coefficient at leading order.

5.3 Matrix Elements at Next-To-Leading Order of \(O(\alpha_s)\)

At next to leading order of \(\alpha_s\), there are the contributions from the operators \(O_2\) and \(O_8\) along with that of the \(O_7\) in \(B \rightarrow K_1\gamma\) decay. Each operator has its vertex contribution and hard spectator contribution terms.

5.3.1 Vertex Contributions

For the operator \((O_7)\), all the subleading contributions are absorbed into the form factor while the corresponding Wilson coefficient \(C_{7}^{\text{eff}}\) contains its NLO part. The vertex corrections shown in Fig. 5-2 and 5-3 are directly proportional to the form factor \(\xi_{\perp}^{(K_1)}\). The detailed calculation
Figure 5-3:
of these corrections is given in Refs. [93, 111] and the corresponding results are

\[
(O_2)_{VC} = \frac{\alpha_s}{4\pi} \langle O_7 \rangle \left( \frac{416}{81} \ln \frac{m_b}{\mu} + r_2 \right), \tag{5.6}
\]

\[
(O_8)_{VC} = \frac{\alpha_s}{4\pi} \langle O_7 \rangle \left[ -\frac{32}{9} \ln \frac{m_b}{\mu} + \frac{4}{27} (33 - 2\pi^2 + 6i\pi) \right], \tag{5.7}
\]

where

\[
r_2 = \frac{2}{243} \left\{ -833 + 144\pi^2 z^{3/2} + \left[ 1728 - 180\pi^2 - 1296\zeta(3) + (1296 - 324\pi^2) L + 108L^2 + 36L^3 \right] z \right.

+ \left[ 648 + 72\pi^2 + (432 - 216\pi^2) L + 36L^3 \right] z^2 + \left[ -54 - 84\pi^2 + 1092L - 756L^2 \right] z^3 \bigg\}

+ \frac{i16\pi}{81} \left\{ -5 + \left[ 45 - 3\pi^2 + 9L + 9L^2 \right] z + \left[ -3\pi^2 + 9L \right] z^2 + \left[ 28 - 12L \right] z^3 \bigg\}, \tag{5.8}
\]

with \( z = m_C^2/m_b^2 \), \( L \equiv \ln z \), and \( \zeta(x) \) being the Liemann \( \zeta \)-function.

### 5.3.2 Hard Spectator Contribution

The Hard spectator contribution is well described by the convolution between the hard kernel \( T_k \) and the light cone distribution amplitudes of the involved mesons, \( \Phi_B \) and \( \Phi_{K_1} \), and can be written as \( \Phi_B \otimes T_k \otimes \Phi_{K_1} \). The corresponding decay amplitude can be calculated in the form of convolution of a formula, whose leading term can be expressed as \([52]\)

\[
\Delta M^{(HSA)} = \frac{4\pi\alpha_s C_F}{N_c} \int_0^1 \int_0^\infty d^+ M^{(B)}_{jk} M^{(K_1)}_{li} T_{ijkl}, \tag{5.9}
\]

where \( N_c \) is the number of colors, and \( C_F = (N_c^2 - 1)/(2N_c) \) is the Casimir operator eigenvalue in the fundamental representation of the color SU\((N_c)\) group. The leading-twist two-particle light-cone projection operators \( M^{(B)}_{jk} \) and \( M^{(K_1)}_{li} \) of \( B \)- and \( K_1 \)-mesons in the momentum
Figure 5.4: Feynman diagrams contributing to the spectator corrections involving the $O_7$ operator in the decay $B \to K_1 \gamma$. The curly (dashed) line here and in subsequent figures represents a gluon (photon).

The corresponding diagrams are [49, 118]

\[ M_{jk}^{(B)} = -\frac{i f_B M}{4} \left[ \frac{1 + \gamma^\mu}{2} \left\{ \phi_+^{(B)}(l_+) \gamma^\mu + \phi_-^{(B)}(l_+) \left( \gamma^\mu - i + \gamma_+^\mu \frac{\partial}{\partial l_+^\mu} \right) \right\} \gamma_5 \right]_{jk} \mid_{l = (l_+/2) n_+}, \quad (5.10) \]

\[ M_{ii}^{(K_1)} = -\frac{i}{4} \left[ f_{\parallel}^{(K_1)}(q^\mu \bar{q}) \gamma_\sigma \phi_\perp^{(K_1)}(u) + f_{\perp}^{(K_1)} \left( \frac{\gamma}{M} \left( \bar{u} q^\sigma \right) \right) \gamma_\tau \phi_\perp^{(K_1)}(u) \right]_{ii}, \quad (5.11) \]

where $f_B$ is the $B$-meson decay constant, $f_{\parallel}^{(K_1)}$ and $f_{\perp}^{(K_1)}$ are the longitudinal and transverse $K_1$-meson decay constants, respectively, and $\epsilon_\mu$ is the $K_1$-meson polarization vector. These projectors include also the leading-twist distribution amplitudes $\phi_+^{(B)}(l_+)$ and $\phi_-^{(B)}(l_+)$ of the $B$-meson and $\phi_\parallel^{(K_1)}(u)$ and $\phi_\perp^{(K_1)}(u)$ of the $K_1$-meson. $T_{ijkl}$ is the hard-scattering amplitude. The kinematical relations are used to calculate the hard spectator contributions are [49]

\[ p_0^\mu \simeq m_0 \epsilon^\mu, \quad l_\mu = \frac{l^\mu}{2} n_+^\mu + l_\perp^\mu n_-^\mu, \]

\[ k_1^\mu \simeq u E n_+^\mu + k_\perp^\mu + O(k_\perp^2) \quad k_2^\mu \simeq \bar{u} E n_-^\mu - k_\perp^\mu + O(k_\perp^2), \]

\[ v^2 = 1, \quad v^\mu = (n_-^\mu + n_+^\mu)/2 \quad E \simeq M/2 \]

\[ q^\mu = \omega n_+^\mu \quad \omega = M/2 \]

To calculate $T_{ijkl}$ let's consider the contribution from all the possible diagrams as done for the $B \to V \gamma$ [52].

**Spectator corrections due to the electromagnetic dipole operator $O_7$**

The corresponding diagrams are presented in Fig. 5.4 and the explicit expression is given by
Figure 5-5: Feynman diagrams contributing to the spectator corrections involving the $O_8$ operator in the decay $B \to K\gamma$. Row a: Photon is emitted from flavor changing quark line. Row b: Photon radiation off the spectator quark line.

\[
T_{ijkl}^{(1)} = -\frac{G_F}{\sqrt{2}} V_{td}^* V_{tb} C_F \tau (\mu) \frac{c m_b(\mu)}{4 \pi^2} \frac{\gamma_{\mu} k l}{(l - k)^2} \times \left[ (q \sigma e^*)(1 + \gamma_5) \frac{y_b l_{\ell} - y_e}{(p_b + l - k)^2 - m_b^2} \gamma_{\mu} + \gamma_{\mu} (k_1 + k_2 + l)^2 (q \sigma e^*)(1 + \gamma_5) \right]_{ij} \tag{5.12}
\]

where the short hand notation is used for $(q \sigma e^*) = \sigma^{\mu \nu} q_{\mu} e^*_\nu$.

**Spectator corrections due to the chromomagnetic dipole operator $O_8$**

The corresponding diagrams are presented in Fig. 5-5. The first two diagrams (Fig. 5-5a) show the corrections for the case when the photon is emitted from the flavor changing quark line and
the result is

\[
\mathcal{T}_{ijkl}^{(3a)} = -i \frac{G_F}{\sqrt{2}} V_{td} V_{tb} C_{8}^{\gamma}(\mu) \frac{e m_b(\mu)}{4 \pi^2} \frac{l - k_2}{(l - k_2)^2} \gamma_{\nu}^{ij} \frac{l - k_2}{(l - k_2)^2} \gamma_{\nu}^{ij} \left[ q^* \left( \frac{p_b + l - k_2}{(p_b + l - k_2)^2} \sigma_{\mu\nu} (1 + \gamma_5) \right) \right. \\
+ \left. \sigma_{\mu\nu} (1 + \gamma_5) \frac{h_1 + h_2 - l f + m_b}{(k_1 + k_2 + l)^2 - m_b^2} q^* \right]_{ij} \tag{5.13}
\]

Fig. 5-5b contains the diagrams with the photon emission from the spectator quark which results into the following hard-scattering amplitude:

\[
\mathcal{T}_{ijkl}^{(2b)} = i \frac{G_F}{\sqrt{2}} V_{td} V_{tb} C_{8}^{\gamma}(\mu) \frac{e Q_{d[ui]} m_b(\mu)}{4 \pi^2} \\
\times \left[ \sigma_{\mu\nu} (1 + \gamma_5) \right]_{ij} \left( \frac{p_b - k_1}{p_b - k_1} \right)_\mu \\
\times \left[ q^* \left( \frac{p_b + l - k_1}{(p_b + l - k_1)^2} \gamma_{\nu}^{ij} \right) + q^* \left( \frac{h_1 + h_2 - p_b}{(k_1 + k_2 + p_b)^2} \gamma_{\nu}^{ij} \right) \right]_{kl} \tag{5.14}
\]

where \( Q_{d[ui]} \) is the charge of the spectator quark.

Spectator corrections involving the penguin-type diagrams and the operator \( O_2 \)

The corresponding diagrams are shown in Figs. 5-6, 5-7 and 5-8. The hard spectator contribution corresponding to the diagrams in Fig. 5-6a is

\[
\mathcal{T}_{ijkl}^{(3a)} = \frac{G_F}{\sqrt{2}} \frac{e}{24 \pi^2} \sum_{f = u, c} V_{fd} V_{tb} C_{2}^{(f)}(\mu) \Delta F_1 \left( z_2^{(f)} \right) \gamma_{\nu}^{ij} \left[ \gamma_{\nu}^{ij} \right]_{kl} \\
\times \left[ \left( \gamma_{\nu} - \frac{(k_2 - l)_\nu (h_2 - h f)}{(k_2 - l)_\nu} \right) (1 - \gamma_5) \frac{h_1 + h_2 - l f + m_b}{(k_1 + k_2 + l)^2 - m_b^2} q^* \right]_{ij} \\
+ q^* \left( \frac{p_b + l - k_2}{(p_b + l - k_2)^2} \left( \gamma_{\nu} - \frac{(k_2 - l)_\nu (h_2 - h f)}{(k_2 - l)_\nu} \right) (1 - \gamma_5) \right)_{ij} \tag{5.15}
\]

81
Figure 5-6: Feynman diagrams contributing to the spectator corrections involving the $O_2$ operator in the decay $B \to K \gamma$. Row a: Photon is emitted from flavor changing quark line. Row b: Photon radiation off the spectator quark line.
Figure 5-7: Feynman diagrams contributing to the spectator corrections in $B \to K_1 \gamma$ decays involving the $O_2$ operator for the case when both the photon and the virtual gluon are emitted from the internal (loop) quark line.

and from the diagrams in Fig. 5-6b, where the photon is emitted from the spectator quark line yield:

$$T^{(3b)}_{ijkl} = \frac{G_F e Q_d^{[u]} \sum_{f=u,c} V_{fd}^* V_{fb} C_2^{(f)}(\mu) \Delta F_1 \left( z_0^{(f)} \right)}{\sqrt{2} \frac{e Q_d^{[u]} \sum_{f=u,c} V_{fd}^* V_{fb} C_2^{(f)}(\mu) \Delta F_1 \left( z_0^{(f)} \right)}{24\pi^2}} \times \left[ \frac{q^\nu \cdot \hat{p}_b - \hat{p}_b}{(k_1 + \hat{k}_1 + \hat{p}_b)q^\nu + \gamma^\nu (p_b + \hat{p}_b - k_1)^2 q^*_{kl}} \right] \times \left[ \frac{(p_b - k_1)_{\mu}}{(p_b - k_1)^2} (p_b - k_1) \right] (1 - \gamma_5) \right]_{ij}$$

(5.16)

The detailed discussion about $\Delta F_1 \left( z_1^{(f)} \right)$ and $\Delta F_1 \left( z_1^{(f)} \right)$ can be found in[52].

The contributions from the diagrams in Fig. 5-7 can be written as

$$T^{(4)}_{ijkl} = -\frac{G_F e}{\sqrt{2} \frac{e Q_d^{[u]} \sum_{f=u,c} V_{fd}^* V_{fb} C_2^{(f)}(\mu)}{6\pi^2 (k_2 - l)^2 (q_k - k_2 - l)}} \sum_{f=u,c} V_{fd}^* V_{fb} C_2^{(f)}(\mu) \times \left[ \begin{array}{c} q^\mu E(k_2 - l, e^*, q) - (q |k_2 - l|) E(\nu, e^* q) \\ + (e^* |k_2 - l|) E(\nu, \nu, k_2 - l) \\ - (q |k_2 - l|) E(e^*, \nu, k_2 - l) \\ + (k_2 - l)^2 E(\nu, e^*, q) \\ + (k_2 - l)^2 E(e^*, k_2 - l, q) \end{array} \right] \left( \begin{array}{c} \Delta i_5 \left( z_0^{(f)}, z_1^{(f)}, 0 \right) \\ \Delta i_{25} \left( z_0^{(f)}, z_1^{(f)}, 0 \right) \end{array} \right) (1 - \gamma_5)$$

(5.17)

where

$$E(\mu, \nu, \rho) \equiv \frac{1}{2} (\gamma_\mu \gamma_\nu \gamma_\rho - \gamma_\rho \gamma_\mu \gamma_\nu) = -i \varepsilon_{\mu\nu\rho\sigma} \gamma^\sigma \gamma_5.$$  

(5.18)
and the form of $\Delta i_6 \left( z_0^{(f)}, z_1^{(f)}, 0 \right)$ and $\Delta i_{25} \left( z_0^{(f)}, z_1^{(f)}, 0 \right)$ along with the detailed discussion is given in [52] and are summarized as:

\[
\begin{align*}
\Delta i_6(z_0, z_1, 0) &= -1 + \frac{z_1}{z_0 - z_1} \left[ Q_0(z_0) - Q_0(z_1) \right] - \frac{2}{z_0 - z_1} \left[ Q_-(z_0) - Q_-(z_1) \right], \\
\Delta i_{25}(z_0, z_1, 0) &= Q_0(z_0) - Q_0(z_1).
\end{align*}
\]  

(5.19)

with

\[
Q_(z) = \int_0^1 \frac{du}{u} \ln \left[ 1 - z u (1 - u) \right]
\]

\[
Q_-(z) = \frac{1}{2} \left( \ln \frac{u_-(z)}{u_+(z)} + i\pi \right)^2
\]

(5.20)

Finally, there are the diagrams where the photon is emitted from the internal quark line due to the effective $b \rightarrow s \gamma$ interaction and a gluon is exchanged between the spectator quark and the $b$ or $s$ quark as shown in Fig. 5-8. For the on shell photon such kind of diagrams do not contribute and hence the contribution comes from the Fig. 5-8 is zero.
\[5.4 \quad \mathcal{O}(\alpha_s)\)-corrected matrix elements for \(B \rightarrow K_1\gamma\) decays \]

Using Eqs. (5.10) and (5.11) along with the hard scattering matrix derived in the Eqs. (5.12-5.17), we can write from Eq. (5.9) as

\[
\Delta M_{sp}^{(K_1)} = \frac{G_F}{\sqrt{2}} \frac{c_{\alpha s} C_F}{4\pi N_c} f_B \int_{\perp}^{(K_1)M} \left[(e^+e^-) + i \epsilon_{\alpha \beta \gamma \delta} \epsilon^{\alpha'} \epsilon^{\beta'} \epsilon^{\gamma'} \epsilon^{\delta'} \right] \sum_{k=1}^{5} \Delta H_k^{(K_1)},
\]

(5.21)

where \(\epsilon_{\mu \nu \rho \sigma} a^\mu b^\nu c^\rho d^\sigma\) and the upper index \(K_1\) characterizes the final axial meson. The dimensionless functions \(\Delta H_k^{(K_1)} (k = 1, 2, 3, 4, 5)\) describe the contributions of the sets of Feynman diagrams presented in Figs. 5.3 - 5.7, respectively. In the leading order of the inverse \(B\)-meson mass (\(\sim \Lambda_{QCD}/M\)), the result reads as follows:

\[
\Delta H_1^{(K_1)}(\mu) \approx V_{tb}^* V_{ub} \frac{C_7^{\text{eff}}(\mu)}{m_b(\mu)} \left[\langle l_{+}^{-1}\rangle_+ \langle \bar{u}^{-1}\rangle_{\perp}^{(K_1)}(\mu) + \langle l_{-}^{-1}\rangle_+ \langle \bar{u}^{-2}\rangle_{\perp}^{(K_1)}(\mu) \right],
\]

(5.22)

\[
\Delta H_2^{(K_1)}(\mu) \approx \frac{1}{3} V_{ts}^* V_{tb} \frac{C_6^{\text{eff}}(\mu)}{m_b(\mu)} \langle l_{+}^{-1}\rangle_+ \langle u^{-1}\rangle_{\perp}^{(K_1)}(\mu),
\]

(5.23)

\[
\Delta H_3^{(K_1)}(\mu) \approx 0,
\]

(5.24)

\[
\Delta H_4^{(K_1)}(\mu) \approx \frac{1}{3} C_2(\mu) M \langle l_{+}^{-1}\rangle_+ \left[V_{ts}^* V_{ub} \langle \bar{u}^{-1}\rangle_{\perp}^{(K_1)}(\mu) + V_{cs}^* V_{cb} h^{(K_1)}(z, \mu) \right],
\]

(5.25)

\[
\Delta H_5^{(K_1)}(\mu) \approx 0,
\]

(5.26)

where \(z = m_c^2/m_b^2\) and the short-hand notation used are for the integrals over the mesons distribution functions:

\[
\langle f \rangle_{\pm,\perp}^{(K_1)}(\mu) \equiv \int_{0}^{\infty} dl_{\pm} f_{\pm}^{(B)}(l_{\pm}),
\]

(5.27)

and for convenience the following function is introduced:

\[
h_{(K_1)}(z, \mu) = \left. \frac{\Delta i_5(z_0^{(e)}(0,0) + 1)}{z} \right|_{\perp}^{(K_1)}
\]

(5.28)
The expressions of $\Delta H^1_{\zeta}(K)$ given in Eqs. (5.22-5.26) are similar to those obtained for $B \to K^* \gamma$ [c.f. Eqs. (4.4-4.6) of Ali et al. [52]] which show that the additional $\gamma$ present in the DA of $K_1$ has no effect on the calculations. Using the above Equations one can write Eq. (5.21) as

$$
\Delta M_{sp} = \frac{G_F}{\sqrt{2}} \frac{V_{tb}^* V_{tb}}{4 \pi} \Delta F^1_{1}(\mu) \left[ (pP) (e^* e^-) + i \epsilon (e^* e^* + pP) \right] \times \left[ C^0_{\alpha} (\mu) + \frac{1}{3} C^0_{8}(\mu) \frac{\langle u^- \rangle^1_{(K_1)}(\mu)}{\langle u^- \rangle^1_{(K_1)}(\mu)} + \frac{1}{3} C_{2}(\mu) \left( 1 + \frac{V_{tb}^* V_{tb}}{V_{ts}^* V_{tb}} \frac{h_{(K_1)}(\mu)}{\langle u^- \rangle^1_{(K_1)}(\mu)} \right) \right] \tag{5.29}
$$

where

$$
\Delta F^1_{1}(\mu) = \frac{8 \pi^2 f_{B} f_{1}(\mu)}{N_c M_{l_{B}}^1} \langle u^- \rangle^1_{(K_1)}(\mu), \tag{5.30}
$$

is the dimensionless quantity. $\lambda_{l_{B}}^{-1} = \langle l^+ \rangle^1_+ \langle l^+ \rangle^1_-$ is the first negative moment of the $B$-meson distribution function $\phi^0_{l^+}(l_+)$ which is typically estimated at $\lambda_{l_{B}}^{-1} = (3 \pm 1) \text{ GeV}$ [49, 119]. In a recent paper by Braun et al. [120], the scale dependence of this moment is worked out at next to leading order and the value obtained is $\lambda_{l_{B}}^{-1}(1 \text{ GeV}) = (2.15 \pm 0.50) \text{ GeV}$. At the scale $\mu_{sp} = \sqrt{\mu_{B} \Lambda_H}$ of the hard-spectator corrections, and for the central values of the parameters are shown in Table 5.1.

The analytical expression for the function $h^{(V)}(z, \mu)$ for the vector meson is given in [52]. We will proceed to give an analytical result for the axial meson. One can write the leading twist distribution amplitude $\phi^0_{l^+}(u, \mu)$ as [118]

$$
\phi^0_{l^+}(u, \mu) = 6 u \bar{u} \left[ 1 + \sum_{n=1}^{\infty} a^{(K_1)}_{\perp n}(\mu) C_n^{3/2}(u - \bar{u}) \right], \tag{5.31}
$$

where $C_n^{3/2}(u - \bar{u})$ are the Gegenbauer polynomials [$C_1^{3/2}(u - \bar{u}) = 3(u - \bar{u}), C_2^{3/2}(u - \bar{u}) = 3 [5(u - \bar{u})^2 - 1]/2$, etc.] and $a^{(K_1)}_{\perp n}(\mu)$ are the corresponding Gegenbauer moments. These moments are scale dependent and so should be evaluated at the scale $\mu$; their scale dependence is governed by [118]:

$$
a^{(K_1)}_{\perp n}(\mu) = \left( \frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} \right)^{\gamma_n/\beta_0} a^{(K_1)}_{\perp n}(\mu_0), \quad \gamma_n = 4 C_F \left( \sum_{k=1}^{n} \frac{1}{k} - \frac{n}{n + 1} \right), \tag{5.32}
$$

where $\beta_0 = (11 N_c - 2 \gamma_f)/3$ and $\gamma_n$ is the one-loop anomalous dimension with $C_F = (N_c^2 - 1)/(2 N_c) = 4/3$. In the limit $\mu \to \infty$ the Gegenbauer moments vanish, $a^{(K_1)}_{\perp n}(\mu) \to 0$, and the
leading-twist transverse distribution amplitude has its asymptotic form:

\[ \phi_{\perp}^{(K_1)}(u, \mu) \to \phi_{\perp}^{(as)}(u) = 6u\bar{u}. \]  

(5.33)

A simple model of the transverse distribution which includes contributions from the first \(a_{11}^{(K_1)}(\mu)\) and the second \(a_{12}^{(K_1)}(\mu)\) Gegenbauer moments only is used here in the analysis. In this approach the quantities \(<u^{-1}>_{\perp}^{(K_1)}\) and \(<\bar{u}^{-1}>_{\perp}^{(K_1)}\) are:

\[ <u^{-1}>_{\perp}^{(K_1)} = 3 \left[ 1 - a_{11}^{(K_1)}(\mu) + a_{12}^{(K_1)}(\mu) \right], \quad <\bar{u}^{-1}>_{\perp}^{(K_1)} = 3 \left[ 1 + a_{11}^{(K_1)}(\mu) + a_{12}^{(K_1)}(\mu) \right]. \]  

(5.34)

and depend on the scale \(\mu\) due to the coefficients \(a_{1n}^{(K_1)}(\mu)\). The calculation for the axial \(K\) meson without Gegenbauer moments is done in detail [89, 90]. In our calculation we will incorporate these effects in the calculations and will check the sensitivity of branching ratio with the LEET form factors in the presence of these moments. The Gegenbauer moments were evaluated at the scale \(\mu_0 = 1\) GeV, yielding [118]: \(a_{11}^{(K^*)}(1\) GeV\) = 0.20 ± 0.05 and \(a_{12}^{(K^*)}(1\) GeV\) = 0.04 ± 0.04 for the \(K^*-\)meson. The value of these two Gegenbauer moments have recently been modified and it has been pointed out that these values are now larger in magnitude, have larger errors and, moreover, the first Gegenbauer moment changes its sign [121]. The new values of these Gegenbauer moments are \(a_{11}^{(K^*)}(1\) GeV\) = -0.34 ± 0.18 and \(a_{12}^{(K^*)}(1\) GeV\) = 0.13 ± 0.08 for the \(K^*-\)meson [122]. We will use the same value for the \(K_1\) because one can see that the value is not changed for the axial meson also because changing the scale has not the noticeable effect on the coupling constants and so on the Gegenbauer moments [118, 123]. In the same manner, the function \(h^{(K_1)}(z, \mu)\) introduced in Eq. (5.29) can be presented as an expansion on the Gegenbauer moments:

\[
h^{(K_1)}(z, \mu) = h_0(z) + a_{11}^{(K_1)}(\mu) h_1(z) + a_{12}^{(K_1)}(\mu) h_2(z) \\
= \left[ 1 + 3a_{11}^{(K_1)}(\mu) + 6a_{12}^{(K_1)}(\mu) \right] \left\langle (\Delta i_5 + 1)/\bar{u} \right\rangle_\perp^{(0)} \\
- 6 \left[ a_{11}^{(K_1)}(\mu) + 5a_{12}^{(K_1)}(\mu) \right] \left\langle (\Delta i_5 + 1) \right\rangle_\perp^{(0)} + 30 a_{12}^{(K_1)}(\mu) \left\langle \bar{u} (\Delta i_5 + 1) \right\rangle_\perp^{(0)}. \]  

(5.35)
where another short-hand notation is introduced for the integral:

\[
\langle f(u) \rangle_{\perp}^{(0)} \approx \int_0^1 du \, f(u) \phi_{\perp}^{(as)}(u). \tag{5.36}
\]

The detail of relevant functions as well as the analytical form of the \(\langle (\Delta i_5 + 1)/\bar{u} \rangle_{\perp}^{(0)}, \langle \Delta i_5 + 1 \rangle_{\perp}^{(0)}\) and \(\langle \bar{u} (\Delta i_5 + 1) \rangle_{\perp}^{(0)}\) is given in ref. [52] and is

\[
\langle \Delta i_5 (\bar{u}/z, 0, 0) + 1 \rangle_{\perp}^{(0)} = -2z \left[ 6 - \ln^3 z + 6Q_0(1/z) \right. \\
+ \frac{3i\pi \ln z}{\sqrt{1 - 4z}} \left[ 2 + Q_0(1/z) \right] - 6(1 - 2z)Q_-(1/z) \\
- 6 \left[ 2 \ln u_+(1/z) - i\pi \right] Li_2(u_+(1/z)) \\
- 6 \left[ 2 \ln u_-(1/z) + i\pi \right] Li_2(u_-(1/z)) \\
+ 12 \left[ Li_3(u_+(1/z)) + Li_3(u_-(1/z)) \right], \tag{5.37}
\]

\[
\langle \Delta i_5 (\bar{u}/z, 0, 0) + 1 \rangle_{\perp}^{(0)} = \frac{3z}{2} (5 - 12z) + 9z (1 - 2z)Q_0(1/z) \\
- 6z \left( 1 - 4z + 6z^2 \right)Q_-(1/z), \tag{5.38}
\]

\[
\langle \bar{u} [\Delta i_5 (\bar{u}/z, 0, 0) + 1] \rangle_{\perp}^{(0)} = \frac{z}{18} (41 + 144z - 720z^2) \\
- \frac{z}{3} (5 + 34z - 120z^2)Q_0(1/z) \\
- 2z \left( 1 - 18z^2 + 40z^3 \right)Q_-(1/z), \tag{5.39}
\]

where \(Q_0(1/z)\) and \(Q_-(1/z)\) are the functions defined in Eq. (5.20), and the dilogarithmic \(Li_2(z)\) and trilogarithmic \(Li_3(z)\) functions have their usual definitions:

\[
Li_2(z) = - \int_0^z \frac{\ln(1-t)}{t} \, dt, \quad Li_3(z) = \int_0^z \frac{Li_2(t)}{t} \, dt.
\]

The real and imaginary parts of the functions \(h_n(z)\) are presented in Figs. 5-9 (for \(n = 0\)) and 5-10 (for \(n = 1\) and \(n = 2\)). The dependence on \(z = m_c^2/m_b^2\) of the function \(h^{(K_1)}(z, \mu)\) at the mass scale \(\mu \approx \mu_{sp} = 1.52\) GeV of hard-spectator corrections is presented in Fig. 5-11. We
Figure 5-9: The function \( h_0(z) \) is plotted against the ratio \( m_f^2/m_b^2 \), where \( m_b \) is the \( b \)-quark mass. The solid curve is the real part of the function and the dashed curve is the imaginary part.

Figure 5-10: The function \( h_1(z) \) (left figure) and \( h_2(z) \) (right figure) are plotted against the ratio \( m_f^2/m_b^2 \), where \( m_b \) is the \( b \)-quark mass. The solid curves are the real parts of the function and the dashed curves are the imaginary parts.
Figure 5-11: The function $h^{(K_1)}(z, \mu_{sp})$ is plotted against the ratio $m_f^2/m_b^2$ at the mass scale of the hard spectator correction $\mu_{sp} = 1.52$ GeV. The solid curve is the real part of the function and the dashed curve is the imaginary part.

We have observed that our plots given in Fig. 5-10 and Fig. 5-11 are different to those given by Ali et al. (c.f. Fig. 7 and Fig. 8) [52]. The authors of the article [52] agreed to this observation [116], pointed out by Gilani [117]. The value of the corresponding Gegenbauer moments used for the evaluation are given in Table 5.1.

The amplitude (5.21) is proportional to the tensor decay constant $f_{\perp}^{(K_1)}$ of the axial meson which is a scale dependent parameter. As for the Gegenbauer moments $a_{\perp n}^{(K_1)}$, there values were defined at the mass scale $\mu_0 = 1$ GeV from the LCSR is [124]: $f_{\perp}^{(K_1)}(1 \text{ GeV}) = 122$ MeV. Their values at an arbitrary scale $\mu$ can be obtained with the help of the evolution equation [118]:

$$f_{\perp}^{(V)}(\mu) = \left( \frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} \right)^{4/3\beta_0} f_{\perp}^{(V)}(\mu_0). \tag{5.40}$$

Central values of the tensor decay constants at the scales $\mu_{sp} = 1.52$ GeV and $m_{b,pole} = 4.65$ GeV are presented in Table I.
<table>
<thead>
<tr>
<th>$K_1(1270)$</th>
<th>$K_1(1400)$</th>
</tr>
</thead>
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<td>$\mu$</td>
<td>$m_{b,\text{pole}}$</td>
</tr>
<tr>
<td>$\mu_1 (\text{GeV})$</td>
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</tr>
<tr>
<td>$a_1(\mu)$</td>
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</tr>
<tr>
<td>$a_{12}(\mu)$</td>
<td>0.118</td>
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<td>$\langle \bar{u}^{-1} \rangle^{(V)}(\mu)$</td>
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</tr>
<tr>
<td>$h^{(V)} / \langle \bar{u}^{-1} \rangle^{(V)}$</td>
<td>$1.21 + i0.73$</td>
</tr>
<tr>
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</tr>
<tr>
<td>$\Delta F_1^{(V)}(\mu)$</td>
<td>0.55</td>
</tr>
</tbody>
</table>

Table 5.1: Input quantities and their values used in the theoretical analysis

Branching Ratio for $B \to K_1 \gamma$

The branching ratio for $B \to K_1 \gamma$ is simply given by

$$
B_{th}(B \to K_1 \gamma) = \tau_B \Gamma_{th}(B \to K_1 \gamma) = \tau_B \frac{G_F^2 \alpha |V_{tb} V_{ts}^*|^2}{32 \pi^4} \frac{m_b}{m_{K_1}} M^3 \left[ \xi^{(K_1)}_1 \right]^2 \left( 1 - \frac{m_{K_1}^2}{M^2} \right)^3 \left| C_7^{(0)} \text{eff} + A^{(1)}(\mu) \right|^2
$$

(3.41)

where $G_F$ is the Fermi coupling constant, $\alpha = \alpha(0) = 1/137$ is the fine-structure constant, $m_{b,\text{pole}}$ is the pole $b$-quark mass, $M$ and $m_{K_1}$ are the $B$- and $K_1$-meson masses, and $\tau_B$ is the lifetime of the $B^0$- or $B^+$-meson. The value of these constants is used from [52] for the numerical analysis. For this study, we consider $\xi^{(K_1)}_1$ as a free parameter and we will extract its value from the current experimental data on $B \to K_1 \gamma$ decays.

The function $A^{(1)}$ in Eq. (3.41) can be decomposed into the following three components:

$$
A^{(1)}(\mu) = A^{(1)}_{C_7}(\mu) + A^{(1)}_{\text{ver}}(\mu) + A^{(1)}_{sp}(\mu_{sp})
$$

(5.42)

Here, $A^{(1)}_{C_7}$ and $A^{(1)}_{\text{ver}}$ are the $O(\alpha_s)$ (i.e. NLO) corrections due to the Wilson coefficient $C_7^{\text{eff}}$ and in the $b \to s \gamma$ vertex, respectively, and $A^{(1)}_{sp}$ is the $O(\alpha_s)$ hard-spectator corrections to the
\[ A^{(1)}_{G_7}(\mu) = \frac{\alpha_s(\mu)}{4\pi} C^{(1)\text{eff}}_G(\mu), \]  
\[ A^{(1)}_{\text{eff}}(\mu) = \frac{\alpha_s(\mu)}{4\pi} \left\{ \frac{32}{81} \left[ 13C^{(0)}_2(\mu) + 27C^{(0)\text{eff}}_7(\mu) - 9C^{(0)\text{eff}}_8(\mu) \right] \ln \frac{m_b}{\mu} - \frac{20}{3} C^{(0)\text{eff}}_7(\mu) + 4 \frac{27}{33 - 2\pi^2 + 6\pi i} C^{(0)\text{eff}}_8(\mu) + r_2(z) C^{(0)}_2(\mu) \right\}, \]  
\[ A^{(1)}_{\text{sp}}(\mu_{\text{sp}}) = \frac{\alpha_s(\mu_{\text{sp}})}{4\pi} \frac{2\Delta F^{(K_1)}(\mu_{\text{sp}})}{g_{\mu}^{(K_1)}} \left\{ 3C^{(0)\text{eff}}_7(\mu_{\text{sp}}) + C^{(0)\text{eff}}_8(\mu_{\text{sp}}) \left[ 1 - \frac{\delta_{a_0}(\mu_{\text{sp}})}{\langle u-1\rangle^{(K_1)}_{\mu_{\text{sp}}} \langle u-1\rangle^{(K_1)}_{\mu_{\text{sp}}}} \right] + C^{(0)}_2(\mu_{\text{sp}}) \left[ 1 - \frac{\delta_{a_0}(\mu_{\text{sp}})}{\langle u-1\rangle^{(K_1)}_{\mu_{\text{sp}}} \langle u-1\rangle^{(K_1)}_{\mu_{\text{sp}}}} \right] \right\}. \]

Actually $C^{(1)\text{eff}}_7(\mu)$ and $A^{(1)}_{\text{eff}}(\mu)$ are process independent and encodes the QCD effects only, where as $A^{(1)}_{\text{sp}}(\mu_{\text{sp}})$ contains the key information about the out going mesons. The factor appearing in the Eq. (5.45) is arising due to the Gegenbauer moments. Our purpose is to see the effect of these Gegenbauer moments on the value of the form factor. As it is mentioned in [52, 89, 90], that the non-asymptotic corrections in the $K_1$ meson wave-function reduces the coefficient of the anomalous choromomagnetic moment $C^{(0)\text{eff}}_8(\mu_{\text{sp}})$ by 20%. Therefore it is viable to calculate the effect of these Gegenbauer moments. The value obtained for the quantity $\left| C^{(0)\text{eff}}_7 + A^{(1)}(\mu) \right|^2$ at different scales is listed in the Table 5.2 for $K_1(1270)$.

<table>
<thead>
<tr>
<th>$m_c/m_b$</th>
<th>0.29</th>
<th>0.29</th>
<th>0.29</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>$\bar{m}_b = 4.27\text{GeV}$</td>
<td>$m_{b,\text{DNC}} = 4.65\text{GeV}$</td>
<td>$m_{b,\text{PS}} = 4.66\text{GeV}$</td>
</tr>
<tr>
<td>$C^{(0)\text{eff}}<em>7 + A^{(1)}(\mu)</em>{\text{sp}}$</td>
<td>$-0.356 - i0.022$</td>
<td>$-0.356 - i0.021$</td>
<td>$-0.356 - i0.021$</td>
</tr>
<tr>
<td>$C^{(0)\text{eff}}<em>7 + A^{(1)}(\mu)</em>{\text{sp}}$</td>
<td>$-0.406 - i0.033$</td>
<td>$\times$</td>
<td>$-0.410 - i0.033$</td>
</tr>
<tr>
<td>$C^{(0)\text{eff}}<em>7 + A^{(1)}(\mu)</em>{\text{sp}}$</td>
<td>0.128</td>
<td>0.127</td>
<td>0.127</td>
</tr>
</tbody>
</table>

Table 5.2. Decay amplitude for $B \rightarrow K_1\gamma$ at different mass scales.

The numbers given for the quantity $\left( C^{(0)\text{eff}}_7 + A^{(1)}(\mu) \right)$ need some comments. The first one is that with the same value of the quark mass ratio $m_c/m_b$, the total amplitude has negligible dependence on the choice of $b$-quark mass or in other words the scale $\mu$. Secondly, if the effects of the Gegenbauer moments are included then one can easily see from third and forth row of the Table 5.2 that the value of the total amplitude reduces as compared to the value given in the literature [89]. In order to calculate the numerical value for the branching ratio we use the
reference scale to be

\[(\mu, \mu_{\text{sp}}) = (4.27 \text{ GeV}, 1.45 \text{ GeV})\]

After calculating the Gegenbauer moments at these scales the only independent parameter which is left in the calculation of the branching ratio is the LEET form factor and it has the biggest theoretical uncertainty. By taking the value of the form factor from the LCSR which is \(\xi_{\perp}^{(K_1)}(0) = 0.14 \pm 0.03\) it was shown some time ago that the value of the branching ratio is very small as compared to the experimental results [89]. Then the value of the form factor is extracted from the experimental measurements (5.2) and it is found that the value is [90]

\[\xi_{\perp}^{(K_1)}(0) = 0.32 \pm 0.03\]

which is much larger than LCSR result for the form factor and is opposite to that for decay \(B \to K^*\gamma\) meson where the value of LEET form factor \(\xi_{\perp}^{(K^*)}\) is smaller compared to the LCSR result. Such discrepancy in case of \(B \to K^*\gamma\) is not yet understood but for the axial \(K\) meson some sources of discrepancies are discussed in [90] and it is said that it will be bad if the Gegenbauer moments increase the value of this form factor. But we have shown that this is not the case.

Now by putting the value of the total amplitude at the scale \((\mu, \mu_{\text{sp}}) = (4.27 \text{ GeV}, 1.45 \text{ GeV})\) calculated in the last row of Table 5.2 and all the other inputs from [89] in Eq. (5.41) one can see that the value extracted for the form factor remains the same. Thus even if we consider the non asymptotic form of the light-cone \(D\lambda\), it has a very small effect on the total decay amplitude and leaves all the other things almost the same. The reason is that the dominant contribution comes from the operator \(O_7\) and so from the Wilson coefficient \(C_7\).

### 5.5 Annihilation Contributions and \(CP\)-Asymmetry

After comparing the LEET based approach with experiment in \(B \to K_1\gamma\) decay, we now present the effect of annihilation and also the \(u\)-quark contribution from the penguin to the branching ratio for the under discussion decay at NLO of \(\alpha_s\). It is pointed out that in the literature (c.f. ref. [112]) on \(B \to K^*\gamma\), the effect of the annihilation contribution to the charmed quark part of the amplitude is numerically small, because only the penguin operator with tiny Wilson coefficients can contribute. On the other hand the annihilation contribution to the up-quark part of the
amplitude contributes significantly, because of the large Wilson coefficients but again the CKM suppression \( \left| \frac{\lambda_{u}^{(s)}}{\lambda_{c}^{(s)}} \right| \approx 0.02 \) puts this large correction for \( B \to K_{1}\gamma \) into perspective. Finally, by incorporating these annihilation and \( u \)-quark contributions we compute the \( CP \)-asymmetry \( A_{CP} (K_{1}^{\pm} \gamma) \) involving the decay \( B \to K_{1}\gamma \). The \( CP \)-asymmetry arises due to the interference of the various penguin amplitudes which have clashing weak phases, with the required strong interaction phase provided by the \( \mathcal{O}(\alpha_{s}) \) corrections entering the penguin amplitudes via the Bander-Silverman-Soni (BSS) mechanism \cite{125}. We find that the hard-spectator corrections reduce the \( CP \)-asymmetry calculated from the vertex contributions alone. The resulting \( CP \)-asymmetry depends rather sensitively on the ratio of the quark masses \( m_{c}/m_{b} \). This parametric dependence, combined with the scale dependence of \( A_{CP} (K_{1}^{\pm} \gamma) \) makes the prediction of direct \( CP \)-asymmetry rather unreliable and the present section will be devoted to the study of this issue.

Since weak annihilation is a power correction, we will content ourselves with the lowest order result \( (\mathcal{O}(\alpha_{s}^{3})) \) for our estimate with a check of its effect on the branching ratio. The reason for including this class of power corrections is that they come with numerical enhancement from the large Wilson coefficients \( C_{1,2} \approx 3C_{7} \) but are CKM suppressed and thus these contributions are expected to be very small for the decay under consideration. The amplitude for charged \( B \)-meson decay in terms of weak annihilation \( A \), charmed penguin \( P_{c} \), gluonic penguin \( M \) and short-distance amplitude \( P_{t} \) can be written as \( \) following the notation of \cite{96} \]

\[
A \left( B^{-} \to K_{1}^{-} \gamma \right) = \lambda_{u}^{(s)}a + \lambda_{t}^{(s)}p \quad (5.46)
\]

\[
A \left( B^{0} \to K_{1}^{0} \gamma \right) = \lambda_{t}^{(s)} \left( P_{t} + \left( M^{(1)} - P_{c}^{(1)} \right) \right) + \frac{2}{3} \left( M^{(2)} - P_{c}^{(2)} \right) \quad (5.47)
\]

where \( \lambda_{q}^{(s)} = V_{qb}V_{qs}^{*} \), \( a = A - P_{c} \) and \( p = P_{t} + M - P_{c} \). As it is known \cite{96}

\[
P_{c} \approx 0.2A, \quad A \approx 0.3P_{t}
\]

i.e. we can safely neglect charmed penguin \( P_{c} \) and gluonic penguin \( M \) amplitudes relative to the
short-distance amplitude \( P_t \) and the weak annihilation amplitude \( A \). Thus Eq. (5.46) becomes

\[
A \left( B^- \rightarrow K_J^- \gamma \right) = \lambda_t^{(s)} p \left( 1 + \frac{\lambda_u^{(s)}}{\lambda_t^{(s)}} \right)
\]

\[
= \lambda_t^{(s)} p \left( 1 + \epsilon_A e^{i\phi_A} \frac{\lambda_u^{(s)}}{\lambda_t^{(s)}} \right)
\]

and

\[
A \left( B^0 \rightarrow K_J^0 \gamma \right) = \lambda_t^{(s)} p
\]

where \( \epsilon_A e^{i\phi_A} \equiv a/p, \phi_A \) is the strong interaction phase which disappears in \( \mathcal{O}(\alpha_s) \) in the chiral limit. Hence we will set it equal to zero in the subsequent calculation. Following the same lines as for the charged \( B \)-meson the ratio of the branching ratios for charged to neutral \( B \)-meson decays can be written as

\[
\frac{\mathcal{B}(B^- \rightarrow K_J^- \gamma)}{\mathcal{B}(B^0 \rightarrow K_J^0 \gamma)} \approx \left| 1 + \epsilon_A e^{i\phi_A} \frac{V_{ub}V_{us}^*}{V_{tb}V_{ts}^*} \right|^2
\]

(5.48)

The estimates in the framework of the light-cone QCD sum rules yield typically [94, 95]: \( \epsilon_A = -0.35 \) and \( \epsilon_A = 0.046 \) for the decays \( B^- \rightarrow K_J^- \gamma \) and \( B^0 \rightarrow K_J^0 \gamma \), respectively. Let us define

\[
\frac{V_{ub}V_{us}^*}{V_{tb}V_{ts}^*} = -\left| \frac{V_{ub}V_{us}^*}{V_{tb}V_{ts}^*} \right| e^{i\gamma} = F_1 + iF_2
\]

(5.49)

where \( \gamma \) is the unitarity triangle phase.

We also recall that the operator basis in \( \mathcal{H}_{eff} \) is larger than what is shown in Eq. (3.1) in which the operators multiplying the CKM factor \( V_{ub}V_{us}^* \) have been neglected. To calculate \( CP \)-asymmetry we have to put them back. Doing this, and using the unitarity relation \( V_{ub}V_{ts}^* = -V_{ub}V_{us}^* - V_{tb}V_{ts}^* \), the effective Hamiltonian reads [126]

\[
\mathcal{H}_{eff} = -\frac{G_F}{\sqrt{2}} \left\{ \begin{array}{c}
V_{ib}V_{is}^*[C_7(\mu)O_7(\mu) + C_8(\mu)O_8(\mu) + C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu)] \\
V_{ub}V_{us}^*[C_1(\mu)(O_{2u}(\mu) - O_1(\mu)) + C_2(\mu)(O_{2u}(\mu) - O_2(\mu)) + \ldots]
\end{array} \right\}
\]

(5.50)

In the above equation the ellipsis denotes the terms proportional to the Wilson coefficients \( C_3 \ldots C_6 \) and we have dropped them because they are very small as compared to \( C_1 \) and \( C_2 \).
The operators $O_{1u}$ and $O_{2u}$ are defined as

\[
O_{1u}(\mu) = (\bar{s}_L \gamma_{\mu} T^u u_L) (\bar{u}_L \gamma^\mu T^u \bar{u}_L)
\]

\[
O_{2u}(\mu) = (\bar{s}_L \gamma_{\mu} u_L) (\bar{u}_L \gamma^\mu b_L)
\]

The values of the Wilson coefficients in Eq. (5.50) are the same as we have already used in Eqs. (5.43-5.45). Thus by including the annihilation contribution and also the effect of the operator $O_{1u}$ and $O_{2u}$, the branching ratio from Eq. (5.41) can be written as

\[
B_{th}(B^\pm \to K_1^{\pm} \gamma) = \tau_{B^+} \Gamma_{th}(B^\pm \to K_1^{\pm} \gamma)
\]

\[
= \tau_{B^+} \frac{G_F^2 \alpha |V_{ts} V_{tb}|^2}{32 \pi^4} \frac{m_b \text{pole}}{M^3} \left( 1 - \frac{m_{K_1}^2}{M^2} \right)^3 \left[ \xi^{(K_1)}(0) \right]^2
\]

\[
\times \left\{ (C_7^{(0)} \text{eff} + A_R^{(1)})^2 + (F_1^2 + F_2^2) (A_R^{(1)} + L_R^{(1)})^2 + 2 F_1 \left[ C_7^{(0)} \text{eff} A_R^{(1)} + L_R^{(1)} \right] + 2 F_2 \left[ C_7^{(0)} \text{eff} A_I^{(1)} + L_I^{(1)} \right] \right\},
\]

(5.51)

where $L_R^{(1)} = \varepsilon A C_7^{(0) \text{eff}}$ and the subscripts $R$ and $I$ denote the real and imaginary parts of the quantities involved. $A^{(1)}$ is the same as defined in Eq. (5.42), and $A^{u}$ corresponds to the contribution from $O_{1u}$ and $O_{2u}$, which can be written as

\[
A^{u}(\mu) = \frac{\alpha_s(\mu)}{4 \pi} C_2^{(0)}(\mu) [r_2(z) - r_2(0)] - \frac{\alpha_s(\mu_{sp})}{18 \pi} C_2^{(0)}(\mu_{sp}) \frac{\Delta F_{1}^{(K_1)}(\mu_{sp})}{\xi^{(K_1)}(0)} \frac{h_{(K_1)}(z, \mu_{sp})}{\langle \bar{u}^{-1} \gamma_{\mu} \rangle_{(K_1)}}. \quad (5.52)
\]

We now proceed to the numerical calculation of the branching ratios for the decay $B^+ \to K_1^{+} \gamma$. Using the value of the CKM elements from [127], the values of $A^{(1)}(\mu)$ from Eq. (5.42) and the value of $C_2^{(0)}(\mu)$ from [93, 111], the branching ratio is plotted with the unitarity triangle phase $\gamma$ as shown in Fig. 5-12.

One can easily see that varying the value of $\gamma$ in the range $55^\circ \leq \gamma \leq 78^\circ$ with $\gamma = 63^\circ$ as the central value, there is a slight change in the value of the branching ratio for the decay $B \to K_1(1270) \gamma$ leaving the value of the form factor unchanged in this range as shown in the Fig. 5-13. We also note that the region of $\gamma$ where the branching ratio is effected is not allowed by the CKM unitarity constraints within the SM which typically yields $55^\circ \leq \gamma \leq 78^\circ$.

We now compute the leading order $CP$-asymmetry $A_{CP}(K_1^{\pm} \gamma)$ for the decay $B^\pm \to K_1^{\pm} \gamma$. 96
Figure 5-12: Branching ratio for $B \to K_1\gamma$ decay vs unitarity triangle phase $\gamma$.

Figure 5-13: Branching ratio for $B \to K_1\gamma$ decay vs LEET form factor for fixed value of $\gamma = 63^0$. 
Figure 5.14: CP-asymmetry ($-\Delta_{CP}$) vs the unitarity triangle phase $\gamma$; dashed line shows the value without hard spectator correction and solid line shows the value with hard spectator correction.

The CP-asymmetry arises from the interference of the penguin operator $O_7$ and the four-quark operator $O_2$ [91, 92]. The direct CP-asymmetry in the $B^\pm \to K^{\mp}_1 \gamma$ is:

$$A_{CP}(K^\pm_1) = \frac{B(B^- \to K^{-}_1 \gamma) - B(B^+ \to K^{+}_1 \gamma)}{B(B^- \to K^{-}_1 \gamma) + B(B^+ \to K^{+}_1 \gamma)}$$

$$= \frac{2F_2 (A_f^\eta - \epsilon_A A^{(1)}_f)}{C_f^{(0)\text{eff}} (1 + 2\epsilon_A [F_1 + \frac{1}{2} \epsilon_A (F_2^2 + F_3^2)])}$$  \hspace{1cm} (5.53)

The dependence of CP-asymmetry on the different parameters involved is shown in Fig. 5.14 and Fig. 5.15. In Fig. 5.14 we have plotted the CP-asymmetry vs the unitarity triangle phase $\gamma$. It is seen that in the SM favored interval of $\gamma$, $55^\circ \leq \gamma \leq 78^\circ$, the CP-asymmetry $|\Delta_{CP}|$ increases and reaches to its maximum value which is $0.75\%$. This reduces to the value $0.45\%$ if one includes the hard-spectator corrections in addition to the vertex corrections and annihilation contributions.
Figure 5-15: CP-asymmetry ($-A_{CP}$) vs the unitarity triangle phase $\gamma$ for different value of the scale $\mu$; dashed line shows the value at $m_{b,pole}/2$; solid line shows the value at $m_{b,pole}$ and the dotted line shows at $2m_{b,pole}$. 

Fig. 5-15 shows the plot of $A_{CP}(K^+_I \gamma)$ with $\gamma$ at different values of the scale $\mu$. It is very clear that the CP-asymmetry has a marked dependence on the scale $\mu$. The value of CP-asymmetry decreases from 0.8% to 0.3% in the interval $m_{b,pole}/2 \leq \mu \leq 2m_{b,pole}$. A similar discussion for $B \rightarrow \rho \gamma$ is given in [52] and we will also apply it to the decays $B \rightarrow (b_1,h_1) \gamma$ in the next chapter.
Chapter 6

Branching Ratios for $B \to 1^1 P_1 \gamma$ decays

6.1 Introduction

The Flavor-Changing-Neutral-Current (FCNC) processes which cause $b \to s\gamma$ and $b \to d\gamma$ decays may contain new physics (NP) effects through penguin amplitudes. As the SM effects represent the background when we search for NP effects, we shall compute these effects. In doing so, we can understand the sensitivity of each NP search.

The first experimental evidence of this FCNC transition process in $B$ decay was observed about a decade ago, where the inclusive process $b \to s\gamma$ and exclusive process $B \to K^*\gamma$ were detected, and their branching ratios were measured [99, 100]. On the other hand, the expected branching ratio for $b \to d\gamma$ is suppressed by $\mathcal{O}(10^{-2})$ with respect to the $b \to s\gamma$, because of the Cabbibo-Kobayashi-Maskawa quark mixing matrix factor (CKM). The world average for $b \to d$ penguin decays are given as follow [128]

$$B(B^0 \to \rho^0 \gamma) = (0.38 \pm 0.18) \times 10^{-6}$$
$$B(B^0 \to \omega \gamma) = (0.54^{+0.23}_{-0.21}) \times 10^{-6}$$
$$B(B^+ \to \rho^+ \gamma) = (0.68^{+0.36}_{-0.31}) \times 10^{-6}$$

Theoretically, $B \to (\rho, \omega) \gamma$ are widely studied both within and beyond the SM [129, 130]. Now after the first measurement of BELLE for the decay $B \to K_1\gamma$, where $K_1$ are the higher
resonances of kaon [131], these higher states become a subject of topical interest for the theoreticians. The detailed calculation for these decays along with the earlier references is giving in the last chapter. Recently, the leading twist LCDAs as well as the first few Gegenbauer moments of $1^{P_1}$ mesons, $b_1 (1235)$ and $h_1 (1170)$, which are the axial vector states of the $\rho$ and $\omega$ mesons have been studied [132]. It is pointed out that these LCDAs are not only important to explore the tensor-type new-physics in $B$ decays but also for $B \rightarrow 1^{P_1} \gamma \gamma$ studies.

In this chapter the branching ratio for $B_d^0 \rightarrow (b_1, h_1) \gamma$ at NLO of $\alpha_s$ are calculated using the LEET approach [12, 14]. We follow the same frame work as we have done for $B \rightarrow K_1 \gamma$ decay in the previous chapter. The reason to use the recipe developed in the previous chapter is quite obvious and that is, as $K_1$ is the orbitally excited state of $K^*$ these $b_1$ and $h_1$ are the corresponding orbitally excited states of $\rho$ and $\omega$ meson respectively. The only difference lies in the two calculations is mainly in the CKM matrix element and wave function of the final state meson. Using $SU(3)$ symmetry for this form factor we relate the branching ratios for $B_d^0 \rightarrow (b_1, h_1) \gamma$ with $B \rightarrow K_1 \gamma$ decay to get some theoretical limit on the branching ratios of these decays. This will be very helpful when we have experimental data on these decays to compare with.

At next-to-leading order of $\alpha_s$, $B \rightarrow (\rho, \omega) \gamma$ and $B_d^0 \rightarrow (b_1, h_1) \gamma$ are characterized by the weak form factor and decay constant, plugged by the common perturbative and kinematical factors. With $B (B \rightarrow (\rho, \omega) \gamma)$ at hand, we can say that the future experiment will check the structure for $B_d^0 \rightarrow (b_1, h_1) \gamma$.

6.2 Effective Hamiltonian

The effective Hamiltonian for the radiative $b \rightarrow d \gamma$ decays (equivalently $B_d^0 \rightarrow b_1 \gamma$ and $B_d^0 \rightarrow h_1 \gamma$ decays) is obtained from the Standard Model (SM) by integrating out the heavy degrees of freedom (the top quark and $W^\pm$-bosons). The resulting expression at the scale $\mu = O(m_b)$, where $m_b$ is the $b$-quark mass, is given by

$$H_{\text{eff}}^{b \rightarrow d} = \frac{G_F}{\sqrt{2}} \left\{ V_{ub} V_{ud}^* \left[ C_1^{(u)}(\mu) O_1^{(u)}(\mu) + C_2^{(u)}(\mu) O_2^{(u)}(\mu) \right] ight. \\
+ V_{cb} V_{cd}^* \left[ C_1^{(c)}(\mu) O_1^{(c)}(\mu) + C_2^{(c)}(\mu) O_2^{(c)}(\mu) \right] \\
- V_{td} V_{td}^* \left[ C_7^{\text{eff}}(\mu) O_7(\mu) + C_8^{\text{eff}}(\mu) O_8(\mu) \right] + \ldots \right\}.$$
where $G_F$ is the Fermi coupling constant and only the dominant terms are shown. This is very similar to the one given in Eq. (3.1) with an obvious replacement of $s$ quark with a $d$ quark. The operators $O_1^{(q)}$ and $O_2^{(q)}$, ($q = u, c$), are the standard four-fermion operators and $O_7$ and $O_8$ are the electromagnetic and chromomagnetic penguin operators, respectively. Their explicit expressions are

$$O_1^{(q)} = (d_\alpha \gamma_\mu (1 - \gamma_5)q_\beta)(\bar{q}_\beta \gamma^\mu (1 - \gamma_5)b_\alpha), \quad O_2^{(q)} = (d_\alpha \gamma_\mu (1 - \gamma_5)q_\alpha)(\bar{q}_\beta \gamma^\mu (1 - \gamma_5)b_\beta). \quad (6.3)$$

$$O_7 = \frac{e m_b}{8\pi^2} (d_\alpha \sigma^{\mu\nu}(1 + \gamma_5)b_\alpha) F_{\mu\nu}, \quad O_8 = \frac{g_s m_b}{8\pi^2} (d_\alpha \sigma^{\mu\nu}(1 + \gamma_5)T^a_{\alpha\beta}b_\beta) G_{\mu\nu}^a. \quad (6.4)$$

Here, $e$ and $g_s$ are the electric and colour charges, $F_{\mu\nu}$ and $G_{\mu\nu}^a$ are the electromagnetic and gluonic field strength tensors, respectively, $T^a_{\alpha\beta}$ are the colour $SU(N_c)$ group generators, and the quark color indices $\alpha$ and $\beta$ and gluonic color index $a$ are written explicitly. Note that in the operators $O_7$ and $O_8$ the $d$-quark mass contributions are negligible and therefore omitted. The coefficients $C_1^{(q)}(\mu)$ and $C_2^{(q)}(\mu)$ in Eq. (6.2) are the usual Wilson coefficients corresponding to the operators $O_1^{(q)}$ and $O_2^{(q)}$, while the coefficients $C_7^{\text{eff}}(\mu)$ and $C_8^{\text{eff}}(\mu)$ include also the effects of the QCD penguin four-fermion operators which are assumed to be present in the effective Hamiltonian (6.2) and denoted by ellipses there. For details and numerical values of these coefficients, see Ref.[134] and also references therein. We use the standard Bjorken-Drell convention [135] for the metric and the Dirac matrices; in particular $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and the totally antisymmetric Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$ is defined as $\epsilon_{0123} = +1$. A point to note is that the three CKM factors shown in $\mathcal{H}_{\text{eff}}^{b\rightarrow d}$ are of the same order of magnitude and, hence, the matrix elements in the decays $b \rightarrow d\gamma$ and $B^0 \rightarrow (b_1, h_1)\gamma$ have non-trivial dependence on the CKM parameters. This is not the case of $b \rightarrow s\gamma$ decay (equivalently the $B \rightarrow K_\gamma$ decays), the effective Hamiltonian $\mathcal{H}_{\text{eff}}^{b\rightarrow s}$ describing the $b \rightarrow s$ transition can be obtained by the replacement of the quark field $d_\alpha$ by $s_\alpha$ in all the operators in Eqs. (6.3) and (6.4) and by replacing the CKM factors $V_{qb}V_{qs}^* \rightarrow V_{qb}V_{qs}^*$ ($q = u, c, t$) in $\mathcal{H}_{\text{eff}}^{b\rightarrow d}$ (6.2). Noting that among the three factors $V_{qb}V_{qs}^*$, the combination $V_{ub}V_{us}^*$ is CKM suppressed, the corresponding contributions to the decay amplitude can be safely neglected.

102
6.3 Theoretical framework for the $B \to 1^1P_1\gamma$ decays

The matrix element for the $B^0 \to 1^1P_1\gamma$ ($1^1P_1 = b_1, h_1$) decays, we need to calculate the matrix elements $(1^1P_1\gamma|\mathcal{O}_1|B)$, where $\mathcal{O}_1$ are the operators appearing in $\mathcal{H}_{\text{eff}}^{b \to s}$ and $\mathcal{H}_{\text{eff}}^{b \to d}$. At the leading order in $\alpha_s$, this involves only the operators $\mathcal{O}_7$, $\mathcal{O}_{1}^{(u)}$ and $\mathcal{O}_{2}^{(u)}$. The contribution from $\mathcal{O}_7$ is termed as the long-distance contribution characterized by the top quark induced amplitude, where $\mathcal{O}_{1}^{(u)}$ and $\mathcal{O}_{2}^{(u)}$ corresponds to the short distance contributions and it includes the penguin amplitude for the $u$ and $c$ quark intermediate states and also the so-called weak annihilation and $W$-exchange contributions. There is also a small contribution from annihilation penguin diagrams, which, however, are small. For detailed discussion about these kind of topologies for $B \to V\gamma$ decays and references to earlier papers, see Ref. [96].

To calculate $O(\alpha_s)$ corrections, all the operators listed in (6.3) and (6.4) have to be included. QCD factorization [11] is most convenient framework to carry out these calculations. This allows to express the hadronic matrix elements in the schematic form:

$$
(1^1P_1\gamma|\mathcal{O}_1|B) = F^{B \to 1^1P_1}T^I_1 + \int \frac{dk_+}{2\pi} \int_0^1 du \, \phi_{B,+}(k_+) T^I_1(k_+, u) \phi^{(1^1P_1)}_\perp(u), \quad (6.5)
$$

where $F^{B \to 1^1P_1}$ are the transition form factors defined through the matrix elements of the operator $\mathcal{O}_7$. $\phi_{B,+}(k_+)$ is the leading-twist $B$-meson wave-function with $k_+$ being a light-cone component of the spectator quark momentum, $\phi^{(1^1P_1)}_\perp(u)$ is the leading-twist light-cone distribution amplitude (LCDA) of the transversely-polarized axial-vector meson, and $u$ is the fractional momentum of the vector meson carried by one of the two partons. The expressions for these wavefunctions are given in Eqs. (5.10) and (5.11), where it was pointed out that vector and axial vector mesons are distinguished by $\gamma_5$ in the gamma structure of the decay amplitude and some non-perturbative parameters. The quantities $T^I_1$ and $T^I_{1I}$ are the hard-perturbative kernels calculated to order $\alpha_s$, with the latter containing the so-called hard-spectator contributions. The factorization formula (6.5) holds in the heavy quark limit, i.e., to order $\Lambda_{\text{QCD}}/M_B$. This factorization framework has been used to calculate the branching fractions and related quantities for the decays $B \to V\gamma$ ($V = K^*, \rho$) [52, 53, 112] and for $B \to K\gamma$ [89, 90, 97]. The isospin violation in the $B \to K^*\gamma$ decays in this framework have also been studied [137]. Very recently, the hard-spectator contribution arising from the chromomagnetic operator $\mathcal{O}_8$ have also been calculated in next-to-next-to-leading order (NNLO) in $\alpha_s$, showing that the spectator
interactions factorize in the heavy quark limit [138]. However, the numerical effect of the resummed NNLO contributions is marginal and we shall not include this in our update.

It was shown in the Chapter 5 that the extra $\gamma_5$ in the DA of axial vector meson in comparison to the vector meson does not alter the calculation, giving the same result for the perturbative part. As for the non-perturbative parameters, the decay constant is most important. The LCDA for $b_1$ and $h_1$ meson has recently been calculated in [132]. The transverse decay constant of these mesons as well as the first few Gegenbauer moments of leading twist LCDA are calculated by using QCD sum rule technique. Their numerical values are given in Table 6.1.

In what follows we shall use the notations and results from previous chapter, to which we refer for detailed derivations for $B \rightarrow K_1 \gamma$: The final state $K_1$ is also the axial vector meson like $b_1$ and $h_1$ mesons. The only difference is in the quark content and we have to change the $s$ quark with $d$ quark every where in the calculation. The branching ratio of the $B_d^0 \rightarrow (b_1, h_1)$ decay corrected to $O(\alpha_s)$ can be written as follows (c. f. Eq. (5.41)):

$$B_{\text{th}}(B_d^0 \rightarrow (b_1, h_1) \gamma) = \tau_B \Gamma_{\text{th}}(B_d^0 \rightarrow (b_1, h_1))$$

$$= \tau_B \frac{G_F^2 \alpha |V_{td}|^2}{32 \pi^4} m_{b, \text{pole}}^2 M^3 \left[ \xi_{(b_1, h_1)}^{(b_1, h_1)} \right]^2 \left( 1 - \frac{m_{(b_1, h_1)}^2}{M^2} \right)^3 \left| C_7^{(0)} + A^{(1)}(\mu) \right|^2$$

(6.6)

where $G_F$ is the Fermi coupling constant, $\alpha = \alpha(0) = 1/137$ is the fine-structure constant, $m_{b, \text{pole}}$ is the pole $b$-quark mass, $M$ and $m_{(b_1, h_1)}$ are the $B$- and axial vector-meson masses, and $\tau_B$ is the lifetime of the $B^0$- or $B^+$-meson. The value of these constants is used from[97] and are collected in Table 6.1, for the numerical analysis. For this study, we consider $\xi_{(b_1, h_1)}^{(b_1, h_1)}$ as a free parameter and we will extract its value from the current experimental data on $B \rightarrow K_1 \gamma$ decays because $K_1$ is also an axial vector meson. This is in analogy with the calculation done for the branching ratio of $B \rightarrow (\rho, \omega) \gamma$ in terms of the branching ratio of $B \rightarrow K^* \gamma$ by Ali et al. [133].

The function $A^{(1)}$ in Eq. (6.6) can be decomposed into the following three components:

$$A^{(1)}(\mu) = A^{(1)}_{\gamma}(\mu) + A^{(1)}_{\omega}(\mu) + A^{(1)1}_{sp}(\mu_{sp})$$

(6.7)

104
<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_W$</td>
<td>80.423 GeV</td>
<td>$M_Z$</td>
<td>91.1876 GeV</td>
</tr>
<tr>
<td>$M_B$</td>
<td>5.279 GeV</td>
<td>$m_{b_1}$</td>
<td>1.229</td>
</tr>
<tr>
<td>$G_F$</td>
<td>$1.166 \times 10^{-5}$ GeV</td>
<td>$m_{h_1}$</td>
<td>1.170</td>
</tr>
<tr>
<td>$\alpha_s(M_Z)$</td>
<td>0.1172</td>
<td>$\alpha$</td>
<td>$1/137.036$</td>
</tr>
<tr>
<td>$m_{t,\text{pole}}$</td>
<td>178 GeV</td>
<td>$\Lambda_h$</td>
<td>0.5 GeV</td>
</tr>
<tr>
<td>$</td>
<td>V_{tb}V_{td}</td>
<td>$</td>
<td>$5 \times 10^{-3}$</td>
</tr>
<tr>
<td>$f_B$</td>
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<td>$\sqrt{s} = m_{\ell}/m_{B_0}$</td>
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</tr>
<tr>
<td>$a_{11}^{(b_1)}(1\text{GeV})$</td>
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<td>$a_{12}^{(b_1)}(1\text{GeV})$</td>
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</tr>
<tr>
<td>$a_{11}^{(h_1)}(1\text{GeV})$</td>
<td>0</td>
<td>$a_{12}^{(h_1)}(1\text{GeV})$</td>
<td>0.35</td>
</tr>
<tr>
<td>$f_{11}^{(b_1)}$</td>
<td>180 MeV</td>
<td>$f_{12}^{(h_1)}$</td>
<td>200 MeV</td>
</tr>
<tr>
<td>$\lambda_{B_1}^-$</td>
<td>$(2.15 \pm 0.50) \text{ GeV}^{-1}$</td>
<td>$\sigma_{B_1}(1 \text{ GeV})$</td>
<td>1.4 $\pm 0.4$</td>
</tr>
</tbody>
</table>

Table 6.1: Input quantities and their values used in the theoretical analysis

Here, $A_{C_1}^{(1)}$ and $A_{\text{ver}}^{(1)}$ are the $O(\alpha_s)$ (i.e. NLO) corrections due to the Wilson coefficient $C_7^{\text{eff}}$ and in the $b \to d\gamma$ vertex, respectively, and $A_{sp}^{(1)1F_1}$ corresponds to the the $O(\alpha_s)$ hard-spectator corrections. The details about the calculation of these quantities is given in the previous chapter and their explicit expressions are as follows:

\[
A_{C_1}^{(1)}(\mu) = \frac{\alpha_s(\mu)}{4\pi} C_7^{(1)\text{eff}}(\mu), \quad (6.8)
\]

\[
A_{\text{ver}}^{(1)}(\mu) = \frac{\alpha_s(\mu)}{4\pi} \left\{ \frac{32}{81} \left[ 13C_2^{(0)}(\mu) + 27C_7^{(0)\text{eff}}(\mu) - 9C_8^{(0)\text{eff}}(\mu) \right] \ln \frac{m_b}{\mu} \\ - \frac{20}{3} C_7^{(0)\text{eff}}(\mu) + \frac{4}{27} \left( 33 - 2\pi^2 + 6\pi i \right) C_8^{(0)\text{eff}}(\mu) + r_2(\mu) C_2^{(0)}(\mu) \right\}, \quad (6.9)
\]

\[
A_{sp}^{(1)1F_1}(\mu_{sp}) = \frac{\alpha_s(\mu_{sp})}{4\pi} \frac{2\Delta F_{11}^{(0)F_1}(\mu_{sp})}{9\xi(K_1)} \left\{ 3C_8^{(0)\text{eff}}(\mu_{sp}) \\ + C_8^{(0)\text{eff}}(\mu_{sp}) \left[ 1 - \frac{6a_{11}^{(11)F_1}(\mu_{sp})}{\langle \bar{u} \gamma^0(11)F_1 \rangle_{\mu_{sp}}} \right] + C_8^{(2)0}(\mu_{sp}) \right\} + \frac{V_{u_1}V_{u_2}}{V_{d_1}^*V_{d_2}} \frac{h^{(1)F_1}(z, \mu_{sp})}{\langle \bar{u} \gamma^0(11)F_1 \rangle_{\mu_{sp}}}, \quad (6.10)
\]

Actually $C_7^{(1)\text{eff}}(\mu)$ and $A_{\text{ver}}^{(1)}(\mu)$ are process independent and encodes the QCD effects only, where as $A_{sp}^{(1)}(\mu_{sp})$ contains the key information about the out going mesons. The factor $\frac{6a_{11}^{(11)F_1}(\mu_{sp})}{\langle \bar{u} \gamma^0(11)F_1 \rangle_{\mu_{sp}}}$ appear in the Eq. (6.8) is arising due to the Gegenbauer moments. The Eq. (6.10) differs from the Eq. (5.45) by the factor $V_{u_1}V_{u_2}/V_{d_1}^*V_{d_2}$ which is $-1$ for $B \to K_1\gamma$ case.
4.1. Branching ratios. We now proceed to calculate numerically the branching ratios for the $B_d^0 \to b_1\gamma$ and $B_d^0 \to h_1\gamma$ decays. The theoretical ratios involving the decay widths on the r.h.s. of these equations can be written in the form:

\[
R_{\text{th}}(b_1\gamma/K_1\gamma) = \frac{B_{\text{th}}(B_d^0 \to b_1\gamma)}{B_{\text{th}}(B_d^0 \to K_1\gamma)} = \frac{1}{2} \left( \frac{V_{td}}{V_{ts}} \right)^2 \frac{(M_B^2 - m_{b_1}^2)^3}{(M_B^2 - m_{K_1}^2)^3} \zeta^2 \left[ 1 + \Delta R(b_1/K_1) \right], \quad (6.11)
\]

\[
R_{\text{th}}(h_1\gamma/K_1\gamma) = \frac{B_{\text{th}}(B_d^0 \to h_1\gamma)}{B_{\text{th}}(B_d^0 \to K_1\gamma)} = \frac{1}{2} \left( \frac{V_{td}}{V_{ts}} \right)^2 \frac{(M_B^2 - m_{h_1}^2)^3}{(M_B^2 - m_{K_1}^2)^3} \zeta^2 \left[ 1 + \Delta R(h_1/K_1) \right], \quad (6.12)
\]

where $m_{b_1}$ and $m_{h_1}$ are the masses of the $b_1$- and $h_1$-mesons, $\zeta$ is the ratio of the transition form factors, which we have assumed to be the same for the $b_0^0$- and $h_1$-mesons. To get the theoretical branching ratios for the decays $B_d^0 \to b_1\gamma$ and $B_d^0 \to h_1\gamma$, the ratios (6.11) and (6.12) should be multiplied with the corresponding experimental branching ratio of the $B_d^0 \to K_1\gamma$ decay.

It is well known that in vector meson case the theoretical uncertainty in the evaluation of the $R_{\text{th}}(\rho\gamma/K^*\gamma)$ and $R_{\text{th}}(\omega\gamma/K^*\gamma)$ ratios is dominated by the imprecise knowledge of $\zeta = T_1^{\rho}(0)/T_1^{K^*}(0)$ characterizing the $SU(3)$ breaking effects in the QCD transition form factors. In the $SU(3)$-symmetry limit, $T_1^{\rho}(0) = T_1^{K^*}(0)$, yielding $\zeta = 1$. We make use of the $SU(3)$ symmetry to relate the form factor of $B \to b_1\gamma$ and $B_d^0 \to h_1\gamma$ with that of $B_d^0 \to K_1\gamma$ decay which is the only unknown parameter involved in the calculation of branching ratio for these decays. We use this symmetry because there is no experimental limit on the branching ratio of these decays. It is reasonable to take $\xi_{\perp}^{b_1,K_1}(0) = 0.32$ together with the values of the other input parameters entering in the calculation of the $B_d^0 \to (b_1,h_1)\gamma$ decay amplitudes and these are given in Table 6.1.

The individual branching ratios $B_{\text{th}}(B_d^0 \to b_1\gamma)$ and $B_{\text{th}}(B_d^0 \to h_1\gamma)$ and their ratios $R_{\text{th}}(b_1\gamma/K_1\gamma)$ and $R_{\text{th}}(h_1\gamma/K_1\gamma)$ with respect to the corresponding $B \to K_1\gamma$ branching ratios are calculated and the corresponding values are:

\[
B_{\text{th}}(B_d^0 \to b_1\gamma) = 0.71 \times 10^{-5} \quad (6.13)
\]

\[
B_{\text{th}}(B_d^0 \to h_1\gamma) = 0.74 \times 10^{-5} \quad (6.14)
\]

\[
R_{\text{th}}(b_1\gamma/K_1\gamma) = 0.0166 \quad (6.15)
\]

\[
R_{\text{th}}(h_1\gamma/K_1\gamma) = 0.0167 \quad (6.16)
\]
Figure 6-1: Branching ratio for $B \to 1^P \gamma$ decay vs LEET form factor. Solid line shows the value for $b_1$ meson and dashed line is for $h_1$ meson.

To calculate these values we have used the experimental value of the branching ratio of $B \to K_1 \gamma$. One can easily see that there is very small difference between $B_d^0 \to b_1 \gamma$ and $B_d^0 \to h_1 \gamma$ branching fractions, and this is due to the slight change in the hadronic parameters of these decays.

The $SU(3)$-breaking effects in $\rho$ and $K^*$ form factors have been evaluated within several approaches, including the LCSR and Lattice QCD. In the earlier calculations of the ratios for $B \to \rho \gamma$ and $B \to K^* \gamma$, the following ranges were used: $\zeta = 0.76 \pm 0.06$ [52] and $\zeta = 0.76 \pm 0.10$ [140], based on the LCSR approach [94, 141–144] which indicate substantial $SU(3)$ breaking in the $B \to K^*$ form factors. There also exists an improved Lattice estimate of this quantity, $\zeta = 0.9 \pm 0.1$ [145]. To see the effect of $SU(3)$ symmetry breaking for these axial meson decays we have plotted the branching ratios of $B_d^0 \to (b_1, h_1) \gamma$ decay with the LEET form factor which is presented in Fig. 6-1: The solid and dashed line show the dependence of the branching ratio of $B_d^0 \to b_1 \gamma$ and $B_d^0 \to h_1 \gamma$ on the LEET form factor $\xi_{1}^{1P_{1}}(0)$ respectively. The graph shows that in the range $0.76 \leq \zeta \leq 0.9$ the value of the branching ratio (in the units of $10^{-6}$) is $0.4 \leq (B_d^0 \to 1^P \gamma) \leq 0.7$. 

107
4.2. CP-violating asymmetries. The direct CP-violating asymmetries in the decay rates for $B^0_d \to (b_1, h_1) \gamma$ decays are defined as follows:

$$A_{CP}^{\text{dir}}(b_1 \gamma) = \frac{\mathcal{B}(\bar{B}^0_d \to b_1 \gamma) - \mathcal{B}(B^0_d \to b_1 \gamma)}{\mathcal{B}(\bar{B}^0_d \to b_1 \gamma) + \mathcal{B}(B^0_d \to b_1 \gamma)},$$

$$A_{CP}^{\text{dir}}(h_1 \gamma) = \frac{\mathcal{B}(\bar{B}^0_d \to h_1 \gamma) - \mathcal{B}(B^0_d \to h_1 \gamma)}{\mathcal{B}(\bar{B}^0_d \to h_1 \gamma) + \mathcal{B}(B^0_d \to h_1 \gamma)}.$$ (6.17)

Before we go for the numerical values of CP-asymmetry, let us discuss the difference in the hadronic parameters involving the $b_1$ and $h_1$ mesons. As these are the axial vector states of $\rho$ and $\omega$ mesons so these are also the maximally mixed superpositions of the $\bar{u}u$ and $\bar{d}d$ quark states: $|b_1\rangle = (|\bar{d}d\rangle - |\bar{u}u\rangle)/\sqrt{2}$ and $|h_1\rangle = (|\bar{d}d\rangle + |\bar{u}u\rangle)/\sqrt{2}$. Neglecting the W-exchange contributions in the decays, the radiative decay widths are determined by the penguin amplitudes which involve only the $|\bar{d}d\rangle$ components of these mesons, leading to identical branching ratios (modulo a tiny phase space difference). The W-exchange diagrams from the $O^{(u)}_1$ and $O^{(u)}_2$ operators (in our approach, we are systematically neglecting the contributions from the penguin operators $O_3, ..., O_6$) yield contributions equal in magnitude but opposite in signs [for detailed calculation please see [96, 133]]. If we use the notations and expressions given in Ref. [96], the LCSR results are: $e^{(b_1)}_A = -e^{(h_1)}_A = 0.07$. The smallness of these numbers reflects both the colour-suppressed nature of the W-exchange amplitudes in $B^0_d \to (b_1, h_1) \gamma$ decays, and the observation that the leading contributions in the weak annihilation and W-exchange amplitudes arise from the radiation off the $d$-quark in the $B^0_d$-meson, which is suppressed due to the electric charge.

The explicit expressions of these asymmetries for the charged axial vector meson in terms of the individual contributions in the decay amplitude can be found in Ref. [136], which for $A_{CP}^{\text{dir}}(b_1 \gamma)$ and $A_{CP}^{\text{dir}}(h_1 \gamma)$ may be obtained by obvious replacements. The calculated values of the CP-asymmetry for the above mentioned decays are summarized in Table 6.2. The CP-asymmetry receives contributions from the hard spectator corrections which tend to decrease its value estimated from the vertex corrections alone. The dependence of the direct CP-asymmetry on the CKM unitarity-triangle angle $\alpha$ is presented in the Fig. 6-2. It should be noted that the predicted direct CP-asymmetries are rather sizable (of order 11%) and is negative like $\rho$ and $\omega$ meson case. It is quite unfortunate that the predicted value of CP-asymmetry is sensitive to both the choice of the scale as well as the quark mass ratio $s = m_s^2/m_b^2$ used in the calculation. In Table 6.2 we have calculated the dependence of CP-asymmetry on different scales. One
Figure 6-2: Unitarity triangle phase $\alpha$ is plotted at different scales. Solid line is for $b_1$ meson and dashed line is for $h_1$ meson. Figures (a), (b) and (c) shows the corresponding value of CP-asymmetry at the scale $\mu = m_b$, $m_b/2$ and $2m_b$ respectively.

<table>
<thead>
<tr>
<th>Scale</th>
<th>$A_{CP}^{b_1} (B_d^0 \rightarrow b_1 \gamma)$</th>
<th>$A_{CP}^{h_1} (B_d^0 \rightarrow h_1 \gamma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = m_b$</td>
<td>$-11%$</td>
<td>$-13%$</td>
</tr>
<tr>
<td>$\mu = m_b/2$</td>
<td>$-25.5%$</td>
<td>$-25.5%$</td>
</tr>
<tr>
<td>$\mu = 2m_b$</td>
<td>$-7%$</td>
<td>$-9%$</td>
</tr>
</tbody>
</table>

Table 6.2: The scale dependence of CP asymmetry

can easily see that this asymmetry decreases with the increase of scale $\mu$. This shows that the typical value lies around 11%, but the uncertainty is rather large and this increases to 25% at the scale $\mu = 2m_b$. 

109
Chapter 7

Conclusions

$B$ physics is among the most active fields in recent particle physics. The main goal of $B$ physics is a precision study of the flavor sector to extract the Standard Model parameters with high precision in order to reveal potential New Physics effects. With in this thesis we have analyzed in detail the rare $B$ meson decays $B \to \gamma \nu \bar{l}$ and $B \to 1^1P_1 \gamma$ ($1^1P_1 \gamma = K_1, b_1, h_1$). Among these rare decays, the semileptonic $B \to \gamma \nu \bar{l}$ decays are very interesting due to their relative cleanliness and sensitivity to new physics.

Preliminary data from the CLEO Collaboration indicate an upper limit on the branching ratio $B(B^+ \to \gamma \nu_e e^+)$ of $2.0 \times 10^{-4}$ at the 90% confidence level [55]. With the better statistics expected from the upcoming $B$ factories, the observation and experimental study of this decay could soon become feasible. It is therefore of some interest to have a good theoretical control over the theoretical uncertainties affecting the relevant matrix elements.

We have studied the $B \to \gamma \nu \bar{l}$-decay using dispersion relations, asymptotic behavior of form factors and Ward identities. The dispersion relation involves the ground state of the $B^*$ and $B^*_\Lambda$ resonances and their radial excitations which model contributions from higher states and a continuum contribution, which is calculated from a quark triangle graph. The asymptotic behavior of form factors and Ward identities fix the normalization of the form factors in terms of the universal function $g_+ (0)$ at $q^2 = 0$ and puts constraints on the residues. Thus in our approach, a parameterization of the $q^2$ dependence of form factors is not approximated by single pole contributions. This parameterization is dictated by considerations mentioned above and also the coupling constants of the $1^-(B^*)$ and $1^+(B^*_\Lambda)$ resonances with the photon are predicted if we restrict ourselves to one radial excitation. By using $\bar{A} = 0.4$ GeV$^{-1}$ we have
calculated $g_A(0) = 0.15$ and predicted the value of $g_{B^*B\gamma} = 5.6 \text{ GeV}^{-1}$ (cf. Eq. (4.76)) and $f_{B^*B\gamma} = 6.5 f_B M_{B\gamma} / f_{B^*}$ GeV$^{-1}$ (cf. Eq. (4.78)) if we restrict to one radial excitation. Taking into account one radial excitation the form factors are summarized in Eq. (4.74, 4.75). The branching ratio for the process is then calculated to be $\mathcal{B}(B \to \gamma \nu\bar{\nu}) = 0.5 \times 10^{-6}$, which lies within the upper limit predicted by the CLEO Collaboration at 90% confidence level [55].

Then we study the effect of a second radial excitation in terms of a single parameter $A$, which in principle is determined once $g_{B^*B\gamma}$ and $f_{B^*B\gamma}$ are known (cf. Eq. (4.80, 4.81)). The resulting form factors are given in Eqs. (4.82, 4.83). By using these form factors the branching ratio is $\mathcal{B}(B \to \gamma \nu\bar{\nu}) = 0.38 \times 10^{-6}$ and $\mathcal{B}(B \to \gamma \nu\bar{\nu}) = 0.32 \times 10^{-6}$ for the two representative cases $A = 3.0$ and $A = 4.8$ respectively. These branching ratios are not sensitive to the value of $A$, in contrast to the radiative coupling constants which give respectively the $B^* \to B\gamma$ width as 23 keV ($A = 0$), 5 keV ($A = 3.0$) and 0.8 keV ($A = 4.8$). One can also predict the radiative widths of the radial excitation in terms of the $B^*$ and $B^*_A$ radiative widths by using relations (4.45), (4.48) and (4.67). The differential decay width versus photon energy is plotted in Fig. 4-5 to compare our results with the existing calculations in the light-cone QCD approach [64, 66] and in the instantaneous Bethe-Salpeter approach [67]. The results for $B \to \gamma \nu\bar{\nu}$ have been reproduced by using Sudakov resummation [66] and have also been shown graphically. In our calculations as well as in [64], the position of the peak of the differential decay width is shifted towards the lower value of the photon energy spectrum. This is due to the double pole in the form factors. The overall effect of radial excitations is to soften the $q^2$-behavior of the differential decay distribution, while in [66] it is due to Sudakov resummation.

Our main inputs have been dispersion relations, asymptotic behavior and Ward identities, all of which have a strong theoretical basis and in these aspects it differs from other approaches. Our approach is closer to the one followed in [72] for $B \to \pi \nu\bar{\nu}$. The only external parameters involved are $f_B$, the resonance masses (which are determined in potential models) and $g_{B^*B\gamma}$ and $f_{B^*B\gamma}$ which are either predicted or on which we have some theoretical information. The radiative widths of the radial excitations are predicted in terms of the above coupling constants. Thus our approach has predictive power and can be tested by future experiments at the B-factories, BaBar at SLAC and Belle at KEK (Japan) and the planned hadronic accelerators which are capable to measure the branching ratio as low as $10^{-8}$ [88].

The exclusive non-leptonic $B$ meson decays are experimentally better accessible but theoretically these are very complicated. The complication is that three fundamental scales are
involved in $B$ decays: the weak interaction scale $M_W$, the $b$ quark mass $m_b$ and the QCD scale $\Lambda_{\text{QCD}}$. By using operator product expansion and renormalization group equations in the framework of an effective theory one can factorize perturbatively calculable short distance Wilson coefficients from the long distance operator matrix elements. The main difficulty is with these long distance contributions because all low-energy contributions below the factorization scale $\mu = \mathcal{O}(m_b)$ are collected into them. Yet, one can take advantage of the fact that $m_b \gg \Lambda_{\text{QCD}}$ and achieve a further separation of short and long-distance part of these matrix elements. In the third part of the thesis we have analyzed the exclusive radiative $B \to (K_1, b_1, h_1) \gamma$ decays using the LEET based factorization approach given in [52].

We have computed the hard spectator corrections in $\mathcal{O}(\alpha_s)$ and leading order in $\Lambda_{\text{QCD}}/M_B$, for these decays, in the leading-twist approximation, using the Large Energy Effective Theory (LEET). This is then combined with the existing contributions from vertex corrections and annihilation contributions to arrive at the NLO expressions for the corresponding decay widths. The matrix elements for these decays in $\mathcal{O}(\alpha_s)$ and to leading power in $\Lambda_{\text{QCD}}/M_B$, are finite. We have also included the intermediate charm quark and up quark contributions from the penguin diagrams.

The branching ratio for the decays in the next-to-leading order are then compared with the current data to determine the value of form factor $\xi_{\perp}^{K_1}(0)$, the only unknown in our calculation, in the LEET approach. Due to the $u$ quark penguin loop the unitarity triangle phase $\gamma$ is involved in the calculation of $B \to K_1\gamma$ decays. It has also been found that the value of the form factor remains unchanged for the value of $\gamma$ favored by the Standard Model. The same formalism is then applied to calculate the branching ratios for the decays $B \to (b_1, h_1) \gamma$. As these decays are not yet observed therefore using the $SU(3)$ symmetry for the form factor, the branching ratio for $B^0_d \to (b_1, h_1) \gamma$ is expressed in terms of the branching ratio of the $B^0_d \to K_1\gamma$ and it is found to be $B(B^0_d \to b_1\gamma) = 0.53 \times 10^{-6}$ and $B(B^0_d \to h_1\gamma) = 0.51 \times 10^{-6}$. Then we have plotted the branching ratio with the LEET form factor which is the only unknown parameter involved in the calculation. It is shown that the corresponding to the range of $SU(3)$ symmetry breaking parameter $\zeta$, $0.76 \leq \zeta \leq 0.9$ the value of the branching ratio $(10^6)$ is $0.4 \leq (B^0_d \to l^1P\gamma) \leq 0.7$. Sizeable uncertainties come from the variation of the nonperturbative input parameters, in particular form factors, decay constant, and information on the $B$ meson wave function, which all are poorly known. But we await results on these quantities from lattice QCD and analytical methods as well as the completion of LHC where
most of these decays are observable.

The \( CP \)-asymmetry \( A_{CP} \) for these decays has also been calculated. This \( CP \)-asymmetry received a contribution from the hard-spectator corrections which tend to decrease its value estimated from the vertex corrections alone. Unfortunately, the predicted value of the \( CP \)-asymmetry is sensitive to the choice of scale \( \mu \) as well as to the quark mass ratio \( \sqrt{2} \equiv m_c/m_b \). The uncertainties relating to the scale \( \mu \) is due to the fact that \( CP \)-asymmetry arises at \( \mathcal{O}(\alpha_s) \) for the first time. This could be reduced by going one order further in perturbation theory, which would amount to a three-loop calculation. The typical value of \( CP \)-asymmetry lies around \(-0.5\%\) for \( B \to K\gamma \) decays where as this value is \(-11\%\) for \( B \to (b_1, h_1)\gamma \) decays. Thus the measurement of \( CP \)-asymmetry will either overconstrain the angles \( \gamma \) and \( \alpha \) of the unitarity triangle, or they may lead to the discovery of physics beyond the SM in the radiative \( b \to s\gamma \) and \( b \to d\gamma \) decays.
Bibliography


[88] G. G. Devidze, The short distance contribution to the $B_s \to \gamma\gamma$ decay in the SM and MSSM, ICTP preprint: IC/2000/71 (2000); The lowest order short-distance contribution to the $B_s \to \gamma\gamma$, [hep-ph/9905431]


M. Beyer, D. Melikov, N. Nikitin, B. Stech, Phys. Rev. D 64 (2001) 094006; S.W. Bosch,


[131] K. Abe et al. (Belle Collaboration), hep-ex/0408138


ph/0405075]

[hep-ph/9512380].


[hep-ph/0309330].


LIST OF PUBLICATIONS

This PhD thesis is based on the following publications.

1- M. Jamil Aslam and Riazuddin; Phys. Rev. D72 (2005) 094019


4- M. Jamil Aslam and Riazuddin; Phys. Rev. D 75, 034004 (2007)