Accelerated flows of viscoelastic fluids with no-slip and partial slip conditions

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PAKISTAN
2004
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A Thesis
Submitted in the Partial Fulfillment of the Requirements for the Degree of DOCTOR OF PHILOSOPHY IN MATHEMATICS

Supervised by

Prof. Dr. Saleem Asghar

Department of Mathematics Quaid-i-Azam University, Islamabad PAKISTAN 2004
Dedicated To

my parents (Late)

my family

&

two lovely daughters

who waited patiently for me
to complete my studies.
The wait is over.
Acknowledgements

Primarily and foremost, all praise for Almighty ALLAH, Who blessed me the ability to fulfill the requirement for this thesis. I offer my humblest words of thanks to the Holy Prophet Muhammad (Peace be upon Him) who is forever a torch of guidance for humanity.

Many people have been a part of my research work, as friends, teachers, and colleagues. Professor Saleem Asghar, first and foremost, has been all of these. The best advisor and teacher I could have wished for, he is actively involved in the work of all his students, and clearly always has their best interest in mind. I wish to express my heartiest gratitude to him. Time after time, his easy grasp of mathematics at its most fundamental level helped me in the struggle for my own understanding. On the personal side, he does not hesitate to share valuable advices to his students, I appreciate this immensely.

I would like to extend my gratitude to Dr. Tasawar Hayat for his support and encouragement during my research. At many stages in the course of my research I benefited from his advice, particularly so when exploring new ideas. His positive outlook and confidence in my research inspired me and gave me confidence. His careful editing contributed enormously to the production of this thesis.

I considered myself lucky to be a member of an incredible group, the Fluid Mechanics Group (FMG) whom I had many productive scientific discussions, and many beers.

Finally, I would like to thank my family, friends, my wife, specially, Mr. Muhammad Mushtaq for supporting me in all respects to complete this work.

Khalid Hanif

December 2004
Abstract

There has been a great deal of interest in understanding the behaviour of non-Newtonian fluids as they are used in various branches of science, engineering and technology: particularly in material processing, chemical industry, geophysics and bio-engineering. The study of non-Newtonian fluid flow is also of significant interest in oil reservoir engineering. Moreover, the non-Newtonian fluids such as mercury amalgams, liquid metals, biological fluids, plastic extrusions, paper coating, lubrication oils and greases have applications in many areas with or without magnetic field. Many magnetohydrodynamic problems of practical interest involving fluids as a working medium have attracted engineers, physicists and mathematicians alike. These problems are challenging because of non-linearity of the governing equations, field coupling, and complex boundary conditions. Further, using Newtonian fluid models to analyse, predict and simulate the behaviour of viscoelastic fluids has been widely adopted in industries. However, the flow characteristics of viscoelastic fluids are quite different from those of Newtonian fluids. This suggests that in practical applications the behaviour of viscoelastic fluids cannot be represented by that of Newtonian fluids. Hence, it is necessary to study the flow behaviour of viscoelastic fluids in order to obtain a thorough cognition and improve the utilization in various manufactures. Due to complexity of fluids in nature, non-Newtonian fluids are classified on the basis of their behaviour in shear. Amongst the many fluid models which have been used to describe the viscoelastic behaviour exhibited by these fluids, the fluids of second and third grades have received a special attention. The major attraction of these fluid models is due to their popularity and the fact that
they are derived from the first principle. Unlike many other phenomenological models, there are no curve-fittings or parameters to adjust for these models. Though, in both of these grade models, there are material properties that need to be measured. Also, the second grade fluid is a subclass of non-Newtonian fluids for which one can reasonably hope to obtain an analytical solution.

Another important aspect in fluid mechanics is the consideration of partial slip condition. One of the cornerstones on which the fluid mechanics is built is the no-slip condition. But, there are situations wherein this condition does not hold. In certain cases, partial slip between the fluid and the moving surface may occur. Mention may be made to the situations when the fluid is particulate such as emulsions, suspensions, foams and polymer solutions. However, literature for non-Newtonian fluids with wall slippage is scarce.

Keeping the above facts in mind, this thesis has been organized offering five chapters. Chapter zero is introductory. In chapter one, the basic equations and mathematical techniques are included for the succeeding chapters. The modelling of the general equation which govern the magnetohydrodynamic (MHD) flow of a third grade fluid is also given. Chapter two deals with the MHD flows due to non-coaxial rotations of a porous disk and a viscous fluid at infinity. Three types of unsteady flows namely, the flows induced by a constant accelerated disk with no-slip and partial slip and the flow due to variable accelerated disk with no-slip. Exact analytical solutions are constructed using Laplace transform technique. It is noted that in presence of partial slip, the reduced shearing force from the boundary causes the velocity to become flatter than that for no-slip case. Moreover, the velocity
profiles in case of constant accelerated flow are greater than for the variable accelerated case for all values of time less than one. However, this situation is quite reverse for all times greater than one. Chapter three is devoted to the flows of a second grade fluid generated by a constant accelerated disk with no-slip and partial slip conditions. The influence of second grade parameters arises in the governing equation and the boundary conditions. Both analytical and numerical solutions are given and are compared for the no-slip case. But only the numerical solution is obtained for the partial slip case. It is worth noting that material parameter of the second grade fluid reduces the velocity profiles. In chapter four, the constant accelerated flows of a third grade fluid with no-slip and partial slip have been presented. The analysis of this chapter involves the solvability of a non-linear equation. Also, the boundary condition in partial slip situation is non-linear. Numerical solutions are given using the Crank-Nicolson scheme with modification. The objective of chapter five is to extend the contents of chapter four to the case of variable accelerated flows. The influence of acceleration against time in third grade fluid is found to be smaller than that of Newtonian fluid.
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Chapter 0

Introduction

With the advent of the 20th century, Lunding Prandtl has given a new dimension to Fluid Mechanics by introducing viscosity in the fluid and thus unifying hydraulics and theoretical hydrodynamics. The Navier Stokes equations were set forth for the complete description of viscous fluids. The equations being nonlinear are difficult to solve and only in a very few cases exact analytical solutions are available. It became known that the flow past a body can be divided into a thin region closed to the body called the boundary layer where viscosity is important and the remaining region where the viscosity can be neglected. The most important application of the boundary layer can be seen as friction drag of bodies in a flow. The boundary layer has its application in lift of an airfoil and heat transfer between a body and the fluid around it. Boundary layer theory is extended to compressible turbulent boundary layers as well. A good account of the fundamental work on boundary layers can be seen from the book by Schlichting et al. [1]. The viscous fluids are
governed by Newton's law of friction

\[ \tau = \mu \frac{du}{dy}. \]

This defines a linear relationship between shear stress \( \tau \) and the velocity gradient \( du/dy \) (\( \mu \) is the dynamic viscosity and \( u \) is the \( x \)-component of velocity). Fluids where there is nonlinear relationship between these quantities are called non-Newtonian fluids. We will be concerned with both Newtonian and non-Newtonian fluids in this study.

It is always of great practical importance to calculate the material moduli of non-Newtonian fluids with the help of an instrument called orthogonal rheometer. This basically consists of two parallel disks rotating with the same angular velocity about two eccentric axes. Besides this the problems at hand have many physical applications in geophysical, astrophysical and cosmic fluid dynamics. Recently, with increasing interest in the production of heavy crude and conventional oil paraffin contents, it has become essential for oil reservoir engineering to have more understanding of the rheological effects of second and third grade fluid models.

Firstly, we would like to present a brief history of non-coaxial rotating disks. The flow due to single disk was done by Karman [2] followed by many authors, e.g., Erdogan [3], [4], Coirier [5] and Siddiqui et al. [6]. For the case of rotating eccentric disks, Berker [7] obtained an exact analytical solution for the flow between two disks rotating with same angular velocity. Berker [8] – [9] further showed that there are infinite number of nontrivial solutions for the rotation of disks about the same or different axes. The inertia effects were included by Abbot et al. [10] while discussing the Newtonian fluid between non coaxial rotating disks assuming small distance between the axes.
The solution for second grade fluid is presented by Rajagopal et al. [11] and Rajagopal [12]. The rotating disk and the fluid at infinity is addressed by Erdogan [13] for an unsteady Newtonian fluid. Since the effects of magnetic field on the fluids for the rotating disks has been a subject of special interest for engineers and scientists, a considerable literature has appeared in this direction as well. Thus an exact analytical solution for a Newtonian fluid between eccentric rotating disks with MHD was presented by Mohanty [14], whereas magnetic field was introduced by Erkman [15]. Murthy and Ram [16] extended this work to evaluate exact analytical solution for non-coaxial rotation of a disk and fluid at infinity introducing transverse magnetic field. The second grade problem was discussed by Kaloni et al. [17]. Recently, Ersoy [18] obtained an exact analytical solution for an Oldroyd-B fluid between two non-coaxially rotating insulated disks in the presence of a uniform transverse magnetic field, and later [19] when the disks are suddenly pulled with constant velocity.

After having discussed the steady state problems, a lot has been accomplished for unsteady cases as well. Smith [20] extended the work of Berker [9] to the unsteady case. Taking into account the non-torsional vibrations of the plates, Kasiviswanathan et al. [21] obtained an exact solution for unsteady Navier Stokes equation. Pop [22] considered sudden motion of the plate and fluid at infinity considering them to be initially at rest. Erdogan [23] made the change in the initial conditions and supposed that the plate and the fluid at infinity, while rotating initially about one axis suddenly shift the positions so as the plate starts rotating about another axis and the fluid at infinity keeps on rotating about its original axis. The exact analytical solution for
With a greater interest in non-Newtonian fluids, due to their applications in engineering and chemical industry, geophysics and bio-engineering, the consideration of these fluids has rather become very important. To understand the behaviour of the non-Newtonian fluids in rotating disks a number of papers have appeared. Reiner-Rivlin fluid between parallel plates and disks was considered by Srivasteva [24] and Bhatnagar et al. [25]. Erdogan [26] studied the flow for a second grade fluid. The flows of Newtonian and non-Newtonian fluids between parallel disks has been reviewed by Rajagopal [27]. To determine the complex viscosity various work was accomplished by Blyler et al.[28] and Bird et al. [29].

Some work in this direction has been accomplished by Tahira [30], in which she considered non-coaxial rotation of a porous disk and a fluid at infinity while taking the disk as stationary in Newtonian and non-Newtonian fluids. Analytical and graphical results are obtained and discussed.

The "no-slip" boundary condition is one of the cornerstones on which the fluid mechanics involving viscous fluid is built. In certain situations, however, the assumption of no-slip does no longer apply and is replaced by a partial slip boundary condition. Examples include situations when the fluid is particulate such as emulsions, suspensions, foams and polymer solutions. Also the extrusate may be rough or porous when the no slip condition is compromised. In these cases, the boundary condition is described by Navier [31], where the amount of relative slip is proportional to the local shear stress. The condition introduced by Navier has been used in studies of fluid flow past permeable walls [32], slotted plates [33], rough and coated surfaces [34], and
gas and fluid flow in microdevices [35]. Rao et al. [36] discussed the effect of the slip boundary condition on the viscous flow in a channel.

The "no-slip" boundary condition is widely used for flows involving non-Newtonian fluids past solid boundaries. However, a class of polymers usually slip or stick-slip on the solid boundaries and the phenomena of spurt and sharkskin have been explained. Related work in this direction has been given by McLeish et al. [37], Malkus et al. [38], Rao [39] and Rao et al. [36]. But the literature on non-Newtonian fluids which involve partial slip is scarce.

It appears that rotating plates with additional velocity or oscillations, particularly non-coaxial rotations, are not visited to a large extent except Erdogan [3] for hydrodynamic fluid and Hayat et al. [40] for Hartmann flow. However, we must keep in mind, that the oscillations of the plates in rotating fluids have been discussed quite extensively. In the literature, we generally encounter the oscillations of the type, harmonic in time, the constant velocity, elliptical oscillations etc. Ersoy's paper [19] is also with a constant velocity situation. Much more meaningful are the situations where the disk is moving with the constant and variable accelerations. These present the difficult situations since the boundary conditions are not only unsteady but do not allow the steady state situation. The effects of the accelerations are quite dominant on the flow and the velocity field increases with time whereas the boundary layer decreases which is of profound interest in Newtonian and non-Newtonian fluid dynamics. Yet another interest lies in the behaviour of different non-Newtonian fluids as a response to the constant and variable accelerations of the disks. Another aspect, in the study of accelerating plates is the behaviour of velocity and the corresponding time in determining the
effects of constant acceleration. To further elaborate, it means that which of the variable velocity or time, has a dominant effect on the overall constant acceleration motion of the plate.

Keeping in view the importance of non-coaxial rotations of disks it is imperative to seek further insight in this important field of fluid dynamics. As noted earlier, non-coaxial rotations in linear and non-linear fluids have been addressed by Hayat et al. [41]. In this, they have, in fact, concentrated on rotations of the disk while discussing the flow field. The disks are generally taken to be quiescent. In the discussion of non-coaxial rotations in third grade, some numerical computations were made. We further note that whatever has been said in the non-coaxial rotations so far no slip boundary conditions have been used. What we plan to present is to consider the non-coaxial rotation in more details in which the disks are not assumed to be stationary but are moving. Further, it is assumed that the disk is accelerating at a constant rate and with variable accelerations. Studies have been made taking into account no slip and partial slip boundary conditions. All these discussions are firstly conducted in Newtonian fluid and is given in chapter two. The exact analytic solutions for viscous fluids are obtained using Laplace transform method when the disk is accelerating with constant and variable accelerations in both no slip and partial slip conditions. The graphical results of the solution are obtained and discussed. Chapter three deals with the analysis for second grade fluid which is a subclass of the differential type fluids. In other words, flows due to non-coaxial rotations with variable and constant accelerations are now extended to the second grade fluid. The governing equations are still linear, however the Laplace trans-
form method does not help to solve the initial value problem in the sense that the complete analytical solution is not workable because of the inverse Laplace transform. Thus, perturbation and Laplace transform methods are applied to obtain approximate analytic solution. The flow field is expanded in terms of asymptotic series in terms of material parameter of the second grade fluid. Two terms are retained in this asymptotic expansion. The influence of partial slip and no-slip conditions are taken into account. The numerical solution of the problem is also given. For this, a complete numerical algorithm is developed to solve the equations. A comparison of analytical and numerical solutions are presented and an excellent agreement has been achieved. At this point we recall that the flow field of a third grade fluid induced by the accelerating (uniform and variable) disk satisfying no slip and partial slip boundary conditions while performing non-coaxial rotations are not accomplished so far. Such attempts have been made in chapters four and five. The analysis for the third grade fluid flows of a uniformly accelerated disk with no-slip and partial slip conditions have been performed in chapter four. Chapter five is prepared to study the influence of variably accelerated disk on the flows with no-slip and partial slip conditions. In these chapters, the governing equations are no more linear and the analytic solutions have their limitations. The perturbation method does not seem to be workable because we are tackling an initial value problem and is thus not tractable analytically even after splitting the original equation to a number of linear equation by perturbation expansion method. Therefore, we completely rely upon the numerical computation of the flow field in a third grade fluid when the disk is making non-coaxial rotation and is being accelerated at the same
time. This, we think a reasonable progress and improvement for (a) non-coaxial rotations of disk (b) satisfying both no-slip and partial slip boundary conditions (c) an initial value problems (d) the disk is being accelerated. The observations made for these problems are given in the respective chapters of the thesis. An excellent agreement has been reached. In the nutshell, both analytical and numerical results are stressed and obtained in a most systematic manner in this thesis. The non-Newtonian fluids are then considered and it has been seen that the analytic results are possible in second grade case as well. Although a lot of effort is required to build up analytical results of these governing equations. Again numerical results are obtained and the parallel development of computational algorithm has been presented in the thesis. Thus, a whole lot of reasonable advancement has been made in the context of non-coaxial rotations both analytically and numerically. It is hoped that the thesis will give further insight to the problems of non-coaxial rotations in Newtonian and non-Newtonian fluids with a great physical applications in the industry.

A paper which contains the contents of chapter two has been submitted in “Meccanica”. The contents of chapter three has been accepted for publication in “Communications in Non-Linear Science and Numerical Simulation”. The other two papers which contain the contents given in chapters four and five have been submitted in “Applied Numerical Mathematics” and ZAMM (Zeitschrift fur Angewandte Mathematik und Mechanik).
Chapter 1

Preliminaries

This chapter consists of some basic concepts and equations. The equations of motion for flow due to non-coaxial rotations involving third grade fluid are also derived.

1.1 Non-Newtonian fluid

A shear stress is a stress state where the shape of a material tends to change without particular volume change. The shape change is evaluated by measure if the change of angle magnitude. In laboratory testing, shear stress is achieved by torsion of specimen. Direct shear of a specimen by a moment induces shear stress, as well as tensile and compressive stress $\tau$ is given by

$$\tau = k \left( \frac{du}{dy} \right)^n$$  \hspace{1cm} (1.1)

where $k$ is flow consistency index, $du/dy$ is the shear rate and the exponent $n$ is called the flow behaviour index. To ensure that $\tau$ has same sign as $du/dy$
in equation (1.1) is rewritten in the form of

\[ \tau = \eta_0 \frac{du}{dy}, \]

where the term \( \eta_0 = k (du/dy)^{n-1} \) is referred as the apparent viscosity or affected viscosity.

A non-Newtonian fluid is a fluid in which apparent viscosity changes with the applied shear force. As a result, non-Newtonian fluid may not have a well-defined viscosity. A fluid property that related the magnitude of fluid shear stress to the fluid strain rate in a non-linear way, or more simply, to the special rate of change in the fluid velocity field. The role of non-Newtonian fluid dynamics is important in connection with plastic manufacture, performance of lubricants, paints, processing of food, and movement of biological fluids which contains higher molecular weight components and are, therefore, non-Newtonian fluid. The distinction between the rheology and the fluid mechanics of viscoelastic materials is vague but worth stating. The former is concerned with constitutive relations between stress and deformation and may involve physical modeling at a molecular level. Controllability of the flow field is essential when making rheological measurements to evaluate material properties, so the kinematics are generally imposed and the momentum equation is not solved. Viscoelastic fluid mechanics, on the other hand, is the study of motion in which the kinematics cannot be established a priori, and the continuity and momentum equations must be solved together with the constitutive equation for the stress. The equations that must be solved for even the most elementary viscoelastic liquids are considerably more complex than the Navier-Stokes equations, and many unsolved issues of a fundamental nature remain. Readers in viscoelastic fluid mechanics have therefore focused
on the use of constitutive equations that capture important qualitative features of material rheology, but the simplistic relative to the formulations believed to characterize real materials.

1.1.1 Thermodynamical differential type fluids

Non-Newtonian fluid mechanics differs from classical Newtonian fluid mechanics in many important respects. The most important difference is that the complexity of non-Newtonian response essentially results in governing equations that vary from one flow type to another whereas in the classical situation the Navier-Stokes equations are accepted without question as the governing equations.

It is possible to characterize a class of fluids, called simple fluids by means of a functional relationship between the stress tensor at time $t$ and the strain history of the material point with respect to its material configuration. Incompressibility is a common assumption in non-Newtonian mechanics, and we may decompose the Cauchy stress tensor as the sum of pressure contribution and extra stress. Due to complexity of fluids there are many models of non-Newtonian fluids available in the literature. Recently, the fluids of differential type have acquired the special status. The simplest subclass of the differential type fluids for which one can reasonably hope to obtain the analytic solution is the second grade fluid model. The constitutive equation for the Cauchy stress tensor of a second grade fluid is of the following form

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2.$$  \hspace{1cm} (1.2)

In the above equation, $-pI$ is the spherical stress due to constraint of incomp-
pressibility, \( \mu \) is the dynamic viscosity, \( \alpha_1 \) and \( \alpha_2 \) are the material moduli and \( A_n(n = 1, 2) \) are the first two \textit{Rivlin - Ericksen} tensors. The expressions for \( A_1 \) and \( A_2 \) are defined through the following equations [42]

\[
A_1 = (\text{gradev}) + (\text{gradev})^*
\]

\[
A_n = \frac{dA_{n-1}}{dt} + A_{n-1} (\text{gradev}) + (\text{gradev})^* A_{n-1}, \quad n \geq 1.
\]

(1.3)

in which \( \mathbf{v} \) is the velocity vector, \((^*)\) signifies the matrix transpose and \(d/dt\) is the material time differentiation. The \textit{Clausius - Duhem} inequality implies that the coefficients \( \mu, \alpha_1 \) and \( \alpha_2 \) must satisfy et al. [43]

\[
\mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0.
\]

Since we are taking second grade fluid so that the above inequality strictly holds. Fosdick et al. [44] showed that if \( \alpha_1 < 0 \), the fluid exhibits anomalous behaviour that is not compatible with any fluid of rheological interest. The relation given in equation (1.2) is capable of modelling a second grade fluid which possess both viscous and elastic properties, while for \( \alpha_1 = 0 \) and \( \mu > 0 \), equation (1.2) reduces to the stress relation for incompressible, homogenous, viscous fluid. In the present thesis we take \( \alpha_1 \geq 0 \).

The constitutive equation for homogeneous third grade fluid is:

\[
T = -p\mathbf{I} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 + \beta_1 \mathbf{A}_3 + \beta_2 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) + \beta_3 (tr\mathbf{A}_1^2) \mathbf{A}_1
\]

(1.4)

where \( \beta_1, \beta_2 \) and \( \beta_3 \) are the material constants, \( tr \) is the trace and \( \mathbf{A}_3 \) is the third \textit{Rivlin - Ericksen} tensor and can be obtained from equation (1.3).
Moreover, the thermodynamics imposes the following conditions [44]:

\[
\mu \geq 0, \alpha_1 \geq 0, \beta_1 = \beta_2 = 0, \beta_3 \geq 0, \\
-\sqrt{24\mu\beta_3} \leq \alpha_1 + \alpha_2 \leq \sqrt{24\mu\beta_3}.
\] (1.5)

Note that the equation (1.4) along with equation (1.5) reduces to the following

\[
T = -pI + \alpha_1A_2 + \alpha_2A_1^2 + \{\mu + \beta_3 (trA_1^2)\}A_1,
\] (1.6)

where the term in the braces is known as the apparent viscosity and is defined by

\[
\mu_{\text{app}} = \mu + \beta_3 (trA_1^2).
\]

### 1.2 Equation of continuity

Let us consider a volume \(V_1\) enclosed by a surface \(S_0\). The normal to the surface is \(dS_0\). If the mass is neither created nor destroyed in the volume, then the mass inside a fixed surface, \(S_0\), bounding the closed volume \(V_1\) will increase if mass (or density) flows into the volume and decrease if it flows out. Mathematically this can be expressed as

\[
\frac{d}{dt} \left\{ \int_{V_1} \rho dV_1 \right\} + \int_{S_0} \rho \mathbf{v} \cdot dS_0 = 0.
\]

By using divergence theorem, we have

\[
\int_{V_1} \frac{\partial \rho}{\partial t} dV_1 + \int_{V_1} \nabla (\rho \mathbf{v}) dV_1 = 0.
\]

This holds for an arbitrary volume and so must be true at each point in space. Thus

\[
\frac{\partial \rho}{\partial t} + \nabla (\rho \mathbf{v}) = 0.
\] (1.7)
This is the mass continuity equation.

For an incompressible fluid \( \rho \) is constant and thus the equation of continuity reduces to

\[
\nabla \cdot \mathbf{v} = 0.
\]  

(1.8)

1.3 Maxwell's equations

In this section we describe the behaviour of electric and magnetic fields, \( \mathbf{E} \) and \( \mathbf{B} \) through the following differential equations

\[
\nabla \times \frac{\mathbf{B}}{\mu_1} = J + \frac{\partial}{\partial t} (\varepsilon \mathbf{E}),
\]  

(1.9)

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},
\]  

(1.10)

\[
\nabla \cdot \mathbf{B} = 0,
\]  

(1.11)

\[
\nabla \cdot \mathbf{E} = \frac{\rho_e}{\varepsilon},
\]  

(1.12)

where \( \mu_1 \) and \( \varepsilon \) are the constants which are magnetic permeability and dielectric constant respectively, \( \mathbf{D} (=\varepsilon \mathbf{E}) \) is the dielectric displacement and \( \rho_e \) is the charge density. The total magnetic field \( \mathbf{B} \) (often referred to as the magnetic field) is related to the magnetic field \( \mathbf{H} \) as \( \mathbf{B}=\mu_1 \mathbf{H} \).

By Ohm's law we can write

\[
\mathbf{J} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}),
\]  

(1.13)

where \( \sigma \) is the electrical conductivity of the fluid assumed finite. In the present study, the polarization effects are negligible \( (E = 0) \) and magnetic Reynolds number is very small i.e. induced magnetic field is also negligible.
This is the case which has applications in aerodynamics. Keeping these facts in view, the Lorentz force is described by the following equation

\[
\frac{1}{\rho} (\mathbf{J} \times \mathbf{B}) = -\frac{\sigma B_0^2}{\rho} \mathbf{v},
\]  

(1.14)

in which \( B_0 \) is the applied magnetic field.

1.4 The momentum equation

The fundamental equation which describes the flow of an incompressible magnetohydrodynamic (MHD) fluid is

\[
\rho \frac{d\mathbf{v}}{dt} = \text{div} \mathbf{T} + \mathbf{J} \times \mathbf{B}.
\]  

(1.15)

In matrix form the expression for the Cauchy stress is

\[
\mathbf{T} = \begin{bmatrix}
\tau_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \tau_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \tau_{zz}
\end{bmatrix}
\]

In above equation \( \tau_{xx}, \tau_{yy}, \tau_{zz} \) are normal stresses and \( \tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz} \) \( \tau_{zx}, \tau_{zy} \) are the shear stresses.

With the help of equation (1.14), equation (1.15) in scalar form gives

\[
\rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] = -\sigma B_0^2 u + \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z},
\]  

(1.16)

\[
\rho \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right] = -\sigma B_0^2 v + \frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z},
\]  

(1.17)

\[
\rho \left[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right] = -\sigma B_0^2 w + \frac{\partial T_{zx}}{\partial x} + \frac{\partial T_{zy}}{\partial y} + \frac{\partial T_{zz}}{\partial z},
\]  

(1.18)

where \( u, v, \) and \( w \) are the velocity components in the \( x, y, \) and \( z \)-directions, respectively.
1.5 General equations of motion for \( MHD \) third grade fluid

Here, we have an interest in the governing equations for thermodynamic, \( MHD \) third grade fluid. For that we use equation (1.6) into equations (1.16) to (1.18) and obtain

\[
\rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] = -\sigma B_0^2 u - \frac{\partial p}{\partial x} + \mu \nabla^2 u \\
+ \alpha_1 \left[ \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \nabla^2 u + \nabla^2 v \frac{\partial v}{\partial x} + \nabla^2 w \frac{\partial w}{\partial x} \right] + 2 \frac{\partial w}{\partial z} \frac{\partial^2 w}{\partial x \partial z} + 2 \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial x \partial y} + \frac{1}{2} \frac{\partial \Psi^2_x}{\partial x} \\
+ 2 \left( \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z \partial y} \right) + (3 \alpha_1 + 2 \alpha_2) \left[ \nabla^2 u \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} \right] \\
+ (\alpha_1 + \alpha_2) \left[ \nabla^2 w \Psi_y + \nabla^2 v \Psi_z + 2 \left( \frac{\partial w}{\partial z} \frac{\partial \Psi_y}{\partial z} + \frac{\partial v}{\partial y} \frac{\partial \Psi_z}{\partial y} \right) \right] \\
+ 2 \left( \frac{\Psi_y}{\partial \Psi_x} + \frac{\Psi_z}{\partial \Psi_y} \right) + \Psi_z \left( \frac{\partial \Psi_x}{\partial z} + \frac{\partial \Psi_y}{\partial y} \right),
\]

\[
(\alpha_1 + \frac{\alpha_2}{2}) \left( \frac{\partial \Psi_x}{\partial x} + \frac{\partial \Psi_y}{\partial y} \right) + \beta_3 \left[ 4 \nabla^2 u \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right) \right] \\
+ 2 \nabla^2 u \left( \Psi_x^2 + \Psi_y^2 + \Psi_z^2 \right) + 4 \frac{\partial u}{\partial x} \left( \frac{\partial \Psi_x^2}{\partial x} + \frac{\partial \Psi_y^2}{\partial x} + \frac{\partial \Psi_z^2}{\partial x} \right) \\
+ 2 \Psi_x \left( \frac{\partial \Psi_x^2}{\partial y} + \frac{\partial \Psi_y^2}{\partial y} + \frac{\partial \Psi_z^2}{\partial y} \right) + 2 \Psi_y \left( \frac{\partial \Psi_x^2}{\partial z} + \frac{\partial \Psi_y^2}{\partial z} + \frac{\partial \Psi_z^2}{\partial z} \right) \\
+ 16 \frac{\partial u}{\partial x} \left( \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial w}{\partial z} \frac{\partial^2 w}{\partial x \partial z} \right) + 8 \Psi_z \left( \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial y} \right) \frac{\partial^2 u}{\partial y \partial^2 y} + \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial z \partial^2 y} \right) + 8 \Psi_y \left( \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial z \partial^2 x} + \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial y \partial^2 z} + \frac{\partial w}{\partial z} \frac{\partial^2 w}{\partial z \partial^2 z} \right) \right], 
\]

\[
\rho \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right] = -\sigma B_0^2 v - \frac{\partial p}{\partial y} + \mu \nabla^2 v
\]
\[ \rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] + \frac{\partial w}{\partial x} \frac{\partial^2 u}{\partial y \partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial z^2} \right] + (3\alpha_1 + 2\alpha_2) \left[ \nabla^2 w \frac{\partial u}{\partial z} + 2 \frac{\partial w}{\partial z} \frac{\partial^2 u}{\partial z^2} \right] \\
+ (\alpha_1 + \alpha_2) \left[ \nabla^2 w \Psi_y + \nabla^2 u \Psi_z + 2 \left( \frac{\partial w}{\partial x} \frac{\partial \Psi_z}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial \Psi_z}{\partial y} \right) \right] \\
+ 2 \left( \Psi_y \frac{\partial^2 w}{\partial x \partial y} + \Psi_z \frac{\partial^2 w}{\partial x \partial y} \right) + 4 \Psi_y \left( \frac{(\partial u)^2}{\partial y} + (\partial v)^2 + (\partial w)^2 \right) \\
+ 2 \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + 3 \Psi_z \left( \frac{\partial u}{\partial x} \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial z^2} + \frac{\partial w}{\partial z} \frac{\partial^2 w}{\partial x \partial z} \right) \right], \quad (1.20) \]
\begin{align*}
+2\nabla^2 w & \left( \Psi_x^2 + \Psi_y^2 + \Psi_z^2 \right) + 2\Psi_y \left( \frac{\partial \Psi_x^2}{\partial x} + \frac{\partial \Psi_y^2}{\partial x} + \frac{\partial \Psi_z^2}{\partial x} \right) \\
+2\Psi_x & \left( \frac{\partial \Psi_y^2}{\partial y} + \frac{\partial \Psi_z^2}{\partial y} + \frac{\partial \Psi_x^2}{\partial y} \right) + 4 \frac{\partial w}{\partial z} \left( \frac{\partial \Psi_x^2}{\partial z} + \frac{\partial \Psi_y^2}{\partial z} + \frac{\partial \Psi_z^2}{\partial z} \right) \\
+8\Psi_y \left( \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial y^2} + \frac{\partial w}{\partial z} \frac{\partial^2 w}{\partial z^2} \right) + 16 \frac{\partial w}{\partial z} \left( \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial y \partial z} \right) \\
+ \frac{\partial w}{\partial z} & \frac{\partial^2 w}{\partial z^2} \right) + 8\Psi_x \left( \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial w}{\partial z} \frac{\partial^2 w}{\partial z \partial y} \right) \right],
\end{align*}
\tag{1.21}

where

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \]
\[ \Psi_x = \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial y} \right), \]
\[ \Psi_y = \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \]
\[ \Psi_z = \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right). \]

\subsection{Equations of motion for flow due to non-coaxial rotations}

In this section we develop the equations for non-coaxial rotations of a porous disk and fluid at infinity. For such flow, the velocity is defined \cite{23} by

\[ u = -\Omega y + f(z, t), \quad v = \Omega x + g(z, t), \quad (1.22) \]

which together with equation (1.8) gives

\[ w = -w_0 \quad (1.23) \]
for uniformly porous boundary. Here $w_0 > 0$ corresponds to the suction velocity and $w_0 < 0$ gives the injection (or blowing) velocity.

Using equations (1.22), (1.23) into equations (1.19) to (1.21) we have

$$\rho \left[ \frac{\partial f}{\partial t} - \Omega^2 x - \Omega g - w_0 \frac{\partial f}{\partial z} \right] = \sigma B_0^2 \Omega y - \sigma B_0^2 f - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 f}{\partial z^2}$$

$$+ \alpha_1 \left[ \frac{\partial^3 f}{\partial t \partial z^2} - w_0 \frac{\partial^3 f}{\partial z^3} + \Omega \frac{\partial^2 g}{\partial z^2} \right] + 2\beta_3 \frac{\partial}{\partial z} \left[ \frac{\partial f}{\partial z} \left\{ \left( \frac{\partial f}{\partial z} \right)^2 + \left( \frac{\partial g}{\partial z} \right)^2 \right\} \right],$$

$$\rho \left[ \frac{\partial g}{\partial t} - \Omega^2 y + \Omega f - w_0 \frac{\partial g}{\partial z} \right] = -\sigma B_0^2 \Omega x - \sigma B_0^2 g - \frac{\partial p}{\partial y} + \mu \frac{\partial^2 g}{\partial z^2}$$

$$+ \alpha_1 \left[ \frac{\partial^3 g}{\partial t \partial z^2} - w_0 \frac{\partial^3 g}{\partial z^3} - \Omega \frac{\partial^2 f}{\partial z^2} \right] + 2\beta_3 \frac{\partial}{\partial z} \left[ \frac{\partial g}{\partial z} \left\{ \left( \frac{\partial f}{\partial z} \right)^2 + \left( \frac{\partial g}{\partial z} \right)^2 \right\} \right],$$

$$\frac{\sigma B_0^2}{\rho} w_0 = \frac{1}{\rho} \frac{\partial P}{\partial z}$$

(1.24)

(1.25)

(1.26)

where the modified pressure $P$ is expressed as

$$P = p - (2\alpha_1 + \alpha_2) \left\{ \left( \frac{\partial f}{\partial z} \right)^2 + \left( \frac{\partial g}{\partial z} \right)^2 \right\}.$$  

(1.27)

Note that for $\beta_3 = 0$, equations (1.24) and (1.25) describe the flow for second grade fluid.

1.6 Partial slip boundary condition

The assumption that a liquid adheres to a solid boundary (no slip boundary condition) is one of the central tenets of the Navier-Stokes theory. However, there are situations wherein this assumption does not hold. Usually, the slip
is assumed to depend on the shear stress at the wall. Mention may be made to the interesting works of Navier [31], Laplace et al. [33], Miksis et al. [34] and Gad-el-Hak [35]. However a number of experiments suggests that the slip velocity also depends on the normal stresses. A detailed discussion of the work in this area can be found in Roux [45].

The no-slip boundary condition is also widely used for flows involving non-Newtonian fluids past solid boundaries. However, it has been found that a large class of rheological materials slip or stick-slip on solid boundaries. This may be an important factor in sharskin; spurt and hysteresis effects. The existence theory for non-Newtonian fluids with wall slippage is scant. However, numerical simulations of non-Newtonian fluids with boundary slip have been performed by Tarunin [46], Debbaut [47], Fower [48] and Tanner [49].

The slip velocity depends strongly on the shear stress, and hence most constitutive equations developed for slip assume that it depends only on the shear stress. Keeping this fact in mind, we consider that slip velocity depends on the shear stress in the present work. Thus from equations, (1.3) and (1.6), we have the following expressions for the shear stresses

$$
\tau_{xz} = \mu \frac{\partial f}{\partial z} + \alpha_1 \left[ \frac{\partial^2 f}{\partial t \partial z} - w_0 \frac{\partial^2 f}{\partial z^2} + \Omega \frac{\partial g}{\partial z} \right] + 2\beta_3 \frac{\partial f}{\partial z} \left[ \left( \frac{\partial f}{\partial z} \right)^2 + \left( \frac{\partial g}{\partial z} \right)^2 \right],
$$

(1.28)

$$
\tau_{yz} = \mu \frac{\partial g}{\partial z} + \alpha_1 \left[ \frac{\partial^2 g}{\partial t \partial z} - w_0 \frac{\partial^2 g}{\partial z^2} - \Omega \frac{\partial f}{\partial z} \right] + 2\beta_3 \frac{\partial g}{\partial z} \left[ \left( \frac{\partial f}{\partial z} \right)^2 + \left( \frac{\partial g}{\partial z} \right)^2 \right],
$$

(1.29)

which would be required later in the definition of partial slip boundary condition.
1.7 Boundary layer

Due its viscosity, when a flowing fluid contacts a solid body, the fluid immediately adjacent to the body takes on the same velocity as the body. Since this velocity is often different from that of the free between the two locations and, thus, a boundary layer is formed. The boundary layer is the region in which flow velocity changes from zero (relative to the wall) to the value of the free stream. Although boundary layers may be extremely thin, they are very important because it is only in this region of the flow that viscous effects are significant. As the fluid proceeds downstream, viscous diffusion occurs further into the free stream and viscous effects are felt at a greater depth from the wall - thus, it is said that the boundary layer 'grows'. For excellent of the boundary layer the readers are referred to the book by Schlichting et al. [1].

Examples

A few examples of the occurrence of boundary layers. Boundary layers can be given as:

- The fluid velocity near a solid wall.
- The velocity at the edge of a jet of fluid.
- The temperature of a fluid near a solid wall.
- Solute concentration near an interface.
- Condensing vapor on a cool surface.
1.8 Mathematical techniques

1.8.1 Perturbation technique

This is one of the most powerful technique to solve nonlinear partial differential equations of fluid mechanics in which one of the physical parameter can be assumed very small. We will present a short review of the perturbation technique. Since this is the underlying concept for the analytical solutions of the regular perturbation can be better understood through the examples where exact problems solution is not possible.

**Example 1:** Consider a non-linear differential equation:

\[
\frac{df}{dt} + f = \varepsilon f^2; \quad (0 < \varepsilon < 1) \quad f(0) = 1
\]

with small parameter \( \varepsilon \) and the given initial condition. Let us expand the function \( f \) and the initial conditions in term of a asymptotic expansion

\[
f(t, \varepsilon) = f_0(t) + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + O(\varepsilon^3),
\]

\[
f_0(0) + \varepsilon f_1(0) + \varepsilon^2 f_2(0) + O(\varepsilon^3) = 1
\]

Now comparing the \( \varepsilon \)-order terms and solving the respective equation, we obtain

\[
\begin{align*}
\varepsilon^0 : \quad & \frac{df_0}{dt} + f_0 = 0; \quad f_0(0) = 1, \quad f_0 = Ae^{-t}, \quad A = 1, \quad f_0 = e^{-t} \\
\varepsilon^1 : \quad & \frac{df_1}{dt} + f_1 = f_0^2; \quad f_1(0) = 0, \quad f_1 = e^{-t} - e^{-2t} \\
\varepsilon^2 : \quad & \frac{df_2}{dt} + f_2 = 2f_0f_1; \quad f_2(0) = 0, \quad f_2 = 2e^{-t}(e^{-t} - e^{-2t})
\end{align*}
\]

The non-linear equation has been written in term of an infinite number of linear differential equations by perturbation expansion. Each has been solved
easily and the perturbation solution is given by

\[ f = e^{-t} + \varepsilon (e^{-t} - e^{-2t}) + \varepsilon^2 (e^{-t} - 2e^{-2t} + e^{-3t}) + \ldots \]

We note that the solution is uniformly valid for all time. This can be seen because each term is of order \( O(e^{-t}) \) and ascending powers of \( \varepsilon \) guarantee the higher terms are small corrections.

**Example 2:** The nonlinear differential equation with given initial condition and small parameter \( \varepsilon \)

\[ \frac{df}{dt} + f = \varepsilon tf^2; \quad f(0) = 1 \quad 0 < \varepsilon < 1 \]

allows the regular perturbation solution and the perturbation solution is given as:

\[ f = e^t + \varepsilon (e^{-t} - e^{-2t} - te^{-2t}) \]

**Convergent and divergent series**

Convergence or divergence of the perturbation expansion series is not important in the study of perturbation expansion. We always deal with finite terms with remainder. Therefore important is the behaviour of remainder in the region of some point of interest. The important question is the behaviour of the remainder for large \( x \). We do not ask does the limit of \( R_{N_0} \) as \( N_0 \to \infty \) equals to zero for fixed \( N_0 \). We ask does the limit \( R_{N_0} \) as \( N_0 \to \infty \) equals to zero for fixed \( N_0 \) and if so does \( R_{N_0} \) approach zero faster than the terms in the truncated series. This series is called asymptotic series.

**Asymptotic expansion**

Let \( R_n \) be the remainder after \( n \) terms of the series. The mathematical
condition for asymptotic expansion is given by

\[ R_{N_0} = o \left( \frac{1}{x^{N_0}} \right) \quad \text{as} \quad x \to \infty \quad \text{is to hold for all} \quad N_0. \]

\[ R_{N_0} = O \left( \frac{1}{x^{N_0+1}} \right) \quad \text{as} \quad x \to \infty. \]

where \( o \) is small oho and \( O \) is big oho.

An example of asymptotic expansion is:

\[ f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + ... + \frac{a_{N_0}}{x^{N_0}} + R_{N_0} \]

This is because

\[ R_{N_0} = O \left( \frac{1}{x^{N_0+1}} \right) \quad \text{as} \quad x \to \infty. \]

\[ f(x) \sim \sum_{N_0=0}^{\infty} \frac{a_{N_0}}{x^{N_0}} \quad \text{as} \quad x \to \infty. \]

**Uniqueness of asymptotic expansion**

The following can be said about uniqueness of asymptotic expansions.

- Function have unique asymptotic expansion
- Expansion does not correspond to a unique function.

**Uniform and non-uniform expansions**

Let \( f(x, \varepsilon) \) be represented by an asymptotic expansion

\[ f(x, \varepsilon) = \sum_{n=0}^{N_0} a_n \delta_n(\varepsilon) + R_{N_0}(x, \varepsilon) \]

\( a_n \) is a function of \( x \), \( \delta_n(\varepsilon) \) is asymptotic sequence.
Asymptotic expansion satisfying the following condition is called uniform.

$$R_{N_0}(x,\varepsilon) = O(\delta_{n+1}(\varepsilon)), \quad \varepsilon \rightarrow 0$$

In other terms the asymptotic expansion is called uniform if

$$|R_{N_0}(x,\varepsilon)| \leq K \delta_{n+1}(\varepsilon).$$

**Regions of non-uniformity**

The expansion

$$f(x,\varepsilon) \sim 1 + \varepsilon + \varepsilon^2 x^2 + ... \quad \text{as } \varepsilon \rightarrow 0$$

becomes non-uniform when subsequent terms are no longer small corrections to previous one. This occurs when subsequent terms are of the same order or of dominant order than previous terms.

**Sources of non-uniformity**

The sources of non-uniformity can be characterized as follows:

- Infinite domains, which allow long term effects of small perturbations to accumulate.

- Singularity in governing equations which lead to localized regions of rapid change. Perturbation expansion techniques provides solution of differential equations and the approximation of integrals.

### 1.8.2 Laplace transform technique

Laplace transform method is very useful in mathematics and engineering. This method solves the differential equations and corresponding initial and boundary value problems.
Laplace transform can be applied on a piecewise continuous function \( F \). There exist a number \( \delta_0 \) and a finite number \( M_1 \) such that

\[
\lim_{t \to \infty} |F(t)| e^{-\delta t} \leq M_1 \quad \text{for } \delta > \delta_0.
\]

This limit does not exist when \( \delta > \delta_0 \), then such function is said to be of exponential order \( \delta_0 \), we can write it as

\[
|F(t)| = O \left(e^{-\delta_0 t}\right).
\]

For any bounded function \( F(t) \) we can say that \( |F(t)| e^{-\delta t} \to 0 \) for all \( \delta > 0 \).

**Definition:** Let \( F(t) \) be the function, defined for all \( t \geq 0 \). We multiply \( F(t) \) by \( e^{-\delta t} \) and integrate with respect to \( t \) from zero to infinity. If the resulting integral exists then it is a function of \( s \), say \( \overline{F}(s) \) defined as

\[
\overline{F}(s) = \mathcal{L}(F) = \int_0^\infty e^{-st} F(t) dt.
\]  \hspace{1cm} (1.30)

The function \( \overline{F}(s) \) of variable \( s \) is called the Laplace transform of the original function \( F(t) \) and is called the inverse transform of the function \( \overline{F}(s) \) and is given by

\[
\mathcal{L}^{-1}(\overline{F}) = F(t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{st} \overline{F}(s) \, dt
\]  \hspace{1cm} (1.31)

### 1.8.3 Numerical technique

Many differential equations cannot be solved by analytical methods. Similarly in many cases, it is very difficult to find the exact solution of non-linear differential equations. Consequently, familiarity with the methods of obtaining numerical solution of differential equations become very important for a modern mathematician.
Finite-difference approximations to derivatives

When a function \( u \) and its derivatives are single-valued, finite, and continuous functions of \( x \), then by Taylor’s theorem,

\[
u(x + h) = u(x) + \frac{h}{1!} u'(x) + \frac{h^2}{2!} u''(x) + \frac{h^3}{3!} u'''(x) + \ldots
\]

and

\[
u(x - h) = u(x) - \frac{h}{1!} u'(x) + \frac{h^2}{2!} u''(x) - \frac{h^3}{3!} u'''(x) + \ldots,
\]

in which prime indicates the differentiation with respect to \( x \). The equations for \( u(x + h) \) and \( u(x - h) \) can also be rewritten as

\[
u(x + h) = u(x) + hu'(x) + \frac{1}{2} h^2 u''(x) + \frac{1}{6} h^3 u'''(x) + O(h^4), \tag{1.32}
\]

\[
u(x - h) = u(x) - hu'(x) + \frac{1}{2} h^2 u''(x) - \frac{1}{6} h^3 u'''(x) + O(h^4), \tag{1.33}
\]

where \( O(h^4) \) denotes terms containing fourth and higher powers of \( h \).

From equation (1.32), we have

\[ u'(x) = \frac{u(x + h) - u(x)}{h} + O(h) \]

or

\[
\left( \frac{du}{dx} \right)_{x=x} \simeq \frac{u(x + h) - u(x)}{h} \tag{1.34}
\]

with an error of order \( h \), assuming second and higher powers of \( h \) are negligible. This is called a forward-difference formula. Similarly, from equation (1.33), we have

\[ u'(x) = \frac{u(x) - u(x - h)}{h} + O(h) \]

or

\[
\left( \frac{du}{dx} \right)_{x=x} \simeq \frac{u(x) - u(x - h)}{h} \tag{1.35}
\]

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with an error on the right-hand side of order \( h \). This is called a \textit{backward-difference} formula.

Subtraction of the equation (1.33) from equation (1.32) gives

\[
u(x + h) - u(x - h) = 2hu'(x) + \frac{1}{3}h^3u'''(x) + O(h^5) \tag{1.36}\]

or

\[
u'(x) = \frac{u(x + h) - u(x - h)}{2h} + O(h^2)
\]

or

\[
\left( \frac{du}{dx} \right) \bigg|_{x=x} \approx \frac{u(x + h) - u(x - h)}{2h} \tag{1.37}
\]

with a leading error on the right-hand side of order \( h^2 \). Consequently, it is preferred to both \textit{forward-difference} and \textit{backward-difference} formulae.

This approximation is called a \textit{central-difference} formula.

Addition of the equations (1.32) and (1.33) yields

\[
u''(x) = \frac{u(x + h) - 2u(x) + u(x - h)}{h^2} + O(h^2)
\]

or

\[
\left( \frac{d^2u}{dx^2} \right) \bigg|_{x=x} \approx \frac{u(x + h) - 2u(x) + u(x - h)}{h^2} \tag{1.38}
\]

with a maximum error of order \( h^2 \) on the right-hand.

To approximate the third order derivative, we replace \( h \) by \( 2h \) in the equation (1.36) to obtain the following equation

\[
u(x + 2h) - u(x - 2h) = 4hu'(x) + \frac{8}{3}h^3u'''(x) + O(h^5) \tag{1.39}
\]

and then subtract the equation (1.36) after multiplying by 2 from equation (1.39) and get

\[
u'''(x) = \frac{u(x + 2h) - 2u(x + h) + 2u(x - h) - u(x - 2h)}{2h^3} + O(h^5)
\]
or
\[
\left( \frac{d^3u}{dx^3} \right)_{x=x} \approx \frac{u(x + 2h) - 2u(x + h) + 2u(x - h) - u(x - 2h)}{2h^3}.
\]
(1.40)

**Finite-difference approximations to partial derivatives**

In this section, we will approximate partial derivatives of \( u \), a function of two independent continuous variables \( x \) and \( t \), such as

\[
\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \text{ and } \frac{\partial u}{\partial t} \text{ etc.}
\]

by finite difference method. For this purpose, the continuous variables \( x \) and \( t \) are discretized, so we consider \( u(x, t) \) to be evaluated only at the intersections \( P_1 \) (mesh points) of the grid lines parallel to \( x \) and \( t \) axes. Let the coordinates \((x, t)\) of the mesh point \( P_1 \) be

\[
x = ih; \quad t = jk,
\]

where \( i, j \) are integers, and \( h, k \) are the (constant) grid spacings in the \( x \) and \( t \) directions respectively. Let the value of \( u \) at \( P_1 \) be denoted by

\[
u_{i,j} = u(ith, jk).
\]

Then borrowing the analogy between ordinary and partial derivatives from calculus and concepts introduced in the previous section, we have the following centre-difference approximations to partial derivatives, each with a leading error of order of \( h^2 \):

\[
\left( \frac{\partial u}{\partial x} \right)_{i,j} \approx \frac{1}{2h} \left[ u((i+1)h, jk) - u((i-1)h, jk) \right]
\]

\[
\approx \frac{1}{2h} \left[ u_{i+1,j} - u_{i-1,j} \right],
\]
(1.41)
\[
\left( \frac{\partial^2 u}{\partial x^2} \right)_{i,j} \simeq \frac{1}{h^2} [u((i+1)h,jk) - 2u(ih,jk) + u((i-1)h,jk)] \\
\simeq \frac{1}{h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}], \quad (1.42)
\]

\[
\left( \frac{\partial^3 u}{\partial x^3} \right)_{i,j} \simeq \frac{1}{2h^3} [u((i+2)h,jk) - 2u((i-2)h,jk) \\
+ 2u((i-1)h,jk) - u((i+1)h,jk)] \\
\simeq \frac{1}{2h^3} [u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}]. \quad (1.43)
\]

With this notation the forward-difference approximation for \( \partial^3 u / \partial x^3 \) and \( \partial u / \partial t \) at \( P_1 \) is
\[
\left( \frac{\partial^3 u}{\partial x^3} \right)_{i,j} \simeq \frac{1}{h^3} [u_{i+2,j} - 3u_{i+1,j} + 3u_{i,j} - u_{i-1,j}] \quad (1.44)
\]
and
\[
\frac{\partial u}{\partial t} \simeq \frac{u_{i,j+1} - u_{i,j}}{k} \quad (1.45)
\]

with leading error of order \( h \) and \( k \) respectively.

Averaged finite difference approximations to partial derivatives

For better approximations to partial derivatives, we develop approximations to partial derivatives at the mid-way between the time levels \( j \) and \( j+1 \). Furthermore, the function \( u \) is also approximated at the midpoint \( (ih, (j + \frac{1}{2})k) \) as the average
\[
u_{i,j+\frac{1}{2}} = \frac{1}{2} [u_{i,j+1} + u_{i,j}]. \quad (1.46)
\]

The temporal first derivative can be approximated at time level \( j + \frac{1}{2} \)
\[
\left( \frac{\partial u}{\partial t} \right)_{i,j+\frac{1}{2}} = \frac{1}{k} [u_{i,j+1} - u_{i,j}]. \quad (1.47)
\]
The spatial first derivative can be determined at the midpoint by averaging the difference approximations at the $j$th and $(j + 1)$th time levels

$$
\left( \frac{\partial u}{\partial x} \right)_{i, j + \frac{1}{2}} = \frac{1}{4h} \left[ (u_{i+1, j+1} - u_{i-1, j+1}) + (u_{i+1, j} - u_{i-1, j}) \right]. \quad (1.48)
$$

Similarly, the spatial second and third derivatives can also be approximated at the midpoint by means of averaging as:

$$
\left( \frac{\partial^2 u}{\partial x^2} \right)_{i, j + \frac{1}{2}} = \frac{1}{2h^2} \left[ (u_{i+1, j+1} - 2u_{i, j+1} + u_{i-1, j+1}) + (u_{i+1, j} - 2u_{i, j} + u_{i-1, j}) \right], \quad (1.49)
$$

$$
\left( \frac{\partial^3 u}{\partial x^3} \right)_{i, j + \frac{1}{2}} = \frac{1}{4h^3} \left[ (u_{i+2, j+1} - 2u_{i+1, j+1} + 2u_{i-1, j+1} - u_{i-2, j+1}) + (u_{i+2, j} - 2u_{i+1, j} + 2u_{i-1, j} - u_{i-2, j}) \right]. \quad (1.50)
$$

To approximate $\partial^3 u/\partial t \partial x^2$, we use central difference approximation to $\partial^2 u/\partial x^2$ at the time level $j + 1$ and forward difference approximation at the time level $j$ in order to achieve convergence of the solutions to problems presented numerically in the present work.

$$
\left( \frac{\partial^3 u}{\partial t \partial x^2} \right)_{i, j + \frac{1}{2}} = \frac{1}{k} \left[ \left( \frac{\partial^2 u}{\partial x^2} \right)_{i, j - 1} - \left( \frac{\partial^2 u}{\partial x^2} \right)_{i, j} \right] = \frac{1}{k} \left[ \frac{1}{h^2} (u_{i+1, j+1} - 2u_{i, j+1} + u_{i-1, j+1}) + \frac{1}{h^2} (u_{i+2, j} - 2u_{i+1, j} + u_{i, j}) \right]

= \frac{1}{kh^2} \left[ (u_{i+1, j+1} - 2u_{i, j+1} + u_{i-1, j+1}) - (u_{i+2, j} - 2u_{i+1, j} + u_{i, j}) \right]. \quad (1.51)
$$

Some of the problems presented in this thesis, involve highly non-linear equations of motion and non-linear boundary conditions and are solved numerically by discretizing using the above approximations to derivatives.
Chapter 2

Unsteady flows of a viscous fluid due to an accelerated disk

This chapter deals with the exact analytical solutions for the three MHD flows induced by the non-coaxial rotations of a porous disk and a viscous fluid at infinity. The partial differential equations governing the flow is solved using Laplace transform method. In section 2.1, we formulate and solve the problem for uniformly accelerated disk with no-slip condition. The flow due to uniformly accelerated disk with partial slip has been analyzed in section 2.2. Section 2.3 deals the flow problem generated by a variable accelerated disk with no-slip condition. In order to describe the salient features of the involved parameters of interest on the velocity profiles, the graphs are sketched and discussed in the respective sections.
2.1 Constant accelerated flow with no-slip condition

We consider an incompressible viscous, MHD fluid filling the semi-infinite space \( z > 0 \) which is in contact with an infinite porous rotating disk at \( z = 0 \). The common angular velocity of the disk and fluid at infinity is \( \Omega \). The fluid is electrically conducting in the presence of an applied constant magnetic field \( B_0 \). Initially the disk and the fluid are rotating about the \( z' \)-axis and at time \( t = 0 \), suddenly the disk starts rotating about the \( z - \)axis and moving with uniform acceleration, while the fluid at infinity continue to rotate about \( z' - \)axis with the same angular velocity \( \Omega \). The axes of rotation of both the disk and that of the fluid at infinity are assumed to be in the plane \( x = 0 \) and the distance between the axes is \( l \). The initial and boundary conditions are

\[
\begin{align*}
    u &= \Omega y + c_0 t, \quad v = \Omega x, \quad \text{at } z = 0, \text{ for } t \to 0, \\
    u &= \Omega (y - l), \quad v = \Omega x, \quad \text{as } z \to \infty, \text{ for all } t, \\
    u &= -\Omega (y - l), \quad v = \Omega x, \quad \text{at } t = 0, \text{ for } z > 0,
\end{align*}
\]

(2.1)

in which \( c_0 \) has dimension \( L/T^2 \).

For viscous fluid \( (\alpha_1 = \beta_3 = 0) \), the equations (1.24) to (1.26) become

\[
\begin{align*}
    \frac{1}{\rho} \frac{\partial \rho}{\partial x} &= \nu \frac{\partial^2 f}{\partial x^2} + w_0 \frac{\partial f}{\partial z} - \frac{\partial f}{\partial t} + \Omega g + \Omega^2 x - \frac{\sigma}{\rho} B_0^2 \left[ f(z, t) - \Omega y \right], \\
    \frac{1}{\rho} \frac{\partial \rho}{\partial y} &= \nu \frac{\partial^2 g}{\partial z^2} + w_0 \frac{\partial g}{\partial z} - \frac{\partial g}{\partial t} - \Omega f + \Omega^2 y - \frac{\sigma}{\rho} B_0^2 \left[ g(z, t) + \Omega x \right],
\end{align*}
\]

(2.2)

(2.3)
\[
\frac{1}{\rho} \frac{\partial p}{\partial z} = \frac{\sigma B_0^2}{\rho} w_0. \tag{2.4}
\]

Using the definitions of \(u\) and \(v\) in equation (1.22), the conditions (2.1) in terms of \(f\) and \(g\) can be written as:

\[
f(0, t) = c_0 t, \quad g(0, t) = 0,
\]
\[
f(\infty, t) = \Omega l, \quad g(\infty, t) = 0, f(z, 0) = \Omega l, \quad g(z, 0) = 0. \tag{2.5}
\]

After eliminating the pressure from equations (2.2) to (2.4), the resulting boundary layer equations can be combined into the following equation

\[
\nu \frac{\partial^2 F^*}{\partial z^2} + w_0 \frac{\partial F^*}{\partial z} - \frac{\partial F^*}{\partial t} - \Omega \left( i + \frac{\sigma B_0^2}{\rho \Omega} \right) F^* = 0, \tag{2.6}
\]

where

\[
F^*(z, t) = f + ig - \Omega l \tag{2.7}
\]

and the conditions (2.5) now become

\[
F^*(0, t) = c_0 t - \Omega l, \quad F^*(\infty, t) = 0, \quad F^*(z, 0) = 0. \tag{2.8}
\]

Introducing the following non-dimensional variables

\[
F = \frac{F^*}{\Omega l}, \quad \eta = \sqrt{\frac{\Omega}{\nu}} Z, \quad S = \frac{w_0}{\sqrt{\nu \Omega}}, \quad N = \frac{\sigma B_0^2}{\rho \Omega}, \quad \tau = \Omega t, \quad c = \frac{c_0}{\Omega l} \tag{2.9}
\]

the governing problem takes the form

\[
\frac{\partial^2 F}{\partial \eta^2} + S \frac{\partial F}{\partial \eta} - \frac{\partial F}{\partial \tau} - (i + N) F = 0, \tag{2.10}
\]
\[
F(0, \tau) = c \tau - 1, \quad F(\infty, \tau) = 0, \quad F(\eta, 0) = 0. \tag{2.11}
\]
2.1.1 Analytical solution

In order to find the solution of the problem consisting of equation (2.10) and conditions (2.11), we use the Laplace transform method. Thus, after taking Laplace transform we have

\[
\frac{d^2 \overline{F}}{d\eta^2} + S \frac{d\overline{F}}{d\eta} - (s + N+i)\overline{F} = 0, \tag{2.12}
\]

\[
\overline{F}(0, s) = \frac{c}{s^2} - \frac{1}{s}, \tag{2.13}
\]

\[
\overline{F}(\infty, s) = 0.
\]

The solution of equation (2.12) satisfying conditions (2.13) is

\[
\overline{F}(\eta, s) = \left(\frac{c}{s^2} - \frac{1}{s}\right) e^{-\eta m}, \tag{2.14}
\]

in which

\[
m = \frac{S}{2} + \sqrt{A + s}, \quad A = \frac{S^2}{4} + N + i. \tag{2.15}
\]

Laplace inversion of equation (2.14) gives

\[
F(\eta, \tau) = e^{-\frac{S}{2} \eta} \left[ e^{-\eta \sqrt{A}} \operatorname{erf} c \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \left\{ c \left( \frac{\tau}{2} - \frac{\eta}{4\sqrt{A}} \right) - \frac{1}{2} \right\} + e^{\eta \sqrt{A}} \operatorname{erf} c \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \left\{ c \left( \frac{\tau}{2} + \frac{\eta}{4\sqrt{A}} \right) - \frac{1}{2} \right\} \right]. \tag{2.16}
\]

From equation (2.16) and equation (2.9), we obtain

\[
\frac{j}{\Omega l} + i \frac{g}{\Omega l} = 1 + e^{-\frac{S}{2} \eta} \left[ e^{-\eta \sqrt{A}} \operatorname{erf} c \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \left\{ c \left( \frac{\tau}{2} - \frac{\eta}{4\sqrt{A}} \right) - \frac{1}{2} \right\} + e^{\eta \sqrt{A}} \operatorname{erf} c \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \left\{ c \left( \frac{\tau}{2} + \frac{\eta}{4\sqrt{A}} \right) - \frac{1}{2} \right\} \right]. \tag{2.17}
\]
Equation (2.17) tells us the effects of $c$, $\tau$, $S$, $N$ on the velocities $f$ and $g$. Now we are interested to see the effects of rotation and non-coaxial parameter on the velocity field. For this, we define dimensionless parameter of the form,

\[
F = \frac{F^*}{(\nu \sigma_0)^{\frac{1}{3}}}, \quad S = \frac{1}{(c_0 \nu)^{\frac{1}{2}}} w_0, \quad \eta = \left(\frac{c_0}{\nu^2}\right)^{\frac{1}{3}} z, \quad \tau = t \left(\frac{c_0^2}{\nu}\right)^{\frac{1}{3}},
\]

\[
N = \left(\frac{c_0}{\nu^2}\right)^{\frac{1}{3}} \frac{\sigma B_0^2}{\rho}, \quad \Omega_1 = \left(\frac{\nu}{c_0}\right)^{\frac{1}{3}} \Omega, \quad l_1 = l \left(\frac{c_0}{\nu^2}\right)^{\frac{1}{3}}.
\]  

(2.18)

Using (2.18), the solution (2.17) can be written as

\[
\frac{f}{(\nu \sigma_0)^{\frac{1}{3}}} + i \frac{g}{(\nu \sigma_0)^{\frac{1}{3}}} = \\
\Omega_1 l_1 + e^{-\frac{2\pi}{3} \eta} \left[ e^{-\eta \sqrt{A_1}} \text{erf} c \left(\frac{\eta}{2\sqrt{\tau}} - \sqrt{A_1} \tau\right) \left\{ \left(\frac{\tau}{2} - \frac{\eta}{4\sqrt{A_1}}\right) - \frac{\Omega_1 l_1}{2} \right\} \right.
\]

\[
+ e^{\eta \sqrt{A_1}} \text{erf} c \left(\frac{\eta}{2\sqrt{\tau}} + \sqrt{A_1} \tau\right) \left\{ \left(\frac{\tau}{2} + \frac{\eta}{4\sqrt{A_1}}\right) - \frac{\Omega_1 l_1}{2} \right\} \right],
\]  

(2.19)

where

\[
A_1 = \frac{S^2}{4} + N + i\Omega_1.
\]  

(2.20)

\[
2.1.2 \quad \text{Graphs and discussion}
\]

We present the flow pattern of velocities for the problem at hand. The numerical graphs for the velocity profiles $f$ and $g$ are given and discussed for various possibilities.

Figures 2.1.1 and 2.1.2 illustrate the variation in time $\tau$ in the graphs of $f$ and $g$ respectively, when $c = 0$, $N = 0$, and $S = 0$. It is seen that $f$ decreases near the disk for large values of $\tau$ while $g$ increases. The figures 2.1.3 and 2.1.4 are plotted for same values of $\tau$ but in the presence of acceleration $c = 0.5$. Here, we see that velocity profiles $f$ and $g$ increases near the disk.
Also it is noted that the layer thickness increases. Figures 2.1.5 and 2.1.6 give the effects of acceleration on the velocity field $f$ and $g$. Interestingly with the increase in acceleration, $f$ increases near the disk and $g$ decreases but in both the cases layer thickness reduces. Now we show the effects of different values of injection(blowing)/suction in the presence of acceleration $c = 0.25$ and time $\tau = 1$. When $S = -1$, we note that velocity $f$ decreases and $g$ increases and layer thickness in both cases increases. On the other hand when we increase the suction velocity $f$ increases and $g$ decreases and consequently the layer thickness reduces, (see figures 2.1.7 and 2.1.8). Figures 2.1.9 and 2.1.10 depict the effects of magnetic field on $f$ and $g$. With the increase in magnetic field the layer thickness in both cases reduces.

Now to see the influence of rotation $\Omega_1$ and non-coaxial parameter $l_1$ on the velocity field, we plot equation (2.19). We note that in figures 2.1.11 and 2.1.12 that increase in rotation reduces the layer thickness when $N = S = l_1 = 0$. With the addition of non-coaxial parameter $l_1$ it is observed that with the increase in rotation the layer thickness decreases and velocity $f$ stabiles at the product of $\Omega_1 l_1$ and the velocity $g$ increases from negative to positive near the disk (see figures 2.1.13 and 2.1.14). Further, increase in $l_1$ reduces layer thickness and velocity $f$ stabiles at the product $\Omega_1 l_1$. On the other hand $g$ increases and becomes positive from negative values.
The variation of the velocity with distance from the disk for various values of time $\tau$ when $S = N = 0$, and $c = 0$.

The variation of the velocity with distance from the disk for various values of time $\tau$ when $S = N = 0$, and $\tau = 1$. 
The variation of the velocity with distance from the disk for various values of acceleration $c$ when $S = N = 0$ and $\tau = 1$.

The variation of the velocity with distance from the disk for various values of injection/suction parameter $S$ when $N = 0$, $c = 0.25$ and $\tau = 1$. 
The variation of the velocity with distance from the disk for various values of magnetic field $N$ when $S = 0$, $c = 0.25$, and $\tau = 1$.

The variation of the velocity with distance from the disk for various values of rotation parameter $\Omega$ when $S = 0$, $l_1 = 0$, and $\tau = 1$. 
The variation of the velocity with distance from the disk for various values of rotation parameter $\Omega$ when $S = 0$, $l_1 = 0.5$, and $\tau = 1$.

The variation of the velocity with distance from the disk for various values of non-coaxial parameter $l_1$ when $S = N = 0$, $\Omega = 1$ and $\tau = 0.5$.
2.2 **Constant accelerated flow with partial slip condition**

This section involves a physical problem which is same to that as given in section 2.1 except that there is a slippage between the disk and the fluid. Thus the governing differential equation, boundary condition at infinity and initial conditions are the same. But the boundary condition at \( z = 0 \) is now given by

\[
\begin{align*}
  u - \lambda_2 \tau_{xz} &= -\Omega y + c_0 t, \\
  v - \lambda_2 \tau_{yz} &= \Omega x, \quad \text{at} \quad z = 0 \quad \text{for} \quad t > 0,
\end{align*}
\]

(2.21)

where \( \lambda_2 = \lambda_1 / \mu \) is slip parameter of dimension of \( L \) and

\[
\begin{align*}
  \tau_{xz} &= \mu \frac{\partial u}{\partial z}, \\
  \tau_{yz} &= \mu \frac{\partial v}{\partial z}.
\end{align*}
\]

Employing the similar procedure as used in section 2.1, the governing problem in terms of non-dimensional variables with new boundary conditions can be written as

\[
\frac{\partial^2 F}{\partial \eta^2} + S \frac{\partial F}{\partial \eta} - \frac{\partial F}{\partial \tau} - (i + N) F = 0,
\]

(2.22)

\[
\begin{align*}
  F(0, \tau) &= \lambda \frac{\partial F}{\partial \eta} - 1 + c \tau, \\
  F(\infty, \tau) &= 0, \\
  F(\eta, 0) &= 0,
\end{align*}
\]

(2.23)

where
\[ \lambda = \lambda_2 \sqrt{\frac{\Omega}{\nu}}. \]  

(2.24)

2.2.1 Analytic solution

Employing the similar procedure as in section 2.2, the transformed solution of the problem is given by

\[ \bar{F}(\eta, s) = \left( \frac{c}{s^2 (1 + \lambda m)} - \frac{1}{s (1 + \lambda m)} \right) e^{-\eta m}, \]  

(2.25)

The inverse of equation (2.25) is

\[ F(\eta, \tau) = \frac{c e^{-\frac{5}{2} \eta}}{\lambda} I_1 - \frac{e^{-\frac{5}{2} \eta}}{\lambda} I_2, \]  

(2.26)
in which

\[ I_1 = \frac{1}{2 \pi i} \int_{-i \infty}^{i \infty} \frac{1}{s^2 \left( a + \sqrt{s + A} \right)} e^{-n \sqrt{s + A} + \eta \tau} ds \]

\[ = -\frac{a}{(A - a^2)} e^{n \alpha + a^2 \tau - A \tau} \text{erf} c \left( \frac{\eta}{2 \sqrt{\tau}} + a \sqrt{\tau} \right) \]

\[ + \frac{1}{4 \sqrt{A} (\sqrt{A} - a)} e^{n \sqrt{A}} \text{erf} c \left( \frac{\eta}{2 \sqrt{\tau}} + \sqrt{A \tau} \right) \]

\[ + \frac{1}{4 \sqrt{A} (\sqrt{A} + a)} e^{-n \sqrt{A}} \text{erf} c \left( -\frac{\eta}{2 \sqrt{\tau}} - \sqrt{A \tau} \right) \]

\[ + \frac{1}{4 \sqrt{A} (\sqrt{A} - a)} \left\{ 2 \sqrt{\pi} e^{-\frac{n^2}{4 \tau} - A \tau} \right. \]

\[ - (\eta + 2 \sqrt{A \tau}) e^{n \sqrt{A}} \text{erf} c \left( \frac{\eta}{2 \sqrt{\tau}} + \sqrt{A \tau} \right) \left\} \right. \]

\[ + \frac{1}{4 \sqrt{A} (\sqrt{A} + a)} \left\{ 2 \sqrt{\pi} e^{-\frac{n^2}{4 \tau} - A \tau} \right. \]

\[ - (\eta - 2 \sqrt{A \tau}) e^{-n \sqrt{A}} \text{erf} c \left( \frac{\eta}{2 \sqrt{\tau}} - \sqrt{A \tau} \right) \left\}, \right. \]

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\[ I_2 = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{s \left( a + \sqrt{s + A} \right)} e^{-\eta \sqrt{s + A + a\tau}} ds \]

\[ = \frac{1}{(a^2 - A)} \left\{ \sqrt{\frac{1}{\pi \tau}} e^{-\frac{\eta^2}{4\tau} - A\tau} - ace^{\eta \sqrt{A} + a^2 \tau - A\tau} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} + a\sqrt{\tau} \right) \right\} \]

\[ + \frac{1}{2\sqrt{A} (\sqrt{A} - a)} \left\{ \frac{1}{\sqrt{\pi \tau}} e^{-\frac{\eta^2}{4\tau} - A\tau} - \sqrt{A} e^{\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \right\} \]

\[ - \frac{1}{2\sqrt{A} (\sqrt{A} + a)} \left\{ \frac{1}{\sqrt{\pi \tau}} e^{-\frac{\eta^2}{4\tau} - A\tau} + \sqrt{A} e^{-\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \right\} . \]

Substituting the values of \( I_1 \) and \( I_2 \) in equation (2.26) and then writing

\[ F = \frac{f}{\Omega l} + \frac{g}{\Omega l} - 1 \]

in the resulting equation we have

\[ \frac{f + ig}{\Omega l} = 1 + ce^{-\frac{\eta^2}{4\tau}} \left[ \frac{-8\lambda^2 (2 + \lambda S)}{(4\lambda^2 A - (2 + \lambda S)^2)} e^{\eta \sqrt{A} + a^2 \tau - A\tau} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} + a\sqrt{\tau} \right) \right] \]

\[ + \frac{\lambda}{\sqrt{A} (2\sqrt{A} - 2 - \lambda S)} \left[ 2\sqrt{\frac{\tau}{\pi}} e^{-\frac{\eta^2}{4\tau} - A\tau} + \sqrt{A} e^{\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \right] \]

\[ + \frac{\lambda}{\sqrt{A} (2\sqrt{A} + 2 + \lambda S)} \left[ 2\sqrt{\frac{\tau}{\pi}} e^{-\frac{\eta^2}{4\tau} - A\tau} - \sqrt{A} e^{-\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \right] \]

\[ + \frac{1}{2\sqrt{A} (2\sqrt{A} - 2 - \lambda S)} \left\{ 2\sqrt{\frac{\tau}{\pi}} e^{-\frac{\eta^2}{4\tau} - A\tau} - (\eta + 2\sqrt{A\tau}) e^{\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \right\} \]

\[ + \frac{1}{2\sqrt{A} (2\sqrt{A} + 2 + \lambda S)} \left\{ 2\sqrt{\frac{\tau}{\pi}} e^{-\frac{\eta^2}{4\tau} - A\tau} - (\eta - 2\sqrt{A\tau}) e^{-\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \right\} \]
\[ -e^{-\frac{\eta}{4}} \left[ \frac{4}{(2 + \lambda S)^2 - 4\lambda A} \left\{ \frac{\lambda}{\sqrt{\pi \tau}} e^{-\frac{\eta^2}{4\tau}} - A\tau \right\} ight. \\
\left. - \frac{(2 + \lambda S)}{2} e^{\eta a + a^2 \tau - A\tau} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} + a\sqrt{\tau} \right) \right] \\
+ \frac{1}{\sqrt{A}} \left( \frac{1}{2\lambda \sqrt{A} - 2 - \lambda S} \right) \left\{ \frac{1}{\sqrt{\pi \tau}} e^{-\frac{\eta^2}{4\tau}} - A\tau \right\} \\
- \sqrt{A} e^{\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \right] \\
+ \frac{1}{\sqrt{A}} \left( \frac{1}{2\lambda \sqrt{A} + 2 + \lambda S} \right) \left\{ \frac{1}{\sqrt{\pi \tau}} e^{-\frac{\eta^2}{4\tau}} - A\tau \right\} \\
+ \sqrt{A} e^{-\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \right], \quad (2.27) \\
\]

where
\[ a = \frac{2 + \lambda S}{2\lambda}. \quad (2.28) \]

In order to see effects of rotations and non-coaxial parameter the solution (2.27) in terms of dimensionless variables (2.18)
\[ \frac{f + ig}{(\nu c_0)^{\frac{1}{2}}} = \Omega_1 l_1 + e^{-\frac{\eta}{4}} \left[ - \frac{8\lambda^2 (2 + \lambda S)}{(4\lambda^2 A - (2 + \lambda S)^2)} e^{\eta a + a^2 \tau - A\tau} \right. \]
\[ + \frac{\lambda}{\sqrt{A}} \left( \frac{1}{2\lambda \sqrt{A} - 2 - \lambda S} \right)^2 e^{\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \]
\[ + \frac{\lambda}{\sqrt{A}} \left( \frac{1}{2\lambda \sqrt{A} + 2 + \lambda S} \right)^2 e^{-\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \]
\[ + \frac{1}{2\sqrt{A}} \left( \frac{1}{2\lambda \sqrt{A} - 2 - \lambda S} \right) \left\{ 2\frac{\sqrt{\frac{\tau}{\pi}} e^{-\frac{\eta^2}{4\tau}} - A\tau} \right\} \\
- \left( \eta + 2\sqrt{A\tau} \right) e^{\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \right] \\
+ \frac{1}{2\sqrt{A}} \left( \frac{1}{2\lambda \sqrt{A} + 2 + \lambda S} \right) \left\{ 2\frac{\sqrt{\frac{\tau}{\pi}} e^{-\frac{\eta^2}{4\tau}} - A\tau} \right\} \\
+ \sqrt{A} e^{-\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \right], \quad (2.27) \\
\]

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\[ -\left( \eta - 2\sqrt{A\tau} \right) e^{-\eta\sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \]

\[ -\Omega_1 l e^{-\frac{s}{2} \eta} \left[ \frac{4}{(2 + \lambda S)^2 - 4\lambda^2 A} \left\{ \frac{\lambda}{\sqrt{\pi \tau}} e^{-\frac{\sqrt{A}}{2\tau}} \right\} \right. \]

\[ -\frac{(2 + \lambda S)}{2} e^{\eta a + a^2 \tau - A\tau} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} + a\sqrt{\tau} \right) \]

\[ + \frac{1}{\sqrt{\lambda}} \left( \frac{1}{2\sqrt{\tau}} \right) \left\{ \frac{1}{\sqrt{\pi \tau}} e^{-\frac{\sqrt{A}}{2\tau}} \right\} \]

\[ -\sqrt{A} e^{\eta\sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \]

\[ + \frac{1}{\sqrt{\lambda}} \left( \frac{1}{2\sqrt{\tau}} \right) \left\{ \frac{1}{\sqrt{\pi \tau}} e^{-\frac{\sqrt{A}}{2\tau}} \right\} \]

\[ + \sqrt{A} e^{-\eta\sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \]. \quad (2.29)

### 2.2.2 Graphs and discussion

In order to study the effects of \( \lambda, S, N, c \) and \( \tau \) on the unsteady flows we have sketched the several graphs and discussed as follows:

Figures 2.2.1 and 2.2.2 give the effect of slip parameter \( \lambda \) on the velocity profiles \( f \) and \( g \). With the increase in \( \lambda \), velocity \( f \) increases near the disk and \( g \) decreases for fixed values of \( \tau, N, S \) and \( c = 0 \). The layer thickness decreases for both \( f \) and \( g \). Now taking \( c = 0.25 \), the velocity \( f \) is found to increase further and \( g \) decreases (see figures 2.2.3 and 2.2.4).

Figures 2.2.5 and 2.2.6 give the influence of acceleration on the velocity field. We observe with the increase in \( c \) increases \( f \) drastically and decreases \( g \) near the disk.
From figures 2.2.7 and 2.2.8, we can see the effect of time on the velocity field when $c = 0$, $\lambda = 0.5$ and $S = N = 0$. We note that with the increase in time the velocity $f$ decreases and $g$ increases near the disk and the layer thickness increases. Now after taking the acceleration $c = 0.5$, it is noted that the velocity increases very near to the disk but after some $\eta$ velocity $f$ reduces and velocity $g$ increases. Also the layer thickness increases (see figures 2.2.9 and 2.2.10).

Figures 2.2.11 and 2.2.12, show the influence of injection/suction parameter $S$ when $\tau = 1$, $c = 0.25$, $\lambda = 0.5$ and $N = 0$. From these figures we see that by increasing suction, velocity $f$ increases, $g$ decreases near the disk and in these cases layer thickness decreases.

Figures 2.2.13 and 2.2.14 are for the variation of magnetic field on the velocity profiles in the presence of slip parameter $\lambda$ when $c = 0.25$. From these figures it is noted that in presence of slip, the layer thickness reduces much when compared with that of no-slip situation.

The variation of the velocity field with distance from the disk for various
values of slip parameter $\lambda$ when $\tau = 1$, $c = 0$, $S = N = 0$.

The variation of the velocity field with distance from the disk for various values of slip parameter $\lambda$ when $\tau = 1$, $c = 0.25$, $S = N = 0$.

The variation of the velocity field with distance from the disk for various values of acceleration parameter $c$ when $\tau = 1$, $\lambda = 0.5$, $S = N = 0$. 

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The variation of the velocity field with distance from the disk for various values of time $\tau$ when $c = 0$, $S = N = 0$ and $\lambda = 0.5$.

The variation of the velocity field with distance from the disk for various values of time $\tau$ when $c = 0.5$, $S = N = 0$ and $\lambda = 0.5$. 

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The variation of the velocity field with distance from the disk for various values of injection/suction parameter $S$ when $c = 0.25$, $\lambda = 0.5$, $N = 0$ and $\tau = 1$. 
2.3 Variable accelerated flow with no-slip condition

In this section, the flow is induced due to non-coaxial rotations of an accelerated disk (moving with variable acceleration) and a fluid at infinity. The boundary condition at infinity and initial conditions are identical to that given in sections 2.1 and 2.2. But the boundary condition at \( z = 0 \) is taken as

\[
u = -\Omega y + \gamma_0 t^2, \quad v = \Omega x, \quad \text{at} \quad z = 0 \quad \text{for} \quad t > 0,
\]

where \( \gamma_0 \) is of the dimension of \( L/T^3 \).

In terms of non-dimensional variables, we can write

\[
\frac{\partial^2 F}{\partial \eta^2} + S \frac{\partial F}{\partial \eta} - \frac{\partial F}{\partial \tau} - (i + N) F = 0,
\]

\[
F(0, \tau) = \gamma \tau^2 - 1, \quad \text{(2.31)}
\]

\[
F(\infty, \tau) = 0, \quad \text{(2.32)}
\]

\[
F(\eta, 0) = 0, \quad \text{(2.33)}
\]

in which

\[
\gamma = \frac{\gamma_0}{\Omega^3 i}.
\]

2.3.1 Analytic solution

By Laplace transform treatment, the solution is

\[
\bar{F}(\eta, s) = \left( \frac{2 \gamma}{s^3} - \frac{1}{s} \right) e^{-\eta m},
\]

\(\text{(2.34)}\)
The Laplace inversion of above equation yields

\[ F(\eta, \tau) = 2\gamma e^{-\frac{\eta^2}{4} I_3} - e^{-\frac{\eta^2}{2} I_4}, \]  \hspace{1cm} (2.35)\]

where

\[ I_3 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{s^3} e^{-n\sqrt{s+A}+\eta\tau} \, ds \]
\[ = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{s^2} \left( \frac{\eta}{4\sqrt{s+A}} \right) e^{-n\sqrt{s+A}+\eta\tau} \, ds \]
\[ + \left( \frac{\tau}{2} + \frac{\eta}{4\sqrt{A}} \right) e^{n\sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \]

\[ -\frac{\eta}{4} \left[ \frac{1}{4\sqrt{A^3}} \left\{ e^{n\sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \right. \right. \]
\[ - e^{-n\sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \]
\[ + \frac{1}{4A} \left\{ 4\sqrt{\frac{\tau}{\pi}} e^{-\frac{\eta^2}{4\pi}} A^\tau - \left( \eta + 2\sqrt{A\tau} \right) e^{n\sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \right. \]
\[ - \left. \left( \eta - 2\sqrt{A\tau} \right) e^{-n\sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \right\} \right], \]

\[ I_4 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{s} e^{-n\sqrt{s+A}+\eta\tau} \, ds \]
\[ = \frac{1}{2} \left\{ e^{n\sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) + e^{-n\sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \right\}.

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From equation (2.7), (2.9) and (2.10), one obtains

\[
\frac{f + ig}{\Omega} = 1 + e^{-\eta \sqrt{A - \frac{\eta}{2}}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \left[ \gamma_A \left( \frac{\tau}{2} - \frac{\eta}{4\sqrt{A}} \right) + \frac{\gamma_A^2}{8\sqrt{A}} \right]
\]

\[= \frac{\gamma_A}{2\sqrt{A}} \left( \frac{\tau}{2} - \frac{\eta}{4\sqrt{A}} \right) - \frac{1}{2} + e^{\eta \sqrt{A - \frac{\eta}{2}}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right)
\times \left[ \gamma_A \left( \frac{\tau}{2} + \frac{\eta}{4\sqrt{A}} \right) - \frac{\gamma_A}{8\sqrt{A}} \frac{\tau}{\frac{\eta}{2} + \frac{\eta}{4\sqrt{A}} - \frac{1}{2}} \right]
\]

\[\frac{\gamma_A}{2\sqrt{A}} \sqrt{\frac{\tau}{\pi}} e^{-\frac{\eta^2}{4\tau} - A\tau - \frac{\eta}{2}}. \tag{2.36}\]

The solution (2.36) in terms of non-coaxial parameters can be easily written as

\[
\frac{f + ig}{(\nu \eta_0)^{\frac{1}{3}}} = \Omega_2 \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \left[ \tau \left( \frac{\tau}{2} - \frac{\eta}{4\sqrt{A}} \right) + \frac{\eta}{8\sqrt{A}} \frac{\tau}{\frac{\eta}{2} + \frac{\eta}{4\sqrt{A}} - \frac{1}{2}} \right]
\]

\[= \left( \frac{\eta}{2A} - \frac{\tau}{\sqrt{A}} \right) - \frac{\Omega_2 l_2}{2} + e^{\eta \sqrt{A - \frac{\eta}{2}}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right)
\times \left[ \tau \left( \frac{\tau}{2} + \frac{\eta}{4\sqrt{A}} \right) - \frac{\eta}{8\sqrt{A}} \left( \frac{\eta}{2A} + \frac{\tau}{\sqrt{A}} \right) - \frac{\Omega_2 l_2}{2} \right]
\]

\[\frac{\eta}{2A} \sqrt{\frac{\tau}{\pi}} e^{-\frac{\eta^2}{4\tau} - A\tau - \frac{\eta}{2}}. \tag{2.37}\]

where

\[
F = \frac{F^*}{(\nu \eta_0)^{\frac{1}{3}}}, \quad S_1 = \frac{1}{(\nu_0\tau)^{\frac{1}{3}}} \omega_0, \quad \eta = \left( \frac{\tau_0}{\nu^2} \right)^{\frac{1}{3}} z,
\]

\[
\tau = t \left( \frac{\tau_0}{\tau} \right)^{\frac{1}{3}}, \quad N = \left( \frac{\tau_0}{\nu^2} \right)^{\frac{1}{3}} \frac{\sigma B_0^2}{\rho},
\]

\[
l_2 = l \left( \frac{\tau_0}{\nu^2} \right)^{\frac{1}{3}}, \quad \Omega_2 = \left( \frac{\nu}{\tau_0} \right)^{\frac{1}{3}} \Omega. \tag{2.38}\]

### 2.3.2 Graphs and discussion

In this section we will discuss the behaviour of the velocity profiles $f$ and $g$ through graphs when disk has variable acceleration.
To see the influence of time on the velocity profiles, the figures 2.3.1 and 2.3.2 are displayed for $c = 0$. Here, it is noted that steady state can be achieved and increase in time decreases $f$ and increases $g$. Now when $c = 0.5$, we observe that for time $\tau < 1$, the increase in $f$ and $g$ is small but on the other hand if $\tau > 1$ velocity profiles are drastically increasing near the disk (see figures 2.3.3 and 2.3.4). On the whole we can say that with the increase in time increases the layer thickness increases.

Figures 2.3.5 to 2.3.8 are depicted for variation in acceleration. From these figures we can see that for fixed change in acceleration, change in $f$ and $g$ is small for $\tau < 1$ on the other hand the change in $f$ and $g$ is very large when $\tau > 1$.

To see the variation of porosity, we prepare figures 2.3.9 and 2.3.10. With the increase in injection velocity, $f$ decreases and $g$ increases and layer thickness increases. However the case of suction, the velocity $f$ increases and $g$ decreases and hence layer thickness decreases.

Further, the influence of magnetic and non-coaxial rotational parameter has been shown in figures 2.3.11 to 2.3.16. The observation noted here are identical to that of the previous section.
The variation of the velocity field with distance from the disk for various values of time $\tau$ when $c = 0$, $N = 0$, and $S = 0$.

The variation of the velocity field with distance from the disk for various values of time $\tau$ when $c = 0.5$, $N = 0$, and $S = 0$. 

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The variation of the velocity field with distance from the disk for various values of time $\tau$ when $\tau = 0.75$, $N = 0$, and $S = 0$.

The variation of the velocity field with distance from the disk for various values of time $c$ when $\tau = 0.25$, $N = 0$, and $S = 0$. 

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The variation of the velocity field with distance from the disk for various values of time $S$ when $\tau = 1.25$, $N = 0$, and $c = 0.25$.

The variation of the velocity field with distance from the disk for various values of time $N$ when $\tau = 1.25$, $c = 0.25$, and $S = 0$. 
The variation of the velocity field with distance from the disk for various values of time $\Omega_2$ when $\tau = 1$, $l_2 = 0.25$, $N = 0$, and $S = 0$.

The variation of the velocity field with distance from the disk for various values of time $\Omega_2$ when $\tau = 2$, $l_2 = 0.25$, $N = 0$, and $S = 0$. 
Chapter 3

Unsteady flows of a second grade fluid due to constant accelerated disk

In this chapter, the MHD flow of a second grade fluid is studied for the following two problems:

i. Constant accelerated flow with no-slip condition.

ii. Constant accelerated flow with partial slip condition.

In problem (i), both analytical and numerical solutions are given. The analytical solution is obtained using Laplace transform treatment and perturbation technique. In problem (ii), the analysis for the numerical simulation has been performed. The Crank-Nicolson scheme with modification has been used for the numerical solution. The comparison between the analytical and
numerical solution has been given. It is found that the large values of material parameter of the second grade fluid are responsible to reduce the velocity profiles.

3.1 Flow with no-slip condition

Here, the description of the flow is same as given in section 2.1 except that the fluid taken is non-Newtonian. For non-Newtonian characteristics, we select a subclass of the differential type fluids namely, the second grade fluids. The Cauchy stress tensor for such fluids is given through the equation (1.2). The governing equations for the flow can be written from equations (1.24) to (1.26) as

\[
\frac{1}{\rho} \frac{\partial P}{\partial x} = v \frac{\partial^2 f}{\partial z^2} + w_0 \frac{\partial f}{\partial z} - \frac{\partial f}{\partial t} + \Omega g + \Omega^2 x - \frac{\sigma}{\rho} B_0^2 [f(z, t) - \Omega y] + \frac{\alpha_1}{\rho} \left[ \frac{\partial^3 f}{\partial t \partial z^2} - w_0 \frac{\partial^3 f}{\partial z^3} + \Omega \frac{\partial^2 g}{\partial z^2} \right], \tag{3.1}
\]

\[
\frac{1}{\rho} \frac{\partial P}{\partial y} = v \frac{\partial^2 g}{\partial z^2} + w_0 \frac{\partial g}{\partial z} - \frac{\partial g}{\partial t} - \Omega f + \Omega^2 y - \frac{\sigma}{\rho} B_0^2 [g(z, t) + \Omega x] + \frac{\alpha_1}{\rho} \left[ \frac{\partial^3 g}{\partial t \partial z^2} - w_0 \frac{\partial^3 g}{\partial z^3} - \Omega \frac{\partial^2 f}{\partial z^2} \right], \tag{3.2}
\]

\[
-\frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{\sigma B_0^2}{\rho} w_0 = 0 \tag{3.3}
\]

and the boundary and initial conditions are given in equation (2.1). Eliminating \( P \) from above equations and using non-dimensionlization, we have the problem of the following form

\[
\alpha \frac{\partial^3 F}{\partial \tau \partial \eta^2} - \alpha S \frac{\partial^3 F}{\partial \eta^3} + (1 - i \alpha) \frac{\partial^2 F}{\partial \eta^2} + S \frac{\partial F}{\partial \eta} - \frac{\partial F}{\partial \tau} - (i + N) F = 0, \tag{3.4}
\]
\[ F(0, \tau) = c\tau - 1, \quad F(\infty, \tau) = 0, \quad F(\eta, 0) = 0, \quad (3.5) \]

where
\[ \alpha = \frac{\alpha_1 \Omega}{\rho \nu}. \]

### 3.1.1 Analytic solution

Before proceeding with the solution of the above problem it would be interesting to remark here that in the classical viscous case \( (\alpha = 0) \), we encounter a differential equation of order two [50]. The analysis of the flow of the second grade fluids, in particular, and the viscoelastic fluids, in general, is more challenging mathematically and computationally, because of a peculiarity in the equations governing the fluid motion; namely the order of the differential equation(s) characterizing the flow of these fluids is more than the number of the available boundary conditions. The difficulty is further accentuated by the fact that a non-Newtonian parameter of fluid (for example \( \alpha \), for a second grade fluid) usually occurs in the coefficient of the highest derivative. The usual attempts to resolve this difficulty centered around seeking a perturbation solution assuming the non-Newtonian fluid parameter to be small; the classical paper being by Beard et al. [51], who considered the two-dimensional stagnation point flow of the Walter’s B fluid. One may also refer, for example, to the works of Shrestha[52], Misra et al. [53], Rajagopal et al. [54], Verma et al. [55], Erdogan [56] and Hayat et al. [57] for other problems in various geometries. In the present analysis, the difficulty is also removed by seeking a solution of the following form

\[ F(\eta, \tau) = F_0(\eta, \tau) + \alpha F_1(\eta, \tau) + O(\alpha^2), \quad (3.6) \]
which is valid for small values of $\alpha$ only.

Now, substituting equation (3.6) in the equation (3.4) and conditions (3.5) and then collecting terms of like powers of $\alpha$, one obtains the following systems of differential equations along with the boundary conditions:

**System of order zero**

\[
\frac{\partial^2 F_0}{\partial \eta^2} + S \frac{\partial F_0}{\partial \eta} - \frac{\partial F_0}{\partial \tau} - (i + N) F_0 = 0, \quad (3.7)
\]

\[
F_0(0, \tau) = c \tau - 1, \quad F_0(\infty, \tau) = 0, \quad F_0(\eta, 0) = 0. \quad (3.8)
\]

**System of order one**

\[
\frac{\partial^2 F_1}{\partial \eta^2} + S \frac{\partial F_1}{\partial \eta} - \frac{\partial F_1}{\partial \tau} - (i + N) F_1 = \frac{\partial^3 F_0}{\partial \tau \partial \eta^2} + S \frac{\partial^3 F_0}{\partial \eta^3} + i \frac{\partial^2 F_0}{\partial \eta^2}, \quad (3.9)
\]

\[
F_1(0, \tau) = 0, \quad F_1(\infty, \tau) = 0, \quad F_1(\eta, 0) = 0. \quad (3.10)
\]

**Zeroth order solution**

We note that the zeroth order system consisting of equation (3.7) along with condition (3.8) is the same as in equations (2.10) and (2.11). In order to avoid repetition, we do not include the detail of calculations. Employing the procedure of Laplace transformation method, the solution is directly written as

\[
F_0(\eta, \tau) = e^{-\frac{\eta}{2}} \left[ \left( \frac{\tau}{2} - \frac{\eta}{4\sqrt{A}} \right) e^{-\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) + \left( \frac{\tau}{2} + \frac{\eta}{4\sqrt{A}} \right) e^{\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \right]
\]

\[
- \frac{1}{2} \left\{ e^{-\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) + e^{\eta \sqrt{A}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \right\}. \quad (3.11)
\]
First order solution

By Laplace transform method, we have from equations (3.9) and (3.10) as

\[ \overline{F''_1} + s \overline{F'_1} - (s + N + i) \overline{F_1} = -(s - i) \overline{F''_0} + s \overline{F'_0}, \]  
(3.12)

\[ \overline{F_1}(0, s) = 0 = \overline{F_1}(\infty, s). \]  
(3.13)

Using equations (2.14) in equation (3.12), we have

\[ \overline{F''_1} + s \overline{F'_1} - (s + N + i) \overline{F_1} = - \left( \frac{c}{s^2} - \frac{1}{s} \right) X e^{-m_\eta}, \]  
(3.14)

where

\[ X = \left\{ \frac{S^4}{2} + \frac{3}{2} S^2 N + 1 + i(S^2 + N) \right\} + s \left\{ 2S^2 + N \right\} \]

\[ + 2S s \sqrt{A + s} + s^2 + S \left\{ S^2 + N \right\} \sqrt{A + s}. \]

The solution of the equation (3.14) satisfying the condition (3.13) is

\[ \overline{F_1}(\eta, s) = \frac{\eta}{2 \sqrt{A + s}} \left( \frac{c}{s^2} - \frac{1}{s} \right) \left[ b_1 + sb_2 + 2S s \sqrt{A + s} + s^2 + b_3 S \sqrt{A + s} \right] e^{-m_\eta}, \]  
(3.15)

in which

\[ b_1 = \frac{S^4}{2} + \frac{3}{2} S^2 N + 1 + i(S^2 - N), \]

\[ b_2 = 2S^2 + N, \]

\[ b_3 = S^2 + N. \]  
(3.16)
Laplace inverse of solution (3.15) yields

\[ F_1(\eta, \tau) = \frac{cb_1 \eta e^{-\frac{s\eta}{2}}}{2} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{s^2 \sqrt{s + A}} e^{-\eta \sqrt{s + A} + s\tau} ds, \]

\[ + \frac{cb_2 \eta e^{-\frac{s\eta}{2}}}{2} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{s \sqrt{s + A}} e^{-\eta \sqrt{s + A} + s\tau} ds \]

\[ + cS \eta e^{-\frac{s\eta}{2}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{s} e^{-\eta \sqrt{s + A} + s\tau} ds \]

\[ + \frac{cn \eta e^{-\frac{s\eta}{2}}}{2} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\sqrt{s + A}} e^{-\eta \sqrt{s + A} + s\tau} ds. \]

\[ + \frac{cb_3 \eta S e^{-\frac{s\eta}{2}}}{2} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{s^2} e^{-\eta \sqrt{s + A} + s\tau} ds \]

\[ - \frac{b_1 \eta e^{-\frac{s\eta}{2}}}{2} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{s \sqrt{s + A}} e^{-\eta \sqrt{s + A} + s\tau} ds \]

\[ - \frac{b_2 \eta e^{-\frac{s\eta}{2}}}{2} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\sqrt{s + A}} e^{-\eta \sqrt{s + A} + s\tau} ds \]

\[ - S \eta e^{-\frac{s\eta}{2}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-\eta \sqrt{s - A} + s\tau} ds \]

\[ - \frac{\eta e^{-\frac{s\eta}{2}}}{2} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{s}{\sqrt{s + A}} e^{-\eta \sqrt{s + A} + s\tau} ds \]

\[ - \frac{b_3 \eta S e^{-\frac{s\eta}{2}}}{2} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{s} e^{-\eta \sqrt{s + A} + s\tau} ds. \]

(3.17)

Note that the values of the integrals in the above expression will complete the solution for \( F_1 \). The integrals appearing here have been solved via a standard
procedure and obtain

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{s^2 \sqrt{s + A}} e^{-\eta \sqrt{s+\Delta+\sigma^2}} d\xi = \frac{1}{2\sqrt{A}} \left[ e^{-\eta \sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \right. \\
\left. - e^{\eta \sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \right].
\]

(3.19)

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{s^2 \sqrt{s + A}} e^{-\eta \sqrt{s+\Delta+\sigma^2}} d\xi = \frac{1}{2} \left[ e^{-\eta \sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \\
+ e^{\eta \sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \right].
\]

(3.20)

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\sqrt{s + A}} e^{-\eta \sqrt{s+\Delta+\sigma^2}} d\xi = \frac{1}{\sqrt{\pi \tau}} e^{-A\tau - \frac{\eta^2}{4\tau}}.
\]

(3.21)

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{s^2} e^{-\eta \sqrt{s-A+\sigma^2}} d\xi = \\
\left[ \left( \frac{\tau}{2} - \frac{\eta}{4\sqrt{A}} \right) e^{-\eta \sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \right. \\
\left. + \left( \frac{\tau}{2} + \frac{\eta}{4\sqrt{A}} \right) e^{\eta \sqrt{A}} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \right].
\]

(3.22)

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-\eta \sqrt{s+\Delta+\sigma^2}} d\xi = \frac{\eta}{2\sqrt{\pi \tau^3}} e^{-A\tau - \frac{\eta^2}{4\tau}}.
\]

(3.23)
\[
\frac{1}{2\pi i} \int_{\infty}^{\infty} \frac{s}{s + A} e^{2\eta \sqrt{s} + \eta s + \eta s^{-\frac{3}{2}}} ds = \frac{1}{\sqrt{\pi \tau}} e^{-A\tau - \eta^2 \frac{\pi^2}{4\tau^2}} \left( \frac{\eta^2}{4\tau^2} - \frac{1}{2\tau} - A \right).
\] (3.24)

With the help of above integrals, the equation (3.17) becomes

\[
F_1(\eta, \tau) = \frac{c b_1}{4} \eta \left[ \left( \frac{1}{2\sqrt{A^3}} - \frac{\tau}{\sqrt{A}} - \frac{\eta}{2A} \right) e^{\eta \sqrt{A}} \text{erf} \left( \eta \frac{2}{2\sqrt{\tau}} + \sqrt{A\tau} \right) 
- \left( \frac{1}{2\sqrt{A^3}} - \frac{\tau}{\sqrt{A}} + \frac{\eta}{2A} \right) e^{-\eta \sqrt{A}} \text{erf} \left( \eta \frac{2}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \right] 
+ \frac{c n b_2}{4\sqrt{A}} e^{-\frac{\eta}{2}} \left[ e^{-\eta \sqrt{A}} \text{erf} \left( \eta \frac{2}{2\sqrt{\tau}} - \sqrt{A\tau} \right) - e^{\eta \sqrt{A}} \text{erf} \left( \eta \frac{2}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \right] 
+ \frac{c n S}{2} e^{-\frac{\eta}{2}} \left[ e^{-\eta \sqrt{A}} \text{erf} \left( \eta \frac{2}{2\sqrt{\tau}} - \sqrt{A\tau} \right) + e^{\eta \sqrt{A}} \text{erf} \left( \eta \frac{2}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \right] 
+ \frac{c n b_3}{2\sqrt{A} \eta} \left( \frac{\tau}{2} - \frac{\eta}{4\sqrt{A}} \right) e^{-\eta \sqrt{A}} \text{erf} \left( \eta \frac{2}{2\sqrt{\tau}} - \sqrt{A\tau} \right)
\]

and thus the equation (3.6) in terms of \( f \) and \( g \) gives

\[
\frac{f + i g}{\Omega l} = 1 + e^{-\frac{\eta}{2}} \eta e^{-\eta \sqrt{A}} \text{erf} \left( \eta \frac{2}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \left[ c \left( \frac{\tau}{2} - \frac{\eta}{4\sqrt{A}} \right) - \frac{1}{2} 
- \frac{c b_1}{4} \left( \frac{1}{2\sqrt{A^3}} - \frac{\tau}{\sqrt{A}} + \frac{\eta}{2A} \right) + \frac{c b_2}{4\sqrt{A}} \eta + \frac{c n S}{2} 
+ \frac{c b_3}{2} (\frac{\tau}{2} - \frac{\eta}{4\sqrt{A}}) - \frac{\eta b_1}{4\sqrt{A}} - \frac{\eta S b_3}{4} \right].
\]
\[ e^{-\frac{\sqrt[3]{\eta}}{2}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \left[ c\left(\frac{\tau}{2} + \frac{\eta}{4\sqrt{A}}\right) - \frac{1}{2} \right] \\
+ c \frac{b_1 \eta}{4} \left( \frac{1}{2\sqrt{A^3}} - \frac{\tau}{\sqrt{A}} - \frac{\eta}{2A} \right) - \alpha \frac{c b_2 \eta}{4\sqrt{A}} + \alpha \frac{c \eta S}{2} \\
+ \alpha \frac{c b_3 \eta S}{2} \left( \frac{\tau}{2} + \frac{\eta}{4\sqrt{A}} \right) + \alpha \frac{\eta b_1}{4\sqrt{A}} - \alpha \frac{\eta S b_3}{4} \\
+ \alpha \frac{\eta}{2\sqrt{\pi \tau}} e^{-A\tau - \frac{\eta^2}{4\tau^2} - \frac{\eta}{2}\tau} \left[ \frac{c b_1 \tau}{A} + c \right. \\
- b_2 - \frac{\eta S}{\tau} - \frac{\eta^2}{4\tau^2} + \frac{1}{2\tau} + A \left. \right] \right]. \tag{3.26} \]

It is worth emphasizing to mention here that for \( \alpha_1 = c = 0 \), the results of Erdogan \{23\} can be recovered. This provides useful mathematical check. In order to see the effects of rotations and non-coaxial parameter we use the non-dimensional variables given in equation (2.18) and obtain

\[ \frac{f + ig}{(\nu \sigma)^{\frac{1}{2}}} = \Omega_1 l_1 + e^{-\frac{\sqrt[3]{\eta}}{2}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} - \sqrt{A\tau} \right) \left[ \left( \frac{\tau}{2} - \frac{\eta}{4\sqrt{A}} \right) - \frac{\Omega_1 l_1}{2} \right] \\
- \alpha \frac{b_1 \eta}{4} \left( \frac{1}{2\sqrt{A^3}} - \frac{\tau}{\sqrt{A}} + \frac{\eta}{2A} \right) + \alpha \frac{b_2 \eta}{4\sqrt{A}} + \alpha \frac{\eta S}{2} \\
+ \alpha \frac{b_3 \eta S}{2} \left( \frac{\tau}{2} - \frac{\eta}{4\sqrt{A}} \right) - \alpha \frac{\eta b_1 \Omega_1 l_1}{4\sqrt{A}} - \alpha \frac{\Omega_1 l_1 \eta S b_3}{4} \\
+ e^{-\frac{\sqrt[3]{\eta}}{2}} \text{erf} \left( \frac{\eta}{2\sqrt{\tau}} + \sqrt{A\tau} \right) \left[ \left( \frac{\tau}{2} + \frac{\eta}{4\sqrt{A}} \right) - \frac{\Omega_1 l_1}{2} \right] \\
+ \alpha \frac{b_1 \eta}{4} \left( \frac{1}{2\sqrt{A^3}} - \frac{\tau}{\sqrt{A}} + \frac{\eta}{2A} \right) - \alpha \frac{b_2 \eta}{4\sqrt{A}} + \alpha \frac{\eta S}{2} \\
+ \alpha \frac{b_3 \eta S}{2} \left( \frac{\tau}{2} + \frac{\eta}{4\sqrt{A}} \right) + \alpha \frac{\eta b_1 \Omega_1 l_1}{4\sqrt{A}} - \alpha \frac{\eta S b_3 \Omega_1 l_1}{4} \\
+ \frac{\alpha \eta}{2\sqrt{\pi \tau}} e^{-A\tau - \frac{\eta^2}{4\tau^2} - \frac{\eta}{2}\tau} \left[ \frac{b_1 \tau}{A} + 1 - \Omega_1 l_1 b_2 \right. \\
- \Omega_1 l_1 \frac{\eta S}{\tau} - \Omega_1 l_1 \left( \frac{\eta^2}{4\tau^2} - \frac{1}{2\tau} - A \right) \left. \right] \right]. \tag{3.27} \]
in which

$$\alpha = \frac{\alpha_1}{\rho} \left( \frac{c_s^2}{\nu^2} \right)^{\frac{1}{3}} .$$  \hspace{1cm} (3.28)

### 3.1.2 Numerical solution

Equation (3.4) is a third order partial differential equation and it is difficult to obtain an exact analytical solution. In the previous section we obtain an approximate solution. In this section, our interest is to present a numerical solution of the complete problem, which besides its own importance, will also help to check the accuracy of the analytic solution obtained by perturbation method. Here, we use implicit formulation for the problem by using forward and central difference with respect to space and time coordinates. For the sake of convergence of the solution, we paid the price in the form of overall order of the scheme, which is $O(h + k)$. The governing equation (3.4) is transformed into an algebraic equation by substituting the approximations to derivatives from equations (1.44), (1.46) to (1.49) and (1.51) as

$$\frac{\alpha}{kh^2} [(F_{i+1,j+1} - 2F_{i,j+1} + F_{i-1,j+1}) - (F_{i+2,j} - 2F_{i+1,j} + F_{i,j})]$$

$$- \frac{\alpha S}{h^3} [F_{i+1,j} - 3F_{i+1,j} + 3F_{i,j} - F_{i-1,j}]$$

$$+ \frac{1}{2h^2} (1 - i\alpha) [(F_{i+1,j+1} - 2F_{i,j+1} + F_{i-1,j+1}) + (F_{i+1,j} - 2F_{i,j} + F_{i-1,j})]$$

$$+ \frac{S}{4h} [(F_{i+1,j+1} - F_{i-1,j+1}) + (F_{i+1,j} - F_{i-1,j})]$$

$$- \frac{1}{k} [F_{i,j+1} - F_{i,j}] = (i + N) \frac{1}{2} [F_{i,j+1} + F_{i,j}] = 0 .$$  \hspace{1cm} (3.29)

The unknown $F_{i,j+1}$ cannot be expressed in term of known quantities at the time level $j$, namely $F_{i-1,j}$, $F_{i,j}$, and $F_{i+1,j}$. Therefore the equation (3.29) rep-
resents an equation in three unknowns, namely $F_{i-1, j+1}$, $F_{i, j+1}$ and $F_{i+1, j+1}$. Hence equation (3.29), when applied at a given mesh point $P_i (ih, jk)$, cannot by itself result in a solution for $F_{i, j+1}$. Therefore the equation must be written at all interior mesh points. Thus we obtain a system of algebraic equations. For that, multiplying equation (3.29) by $k$, we get

$$A_i F_{i-1, j+1} + B_i F_{i, j+1} + C_i F_{i+1, j+1} = D_i,$$  \( (3.30) \)

where

$$A_i = \frac{\alpha}{h^2} + \frac{k(1 - i\alpha)}{2h^2} - \frac{kS}{4h},$$  \( (3.31) \)

$$B_i = -\frac{2\alpha}{h^2} - \frac{k(1 - i\alpha)}{h^2} - 1 - \frac{k(i + N)}{2},$$  \( (3.32) \)

$$C_i = \frac{\alpha}{h^2} + \frac{k(1 - i\alpha)}{2h^2} + \frac{kS}{4h},$$  \( (3.33) \)

$$D_i = \frac{\alpha}{h^2} [F_{i+2, j} - 2F_{i+1, j} + F_{i, j}] + \frac{k\alpha S}{h} [F_{i+2, j} - 3F_{i+1, j} + 3F_{i, j} - F_{i-1, j}]$$

$$- \frac{k(1 - i\alpha)}{2h^2} [F_{i+1, j} - 2F_{i, j} + F_{i-1, j}]$$

$$- \frac{kS}{4h} [F_{i+1, j} - F_{i-1, j}] - F_{i, j} + \frac{k}{2}(i + N) F_{i, j}.$$  \( (3.34) \)

The initial and boundary conditions can be written in the following form

$$F_{0, j} = c j k - 1, \quad F_{M, j} = 0, \quad F_{i, 0} = 0, \quad i = 0, 1, 2, \ldots, M.$$

Here $M$ denotes an integer larger enough such that $Mh$ approximates infinity.

To find the value of $D_{M-1}$ at each time level $j$, we need the value of $F_{M+1, j}$. Since our governing equation (3.4) is of order three while given

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boundary conditions are two, therefore we introduce an augmented boundary condition
\[ \frac{\partial F(\infty, \tau)}{\partial \eta} = 0 \] (3.35)
and consequently the problem becomes well-posed. This boundary condition is discretized to give
\[ \frac{F_{M+1,j} - F_{M,j}}{h} = 0, \]
i.e. \[ F_{M+1,j} = F_{M,j}. \]

For \( i = 1 \):
\[ B_i F_{1,j-1} + C_i F_{2,j+1} = D'_1, \]
where \[ D'_1 = D_1 - A_1 F_{0,j+1}. \]

For \( 2 \leq i \leq M - 2 \) the equations are given by
\[ A_i F_{i-1,j+1} + B_i F_{i,j+1} + C_i F_{i+1,j+1} = D'_i. \]

For \( i = M - 1 \):
\[ A_{M-1} F_{M-2,j+1} + B_{M-1} F_{M-1,j-1} = D'_{M-1}, \]
where \[ D'_{M-1} = D_{M-1} - C_{M-1} F_{M,j+1}. \]

In matrix form, the above set of \( M - 1 \) equations can be written as
The above system of equations has been solved and the results are given in the tables.

3.1.3 **Comparison of analytical and numerical solution**

Now we compare the analytical result with numerical result through the following tables and see that a very good accuracy is obtained.
Table 1

\( \tau = 1, \ c = 0.5, \ \alpha = 0.1, \ S = 0.5, \ N = 0. \)

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>Analytic Solution</th>
<th>Numerical Solution</th>
<th>Difference</th>
<th>Analytic Solution</th>
<th>Numerical Solution</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.52693</td>
<td>0.67725</td>
<td>0.05032</td>
<td>0.01381</td>
<td>0.03132</td>
<td>0.01751</td>
</tr>
<tr>
<td>0.5</td>
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<td>0.65037</td>
<td>0.00158</td>
<td>0.10702</td>
<td>0.10249</td>
<td>0.00453</td>
</tr>
<tr>
<td>1.0</td>
<td>0.78860</td>
<td>0.79018</td>
<td>0.00158</td>
<td>0.12261</td>
<td>0.11269</td>
<td>0.00992</td>
</tr>
<tr>
<td>1.5</td>
<td>0.88596</td>
<td>0.88740</td>
<td>0.00144</td>
<td>0.09964</td>
<td>0.08718</td>
<td>0.01246</td>
</tr>
<tr>
<td>2.0</td>
<td>0.94275</td>
<td>0.94479</td>
<td>0.00204</td>
<td>0.06850</td>
<td>0.05679</td>
<td>0.01171</td>
</tr>
<tr>
<td>2.5</td>
<td>0.97238</td>
<td>0.97496</td>
<td>0.00258</td>
<td>0.04184</td>
<td>0.03310</td>
<td>0.00874</td>
</tr>
<tr>
<td>3.0</td>
<td>0.98702</td>
<td>0.98944</td>
<td>0.00242</td>
<td>0.02296</td>
<td>0.01778</td>
<td>0.00518</td>
</tr>
</tbody>
</table>

Table 2

\( \tau = 1, \ c = 0.5, \ \alpha = 0.1, \ S = 0, \ N = 0.5. \)

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>Analytic Solution</th>
<th>Numerical Solution</th>
<th>Difference</th>
<th>Analytic Solution</th>
<th>Numerical Solution</th>
<th>Difference</th>
</tr>
</thead>
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<td>0.52694</td>
<td>0.00208</td>
<td>0.02614</td>
<td>0.02795</td>
<td>0.00181</td>
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<tr>
<td>0.5</td>
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<td>0.64643</td>
<td>0.01059</td>
<td>0.09116</td>
<td>0.09442</td>
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<tr>
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<td>0.10742</td>
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</tr>
<tr>
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<tr>
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<tr>
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<tr>
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<td>0.00043</td>
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<td>0.01838</td>
<td>0.00563</td>
</tr>
</tbody>
</table>

Table 3

\( \tau = 1, \ c = 0.5, \ \alpha = 0.1, \ S = 0.5, \ N = 0.5. \)

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>Analytic Solution</th>
<th>Numerical Solution</th>
<th>Difference</th>
<th>Analytic Solution</th>
<th>Numerical Solution</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
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<td>0.53948</td>
<td>0.53962</td>
<td>0.00014</td>
<td>0.02484</td>
<td>0.02704</td>
<td>0.00220</td>
</tr>
<tr>
<td>0.5</td>
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<td>0.68766</td>
<td>0.00280</td>
<td>0.07927</td>
<td>0.08413</td>
<td>0.00486</td>
</tr>
<tr>
<td>1.0</td>
<td>0.82211</td>
<td>0.82707</td>
<td>0.00496</td>
<td>0.08659</td>
<td>0.08767</td>
<td>0.00108</td>
</tr>
<tr>
<td>1.5</td>
<td>0.90797</td>
<td>0.91294</td>
<td>0.00497</td>
<td>0.06723</td>
<td>0.06475</td>
<td>0.00248</td>
</tr>
<tr>
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<td>0.95856</td>
<td>0.00327</td>
<td>0.04478</td>
<td>0.04049</td>
<td>0.00429</td>
</tr>
<tr>
<td>2.5</td>
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<td>0.00339</td>
<td>0.02672</td>
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<td>0.00241</td>
<td>0.01442</td>
<td>0.01181</td>
<td>0.00261</td>
</tr>
</tbody>
</table>
3.1.4 Graphs and discussion

In order to show the flow pattern of velocities for second grade fluid, we present the velocity profiles $f$ and $g$ through the graphs by separating $F$ into real and imaginary parts.

Figures 3.1.1. and 3.1.2 show the effect of acceleration on the velocity profiles. We see that with the increase in $c$ the velocity $f$ increases near the disk while $g$ decreases. Figures 3.1.3 to 3.1.6 represent the variation in time $\tau$. Here we note that in the absence of $c$, $N$ and $S$, the velocity $f$ decreases and $g$ increases and the boundary layer thickness increases in all the cases. Also it is noted that for $c = 0$, the steady state situation can be achieved. On the other hand if we take $c = 0.5$ then steady state situation cannot be achieved and with the increase in $c$, the velocities increases.

Now we want to see the effects of second grade parameter $\alpha$ in the absence of motion of the disk. We observe that $f$ decreases and $g$ increases with the increase in $\alpha$ and overall layer thickness increases. Now we choose $c = 0.5$ and note that the velocity profiles $f$ and $g$ both decrease and no steady state situation can be achieved.

Figures 3.1.11 to 3.1.16 are sketched to see the influence of porosity, magnetic field and rotation. Interestingly, the layer thickness reduces for large values of suction, magnetic field and rotation. The layer thickness in injection case is larger than that of suction.
The variation of the velocity field with distance from the disk for various values of acceleration parameter $c$, when $\alpha = 0.25$, $\tau = 1$, $N = 0$, and $S = 0$.

The variation of the velocity field with distance from the disk for various values of time $\tau$, when $c = 0$, $\alpha = 0.25$, $N = 0$, and $S = 0$. 

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The variation of the velocity field with distance from the disk for various values of time $\tau$, when $c = 0.5$, $\alpha = 0.25$, $N = 0$, and $S = 0$.

The variation of the velocity field with distance from the disk for various values of second grade parameter $\alpha$, when $c = 0$, $\tau = 1$, $N = 0$, and $S = 0$. 
The variation of the velocity field with distance from the disk for various values of second grade parameter $\alpha$, when $\epsilon = 0.5$, $\tau = 1$, $N = 0$, and $S = 0$.

The variation of the velocity field with distance from the disk for various values of suction parameter $S$, when $\epsilon = 0.25$, $\tau = 1$, $N = 0$, and $\alpha = 0.25$. 

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The variation of the velocity field with distance from the disk for various values of magnetic field $N$, when $c = 0.25$, $\tau = 1$, $S = 0$, and $\alpha = 0.25$.

The variation of the velocity field with distance from the disk for various values of rotation $\Omega_1$ when $l_1 = 0.25$, $\tau = 1$, $N = S = 0$, and $\alpha = 0.25$. 
3.2 Flow with partial slip condition

In this section, the physical model of the problem is the same as in section (2.3) except that the second grade fluid adheres the slip condition. The governing differential equation is (2.22). The initial and boundary condition at infinity are also the same and are given in equation (2.23). But the boundary condition at \( z = 0 \) is different due to shear stress of the second grade fluids. Here, the boundary condition is of the following type

\[
 u - \lambda_2 \tau_{zz} = -\Omega y + c_0 t, \quad v - \lambda_2 \tau_{yzz} = \Omega x, \quad \text{at} \quad z = 0 \quad \text{for} \quad t > 0, \quad (3.36)
\]

where

\[
 \tau_{zz} = \mu \frac{\partial u}{\partial z} + \alpha_1 \left[ \frac{\partial^2 u}{\partial t \partial z} + \frac{w}{\partial z^2} + \frac{\partial v}{\partial z} \frac{\partial u}{\partial y} + \frac{2}{\partial x} \frac{\partial u}{\partial z} \right], \quad (3.37)
\]

\[
 \tau_{yzz} = \mu \frac{\partial v}{\partial z} + \alpha_1 \left[ \frac{\partial^2 v}{\partial t \partial z} + \frac{w}{\partial z^2} + \frac{2}{\partial y} \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial y} \right]. \quad (3.38)
\]

The above conditions in terms of \( f \) and \( g \) is of the following type

\[
 f(0, t) = c_0 t + \lambda_1 \left[ \frac{\partial f}{\partial z} + \frac{\alpha_1}{\mu} \left\{ \frac{\partial^2 f}{\partial t \partial z} - w_0 \frac{\partial^2 f}{\partial z^2} + \Omega \frac{\partial g}{\partial z} \right\} \right],
\]

\[
 g(0, t) = \lambda_1 \left[ \frac{\partial g}{\partial z} + \frac{\alpha_1}{\mu} \left\{ \frac{\partial^2 g}{\partial t \partial z} - w_0 \frac{\partial^2 g}{\partial z^2} - \Omega \frac{\partial f}{\partial z} \right\} \right]. \quad (3.39)
\]

Now, the governing dimensionless problem consists of equation (3.4) along-with the following conditions

\[
 F(0, r) = c_\tau - 1 + \lambda \left[ \frac{\partial F}{\partial \eta} + \alpha \left\{ \frac{\partial^2 F}{\partial \tau \partial \eta} - S \frac{\partial^2 F}{\partial \eta^2} - \frac{\partial F}{\partial \eta} \right\} \right],
\]

\[
 F(\infty, \tau) = 0, \quad F(\eta, 0) = 0. \quad (3.40)
\]
3.2.1 Numerical solution

Here, we note that although the governing problem is linear but it is more complicated than discussed in section 3.1. Thus, we avoid the much details and using the procedure given in section 3.1, we have

\[ A_i F_{i-1,j+1} + B_i F_{i,j+1} + C_i F_{i+1,j+1} = D_i, \]
\[ F_{M,j} = 0, \quad F_{i,0} = 0, \quad i = 0, 1, 2, ..., M, \]

where \( A_i, B_i, C_i \) and \( D_i \) are given in equations (3.31) to (3.34).

The boundary condition (3.40) is discretized by using forward and backward differences for the derivatives involved with respect to space and time coordinate respectively. This yields

\[ F_{0,j} = r_1 F_{1,j} + r_2 F_{2,j} + r_3 [F_{0,j-1} - F_{1,j-1}] + r_4 [cjk - 1], \quad (3.41) \]

in which

\[ r_0 = h^2 k + \lambda (1 - i \alpha) h k + \alpha \lambda h + \alpha S \lambda k, \]
\[ r_1 = [\lambda (1 - i \alpha) h k + \alpha \lambda h + 2 \alpha S \lambda k] / r_0, \]
\[ r_2 = -[\alpha S \lambda k] / r_0, \]
\[ r_3 = [\alpha \lambda h] / r_0, \]
\[ r_4 = [h^2 k] / r_0, \]

and an augmented boundary condition is

\[ \frac{\partial F(\infty, \tau)}{\partial \eta} = 0 \]
which makes the problem well-posed and also yields $F_{M+1,j} = F_{M,j}$. Consequently, we are able to find $D_{M,j}$ at each time level $j$. $F_{0,0}$ is approximated with the help of (3.41) by letting $F_{0,-1} = F_{1,-1} = 0$.

For $i = 1$, $2 \leq i \leq M - 2$, and $i = M - 1$ we respectively have

$$B'_i F_{1,j+1} + C'_i F_{2,j+1} = D'_i,$$

$$A_i F_{i-1,j-1} + B_i F_{i,j+1} + C_i F_{i+1,j+1} = D_i,$$

$$A_{M-1} F_{M-2,j+1} + B_{M-1} F_{M-1,j+1} = D'_{M-1}.$$

In above equations

$$B'_i = B_i + r_1 A_i,$$

$$C'_i = C_i + r_2 A_i,$$

$$D'_i = D_i - A_1 \left[ r_3 (F_{0,j} - F_{1,j}) + r_4 (c (j + 1) k - 1) \right],$$

$$D'_{M-1} = D_{M-1} - C_{M-1} F_{M,j+1}.$$ 

From above $M - 1$ equations, one can write
\[
\begin{bmatrix}
B'_1 & C'_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
A_2 & B_2 & C_2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & A_3 & B_3 & C_3 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & A_i & B_i & C_i & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & A_{M-3} & B_{M-3} & C_{M-3} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & A_{M-2} & B_{M-2} & C_{M-2} & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & A_{M-1} & B_{M-1} & \cdots \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
\vdots \\
F_i \\
F_{M-3} \\
F_{M-2} \\
F_{M-1}
\end{bmatrix}
\times
\begin{bmatrix}
D'_1 \\
D_2 \\
D_3 \\
\vdots \\
D_i \\
D_{M-3} \\
D_{M-2} \\
D'_{M-1}
\end{bmatrix}
\]

### 3.2.2 Graphs and discussion

In this section we want to discuss the effects of partial slip condition on the velocity profiles \(f\) and \(g\) for second grade fluid. Figures 3.2.1 and 3.2.2 give the effects of partial slip parameter \(\lambda\). We note that increase in \(\lambda\) enhances the velocity profiles \(f\) and \(g\) near the disk and after some \(\eta\), \(g\) decreases.
with the increase in $\lambda$, when $\alpha = 0.2$, $c = 0$, $S = N = 0$, and $\tau = 1$. In these cases layer thickness decreases. In figures 3.2.3 and 3.2.4, we take parameter $c = 0.5$, $\alpha = 0.2$, $S = N = 0$, and $\tau = 1$. We note that $f$ further increases and $g$ reduces in comparison to the previous case. For the effects of acceleration, the figure 3.2.5 and 3.2.6 are sketched when $\lambda = 0.5$. Here it is revealed that $f$ increases and $g$ decreases for large values of $c$. Figures 3.2.7 and 3.2.8 are for the effects of time on the velocities. It is found that by increasing the time $f$ decreases and $g$ increases and as a result layer thickness increases. The effects of time in the presence of acceleration $c = 0.5$ can be seen from figures 3.2.9 and 3.2.10. Here the velocity profiles $f$ and $g$ increase near the disk and reduces after some $\eta$ when $\lambda = 0.5$, $c = 0.5$, $\alpha = 0.2$. Figures 3.2.11 and 3.2.12 are depicted to examine the behaviour of second grade parameter $\alpha$ when $c = 0$, $\lambda = 0.5$, $S = 0$, $N = 0$, and $\tau = 1$. We observe that with the increase in $\alpha$ causes reduction in the velocities $f$ and $g$. Figures 3.2.13 and 3.2.14 are made in presence of $c = 0.5$ and $\lambda = 0.5$ when $\tau = 1$. Here $f$ increases and $g$ decreases near the disk in comparison to previous figures for $c = 0$. The effects of injection/suction can be seen from figures 3.2.15 and 3.2.16. The fluid velocity $f$ is found to increase and $g$ to decrease at the disk for fixed value of $\lambda = 0.5$ and values of $f$ and $g$ do not change at the disk in the absence of slip parameter. With the increase in suction parameter layer thickness decreases. The effects of magnetic field are identical to that of suction (see figures 3.2.17 and 3.2.18).

Now to see the effects of rotation and non-coaxial parameter we prepare figures 3.2.19 and 3.2.20 when $\lambda = 0.5$, $\alpha = 0.2$, $S = N = 0$, $l_1 = 0$ and $\tau = 1$. It is found that for large values of $\Omega_1$ the velocity profiles $f$ and $g$
decrease. Moreover, the velocity profiles \( f \) and \( g \) increases at the disk and \( f \) stables at the product \( \Omega_1 \lambda \) and \( g \) becomes positive from negative with the increase in \( \Omega_1 \).

The variation of the velocity with distance from the disk for various values of slip parameter \( \lambda \) when \( \alpha = 0.2, \ c = 0, \ S = 0, \ N = 0, \) and \( \tau = 1. \)
The variation of the velocity with distance from the disk for various values of slip parameter $\lambda$ when $\alpha = 0.2$, $c = 0.5$, $S = 0$, $N = 0$, and $\tau = 1$.

The variation of the velocity with distance from the disk for various values of acceleration parameter $c$ when $\alpha = 0.2$, $\lambda = 0.5$, $S = 0$, $N = 0$, and $\tau = 1$.

The variation of the velocity with distance from the disk for various values
of time $\tau$ when $\alpha = 0.2$, $c = 0.0$, $\lambda = 0.5$, $S = 0$, and $N = 0$.

The variation of the velocity with distance from the disk for various values of time $\tau$ when $\alpha = 0.2$, $c = 0.5$, $\lambda = 0.5$, $S = 0$, and $N = 0$.

The variation of the velocity with distance from the disk for various values of second grade parameter $\alpha$ when $c = 0$, $\lambda = 0.5$, $S = 0$, $N = 0$ and $\tau = 1$. 
The variation of the velocity with distance from the disk for various values of second grade parameter $\alpha$ when $c = 0.5$, $\lambda = 0.5$, $S = 0$, $N = 0$ and $\tau = 1$.

The variation of the velocity with distance from the disk for various values of injection/suction parameter $S$ when $\alpha = 0.2$, $c = 0.5$, $\lambda = 0.5$, $N =$
0, and \( \tau = 1 \).

The variation of the velocity with distance from the disk for various values of magnetic field parameter \( N \) when \( a = 0.2, \ c = 0.5, \ \lambda = 0.5, \ S = 0, \) and \( \tau = 1 \).

The variation of the velocity with distance from the disk for various values
of rotation parameter $\Omega_1$ when $\alpha = 0.2$, $\lambda = 0.5$, $S = 0$, $N = 0$, $l_1 = 0$ and $\tau = 1$.

The variation of the velocity with distance from the disk for various values of rotation parameter $\Omega_1$ when $\alpha = 0.2$, $\lambda = 0.5$, $S = 0$, $N = 0$, $l_1 = 0.5$ and $\tau = 1$.

The variation of the velocity with distance from the disk for various values
of non-coaxial parameter $l_1$ when $\alpha = 0.2$, $\lambda = 0.5$, $S = 0$, $N = 0$, $\Omega_1 = 1$ and $\tau = 1$. 
Chapter 4

Transient flows of a third grade fluid due to constant accelerated disk

In this chapter, we develop a mathematical model to predict the velocity profile for the $MHD$ flow due to non-coaxial rotation of a constant accelerated porous disk and a third grade fluid at infinity. The governing equation is non-linear. Two types of unsteady flows are considered. The first problem involves the no slip condition whereas the second problem deal with the partial slip situation. The boundary condition at the disk is linear (in the first problem) and highly non-linear (in the second problem). The governing non-linear equation is solved numerically subject to boundary and initial conditions. The influence of the physical parameters involved in the problems such as the applied magnetic field, the uniform acceleration, the material constants and the slip parameter is shown for salient features of the
velocity profile. Numerical results indicate that the effects of acceleration is significantly more than the influence of time in the accumulated effect of velocity. Also the velocity is smaller in a third grade fluid when compared to the viscous fluid and the effect of acceleration versus time is much smaller in third grade fluid than that of viscous fluid.

4.1 Flow with no-slip condition

In this section, the analysis of section 3.1 is extended to the case of third grade fluid. The involved partial differential equation is different to that of second grade fluid. This is due to the difference in the Cauchy stress tensors of the two fluids. The constitutive equation for the flow of third grade fluid is given in equation (1.6). The governing equation for thermodynamical third grade fluid can be written as

\[
v \frac{\partial^2 f}{\partial z^2} + w_0 \frac{\partial f}{\partial z} - \frac{\partial f}{\partial t} + \Omega g + \Omega^2 x - \frac{\sigma}{\rho} B_0^2 [f(z, t) - \Omega y] \\
+ \frac{\alpha_1}{\rho} \left[ \frac{\partial^3 f}{\partial t \partial z^2} - w_0 \frac{\partial^3 f}{\partial z^3} + \Omega \frac{\partial^2 g}{\partial z^2} \right] \\
+ 2 \beta_3 \frac{\partial}{\partial z} \left[ \frac{\partial f}{\partial z} \left( \left( \frac{\partial f}{\partial z} \right)^2 + \left( \frac{\partial g}{\partial z} \right)^2 \right) \right] - \frac{1}{\rho} \frac{\partial P}{\partial x} = 0, \quad (4.1)
\]

\[
v \frac{\partial^2 g}{\partial z^2} + w_0 \frac{\partial g}{\partial z} - \frac{\partial g}{\partial t} - \Omega f + \Omega^2 y - \frac{\sigma}{\rho} B_0^2 [g(z, t) + \Omega x] \\
+ \frac{\alpha_1}{\rho} \left[ \frac{\partial^3 g}{\partial t \partial z^2} - w_0 \frac{\partial^3 g}{\partial z^3} + \Omega \frac{\partial^2 f}{\partial z^2} \right] \\
+ 2 \beta_3 \frac{\partial}{\partial z} \left[ \frac{\partial g}{\partial z} \left( \left( \frac{\partial f}{\partial z} \right)^2 + \left( \frac{\partial g}{\partial z} \right)^2 \right) \right] - \frac{1}{\rho} \frac{\partial P}{\partial y} = 0, \quad (4.2)
\]

\[
\frac{\sigma B_0^3}{\rho} w_0 - \frac{1}{\rho} \frac{\partial P}{\partial z} = 0. \quad (4.3)
\]
The above equation indicates that \( P \) is linear in \( z \) and combination of equations (4.1) and (4.2) yields after elimination of \( P \)

\[
\frac{\alpha_3}{\rho} \left[ \frac{\partial^3 F^*}{\partial t \partial z^2} - w_0 \frac{\partial^3 F^*}{\partial z^3} \right] + \left( \frac{v - i\alpha_1 \Omega}{\rho} \right) \frac{\partial^2 F^*}{\partial z^2} + w_0 \frac{\partial F^*}{\partial z} - \frac{\partial F^*}{\partial t} = 0
\]

\[-\Omega \left( i + \frac{\sigma B_0^2}{\rho \Omega} \right) F^* + 2\beta_3 \frac{\partial}{\partial z} \left\{ \left( \frac{\partial F^*}{\partial z} \right)^2 \frac{\partial F^*}{\partial z} \right\} = 0, \quad (4.4)\]

\[F^*(0, t) = c_0 t - \Omega t, \quad F^*(\infty, t) = 0, \quad F^*(z, 0) = 0, \quad (4.5)\]

where

\[F^* = f - i g - \Omega l.\]

In dimensionless form, the problem defined through equation (4.4) is

\[
\frac{\alpha}{\beta^3} \frac{\partial^3 F}{\partial \tau \partial \eta^2} - \alpha S \frac{\partial^3 F}{\partial \eta^3} + (1 - i\alpha) \frac{\partial^2 F}{\partial \eta^2} + S \frac{\partial F}{\partial \eta} - \frac{\partial F}{\partial \tau} - (i + N) F
\]

\[+ \beta \frac{\partial}{\partial \eta} \left\{ \left( \frac{\partial F}{\partial \eta} \right)^2 \frac{\partial F}{\partial \eta} \right\} = 0, \quad (4.6)\]

\[F(0, \tau) = c \tau - 1, \quad F(\infty, \tau) = F(\eta, 0) = 0\]

in which

\[\beta = \frac{\Omega^2 \beta_3}{\rho v^2}. \quad (4.7)\]

### 4.1.1 Numerical solution

One of the most important applications of numerical methods of solution is to nonlinear partial differential equations. These type of nonlinear partial differential equations have always been a challenge to mathematicians. In
many cases, it is difficult to obtain analytical solution in closed form for arbitrary values of all parameters. The equation (4.6) is highly non-linear partial differential equation and the dependent variable has mixed derivatives with respect to time and space coordinates. Therefore, we need an implicit numerical technique to transform the partial differential equation (4.6) into a system of algebraic equations. We formulate the problem implicitly by using finite difference approximations to derivatives which are centered midway in time between the known and the unknown levels and we take central differences to approximate the derivatives for unknown level and for known level we also use central differences to approximate the derivatives except for the second order derivative. Furthermore, we approximate the nonlinear term only at the known level by using central difference for the first order derivative and forward difference for the second order derivative. The equation (4.6) is transformed into algebraic equation by substituting the approximations to derivatives using equations (1.46) to (1.48), (1.50) and (1.51) as follows:

\[
\frac{\alpha}{k h^2} [(F_{i+1,j+1} - 2F_{i,j+1} + F_{i-1,j+1}) - (F_{i+2,j} - 2F_{i+1,j} + F_{i,j})] \\
- \frac{\alpha S}{4 h^3} \left[ (F_{i+2,j+1} - 2F_{i+1,j+1} + 2F_{i-1,j+1} - F_{i-2,j+1}) \right. \\
\left. + (F_{i+2,j} - 2F_{i+1,j} + 2F_{i-1,j} - F_{i-2,j}) \right] \\
+ \frac{1}{2h^2} (1 - i\alpha) [(F_{i+1,j+1} - 2F_{i,j+1} + F_{i-1,j+1}) + (F_{i+2,j} - 2F_{i+1,j} + F_{i,j})] \\
+ \frac{S}{4h} [(F_{i+1,j+1} - F_{i-1,j+1}) + (F_{i+1,j} - F_{i-1,j})] \\
- \frac{1}{h} [F_{i,j+1} - F_{i,j}] - \frac{1}{2}(i + N) [F_{i,j+1} + F_{i,j}]
\]
\[
\begin{align*}
\frac{\beta}{4h^4} \left[ 2(F_{i+1,j} - F_{i-1,j})(F_{i-2,j} - 2F_{i+1,j} + F_{i,j}) (\bar{F}_{i+1,j} - \bar{F}_{i-1,j}) + \\
(F_{i+1,j} - F_{i-1,j})^2 (\bar{F}_{i+2,j} - 2\bar{F}_{i+1,j} + \bar{F}_{i,j}) \right] &= 0. \\
\text{(4.8)}
\end{align*}
\]

The problem consisting of above equation along with initial and boundary conditions become

\[
A_i F_{i-2,j+1} + B_i F_{i-1,j+1} + C_i F_{i,j+1} + D_i F_{i+1,j+1} + E_i F_{i+2,j+1} &= G_i, \\
F_{0,j} &= cjk - 1, \quad F_{M,j} = 0, \quad F_{i,0} = 0, \quad i = 0, 1, 2, ..., M, \\
\text{(4.9)}
\]

where

\[
\begin{align*}
A_i &= \frac{k\alpha S}{4h^3}, \\
B_i &= -\frac{k\alpha S}{2h^3} + \frac{\alpha}{h^2} + \frac{k(1 - i\alpha)}{2h^2} - \frac{kS}{4h}, \\
C_i &= -\frac{2\alpha}{h^2} - \frac{k(1 - i\alpha)}{2h^2} - 1 - \frac{k}{2}(i + N), \\
D_i &= \frac{k\alpha S}{2h^3} + \frac{\alpha}{h^2} + \frac{k(1 - i\alpha)}{2h^2} + \frac{kS}{4h}, \\
E_i &= -\frac{k\alpha S}{4h^3}, \\
G_i &= \frac{\alpha}{h^2} [F_{i+2,j} - 2F_{i+1,j} + F_{i,j}] + \frac{k\alpha S}{4h^3} [F_{i+2,j} - 2F_{i+1,j} + 2F_{i-1,j} - F_{i-2,j}] \\
&- \frac{k(1 - i\alpha)}{2h^2} [F_{i+2,j} - 2F_{i+1,j} + F_{i,j}] \\
&- \frac{kS}{4h} [F_{i+1,j} - F_{i-1,j}] - \frac{k}{2}(i + N) F_{i,j} \\
&- \frac{k\beta}{4h^4} \left[ 2(F_{i+1,j} - F_{i-1,j}) (F_{i+2,j} - 2F_{i+1,j} + F_{i,j}) (\bar{F}_{i+1,j} - \bar{F}_{i-1,j}) \\
+ (F_{i+1,j} - F_{i-1,j})^2 (\bar{F}_{i+2,j} - 2\bar{F}_{i+1,j} + \bar{F}_{i,j}) \right]. \\
\text{(4.15)}
\end{align*}
\]
For $i = 1$ we have from equation (4.9) as

$$A_1 F_{-1,j+1} + B_1 F_{0,j+1} - C_1 F_{1,j+1} + D_1 F_{2,j+1} + E_1 F_{3,j+1} = G_1. \quad (4.16)$$

The value of $F$ at the fictitious point $\eta_{-1}$ is approximated by means of the Lagrange polynomial of third degree

$$F_{-1,j+1} = L_0 F_{0,j+1} + L_1 F_{1,j+1} + L_2 F_{2,j+1} + L_3 F_{3,j+1}, \quad (4.17)$$

where

$$L_q = \prod \frac{(\eta_{-1} - \eta_p)}{(\eta_q - \eta_p)}, \quad p = 0, 1, 2, 3 \quad \text{and} \quad p \neq q.$$  

Using equation (4.17) in equation (4.16), we have

$$\begin{align*}
(A_1 L_0 + B_1) F_{0,j+1} + (A_1 L_1 + C_1) F_{1,j+1} \\
+ (A_1 L_2 + D_1) F_{2,j+1} + (A_1 L_3 + E_1) F_{3,j+1} &= G_1. \quad (4.18)
\end{align*}$$

Since $F_{0,j+1}$ is known, so the equation (4.18) must be written as

$$C'_1 F_{1,j+1} + D'_1 F_{2,j+1} + E'_1 F_{3,j+1} = G'_1,$$

in which

$$\begin{align*}
C'_1 &= A_1 L_1 + C_1, \quad D'_1 = A_1 L_2 + D_1, \\
E'_1 &= A_1 L_3 + E_1, \quad G'_1 = G_1 - (A_1 L_0 + B_1) F_{0,j+1}. \quad (4.19)
\end{align*}$$

For $i = 2$ : 

$$A_2 F_{0,j+1} + B_2 F_{1,j+1} + C_2 F_{2,j+1} + D_2 F_{3,j+1} + E_2 F_{4,j+1} = G_2. \quad (4.20)$$
Since $F_{0,j+1}$ is known thus from above equation, we have

$$B_2 F_{1,j+1} + C_2 F_{2,j+1} + D_2 F_{3,j+1} + E_2 F_{4,j+1} = G'_2,$$

where

$$G'_2 = G_2 - A_2 F_{0,j+1}. \quad (4.21)$$

For $3 \leq i \leq M - 3$, the equations are given by

$$A_i F_{i-2,j+1} + B_i F_{i-1,j+1} + C_i F_{i,j+1} + D_i F_{i+1,j+1} + E_i F_{i+2,j+1} = G_i.$$

For $i = M - 2$ we obtain

$$A_{M-2} F_{M-4,j+1} + B_{M-2} F_{M-3,j+1} + C_{M-2} F_{M-2,j+1} + D_{M-2} F_{M-1,j+1} + E_{M-2} F_{M,j+1} = G_{M-2}. \quad (4.22)$$

Since $F_{M,j+1}$ is known, so the equation (4.22) should be written as

$$A_{M-2} F_{M-4,j+1} + B_{M-2} F_{M-3,j+1} + C_{M-2} F_{M-2,j+1} + D_{M-2} F_{M-1,j+1} = G'_{M-2},$$

in which

$$G'_{M-2} = G_{M-2} - E_{M-2} F_{M,j+1}. \quad (4.23)$$

For $i = M - 1$, we can write

$$A_{M-1} F_{M-3,j+1} + B_{M-1} F_{M-2,j+1} + C_{M-1} F_{M-1,j+1} + D_{M-1} F_{M,j-1} + E_{M-1} F_{M+1,j+1} = G_{M-1}. \quad (4.24)$$

To find the value of $G_1$ at the time level $j$, we must have the value of $F_{M+1,j}$. Now augmentation of the boundary condition

$$\frac{\partial F(\infty, \tau)}{\partial \eta} = 0 \quad (4.25)$$

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impose a well-posed problem. The boundary condition is discretized to give

\[ F_{M-1,j} = F_{M,j} \]  \hspace{1cm} (4.26)

Thus for \( i = M - 1 \), the equation (4.24) takes the form

\[ A_{M-1} F_{M-3,i+1} + B_{M-1} F_{M-2,i+1} + C_{M-1} F_{M-1,i+1} = G'_{M-1}, \]

where

\[ G'_{M-1} = G_{M-1} - (D_{M-1} + E_{M-1}) F_{M,i+1}. \]  \hspace{1cm} (4.27)

It is noted that there are \( M - 1 \) equations and these in matrix form can be expressed as
\[
\begin{bmatrix}
C'_1 & D'_1 & E'_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
B_2 & C_2 & D_2 & E_2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
A_3 & B_3 & C_3 & D_3 & E_3 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & A_i & B_i & C_i & D_i & E_i & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & A_{M-3} & B_{M-3} & C_{M-3} & D_{M-3} & E_{M-3} & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & A_{M-2} & B_{M-2} & C_{M-2} & D_{M-2} & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{M-1} & B_{M-1} & C_{M-1} & \cdots & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
\vdots \\
F_i \\
\vdots \\
F_{M-3} \\
F_{M-2} \\
F_{M-1}
\end{bmatrix}
\times
\begin{bmatrix}
G'_1 \\
G'_2 \\
G'_3 \\
\vdots \\
G_i \\
\vdots \\
G'_{M-3} \\
G'_{M-2} \\
G'_{M-1}
\end{bmatrix}
= 
\begin{bmatrix}
G'_1 \\
G'_2 \\
G'_3 \\
\vdots \\
G_i \\
\vdots \\
G'_{M-3} \\
G'_{M-2} \\
G'_{M-1}
\end{bmatrix}
\tag{4.28}
\]

We observe that the matrix involved in equation (4.28) is pentadiagonal.

### 4.1.2 Graphs and discussions

The aim of this section is to examine the influence of magnetic, porosity and material parameters of the third grade fluid on the flow in the absence of slip parameter. To achieve this, we plotted figures 1 to 16.
Figures 4.1.1 and 4.1.2 carry the effects which show the variation in acceleration on the velocity profile $f$ and $g$. As might be expected that large values of acceleration parameter $c$ are responsible to increase $f$ and decrease $g$ near the disk. However, the boundary layer thickness is found to decrease. The influence of third grade parameter is depicted in figures 4.1.3 to 4.1.6. These figures show clearly that $f$ decreases and $g$ increases when $\beta$ is increased. Further, figures 4.1.7 to 4.1.10 are sketched to see the influence of $c$ for different time. These figures indicate that near the disk $f$ increases and $g$ decreases in the presence of $c$. The effects of porosity is shown in figures 4.1.11 and 4.1.12 when $\alpha = 0.1$, $\beta = 0.5$, $c = 0.25$, $N = 0$ and $\tau = 1$. It is noted that $f$ decreases and $g$ increases when $S = -1$ case is compared with $S = 0$. However, for $S = 1$, $f$ increases and $g$ decreases. This leads to conclude that suction causes reduction in the boundary layer thickness and blowing (or injection) enhances the layer thickness. Further, figures 4.1.13 and 4.1.14 display the variation of the magnetic parameter on $f$ and $g$. It is interesting to note that the boundary layer thickness decreases for large values of the magnetic parameter.
The variation of the velocity with distance from the disk for various values of acceleration $c$ when $\alpha = 0.2$, $\beta = 0.5$, $S = N = 0$ and $\tau = 1$.

The variation of the velocity with distance from the disk for various values of third grade parameter $\beta$ when $c = 0$, $\alpha = 0.1$, $S = N = 0$ and $\tau = 1$. 
The variation of the velocity with distance from the disk for various values of third grade parameter $\beta$ when $c = 0.25$, $\alpha = 0.1$, $S = N = 0$ and $\tau = 1$.

The variation of the velocity with distance from the disk for various values of time $\tau$ when $\alpha = 0.1$, $\beta = 0.5$, $S = N = 0$ and $c = 0$. 
The variation of the velocity with distance from the disk for various values of time $\tau$ when $\alpha = 0.1$, $\beta = 0.5$, $S = N = 0$ and $c = 0.25$.

The variation of the velocity with distance from the disk for effect of injection/suction parameter $S$ when $\alpha = 0.1$, $\beta = 0.5$, $N = 0$ and $\tau = 1$. 

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The variation of the velocity with distance from the disk for various values of magnetic field $N$ when $\alpha = 0.1$, $\beta = 0.5$, $S = 0$ and $\tau = 1$.

The variation of the velocity with distance from the disk for various values of rotation parameter $\Omega$ when $\alpha = 0.1$, $\beta = 0.5$, $S = N = 0$, $l_t = 0$ and $\tau = 1$. 
The variation of the velocity with distance from the disk for various values of rotation parameter $\Omega$ when $\alpha = 0.1$, $\beta = 0.5$, $S = N = 0$, $l_1 = 0.5$ and $\tau = 1$.

The effect of acceleration and time on the velocity with distance from the disk when viscous and third grade taking into account.
4.2 Flow with partial slip condition

In this section, the governing problem in section 4.1 is extended to the case of partial slip instead of no-slip condition. Thus, the difference in both the section is lies in the boundary condition at the disk. In the present case, the boundary condition at $z = 0$ is of the following form

$$u - \lambda_2 \tau_{xz} = -\Omega y + c_0 t, \quad v - \lambda_2 \tau_{yz} = \Omega x, \quad \text{at} \quad z = 0 \quad \text{for} \quad t > 0, \quad (4.29)$$

where for third grade fluid the shear stress $\tau_{xy}$ and $\tau_{xz}$ are given by

$$\tau_{xz} = \mu \frac{\partial u}{\partial z} + \alpha_1 \left[ \frac{\partial^2 u}{\partial t \partial z} + w \frac{\partial^2 u}{\partial z^2} + \frac{\partial v}{\partial z} \frac{\partial u}{\partial y} + 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial z} \right]$$

$$+ 2\beta_3 \left[ \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \frac{\partial u}{\partial z} + \left( \frac{\partial u}{\partial z} \right)^3 + \frac{\partial u}{\partial z} \left( \frac{\partial v}{\partial z} \right)^2 \right], \quad (4.30)$$

$$\tau_{yz} = \mu \frac{\partial v}{\partial z} + \alpha_1 \left[ \frac{\partial^2 v}{\partial t \partial z} + w \frac{\partial^2 v}{\partial z^2} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial z} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial x} \right]$$

$$+ 2\beta_3 \left[ \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \frac{\partial v}{\partial z} + \left( \frac{\partial v}{\partial z} \right)^3 + \frac{\partial v}{\partial z} \left( \frac{\partial u}{\partial z} \right)^2 \right]. \quad (4.31)$$

The boundary condition (4.29) in dimensionless form can be written finally as

$$F(0, \tau) = c\tau - 1 + \lambda \left[ \frac{\partial F}{\partial \eta} + \alpha \left\{ \frac{\partial^2 F}{\partial \tau \partial \eta} - S \frac{\partial^2 F}{\partial \eta^2} - \frac{\partial F}{\partial \eta} \right\} \right]$$

$$+ \beta \left\{ \left( \frac{\partial F}{\partial \eta} \right)^2 \frac{\partial F}{\partial \eta} \right\}. \quad (4.32)$$

4.2.1 Numerical solution

Here it is noted that the governing equation (4.6) is highly non-linear partial differential equation together with initial and boundary conditions (4.32).
Also, the boundary conditions at $z = 0$ is non-linear. The problem here is an extension of the problem presented in section 4.1 and hence it may be fair to avoid rewriting the detailed steps. Thus, following the same procedure as used in section 4.1 we are including some major steps for the convenience of the readers. The discretized form of the governing equation (4.6) is given in equation (4.9). However, the boundary condition (4.32) is discretized by using forward and backward differences for the derivatives involved with respect to space and time coordinate respectively. This results in the form

$$F_{0,j} = r_1 F_{1,j} + r_2 F_{2,j} + r_3 [F_{0,j-1} - F_{2,j-1}] + r_4 [BT_j + cjk - 1].$$

(4.33)

In above equations

$$r_0 = h^2 k + \lambda (1 - i\alpha) h k + \alpha \lambda h + \alpha S \lambda k, \quad (4.34)$$

$$r_1 = [\lambda (1 - i\alpha) h k + \alpha \lambda h + 2\alpha S \lambda k] / r_0, \quad (4.35)$$

$$r_2 = [-\alpha S \lambda k] / r_0, \quad (4.36)$$

$$r_3 = [\alpha \lambda h] / r_0, \quad (4.37)$$

$$r_4 = [h^2 k] / r_0, \quad (4.38)$$

$$BT_j = \frac{\beta \lambda}{h^3} \left[ (F_{1,j} - F_{0,j})^2 (\bar{F}_{1,j} - \bar{F}_{0,j}) \right]. \quad (4.39)$$

To evaluate $F_{0,j+1}$, we firstly take $BT_{j+1} = BT_j$ in the system of algebraic equations and the solution of the system is sought which results in known values of $F_{i,j+1}; i = 1, 2, 3, ..., M - 1$. Secondly, we update $F_{0,j+1}$ by using iterative method as follows:

$$F_{0,j+1}^{k+1} = r_1 F_{1,j+1} + r_2 F_{2,j+1} + r_3 [F_{0,j}^k - F_{1,j}^k]
+ r_4 \frac{\beta \lambda}{h^3} \left[ (F_{1,j+1} - F_{0,j+1}^k)^2 (\bar{F}_{1,j+1} - \bar{F}_{0,j+1}^k) \right]
+ r_4 [c(j + 1)k - 1], \quad (4.40)$$

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where

\[ F_{0,j+1}^0 = F_{0,j+1} \]

and this iterative procedure is continued until \( F_{0,j+1}^k \approx F_{0,j+1}^k \). Furthermore, \( F_{0,0}^0 \) is evaluated by letting \( F_{0,-1} = F_{1,-1} = 0 \) and by using iterative method as described above with \( F_{0,0}^0 = 0 \) as initial guess.

For \( i = 1, i = 2, 3 \leq i \leq M - 3, i = M - 2 \) and \( i = M - 1 \), we respectively have

\[
C'_1 F_{1,j+1} + D'_1 F_{2,j+1} + E'_1 F_{3,j+1} = G'_1, \\
B'_2 F_{1,j+1} + C'_2 F_{2,j+1} + D'_2 F_{3,j+1} + E'_2 F_{4,j+1} = G'_2, \\
A_i F_{i-2,j+1} + B_i F_{i-1,j+1} + C_i F_{i,j+1} + D_i F_{i+1,j+1} + E_i F_{i+2,j+1} = G_i \\
A_{M-2} F_{M-4,j+1} + B_{M-2} F_{M-3,j+1} + C_{M-2} F_{M-2,j+1} + D_{M-2} F_{M-1,j+1} = G'_{M-2}, \\
A_{M-1} F_{M-3,j+1} + B_{M-1} F_{M-2,j+1} + C_{M-1} F_{M-1,j+1} = G'_{M-1},
\]

where \( A_i, B_i, C_i, D_i, E_i \) and \( G_i \) are given through equations (4.10) to (4.15).

In the above equations

\[
C'_1 = A_1 L_1 + C_1 + r_1 (B_1 + L_0 A_1), \\
D'_1 = A_1 L_2 + D_1 + r_2 (B_1 + L_0 A_1), \\
E'_1 = A_1 L_3 + E_1, \\
G'_1 = G_1 - (A_1 L_0 + B_1) \left[ r_3 (F_{0,j} - F_{1,j}) + r_4 (B T_{j+1} + c (j + 1) k - 1) \right],
\]

\[
B'_2 = B_2 + r_1 A_2, \\
C'_2 = C_2 + r_2 A_2, \\
G'_2 = G_2 - A_2 \left[ r_3 (F_{0,j} - F_{1,j}) + r_4 (B T_{j+1} + c (j + 1) k - 1) \right],
\]

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\[ G'_{M-2} = G_{M-2} - E_{M-2} F_{M,j+1}, \] (4.46)

\[ G'_{M-1} = G_{M-1} - (D_{M-1} + E_{M-1}) F_{M,j+1}. \] (4.47)

The matrix form of the above set of \( M - 1 \) equations is

\[
\begin{bmatrix}
    C'_1 & D'_1 & E'_1 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & 0 \\
    B'_2 & C'_2 & D'_2 & E'_2 & 0 & 0 & 0 & 0 & \ldots & \ldots & 0 \\
    A'_3 & B'_3 & C'_3 & D'_3 & E'_3 & 0 & 0 & 0 & \ldots & \ldots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & A_i & B_i & C_i & D_i & E_i & 0 & \ldots & \ldots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \ldots & 0 & A_{M-3} & B_{M-3} & C_{M-3} & D_{M-3} & E_{M-3} \\
    0 & 0 & 0 & 0 & \ldots & 0 & A_{M-2} & B_{M-2} & C_{M-2} & D_{M-2} \\
    0 & 0 & 0 & 0 & 0 & \ldots & 0 & A_{M-1} & B_{M-1} & C_{M-1} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    F_1 \\
    F_2 \\
    F_3 \\
    \vdots \\
    F_i \\
    \vdots \\
    F_{M-3} \\
    F_{M-2} \\
    F_{M-1}
\end{bmatrix}
\times
\begin{bmatrix}
    G'_1 \\
    G'_2 \\
    G'_3 \\
    \vdots \\
    G'_i \\
    \vdots \\
    G'_{M-3} \\
    G'_{M-2} \\
    G'_{M-1}
\end{bmatrix}
\]
4.2.2 Graphs and discussion

In this section, we present the effects of partial slip condition on the flow pattern for third grade fluid problem. Of particular interest here are the effects of slip parameter $\lambda$, constant acceleration parameter $c$, material parameter of third grade fluid $\beta$, porosity parameter $S$ and magnetic parameter $N$. For this purpose, we prepare figures 4.2.1 to 4.2.18 (for the real part of velocity $f$ and imaginary part of the velocity $g$).

Figures 4.2.1 and 4.2.2 illustrate the effects of $\lambda$ for $c = 0$ while figures 4.2.3 and 4.2.4 for $c = 0.25$. It should be noted from these figures that with the increase in $\lambda$ both $f$ and $g$ increases near the disk. However, far away from the disk the velocity $g$ decreases with the increase of $\lambda$. Moreover, the figures 4.2.1 to 4.2.6 indicate that $f$ increases and $g$ decreases near the disk for large values of the acceleration. It is further observed from these figures that partial slip causes the reduction in the boundary layer thickness which is in quite agreement with the physical expectations.

The behaviour of the velocity profiles $f$ and $g$ for various values of $\tau$ in the absence and presence of $c$ is shown in figures 4.2.7 to 4.2.10. For $c = 0$, it is revealed that $f$ decreases and $g$ increases for large values of $\tau$. However it is also observed that the variation of $f$ and $g$ in case of $c = 0.25$ is smaller when we compared with the case $c = 0$.

In order to illustrate the effect of material parameter of third grade fluid, we made figures 4.2.11 and 4.2.12. These figures show that velocity profiles increase near the disk for large values of $\beta$.

To see the variation of porosity and magnetic field parameters, the figures 4.2.13 to 4.2.16 are plotted. These show that with the increase in suction,
there is a reduction in the boundary layer thickness. Magnetic parameter has the same behaviour as that of suction parameter. However, the boundary layer thickness reduces much in case of magnetic parameter. Finally, figures 4.2.17 and 4.2.18 are made in order to compare the velocity profiles in viscous and third grade fluid. It can be seen that near the disk velocity profiles for viscous fluid are greater than that of third grade fluid.

The variation of the velocity with distance from the disk for various values of slip parameter \( \lambda \) when \( \alpha = 0.2, \beta = 0.5, c = S = N = 0 \) and \( \tau = 1 \).
The variation of the velocity with distance from the disk for various values of slip parameter $\lambda$ when $\alpha = 0.2$, $\beta = 0.5$, $c = 0.25$, $S = N = 0$ and $\tau = 1$.

The variation of the velocity with distance from the disk for various values of acceleration parameter $c$ when $\alpha = 0.2$, $\beta = 0.5$, $S = N = 0$, $\lambda = 0.5$ and $\tau = 1$. 

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The variation of the velocity with distance from the disk for various values of time $\tau$ when $\alpha = 0.2$, $\beta = 0.5$, $S = N = 0$, $\lambda = 0.5$, and $c = 0$.

The variation of the velocity with distance from the disk for various values of time $\tau$ when $\alpha = 0.2$, $\beta = 0.5$, $S = N = 0$, $\lambda = 0.5$, and $c = 0.25$. 
The variation of the velocity with distance from the disk for various values of third grade parameter $\beta$ when $S = N = 0$, $\alpha = 0.2$, $c = 0.25$ and $\tau = 1$.

The variation of the velocity with distance from the disk for various injection/suction parameter $S$ when $\alpha = 0.1$, $\beta = 0.5$, $N = 0$, $c = 0.25$, $\lambda = 0.5$ and $\tau = 1$. 
The variation of the velocity with distance from the disk for various values of magnetic parameter $N$ when $\lambda = 0.5$, $c = 0.25$, $\alpha = 0.2$, $\beta = 0.5$, $S = 0$ and $\tau = 1$.

The variation of the velocity with distance from the disk for various fluids in the presence of slip parameter $\lambda = 0.5$ and $l_1 = 0.25$ when $S = N = 0$, $\tau = 1$. 
and \( r = 1 \).

The variation of the velocity with distance from the disk for effect of various fluids when we fix the value of \( \lambda = 0.5, S = N = 0 \) and \( c = 0.25 \).
Chapter 5

Transient flows of a third grade fluid due to variable accelerated disk

The objective of this chapter is to extend the analysis of chapter four when disk is variably accelerated. Numerical results are given and illustrated graphically. Finally, the physical description of the graphs is given.

5.1 Flow with no-slip condition

Here, the physical description of the governing problem is the same as in section 4.1 except that the disk is moving with variable acceleration. Thus the governing problem consists of differential equation (4.6), initial and boundary conditions (2.30). Our interest now lies to give numerical solution in the next subsection.
5.1.1 Numerical solution

We note that exact or approximate analytical solutions of the governing non-linear problem is not amenable for every involved parameter. For that we present a numerical solution. The choice of an appropriate numerical technique is closely related to the mathematical behaviour of non-linear partial differential equation. Note that equation (4.6) contains mixed derivatives with respect to time and space coordinates. Thus, implicit numerical technique has been used. The implicit scheme for non-linear equations is not straightforward as for linear equations. The governing equation (4.6) is exactly the same as that of the problem presented in this section. Therefore, the same numerical technique is used to solve the problem and the discretized form of the involved problem is

\[ A_i F_{i-2,j+1} + B_i F_{i-1,j+1} + C_i F_{i,j+1} + D_i F_{i+1,j+1} + E_i F_{i+2,j+1} = G_i, \]  

\( i = 1, \ldots, M, j = 0, 1, 2, \ldots, \)  

\( F_{0,j} = c_j^2 k^2 - 1, \quad F_{M,j} = 0, \quad F_{i,0} = 0, \quad j = 0, 1, 2, \ldots, \)  

\( M, \)  

where \( A_i, B_i, C_i, D_i, E_i \) and \( G_i \) are given by (4.10) to (4.15).

We note that the difference between the problem (4.9) and (5.1) is in the boundary condition. Thus, for the solution, we adopt the same procedure as for problem (4.9). However, some major steps are given for the convenience of the readers. Adopting the procedure used in section 4.1 we obtain

\[ C_i' F_{i,j+1} + D_i' F_{i+1,j+1} + E_i' F_{i+2,j+1} = G_i', \quad i = 1, \ldots, M, \]
\[ B_2 F_{1, j+1} + C_2 F_{2, j+1} + D_2 F_{3, j+1} + E_2 F_{4, j+1} = G_{2}', \quad i = 2, \quad (5.4) \]
\[ A_i F_{i-2, j+1} + B_i F_{i-1, j+1} + C_i F_{i, j+1} + D_i F_{i+1, j+1} \]
\[ - E_i F_{i+2, j+1} = G_i, \quad 3 \leq i \leq M - 3, \quad (5.5) \]
\[ A_{M-2} F_{M-4, j+1} + B_{M-2} F_{M-3, j+1} + C_{M-2} F_{M-2, j+1} \]
\[ + D_{M-2} F_{M-1, j+1} = G_{M-2}', \quad i = M - 2, \quad (5.6) \]

\[ A_{M-1} F_{M-3, j+1} + B_{M-1} F_{M-2, j+1} + C_{M-1} F_{M-1, j+1} = G_{M-1}', \quad i = M - 1 \]

where \( C'_1, \ D'_1, \ E'_1, \ G'_1, \ G'_2, \ G'_{M-2} \) and \( G'_{M-1} \) are given through equations (4.19), (4.21), (4.23) and (4.27) respectively.

The resultant algebraic equations in matrix form give
\[
\begin{bmatrix}
C'_1 & D'_1 & E'_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_2 & C_2 & D_2 & E_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_3 & B_3 & C_3 & D_3 & E_3 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & \\
0 & . & 0 & A_i & B_i & C_i & D_i & E_i & 0 & 0 \\
& & & & & & & & & \\
0 & 0 & 0 & . & 0 & A_{M-3} & B_{M-3} & C_{M-3} & D_{M-3} & E_{M-3} \\
0 & 0 & 0 & 0 & . & 0 & A_{M-2} & B_{M-2} & C_{M-2} & D_{M-2} \\
0 & 0 & 0 & 0 & 0 & . & 0 & A_{M-1} & B_{M-1} & C_{M-1} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
& \\
F_i \\
& \\
F_{M-3} \\
F_{M-2} \\
F_{M-1}
\end{bmatrix}
\times
\begin{bmatrix}
G'_1 \\
G'_2 \\
G_3 \\
& \\
G_i \\
& \\
G_{M-3} \\
G'_{M-2} \\
G'_{M-1}
\end{bmatrix}
\]
The variation of the velocity with distance from the disk for various values of acceleration parameter $\gamma$ when $N = 0, \alpha = 0.2, \beta = 0.5, S = 0$ and $\tau = 0.5$.

The variation of the velocity with distance from the disk for various values of acceleration parameter $\gamma$ when $N = 0, \alpha = 0.2, \beta = 0.5, S = 0$ and $\tau = 1.5$. 

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The variation of the velocity with distance from the disk for various values of third grade parameter $\beta$ when $N = 0$, $\alpha = 0.2$, $\gamma = 0.25$, $S = 0$ and $\tau = 0.5$.

The variation of the velocity with distance from the disk for various values of third grade parameter $\beta$ when $N = 0$, $\alpha = 0.2$, $\gamma = 0.25$, $S = 0$ and
\( \tau = 1.5. \)

The variation of the velocity with distance from the disk for various values of time \( \tau \) when \( N = 0, \alpha = 0.2, \beta = 0.5, \gamma = .25, \) and \( S = 0. \)

The variation of the velocity with distance from the disk for various values of time \( \tau \) when \( N = 0, \alpha = 0.2, \beta = 0.5, \gamma = .25, \) and \( S = 0. \)
The variation of the velocity with distance from the disk for various values of magnetic field parameter \(N\) when \(\alpha = 0.2, \beta = 0.5, \gamma = 0.25, S = 0,\) and \(\tau = 1.5.\)

The variation of the velocity with distance from the disk for various values of suction parameter \(S\) when \(\alpha = 0.2, \beta = 0.5, \gamma = 0.25, N = 0,\) and \(\tau = 1.5.\)
The variation of the velocity with distance from the disk for various values of rotation $\Omega$ when $\alpha = 0.2$, $\beta = 0.5$, $N = 0$, and $\tau = 1.5$.

The variation of the velocity with distance from the disk for various values of rotation $\Omega$ when $\alpha = 0.2$, $\beta = 0.5$, $N = 0$, and $\tau = 1.5$. 
5.2 Flow with partial slip condition

Here, the physical problem is similar to that presented in section 4.2 but now the disk is moving with variable acceleration. The governing differential equation, boundary condition at infinity and initial conditions are the same. The boundary condition at \( z = 0 \) is given by

\[
\begin{align*}
\frac{\partial u}{\partial z} - \lambda_2 \frac{\partial^2 u}{\partial z^2} &= -\Omega y - \gamma_0 t^2, \\
\frac{\partial v}{\partial z} &= \Omega x, \text{ at } z = 0 \text{ for } t > 0,
\end{align*}
\]

where \( \tau_{xz} \) and \( \tau_{yz} \) are given by equations (4.30) and (4.31). This problem is in fact equal to our earlier work in chapter four and hence it may be fair to avoid rewriting the constitutive equations and the assumptions made. Thus, employing the same method, the non-dimensional boundary condition at \( \varphi = 0 \) is given as

\[
F(0, \tau) = \gamma \tau^2 - 1 + \lambda \left[ \frac{\partial F}{\partial \eta} + \alpha \left\{ \frac{\partial^2 F}{\partial \tau \partial \eta} - \frac{S \partial^2 F}{\partial \eta^2} - \frac{i \partial F}{\partial \eta} \right\} \right] + \beta \left\{ \left( \frac{\partial F}{\partial \eta} \right)^2 + \frac{\partial F}{\partial \eta} \right\}. \tag{5.7}
\]

5.2.1 Numerical solution

In this section, we seek numerical solution to the problem given by the equation (4.6), initial (2.33), boundary conditions (2.32) and (5.7). The discretized form of the governing equation (4.6) is given in (4.9). However, the boundary condition (5.7) is discretized as

\[
F_{0, j} = r_1 F_{1, j} + r_2 F_{2, j} + r_3 [F_{0, j-1} - F_{1, j-1}] + r_4 \left[ BT_j + c j^2 k^2 - 1 \right],
\]

where \( r_1, r_2, r_3, r_4 \) and \( BT_j \) are given in equations (4.35) to (4.39).
To evaluate $F_{0,j+1}$, the iterative procedure will be as follows:

\[
F_{0,j+1}^{k+1} = r_1 F_{1,j+1} + r_2 F_{2,j+1} + r_3 \left[ F_{0,j}^k - F_{1,j}^k \right] \\
+ r_4 \beta^2 \lambda k^2 \left[ (F_{1,j+1} - F_{0,j+1}^k) \right]^2 \left( F_{1,j+1} - F_{0,j+1}^k \right) \\
+ r_4 \left[ c (j + 1)^2 k^2 - 1 \right]
\]

(5.8)

with

\[
F_{0,j+1}^0 = F_{0,j+1}.
\]

For $i = 1, i = 2, 3 \leq i \leq M - 3, i = M - 2$ and $i = M - 1$, we respectively have

\[
C_i' F_{1,j+1} + D_i' F_{2,j+1} + E_i' F_{3,j+1} = G_i',
\]

\[
B_1' F_{1,j+1} + C_2' F_{2,j+1} + D_2' F_{3,j+1} + E_2' F_{4,j+1} = G_2',
\]

\[
A_i F_{i-2,j+1} + B_i F_{i-1,j+1} + C_i F_{i,j+1} + D_i F_{i+1,j+1} + E_i F_{i+2,j+1} = G_i,
\]

\[
A_{M-2} F_{M-4,j+1} + B_{M-2} F_{M-3,j+1} + C_{M-2} F_{M-2,j+1} + D_{M-2} F_{M-1,j+1} = G_{M-2},
\]

\[
A_{M-1} F_{M-3,j+1} + B_{M-1} F_{M-2,j+1} + C_{M-1} F_{M-1,j+1} = G_{M-1},
\]

where $A_i, B_i, C_i, D_i, E_i, G_i, C_i', D_i', E_i', B_i', C_i', G_i'$, and $G_i'$ are given through equations (4.10) to (4.15), (4.41) to (4.47) and

\[
G_1' = G_1 - (A_1 L_0 + B_1) \left[ r_3 (F_{0,j} - F_{1,j}) + r_4 (BT_{j+1} + c (j + 1)^2 k^2 - 1) \right],
\]

\[
G_2' = G_2 - A_2 \left[ r_3 (F_{0,j} - F_{1,j}) + r_4 (BT_{j+1} + c (j + 1)^2 k^2 - 1) \right].
\]

In matrix form, the above set of $M - 1$ equations can be finally written as

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We present the flow pattern of velocities for the problem at hand. The numerical graphs for the velocity field \((f, g)\) are given for various possibilities.
The variation of the velocity with distance from the disk for various values of slip parameter $\lambda$ when $c = 0.25, \alpha = 0.2, \beta = 1, S = N = 0$ and $\tau = 0.5$.

The variation of the velocity with distance from the disk for various values of slip parameter $\lambda$ when $c = 0.25, \alpha = 0.2, \beta = 1, S = N = 0$ and $\tau = 1.5$. 

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The variation of the velocity with distance from the disk for various values of acceleration \(c\) when \(\lambda = 0.5, \; \alpha = 0.2, \; \beta = 1, \; S = N = 0\) and \(\tau = 1.5\).

The variation of the velocity with distance from the disk for various values of time \(\tau\) when \(c = 0.25, \; \lambda = 0.5, \; \alpha = 0.2, \; \beta = 1, \; S = 0, \) and \(N = 0\).
The variation of the velocity with distance from the disk for various values of third grade parameter $\beta$ when $c = 0.25$, $\lambda = 0$, $\alpha = 0.2$, $\beta = 1$, $S = N = 0$ and $\tau = 0.5$.

The variation of the velocity with distance from the disk for various values of third grade parameter $\beta$ when $c = 0.25$, $\lambda = 0.25$, $\alpha = 0.2$, $\beta = 1$, $S = N = 0$ and $\tau = 0.5$. 

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The variation of the velocity with distance from the disk for various values of suction parameter $S$ when $c = 0.25$, $\lambda = 0.5$, $\alpha = 0.2$, $\beta = 1$, $N = 0$ and $\tau = 0.5$.

The variation of the velocity with distance from the disk for various values of magnetic field $N$ when $c = 0.25$, $\lambda = 0.5$, $\alpha = 0.2$, $\beta = 1$, $S = 0$ and $\tau = 0.5$. 

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The variation of the velocity with distance from the disk for various values of rotation parameter $\Omega_2$ when $l_2 = 0.25, \lambda = 0, \alpha = 0.2, \beta = 1, S = N = 0$ and $\tau = 1.5$.

The variation of the velocity with distance from the disk for various values of rotation parameter $\Omega_2$ when $l_2 = 0.25, \lambda = 0.25, \alpha = 0.2, \beta = 1, S = N = 0$ and $\tau = 1.5$. 

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5.3 Concluding Remarks

In this chapter, the numerical solutions for the two problems of a third grade fluid are given. The flows here are generated by a disk moving with variable acceleration. It is noted that all the observations of the involved physical parameters are similar to that given in chapter four i.e. for constant accelerated disk. However, it is found that for the $\tau < 1$, the velocity profiles for constant accelerated flow are greater to that of variable accelerated flow. For $\tau > 1$, the velocity profiles for variable accelerated flow are much larger when compared with constant accelerated flow.
Bibliography


