SYMOMETRY ANALYSIS AND CONSERVATION LAWS OF
PHYSICAL MODELS ON CURVED SURFACES

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A THESIS
SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

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LAHORE UNIVERSITY OF MANAGEMENT SCIENCES
LAHORE PAKISTAN
APRIL 2011
This work is submitted as a Thesis in the partial fulfilment of the requirement for degree of Doctor of Philosophy in Mathematics, to the Department of Mathematics, Lahore University of Management Sciences, Lahore, Pakistan.
DEDICATED

to

My Family
Abstract

Physical models with non-flat background are important in biological mathematics. Most of the biological membranes are not flat in general. For example, membranes which convert energy in mitochondria and chloroplasts are tubes, buds and may be sheets. In most of the biological processes, the geometry of membranes is very important. The organization and shape of the membranes play a vital role in biological processes such as shape change, fusion-division, ion adsorption etc. A cell membrane is a system for exchange of energy and matter from the neighbourhood. Absorption and transformation of conserved quantities such as energy and matter from the environment are one of the characteristics of membranes. The shape of proteins, non zero curvature of membranes and involvement of conserved quantities lead one to discuss physical models on curved surfaces.

Conservation laws play a vital role in science and also helpful to construct potential systems which can be used to calculate exact solutions of differential equations. Physical models on curved surfaces govern partial differential equation which need not to be derivable from variational principle. The partial Noether approach is the systematic way to construct the conservation laws for non-variational problems.

The group classification and conservation laws for some partial differential equation on curved surfaces are presented in this dissertation. In particular some linear and nonlinear models of heat and wave equation on plane, cone, sphere are classified. The conservation laws for the (1 + 2)-dimensional heat equation on different surfaces are constructed via partial Noether approach and then the results are generalized for the (1+n)-dimensional case. The symmetry conservation laws relation is used to simplify the derived conserved vectors and exact solutions are constructed. We also extend these results to a special type of (1 + n)-dimensional
linear evolution equation. Potential systems of some models from different sciences are also given. The similar analysis is performed for the \((1 + 2)\)-dimensional wave equation on the sphere, cone and on flat surface.

Furthermore, the nonlinear heat equation on curved surfaces is considered. A class of functions is found on the plane, sphere and torus, which is not only independent of the number of independent variables but also independent of the background metric. We consider whether the background metric or the nonlinearity have the dominant role in the infinitesimal generators of heat equation on curved manifolds. Then a complete Lie analysis of the time dependent Ginzburg-Landau equation (TDGL model) is presented on the sphere and torus.

In addition, for the \((1 + n)\)-dimensional nonlinear wave equation (Klein Gordon Equation) it is proved that there is a class of functions which is independent from number of independent variables. Then for the \((1 + 2)\)-dimensional wave equation it is proved that there is a class of functions which is invariant either the underlying space is a plane, sphere or torus.
Acknowledgements

I begin with the name of Almighty Allah, Who is most beneficent and merciful. He bestowed His blessings on me to continue my research work and to achieve this goal. May His peace and blessings be upon His prophet Muhammad.

The success of this study could not have come together without the help, encouragement and motivation of many people. It gives me great pleasure to acknowledge my special thanks to my mentors Dr. Imran Naeem and Dr. Sultan Sial for their valuable suggestions, guidelines and their personal commitments. I would also like to thank Dr. Mohammad Naeem Qureshi, Chairman of Department of Mathematics, Azad and Jammu Kashmir University, for his guidance and useful discussions. My special thanks are due to Dr. Fazal M. Mahomed for his valuable comments.

I am grateful to all of the teachers in LUMS for their support and encouragement during my stay as a Ph.D. student. I would also like to thank Nauman Raza, Abdul Majid, Asma Rashid Butt, Kashif Nazar, Kashif A. Khan and Basit Ali, for their help during my studies.

And of course, my deepest thanks go to my parents and wife for their enormous contribution and moral support without which its impossible to attain this milestone. My sincere gratitude is due to my sisters, brothers and all other family members.

Lahore

April, 2011

Adil Jhangeer
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Chapter 1

Introduction & Preliminaries

1.1 Introduction

Symmetry analysis is considered to be a handy tool for solving linear and nonlinear differential equations and it is one of the great discoveries of the nineteenth century made by Sophus Lie. Similarity solutions, conservation laws and group classification are some of the important aspects of this approach. By using this method, one can find symmetries of almost any differential equation (if these exist) and these symmetries can simplify the analysis of physical problems. Differential equations have great importance in almost all fields of science. Many physical phenomena and engineering problems are represented by differential equations.

1.1.1 Brief history of differential equations

The history of differential equations starts when Newton discovered Calculus as integrals in 1665 – 1666. After that, many mathematicians of that century, for example Leibniz and Bernoulli, contributed to the field of differential equations. They worked on the solution of various types of equations.

In 1712, Ricatti introduced a method for the solution of a differential equation, now known
as the Ricatti equation, while Clairaut studied and solved a special type of differential equation now familiar as the Clairaut differential equation. He also gave some remarkable results on the existence of an integrating factor for first order differential equations.

Solution by series methods and variation of parameters are the invention of Euler. The Laplacian of a function and the Fourier transformation play a vital role in calculating the solution of differential equations. These were developed by P. S. Laplace and J. Fourier respectively. The efforts of Taylor, d’Alembert, Lagrange, Legendre and Bessel in this field cannot be ignored without their contributions we could not fill the gaps covered by their remarkable work.

Cauchy did work on the existence and uniqueness of the solution of differential equations in the 19th century, while Lipschitz developed the existence theorem for first order differential equations. Other notable contributions were made by Hermite, Liouville, Riemann, Kovalevski, Laguerre, Noether, Gauss and Lie.

The history of partial differential equation started in the 18th century after Euler’s work in this era. Before Euler’s contribution, the history of partial differential equations is difficult to trace. After that, d’Alembert, Lagrange, Laplace and Riemann made efforts for the development of this field. The vast use of partial differential equations in almost all the branches of sciences makes this theory valuable for research.

The $(1 + 1)$-dimensional wave equation was studied by d’Alembert in 1752, in 1759 it was extended to the $(1 + 2)$-dimensional wave equation by Euler and further work was done by Daniel Bernoulli in 1762 to extend it to the $(1 + 3)$-dimensional wave equation. The Euler equation for incompressible fluid flows was modelled by Euler in 1755. The Monge-Ampère
equation was studied by Monge in 1775. The Laplace equation was discussed in 1780 by Laplace. The heat equation was introduced by Fourier between 1810 and 1822. Similarly, the Laplace and Poisson equations, Navier-Stokes equations, Maxwell’s equations, the Helmholtz equation, Kdv equation and many more are the discoveries of 19th century.

In 1747, d’Alembert and in 1748 Euler introduced the method of separation of variables and superposition solution of linear equations for the wave equation. In the 19th century, many powerful tools were introduced and mathematicians worked on the solution of partial differential equations. The method of Green’s functions was introduced in 1835 for Laplace’s equation. The power series methods had been used by Euler, d’Alembert, Laplace and others. Existence and uniqueness of the solutions for Laplace’s equation for any continuous Dirichlet boundary condition was proven in 1880 by Poincaré. The successive approximation method to obtain solutions of nonlinear partial differential equation was applied by Picard in early 1880 while in 1898, Poincaré proved a remarkable result about the existence of the solution of a nonlinear equation.

At the end of 19th century, Lie discovered a systematic method for the solution of differential equations. He used the theory of groups for this purpose. He considered a differential equation as a surface in the space of independent and dependent variables together with the its derivatives. In 1870, he presented a mechanism for finding a transformation which maps solutions of a differential equation to another solution of the same differential equation. These transformations satisfy all the axioms of a group and hence are called symmetry or Lie groups. One of the amazing properties of these symmetry groups is that when a symmetry is applied to the partial differential equation, it provides an extra constraint on that partial differential equation which reduces the dimension (number of independent variable) of the partial differential equation by one. A second remarkable aspect is the connection between
conserved quantities and symmetries. The theorem which links symmetry and conserved quantity was discovered by Noether [9, 49]. It opened a new era of mathematics.

Lie contribution in the field of differential equation is remarkable. It is considered to be one of the greatest discoveries of nineteenth century. Reduction, analysis and classification of a differential equation are the revolutionary aspects of Lie theory. Lie’s contribution is considered to be the one of the important chapter of the modern theory of differential equation. In the meanwhile we cannot ignore the contribution of Poincaré. Lot of theorems related to Hamilton-Jacobi equation and principles are because of Poincaré. Hamilton-Jacobi equation is frequently used in analytic mechanics. Formation and derivation of Hamilton-Jacobi theorems was the big achievement of Poincaré. Yet there are lot of names who did revolutionary contribution in the theory of modern theory of differential equation, Painleve and Brickhoff are one of them.

Idea of a conservation law

Consider a $p$th-order system of PDEs

$$G_{\theta}(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(p)}), \quad \theta = 1, \ldots, m,$$

where we have used $u = (u^1, \ldots, u^m)$ as dependent variables, $x = (x^1, \ldots, x^n)$ as independent variables and $u^{(1)}, u^{(2)}, \ldots, u^{(p)}$ are first up to $p$th-order derivatives.

A conserved vector

$$T = (T^1, T^2, \ldots, T^n),$$

satisfying the relation $D_i T^i = 0$ for every solution of Eq. (6.1.2) is called a conservation law, in which $D_i$ is defined as

$$D_i = \frac{\partial}{\partial x^i} + u_\theta \frac{\partial}{\partial u^\theta} + u_{ij} \frac{\partial}{\partial u^j} + \cdots, \quad i = 1, \ldots, n.$$
1.1.2 Different approaches for the conservation laws

There are several methods for the construction of conservation laws. The first and most elegant approach for a construction of a conservation law was given by Noether herself [9, 49]. Roughly speaking this theorem states that for every Euler-Lagrange equation, there corresponds a conservation law to each Noether symmetry associated with the Lagrangian.

A second approach is the direct method [45] which does not depend on the Lagrangian and so conservation laws can be obtained for the class of equations which do not admit a Lagrangian. In a third approach [58], conservation laws are written in characteristic form which are multipliers of the differential equation. The fourth approach [50] is linked to the third, variational derivatives are used in it. In the fifth approach, the variational derivatives are computed in a solution space.

Calculation of conservation laws by means of computer packages for direct method and multiplier approaches was done by Wolf [63], Wolf et al. [64], Goktas et al. [29] and Hereman et al. [30, 31, 32]. The Maple code to compute conservation laws based on the multipliers approach was introduced by Cheviakov [14]. The sixth method was discussed by Kara et al. They calculated conservation laws by using symmetry condition on the direct method. Finding conservation laws by known characteristic function is the seventh approach [2, 3], given by Anco and Bluman. The eighth approach [39] is given by Kara et al. which depends on the term partial Lagrangian. The ninth approach is given by Ibragimov [35] which depends on integrating factors and adjoint equations.
1.2 Preliminaries

In this section, we will discuss the basic operators and definitions that will be referred to throughout this work.

1.2.1 Conserved quantities by using partial Lagrangian

If (6.1.2) is a $p$th order system of partial differential equation then the Euler operator is

$$\frac{\delta}{\delta u^\theta} = \frac{\partial}{\partial u^\theta} + \sum_{l \geq 1} (-1)^l D_{i_1} \cdots D_{i_l} \frac{\partial}{\partial u_{i_1 i_2 \cdots i_l}^\theta}, \quad \theta = 1, \cdots, m,$$

(1.2.1)

where

$$D_i = \frac{\partial}{\partial x^i} + u_{i}^\theta \frac{\partial}{\partial u^\theta} + u_{ij}^\theta \frac{\partial}{\partial u_{j}^\theta} + \cdots, \quad i = 1, \cdots, n$$

(1.2.2)

is known as the total derivative operator. Note that the summation convention is used throughout. The generalized or Lie-Bäcklund operator is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\theta \frac{\partial}{\partial u^\theta} + \sum_{l \geq 1} \zeta^\theta_{i_1 i_2 \cdots i_l} \frac{\partial}{\partial u_{i_1 i_2 \cdots i_l}^\theta}.$$

(1.2.3)

In Eq. (1.2.3) the additional coefficients $\zeta^\theta_{i_1 i_2 \cdots i_l}$ are given by

$$\zeta^\theta_i = D_i(W^\theta) + \xi^i u_{ij}^\theta,$$

(1.2.4)

$$\zeta^\theta_{i_1 i_2 \cdots i_l} = D_{i_1} \cdots D_{i_l} (W^\theta) + \xi^j u_{j i_1 i_2 \cdots i_l}^\theta, \quad l \geq 1,$$

(1.2.5)

where $W^\theta$ is known as the Lie characteristic function and can be found from

$$W^\theta = \eta^\theta - \xi^j u_{j}^\theta.$$  

(1.2.6)

Eq. (1.2.3) can also be expressed in terms of characteristic functions as

$$X = \xi^i D_i + W^\theta \frac{\partial}{\partial u^\theta} + \sum_{l \geq 1} D_{i_1} \cdots D_{i_l} (W^\theta) \frac{\partial}{\partial u_{i_1 i_2 \cdots i_l}^\theta}.$$

(1.2.7)
The Noether operator associated with a Lie-Bäcklund or generalized operator \( X \) is defined by

\[
N^i = \xi^i + W^\theta \frac{\delta}{\delta u_i^\theta} + \sum_{l \geq 1} D_{i_1} \cdots D_{i_l} (W^\theta) \frac{\delta}{\delta u_{i_1 i_2 \cdots i_l}} , \quad i = 1, \cdots n,
\]

(1.2.8)

where

\[
\frac{\delta}{\delta u_i^\theta} = \frac{\partial}{\partial u_i^\theta} + \sum_{l \geq 1} (-1)^l D_{j_1} \cdots D_{j_l} \frac{\partial}{\partial u_{j_1 \cdots j_l}} , \quad i = 1, \cdots n, \quad \theta = 1, \cdots m.
\]

(1.2.9)

We assume that the system (6.1.2) in the split form can be expressed as

\[
G_\theta = G_\theta^0 + G_\theta^1 = 0 , \quad \theta = 1, \cdots m.
\]

(1.2.10)

As we know \( C^\infty(\mathbb{R}^n) \) is a space of differentiable functions. For simplicity let us take \( A = C^\infty(\mathbb{R}^n) \). Then one can take \( L = L(x, u, u(1), u(2), \cdots, u(l)) \in A \) where \( l \leq p \), is a differential function such that

\[
\frac{\delta L}{\delta u_i^\theta} = f_\beta^\theta G_\beta^1 ,
\]

(1.2.11)

the matrix \( f_\beta^\theta \) is invertible. If \( G_\beta^1 = 0 \) in Eq. (1.2.11) then \( L \) is said to be a standard Lagrangian and \( \frac{\delta L}{\delta u^\theta} = 0 \) are called Euler-Lagrange equations. On the other hand, if \( G_\beta^1 \neq 0 \) in Eq. (1.2.11) then \( L \) is called a partial Lagrangian (see e.g [39]) and Eq. (1.2.11) are called partial Euler-Lagrange equations.

The generalized operator given in Eq. (1.2.3) is known as the partial Noether operator associated with a partial Lagrangian \( L \) if it satisfies

\[
X(L) + LD_i (\xi^i) = W^\theta \frac{\delta L}{\delta u^\theta} + D_i (B^i) ,
\]

(1.2.12)

where \( B^i \)’s are known as the gauge terms and \( W^\theta \) are the characteristic functions defined in Eq. (1.2.6).
Partial Noether theorem

The operator \( X \) in Eq. (1.2.7) is a partial Noether operator of a partial Lagrangian \( L \) corresponding to Eq. (1.2.11) if the characteristic \( W = (W^1, \cdots, W^m) \), \( W^\theta \in A \) of \( X \) is also the characteristic of the conservation laws \( D_i T^i = 0 \), where

\[
T^i = B^i - N^i(L), \quad i = 1, \cdots, n , \tag{1.2.13}
\]

of the Eq. (1.2.11). In Eq. (1.2.13) \( N^i(L) \) is defined by Eq. (1.2.8).

Symmetry, conservation law relation

Suppose \( X \) is a Lie-Bäcklund symmetry of system (6.1.2), then \( X \) is associated with \( T^i \) if

\[
X(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i) = 0, \quad i = 1, \cdots, n , \tag{1.2.14}
\]

where \( T^i \) are the components of the conserved vector of system (6.1.2).

1.2.2 Geometry

Manifolds are useful in almost all the branches of applied sciences. It is the generalization of curves and surfaces in higher dimension.

Manifold

A topological space \( M \) is said to be manifold if neighborhood of \( M \) at every point is topologically the same as the open unit ball in \( \mathbb{R}^n \), which implies \( x \in M, \ \exists \ T_x(M) \) diffeomorphic to \( \mathbb{R}^n \).

Sub-Manifold

A subset \( \hat{M} \) of a manifold \( M \) is said to be sub-manifold which itself has the structure of manifold.
Embedded n-Manifold

A sub-manifold of the type

\[ F(u_1, u_2, \cdots, u_n) = 0 \]  

where \( u_1, u_2, \cdots, u_n \) are variables, is called a surface.

Parametrization

The parameterizations for a surface (1.2.15) contains \( n \) variables and can be written as

\[
\begin{align*}
  u_1 &= u_1(x_1, x_2, \cdots x_{n-1}) \\
  u_2 &= u_2(x_1, x_2, \cdots x_{n-1}) \\
  &\vdots \\
  u_n &= u_n(x_1, x_2, \cdots x_{n-1})
\end{align*}
\]

\[
\sigma(u_1, u_2, \cdots, u_n) = \left( u_1(x_1, x_2, \cdots x_{n-1}), u_2(x_1, x_2, \cdots x_{n-1}), \cdots, u_n(x_1, x_2, \cdots x_{n-1}) \right)
\]

First Fundamental Form (FFF)

The FFF can be written as

\[
ds^2 = E_{11}dx_1^2 + E_{12}dx_1dx_2 + E_{22}dx_2^2 + \cdots + E_{ij}dx_idx_j
\]

where

\[
E_{ij} = \sigma_{x_i}\sigma_{x_j}, \quad i, j = 1, 2, \cdots, n-1, \quad i \neq j
\]

and

\[
\sigma_{x_i} = \frac{\partial \sigma}{\partial x_i}
\]
1.2.3 \( (1 + n) \)-dimensional Laplacian on curved surfaces

In this section, we will compute the \((1 + n)\)-dimensional Laplacian on curved surfaces.

Firstly, let us define a derivative on curved surface which is the generalization of directional derivative i.e.

\[
D_{ij} \mu_j = \frac{\partial \mu_j}{\partial x_i} - \Gamma_{ij}^k \mu_k, \tag{1.2.16}
\]

where \(\Gamma_{ij}^k\) is the Christoffel symbol defined as

\[
\Gamma_{ij}^k = \frac{1}{2} E^{kl} \left( \partial_i E_{jl} + \partial_j E_{il} - \partial_l E_{ij} \right),
\]

where

\[
\partial_i = \frac{\partial}{\partial x_i},
\]

Note that

\[
\Gamma_{jk}^i = \Gamma_{kj}^i
\]

and

\[
E_{ij} E^{jk} = \delta^k_i
\]

where \(\delta\) is the Dirac delta function.

Using Eq. (1.2.16) we can write

\[
D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial u}{\partial x_k}. \tag{1.2.17}
\]

The Laplacian in \(\mathbb{R}^n\), in Cartesian coordinates,

\[
\nabla^2 u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}. \tag{1.2.18}
\]
On a general n-manifold Eq. (1.2.18) becomes the Beltrami operator viz.

\[ \Delta_2 u = E^{ij} \left[ D_{ij} u \right]. \] (1.2.19)

By using Eq. (1.2.17) we get

\[ \Delta_2 u = E^{ij} \left[ \frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma^{k}_{ij} \frac{\partial u}{\partial x_k} \right]. \] (1.2.20)

In two dimensions, Eq. (1.2.20) yields

\[ \Delta_2 u = \frac{1}{E_{11}} \left( \frac{\partial^2 u}{\partial x_1^2} \right) + \frac{1}{E_{22}} \left( \frac{\partial^2 u}{\partial x_2^2} \right) - \left[ \Gamma^{1}_{11} \frac{E_{11}}{E_{11}} + \Gamma^{1}_{22} \frac{E_{11}}{E_{22}} \right] \frac{\partial u}{\partial x_1} - \left[ \Gamma^{2}_{11} \frac{E_{11}}{E_{11}} + \Gamma^{2}_{22} \frac{E_{22}}{E_{22}} \right] \frac{\partial u}{\partial x_2}. \] (1.2.21)

As we know that

\[ \Gamma^{i}_{ii} = \left( \frac{1}{2E_{ii}} \right) \frac{\partial E_{ii}}{\partial x_i} \] (1.2.22)

\[ \Gamma^{j}_{ii} = \left( -\frac{1}{2E_{jj}} \right) \frac{\partial E_{ij}}{\partial x_j}, \quad i \neq j. \] (1.2.23)

Using the identities (1.2.22) and (1.2.23) in Eq. (1.2.21) we obtain

\[ \Delta_2 u = \frac{u_{x_1x_1}}{E_{11}} + \frac{u_{x_2x_2}}{E_{22}} - \left( \frac{(E_{11})_{x_1}}{2E_{11}^2} - \frac{(E_{22})_{x_1}}{2E_{22}E_{11}} \right) u_{x_1} - \left( \frac{(E_{22})_{x_2}}{2E_{22}^2} - \frac{(E_{11})_{x_2}}{2E_{22}E_{11}} \right) u_{x_2}. \] (1.2.24)

Similarly for \( i, j = 1, 2, 3, \ldots, n \),

\[ \Delta_2 u = \frac{u_{x_1x_1}}{E_{11}} + \frac{u_{x_2x_2}}{E_{22}} + \frac{u_{x_3x_3}}{E_{33}} - \left( \frac{(E_{11})_{x_1}}{2E_{11}^2} - \frac{(E_{22})_{x_1}}{2E_{11}E_{22}} - \frac{(E_{33})_{x_1}}{2E_{22}E_{33}} \right) u_{x_1} - \left( \frac{(E_{22})_{x_2}}{2E_{22}^2} - \frac{(E_{33})_{x_2}}{2E_{22}E_{33}} - \frac{(E_{11})_{x_2}}{2E_{11}E_{22}} \right) u_{x_2} - \left( \frac{(E_{33})_{x_3}}{2E_{33}^2} - \frac{(E_{11})_{x_3}}{2E_{11}E_{33}} - \frac{(E_{22})_{x_3}}{2E_{22}E_{33}} \right) u_{x_3}. \] (1.2.25)

The extension for \( i, j = 1, 2, 3, \ldots, n \) is

\[ \Delta_2 u = \sum_{i=1}^{n} \frac{u_{x_ix_i}}{E_{ii}} + \sum_{i=1}^{n} \left( \frac{(E_{ii})_{x_i}}{2E_{ii}^2} - \sum_{j=1, i \neq j}^{n} \frac{(E_{jj})_{x_i}}{2E_{ii}E_{jj}} \right) u_{x_i}. \] (1.2.26)
1.3 Outline of the thesis

This thesis has been organized as follows: In Chapter 1, introduction and preliminaries have been presented. Chapter 2 is devoted to the conservation laws of the \((1 + n)\)-dimensional heat equation on curved surfaces. In Chapter 3, the derivation of conserved quantities of the \((1 + 2)\)-dimensional wave equation on different surfaces is discussed. Chapter 4 deals with conserved quantities of the \((1 + n)\)-dimensional heat equation. The effect of a background metric on the symmetries of the nonlinear \((1 + 2)\)-dimensional heat equation is discussed in Chapter 5, then all possible reduced equations up to conjugacy classes of the time dependent Ginzburg-Landau model are calculated and then the \((1 + 2)\)-dimensional heat equation on the torus is considered, subalgebras are classified up to conjugacy classes and finally some solutions are calculated. A complete group classification of the \((1 + n)\)-dimensional Klein-Gordon equation and nonlinear \((1 + 2)\)-dimensional wave equation on the sphere and torus is presented in Chapter 6. All possible reduced equations of the \((1 + 2)\)-dimensional wave equation on a sphere and a torus are calculated via similarity variables in Chapter 6.
Chapter 2

Conservation Laws for Heat Equation on Curved Surfaces

In this chapter derivation of conservation laws for the \((1 + n)\)-dimensional heat equation on curved surfaces is considered using the partial Noether approach. The partial Noether theorem for the general case \(n \geq 2\) gives infinitely many conservation laws. The analysis is then applied to the heat equation on the cone, sphere, torus and plane. Moreover, we use the symmetry/conservation laws relation and associated conservation laws are deduced.

2.1 Introduction

Many physical models are analyzed by Lie symmetry groups assuming a flat background [11, 33, 34] and non flat surfaces [55, 59]. The conservation laws for the heat equation in higher dimensions haven’t yet been discussed and are the subject of this chapter. The importance of conservation laws has been discussed in Chapter 1.

The \((1 + n)\)-dimensional heat equation,

\[
\frac{u_t}{E_{ii}} = \sum_{i=1}^{n} \frac{u_{xi}}{E_{ii}} + \sum_{i=1}^{n} \left( \frac{(E_{ii})_{x_i}}{2E_{ii}^2} - \frac{\sum_{j=1,i\neq j}^{n} (E_{jj})_{x_i}}{2E_{ii}E_{jj}} \right) u_{x_i}, \tag{2.1.1}
\]
where

\[ ds^2 = E_{11}dx_1^2 + E_{12}dx_1^2 + E_{22}dx_2^2 + \cdots \]

is the FFF of the n-dimensional surface.

Eq. (2.1.1) does not admit a standard Lagrangian but it does have a partial Lagrangian. The conservation laws can not be constructed with the help of Noether’s approach as it relies on the existence of a standard Lagrangian but the partial Noether approach can still be used. Hence it is of great interest to study Eq. (2.1.1) with respect to its conservation laws. Moreover we derive associated conservation laws by using the symmetry conservation law relation given in [38].

The layout of this chapter is as follows. In Section 2.2, the conditions on gauge terms and conserved vectors are derived for the (1 + n)-dimensional heat equation. Section 2.3 gives the conservation laws for the heat equation on the cone, sphere, torus, hyper-sphere and on hyper-plane.

## 2.2 Partial Noether operators of (1 + n)-dimensional heat equation

In this section, we will compute the partial Noether operators of Eq. (2.1.1) for n = 2 and n = 3 and then we will generalize the results for n ≥ 2.

### 2.2.1 (1 + 2)-dimensional heat equation

For n = 2 the Eq. (2.1.1) reduces to

\[
u_t + \frac{u_{x_1 x_1}}{E_{11}} + \frac{u_{x_2 x_2}}{E_{22}} - \left( \frac{(E_{11})_{x_1}}{2E_{11}^2} - \frac{(E_{22})_{x_1}}{2E_{11}E_{22}} \right) u_{x_1} - \left( \frac{(E_{22})_{x_2}}{2E_{22}^2} - \frac{(E_{11})_{x_2}}{2E_{11}E_{22}} \right) u_{x_2}, \quad (2.2.1)
\]
which admits the partial Lagrangian

\[
L = -\frac{u_{x_1}^2}{2E_{11}} - \frac{u_{x_2}^2}{2E_{22}}
\]  

(2.2.2)

and associated partial Euler-Lagrange equation is

\[
\frac{\delta L}{\delta u} = u_t - \left( \frac{(E_{11})_{x_1} u_{x_1}^2}{2E_{11}^2} + \frac{(E_{22})_{x_1} u_{x_1}^2}{2E_{22}^2} \right) u_{x_1} - \left( \frac{(E_{11})_{x_2} u_{x_2}^2}{2E_{11}^2} + \frac{(E_{22})_{x_2} u_{x_2}^2}{2E_{22}^2} \right) u_{x_2}.
\]  

(2.2.3)

Suppose

\[
X = \tau \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial x_1} + \xi^2 \frac{\partial}{\partial x_2} + \eta \frac{\partial}{\partial u}
\]

is the infinitesimal operator of Eq. (2.2.1). Substituting in Eq. (1.2.12) yields

\[
\left( \frac{(E_{11})_{x_1} u_{x_1}^2}{2E_{11}^2} + \frac{(E_{22})_{x_1} u_{x_1}^2}{2E_{22}^2} \right) \xi^1 + \left( \frac{(E_{11})_{x_2} u_{x_2}^2}{2E_{11}^2} + \frac{(E_{22})_{x_2} u_{x_2}^2}{2E_{22}^2} \right) \xi^2 - \frac{\eta u_{x_1}}{E_{11}}
\]

\[-\frac{\eta u_{x_2}}{E_{22}} - \left( \frac{u_{x_1}^2}{2E_{11}} + \frac{u_{x_2}^2}{2E_{22}} \right) \left( \tau_t + \tau u_t + \xi^1_{x_1} u_{x_1} + \xi^2_{x_2} u_{x_2} \right) = \left( \eta - \tau u_t \right)
\]

\[-\xi^1 u_{x_1} - \xi^2 u_{x_2} \left[ u_t - \left( \frac{(E_{11})_{x_1}}{2E_{11}^2} + \frac{(E_{22})_{x_1}}{2E_{11}E_{22}} \right) u_{x_1} - \left( \frac{(E_{11})_{x_2}}{2E_{11}^2} + \frac{(E_{22})_{x_2}}{2E_{11}E_{22}} \right) u_{x_2} \right]
\]

\[+ B_1^1 + B_1^2 u_t + B_2^2 u_{x_1} + B_3^2 u_{x_2} + B_4^3 u_{x_2}.
\]

(2.2.4)

From Eq. (2.2.4) we find that

\[
\tau = 0, \quad \xi^1 = 0, \quad \xi^2 = 0
\]

(2.2.5)

with

\[
B_1^1 = -\eta u + \alpha(t, x_1, x_2),
\]

(2.2.5)

\[
B_2^2 = \left[ \left( \frac{(E_{11})_{x_1}}{2E_{11}^2} + \frac{(E_{22})_{x_1}}{2E_{11}E_{22}} \right) \eta - \frac{1}{E_{11}} \eta u_{x_1} \right] u + \beta(t, x_1, x_2),
\]

(2.2.6)
\[ B^3 = \left[ \left( \frac{(E_{22})_x}{2E_{22}} + \frac{(E_{11})_x}{2E_{11}E_{22}} \right) \eta - \frac{1}{E_{22}} \eta_x \right] u + \gamma(t, x_1, x_2) \]  

(2.2.7)

and

\[ B^1_t + B^2_{x_1} + B^3_{x_2} = 0. \]  

(2.2.8)

After substituting Eqs. (2.2.5 – 2.2.7) in Eq. (2.2.8), we obtain

\[ \eta_t + \frac{\eta_{x_1}x_1}{E_{11}} + \frac{\eta_{x_2}x_2}{E_{22}} - \left( \frac{3(E_{11})_x}{2E_{11}} + \frac{(E_{22})_x}{2E_{11}E_{22}} \right) \eta_{x_1} - \left( \frac{3(E_{22})_x}{2E_{22}} + \frac{(E_{11})_x}{2E_{11}E_{22}} \right) \eta_{x_2} \]

\[ - \left( \frac{(E_{11})_x}{2E^2_{11}} + \frac{(E_{22})_x}{2E^2_{22}} \right) \eta - \left[ \frac{(E_{11})_x^2}{E_{11}^2} - \frac{(E_{22})_x^2}{E_{22}^2} + \frac{(E_{11})_{x_2}x_2}{E_{11}E_{22}} + \frac{(E_{22})_{x_1}x_1}{E_{11}E_{22}} \right] \eta = 0, \]  

(2.2.9)

\[ \alpha_t + \beta_{x_1} + \gamma_{x_2} = 0. \]  

(2.2.10)

Without loss of generality, one can take

\[ \alpha = 0, \quad \beta = 0, \quad \gamma = 0. \]

Now by using partial Noether theorem (see [39]), the conserved vectors in terms of the coefficients of the FFF are

\[ T^1 = -\eta u, \]

\[ T^2 = \left[ \left( \frac{(E_{11})_x}{2E^2_{11}} + \frac{(E_{22})_x}{2E_{11}E_{22}} \right) \eta - \frac{\eta_{x_1}}{E_{11}} \right] u + \frac{\eta_{x_1}u_{x_1}}{E_{11}}, \]  

(2.2.11)

\[ T^3 = \left[ \left( \frac{(E_{22})_x}{2E^2_{22}} + \frac{(E_{11})_x}{2E_{11}E_{22}} \right) \eta - \frac{\eta_{x_2}}{E_{22}} \right] u + \frac{\eta_{x_2}u_{x_2}}{E_{22}}, \]

where \( \eta \) is a solution of Eq. (2.2.9). For the (1 + 2)-dimensional heat equation, we obtain an infinite number of conservation laws.
### 2.2.2 \((1+3)\)-dimensional heat equation

Eq. (2.1.1) for \((1+3)\)-dimensional heat equation case be expressed as

\[
u_t = \frac{u_{x_1 x_1}}{E_{11}} + \frac{u_{x_2 x_2}}{E_{22}} + \frac{u_{x_3 x_3}}{E_{33}} - \left(\frac{(E_{11})_{x_1}}{2E_{11}^2} - \frac{(E_{22})_{x_1}}{2E_{11} E_{22}} - \frac{(E_{33})_{x_1}}{2E_{11} E_{33}}\right) u_{x_1} - \left(\frac{(E_{22})_{x_2}}{2E_{22}^2}\right)
\]

\[-\left(\frac{(E_{11})_{x_2}}{2E_{11} E_{22}} - \frac{(E_{33})_{x_2}}{2E_{22} E_{33}}\right) u_{x_2} - \left(\frac{(E_{33})_{x_3}}{2E_{33}^2} - \frac{(E_{11})_{x_3}}{2E_{11} E_{33}} - \frac{(E_{22})_{x_3}}{2E_{22} E_{33}}\right) u_{x_3}\].

(2.2.12)

The partial Lagrangian for Eq. (2.2.12) is

\[
L = -\frac{u_{x_1}^2}{2E_{11}} - \frac{u_{x_2}^2}{2E_{22}} - \frac{u_{x_3}^2}{2E_{33}}
\]

(2.2.13)

and the corresponding partial Euler-Lagrange equation is

\[
\frac{\delta L}{\delta u} = u_t - \left(\frac{(E_{11})_{x_1}}{2E_{11}^2} + \frac{(E_{22})_{x_1}}{2E_{11} E_{22}} + \frac{(E_{33})_{x_1}}{2E_{11} E_{33}}\right) u_{x_1} - \left(\frac{(E_{22})_{x_2}}{2E_{22}^2} + \frac{(E_{11})_{x_2}}{2E_{11} E_{22}} + \frac{(E_{33})_{x_2}}{2E_{22} E_{33}}\right) u_{x_2}
\]

\[-\left(\frac{(E_{33})_{x_3}}{2E_{33}^2} + \frac{(E_{11})_{x_3}}{2E_{11} E_{33}} + \frac{(E_{22})_{x_3}}{2E_{22} E_{33}}\right) u_{x_3}.

(2.2.14)

Let

\[
X = \tau \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial x_2} + \xi^2 \frac{\partial}{\partial x_2} + \xi^3 \frac{\partial}{\partial x_3} + \eta \frac{\partial}{\partial u}
\]

be the infinitesimal operator of Eq. (2.2.12). Finally partial Noether operator determining equation (1.2.12) gives rise to

\[
\tau = 0, \quad \xi^1 = 0, \quad \xi^2 = 0, \quad \xi^3 = 0
\]

and

\[
B^1 = -\eta u + \alpha(t, x_1, x_2),
\]

(2.2.15)

\[
B^2 = \left[\left(\frac{(E_{11})_{x_1}}{2E_{11}^2} + \frac{(E_{22})_{x_1}}{2E_{11} E_{22}} + \frac{(E_{33})_{x_1}}{2E_{11} E_{33}}\right) \eta - \frac{\eta x_1}{E_{11}}\right] u + \beta(t, x_1, x_2),
\]

(2.2.16)
\[ B^3 = \left[ \frac{(E_{22})_{x_2}}{2E_{22}} + \frac{(E_{11})_{x_2}}{2E_{11}E_{22}} + \frac{(E_{33})_{x_2}}{2E_{22}E_{33}} \right] \eta - \frac{\eta_{x_2}}{E_{22}} u + \gamma(t, x_1, x_2), \]  
\[ \text{(2.2.17)} \]

\[ B^4 = \left[ \frac{(E_{33})_{x_3}}{2E_{33}} + \frac{(E_{11})_{x_3}}{2E_{11}E_{33}} + \frac{(E_{22})_{x_3}}{2E_{22}E_{33}} \right] \eta - \frac{\eta_{x_3}}{E_{33}} u + \delta(t, x_1, x_2), \]  
\[ \text{(2.2.18)} \]

\[ B_t^1 + B_{x_1}^2 + B_{x_2}^3 + B_{x_3}^4 = 0. \]  
\[ \text{(2.2.19)} \]

Without loss of generality, we use

\[ \alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0. \]

After substituting Eqs. (2.2.15-2.2.18) in Eq. (2.2.19) the following condition on \( \eta \) is derived

\[ -\eta_t - \frac{\eta_{x_1x_1}}{E_{11}} - \frac{\eta_{x_2x_2}}{E_{22}} - \frac{\eta_{x_3x_3}}{E_{33}} + \left( \frac{3(E_{11})_{x_1}}{2E_{11}^2} + \frac{(E_{22})_{x_1}}{2E_{11}E_{22}} + \frac{(E_{33})_{x_1}}{2E_{22}E_{33}} \right) \eta_{x_1} + \left( \frac{3(E_{22})_{x_2}}{2E_{22}^2} + \frac{(E_{11})_{x_2}}{2E_{11}E_{22}} + \frac{(E_{33})_{x_2}}{2E_{22}E_{33}} \right) \eta_{x_2} + \left( \frac{3(E_{33})_{x_3}}{2E_{33}^2} + \frac{(E_{22})_{x_3}}{2E_{22}E_{33}} + \frac{(E_{11})_{x_3}}{2E_{11}E_{33}} \right) \eta_{x_3} \]

\[ + \frac{(E_{11})_{x_1}}{2E_{11}E_{22}} + \frac{(E_{33})_{x_1}}{2E_{11}E_{33}} + \frac{(E_{22})_{x_2}}{2E_{22}^2} + \frac{(E_{22})_{x_2}}{2E_{22}E_{33}} + \frac{(E_{33})_{x_2}}{2E_{22}E_{33}} + \frac{(E_{33})_{x_2}}{2E_{11}E_{33}} + \frac{(E_{11})_{x_3}}{2E_{11}E_{33}} \]

\[ - \frac{(E_{22})_{x_3}}{2E_{22}E_{33}} - \frac{(E_{11})_{x_1}}{E_{11}^2} - \frac{(E_{22})_{x_1}}{E_{22}^2} - \frac{(E_{33})_{x_1}}{E_{33}^2} - \frac{(E_{22})_{x_2}}{E_{22}^2} - \frac{(E_{11})_{x_2}}{E_{11}^2} - \frac{(E_{33})_{x_2}}{E_{33}^2} - \frac{(E_{22})_{x_3}}{E_{22}^2} - \frac{(E_{33})_{x_3}}{E_{33}^2} \]

\[ - \frac{(E_{33})_{x_3}}{E_{33}^2} - \frac{(E_{22})_{x_3}}{E_{22}^2} - \frac{(E_{33})_{x_3}}{E_{33}^2} - \frac{(E_{22})_{x_3}}{E_{22}^2} - \frac{(E_{33})_{x_3}}{E_{33}^2} - \frac{(E_{22})_{x_3}}{E_{22}^2} - \frac{(E_{33})_{x_3}}{E_{33}^2} \]

\[ = 0. \]  
\[ \text{(2.2.20)} \]

Now by using partial Noether theorem, the following components of conserved vector are obtained

\[ T^1 = -\eta u, \]
\[ T^2 = \left[ \left( \frac{(E_{11})_{x_1}}{2E_{11}^2} + \frac{(E_{22})_{x_1}}{2E_{11}E_{22}} + \frac{(E_{33})_{x_1}}{2E_{11}E_{33}} \right) \eta - \frac{\eta_{x_1}}{E_{11}} \right] u + \frac{\eta u_{x_1}}{E_{11}}, \quad (2.2.21) \]

\[ T^3 = \left[ \left( \frac{(E_{22})_{x_2}}{2E_{22}^2} + \frac{(E_{11})_{x_2}}{2E_{11}E_{22}} + \frac{(E_{33})_{x_2}}{2E_{22}E_{33}} \right) \eta - \frac{\eta_{x_2}}{E_{22}} \right] u + \frac{\eta u_{x_2}}{E_{22}}, \]

\[ T^4 = \left[ \left( \frac{(E_{33})_{x_3}}{2E_{33}^2} + \frac{(E_{11})_{x_3}}{2E_{11}E_{33}} + \frac{(E_{22})_{x_3}}{2E_{22}E_{33}} \right) \eta - \frac{\eta_{x_3}}{E_{33}} \right] u + \frac{\eta u_{x_3}}{E_{33}}, \]

where \( \eta \) is the solution of Eq. (2.2.20). An infinite number of conservation laws are derived for the \((1 + 3)\)-dimensional heat equation.

### 2.2.3 \((1 + n)\)-dimensional heat equation

By following the same procedure as we have done for Eqs. (2.2.1) and (2.2.12), now we will generalize the results for \((1 + n)\)-dimensional heat equation. Eq. (2.1.1) does not admit a standard Lagrangian but it does have a partial Lagrangian

\[ L = -\sum_{i=1}^{n} \frac{u_{x_i}^2}{E_{ii}}, \quad (2.2.22) \]

and the partial Euler-Lagrangian equation is

\[ \frac{\delta L}{\delta u} = u_t - \sum_{i=1}^{n} \left( \frac{(E_{ii})_{x_i}}{2E_{ii}^2} + \frac{\sum_{j=1}^{n} \eta_{x_j}}{2E_{ii}E_{jj}} \right) u_{x_i}, \quad (2.2.23) \]

The infinitesimal operator

\[ X = \tau \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial x_1} + \xi^2 \frac{\partial}{\partial x_2} + \cdots + \xi^n \frac{\partial}{\partial x_n} + \eta \frac{\partial}{\partial u} \]

is a partial Noether operator corresponding to a partial Lagrangian (2.2.22) if it satisfies Eq. (1.2.12) which finally yields

\[ \tau = 0, \quad \xi^i = 0, \quad i = 1, \cdots, n, \]

\[ B^1 = -\eta u, \quad (2.2.24) \]
\[ B^{i+1} = \left[ \left( \frac{(E_{ii})_{x_i}}{2E_{ii}^2} + \sum_{j=1, j \neq i}^{n} \frac{(E_{jj})_{x_i}}{2E_{ii}E_{jj}} \right) \eta - \frac{\eta x_i}{E_{ii}} \right] u, \quad i = 1, \cdots, n, \]  

(2.2.25)

where

\[ (B^1)_t + \sum_{i=1}^{n} (B^{i+1})_{x_i} = 0. \]

The conserved vectors for \((1 + n)\)-dimensional heat equation are

\[ T^1 = -\eta u, \]  

(2.2.26)

\[ T^{i+1} = \left[ \left( \frac{(E_{ii})_{x_i}}{2E_{ii}^2} + \sum_{j=1, j \neq i}^{n} \frac{(E_{jj})_{x_i}}{2E_{ii}E_{jj}} \right) \eta - \frac{\eta x_i}{E_{ii}} \right] u + \frac{\eta u x_i}{E_{ii}}, \quad i = 1, \cdots, n, \]  

(2.2.27)

where \( \eta \) satisfies

\[ -\eta_t + \sum_{j=1}^{n} \sum_{i=1}^{n} \left[ -\frac{\eta x_i x_i}{E_{ii}} + \left( \frac{3(E_{ii})_{x_i}}{2E_{ii}^2} + \frac{(E_{jj})_{x_i}}{2E_{ii}E_{jj}} \right) \eta x_i + \left( \frac{(E_{ii})_{x_i x_i}}{2E_{ii}^2} + \frac{(E_{jj})_{x_i x_i}}{2E_{ii}E_{jj}} \right) \eta \right] = 0, \quad i \neq j. \]  

(2.2.28)

For the \((1 + n)\)-dimensional heat equation (for \( n \geq 2 \)), an infinite number of conserved vectors are obtained.

### 2.3 Conservation laws for heat equation on different curved surfaces

In this section, we will calculate the conservation laws for the heat equation on different surfaces by using the results obtained in Section 2.2, then we will use the relationship between symmetries and conservation laws to generate new conserved vectors. Lie point symmetries in this section are calculated by using the standard method given by Lie [11, 50, 57].
2.3.1 Conservation laws for heat equation on cone

The coefficients of the FFF for the heat equation on a cone are

\[ E_{11} = (x_2)^2, \quad E_{22} = 2. \tag{2.3.1} \]

After substituting these, Eq. (2.1.1) becomes

\[ u_t = \frac{u_{x_1x_1}}{(x_2)^2} + \frac{u_{x_2x_2}}{2} + \frac{u_{x_2}}{2x_2}. \tag{2.3.2} \]

The substitution of coefficients of FFF in Eq. (2.2.11) followed by some simple manipulations yields the following conserved vectors

\[ T^1 = -\mu u, \]

\[ T^2 = \frac{1}{(x_2)^2} \left( \mu u_{x_1} - \mu_{x_1} u \right), \tag{2.3.3} \]

\[ T^3 = \frac{1}{2} \left( \mu u_{x_2} - \mu_{x_2} u \right) + \frac{\mu u}{2x_2}, \]

where \( \mu \) satisfies

\[ \mu_t + \frac{\mu_{x_1x_1}}{(x_2)^2} + \frac{\mu_{x_2x_2}}{2} - \frac{\mu_{x_2}}{2x_2} + \frac{\mu}{2(x_2)^2} = 0. \tag{2.3.4} \]

Hence there are infinitely many conservation laws for Eq. (2.3.2). By using relation (1.2.14) we can associate Lie point symmetries with conserved vectors in Eq. (2.3.3) and we will compute new conserved vectors. The Lie symmetries of Eq. (2.3.2) are

\[ Y_1 = -\frac{2}{\sqrt{2}x_2} \sin\left(\frac{x_1}{\sqrt{2}}\right) \frac{\partial}{\partial x_1} + \cos\left(\frac{x_1}{\sqrt{2}}\right) \frac{\partial}{\partial x_2}, \quad Y_2 = \frac{2}{\sqrt{2}x_2} \cos\left(\frac{x_1}{\sqrt{2}}\right) \frac{\partial}{\partial x_1} + \sin\left(\frac{x_1}{\sqrt{2}}\right) \frac{\partial}{\partial x_2}, \]

\[ Y_3 = \frac{\partial}{\partial x_1}, \quad Y_4 = t^2 \frac{\partial}{\partial t} + tx_2 \frac{\partial}{\partial x_2} - \left(\frac{y^2}{4} + \frac{t}{2}\right) u \frac{\partial}{\partial u}, \tag{2.3.5} \]

\[ Y_5 = 2t \frac{\partial}{\partial t}, \quad Y_6 = u \frac{\partial}{\partial u}, \quad Y_7 = \frac{\partial}{\partial t}. \]
Translational symmetries associated with the conservation laws

In this section, we will use relation (1.2.14) and associate symmetry generators with the conserved vectors. Consider first the translational symmetry generators $Y_3$ and $Y_7$ admitted by Eq. (2.3.2).

(I): The symmetry condition (1.2.14) with the use of $< Y_3 >$ and conserved vectors from Eq. (2.3.3) yields

$$T^1 = -\mu u,$$

$$T^2 = \frac{\mu u x_1}{(x_2)^2},$$

$$T^3 = \frac{1}{2} \left( \mu u x_2 - \mu x_2 u \right) + \frac{\mu u}{2x_2},$$

where $\mu$ is a solution of the equations

$$\mu_t + \frac{\mu x_1 x_2}{2} - \frac{\mu x_2}{2x_2} + \frac{\mu}{2(x_2)^2} = 0, \quad \mu u x_1 = 0. \quad (2.3.7)$$

(II): One can easily construct the conserved vectors associated with the translational symmetry $< Y_7 >$:

$$T^1 = -\mu u,$$

$$T^2 = \frac{1}{(x_2)^2} \left( \mu u x_1 - \mu x_1 u \right),$$

$$T^3 = \frac{1}{2} \left( \mu u x_2 - \mu x_2 u \right) + \frac{\mu u}{2x_2},$$

where $\mu$ is the solution of

$$\mu_t = 0, \quad \frac{\mu x_1 x_1}{(x_2)^2} + \frac{\mu x_2 x_2}{2} - \frac{\mu x_2}{2x_2} + \frac{\mu}{2(x_2)^2} = 0. \quad (2.3.9)$$
In above two cases we get a pair of PDEs which govern the unknown parameter $\mu$. Both equations in that pair have less independent variables as compared to Eq. (2.3.4).

(III): Association of a two dimensional algebra reduces the Eq. (2.3.4) into an ODE e.g. if we consider the two dimensional algebra $< Y_3, Y_7 >$ then we get following components of conserved vectors

\[
T^1 = -\mu u, \\
T^2 = \frac{\mu u x_1}{(x_2)^2}, \\
T^3 = \frac{1}{2} \left( \mu u x_2 - \mu x_2 u \right) + \frac{\mu u}{2x_2}
\]

where

\[
\mu = ax_2 + bx_2 \ln x_2,
\]

in which $a$ and $b$ are real constants.

The remaining two or more dimensional algebras neither reduce the dimension of Eq. (2.3.4) nor give the nontrivial results.

### 2.3.2 Conservation laws for heat equation on a sphere

The coefficients of the FFF of sphere are

\[
E_{11} = 1, \quad E_{22} = \sin^2 x_1.
\]

With the substitution of these into Eq. (2.1.1), the heat equation on a sphere becomes

\[
u_t = u x_1 x_1 + \cot x_1 u x_1 + \csc^2 x_1 u x_2 x_2.
\]

Now the conserved vectors corresponding to the Eq. (2.3.12) are

\[
T^1 = -\nu u,
\]
\[ T^2 = \left( \nu \cot x_1 - \nu x_1 \right) u + \nu u_x, \quad (2.3.13) \]
\[ T^3 = -\csc^2 x_1 \left( \nu x_2 u - \nu u_x \right), \]

where \( \nu \) satisfies the following equation
\[ \nu_t + \nu x_1 + \csc^2 x_1 \nu x_2 x_2 - \cot x_1 \nu x_1 + \nu \csc^2 x_1 \nu = 0. \quad (2.3.14) \]

Lie point symmetries of Eq. (2.3.12) are
\[ Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = u \frac{\partial}{\partial u}, \quad Y_3 = \cos x_2 \frac{\partial}{\partial x_1} - \cot x_1 \sin x_2 \frac{\partial}{\partial x_2}, \]
\[ Y_4 = \sin x_2 \frac{\partial}{\partial x_1} + \cot x_1 \cos x_2 \frac{\partial}{\partial x_2}, \quad Y_5 = \frac{\partial}{\partial x_2}. \quad (2.3.15) \]

**Translational symmetries associated with the conservation laws**

In this section, we will associate the algebras of translational symmetries of Eq. (2.3.12) with the components of conserved vectors in Eq. (2.3.13) by using the relation (1.2.14).

(I): The conserved vectors associated with \( < Y_1 > \) are
\[ T^1 = -\nu u, \]
\[ T^2 = \left( \nu \cot x_1 - \nu x_1 \right) u + \nu u_x, \quad (2.3.16) \]
\[ T^3 = -\csc^2 x_1 \left( \nu x_2 u - \nu u_x \right), \]

where \( \nu \) satisfies
\[ \nu_{x_1} x_1 + \csc^2 x_1 \nu x_2 x_2 - \cot x_1 \nu x_1 + \csc^2 x_1 \nu = 0, \quad \nu_t = 0. \quad (2.3.17) \]

(II): By associating \( < Y_5 > \) with the conserved vectors (2.3.13) one can obtain
\[ T^1 = -\nu u, \]
\[ T^2 = \left( \nu \cot x_1 - \nu x_1 \right) u + \nu u_{x_1}, \]  
\[ T^3 = \nu u_{x_2} \csc^2 x_1, \]

where \( \nu \) satisfies the equations

\[ \nu_t + \nu_{x_1 x_1} - \nu_{x_1} \cot x_1 + \nu \csc^2 x_1 = 0, \quad \nu_{x_2} = 0. \]

(III): The two dimensional algebra \( \langle Y_1, Y_5 \rangle \) permits a double reduction and Eq. (2.3.14) reduces to an ODE. Hence the associated components of conserved vectors in this case are

\[ T^1 = -\nu u, \]

\[ T^2 = \left( \nu \cot x_1 - \nu x_1 \right) u + \nu u_{x_1}, \]

\[ T^3 = \nu u_{x_2} \csc^2 x_1, \]

where

\[ \nu = \left[ a + b \sin^{-1}(\csc x_1) \right] \sin x_1 \]

with \( a, b \in \mathbb{R} \).

The rest of the two or more dimensional algebras neither reduce the dimension of Eq. (2.3.14) nor give the nontrivial results.

### 2.3.3 Conservation laws for heat equation on a torus

The heat equation on a torus is obtained by substituting

\[ E_{11} = (1 + \cos x_2)^2, \quad E_{22} = 1 \]
into Eq. (2.1.1), i.e.

\[ u_t = \frac{u_{x_1 x_1}}{(1 + \cos x_2)^2} - \frac{\sin x_2 u_{x_2}}{(1 + \cos x_2)} + u_{x_2 x_2}. \tag{2.3.22} \]

The conservation laws for Eq. (2.3.22) are

\[ T^1 = -\theta u, \]

\[ T^2 = \frac{1}{(1 + \cos x_2)^2} (\theta u_{x_1} - \theta x_1 u), \tag{2.3.23} \]

\[ T^3 = \left[ \left( \frac{\sin x_2}{1 + \cos x_2} \right) \theta - \theta x_2 \right] u + \theta u_{x_2}, \]

where \( \theta \) satisfies

\[ \theta_t + \frac{\theta_{x_1 x_1}}{(1 + \cos x_2)^2} + \theta_{x_2 x_2} + \left( \frac{\sin x_2}{1 + \cos x_2} \right) \theta_{x_2} + \frac{\theta}{1 + \cos x_2} = 0. \tag{2.3.24} \]

The following are the infinitesimal generators for Eq. (2.3.22)

\[ Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = u \frac{\partial}{\partial u}, \quad Y_3 = \cos x_1 \frac{\partial}{\partial x_2} + \frac{1}{2} \tan \left( \frac{x_2}{2} \right) \sin x_1 \frac{\partial}{\partial x_1}, \]

\[ Y_4 = \sin x_1 \frac{\partial}{\partial x_2} - \frac{1}{2} \tan \left( \frac{x_2}{2} \right) \cos x_1 \frac{\partial}{\partial x_1}, \quad Y_5 = \frac{\partial}{\partial x_1}. \tag{2.3.25} \]

**Translational symmetries associated with the conservation laws**

(I): Associated components of conserved vector with the translational symmetry \( < Y_1 > \) are given below

\[ T^1 = -\theta u, \]

\[ T^2 = \frac{1}{(1 + \cos x_2)^2} (\theta u_{x_1} - \theta x_1 u), \tag{2.3.26} \]

\[ T^3 = \left[ \left( \frac{\sin x_2}{1 + \cos x_2} \right) \theta - \theta x_2 \right] u + \theta u_{x_2}, \]
where $\theta$ satisfies
\[
\frac{\theta_{x_1}}{(1 + \cos x_2)^2} + \theta_{x_2} + \left( \frac{\sin x_2}{1 + \cos x_2} \right) \theta_{x_2} + \frac{\theta}{1 + \cos x_2} = 0, \quad \theta_t = 0.
\] (2.3.27)

(II): For translational symmetry $< Y_5 >$ we obtain the components of conserved vectors which satisfy the relation (1.2.14)

\[
T^1 = -\theta u,
\]
\[
T^2 = \frac{\theta u_{x_1}}{(1 + \cos x_2)^2},
\] (2.3.28)
\[
T^3 = \left[ \left( \frac{\sin x_2}{1 + \cos x_2} \right) \theta - \theta_{x_2} \right] u + \theta u_{x_2},
\]
where $\theta$ satisfies
\[
\theta_t + \theta_{x_2} + \frac{\sin x_2}{(1 + \cos x_2)} \theta_{x_2} + \left( \frac{\theta}{1 + \cos x_2} \right) = 0, \quad \theta_{x_1} = 0.
\] (2.3.29)

(III): The two dimensional algebra $< Y_1, Y_5 >$ reduces Eq. (2.3.24) to an ODE. Hence the associated components of conserved vectors are

\[
T^1 = -\theta u,
\]
\[
T^2 = \frac{\theta u_{x_1}}{(1 + \cos x_2)^2},
\] (2.3.30)
\[
T^3 = \left[ \left( \frac{\sin x_2}{1 + \cos x_2} \right) \theta - \theta_{x_2} \right] u + \theta u_{x_2},
\]
where
\[
\theta = a \sin x_2 - b \sin x_2 (\cot x_2 + \csc x_2).
\] (2.3.31)

The remaining algebras neither reduce the dimension of Eq. (2.3.24) nor give non-trivial results.
2.3.4 Conservation laws for heat equation on 2-dimensional flat surface

The coefficients of FFF for 2-dimensional flat surface are

\[ E_{11} = 1, \quad E_{22} = 1. \]  (2.3.32)

Substituting of these in Eq. (2.1.1) results in

\[ u_t = u_{x_1 x_1} + u_{x_2 x_2}. \]  (2.3.33)

After plug in the values of coefficients of FFF in Eq. (2.2.9) and Eq. (2.2.11) the components of the conserved vectors for Eq. (2.3.33) are

\[ T^1 = -\eta u, \]

\[ T^2 = \eta u_{x_1} - \eta_{x_1} u, \]  (2.3.34)

\[ T^3 = \eta u_{x_2} - \eta_{x_2} u, \]

where

\[ \eta_t + \eta_{x_1 x_1} + \eta_{x_2 x_2} = 0. \]  (2.3.35)

Hence there are infinite many conservation laws for the Eq. (2.3.33). Lie point symmetries of the Eq. (2.3.33) are calculated in [34, 50]. The following are the translational symmetry generators for Eq. (2.3.33)

\[ Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x_1}, \quad Y_3 = \frac{\partial}{\partial x_2}. \]  (2.3.36)
Translational symmetries associated with the conservation laws

(I): The relation (1.2.14) with the help of $< Y_1 >$ and conserved vectors (2.3.34) yields

$$
\eta_t = 0, \quad \eta_{x_1 x_1} + \eta_{x_2 x_2} = 0. \tag{2.3.37}
$$

In this case we get the same conserved vectors as in (2.3.34) in which $\eta$ is the solution of the system (2.3.37).

(II): The following are the associated components of conserved vector for $< Y_2 >$

$$
T^1 = -\eta u,
$$

$$
T^2 = \eta u_{x_1}, \tag{2.3.38}
$$

$$
T^3 = \eta u_{x_2} - \eta_{x_2} u,
$$

where $\eta$ satisfies

$$
\eta_{x_1} = 0, \quad \eta_t + \eta_{x_2 x_2} = 0. \tag{2.3.39}
$$

(III): The associated conserved vectors for $< Y_3 >$ are

$$
T^1 = -\eta u,
$$

$$
T^2 = \eta u_{x_1} - \eta_{x_1} u, \tag{2.3.40}
$$

$$
T^3 = \eta u_{x_2},
$$

where $\eta$ can be found by the following pair of equations

$$
\eta_{x_2} = 0, \quad \eta_t + \eta_{x_1 x_1} = 0. \tag{2.3.41}
$$

In all the above cases we use a one-dimensional algebra and get a pair of PDEs in which both equations have less independent variables as compared to Eq. (2.3.35). Next we will
derive associated components of conserved vectors with two dimensional algebras.

(IV): The associated components of conserved vectors with the two dimensional algebra 
\( < Y_1, Y_2 > \) are

\[
T^1 = -(ax_2 + b)u, \\
T^2 = (ax_2 + b)u_{x_1}, \\
T^3 = (ax_2 + b)u_{x_2} - au,
\]

where \( a \) and \( b \) are real constants.

(V): The following are the associated components of conserved vectors with 
\( < Y_2, Y_3 > \)

\[
T^1 = -au, \\
T^2 = au_{x_1}, \\
T^3 = au_{x_2},
\]

where \( a \) is real constant.

(VI): The relation (1.2.14) yields the conserved vectors associated with \( < Y_1, Y_3 > \)

\[
T^1 = -(ax_1 + b)u, \\
T^2 = (ax_1 + b)u_{x_1} - au, \\
T^3 = (ax_1 + b)u_{x_2},
\]

with \( a \) and \( b \) real constant.

In the next section we will consider Eq. (2.1.1) on three dimensional surfaces.
2.3.5 Conservation laws for heat equation on 3-dimensional sphere

The coefficients of the FFF are

\[ E_{11} = \sin^2 x_3, \quad E_{22} = \sin^2 x_3 \sin^2 x_1, \quad E_{33} = 1 \]  \hspace{1cm} (2.3.45)

which reduces Eq. (2.1.1) to

\[ u_t = \csc^2 x_3 u_{x_1 x_1} + \csc^2 x_1 \csc^2 x_3 u_{x_2 x_2} + u_{x_3 x_3} - \cot x_1 \csc^2 x_3 u_{x_1} - 2 \cot x_3 u_{x_3}. \]  \hspace{1cm} (2.3.46)

Using Eq. (2.3.45) in Eq. (2.2.21) gives rise to

\[ T^1 = -\eta u, \]
\[ T^2 = \csc^2 x_3 \cot x_1 \eta u - (\eta u_{x_1} - \eta u_{x_1}) \csc^2 x_1, \]
\[ T^3 = \csc^2 x_1 \csc^2 x_3 (\eta u_{x_2} - \eta u_{x_2} u), \]  \hspace{1cm} (2.3.47)
\[ T^4 = 2 \cot x_3 \eta u + \eta u_{x_3} - \eta_{x_3} u, \]

where \( \eta \) can be obtained from

\[ -\eta_t - \csc^2 x_3 x_3 x_1 x_1 - \csc^2 x_1 \csc^2 x_3 x_3 x_2 x_2 - \eta_{x_3 x_3} + \cot x_1 \csc^2 x_3 x_3 x_1 + 2 \cot x_3 x_3 x_3 \]
\[ + \eta \csc^2 x_1 \csc^2 x_3 (\cot 2 x_1 - 2) = 0. \]  \hspace{1cm} (2.3.48)

Eq. (2.3.48) is a nonlinear PDE which can be simplified by associating the conserved vectors with the translational symmetries. So translational symmetry generators for (2.3.46) are [21]

\[ Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x_2}. \]  \hspace{1cm} (2.3.49)
Translational symmetries associated with conservation laws

After associating one dimensional Lie algebras with components of conserved vectors we obtain the equations in reduced form as below:

(I): The relation (1.2.14) with the help of \(< Y_1 >\) and conserved vectors (2.3.47) yields

\[ \eta_t = 0, \quad -\csc^2 x_3 \eta_{x_1 x_1} - \csc^2 x_4 \csc^2 x_3 \eta_{x_2 x_2} - \eta_{x_3 x_3} + \cot x_1 \csc^2 x_3 \eta_{x_1} + 2 \cot x_3 \eta_{x_3} \]

\[ + \eta \csc^2 x_1 \csc^2 x_3 \left( \cot 2 x_1 - 2 \right) = 0. \quad (2.3.50) \]

In this case we get the same conserved vectors (2.3.47) in which \( \eta \) satisfies system (2.3.50).

(II): The associated components of the conserved vector by using the translational symmetry of \( < Y_2 > \) are

\[ T^1 = -\eta u, \]

\[ T^2 = \csc^2 x_3 \cot x_1 \eta u - (\eta_{x_1} u - \eta u_{x_1}) \csc^2 x_1, \]

\[ T^3 = \csc^2 x_1 \csc^2 x_3 (\eta u_{x_2}), \quad (2.3.51) \]

\[ T^4 = 2 \cot x_3 \eta u + \eta u_{x_3} - \eta_{x_3} u, \]

where \( \eta \) is the solution of

\[ \eta_{x_2} = 0, \quad -\eta_t - \csc^2 x_3 \eta_{x_1 x_1} - \eta_{x_3 x_3} + \cot x_1 \csc^2 x_3 \eta_{x_1} + 2 \cot x_3 \eta_{x_3} \]

\[ + \eta \csc^2 x_1 \csc^2 x_3 \left( \cot 2 x_1 - 2 \right) = 0. \quad (2.3.52) \]

(III): We obtain these results by associating with \( < Y_1, Y_2 > \) the conserved vectors in Eq. (2.3.47)

\[ T^1 = -\eta u, \]
\[ T^2 = \csc^2 x_3 \cot x_1 \eta u - (\eta x_1 u - \eta u x_1) \csc^2 x_1, \]

\[ T^3 = \csc^2 x_1 \csc^2 x_3 (\eta u x_2 - \eta x_2 u), \quad (2.3.53) \]

\[ T^4 = 2 \cot x_3 \eta u + \eta u x_3 - \eta x_3 u, \]

where \( \eta \) satisfies

\[ \eta_t = 0, \quad \eta_{x_2} = 0, \quad - \csc^2 x_3 \eta_{x_1 x_1} - \eta x_3 x_3 + \cot x_1 \csc^2 x_3 \eta_{x_1} + 2 \cot x_3 \eta_{x_3} \]

\[ + \eta \csc^2 x_1 \csc^2 x_3 \left( \cot 2 x_1 - 2 \right) = 0. \quad (2.3.54) \]

It is interesting to note that Eq. (2.3.48) can not be converted into an ODE.

### 2.3.6 Conservation laws for heat equation on 3-dimensional flat surface

Eq. (2.1.1) after substituting

\[ E_{11} = 1, \quad E_{22} = 1, \quad E_{33} = 1 \quad (2.3.55) \]

becomes

\[ u_t = u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3}, \quad (2.3.56) \]

The replacement of coefficients of the FFF in Eq. (2.2.21) yields the conserved components

\[ T^1 = -\eta u, \]

\[ T^2 = \eta u_{x_1} - \eta_{x_1} u, \]

\[ T^3 = \eta u_{x_2} - \eta_{x_2} u, \quad (2.3.57) \]
\[ T^4 = \eta u_{x_3} - \eta x_3 u, \]

where

\[ \eta_t + \eta x_1 + \eta x_2 + \eta x_3 = 0. \quad (2.3.58) \]

In this case, we also obtain infinitely many conservation laws for Eq. (2.3.56). One can easily check that Eq. (2.3.56) admits the translational symmetry generators [11, 50]

\[
Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x_1}, \quad Y_3 = \frac{\partial}{\partial x_2}, \quad Y_4 = \frac{\partial}{\partial x_3}. \quad (2.3.59)
\]

For this case, only three-dimensional algebras are associated with conserved vectors for Eq. (2.3.57), which reduce Eq. (2.3.58) to an ODE.

**Three dimensional algebras associated with conservation laws**

(I): The associated conserved vectors for \(< Y_1, Y_2, Y_3 >\) are

\[ T^1 = -(ax_3 + b)u, \]

\[ T^2 = (ax_3 + b)u_{x_1}, \]

\[ T^3 = (ax_3 + b)u_{x_2}, \quad (2.3.60) \]

\[ T^4 = (ax_3 + b)u_{x_3} - au. \]

(II): For \(< Y_1, Y_2, Y_4 >\) the associated components of conserved vectors are

\[ T^1 = -(ax_2 + b)u, \]

\[ T^2 = (ax_2 + b)u_{x_1}, \]

\[ T^3 = (ax_2 + b)u_{x_2} - au, \quad (2.3.61) \]
\[ T^4 = (ax_2 + b)u_{x_3}. \]

(III): The components of conserved vectors associated with \( \langle Y_1, Y_3, Y_4 \rangle \) yield

\[ T^1 = -(ax_1 + b)u, \]

\[ T^2 = (ax_1 + b)u_{x_1} - au, \]

\[ T^3 = (ax_1 + b)u_{x_2}, \]

\[ T^4 = (ax_1 + b)u_{x_3}. \tag{2.3.62} \]

(IV): Following are the components of the conserved vector associated with \( \langle Y_2, Y_3, Y_4 \rangle \):

\[ T^1 = -au, \]

\[ T^2 = au_{x_1}, \]

\[ T^3 = au_{x_2}, \]

\[ T^4 = au_{x_3}. \tag{2.3.63} \]

Note that \( a, b, c \) are real constants.

### 2.4 Concluding Remarks

The partial Noether operators and conservation laws for the heat equation on different surfaces were computed by using the partial Noether approach. For the \((1 + n)\)-dimensional heat equation, the partial Noether determining equation yielded the gauge terms and the conserved vectors in terms of unknown functions which satisfy certain conditions. Then
these conserved vectors were used to construct conservation laws for the heat equation on particular surfaces such as the sphere, cone, torus and plane.

The relationship between symmetries and conservation laws was used to simplify the conserved vectors and conditions on unknowns. All classes of algebras for translational symmetries which simplify the conserved vectors were studied. We also commented on those algebras which neither reduce the dimension of the heat equation nor give nontrivial results. An infinite number of conservation laws were derived for the \((1 + n)\)-dimensional heat equation. The heat equation on the \(S^3\) and \(R^3\) were also discussed.
Chapter 3

Conserved Quantities for a Class of \((1 + n)\)-Dimensional Heat Equation

In this chapter, we will extend the results presented in chapter 2. For this, a special type of \((1 + n)\)-dimensional heat equation is considered. Then conservation laws are derived in terms of coefficients using the partial Lagrangian approach. The derived results are applied to different models from different sciences. We also discuss the conservation laws of the heat equation in different coordinate systems. A potential system is also obtained for some models.

3.1 Introduction

Conservation laws have great importance in the field of differential equations, e.g. in jet problems conserved quantities are important in solution processes and are used to calculate the unknown exponent appearing in similarity solutions. Conservation laws are identified with conserved quantities in a physical system. But here we will discuss conservation laws from a theoretical perspective. Krasil’schik et al. [42] defined nonlocal symmetries of an associated auxiliary system of differential equations, these nonlocal symmetries are called potential symmetries and the auxiliary system known a potential system. For a potential
system one has to write down the equation in conserved form. So one can use the compon-
ents of the conserved vector to convert the equation in its conserved form.

In this chapter we will also present one of the potential systems for some models. In fact
from the results given in this chapter one can directly construct the potential systems, which
can be further used to obtain an exact solution. Firstly, we will present some well-known
models which will be considered here.

(i): The Fokker-Planck equation is a very useful tool in applied sciences and plays a central
role in diverse fields. It has been used frequently by scientists, engineers and experts in different
fields, including mathematical finance, laser physics and electronics [4, 8, 10, 17, 40, 47].
Many research articles discuss the nature, solutions and applications of the Fokker-Planck
equation. Mathematically, the Fokker-Planck equation is a parabolic second order linear par-
tial differential equation, which was introduced by Fokker and Planck [22, 52] to study the
distribution function describing Brownian motion. The $(1 + n)$-dimensional Fokker-Planck
equation in its general form can be written as

$$u_t = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[ A_i^1 u \right] + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left[ B_{ij}^1 u \right],$$

(3.1.1)

where $u(t, x_1, \cdots, x_n)$ is probability density function, $A_i^1$ and $B_{ij}^1$ are smooth functions of
t, $x_1, \cdots, x_n$ and represent the coefficients of drift and diffusion, respectively.

(ii): The Black-Scholes model [10] which is frequently used in probability theory or in finance
and has the following form:

$$u_t + \frac{\sigma^2 S^2}{2} u_{SS} + rSu_S - cu = 0,$$

(3.1.2)

where $S$ is the price of the stock, $u$ represents the price of a derivative which is a function
of time and stock price, $\sigma$ is the volatility of the stock (volatility is a measure variation
of price of a financial instrument over time), \( r \) represents the risk-free interest rate that is continuously compounded and \( u \) is the “fair” price of the option.

(iii): The Basket option model has significant importance in probability theory and is modeled by the following equation

\[
    u_t = \frac{A^2 x_1^2}{2} u_{x_1 x_1} + \frac{B^2 x_2^2}{2} u_{x_2 x_2} + \rho A B x_1 x_2 u_{x_1 x_2} + r x_1 u_{x_1} + r x_2 u_{x_2} - r u,
\]

where \( A, B, r \) and \( \rho \) are real constants.

In past decades it’s become common to mix geometry and theoretical physics [55, 59]. It is also an interesting topic of research in mathematical biology where membranes are not flat in general. Now we will discuss two special classes of partial differential equations, one the class where partial differential equations have non-flat background, while the second class contains partial differential equations which have non-Cartesian variables e.g. cylindrical or spherical.

(iv): The \((1 + 2)\)-dimensional heat equation on a curved surface is

\[
    u_t = \frac{u_{x_1 x_1}}{E_{11}} + \frac{u_{x_2 x_2}}{E_{22}} - \left( \frac{(E_{11})_x}{2 E_{11}} - \frac{(E_{22})_x}{2 E_{22}} \right) u_{x_1} - \left( \frac{(E_{22})_x}{2 E_{22}} - \frac{(E_{11})_x}{2 E_{11}} \right) u_{x_2},
\]

where

\[
    ds^2 = E_{11} dx_1^2 + E_{12} dx_1 dx_2 + E_{22} dx_2^2
\]

is the FFF of the 2-dimensional surface.


\[
    u_t = u_{\rho \rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\theta \theta} + \frac{1}{\rho^2} \cot \theta u_\theta + \frac{1}{\rho^2} \csc^2 \theta u_{\phi \phi} + (p - k) u,
\]

where \( p \) is the proliferation rate and \( k \) represents the therapy-dependent killing rate. Here we consider both \( p \) and \( k \) as arbitrary constants.
The models (3.1.1) - (3.1.6) are subsumed in an equation of the form

\[ u_t = \sum_{j=1}^{n} \sum_{i=1}^{n} \left( a_{ii} u_{x_i x_i} + a_{ij} u_{x_i x_j} + a_i u_{x_i} \right) + cu, \quad a_{ii} \neq 0, \quad i < j \]  

(3.1.7)

where \( a_{ii}, a_{ij}, a_i \) and \( c \) are smooth functions of the independent variables \((t, x_1, x_2, \cdots x_n)\), and \( u \) is considered the dependent variable. There are number of models which are included in the family (3.1.7), e.g. the Kolomogrov equation [41], the Longstaff equation [46], the Vasicek equation [62], the Cox-Ingersoll-Ross equation [16] and others listed in [34]. Higher order forms of Eq. (3.1.7) is useful in finance [40, 56], mathematical biology and in fact in all fields where processes depend on more than one variable.

For simplicity and generality of the results we will use Eq. (3.1.7). Further Eq. (3.1.7) is a second-order linear parabolic partial differential equation, its standard Lagrangian does not exist and hence approaches for the calculation of conservation laws that depend on the standard Lagrangian can not be applied here. There are a number of methods to obtain conservation laws for a partial differential equation [9, 39, 45, 49, 58] and many authors calculated conservation laws by using computing utility [14, 29, 30, 31, 32, 63, 64].

The wide use of Eq. (3.1.7) in diverse fields and the importance of conservation laws in differential equations make this research work meaningful. Jhangeer et al. derived conservation laws for \((1 + n)\)-dimensional heat equation on curved surfaces [36] but the equation considered there is the subcase of Eq. (3.1.7). To the best of our knowledge, conservation
laws of Eq. (3.1.7) are neither discussed nor derived yet and are given here for the first time.

The outline of this chapter is as follow: Section 3.2 is for the conservation laws of Eq. (3.1.7), in Section 3.3 we present illustrative examples from different fields, while in Section 3.4 a brief conclusion is given.

### 3.2 Conserved vectors for 
\((1+n)\)-dimensional linear evolution equation

In this section, our objective is to obtain the conserved quantities of Eq. (3.1.7). For this we will investigate the conservation laws for \(n = 2\) and \(n = 3\) and then we will generalize the results for Eq. (3.1.7). It should be remarked Eq. (3.1.7) is not derivable from a variational principle because it does not posses a standard Lagrangian.

#### 3.2.1 The \((1 + 1)\)-dimensional linear diffusion equation

For \(n = 1\) Eq. (3.1.7) reduces to

\[
\frac{\partial u}{\partial t} = a_{11}u_{x_1x_1} + a_1u_{x_1} + cu, \quad a_{11} \neq 0. \tag{3.2.1}
\]

The Eq. (3.2.1) has the partial Lagrangian

\[
L = \frac{a_{11}u_{x_1}^2}{2} \tag{3.2.2}
\]

associated with the partial Euler-Lagrange equation, viz.

\[
\frac{\delta L}{\delta u} = \left[ a_1 - (a_{11})_{x_1} \right] u_{x_1} - \frac{\partial u}{\partial t} + cu. \tag{3.2.3}
\]

Consider

\[
X = \tau \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial x_1} + \eta \frac{\partial}{\partial u}
\]
is the partial Noether operator for Eq. (3.2.1). Substituting in Eq. (1.2.12) and separating with respect to the derivatives of \( u \) we obtain the following set of determining equations

\[
\tau = 0, \quad \xi^1 = 0, \quad \eta_u = 0
\]  

(3.2.4)

with

\[
B^1 = \eta u + \alpha(t, x_1),
\]

(3.2.5)

\[
B^2 = \left[ (a_{11})_{x_1} - a_1 \right] \eta + a_{11} \eta x_1 \right] u + \beta(t, x_1)
\]

(3.2.6)

and

\[
B^1_t + B^2_{x_1} + cu = 0.
\]

(3.2.7)

After substituting Eqs. (3.2.5-3.2.6) in Eq. (3.2.7) and comparing the coefficients of \( u^1 \) and \( u^0 \), we obtain the following equations respectively

\[
\eta_t + a_{11} \eta_{x_1 x_1} + \left[ 2(a_{11})_{x_1} - a_1 \right] \eta_{x_1} + \left[ (a_{11})_{x_1} x_1 - (a_1)_{x_1} + c \right] \eta = 0,
\]

(3.2.8)

\[
\alpha_t + \beta_{x_1} = 0.
\]

(3.2.9)

Without loss of generality one can take

\[
\alpha = 0, \quad \beta = 0.
\]

Now by using Eq. (1.2.2) (see [39]), the conserved vectors are

\[
T^1 = \eta u,
\]

(3.2.10)

where \( \eta \) is a solution of Eq. (3.2.8). For the (1 + 1)-dimensional heat equation we obtain an infinite number of conservation laws.
3.2.2 For the \((1 + 2)\)-dimensional linear evolution equation

For \(n = 2\) Eq. (3.1.7) becomes

\[
\begin{align*}
  u_t &= a_{11}u_{x_1 x_1} + a_{22}u_{x_2 x_2} + a_{12}u_{x_1 x_2} + a_1 u_{x_1} + a_2 u_{x_2} + cu, \quad a_{11} \neq 0, \quad a_{22} \neq 0, \\
  \end{align*}
\]

(3.2.11)

which admits the partial Lagrangian

\[
L = \frac{a_{11}u_{x_1}^2}{2} + \frac{a_{22}u_{x_2}^2}{2} + \frac{a_{12}u_{x_1}u_{x_2}}{2}.
\]

(3.2.12)

The partial Euler-Lagrange equation Eq. (3.2.11) with the help of the partial Lagrangian (3.2.12) is

\[
\frac{\delta L}{\delta u} = \left[ a_1 - (a_{11})_{x_1} - \frac{(a_{12})_{x_2}}{2} \right] u_{x_1} + \left[ a_2 - (a_{22})_{x_2} - \frac{(a_{12})_{x_1}}{2} \right] u_{x_2} - u_t + cu.
\]

(3.2.13)

Eq. (3.2.12) represents the partial Lagrangian of Eq. (3.2.11) for which the partial Noether operator

\[
X = \tau \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial x_1} + \xi^2 \frac{\partial}{\partial x_2} + \eta \frac{\partial}{\partial u},
\]

which satisfies Eq. (1.2.12). The usual separation of Eq. (1.2.12) with respect to the derivatives of \(u\) yields the following over-determined linear system

\[
\tau = 0, \quad \xi^1 = 0, \quad \xi^2 = 0, \quad \eta_u = 0
\]

(3.2.14)

with

\[
B^1 = \eta u + \alpha(t, x_1, x_2),
\]

(3.2.15)

\[
B^2 = \left[ \left( (a_{11})_{x_1} - a_1 + \frac{(a_{12})_{x_2}}{2} \right) \eta + \frac{a_{12}}{2} \eta_{x_2} + a_{11} \eta_{x_1} \right] u + \beta(t, x_1, x_2),
\]

(3.2.16)

\[
B^3 = \left[ \left( (a_{22})_{x_2} - a_2 + \frac{(a_{12})_{x_1}}{2} \right) \eta + \frac{a_{12}}{2} \eta_{x_1} + a_{22} \eta_{x_2} \right] u + \gamma(t, x_1, x_2)
\]

(3.2.17)
and

\[ B^1_t + B^2_{x_1} + B^2_{x_2} + cu = 0. \] (3.2.18)

After substituting Eqs. (3.2.15-3.2.17) into Eq. (3.2.18) and comparing the coefficients of \( u^1 \) and \( u^0 \), we obtain

\[
\eta_t + a_{11} \eta_{xx_1} + a_{22} \eta_{xx_2} + a_{12} \eta_{x_1 x_2} + \left( 2(a_{11})_{x_1} - a_1 + (a_{12})_{x_2} \right) \eta_{x_1} \\
+ \left( 2(a_{22})_{x_2} - a_2 + (a_{12})_{x_1} \right) \eta_{x_2} + \left( (a_{11})_{x_1 x_1} + (a_{22})_{x_2 x_2} + (a_{12})_{x_1 x_2} \right) \eta_{x_2} \\
-(a_1)_{x_1} - (a_2)_{x_2} + c \eta = 0, \] (3.2.19)

\[
\alpha_t + \beta_{x_1} + \gamma_{x_2} = 0. \] (3.2.20)

Without loss of generality one can assume

\[
\alpha = 0, \quad \beta = 0, \quad \gamma = 0. \]

From Eq. (1.2.2) we obtain

\[
T^1 = \eta u, \\
T^2 = \left[ \left( (a_{11})_{x_1} - a_1 + \frac{(a_{12})_{x_2}}{2} \right) \eta + \frac{a_{12}}{2} \eta_{x_2} + a_{11} \eta_{x_1} \right] u - \left[ a_{11} u_{x_1} + \frac{a_{12} u_{x_2}}{2} \right] \eta, \] (3.2.21)

\[
T^3 = \left[ \left( (a_{22})_{x_2} - a_2 + \frac{(a_{12})_{x_1}}{2} \right) \eta + \frac{a_{12}}{2} \eta_{x_1} + a_{22} \eta_{x_2} \right] u - \left[ a_{22} u_{x_2} + \frac{a_{12} u_{x_1}}{2} \right] \eta, \]

where \( \eta \) is a solution of Eq. (3.2.19). For the \((1 + 2)\)-dimensional heat equation we also obtain an infinite number of conservation laws.
3.2.3 Extension to the \((1 + n)\)-dimensional linear evolution equation

By following the same procedure as we have done for Eq. (3.2.1) and Eq. (3.2.11), we will generalize the results for the \((1 + n)\)-dimensional heat equation. Eq. (3.1.7) does not admit a standard Lagrangian but it does have a partial Lagrangian, viz.

\[
L = \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} \left( \frac{a_{ii}u_{x_i}^2}{2} + \frac{a_{ij}u_{x_i}u_{x_j}}{2} \right) \right], \quad i < j, \quad i, j = 1, 2, \cdots, n \tag{3.2.22}
\]

whose associated partial Euler-Lagrange equation is

\[
\frac{\delta L}{\delta u} = \sum_{j=1}^{n} \left[ \sum_{i=1}^{n} \left( a_i - (a_{ii})_{x_i} - \frac{(a_{ij})_{x_i}}{2} \right) u_{x_i} \right] - u_t + cu, \quad i < j, \quad i, j = 1, 2, \cdots, n. \tag{3.2.23}
\]

After a routine calculation, we obtain the following conserved vectors

\[
T^1 = \eta u, \tag{3.2.24}
\]

\[
T^{i+1} = \sum_{j=1}^{n} \left[ \left( (a_{ii})_{x_i} - a_i + \frac{(a_{ij})_{x_j}}{2} \right) \eta u + a_{ii} \left( \eta_{x_i}u - \eta u_{x_i} \right) + \frac{a_{ij}}{2} \left( \eta_{x_j}u - \eta u_{x_j} \right) \right], \quad i < j, \quad i = 1, 2, \cdots, n \tag{3.2.25}
\]

where \(\eta\) satisfies

\[
\eta_t + \sum_{j=1}^{n} \sum_{i=1}^{n} \left[ \left( a_{ii} \eta_{x_i} + a_{ij} \eta_{x_ix_j} \right) + \left( 2(a_{ii})_{x_i} - a_i + (a_{ij})_{x_j} \right) \eta_{x_j} \right.
\]

\[
\left. + \left( (a_{ii})_{x_i} - (a_i)_{x_i} + (a_{ij})_{x_ix_j} + c \right) \eta \right] = 0, \quad i < j. \tag{3.2.26}
\]

For the \((1+n)\)-dimensional heat equation an infinite number of conserved vectors are derived.

3.3 Application

In this section, we will derive the conservation laws for different models by using the results obtained in Section 3.
Example 1 (Fokker-Planck Equation)

The \((1 + 1)\)-dimensional Fokker-Planck equation with \(n = 1\) in Eq. (3.1.1) becomes

\[
    u_t = -\frac{\partial(A(t, x_1)u)}{\partial x_1} + \frac{1}{2} \frac{\partial^2 (B(t, x_1)u)}{\partial x_1^2},
\]

(3.3.1)

where \(u\) is the probability density function, \(A\) represents the coefficient of drift and \(B\) is the coefficient of diffusion.

Expanding Eq. (3.3.1) we obtain

\[
    u_t = \frac{B}{2} u_{x_1x_1} + \left( B_{x_1} - A \right) u_{x_1} + \left( \frac{B_{x_1x_1}}{2} - A_{x_1} \right) u.
\]

(3.3.2)

After comparing it with Eq. (3.2.1) we get

\[
    a_{11} = \frac{B}{2}, \quad a_1 = B_{x_1} - A, \quad c = \frac{B_{x_1x_1}}{2} - A_{x_1}.
\]

(3.3.3)

Using Eq. (3.3.3) in Eq. (3.2.2), the partial Lagrangian is obtained for Eq. (3.3.2) i.e.

\[
    L = \frac{Bu_{x_1}^2}{2}.
\]

(3.3.4)

Eq. (3.2.10) after using Eq. (3.3.3) gives components of conserved vector i.e.

\[
    T^1 = \eta u,
\]

\[
    T^2 = (A - B_{x_1})\eta u + \frac{B}{2} \left[ \eta_{x_1} u - \eta u_{x_1} \right],
\]

(3.3.5)

where \(\eta\) is the solution of

\[
    \eta_t + \frac{B}{2} \eta_{x_1x_1} - (B_{x_1} - A)\eta_{x_1} - \frac{B_{x_1x_1}}{2} \eta = 0,
\]

(3.3.6)

which is obtained from Eq. (3.2.8) after substituting Eq. (3.3.3).
Example 2 (Population Genetics Model)

Consider the equation describing population genetics [47], which is of the form

\[ u_t = (1 - x_1^2)^2 u_{x_1 x_1} - 8x_1(1 - x_1^2)u_{x_1} - 4(1 - 3x_1^2)u. \]  

(3.3.7)

For Eq. (3.3.7) the partial Lagrangian is

\[ L = \frac{(1 - x_1^2)^2}{2} u_{x_1}^2. \]  

(3.3.8)

Substituting \( a_{11} = (1 - x_1^2)^2, a_1 = -8x_1(1 - x_1^2), c = -4(1 - 3x_1^2) \) into Eq. (3.2.8) and Eq. (3.2.10) the following results are obtained

\[ T^1 = \eta u, \]

(3.3.9)

where \( \eta \) can be determined from

\[ \eta_t + (1 - x_1^2)^2 \eta_{x_1 x_1} = 0. \]  

(3.3.10)

The potential system for Eq. (3.3.7) is

\[ v_{x_1} = (ax_1 + b)u, \]

\[ v_t = 4x_1(1 - x_1^2)(ax_1 + b)u + (1 - x_1^2) \left[ au - (ax_1 + b)u_{x_1} \right]. \]  

(3.3.11)

One can easily check the compatibility condition of the system (3.3.11) is Eq. (3.3.7).

Example 3 (Rayleigh-type Process Model)

A Rayleigh-type process [56] is governed by the following equation

\[ u_t = \frac{a}{2} u_{x_1 x_1} + \left( kx_1 - \frac{a}{x_1} \right)u_{x_1} + \left( b + \frac{a}{x_1^2} \right)u, \]  

(3.3.12)
where $a$ and $b$ are real constant. The partial Lagrangian for Eq. (3.3.12) is

$$L = \frac{a}{2} u_{x_1}^2.$$  \hfill (3.3.13)

Substituting $a_{11} = a/2$, $a_1 = (bx_1 - a/x_1)$, $c = (b + a/(x_1)^2)$ into Eq. (3.2.8) and Eq. (3.2.10) the following results are derived

$$T^1 = \eta u,$$

$$T^2 = -(bx_1 - \frac{a}{x_1})\eta u + \frac{a}{2} \left[ \eta_{x_1} u - \eta u_{x_1} \right],$$  \hfill (3.3.14)

where $\eta$ is a solution of

$$\eta_t + \frac{a}{2} \eta_{x_1 x_1} - (bx_1 - \frac{a}{x_1})\eta_{x_1} = 0.$$  \hfill (3.3.15)

One of the potential system for Eq. (3.3.12) is

$$v_{x_1} = ku,$$

$$v_t = -(bx_1 - \frac{a}{x_1})u + \frac{a}{2} u_{x_1} k,$$  \hfill (3.3.16)

where $k$ is a real constant.

**Example 4 (Black-Scholes Model)**

For the Black-Scholes model (3.1.2) after comparing it with Eq. (3.2.1) we have

$$a_{11} = -\sigma^2 S^2/2, \quad a_1 = -r S, \quad a_{12} = 0, \quad c = c.$$  \hfill (3.3.17)

From Eq. (3.2.10) with the use of Eq. (3.3.17) the following partial Lagrangian is obtained for Eq. (3.1.2)

$$L = -\frac{\sigma^2 S^2}{4} u_S^2.$$  \hfill (3.3.18)
Eq. (3.2.8) with the help of Eq. (3.3.17) yields the following components of the conserved vectors

\[
T^1 = \eta u, \\
T^2 = \frac{\sigma^2 S^2}{2} \eta u_S + \left[ (r - \sigma^2) \eta S - \frac{\sigma^2 S^2}{2} \eta_S \right] u,
\]

(3.3.19)

where \( \eta \) is a solution of

\[
\eta_t - \frac{\sigma^2 S^2}{2} \eta_{SS} + (r - 2\sigma^2) S \eta_S + (r - \sigma^2 + c) \eta = 0.
\]

(3.3.20)

For \( \sigma = \sqrt{2}, \ r = 1, \ c = 1 \) and taking \( \eta = 1 \), the potential system arises

\[
v_S = u, \\
v_t = S^2 u_S - Su,
\]

(3.3.21)

which is similar to that of [19].

By taking \( \eta = e^{(\sigma^2 - r - c)t} \) one can obtain the potential system i.e.

\[
v_S = e^{(\sigma^2 - r - c)t} u, \\
v_t = \left( \frac{\sigma^2 S^2}{2} \eta u_S + (r - \sigma^2) S \eta u \right) e^{(\sigma^2 - r - c)t}.
\]

(3.3.22)

It is interesting to note that potential system (3.3.22) is more general compared to (3.3.21). Hence by using the potential system (3.3.22) and following the technique given in [19] one can obtain the general solution.

**Example 5 (Basket Option Model)**

We use

\[
a_{11} = -A^2 x_1^2/2, \ a_{22} = -B^2 x_2^2/2, \ a_{12} = -\rho AB x_1 x_2,
\]
\[ a_1 = -rx_1, \quad a_2 = -rx_2 \text{ and } c = r \] 

(3.3.23)

for the basket option model (3.1.3) after comparing the coefficients with the Eq. (3.2.11). Using values from (3.3.23) in Eq. (3.2.12), Eq. (3.2.21) and Eq. (3.2.19) yields

\[ L = -\frac{A^2 x_1^2}{4} u_{x_1}^2 - \frac{B^2 x_2^2}{4} u_{x_2}^2 - \rho AB x_1 x_2 u_{x_1} u_{x_2}, \] 

(3.3.24)

\[ T_1^1 = \eta u, \]

\[ T^2 = x_1 \left( r - A^2 - \frac{\rho AB}{2} \right) \eta u + \frac{\rho AB x_1 x_2}{2} \left[ \eta u_{x_2} - \eta_{x_2} u \right] + \frac{A^2 x_1^2}{2} \left[ \eta u_{x_1} - \eta_{x_1} u \right], \]

(3.3.25)

\[ T^3 = x_2 \left( r - B^2 - \frac{\rho AB}{2} \right) \eta u + \frac{\rho AB x_1 x_2}{2} \left[ \eta u_{x_1} - \eta_{x_1} u \right] + \frac{B^2 x_2^2}{2} \left[ \eta u_{x_2} - \eta_{x_2} u \right], \]

where \( \eta \) can be obtained from

\[ \eta_t - \frac{A^2 x_1^2}{2} \eta_{x_1 x_1} - \frac{B^2 x_2^2}{2} \eta_{x_2 x_2} - \rho AB x_1 x_2 \eta_{x_1 x_2} + \left( r - \frac{\rho AB}{2} \right) x_1 \eta_{x_1} + \]

\[ \left( r - \frac{\rho AB}{2} \right) x_2 \eta_{x_2} + \left( 3r - A^2 - B^2 - \rho AB \right) \eta = 0. \] 

(3.3.26)

One of the potential systems for Eq. (3.1.3) can be derived by using the components of the conserved vector as below,

\[ T^1 = e^{(A^2 + B^2 + \rho AB - 3r)} t u, \]

(3.3.27)

\[ T^2 = \left[ x_1 \left( r - A^2 - \frac{\rho AB}{2} \right) u + \left( \frac{\rho AB x_1 x_2 u_{x_2}}{2} + \frac{A^2 x_1^2 u_{x_1}}{2} \right) \right] e^{(A^2 + B^2 + \rho AB - 3r)} t, \]

\[ T^3 = \left[ x_2 \left( r - B^2 - \frac{\rho AB}{2} \right) u + \left( \frac{\rho AB x_1 x_2 u_{x_1}}{2} + \frac{B^2 x_2^2 u_{x_2}}{2} \right) \right] e^{(A^2 + B^2 + \rho AB - 3r)} t. \]
Example 6 ((1 + 2)-dimensional heat equation on curved manifolds)

The (1 + 2)-dimensional heat equation on curved surfaces (3.1.4) compared with Eq. (3.2.11) gives

\[ a_{11} = 1/E_{11}, \quad a_{22} = 1/E_{22}, \quad a_{12} = 0, \quad c = 0, \]

\[ a_1 = -\left( \frac{(E_{11})_{x_1}}{2E_{11}^2} - \frac{(E_{22})_{x_1}}{2E_{11}E_{22}} \right), \quad a_2 = -\left( \frac{(E_{22})_{x_2}}{2E_{22}^2} - \frac{(E_{11})_{x_2}}{2E_{11}E_{22}} \right). \]  \hspace{1cm} (3.3.28)

Using the results given in (3.3.28) and Eq. (3.2.12) we obtain

\[ L = \frac{u_{x_1}^2}{2E_{11}} + \frac{u_{x_2}^2}{2E_{22}}, \]  \hspace{1cm} (3.3.29)

while putting (3.3.28) in Eq. (3.2.21), the conserved vectors in terms of the coefficients of the FFF are

\[ T^1 = \eta u, \]
\[ T^2 = \left[ \frac{\eta_{x_1}}{E_{11}} - \left( \frac{(E_{11})_{x_1}}{2E_{11}^2} + \frac{(E_{22})_{x_1}}{2E_{11}E_{22}} \right) \eta \right] u - \frac{\eta u_{x_1}}{E_{11}}, \]  \hspace{1cm} (3.3.30)
\[ T^3 = \left[ \frac{\eta_{x_2}}{E_{22}} - \left( \frac{(E_{22})_{x_2}}{2E_{22}^2} + \frac{(E_{11})_{x_2}}{2E_{11}E_{22}} \right) \eta \right] u - \frac{\eta u_{x_2}}{E_{22}}, \]

where \( \eta \) satisfies

\[ \eta_t + \frac{\eta_{x_1}x_1}{E_{11}} + \frac{\eta_{x_2}x_2}{E_{22}} - \left( \frac{3(E_{11})_{x_1}}{2E_{11}^2} + \frac{(E_{22})_{x_1}}{2E_{11}E_{22}} \right) \eta_{x_1} - \left( \frac{3(E_{22})_{x_2}}{2E_{22}^2} + \frac{(E_{11})_{x_2}}{2E_{11}E_{22}} \right) \eta_{x_2} \]
\[ - \left[ \frac{(E_{11})_{x_1}x_1}{2E_{11}^2} + \frac{(E_{22})_{x_2}x_2}{2E_{22}^2} - \frac{[(E_{11})_{x_1}]^2}{E_{11}^3} - \frac{[(E_{22})_{x_2}]^2}{E_{22}^3} + \frac{(E_{11})_{x_2}x_2}{2E_{11}E_{22}} + \frac{(E_{22})_{x_1}x_1}{2E_{11}E_{22}} \right] \eta = 0, \]  \hspace{1cm} (3.3.31)

which is obtained by substituting (3.3.28) into Eq. (3.2.19).

These results are the same as those obtained in [36].
Example 7 ((1 + 2)-dimensional heat equation on a sphere)

The coefficients of the FFF of a sphere are

\[ E_{11} = 1, \quad E_{22} = \sin^2 x_1. \]

The substitution of the above coefficients of FFF into Eq. (3.1.4) gives us the heat equation on a sphere,

\[ u_t = u_{x_1 x_1} + \cot x_1 u_{x_1} + \csc^2 x_1 u_{x_2 x_2}. \]  \( (3.3.32) \)

Now the conserved vectors corresponding to Eq. (3.3.32) by using the results obtained in Eq. (3.3.30) are

\[ T^1 = \eta u, \]

\[ T^2 = \left( \eta_{x_1} - \eta \cot x_1 \right) u - \eta u_{x_1}, \]  \( (3.3.33) \)

\[ T^3 = \csc^2 x_1 \left( \eta_{x_2} u - \eta u_{x_2} \right), \]

where \( \eta \) satisfies the following equation

\[ \eta_t + \eta_{x_1 x_1} + \csc^2 x_1 \eta_{x_2 x_2} - \cot x_1 \eta_{x_1} + \eta \csc^2 x_1 = 0. \]  \( (3.3.34) \)

Example 8 (Model for tumor growth in the brain in terms of spherical coordinates)

The model for tumor growth in the brain in terms of spherical coordinates \([13]\) given in (3.1.5) has the values

\[ a_{11} = 1, \quad a_{22} = 1/\rho^2, \quad a_{33} = 1/\rho^2 \sin^2 \theta, \quad a_{12} = 0, \quad c = p - k = K \text{(say)}, \]

\[ a_1 = 2/\rho, \quad a_2 = \cot \theta/\rho^2, \quad a_3 = 0. \]  \( (3.3.35) \)
Eq.(3.3.35) gives the following results

\[ L = \frac{u_\rho^2}{2} + \frac{u_\theta^2}{2r^2} + \frac{u_\phi^2}{2r^2\sin \theta}, \]  

(3.3.36)

\[ T^1 = \eta u, \]

\[ T^2 = \eta \rho u - \eta u_\rho - \frac{2\eta u}{\rho}, \]

(3.3.37)

\[ T^3 = \frac{1}{\rho^2} \left( \eta \theta u - \eta u_\theta \right) - \frac{\cot \theta}{\rho^2}, \]

\[ T^4 = \frac{1}{\rho^2 \sin^2 \theta} \left( \eta \phi u - \eta u_\phi \right), \]

where \( \eta \) is the solution of the following equation

\[ \eta_t + \eta_{\rho \rho} - \frac{2}{\rho} \eta_{\rho} + \frac{1}{\rho^2} \eta_{\theta \theta} - \frac{\cot \theta}{\rho^2} \eta_{\theta} + \frac{1}{\rho^2} \csc^2 \theta \eta_{\phi \phi} + \left( K + \frac{2}{\rho^2} + \frac{1}{\rho^2} \csc^2 \theta \right) \eta = 0. \]  

(3.3.38)

It should be noted that in the above example, the following assumptions were made: \( x_1 = \rho, \ x_2 = \theta \) and \( x_3 = \phi. \)

**Example 9 \((1 + 3)\)-dimensional heat equation in cylindrical coordinates**

After doing a routine calculation for the \((1 + 3)\)-dimensional heat equation in cylindrical coordinates (3.1.6) we get

\[ a_{11} = 1, \quad a_{22} = 1/\rho^2, \quad a_{33} = 1, \quad a_{12} = 0, \quad c = 0, \]

(3.3.39)

\[ a_1 = 1/\rho, \quad a_2 = a_3 = 0. \]

Use of Eq. (3.2.22) gives the following results after substituting the coefficients from Eq.(3.3.39)

\[ L = \frac{u_\rho^2}{2} + \frac{u_\phi^2}{2\rho^2} + \frac{u_z^2}{2}, \]  

(3.3.40)
After using the results given in Eq.(3.2.25) with Eq.(3.3.39) the following components of conserved vector are obtained

\[
T^1 = \eta u,
\]

\[
T^2 = \eta_\rho u - \eta u_\rho - \frac{\eta u}{r},
\]

\[
T^3 = \frac{1}{\rho^2} \left( \eta_\phi u - \eta u_\phi \right),
\]

\[
T^4 = \eta_z u - \eta u_z,
\]

where \( \eta \) is the solution of the following equation

\[
\eta_t + \eta_{\rho\rho} - \frac{1}{\rho} \eta_\rho + \frac{1}{\rho^2} \eta_{\phi\phi} + \eta_{zz} + \frac{1}{\rho^2} \eta = 0.
\]

For Eq.(3.1.6) we used \((\rho, \phi, z) = (x_1, x_2, x_3)\). It is worth mentioning that the independent variables \(x_i\)s used in section 3.2 are arbitrary. The last two problems are examples of this fact.

### 3.4 Concluding Remarks

In this chapter, conservation laws were constructed for the \((1 + n)\)-dimensional linear evolution equation by using the approach given in [39] where the Fokker-Planck equation is the subcase of the considered equation. The partial Lagrangian approach when applied to the \((1 + n)\)-dimensional linear evolution equation yields infinitely many conservation laws depending on the coefficients of the equation for each \(n\). Then some models from different branches of sciences were considered and conserved vectors for each case were calculated. These results gave potential systems for the family (3.1.7). These potential systems can be used to calculate potential symmetries which can be further used to calculate exact solutions.
Chapter 4

Derivation of Conservation Laws for
$(1 + 2)$-dimensional Wave Equation on Curved Surfaces

In this chapter, conservation laws for the wave equation on the sphere, cone and on flat space are derived. The partial Noether approach is applied to the wave equation on curved surfaces in terms of the coefficients of the first fundamental form (FFF) and the partial Noether operator’s determining equations are derived. These determining equations are then used to construct the partial Noether operators and conserved vectors for the wave equation on different surfaces. The conserved vectors for the wave equation on the sphere, cone and flat space are simplified using the Lie point symmetry generators of the equation and conserved vectors with the help of the symmetry conservation laws relation.

4.1 Introduction

The conservation laws of the nonlinear $(1 + 1)$-dimensional wave equation were discussed in [37]. Bokhari et al. [12] derived the conservation laws for a class of nonlinear $(1 + n)$-dimensional wave equations. To the best of our knowledge, conservation laws for the wave equation on curved surfaces were neither derived nor discussed before and will be considered here.
The wave equation on a curved surface is

\[ u_{tt} = \frac{u_{x_1 x_1}}{E_{11}} + \frac{u_{x_2 x_2}}{E_{22}} - \left( \frac{(E_{11})_{x_1}}{2E_{11}^2} - \frac{(E_{22})_{x_1}}{2E_{11}E_{22}} \right) u_{x_1} - \left( \frac{(E_{22})_{x_2}}{2E_{22}^2} - \frac{(E_{11})_{x_2}}{2E_{11}E_{22}} \right) u_{x_2}, \]  

(4.1.1)

where \( E_{11} \) and \( E_{22} \) are the coefficients of the FFF.

In this chapter the partial Noether approach is used to construct the conservation laws for the \((1 + 2)\)-dimensional wave equation on particular curved surfaces. The symmetry conservation laws relation is then used to obtain associated conservation laws from symmetries and known conserved vectors of the underlying equations.

The outline of the chapter is as follows: in Section 4.2 the determining equations for the \((1 + 2)\)-dimensional wave equation are derived. The conservation laws for the wave equation on particular curved surfaces are also constructed in Section 4.2. Finally, in Section 4.3 the conclusions are summarized.

### 4.2 Determining equations for the \((1 + 2)\)-dimensional wave equation on curved surfaces

The partial Noether approach is employed to derive the determining equations for Eq. (4.1.1) on curved surfaces.

A partial Lagrangian for Eq. (4.1.1) is

\[ L = \frac{u_{x_1}^2}{2E_{11}} + \frac{u_{x_2}^2}{2E_{22}} - \frac{u_t^2}{2}, \]  

(4.2.1)

and Eq. (4.1.1) can be expressed as

\[ \frac{\delta L}{\delta u} = \left( \frac{(E_{11})_{x_1}}{2E_{11}^2} + \frac{(E_{22})_{x_1}}{2E_{11}E_{22}} \right) u_{x_1} + \left( \frac{(E_{22})_{x_2}}{2E_{22}^2} + \frac{(E_{11})_{x_2}}{2E_{11}E_{22}} \right) u_{x_2}. \]  

(4.2.2)
In Eq. (4.2.2) $\delta/\delta u$ is the Euler operator defined in Eq. (6.2.3). Let us consider the infinitesimal operator for Eq. (4.1.1)

$$X_1 = \tau \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial x_1} + \xi^2 \frac{\partial}{\partial x_2} + \eta \frac{\partial}{\partial u}.$$ 

Using Eq. (1.2.12) and separation with respect to derivatives of $u$ results in

$$\tau_u = 0, \quad \xi^1_u = 0, \quad \xi^2_u = 0,$$

(4.2.3)

$$\xi^1_t - \frac{\tau x_1}{E_{11}} + \tau \left( \frac{(E_{11})_x}{2E_{11}} + \frac{(E_{22})_x}{2E_{11}E_{22}} \right) = 0,$$

(4.2.4)

$$\xi^2_t - \frac{\tau x_2}{E_{22}} + \tau \left( \frac{(E_{22})_x}{2E_{22}} + \frac{(E_{11})_x}{2E_{11}E_{22}} \right) = 0,$$

(4.2.5)

$$\eta_u + \frac{\tau_t}{2} + \frac{\xi^1_{x_1}}{2} + \frac{\xi^2_{x_2}}{2} = \tau,$$

(4.2.6)

$$\eta_u + \frac{\tau_t}{2} + \frac{\xi^1_{x_1}}{2} + \frac{\xi^2_{x_2}}{2} = \xi^1_{x_1} + \frac{(E_{11})_x \xi^2}{2E_{11}} - \frac{(E_{22})_x \xi^1}{2E_{22}},$$

(4.2.7)

$$\eta_u + \frac{\tau_t}{2} + \frac{\xi^1_{x_1}}{2} + \frac{\xi^2_{x_2}}{2} = \xi^2_{x_2} + \frac{(E_{22})_x \xi^1}{2E_{22}} - \frac{(E_{11})_x \xi^2}{2E_{11}},$$

(4.2.8)

$$\left( \frac{(E_{22})_x}{2E_{22}} + \frac{(E_{11})_x}{2E_{11}E_{22}} \right) \xi^1 + \left( \frac{(E_{11})_x}{2E_{11}} + \frac{(E_{22})_x}{2E_{11}E_{22}} \right) \xi^2 - \frac{\xi^1_{x_2}}{E_{22}} - \frac{\xi^2_{x_1}}{E_{11}} = 0,$$

(4.2.9)

$$B^1 = -\eta u + \alpha(t, x_1, x_2),$$

(4.2.10)

$$B^2 = \left[ \frac{\eta x_1}{E_{11}} - \eta \left( \frac{(E_{11})_x}{2E_{11}^2} + \frac{(E_{22})_x}{2E_{11}E_{22}} \right) \right] u + \beta(t, x_1, x_2),$$

(4.2.11)

$$B^3 = \left[ \frac{\eta x_2}{E_{22}} - \eta \left( \frac{(E_{22})_x}{2E_{22}^2} + \frac{(E_{11})_x}{2E_{11}E_{22}} \right) \right] u + \gamma(t, x_1, x_2),$$

(4.2.12)

$$B^1_t + B^2_{x_1} + B^3_{x_2} = 0.$$  

(4.2.13)
The substitution of Eqs. (4.2.10 – 4.2.12) into Eq. (4.2.13) and some lengthy manipulations yield

\[ \eta_{tt} = \frac{1}{E_{11}} \eta_{x_1 x_1} + \frac{1}{E_{22}} \eta_{x_2 x_2} - \left( \frac{3(E_{11})_{x_1}}{2E_{11}^2} + \frac{(E_{22})_{x_1}}{2E_{11}E_{22}} \right) \eta_{x_1} - \left( \frac{3(E_{22})_{x_2}}{2E_{22}^2} + \frac{(E_{11})_{x_2}}{2E_{11}E_{22}} \right) \eta_{x_2} \]

\[ -\left( \frac{(E_{11})_{x_1 x_1}}{E_{11}^2} - \frac{2[(E_{11})_{x_1}]^2}{E_{11}^3} + \frac{(E_{22})_{x_1 x_1}}{E_{11}E_{22}} - \frac{(E_{22})_{x_1}(E_{11})_{x_1}}{E_{11}^2E_{22}} - \frac{[(E_{22})_{x_1}]^2}{E_{11}E_{22}^2} + \frac{(E_{22})_{x_2 x_2}}{E_{22}^2} \right) \eta + \frac{2[(E_{22})_{x_2}]^2}{E_{22}^3} + \frac{(E_{11})_{x_2 x_2}}{E_{11}E_{22}} - \frac{[(E_{11})_{x_2}]^2}{E_{11}^2E_{22}^2} - \frac{(E_{11})_{x_2}(E_{22})_{x_2}}{E_{11}E_{22}^2} \right) \eta \]

\[ \alpha_t + \beta_{x_1} + \gamma_{x_2} = 0. \]  

(4.2.14)

4.2.1 Partial Noether operators and conservation laws for Eq. (4.1.1) on different curved surfaces

In this section, we will discuss Eq. (4.1.1) on different surfaces. By using results derived in Section 3 we will calculate partial Noether operators on different surfaces. Then we will use the symmetry conservation laws relation (1.2.14) to derive associated components of conserved vectors with the partial Lagrangian which simplify the conservation laws for the wave equation on different surfaces.

4.2.2 Conservation laws for wave equation on sphere

After substituting the coefficients of the first fundamental form of the sphere

\[ E_{11} = 1, \quad E_{22} = \sin^2 x_1 \]  

(4.2.16)

into Eq. (4.1.1), we obtain

\[ u_{tt} = u_{x_1 x_1} + \cot x_1 u_{x_1} + \csc^2 x_1 u_{x_2 x_2}. \]  

(4.2.17)
The determining equations Eqs. (4.2.3 – 4.2.14), using Eq. (4.2.16), give rise to

\begin{align*}
\tau_u &= 0, \quad \xi_u^1 = 0, \quad \xi_u^2 = 0, \quad (4.2.18) \\
\xi_t^1 - \tau x_1 + \cot x_1 \tau &= 0, \quad (4.2.19) \\
\xi_t^2 - \csc^2 x_1 \tau x_2 &= 0, \quad (4.2.20) \\
\eta_u + \frac{\tau t}{2} + \frac{\xi_t^1 x_1}{2} + \frac{\xi_t^2 x_2}{2} &= \tau, \quad (4.2.21) \\
\eta_u + \frac{\tau t}{2} + \frac{\xi_t^1 x_1}{2} + \frac{\xi_t^2 x_2}{2} &= \xi_t^1 + \cot x_1 \xi^1, \quad (4.2.22) \\
\eta_u + \frac{\tau t}{2} + \frac{\xi_t^1 x_1}{2} + \frac{\xi_t^2 x_2}{2} &= \xi_t^2 + \cot x_1 \xi^1, \quad (4.2.23) \\
\cot x_1 \xi^2 - \csc^2 x_1 \xi^1 - \xi^2 x_1 &= 0 \quad (4.2.24)
\end{align*}

and

\begin{align*}
B^1 &= -\eta u + \alpha_1(t, x_1, x_2), \quad (4.2.25) \\
B^2 &= (\eta x_1 - \eta \cot x_1)u + \beta_1(t, x_1, x_2), \quad (4.2.26) \\
B^3 &= \eta x_2 u \csc^2 x_1 + \gamma_1(t, x_1, x_2), \quad (4.2.27)
\end{align*}

where \( \eta \) satisfies

\begin{align*}
\eta_{tt} = \eta x_1 x_1 - \cot x_1 \eta x_1 + \csc^2 x_1 \eta x_2 x_2 + \csc^2 x_1 \eta. \quad (4.2.28)
\end{align*}

Solving Eqs. (4.2.18-4.2.24), we arrive at

\begin{align*}
\tau &= a_1 \sin x_1,
\end{align*}
\[ \xi^1 = \sin x_1 (a_2 \cos x_2 + a_3 \sin x_2), \]
\[ \xi^2 = \cos x_1 (a_3 \cos x_2 - a_2 \sin x_2), \] (4.2.29)

\[ \eta = \eta(t, x_1, x_2), \]

where \( \eta \) satisfies Eq. (4.2.28). Without loss of generality we can take

\[ (\alpha_1) = (\beta_1) = (\gamma_1) = 0 \]

in Eqs. (4.2.25 – 4.2.27) which reduces to

\[ B^1 = -\eta u, \] (4.2.30)

\[ B^2 = (\eta x_1 - \eta \cot x_1)u, \] (4.2.31)

\[ B^3 = \eta x_2 u \csc^2 x_1. \] (4.2.32)

The partial Noether operators are constructed by setting one constant equal to unity and the remaining constants to zero:

\[ X_1 = \sin x_1 \frac{\partial}{\partial t}, \quad X_2 = \cos x_2 \sin x_1 \frac{\partial}{\partial x_1} - \cos x_1 \sin x_2 \frac{\partial}{\partial x_2}, \]

\[ X_3 = \sin x_2 \sin x_1 \frac{\partial}{\partial x_1} + \cos x_1 \cos x_2 \frac{\partial}{\partial x_2}, \quad X_\mu = \mu \frac{\partial}{\partial u} . \] (4.2.33)

I: The components of the conserved vector corresponding to \( X_1 \) by using Eq. (1.2.13) are

\[ T^1_1 = -\frac{1}{2} \left( \sin x_1 u^2_{x_1} + \csc x_1 u^2_{x_2} + \sin x_1 u^2_t \right), \]

\[ T^2_1 = \sin x_1 u_t u_{x_1}, \] (4.2.34)

\[ T^3_1 = \csc x_1 u_t u_{x_2}. \]
II: Similarly for $X_2$ we obtain the following components of the conserved vector

\[ T_1^1 = -\cos x_2 \sin x_1 u_t u_{x_1} + \cos x_1 \sin x_2 u_t u_{x_2}, \]

\[ T_2^2 = \frac{1}{2} \left( \cos x_2 \sin x_1 u_t^2 - \cos x_2 \csc x_1 u_{x_2}^2 + \cos x_1 \sin x_1 u_{x_1}^2 \right) \]

\[ + \sin x_2 \cos x_1 u_{x_1} u_{x_2}, \quad (4.2.35) \]

\[ T_3^3 = \frac{1}{2} \left( \cos x_1 \sin x_2 u_{x_1}^2 - \cos x_1 \sin x_2 u_t^2 - \cos x_1 \csc^2 x_1 \sin x_2 u_{x_2}^2 \right) \]

\[ + \cos x_2 \csc x_1 u_{x_1} u_{x_2}. \]

III: Routine calculations show that the components of the conserved vector corresponding to $X_3$ are

\[ T_1^1 = -\sin x_2 \sin x_1 u_t u_{x_1} - \cos x_2 \cos x_1 u_t u_{x_2}, \]

\[ T_2^2 = -\frac{1}{2} \left( \sin x_2 \csc x_1 u_{x_2}^2 - \sin x_2 \sin x_1 u_t^2 - \sin x_2 \sin x_1 u_{x_1}^2 \right) + \cos x_2 \cos x_1 u_{x_1} u_{x_2}, \quad (4.2.36) \]

\[ T_3^3 = -\frac{1}{2} \left( \cos x_1 \cos x_2 u_{x_1}^2 - \cos x_1 \cos x_2 u_t^2 - \cos x_1 \csc^2 x_1 \cos x_2 u_{x_2}^2 \right) + \sin x_2 \csc x_1 u_{x_1} u_{x_2}. \]

IV: The components of the conserved vectors for $X_\eta$ ($\eta \neq 0$) can be written as

\[ T_1^\eta = -\eta u + \eta u_t, \]

\[ T_2^\eta = \left( \eta x_1 - \eta \cot x_1 \right) u - \eta u_{x_1}, \quad (4.2.37) \]

\[ T_3^\eta = \left( \eta x_2 u - \eta u_{x_2} \right) \csc^2 x_1, \]
where $\eta$ is a solution of Eq. (4.2.28). By using relation (1.2.14) one can associate Lie point symmetries of an equation with its conserved components which simplify the conserved vectors and conditions on unknowns. Note that it is interesting to employ Lie algebras to get some valuable results. The following are the infinitesimal generators of Eq. (4.2.17) (see [5]).

\[
Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = u \frac{\partial}{\partial u}, \quad Y_3 = \cos x_2 \frac{\partial}{\partial x_1} - \cot x_1 \sin x_2 \frac{\partial}{\partial x_2},
\]

\[
Y_4 = \sin x_2 \frac{\partial}{\partial x_1} + \cot x_1 \cos x_2 \frac{\partial}{\partial x_2}, \quad Y_5 = \frac{\partial}{\partial x_2}.
\] (4.2.38)

The conserved vectors (4.2.37) yield infinitely many conservation laws for the unknown parameter $\eta$ which satisfies Eq. (4.2.28). Now our next task is to calculate all possible conserved vectors from (4.2.37) which are associated with the translational symmetries given in (4.2.38) by using the relation (1.2.14). Translational symmetries are very helpful in calculating the associated conserved vectors from (4.2.37) because they directly reduce the dimension of Eq. (4.2.28), while two-dimensional algebras of these symmetries convert Eq. (4.2.28) into an ODE.

**Lie algebras associated with the conservation laws**

We consider the one- and two-dimensional algebras of translational symmetries of the wave equation on the sphere and we will associate with its conserved vectors.

I. Using $X = Y_1$ and the conserved vectors from (4.2.37) in (1.2.14) the components of the conserved vectors are

\[
T^1_\eta = \eta u_t,
\]

\[
T^2_\eta = \left( \eta_{x_1} - \cot x_1 \right) u - \eta u_{x_1},
\] (4.2.39)
$T^3_\eta = \left( \eta_x u - \eta u_x \right) \csc^2 x_1,$

where $\eta$ is the solution of the following system of PDEs

\[
\eta_t = 0, \quad \eta_{x_1 x_1} - \cot x_1 \eta_{x_1} + \csc^2 x_1 \eta_{x_2 x_2} + \csc^2 x_1 \eta = 0. \tag{4.2.40}
\]

II. The associated components of conserved vectors after substituting $X = Y_5$ in (1.2.14) are

\[
T^1_\eta = -\eta u + \eta u_t,
\]

\[
T^2_\eta = \left( \eta_{x_1} - \cot x_1 \right) u - \eta u_{x_1}, \tag{4.2.41}
\]

\[
T^3_\eta = -\eta u_{x_2} \csc^2 x_1,
\]

where $\eta$ satisfies

\[
\eta_{x_2} = 0, \quad \eta_{tt} = \eta_{x_1 x_1} - \eta_{x_1} \cot x_1 + \eta \csc^2 x_1. \tag{4.2.42}
\]

Note that without the association of translational symmetries with the conserved vectors (4.2.37), it is not possible to convert Eq. (4.2.28) into a system of PDEs.

III. Following routine calculations the two dimensional algebra $< Y_1, Y_5 >$ reduces Eq. (4.2.28) to an ODE so that the following components of the conserved vector are obtained

\[
T^1_\eta = \eta u_t,
\]

\[
T^2_\eta = \left( \eta_{x_1} - \cot x_1 \right) u - \eta u_{x_1}, \tag{4.2.43}
\]

\[
T^3_\eta = -\eta u_{x_2} \csc^2 x_1,
\]

where

\[
\eta = b \sin x_1 + a \left[ \ln \left( \sin \left( \frac{x_1}{2} \right) \right) - \ln \left( \cos \left( \frac{x_1}{2} \right) \right) \right] \sin x_1,
\]

with $a$ and $b$ real constants.

There is no associated non-trivial conservation law for the remainder of the algebras.
4.2.3 Conservation laws for the wave equation on a cone

The wave equation on a cone is obtained by substituting

\[ E_{11} = x_2^2, \quad E_{22} = 2 \]  \hspace{1cm} (4.2.44)

in Eq. (4.1.1) i.e

\[ u_{tt} = \frac{u_{x_1 x_1}}{x_2^2} + \frac{u_{x_2 x_2}}{2x_2} + \frac{u_{x_2}}{2x_2}. \]  \hspace{1cm} (4.2.45)

The determining Eqs. (4.2.3 - 4.2.13) after substituting the values of \( E_{11} \) and \( E_{22} \) become

\[ \tau_u = 0, \quad \xi_u^1 = 0, \quad \xi_u^2 = 0, \]  \hspace{1cm} (4.2.46)

\[ \xi_t^1 - \frac{\tau_{x_1}}{x_2^2} = 0, \]  \hspace{1cm} (4.2.47)

\[ \xi_t^2 - \frac{\tau_{x_2}}{2} + \frac{\tau}{2x_2} = 0, \]  \hspace{1cm} (4.2.48)

\[ \eta_u + \frac{\tau_t}{2} + \frac{\xi_{x_1}}{2} + \frac{\xi_{x_2}}{2} = \tau_t, \]  \hspace{1cm} (4.2.49)

\[ \eta_u + \frac{\tau_t}{2} + \frac{\xi_{x_1}}{2} + \frac{\xi_{x_2}}{2} = \xi_{x_1} + \frac{\xi_{x_2}}{x_2}, \]  \hspace{1cm} (4.2.50)

\[ \eta_u + \frac{\tau_t}{2} + \frac{\xi_{x_1}}{2} + \frac{\xi_{x_2}}{2} = \xi_{x_2} - \frac{\xi_{x_1}}{x_2}, \]  \hspace{1cm} (4.2.51)

\[ \frac{\xi_{x_1}}{2x_2} - \frac{\xi_{x_2}}{2} + \frac{\xi_{x_1}}{x_2} = 0, \]  \hspace{1cm} (4.2.52)

\[ B^1 = -\eta_t u + \alpha(t, x_1, x_2), \]  \hspace{1cm} (4.2.53)

\[ B^2 = \eta_{x_2} + \beta(t, x_1, x_2), \]  \hspace{1cm} (4.2.54)
\[ B^3 = \left( \frac{\eta_{x_2}}{2} - \frac{\eta}{2x_2} \right) u + \gamma(t, x_1, x_2), \]  
(4.2.55)

\[ B_1^1 + B_2^2 + B_3^3 = 0. \]  
(4.2.56)

The solution of Eqs. (4.2.46 – 4.2.52) is

\[ \tau = C_3x_2 + x_2^2 \left( C_1 \cos\left( \frac{x_1}{\sqrt{2}} \right) + C_2 \sin\left( \frac{x_1}{\sqrt{2}} \right) \right), \]

\[ \xi^1 = \frac{t}{\sqrt{2}} \left( C_2 \cos\left( \frac{x_1}{\sqrt{2}} \right) - C_1 \sin\left( \frac{x_1}{\sqrt{2}} \right) \right) - \sqrt{2} \left( C_4 \sin\left( \frac{x_1}{\sqrt{2}} \right) - C_5 \cos\left( \frac{x_1}{\sqrt{2}} \right) \right) + C_6x_2, \]

\[ \xi^2 = \frac{x_2t}{2} \left( C_1 \cos\left( \frac{x_1}{\sqrt{2}} \right) + C_2 \sin\left( \frac{x_1}{\sqrt{2}} \right) \right) + x_2 \left( C_4 \cos\left( \frac{x_1}{\sqrt{2}} \right) + C_5 \sin\left( \frac{x_1}{\sqrt{2}} \right) \right), \]  
(4.2.57)

\[ \eta = \eta(t, x_1, x_2), \]

where \( \eta \) can be obtained from

\[ \eta_{tt} = \frac{\eta_{x_1x_1}}{x_2^2} + \frac{\eta_{x_2x_2}}{2} - \frac{\eta_{x_2}}{2x_2} + \frac{\eta}{2x_2^2}. \]  
(4.2.58)

The partial Noether operators are obtained by setting one constant to unity and the other constants to zero.

\[ X_1 = x_2^2 \cos\left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial t} - \frac{t}{\sqrt{2}} \sin\left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_1} + \left( \frac{x_2t}{2} \right) \cos\left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_2}, \]

\[ X_2 = x_2^2 \sin\left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial t} + \frac{t}{\sqrt{2}} \cos\left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_1} + \left( \frac{x_2t}{2} \right) \sin\left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_2}, \]  
(4.2.59)

\[ X_3 = x_2 \frac{\partial}{\partial t}, \quad X_4 = -\sqrt{2} \sin\left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_1} + x_2 \cos\left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_2}, \]

\[ X_5 = \sqrt{2} \cos\left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_1} + x_2 \sin\left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_2}, \quad X_6 = x_2 \frac{\partial}{\partial x_1}, \quad X_\eta = \eta \frac{\partial}{\partial u}, \]  
(4.2.60)
where $\eta$ is a solution of Eq. (4.2.58).

The conserved vectors computed for Eq. (4.2.45) corresponding to each $X_i$ by using relation (1.2.14), are summarized below

I: The components of the conserved vector corresponding to $X_1$ are

\[
T_1^1 = -x_2^2 \cos\left(\frac{x_1}{\sqrt{2}}\right)\left(\frac{u_{x_1}^2}{2x_2^2} + \frac{u_{x_2}^2}{4} - \frac{u_t^2}{2}\right) - \left(x_2^2 \cos\left(\frac{x_1}{\sqrt{2}}\right)u_t - \frac{t}{\sqrt{2}} \sin\left(\frac{x_1}{\sqrt{2}}\right)u_{x_1}\right)
+ \left(x_2t \cos\left(\frac{x_1}{\sqrt{2}}\right)u_{x_2}\right)u_t,
\]

\[
T_1^2 = \frac{t}{\sqrt{2}} \sin\left(\frac{x_1}{\sqrt{2}}\right)\left(\frac{u_{x_1}^2}{2x_2^2} + \frac{u_{x_2}^2}{4} - \frac{u_t^2}{2}\right) + \left(x_2^2 \cos\left(\frac{x_1}{\sqrt{2}}\right)u_t - \frac{t}{\sqrt{2}} \sin\left(\frac{x_1}{\sqrt{2}}\right)u_{x_1}\right)
+ \left(x_2t \cos\left(\frac{x_1}{\sqrt{2}}\right)u_{x_2}\right)\frac{u_{x_1}}{x_2^2},
\]

\[
T_1^3 = -\frac{x_2t}{2} \cos\left(\frac{x_1}{\sqrt{2}}\right)\left(\frac{u_{x_1}^2}{2x_2^2} + \frac{u_{x_2}^2}{4} - \frac{u_t^2}{2}\right) + \left(x_2^2 \cos\left(\frac{x_1}{\sqrt{2}}\right)u_t - \frac{t}{\sqrt{2}} \sin\left(\frac{x_1}{\sqrt{2}}\right)u_{x_1}\right)
+ \left(x_2t \cos\left(\frac{x_1}{\sqrt{2}}\right)u_{x_2}\right)\frac{u_{x_2}}{x_2^2}.
\]

II: For $X_2$ we obtain

\[
T_2^1 = -x_2^2 \sin\left(\frac{x_1}{\sqrt{2}}\right)\left(\frac{u_{x_1}^2}{2x_2^2} + \frac{u_{x_2}^2}{4} - \frac{u_t^2}{2}\right) - \left(x_2^2 \sin\left(\frac{x_1}{\sqrt{2}}\right)u_t + \frac{t}{\sqrt{2}} \cos\left(\frac{x_1}{\sqrt{2}}\right)u_{x_1}\right)
+ \left(x_2t \sin\left(\frac{x_1}{\sqrt{2}}\right)u_{x_2}\right)u_t,
\]

\[
T_2^2 = -\frac{t}{\sqrt{2}} \cos\left(\frac{x_1}{\sqrt{2}}\right)\left(\frac{u_{x_1}^2}{2x_2^2} + \frac{u_{x_2}^2}{4} - \frac{u_t^2}{2}\right) + \left(x_2^2 \sin\left(\frac{x_1}{\sqrt{2}}\right)u_t + \frac{t}{\sqrt{2}} \cos\left(\frac{x_1}{\sqrt{2}}\right)u_{x_1}\right)
+ \left(x_2t \sin\left(\frac{x_1}{\sqrt{2}}\right)u_{x_2}\right).
\[ T^3_2 = -\frac{x_2}{2} \sin\left(\frac{x_1}{\sqrt{2}}\right)\left(\frac{u^2_{x_1}}{2x_2^2} + \frac{u^2_{x_2}}{4} - \frac{u^2_t}{2}\right) + \left(\frac{x_2}{2} \sin\left(\frac{x_1}{\sqrt{2}}\right)u_t + \frac{t}{\sqrt{2}} \cos\left(\frac{x_1}{\sqrt{2}}\right)u_{x_1}\right) \]
\[ + \left(\frac{x_2^2}{2} \sin\left(\frac{x_1}{\sqrt{2}}\right)u_{x_2}\right)^2. \]  

III: By using \( X_3 \) following results are obtained
\[ T^1_3 = -x_2 \left(\frac{u^2_{x_1}}{2x_2^2} + \frac{u^2_{x_2}}{4} - \frac{u^2_t}{2}\right) - x_2u^2_t, \]
\[ T^2_3 = \frac{x_2u_tu_{x_1}}{x_2^2}, \]
\[ T^3_3 = \left(x_2u_t\right)^2 \frac{u_{x_2}}{2}. \]

IV: For \( X_4 \) the conserved vectors are
\[ T^1_4 = \left(\sqrt{2} \sin\left(\frac{x_1}{\sqrt{2}}\right)u_{x_1} - x_2 \cos\left(\frac{x_1}{\sqrt{2}}\right)u_{x_2}\right) u_t, \]
\[ T^2_4 = \sqrt{2} \sin\left(\frac{x_1}{\sqrt{2}}\right)\left(\frac{u^2_{x_1}}{2x_2^2} + \frac{u^2_{x_2}}{4} - \frac{u^2_t}{2}\right) - \left(\sqrt{2} \sin\left(\frac{x_1}{\sqrt{2}}\right)u_{x_1} - x_2 \cos\left(\frac{x_1}{\sqrt{2}}\right)u_{x_2}\right) \frac{u_{x_1}}{x_2^2}, \]
\[ T^3_4 = -x_2 \cos\left(\frac{x_1}{\sqrt{2}}\right)\left(\frac{u^2_{x_1}}{2x_2^2} + \frac{u^2_{x_2}}{4} - \frac{u^2_t}{2}\right) - \left(\sqrt{2} \sin\left(\frac{x_1}{\sqrt{2}}\right)u_{x_1} - x_2 \cos\left(\frac{x_1}{\sqrt{2}}\right)u_{x_2}\right) \frac{u_{x_2}}{2}. \]

V: The conserved vectors for \( X_5 \) are
\[ T^1_5 = -\left(\sqrt{2} \cos\left(\frac{x_1}{\sqrt{2}}\right)u_{x_1} + x_2 \sin\left(\frac{x_1}{\sqrt{2}}\right)u_{x_2}\right) u_t, \]
\[ T^2_5 = -\sqrt{2} \cos\left(\frac{x_1}{\sqrt{2}}\right)\left(\frac{u^2_{x_1}}{2x_2^2} + \frac{u^2_{x_2}}{4} - \frac{u^2_t}{2}\right) + \left(\sqrt{2} \cos\left(\frac{x_1}{\sqrt{2}}\right)u_{x_1} + x_2 \sin\left(\frac{x_1}{\sqrt{2}}\right)u_{x_2}\right) \frac{u_{x_1}}{x_2^2}, \]
\[ T^3_5 = -x_2 \sin\left(\frac{x_1}{\sqrt{2}}\right)\left(\frac{u^2_{x_1}}{2x_2^2} + \frac{u^2_{x_2}}{4} - \frac{u^2_t}{2}\right) + \left(\sqrt{2} \cos\left(\frac{x_1}{\sqrt{2}}\right)u_{x_1} + x_2 \sin\left(\frac{x_1}{\sqrt{2}}\right)u_{x_2}\right) \frac{u_{x_2}}{2}. \]
VI: Similarly \( X_6 \) yields the following components of the conserved vector

\[
T_1^3 = -x_2 u_x u_t,
\]

\[
T_2^3 = -x_2 \left( \frac{u_x^2 + u_{x1}^2}{2x_2^2} - \frac{u_{x1}^2}{2} \right) + \frac{u_{x1}^2}{x_2},
\]

\[
T_3^3 = \frac{x_2 u_{x1} u_{x2}}{2}.
\]

The above conserved vectors were obtained when \( \eta = 0 \). For \( \eta \neq 0 \), Eq. (1.2.13) with the help of partial Noether operator yields

\[
T_1^\eta = \eta u_t - \eta u,
\]

\[
T_2^\eta = \frac{\eta x_1 u - \eta u x_1}{x_2^2},
\]

\[
T_3^\eta = \left( \eta x_2 - \frac{\eta}{x_2} \right) \frac{u}{2} - \frac{\eta u x_2}{2},
\]

where \( \eta \) satisfies Eq. (4.2.58).

One can easily compute the finite Lie point symmetries for Eq. (4.2.45) (see [11, 34, 50])

\[
Y_1 = tx_2 \cos \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial t} - \left( \frac{t^2 - 2x_2^2}{2\sqrt{2}x_2^2} \right) \sin \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_1} + \left( \frac{t^2 + 2x_2^2}{4} \right) \cos \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_2} - \frac{x_2}{2} \cos \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial u},
\]

\[
Y_2 = x_2 \cos \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial t} - t \sin \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_1} + \frac{t}{2} \cos \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_2},
\]

\[
Y_3 = tx_2 \sin \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial t} - \left( \frac{t^2 - 2x_2^2}{2\sqrt{2}x_2^2} \right) \cos \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_1} + \left( \frac{t^2 + 2x_2^2}{4} \right) \sin \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_2} - \frac{x_2}{2} \sin \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial u},
\]

\[
Y_4 = x_2 \sin \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial t} - \frac{t}{x_2\sqrt{2}} \cos \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_1} + \frac{t}{2} \sin \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_2},
\]

\[
Y_5 = \left( \frac{t^2 + 2x_2^2}{2} \right) \frac{\partial}{\partial t} + tx_2 \frac{\partial}{\partial x_2} - \frac{t}{2} \frac{\partial}{\partial u}, \quad Y_6 = t \frac{\partial}{\partial t} + x_2 \frac{\partial}{\partial x_2},
\]

\[
Y_7 = \frac{\partial}{\partial t}, \quad Y_8 = \frac{\partial}{\partial x_1}, \quad Y_9 = -\sqrt{2} \frac{x_1}{x_2} \sin \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_1} + \cos \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_2},
\]

\[
Y_{10} = \sqrt{2} \frac{x_1}{x_2} \cos \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_1} + \sin \left( \frac{x_1}{\sqrt{2}} \right) \frac{\partial}{\partial x_2}, \quad Y_{11} = \frac{\partial}{\partial u}.
\]
Lie algebras associated with the conservation laws

We will use the symmetry conservation laws relation (1.2.14) to derive the components of the conserved vectors from (4.2.67) associated with the symmetries $Y_6$, $Y_7$ and $Y_8$ from (4.2.68). The reduced conserved vectors will be of lower dimension when compared to the original conserved vectors.

I: The components of the conserved vector (4.2.67) associated with $<Y_6>$ are the same as (4.2.67) in which $\eta$ satisfies Eq. (4.2.58) and

$$t\eta_t + x_2 \eta_{x_2} = 0.$$ \hspace{1cm} (4.2.69)

II: For $<Y_7>$ the following are the associated components of conserved vectors

$$T^1_\eta = \eta u_t,$$

$$T^2_\eta = \frac{(\eta_{x_1} u - \eta u_{x_1})}{x_2^2},$$ \hspace{1cm} (4.2.70)

$$T^3_\eta = (\eta_{x_2} - \frac{\eta}{x_2}) \frac{u}{2} - \frac{\eta u_{x_2}}{2},$$

where $\eta$ is a solution of

$$\eta_t = 0, \quad \frac{\eta_{x_1} x_1}{x_2^2} + \frac{\eta_{x_2} x_2}{2} - \frac{\eta_{x_2}}{2 x_2} + \frac{\eta}{2 x_2^2} = 0.$$ \hspace{1cm} (4.2.71)

III: The use of $<Y_8>$ in (1.2.14) gives the following components of conserved vectors

$$T^1_\eta = \eta u_t - \eta_t u,$$

$$T^2_\eta = \frac{\eta u_{x_1}}{x_2^2},$$ \hspace{1cm} (4.2.72)

$$T^3_\eta = (\eta_{x_2} - \frac{\eta}{x_2}) \frac{u}{2} - \frac{\eta u_{x_2}}{2},$$
where $\eta$ can be calculated from

\[ \eta_{x_1} = 0, \quad \eta_{tt} = \frac{\eta_{x_2} x_2}{2} - \frac{\eta_{x_2}}{2 x_2} + \frac{\eta}{2 x_2^2}. \]  \hspace{1cm} (4.2.73)

Now we will compute the associated components of the conserved vectors with two-dimensional algebra.

I: For the two-dimensional algebra $< Y_7, Y_8 >$ we obtain the associated components of the conserved vector as

\[ T^1_\eta = \eta u_t, \]

\[ T^2_\eta = -\frac{\eta u_{x_1}}{x_2^2}, \]

\[ T^3_\eta = \left( \eta_{x_2} - \frac{\eta}{x_2} \right) u - \frac{\eta u_{x_2}}{2}, \]

where

\[ \eta = (a + b \ln x_2)x_2, \]

with $a$ and $b$ real constants.

II: The components of the conserved vectors (4.2.67) associated with $< Y_6, Y_8 >$ are

\[ T^1_\eta = \eta u_t - \eta_t u, \]

\[ T^2_\eta = -\frac{\eta u_{x_1}}{x_2^2}, \]

\[ T^3_\eta = \eta_{x_2} u - \eta u_{x_2} - \frac{\eta u}{x_2}, \]

where $\eta$ can be calculated from

\[ t\eta_t + x_2 \eta_{x_2} = 0, \quad \eta_{x_1} = 0, \]
\( \eta_{tt} = \eta_{x_2 x_2} - \frac{\eta_{x_2}}{x_2} + \frac{\eta}{2x_2^2}. \)  

(4.2.76)

III: The components of the conserved vectors (4.2.67) associated with the two-dimensional algebra \(< Y_6, Y_7 >\) are

\[
T^1_\eta = \eta u_t, \\
T^2_\eta = \frac{\eta_{x_1} u - \eta u_{x_1}}{x_2^2}, \\
T^3_\eta = -\frac{\eta u}{2x_2} - \frac{\eta u_{x_2}}{2},
\]

(4.2.77)

where

\[
\eta = a \cos \left( \frac{x_1}{\sqrt{2}} \right) + b \sin \left( \frac{x_1}{\sqrt{2}} \right)
\]

(4.2.78)

with \(a\) and \(b\) real constants.

Conserved vectors (4.2.77) cannot be obtained directly without using Eq. (1.2.14).

**Three-dimensional algebras**

The association of the three-dimensional algebra \(< Y_6, Y_7, Y_8 >\) with the conserved vectors (4.2.67) yields only trivial conservation laws.

The remaining algebras yield either known or trivial results.

**4.2.4 The wave equation on a flat surface**

We consider the wave equation on a flat surface. For this we take

\[
E_{11} = 1, \quad E_{22} = 1.
\]

(4.2.79)
After substituting Eq. (4.2.79) in Eq. (4.1.1), we obtain

\[ u_{tt} = u_{x_1 x_1} + u_{x_2 x_2}. \]  

(4.2.80)

The replacement of \( E_{11}, E_{22} \) from Eq. (4.2.79) in Eqs. (4.2.3–4.2.9) yields the set of equations

\[ \tau_u = 0, \quad \xi_u^1 = 0, \quad \xi_u^2 = 0, \]  

(4.2.81)

\[ \xi_t^1 - \tau_{x_1} = 0, \]  

(4.2.82)

\[ \xi_t^2 - \tau_{x_2} = 0, \]  

(4.2.83)

\[ \eta_u + \frac{\tau_t}{2} + \frac{\xi_u^1}{2} x_1 + \frac{\xi_u^2}{2} x_2 = \tau_t, \]  

(4.2.84)

\[ \eta_u + \frac{\tau_t}{2} + \frac{\xi_u^1}{2} x_1 + \frac{\xi_u^2}{2} x_2 = \xi_u^1, \]  

(4.2.85)

\[ \eta_u + \frac{\tau_t}{2} + \frac{\xi_u^1}{2} x_1 + \frac{\xi_u^2}{2} x_2 = \xi_u^2, \]  

(4.2.86)

\[ \xi_{x_2}^1 + \xi_{x_1}^2 = 0. \]  

(4.2.87)

From Eqs. (4.2.81–4.2.87) we conclude that

\[ \tau = C_1 + C_4 t + C_6 x_1 + C_7 x_2 + (t^2 + x_1^2 + x_2^2)C_8 + C_9 t x_1 + C_{10} t x_2, \]  

(4.2.88)

\[ \xi^1 = C_2 + C_4 x_1 + C_5 x_2 + C_6 t + (t^2 + x_1^2 - x_2^2) \frac{C_9}{2} + 2C_8 t x_1 + C_{10} x_1 x_2, \]  

(4.2.88)

\[ \xi^2 = C_3 + C_4 x_2 - C_5 x_1 + C_7 t + 2C_8 t x_2 + C_9 x_1 x_2 + (t^2 - x_1^2 + x_2^2) \frac{C_{10}}{2}, \]  

(4.2.88)

\[ \eta = \frac{-u}{2} (C_4 + 2tC_8 + x_1 C_9 + x_2 C_{10}) + \eta(t, x_1, x_2), \]
where η is the solution of Eq. (4.2.80).

For this case the partial Noether operators are

\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x_1}, \quad X_3 = \frac{\partial}{\partial x_2}, \quad X_4 = t \frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - \frac{u}{2} \frac{\partial}{\partial u}, \]
\[ X_5 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}, \quad X_6 = x_1 \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_1}, \quad X_7 = x_2 \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_2}, \]
\[ X_8 = (t^2 + x_1^2 + x_2^2) \frac{\partial}{\partial t} + 2tx_1 \frac{\partial}{\partial x_1} + 2tx_2 \frac{\partial}{\partial x_2} - tu \frac{\partial}{\partial u}, \quad (4.2.89) \]
\[ X_9 = tx_1 \frac{\partial}{\partial t} + \frac{1}{2}(t^2 + x_1^2 + x_2^2) \frac{\partial}{\partial x_1} + x_1x_2 \frac{\partial}{\partial x_2} - \frac{x_1u}{2} \frac{\partial}{\partial u}, \]
\[ X_{10} = tx_2 \frac{\partial}{\partial t} + x_1x_2 \frac{\partial}{\partial x_1} + \frac{1}{2}(t^2 - x_1^2 + x_2^2) \frac{\partial}{\partial x_2} - \frac{x_2u}{2} \frac{\partial}{\partial u}, \]
\[ X_\eta = \eta \frac{\partial}{\partial u}, \]

where η is the solution of Eq. (4.2.58). The partial Noether approach for the wave equation on a flat surface yields infinite partial Noether operators. The finite operators \( X_1 \) to \( X_{10} \) are studied in [34]. The conserved vectors generated due to infinite-dimensional operators are discussed below.

Let us take

\[ X_\eta = \eta \frac{\partial}{\partial u}, \quad (4.2.90) \]

and gauge terms corresponding to partial Noether operator (4.2.90) are

\[ B^1 = -\eta u + \alpha_3, \quad (4.2.91) \]
\[ B^2 = \eta x_1 u + \beta_3, \quad (4.2.92) \]
\[ B^3 = \eta x_2 u + \gamma_3, \quad (4.2.93) \]

where

\[ (\alpha_3)_t + (\beta_3)_{x_1} + (\gamma_3)_{x_2} = 0. \]

For simplicity, one can choose

\[ \alpha_3 = \beta_3 = \gamma_3 = 0. \]

By using relation (1.2.13) the components of the conserved vector are

\[ T^1 = -\eta_t u + \eta u_t, \]
\[ T^2 = \eta_{x_1} u - \eta u_{x_1}, \quad (4.2.94) \]
\[ T^3 = \eta_{x_2} u - \eta u_{x_2}. \]

Finite dimensional Lie point symmetries for Eq. (4.2.79) are given below (see [11, 34, 50]):

\[ Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x_1}, \quad Y_3 = \frac{\partial}{\partial x_2}, \quad Y_4 = t \frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \]
\[ Y_5 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}, \quad Y_6 = x_1 \frac{\partial}{\partial t} - t \frac{\partial}{\partial x_1}, \quad Y_7 = x_2 \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_2}, \]
\[ Y_8 = (t^2 + x_1^2 + x_2^2) \frac{\partial}{\partial t} + 2tx_1 \frac{\partial}{\partial x_1} + 2tx_2 \frac{\partial}{\partial x_2} - tu \frac{\partial}{\partial u}, \quad (4.2.95) \]
\[ Y_9 = tx_1 \frac{\partial}{\partial t} + \frac{1}{2}(t^2 + x_1^2 + x_2^2) \frac{\partial}{\partial x_1} + x_1x_2 \frac{\partial}{\partial x_2} - x_1u \frac{\partial}{\partial u}, \]
\[ Y_{10} = tx_2 \frac{\partial}{\partial t} + x_1x_2 \frac{\partial}{\partial x_1} + \frac{1}{2}(t^2 - x_1^2 + x_2^2) \frac{\partial}{\partial x_2} - x_2u \frac{\partial}{\partial u}, \]
\[ Y_{11} = u \frac{\partial}{\partial u}. \]
Lie algebras associated with the conservation laws

In this section using symmetry conservation law relation (1.2.14) one, two and three dimensional Lie algebras of translational symmetries from (4.2.95) are associated with components of the conserved vector (4.2.94).

I: The components of the conserved vector (4.2.94) associated with $X = Y_1$ by using (1.2.14) yield

$$T^1_\eta = \eta u_t,$$

$$T^2_\eta = \eta_{x_1} u - \eta u_{x_1},$$

$$T^3_\eta = \eta_{x_2} u - \eta u_{x_2},$$

where $\eta$ can be calculated from

$$\eta_t = 0, \quad \eta_{x_1 x_1} + \eta_{x_2 x_2} = 0.$$  \hspace{1cm} (4.2.97)

II: Associated components of the conserved vector (4.2.94) after substituting $X = Y_2$ in (1.2.14) are

$$T^1_\eta = -\eta u + \eta u_t,$$

$$T^2_\eta = -\eta_{x_1},$$

$$T^3_\eta = \eta_{x_2} u - \eta u_{x_2},$$

where $\eta$ is the solution of

$$\eta_{x_1} = 0, \quad \eta_{tt} - \eta_{x_2 x_2} = 0.$$  \hspace{1cm} (4.2.99)
III: Eq. (1.2.13) with the use of $X = Y_3$ from (4.2.95) and the conserved vector from (4.2.94) give rise to

$$T^1_\eta = -\eta u + \eta u_t,$$

$$T^2_\eta = \eta_{x1} u - \eta u_{x1},$$

$$T^3_\eta = -\eta u_{x2},$$

where \( \eta \) can be calculated from the following equations

$$\eta_{x2} = 0, \quad \eta_{tt} - \eta_{xx_1} = 0.$$  

(4.2.101)

All one-dimensional algebras can be associated. Similarly now we will associate two-dimensional Lie algebras.

IV: Simple calculations show that the two dimensional algebra \( < Y_1, Y_2 > \) associated with the conserved vector (4.2.94) reduces to

$$T^1 = (ax_2 + b)u_t,$$

$$T^2 = -(ax_2 + b)u_{x_1},$$

$$T^3 = au - (ax_2 + b)u_{x_2}.$$  

(4.2.102)

V: The associated conserved vector (4.2.94) with \( < Y_1, Y_3 > \) is

$$T^1 = (ax_1 + b)u_t,$$

$$T^2 = au - (ax_1 + b)u_{x_1}.$$  

(4.2.103)
\[ T^3 = -(ax_1 + b)u_{x_2}, \]

VI: For \( < Y_2, Y_3 > \) the associated components of the conserved vector are

\[ T^1 = -au + (at + b)u_t, \]

\[ T^2 = -(at + b)u_{x_1}, \]  \hspace{1cm} (4.2.104)

\[ T^3 = -(at + b)u_{x_2}, \]

where \( a, b \) and \( c \) are real constants.

VII: \( < Y_1, Y_2, Y_3 > \)

In this case we obtain the conserved vector below

\[ T^1 = du_t, \]

\[ T^2 = -du_{x_1}, \]  \hspace{1cm} (4.2.105)

\[ T^3 = -du_{x_2}. \]

where \( d \) is a real constant. For the remaining algebras, we get either the same or trivial results. It should be noted that above results cannot be obtained without the association of translational symmetries from (4.2.95) with the conserved vector (4.2.94).

4.3 Concluding Remarks

The partial Noether operators and components of the conserved vectors for the wave equation on different surfaces were computed by using the partial Noether approach. The determining equations for the \( (1+2) \)-dimensional wave equation on curved surfaces were constructed in
general form with the help of the partial Noether determining equation associated with a partial Lagrangian. Then these results were used to derive the conservation laws for the wave equation on a sphere, cone and flat space.

Moreover, we have used the relationship between symmetries and conservation laws given in Eq. (1.2.14), for which the conserved vectors and the PDE with unknown parameters become simpler. All classes of algebras of translational symmetries which simplify the conserved vectors were studied. We also commented those algebras which neither reduce the dimension of the wave equation nor give nontrivial results.
Chapter 5

Effect of Background Geometry on Symmetries of the Nonlinear (1 + 2)-dimensional Heat Equation and Reductions of the TDGL Model

This chapter is devoted to the investigation of the effect of changing the underlying space on the group classification of the nonlinear heat equation on curved surfaces. To this end, the nonlinear heat equation on a plane, sphere and torus is discussed. Further group classification of the heat equation is presented for each surface respectively. This leads to a class of functions which is not only independent of the number of independent variables but also independent of the underlying geometry. We also show that for the infinitesimal generators, the underlying geometry is more important than the nonlinearity. To finish, the complete symmetry analysis of the time-dependent Ginzburg-Landau equation (TDGL model) on the sphere and torus is presented.

5.1 Introduction

The study of differential equations on curved surfaces is the corner stone of the geometric analysis. From the last decades, it is a powerful tendency to involve geometry in the field of differential equation. Inter-membrane, protein diffusion, shape instabilities, ion absorption
etc. are one of the biological processes where shape, composition and functionality of the object play an important role. Thus to discuss the physical phenomena in these objects one cannot ignore the shape of these objects.

The two major types of membranes are vesicles and tubules, which can be differentiated by their geometry. Vesicles are ideally spherical objects whereas tubules is identical to circular cylinder. It is noted worthy that sphere has positive curvature while circular cylinder is isometric to plane thus has zero curvature. The geometry of torus is different from other surfaces it has variable curvature.

In [36], authors discussed the effect of background metric on conserved quantities for heat equation. The relationship between underlying space of the \((1 + 2)\)-dimensional heat equation and group classification is yet not reported in past. For this let us consider the nonlinear \((1 + 2)\)-dimensional heat equation on a curved surface \(\Omega\) (see [36, 43])

\[
    u_t = \frac{u_{x_1 x_1}}{E_{11}} + \frac{u_{x_2 x_2}}{E_{22}} - \left( \frac{(E_{11})_{x_1}}{2E_{11}^2} - \frac{(E_{22})_{x_1}}{2E_{11}E_{22}} \right) u_{x_1} - \left( \frac{(E_{22})_{x_2}}{2E_{22}^2} - \frac{(E_{11})_{x_2}}{2E_{11}E_{22}} \right) u_{x_2} + f(u), \quad (5.1.1)
\]

where \(f_{uu} \neq 0\) and

\[
    ds^2 = E_{11}dx_1^2 + F_{12}dx_1dx_2 + E_{22}dx_2^2
\]

is the FFF of the 2-dimensional surface \(\Omega\). We note in passing that the coefficients of \(u_{x_j x_j}\) and \(u_{x_j} (j = 1, 2)\) represent the background metric.

Group classification is an important aspect of Lie analysis. By group classification, one can find a class of functions for which the equation has nontrivial symmetries. Using nontrivial symmetries, one can either reduce the dimension of the equation or find a nontrivial solution.
Previously, the group classification was carried out for

$$u_t = \sum_{i=1}^{n} [b(u)u_{x_i}]_{x_i} + c(u).$$

(5.1.2)

Dorodnitsyn et al. [18] presented the group classification of Eq. (5.1.2) for \( n = 2 \) and \( n = 3 \). After that, Galaktionov et al. [28] carried out the group classification of Eq. (5.1.2) for arbitrary \( n \) and got the same results as obtained in [18]. The class of functions calculated for Eq. (5.1.2) in [18, 28] is independent of the number of independent variables.

The duplication of results in [18, 28] raises the following questions:

(i) Is there any class of functions for Eq. (5.1.1) independent of the background metric? (independent of the background metric means a common class of functions that is invariant from the different choice of background space.)

(ii) How does the underlying space of the equation affect the Lie algebra of the equation?

In this chapter, we will compute a class of \( f(u) \) for which Eq. (5.1.1) has nontrivial symmetries by considering Eq. (5.1.1) on plane, sphere and torus respectively. Moreover it is observed that a class of functions for Eq. (5.1.1) on sphere and torus is identical. After this result, we investigate the relative importance of the background metric and the nonlinearity of \( f(u) \) in the Lie symmetry algebra of Eq. (5.1.1). By these results one can easily show that the nonlinearity does not play any role in Lie symmetry generators of the TDGL model either the underlying space is sphere or torus. Some excellent books and work are [11, 33, 50, 55, 59].

The outline of this chapter is as follows: Section 2 is devoted to the calculation of a class of functions for Eq. (5.1.1) on plane, sphere and torus. The effect of both background geometry and \( f(u) \) on the infinitesimal generators of the corresponding equations is also given in this
section. In Section 3, classification of symmetry algebras of TDGL model on sphere and torus is carried out, which is then utilized for symmetry reductions of the TDGL model.

5.2 Group classification of Eq. (5.1.1)

In this section, we will perform the group classification of Eq. (5.1.1) on three different type of surfaces i.e. plane, sphere and torus respectively.

5.2.1 The nonlinear (1 + 2)-dimensional heat equation on plane

The nonlinear (1 + 2)-dimensional heat equation on plane:

\[ u_t = u_{x_1} + u_{x_2} + f(u), \quad f_{uu} \neq 0. \quad (5.2.1) \]

can be obtained by substituting the coefficients of \( F \) viz \( E_{11} = E_{22} = 1 \) in Eq. (5.1.1).

If

\[ Y = \tau \frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + \phi \frac{\partial}{\partial u} \]

is the symmetry generator for Eq. (5.2.1). Then applying the Lie’s symmetry criteria i.e.

\[ Y^{[2]} \left( u_t - u_{x_1} - u_{x_2} - f(u) \right) \big|_{u_t = u_{x_1} + u_{x_2} + f(u) = 0} = 0 \quad (5.2.2) \]

where \( Y^{[2]} \) is the second prolongation operator, gives the following determining equations that give the Lie point symmetries for Eq. (5.2.1) (see [11, 33, 50])

\[ \xi_u^1 = 0, \quad \xi_u^2 = 0, \quad \tau_u = 0, \quad \phi_{uu} = 0, \quad \tau_{x_1} = 0, \quad \tau_{x_2} = 0, \quad (5.2.3) \]

\[ \phi_t - \phi_{x_1} - \phi_{x_2} - 2f\xi_{x_1} + f\phi_u - f_u\phi = 0, \quad (5.2.4) \]

\[ -\tau_t + 2\xi_{x_1}^1 = 0, \quad -\tau_t + 2\xi_{x_2}^2 = 0, \quad (5.2.5) \]
\[ -\xi_1^1 + \xi_{x_1x_1}^1 + \xi_{x_2x_2}^1 - 2\phi_{x_1u} = 0, \quad (5.2.6) \]

\[ -\xi_2^2 + \xi_{x_1x_1}^2 + \xi_{x_2x_2}^2 - 2\phi_{x_2u} = 0, \quad (5.2.7) \]

Using Eq. (5.2.5) and \( \tau_{x_1} = 0, \tau_{x_2} = 0 \), from Eq. (5.2.3) one can easily conclude that

\[ \xi_{x_1x_1}^1 = 0, \quad \xi_{x_2x_2}^2 = 0. \quad (5.2.8) \]

Eq. (5.2.8) reduces Eqs. (5.2.6-5.2.7)

\[ -\xi_t^1 + \xi_{x_2x_2}^1 - 2\phi_{x_1u} = 0, \quad (5.2.9) \]

\[ -\xi_t^2 + \xi_{x_1x_1}^2 - 2\phi_{x_2u} = 0, \quad (5.2.10) \]

Twice differentiating Eq. (5.2.4) with respect to \( u \) and using the results given in Eqs. (5.2.9-5.2.10) yields

\[ \phi f_{uuu} + \phi_u f_u + 2\xi_{x_1}^1 f_{uu} = 0, \quad (5.2.11) \]

\[ \phi \left[ -f_{uu}f_{uuu} + 2(f_{uuu})^2 \right] + 2f_{uu}f_{uuu}\xi_{x_1}^1 = 0. \quad (5.2.12) \]

Differentiating Eq. (5.2.12) with respect to \( u \) and using Eqs. (5.2.11 - 5.2.12) one obtains

\[ \xi_{x_1}^1 (f_{uu})^2 \left[ f_{uuu}(f_{uuu})^2 - 2f_{uu}(f_{uuu})^2 + f_{uu}f_{uuu}f_{uuuu} \right] = 0. \quad (5.2.13) \]

The solutions of Eq. (5.2.13) [15] are

(i) \( f(u) = au^3 + bu^2 + cu + d \)

(ii) \( f(u) = (au + b)^{n+2} + cu + d \quad \text{for} \quad n \neq 0, 1, 2, 3 \)
(iii) \( f(u) = ae^{bu} + cu + d \)

(iv) \( f(u) = \left[ \ln(au + b) \right] / \left( a + cu + d \right) \)

(v) \( f(u) = \left[ (au + b) \ln(au + b) - (au + b) \right] / \left( a^2 + cu + d \right) \) or \( f(u) \) arbitrary if \( \xi_{x_1}^1 = 0 \).

The above results are the same as those obtained in [28].

5.2.2 The nonlinear \((1 + 2)\)-dimensional heat equation on sphere

After substituting the coefficients of first fundamental form of sphere

\[ E_{11} = 1, \quad E_{22} = \sin^2 x_1 \] (5.2.14)

in Eq. (5.1.1), we obtain

\[ u_t = u_{x_1 x_1} + \cot x_1 u_{x_1} + \csc^2 x_1 u_{x_2 x_2} + f(u), \quad f_{uu} \neq 0. \] (5.2.15)

The determining equations for Eq. (5.2.15) are

\[ \xi_{u}^1 = 0, \quad \xi_{u}^2 = 0, \quad \tau_u = 0, \quad \phi_{uu} = 0, \] (5.2.16)

\[ -f_u \phi - 2 f \tau_t + f \phi_u + \phi_t - \phi_{x_1 x_1} - \cot x_1 \phi_{x_1} - \csc^2 x_1 \phi_{x_2 x_2} = 0, \] (5.2.17)

\[ \csc^2 x_1 \xi_{x_1}^1 - \xi_{t}^1 + \xi_{x_1 x_1}^1 + \cot x_1 \xi_{x_1}^1 + \csc^2 x_1 \xi_{x_2 x_2}^1 - 2 \cot x_1 \tau_t - 2 \phi_{x_1 u} = 0, \] (5.2.18)

\[ \cot x_1 \xi_{x_1}^2 + \csc^2 x_1 \xi_{x_2 x_2}^2 + \xi_{x_1 x_1}^2 - \xi_{t}^2 - 2 \csc^2 x_1 \phi_{x_2 u} = 0, \] (5.2.19)

\[ \xi_{x_1}^1 - \tau_t = 0, \quad \tau_{x_1} = 0, \quad \tau_{x_2} = 0, \] (5.2.20)
\[ 2 \cot x_1 \xi_1 + 2 \xi_2^2 - \tau_t = 0, \quad \csc^2 x_1 \xi_1^2 + \xi_2^2 = 0. \quad (5.2.21) \]

After solving Eqs. (5.2.20-5.2.21) one gets

\[ \xi_1 = b_2 \cos x_2 + b_3 \sin x_2, \]

\[ \xi_2 = \cot x_1 [b_3 \cos x_2 - b_2 \sin x_2] + b_4, \quad (5.2.22) \]

\[ \tau = b_1. \]

The substitution of Eq. (5.2.22) into Eqs. (5.2.18-5.2.19), yields the following results

\[ \phi_{x_1 u} = 0, \quad \phi_{x_2 u} = 0. \quad (5.2.23) \]

Differentiating Eq. (5.2.17) with respect to \( u \) and using Eq. (5.2.16) and Eq. (5.2.22) we get the following equation

\[ -f_{uu} \phi + \phi_{tu} = 0. \quad (5.2.24) \]

In order to eliminate \( \phi_{tu} \), differentiating Eq. (5.2.24) with respect to \( u \) to obtain

\[ \phi f_{uuu} + \phi_u f_{uu} = 0, \quad (5.2.25) \]

Eq. (5.2.25) after some manipulations yields

\[ \phi \left[ -f_{uu} f_{uuu} + 2 (f_{uuu})^2 \right] = 0. \quad (5.2.26) \]

The solutions of Eq. (5.2.26) [15] are

(i) \( f(u) = au^2 + bu + c \)

(ii) \( f(u) = \left( (au + b) \ln(au + b) - (au + b) \right) / \left( a^2 + cu + d \right) \) or
$f(u)$ arbitrary if $\phi = 0$.

It is important to mention that symmetry generators (5.2.20) are invariant from the choice of $f(u)$ in Eq. (5.2.15).

### 5.2.3 The nonlinear $(1 + 2)$-dimensional heat equation on torus

The coefficients of the first fundamental form of torus are

$$E_{11} = (1 + \cos x_2)^2, \quad E_{22} = 1. \quad (5.2.27)$$

Substituting Eq. (5.2.27) into Eq. (5.1.2) yields

$$u_t = \frac{u_{x_1 x_1}}{(1 + \cos x_2)^2} - \left( \frac{\sin x_2}{1 + \cos x_2} \right) u_{x_2} + u_{x_2 x_2} + f(u), \quad f_{uu} \neq 0. \quad (5.2.28)$$

Following the same procedure as for the above cases we get the following results

$$\xi^1 = \frac{1}{2} \tan \left( \frac{x_2}{2} \right) \left( c_2 \sin x_1 - c_3 \cos x_1 \right) + c_4,$$

$$\xi^2 = c_2 \cos x_1 - c_3 \sin x_1, \quad (5.2.29)$$

$$\tau = c_1,$$

which yields

$$\phi_{x_1 u} = 0, \quad \phi_{x_2 u} = 0. \quad (5.2.30)$$

Hence by using the results given in Eq. (5.2.30) and doing routine calculations one can conclude that Eq. (5.2.28) belongs to the same class of functions as obtained above for the $(1 + 2)$-dimensional heat equation on sphere.
5.3 “f(u)” vs “underlying geometry”

In the next section, we will analyze the symmetry generators of the heat equation on different surfaces one by one and then discuss the role of the underlying space in the derivation of these generators.

5.3.1 For sphere

The infinitesimal coefficients (5.2.22) of Eq. (5.2.15) yield the following symmetry generators

\[ Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \cos x_2 \frac{\partial}{\partial x_1} - \cot x_1 \sin x_2 \frac{\partial}{\partial x_2}, \]

\[ Y_3 = \sin x_2 \frac{\partial}{\partial x_1} + \cot x_1 \cos x_2 \frac{\partial}{\partial x_2}, \quad Y_4 = \frac{\partial}{\partial x_2}. \]

Here \( Y_1 \) arises due to the fact that Eq. (5.2.15) is invariant under time translation, \( Y_2, Y_3 \) and \( Y_4 \) because of the fact that sphere is invariant under rotational symmetries. In this case, geometry of the underlying space is dominant during the calculation of the infinitesimal symmetries and no single symmetry generator there represents nonlinearity of the equation.

5.3.2 For torus

\[ Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \cos x_1 \frac{\partial}{\partial x_2} + \frac{1}{2} \tan \left( \frac{x_2}{2} \right) \sin x_1 \frac{\partial}{\partial x_1}, \]

\[ Y_3 = \sin x_1 \frac{\partial}{\partial x_2} - \frac{1}{2} \tan \left( \frac{x_2}{2} \right) \cos x_1 \frac{\partial}{\partial x_1}, \quad Y_4 = \frac{\partial}{\partial x_1}. \]

One can easily make some observations about these symmetries. \( Y_1 \) is caused by the fact that Eq. (5.2.28) is invariant under time translations, \( Y_2, Y_3 \) and \( Y_4 \) are by the symmetries of torus. The geometry of the background metric is more powerful than the nonlinearity of \( f(u) \).
Following the same reasoning, it can be proved easily that for the \((1 + n)\)-dimensional non-linear heat equation, the geometry of \(n\)-dimensional sphere and \(n\)-dimensional torus is more important than the nonlinearity of \(f(u)\).

### 5.4 The TDGL model on curved surfaces

The TDGL model on general surface \(\Omega\) without external field is

\[
\psi_t = \frac{\psi_{x_1x_1}}{E_{11}} + \frac{\psi_{x_2x_2}}{E_{22}} - \left( \frac{(E_{11})_{x_1}}{2E_{11}^2} - \frac{(E_{22})_{x_1}}{2E_{11}E_{22}} \right) \psi_{x_1} - \left( \frac{(E_{22})_{x_2}}{2E_{22}^2} - \frac{(E_{11})_{x_2}}{2E_{11}E_{22}} \right) \psi_{x_2} + \frac{dV}{d\psi}, \quad (5.4.1)
\]

Here \(V(\psi) = -a\psi^2/2 + b\psi^4/4\).

The simplified Ginzburg-Landau time evolution comes from

\[
\psi_t = -\frac{\delta F[\psi]}{\delta \psi},
\]

where “\(F\)” is the Ginzburg-Landau free energy functional s.t.

\[
F[\psi] = \int_{\Omega} \left( V(\psi) + \frac{1}{2} |\nabla_{LB}\psi|^2 \right) \sqrt{g} d^2x,
\]

in which \(\sqrt{g} d^2x\) is the area element and \(\nabla_{LB}\) is the Laplace-Beltrami operator.

Ignoring the spatial gradient term, \(a\) negative gives a one-state equilibrium and \(a\) positive gives a two-state equilibrium, \(b\) is strictly positive. The gradient term imposes an energy penalty for diffusing values of \(\psi\).

In this section, we will calculate all possible reduced equation of Eq. (5.4.1) on sphere and torus via the corresponding symmetry algebra up to conjugacy classes. We refer the reader to [55, 59].

#### 5.4.1 The TDGL model on sphere

The TDGL model on sphere is given below

\[
\psi_t = \psi_{x_1x_1} + \cot x_1 \psi_{x_1} + \csc^2 x_1 \psi_{x_2x_2} - a\psi + b\psi^3. \quad (5.4.2)
\]
The infinitesimal generators of Eq. (5.4.2) are

\[ Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \cos x_2 \frac{\partial}{\partial x_1} - \cot x_1 \sin x_2 \frac{\partial}{\partial x_2}, \]

\[ Y_3 = \sin x_2 \frac{\partial}{\partial x_1} + \cot x_1 \cos x_2 \frac{\partial}{\partial x_2}, \quad Y_4 = \frac{\partial}{\partial x_2}. \]  

(5.4.3)

Classification of subalgebra up to conjugacy classes

In this section, we will classify Lie algebra \( L = \{Y_1, Y_2, Y_3, Y_4\} \) into its subalgebras up to conjugacy classes. For this we will adopt some definitions from literature.

1: A subspace \( \mathcal{L}_i \) of the Lie algebra \( \mathcal{L} \) is said to be Lie subalgebra if it is closed under the Lie bracket or Lie commutator which is defined as

\[ [Y_i, Y_j] = Y_j \left( Y_i \right) - Y_i \left( Y_j \right), \]

where \( Y_i \) and \( Y_j \) are Lie point symmetry generators (see [50]).

The commutator table for \( L \) is

<table>
<thead>
<tr>
<th></th>
<th>( Y_1 )</th>
<th>( Y_2 )</th>
<th>( Y_3 )</th>
<th>( Y_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( Y_2 )</td>
<td>0</td>
<td>0</td>
<td>(-Y_4/4)</td>
<td>( Y_3 )</td>
</tr>
<tr>
<td>( Y_3 )</td>
<td>0</td>
<td>( Y_4/4 )</td>
<td>0</td>
<td>( Y_2 )</td>
</tr>
<tr>
<td>( Y_4 )</td>
<td>0</td>
<td>( -Y_3 )</td>
<td>( -Y_2 )</td>
<td>0</td>
</tr>
</tbody>
</table>

By the definition of a subalgebra, one can easily conclude that there are infinite number of a one-dimensional subalgebras of \( L \). The greatest or best representative of these algebras is called optimal system of one-dimensional subalgebra. Here \( \{Y_2, Y_3, Y_4\} \) forms \( \text{so}(3) \) subalgebra. By taking \( Y_4 \) as a representative of \( \text{so}(3) \), \( L \) reduces to \( L_1 = \{Y_1, Y_4\} \). In view of all
these we can write an arbitrary element from $L_1$ i.e.

$$
Y = dY_1 + eY_4,
$$

(5.4.4)

where $d$ and $e$ are real constants. Before going to write down optimal system for the one-dimensional algebra it should be noted that $\mathcal{L}_i$ and $\mathcal{L}_j$ are said to be equivalent conjugacy classes if

$$
\mathcal{L}_i = \text{Ad} \ Y_i \left( \mathcal{L}_j \right)
$$

where $Y_i \in L_1$ and

$$
\text{Ad} \left[ \exp \left( tY_i \right) \right] Y_j = Y_j - t[Y_i, \ Y_j] + \frac{t^2}{2} \left[ [Y_i, [Y_i, \ Y_j]] \right] - \cdots.
$$

Now our next task is to simplify (5.4.4) by applying a carefully chosen adjoint transformation that gives

$$
\mathcal{L}_1 = \left< dY_1 + Y_4 \right>, \quad \mathcal{L}_2 = \left< Y_1 \right>.
$$

Two- and higher-dimensional subalgebras are finite in number hence one can easily conclude the following results

$$
\mathcal{L}_3 = \left< Y_1, Y_4 \right>, \quad \mathcal{L}_4 = \left< Y_2, Y_3, Y_4 \right>, \quad \mathcal{L}_5 = \left< Y_1, Y_2, Y_3, Y_4 \right>.
$$

**Reduction by calculating similarity variables**

$$
\mathcal{L}_1 = \left< dY_1 + Y_4 \right>
$$

I. $d = 0$

For this case, we have

$$
\xi_1 = x_1, \quad \xi_2 = t, \quad W(\xi_1, \xi_2) = \psi,
$$

where $W$ satisfies the following reduced PDE for Eq. (5.4.2)

$$
W_{\xi_2} = W_{\xi_1 \xi_1} + \cot \xi_1 W_{\xi_1} - aW + bW^3.
$$

(5.4.5)
II. \( d \neq 0 \)

It yields

\[ \xi_1 = x_1, \quad \xi_2 = x_2 - \frac{t}{d}, \quad W(\xi_1, \xi_2) = \psi. \]

After substituting into Eq. (5.4.2) we get

\[ -\frac{1}{d} W_{\xi_2} = W_{\xi_1\xi_1} + \cot \xi_1 W_{\xi_1} + \csc^2 \xi_1 W_{\xi_2\xi_2} - aW + bW^3. \]  

(5.4.6)

\[ \mathcal{L}_2 = \langle Y_1 \rangle \]

This gives

\[ \xi_1 = x_1, \quad \xi_2 = x_2, \quad \psi = W(\xi_1, \xi_2), \]

which converts Eq. (5.4.2) into the following equation

\[ W_{\xi_1\xi_1} + \cot \xi_1 W_{\xi_1} + \csc^2 \xi_1 W_{\xi_2\xi_2} - aW + bW^3 = 0. \]  

(5.4.7)

\[ \mathcal{L}_3 = \langle Y_1, Y_4 \rangle \]

For this case, we have

\[ \xi = x_1, \quad W(\xi) = \psi \]

that reduces Eq. (5.4.2) into

\[ W_{\xi_1\xi_1} + \cot \xi_1 W_{\xi_1} - aW + bW^3 = 0. \]  

(5.4.8)

It is noteworthy that Eq. (5.4.8) is a second-order nonlinear ordinary differential equation. This cannot be obtained without doing the reduction through similarity variables.
\[ \mathcal{L}_4 = \langle Y_2, Y_3, Y_4 \rangle \]

That gives

\[ \frac{d\psi}{dt} = -a\psi + b\psi^3. \]

It is a first order nonlinear ordinary differential equation which can be solved easily for different values of \( a \) and \( b \).

\[ \mathcal{L}_5 = \langle Y_1, Y_2, Y_3, Y_4 \rangle \]

It gives a trivial result.

### 5.4.2 The TDGL model on torus

The TDGL model on torus is given below

\[ \psi_t = \frac{\psi_{x_1 x_1}}{(1 + \cos x_2)^2} - \left( \frac{\sin x_2}{1 + \cos x_2} \right) \psi_{x_2} + \psi_{x_2 x_2} - a\psi + b\psi^3. \] (5.4.9)

The infinitesimal generators of Eq. (5.4.9) are

\[ Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \cos x_1 \frac{\partial}{\partial x_2} + \frac{1}{2} \tan \left( \frac{x_2}{2} \right) \sin x_1 \frac{\partial}{\partial x_1}, \] (5.4.10)

\[ Y_3 = \sin x_1 \frac{\partial}{\partial x_2} - \frac{1}{2} \tan \left( \frac{x_2}{2} \right) \cos x_1 \frac{\partial}{\partial x_1}, \quad Y_4 = \frac{\partial}{\partial x_1}. \]

Following the same reasoning used earlier, we have the following optimal systems of one-, two-, three- and four-dimensional subalgebras:

\[ \mathcal{L}_1 = \langle dY_1 + Y_4 \rangle, \mathcal{L}_2 = \langle Y_1 \rangle, \mathcal{L}_3 = \langle Y_1, Y_4 \rangle, \mathcal{L}_4 = \langle Y_2, Y_3, Y_4 \rangle, \mathcal{L}_5 = \langle Y_1, Y_2, Y_3, Y_4 \rangle. \]
Reduction by calculating similarity variables

\[ \mathcal{L}_1 = < dY_1 + Y_4 > \]

I. \( d = 0 \)

This gives

\[ \xi_1 = t, \quad \xi_2 = x_2, \quad W(\xi_1, \xi_2) = \psi, \]

where \( W \) satisfies the following reduced PDE for Eq. (5.4.9)

\[ W_{\xi_1} = W_{\xi_2\xi_2} - \left( \frac{\sin \xi_2}{1 + \cos \xi_2} \right) W_{\xi_2} - aW + bW^3. \quad (5.4.11) \]

II. \( d \neq 0 \)

This case leads to

\[ \xi_1 = dx_1 - t, \quad \xi_2 = x_2, \quad W(\xi_1, \xi_2) = \psi. \]

After substituting in Eq. (5.4.9) we get

\[ W_{\xi_1} = \frac{-d^2 W_{\xi_1\xi_1}}{(1 + \cos \xi_2)^2} + \left( \frac{\sin \xi_2}{1 + \cos \xi_2} \right) W_{\xi_2} - W_{\xi_2\xi_2} + aW - bW^3. \quad (5.4.12) \]

\[ \mathcal{L}_2 = < Y_1 > \]

It yields

\[ \xi_1 = x_1, \quad \xi_2 = x_2, \quad \psi = W(\xi_1, \xi_2) \]

which converts Eq. (5.4.9) to the following reduced equation

\[ \frac{W_{\xi_1\xi_1}}{(1 + \cos \xi_2)^2} + W_{\xi_2\xi_2} - \left( \frac{\sin \xi_2}{1 + \cos \xi_2} \right) W_{\xi_2} - aW + bW^3 = 0. \quad (5.4.13) \]
\[ \mathcal{L}_3 = \langle Y_1, Y_4 \rangle \]

It gives

\[ \xi = x_2, \quad W(\xi) = \psi \]

\[ W_{\xi \xi} - \left( \frac{\sin \xi}{1 + \cos \xi} \right) W_\xi - aW + bW^3 = 0. \quad (5.4.14) \]

Double reduction converts Eq. (5.4.9) to a second-order nonlinear ordinary differential equation.

\[ \mathcal{L}_4 = \langle Y_2, Y_3, Y_4 \rangle \]

This gives

\[ \frac{d\psi}{dt} = -a\psi + b\psi^3. \]

It is interesting to note that the reduced equation is a first-order nonlinear ordinary differential equation which cannot be obtained without using the above procedure.

\[ \mathcal{L}_5 = \langle Y_1, Y_2, Y_3, Y_4 \rangle \]

It gives a trivial result.

### 5.5 Conclusion

It is interesting to note that Eq. (5.1.1) can be transformed from one surface to another if and only if both surfaces have the same coefficients of FFF. Duplication of the result for Eq. (5.1.1) on sphere and torus by knowing the above fact is important.

It should be noted from Eq. (5.2.22) that \( \xi^1 \) is purely a function of \( x_2 \) that implies that the geometry of sphere is more important than the nonlinearity of \( f(u) \). Same argument can
be made for the torus. So symmetry generators neglect the effect of the nonlinear function: $f(u)$. 
Chapter 6

Group Classification of
(1 + n)-dimensional Klein-Gordon
Equation and the Nonlinear Wave
Equation on Curved Surfaces

This chapter is devoted to investigating the connection between the underlying space and
a class of functions for the nonlinear wave equation. For the (1 + n)-dimensional nonlinear
wave equation (Klein-Gordon equation) we prove that there is a class of functions which is
independent of the number of independent variables. We also show that the class of functions
is invariant for the (1+2)-dimensional wave equation whether the underlying space is a plane,
sphere or torus. A complete group classification is presented for the (1 + n)-dimensional
nonlinear wave equation (Klein-Gordon equation) when \( n = 2 \) or 3. Then using these results
we discuss the classification for the general case when \( n \) is arbitrary. Maximum symmetry
generators are calculated for each \( f(u) \); after that the (1 + 2)-dimensional wave equation on
the sphere and torus is discussed for \( f(u) \) and symmetry generators are calculated for each
equation. A classification of these symmetry algebras is obtained up to conjugacy classes
and similarity reductions for each class are given. The effect of \( f(u) \) and the underlying
space on the infinitesimal generators is also discussed.
6.1 Introduction

The $(1 + n)$-dimensional Klein-Gordon equation:

$$u_{tt} = \Delta_2 u + f(u), \quad f_{uu} \neq 0,$$

(6.1.1)

where

$$u = u(t, x_1, ..., x_n) \quad \text{with} \quad \Delta_2 u = \sum_{i=1}^{n} u_{x_i x_i},$$

has its significance importance in theoretical and mathematical physics.

In past, many authors [7, 25] discussed Eq. (6.1.1) for different values of $n$, by means of exact solutions, compatibility of reduction conditions and reduced equations by considering ansatz formula which reduces the dimension of the corresponding PDE (see [26]). In [24], author discussed the symmetry properties and found particular solutions for some cases of Eq. (6.1.1). Tajiri [60] purposed some similarity and soliton solutions for the three-dimensional Klein-Gordon equation by means of similarity variables. Fushchych et al. [27] discussed the reductions and solutions by using the broken symmetry for Eq. (6.1.1) with $n = 3$. In [20], Fedruchuk discussed the reductions of Eq. (6.1.1) for $n = 4$ by using decomposable subgroups of the generalized Poincaré group $P(1, 4)$. For the Eq. (6.1.1) Fushchych [25] purposed an ansatz of the form $u = f(x)\phi(\omega) + g(x)$. Description of such ansatz for Eq. (6.1.1) is an extremely difficult nonlinear problem. That problem can be simplified by using Lie theory.

Lie symmetry analysis is a systematic way to construct ansatz which further reduces the dimension of the differential equation. The symmetry method also plays a central role in the analysis of differential equation. It is equally useful for the linear and nonlinear differential equations which posses nontrivial Lie symmetries. There are nonlinear equations which do
not possess nontrivial Lie point symmetries. Thus nonlinear differential equations can
be classified with respect to their functions which have nontrivial Lie point symmetries. This
classification is known as group classification. The problem of group classification is one of
the central aspect of the modern symmetry analysis of differential equations. Firstly, it was
carried out by Ovsiannikov [51] for the nonlinear heat equation: $u_t = (f(u)u_{x_1})_{x_1}$. For the
nonlinear wave equation: $u_{tt} = (f(u)u_{x_1})_{x_1}$, group properties were discussed by Ames [1].
Pucci [53] discussed the group classification of $u_{tt} + u_{x_1x_1} = f(u, u_{x_1})$. A list of symmetries
of the equations: $u_{tt} = u_{x_1x_1} + f(t, x_1, u, u_{x_1})$ is presented in [44]. Furthermore, the group
classification of $u_{tt} = u_{x_1x_1} + u_{x_2x_2} + f(u)$ was discussed by Rudra et al. [54]. A new method
for the group classification by using Gröbner basis was invented by Clarkson et al. [15].
The authors in [6] used that method for the group classification of the $(1 + 1)$-dimensional
Klein-Gordon equation:

$$u_{tt} = u_{x_1x_1} + f(u), \quad f_{uu} \neq 0.$$  

To the best of our knowledge no one discussed the group classification of Eq. (6.1.1) for
$n = k$. Here we will investigate the group classification of Eq. (6.1.1) for $n = k$. A list of Lie
point symmetries is presented for each $f(u)$. It is worthwhile to mention that this complete
list of Lie point symmetries for Eq. (6.1.1) for each $f(u)$ is not discussed in past.

In the past decade its a powerful tendency to involve geometry and theoretical physics.
It is also the hot topic of research in mathematical biology where the membranes are not
flat in general. Physical models with non-flat background are important in biological math-
ematics. Most of the biological membranes are rarely flat [23, 61]. For example, membranes
which convert energy in mitochondria and chloroplasts are tubes, buds and may be sheets.
In most of the biological processes, the geometry of membranes is very important. The
organization and shape of the membranes play a vital role in biological processes such as
shape change, fusion-division, ion adsorption etc. A cell membrane is a system for exchange
of energy and matter from the neighborhood. Absorption and transformation of conserved quantities such as energy and matter from the environment are one of the characteristics of membranes. The shape of proteins, non zero curvature of membranes and involvement of conserved quantities lead one to discuss physical models on curved surfaces (see [55, 59]).

The \((1 + n)\)-dimensional Klein-Gordon equation on curved surfaces \(\Omega\) [43] is

\[
 u_{tt} = \frac{u_{x_1 x_1}}{E_{11}} + \frac{u_{x_2 x_2}}{E_{22}} - \left( \frac{(E_{11})_{x_1} x_1}{2E_{11}^2} - \frac{(E_{22})_{x_1} x_1}{2E_{11}E_{22}} \right)u_{x_1} - \left( \frac{(E_{22})_{x_2} x_2}{2E_{22}^2} - \frac{(E_{11})_{x_2} x_2}{2E_{11}E_{22}} \right)u_{x_2} + f(u), \tag{6.1.2}
\]

with \(f_{uu} \neq 0\), where

\[
 ds^2 = E_{11}dx_1^2 + E_{12}dx_1dx_2 + E_{22}dx_2^2
\]

is the first fundamental form (FFF) of 2-dimensional surface. It is also important to mention here that Eq. (6.1.2) is not the subcase of any equation in the literature whose group classification was carried out in past. Note that Eq. (6.1.2) on flat surface with \(E_{11} = 1 = E_{22}\) becomes the \((1 + 2)\)-dimensional Klein-Gordon equation. The group classification of the \((1 + 2)\)-dimensional Klein-Gordon equation on sphere and torus is yet not given in past and presented here.

We also investigated following important questions related to the group classification of Eq. (6.1.1):

Is there any class of functions related to Eq. (6.1.1) independent from number of independent variables? We will also investigate how the group classification of Eq. (6.1.2) relates to the background metric of Eq. (6.1.2). For this we will discuss three different surfaces with respect to Gaussian curvature i.e plane, sphere and torus. The plane has zero, sphere has positive and torus has variable Gaussian curvature.

We investigated the connection between some underlying spaces and a class of functions
for the $(1 + 2)$-dimensional Klein-Gordon equation. It is proved that the geometry of highly symmetrical surfaces: sphere and torus, play a more dominant role than the nonlinearity of $f(u)$ in the calculation of infinitesimal generators of Lie symmetry. We also present similarity reductions for equations under consideration.

The outline of the chapter is as follows. In Section 2 group classification of the $(1 + n)$-dimensional Klein-Gordon equation is given. The largest set of symmetry generators for each $f(u)$ is calculated in Section 3. In Section 4, group classification of the $(1 + 2)$-dimensional Klein-Gordon equation on sphere and torus is carried out. For both equations similarity reductions are performed for each conjugacy class of their subalgebras. Section 5 is devoted to brief remark and conclusion.

### 6.2 Group classification of the $(1+n)$-dimensional Klein-Gordon equation

In this section, we will discuss the group classification of Eq. (6.1.1) for different values of $n$ one by one and then by using these results we will generalize it for $n = k$. Here we will follow the technique by Clarkson et al. [15] because of the vast use of Gröbner basis and their potentials.

#### 6.2.1 Case 1: $n = 1$

For $n = 1$, Eq. (6.1.1) reduces to the equation: $u_{tt} = u_{x_1 x_1} + f(u)$, for which group classification is carried out in [6] by using the same technique studied in [15]. So here we will use the results given in [6].
6.2.2 Case 2: \( n = 2 \)

The \((1 + 2)\)-dimensional Klein-Gordon equation is:

\[
u_{tt} = u_{x_1 x_1} + u_{x_2 x_2} + f(u), \quad f_{uu} \neq 0.
\] (6.2.1)

We search for classical Lie symmetries of Eq. (6.2.1) which is described in many books. The reader who is not familiar with the definition and properties of Lie point symmetries we recommend some references (see [11, 33, 50]).

If

\[
Y = \tau \frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + \phi \frac{\partial}{\partial u}
\] (6.2.2)

is symmetry generator for Eq. (6.2.1) then applying the Lie’s symmetry criteria:

\[
Y^{[2]}(u_{tt} - u_{x_1 x_1} - u_{x_2 x_2} - f(u)) \bigg|_{u_{tt}=u_{x_1 x_1}+u_{x_2 x_2}+f(u)} = 0,
\] (6.2.3)

where \( Y^{[2]} \) is the second prolongation of (6.2.2).

Eq. (6.2.3) after some lengthy manipulation gives the following set of determining equations that determines the Lie point symmetries of Eq. (6.2.1):

\[
(i) : \xi_u^1 = 0, \quad (ii) : \xi_u^2 = 0, \quad (iii) : \tau_u = 0,
\] (6.2.4)

\[
(iv) : \phi_{uu} = 0, \quad (v) : \xi_{x_2}^1 + \xi_{x_1}^2 = 0,
\] (6.2.5)

\[
\phi_{tt} - \phi_{x_1 x_1} - \phi_{x_2 x_2} - 2f\tau_t + f\phi_u - f_u\phi = 0,
\]

\[
-\tau_t + \xi_{x_1}^1 = 0, \quad -\xi_{t}^1 + \tau_{x_1} = 0,
\] (6.2.6)

\[
-\tau_t + \xi_{x_2}^2 = 0, \quad -\xi_{t}^2 + \tau_{x_2} = 0,
\] (6.2.7)
\[-\xi_1^{tt} + \xi_1^{x_1x_1} + \xi_1^{x_2x_2} - 2\phi_{x_1u} = 0, \tag{6.2.8}\]

\[-\xi_2^{tt} + \xi_2^{x_1x_1} + \xi_2^{x_2x_2} - 2\phi_{x_2u} = 0, \tag{6.2.9}\]

\[-\tau_{tt} + \tau_{x_1x_1} + \tau_{x_2x_2} + 2\phi_{tu} = 0. \tag{6.2.10}\]

From Eq. (6.2.6) we conclude that

\[\tau_{tt} - \tau_{x_1x_1} = 0, \quad \xi_1^{x_1x_1} - \xi_1^{tt} = 0. \tag{6.2.11}\]

Using Eq. (6.2.7) one can easily obtain

\[\tau_{tt} - \tau_{x_2x_2} = 0, \quad \xi_2^{tt} - \xi_2^{x_2x_2} = 0. \tag{6.2.12}\]

Eqs. (6.2.11-6.2.12) with the help of Eqs. (6.2.8-6.2.10) yields

\[\xi_1^{x_2x_2} + 2\phi_{x_1u} = 0, \tag{6.2.13}\]

\[\xi_2^{x_1x_1} + 2\phi_{x_2u} = 0, \tag{6.2.14}\]

\[\tau_{tt} + 2\phi_{tu} = 0. \tag{6.2.15}\]

The solution of Eqs. (6.2.4 \((i, ii, iii, v)\)) and Eqs. (6.2.6-6.2.7) gives

\[\xi_1^{1} = a_1 + a_4x_1 - a_5x_2 + a_6t + a_8(x_1^2 - x_2^2 + t^2) + 2a_9x_1x_2 + 2a_{10}x_1t, \tag{6.2.16}\]

\[\xi_2^{2} = a_2 + a_4x_2 + a_5x_1 + a_7t + 2a_8x_1x_2 + a_9(-x_1^2 + x_2^2 + t^2) + 2a_{10}x_2t, \tag{6.2.16}\]

\[\tau = a_3 + a_4t + a_6x_1 + a_7x_2 + 2a_8x_1t + 2a_9x_2t + a_{10}(x_1^2 + x_2^2 + t^2). \tag{6.2.16}\]

The substitution of \(\xi_1^{1}, \xi_2^{2}, \tau\) from Eq. (6.2.16) in Eqs. (6.2.13-6.2.15) gives rise to

\[\phi_{x_1u} = \text{constant}, \quad \phi_{x_2u} = \text{constant}, \quad \phi_{tu} = \text{constant}. \tag{6.2.17}\]
Differentiating Eq. (6.2.5) with respect to \( u \) and using the results given in Eq. (6.2.17) and Eq. (6.2.4, (iii)), one obtains

\[
\phi f_{uu} + 2\tau_t f_u = 0. \tag{6.2.18}
\]

Differentiation of Eq. (6.2.18) with respect to \( u \) results in

\[
\phi f_{uuu} + f_{uuu}\phi_u + 2f_{uu}\tau_t = 0. \tag{6.2.19}
\]

Differentiating Eq. (6.2.19) with respect to \( u \) and using Eqs. (6.2.18 − 6.2.19), we arrive at

\[
\tau_t \left[ (f_{uu})^2 f_{uuu} - 2f_u (f_{uu})^2 + f_u f_{uu} f_{uuu} \right] = 0. \tag{6.2.20}
\]

The following cases arises from Eq. (6.2.20) namely \( \tau_t = 0 \) and \( \tau_t \neq 0 \).

\( \tau_t = 0 \)

This implies \( f(u) \) will be arbitrary.

\( \tau_t \neq 0 \)

This results in

\[
\left[ (f_{uu})^2 f_{uuu} - 2f_u (f_{uu})^2 + f_u f_{uu} f_{uuu} \right] = 0. \tag{6.2.21}
\]

The solutions of Eq. (6.2.21) yield the following functions [6]

(i) \( f(u) = au^2 + bu + c \)

(ii) \( f(u) = (au + b)^m + c \) \quad for \quad \( m \neq 0, 1, 2 \)

(iii) \( f(u) = \frac{\ln(au+b)}{a} + c \)
(iv) $f(u) = ae^{bu} + c$.

Now differentiating Eq. (6.2.18) with respect to $t$ and simplifying to obtain

$$\tau_{tt} \left[(f_{uu})^2 - f_u f_{uuu}\right] = 0 \quad \text{if} \quad \tau_t \neq 0.$$  \hfill (6.2.22)

From Eq. (6.2.22) further two cases arises which finally yields

(i) $f(u) = ae^{bu} + c$ or $\tau_{tt} = 0$.

(ii) $f(u) \neq ae^{bu} + c$, then there must be $\tau_{tt} = 0$.

Above class of functions is same as obtained in [54].

6.2.3 Case 3: $n = 3$

The substitution of $n = 3$ in Eq. (6.1.1) reduces to the $(1 + 3)$-dimensional Klein-Gordon equation:

$$u_{tt} = u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3} + f(u), \quad f_{uu} \neq 0.$$  \hfill (6.2.23)

Following the routine calculation as we performed for $n = 2$ following determining equations are obtained

\begin{align*}
(i) : & \quad \xi_u^1 = 0, \quad (ii) : \quad \xi_u^2 = 0, \quad (iii) : \quad \xi_u^3 = 0, \\
(iv) : & \quad \tau_u = 0, \quad (v) : \quad \phi_{uu} = 0, \quad (vi) : \quad \phi_{tt} - \phi_{x_1 x_1} - \phi_{x_2 x_2} - \phi_{x_3 x_3} - 2f\tau_t + f\phi_u - f_u \phi = 0, \quad (6.2.24)\\
-\tau_t + \xi_{x_1}^1 = 0, \quad -\xi_t^1 + \tau_{x_1} = 0, \quad (6.2.25)\end{align*}
\[-\tau_t + \xi^2_{x_2} = 0, \quad -\xi^2_t + \tau_{x_2} = 0, \quad (6.2.27)\]

\[-\tau_t + \xi^3_{x_3} = 0, \quad -\xi^3_t + \tau_{x_3} = 0, \quad (6.2.28)\]

\[\xi^1_{x_3} + \xi^3_{x_1} = 0, \quad \xi^1_{x_2} + \xi^2_{x_1} = 0, \quad \xi^2_{x_3} + \xi^3_{x_2} = 0, \quad (6.2.29)\]

\[-\xi^1_{tt} + \xi^1_{x_1x_1} + \xi^1_{x_2x_2} + \xi^1_{x_3x_3} - 2\phi_{x_1u} = 0, \quad (6.2.30)\]

\[-\xi^2_{tt} + \xi^2_{x_1x_1} + \xi^2_{x_2x_2} + \xi^2_{x_3x_3} - 2\phi_{x_2u} = 0, \quad (6.2.31)\]

\[-\xi^3_{tt} + \xi^3_{x_1x_1} + \xi^3_{x_2x_2} + \xi^3_{x_3x_3} - 2\phi_{x_3u} = 0, \quad (6.2.32)\]

\[-\tau_{tt} + \tau_{x_1x_1} + \tau_{x_2x_2} + \tau_{x_3x_3} + 2\phi_{tu} = 0. \quad (6.2.33)\]

From Eq. (6.2.26) and Eq. (6.2.27), we obtain

\[\tau_{tt} - \tau_{x_1x_1} = 0, \quad \xi^1_{tt} - \xi^1_{x_1x_1} = 0, \quad (6.2.34)\]

\[\tau_{tt} - \tau_{x_2x_2} = 0, \quad \xi^2_{tt} - \xi^2_{x_2x_2} = 0. \quad (6.2.35)\]

Eq. (6.2.28) implies that

\[\tau_{tt} - \tau_{x_3x_3} = 0, \quad \xi^3_{tt} - \xi^3_{x_3x_3} = 0. \quad (6.2.36)\]

Eqs. (6.2.34-6.2.36) with the help of Eqs. (6.2.30-6.2.33) reduce to

\[\xi^1_{x_2x_2} + \xi^1_{x_3x_3} - 2\phi_{x_1u} = 0, \quad \xi^2_{x_1x_1} + \xi^2_{x_3x_3} - 2\phi_{x_2u} = 0, \quad (6.2.37)\]

\[\xi^3_{x_1x_1} + \xi^3_{x_2x_2} - 2\phi_{x_3u} = 0, \quad \tau_{x_2x_2} + \tau_{x_3x_3} + 2\phi_{tu} = 0. \quad (6.2.38)\]
Solving Eqs. (6.2.26-6.2.29) and Eq. (6.2.24, (i ii, iii, iv)), we obtain (see [34])

\[ \xi^1 = a_2 + a_5x_1 + a_6x_2 + a_7x_3 + a_9t + a_{12}x_1t + \frac{a_{13}}{2}(x_1^2 - x_2^2 - x_3^2 + t^2) \]

\[ + a_{14}x_1x_2 + a_{15}x_1x_3, \]

\[ \xi^2 = a_3 + a_5x_2 - a_6x_1 + a_8x_3 + a_{10}t + a_{12}x_2t + a_{13}x_1x_2 \]

\[ + \frac{a_{14}}{2}(x_2^2 - x_1^2 - x_3^2 + t^2) + a_{15}x_2x_3, \]

\[ \xi^3 = a_4 + a_5x_3 - a_7x_1 - a_8x_2 + a_{11}t + a_{12}tx_3 + a_{13}x_1x_3 \]

\[ + a_{14}x_2x_3 + \frac{a_{15}}{2}(-x_1^2 - x_2^2 + x_3^2 + t^2), \]

\[ \tau = a_1 + a_5t + a_9x_1 + a_{10}x_2 + a_{11}x_3 + \frac{a_{12}}{2}(x_1^2 + x_2^2 + x_3^2 + t^2) \]

\[ + a_{13}x_1t + a_{14}x_2t + a_{15}x_3t. \]

The substitution of \( \xi^1, \xi^2, \xi^3, \tau \) from (6.2.39) in Eqs. (6.2.37-6.2.38) leads to

\[ \phi_{x_1u} = \text{constant}, \quad \phi_{x_2u} = \text{constant}, \]

\[ \phi_{x_3u} = \text{constant}, \quad \phi_{tu} = \text{constant}. \]

Differentiating Eq. (6.2.25) with respect to \( u \) and using the facts given in (6.2.40), one can easily obtain

\[ \phi_{fuu} + 2\tau_{fu} = 0. \]

Eq. (6.2.41) and Eq. (6.2.18) are identical. Hence we will get the same class of functions as we have obtained for the (1 + 2)-dimensional Klein-Gordon equation.
6.2.4 General Case

In this section, we will discuss the group classification for the \((1 + n)\)-dimensional Klein-Gordon equation i.e. Eq. (6.1.1). Now using the technique mentioned in literature for the calculation of symmetry generators we apply \(n\)th prolongation vector (see \([11, 33, 50]\))

\[
Y^{[n]} \left( u_{tt} - \sum_{i=1}^{n} u_{x_ix_i} - f(u) \right) \bigg|_{u_{tt} = \sum_{i=1}^{n} u_{x_ix_i} + f(u) = 0} = 0. \tag{6.2.42}
\]

Eq. (6.2.42) yields the following determining equations

\((i)\) : \(\xi^i_u = 0, \ i = 1, 2, \cdots, n, \ (ii)\) : \(\tau_u = 0, \ (iii)\) : \(\phi_{uu} = 0, \tag{6.2.43}\)

\[
\phi_{tt} - \sum_{i=1}^{n} \phi_{x_ix_i} - 2f\tau_t + f\phi_u - fu\phi = 0, \tag{6.2.44}\]

\[
-\tau_t + \xi^i_{x_i} = 0, \ -\xi^i_t + \tau_{x_i} = 0, \ i = 1, 2, \cdots, n, \tag{6.2.45}\]

\[
-\xi^i_{tt} + \sum_{j=1}^{n} \xi^i_{x_jx_j} - 2\phi_{x_iu} = 0, \ i = 1, 2, \cdots, n, \tag{6.2.46}\]

\[
-\tau_{tt} + \sum_{i=1}^{n} \tau_{x_ix_i} + 2\phi_{tu} = 0. \tag{6.2.47}\]

From (6.2.45) we obtain

\[
\tau_{tt} - \tau_{x_ix_i} = 0, \ \xi^i_{x_ix_i} - \xi^i_{tt} = 0, \ i = 1, 2, \cdots, n. \tag{6.2.48}\]

Eq. (6.2.45) forms a set of equations for an infinitesimal conformal transformation on \(\mathbb{R}^{n+1}\) with Lorentz metric and thus the unknowns appeared in these equations are quadratic polynomials of \(t, x_1, \cdots, x_n\) (see \([50]\)) and hence Eq. (6.2.46) and Eq. (6.2.47) implies that

\[
\phi_{x_iu} = \text{constant}, \ \phi_{tu} = \text{constant}, \ i = 1, 2, \cdots, n. \tag{6.2.49}\]
Differentiating Eq. (6.2.44) with respect to $u$ and using the results given in Eq. (6.2.49), yields

$$\phi f_{uu} + 2\tau f_u = 0.$$ \hfill (6.2.50)

Hence after doing the routine calculations one can easily conclude that the $(1+n)$-dimensional Klein-Gordon equation belongs to the same class of functions obtained by Eq. (6.2.20) and Eq. (6.2.22). Repetitions in the results shows that obtained class is invariant with the choice of $n$.

### 6.3 Lie point symmetry generators

In this section, we will discuss the different forms of $f(u)$ which lead to the maximum Lie point symmetry generators for Eq. (6.1.1) for $n = 2, 3, \cdots, k$, as for $n = 1$, results are presented in [6].

#### 6.3.1 $f(u) \neq ae^{bu} + c$

The subcases of case 3.1 are:

$$f(u) = au^2 + bu + c$$

In order to get maximum Lie point symmetry generators of Eq. (6.1.1) with $f(u) = au^2 + bu + c$, Eq. (6.2.44) and Eq. (6.2.50) imply that $f(u)$ should be a perfect square.

For particular $f(u) = u^2$, the symmetry generators will be

$$Y_{i1} = \frac{\partial}{\partial t}, \quad Y_{2i} = \frac{\partial}{\partial x_i}, \quad i = 1, \cdots, n,$$ \hfill (6.3.1)

$$Y_{3i} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad j > i, \quad i, j = 1, \cdots, n.$$ \hfill (6.3.2)
\[ Y_{4i} = x_i \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_i}, \quad i = 1, \cdots, n, \quad (6.3.3) \]

\[ Y_{5i} = t \frac{\partial}{\partial t} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} - 2u \frac{\partial}{\partial u}, \quad (6.3.4) \]

\[ Y_{6i} = 2x_i t \frac{\partial}{\partial t} + \left( x_i^2 + t^2 - \sum_{j=1}^{n} x_j^2 \right) \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} 2x_i x_j \frac{\partial}{\partial x_j} - 4x_i u \frac{\partial}{\partial u}, \quad (6.3.5) \]

\[ Y_{7i} = \left( t^2 + \sum_{i=1}^{n} x_i^2 \right) \frac{\partial}{\partial t} + 2t \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} - 4tu \frac{\partial}{\partial u}. \quad (6.3.6) \]

After replacing \( f(u) = (u)^m + c \) in Eq. (6.2.50) and Eq. (6.2.44) we conclude that \( c = 0 \) leads to a larger set of Lie point symmetry generators. By replacing \( f(u) = (u)^m \) in Eq. (6.1.1) following generators are obtained.

For particular \( f(u) = u^2 \), the symmetry generators will be

\[ Y_{1i} = \frac{\partial}{\partial t}, \quad Y_{2i} = \frac{\partial}{\partial x_i}, \quad i = 1, \cdots, n, \quad (6.3.7) \]

\[ Y_{3i} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad j > i, \quad i, j = 1, \cdots, n, \quad (6.3.8) \]

\[ Y_{4i} = x_i \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_i}, \quad i = 1, \cdots, n, \quad (6.3.9) \]

\[ Y_{5i} = t \frac{\partial}{\partial t} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} - \frac{2u}{m-1} \frac{\partial}{\partial u}, \quad (6.3.10) \]

\[ Y_{6i} = 2x_i t \frac{\partial}{\partial t} + \left( x_i^2 + t^2 - \sum_{j=1}^{n} x_j^2 \right) \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} 2x_i x_j \frac{\partial}{\partial x_j} - \frac{4x_i u}{m-1} \frac{\partial}{\partial u}, \quad (6.3.11) \]

\[ Y_{7i} = \left( t^2 + \sum_{i=1}^{n} x_i^2 \right) \frac{\partial}{\partial t} + 2t \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} - \frac{4tu}{m-1} \frac{\partial}{\partial u}. \quad (6.3.12) \]
\[ f(u) = \frac{\ln(a u + b)}{a} + c \]

Eq. (6.2.50) and Eq. (6.2.44) with \( f(u) = \frac{\ln(a u + b)}{a} + c \) give an extra condition i.e. \( \tau_t = 0 \) which yields the minimum Lie point symmetry generators. Since we are only going to discuss the larger symmetry generators so we will not pursue this case further here and discussed later for \( \tau_t = 0 \).

### 6.3.2 \( f(u) = a e^{bu} + c \)

Eq. (6.2.50) and Eq. (6.2.44) with \( f(u) = a e^{bu} + c \) implies \( c = 0 \) for the larger set of symmetry generators. Hence for particular case \( f(u) = e^{2u} \) we obtain

\[
\phi = -\tau_t,
\]

\[
\phi = \alpha(t, x_1, \cdots, x_n),
\]

\[
\tau_t = -\alpha(t, x_1, \cdots, x_n),
\]

\[
\xi^i_{x_i} = -\alpha(t, x_1, \cdots, x_n),
\]

where \( \alpha \) satisfies the following PDE

\[
\alpha_{tt} - \sum_{i=1}^{n} \alpha_{x_i x_i} = 0. \quad (6.3.13)
\]

For different choices of \( \alpha \) symmetry generators can be obtained. Here we will discuss some cases for \( \alpha \) which yield larger set of symmetry generators for \( n = 2, 3, \ldots, k \).
\( \alpha = \text{constant} \)

In this case

\[
Y_{1i} = -t \frac{\partial}{\partial t} - \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial u}, 
\]

\( (6.3.14) \)

\[
Y_{2i} = x_i \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n, 
\]

\( (6.3.15) \)

\[
Y_{3i} = \frac{\partial}{\partial t}, \quad Y_{4i} = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n. 
\]

\( (6.3.16) \)

\( \alpha = \alpha(t) \)

The symmetry generators for this case are

\[
Y_{1i} = \left( t^2 + \sum_{i=1}^{n} x_i^2 \right) \frac{\partial}{\partial t} + 2t \left( \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \right) + 2t \frac{\partial}{\partial u}, 
\]

\( (6.3.17) \)

\[
Y_{2i} = t \frac{\partial}{\partial t} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} - u \frac{\partial}{\partial u}, 
\]

\( (6.3.18) \)

\[
Y_{3i} = x_i \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n, 
\]

\( (6.3.19) \)

\[
Y_{4i} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}, \quad j > i, \quad i, j = 1, \ldots, n, 
\]

\( (6.3.20) \)

\[
Y_{5i} = \frac{\partial}{\partial t}, \quad Y_{6i} = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n. 
\]

\( (6.3.21) \)

\( \alpha = \alpha(x_i) \)

We obtain

\[
Y_{1i} = 2tx_i \frac{\partial}{\partial t} + \left( t^2 + x_i^2 + \sum_{j=1}^{n} x_j^2 \right) \frac{\partial}{\partial x_i} + 2x_i \left( \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} \right) - 2x_i \frac{\partial}{\partial u}, 
\]

\( (6.3.22) \)
where \( i \neq j \) and \( i, j = 1, 2, \ldots n \),

\[
Y_{2i} = t \frac{\partial}{\partial t} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial u},
\]

(6.3.23)

\[
Y_{3i} = x_i \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n,
\]

(6.3.24)

\[
Y_{4i} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad j > i, \quad i, j = 1, \ldots, n,
\]

(6.3.25)

\[
Y_{5i} = \frac{\partial}{\partial t}, \quad Y_{6i} = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n.
\]

(6.3.26)

### 6.3.3 \( \tau_t = 0 \)

In this section, we will calculate maximum symmetry generators for arbitrary \( f(u) \). Eq. (6.2.50) with the substitution of \( \tau_t = 0 \) gives \( \phi = 0 \). The symmetry generators in this case are:

\[
Y_{1i} = \frac{\partial}{\partial t}, \quad Y_{2i} = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n,
\]

(6.3.27)

\[
Y_{3i} = x_i \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n,
\]

(6.3.28)

\[
Y_{4i} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad j > i, \quad i, j = 1, \ldots, n.
\]

(6.3.29)

### 6.4 Group classification of the \((1+2)\)-dimensional Klein-Gordon equation on curved surfaces

In this section, we will investigate how a class of function is affected from the non-flat underlying space and which factor is more dominant in the calculation of infinitesimal generators, non-flat background metric or \( f(u) \)? For this we will consider sphere and torus as an underlying space. For flat background Eq. (6.1.2) becomes Eq. (6.2.1) which has already been discussed in Section 2.2.
6.4.1 The \((1 + 2)\)-dimensional Klein-Gordon equation on a sphere

The \((1 + 2)\)-dimensional Klein-Gordon equation on sphere is

\[
 u_{tt} = u_{x_1 x_1} + \cot x_1 u_{x_1} + \csc^2 x_1 u_{x_2 x_2} + f(u), \quad f_{uu} \neq 0, \tag{6.4.1}
\]

which is obtained by substituting the coefficients of first fundamental form of sphere:

\[
 E_{11} = 1, \quad E_{22} = \sin^2 x_1 \tag{6.4.2}
\]

in Eq. (6.1.2). The determining equations for Eq. (6.4.1) are

\[
(i): \quad \xi^1_u = 0, \quad (ii): \quad \xi^2_u = 0, \quad (iii): \quad \tau_u = 0, \quad (iv): \quad \phi_{uu} = 0, \tag{6.4.3}
\]

\[
-f_u \phi - 2 f_t + f \phi_u + \phi_{tt} - \phi_{x_1 x_1} - \cot x_1 \phi_{x_1} - \csc^2 x_1 \phi_{x_2 x_2} = 0, \tag{6.4.4}
\]

\[
csc^2 x_1 \xi^1 - \xi^1_{tt} + \xi^1_{x_1 x_1} + \cot x_1 \xi^1_{x_1} + \csc^2 x_1 \xi^1_{x_2 x_2} - 2 \cot x_1 \tau_t - 2 \phi_{x_1 u} = 0, \tag{6.4.5}
\]

\[
cot x_1 \xi^2 + \csc^2 x_1 \xi^2_{x_2 x_2} + \xi^2_{x_1 x_1} - \xi^2_{tt} - 2 \csc^2 x_1 \phi_{x_2 u} = 0, \tag{6.4.6}
\]

\[
\cot x_1 \tau_{x_1} + \csc^2 x_1 \tau_{x_2 x_2} + \tau_{x_1 x_1} - \tau_{tt} + 2 \phi_{tu} = 0, \tag{6.4.7}
\]

\[
\frac{\xi^1_t}{\tau_{x_1}} = 0, \quad \frac{\xi^1_{x_1}}{\tau_t} = 0, \quad \frac{\xi^2_t}{\csc^2 x_1 \tau_{x_2}} = 0, \tag{6.4.8}
\]

\[
\cot x_1 \xi^1 + \xi^2_{x_1} - \tau_t = 0, \quad \csc^2 x_1 \xi^1_{x_2} + \xi^2_{x_1} = 0. \tag{6.4.9}
\]

The solution of Eqs. (6.4.8-6.4.9) with the help of Eqs. (6.4.3, \(i, \ ii, \ iii\)) is

\[
\xi^1 = b_2 \cos x_2 + b_3 \sin x_2, \tag{6.4.10}
\]

\[
\xi^2 = \cot x_1 [b_3 \cos x_2 - b_2 \sin x_2] + b_4,
\]
\[ \tau = b_1. \]

After substituting (6.4.10) in Eqs. (6.4.5-6.4.7) we arrive at

\[ \phi_{x_1 u} = 0, \quad \phi_{x_2 u} = 0, \quad \phi_{tu} = 0 \quad \text{and} \quad \tau_t = 0. \]  \hspace{1cm} (6.4.11)

Now using Eq. (6.4.3, (iii)) and Eq. (6.4.11) in Eq. (6.4.4) after differentiating with respect to \( u \) we obtain

\[ \phi_{fuu} = 0. \]  \hspace{1cm} (6.4.12)

From Eq. (6.4.12) we get \( \phi = 0 \) for arbitrary choice of \( f \) as \( f_{uu} \neq 0 \). This result shows that there is no contribution of \( f(u) \) in the derivation of symmetry generators. Geometry of the underlying space is more dominant than the nonlinearity of \( f(u) \) in the calculation of infinitesimal symmetries of Eq. (6.4.1).

**Symmetry generators for Eq. (6.4.1)**

The infinitesimal generators of Eq. (6.4.1) are

\[ Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \cos x_2 \frac{\partial}{\partial x_1} - \cot x_1 \sin x_2 \frac{\partial}{\partial x_2}, \]
\[ Y_3 = \sin x_2 \frac{\partial}{\partial x_1} + \cot x_1 \cos x_2 \frac{\partial}{\partial x_2}, \quad Y_4 = \frac{\partial}{\partial x_2}. \]  \hspace{1cm} (6.4.13)

Invariance of Eq. (6.4.1) under time translation is represented by \( Y_1 \), where \( Y_2, Y_3 \) and \( Y_4 \) are due to the fact that sphere is invariant from rotation. There is no contribution of nonlinearity of \( f(u) \).

**Optimal systems**

Next aim is to classify Lie algebra \( L = \{Y_1, Y_2, Y_3, Y_4\} \) into its subalgebras up to conjugacy classes. For this we present some basic definitions from literature.
1: A subspace \( \mathcal{L}_i \) of a Lie algebra \([50]\) \( \mathcal{L} \) is said to be Lie subalgebra if it is closed under Lie bracket or Lie commutator which is

\[
[Y_i, Y_j] = Y_j \left( Y_i \right) - Y_i \left( Y_j \right),
\]

where \( Y_i \) and \( Y_j \) are Lie point generators.

The commutator table for \( L \) is

<table>
<thead>
<tr>
<th></th>
<th>( Y_1 )</th>
<th>( Y_2 )</th>
<th>( Y_3 )</th>
<th>( Y_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( Y_2 )</td>
<td>0</td>
<td>0</td>
<td>(-\frac{1}{4}Y_4)</td>
<td>( Y_3 )</td>
</tr>
<tr>
<td>( Y_3 )</td>
<td>0</td>
<td>( \frac{1}{4}Y_4 )</td>
<td>0</td>
<td>( Y_2 )</td>
</tr>
<tr>
<td>( Y_4 )</td>
<td>0</td>
<td>(-Y_3)</td>
<td>(-Y_2)</td>
<td>0</td>
</tr>
</tbody>
</table>

Using the definition of subalgebra one can easily conclude that there are infinite number of one dimensional subalgebras of \( L \). So optimal system of one dimensional subalgebra is the greatest or best representative of these algebras. Here \( \{Y_2, Y_3, Y_4\} \) forms \( so(3) \) subalgebra.

By taking \( Y_4 \) as a representative of \( so(3) \), \( L \) reduces to \( L_1 = \{Y_1, Y_4\} \). In view of all these we can write an arbitrary element from \( L_1 \) i.e.

\[
Y = dY_1 + eY_4,
\]

where \( d \) and \( e \) are real constants. Before going to write down optimal systems for one-dimensional algebra it should be noted that \( \mathcal{L}_i \) and \( \mathcal{L}_j \) are said to be equivalent conjugacy classes if

\[
\mathcal{L}_i = Ad Y_i \left( \mathcal{L}_j \right),
\]

where \( Y_i \in L_1 \) and

\[
Ad \left[ \exp \left( tY_i \right) \right] Y_j = Y_j - t[Y_i, Y_j] + \frac{t^2}{2} [Y_i, [Y_i, Y_j]] - \cdots.
\]
As $L_1$ forms solvable Lie algebra hence

$$Ad \left[ \exp \left( tY_i \right) \right] Y_j = Y_j, \quad i, j = 1, 4$$

Now our next task is to simplify (6.4.14) by applying carefully chosen adjoint transformation. This gives

$$\mathcal{L}_1 = \langle dY_1 + Y_4 \rangle, \quad \mathcal{L}_2 = \langle Y_1 \rangle.$$  

For two- and higher-dimensional algebras there are finite number of subalgebras so it’s not difficult to get the following results

$$\mathcal{L}_3 = \langle Y_1, Y_4 \rangle, \quad \mathcal{L}_4 = \langle Y_2, Y_3, Y_4 \rangle, \quad \mathcal{L}_5 = \langle Y_1, Y_2, Y_3, Y_4 \rangle.$$  

**Reduction by using optimal systems**

In this section, we will find all possible similarity reductions via similarity variables.

$$\mathcal{L}_1 = \langle dY_1 + Y_4 \rangle$$

I. $d = 0$

For this case we have

$$\xi_1 = x_1, \quad \xi_2 = t, \quad W(\xi_1, \xi_2) = u,$$

where $W$ satisfies the following reduced PDE

$$W_{\xi_2\xi_2} = W_{\xi_1\xi_1} + \cot \xi_1 W_{\xi_1} + f(W), \quad f_{WW} \neq 0. \quad (6.4.15)$$

II. $d \neq 0$

We obtain

$$\xi_1 = x_1, \quad \xi_2 = x_2 - \frac{t}{d}, \quad W(\xi_1, \xi_2) = u,$$
which further reduces Eq. (6.4.1) to
\[
\frac{1}{d^2} W_{\xi_2 \xi_2} = W_{\xi_1 \xi_1} + \cot \xi_1 W_{\xi_1} + \csc^2 \xi_1 W_{\xi_2 \xi_2} + f(W), \quad f_{WW} \neq 0. \tag{6.4.16}
\]

\[\mathcal{L}_2 = \langle Y_1 \rangle\]

In this case
\[\xi_1 = x_1, \quad \xi_2 = x_2, \quad u = W(\xi_1, \xi_2),\]

which converts Eq. (6.4.1) into the following equation
\[
W_{\xi_1 \xi_1} + \cot \xi_1 W_{\xi_1} + \csc^2 \xi_1 W_{\xi_2 \xi_2} + f(W) = 0, \quad f_{WW} \neq 0. \tag{6.4.17}
\]

It should be noted that Eqs. (6.4.15 - 6.4.17) have one dimension less as compared to Eq. (6.4.1).

Further we use two- or higher-dimensional algebras for reduction. These algebras convert Eq. (6.4.1) into ordinary differential equation.

\[\mathcal{L}_3 = \langle Y_1, Y_4 \rangle\]

For this class we obtain
\[\xi = x_1, \quad W(\xi) = u,\]

where \(W\) is the solution of
\[
W_{\xi_1 \xi_1} + \cot \xi_1 W_{\xi_1} + f(W) = 0, \quad f_{WW} \neq 0. \tag{6.4.18}
\]

\[\mathcal{L}_4 = \langle Y_2, Y_3, Y_4 \rangle\]

This leads to
\[\xi = t, \quad W(\xi) = u,\]
where $W$ can be obtained by solving ordinary differential equation:

$$
\frac{d^2 W}{dt^2} = f(W), \quad f_{WW} \neq 0.
$$

(6.4.19)

$L_5 = \langle Y_1, Y_2, Y_3, Y_4 \rangle$

It gives a trivial result.

### 6.4.2 The $(1 + 2)$-dimensional Klein-Gordon equation on torus

The coefficients of the first fundamental form of torus are

$$E_{11} = (1 + \cos x_2)^2, \quad E_{22} = 1. \quad (6.4.20)$$

The substitution of $E_{11}$ and $E_{22}$ from Eq. (6.4.20) reduces Eq. (6.1.2) to

$$u_{tt} = \frac{u_{x_1 x_1}}{(1 + \cos x_2)^2} - \left( \frac{\sin x_2}{1 + \cos x_2} \right) u_{x_2} + u_{x_2 x_2} + f(u), \quad f_{uu} \neq 0. \quad (6.4.21)$$

Following the same procedure as discussed in the previous sections, the determining equations for the symmetry generators are:

$$\tau_u = 0, \quad \xi^1_u = 0, \quad \xi^2_u = 0, \quad \phi_{uu} = 0, \quad (6.4.22)$$

$$\sec^4 \left( \frac{x_2}{2} \right) \tau_{x_1} - 4 \xi^1_{x_1} = 0, \quad \tau_{x_2} - \xi_2 = 0, \quad -\tau_t + \xi^2 = 0, \quad (6.4.23)$$

$$4\xi^1_{x_2} + \xi^2_{x_1} \sec^4 \left( \frac{x_2}{2} \right) = 0, \quad \xi^2 \tan \left( \frac{x_2}{2} \right) + \tau_t - \xi^1_{x_1} = 0, \quad (6.4.24)$$

$$2\xi^2 \sec^2 \left( \frac{x_2}{2} \right) - 4 \xi^2_{tt} + \xi^2_{x_1 x_1} \sec^4 \left( \frac{x_2}{2} \right) + 4 \xi^2_{x_2 x_2} + 8 \tau_t \tan \left( \frac{x_2}{2} \right)$$

$$-4 \xi^1_{x_2} \tan \left( \frac{x_2}{2} \right) - 8 \phi_{x_2 u} = 0, \quad (6.4.25)$$

$$-4 \xi^1_{x_2} \tan \left( \frac{x_2}{2} \right) + \xi^1_{x_1 x_1} \sec^4 \left( \frac{x_2}{2} \right) - 4 \xi^1_{tt} + 4 \xi^1_{x_2 x_2} - 2 \phi_{x_1 u} \sec^4 \left( \frac{x_2}{2} \right) = 0, \quad (6.4.26)$$
\[-4\tau x_2 \tan(\frac{x_2}{2}) + \tau x_1 x_1 \sec^4(\frac{x_2}{2}) - 4\tau \tau + 4\tau x_2 x_2 + 8\phi u = 0, \quad (6.4.27)\]

\[-4f_u \phi - 8f \tau + 4f \phi_x \tan(\frac{x_2}{2}) - \phi x_1 x_1 \sec^4(\frac{x_2}{2}) + 4\phi \tau + 4\phi x_2 x_2 = 0. \quad (6.4.28)\]

After some tedious calculations, Eqs. (6.4.22 – 6.4.24) results in

\[\xi^1 = \frac{1}{2} \tan(\frac{x_2}{2}) \left( c_2 \sin x_1 - c_3 \cos x_1 \right) + c_4,\]

\[\xi^2 = c_2 \cos x_1 - c_3 \sin x_1, \quad (6.4.29)\]

\[\tau = c_1.\]

Substituting (6.4.29) in Eq. (6.4.25 – 6.4.27) we finally get

\[\phi_{x_1 u} = 0, \quad \phi_{x_2 u} = 0, \quad \phi_{tu} = 0 \text{ and } \tau_t = 0. \quad (6.4.30)\]

Hence by using the results given in Eq. (6.4.30) one can conclude that remaining results for Eq. (6.4.21) is exactly the same as obtained above for the \((1 + 2)\)-dimensional Klein-Gordon equation on sphere.

**Optimal systems**

The infinitesimal generators of Eq. (6.4.21) are

\[Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \cos x_1 \frac{\partial}{\partial x_2} + \frac{1}{2} \tan(\frac{x_2}{2}) \sin x_1 \frac{\partial}{\partial x_1}, \quad (6.4.31)\]

\[Y_3 = \sin x_1 \frac{\partial}{\partial x_2} - \frac{1}{2} \tan(\frac{x_2}{2}) \cos x_1 \frac{\partial}{\partial x_1}, \quad Y_4 = \frac{\partial}{\partial x_1}.\]

One can easily make some observations about these Lie point symmetry generators. \(Y_1\) is caused by the fact that Eq. (6.4.21) is invariant under time translations, \(Y_2, Y_3\) and \(Y_4\) are because of symmetries of torus. While \(f(u)\) has no influence in symmetry generators.
Geometry of background metric is more powerful as compared to the nonlinearity of $f(u)$.

The routine calculations show that

$$\mathcal{L}_1 = \langle dY_1 + Y_4 \rangle, \quad \mathcal{L}_2 = \langle Y_1 \rangle, \quad \mathcal{L}_3 = \langle Y_1, Y_4 \rangle,$$

$$\mathcal{L}_4 = \langle Y_2, Y_3, Y_4 \rangle, \quad \mathcal{L}_5 = \langle Y_1, Y_2, Y_3, Y_4 \rangle.$$

**Reduction by optimal systems**

$$\mathcal{L}_1 = \langle dY_1 + Y_4 \rangle$$

I. $d = 0$

This gives

$$\xi_1 = t, \quad \xi_2 = x_2, \quad W(\xi_1, \xi_2) = u,$$

where $W$ is the solution of following reduced PDE

$$W_{\xi_1 \xi_1} = W_{\xi_2 \xi_2} - \left( \frac{\sin \xi_2}{1 + \cos \xi_2} \right) W_{\xi_2} + f(W), \quad f_{WW} \neq 0. \quad (6.4.32)$$

II. $d \neq 0$

This case leads to

$$\xi_1 = dx_1 - t, \quad \xi_2 = x_2, \quad W(\xi_1, \xi_2) = u,$$

which converts Eq. (6.4.21) into

$$W_{\xi_1 \xi_1} = \frac{d^2 W_{\xi_1 \xi_1}}{(1 + \cos \xi_2)^2} - \left( \frac{\sin \xi_2}{1 + \cos \xi_2} \right) W_{\xi_2} + W_{\xi_2 \xi_2} + f(W), \quad f_{WW} \neq 0. \quad (6.4.33)$$

$$\mathcal{L}_2 = \langle Y_1 \rangle$$

This class yields

$$\xi_1 = x_1, \quad \xi_2 = x_2, \quad u = W(\xi_1, \xi_2),$$
which reduces Eq. (6.4.21) to

\[
\frac{W_{\xi_1\xi_1}}{(1 + \cos \xi_2)^2} + W_{\xi_2\xi_2} - \left(\frac{\sin \xi_2}{1 + \cos \xi_2}\right)W_{\xi_2} + f(W) = 0, \quad f_{WW} \neq 0.
\] (6.4.34)

Next we will use two- or higher-dimensional algebras for the reduction of Eq. (6.4.21).

\[\mathcal{L}_3 = < Y_1, Y_4 >\]

In this case

\[\xi = x_2, \quad W(\xi) = u,\]

where \(W\) satisfies

\[
W_{\xi\xi} - \left(\frac{\sin \xi}{1 + \cos \xi}\right)W_\xi + f(W) = 0, \quad f_{WW} \neq 0.
\] (6.4.35)

\[\mathcal{L}_4 = < Y_2, Y_3, Y_4 >\]

This results in

\[\xi = t, \quad W(\xi) = u,\]

where \(W\) is the solution of following equation

\[
\frac{d^2W}{dt^2} = f(W), \quad f_{WW} \neq 0
\] (6.4.36)

\[\mathcal{L}_5 = < Y_1, Y_2, Y_3, Y_4 >\]

It leads to a trivial result.
6.5 Remarks

It should be noted that in all Lie point symmetries given in (6.3.7)-(6.3.12), (6.3.14)-(6.3.29), (6.4.13) and (6.4.31), the coefficients of $\partial_t$, $\partial_{x_1}$ and $\partial_{x_2}$ are independent from the dependent variable $u$ and hence obtained symmetries are fiber preserving or projective transformations. Such transformations allow one to calculate the expression for the group transformation on the actual function $u(t, x_1, x_2)$ with less difficulty.

It is also worthwhile to mention here that after doing the routine calculations it can be proved easily that the results obtained for Eq. (6.1.1) on sphere or torus are unchanged for the $(1 + n)$-dimensional Klein-Gordon equation on $n$-dimensional sphere or $n$-dimensional torus. The geometry of the hyper sphere and torus: $n$-dimensional sphere and $n$-dimensional torus respectively, prevailing over the nonlinearity of $f(u)$ as observed above for sphere and torus.

6.5.1 Conclusion

We have presented a complete group classification of the $(1 + n)$-dimensional Klein-Gordon equation. Firstly the procedure is carried out for $n = 2$ & 3 and then it is generalized for $n = k$. A class of functions for $n = 1, 2, \ldots, k$ is same which shows that this class is independent from number of independent variables. Maximal symmetry algebras are constructed for each $f(u)$ for $(1 + n)$-dimensional Klein-Gordon equation. These maximal symmetry algebras given in (6.3.1)-(6.3.12) and (6.3.14)-(6.3.29), can be further used to construct ansatz or similarity variables.

The $(1 + 2)$-dimensional Klein-Gordon equation on sphere and torus is also studied. It is proved that underlying geometry of the $(1 + 2)$-dimensional Klein-Gordon equation is
dominant over the nonlinearity of $f(u)$ whether the underlying geometry is sphere or torus. These two surfaces are considered to be most symmetrical surfaces in mathematics. In both the cases generators are unchanged for any arbitrary choice of $f(u)$ ($f_{uu} \neq 0$). At the last, similarity reductions for each conjugacy class have been made for both equations. We also remarked about the group classification of the $(1 + n)$-dimensional Klein-Gordon equation on $n$-dimensional sphere and $n$-dimensional torus.
Chapter 7

Conclusions

In this work we have performed an intensive analysis of the heat and wave equations on curved manifolds via Lie group theory. In this analysis, we discussed the two main aspects of Lie theory: conservation laws and group classification. We also discussed the importance of the underlying space for a partial differential equation.

We presented the conservation laws for the \((1 + n)\)-dimensional heat equation on a variety of curved surfaces in terms of the coefficients of the FFF. For the \((1 + n)\)-dimensional heat equation, the partial Noether determining equation yielded the gauge terms and the conserved vectors in terms of unknown functions which satisfy certain conditions. Then we extended the results to the \((1 + n)\)-dimensional linear evolution equation. From the application of these results to different models from different sciences, we have provided the potential systems for each model.

The conservation laws of the nonlinear \((1 + 1)\)-dimensional wave equation were discussed in [37]. Bokhari et al. [12] derived the conservation laws for a class of nonlinear \((1 + n)\)-dimensional wave equations. We illustrated the partial Noether approach to the \((1 + 2)\)-dimensional wave equation on a sphere, cone and flat surface.

Surfaces can be classified with respect to their Gaussian curvature [43]. The plane, sphere and one-sheeted hyperboloid have zero, constant positive and constant negative Gaussian curvature respectively. In this respect, the geometry of the torus is more interesting because it has variable Gaussian curvature. The group classification of the heat and wave equations on curved surfaces has never before been studied.

In this work, the group classification of the nonlinear heat equation on the plane, sphere and torus is discussed. It is proved that a class of functions for which the nonlinear heat
equation has nontrivial symmetries, is independent of the underlying surface.

We derived a class of functions for the $(1 + 2)$-dimensional nonlinear wave equation on the plane, sphere and torus. For this class of functions, the $(1 + 2)$-dimensional nonlinear wave equation has nontrivial Lie point symmetries. The group classification of the $(1 + 1)$-dimensional Klein-Gordon equation was discussed in [6], while for the $(1 + 2)$-dimensional Klein-Gordon equation the group classification was done by Rudra [54]. A complete group classification of the $(1 + n)$-dimensional Klein-Gordon equation is given in this work. A class of the functions was obtained for $(1 + 2)$-dimensional wave equation which is not only independent from the number of independent variables but also remains unchanged either the background metric is plane, sphere or torus.

7.1 Future Directions

We conclude by mentioning some further work. There are still many directions which can be explored; some ideas are presented here:

- An important line of inquiry is to derive the conserved quantities for the wave equation on higher dimensional surfaces, e.g. $S^n$ and $\mathbb{R}^n$.

- It would be of interest to discuss the conservation laws of the heat and wave equations on a variable geometry or less symmetrical surfaces, e.g. ellipsoid.

- The solution for the $(1 + 2)$-dimensional heat and wave equations on curved surfaces by using potential systems could be the further line of inquiry.

- The derivation of conserved quantities for a system of partial differential equations on curved manifolds could be a future direction.

- It would be of great interest to discuss the conserved quantities for the heat and wave equations with a small parameter.

- In the group classification, how a class of functions is related to the geometry of the
underlying space of the nonlinear wave equation could be a possible line of investigation.

• The group classification of a system of partial differential equation using a Gröbner basis could be a line of interest.

• Conserved quantities and the group classification of differential equations in different coordinate systems could be studied; such equations are important in mathematical biology and represent many biological phenomena.

• The theoretical aspect of the conserved quantities of many differential equations are discussed here, the physical interpretation of these conserved quantities (especially for the equations used in finance) could be a subject of great interest.
Bibliography


[53] E. Pucci, Group analysis of the equation \( u_{tt} + \lambda u_{xx} = g(u, u_x) \), Riv. Mat. Univ. Parma, 4 (1987) 71 – 87.


