KERNEL OPERATORS IN SOME NEW FUNCTION SPACES

Name : **Muhammad Asad Zaighum**

Year of Admission : **2009**

Registration No. : **103-GCU-PHD-SMS-09**

Abdus Salam School of Mathematical Sciences

GC University Lahore, Pakistan

KERNEL OPERATORS IN SOME NEW FUNCTION SPACES

Submitted to

Abdus Salam School of Mathematical Sciences

GC University Lahore, Pakistan

in the partial fulfillment of the requirements for the award of degree of

Doctor of Philosophy

in

Mathematics

By

Name : Muhammad Asad Zaighum

Year of Admission : 2009

Registration No. : 103-GCU-PHD-SMS-09

Abdus Salam School of Mathematical Sciences

GC University Lahore, Pakistan

DECLARATION

I, **Mr. Muhammad Asad Zaighum** Registration No. **103-GCU-PHD-SMS-09** student at **Abdus Salam School of Mathematical Sciences GC University** in the subject of **Mathematics** year of admission **2009**, hereby declare that the matter printed in this thesis titled

"KERNEL OPERATORS IN SOME NEW FUNCTION SPACES"

is my own work and that

(i) I am not registered for the similar degree elsewhere contemporaneously.

(ii) No direct major work had already been done by me or anybody else on this topic; I worked on, for the Ph. D. degree.

(iii) The work, I am submitting for the Ph. D. degree has not already been submitted elsewhere and shall not in future be submitted by me for obtaining similar degree from any other institution.

Dated: ------------------------- ------------------------------------

Signature

RESEARCH COMPLETION CERTIFICATE

Certified that the research work contained in this thesis titled

"KERNEL OPERATORS IN SOME NEW FUNCTION SPACES"

has been carried out and completed by **Mr. Muhammad Asad Zaighum** Registration No. **103-GCU-PHD-SMS-09** under my supervision.

----------------------------	------------------------------
Date	**Alexander Meskhi**
	Supervisor

Submitted Through

Prof. Dr. A. D. Raza Choudary	**-------------------------------**
Director General	Controller of Examination
Abdus Salam School of Mathematical Sciences	GC University Lahore
GC University Lahore	Pakistan.
Pakistan.	

Dedicated to

My loving parents

Mr. Muhammad Iqbal Qureshi

and

Mrs. Rafia Iqbal

Table of Contents

Abstract

The thesis is devoted to the weighted criteria for integral operators with positive kernels in variable exponent Lebesgue and amalgam spaces. Similar results for multiple kernel operators defined with respect to a Borel measure in the classical Lebesgue spaces are also obtained. More precisely, we established necessary and sufficient conditions on a weight function v governing the boundedness/compactness of the weighted positive kernel operator $K_v f(x) = v(x) \int_0^x k(x,y) f(y) dy$ from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$ under the local log-Hölder continuity condition and the decay condition at infinity on the exponents p and q. In the case when K_v is bounded but not compact, two-sided estimates of the measure of non-compactness (essential norm) for K_v are obtained in terms of the weight v and kernel k. Criteria guaranteeing the boundedness /compactness of weighted kernel operators defined on $\mathbb{R}_+$ (resp. on $\mathbb{R}$) in variable exponent amalgam spaces are found. The kernel operators under consideration involve, for example, the Riemann-Liouville transform $R_\alpha f(x) = \int\limits_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt$, $0 < \alpha < 1$. Necessary and sufficient conditions ensuring weighted estimates for maximal and potential operators in variable exponent amalgam spaces are also established under the local log-Hölder continuity condition on exponent of spaces. Further, we establish criteria on measures governing the boundedness of integral operators with product positive kernels defined with respect to a Borel measure in the classical Lebesgue spaces. Finally, we point out that Fefferman-Stein type inequality for the multiple Riemann-Liouville transform defined with respect to a product Borel measure is derived.

Acknowledgements

All praise unto Allah, Who is the most Gracious and Merciful, Lord of all the worlds.

I am pleased to acknowledge my deepest gratitude, heartfelt regards and obligation to my respected supervisor Prof. Dr. Alexander Meskhi whose guidance, creativity and keen interest enabled me to complete the present thesis. He became a remarkable role model to me, both as a professional and as a teacher, yet most importantly as well as a human being for me. He has always been a source of inspiration for me.

Secondly, I would also like to acknowledge Prof. Dr. A. D. Raza Choudary (Director General ASSMS) for his invaluable guidance, constant help and encouragement over the last four and half years. I would like to thank him for providing us with excellent research and study facilities and atmosphere at Abdus Salam School of Mathematical Sciences.

Moreover, I wish to thank all the foreign professors at ASSMS (for they made ASSMS, a real place of learning), here I should mention Prof. Dr. Oleg I Reinov, Prof. Dr. Rein Zeinstra, Prof. Dr. Barbu Berceanu, Prof. Dr. Vasile Lupulescu, Prof. Dr. Constantin P. Niculescu, Prof. Dr. G. E. Karadzhov and Prof. Dr. Malkhaz Shashiashvili who taught me so much valuable information during my stay in ASSMS.

My special thanks are for Prof. Dr. Vakhtang Kokilashvili. I am thankful to Prof. Dr. Elisabeta Zelinka for her valuable advices during the preparation of the present thesis. I would also like to thank all the staff members, especially Awais Naeem.

Further on, I also express my gratitude to all of my colleagues in ASSMS for giving me a chance to improve my academic competence. I am especially thankful to colleagues in Batch IV, V, VI for their cooperation during the past years in ASSMS.

At this point I would like to extend my heartiest thank to my two colleague Azhar Ali Zafar and Asif Raza Khan for providing me support and advice whenever I required. I would also like to thank my friends Bismah Jamil, Mohsin Raza and Iftikhar Ahmed for their continuous encouragement and help during all these years. I would like to extend my thanks to Mr. Muhammad Jamil and Ms Sadia Jamil for their good wishes and prayers.

At this stage I cannot forget my entire family, especially my parents and my sisters: Nafeesa Saddaf, Tamseela Iffat, Alveena Riffat, Qurat ul Ain Kashifa for their prayers, continuous help and love. Their prayers lightened my sprit to finish this thesis.

It is my duty to acknowledge the financial support given by Higher Education Commission and Government of the Punjab throughout the process of completing the present thesis. Lastly, I offer my regards and blessings to all those who supported me in any respect during the completion of the thesis.

Muhammad Asad Zaighum,
Abdus Salam School of Mathematical Sciences,
GCU Lahore, Pakistan,
2013.

Notations

$\mathbb{R} = (-\infty, \infty)$	Set of real numbers
$\mathbb{R}_+ = (0, \infty)$	Set of positive real numbers
$\mathbb{R}_+^n$	$\underbrace{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+}_{n-factors}$
$\mathbb{Z}$	Set of Integers
$\mathbb{Z}_-$	Set of negative Integers
$\lvert E \rvert$	Lebesgue measure of a measurable set E
E_n	Denotes the interval $[2^n, 2^{n+1})$ $n \in \mathbb{N}$
I_n	Denotes the interval $[2^{n-1}, 2^{n+1})$ $n \in \mathbb{N}$
V_n	Denotes the interval $[a2^n, a2^{n+1})$ $n \in \mathbb{N}$, $a \in \mathbb{R}_+$
$I_k^{a,b}$	Denotes the interval $\left[a + \frac{b-a}{2^{k+1}}, a + \frac{b-a}{2^{k-1}}\right), \quad k \in \mathbb{N}$, for $b < \infty$
$I_k^{a,\infty}$	Denotes the interval $\left[a + 2^{k-1}, a + 2^{k+1}\right), \quad k \in \mathbb{Z}$
$a \approx b$	$c_1 a \leq b \leq c_2 a$ for some positive constants c_1 and c_2

Introduction

One of the important problems of modern Harmonic Analysis is to establish mapping properties of integral operators, generally speaking, in weighted Banach function spaces. The thesis is focused on weighted criteria for integral operators with positive kernels in variable exponent Lebesgue and amalgam spaces. Two-weighted (two-measured) estimates for the boundedness of multiple kernel operators defined with respect to a Borel measure in the classical Lebesgue spaces are also investigated. More precisely, we established necessary and sufficient conditions on a weight function v governing the boundedness/compactness of the weighted positive kernel operator $K_v f(x) = v(x)\int_0^x k(x,y)f(y)dy$ from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$ under the local log-Hölder continuity condition and the decay condition at infinity on the exponents p and q. In the case when K_v is bounded but not compact, two-sided estimates of the measure of non-compactness (essential norm) are obtained in terms of the weight v and kernel k. Criteria guaranteeing the boundedness/ compactness of weighted kernel operators defined on $\mathbb{R}_+$ (resp. on $\mathbb{R}$) in variable exponent amalgam spaces are found. It should be emphasized that the latter result is new even for constant exponent amalgam spaces. The kernel operators under consideration involve, for example, the Riemann-Liouville transform $R_\alpha f(x) = \int\limits_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt,\ 0 < \alpha < 1$. Necessary and sufficient conditions ensuring weighted estimates for maximal and potential operators in variable exponent amalgam spaces are also established under the local log-Hölder continuity condition on exponent of spaces. In the thesis, criteria on measures governing the boundedness of integral operators with product positive kernels defined with respect to a Borel measure in the classical Lebesgue spaces are also established. Finally, Fefferman-Stein type inequality for the multiple Riemann-Liouville transform defined with respect to a product Borel measure is derived.

In recent years, many researchers were involved to investigate mapping properties of integral operators in, so called, new function spaces. Examples of such spaces are the variable exponent Lebesgue and amalgam spaces. Lebesgue and Sobolev spaces with non-standard growth naturally arise, for example, in mechanics of the continuum medium [71] and Image restoration (see e.g. [1], [29]). In some variational problems [87, 88, 89], where the Lagrangians changes its non–linear character from point to point (examples of such Lagrangians can be found in plasticity theory, in study of rheological fluids and others). The space $L^{p(\cdot)}$ is a particular case of the Musielak-Orlicz space (see [66], [67]). H. Nakano [68] was the the first one who conducted a well–ordered study of modular spaces. Some essential properties of these spaces were studied by Kovácik–Rákosník [51], S. Samko [74, 75], I. Sharapudinov [81] etc.

The first breakthrough in the direction of the boundedness of integral operators in these spaces is due to L. Diening (see [20]). In that paper the author proved the boundedness of the Hardy–Littlewood maximal operator in $L^{p(\cdot)}(\Omega)$, where Ω is a bounded domain, under the local log-Hölder continuity condition on p. The mapping properties of potentials, singular integrals and maximal functions in variable exponent Lebesgue spaces (VELS) were studied by S. Samko; A. Nekvinda; P. Harjulehto, P. Hästö and M. Pere; C. Capone, D. Cruz–Uribe; D. Cruz–Uribe, A. Fiorenza and

Neugebauer; D. E. Edmunds, V. Kokilashvili and A. Meskhi; D. E. Edmunds and Rákosník; L. Diening and Ružička; and A. Fiorenza; A. Almeida and S. Samko; H. Rafeiro and S. Samko; T. Futamura and Y. Mizuta; Y. Sawano and T. Shimomura; D. Cruz–Uribe, A. Fiorenza, J. M. Martell and C. Perez, etc.

Historically, the boundedness problem for the two–weighted Hardy transform $(H_{v,w}f)(x) = v(x)\int\limits_0^x f(t)w(t)dt$ from $L^{p(\cdot)}$ to $L^{q(\cdot)}$ was studied in the papers [24] and [50] under different terms on weights (see also [13] for related results). The compactness problem for $H_{v,w}$ was investigated in the paper [24]. The boundedness of maximal and fractional integral operators in unweighted/weighted variable exponent Lebesgue spaces defined on $\mathbb{R}^n$ was studied by many authors. We mention some of the important works in this direction: [3], [10], [14], [15], [16], [17], [20], [21], [22], [26], [28], [38], [39], [40], [42], [45], [47], [49], [54], [57], [61], [75], etc. The compactness (resp. non-compactness) problem for singular and fractional integrals in weighted VELS was explored by A. Meskhi in his monograph [61]. We refer also to the monographs [12] and [19] for these and other relevant results. It should be emphasized that in [15] and [22] a complete characterization of the one–weight inequality for the Hardy–Littlewood maximal operator is given under the Muckhenhoupt–type conditions (see also the paper [14], and the monographs [12] and [19] for related topics).

Another space which is considered in this thesis is the amalgam space. The idea of considering amalgam spaces, as opposed to the classical Lebesgue spaces, is a natural one because it allows us to separate the global behavior from local behavior of a function. This idea goes back to N. Wiener [84, 85] who considered the special cases of amalgam spaces. Other cases have appeared sporadically since then (see e.g. [82]). The first systematic study of amalgam space on the real line was conducted by F.

Holland [32]. For other structural properties of amalgam space we refer to [27], [82].

Two–weight problem for the Hardy transform $(\mathcal{H}f)(x) = \int_{-\infty}^{x} f(t)dt$ in amalgam spaces defined on $\mathbb{R}$ was solved by C. Carton–Lebrun, H. P. Heinig and S. C. Hofmann [11] (see also [31], [72] for related topics). One–weighted inequality for the Hardy-Littlewood maximal operator was also established in [11]. Two–weighted inequalities for the Hardy–Littlewood,fractional maximal operators and fractional integrals in amalgam spaces defined on $\mathbb{R}$ were established by Y. Rakotondratsimba [70]. In the paper [9] authors studied the two–weight criteria for generalized Hardy–type kernel operators which includes fractional integrals without singularity (i.e. of order greater than one).

It is worth mentioning that the contents of the main results of thesis are the part of the following papers [36], [37], [58], [59], [60]. Some of them were presented at the sixth world conference on 21^{st} century mathematics, Lahore, 2013.

Chapter 1

Kernel Operators in Variable Exponent Lebesgue Spaces

In this chapter, necessary and sufficient conditions on a weight v governing the boundedness/compactness of the weighted kernel operator $K_v f(x) = v(x) \int_0^x k(x,t)f(t)dt$ from the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}_+)$ into another one $L^{q(\cdot)}(\mathbb{R}_+)$ are established under the local log–Hölder continuity condition and the decay condition at infinity on exponents. The distance between K_v and the class of compact integral operators acting from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$ (measure of non–compactness) is also estimated from above and below.

1.1 Preliminaries

Let $J \subseteq \mathbb{R}$ be a measurable set with positive measure. For a measurable function p on J we denote:

$$p_+(J) := \sup_J p, \qquad p_-(J) := \inf_J p.$$

By $\mathcal{P}(J)$ we denote the class of all measurable function p for which $1 < p_-(J) \leq p_+(J) < \infty$. By the symbol $p'(x)$ we denote the function $\frac{p(x)}{p(x)-1}$, $1 < p(x) < \infty$.

We say that a real-valued measurable function g defined on J belongs to $L^{p(\cdot)}(J)$ (or to $L^{p(x)}(J)$) if

$$S_{p(\cdot)}(g) = \int\limits_J \left|g(x)\right|^{p(x)} dx < \infty.$$

The space $L^{p(\cdot)}(J)$ is a Banach space with respect to the norm (see e.g. [44], [51])

$$\|g\|_{L^{p(\cdot)}(J)} = \inf \left\{\lambda > 0 : S_{p(\cdot)}\big(g/\lambda\big) \le 1\right\}.$$

We begin with the following propositions:

Proposition A ([51], [74], [81])**.** *Let $J \subseteq \mathbb{R}$ be a measurable subset. Suppose that $p \in \mathcal{P}(J)$. Then*

$$\text{(i)} \qquad \|g\|^{p_+(J)}_{L^{p(\cdot)}(J)} \le S_{p(\cdot)}(g\chi_J) \le \|g\|^{p_-(J)}_{L^{p(\cdot)}(J)}, \ \ \|g\|_{L^{p(\cdot)}(J)} \le 1;$$

$$\|g\|^{p_-(J)}_{L^{p(\cdot)}(J)} \le S_{p(\cdot)}(g\chi_J) \le \|g\|^{p_+(J)}_{L^{p(\cdot)}(J)}, \ \ \|g\|_{L^{p(\cdot)}(J)} \ge 1;$$

(ii) *Hölder's inequality*

$$\Big|\int\limits_J g_1(x)g_2(x)dx\Big| \le \Big(\frac{1}{p_-(J)} + \frac{1}{(p_+(J))'}\Big) \ \|g_1\|_{L^{p(\cdot)}(J)} \ \|g_2\|_{L^{p'(\cdot)}(J)}$$

holds, where $g_1 \in L^{p(\cdot)}(J)$, $g_2 \in L^{p'(\cdot)}(J)$.

Proposition B ([51], [74])**.** *Let $1 \le s(x) \le r(x)$, $x \in J$. Then the following inequality*

$$\|g\|_{L^{s(\cdot)}(J)} \le (|J| + 1)\|g\|_{L^{r(\cdot)}(J)}$$

holds.

Definition 1.1.1. A measurable function p satisfies the log-Hölder continuity (weak Lipschitz) condition on J ($p \in \mathcal{P}^{log}(J)$), if there is a positive constant c_1 such that for all x_1 and x_2 in J with $0 < |x_1 - x_2| < 1/2$, the inequality

$$|p(x_1) - p(x_2)| \le c_1/(-\ln|x_1 - x_2|)$$

holds.

Definition 1.1.2. A measurable function p satisfies the decay condition on J at infinity ($p \in \mathcal{P}_\infty(J)$), if there are constants $c_\infty > 0$ and $p_\infty \in (1, \infty)$ such that for all x in J the inequality

$$|p(x) - p_\infty| \leq \frac{c_\infty}{\ln(e + |x|)}$$

holds.

Let us use the following notation:

$$\mathcal{P}^{log}(J) \cap \mathcal{P}_\infty(J) =: \mathcal{P}^{log}_\infty(J).$$

It is known (see [20]) that if $p \in \mathcal{P}^{log}$, then the Hardy–Littlewood maximal operator M is bounded of in $L^{p(x)}$ spaces defined on a bounded domain, while the condition $p \in \mathcal{P}^{log}_\infty$ implies the boundedness of M in $L^{p(x)}$ space on unbounded domains (see [17]).

Lemma A ([20])**.** *Let $I \subseteq \mathbb{R}$ be an interval. Then $p \in \mathcal{P}^{log}(I)$ if and only if there exists a positive constant c_1 such that*

$$|I_1|^{p_-(I_1) - p_+(I_1)} \leq c_1$$

for all intervals $I_1 \subseteq I$ with $|I_1| > 0$.

Remark 1. If $p \in \mathcal{P}^{log}_\infty(\mathbb{R}_+)$, then following conditions are satisfied at 0 and ∞:

$$|p(x) - p(0)| \leq \frac{A_0}{|ln|x||} \qquad |x| \leq 1, \tag{1.1.1}$$

$$|p(x) - p_\infty| \leq \frac{A_\infty}{|ln|x||} \qquad |x| > 1. \tag{1.1.2}$$

Remark 2. Let $I = \mathbb{R}_+$. It is known that $\|\chi_{(0,r)}\| \approx r^{1/p(0)}$ as $r \to 0$ if $p \in \mathcal{P}^{log}(I)$, and $\|\chi_{(0,r)}\| \approx r^{1/p_\infty}$ as $r \to \infty$, if $p \in \mathcal{P}^{\log}_\infty(I)$.

Lemma B. *Let D be a constant greater than 1 and $p \in \mathcal{P}^{log}_\infty(\mathbb{R}_+)$. Then*

$$\frac{1}{c_0} r^{\frac{1}{p(0)}} \leq \|\chi_{(r,Dr)}\|_{L^{p(\cdot)}} \leq c_0 r^{\frac{1}{p(0)}} \qquad for \qquad 0 < r \leq 1 \tag{1.1.3}$$

and

$$\frac{1}{c_\infty} r^{\frac{1}{p_\infty}} \leq \|\chi_{(r,Dr)}\|_{L^{p(\cdot)}} \leq c_\infty r^{\frac{1}{p_\infty}} \qquad for \qquad r \geq 1 \tag{1.1.4}$$

holds, where $c_0 \geq 1$ and $c_\infty \geq 1$ depend on D, but do not depend on r.

Proof. We follow the proof of Lemma 4.6 in [73]. We prove only (1.1.4). The proof for (1.1.3) is similar. Recall that $\int\limits_{\mathbb{R}_+} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leq 1 \Leftrightarrow \|f\|_{L^{p(\cdot)}} \leq \lambda$ for $\lambda > 0$; $\int\limits_{\mathbb{R}_+} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \geq 1 \Leftrightarrow \|f\|_{L^{p(\cdot)}} \geq \lambda$ for $\lambda > 0$. Therefore the right–hand side inequality of (1.1.4) holds if and only if

$$\int\limits_r^{Dr} \frac{dx}{(c_\infty r^{\frac{1}{p_\infty}})^{p(x)}} \leq 1 \tag{1.1.5}$$

holds.

The left–hand side of (1.1.5) is estimated as follows

$$\begin{aligned} \int\limits_r^{Dr} \frac{dx}{(c_\infty r^{\frac{1}{p_\infty}})^{p(x)}} &\leq \frac{1}{c_\infty^{p_-}} \int\limits_r^{Dr} \frac{dx}{\left(\frac{x}{D}\right)^{\frac{p(x)}{p_\infty}}} \\ &\leq \frac{D^{\frac{p_+}{p_\infty}}}{c_\infty^{p_-}} \int\limits_r^{Dr} \frac{dx}{x^{\frac{p(x)}{p_\infty}}}. \end{aligned}$$

By (1.1.2) we have $e^{\frac{-A_\infty}{p_\infty}} x \leq x^{\frac{p(x)}{p_\infty}} \leq e^{\frac{A_\infty}{p_\infty}} x$ for $x \geq 1$.
Therefore,

$$\int\limits_r^{Dr} \frac{dx}{(c_\infty r^{\frac{1}{p_\infty}})^{p(x)}} \leq \frac{e^{\frac{A_\infty}{p_\infty}} D^{\frac{p_+}{p_\infty}}}{c_\infty^{p_-}} \int\limits_r^{Dr} \frac{dx}{x} = \frac{e^{\frac{A_\infty}{p_\infty}} D^{\frac{p_+}{p_\infty}}}{c_\infty^{p_-}} lnD.$$

Hence choosing $c_\infty^{p_-} = e^{\frac{A_\infty}{p_\infty}} D^{\frac{p_+}{p_\infty}} lnD$ proves the right–hand side of the inequality. The proof for the left–hand side inequality is similar. ■

For the next statement we refer to [49].

Proposition C. *Let p and q be measurable functions on $I := (a, b)$ $(-\infty < a < b \leq +\infty)$ such that $p, q \in \mathcal{P}(I)$ and $p(x) \leq q(x)$, $x \in I$. Let $p,q \in \mathcal{P}_\infty^{log}(I)$. Then there is a positive constant c depending only on p and q such that for all $f \in L^{p(\cdot)}(I)$, $g \in L^{q'(\cdot)}(I)$ and all sequences of intervals $S_k := [x_k, x_{k+1})$, where $[x_k, x_{k+1})$ are disjoint intervals satisfying the condition $\cup_k [x_k, x_{k+1}) = I$, the inequality*

$$\sum_k \|f\chi_{S_k}\|_{L^{p(\cdot)}(I)} \|g\chi_{S_k}\|_{L^{q'(\cdot)}(I)} \leq c\|f\|_{L^{p(\cdot)}(I)} \|g\|_{L^{q'(\cdot)}(I)}$$

holds.

In the next statement, we assume that the exponents are constant outside some large interval; this statement would be useful because it allows us to give values of constants. For this statement we refer to [50] in the case of finite interval, and [4] for infinite interval.

Proposition D. *Let p and q be measurable functions on $I := (a,b)$ $(-\infty < a < b \le +\infty)$ such that $p, q \in \mathcal{P}(I)$ and $p(x) \le q(x)$, $x \in I$. Let p,$q \in \mathcal{P}^{log}(I)$. Suppose also that if $b = \infty$, then $p(x) \equiv p_c \equiv const$, $q(x) \equiv q_c \equiv const$ outside some large interval (a,d) . Then there is a positive constant c depending only on p and q such that for all $f \in L^{p(\cdot)}(I)$, $g \in L^{q'(\cdot)}(I)$ and all sequences of intervals $S_k := [x_{k-1}, x_{k+1})$, where $[x_k, x_{k+1})$ are disjoint intervals satisfying the condition $\cup_k [x_k, x_{k+1}) = I$, the inequality*

$$\sum_k \|f\chi_{S_k}\|_{L^{p(\cdot)}(I)} \|g\chi_{S_k}\|_{L^{q'(\cdot)}(I)} \le cC_{a,b} \|f\|_{L^{p(\cdot)}(I)} \|g\|_{L^{q'(\cdot)}(I)}$$

holds. Moreover, the value of $C_{a,b}$ is defined as follows: $C_{a,b} = [(b-a)+1]^2$ if $b < \infty$ and $C_{a,\infty} = [(d-a)+1]^2 + 1$ if $b = \infty$.

In the next statement the intervals S_k are replaced by $I_k^{a,b}$.

Proposition E. *Let p and q be measurable functions on $I := (a,b)$ $(-\infty < a < b \le +\infty)$ such that $p, q \in \mathcal{P}(I)$ and $p(x) \le q(x)$, $x \in I$. Let p,$q \in \mathcal{P}^{log}_{\infty}(I)$. Then there is a positive constant c depending only on p and q such that for all $f \in L^{p(\cdot)}(I)$, $g \in L^{q'(\cdot)}(I)$ and all intervals $I_k^{a,b}$, the inequality*

$$\sum_k \|f\chi_{I_k^{a,b}}\|_{L^{p(\cdot)}(I)} \|g\chi_{I_k^{a,b}}\|_{L^{q'(\cdot)}(I)} \le c\|f\|_{L^{p(\cdot)}(I)} \|g\|_{L^{q'(\cdot)}(I)}$$

holds.

Proof. The proof in the case of $I = (0,1)$ can be found in [4]. For simplicity let us assume that $I = \mathbb{R}_+$. In this case $a = 0$, $b = \infty$ and consequently, $I_k^{0,\infty} = I_k$. Now the proof follows in same manner as in [49] Proposition 3.4, since the map $g := I \to (-1/2, 1/2)$ defined by $g(x) = \frac{\arctan x}{\pi}$ keeps the property $\sum\limits_k \chi_{g(I_k)}(x) \le 2$. Details are omitted. ∎

Let w and v be a.e. positive measurable function on $[a,b)$, $-\infty < a < b \le \infty$, and let

$$(H^{(a,b)}_{v,w}g)(x) = v(x)\int\limits_a^x g(t)w(t)dt, \qquad x \in [a,b).$$

Further, we denote

$$(H_{v,w}g)(x) = v(x)\int\limits_0^x g(t)w(t)dt, \qquad x > 0,$$

$$(\mathcal{H}_{v,w}g)(x) = v(x)\int\limits_{-\infty}^x g(t)w(t)dt, \qquad x \in \mathbb{R}.$$

For the boundedness of Hardy operator in classical Lebesgue spaces we have the following two–weight criterion.

Theorem A ([48], [65]). *Let p and q be constants such that $1 < p \le q < \infty$. Suppose that $0 \le c < d \le \infty$. Let w and v be non-negative measurable functions on $[c,d)$. Then the inequality*

$$\left(\int_c^d v(x)\left(\int_c^x g(t)dt\right)^q dx\right)^{1/q} \le c_1\left(\int_c^d w(t)(g(t))^p dt\right)^{1/p}, \;\; g \ge 0,$$

holds if and only if

$$A_1 := \sup_{c\le t\le d}\left(\int_t^d v(x)dx\right)^{1/q}\left(\int_c^t w^{1-p'}(x)dx\right)^{1/p'} < \infty.$$

Moreover, if c_1 is the best constant in the inequality, then $c_1 \approx A_1$.

For more literature on Hardy inequalities we refer to the books [53], [56].

The next statement was proved in [39] for the case of infinite interval and in [50] for finite interval but we present the proof because of the lower and upper bound of $\|H_{v,w}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)}$.

Theorem B. *Let $-\infty < a < b < +\infty$ and let p and q be measurable functions on $I := (a,b)$ satisfying the conditions: $p, q \in \mathcal{P}(I)$ and $p(x) \le q(x)$, $x \in I$. Let $p, q \in \mathcal{P}^{log}(I)$. Further, assume that $p \equiv p_c \equiv const$, $q \equiv q_c \equiv const$ outside some large interval (a,d) if $b = \infty$. Then $H^{(a,b)}_{v,w}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$ if and only if*

$$A_{a,b} := \sup_{a<t<b} \|\chi_{(t,b)(\cdot)} v(\cdot)\|_{L^{q(\cdot)}(I)} \|\chi_{(a,t)(\cdot)} w(\cdot)\|_{L^{p'(\cdot)}(I)} < \infty.$$

Moreover, there are positive constants c_1 and c_2 independent of the interval I such that

$$c_1 A_{a,b} \le \|H^{(a,b)}_{v,w}\|_{L^{p(\cdot)}(I) \to L^{q(\cdot)}(I)} \le c_2 C_{a,b} A_{a,b},$$

where the constant $C_{a,b}$ is defined in Proposition D.

Proof. Sufficiency. Let $f \ge 0$. Suppose that $b < \infty$ and that $\int\limits_a^b f(t)dt \in [2^{m_0}, 2^{m_0+1})$ for some integer m_0. We construct a sequence $\{x_k\}$ so that

$$\int\limits_a^{x_k} fw = \int\limits_{x_k}^{x_{k+1}} fw = 2^k.$$

It is easy to check that $(a,b) = \cup_k [x_k, x_{k+1})$. Let h be a function such that $\|h\|_{L^{q'(\cdot)}([a,b])} \le 1$. Applying Hölder's inequality for VELS (see Proposition A) and Proposition D we have that

$$\begin{aligned}
\int\limits_a^b (H_{v,w}f)h &\le \sum_k \Big(\int\limits_{x_k}^{x_{k+1}} hv \Big) \Big(\int\limits_0^{x_{k+1}} fw \Big) \\
&= 4 \sum_k \Big(\int\limits_{x_k}^{x_{k+1}} hv \Big) \Big(\int\limits_{x_{k-1}}^{x_k} fw \Big) \\
&\le 4 \sum_k \|\chi_{(x_k, x_{k+1})}(\cdot) h(\cdot)\|_{L^{q'(\cdot)}(I)} \|\chi_{(x_k, x_{k+1})}(\cdot) v(\cdot)\|_{L^{q(\cdot)}(I)} \\
&\quad \times \|\chi_{(x_{k-1}, x_k)}(\cdot) f(\cdot)\|_{L^{p(\cdot)}(I)} \|\chi_{(x_{k-1}, x_k)}(\cdot) w(\cdot)\|_{L^{p'(\cdot)}(I)} \\
&\le 4A_{a,b} \sum_k \|\chi_{(x_k, x_{k+1})}(\cdot) h(\cdot)\|_{L^{q'(\cdot)}(I)} \|\chi_{(x_{k-1}, x_k)}(\cdot) f(\cdot)\|_{L^{p(\cdot)}(I)} \\
&\le 4C_{a,b} A_{a,b} \|f\|_{L^{p(\cdot)}(I)} \|h\|_{L^{q'(\cdot)}(I)},
\end{aligned}$$

where $C_{a,b}$ is the constant defined in Proposition D. Taking now the supremum with respect to h we have sufficiency for $b<\infty$.

Let now $b=\infty$. Then

$$\begin{aligned}\| \quad & H_{v,w}^{(a,\infty)} f\|_{L^{q(\cdot)}((a,+\infty))} \leq \Big\| v(x)\int_a^x fw \Big\|_{L^{q(\cdot)}((a,d))} + \Big\| v(x)\int_a^x fw \Big\|_{L^{q_c}([d,+\infty))} \\ := \quad & I_1 + I_2.\end{aligned}$$

By applying already used arguments we have that $I_1 \leq 4C_{a,\infty}A_{a,+\infty}$, where $C_{a,\infty} = [(d-a)+1]^2$. Further, due to Hölder's inequality and Theorem A we find that

$$\begin{aligned} I_2 \;&\leq\; \|v(x)\int_a^d fw\|_{L^{q_c}([d,+\infty))} + \|v(x)\int_d^x fw\|_{L^{q_c}([d,+\infty))} \\ &\leq\; \|v(\cdot)\chi_{[d,+\infty)}(\cdot)\|_{L^{q(\cdot)}} \|w(\cdot)\chi_{[a,d)}(\cdot)\|_{L^{p'(\cdot)}} \|f\|_{L^{p(\cdot)}} \\ &+\; 4A_{a,+\infty}\|f\|_{L^{p(\cdot)}(I)} \leq 5A_{a,+\infty}\|f\|_{L^{p(\cdot)}(I)}. \quad \Box \end{aligned}$$

The lower bound for the norm $\|H_{v,w}^{(a,b)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)}$ can be obtained by choosing the test function $f(x) = \chi_{(a,t)}(x)$, $a<t<b$ in the inequality

$$\|H_{v,w}^{(a,b)} f\|_{L^{q(\cdot)}(I)} \leq c\|f\|_{L^{p(\cdot)}(I)}.$$

■

The following statement deals with the unbounded interval $I=\mathbb{R}_+$.

Theorem C. *Let $I=\mathbb{R}_+$. Let $p,q\in\mathcal{P}(I)$ and $p(x)\leq q(x)$, $x\in I$. Suppose that $p,q\in\mathcal{P}_\infty^{log}(I)$. Then $H_{v,w}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$ if and only if*

$$D_\infty := \sup_{t>0} \|\chi_{(t,\infty)}(\cdot)v(\cdot)\|_{L^{q(\cdot)}(I)} \|\chi_{(0,t)}(\cdot)w(\cdot)\|_{L^{p'(\cdot)}(I)} < \infty.$$

Proof. Sufficiency. Let $f\geq 0$, and $\int_0^\infty f(t)dt=\infty$. We construct a sequence $\{x_k\}$ so that

$$\int_0^{x_k} fw = \int_{x_k}^{x_{k+1}} fw = 2^k.$$

It is easy to check that $[0,\infty) = \cup_k [x_k, x_{k+1})$. Let h be a function satisfying the condition, $\|h\|_{L^{q'(\cdot)}([a,b])} \leq 1$. By applying Hölder's inequality for variable exponent Lebesgue spaces and Proposition E we have that

$$\int_0^\infty (H_{v,w}f)h \leq \sum_k \Big(\int_{x_k}^{x_{k+1}} hv\Big)\Big(\int_0^{x_{k+1}} fw\Big)$$

$$\begin{aligned}
&= 4\sum_k \Big(\int_{x_k}^{x_{k+1}} hv\Big)\Big(\int_{x_{k-1}}^{x_k} fw\Big)\\
&\leq 4\sum_k \|\chi_{(x_k,x_{k+1})}(\cdot)h(\cdot)\|_{L^{q'(\cdot)}(I)}\|\chi_{(x_k,x_{k+1})}(\cdot)v(\cdot)\|_{L^{q(\cdot)}(I)}\\
&\quad\times\|\chi_{(x_{k-1},x_k)}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)}\|\chi_{(x_{k-1},x_k)}(\cdot)w(\cdot)\|_{L^{p'(\cdot)}(I)}\\
&\leq 4D_\infty\sum_k \|\chi_{(x_k,x_{k+1})}(\cdot)h(\cdot)\|_{L^{q'(\cdot)}(I)}\|\chi_{(x_{k-1},x_k)}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)}\\
&\leq 4D_\infty\|f\|_{L^{p(\cdot)}(I)}\|h\|_{L^{q'(\cdot)}(I)}.
\end{aligned}$$

Taking now the supremum with respect to h gives sufficiency.
Necessity can be obtained by the standard way taking the test function f supported in $(0,t)$ with $\|f\|_{L^{p(\cdot)}} \leq 1$. ∎

For the characterization of two–weight inequality for the Hardy transform in the classical Lebesgue spaces we refer e.g. to the papers [48], [56].

Remark 3. (a) The norm $\|\chi_{(0,2^n)}\|_{L^{p'(\cdot)}(I)}$ can be replaced by $\|\chi_{E_n}(\cdot)\|_{L^{p'(\cdot)}(I)}$. This follows from Lemma B and Remark 2.

(b) If w is constant and $p \in \mathcal{P}^{log}_\infty(I)$, then $D_\infty < \infty$ is equivalent to the condition:

$$\bar{D}_\infty := \sup_{n\in\mathbb{Z}} \|\chi_{E_n}(\cdot)v(\cdot)\|_{L^{q(\cdot)}(I)}\|\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)} < \infty.$$

The fact that $D_\infty < \infty$ implies $\bar{D}_\infty < \infty$ is obvious.
Conversely, let $\bar{D}_\infty < \infty$. Let us now take $t \in I$. Then $t \in [2^m, 2^{m+1})$ for some $m \in \mathbb{Z}$. Consequently,

$$\begin{aligned}
D_\infty(t) &\leq \sum_{n=m}^\infty \|\chi_{E_n}(x)v(x)\|_{L^{q(x)}(I)}\|\chi_{(0,2^{m+1})}\|_{L^{p'(\cdot)}(I)}\\
&\leq \bar{D}_\infty\Big(\sum_{n=m}^\infty \|\chi_{(0,2^n)}\|^{-1}_{L^{p'(\cdot)}(I)}\Big)\|\chi_{(0,2^{m+1})}\|_{L^{p'(\cdot)}(I)}.
\end{aligned}$$

Hence,

$$D_\infty(t) \le \begin{cases} \bar{D}_\infty[(\sum\limits_{n=m}^{0} 2^{-n/p'(0)})2^{m/p'(0)} + (\sum\limits_{n=0}^{\infty} 2^{-n/p'_\infty})2^{m/p'_\infty}] \le c_1(p)\bar{D}_\infty & \text{if } m < 0, \\ \bar{D}_\infty(\sum\limits_{n=m}^{\infty} 2^{-n/p'_\infty})2^{m/p'_\infty} \le c_2(p)\bar{D}_\infty & \text{if } m \ge 0. \end{cases}$$

where $c_1(p)$ and $c_2(p)$ are constants depending only on p. Finally, $D_\infty < c\bar{D}_\infty$.

Theorem D ([26]). *Let $r(x)$ and $s(x)$ be measurable functions on an interval $I \subseteq \mathbb{R}_+$. Suppose that $r, s \in \mathcal{P}(I)$. If*

$$\Big\| \|k(x,y)\|_{L^{r'(y)}(I)} \Big\|_{L^{s(x)}(I)} < \infty,$$

where k is a non-negative kernel, then the operator

$$Kf(x) = \int\limits_I k(x,y)f(y)dy$$

is compact from $L^{r(\cdot)}(I)$ to $L^{s(\cdot)}(I)$.

Definition 1.1.3. Let $I := (0,a)$, $0 < a \le \infty$. A kernel $k : \{(x,y) : 0 < y < x < a\} \to (0,\infty)$ belongs to the class $V(I)$ ($k \in V(I)$) if there exists a constant c_1 such that for all x, y, t with $0 < y < t < x < a$, the inequality

$$k(x,y) \le c_1 k(x,t)$$

holds.

Definition 1.1.4. Let r be a measurable function on $I = (0,a)$, $0 < a \le \infty$ with values in $(1,+\infty)$. A kernel k belongs to the class $V_{r(\cdot)}(I)$ if there exists a positive constant c_2 such that for a.e. $x \in (0,a)$, the inequality

$$\|\chi_{(\frac{x}{2},x)}(\cdot)k(x,\cdot)\|_{L^{r(\cdot)}(I)} \le c_2 \|\chi_{(\frac{x}{2},x)}(\cdot)\|_{L^{r(\cdot)}} k\Big(x,\frac{x}{2}\Big) \tag{1.1.6}$$

is fulfilled.

These conditions on a kernel k were introduced by the first author in the paper [63] for the constant p.

Remark 4. Using Lemmas A and B we have $\|\chi_{(\frac{x}{2},x)}\|_{L^{r(\cdot)}} \approx x^{1/r(0)} \approx x^{1/r(x)}$ near zero. Therefore (1.1.6) coincides with the following condition

$$\|\chi_{(\frac{x}{2},x)}(\cdot)k(x,\cdot)\|_{L^{r(\cdot)}(I)} \leq cx^{\frac{1}{r(x)}}k\Big(x,\frac{x}{2}\Big).$$

Similarly, by Lemma B we see that $\|\chi_{(\frac{x}{2},x)}\|_{L^{r(\cdot)}} \approx x^{1/r_\infty}$ near infinity.

Example 1. *Let $I := \mathbb{R}_+$. Let α be a measurable function on I satisfying the condition $0 < \alpha_-(I) \leq \alpha_+(I) \leq 1$. Let $r \in \mathcal{P}^{log}_\infty(I)$. Suppose that r be non-increasing on (a,∞) for some large $a > 0$. Then $k(x,t) = (x-t)^{\alpha(x)-1} \in V(I) \cap V_{r(\cdot)}(I)$ when $(\alpha r')_+(I) > 1$.*

Indeed, first it is easy to check that $k \in V(I)$. Further to prove that $k \in V_{r(\cdot)}(I)$ we need to show

$$I(x) := \|(x-\cdot)^{\alpha(x)-1}\chi_{(x/2,x)}(\cdot)\|_{L^{r(\cdot)}} \leq c\|\chi_{(x/2,x)}(\cdot)\|_{L^{r(\cdot)}}x^{\alpha(x)-1}, \tag{1.1.7}$$

where the constant c in (1.1.7) is independent of x. Since $r \in \mathcal{P}^{\log}_\infty(I)$, by Lemma A for $x - t < 1$, we have

$$(x-t)^{r(t)} \leq c_1(x-t)^{r(x)} \leq c_2(x-t)^{r(t)} \tag{1.1.8}$$

where c_1 and c_2 does not depend on x.

Since r is non-increasing, for $x - t \geq 1$, we have

$$(x-t)^{r(t)} \geq (x-t)^{r(x)}. \tag{1.1.9}$$

Consequently,

$$\begin{aligned} S(x) := \int_{x/2}^{x} (x-t)^{(\alpha(x)-1)r(t)}dt &= \int\limits_{\{t:\, t\in(x/2,x),\,(x-t)<1\}} \cdots + \int\limits_{\{t:\, t\in(x/2,x),\,(x-t)\geq 1\}} \cdots \\ &=: S_1(x) + S_2(x). \end{aligned}$$

First we estimate $S_1(x)$. Taking into account (1.1.8) we have the following pointwise estimate

$$\begin{aligned} S_1(x) &\leq \int\limits_{\{t:\, t\in(x/2,x),\,(x-t)<1\}} (x-t)^{(\alpha(x)-1)r(x)}dt \\ &\leq \int\limits_{x/2}^{x} (x-t)^{(\alpha(x)-1)r(x)}dt = cx^{(\alpha(x)-1)r(x)+1} \end{aligned}$$

By using (1.1.9) for $S_2(x)$, we have the following pointwise estimate

$$\begin{aligned} S_2(x) &\leq \int\limits_{\{t:\, t\in(x/2,x),\,(x-t)\geq 1\}} (x-t)^{(\alpha(x)-1)r(x)}dt \\ &\leq \int\limits_{x/2}^{x} (x-t)^{(\alpha(x)-1)r(x)}dt = cx^{(\alpha(x)-1)r(x)+1} \end{aligned}$$

Since $I(x) \geq d$ for some positive constant d, by Lemma B and Proposition A we have

$$\begin{aligned} \frac{I(x)}{d} &\leq cS(x)^{1/r_-([x/2,x])} = cS(x)^{1/r(x)} \\ &= cx^{\alpha(x)-1+\frac{1}{r(x)}} \leq cx^{\alpha(x)-1+\frac{1}{r_\infty}} \\ &= c\|\chi_{(x/2,x)}(\cdot)\|_{L^{r(\cdot)}(I)}k(x/2,x). \end{aligned}$$

Hence, we have estimate (1.1.7).

The following examples of kernels can be verified easily:

Example 2 (Lemma 3 of [4])**.** *Let $I := (0,a)$, where $0 < a \leq \infty$. Let α be a measurable function on I satisfying the condition $0 < \alpha_-(I) \leq \alpha_+(I) \leq 1$. Suppose that r is a function on I such that $r \in \mathcal{P}(I)$ and $r \in \mathcal{P}^{log}(I)$. Suppose that $r(x) \equiv r_0 \equiv const$ outside some interval $(0,b)$ when $a = +\infty$. Then $k(x,t) = (x-t)^{\alpha(x)-1} \in V(I) \cap V_{r(\cdot)}(I)$ when $r(x) < \frac{1}{1-\alpha(x)}$.*

Example 3. *Let $I := (0,a)$, where $0 < a \leq \infty$. Suppose that α is a measurable function on I satisfying the condition $0 < \alpha_-(I) \leq \alpha_+(I) \leq 1$. Let r be a function* *on I such that $r \in \mathcal{P}(I)$ and $r, \bar{r} \in \mathcal{P}^{log}(I)$ where $\overline{r}(t) = r(t^{1/\sigma})$. Suppose that $r(x) \equiv r_0 \equiv const$ outside some interval $(0,b)$ when $a = +\infty$. Then $k(x,y) = (x^\sigma - y^\sigma)^{\alpha(x)-1} \in V(I) \cap V_{r(\cdot)}(I)$ when $r(x) < \frac{1}{1-\alpha(x)}$ and $\sigma > 0$.*

Example 4. *Let $I := (0,a)$, $0 < a \le \infty$. Let r be a function on I such that $r \in \mathcal{P}(I)$ and $r \in \mathcal{P}^{log}(I)$. Let r be increasing on I. Suppose that $r(x) \equiv r_0 \equiv const$ outside some interval $(0,b)$ when $a = +\infty$. Further, let $0 < \alpha_- \le \alpha(x) \le 1$ and $\alpha(x) + \beta(x) > 2 - \frac{1}{r(x)}$. Then $k(x,y) = (x-y)^{\alpha(x)-1} ln^{\beta(x)-1} \frac{x}{y} \in V(I) \cap V_{r(\cdot)}(I)$.*

Remark 5. It is easy to check that if $k \in V(I) \cap V_{r(\cdot)}(I)$, then the kernel $\varphi(x)k(x,y)$ belongs to $V(I) \cap V_{r(\cdot)}(I)$, where φ is a measurable function defined on I.

For other examples of kernels in the classical Lebesgue space, we refer to [63].

1.2 Boundedness Criteria

This section is devoted to boundedness criteria for the operator

$$K_v f(x) = v(x) \int_0^x f(t)dt, \qquad x \in \mathbb{R}_+,$$

from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$. If $v \equiv const$ then we have the integral operator defined by,

$$Kf(x) = \int_0^x f(t)dt, \qquad x \in \mathbb{R}_+.$$

We begin with boundedness result for the integral operator defined on a bounded interval.

Theorem 1.2.1. *Let $I := (0,a)$ be a bounded interval and let $p, q \in \mathcal{P}(I)$, $p(x) \le q(x)$, $x \in I$. Suppose that $k \in V(I) \cap V_{p'(\cdot)}(I)$. Further, assume that $p, q \in \mathcal{P}^{log}(I)$. Then the following statements are equivalent:*

(i) $\quad \|K_v f\|_{L^{q(\cdot)}(I)} \le c\|f\|_{L^{p(\cdot)}(I)}, \quad f \in L^{p(\cdot)}(I),$

(ii) $\bar{C}_a := \sup_{n\in\mathbb{Z}_-} \bar{C}_a(n) := \sup_{n\in\mathbb{Z}_-} \left\|\chi_{V_{n-1}}(x)v(x)k(x,\tfrac{x}{2})\right\|_{L^{q(x)}(I)}(a2^n)^{1/p'(0)} < \infty.$

(iii) $C_a := \sup_{0<t<a} C_a(t) := \sup_{0<t<a} \left\|\chi_{(t,a)}(x)v(x)k(x,\tfrac{x}{2})\right\|_{L^{q(x)}(I)} t^{1/p'(0)} < \infty,$

Moreover, $\|K_v\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} \approx C_a \approx \bar{C}_a$.

Proof. Taking into account Remark 5, without any loss of generality we give the proof for $v \equiv$ *const*. For simplicity we take $a = 1$. In this case $C_a = C_1$ and $\overline{C}_a = \overline{C}_1$. First we prove the implication (iii) ⇒ (i). Suppose that $f \geq 0$.

$$\begin{aligned}(Kf)(x) &= \int_0^{x/2} k(x,t)f(t)dt + \int_{x/2}^x k(x,t)f(t)dt \\ &=: (K^{(1)}f)(x) + (K^{(2)}f)(x).\end{aligned}$$

Hence,

$$\|(Kf)(x)\|_{L^{q(x)}(I)} \leq c\|(K^{(1)}f)(x)\|_{L^{q(x)}(I)} + \|(K^{(2)}f)(x)\|_{L^{q(x)}(I)} =: S^{(1)} + S^{(2)}.$$

It is easy to see that if $k \in V(I)$ and $0 < t < x/2$, then $k(x,t) \leq ck(x,\frac{x}{2})$. Hence, taking Theorem B into account we have that

$$S^{(1)} \leq c\left\|k(x,\frac{x}{2})\Big(\int_0^x f(t)dt\Big)\right\|_{L^{q(x)}(I)} \leq cC_1\|f\|_{L^{p(\cdot)}(I)}.$$

Suppose now that $\|h\|_{L^{q'(\cdot)}(I)} \leq 1$. Applying Hölder's inequality twice with respect to the pairs of exponents $(p(\cdot),p'(\cdot))$, $(q(\cdot),q'(\cdot))$ (see (ii) of Proposition A), Lemma A and the condition $k \in V_{p'(\cdot)}(I)$ we find that

$$\begin{aligned}&\int_0^1\Big(\int_{x/2}^x k(x,t)f(t)dt\Big)h(x)dx \\ \leq\ & c\sum_{n\in\mathbb{Z}_-}\int_{E_{n-1}} \|\chi_{(x/2,x)}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)}\|\chi_{(x/2,x)}(\cdot)k(x,\cdot)\|_{L^{p'(\cdot)}(I)}h(x)dx \\ \leq\ & c\sum_{n\in\mathbb{Z}_-}\|\chi_{I_{n-1}}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)}\int_{E_{n-1}} x^{1/p'(x)}k(x,\frac{x}{2})h(x)dx\end{aligned}$$

$$\begin{aligned}
&\leq c\sum_{n\in\mathbb{Z}_-}\|\chi_{I_{n-1}}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)}\Big\|\chi_{E_{n-1}}(x)x^{1/p'(x)}k(x,\frac{x}{2})\Big\|_{L^{q(x)}(I)}\\
&\qquad\times\|\chi_{E_{n-1}}(\cdot)h(\cdot)\|_{L^{q'(\cdot)}(I)}\\
&\leq c\sum_{n\in\mathbb{Z}_-}2^{n/p'(0)}\|k(x,\frac{x}{2})\chi_{E_{n-1}}(x)\|_{L^{q(x)}(I)}\|\chi_{I_{n-1}}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)}\\
&\qquad\times\|\chi_{E_{n-1}}(\cdot)h(\cdot)\|_{L^{q'(\cdot)}(I)}\leq cC_1\|f\|_{L^{p(\cdot)}(I)}\|h\|_{L^{q'(\cdot)}(I)}\leq cC_1\|f\|_{L^{p(\cdot)}(I)}.
\end{aligned}$$

Taking the supremum with respect to h and summarizing the estimates for $S^{(1)}$ and $S^{(2)}$ we have the desired result.

(i)⇒(ii): Let us take $f_n(x)=\chi_{(0,2^n]}(x)$, where $n\in\mathbb{Z}_-$. Then by Proposition A (part (i)) and Lemma A we have that

$$\|f_n\|_{L^{p(\cdot)}(I)}\leq c2^{n/p_+((0,2^k])}\leq c2^{n/p(0)}.$$

On the other hand, since $k\in V(I)$,

$$\begin{aligned}
\|Kf\|_{L^{q(x)}(I)}&\geq c\|\chi_{E_{n-1}}(x)k(x,\frac{x}{2})x\|_{L^{q(x)}(I)}\\
&\geq c2^n\|\chi_{E_{n-1}}(x)k(x,\frac{x}{2})\|_{L^{q(x)}(I)}.
\end{aligned}$$

Hence, by the boundedness of K we conclude that

$$\bar{C}_1:=\sup_{n\in\mathbb{Z}_-}\bar{C}_1(n):=\sup_{n\in\mathbb{Z}_-}\|\chi_{E_{n-1}}(x)k(x,\frac{x}{2})\|_{L^{q(x)}(I)}2^{n/p'(0)}<\infty.$$

(ii)⇒ (iii): Let us now take $t\in I$. Then $t\in[2^{m-1},2^m)$ for some $m\in\mathbb{Z}_-$. Consequently,

$$\begin{aligned}
C_1(t)&\leq\sum_{n=m}^{0}\|\chi_{E_{n-1}}(x)k(x,\frac{x}{2})\|_{L^{q(x)}(I)}2^{m/p'(0)}\leq\bar{C}_12^{m/p'(0)}\sum_{n=m}^{0}2^{-n/p'(0)}\\
&\leq c\bar{C}_1.
\end{aligned}$$

Hence, $C_1<c\bar{C}_1$. ■

Next boundedness result is for the integral operator defined on unbounded interval $I=\mathbb{R}_+$.

Theorem 1.2.2. *Let $I:=\mathbb{R}_+$ and let $p,q\in\mathcal{P}(I)$, $p(x)\leq q(x)$, $x\in I$. Suppose that $k\in V(I)\cap V_{p'(\cdot)}(I)$. Further, assume that $p,q\in\mathcal{P}^{log}_\infty(I)$. Then the following statements are equivalent*

(i) $\quad \|K_v f\|_{L^{q(\cdot)}(I)} \le c\|f\|_{L^{p(\cdot)}(I)}, \quad f \in L^{p(\cdot)}(I),$

(ii) $\quad \bar{C}_\infty := \sup_{n\in\mathbb{Z}} \bar{C}_\infty(n) := \sup_{n\in\mathbb{Z}} \Big\|\chi_{E_n}(x)v(x)k(x,\tfrac{x}{2})\Big\|_{L^{q(x)}(I)} \|\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}} < \infty,$

(iii) $\quad C_\infty := \sup_{t>0} C_\infty(t) := \sup_{t>0} \Big\|\chi_{(t,\infty)}(x)v(x)k(x,\tfrac{x}{2})\Big\|_{L^{q(x)}(I)} \|\chi_{(0,t)}(\cdot)\|_{L^{p'(\cdot)}(I)} < \infty.$

Moreover, $\|K_v\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} \approx C_\infty \approx \bar{C}_\infty.$

Proof. Taking into account Remark 5, without any loss of generality we give the proof for $v \equiv$ *const*.

(iii)⇒(i): Suppose that $f \ge 0$.

$$\begin{aligned}(Kf)(x) &= \int_0^{x/2} k(x,t)f(t)dt + \int_{x/2}^{x} k(x,t)f(t)dt \\ &=: (K^{(1)}f)(x) + (K^{(2)}f)(x).\end{aligned}$$

Hence,

$$\|(Kf)(x)\|_{L^{q(x)}(I)} \le \|(K^{(1)}f)(x)\|_{L^{q(x)}(I)} + \|(K^{(2)}f)(x)\|_{L^{q(x)}(I)} =: S^{(1)} + S^{(2)}.$$

It can be verified easily that if $0 < t < x/2$, then $k(x,t) \le c_1 k(x,\frac{x}{2})$. Hence, taking Theorem C into consideration we have that

$$S^{(1)} \le c\bigg\|k(x,\frac{x}{2})\Big(\int_0^x f(t)dt\Big)\bigg\|_{L^{q(x)}(I)} \le cC_\infty\|f\|_{L^{p(\cdot)}(I)}.$$

Suppose now that $h \ge 0$, $\|h\|_{L^{q'(\cdot)}(I)} \le 1$. Applying Hölder's inequality twice with respect to the pairs of exponents $(p(\cdot),p'(\cdot))$, $(q(\cdot),q'(\cdot))$ (see (ii) of Proposition A), Lemmas A, B, Proposition E and the condition $k \in V_{p'(\cdot)}(I)$ we have that

$$\begin{aligned}&\int_0^\infty \Big(\int_{x/2}^x k(x,t)f(t)dt\Big)h(x)dx \\ \le\ & c\sum_{n\in\mathbb{Z}}\int_{E_n} \|\chi_{(x/2,x)}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)}\|\chi_{(x/2,x)}(\cdot)k(x,\cdot)\|_{L^{p'(\cdot)}(I)}h(x)dx \\ \le\ & c\sum_{n\in\mathbb{Z}}\|\chi_{I_n}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)}\int_{E_n}\|\chi_{(x/2,x)}(\cdot)\|_{L^{p'(\cdot)}}k(x,\frac{x}{2})h(x)dx\end{aligned}$$

$$
\begin{aligned}
&\le c\sum_{n\in\mathbb{Z}}\|\chi_{I_n}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)}\|\chi_{I_n}(\cdot)\|_{L^{p'(\cdot)}}\int\limits_{E_n}k(x,\frac{x}{2})h(x)dx\\
&\le c\sum_{n\in\mathbb{Z}}\|\chi_{I_n}(\cdot)f(\cdot)\|_{L^{p(\cdot)}(I)}\|\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}}\Big\|\chi_{E_n}(x)k(x,\frac{x}{2})\Big\|_{L^{q(x)}(I)}\\
&\qquad\times\|\chi_{E_n}(\cdot)h(\cdot)\|_{L^{q'(\cdot)}(I)}\\
&\le cC_\infty\|f\|_{L^{p(\cdot)}(I)}\|h\|_{L^{q'(\cdot)}(I)}\le cC_\infty\|f\|_{L^{p(\cdot)}(I)}.
\end{aligned}
$$

Taking the supremum with respect to h and summarizing the estimates for $S^{(1)}$ and $S^{(2)}$ we have the desired implication.

(i)⇒(ii): For necessity take the test function $f_n(x)=\chi_{(0,2^n)}(x)$. Then by Remark 2 we see that

$$
\begin{aligned}
\|f_n\|_{L^{p(\cdot)}}&\approx 2^{n/p(0)}\qquad n<0,\\
\|f_n\|_{L^{p(\cdot)}}&\approx 2^{n/p_\infty}\qquad n\ge 0.
\end{aligned}
$$

Hence,

$$\|Kf\|_{L^{q(\cdot)}}\ge c2^n\|\chi_{E_{n-1}}(x)k(x,\frac{x}{2})\|_{L^{q(\cdot)}}$$

Using the boundedness we have

$$\|\chi_{E_{n-1}}(x)k(x,\frac{x}{2})\|_{L^{q(\cdot)}}2^{n/p'(0)}<\infty\qquad for\qquad n<0 \tag{1.2.1}$$

$$\|\chi_{E_{n-1}}(x)k(x,\frac{x}{2})\|_{L^{q(\cdot)}}2^{n/p'_\infty}<\infty\qquad for\qquad n\ge 0. \tag{1.2.2}$$

Combining (1.2.1) and (1.2.2) we have the required conclusion. The implication (ii)⇒(iii) can be proved in similar manner as in Remark 3; therefore we omit details. ∎

1.3 Compactness Criteria

This section is devoted to weighted criteria for the compactness of kernel operator K_v from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$. The following result is for kernel operator defined on a bounded interval.

Theorem 1.3.1. *Let $I=(0,a)$, where $0<a<\infty$. Suppose that $p,q\in\mathcal{P}(I)$ and $p(x)\le q(x)$, $x\in I$. Suppose also that $k\in V(I)\cap V_{p'(\cdot)}(I)$. Further, assume that $p,q\in\mathcal{P}^{log}(I)$. Then the following statements are equivalent:*

(i) K_v *is compact from* $L^{p(\cdot)}(I)$ *to* $L^{q(\cdot)}(I)$*;*

(ii) $\bar{C}_a < \infty$ *and* $\lim\limits_{j\to-\infty} \bar{C}_a(j) = 0$*;*

(iii) $C_a < \infty$ *and* $\lim\limits_{d\to 0+} C_d = 0$*, where*

$$C_d := \sup_{0<t<d} C_d(t) := \sup_{0<t<d} \Big\| \chi_{(t,d)}(x)v(x)k(x,\frac{x}{2})\Big\|_{L^{q(x)}(I)} t^{1/p'(0)};$$

where $\bar{C}_a$ *and* $\bar{C}_a(j)$ *are defined in Theorem 1.2.1.*

Proof. Taking into account Remark 5, without any loss of generality we give the proof for $v \equiv$ *const*.
(iii) ⇒ (i): For the simplicity assume that $a = 1$. Thus, $C_a = C_1$. We represent K_v as follows:

$$Kf(x) = K^{(1)}f(x) + K^{(2)}f(x),$$

where

$$K^{(1)}f(x) = \chi_{(0,\beta]}(x)Kf(x), \quad K^{(2)}f(x) = \chi_{(\beta,1]}(x)Kf(x)$$

and $0 < \beta < 1$.

From Proposition A and Lemma A we have,

$$\|\chi_{(x/2,x)}\|_{L^{p(\cdot)}(I)} \approx x^{1/p'(0)}.$$

Hence, the condition $k \in V(I) \cap V_{p'(\cdot)}(I)$ yields:

$$\begin{aligned}
&\Big\|\chi_{(\beta,1]}(x)\big\|\chi_{(0,x]}(y)k(x,y)\big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)} \\
&\le \Big\|\chi_{(\beta,1]}(x)\big\|\chi_{(0,x/2]}(y)k(x,y)\big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)} \\
&\quad + \Big\|\chi_{(\beta,1]}(x)\big\|\chi_{(x/2,x]}(y)k(x,y)\big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)} \\
&\le c\big\|\chi_{(\beta,1]}(x)x^{1/p'(0)}k(x,\frac{x}{2})\big\|_{L^{q(x)}(I)} \\
&\le c\big\|\chi_{(\beta,1]}(x)k(x,\frac{x}{2})\big\|_{L^{q(x)}(I)} < \infty
\end{aligned}$$

because $C_1 < \infty$. Consequently, by Theorem D, $K^{(1)}$ is compact. Further, according to Theorem 1.2.1 we have

$$\|K - K^{(1)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} \le \|K^{(2)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} \le c \sup_{0<t<\beta} C_1(t),$$

where the positive constant c depends only on p, q and α. Letting β to 0 we have that K is compact as a limit of compact operators.

(i)$\Rightarrow$ (ii): Suppose that $f_n(x) = 2^{-n/p(0)}\chi_{(0,2^n)}(x)$, $n \in \mathbb{Z}_-$. Hence by Proposition A and Lemma A we have that

$$\begin{aligned}\left|\int_0^1 f_n(x)\varphi(x)dx\right| &\leq c_p\|f_n(\cdot)\|_{L^{p(\cdot)}(I)}\|\varphi(\cdot)\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)}\\ &\leq c2^{-n/p(0)}2^{n/p_+((0,2^n))}\|\varphi(\cdot)\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)}\\ &\leq c\|\varphi(\cdot)\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)} \to 0\end{aligned}$$

for all $\varphi \in L^{p'(x)}(I)$ as $n \to -\infty$. Hence, f_n converges weakly to 0 as $n \to -\infty$. Further, it is easy to see that

$$\|Kf_n\|_{L^{q(\cdot)}(I)} \geq c2^{n/p'(0)}\left\|\chi_{E_{n-1}}(x)k(x,\frac{x}{2})\right\|_{L^{q(x)}(I)}.$$

Finally we conclude that $\lim_{n\to-\infty}\bar{C}_1(n) = 0$ because a compact operator maps weakly convergent sequence into strongly convergent one.

(ii)$\Rightarrow$(iii): Let $d \in (0,1)$. Then there exists an integer m such that $d \in [2^{m-1}, 2^m)$ and consequently,

$$C_d \leq \sup_{0<t<2^m}\left\|\chi_{(t,2^m)}(x)k(x,\frac{x}{2})\right\|_{L^{q(x)}(I)} t^{1/p'(0)} =: \sup_{0<t<2^m} A_{2^m}(t).$$

If $t \in (0, 2^m)$, then $t \in [2^{n-1}, 2^n)$ for some integer $n \leq m$. Consequently, we obtain

$$\begin{aligned}A_{2^m}(t) &\leq \left\|\chi_{(2^{n-1},2^m)}(x)k(x,\frac{x}{2})\right\|_{L^{q(x)}(I)} 2^{n/p'(0)}\\ &\leq 2^{n/p'(0)}\sum_{j=n}^{m}\left\|\chi_{E_{j-1}}(x)k(x,\frac{x}{2})\right\|_{L^{q(x)}(I)}\\ &\leq 2^{n/p'(0)}\sum_{j=n}^{m}\bar{C}_1(j).2^{-j/p'(0)}\\ &\leq c\sup_{j\leq m}\bar{C}_1(j).\end{aligned}$$

If $d \to 0^+$, then $2^m \to 0^+$. Therefore $\lim_{d\to0^+} C_d = 0$ as $\lim_{j\to-\infty}\bar{C}_1(j) = 0$. ■

The next result is for the compactness of kernel operators defined on unbounded interval $I = \mathbb{R}_+$.

Theorem 1.3.2. *Let $I = \mathbb{R}_+$. Suppose that $p, q \in \mathcal{P}(I)$ and $p(x) \le q(x)$, $x \in I$. Suppose also that $k \in V(I) \cap V_{p'(\cdot)}(I)$. Further, assume that $p, q \in \mathcal{P}^{log}_{\infty}(I)$. Then the following statements are equivalent:*

(i) K_v *is compact from* $L^{p(\cdot)}(I)$ *to* $L^{q(\cdot)}(I)$;

(ii) $\bar{C}_\infty < \infty$ *and* $\lim\limits_{n\to-\infty} \bar{C}_\infty(n) = 0 = \lim\limits_{n\to\infty} \bar{C}_\infty(n)$,
where $\bar{C}_\infty$ *and* $\bar{C}_\infty(n)$ *are defined in Theorem 1.2.2.*

(iii) $C_\infty < \infty$ *and* $\lim\limits_{d\to 0^+} C_d = \lim\limits_{b\to+\infty} C_b = 0$,
where C_∞ *is defined in Theorem 1.2.2 and*

$$C_d := \sup_{0<t<d} C_d(t) := \sup_{0<t<d} \left\| \chi_{(t,\infty)}(x) v(x) k(x, \frac{x}{2}) \right\|_{L^{q(x)}(I)} \|\chi_{(0,t)}(\cdot)\|_{L^{p'(\cdot)}};$$

$$C_b := \sup_{t\ge b} C_b(t) := \sup_{t\ge b} \left\| \chi_{(t,\infty)}(x) v(x) k(x, \frac{x}{2}) \right\|_{L^{q(x)}(I)} \|\chi_{(0,t)}(\cdot)\|_{L^{p'(\cdot)}}.$$

Proof. Taking into account Remark 5, without any loss of generality we give the proof for $v \equiv$ *const*.

First we show that the implication (iii)⇒(i) holds. We represent $Kf = \sum_{n=1}^{4} K^{(n)} f$, where

$$\begin{aligned} K^{(1)} f(x) &= \chi_{(0,d)}(x) K(\chi_{(0,d)} f)(x), \\ K^{(2)} f(x) &= \chi_{[d,b)}(x) K(\chi_{(0,b)} f)(x), \\ K^{(3)} f(x) &= \chi_{[b,\infty)}(x) K(\chi_{(0,b/2]} f)(x), \\ K^{(4)} f(x) &= \chi_{[b,\infty)}(x) K(\chi_{(b/2,\infty)} f)(x), \end{aligned}$$

where $0 < d < 1 < b < \infty$. Now observe that

$$K^{(2)} f(x) = \int_I k^{(2)}(x,y) f(y) dy,$$

where $k^{(2)}(x,y)=\chi_{[d,b)}(x)k(x,y)$ when $0<y<x<\infty$ and $k^{(2)}(x,y)=0$ if $0<x\le y<\infty$. Consequently, since $k\in V(I)\cap V_{p'(\cdot)}(I)$, we have for $K^{(2)}$,

$$\begin{aligned}
&\Big\|\chi_{[d,b]}(x)\Big\|k^{(2)}(x,y)\Big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)}\\
=&\ \Big\|\chi_{[d,b]}(x)\Big\|\chi_{(0,x)}(y)k(x,y)\Big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)}\\
\le&\ \Big\|\chi_{[d,b]}(x)\Big\|\chi_{(0,x/2)}(y)k(x,y)\Big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)}\\
+&\ \Big\|\chi_{[d,b]}(x)\Big\|\chi_{[x/2,x)}(y)k(x,y)\Big\|_{L^{p'(y)}(I)}\Big\|_{L^{q(x)}(I)}\\
\le&\ \Big\|\chi_{[d,b]}(x)k(x,\frac{x}{2})\Big\|_{L^{q(x)}(I)}\Big\|\chi_{(0,b/2)}(y)\Big\|_{L^{p'(y)}(I)}\\
+&\ \Big\|\chi_{[d,b]}(x)k(x,\frac{x}{2})\Big\|_{L^{q(x)}(I)}\Big\|\chi_{(d/2,b)}(y)\Big\|_{L^{p'(y)}(I)}\\
\le&\ 2\Big\|\chi_{[d,b]}(x)k(x,\frac{x}{2})\Big\|_{L^{q(x)}(I)}\Big\|\chi_{(0,b)}(y)\Big\|_{L^{p'(y)}(I)}=:S.
\end{aligned}$$

It is easy to see that $S<\infty$ because $C_\infty<\infty$. Hence, by Theorem D we conclude that $K^{(2)}$ is compact. Similarly we can show that $K^{(3)}$ is compact. Applying now Theorem 1.2.2 for the interval $(0,d)$ (see also [36]) we find that

$$\|K^{(1)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)}=\|K\|_{L^{p(\cdot)}([0,d))\to L^{q(\cdot)}([0,d))}\le c\sup_{0<t<d}C_d(t)$$

as $d\to 0^+$, where the positive constant c depends only on p, q. Further following the proof of Theorem 1.2.2 we have

$$\|K^{(4)}\|_{L^{p(x)}([b,\infty))\to L^{q(x)}([b,\infty))}\le c\sup_{t\ge b}\|\chi_{(t,\infty)}(x)k(x,\frac{x}{2})\|_{L^{q(\cdot)}}\|\chi_{(0,t)}(\cdot)\|_{L^{p'(\cdot)}}=c\sup_{t\ge b}C_b(t).$$

Further,

$$\begin{aligned}
\|K-K^{(2)}-K^{(3)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} &\le \|K^{(1)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)}+\|K^{(4)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)}\\
&\le c\Big(\sup_{0<t<d}C_d(t)+\sup_{t\ge b}C_b(t)\Big)
\end{aligned}$$

where the positive constant c depends only on p, q and α. Passing d to 0^+ and b to $+\infty$ we have that K is compact.

(i)⇒(ii): Suppose that $f_n(x) = 2^{-n/p(0)}\chi_{I_n}(x)$, $n \in \mathbb{Z}_-$ and $f_n(x) = 2^{-n/p_\infty}\chi_{I_n}(x)$, $n > 0$. Let us denote $p_n = p(0)$ for $n < 0$ and $p_n = p_\infty$ for $n \geq 0$. Hence by the condition $k \in V(I)$ and Proposition A, Lemmas A, B we have that

$$\begin{aligned} \left| \int_0^\infty f_n(x)\varphi(x)dx \right| &\leq c_p \|f_n(\cdot)\|_{L^{p(\cdot)}(I)} \|\varphi(\cdot)\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)} \\ &\leq c2^{-n/p_n} \|\chi_{I_n}(\cdot)\|_{L^{p(\cdot)}} \|\varphi(\cdot)\chi_{(0,2^n)}(\cdot)\|_{L^{p'(\cdot)}(I)} \\ &\leq c\|\varphi(\cdot)\chi_{I_n}(\cdot)\|_{L^{p'(\cdot)}(I)} \to 0 \end{aligned}$$

for all $\varphi \in L^{p'(x)}(I)$ as $n \to \pm\infty$. Hence, f_n converges weakly to 0 as $n \to \pm\infty$. Further, it is obvious that

$$\|Kf_n\|_{L^{q(\cdot)}(I)} \geq c2^{n/p'(0)} \left\| \chi_{E_n}(x)k(x, \frac{x}{2}) \right\|_{L^{q(x)}(I)}.$$

for $n \leq -1$,

$$\|Kf_n\|_{L^{q(\cdot)}(I)} \geq c2^{n/p'_\infty} \left\| \chi_{E_n}(x)k(x, \frac{x}{2}) \right\|_{L^{q(x)}(I)}.$$

for $n > 1$.

Finally we conclude that $\lim\limits_{n\to\pm\infty} \bar{C}_\infty(n) = 0$ because a compact operator maps weakly convergent sequence into strongly convergent one. The implication (ii)⇒(iii) follows from estimates similar to those given in Remark 3; therefore we omit details.

■

1.4 Measure of Non–Compactness

This section is devoted to the two-sided estimates of the distance between the operator K_v and the class of all compact linear operators from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.

Let X_1 and X_2 be Banach spaces. Suppose that $\mathcal{K}(X_1, X_2)$ (resp. $F_R(X_1, X_2)$) denotes the class of compact linear operators (resp. finite rank operators) acting from X_1 to X_2. Let

$$\|T\|_{\mathcal{K}(X_1,X_2)} := \operatorname{dist}\{T, \mathcal{K}(X_1, X_2)\}; \quad \overline{\alpha}(T) := \operatorname{dist}\{T, F_R(X_1, X_2)\},$$

where T is a bounded linear operator from X_1 to X_2.

Theorem E. [61, p. 80] *Let* $I := (0, a)$*, where* $0 < a \leq \infty$*. Let* $q \in \mathcal{P}^{log}_{\infty}(I)$*. Assume that* X_1 *is a Banach space. Suppose that* $q \in \mathcal{P}(I)$*. Then*

$$\|T\|_{\mathcal{K}\left(X_1, L^{q(\cdot)}(I)\right)} = \overline{\alpha}(T),$$

where T *is a bounded linear operator from* X_1 *to* $L^{q(\cdot)}(I)$*.*

The next statement is for kernel operators defined on a finite interval.

Theorem 1.4.1. *Let* $I := (0, a)$*, where* $0 < a < \infty$*. Suppose that* $p, q \in \mathcal{P}(I)$ *and* $p(x) \leq q(x)$*,* $x \in I$*. Let* $p, q \in \mathcal{P}^{log}(I)$ *and let* $C_a < \infty$ *(see Theorem 1.2.1). Then there exist two positive constants* b_1 *and* b_2 *such that*

$$b_1\mathcal{C} \leq \|K_v\|_{\mathcal{K}(L^{p(\cdot)}(I), L^{q(\cdot)}(I))} \leq b_2\mathcal{C},$$

where $\mathcal{C} := \lim\limits_{\beta \to 0} C_\beta$*,* $C_\beta := \sup\limits_{0<t<\beta} C_a(t)$ *and* $C_a(t)$ *is defined in Theorem 1.2.1.*

Proof. Taking into account Remark 5, without any loss of generality we give the proof for $v \equiv$ *const*.

For simplicity we assume that $a = 1$. In this case, by our notation, $E_n = V_n$. The upper estimate follows immediately from the estimate

$$\|K - K^{(2)}\|_{L^{p(\cdot)}(I) \to L^{q(\cdot)}(I)} \leq \|K^{(1)}\|_{L^{p(\cdot)}(I) \to L^{q(\cdot)}(I)} \leq cC_\beta,$$

where $K^{(1)} = \chi_{(0,\beta]}(x)Kf(x)$, $K_v^{(2)} = \chi_{[\beta,1]}(x)Kf(x)$, $0 < \beta < 1$ (see the proof of Theorem 1.3.1 for the details) and the fact that $K^{(2)}$ is compact (by Theorem D). Now for lower estimate take $\lambda > 0$ such that $\lambda > \|K\|_{\mathcal{K}(L^{p(\cdot)}(I), L^{q(\cdot)}(I))}$. Consequently, by Theorem E we have that $\lambda > \overline{\alpha}(K)$. Thus, there exist $h_1, \ldots, h_N \in L^{q(\cdot)}(I)$ such that

$$\overline{\alpha}(K) \leq \|K - F\| < \lambda,$$

where $Ff(x) = \sum_{j=1}^{N} \beta_j(f)h_j(x)$, β_j are linear bounded functionals in $L^{p(\cdot)}(I)$ and h_i are linearly independent. Further, there exist $\bar{h}_1, \ldots, \bar{h}_N$ such that supports of $\bar{h}_i$ are in $[\tau_i, a]$, $0 < \tau_i < a$, and

$$\|K - F_0\| < \lambda,$$

where $F_0f(x) = \sum_{j=1}^{N} \beta_j(f)\bar{h}_j(x)$. Suppose that $\tau = \min\{\tau_j\}$. Then obviously, $\mathrm{supp}F_0f \subset [\tau, a]$. Let $f_n := \chi_{(2^{n-1}, 2^{n+1})}$. Then for negative integer n chosen so that $2^{n+1} < \tau$, we find that

$$\begin{aligned}
\lambda & \quad \|f_n\|_{L^{p(\cdot)}(I)} \geq \|\chi_{E_n}(x)(Kf_n(x) - F_0 f_n(x))\|_{L^{q(x)}(I)} \\
\geq & \quad \|\chi_{E_n}(x)(Kf_n)(x)\|_{L^{q(x)}(I)} \\
\geq & \quad \left\|\chi_{E_n}(x)\int_{x/2}^{x} k(x,y)f_n(y)dy\right\|_{L^{q(x)}(I)} \\
\geq & \quad c\left\|\chi_{E_n}(x)xk(x,x/2)\right\|_{L^{q(x)}(I)} \\
\geq & \quad c2^n \cdot \left\|\chi_{E_n}(x)k(x,x/2)\right\|_{L^{q(x)}(I)}
\end{aligned}$$

Further, by using the condition $p \in \mathcal{P}^{log}(I)$ and Lemma A we find that

$$\lambda \geq c\left\|\chi_{E_n}(x)xk(x,x/2)\right\|_{L^{q(x)}(I)} 2^{(n+1)/p'(0)} = c\overline{C}_1(n+1),$$

where the positive constant c depends only on q and p, and $\overline{C}_1(n)$ is defined in Theorem 1.2.1. Since λ is arbitrarily close to $\|K\|_{\mathcal{K}(L^{p(\cdot)}(I),L^{q(\cdot)}(I))}$ we conclude that

$$c \lim_{n\to-\infty} \overline{C}_1(n) \leq \|K\|_{\mathcal{K}(L^{p(\cdot)}(I),L^{q(\cdot)}(I))}.$$

Further, it is easy to check that (see also the part of (iii) ⇒ (ii) in the proof Theorem 1.2.1)

$$\mathcal{C} \leq c \lim_{n\to-\infty} \overline{C}_1(n),$$

where the positive constant c depends only on p and q. Now the result follows. ■

The following statement gives the estimate for the measure of non compactness for kernel operators defined on unbounded interval $\mathbb{R}_+$.

Theorem 1.4.2. *Let $I := \mathbb{R}_+$. Suppose that $p, q \in \mathcal{P}(I)$ and $p(x) \leq q(x)$, $x \in I$. Let $p, q \in \mathcal{P}^{log}_{\infty}(I)$ and let $\bar{C}_\infty < \infty$ (see Theorem 1.2.2). Then there exist two positive constants b_1 and b_2 depending only on p, q and the constants c_1 and c_2 defined in Definitions 1.1.3 and 1.1.4 respectively such that*

$$b_1 J \leq \|K_v\|_{\mathcal{K}(L^{p(\cdot)}(I),L^{q(\cdot)}(I))} \leq b_2 J, \tag{1.4.1}$$

where,

$$J = \overline{\lim_{n\to\infty}} \bar{C}_\infty(n) + \overline{\lim_{n\to-\infty}} \bar{C}_\infty(n).$$

Proof. Taking into account Remark 5, without any loss of generality we give the proof for $v \equiv$ *const.*

The upper estimate follows immediately from the inequalities

$$\begin{aligned}\|K - K^{(2)} - K^{(3)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} &\le \|K^{(1)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} + \|K^{(4)}\|_{L^{p(\cdot)}(I)\to L^{q(\cdot)}(I)} \\ &\le c[\sup_{\substack{i\le m\\ i\in\mathbb{Z}}} \bar{C}_\infty(i) + \sup_{\substack{j\ge n\\ j\in\mathbb{Z}}} \bar{C}_\infty(j)]\end{aligned}$$

where $K^{(i)}$, $i = 1\cdots 4$ are defined in Theorem 1.3.2 assuming $d = 2^m$, $b = 2^n$, $m < 0$ and $n > 0$ (see the proof of Theorem 1.3.2 for the details) and the fact that $K^{(2)}$ and $K^{(3)}$ are compact according to Theorem D.

Now for lower estimate take $\lambda > 0$ so that $\lambda > \|K\|_{\mathcal{K}(L^{p(\cdot)}(I),L^{q(\cdot)}(I))}$. Consequently, by Theorem E we have that $\lambda > \overline{\alpha}(K)$. Hence, there exist $h_1, \dots, h_N \in L^{q(\cdot)}(I)$ such that

$$\overline{\alpha}(K) \le \|K - F\| < \lambda,$$

where $Ff(x) = \sum_{j=1}^{N} \beta_j(f) g_j(x)$, β_j are linear bounded functionals in $L^{p(\cdot)}(I)$ and h_i are linearly independent. Further, there exist $\bar{h}_1, \dots, \bar{h}_N$ such that supports of $\bar{h}_i$ are in $[\tau_i, \eta_i]$, $0 < \tau_i < \eta_i < \infty$, and

$$\|K - F_0\| < \lambda,$$

where $F_0 f(x) = \sum_{j=1}^{N} \beta_j(f)\bar{h}_j(x)$. Suppose that $\tau = \min\{\tau_j\}$, $\eta = \max\{\eta_j\}$. Then obviously, $\mathrm{supp} F_0 f \subset [\tau, \eta]$. Let $f_n := \chi_{(2^{n-1}, 2^{n+1})}$. Then by applying the condition $k \in V(I)$ for a negative integer n chosen so that $2^{n+1} < \tau$, we find that

$$\begin{aligned}\lambda \quad & \|f_n\|_{L^{p(\cdot)}(I)} \ge \|\chi_{E_n}(x)(Kf_n(x) - F_0 f_n(x))\|_{L^{q(x)}(I)} \\ \ge \quad & \|\chi_{E_n}(x)(Kf_n)(x)\|_{L^{q(x)}(I)} \\ \ge \quad & \left\|\chi_{E_n}(x)\int_{x/2}^{x} k(x,y) f_n(y)dy\right\|_{L^{q(x)}(I)} \\ \ge \quad & c_1\left\|\chi_{E_n}(x)xk(x,x/2)\right\|_{L^{q(x)}(I)} \\ \ge \quad & c_1 2^n \cdot \left\|\chi_{E_n}(x)k(x,x/2)\right\|_{L^{q(x)}(I)}\end{aligned}$$

Further, by using the condition $p \in \mathcal{P}_\infty^{log}(I)$ the condition $k \in V(I)$, Lemma B we find that

$$\lambda \ge d_1\left\|\chi_{E_n}(x)k(x,x/2)\right\|_{L^{q(x)}(I)} 2^{n/p'(0)},$$

where the positive constant d_1 depends only on p, q and the constant c_1 from Definition 1.1.3. Consequently, we have $\lambda \geq d_1 \overline{\lim\limits_{n\to-\infty}} \bar{C}_\infty(n)$.

Similarly let $f_m := \chi_{(2^{m-1},2^{m+1})}$. Now choosing a positive integer m so that $2^{m+1} > \eta$ and using Lemma A we find that

$$\lambda \geq d_2 \Big\| \chi_{E_m}(x) k(x, x/2) \Big\|_{L^{q(x)}(I)} 2^{m/p'_\infty}.$$

where the positive constant d_2 depends only on p, q and the constant c_1 from Definition 1.1.3. Hence, we have $\lambda \geq d_2 \overline{\lim\limits_{m\to+\infty}} \bar{C}_\infty(m)$.

Since λ is arbitrarily close to $\|K\|_{\mathcal{K}(L^{p(\cdot)}(I),L^{q(\cdot)}(I))}$, hence we conclude that the lower estimate of (1.4.1) holds. ■

Analogously follows the next statement, proof of which is omitted.

Theorem 1.4.3. *Let $I := \mathbb{R}_+$. Suppose that $1 < p_-(I) \leq p(x) \leq q(x) \leq q_+(I) < \infty$. Let $p, q \in \mathcal{P}^{log}_\infty(I)$ and let $C_\infty < \infty$ (see Theorem 1.2.2). Then there exist two positive constants e_1 and e_2 depending only on p, q and the constants c_1 and c_2 defined in Definitions 1.1.3 and 1.1.4 respectively such that*

$$e_1 U \leq \|K_v\|_{\mathcal{K}(L^{p(\cdot)}(I),L^{q(\cdot)}(I))} \leq e_2 U,$$

where,

$$U = \lim_{d\to 0^+} C_d + \lim_{b\to+\infty} C_b,$$

C_b and C_d are defined in Theorem 1.3.2.

For two-sided estimate of the measure of non-compactness of kernel operators in the classical Lebesgue spaces we refer e.g. to [25], [61], [63] and references cited therein.

Chapter 2

Integral Operators in Variable Exponent Amalgam Spaces

In this chapter, we present weighted inequalities for positive kernel operators in variable exponent amalgam spaces (VEAS). In particular, a characterization of a weight v governing the boundedness/ compactness of the weighted kernel operators

$$K_v f(x) = v(x) \int_0^x f(t)dt, \qquad x \in \mathbb{R}_+,$$

$$\mathcal{K}_v f(x) = v(x) \int_{-\infty}^x f(t)dt, \qquad x \in \mathbb{R},$$

under the log–Hölder continuity condition on exponents of spaces is established. These operators involve, for example, weighted variable parameter fractional integrals. We also establish two-weight estimates for maximal operators in these spaces.

2.1 Variable Exponent Amalgam Spaces

We start this section by the definition of weighted variable exponent Lebesgue space. Let f be a measurable function on an interval $I \subseteq \mathbb{R}_+$, for a weight u on I, let

$$\|f\|_{L^{p(\cdot)}_u(I)} := \|uf\|_{L^{p(\cdot)}(I)},$$

the space $L_u^{p(\cdot)}(I)$ is defined as

$$L_u^{p(\cdot)}(I) := \{f : \|f\|_{L_u^{p(\cdot)}(I)} < \infty\}.$$

Let I be $\mathbb{R}$ or $\mathbb{R}_+$ and $\alpha = \{I_n; n \in \mathbb{Z}\}$ be a cover of I consisting of disjoint half-open intervals I_n, each of the form $[a_1, a_2)$, whose union is I. Let

$$\|f\|_{(L_u^{p(\cdot)}(I),l^q)_\alpha} := \left(\sum_{n\in\mathbb{Z}} \|\chi_{I_n}(\cdot)f(\cdot)\|^q_{L_u^{p(\cdot)}(I)} \right)^{1/q},$$

we define the general amalgams with variable exponent

$$(L_u^{p(\cdot)}(I), l^q)_\alpha = \{f : \|f\|_{(L_u^{p(\cdot)}(I),l^q)_\alpha} < \infty\}.$$

If $u \equiv const$, then $(L_u^{p(\cdot)}(I), l^q)_\alpha$ is denoted by $(L^{p(\cdot)}(I), l^q)_\alpha$.

Let $p \equiv p_c \equiv const$ and $u \equiv const$. Then we have the usual irregular amalgam (see [82]); if $I = \mathbb{R}$ and $I_n = [n, n+1)$, then $(L^{p_c}(I), l^q)_\alpha$ is the amalgam space introduced by N. Wiener (see [84], [85]) in connection with the development of the theory of generalized harmonic analysis.

We call $(L_u^{p(\cdot)}(I), l^q)_\alpha$ irregular weighted amalgam spaces with variable exponent. If $I_n = [n, n+1)$, then $(L_u^{p(\cdot)}(I), l^q)_\alpha$ will be denoted by $(L_u^{p(\cdot)}(I), l^q)$.

Let $d = \{[2^n, 2^{n+1}); \ \ n \in \mathbb{Z}\}$ and $I = \mathbb{R}_+$. We denote weighted dyadic amalgam space with variable exponent by $(L_u^{p(\cdot)}(I), l^q)_d$. Some properties regarding general amalgams with variable exponent can be derived in the same way as for usual irregular amalgams $(L_u^p(\mathbb{R}), l^q)_\alpha$, where p is constant. Irregular amalgams were introduced in [33] and studied in [82].

Theorem F. *Let p be a measurable function on I such that $p \in \mathcal{P}(I)$. Let q be constant with $1 < q < \infty$. The irregular amalgams with variable exponent $(L^{p(\cdot)}(I), l^q)_\alpha$ is a Banach space whose dual space is $(L^{p(\cdot)}(I), l^q)^*_\alpha = (L^{p'(\cdot)}(I), l^{q'})_\alpha$. Further, Hölder's inequality holds in the following form*

$$\left| \int_I f(t)g(t)dt \right| \le c\|f\|_{(L^{p(\cdot)}(I),l^q)_\alpha} \|g\|_{(L^{(p(\cdot))'}(I),l^{q'})_\alpha},$$

where c is positive constant depending on p.

Proof. Since $L^{p(\cdot)}$ is a Banach space and $\left(L^{p(\cdot)}\right)^* = L^{p'(\cdot)}$ (see [51]), from general arguments (see [18], [35], [27], [82]) we have the desired result. ■

The next statement for more general case, i.e. when amalgams are defined with respect to Banach spaces, we refer to [82].

Theorem G. *Let s be measurable function on I and* $1 \le q_1 \le q_2$. *Then*

$$(L^{s(\cdot)}(I), l^{q_1})_\alpha \subset (L^{s(\cdot)}(I), l^{q_2})_\alpha.$$

Other structural properties of amalgams are investigated e.g., in [27] and [82].

The next statement is a generalization of Theorem 4 in [82] for variable exponent amalgams with weights.

Proposition F. *Let p, q be measurable functions on I such that* $1 \le q_-(I) \le q(x) < p(x) \le p_+(I)$ *and* $1 \le r < \infty$. *Then the space* $(L^{p(\cdot)}_w(I), l^r)_\alpha$ *is continuously embedded in* $(L^{q(\cdot)}_v(I), l^r)_\alpha$ *if*

$$S := \sup_{n\in\mathbb{Z}} \int\limits_{I_n} \left(\frac{v(x)^{q(x)}}{w(x)^{p(x)}}\right)^{\frac{p(x)}{p(x)-q(x)}} dx < \infty. \tag{2.1.1}$$

Conversely, if $1 < q_-(I) \le q_+(I) < p_-(I) \le p_+(I) < \infty$, *then condition* (2.1.1) *is also necessary for the continuous embedding of* $(L^{p(\cdot)}_w(I), l^r)_\alpha$ *into* $(L^{q(\cdot)}_v(I), l^r)_\alpha$.

Proof. It is known (see [23]) that the continuous embedding $L^{p(\cdot)}_w(I) \hookrightarrow L^{q(\cdot)}_v(I)$ $(q(x) < p(x))$ holds if and only if

$$\int_I \left(\frac{v(x)^{q(x)}}{w(x)^{p(x)}}\right)^{\frac{p(x)}{p(x)-q(x)}} dx < \infty.$$

Moreover, the estimate

$$\frac{\left\| \left(v^{q(\cdot)}/w^{p(\cdot)}\right)^{1/(p(\cdot)-q(\cdot))} \right\|_{L^{q(\cdot)}_v}}{\left\| \left(v^{q(\cdot)}/w^{p(\cdot)}\right)^{1/(p(\cdot)-q(\cdot))} \right\|_{L^{p(\cdot)}_w}} \le \|Id\|_{L^{p(\cdot)}_w \to L^{q(\cdot)}_v}$$

$$\leq c \max \left\{ 1, \left\| v^{q(\cdot)}/w^{p(\cdot)} \right\|_{L_w^{\left(p(\cdot)/q(\cdot)\right)'}} \right\} \tag{2.1.1$'$}$$

holds, where $c > 0$ is constant that depends only on p and q; Id is the identity operator.

Let condition (2.1.1) hold. Then

$$\|Id\|_{L_w^{p(\cdot)}(I_n) \to L_v^{q(\cdot)}(I_n)} \leq \|Id\|_{L_w^{p(\cdot)}(I) \to L_v^{q(\cdot)}(I)} < \infty.$$

Hence, $(L_w^{p(\cdot)}, l^r)_\alpha \hookrightarrow (L_v^{q(\cdot)}, l^r)_\alpha$.

Conversely, let the continuous embedding $(L_w^{p(\cdot)}, l^r)_\alpha \hookrightarrow (L_v^{q(\cdot)}, l^r)_\alpha$ hold and let $q_+(I) < p_-(I)$ and $p, q \in \mathcal{P}(I)$. By taking functions supported in I_n we can derive the estimate

$$\sup_{n \in \mathbb{Z}} \|Id\|_{L_w^{p(\cdot)}(I_n) \mapsto L_v^{q(\cdot)}(I_n)} \leq \|Id\|_{(L_w^{p(\cdot)}(I), l^r)_\alpha \mapsto (L_v^{q(\cdot)}(I), l^r)_\alpha}.$$

By applying the left–hand side inequality of (2.1.1$'$) and Proposition A we have that (2.1.1) is satisfied. ■

2.2 Some Well–Known Results Regarding Fractional Integrals in $L^{p(\cdot)}$ Spaces

We start this section by definition of doubling measure.

Definition 2.2.1. Let I be a bounded interval in $\mathbb{R}$. A measure μ satisfies the doubling condition on I ($\mu \in DC(I)$) if there is a positive constant c such that for all $x \in I$ and all s, $0 < s < |I|$, the inequality

$$\mu\big((x - 2s, x + 2s) \cap I\big) \leq c\mu\big((x - s, x + s) \cap I\big)$$

holds.

For a weight function w, we sometimes denote:

$$w(E) := \int_E w(x)dx, \quad E \subseteq \mathbb{R}.$$

Lemma C ([30], [39])**.** *Let J be a finite interval and let μ be a doubling measure on J. Suppose that $p \in \mathcal{P}(J)$ and $p \in \mathcal{P}^{log}(J)$. Then there is a positive constant C depending only on doubling constant d such that for all subintervals I of J,*

$$(\mu(I))^{p_-(I)-p_+(I)} \leq C.$$

Let J be an interval in $\mathbb{R}$, $J \subseteq \mathbb{R}$ and let

$$(M_\alpha^{(J)} f)(x) = \sup_{\substack{I \ni x \\ I \subset J}} \frac{1}{|I|^{1-\alpha}} \int_I |f(y)|dy, \quad x \in J,$$

where $x \in J$ and α is a constant satisfying the condition $0 \leq \alpha < 1$.

When $\alpha = 0$, then we have the Hardy–Littlewood maximal operator. In this case we denote $M_\alpha^{(J)}$ by $M^{(J)}$.

The next statement is a solution of the one–weight problem for the Hardy–Littlewood maximal operator (see [15]). We formulate the result for a bounded interval.

Proposition 2.2.1. *The operator $M^{(J)}$ is bounded in $L_w^{p(\cdot)}(J)$ if and only if $w \in A_{p(\cdot)}(J)$, i.e.*

$$\sup_{I \subseteq J} |I|^{-1} \|w\chi_I\|_{L^{p(\cdot)}} \|w^{-1}\chi_I\|_{L^{p'(\cdot)}} < \infty$$

provided that $p \in \mathcal{P}(J)$ and $p \in \mathcal{P}^{log}(J)$.

Now we formulate Sawyer [78] type results for maximal operators in variable exponent Lebesgue spaces.

The next statements (Propositions 2.2.2–2.2.3 and Corollary 2.2.1) are taken from [39].

Proposition 2.2.2. *Let an exponent p be defined on a finite interval J and let $p \in \mathcal{P}(J)$. Suppose that v and w are weight functions on J and that $d\nu(x) = w(x)^{-p'(x)}dx$ belongs to $DC(J)$. Suppose also that $0 \leq \alpha < 1$ and that $p \in \mathcal{P}^{log}(J)$. Then the inequality*

$$\|v(\cdot)M_\alpha^{(J)} f\|_{L^{p(\cdot)}(J)} \leq c\|w(\cdot)f(\cdot)\|_{L^{p(\cdot)}(J)}$$

holds, if and only if there exists a positive constant c such that for all intervals I, $I \subset J$,

$$\int_I (v(x))^{p(x)} (M_\alpha^{(J)}(w(\cdot)^{-p'(\cdot)}\chi_{I(\cdot)}))^{p(x)} dx \le c \int_I w^{-p'(x)} dx < \infty.$$

Corollary 2.2.1. *Let J be a bounded interval and let $p \in \mathcal{P}(J)$. Suppose that $0 \le \alpha < 1$. Assume that $p \in \mathcal{P}^{log}(J)$. Then the inequality*

$$\big\| v(\cdot)\big(M_\alpha^{(J)} f\big)(\cdot)\big\|_{L^{p(\cdot)}(J)} \le c\|f\|_{L^{p(\cdot)}(J)} \quad (\textit{Trace inequality})$$

holds if and only if

$$\sup_{I, I\subset J} \frac{1}{|I|} \int_I (v(x))^{p(x)} |I|^{\alpha p(x)} dx < \infty,$$

where the supremum is taken over all subintervals I of J.

Proposition 2.2.3. *Let $0 \le \alpha < 1$, $p \in \mathcal{P}(\mathbb{R})$, and let $p \in \mathcal{P}^{log}(\mathbb{R})$. Suppose that there is a positive number a such that $w^{-p'(\cdot)}(\cdot) \in DC([-a, a])$ and $p \equiv p_c \equiv$const outside $[-a, a]$. Then the inequality*

$$\|vM_\alpha^{(\mathbb{R})} f\|_{L^{p(\cdot)}(\mathbb{R})} \le \|wf\|_{L^{p(\cdot)}(\mathbb{R})},$$

holds if and only if there is a positive constant c such that for all bounded intervals $I \subset \mathbb{R}$,

$$\|vM_\alpha^{(\mathbb{R})}(w^{-p'(\cdot)}\chi_I)\|_{L^{p(\cdot)}(\mathbb{R})} \le c\|w^{1-p'(\cdot)}\|_{L^{p(\cdot)}(I)} < \infty.$$

To formulate the next statement we need the following definition.

Definition 2.2.2. Let μ be a measure on $\mathbb{R}$. A measure μ satisfies the reverse doubling condition on $\mathbb{R}$ ($\mu \in RD(\mathbb{R})$) if there is a constant $b > 1$ such that

$$\mu(x - 2s, x + 2s) \ge b\mu(x - s, x + s).$$

It is well-known that the reverse doubling condition implies the doubling condition.

Proposition 2.2.4 ([42])**.** *Suppose that $p = const$; $1 < p < q_-(\mathbb{R}) \le q_+(\mathbb{R}) < \infty$; $0 < \alpha < 1$. Assume that $w^{-p'} \in RD(\mathbb{R})$. Then the inequality*

$$\|vM_\alpha f\|_{L^{q(\cdot)}(\mathbb{R})} \le c\|wf\|_{L^p(\mathbb{R})} \tag{2.2.1}$$

holds if and only if

$$\sup_{I\subset\mathbb{R}} \|v\chi_I|I|^{\alpha-1}\|_{L^{q(\cdot)}(\mathbb{R})} \|w^{-1}\chi_I\|_{L^{p'}(\mathbb{R})} < \infty. \tag{2.2.2}$$

Let

$$(I_\alpha f)(x) := \int\limits_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\alpha}} dy, \quad x \in \mathbb{R}$$

be the fractional integral operator defined on $\mathbb{R}$, where $0 < \alpha < 1$.

The next statement is a generalization of the result by D. Adams [2] for variable exponent Lebesgue spaces:

Proposition 2.2.5 ([42]). *Let s be a measurable function on $\mathbb{R}$ such that $s \in \mathcal{P}(\mathbb{R})$. Suppose that r and α are constants satisfying the conditions: $1 < r < s_-(\mathbb{R})$, $0 < \alpha < 1/r$. Then the following statements are equivalent:*

(i) *I_α is bounded from $L^r(\mathbb{R})$ to $L_v^{s(\cdot)}(\mathbb{R})$;*

(ii)

$$\sup_{I; I \subset \mathbb{R}} \|\chi_I\|_{L_v^{s(\cdot)}(\mathbb{R})} |I|^{\alpha - 1/r} < \infty,$$

where the supremum is taken over all bounded intervals I in $\mathbb{R}$.

2.3 Some Inequalities for Discrete Potentials

In this section, we give some inequalities for discrete potentials.

Let

$$\begin{aligned}
\big(\mathcal{I}_\alpha(\{g_k\})\big)_n &= \sum_{k \in \mathbb{Z}, k \neq n} \frac{g_k}{|n-k|^{1-\alpha}}\ , \ n \in \mathbb{Z} \\
\big(\mathcal{R}_\alpha(\{g_k\})\big)_n &= \sum_{k=-\infty}^{n} \frac{g_k}{(n-k+1)^{1-\alpha}}\ , \ n \in \mathbb{Z}, \\
\big(\mathcal{W}_\alpha(\{g_k\})\big)_n &= \sum_{k=n}^{\infty} \frac{g_k}{(k-n+1)^{1-\alpha}}\ , \ n \in \mathbb{Z},
\end{aligned}$$

be discrete fractional integral operators, where $0 < \alpha < 1$.

It is easy to check that

$$\frac{1}{2}\Big(\big(\mathcal{R}_\alpha(\{g_k\})\big)_{n-1} + \big(\mathcal{W}_\alpha(\{g_k\})\big)_{n+1}\Big) \le \big(\mathcal{I}_\alpha(\{g_k\})\big)_n = \big(\mathcal{R}_\alpha(\{g_k\})\big)_{n-1} + \big(\mathcal{W}_\alpha(\{g_k\})\big)_{n+1}.$$

Let $\{u_n\}_{n\in\mathbb{Z}}$ be a positive (weight) sequence. In the sequel by $l^p(\mathbb{Z}, u_n)$, $1 < p < \infty$, will denote the class of all sequences $\{g_k\}_{k\in\mathbb{Z}}$ for which

$$\|g_k\|_{l^p(\mathbb{Z},u_n)} = \left(\sum_{k\in\mathbb{Z}} |g_k|^p u_k\right)^{1/p} < \infty.$$

If u_n is a constant sequence, then we denote $l^p(\mathbb{Z}, u_n)$ by $l^p(\mathbb{Z})$.

Sometimes we use the symbol $T(\{g_k\})(n)$ instead of $T(\{g_k\})_n$ for a discrete operator T.

Let $(X, \mathcal{U}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces with ν being σ- finite. Suppose that $k(x, y)$ is a non–negative real–valued $\mathcal{U} \times \mathcal{B}$– measurable function and that

$$Kf(y) = \int\limits_X k(x, y)f(x)d\mu(x)$$

is the kernel operator.

Denote:

$$e_\lambda(x) := \{y \in Y : k(x, y) > \lambda\}, \;\; e_\lambda(y) := \{x \in X : k(x, y) > \lambda\},$$

where λ is a positive number;

$$M_r(\mu)(y) := \sup_{\lambda>0} \lambda^r \mu\big(e_\lambda(y)\big); \;\; M_s(\nu)(x) := \sup_{\lambda>0} \lambda^s \nu\big(e_\lambda(x)\big),$$

where r and s are real numbers.

To prove the statements regarding fractional integrals we use the following statement which is a corollary of part (ii) of Theorem A in [2].

Theorem H. *Suppose that $1 < p < q < \infty$, $\frac{s}{q} = \frac{r}{p} + 1 - r$, where $r > 0$. If $M_r(\mu)(y) \le A < \infty$ for all $y \in Y$; $M_s(\nu)(x) \le B < \infty$ for all $x \in X$, then the operator K is bounded from $L^p(X, \mu)$ to $L^q(Y, \nu)$, where $L^p(X, \mu)$ $L^q(Y, \nu)$ are Lebesgue spaces defined with respect to the measures μ and ν, respectively.*

Proposition 2.3.1. *Suppose that p, q and α are constants satisfying the conditions: $1 < p < q < \infty$, $0 < \alpha < 1/p$. Then the following statements are equivalent:*

(i) $\mathcal{R}_\alpha$ *is bounded from* $l^p(\mathbb{Z})$ *to* $l^q(\mathbb{Z}, v_k)$*;*

(ii) $\mathcal{W}_\alpha$ *is bounded from* $l^p(\mathbb{Z})$ *to* $l^q(\mathbb{Z}, v_k)$*;*

(iii) $\mathcal{I}_\alpha$ *is bounded from* $l^p(\mathbb{Z})$ *to* $l^q(\mathbb{Z}, v_k)$*;*

(iv)

$$B := \sup_{m\in\mathbb{Z}, j\in\mathbb{N}} \left(\sum_{k=m}^{m+j} v_k\right)^{1/q} (j+1)^{\alpha-1/p} < \infty.$$

Proof. (iv)$\Rightarrow$ (i). Suppose that $X = Y = \mathbb{Z}$, μ is the counting measure on $\mathbb{Z}$ and that $d\nu(n) = v_n d\mu(n)$, where $\{v_n\}_{n\in\mathbb{Z}}$ is the weight sequence. In our case the kernel operator is given by

$$\{\mathcal{R}_\alpha\{g_m\}\}_n = \sum_{m=-\infty}^{\infty} k(m,n) g_m, \quad n \in \mathbb{Z},$$

where

$$k(m,n) = \chi_{\{m\in\mathbb{Z}: m\le n\}}(n-m+1)^{\alpha-1}.$$

Let $r = \frac{1}{1-\alpha}$ and let $\frac{s}{q} = \frac{r}{p} + 1 - r$. That is $s = \frac{q(\alpha-1/p)}{\alpha-1} > 0$. We have

$$\begin{aligned}
\sup_{n\in\mathbb{Z}} M_r(\mu)(n) &= \sup_{\lambda\le 1, n\in\mathbb{Z}} \lambda^r \mu\{m\in\mathbb{Z} : m \le n; (n-m+1)^{\alpha-1} > \lambda\} \\
&= \sup_{\lambda\ge 1, n\in\mathbb{Z}} \lambda^{r(\alpha-1)} \mu\{m\in\mathbb{Z} : m\le n; n-m+1 < \lambda\} \\
&\le \sup_{k\in\mathbb{N}, n\in\mathbb{Z}} k^{-1} \sum_{m=n-k}^{n} 1 \le c.
\end{aligned}$$

Further,

$$\begin{aligned}
\sup_{m\in\mathbb{Z}} M_s(\nu)(m) &= \sup_{\lambda\le 1, m\in\mathbb{Z}} \lambda^s \nu\{n\in\mathbb{Z} : m\le n; (n-m+1)^{\alpha-1} > \lambda\} \\
&= \sup_{\lambda\ge 1, m\in\mathbb{Z}} \lambda^{s(\alpha-1)} \nu\{n\in\mathbb{Z} : m\le n; n-m+1 < \lambda\} \\
&\le \sup_{k\in\mathbb{N}, m\in\mathbb{Z}} k^{s(\alpha-1)} \sum_{n=m}^{m+k} v_n \le cB^q.
\end{aligned}$$

(i) $\Rightarrow$ (iv). Let

$$(\beta^{(m)})_k = \begin{cases} 1 & \text{if } m-j < k \le m; \\ 0 & \text{otherwise,} \end{cases}$$

where m, j are positive integers such that $j \le m$. Then we have

$$\begin{aligned}\left(\sum_{n=1}^{\infty} v_n \left(\sum_{k=-\infty}^{n} \frac{(\beta^{(m)})_k}{(n-k+1)^{1-\alpha}}\right)^q\right)^{1/q} &\ge \left(\sum_{n=m}^{m+j} v_n \left(\sum_{k=m-j}^{m} \frac{1}{(n-k+1)^{1-\alpha}}\right)^q\right)^{1/q} \\ &\ge c\left(\sum_{n=m}^{m+j} v_n\right)^{1/q} j^{\alpha}.\end{aligned}$$

Since $\mathcal{R}_\alpha$ is a bounded operator, hence we conclude that

$$\left(\sum_{n=m}^{m+j} v_n\right)^{1/q} j^{\alpha-1/p} \le c, \quad 1 \le j \le m.$$

(i) $\Rightarrow$ (ii). Let

$$(\beta^{(m)})_k = \begin{cases} 1 & \text{if } m-j < k \le m; \\ 0 & \text{otherwise,} \end{cases}$$

where $m \in \mathbb{Z}$ and $j \in \mathbb{Z}$. Then we have

$$\begin{aligned}\left(\sum_{n\in\mathbb{Z}} v_n \left(\sum_{k=-\infty}^{n} \frac{(\beta^{(m)})_k}{(n-k+1)^{1-\alpha}}\right)^q\right)^{1/q} &\ge \left(\sum_{n=m}^{m+j} v_n \left(\sum_{k=m-j}^{m} \frac{1}{(n-k+1)^{1-\alpha}}\right)^q\right)^{1/q} \\ &\ge c\left(\sum_{n=m}^{m+j} v_n\right)^{1/q} j^{\alpha}.\end{aligned}$$

Therefore, by the boundedness of $\mathcal{R}_\alpha$ we conclude that

$$\left(\sum_{n=m}^{m+j} v_n\right)^{1/q} j^{\alpha-1/p} \le c, \quad m \in \mathbb{Z}, \;\; j \in \mathbb{Z}.$$

The remaining parts (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv) follows similarly; therefore we omit proofs.

$\blacksquare$

The next statement gives criteria guaranteeing the trace inequality in the diagonal case, i.e., when $p = q$ for the discrete potential operators. Criteria are of Maz'ya-Verbitsky [55] type.

Proposition 2.3.2. *Let $1 < p < \infty$ and let $0 < \alpha < 1/p$.*

(i) *The inequality*

$$\sum_{i=-\infty}^{+\infty} \Big(\mathcal{R}_\alpha g_j\Big)_i^p v_i \le c \sum_{i=-\infty}^{+\infty} g_i^p \tag{2.3.1}$$

holds for all non-negative sequences $\{g_i\}_i$ if and only if $\{\mathcal{W}_\alpha v_i\}_i < \infty$ for all $i \in \mathbb{Z}$ and there is a positive constant c such that for all $j \in \mathbb{Z}$

$$\Big\{\mathcal{W}_\alpha [\mathcal{W}_\alpha v_j]^{p'}\Big\}_i \le c \Big\{\mathcal{W}_\alpha v_i\Big\}_i. \tag{2.3.2}$$

(ii) *The inequality*

$$\sum_{i=-\infty}^{+\infty} \Big(\mathcal{W}_\alpha g_j\Big)_i^p v_i \le c \sum_{i=-\infty}^{+\infty} g_i^p \tag{2.3.3}$$

holds for all non-negative sequences $\{g_i\}_i$ if and only if $\{\mathcal{R}_\alpha v_i\}_i < \infty$ for all $i \in \mathbb{Z}$ and there is a positive constant c such that for all $j \in \mathbb{Z}$

$$\Big\{\mathcal{R}_\alpha [\mathcal{R}_\alpha v_j]^{p'}\Big\}_i \le c \Big\{\mathcal{R}_\alpha v_i\Big\}_i. \tag{2.3.4}$$

To prove Proposition 2.3.2 we need some auxiliary statements.

Proposition G. *Let $1 < p < \infty$, and let $0 < \alpha < 1/p$. If $\mathcal{R}_\alpha$ is bounded from $l^p(\mathbb{Z})$ to $l^p(\mathbb{Z}, v_i)$ then there exist a positive constant c such that*

$$\sum_{i=m}^{m+h} v_i \le c\, h^{1-\alpha p} \tag{2.3.5}$$

holds for all $m \in \mathbb{Z}$ and $h \in \mathbb{N}$.

Proposition G follows just in the same way as in the proof of the implication (i)$\Rightarrow$ (iv) of Proposition 2.3.1; therefore it is omitted.

We will prove the first part of Proposition 2.3.2. The second part follows analogously.

Proof of (i) of Proposition 2.3.2. Let us first show that, from (2.3.1) it follows that $\{\mathcal{W}_\alpha v_k\}_k < \infty$ for all $k \in \mathbb{Z}$. By the duality arguments (2.3.1) is equivalent to the inequality

$$\sum_{i=1}^{\infty} \Big(\mathcal{W}_\alpha g_j\Big)_i^{p'} \le c \sum_{i=1}^{\infty} g_i^{p'} v_i^{1-p'}. \tag{2.3.6}$$

Let $v_i^{(1)} = v_i\chi_{\{i:\, m\le i<m+2h\}}$ and $v_i^{(2)} = v_i\chi_{\{i:\, i<m \text{ or } i\ge m+2h\}}$, where $m\in\mathbb{Z}$ and $h\in\mathbb{N}$. Note that for $k\ge m+2h-1$ and $m\le i\le m+h$, we have that $k-m+1\le 2(k-i+1)$. Further, by using (2.3.5), we arrive to the estimates:

$$\begin{aligned}
\{\mathcal{W}_\alpha v_j^{(2)}\}_i &\le \sum_{k=m+2h-1}^{\infty} v_k(k-i+1)^{\alpha-1} \le c\sum_{k=m+h}^{\infty} v_k(k-m+1)^{\alpha-1}\\
&\le c\sum_{k=m+h}^{\infty} v_k\Big(\sum_{j=k-m+1}^{\infty} j^{\alpha-2}\Big) \le c\sum_{j=h+1}^{\infty} j^{\alpha-2}\Big(\sum_{k=m}^{j+m-1} v_k\Big)\\
&\le c\sum_{j=h+1}^{\infty} j^{\alpha-2} j^{1-\alpha p} < \infty.
\end{aligned}$$

Therefore $\big(\mathcal{W}_\alpha v_j^{(2)}\big)_i < \infty$. The fact that $\big(\mathcal{W}_\alpha v_j^{(1)}\big)_i < \infty$ is obvious. Thus, $\big(\mathcal{W}_\alpha v_j\big)_i < \infty$ for every $i\in\mathbb{Z}$ because m and h are taken arbitrarily.

Now we prove that (2.3.1) yields (2.3.2). For this we need the next lemmas.

Lemma D. *Let $0<\alpha<1$. Then there are positive constants $c_\alpha^{(1)}$ and $c_\alpha^{(2)}$ depending only on α such that for all, $m\in\mathbb{Z}$ the inequality*

$$(\mathcal{W}_\alpha\beta_m)_m \le c_\alpha^{(1)}\sum_{j=1}^{\infty} j^{\alpha-2}\Big(\sum_{k=m}^{m+j-1}\beta_k\Big) \le c_\alpha^{(2)}\,(\mathcal{W}_\alpha\beta_m)_m$$

holds, where $\beta_m\ge 0$.

Proof. The proof follows easily if we observe that there are positive constants $b_\alpha^{(1)}$ and $b_\alpha^{(2)}$ independent of k and m such that

$$\sum_{j=k-m+1}^{\infty} j^{\alpha-2} \le b_\alpha^{(1)}(k-m+1)^{\alpha-1} \le b_\alpha^{(2)}\sum_{j=k-m+1}^{\infty} j^{\alpha-2}.$$

It remains to change the order of summation.

■

Corollary A. *Let $0<\alpha<1$, $\beta_m\ge 0$. Then there are positive constants $c_\alpha^{(1)}$ and $c_\alpha^{(2)}$ such that for all, $m\in\mathbb{Z}$ the inequality*

$$\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha\beta_m]^{p'}\Big\}_m \le c_\alpha^{(1)}\sum_{j=1}^{\infty} j^{\alpha-2}\Big(\sum_{k=m}^{m+j-1}\{\mathcal{W}_\alpha\beta_m\}^{p'}\Big) \le c_\alpha^{(2)}\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha\beta_m]^{p'}\Big\}_m$$

holds.

Let $v_i^{(1)}$ and $v_i^{(2)}$ be defined as above. Then by using (2.3.6) we have that

$$\sum_{i=m}^{m+h} \Big(\mathcal{W}_\alpha v_j^{(1)}\Big)_i^{p'} \le c \sum_{i=m}^{m+h} v_i.$$

Thus, by Corollary A we conclude that

$$\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i^{(1)}]^{p'}\Big\}_i \le c \sum_{j=1}^{\infty} j^{\alpha-2}\Big(\sum_{k=i}^{i+2(j-1)} v_k\Big) \le c\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i]\Big\}_i.$$

For the estimate of $\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i^{(2)}]^{p'}\Big\}_i$ we need some auxiliary statements.

Lemma E. *Let $0<\alpha<1$. Then there is a positive constant c such that for all natural numbers m, k and an integer j satisfying the condition $m \le k \le m+j-1$, the inequality*

$$\Big\{\mathcal{W}_\alpha v_j^{(2)}\Big\}_k \le c \sum_{s=j}^{\infty} s^{\alpha-2}\Big(\sum_{t=m}^{m+s-1} v_t\Big)$$

holds.

Proof. We recall that $v_k^{(2)} = v_k \chi_{\{k:\, k<m \text{ or } k\ge m+2j\}}$. Using the arguments of the proof of Lemma D and the fact that

$$\Big(\mathcal{W}_\alpha v_j^{(2)}\Big)_k = \sum_{s=m+2j}^{\infty} v_s(s-k+1)^{\alpha-1}$$

we have

$$\begin{aligned}\Big(\mathcal{W}_\alpha v_j^{(2)}\Big)_k &\le c \sum_{s=m+2j}^{\infty} v_s(s-m+1)^{\alpha-1} \\ &\le c \sum_{s=m+2j}^{\infty} v_s \sum_{t=s-m+1}^{\infty} t^{\alpha-2} \le c\sum_{t=j}^{\infty} t^{\alpha-2}\Big(\sum_{s=m}^{m+t-1} v_s\Big).\end{aligned}$$

■

Lemma F. *Let $0<\alpha<1$. Then there is a positive constant c such that for all $m\in\mathbb{Z}$,*

$$\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i^{(2)}]^{p'}\Big\}_m \le c\sum_{t=1}^{\infty} t^{\alpha-1}\Big(\sum_{s=t}^{\infty} s^{\alpha-2}\Big(\sum_{j=m}^{m+s-1} v_j\Big)\Big)^{p'}.$$

Proof. Using Lemma E in Corollary A we have that

$$\begin{aligned}
\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i^{(2)}]^{p'}\Big\}_m &\le c\sum_{t=1}^{\infty} t^{\alpha-2}\Big(\sum_{k=m}^{m+t-1}\{\mathcal{W}_\alpha v_k\}^{p'}\Big)\\
&\le c\sum_{t=1}^{\infty} t^{\alpha-2}\sum_{k=m}^{m+t-1}\Big(\sum_{s=t}^{\infty} s^{\alpha-2}\sum_{\epsilon=m}^{m+s-1} v_\epsilon\Big)^{p'}\\
&\quad \text{(the inner sum does not depend on } k\text{)}\\
&= c\sum_{t=1}^{\infty} t^{\alpha-2}\Big(\sum_{s=t}^{\infty} s^{\alpha-2}\sum_{\epsilon=m}^{m+s-1} v_\epsilon\Big)^{p'}\Big(\sum_{k=m}^{m+t-1} 1\Big)\\
&= c\sum_{t=1}^{\infty} t^{\alpha-2}\Big(\sum_{s=t}^{\infty} s^{\alpha-2}\sum_{\epsilon=m}^{m+s-1} v_\epsilon\Big)^{p'}.
\end{aligned}$$

■

Lemma G. *Let $0 < \alpha < 1$. Then there is a positive constant c such that for all $m \in \mathbb{Z}$,*

$$\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i^{(2)}]^{p'}\Big\}_m \le c\sum_{t=1}^{\infty} t^{\alpha}\Big(\sum_{s=t}^{\infty} s^{\alpha-2}\sum_{\epsilon=m}^{m+s+1} v_\epsilon\Big)^{p'-1}\Big(t^{\alpha-2}\sum_{j=m}^{m+t+1} v_j\Big).$$

Proof. We will deduce the discrete case from the continuous case. Let $v(x) = v_j$, $j \le x < j+1$. Then $\int_j^{j+1} v(x)dx = v_j$. Hence, by using lemmas proved above and integration by parts, we find that

$$\begin{aligned}
\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i^{(2)}]^{p'}\Big\}_m &\le c\sum_{n=1}^{\infty} n^{\alpha-1}\Big(\sum_{j=n}^{\infty} j^{\alpha-2}\Big(\sum_{k=m}^{m+2j} v_k\Big)\Big)^{p'}\\
&\le c\sum_{n=1}^{\infty}\int_n^{n+1} x^{\alpha-1}\Big(\sum_{i=2n}^{\infty}\int_i^{i+1} y^{\alpha-2}\Big(\sum_{k=m}^{m+y} v_k\Big)dy\Big)^{p'}dx\\
&\le c\int_1^{\infty} x^{\alpha-1}\Big(\int_x^{\infty} y^{\alpha-2}\Big(\sum_{k=m}^{m+y} v_k\Big)dy\Big)^{p'}dx
\end{aligned}$$

$$
\begin{aligned}
&= c\Big[\frac{x^\alpha}{\alpha}\Big(\int_x^\infty \cdots\Big)^{p'}\Big|_1^\infty + \int_1^\infty x^\alpha\Big(\int_x^\infty \cdots\Big)^{p'-1} x^{\alpha-2}\Big(\sum_{k=m}^{m+x} v_k\Big)dx\Big] \\
&\le c\int_1^\infty x^\alpha\Big(\int_x^\infty \cdots\Big)^{p'-1} x^{\alpha-2}\Big(\sum_{k=m}^{m+x} v_k\Big)dx \\
&= c\sum_{n=1}^\infty \int_n^{n+1} x^\alpha\Big(\int_x^\infty \cdots\Big)^{p'-1} x^{\alpha-2}\Big(\sum_{k=m}^{m+n+1} v_k\Big)dx \\
&\le c\sum_{n=1}^\infty n^\alpha\Big(\int_n^\infty \cdots\Big)^{p'-1} n^{\alpha-2}\Big(\sum_{k=m}^{m+n+1} v_k\Big) \\
&\le c\sum_{n=1}^\infty n^\alpha\Big(\sum_{k=n}^\infty \int_k^{k+1} k^{\alpha-2}\Big(\sum_{i=m}^{m+k+1} v_i\Big)dy\Big)^{p'-1} n^{\alpha-2}\Big(\sum_{k=m}^{m+n+1} v_k\Big) \\
&= c\sum_{n=1}^\infty n^\alpha\Big(\sum_{k=n}^\infty k^{\alpha-2}\Big(\sum_{i=m}^{m+k+1} v_i\Big)\Big)^{p'-1} n^{\alpha-2}\Big(\sum_{k=m}^{m+n+1} v_k\Big).
\end{aligned}
$$

■

Now necessity of Proposition 2.3.2 follows easily because of Proposition G. Indeed, by using Proposition G we have that

$$
\begin{aligned}
\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_j^{(2)}]^{p'}\Big\}_m &\le c\sum_{n=1}^\infty n^\alpha\Big(\sum_{k=n}^\infty k^{\alpha-2}(k+2)^{1-\alpha p}\Big)^{p'-1}\Big(n^{\alpha-2}\sum_{k=m}^{m+n+1} v_k\Big) \\
&\le c\sum_{n=1}^\infty n^{\alpha-2}\sum_{k=m}^{m+n+1} v_k \le c\Big\{\mathcal{W}_\alpha v_m\Big\}_m.
\end{aligned}
$$

In the last inequality we used Lemma D, in particular, the right-hand side inequality.

Necessity of Proposition 2.3.2 is proved.

Now we prove sufficiency of Proposition 2.3.2. We need some auxiliary statements.

Lemma H. *Let $1 < p < \infty$ and $0 < \alpha < 1$. Then there exists a positive constant c such that for all non-negative sequences $\{g_i\}_{i\in\mathbb{Z}}$ and for all $i \in \mathbb{Z}$, the following inequality holds*

$$\{\mathcal{R}_\alpha g_k\}_i^p \le c\{\mathcal{R}_\alpha[\mathcal{R}_\alpha g_k]_j^{p-1} g_m\}_i. \tag{2.3.7}$$

Proof. First we assume that $\{V_\alpha g_i\}_i := \{\mathcal{R}_\alpha[\mathcal{R}_\alpha g_k]^{p-1}g_j\}_i$ and

$$\{V_\alpha g_j\}_i \leq \{\mathcal{R}_\alpha g_j\}_i^p.$$

Otherwise (2.3.7) is obvious for $c=1$. Now let us assume that $1<p\leq 2$. Then we have

$$\begin{aligned}
\{\mathcal{R}_\alpha g_k\}_i^p &= \sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=-\infty}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{p-1}\\
&\leq \sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=-\infty}^{k}(i-j+1)^{\alpha-1}g_j\Big)^{p-1}\\
&\quad + \sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{p-1} =: I_i^{(1)}+I_i^{(2)}.
\end{aligned}$$

It is obvious that if $j\leq k\leq i$, then $k-j+1\leq i-j+1$. Consequently,

$$I_i^{(1)} \leq \sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=-\infty}^{k}(k-j+1)^{\alpha-1}g_j\Big)^{p-1} = \{V_\alpha g_i\}_i.$$

Now we use Hölder's inequality with respect to the exponents $\frac{1}{p-1}$, $\frac{1}{2-p}$ and measure $d\mu(k)=(i-k+1)^{\alpha-1}g_k\,d\mu_c(k)$ (here μ_c is the counting measure on $\mathbb{Z}$). We have

$$\begin{aligned}
I_i^{(2)} &\leq \Big(\sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1}g_k\Big)^{2-p}\Big(\sum_{k=-\infty}^{i}\Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\Big)(i-k+1)^{\alpha-1}g_k\Big)^{p-1}\\
&= \{\mathcal{R}_\alpha g_i\}_i^{2-p}(J_i)^{p-1},
\end{aligned}$$

where

$$J_i \equiv \sum_{k=-\infty}^{i}\Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\Big)(i-k+1)^{\alpha-1}g_k.$$

Using Fubini's Theorem we have

$$J_i = \sum_{j=-\infty}^{i}(i-j+1)^{\alpha-1}g_j\Big(\sum_{k=-\infty}^{j}(i-k+1)^{\alpha-1}g_k\Big).$$

Further, it is obvious that the following simple inequality

$$\begin{aligned}
\sum_{k=-\infty}^{j}(i-k+1)^{\alpha-1}g_k &\leq \Big(\sum_{k=-\infty}^{j}(i-k+1)^{\alpha-1}g_k\Big)^{p-1}\Big\{\mathcal{R}_\alpha g_i\Big\}_i^{2-p}\\
&\leq \{\mathcal{R}_\alpha g_j\}_j^{p-1}\{\mathcal{R}_\alpha g_i\}_i^{2-p}
\end{aligned}$$

holds, where $j \le i$. Taking into account the last estimate, we obtain

$$J_i \;\le\; \Big(\sum_{j=-\infty}^{i}(i-j+1)^{\alpha-1}g_j\,\{\mathcal{R}_\alpha g_j\}_j^{p-1}\Big)\Big\{\mathcal{R}_\alpha g_i\Big\}_i^{2-p} = \{V_\alpha g_i\}_i\{\mathcal{R}_\alpha g_i\}_i^{2-p}.$$

Thus,

$$I_i^{(2)} \;\le\; \{\mathcal{R}_\alpha g_i\}_i^{2-p}\{\mathcal{R}_\alpha g_i\}_i^{(2-p)(p-1)}\{V_\alpha g_i\}_i^{p-1} = \{\mathcal{R}_\alpha g_i\}_i^{p(2-p)}\{V_\alpha g_i\}_i^{p-1}.$$

Combining the estimate for $I^{(1)}$ and $I^{(2)}$ we derive

$$\{\mathcal{R}_\alpha g_i\}_i^p \le \{V_\alpha g_i\}_i + \{\mathcal{R}_\alpha g_i\}_i^{p(2-p)}\{V_\alpha g_i\}_i^{p-1}.$$

As we have assumed that $\{V_\alpha g_i\}_i \le \{\mathcal{R}_\alpha g_i\}_i^p$, we obtain

$$\{V_\alpha g_i\}_i = \{V_\alpha g_i\}_i^{2-p}\{V_\alpha g_i\}_i^{p-1} \le \{V_\alpha g_i\}_i^{p-1}\{\mathcal{R}_\alpha g_i\}_i^{p(2-p)}.$$

Hence

$$\begin{aligned}\{\mathcal{R}_\alpha g_i\}_i^p &\le \{V_\alpha g_i\}_i^{p-1}\{\mathcal{R}_\alpha g_i\}_i^{p(2-p)} + \{V_\alpha g_i\}_i^{p-1}\{\mathcal{R}_\alpha g_i\}_i^{p(2-p)}\\ &= 2\{V_\alpha g_i\}_i^{p-1}\{\mathcal{R}_\alpha g_i\}_i^{p(2-p)}.\end{aligned}$$

Applying the fact $\big(\mathcal{R}_\alpha g_j\big)_i < \infty$ we find that

$$\{\mathcal{R}_\alpha g_i\}_i^p \le 2^{\frac{1}{p-1}}\{V_\alpha g_i\}_i.$$

Now we shall deal with the case $p > 2$. Let us assume again that

$$\{V_\alpha g_j\}_i \le \{\mathcal{R}_\alpha g_j\}_i^p.$$

Since $p > 2$ we have

$$\begin{aligned}\{\mathcal{R}_\alpha g_i\}_i^p &= \sum_{k=1}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=1}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{p-1}\\ &\le 2^{p-1}\sum_{k=1}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=1}^{k}(i-j+1)^{\alpha-1}g_j\Big)^{p-1}\\ &\quad +2^{p-1}\sum_{k=1}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{p-1}\\ &=: 2^{p-1}I_i^{(1)} + 2^{p-1}I_i^{(2)}.\end{aligned}$$

It is clear that if $j \le k \le i$, then $(i-j+1)^{\alpha-1} \le (k-j+1)^{\alpha-1}$. Therefore $I_i^{(1)} \le \{V_\alpha g_i\}_i$. Now we estimate $I_i^{(2)}$. We obtain

$$\Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{p-1} = \Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{p-2}\Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\Big)$$
$$\le \Big\{\mathcal{R}_\alpha g_i\Big\}_i^{p-2}\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j.$$

Using Fubini's theorem and the last estimate we have

$$\begin{aligned} I_i^{(2)} &\le \Big\{\mathcal{R}_\alpha g_i\Big\}_i^{p-2}\sum_{k=-\infty}^{i}(i-k+1)^{\alpha-1}g_k\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j \\ &= \Big\{\mathcal{R}_\alpha g_i\Big\}_i^{p-2}\sum_{j=-\infty}^{i}(i-j+1)^{\alpha-1}g_j\sum_{k=-\infty}^{j}(i-k+1)^{\alpha-1}g_k \\ &\le \Big\{\mathcal{R}_\alpha g_i\Big\}_i^{p-2}\sum_{j=-\infty}^{i}(i-j+1)^{\alpha-1}g_j\sum_{k=-\infty}^{j}(j-k+1)^{\alpha-1}g_k. \end{aligned}$$

Applying Hölder's inequality with respect to the exponents $\big\{p-1, \frac{p-1}{p-2}\big\}$ and the measure $d\mu(j) = (i-j+1)^{\alpha-1}g_j d\mu_c(j)$ (μ_c is the counting measure on $\mathbb{Z}$) we derive

$$\sum_{j=-\infty}^{i}(i-j+1)^{\alpha-1}g_j\sum_{k=-\infty}^{j}(j-k+1)^{\alpha-1}g_k \le \Big(\sum_{j=-\infty}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{\frac{p-2}{p-1}}$$
$$\times\Big(\sum_{j=-\infty}^{i}\Big(\sum_{k=-\infty}^{j}(j-k+1)^{\alpha-1}g_k\Big)^{p-1}(i-j+1)^{\alpha-1}g_j\Big)^{\frac{1}{p-1}} = \{\mathcal{R}_\alpha g_i\}_i^{\frac{p-2}{p-1}}\{V_\alpha g_i\}_i^{\frac{1}{p-1}}.$$

Combining these estimates we obtain

$$\{\mathcal{R}_\alpha g_i\}_i^p \le 2^{p-1}\{V_\alpha g_i\}_i + 2^{p-1}\{\mathcal{R}_\alpha g_i\}_i^{\frac{p(p-2)}{p-1}}\{V_\alpha g_i\}_i^{\frac{1}{p-1}}.$$

By virtue of the inequality $\{V_\alpha g_i j\}_i \le \{\mathcal{R}_\alpha g_j\}_i^p$ it follows that

$$\{V_\alpha g_j\}_i = \{V_\alpha g_j\}_i^{\frac{1}{p-1}}\{V_\alpha g_j\}_i^{\frac{p-2}{p-1}} \le \{V_\alpha g_j\}_i^{\frac{1}{p-1}}\{\mathcal{R}_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}}.$$

Hence

$$\{\mathcal{R}_\alpha g_j\}_i^p \le 2^{p-1}\Big(\{V_\alpha g_j\}_i^{\frac{1}{p-1}}\{\mathcal{R}_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}} + \{V_\alpha g_j\}_i^{\frac{1}{p-1}}\{\mathcal{R}_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}}\Big)$$
$$= 2^p\{V_\alpha g_j\}_i^{\frac{1}{p-1}}\{\mathcal{R}_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}}.$$

Further, from the last estimate we conclude that

$$\{\mathcal{R}_\alpha g_j\}_i^p \le 2^{p(p-1)}\{V_\alpha g_j\}_i,$$

where $2 < p < \infty$. ∎

Lemma I. *Let $1 < p < \infty$, $0 < \alpha < 1$ and v_i be a sequence of positive numbers on $\mathbb{Z}$. Let there exist a constant $c > 0$ such that the inequality*

$$\|\mathcal{R}_\alpha\{g_i\}\|_{l^p(\mathbb{Z},v_i^{(1)})} \le c_1 \|g_i\|_{l^p(\mathbb{Z})}, \quad \{v_i^{(1)}\}_i = \{\mathcal{W}_\alpha v_i\}_i^{p'}$$

holds for all sequences $g_i \in l^p(\mathbb{Z})$. Then

$$\|\mathcal{R}_\alpha\{g_i\}\|_{l^p(\mathbb{Z},v_i)} \le c_2 \|g_i\|_{l^p(\mathbb{Z})}, \quad g_i \in l^p(\mathbb{Z}),$$

where $c_2 = c_1^{1/p'}c^{1/p}$.

Proof. Let $g_i \ge 0$. Using Lemma H, Fubini's theorem and Hölder's inequality we have the following inequalities:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}^{\infty}\{\mathcal{R}_\alpha g_k\}_k^p v_k &\le c\sum_{k\in\mathbb{Z}}^{\infty}\sum_{i=-\infty}^{k}\{\mathcal{R}_\alpha g_j\}_i^{p-1} g_i (k-i+1)^{\alpha-1} v_k \\
&= c\sum_{i\in\mathbb{Z}}^{\infty}\{\mathcal{R}_\alpha g_j\}_i^{p-1} g_i \{\mathcal{R}_\alpha v_j\}_i \le c\Big(\sum_{i=1}^{\infty} g_i^p\Big)^{1/p}\Big(\sum_{i=1}^{\infty}\{\mathcal{R}_\alpha g_j\}_i^p v_i^{(1)}\Big)^{1/p'} \\
&= c\|g_i\|_{l^p(\mathbb{Z})}\|\mathcal{R}_\alpha g_i\|_{l^p(\mathbb{Z},v_i^{(1)})}^{p-1} \le c_1^{p-1} c \|g_i\|_{l^p(\mathbb{Z})}\|g_i\|_{l^p(\mathbb{Z})}^{p-1} = c_1^{p-1} c \|g_i\|_{l^p(\mathbb{Z})}^p .
\end{aligned}$$

Hence,

$$\|\mathcal{R}_\alpha g_j\|_{l^p(\mathbb{Z},v_i)} \le c_1^{1/p'} c^{1/p}\|g_j\|_{l^p(\mathbb{Z})} .$$

∎

Lemma J. *Let $0 < \alpha < 1$ and $1 < p < \infty$. Suppose that $\{\mathcal{W}_\alpha v_i\}_i < \infty$ and*

$$\Big\{\mathcal{W}_\alpha[\mathcal{W}_\alpha v_i]^{p'}\Big\}_i \le c\Big\{\mathcal{W}_\alpha v_i\Big\}_i$$

for all $i \in \mathbb{Z}$. Then we have

$$\|\mathcal{R}_\alpha\{g_i\}\|_{l^p(\mathbb{Z},v_i^{(1)})} \le c\|g_i\|_{l^p(\mathbb{Z})}, \quad g_i \in l^p(\mathbb{Z}), \tag{2.3.8}$$

where $\{v_i^{(1)}\}_i = \{\mathcal{W}_\alpha v_i\}_i^{p'}$.

Proof. Let $g_i \geq 0$ and let g_i be supported on the set $E_{m,l} := \{i : l \leq i \leq m\}$, where $m,\ l \in \mathbb{Z}$. Let $t_{i,j}^{(n)} = \chi_{\{j:j\leq i\}} \min\{(i-j+1)^{\alpha-1}, n\},\ n \in \mathbb{Z}$. Then using Lemma H (which is true also for the kernel $t_{i,j}^{(n)}$), Fubini's theorem and Hölder's inequality we obtain the following inequalities:

$$\begin{aligned}
\sum_{i=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{i} t_{i,j}^{(n)} g_j\Big)^p v_i^{(1)} &\leq c\sum_{i=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{i} t_{i,j}^{(n)}\Big(\sum_{k=1}^{j} t_{j,k}^{(n)} g_k\Big)^{p-1} g_j\Big) v_i^{(1)} \\
&\leq c\sum_{j=-\infty}^{\infty} g_j\Big(\sum_{k=-\infty}^{j} t_{j,k}^{(n)} g_k\Big)^{p-1}\Big(\sum_{i=j}^{\infty} t_{i,j}^{(n)} v_i^{(1)}\Big) \\
&\leq c\,\|g_i\|_{l^p(\mathbb{Z})}\Big(\sum_{j=-\infty}^{m}\Big(\sum_{k=1}^{j} t_{j,k}^{(n)} g_k\Big)^p\Big\{\mathcal{R}_\alpha[\mathcal{R}_\alpha v_j]^{p'}\Big\}_j^{p'}\Big)^{1/p'} \\
&\leq c\,\|g_i\|_{l^p(\mathbb{Z})}\Big(\sum_{j=1}^{m}\Big(\sum_{k=1}^{j} t_{j,k}^{(n)} g_k\Big)^p\Big\{\mathcal{R}_\alpha v_j\Big\}_j^{p'}\Big)^{1/p'}.
\end{aligned}$$

Since $\sum_{k=1}^{j} t_{j,k}^{(n)} g_k < \infty$ and $\{\mathcal{W}v_j\}_j < \infty$ for all j, therefore we have that

$$\Big(\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{i} t_{i,j}^{(n)} g_j\Big)^p v_i^{(1)}\Big)^{1/p} \leq c\,\|g_i\|_{l^p(\mathbb{Z})}.$$

Passing now by to the limits as m and n to $+\infty$, and by l to $-\infty$ we derive (2.3.8). ■

Combining these lemmas we have also sufficiency of Proposition 2.3.2. Proposition 2.3.2 is completely proved.

The next lemma will also be useful for us:

Lemma K. *Let $1 < r, s < \infty$ and let g_n be a non-negative sequence. Suppose that u_n be a positive sequence on $\mathbb{Z}$.*

(i) *The following two inequalities are equivalent*

$$\Big(\sum_{n\in\mathbb{Z}}\Big[\sum_{m=-\infty}^{n-1}(n-m)^{\alpha-1} g_m\Big]^r u_n\Big)^{1/r} \leq c_1\|g_k\|_{l^s(\mathbb{Z})}$$

and

$$\left(\sum_{n\in\mathbb{Z}}(\mathcal{R}_\alpha g_k)_n]^r u_{n+1}\right)^{1/r} \le c_1\|g_k\|_{l^s(\mathbb{Z})},$$

where the positive constant c_1 does depend on g_k;

(ii) *The following two inequalities are equivalent*

$$\left(\sum_{n\in\mathbb{Z}}\left[\sum_{m=n+3}^{\infty}(m-n)^{\alpha-1}g_m\right]^r u_n\right)^{1/r} \le c_2\|g_k\|_{l^s(\mathbb{Z})}$$

and

$$\left(\sum_{n\in\mathbb{Z}}(\mathcal{W}_\alpha g_k)_n]^r u_{n-3}\right)^{1/r} \le c_2\|g_k\|_{l^s(\mathbb{Z})},$$

where again the positive constant c_2 does depend on g_k.

2.4 Boundedness of Positive Kernel Operators

In this section, we establish the boundedness of positive kernel operators on VEAS. We start this section by following lemma.

Lemma L (see, e.g. [8])**.** *Let $1 < q < \bar{q} < \infty$ and $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$. Suppose that $\{v_n\}$ and $\{w_n\}$ are sequences of positive real numbers. The following statements are equivalent:*
(i) *There exists $C > 0$ such that the inequality*

$$\left\{\sum_{n\in\mathbb{Z}}(|a_n|v_n)^q\right\}^{1/q} \le C\left\{\sum_{n\in\mathbb{Z}}(|a_n|w_n)^{\bar{q}}\right\}^{1/\bar{q}}$$

holds for all sequences $\{a_n\}$ of real numbers.

(ii) $\left\{\sum_{n\in\mathbb{Z}}(v_n {w_n}^{-1})^s\right\}^{1/s} < \infty.$

The next statements gives the boundedness of the discrete Hardy operator (see e.g., [5], [11], [80, pp. 70-71], [52, Ch. 6]).

Lemma M. *Let p, q be constants such that $1 < p, q < \infty$. Suppose that $v_k \geq 0$, $w_k > 0$, $k \in \mathbb{Z}$. Then there exists a constant $c > 0$ such that*

$$\left\{ \sum_{n\in\mathbb{Z}} \left(\sum_{k=-\infty}^{n} a_k v_n \right)^q \right\}^{1/q} \leq c \left(\sum_{n\in\mathbb{Z}} (a_n w_n)^p \right)^{1/p}$$

holds for all nonnegative sequence $\{a_k\} \in l^p(\mathbb{Z}, w_k^p)$, if and only if
(i) *in case $1 < p \leq q < \infty$,*

$$A_1 := \sup_{m\in\mathbb{Z}} \left(\sum_{n=m}^{\infty} v_n^q \right)^{1/q} \left(\sum_{n=-\infty}^{m} w_n^{-p'} \right)^{1/p'} < \infty;$$

(ii) *in case $1 < q < p < \infty$,*

$$A_2 := \left\{ \sum_{m\in\mathbb{Z}} \left(\sum_{n=m}^{\infty} v_n^q \right)^{r/q} \left(\sum_{n=-\infty}^{m} w_n^{-p'} \right)^{r/q'} w_m^{-p'} \right\}^{1/r} < \infty,$$

where $1/r = 1/q - 1/p$.

The following lemma gives a uniform bound for the norm of integral operator $H_{v,w}^{(2^n,2^{n+1})}$.

Lemma N. *Let p and q be defined on $\mathbb{R}_+$ and satisfy the conditions of Theorem B. Then for all $n \in \mathbb{Z}$,*

$$\left\| v(x) \int_{2^n}^{x} f(t)w(t)dt \right\|_{L^{q(\cdot)}([2^n,2^{n+1}])} \leq D\|f\|_{L^{p(\cdot)}([2^n,2^{n+1}])}$$

where $D = \max\{c(2d+1)^2, 4\} \sup_{n\in\mathbb{Z}} A_{2^n,2^{n+1}}$, $A_{2^n,2^{n+1}}$ is defined in Theorem B and the constant c depends only on p and q.

Proof. By the hypothesis, p and q are constant outside some large interval $(0, d)$. Let $d \in [2^{m_0-1}, 2^{m_0})$ for some integer m_0. Then by Theorem B for $n \leq m_0$, we have

$$\begin{aligned} \|H_{v,w}^{(2^n,2^{n+1})}\|_{L^{p(\cdot)}([2^n,2^{n+1}])\to L^{q(\cdot)}([2^n,2^{n+1}])} &\leq c(2^n+1)^2 A_{2^n,2^{n+1}} \\ &\leq c(2^{m_0}+1)^2 A_{2^n,2^{n+1}} \\ &\leq c(2d+1)^2 \sup_{n\in\mathbb{Z}} A_{2^n,2^{n+1}}, \end{aligned}$$

where the positive constant c depends only on p and q. If $n > m_0$, then p and q are constants on the intervals $[2^n, 2^{n+1})$. In this case taking the proof of Theorem B into account we find that

$$\sup_{n>m_0} \left\| H_{v,w}^{(2^n,2^{n+1})} \right\|_{L^{p(\cdot)}([2^n,2^{n+1}]) \to L^{q(\cdot)}([2^n,2^{n+1}])} \leq 4 \sup_{n\in\mathbb{Z}} A_{2^n,2^{n+1}}.$$

∎

The next statement gives sufficient condition for boundedness of Hardy transform in VEAS.

Proposition 2.4.1. *Let $I := \mathbb{R}_+$ and let $\bar{p}(x) \leq p(x)$ and $p, \bar{p} \in \mathcal{P}(I)$. Let $1 < \bar{q}, q < \infty$. Suppose that $p, \bar{p} \in \mathcal{P}^{log}(\mathbb{R}_+)$ and that $p \equiv p_c \equiv const$ outside some large interval $(0, b)$. Then the inequality*

$$\|H_{v,w} f\|_{(L^{p(\cdot)}(I), l^q)_d} \leq c \|f\|_{(L^{\bar{p}(\cdot)}, l^{\bar{q}})_d}$$

with a positive constant independent of f holds if

(i) *in the case $1 < \bar{q} \leq q < \infty$,*

(a) $$\sup_{m\in\mathbb{Z}} \left\{ \sum_{n=m}^{\infty} \|\chi_{[2^n,2^{n+1})(\cdot)} v(\cdot)\|^q_{L^{p(\cdot)}} \right\}^{1/q} \left\{ \sum_{n=-\infty}^{m} \|\chi_{[2^{n-1},2^n)(\cdot)} w(\cdot)\|^{\bar{q}'}_{L^{\bar{p}'(\cdot)}} \right\}^{1/\bar{q}'} < \infty,$$

(b) $$\sup_{n\in\mathbb{Z}} \sup_{0<\alpha<1} \|\chi_{[2^{n+\alpha},2^{n+1})(\cdot)} v(\cdot)\|_{L^{p(\cdot)}} \|w(\cdot)\chi_{(2^n,2^{n+\alpha})(\cdot)}\|_{L^{\bar{p}'(\cdot)}} < \infty;$$

(ii) *in the case $1 < q < \bar{q} < \infty$,*

(a) $\{C_n\} \in l^s$, *where*

$$C_n = \sup_{\beta\in(0,1)} \|\chi_{[2^{n+\beta},2^{n+1})} v(\cdot)\|_{L^{p(\cdot)}} \|w(\cdot)\chi_{[2^n,2^{n+\beta})}\|_{L^{\bar{p}(\cdot)}},$$

(b)

$$\left\{ \sum_{n\in\mathbb{Z}} \left(\sum_{k=n}^{\infty} \|\chi_{[2^k,2^{k+1})} v(\cdot)\|^q_{L^{p(\cdot)}} \right)^{s/q} \left(\sum_{k=-\infty}^{n} \|\chi_{[2^{k-1},2^k)}(\cdot) w(\cdot)\|^{1-\bar{q}'}_{L^{\bar{p}'(\cdot)}} \right)^{s/\bar{q}'} \times \|\chi_{[2^n,2^{n+1})} v(\cdot)\|^q_{L^{p(\cdot)}} \right\}^{1/s} < \infty,$$

where $\frac{1}{s} = \frac{1}{\bar{q}} - \frac{1}{q}$.

Proof. Let $1<\bar{q}\le q<\infty$. Suppose that $f\ge 0$. We represent:

$$\begin{aligned}\big(H_{v,w}f\big)(x) &= v(x)\int\limits_0^{2^n} f(t)w(t)dt + v(x)\int\limits_{2^n}^{x} f(t)w(t)dt\\ &=: (H^{(1)}_{v,w}f)(x)+(H^{(2)}_{v,w}f)(x),\quad x\in[2^n,2^{n+1}]. \qquad (2.4.1)\end{aligned}$$

We have

$$\begin{aligned}\|(H_{v,w}f)\chi_{[2^n,2^{n+1})}(\cdot)\|_{L^{p(\cdot)}} &\le \|v(\cdot)\chi_{[2^n,2^{n+1})(\cdot)}\|_{L^{p(\cdot)}}\bigg(\int\limits_0^{2^n} f(t)w(t)dt\bigg)\\ &+ \|v(x)\int\limits_{2^n}^{x} f(t)w(t)dt\|_{L^{p(\cdot)}([2^n,2^{n+1}))}\\ &=: S_1^{(n)}+S_2^{(n)}.\end{aligned}$$

Let $a_k:=\int\limits_{2^{k-1}}^{2^k} fw$. Then by the discrete Hardy inequality (see Lemma M) and Hölder's inequality with respect to the exponents $\bar{p}(\cdot)$ and $(\bar{p}(\cdot))'$ we derive

$$\begin{aligned}\bigg(\sum_{n\in\mathbb{Z}}\big(S_1^{(n)}\big)^q\bigg)^{1/q} &= \bigg[\sum_{n\in\mathbb{Z}}\|v(\cdot)\chi_{[2^n,2^{n+1})}(\cdot)\|^q_{L^{p(\cdot)}}\bigg(\sum_{k=-\infty}^{n}\int\limits_{2^{k-1}}^{2^k} f(t)w(t)dt\bigg)^q\bigg]^{1/q}\\ &\le c\bigg[\sum_{n\in\mathbb{Z}}\bigg(\int\limits_{2^{n-1}}^{2^n} f(t)w(t)dt\bigg)^{\bar{q}}\|w(\cdot)\chi_{[2^{n-1},2^n)}(\cdot)\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}}\bigg]^{1/q}\\ &\le c\bigg[\sum_{n\in\mathbb{Z}}\|\chi_{[2^{n-1},2^n)}(\cdot)f(\cdot)\|^{\bar{q}}_{L^{\bar{p}(\cdot)}}\bigg]^{1/\bar{q}} = c\|f\|_{(L^{\bar{p}(\cdot)},l^{\bar{q}})_d}\end{aligned}$$

Further, by Lemma N and Theorem G we have that

$$\begin{aligned}\bigg(\sum_{n\in\mathbb{Z}}\big(S_2^{(n)}\big)^q\bigg)^{1/q} &= \bigg[\sum_{n\in\mathbb{Z}}\|v(x)\int\limits_{2^n}^{x} f(t)w(t)dt\|^q_{L^{p(\cdot)}(2^n,2^{n+1})}\bigg]^{1/q}\\ &\le c\bigg[\sum_{n\in\mathbb{Z}}\|f(\cdot)\chi_{(2^n,2^{n+1})}(\cdot)\|^q_{L^{\bar{p}(\cdot)}(2^n,2^{n+1})}\bigg]^{1/q}\\ &\le c\bigg[\sum_{n\in\mathbb{Z}}\|f(\cdot)\chi_{(2^n,2^{n+1})}(\cdot)\|^{\bar{q}}_{L^{\bar{p}(\cdot)}(2^n,2^{n+1})}\bigg]^{1/\bar{q}}\\ &= c\|f\|_{(L^{\bar{p}(\cdot)},l^{\bar{q}})_d}.\end{aligned}$$

Let $1<q<\bar{q}<\infty$. Using representation (2.4.1) we derive

$$\begin{aligned}\|(H_{v,w}f)\|_{(L^{p(\cdot)}(\mathbb{R}_+),l^q)_d} &\leq \Big[\sum_{n\in\mathbb{Z}}\|\chi_{[2^n,2^{n+1})}H^{(1)}_{v,w}f\|^q_{L^{p(\cdot)}}\Big]^{1/q}\\ &\quad+\Big[\sum_{n\in\mathbb{Z}}\|\chi_{[2^n,2^{n+1})}H^{(2)}_{v,w}f\|^q_{L^{p(\cdot)}}\Big]^{1/q}\\ &=: S_1+S_2.\end{aligned}$$

We estimate S_1 and S_2.

$$\begin{aligned}S_1 &= \Big[\sum_{n\in\mathbb{Z}}\|\chi_{[2^n,2^{n+1})}(\cdot)v(\cdot)\|^q_{L^{p(\cdot)}}\Big(\int_0^{2^n} fw\Big)^q\Big]^{1/q}\\ &= \Big[\sum_{n\in\mathbb{Z}}\|\chi_{[2^n,2^{n+1})}(\cdot)v(\cdot)\|^q_{L^{p(\cdot)}}\Big(\sum_{k=-\infty}^{n}\int_{2^{k-1}}^{2^k} fw\Big)^q\Big]^{1/q}.\end{aligned}$$

By the two–weight inequality for the discrete Hardy transform (see Lemma M), we have

$$\begin{aligned}S_1 &\leq c\ \Big[\sum_{n\in\mathbb{Z}}\|\chi_{[2^{n-1},2^n)}(\cdot)w(\cdot)\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}}\Big(\int_{2^{n-1}}^{2^n} fw\Big)^{\bar{q}}\Big]^{1/\bar{q}}\\ &\leq c\ \Big[\sum_{n\in\mathbb{Z}}\|\chi_{[2^{n-1},2^n)}(\cdot)w(\cdot)\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}}\|\chi_{[2^{n-1},2^n)}f\|^{\bar{q}}_{L^{\bar{p}(\cdot)}}\|\chi_{[2^{n-1},2^n)}w\|^{\bar{q}}_{L^{\bar{p}'(\cdot)}}\Big]^{1/\bar{q}}\\ &\leq c\|f\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}_+),l^{\bar{q}})_d}.\end{aligned}$$

Now we estimate S_2. Using Lemma N for intervals $(2^n,2^{n+1}]$ and Hölder's inequality, we find that

$$\begin{aligned}S_2 &\leq c\Big\{\sum_{n\in\mathbb{Z}}C_n^q\|\chi_{[2^n,2^{n+1})}f\|^q_{L^{\bar{p}(\cdot)}}\Big\}^{1/q}\\ &\leq c\Big\{\Big(\sum_{n\in\mathbb{Z}}\|\chi_{[2^n,2^{n+1})}f\|^{\bar{q}}_{L^{\bar{p}(\cdot)}}\Big)^{q/\bar{q}}\Big(\sum_{n\in\mathbb{Z}}C_n^{\frac{q\bar{q}}{\bar{q}-q}}\Big)^{\frac{\bar{q}-q}{q}}\Big\}^{1/q}\\ &\leq c\Big(\sum_{n\in\mathbb{Z}}C_n^s\Big)^{1/s}\|f\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}_+),l^{\bar{q}})_d}.\end{aligned}$$

■

Our aim is to characterize a class of weights v governing the boundedness of K_v from $(L^{\bar{p}(\cdot)}, l^{\bar{q}})_d$ to $(L^{p(\cdot)}, l^q)_d$. We will use the notation:

$$\begin{aligned} B_1 := \sup_{m\in\mathbb{Z}} & \left[\sum_{n=m}^{\infty} \|\chi_{(2^n,2^{n+1}]}(x) k(x, \tfrac{x}{2}) v(x)\|^q_{L^{p(\cdot)}} \right]^{1/q} \\ & \times \left[\sum_{n=-\infty}^{m} \|\chi_{(2^{n-1},2^n]}\|^{\bar{q}'}_{L^{\bar{p}'(\cdot)}} \right]^{1/\bar{q}'}; \end{aligned} \tag{2.4.2}$$

$$B_2 := \sup_{n\in\mathbb{Z}} \sup_{0<\alpha<1} \|\chi_{(2^{n+\alpha},2^{n+1}]} k(x, x/2) v(x)\|_{L^{p(\cdot)}} \|\chi_{(2^n,2^{n+\alpha}]}\|_{L^{\bar{p}'(\cdot)}}. \tag{2.4.3}$$

Theorem 2.4.1. *Let $I := \mathbb{R}_+$, $1 < \bar{p}_-(I) \le \bar{p}(\cdot) \le p(\cdot) \le p_+(I) < \infty$ and let $\bar{p}, p \in \mathcal{P}^{log}(I)$. Suppose that $\bar{q}$ and q are constants such that $1 < \bar{q} \le q < \infty$. Let $p(x) \equiv p_c \equiv const$ and $\bar{p}(x) \equiv \bar{p}_c \equiv const$ outside some large interval $(0, 2^{m_0})$, $m_0 \in \mathbb{Z}$. Let $k \in V(I) \bigcap V_{\bar{p}'(\cdot)}(I)$. Then K_v is bounded from $(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_d$ to $(L^{p(\cdot)}(I), l^q)_d$ if and only if $B < \infty$, where $B = \max\{B_1, B_2\}$.*

Proof. Sufficiency. Using the representation:

$$\begin{aligned} (K_v f)(x) &= v(x) \int_0^{x/2} k(x,t) f(t) dt + v(x) \int_{x/2}^{x} k(x,t) f(t) dt \\ &=: (K_v^{(1)} f)(x) + (K_v^{(2)} f)(x) \end{aligned}$$

we have that

$$\|K_v f\|_{(L^{p(\cdot)}, l^q)_d} \le \|K_v^{(1)} f\|_{(L^{p(\cdot)}, l^q)_d} + \|K_v^{(2)} f\|_{(L^{p(\cdot)}, l^q)_d}.$$

Further, taking Proposition 2.4.1 and the condition $k \in V(I)$ into account we find that

$$\begin{aligned} \left\| K_v^{(1)} f \right\|_{(L^{p(\cdot)}(I), l^q)_d} &\le c \left\| v(x) k(x, \frac{x}{2}) \int_0^x f(t) dt \right\|_{(L^{p(\cdot)}, l^q)_d} \\ &\le cB \|f\|_{(L^{\bar{p}(\cdot)}(I), l^q)_d}. \end{aligned}$$

Now observe that by the condition $k \in V_{\bar{p}'(\cdot)}(I)$, Proposition A and Lemma A we obtain

$$\|K_v^{(2)}f\|_{\left(L^{p(\cdot)}([0,2^{m_0}+1)),l^q\right)_d} \leq \left[\sum_{k=-\infty}^{+\infty}\|\chi_{(2^k,2^{k+1}]}(x)v(x)\Big(\int\limits_{x/2}^{x} f(t)k(x,t)dt\Big)\|^q_{L^{p(x)}}\right]^{1/q}$$

$$\leq \left[\sum_{k=-\infty}^{+\infty}\left\|\chi_{(2^k,2^{k+1}]}(x)v(x)\|\chi_{(x/2,x)}(\cdot)f(\cdot)\|_{L^{\bar{p}(\cdot)}}\|\chi_{(x/2,x)}k(x,\cdot)\|_{L^{\bar{p}'(\cdot)}}\right\|^q_{L^{p(x)}}\right]^{1/q}$$

$$\leq \left[\sum_{k=-\infty}^{+\infty}\left\|\chi_{(2^k,2^{k+1}]}(x)v(x)x^{\frac{1}{\bar{p}'(x)}}k(x,x/2)\right\|^q_{L^{p(x)}}\|\chi_{(2^{k-1},2^{k+1})}(\cdot)f(\cdot)\|^q_{L^{\bar{p}(\cdot)}}\right]^{1/q}$$

$$\leq c\left[\sum_{k=-\infty}^{+\infty}2^{kq/(\bar{p})'(2^k)}\left\|\chi_{(2^k,2^{k+1}]}(x)v(x)k(x,x/2)\right\|^q_{L^{p(x)}}\|\chi_{(2^{k-1},2^{k+1})}(\cdot)f(\cdot)\|^q_{L^{\bar{p}(\cdot)}}\right]^{1/q}$$

$$\leq c\bar{B}_1\left[\sum_{k=-\infty}^{+\infty}\|\chi_{(2^{k-1},2^k)}(\cdot)f(\cdot)\|^q_{L^{\bar{p}(\cdot)}}\right]^{1/q}+c\bar{B}_1\left[\sum_{k=-\infty}^{+\infty}\|\chi_{(2^k,2^{k+1})}(\cdot)f(\cdot)\|^q_{L^{\bar{p}(\cdot)}}\right]^{1/q}$$

$$\leq c\bar{B}_1\|f\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}_+),l^{\bar{q}})_d},$$

where

$$\bar{B}_1 := \sup_{n\in\mathbb{Z}}\|\chi_{(2^n,2^{n+1}]}(x)k(x,\frac{x}{2})v(x)\|_{L^{p(x)}}2^{1/(\bar{p}_n)'},$$

$$\bar{p}_n := \begin{cases}\bar{p}(2^n), & n\leq m_0,\\ \bar{p}_c, & n>m_0\end{cases}.$$

Let us now observe that by Proposition A and Lemma A, $\bar{B}_1\approx\bar{A}\leq cB_1$, where

$$\bar{A} := \sup_{k\in\mathbb{Z}}\|v(\cdot)k(x,x/2)\chi_{(2^k,2^{k+1}]}\|_{L^{p(\cdot)}}\|\chi_{(2^{k-1},2^k]}(\cdot)\|_{L^{\bar{p}'(\cdot)}}. \tag{2.4.4}$$

Necessity. Let $\bar{p}_n$ be the sequence defined above. Considering the test function $f_n=\chi_{(2^n,2^{n+1}]}2^{-n/\bar{p}_n}$ in the boundedness of K_v from $(L^{\bar{p}(\cdot)}(I),l^{\bar{q}})_d$ to $(L^{p(\cdot)}(I),l^q)_d$ and taking the condition $k\in V(I)$ into account we have that

$$I_n := \|\chi_{(2^n,2^{n+1}]}(x)v(x)k(x,x/2)\|_{L^{p(x)}}\leq c2^{-n/(\bar{p}_n)'}. \tag{2.4.5}$$

It is easy to see that

(i)

$$\sum_{n=m}^{\infty} I_n \leq c\Big(2^{-m/\bar{p}'(0)} + 2^{-m_0/\bar{p}'_c}\Big) \tag{2.4.6}$$

for $m \leq m_0$;

(ii)

$$\sum_{n=m}^{\infty} I_n \leq c2^{-m_0/\bar{p}'_c} \tag{2.4.7}$$

for $m \geq m_0 + 1$.

Denoting $S_m := \left[\sum\limits_{n=m}^{\infty} I_n^q\right]^{1/q} \left[\sum\limits_{n=-\infty}^{m-1} \|\chi_{(2^n,2^{n+1}]}\|^{\bar{q}}_{L^{\bar{p}'(\cdot)}}\right]^{1/\bar{q}}$ and taking (2.4.6), Proposition A and Lemma A into account we have for $m \leq m_0$,

$$\begin{aligned} S_m &\leq \left[\sum_{n=m}^{\infty} I_n^q\right]^{1/q} 2^{m/\bar{p}'(0)} \leq \left[2^{-m/\bar{p}'(0)} + 2^{-m_0/\bar{p}'_c}\right] 2^{m/\bar{p}'(0)} \\ &\leq 1 + 2^{m/\bar{p}'(0)} 2^{-m_0/\bar{p}'_c} \leq 1 + 2^{m_0/\bar{p}'(0)} 2^{-m_0/\bar{p}'_c} < \infty. \end{aligned}$$

Similarly if $m \geq m_0 + 1$, then by (2.4.7),

$$\begin{aligned} S_m &\leq \left[\sum_{n=m}^{\infty} I_n^q\right]^{1/q} [2^{m_0/\bar{p}'(0)} + 2^{m/\bar{p}'_c}] \leq 2^{-m/\bar{p}'_c}[2^{m_0/\bar{p}'(0)} + 2^{m/\bar{p}'_c}] \\ &\leq 1 + 2^{m_0/\bar{p}'(0)} 2^{-m_0/\bar{p}'_c} < \infty. \end{aligned}$$

Hence, $B_1 < \infty$.

Let now f be a function supported in $(2^m, 2^{m+1}]$. Then due to the boundedness of K_v from $(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_d$ to $(L^{p(\cdot)}(I), l^q)_d$ and the condition $k \in V(I)$ we have that

$$\left\|\chi_{(2^m,2^{m+1}]} v(x) k(x, x/2) \Bigg(\int_{2^m}^{x} f(y)dy\Bigg)\right\|_{(L^{p(\cdot)}(I),l^q)_d} \leq c\left\|\chi_{(2^m,2^{m+1}]} f\right\|_{(L^{\bar{p}(\cdot)}(I),l^{\bar{q}})_d},$$

where the positive constant c does not depend on n. Using Theorem B with respect to the intervals $[2^m, 2^{m+1})$ and the weight pair $(\bar{v}, w)$, where $\bar{v}(x) = v(x)k(x, x/2)$ $\chi_{(2^m,2^{m+1}]}$ and $\bar{w} \equiv \text{const}$, it follows that $B_2 < \infty$. ■

Remark 6. We have noticed in the proof of Theorem 2.4.1 that $B_1 \approx \bar{A}$, where $\bar{A}$ is defined in the same proof.

Now we formulate the boundedness criteria for the kernel operator

$$(\mathcal{K}_v f) = v(x) \int_{-\infty}^{x} k(x,t) f(t) dt \qquad x \in \mathbb{R},$$

on amalgams defined on $\mathbb{R}$.

Let $k(x,y)$ be a kernel on $\{(x,y) : y < x\}$ and v, p, $\bar{p}$ be defined on $\mathbb{R}$. For the next statement we define $\tilde{k}$, $\tilde{v}$, p_0 and $\bar{p}_0$ as follows:

$$\begin{aligned}
\tilde{k}(x,t) &:= \left(\frac{t^{-1/\bar{p}'(log_2 t)}}{x^{1/p(log_2 x)}} \right) k(log_2 x, log_2 t) \\
\tilde{v}(x) &:= v(log_2 x) \\
\bar{p}_0(x) &:= \bar{p}(log_2 x), \qquad\qquad p_0(x) := p(log_2 x).
\end{aligned}$$

Theorem 2.4.2. *Let $1 < \bar{p}_-(\mathbb{R}) \leq \bar{p}(x) \leq p(x) \leq p_+(\mathbb{R}) < \infty$ and let $\bar{p}_0, p_0 \in \mathcal{P}^{log}(\mathbb{R}_+)$. Let $\bar{q}$ and q are constants such that $1 < \bar{q} \leq q < \infty$. Assume that $\bar{p}(x) \equiv \bar{p}_c \equiv const$ and $p(x) \equiv p_c \equiv const$ outside some large interval $(-\infty, b)$. Let $\tilde{k} \in V(\mathbb{R}_+) \bigcap V_{(\bar{p}_0(\cdot))'}(\mathbb{R}_+)$. Then $\mathcal{K}_v$ is bounded from $(L^{\bar{p}(\cdot)}(\mathbb{R}), l^{\bar{q}})$ to $(L^{p(\cdot)}(\mathbb{R}), l^{q})$ if and only if*

$$\begin{aligned}
D_1 &:= \sup_{m \in \mathbb{Z}} \left[\sum_{n=m}^{\infty} \|\chi_{[2^n, 2^{n+1})}(x) \tilde{k}(x, \frac{x}{2}) \tilde{v}(x)\|^q_{L^{p_0(\cdot)}(\mathbb{R}_+)} \right]^{1/q} \\
&\quad \times \left[\sum_{n=-\infty}^{m} \|\chi_{[2^{n-1}, 2^n)}\|^{\bar{q}'}_{L^{(\bar{p}_0(\cdot))'}(\mathbb{R}_+)} \right]^{1/\bar{q}'} < \infty
\end{aligned}$$

$$D_2 := \sup_{n \in \mathbb{Z}} \sup_{0<\alpha<1} \|\chi_{[2^{n+\alpha}, 2^{n+1})} \tilde{k}(x, \frac{x}{2}) \tilde{v}(x)\|_{L^{p_0(\cdot)}(\mathbb{R}_+)} \|\chi_{[2^n, 2^{n+\alpha})}\|_{L^{(\bar{p}_0(\cdot))'}(\mathbb{R}_+)} < \infty.$$

Proof. The proof follows from Theorem 2.4.1 by the change of variable $z \to log_2 t$. ■

Let

$$(\mathcal{R}_{\alpha(\cdot)} f)(x) = v(x) \int_{-\infty}^{x} \frac{2^t f(t)}{(x-t)^{1-\alpha(x)}} dt,$$

where $0 < \inf \alpha \leq \sup \alpha < 1$ and $x \in \mathbb{R}_+$.

By virtue of Theorem 2.4.2 and Example 2 we can easily deduce the next statement:

Corollary 2.4.3. *Let p, $\bar{p}$, q and $\bar{q}$ be constants. Suppose that α is a measurable function on $\mathbb{R}$ and that $1 < \bar{p} \leq p < \infty$, $1 < \bar{q} \leq q < \infty$, $\frac{1}{\bar{p}} < \alpha(x) < 1$. Then the operator $\mathcal{R}_{\alpha(\cdot)}$ is bounded from $(L^{\bar{p}}, l^{\bar{q}})$ to (L^p, l^q) if and only if*

$$\tilde{D}_1 := \sup_{m \in \mathbb{Z}} \left[\sum_{n=m}^{\infty} \left(\int_n^{n+1} (2^u)^{\frac{p}{\bar{p}}} v^p(u) du \right)^{q/p} \right]^{1/q} 2^{m/\bar{p}'} < \infty$$

$$\tilde{D}_2 := \sup_{n \in \mathbb{Z}} \sup_{0<\beta<1} \left(\int_{n+\beta}^{n+1} (2^u)^{\frac{p}{\bar{p}}} v^p(u) du \right)^{1/p} \left(2^n (2^\beta - 1) \right)^{1/\bar{p}'} < \infty.$$

2.5 Compactness of Positive Kernel Operators

This section is devoted to the compactness criteria for the kernel operators on VEAS. Since for the amalgam norm we have the property $\|f_n\|_{(L^{p(\cdot)}(I), l^q)_\alpha} \downarrow 0$ when $f_n \downarrow 0$ a.e. ($f_n \in (L^{p(\cdot)}(I), l^q)_\alpha$), therefore the following statement holds (see [34], Ch. XI).

Proposition 2.5.1. *Let $p, \bar{p} : I \to \mathbb{R}$, be measurable functions such that $\bar{p}, p \in \mathcal{P}(I)$. Let q, $\bar{q}$ be constants satisfying the condition $1 < q, \bar{q} < \infty$. Then the set of all functions of the form*

$$k_n(s,t) \equiv \sum_{i=1}^{n} \eta_i(s) \lambda_i(t) \qquad s, t \in I.$$

is dense in the mixed norm space $(L^{p(\cdot)}(I), l^q)_\alpha [(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_\alpha]$, where $\lambda_i \equiv \chi_{B_i}$, $\chi_{B_i} \in (L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_\alpha$ (B_i are measurable disjoint sets of I) and $\eta_i \in (L^{p(\cdot)}(I), l^q)_\alpha \bigcap L^\infty(I)$.

The next statement gives sufficient condition for the kernel operator to be compact on amalgams defined on $\mathbb{R}_+$.

Proposition 2.5.2. *Let $p(x)$ and $\bar{p}(x)$ be measurable functions on an interval $I \subseteq \mathbb{R}_+$. Suppose that $p, \bar{p} \in \mathcal{P}(I)$. Let $q, \bar{q}$ be constants such that $1 < \bar{q}, q < \infty$. If*

$$M := \left\| \|k(x,y)\|_{(L^{(\bar{p}(y))'}(I), l^{(\bar{q})'})_\alpha} \right\|_{(L^{p(x)}(I), l^q)_\alpha} < \infty,$$

where k is a non-negative kernel, then the operator

$$Kf(x) = \int_I k(x,y) f(y) dy$$

is compact from $(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_\alpha$ to $(L^{p(\cdot)}(I), l^q)_\alpha$.

Proof. By Proposition 2.5.1 the set of functions

$$k_m(s,t) = \sum_{i=1}^{m} \eta_i(s) \lambda_i(t), \qquad s, t \in I,$$

is dense in $(L^{p(\cdot)}(I), l^q)_\alpha[(L^{\bar{p}'(\cdot)}(I), l^{\bar{q}'})_\alpha]$. By Hölder's inequality for amalgam spaces (see Theorem F) we have

$$|Kf(x)| = \left| \int_I k(x,y) f(y) dy \right| \leq \|f\|_{(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_\alpha} \|k(x,y)\|_{(L^{(\bar{p}(\cdot))'}(I), l^{(\bar{q})'})_\alpha}.$$

Hence,

$$\begin{aligned} \|Kf\|_{(L^{p(\cdot)}(I), l^q)_\alpha} &\leq \left\| \|k(x,y)\|_{(L^{(\bar{p}(y))'}(I), l^{(\bar{q})'})_\alpha} \right\|_{(L^{p(x)}(I), l^q)_\alpha} \|f\|_{(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_\alpha} \\ &\leq M \|f\|_{(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_\alpha}. \end{aligned}$$

This means that $\|K\| \leq M$.

Now we prove the compactness of K. For each $n \in \mathbb{N}$, let

$$(K_n \phi)(x) = \int_I k_n(x,y) \phi(y) dy.$$

Note that,

$$\begin{aligned} (K_n \phi)(x) &= \int_I k_n(x,y) \phi(y) dy \\ &= \sum_{i=1}^{n} \eta_i(x) \int_I \lambda_i(y) \phi(y) dy =: \sum_{i=1}^{n} \eta_i(x) b_i, \end{aligned}$$

where,

$$b_i = \int_I \lambda_i(y)\phi(y)dy.$$

This means that K_n is a finite rank operator, i.e it is compact. Further, let $\epsilon > 0$. Using the above–mentioned arguments, we have that there is $N_0 \in \mathbb{N}$ such that for $n > N_0$,

$$\|K - K_n\| \leq \left\| \|k(x,y) - k_n(x,y)\|_{(L^{(\bar{p}(y))'}(I), l^{(\bar{q})'})_\alpha} \right\|_{(L^{p(x)}(I), l^q)_\alpha} < \epsilon.$$

Thus K can be represented as a limit of finite rank operators. Hence, K is compact.

■

Theorem 2.5.1. *Let* $1 < \bar{p}_-(\mathbb{R}_+) \leq \bar{p}(x) \leq p(x) \leq p_+(\mathbb{R}_+) < \infty$ *and let* $\bar{p}, p \in \mathcal{P}^{log}(\mathbb{R}_+)$. *Let* $\bar{q}$ *and* q *be constants such that* $1 < \bar{q} \leq q < \infty$. *Assume that* $k \in V(\mathbb{R}_+) \bigcap V_{(\bar{p}(\cdot))'}(\mathbb{R}_+)$. *Suppose that* $\bar{p}(x) \equiv \bar{p}_c \equiv const$ *and* $p(x) \equiv p_c \equiv const$ *outside some large interval* $(0, 2^{m_0})$. *Then* K_v *is compact from* $(L^{\bar{p}(\cdot)}, l^{\bar{q}})_d$ *to* $(L^{p(\cdot)}, l^q)_d$ *if and only if*

$$\begin{array}{ll} \text{(i)} & B_1 < \infty; \qquad B_2 < \infty, \\ \text{(ii)} & \lim\limits_{m\to-\infty} B_1(m) = \lim\limits_{m\to+\infty} \mathbb{B}_1(m) = 0, \\ \text{(iii)} & \lim\limits_{n\to-\infty} B_2(n) = \lim\limits_{n\to+\infty} B_2(n) = 0, \end{array}$$

where, B_1 *and* B_2 *are defined by* (2.4.2) *and* (2.4.3) *respectively, and*

$$B_1(m) := \|\chi_{[2^m, 2^{m+1})} k(x, x/2) v(x)\|_{L^{p(\cdot)}} 2^{m/\bar{p}'(0)};$$

$$\mathbb{B}_1(m) := \left[\sum_{n=m}^{\infty} \|\chi_{[2^n, 2^{n+1})} k(x, x/2) v(x)\|^q_{L^{p(\cdot)}} \right]^{1/q} \left[\sum_{n=-\infty}^{m} \|\chi_{[2^{n-1}, 2^n)}(\cdot)\|^{(\bar{q})'}_{L^{(\bar{p}(\cdot))'}} \right]^{1/(\bar{q})'},$$

$$B_2(n) := \sup_{0<\alpha<1} \|\chi_{[2^{n+\alpha}, 2^{n+1})}(x) v(x) k(x, x/2)\|_{L^{p(\cdot)}} \|\chi_{(2^n, 2^{n+\alpha})}(\cdot)\|_{L^{(\bar{p}(\cdot))'}}.$$

Proof. Sufficiency. Let k_0, n_0 be integers such that $k_0 < m_0 < n_0$. Then we represent K_v as follows:

$$\begin{aligned}(K_v f)(x) &= \chi_{[0,2^{k_0}]}(x)K_v(f\chi_{[0,2^{k_0})})(x) + \chi_{(2^{k_0},2^{n_0})}(x)K_v(f\chi_{[0,2^{n_0})})(x)\\ &\quad +\chi_{[2^{n_0},\infty)}(x)K_v(f\chi_{[0,2^{n_0-1})})(x) + \chi_{[2^{n_0},\infty)}(x)K_v(f\chi_{(2^{n_0-1},\infty)})(x)\\ &=: \big(K_v^{(1)}f\big)(x) + \big(K_v^{(2)}f\big)(x) + \big(K_v^{(3)}f\big)(x) + \big(K_v^{(4)}f\big)(x).\end{aligned}$$

It is clear that

$$(K_v^{(2)}f)(x) = \int\limits_{\mathbb{R}_+} k_2(x,y)f(y)dy,$$

where $k_2(x,y) = v(x)\chi_{(2^{k_0},2^{n_0})}(x)k(x,y)$ if $y < x$ and $k_2(x,y) = 0$ if $y \geq x$. Then

$$\begin{aligned}&\Big\|\|k_2(x,y)\|_{(L^{(\bar{p})'(y)}(I),l^{(\bar{q})'})_d}\Big\|_{(L^{p(x)}([2^{k_0},2^{m_0})),l^q)_d}\\ &= \left\{\sum_{m=k_0}^{n_0-1}\Big\|\chi_{(2^m,2^{m+1})}(x)v(x)\Big(\sum_{n=-\infty}^{m}\|\chi_{(2^n,2^{n+1})}k(x,y)\|^{(\bar{q})'}_{L^{(\bar{p})'(y)}}\Big)^{1/(\bar{q})'}\Big\|^q_{L^{p(x)}}\right\}^{1/q}\\ &=: J(x).\end{aligned}$$

Denoting $I(x) := \sum\limits_{n=-\infty}^{m}\|\chi_{(2^n,2^{n+1})}k(x,y)\|^{(\bar{q})'}_{L^{(\bar{p})'(y)}}$, $x \in [2^m, 2^{m+1})$, $k_0 \leq m \leq n_0 - 1$, we represent $I(x)$ as

$$\begin{aligned}I(x) &= \sum_{n=-\infty}^{m-2}\|\chi_{(2^n,2^{n+1})}(y)k(x,y)\|^{(\bar{q})'}_{L^{(\bar{p})'(y)}}\\ &\quad + \|\chi_{(2^{m-1},2^m)}(y)k(x,y)\|^{(\bar{q})'}_{L^{(\bar{p})'(y)}} + \|\chi_{(2^m,x)}(y)k(x,y)\|^{(\bar{q})'}_{L^{(\bar{p})'(y)}}\\ &=: I_1(x) + I_2(x) + I_3(x).\end{aligned}$$

Now we estimate $I_1(x)$, $I_2(x)$ and $I_3(x)$ separately.

$$\begin{aligned}I_1(x) &\leq ck^{(\bar{q})'}(x,\frac{x}{2})\sum_{n=-\infty}^{m-2}\|\chi_{[2^n,2^n+1)}(y)\|^{(\bar{q})'}_{L^{(\bar{p})'(\cdot)}}\\ &\leq ck^{(\bar{q})'}(x,\frac{x}{2})\Bigg[\sum_{n=-\infty}^{m_0}\|\chi_{[2^n,2^n+1)}(\cdot)\|^{(\bar{q})'}_{L^{(\bar{p})'(\cdot)}} + \sum_{n=m_0+1}^{m-2}\|\chi_{[2^n,2^n+1)}(y)\|^{(\bar{q})'}_{L^{(\bar{p})'(y)}}\Bigg]\\ &\leq ck^{\bar{q}'}(x,\frac{x}{2})\Bigg[\sum_{n=-\infty}^{m_0}(2^n)^{(\bar{q})'/(\bar{p})'(0)} + \sum_{m_0+1}^{n_0}(2^n)^{(\bar{q})'/(\bar{p})'_c}\Bigg]\\ &\leq ck^{(\bar{q})'}(x,\frac{x}{2})\Bigg[(2^{m_0})^{(\bar{q})'/(\bar{p})'(0)} + (2^{n_0})^{(\bar{q})'/(\bar{p})'_c}\Bigg].\end{aligned}$$

Further,

$$\begin{aligned} I_2(x)+I_3(x) &\le 2\|\chi_{(0,x)}k(x,y)\|^{(\bar q)'}_{L^{(\bar p)'(y)}} \\ &\le c\|\chi_{(0,x/2)}k(x,y)\|^{(\bar q)'}_{L^{(\bar p)'(y)}} + c\|\chi_{(x/2,x)}k(x,y)\|^{(\bar q)'}_{L^{(\bar p)'(y)}} \\ &\le k^{(\bar q)'}(x,\frac{x}{2})\left[\|\chi_{(0,2^m)}(y)\|^{(\bar q)'}_{L^{\bar p(y)}} + x^{(\bar q)'/(\bar p)'(x)}\right]. \end{aligned}$$

Considering separately the cases $m \le m_0$ and $m > m_0$, by using Proposition A and Lemma A we find that

$$I_2(x)+I_3(x) \le ck^{(\bar q)'}(x,\frac{x}{2})\left[(2^m)^{(\bar q)'/(\bar p)'(0)} + (2^m)^{(\bar q)'/(\bar p)'_c}\right].$$

Consequently, since $k_0 \le m < n_0-1$, we have

$$I(x) \le ck^{(\bar q)'}(x,\frac{x}{2})\left[(2^{n_0})^{(\bar q)'/(\bar p)'(0)} + (2^{n_0})^{(\bar q)'/(\bar p)'_c}\right] =: ck^{(\bar q)'}(x,\frac{x}{2})B_{n_0}.$$

Since $B_1 < \infty$ we find that

$$J(x) \le B_{n_0}^{1/(\bar q)'}\left[\sum_{m=k_0}^{n_0-1}\|\chi_{[2^n,2^{n+1})}k(x,x/2)v(x)\|^q_{L^{p(\cdot)}}\right]^{1/q} < \infty.$$

So by Proposition 2.5.2 we conclude that $K_v^{(2)}$ is a compact operator. Further, write $K_v^{(3)}$ as follows:

$$K_v^{(3)}f(x) = \int\limits_{\mathbb{R}_+} k_3(x,y)f(y)dy,$$

where $k_3(x,y) = k(x,y)\chi_{(0,2^{n_0-1})}(y)\chi_{[2^{n_0},\infty)}(x)v(x)$ if $y<x$ and $k_3(x,y)=0$ if $y \ge x$. Then we have

$$\begin{aligned} &\left\|\|k_3(x,y)\|_{(L^{(\bar p)'(y)}(I),l^{(\bar q)'})_d}\right\|_{(L^{p(x)}(I),l^q)_d} \\ = &\left\{\sum_{m=n_0}^{\infty}\|\chi_{(2^m,2^{m+1})}(x)v(x)\left(\sum_{n=-\infty}^{n_0-2}\|\chi_{(2^n,2^{n+1})}(y)k(x,y)\|^{(\bar q)'}_{L^{(\bar p)'(y)}}\right)^{1/(\bar q)'}\|^q_{L^{p(x)}}\right\}^{1/q} \\ \le &\left\{\sum_{m=n_0}^{\infty}\|\chi_{(2^m,2^{m+1})}(x)v(x)k(x,x/2)\|^q_{L^{p(x)}}\right\}^{1/q}\left(\sum_{n=-\infty}^{n_0-2}\|\chi_{(2^n,2^{n+1})}(y)\|^{(\bar q)'}_{L^{(\bar p)'(y)}}\right)^{1/(\bar q)'} \\ &=: G. \end{aligned}$$

Denoting $F := \left(\sum_{n=-\infty}^{n_0-1} \|\chi_{(2^n,2^{n+1})}(y)\|_{L^{\bar{p}(\cdot)}}^{(\bar{q})'} \right)^{1/(\bar{q})'}$ and considering both cases when $m_0 \le n_0 - 2$ and $m_0 > n_0 - 2$ separately, we derive as previously, that

$$F \le c\Big[(2^{m_0})^{(\bar{q})'/(\bar{p})'(0)} + (2^{n_0})^{(\bar{q})'/(\bar{p})'_c}\Big]^{1/(\bar{q})'} =: B_{n_0,m_0},$$

and since $B_1 < \infty$ we have

$$G \le B_{n_0,m_0}\Big[\sum_{m=n_0}^{\infty} \|\chi_{[2^n,2^{n+1})}(x)k(x,x/2)v(x)\|_{L^{p(x)}}^q\Big]^{1/q} < \infty.$$

Hence, by Proposition 2.5.2, $K_v^{(3)}$ is compact.

Let us denote:

$$I_m := \|\chi_{[2^m,2^{m+1})}(x)k(x,x/2)v(x)\|_{L^{p(\cdot)}}. \tag{2.5.1}$$

Following the proofs of Theorems 2.4.1, 2.4.2 and applying Proposition A, Lemma A we have that

$$\begin{aligned}
&\|K_v^{(1)}\|_{\left(L^{\bar{p}(\cdot)}(I),l^{\bar{q}}\right)\to\left(L^{p(\cdot)}(I),l^q\right)_d} \\
\le\ & \max\Big\{\sup_{n\le k_0}\big[\sum_{m=n}^{k_0} I_m^q\big]^{1/q}\big[\sum_{m=-\infty}^{n}\|\chi_{[2^{m-1},2^m)}(\cdot)\|_{L^{(\bar{p}(\cdot))'}}^{(\bar{q})'}\big]^{1/(\bar{q})'}, \sup_{m\le k_0} B_2(m)\Big\} \\
\le\ & c\max\Big\{\Big[\sup_{m\le k_0} I^m 2^{m/\bar{p}'(0)}\Big]\sup_{n\le k_0}\Big[\sum_{m=n}^{\infty}2^{-m/\bar{p}'(0)}\Big]\Big[\sum_{m=-\infty}^{n}2^{m/\bar{p}'(0)}\Big], \sup_{m<k_0} B_2(m)\Big\} \\
\le\ & c\max\Big\{\sup_{m\le k_0} I^m 2^{m/\bar{p}'(0)}, \sup_{m<k_0} B_2(m)\Big\} \to 0
\end{aligned}$$

as $k_0 \to 0$ because $\lim_{m\to-\infty} B_1(m) = \lim_{m\to-\infty} B_2(m) = 0$. Further, applying Theorem 2.4.1 we find that

$$\|K_v^{(4)}\|_{\left(L^{\bar{p}(\cdot)}(I),l^{\bar{q}}\right)\to\left(L^{p(\cdot)}(I),l^q\right)} \le \max\big\{\sup_{m\ge n_0}\mathbb{B}_1(m), \sup_{m\ge n_0} B_2(m)\big\} \to 0$$

as $n_0 \to +\infty$.

Hence,

$$\|K_v f - K_v^{(2)}f - K_v^{(3)}f\| \le \|K_v^{(1)}f\| + \|K_v^{(4)}f\| \to 0$$

as $\mathbb{B}_1(m) \to 0$, $B_i(m) \to 0$, $i = 1,2$. Hence K_v is compact, since it is the limit of compact operators.

Necessity. First we show that $\lim_{m\to-\infty} B_1(m) = 0$. Let $f_n = \chi_{(2^{n-1},2^{n+1})}2^{-n/\bar{p}_n}$, where $\bar{p}_n$ is defined in the proof of Theorem 2.4.1. Then, $f_n \to 0$ weakly in $(L^{\bar{p}(\cdot)}(I), l^{\bar{q}})_d$ as $n \to -\infty$. Indeed, let $\phi \in (L^{(\bar{p}(\cdot))'}(I), l^{(\bar{q})'})_d$. Then

$$\left|\int_0^\infty f_n(y)\phi(y)dy\right| \leq \left(\|\chi_{(2^{n-1},2^n]}\|^{\bar{q}}_{L^{\bar{p}(\cdot)}} + \|\chi_{(2^n,2^{n+1}]}\|^{\bar{q}}_{L^{\bar{p}(\cdot)}}\right)^{1/\bar{q}} 2^{-n/\bar{p}_c}$$
$$\times\left(\|\phi\chi_{(2^{n-1},2^n]}\|^{\bar{q}}_{L^{(\bar{p}(\cdot))'}} + \|\phi\chi_{(2^{n-1},2^n]}\|^{\bar{q}}_{L^{(\bar{p}(\cdot))'}}\right)^{1/\bar{q}} \to 0$$

as $n \to -\infty$.

Observe now that

$$\|K_v f_n\|_{(L^{\bar{p}(\cdot)}(I),l^{\bar{q}})_d} \geq \|\chi_{(2^n,2^{n+1})}(x)v(x)k(x,x/2)\|_{L^{p(\cdot)}}2^{n/\bar{p}'_n}, \quad n \in \mathbb{Z}. \tag{2.5.2}$$

Hence $\lim_{n\to-\infty} B_1(n) \to 0$ because K_v is compact and $\bar{p}_n = \bar{p}(0)$ if $n < m_0$.

Further, (2.5.2) implies that

$$\|\chi_{(2^n,2^{n+1})}(x)v(x)k(x,x/2)\|_{L^{p(\cdot)}}2^{n/(\bar{p}_c)'} \to 0$$

as $n \to +\infty$.

To show that $\lim_{n\to+\infty} \mathbb{B}_1(n) \to 0$ we represent $\mathbb{B}_1(n)$ as follows:

$$\begin{aligned}
\mathbb{B}_1(n) &= \left(\sum_{m=n}^\infty I_m^q\right)^{1/q}\left(\sum_{m=-\infty}^{n-1} \|\chi_{(2^m,2^{m+1}]}\|^{\bar{q}}_{L^{(\bar{p}(\cdot))'}}\right)^{1/\bar{q}} \\
\leq &\left(\sum_{m=n}^\infty I_m^q\right)^{1/q}\left(\sum_{m=-\infty}^{m_0-1} 2^{m\bar{q}/(\bar{p}(0))'}\right)^{1/\bar{q}} + \left(\sum_{m=n}^\infty I_m^q\right)^{1/q}\left(\sum_{m=m_0}^{n-1} 2^{m\bar{q}/(\bar{p}_c)'}\right)^{1/\bar{q}} \\
&=: J_n^{(1)} + J_n^{(2)},
\end{aligned}$$

where $n \geq m_0$ and I_m is defined by (2.5.1). Observe now that

$$J_n^{(1)} = \left(\sum_{m=n}^\infty I_m^q\right)^{1/q} 2^{m_0/(\bar{p}(0))'} \to 0$$

as $n \to +\infty$ because $\left(\sum\limits_{m=n}^{\infty} I_m^q\right)^{1/q} \to 0$ as $n \to +\infty$. The latter convergence follows from the convergence of the series.

Further,

$$\begin{aligned} J_n^{(2)} &\leq c \sup_{m\geq n} \left(I_m 2^{m/(\bar{p}_c)'} \right) 2^{-n/(\bar{p}_c)'} 2^{n/(\bar{p}_c)'} \\ &\leq c \sup_{m\geq n} I_m 2^{m/(\bar{p}_c)'} \to 0 \end{aligned}$$

as $n \to +\infty$ because $I_m 2^{m/(\bar{p}_c)'} \to 0$ as $m \to +\infty$ (see (2.5.2)). Hence, $\lim\limits_{m\to+\infty} \mathbb{B}_1(m) = 0$.

Further, it is easy to see that for $0 < \alpha < 1$ and f_n,

$$\begin{aligned} \|K_v f_n\|_{(L^{p(\cdot)},l^q)_d} &\geq 2^{-n/\bar{p}_n} \|\chi_{(2^n,2^{n+1})}(x) v(x) k(x, x/2) x\|_{L^{p(\cdot)}} \\ &\geq 2^{n/(\bar{p}_n)'} \|\chi_{(2^n,2^{n+1})}(x) v(x) k(x, x/2)\|_{L^{p(\cdot)}} \\ &\geq c\big(2^n(2^\alpha - 1)\big)^{1/(\bar{p}_n)'} \|\chi_{(2^{n+\alpha},2^{n+1})}(x) v(x) k(x, x/2)\|_{L^{p(\cdot)}}. \end{aligned}$$

Hence

$$\|K_v f_n\|_{(L^{p(\cdot)},l^q)_d} \geq \sup_{0<\alpha<1} \big(2^n(2^\alpha - 1)\big)^{1/(\bar{p}_n)'} \|\chi_{(2^{n+\alpha},2^{n+1})}(x) v(x) k(x, x/2)\|_{L^{p(\cdot)}} \to 0$$

as $n \to +\infty$ or $n \to -\infty$.

The conditions $B_1 < \infty$ and $B_2 < \infty$ follow from the fact that every compact operator is bounded. ■

Now we formulate the compactness criteria for the kernel operator $\mathcal{K}_v$ defined on $\mathbb{R}$.

Theorem 2.5.2. *Let* $1 < \bar{p}_-(\mathbb{R}) \leq \bar{p}(x) \leq p(x) \leq p_+(\mathbb{R}) < \infty$ *and let* $\bar{p}_0, p_0 \in \mathcal{P}^{log}(\mathbb{R}_+)$. *Let* $\bar{q}$ *and* q *be constants such that* $1 < \bar{q} \leq q < \infty$. *Assume that* $\bar{p}(x) \equiv \bar{p}_c \equiv const$ *and* $p(x) \equiv p_c \equiv const$ *outside some large interval* $(-\infty, 2^{m_0})$. *Let* $\tilde{k} \in V(\mathbb{R}_+) \bigcap V_{(\bar{p}_0(\cdot))'}(\mathbb{R}_+)$. *Then* $\mathcal{K}_v$ *is compact from* $(L^{\bar{p}(\cdot)}, l^{\bar{q}})$ *to* $(L^{p(\cdot)}, l^q)$ *if and only if*

$$\begin{array}{ll} \text{(i)} & D_1 = \sup_{m\in\mathbb{Z}} \mathbb{D}_1(m) < \infty \quad ; \quad D_2 = \sup_{n\in\mathbb{Z}} D_2(n) < \infty, \\ \text{(ii)} & \lim_{m\to-\infty} D_1(m) = \lim_{m\to\infty} D_1(m) = 0, \\ \text{(iii)} & \lim_{n\to-\infty} D_2(n) = \lim_{n\to\infty} D_2(n) = 0, \end{array}$$

where

$$D_1(m) := \|\chi_{[2^m,2^{m+1})}\tilde{k}(x,x/2)\tilde{v}(x)\|_{L^{p_0(\cdot)}} 2^{m/\bar{p}_0'(0)};$$

$$\begin{aligned} \mathbb{D}_1(m) &:= \left[\sum_{n=m}^{\infty} \|\chi_{[2^n,2^{n+1})}\tilde{k}(x,x/2)\tilde{v}(x)\|^q_{L^{p_0(\cdot)}}\right]^{1/q} \\ &\quad \times \left[\sum_{n=-\infty}^{m} \|\chi_{[2^{n-1},2^n)}(\cdot)\|^{(\bar{q})'}_{L^{\bar{p}_0'(\cdot)}}\right]^{1/(\bar{q})'}; \\ D_2(n) &:= \sup_{0<\alpha<1} \|\chi_{[2^{n+\alpha},2^{n+1})}(x)\tilde{k}(x,x/2)\tilde{v}(x)\|_{L^{p_0(\cdot)}} \|\chi_{(2^n,2^{n+\alpha})}(\cdot)\|_{L^{(\bar{p}_0(\cdot))'}}; \end{aligned}$$

$\tilde{k}$, $\tilde{v}$ and p_0 and $\bar{p}_0$ are defined in Section 2.4.

Proof. The proof follows from Theorem 2.5.1 by the change of variable $z \to log_2 t$. ■

2.6 Maximal and Potential Operators

This section is devoted to the boundedness criteria for maximal and potential operators in VEAS.

2.6.1 General Operators in VEAS

We begin this subsection by the following definition:

Definition 2.6.1 ([8]). Let T be an operator defined on a set of real measurable functions f on $\mathbb{R}$. Define a sequence of local operators

$$(T_n f)(x) := T(f\chi_{(n-1,n+2)})(x), \quad x \in (n-1,n+2), \qquad n \in \mathbb{Z}.$$

Let us assume that there is a discrete operator T^d satisfying the following conditions:
(i) There exists a positive constant c such that for all non-negative functions f, all $n \in \mathbb{Z}$ and all $x \in (n, n+1)$, the inequality

$$T(f\chi_{(-\infty,n-1)} + f\chi_{(n+2,\infty)})(x) \le cT^d\bigg(\int_{m-1}^{m} f\bigg)(n)$$

holds.
(ii) There is $c > 0$ such that for all sequences $\{a_k\}$ of non-negative real numbers and $n \in \mathbb{Z}$, the inequality

$$T^d(\{a_k\})(n) \le cTf(y)$$

holds for all $y \in (n, n+1)$ and all non-negative f, where $\int_{m-1}^{m} f =: a_m$, $m \in \mathbb{Z}$. It is also assumed that T satisfies the conditions

$$Tf = T|f|, \qquad T(\lambda f) = |\lambda| Tf, \qquad T(f+g) \le Tf + Tg, \qquad Tf \le Tg \;\text{ if }\; f \le g.$$

We will say that an operator T satisfying all the above– mentioned conditions is admissible on $\mathbb{R}$. For example, Hardy operators, Hardy -Littlewood maximal operators, fractional integral operators, fractional maximal operators are admissible on $\mathbb{R}$ (see [8]).

General type results for admissible operators read as follows:

Theorem I ([8])**.** *Let $1 < p, \bar{p}, q, \bar{q} < \infty$, and let v and w be weight functions on $\mathbb{R}$. Suppose that T is an admissible operator on $\mathbb{R}$. Then the inequality*

$$\|vTf\|_{(L^p(\mathbb{R}),l^q)} \le c\|wf\|_{(L^{\bar{p}}(\mathbb{R}),l^{\bar{q}})}$$

holds for all measurable f if and only if

(i) *T^d is bounded from $l^{\bar{q}}(\{w_n\})$ to $l^q(\{v_n\})$, where $w_n := \left(\int_{n-1}^{n} w^{-\bar{p}'}\right)^{\frac{-\bar{q}}{\bar{p}'}}$, $v_n := \left(\int_{n}^{n+1} v\right)^{\frac{q}{p}}$.*
(ii)
 (a) $\sup_{n\in\mathbb{Z}} \|T_n\|_{[L^{\bar{p}}_w(n-1,n+2)\to L^p_v(n-1,n+2)]} < \infty$ *for $1 < \bar{q} \le q < \infty$.*
 (b) $\|T_n\|_{[L^{\bar{p}}_w(n-1,n+2)\to L^p_v(n-1,n+2)]} \in l^s$*, where $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$ for $1 < q < \bar{q} < \infty$.*

Let $X(\mathbb{R})$ be a Banach function space defined with respect to the Lebesgue measure on $\mathbb{R}$ (see [6], p.9). We establish the statement similar to Theorem I for amalgam spaces defined with respect to a Banach function space i.e., in the amalgam spaces, where instead of the $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R})}$ norm is taken Banach function norm $\|\cdot\|_{X(\mathbb{R})}$. This general amalgam space will be denoted by $(X(\mathbb{R}), l^q)$. Associate space of $X(\mathbb{R})$ is denoted by $X'(\mathbb{R})$. In a Banach function spaces Hölder's inequality holds (see [6], p. 9):

$$\int |fg| \leq \|f\|_X \|g\|_{X'}, \quad f \in X, \;\; g \in X'. \tag{2.6.1}$$

Let, as before, T be an operator defined on a set of measurable functions on $\mathbb{R}$ and let $T_{v,w}$ be an operator defined by

$$T_{v,w}f = vT(wf),$$

where v and w are a.e. positive functions on $\mathbb{R}$.

Theorem 2.6.1. *Let $X(\mathbb{R})$ and $Y(\mathbb{R})$ be Banach function spaces. Suppose that q and $\bar{q}$ are constants satisfying $1 < q, \bar{q} < \infty$. Suppose that w and v are weight functions on $\mathbb{R}$ and that T is an admissible operator on $\mathbb{R}$. Then the inequality*

$$\|T_{v,w}f\|_{(Y(\mathbb{R}),l^q)} \leq c\|f\|_{(X(\mathbb{R}),l^{\bar{q}})} \tag{2.6.2}$$

holds if

(i) *T^d is bounded from $l^{\bar{q}}(\{\bar{w}_n\})$ to $l^q(\{\bar{v}_n\})$ where* $\bar{w}_n := \|\chi_{(n-1,n)}(\cdot)w(\cdot)\|^{-\bar{q}}_{X(\mathbb{R})}$, $\bar{v}_n := \|\chi_{(n,n+1)}(\cdot)v(\cdot)\|^{q}_{Y(\mathbb{R})}$.

(ii) (a) $\sup\limits_{n\in\mathbb{Z}} \|(T_n)_{v,w}\|_{[X(n-1,n+2)\to Y(n-1,n+2)]} < \infty$ *for* $1 < \bar{q} \leq q < \infty$.

(b) $\|(T_n)_{v,w}\|_{[X(n-1,n+2)\to Y(n-1,n+2)]} \in l^s$ *with* $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$ *for* $1 < q < \bar{q} < \infty$.

Conversely, let (2.6.2) hold. Then

1) *conditions* (ii) *are satisfied;*

2) *condition* (i) *is satisfied for* $w \equiv const$.

Proof. Let (i) and (ii) hold. We have

$$\begin{aligned}\|vTf\|_{(Y(\mathbb{R}),l^q)} &\leq c\Big\{\sum_{n\in\mathbb{Z}}\|T[wf(\chi_{(-\infty,n-1)}+\chi_{(n+2,\infty)})]v(\cdot)\|^q_{Y(n,n+1)}\Big\}^{1/q}\\ &\quad +c\Big\{\sum_{n\in\mathbb{Z}}\|vT_n(fw)\|^q_{Y(n,n+1)}\Big\}^{1/q} =: S_1+S_2.\end{aligned}$$

Let $a_m := \int_{m-1}^m fw$. By the hypothesis and Hölder's inequality (see (2.6.1)) we have that

$$\begin{aligned}S_1 &\leq c\Big\{\sum_{n\in\mathbb{Z}}(T^d(\{a_m\})(n))^q\|\chi_{(n,n+1)}v\|^q_{Y(n,n+1)}\Big\}^{1/q}\\ &\leq c\Big\{\sum_{n\in\mathbb{Z}}a_n^{\bar{q}}\|\chi_{(n-1,n)}w\|^{-\bar{q}}_{X'(n-1,n)}\Big\}^{1/\bar{q}} \leq c\|f\|_{(X(\mathbb{R}),l^q)}.\end{aligned}$$

Let us estimate S_2. Suppose that $1<\bar{q}\leq q<\infty$. Since the operators $(T_n)_{v,w}$ are uniformly bounded we find that

$$\begin{aligned}S_2 &\leq c\Big\{\sum_{n\in\mathbb{Z}}\|f\|^q_{X(n-1,n+2)}\Big\}^{1/q} \leq c\Big\{\sum_{n\in\mathbb{Z}}\|f\|^{\bar{q}}_{X(n-1,n+2)}\Big\}^{1/\bar{q}}\\ &\leq c\|f\|_{(X(\mathbb{R}),l^{\bar{q}})}.\end{aligned}$$

If $1<q<\bar{q}<\infty$, then by using Hölder's inequality (see (2.6.1)) we find that

$$\begin{aligned}S_2 &\leq c\Big\{\sum_{n\in\mathbb{Z}}\|(T_n)_{v,w}\|^q_{\big[X(n-1,n+2)\to Y(n-1,n+2)\big]}\|\chi_{(n-1,n+2)}f\|^q_{X(\mathbb{R})}\Big\}^{1/q}\\ &\leq c\Bigg[\Big\{\sum_{n\in\mathbb{Z}}\|(T_n)_{v,w}\|^{\frac{q\bar{q}}{\bar{q}-q}}\Big\}^{\frac{\bar{q}-q}{q}}\Big\{\sum_{n\in\mathbb{Z}}\|\chi_{(n-1,n+2)}f\|^{\bar{q}}_{X(\mathbb{R})}\Big\}^{\frac{q}{\bar{q}}}\Bigg]^{1/q} \leq c\|f\|_{(X(\mathbb{R}),l^{\bar{q}})}.\end{aligned}$$

Conversely, let (2.6.2) holds. Suppose that $n\in\mathbb{Z}$ and f is a non-negative function supported in $(n-1,n+2)$. Then

$$\|f\|_{(X(\mathbb{R}),l^{\bar{q}})} \leq 3\|f\chi_{(n-1,n+2)}\|_{(X(\mathbb{R}))}.$$

On the other hand,

$$\begin{aligned}\|T_{v,w}f\|_{(Y(\mathbb{R}),l^q)} &\geq \|v\chi_{(n-1,n+2)}T(fw)\|_{Y(\mathbb{R})}\\ &\geq \|vT_n(fw)\|_{Y(\mathbb{R})}.\end{aligned}$$

By the two–weight inequality we conclude that (a) of (ii) holds. Let us now show that if $1 < q < \bar{q} < \infty$, then (b) of (ii) is satisfied.

Since $\|(T_n)_{v,w}\|_{[X(\mathbb{R})\to Y(\mathbb{R})]} = \sup\limits_{\{f:\|f\|_{X(\mathbb{R})}=1\}} \|vT_n(fw)\|_{Y(\mathbb{R})}$ we have that for each n, there exists a non-negative measurable function f_n, with the support in $(n-1, n+2)$ and with $\|\chi_{(n-1,n+2)}f_n\|_{X(\mathbb{R})} = 1$, such that $\|(T_n)_{v,w}\|_{X(\mathbb{R})\to Y(\mathbb{R})} < \|vT_n(f_n w)\|_{Y(\mathbb{R})} + \frac{1}{2^{|n|}}$. So it is sufficient to prove that $\|vT_n(f_n w)\|_{X(\mathbb{R})} \in l^s$.

Let $\{a_n\}$ be a sequence of non-negative real numbers and $f = \sum\limits_n a_n f_n$. For each $n \in \mathbb{Z}$, $f(x) > a_n f_n(x)$ and then $v(x)T(fw)(x) \geq a_n v(x) T_n(f_n w)(x)$ for all $x \in (n-1, n+2)$.

Thus,

$$\|T_{v,w}f\|_{(Y(\mathbb{R}),l^q)} \geq \left\{ \sum_{n\in\mathbb{Z}} c a_n^q \|\chi_{(n-1,n+2)} vT_n(fw)\|^q_{Y(\mathbb{R})} \right\}^{1/q} = c\Big\{ \sum_{n\in\mathbb{Z}} a_n^q \|vT_n(f_n w)\|^q_{Y(\mathbb{R})} \Big\}^{1/q}.$$

Hence, the two– weight inequality yields that

$$\begin{aligned}
\Big\{ \sum_{n\in\mathbb{Z}} a_n^q \|vT_n(f_n w)\|^q_{Y(\mathbb{R})} \Big\}^{1/q} &\leq c\Big\{ \sum_{n\in\mathbb{Z}} \|\chi_{(n-1,n+2)} f\|^{\bar{q}}_{X(\mathbb{R})} \Big\}^{1/\bar{q}} \\
&\leq c\Big\{ \sum_{n\in\mathbb{Z}} a_n^{\bar{q}} \|\chi_{(n-1,n+2)} f_n\|^{\bar{q}}_{X(\mathbb{R})} \Big\}^{1/\bar{q}} = c\{\sum_{n\in\mathbb{Z}} a_n^{\bar{q}}\}.
\end{aligned}$$

Finally, by Lemma L we see that (b) of (ii) holds.
Now let us prove that (i) holds when $w \equiv const$. If $\{a_m\}$ is a sequence of non-negative real numbers and if $f := \sum\limits_{m\in\mathbb{Z}} a_m \chi_{(m-1,m)}$, then $\int\limits_{m-1}^{m} f = a_m$, and $\|\chi_{(n,n+1)} f\|^{\bar{q}}_{X(\mathbb{R})} = a_n^{\bar{q}} \|\chi_{(n,n+1)}\|^{\bar{q}}_{X(\mathbb{R})} = a_n^{\bar{q}}$. By the properties of T we have,

$$\begin{aligned}
\|vTf\|_{(Y(\mathbb{R}),l^q)} &= \Big\{ \sum_{n\in\mathbb{Z}} \|\chi_{(n,n+1)} vTf\|^q_{Y(\mathbb{R})} \Big\}^{1/q} \\
&\geq \Big\{ \sum_{n\in\mathbb{Z}} \|\chi_{(n,n+1)} vT^d \Big(\int_{m-1}^{m} f \Big) \|^q_{Y(\mathbb{R})} \Big\}^{1/q} \\
&\geq c\Big\{ \sum_{n\in\mathbb{Z}} T^d(a_m)^q(n) \|\chi_{(n,n+1)} v\|^q_{Y(\mathbb{R})} \Big\}^{1/q} = \|\bar{v}_n T^d \{a_m(n)\}\|_{l^q}.
\end{aligned}$$

Applying the two-weight inequality we have that

$$\|\bar{v}_n T^d\{a_m(n)\}\|_{l^q} \;\leq\; c\Big\{\sum_{n\in\mathbb{Z}}\|\chi_{(n,n+1)}f\|^{\bar{q}}_{X(\mathbb{R})}\Big\}^{1/\bar{q}} = c\Big\{\sum_{n\in\mathbb{Z}} a_n^{\bar{q}}\Big\}^{1/\bar{q}} = \|a_n\|_{l^{\bar{q}}}.$$

Hence (i) holds. ■

Theorem 2.6.1 implies the following statement:

Theorem 2.6.2. *Let $\bar{p}(\cdot)$, $p(\cdot)$ be measurable functions on $\mathbb{R}$ satisfying $1 < p_-(\mathbb{R}) \leq p_+(\mathbb{R}) < \infty$, $1 < \bar{p}_-(\mathbb{R}) \leq \bar{p}_+(\mathbb{R}) < \infty$. Suppose that q and $\bar{q}$ are constants satisfying $1 < q, \bar{q} < \infty$. Suppose that w and v are weight functions on $\mathbb{R}$ and that T is an admissible operator on $\mathbb{R}$. Then the inequality*

$$\|vTf\|_{(L^{p(\cdot)}(\mathbb{R}),l^q)} \leq c\|wf\|_{(L^{\bar{p}(\cdot)}(\mathbb{R}),l^{\bar{q}})} \tag{2.6.3}$$

holds if

(i) *T^d is bounded from $l^{\bar{q}}(\{\bar{w}_n\})$ to $l^q(\{\bar{v}_n\})$ where $\bar{w}_n := \|\chi_{(n-1,n)}(\cdot)w^{-1}(\cdot)\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}}$, $\bar{v}_n := \|\chi_{(n,n+1)}(\cdot)v(\cdot)\|^q_{L^{p(\cdot)}}$.*

(ii) (a) $\sup_{n\in\mathbb{Z}}\|T_n\|_{[L^{\bar{p}(\cdot)}_w(n-1,n+2)\to L^{p(\cdot)}_v(n-1,n+2)]} < \infty$ *for $1 < \bar{q} \leq q < \infty$.*

(b) $\|T_n\|_{[L^{\bar{p}(\cdot)}_w(n-1,n+2)\to L^{p(\cdot)}_v(n-1,n+2)]} \in l^s$ *with $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$ for $1 < q < \bar{q} < \infty$.*

Conversely, let (2.6.3) hold. Then

1) *conditions* (ii) *are satisfied;*

2) *condition* (i) *is satisfied for $w \equiv const$ or for p and $\bar{p}$ being constant outside some large interval $[-m_0, m_0]$, $m_0 \in \mathbb{Z}$.*

Proof. Proof follows from Theorem 2.6.1. We only need to show that if (2.6.3) holds, then condition (i) is satisfied for p and $\bar{p}$ being constant outside some large interval $[-m_0, m_0]$, $m_0 \in \mathbb{Z}$.

Suppose now that w is a general weight and there is a positive integer m_0 such that p, $\bar{p}$ are constants outside $[-m_0, m_0]$. Taking

$$f(x) = \sum_{m\in\mathbb{Z}} a_m\chi_{(m-1,m)}(x)\Big(\int_{m-1}^{m} w^{-\bar{p}'(y)}(y)dy\Big)^{-1} w^{-\bar{p}'(x)}(x)$$

it is easy to see that $\int_{m-1}^{m} f = a_m$. Moreover, by Proposition A and the fact that

$$\int_{m-1}^{m} w^{-\bar{p}'(y)}(y)dy \le \int_{-m_0}^{m_0} w^{-\bar{p}'(y)}(y)dy < \infty, \;\; [m-1,m] \subset [-m_0, m_0],$$

we have for $m \le m_0 + 1$,

$$\|\chi_{(m-1,m)} f w\|_{L^{\bar{p}(\cdot)}} = a_m \left(\int_{m-1}^{m} w^{-\bar{p}'(y)}(y)dy \right)^{-1} \|\chi_{(m-1,m)} w^{1-\bar{p}'(\cdot))}\|_{L^{\bar{p}(\cdot)}}$$

$$\le c a_m \left(\int_{m-1}^{m} w^{-\bar{p}'(y)}(y)dy \right)^{-1/\bar{p}_+\left([m-1,m)\right)},$$

where the positive constant c depends on m_0. Since

$$\|vTf\|_{(L^{p(\cdot)}(\mathbb{R}),l^q)} \ge C\|\bar{v}_n(T^d\{a_m\})(n)\|_{l^q},$$

using again Proposition A we find that

$$\|\bar{v}_n(T^d\{a_m\})(n)\|_{l^q} \le C\left[\sum_m \left\|\chi_{(m-1,m)} f w\right\|^{\bar{q}}_{L^{\bar{p}(\cdot)}(\mathbb{R})} \right]^{1/\bar{q}}$$

$$\le c\left[\sum_m a_m^{\bar{q}} \left(\int_{m-1}^{m} w^{-\bar{p}'(y)}(y)dy \right)^{-\bar{q}/\bar{p}_+([m-1,m))} \right]^{1/\bar{q}} = \|a_m \bar{w}_m\|_{l^{\bar{q}}}.$$

■

2.6.2 Maximal Operators in Amalgams $(L^{p(\cdot)}(\mathbb{R}), l^q)$

In this section we establish criteria for the boundedness of maximal operators in variable exponent amalgam spaces.

Recall the E. Sawyer [78] result for the discrete fractional maximal operator

$$M_\alpha^d(\{a_n\})(j) = \sup_{r \le j \le k} \frac{1}{(k-r+1)^{1-\alpha}} \sum_{i=r}^{k} |a_i|, \;\; 0 < \alpha < 1.$$

which is a consequence of more general result regarding two-weight criteria for maximal operators defined on spaces of homogeneous type (see [76]).

Theorem J. *Let r and s be constants satisfying the condition $1 < r \le s < \infty$ and let α_n, β_n be positive sequences on $\mathbb{Z}$. Then the two–weight inequality*

$$\Big(\sum_{n\in\mathbb{Z}} \big(M^d(\{a_n\})\big)^s_n \alpha_n\Big)^{1/s} \le c\Big(\sum_{n\in\mathbb{Z}} |a_n|^r \beta_n\Big)^{1/r},$$

holds if and only if there is a positive constant c such that for all $r, k \in \mathbb{Z}$ with $r \le k$,

$$\Big(\sum_{j=r}^{k} \big(M^d(\{\beta_n^{1-r'}\}\chi_{[r,k]})\big)^s(j)\alpha_j\Big)^{1/s} \le c\Big(\sum_{j=r}^{k} \beta_n^{1-r'}\Big)^{1/r}.$$

Corollary B. *Let $1 < r \le s < \infty$ and let α_n be a positive sequences on $\mathbb{Z}$. Then the weighted inequality*

$$\Big(\sum_{n\in\mathbb{Z}} \big(M^d_\alpha(\{a_n\})\big)^s_n \alpha_n\Big)^{1/s} \le c\Big(\sum_{n\in\mathbb{Z}} |a_n|^r\Big)^{1/r} \tag{2.6.4}$$

holds if and only if

$$\sup_{k,r\in\mathbb{Z}, r<k} \Big(\sum_{j=r}^{k} \alpha_j\Big)^{1/s} (k-r+1)^{\alpha-1/r} \le c, \tag{2.6.5}$$

where the positive constant c is independent of $\{a_n\}$.

Theorem K ([83])**.** *Let s and r be constants satisfying the condition $1 < s < r < \infty$ and let α_n be a positive sequence on $\mathbb{Z}$. We set $h_j := \sup_{r\le i\le k} \frac{1}{(k-r+1)^{1-\alpha r}} \sum_{i=r}^{k} \alpha_j$. Then the inequality* (2.6.4) *holds if and only if $\{h_j\} \in l_{\alpha_j}^{\frac{s}{r-s}}$.*

Now we formulate our result regarding variable exponent amalgam spaces.

Theorem 2.6.3. *Let p be continuous function defined on $\mathbb{R}$ satisfying the conditions $1 < p_-(\mathbb{R}) \le p(x) \le p_+(\mathbb{R}) < \infty$. Suppose that $p \in \mathcal{P}^{log}(\mathbb{R})$. If*

(a) *$w \in A_{p(\cdot)}([n-1, n+2))$ uniformly with respect to n;*

(b) *the pair of discrete weights $(\{\bar{w}_n\}, \{\bar{v}_n\})$ satisfies the condition: there is a positive constant c such that for all $r, k \in \mathbb{Z}$ with $r \le k$,*

$$\sum_{j=r}^{k} \big(M^d(\{\bar{w}_n^{1-q'}\}\chi_{[r,k]})\big)^q(j)\bar{v}_j \le c\sum_{j=r}^{k} \bar{w}_j^{1-q'}, \tag{2.6.6}$$

where

$$\bar{w}_n := \|\chi_{(n-1,n)}(\cdot)w^{-1}(\cdot)\|^{-q}_{L^{p'(\cdot)}(\mathbb{R})},\ \bar{v}_n := \|\chi_{(n,n+1)}(\cdot)w(\cdot)\|^{q}_{L^{p(\cdot)}(\mathbb{R})}.$$

Then $M^{(\mathbb{R})}$ is bounded in $(L^{p(\cdot)}_w(\mathbb{R}), l^q)$.

Conversely, let $M^{(\mathbb{R})}$ be bounded in $(L^{p(\cdot)}_w(\mathbb{R}), l^q)$. Then (a) holds. If, in addition, there is a large positive integer m_0 such that p is constant outside $[-m_0, m_0]$, then condition (b) is also satisfied.

Proof. Observe that the Hardy–Littlewood maximal operator $M^{(\mathbb{R})}$ is admissible (see [70]) and associated discrete operator is given by

$$M^d(\{a_n\})(j) = \sup_{r\le j\le k} \frac{1}{k-r+1}\sum_{i=r}^{k}|a_i|.$$

Also, $(M^{(\mathbb{R})}f)_n = (M^{([n-1,n+2)}f)(x)$, $x \in [n-1, n+2)$.

Now by Theorems J, 2.6.2 and Proposition 2.2.1 we have the desired result. ■

Theorem 2.6.4. *Let p be a continuous function defined on $\mathbb{R}$ satisfying the condition $1 < p_-(\mathbb{R}) \le p_+(\mathbb{R}) < \infty$. Let $0 \le \alpha < 1$. Suppose that v, w are weight functions on $\mathbb{R}$ and that $d\nu(x) := w(x)^{-p'(x)}dx$ belongs to $DC([n-1, n+2))$ uniformly with respect to n. Suppose also that $p \in \mathcal{P}^{log}(\mathbb{R})$. Then the operator $M^{(\mathbb{R})}_\alpha$ is bounded from $(L^{p(\cdot)}_w(\mathbb{R}), l^q)$ to $(L^{p(\cdot)}_v(\mathbb{R}), l^q)$ if*

(i) there is a positive constant c such that for all n and all intervals $I \subseteq [n-1, n+2)$ the inequality

$$\int_I (v(x))^{p(x)} M_\alpha^{[n-1,n+2)}\big(w(\cdot)^{-p'(\cdot)}\chi_I(\cdot)\big)\big)^{p(x)}dx \le c\int_I w^{-p'(x)}dx < \infty$$

holds*;*

(ii) there is a positive constant c such that for all $r, k \in \mathbb{Z}$ with $r \le k$,

$$\sum_{j=r}^{k}\big((M_\alpha)^d(\{\bar{w}_n^{1-q'}\}\chi_{[r,k]})\big)^q(j)\bar{v}_j \le c\sum_{j=r}^{k}\bar{w}_j^{1-q'}, \tag{2.6.7}$$

where

$$\bar{w}_n := \|\chi_{(n-1,n)}(\cdot)w^{-1}(\cdot)\|^{-\bar{q}}_{L^{\bar{p}'(\cdot)}(\mathbb{R})}, \quad \bar{v}_n := \|\chi_{(n,n+1)}(\cdot)v(\cdot)\|^{q}_{L^{p(\cdot)}(\mathbb{R})}.$$

Conversely, let $M^{(\mathbb{R})}_\alpha$ be bounded from $(L^{p(\cdot)}_{w(\cdot)}(\mathbb{R}), l^q)$ to $(L^{p(\cdot)}_{v(\cdot)}(\mathbb{R}), l^q)$. Then (i) holds. If, in addition, there is a large positive integer m_0 such that p is constant outside $[-m_0, m_0]$, then condition (ii) is also satisfied.

Proof. It is known (see [70]) that the operator $M_\alpha^{\mathbb{R}}$ is admissible and that its discrete analog is M_α^d.

By Proposition 2.2.2 and Theorems J, 2.6.2 we have the desired result. ∎

Theorem 2.6.5. *Let p be a continuous function defined on $\mathbb{R}$ satisfying the condition $1 < p_-(\mathbb{R}) \le p(x) \le p_+(\mathbb{R}) < \infty$. Assumed that $0 < \alpha < 1$. Suppose that v is a weight function on $\mathbb{R}$. Suppose also that $p \in \mathcal{P}^{log}(\mathbb{R})$. Then the operator $M_\alpha^{(\mathbb{R})}$ is bounded from $(L^{p(\cdot)}(\mathbb{R}), l^{\bar{q}})$ to $(L_v^{p(\cdot)}(\mathbb{R}), l^q)$ if and only if*

(i) *in the case $1 < \bar{q} \le q < \infty$,*

$$\sup_{\substack{n\in\mathbb{Z}\\ I\subset(n-1,n+2)}} \frac{1}{|I|}\int_I (v(x))^{p(x)}|I|^{\alpha p(x)}dx < \infty$$

and

$$\sup_{k,r\in\mathbb{Z},r<k}\left(\sum_{j=r}^{k}\bar{v}_j\right)(k-r+1)^{\alpha q-1} \le c, \tag{2.6.8}$$

where $\bar{v}_n = \|\chi_{[n,n+1)}v\|^q_{L^{p(\cdot)}(\mathbb{R})}$;

(ii) *in the case $1 < q < \bar{q} < \infty$, $\{J_n\} \in l^s$, where $\frac{1}{s} = \frac{1}{q} - \frac{1}{\bar{q}}$, and $\{H_j\} \in l_{\bar{v}_j}^{\frac{q}{\bar{q}-q}}$, where*

$$J_n := \sup_{\substack{n\in\mathbb{Z}\\ I\subset(n-1,n+2)}} \frac{1}{|I|}\int_I (v(x))^{p(x)}|I|^{\alpha p(x)}dx,$$

$$H_j := \sup_{r\le i\le k}\frac{1}{(k-r+1)^{1-\alpha\bar{q}}}\sum_{i=r}^{k}\bar{v}_j, \quad \bar{v}_n := \|\chi_{(n,n+1)}(\cdot)v(\cdot)\|^q_{L^{p(\cdot)}(\mathbb{R})}.$$

Proof. Part (i) follows in the same way as Theorem 2.6.4 was proved. We observe that in this case we use Corollary B. The proof of Part (ii) is similar by applying Theorems 2.6.2, K and Corollary 2.2.1. ∎

Theorem 2.6.6. *Let p be a measurable function on $\mathbb{R}$ such that $1 < p_-(\mathbb{R}) \le p_+(\mathbb{R}) < \infty$. Let $\bar{p}$, q, $\bar{q}$ and α be constants satisfying the condition $1 < \bar{p} < p_-$, $1 < \bar{q} \le q < \infty$, $0 < \alpha < 1$. Suppose that $w^{-\bar{p}'} \in RD(\mathbb{R})$. Then the M_α is bounded from $(L_w^{\bar{p}}(\mathbb{R}), l^{\bar{q}})$ to $(L_v^{p(\cdot)}(\mathbb{R}), l^q)$ if and only if*

(i)

$$\sup_{\substack{n\in\mathbb{Z}\\ I\subset[n-1,n+2)}} \|v\chi_I|I|^{\alpha-1}\|_{L^{p(\cdot)}(\mathbb{R})}\|w^{-1}\chi_I\|_{L^{\bar{p}'}(\mathbb{R})} < \infty. \tag{2.6.9}$$

(ii)

$$\left(\sum_{j=r}^{k}\big(M^d(\{\bar{w}_n^{1-\bar{q}'}\}\chi_{[r,k]})\big)^q(j)\bar{v}_j\right)^{1/q} \le c\left(\sum_{j=r}^{k}\bar{w}_j^{1-\bar{q}'}\right)^{1/\bar{q}}, \tag{2.6.10}$$

where

$$\bar{w}_n := \|\chi_{(n-1,n)}(\cdot)w^{-1}(\cdot)\|^{-\bar{q}}_{L^{\bar{p}'}(\mathbb{R})}, \quad \bar{v}_n := \|\chi_{(n,n+1)}(\cdot)w(\cdot)\|^{q}_{L^{p(\cdot)}(\mathbb{R})}.$$

Theorem 2.6.6 is a direct consequence of Proposition 2.2.4 and Theorems J, 2.6.2.

2.6.3 Fractional Integrals: Trace Inequality

In this subsection we discuss trace inequality criteria for the fractional integrals operators I_α, R_α and W_α in weighted VEAS defined on $\mathbb{R}$. In particular, we show that the following statement holds.

Lemma O (see the proof of Theorem 3.1 in [70])**.** *The following equivalences hold:*

$$(I_\alpha f\chi_{(-\infty,n-1)})(x) \approx \sum_{m=-\infty}^{n-1}(n-m)^{\alpha-1}\mathcal{G}(m); \tag{2.6.11}$$

$$(I_\alpha f\chi_{(n+2,\infty)})(x) \approx \sum_{m=n+3}^{\infty}(m-n)^{\alpha-1}\mathcal{G}(m) \tag{2.6.12}$$

where $x \in [n, n+1)$ *and* $\mathcal{G}(m) = \int_{m-1}^{m} f(y)dy$.

Theorem 2.6.7. *Let* p *be a measurable function on* $\mathbb{R}$ *such that* $1 < p_-(\mathbb{R}) \le p_+(\mathbb{R}) < \infty$*. Let* $\bar{p}$*,* q*,* $\bar{q}$ *and* α *be constants satisfying the condition* $1 < \bar{p} < p_-(\mathbb{R})$*,* $1 < \bar{q} < q < \infty$*,* $0 < \alpha < \min\{1/\bar{p}, 1/\bar{q}\}$*. Then the following statements are equivalent:*

(i) I_α *is bounded from* $(L^{\bar{p}}(\mathbb{R}), l^{\bar{q}})$ *to* $(L^{p(\cdot)}_v(\mathbb{R}), l^q)$*;*

(ii) (a)

$$\sup_{\substack{n\in\mathbb{Z}\\ I\subset[n-1,n+2)}} \|\chi_I\|_{L^{p(\cdot)}_v(I)}|I|^{\alpha-1/\bar{p}} < \infty; \tag{2.6.13}$$

(b)

$$\sup_{m\in\mathbb{Z}, j\in\mathbb{N}}\left(\sum_{k=m}^{m+j}\bar{v}_k\right)^{1/q}(j+1)^{\alpha-1/\bar{q}} < \infty, \tag{2.6.14}$$

where $\bar{v}_n := \|\chi_{[n,n+1)}(\cdot)\|^q_{L^{p(\cdot)}_v(\mathbb{R})}$.

Theorem 2.6.8. *Let p be a measurable function on $\mathbb{R}$ such that $1 < p_-(\mathbb{R}) \le p_+(\mathbb{R}) < \infty$. Let $\bar{p}$, q and α be constants satisfying the condition $1 < \bar{p} < p_-(\mathbb{R})$, $1 < q < \infty$, $0 < \alpha < \min\{1/\bar{p}, 1/q\}$. Then the following statements are equivalent:*

(i) *I_α is bounded from $(L^{\bar{p}}(\mathbb{R}), l^q)$ to $L_v^{p(\cdot)}(\mathbb{R})), l^q)$;*

(ii)

(a)

$$\sup_{\substack{n\in\mathbb{Z}\\ I\subset[n-1,n+2)}} \|\chi_I\|_{L_v^{p(\cdot)}(I)} |I|^{\alpha-1/\bar{p}} < \infty;$$

(b) *$\{\mathcal{W}_\alpha \bar{v}_i\}_i < \infty$ for all $i \in \mathbb{Z}$ and there is a positive constant c such that*

$$\left\{\mathcal{W}_\alpha[\mathcal{W}_\alpha(\bar{v}_j)]^{q'}\right\}_k \le c\left\{\mathcal{W}_\alpha(\bar{v}_j)\right\}_k \tag{2.6.15}$$

for all $k \in \mathbb{Z}$, where $\bar{v}_n$ is the same as in Theorem 2.6.7;

$\{\mathcal{R}_\alpha \bar{v}_i\}_i < \infty$ for all $i \in \mathbb{Z}$ and there is a positive constant c such that

$$\left\{\mathcal{R}_\alpha[\mathcal{R}_\alpha(\bar{v}_j)]^{q'}\right\}_k \le c\left\{\mathcal{R}_\alpha(\bar{v}_j)\right\}_k \tag{2.6.16}$$

for all $k \in \mathbb{Z}$, where $\bar{v}_n$ is defined in Theorem 2.6.7.

Proof of Theorem 2.6.7.

First observe that

$$(I_\alpha)_n f(x) = \int_{n-1}^{n+2} \frac{f(t)}{|x-t|^{1-\alpha}} dt, \quad x \in [n-1, n+2).$$

Due to Proposition 2.2.5, uniform boundedness of $(I_\alpha)_n$ is equivalent to (2.6.13). Further, it is easy to check that condition (2.6.14) is equivalent to each of the following two conditions:

$$\sup_{m\in\mathbb{Z}, j\in\mathbb{N}} \left(\sum_{k=m}^{m+j} \bar{v}_k^{(i)}\right)^{1/q} (j+1)^{\alpha-1/\bar{q}} < \infty, \quad i = 1, 2, \tag{2.6.17}$$

where $\bar{v}_k^{(1)} = \bar{v}_{k+1}$, $\bar{v}_k^{(2)} = \bar{v}_{k-3}$.

Since (see [70])

$$(I_\alpha)^d(\{a_j\})(n) \approx \sum_{k=-\infty}^{n-1} \frac{a_k}{(k-n)^{1-\alpha}} + \sum_{k=n+3}^{+\infty} \frac{a_k}{(k-n+1)^{1-\alpha}}, \tag{2.6.18}$$

by Theorem 2.6.2, Lemma O, Lemma K and Proposition 2.3.1 we have the desired result. ■

Proof of Theorem 2.6.8 follows similarly by applying Proposition 2.2.5, Proposition 2.3.2, Lemma O, Lemma K and Theorem 2.6.2. ■

Chapter 3

Multiple Integral Operators in Classical Lebesgue Spaces

In this chapter, we characterize $L^p(\mathbb{R}^n_+, \mu) \to L^q(\mathbb{R}^n_+, \nu)$ boundedness for multiple kernel operators K_μ defined with respect to a product Borel measure μ on $\mathbb{R}^n_+$. The derived results involve necessary and sufficient conditions for the two–weight (two–measure) inequalities for the multiple Hardy and Riemann-Liouville transforms defined by product measures. The similar result for the strong one-sided fractional maximal operator $\mathcal{M}^\mu_{\alpha_1,\cdots,\alpha_n}$ is also derived. In all cases the target Lebesgue spaces are defined by a measure having, generally speaking, non-product form. As a corollary we have, for example, two–weight criteria for discrete multiple Hardy transforms. Fefferman-Stein type inequality for the multiple Riemann-Liouville transform defined with respect to a measure is also derived in the diagonal case. It should be underlined that all main theorems of this chapter are formulated for $n \geq 2$ but for simplicity throughout the chapter we give the proofs for $n = 2$. Proofs of other cases can be derived in the same manner, and therefore are omitted.

3.1 Preliminaries

Let μ be positive Borel measure on a set $\Omega \subset \mathbb{R}^n$. We denote by $L^p(\Omega,\mu)$, $1<p<\infty$, the set of all measurable functions $f:\Omega\to\mathbb{R}$ for which the norm

$$\|f\|_{L^p(\Omega,\mu)} = \Big(\int\limits_{\Omega} |f(x)|^p d\mu(x)\Big)^{\frac{1}{p}}$$

is finite.

The two–weight problem for the classical Hardy-transform $Hf(x)=\int\limits_0^x f(t)dt$ was solved by B. Muckhenhoupt [65] for $p=q$; by V. Kokilashivili [48], J. Bradley [7] (see also V. Maz'ya [56, Ch. 1]) for $p\le q$. A complete characterization of a class of weight pairs (v,w) for which the operator

$$H_2 f(x,y) = \int\limits_0^x \int\limits_0^y f(t,\tau)dtd\tau, \ \ x,y>0,$$

is bounded from $L^p(\mathbb{R}^2_+, wdx)$ to $L^q(\mathbb{R}^2_+, vdx)$ was given by E. Sawyer in terms of three independent conditions. Namely he proved the following statement:

Theorem L ([77])**.** *Let $1<p\le q<\infty$. Then for the boundedness of the operator H_2 from $L^p(\mathbb{R}^2_+, wdx)$ to $L^q(\mathbb{R}^2_+, vdx)$ it is necessary and sufficient that the following conditions are satisfied:*
(i)

$$A := \sup_{a,b>0} (H_2^{'}v(a,b))^{\frac{1}{q}} (H_2\sigma(a,b))^{\frac{1}{p'}} < \infty,$$

where $\sigma \equiv w^{1-p'}$, $p^{'}=\frac{p}{p-1}$;
(ii)

$$\int\limits_0^a \int\limits_0^b (H_2^{'}\sigma)^q v \le A^q [H_2\sigma(a,b)]^{\frac{q}{p}},$$

for all $a,b>0$;
(iii)

$$\int\limits_a^\infty \int\limits_b^\infty (H_2^{'}v)^{p'}\sigma \le A^{p'} [H_2^{'}v(a,b)]^{\frac{p'}{q'}},$$

for all $a, b > 0$; where

$$H_2^{'} f(x,y) = \int\limits_x^{\infty}\int\limits_y^{\infty} f(t,\tau)dtd\tau, \;\; x,y>0.$$

If the right–hand side weight w is a product of two weights of separate variables, then the $L^p(\mathbb{R}_+, wdx) \to L^q(\mathbb{R}_+, vdx)$ boundedness can be characterized just by one condition (see [86], [62], [41, Ch. 1]). The Muckenhoupt–type characterization reads as follows:

Theorem M ([62])**.** *Let $1 < p \le q < \infty$ and let $w(x,y) = w_1(x)w_2(y)$. Then the operator H_2 is bounded from $L^p(\mathbb{R}^2_+, wdx)$ to $L^q(\mathbb{R}^2_+, vdx)$ if and only if*

$$B := \sup_{a,b>0} \Big(\int\limits_0^a w_1^{1-p'}(x)dx\Big)^{\frac{1}{p'}} \Big(\int\limits_0^b w_2^{1-p'}(y)dy\Big)^{\frac{1}{p'}} \Big(\int\limits_a^{\infty}\int\limits_b^{\infty} v(x,y)dxdy\Big)^{\frac{1}{q}} < \infty.$$

Moreover, $\|H_2\|_{L^p(\mathbb{R}^2_+, wdx)\to L^q(\mathbb{R}^2_+, vdx)} \approx B$.

For the next statements regarding the two–weight (two–measured) inequality for the Hardy operators

$$\big(\mathcal{H}f\big)(x) := \int\limits_{(0,x]} f(t)d\mu(t), \;\; x>0; \qquad \big(\widetilde{\mathcal{H}}f\big)(x) := \int\limits_{(-\infty,x]} f(t)d\mu(t), \;\; x \in \mathbb{R},$$

defined with respect to a general measure μ we refer e.g., to the PhD thesis of G. Sinnamon [80].

Theorem N. *Suppose that $1 < p \le q < \infty$ and μ, ν are non–negative regular Borel measure on $\mathbb{R}_+$. Then there exists a constant $c > 0$ such that*

$$\Big(\int\limits_{\mathbb{R}_+} \big(\mathcal{H}g\big)^q(x)d\nu(x)\Big)^{\frac{1}{q}} \le c\Big(\int\limits_{\mathbb{R}_+} g^p d\mu\Big)^{\frac{1}{p}} \tag{3.1.1}$$

holds for all non–negative $g \in L^p(\mathbb{R}, \mu)$, if and only if

$$c_1 := \sup_{y\in\mathbb{R}_+} \Big(\int\limits_{[y,\infty)} d\nu\Big)^{1/q} \Big(\int\limits_{(0,y]} d\mu\Big)^{1/p'} < \infty$$

Furthermore, if c is the smallest constant such that (3.1.1) *holds, then* $c \approx c_1$.

Theorem O. *Suppose that $1 < p \le q < \infty$ and μ, ν are non–negative regular Borel measure on $\mathbb{R}$. Then there exists a constant $c > 0$ such that*

$$\left(\int\limits_{\mathbb{R}} \big(\widetilde{\mathcal{H}}g\big)^q (x) d\nu(x) \right)^{\frac{1}{q}} \le c \left(\int\limits_{\mathbb{R}} g^p d\mu \right)^{\frac{1}{p}} \tag{3.1.2}$$

holds for all non–negative $g \in L^p(\mathbb{R}, \mu)$, if and only if

$$c_2 := \sup_{y \in \mathbb{R}} \left(\int\limits_{[y,\infty)} d\nu \right)^{1/q} \left(\int\limits_{(-\infty,y]} d\mu \right)^{1/p'} < \infty.$$

Furthermore, if c is the smallest constant such that (3.1.2) *holds, then* $c \approx c_2$.

We recall that criteria for the boundedness of $R_\alpha f(x) := \int\limits_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt$ from $L^p(\mathbb{R}_+)$ to $L^q(\mathbb{R}_+, vdx)$, when $1 < p \le q < \infty$ and $\frac{1}{p} < \alpha < 1$ have been obtained in [64] (see also [25, Ch. 2], for this and related results).

Theorem P ([64])**.** *Let $1 < p \le q < \infty$ and let $1/p < \alpha < 1$. Then the following statements are equivalent:*

(i) *Operator R_α is bounded from $L^p(\mathbb{R}_+)$ to $L^q(\mathbb{R}_+, vdx)$;*

(ii)

$$\sup_{t>0} \left(\int\limits_t^\infty \frac{v(x)}{x^{(1-\alpha)q}} dx \right)^{1/q} t^{1/p'} < \infty.$$

The following proposition will be useful for us.

Proposition H. ([80, Lemma 2.4]) *Let $1 < p < \infty$. Then there exists a constant $c > 0$ such that for all $x \in \mathbb{R}$,*

$$\int\limits_{[x,\infty)} \big(\mu(-\infty, t]\big)^{-p} d\mu(t) \le c\big(\mu(-\infty, x]\big)^{1-p}.$$

Proposition I. *Let $1 < p < \infty$. Then there exists a constant $c > 0$ such that for all $x \in \mathbb{R}_+$,*

$$\int_{[x,\infty)} \big(\mu(0,t]\big)^{-p} d\mu(t) \leq c\big(\mu(0,x]\big)^{1-p}. \tag{3.1.3}$$

Proof. We follow the proof of Lemma 2.4 in [80]. If $\mu((0,x]) = 0$ or $\mu((0,x]) = \infty$, then the result holds trivially. The non trivial case arises when $0 < \mu((0,x]) < \infty$. Fix $x \in \mathbb{R}_+$ and $a > 1$, define

$$F_n := \Big\{ t \in \mathbb{R}_+ : \mu((0,t]) \leq a^n \mu((0,x]) \Big\} \text{ for } n = 0, 1, 2, \cdots ,$$

and $F_{-1} := (0,x)$. It follows then that $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{Z}$, and $\bigcup_{n\geq -1} F_n = \mathbb{R}_+$. Since $\mu((0,t])$ is non-decreasing as a function of t. So there exists a real number t_n such that $F_n = (0,t_n)$ or $F_n = (0,t_n]$.

If $F_n = (0,t_n)$, then we have $\mu(F_n) = \lim_{\epsilon \to 0^+} \mu((0,t_n - \epsilon]) \leq a^n \mu((0,x])$. In the latter case if $F_n = (0,t_n]$ we have $\mu(F_n) \leq a^n \mu((0,x])$. So we have the following estimate for F_n:

$$\mu(F_n) \leq a^n \mu((0,x]).$$

Further, if $t \notin F_{n-1}$ then $\mu((0,t]) > a^{n-1}\mu((0,x])$. Consequently,

$$\begin{aligned} \int_{[x,\infty)} \mu\big((0,t]\big)^{-p} d\mu(t) &= \sum_{n=0}^{\infty} \int_{F_n \setminus F_{n-1}} \big(\mu(0,t]\big)^{-p} d\mu(t) \\ &\leq \sum_{n=0}^{\infty} \int_{F_n \setminus F_{n-1}} \big(a^{n-1}\mu(0,x]\big)^{-p} d\mu(t) \\ &\leq \big(\mu(0,x]\big)^{1-p} a^p \sum_{n=0}^{\infty} a^{(1-p)n} \leq c\big(\mu(0,x]\big)^{1-p}. \end{aligned}$$

■

In order to state our main results, we introduce some definitions.

Definition 3.1.1. A sequence $\{u_{i_1,\cdots,i_n}\}_{i_1,\cdots,j_n=1}^{\infty}$ is a product sequence if there are sequences $\{u_{1,i_1}\}_{i_1=1}^{\infty}$, $\cdots$, $\{u_{n,i_n}\}_{i_n=1}^{\infty}$ such that $u_{1,\cdots,i_n} = u_{1,i_1} \times \cdots \times u_{n,i_n}$.

Proposition 3.1.1. *Let $1 < p < \infty$. Then for any regular Borel measure μ on $\mathbb{R}$ there exists $c > 0$ such that*

$$\int\limits_{\mathbb{R}} \left(\frac{1}{\mu((-\infty, x])} \int\limits_{(-\infty,x]} f d\mu \right)^p d\mu(x) \leq c \int\limits_{\mathbb{R}} f^p d\mu \tag{3.1.4}$$

holds for all non-negative f, $f \in L^p(\mathbb{R}, \mu)$.

Proof. By taking the measure $d\nu(x) = \frac{1}{\mu(-\infty,x]^p} d\mu(x)$ in Theorem O and using Proposition H we get

$$\int\limits_{\mathbb{R}} \left(\frac{1}{\mu(-\infty, x]} \int\limits_{(-\infty,x]} f d\mu \right)^p d\mu(x) \leq c \int\limits_{\mathbb{R}} f^p d\mu. \qquad \blacksquare$$

Proposition 3.1.2. *Let $1 < p < \infty$. Then for any measure μ on $\mathbb{R}_+$ there exist $c > 0$ such that*

$$\int\limits_{\mathbb{R}_+} \left(\frac{1}{\mu((0, x])} \int\limits_{(0,x]} f d\mu \right)^p d\mu(x) \leq c \int\limits_{\mathbb{R}_+} f^p d\mu \tag{3.1.5}$$

holds for all non-negative f, $f \in L^p(\mathbb{R}_+, \mu)$.

Proof. The proof of the statement is similar to Proposition 3.1.1 ; in this case we use Theorem N instead of Theorem O. $\blacksquare$

3.2 Multiple Hardy Operators Defined with Respect to a Borel Measure

This section is devoted to $L^p(\mathbb{R}^n_+, \mu) \to L^q(\mathbb{R}^n_+, \nu)$ boundedness of multiple Hardy operators defined with respect to a Borel measure.

Let us denote by

$$(\mathcal{H}_n f)(x_1, \cdots, x_n) = \int\limits_{(0,x_1]\times\cdots\times(0,x_n]} \cdots \int f(t_1, \cdots, t_n) d\mu(t_1 \cdots t_n),$$

$$(\widetilde{\mathcal{H}}_n f)(x_1,\cdots,x_n) = \int\limits_{(-\infty,x_1]\times\cdots\times(-\infty,x_n]} \cdots \int f(t_1,\cdots,t_n)d\mu(t_1\cdots t_n). \tag{3.2.1}$$

Theorem 3.2.1. *Let $1 < p \le q < \infty$ and let μ, ν be regular Borel measures on $\mathbb{R}^n$. Suppose that $\mu = \mu_1 \times \cdots \times \mu_n$, where μ_i are regular Borel measures on $\mathbb{R}$ for $i = 1 \cdots n$. Then there exists a constant $c > 0$ such that the inequality*

$$\Bigg(\int\limits_{\mathbb{R}^n}\Bigg|\int\limits_{(-\infty,x_1]}\cdots\int\limits_{(-\infty,x_n]} f(t_1,\cdots,t_n)d\mu(t_1,\cdots,t_n)\Bigg|^q d\nu(x_1,\cdots,x_n)\Bigg)^{\frac{1}{q}}$$

$$\le c\Bigg(\int\limits_{\mathbb{R}^n}|f(x_1,\cdots,x_n)|^p d\mu(x_1,\cdots,x_n)\Bigg)^{\frac{1}{p}} \tag{3.2.2}$$

holds for all $f \in L^p(\mathbb{R}^n,\mu)$, if and only if

$$\begin{aligned} B_1 := \sup_{a_1,\cdots,a_n\in\mathbb{R}} & \Bigg(\nu\Big([a_1,\infty)\times\cdots\times[a_n,\infty)\Big)\Bigg)^{\frac{1}{q}} \\ \times & \Bigg(\mu_1(-\infty,a_1]\cdots\mu_n(-\infty,a_n]\Bigg)^{\frac{1}{p'}} < \infty. \end{aligned}$$

Moreover, if c is the best constant in (3.2.2)*, then $c \approx B_1$.*

Proof. Necessity. Let the operator $\widetilde{\mathcal{H}}_2$ defined by (3.2.1) be bounded from $L^p(\mathbb{R}^2,\mu)$ to $L^q(\mathbb{R}^2,\nu)$ and let us take the test function $f_{ab}(x,y) = \chi_{(-\infty,a]\times(-\infty,b]}(x,y)$, $a, b \in \mathbb{R}$. Then

$$\|f_{ab}\|_{L^p(\mathbb{R}^2,\mu)} = (\mu_1(-\infty,a]\times\mu_2(-\infty,b])^{\frac{1}{p}} < \infty.$$

On the other hand,

$$\begin{aligned} \|\widetilde{\mathcal{H}}_2 f_{ab}\|_{L^q(\mathbb{R}^2,\nu)} &\ge \Bigg(\int\limits_{[a,\infty)}\int\limits_{[b,\infty)}\Bigg(\int\limits_{(-\infty,a]}\int\limits_{(-\infty,b]} d\mu(t,\tau)\Bigg)^q d\nu(x,y)\Bigg)^{\frac{1}{q}} \\ &= (\nu([a,\infty)\times[b,\infty)))^{\frac{1}{q}}\mu_1(-\infty,a]\mu_2(-\infty,b]. \end{aligned}$$

By the boundedness of $\widetilde{\mathcal{H}}_2$ we conclude that $B_1 < \infty$.

Sufficiency. Suppose that $f \ge 0$ and $\|f\|_{L^p(\mathbb{R}^2,\mu)} \le 1$. Define:

$x_k := \inf\{x \in \mathbb{R} : \int_{(-\infty,x]} d\mu_1 \geq 2^k\}$, $\quad y_j := \inf\{y \in \mathbb{R} : \int_{(-\infty,y]} d\mu_2 \geq 2^j\}$;
$K := \{k \in \mathbb{Z} : x_k < x_{k+1}\}$, $\quad J := \{j \in \mathbb{Z} : y_j < y_{j+1}\}$
and denote:
$E_k := (x_k, x_{k+1}]$, $\quad F_j := (y_j, y_{j+1}]$.

Then it is easy to see that, $\mathbb{R} = \bigcup_{k\in K}(x_k, x_{k+1}] = \bigcup_{j\in J}(y_j, y_{j+1}]$ and
$\mathbb{R}^2 = \bigcup_{k\in K, j\in J}(x_k, x_{k+1}] \times (y_j, y_{j+1}]$.

Now observe that the following estimates holds for $i = 1, 2$:

$$\mu_i(-\infty, x_k] = \lim_{x\to x_k^+} \mu_i(-\infty, x] \geq 2^k;$$

$$\mu_i(-\infty, x_k) = \lim_{x\to x_k^-} \mu_i(-\infty, x] \leq 2^k;$$

$$\mu_i(-\infty, x_k] \geq 4^{-1}\mu_i(-\infty, x_{k+2});$$

$$\mu_i[x_{k+1}, x_{k+2}] \geq \mu_i(-\infty, x_{k+1}) \geq \mu_i(-\infty, x_k].$$

Taking into account these estimates we find that,

$$\|\widetilde{\mathcal{H}}_2 f\|^q_{L^q(\mathbb{R}^2,\nu)} = \iint\limits_{\mathbb{R}^2} \Bigg(\int\limits_{(-\infty,x]} \int\limits_{(-\infty,y]} f(t,\tau)d\mu(t,\tau)\Bigg)^q d\nu(x,y)$$

$$= \sum_{k\in K, j\in J} \Bigg(\iint\limits_{E_k\times F_j} \int\limits_{(-\infty,x]} \int\limits_{(-\infty,y]} f(t,\tau)d\mu(t,\tau)\Bigg)^q d\nu(x,y)$$

$$\leq \sum_{k\in K, j\in J} \Bigg(\iint\limits_{E_k\times F_j} d\nu(x,y)\Bigg)\Bigg(\int\limits_{(-\infty,x_{k+1}]} \int\limits_{(-\infty,y_{j+1}]} f(t,\tau)d\mu(t,\tau)\Bigg)^q$$

$$\leq \sum_{k\in K, j\in J} \Bigg(\int\limits_{[x_k,\infty)} \int\limits_{[y_j,\infty)} d\nu(x,y)\Bigg)\Bigg(\int\limits_{(-\infty,x_{k+1}]} \int\limits_{(-\infty,y_{j+1}]} f(t,\tau)d\mu(t,\tau)\Bigg)^q.$$

$$\leq B_1^q \sum_{k\in K, j\in J} \Bigg(\mu_1(-\infty, x_k]\mu_2(-\infty, y_j])\Bigg)^{\frac{-q}{p'}} \Bigg(\iint\limits_{(-\infty,x_{k+1}](-\infty,y_{j+1}]} f(t,\tau)d\mu(t,\tau)\Bigg)^q$$

$$\leq B_1^q \sum_{k\in K, j\in J} \Big(\mu_1(-\infty, x_k]\mu_2(-\infty, y_j])\Big)^{\frac{q}{p}} \Big(\frac{1}{\mu_1(-\infty, x_k]\mu_2(-\infty, y_j]}$$
$$\times \int\limits_{(-\infty, x_{k+1}]}\int\limits_{(-\infty, y_{j+1}]} f(t,\tau)d\mu(t,\tau)\Big)^q$$

$$\leq cB_1^q \sum_{k\in K, j\in J} \Big(\mu_1[x_{k+1}, x_{k+2}]\mu_2[y_{j+1}, y_{j+2}])\Big)^{\frac{q}{p}} \Big(\frac{1}{\mu_1(-\infty, x_k]\mu_2(-\infty, y_j]}$$
$$\times \int\limits_{(-\infty, x_{k+1}]}\int\limits_{(-\infty, y_{j+1}]} f(t,\tau)d\mu(t,\tau)\Big)^q$$

$$\leq cB_1^q \sum_{k\in K, j\in J} \Big(\int\int\limits_{[x_{k+1}, x_{k+2}]\times[y_{j+1}, y_{j+2}]} \Big(\frac{1}{\mu_1(-\infty, x]\mu_2(-\infty, y]}$$
$$\times \int\limits_{(-\infty, x_{k+1}]}\int\limits_{(-\infty, y_{j+1}]} f(t,\tau)d\mu(t,\tau)\Big)^p d\mu(x,y)\Big)^{\frac{q}{p}}$$

$$\leq \quad cB_1^q \Big(\sum_{k\in K, j\in J} \int\int\limits_{[x_{k+1}, x_{k+2}]\times[y_{j+1}, y_{j+2}]} \Big(\frac{1}{\mu_1(-\infty, x]\mu_2(-\infty, y]}$$
$$\times \int\limits_{(-\infty, x_{k+1}]}\int\limits_{(-\infty, y_{j+1}]} f(t,\tau)d\mu(t,\tau)\Big)^p d\mu(x,y)\Big)^{\frac{q}{p}}$$

$$\leq \quad cB_1^q \Big(\int\int\limits_{\mathbb{R}^2} \Big(\frac{1}{\mu_1(-\infty, x]\mu_2(-\infty, y]} \int\limits_{(-\infty, x_{k+1}]}\int\limits_{(-\infty, y_{j+1}]} f(t,\tau)$$
$$\times \quad d\mu(t,\tau)\Big)^p d\mu(x,y)\Big)^{\frac{q}{p}} := cB_1^q S,$$

where,

$$S := \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big(\frac{1}{\mu_1(-\infty, x]\mu_2(-\infty, y]} \int\limits_{(-\infty, x_{k+1}]} \int\limits_{(-\infty, y_{j+1}]} f(t,\tau)d\mu(t,\tau)\Big)^p d\mu(x,y)\Big)^{\frac{q}{p}}.$$

By Proposition 3.1.1 we find that

$$S \le c\Bigg(\int_{-\infty}^{\infty} \frac{1}{\mu_1(-\infty, y]^p} \Bigg(\int_{-\infty}^{\infty} \Bigg(\int_{(-\infty,y]} f(x,\tau)d\mu_2(\tau)\Bigg)^p d\mu_1(x)\Bigg) d\mu_2(y)\Bigg)^{\frac{q}{p}}.$$

Using generalized Minkowski inequality and Proposition 3.1.1 again we have that

$$\begin{aligned} S &\le c\Bigg(\int_{-\infty}^{\infty} \frac{1}{\mu_1(-\infty, y]^p} \Bigg(\int_{(-\infty,y]} \Bigg(\int_{-\infty}^{\infty} f^p(x,\tau)d\mu_1(x)\Bigg)^{\frac{1}{p}} d\mu_2(\tau)\Bigg)^p d\mu_2(y)\Bigg)^{\frac{q}{p}} \\ &\le c\Bigg(\int_{-\infty}^{\infty} \Bigg(\int_{-\infty}^{\infty} f^p(x,y)d\mu_1(x)\Bigg)^{\frac{p}{p}} d\mu_2(y)\Bigg)^{\frac{q}{p}} \\ &\le c\Bigg(\iint_{\mathbb{R}^2} f^p(x,y)d\mu_1(x)d\mu_2(y)\Bigg)^{\frac{q}{p}} \le c\|f\|^q_{L^p(\mathbb{R}^2,\mu)}. \end{aligned}$$

Hence,

$$\|\widetilde{\mathcal{H}}_2 f\|_{L^q(\mathbb{R}^2,\nu)} \le c\|f\|_{L^p(\mathbb{R}^2,\mu)}.$$

■

Theorem 3.2.2. *Let $1 < p \le q < \infty$ and let μ, ν be regular Borel measures on $\mathbb{R}^n_+$. Suppose that $\mu = \mu_1 \times \cdots \times \mu_n$, where μ_i are regular Borel measures on $\mathbb{R}_+$ for $i = 1 \cdots n$. Then there exists a constant $c > 0$ such that the inequality*

$$\Bigg(\int_{\mathbb{R}^n_+} \Bigg| \int_{(0,x_1]} \cdots \int_{(0,x_n]} f(t_1, ..., t_n) d\mu(t_1, \cdots, t_n)\Bigg|^q d\nu(x_1, \cdots, x_n)\Bigg)^{\frac{1}{q}}$$

$$\le c\Bigg(\int_{\mathbb{R}^n_+} |f(x_1, \cdots, x_n)|^p d\mu(x_1, \cdots, x_n)\Bigg)^{\frac{1}{p}} \tag{3.2.3}$$

holds for all $f \in L^p(\mathbb{R}^n_+, \mu)$ if and only if

$$B_2 := \sup_{a_1,\cdots,a_n>0} \Bigg(\nu\Big([a_1, \infty) \times \cdots \times [a_n, \infty)\Big)\Bigg)^{\frac{1}{q}} \Bigg(\mu_1(0, a_1] \cdots \mu_n(0, a_n]\Bigg)^{\frac{1}{p'}} < \infty.$$

Moreover, if c is the best constant in (3.2.3), then $c \approx B_2$.

Proof. It follows just in the same way as Theorem 3.2.1 was; therefore we omit details. ■

For the discrete case we have the following statement:

Corollary 3.2.3. *Let* $1 < p \le q < \infty$, $w_{k_1,\cdots,k_1} > 0$, $v_{k_1,\ldots,k_n} \ge 0$ *for all* $k_1, \cdots, k_n \in \mathbb{Z}$, *and* $w_{k_1,\cdots,k_n} = w_{1,k_1} \cdots w_{n,k_n}$. *Then there exists a constant* $c > 0$ *such that*

$$\left(\sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_n=-\infty}^{\infty} \left| \sum_{k_1=-\infty}^{m_1} \cdots \sum_{k_n=-\infty}^{m_n} a_{k_1,\cdots,k_n} \right|^q v_{m_1,\cdots,m_n} \right)^{\frac{1}{q}} \le c \left(\sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_n=-\infty}^{\infty} \left| a_{m_1,\cdots,m_n} \right|^p w_{m_1,\cdots,m_n} \right)^{\frac{1}{p}} \quad (3.2.4)$$

holds for all sequence $\{a_{k_1,\cdots,k_n}\} \in l^p(\mathbb{Z}^n, w_{k_1,\cdots,k_n})$ *if and only if*

$$B_4 := \sup_{k_1,\cdots,k_n \in \mathbb{Z}} \left(\sum_{l_1=k_1}^{\infty} \cdots \sum_{l_n=k_n}^{\infty} v_{l_1,\cdots,l_n} \right)^{\frac{1}{q}} \times \left(\sum_{l_1=-\infty}^{k_1} w_{1,l_1}^{1-p'} \ldots \sum_{l_n=-\infty}^{k_n} w_{n,l_n}^{1-p'} \right)^{\frac{1}{p'}} < \infty.$$

Proof. Let $\delta_{k,j}$ denote the Dirac measure concentrated at $(k, j) \in \mathbb{Z} \times \mathbb{Z}$ and δ_i denote the Dirac measure concentrated $i \in \mathbb{Z}$. Considering the measures $\mu_1 = \sum_{k \in \mathbb{Z}} w_{1,k}^{1-p'} \delta_k$, $\mu_2 = \sum_{j \in \mathbb{Z}} w_{2,j}^{1-p'} \delta_j$ and $\nu = \sum_{k,j \in \mathbb{Z}} v_{k,j} \delta_{k,j}$ in Theorem 3.2.1, the inequality

$$\left(\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left| \sum_{k=-\infty}^{n} \sum_{j=-\infty}^{m} f(k,j) w_{1,k}^{1-p'} w_{2,j}^{1-p'} \right|^q v_{n,m} \right)^{\frac{1}{q}} \le c \left(\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left| f(n,m) w_{1,n}^{1-p'} w_{2,m}^{1-p'} \right|^p w_{n,m} \right)^{\frac{1}{p}}$$

holds for all $f \in L^p(\mathbb{Z}^2, \mu) = l^p(\mathbb{Z}^2, w_{n,m}^{1-p'})$, if and only if $B_4 < \infty$. Now letting $a_{k,j} = f(k,j) w_{k,j}^{1-p'}$ we have the required result. ■

Corollary 3.2.4. *Let* $1 < p \le q < \infty$, $w_{k_1,\ldots,k_1} > 0$, $v_{k_1,\cdots,k_n} \ge 0$ *for all* $k_1, \cdots, k_n \in \mathbb{N}$, *and* $w_{k_1,\cdots,k_n} = w_{1,k_1} \cdots w_{n,k_n}$. *Then there exists a constant* $c > 0$ *such that*

$$\left(\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \left| \sum_{k_1=1}^{m_1} \cdots \sum_{k_n=1}^{m_n} a_{k_1,\cdots,k_n} \right|^q v_{m_1,\cdots,m_n} \right)^{\frac{1}{q}} \le c \left(\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \left| a_{m_1,\cdots,m_n} \right|^p w_{m_1,\cdots,m_n} \right)^{\frac{1}{p}} \tag{3.2.5}$$

holds for all sequence $\{a_{k_1,\cdots,k_n}\} \in l^p(\mathbb{N}^n, w_{k_1,\cdots,k_n})$ *if and only if*

$$B_6 := \sup_{k_1,\ldots,k_n \in \mathbb{N}} \left(\sum_{l_1=k_1}^{\infty} \cdots \sum_{l_n=k_n}^{\infty} v_{l_1,\cdots,l_n} \right)^{\frac{1}{q}} \times \left(\sum_{l_1=1}^{k_1} w_{1,l_1}^{1-p'} \cdots \sum_{l_n=1}^{k_n} w_{n,l_n}^{1-p'} \right)^{\frac{1}{p'}} < \infty.$$

Proof. It is similar to that of Corollary 3.2.3 and is omitted. ■

Another characterization of the two-weight double discrete Hardy inequality was given in [69].

3.3 Boundedness of Positive Kernel Operators Defined with Respect to a Borel Measure

In this section, we give necessary and sufficient conditions governing $L^p(\mathbb{R}^n_+, \mu) \to L^q(\mathbb{R}^n_+, \nu)$ boundedness of the positive multiple kernel operator

$$(K_\mu f)(x_1 \cdots, x_n) = \int\limits_{(0,x_1]} \cdots \int\limits_{(0,x_n]} \left(\prod_{i=1}^{n} k_i(x_i, t_i) \right) f(t_1, \cdots, t_n) d\mu(t_1, \cdots, t_n), \; x_i > 0.$$

To formulate the main results for positive kernel operators K_μ we need some definitions.

Definition 3.3.1. A kernel k belongs to $V_\lambda(\mu)$, where μ is a Borel measure on $\mathbb{R}_+$ and $1<\lambda<\infty$, if there exists a positive constant d_2 such that for a.e. $x>0$, the inequality

$$\int\limits_{(\frac{x}{2},x]} k^{\lambda'}(x,y)d\mu(y) \leq d_2\mu(0,x]k^{\lambda'}\Big(x,\frac{x}{2}\Big)$$

is fulfilled, where $\lambda' = \lambda/(\lambda-1)$.

Lemma 3.3.1. *Let $1<p<\infty$, and $\frac{1}{p}<\alpha<1$. Then for any Borel measure μ on $\mathbb{R}_+$, there exists a positive constant c such that for all $x\in\mathbb{R}_+$ the following inequality holds:*

$$I(x) := \int\limits_{(\frac{x}{2},x]} (\mu(t,x])^{(\alpha-1)p'}d\mu(t) \leq c\Big(\mu(\frac{x}{2},x]\Big)^{(\alpha-1)p'+1}.$$

Proof. We have

$$\begin{aligned} I(x) &= \int\limits_0^\infty \mu\big(\{t\in(\frac{x}{2},x] : (\mu(t,x])^{(\alpha-1)p'} > \lambda\}\big)d\lambda \\ &= \int\limits_0^{A(x,p,\alpha)} \Big(\cdots\Big)\,d\lambda + \int\limits_{A(x,p,\alpha)}^\infty \Big(\cdots\Big)\,d\lambda := I_1(x)+I_2(x), \end{aligned}$$

where $A(x,p,\alpha) := \mu((\frac{x}{2},x])^{(\alpha-1)p'}$.

Observe that

$$I_1(x) \leq \Big(\mu(\frac{x}{2},x]\Big)A(x,p,\alpha) = \Big(\mu(\frac{x}{2},x]\Big)^{(\alpha-1)p'+1}.$$

Now we estimate $I_2(x)$. For this we show that

$$E_\lambda(x) := \mu\big(\{t\in(\frac{x}{2},x] : (\mu(t,x])^{(\alpha-1)p'} > \lambda\}\big) \leq \lambda^{\frac{1}{(\alpha-1)p'}}.$$

Indeed, let

$$t_0 := \inf\Big\{t : \mu\big(\{t\in(\frac{x}{2},x] : (\mu(t,x])^{(\alpha-1)p'} > \lambda\}\big)\Big\}.$$

It is easy to see that

$$\mu(t_0,x] \leq \lambda^{\frac{1}{(\alpha-1)p'}}.$$

Hence,

$$E_\lambda(x) \leq \mu(t_0,x] \leq \lambda^{\frac{1}{(\alpha-1)p'}}.$$

Using this estimate we find that

$$I_2(x) \le \int\limits_{A(x,p,\alpha)}^{\infty} \lambda^{\frac{1}{(\alpha-1)p'}} d\lambda \le c\Big(\mu(\frac{x}{2}, x]\Big)^{(\alpha-1)p'+1}.$$

■

Remark 7. If I is $\mathbb{R}$ or $\mathbb{R}_+$ in Definition 1.1.3, then we denote $V(I)$ by V.

Example 5. *Let $1 < p < \infty$ and let $k(x,t) = \mu(t,x]^{\alpha-1}$, where $\frac{1}{p} < \alpha \le 1$. Then $k \in V \cap V_p(\mu)$.*

Indeed, it is easy to check that $k \in V$. Further, the fact that $k \in V_p(\mu)$ follows from Lemma 3.3.1.

Remark 8. Examples of the appropriate kernel $k(x,y) = \big(\mu(t,x]\big)^{\alpha-1}$ are $k(x,y) = (x-y)^{\alpha-1}$, $\frac{1}{p} < \alpha \le 1$; $k(x,y) = (x^\sigma - y^\sigma)^{\alpha-1}$, $\frac{1}{p} < \alpha \le 1$ and $\sigma > 0$.

For other examples of the kernel k satisfying the condition $k \in V \cap V_p(dx)$ with respect to the Lebesgue measure dx we refer to [63] (see also [25], Ch.1).

Let us recall the definition of the doubling measure.

Definition 3.3.2. A measure μ defined on $\mathbb{R}_+$ satisfies the doubling condition ($\mu \in DC(\mathbb{R}_+)$) if there exists a constant $c > 0$ such that for all $a > 0$, $\mu(0, 2a] \le c\mu(0, a]$.

Definition 3.3.3. A measure μ defined on $\mathbb{R}_+$ satisfies the strong doubling condition ($\mu \in SDC(\mathbb{R}_+)$) if μ satisfies the doubling condition ($\mu \in DC(\mathbb{R}_+)$) and there exists a constant $c > 0$ such that for all $a > 0$, $\mu(0, a] \le c\mu\big(\frac{a}{2}, a]$.

To prove the next result we need the following Lemma.

Lemma 3.3.2. *Let $1 < p \le q < \infty$ also let μ be regular Borel measure on $\mathbb{R}_+$. Then there exists a constant $c > 0$ such that following inequality*

$$\Bigg(\int\limits_{\mathbb{R}_+} \Big(\int_{(0,x]} f(t) d\mu(t)\Big)^q \big(\mu(0,x]\big)^{\frac{-q}{p'}-1} d\mu(x)\Bigg)^{\frac{1}{q}} \le c\Bigg(\int\limits_{\mathbb{R}_+} (f(x))^p d\mu(x)\Bigg)^{\frac{1}{p}} \qquad (3.3.1)$$

holds for all non-negative $f \in L^p(\mathbb{R}_+, \mu)$.

Proof. By Theorem N, (3.3.1) holds if

$$\sup_{y>0}\Bigg(\int\limits_{[y,\infty)} (\mu(0,t])^{\frac{-q}{p'}-1} d\mu(t)\Bigg)^{\frac{1}{q}} \mu(0,y]^{\frac{1}{p'}} < \infty.$$

Using Proposition I (for $\frac{q}{p'}+1>1$) we find that

$$\Bigg(\int\limits_{[y,\infty)} (\mu(0,t])^{\frac{-q}{p'}-1} d\mu(t)\Bigg)^{\frac{1}{q}} \mu(0,y]^{\frac{1}{p'}} \le c\mu(0,y]^{-\frac{1}{p'}}\mu(0,y]^{\frac{1}{p'}} = c.$$

Hence, (3.3.1) holds. ∎

Theorem 3.3.3. *Let $1<p\le q<\infty$. Suppose that ν and μ are regular Borel measures on $\mathbb{R}^n_+$. Suppose also that $\mu=\mu_1\times\cdots\times\mu_n$, where μ_i are regular Borel measures on $\mathbb{R}_+$ such that $\mu_i\in SDC(\mathbb{R}_+)$. Assume that the kernels k_i belong to $V\cap V_p(\mu_i)$ for $i=1\cdots n$. Then the operator K_μ is bounded from $L^p(\mathbb{R}^n_+,\mu)$ to $L^q(\mathbb{R}^n_+,\nu)$ if and only if*

$$\begin{aligned}\tilde{B}_{\mu,\nu} := \sup_{a_1,\cdots,a_n>0} &\Bigg(\int\limits_{[a_1,\infty)}\cdots\int\limits_{[a_n,\infty)} \prod_{i=1}^n k_i^q\Big(x_i,\frac{x_i}{2}\Big) d\nu(x_1,\cdots,x_n)\Bigg)^{\frac{1}{q}}\\ \times &\Big(\mu_1(0,a_1]\cdots\mu_2(0,a_n]\Big)^{\frac{1}{p'}} < \infty.\end{aligned}$$

Proof. Sufficiency. Let $f\ge 0$. Represent $K_\mu f(x,y)$ as a sum of four two-dimensional integrals:

$$\begin{aligned}K_\mu f(x,y) &= \int\limits_{(0,\frac{x}{2}]}\int\limits_{(0,\frac{y}{2}]}(\cdots)d\mu(t,\tau) + \int\limits_{(0,\frac{x}{2}]}\int\limits_{(\frac{y}{2},y]}(\cdots)d\mu(t,\tau)\\ &+ \int\limits_{(\frac{x}{2},x]}\int\limits_{(0,\frac{y}{2}]}(\cdots)d\mu(t,\tau) + \int\limits_{(\frac{x}{2},x]}\int\limits_{(\frac{y}{2},y]}(\cdots)d\mu(t,\tau)\\ &=: K_\mu^{(1)}f(x,y)+K_\mu^{(2)}f(x,y)+K_\mu^{(3)}f(x,y)+K_\mu^{(4)}f(x,y).\end{aligned}$$

For $t\le\frac{x}{2}$, the condition $k_1,k_2\in V$ gives $k_i(x,t)\le d_ik_i(x,\frac{x}{2})$, $i=1,2$. Using Theorem 3.2.2 we have

$$\|K_\mu^{(1)}f\|^q_{L^q(\mathbb{R}^2_+,\nu)} \le c\tilde{B}^q_{\mu,\nu}\|f\|^q_{L^p(\mathbb{R}^2_+,\mu)}.$$

Applying Hölder's inequality, the conditions $k_i \in V_p(\mu_i)$, $\mu_i \in SDC(\mathbb{R}_+)$, $i = 1, 2$, we have that

$$\begin{aligned}
\|K^{(4)}_\mu f\|^q_{L^q(\mathbb{R}^2_+,\nu)} &\le \int\int_{\mathbb{R}^2_+} \Bigg(\int_{(\frac{x}{2},x]} \int_{(\frac{y}{2},y]} \big(f(t,\tau)\big)^p d\mu(t,\tau) \Bigg)^{\frac{q}{p}} \\
&\times \Bigg(\int_{(\frac{x}{2},x]} k_1^{p'}(x,t) d\mu_1(t) \Bigg)^{\frac{q}{p'}} \Bigg(\int_{(\frac{y}{2},y]} k_2^{p'}(y,\tau) d\mu_2(\tau) \Bigg)^{\frac{q}{p'}} d\nu(x,y) \\
&\le c \int\int_{\mathbb{R}^2_+} \Bigg(\int_{(\frac{x}{2},x]} \int_{(\frac{y}{2},y]} \big(f(t,\tau)\big)^p d\mu(t,\tau) \Bigg)^{\frac{q}{p}} \\
&\times \Bigg(\mu_1(0,x] k_1^{p'}(x,\frac{x}{2}) \mu_2(0,y] k_2^{p'}(y,\frac{y}{2}) \Bigg)^{\frac{q}{p'}} d\nu(x,y) \\
&\le c \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \Bigg(\int_{(2^k,2^{k+1}]} \int_{(2^j,2^{j+1}]} \Bigg(\mu_1(0,x]\mu_2(0,y] \Bigg)^{\frac{q}{p'}} k_1^q(x,\frac{x}{2}) k_2^q(y,\frac{y}{2}) \Bigg) \\
&\times \Bigg(\int_{(\frac{x}{2},x]} \int_{(\frac{y}{2},y]} \big(f(t,\tau)\big)^p d\mu(t,\tau) \Bigg)^{\frac{q}{p}} d\nu(x,y) \\
&\le c\tilde{B}^q_{\mu,\nu} \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \Bigg(\int_{(2^{k-1},2^{k+1}]} \int_{(2^{j-1},2^{j+1}]} \big(f(t,\tau)\big)^p d\mu(t,\tau) \Bigg)^{\frac{q}{p}} \\
&\le c\tilde{B}^q_{\mu.\nu} \|f\|^q_{L^p(\mathbb{R}^2_+,\mu)}.
\end{aligned}$$

Now we estimate $\|K^{(2)}_\mu f\|^q_{L^q(\mathbb{R}^2_+,\nu)}$. Using Hölder's inequality for the integral $\int_{(\frac{y}{2},y]}$, the conditions $k_1 \in V$, $k_2 \in V_p(\mu_2)$, $\mu_2 \in SDC(\mathbb{R})$ and Lemma 3.3.2 we have that

$$\|K^{(2)}_\mu f\|^q_{L^q(\mathbb{R}^2_+,\nu)} \le c \int\int_{\mathbb{R}^2_+} k_1^q(x,\frac{x}{2}) \Bigg(\int_{(0,\frac{x}{2}]} \int_{(\frac{y}{2},y]} f(t,\tau) k_2(y,\tau) d\mu_1(t) d\mu_2(\tau) \Bigg)^q d\nu(x,y)$$

$$\leq \quad c \iint\limits_{\mathbb{R}^2_+} k_1^q(x, \frac{x}{2}) \Big(\int\limits_{(0,\frac{x}{2}]} \Big(\int\limits_{(\frac{y}{2},y]} (f(t,\tau))^p d\mu_2(\tau) \Big)^{\frac{1}{p}}$$

$$\times \quad \Big(\int\limits_{(\frac{y}{2},y]} k_2^{p'}(y,\tau) d\mu_2(\tau) \Big)^{\frac{1}{p'}} d\mu_1(t) \Big)^q d\nu(x,y)$$

$$\leq \quad c \iint\limits_{\mathbb{R}^2_+} k_1^q(x, \frac{x}{2}) k_2^q(y, \frac{y}{2}) \mu_2^{\frac{q}{p'}}(0,y]$$

$$\times \quad \Big(\int\limits_{(0,\frac{x}{2}]} \Big(\int\limits_{(\frac{y}{2},y]} (f(t,\tau))^p d\mu_2(\tau) \Big)^{\frac{1}{p}} d\mu_1(t) \Big)^q d\nu(x,y)$$

$$\leq \quad c \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \Big(\int\limits_{(2^k,2^{k+1}]} \int\limits_{(2^j,2^{j+1}]} k_1^q(x, \frac{x}{2}) k_2^q(y, \frac{y}{2}) \mu_2^{\frac{q}{p'}}(0,y] d\nu(x,y) \Big)$$

$$\times \quad \Big(\int\limits_{(0,2^k]} \Big(\int\limits_{(2^{j-1},2^{j+1}]} (f(t,\tau))^p d\mu_2(\tau) \Big)^{\frac{1}{p}} d\mu_1(t) \Big)^q$$

$$\leq \quad c \tilde{B}^q_{\mu,\nu} \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} (\mu_1(0,2^k])^{\frac{-q}{p'}}$$

$$\times \quad \Big(\int\limits_{(0,2^k]} \Big(\int\limits_{(2^{kj},2^{j+1}]} (f(t,\tau))^p d\mu_2(\tau) \Big)^{\frac{1}{p}} d\mu_1(t) \Big)^q =: A.$$

Observe now that the condition $\mu_1 \in SDC(\mathbb{R}_+)$ implies that

$$\mu_1(0,2^{k+1}] \leq c \ \min\{\mu_1(0,2^k], \mu_1(2^k,2^{k+1}]\}$$

where the positive constant c is independent of x. Hence,

$$
\begin{aligned}
A &\le c\tilde{B}^q_{\mu,\nu} \sum_{k\in\mathbb{Z}} \int\limits_{(2^k,2^{k+1}]} (\mu_1(0,2^{k+1}])^{\frac{-q}{p'}-1} \\
&\times \Bigg(\int\limits_{(0,x]} \Bigg(\int\limits_{(2^{k-1},2^{k+1}]} (f(t,\tau))^p d\mu_2(\tau) \Bigg)^{\frac{1}{p}} d\mu_1(t) \Bigg)^q d\mu_1(x) \\
&\le c\tilde{B}^q_{\mu,\nu} \sum_{k\in\mathbb{Z}} \Bigg(\int\limits_{\mathbb{R}_+} \int\limits_{(2^{k-1},2^{k+1}]} (f(x,\tau))^p d\mu_2(\tau) d\mu_1(x) \Bigg)^{\frac{q}{p}} \le c\tilde{B}^q_{\mu,\nu} \|f\|^q_{L^p(\mathbb{R}^2_+,\mu)}.
\end{aligned}
$$

Similarly, the conditions $\mu_1 \in DC(\mathbb{R}_+)$, $\mu_2 \in SDC(\mathbb{R}_+)$ yield that

$$\|K^{(3)}_\mu f\|^q_{L^q(\mathbb{R}^2_+,\nu)} \le c\tilde{B}^q_{\mu,\nu} \|f\|^q_{L^p(\mathbb{R}^2_+,\mu)}$$

Taking into account the estimates for $K^{(j)}_\mu f$, $j = 1,2,3,4$, we conclude that

$$\|K_\mu f\|^q_{L^q(\mathbb{R}^2_+,\nu)} \le c\tilde{B}^q_{\mu,\nu} \|f\|^q_{L^p(\mathbb{R}^2_+,\mu)}.$$

Necessity. Taking the test function $f_{a,b}(x,y) = \chi_{(0,a]}(x)\chi_{(0,b]}(y)$, $a,b > 0$, we find that $\|f_{a,b}\|_{L^p(\mathbb{R}^2_+,\mu)} = \Big(\mu_1(0,a]\mu_2(0,b]\Big)^{\frac{1}{p}}$. On the other hand, by the conditions $k_i \in V$, $i = 1,2$, we have

$$
\begin{aligned}
\|K_\mu f_{a,b}\|^q_{L^q(\mathbb{R}^2_+,\nu)} &\ge \Bigg(\int\limits_{[a,\infty)} \int\limits_{[b,\infty)} \Bigg(\int\limits_{(\frac{x}{2},x]} \int\limits_{(\frac{y}{2},y]} k_1(x,t)k_2(y,\tau)d\mu_1(t)d\mu_2(\tau) \Bigg)^q d\nu(x,y) \Bigg)^{\frac{1}{q}} \\
&\ge c\Bigg(\int\limits_{[a,\infty)} \int\limits_{[b,\infty)} k_1^q(x,\frac{x}{2})k_2^q(y,\frac{y}{2})(\mu_1(\frac{x}{2},x]\mu_2(\frac{y}{2},y])^q d\nu(x,y) \Bigg)^{\frac{1}{q}}.
\end{aligned}
$$

Observe that if $x \ge a$ and $\mu_1 \in SDC(\mathbb{R}_+)$, then

$$\mu_1(\frac{x}{2},x] \ge c\mu_1(0,x] \ge c\mu_1(0,a].$$

Similarly, we have that

$$\mu_2(\frac{y}{2},y] \ge c\mu_2(0,b]$$

for $y \ge b$. Using these estimates in the inequality above we conclude that,

$$\|K_\mu f_{a,b}\|^q_{L^q(\mathbb{R}^2_+,\nu)} \ge c\Bigg(\int\limits_{[a,\infty)} \int\limits_{[b,\infty)} k_1^q(x,\frac{x}{2})k_2^q(y,\frac{y}{2})d\nu(x,y) \Bigg)^{\frac{1}{q}} (\mu_1(0,a]\mu_2(0,b])$$

holds for a positive constant c independent of a and b. By the boundedness of K_μ we finally have that $\tilde{B}_{\mu,\nu} < \infty$.

■

Let us denote the fractional integral operator with product kernels

$$(\mathcal{R}^{\mu}_{\alpha_1,\cdots\alpha_n} f)(x_1,\cdots,x_n) = \int\limits_{(0,x_1]\times\cdots\times(0,x_n]}\cdots\int \frac{f(t_1,\cdots,t_n)}{\prod_{i=1}^{n}(\mu_i(t_i,x_i])^{1-\alpha_i}} d\mu(t_1,\cdots,t_n)$$

and strong one-sided fractional maximal operator defined with respect to measure μ:

$$(\mathcal{M}^{\mu}_{\alpha_1,\cdots\alpha_n} f)(x_1,\cdots,x_n) = \sup_{\substack{0<r_i\le xi\\1\le i\le n}} \frac{1}{\prod_{i=1}^{n}(\mu_i(x_i-r_i,x_i])^{1-\alpha_i}} \times \int\limits_{(x_1-r_1,x_1]\times\cdots\times(x_n-r_n,x_n]}\cdots\int |f(t_1,\cdots,t_n)| d\mu(t_1,\cdots,t_n),$$

where $0<\alpha_i<1$, $i=1\cdots,n$. Formal dual of $\mathcal{R}^{\mu}_{\alpha_1,\cdots\alpha_n}$ is given by

$$(\mathcal{W}^{\mu}_{\alpha_1,\cdots\alpha_n} f)(x_1,\cdots,x_n) = \int\limits_{[x_1,\infty)\times\cdots\times[x_n,\infty)}\cdots\int \frac{f(t_1,\cdots,t_n)}{\prod_{i=1}^{n}(\mu_i(x_i,t_i])^{1-\alpha_i}} d\mu(t_1,\cdots,t_n).$$

Theorem 3.3.3 and Example 5 immediately imply the next statement:

Corollary 3.3.4. *Let $1<p\le q<\infty$ and $\frac{1}{p}<\alpha_i<1$ for $i=1\cdots n$. Suppose that ν and μ are regular Borel measures on $\mathbb{R}^n_+$. Suppose also that $\mu=\mu_1\times\cdots\mu_n$, where μ_i are regular Borel measures on $\mathbb{R}_+$ such that $\mu_i\in SDC(\mathbb{R}_+)$. Then the operator $\mathcal{R}^{\mu}_{\alpha_1,\cdots,\alpha_n}$ is bounded from $L^p(\mathbb{R}^n_+,\mu)$ to $L^q(\mathbb{R}^n_+,\nu)$ if and only if*

$$\bar{B}_{\mu,\nu} := \sup_{a_1,\cdots,a_n>0} \Bigg(\int\limits_{[a_1,\infty)}\cdots\int\limits_{[a_n,\infty)} \prod_{i=1}^{n}(\mu_i(\frac{x_i}{2},x_i])^{(\alpha_i-1)q} d\nu(x_1,\cdots,x_n)\Bigg)^{\frac{1}{q}} \times\Bigg(\prod_{i=1}^{n}\mu_i(0,a_i]\Bigg)^{\frac{1}{p'}} < \infty. \tag{3.3.2}$$

Corollary 3.3.5. *Let* $1 < p \le q < \infty$ *and* $\frac{1}{p} < \alpha_i < 1$ *for* $i = 1 \cdots n$. *Suppose that* ν *and* μ *are regular Borel measures on* $\mathbb{R}^n_+$. *Suppose also that* $\mu = \mu_1 \times \cdots \mu_n$, *where* μ_i *are regular Borel measures on* $\mathbb{R}_+$ *such that* $\mu_i \in SDC(\mathbb{R}_+)$. *Then the operator* $\mathcal{M}^{\mu}_{\alpha_1,\cdots,\alpha_n}$ *is bounded from* $L^p(\mathbb{R}^n_+,\mu)$ *to* $L^q(\mathbb{R}^n_+,\nu)$ *if and only if* (3.3.2) *holds.*

Finally we point out that Corollaries 3.3.4 and 3.3.5 are new even for $n = 1$.

Proof. of Corollary 3.3.5. Sufficiency is the consequence of Corollary 3.3.4 and the estimate $\mathcal{R}^{\mu}_{\alpha_1,\alpha_2} f \ge \mathcal{M}^{\mu}_{\alpha_1,\alpha_2} f$ where $f \ge 0$.

Necessity follows by taking the test function $f_{a,b}(x_1,x_2) = \chi_{(0,a]}(x_1)\chi_{(0,b]}(x_2)$, $a,b > 0$, in the two-weight inequality. Observe that for $a, b > 0$,

$$\begin{aligned} \|\mathcal{M}^{\mu}_{\alpha_1,\alpha_2} f_{a,b}\|_{L^q_{(\mathbb{R}^2_+,\nu)}} &\ge \mu_1(0,a]\mu_2(0,b]\Big(\int\limits_{[a,\infty)}\int\limits_{[b,\infty)}\prod_{i=1}^{2}\mu_i(0,x_i])^{(\alpha_i-1)q}d\nu(x_1,x_2)\Big)^{\frac{1}{q}} \\ &\ge c\mu_1(0,a]\mu_2(0,b]\Big(\int\limits_{[a,\infty)}\int\limits_{[b,\infty)}\prod_{i=1}^{2}\mu_i(\frac{x_i}{2},x_i])^{(\alpha_i-1)q}d\nu(x_1,x_2)\Big)^{\frac{1}{q}}, \end{aligned}$$

where we applied the condition $\mu_i \in SDC(\mathbb{R}_+)$, $i = 1, 2$. ∎

3.4 Fefferman-Stein Type Inequality for the Measured Multiple Riemann-Liouville Transform

In this section, we derive the Fefferman-Stein type inequality for the multiple Riemann-Liouville transform $\mathcal{R}^{\mu}_{\alpha_1,\cdots,\alpha_n}$, where μ is a product measure. We start this section by the following lemma.

Lemma 3.4.1. *Let* $0 < \alpha < 1$ *and* μ *be a regular Borel measure on* $\mathbb{R}_+$. *Then there exists a positive constant* c *such that for all* $x \in \mathbb{R}_+$ *the following inequality holds:*

$$J(x) := \int\limits_{(0,x]} (\mu(t,x])^{\alpha-1} d\mu(t) \le c\big(\mu(0,x]\big)^{\alpha}.$$

Proof. We have

$$
\begin{aligned}
J(x) &= \int_0^\infty \mu\Big(\{t \in \big(0,x] : (\mu(t,x])^{\alpha-1} > \lambda\}\Big) d\lambda \\
&= \int_0^{A(x,\alpha)} \Big(\cdots\Big)\, d\lambda + \int_{A(x,\alpha)}^\infty \Big(\cdots\Big)\, d\lambda := J_1(x) + J_2(x),
\end{aligned}
$$

where $A(x,\alpha) := \mu((0,x])^{\alpha-1}$.

Observe that

$$J_1(x) \leq \Big(\mu(0,x]\Big) A(x,\alpha) = \Big(\mu(0,x]\Big)^\alpha.$$

Now we estimate $J_2(x)$. For this we show that the inequality

$$E_\lambda(x) := \mu\Big(\{t \in \big(0,x] : (\mu(t,x])^{\alpha-1} > \lambda\}\Big) \leq \lambda^{\frac{1}{\alpha-1}}.$$

holds. Indeed, let

$$t_0 := \inf\Big\{t : \mu\Big(\{t \in \big(0,x] : (\mu(t,x])^{\alpha-1} > \lambda\}\Big)\Big\}.$$

It is easy to see that

$$\mu(t_0,x] \leq \lambda^{\frac{1}{\alpha-1}}.$$

Hence,

$$E_\lambda(x) \leq \mu(t_0,x] \leq \lambda^{\frac{1}{\alpha-1}}.$$

Using this estimate we find that

$$J_2(x) \leq \int_{A(x,\alpha)}^\infty \lambda^{\frac{1}{\alpha-1}} d\lambda \leq c\Big(\mu(0,x]\Big)^\alpha.$$

■

Theorem 3.4.2. *Let $1 < p < \infty$, $0 < \alpha_i < 1$ and μ be a measure on $\mathbb{R}^n_+$ such that $\mu = \mu_1 \times \cdots \times \mu_n$ where μ_i are Borel measure on $\mathbb{R}_+$ for $i = 1 \cdots n$. We set*

$$
\begin{aligned}
d\nu(x_1,\cdots,x_n) &= v(x_1,\cdots,x_n) d\mu(x_1,\cdots,x_n) \\
d\nu_1(x_1,\cdots,x_n) &= \big(\mathcal{W}^{\mu_1\times\cdots\times\mu_n}_{\alpha_1,\cdots,\alpha_n} v\big)(x_1,\cdots,x_n) d\mu(x_1,\cdots,x_n)
\end{aligned}
$$

where $x_i > 0$ and v is a non-negative $\mu-$measurable function on $\mathbb{R}^n_+$. Then there exists a positive constant c such that the following inequality holds.

$$\|(\mu_1(0,x])^{-\alpha_1} \times \cdots \times (\mu_n(0,x])^{-\alpha_n} \mathcal{R}^{\mu_1\times\cdots\times\mu_n}_{\alpha_1,\cdots,\alpha_n} f\|_{L^p(\mathbb{R}^n_+,\nu)} \leq c\|f\|_{L^p(\mathbb{R}^n_+,\nu_1)}.$$

Proof. Let $\|g\|_{L^{p'}(\mathbb{R}^2_+,\overline{\nu})} \leq 1$, where $d\overline{\nu}(x,y) = (\mu(0,x])^{\alpha}(\mu(0,y])^{\beta} d\nu(x,y)$.

Using Hölder's inequality twice, Fubini theorem and Lemma 3.4.1 we have that

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty \big(\mathcal{R}^{\mu_1\times\mu_2}_{\alpha,\beta} f\big)(x,y) g(x,y) v(x,y) d\mu_1(x) d\mu_2(y) \\
=& \int_0^\infty\int_0^\infty f(x,y)\big(\mathcal{W}^{\nu}_{\alpha,\beta} g\big)(x,y) d\mu_1(x) d\mu_2(y) \\
\leq& \int_0^\infty\int_0^\infty f(x,y)\Big(\int_{[x,\infty)}\int_{[y,\infty)} \frac{1}{(\mu_1(x,t])^{1-\alpha}(\mu_2(y,\tau])^{1-\beta}} d\nu(t,\tau)\Big)^{\frac{1}{p}} \\
\times& \Big(\int_{[x,\infty)}\int_{[y,\infty)} \frac{g^{p'}(t,\tau) d\nu(t,\tau)}{(\mu_1(x,t])^{1-\alpha}(\mu_2(y,\tau])^{1-\beta}}\Big)^{\frac{1}{p'}} d\mu_1(x) d\mu_2(y)
\end{aligned}
$$

$$
\begin{aligned}
\leq& \Big(\int_0^\infty\int_0^\infty f^p(x,y)\Big(\int_{[x,\infty)}\int_{[y,\infty)} \frac{d\nu(t,\tau)}{(\mu_1(x,t])^{1-\alpha}(\mu_2(y,\tau])^{1-\beta}}\Big) d\mu_1(x) d\mu_2(y)\Big)^{\frac{1}{p}} \\
\times& \Big(\int_0^\infty\int_0^\infty \Big(\int_{[x,\infty)}\int_{[y,\infty)} \frac{g^{p'}(t,\tau) d\nu(t,\tau)}{(\mu_1(x,t])^{1-\alpha}(\mu_2(y,\tau])^{1-\beta}}\Big) d\mu_1(x) d\mu_2(y)\Big)^{\frac{1}{p'}}
\end{aligned}
$$

$$
\begin{aligned}
=& \Big(\int_0^\infty\int_0^\infty f^p(x,y)\big(\mathcal{W}^{\nu}_{\alpha,\beta} 1\big) d\mu_1(x) d\mu_2(y)\Big)^{\frac{1}{p}} \\
\times& \Big(\int_0^\infty\int_0^\infty g^{p'}(t,\tau)\Big(\int_{(0,t]}\int_{(0,\tau]} \frac{d\mu_1(x) d\mu_2(y)}{(\mu_1(t,x])^{1-\alpha}(\mu_2(\tau,y])^{1-\beta}}\Big) d\nu(t,\tau)\Big)^{\frac{1}{p'}} \\
\leq& c_{\alpha,\beta}\|f\|_{L^p(\mathbb{R}^2_+,\nu_1)}\Big(\int_0^\infty\int_0^\infty g^{p'}(t,\tau)(\mu_1(0,t])^{\alpha}(\mu_2(0,\tau])^{\beta} d\nu(t,\tau)\Big)^{\frac{1}{p'}} \\
=& c_{\alpha,\beta}\|f\|_{L^p(\mathbb{R}^2_+,\nu_1)}\|g\|_{L^{p'}(\mathbb{R}^2_+,\overline{\nu})} \\
\leq& c_{\alpha,\beta}\|f\|_{L^p(\mathbb{R}^2_+,\nu_1)}.
\end{aligned}
$$

Taking supremum over all g satisfying $\|g\|_{L^{p'}(\mathbb{R}^2_+,\overline{\nu})} \leq 1$ we have the desired result. ∎

Corollary 3.4.3. *Let* $1 < p < \infty$, $0 < \alpha_i < 1$ *for* $i = 1 \cdots n$, *and let* v *be a non-negative Lebesgue measurable function on* $\mathbb{R}^n_+$. *Then there exists a positive constant* c *such that the following inequality holds.*

$$\|x_1^{-\alpha_1} \cdots x_n^{-\alpha_n} R_{\alpha_1,\cdots,\alpha_n} f\|_{L^p(\mathbb{R}^n_+, vdx)} \leq c \|f\|_{L^p(\mathbb{R}^n_+, W_{\alpha_1,\cdots,\alpha_n} vdx)},$$

where $R_{\alpha_1,\cdots,\alpha_n}$ *and* $W_{\alpha_1,\cdots,\alpha_n}$ *denote the operators* $\mathcal{R}^\mu_{\alpha_1,\cdots,\alpha_n}$ *and* $\mathcal{W}^\mu_{\alpha_1,\cdots,\alpha_n}$ *respectively in the case when* μ *is an* n*-dimensional Lebesgue measure on* $\mathbb{R}^n_+$.

Bibliography

[1] R. Aboulaich, D. Meskine and A. Souissi, New diffusion models in image processing, *Comput. Math. Appl.* **56**, no. 4, 874–882 (2008).

[2] D. Adams, A trace inequality for generalized potentials, *Studia Math.* **48**, 99–105 (1973).

[3] A. Almeida and S. Samko, Characterization of Riesz and Bessel potentials on variable Lebesgue spaces, *J. Funct. Spaces Appl.* **4**, no. 2, 113–144 (2006).

[4] U. Ashraf, V. Kokilashvili and A. Meskhi, Weight characterization of the trace inequality for the generalized Riemann-Liouville transform in $L^{p(x)}$ spaces, *Math. Inequal. Appl.* **13**, no.1, 63–81 (2010).

[5] G. Bennett, Some elementary inequalities III, *Quart. J. Math. Oxford.* **42**, no. 2, 149–174 (1991).

[6] G. Bennett and R. Sharply, *Interpolation of operators*, Pure and Appl. Math. **129**, Academic press, (1988).

[7] J. Bradley, Hardy inequality with mixed norms, *Canad. Math. Bull.* **21**, 405–408 (1978).

[8] M. I. Aguilar Cañestro and P. Ortega Salvador, Boundedness of positive operators on weighted amalgams, *J. Inequal. Appl.* **2011**, 2011:13, 12 pp (2011).

[9] M. I. Aguilar Cañestro and P. Ortega Salvador, Boundedness of generalized Hardy operators on weighted amalgam spaces, *Math. Inequal. Appl.* **13**, no.2, 305–318 (2010).

[10] C. Capone, D. Cruz-Uribe and A. Fiorenza, The fractional maximal operator on variable L^p spaces, *Revista Mat. Iberoamericana.* **23**, no. 3, 743–770 (2007).

[11] C. Carton-Lebrun, H. P. Heinig and S. C. Hofmann, Integral operators on weighted amalgams, *Studia Math.* **109**, no. 2, 133–175 (1994).

[12] D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue spaces*, Foundations and Harmonic Analysis, Applied and Numerical Harmonic Analysi, Birkhauser, Basel, (2013).

[13] D. Cruz-Uribe and F. Mamedov, On a general weighted Hardy type inequality in the variable exponent Lebesgue spaces, *Rev. Mat. Complut.* **25**, no. 2, 335–367 (2012).

[14] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, Weighted norm inequalities for the maximal operator on variable Lebesgue spaces, *J. Math. Anal. Appl.* **394**, 744–760 (2012).

[15] D. Cruz-Uribe, L. Diening and P. Hästö, The maximal operator on weighted variable Lebesgue spaces, *Frac. Calc. Appl. Anal.* **14**, no. 3, 361–374 (2011).

[16] D. Cruz-Uribe, A. Fiorenza, J. M. Martell and C. Pérez, The boundedness of classical operators on variable L^p spaces, *Ann. Acad. Sci. Fenn. Math.* **31**, 239–264 (2006).

[17] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, The maximal function on variable L^p spaces, *Ann. Acad. Sci. Fenn. Math.* **28**, no. 1, 223–238 (2003).

[18] M. Day, Some more uniformly convex spaces, *Bull. Amer. Math. Soc.* **47**, 504–507 (1941).

[19] L. Diening, P. Harjulehto, P. Hästö and M. Ružička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol.2017, Springer-Verlag, Berlin, (2011).

[20] L. Diening, Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, *Math. Inequal. Appl.* **7**, no. 2, 245–253 (2004).

[21] L. Diening, Riesz potentials and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$, *Math. Nachr.* **268**, 31–34 (2004).

[22] L. Diening and P. Hästö, Muckenhoupt weights in variable exponent spaces, *Preprint, Available at* http://www.helsinki.fi/ pharjule/varsob/publications.shtml .

[23] D. E. Edmunds, A. Fiorenza and A. Meskhi, On the measure of non-compactness for some classical operators, *Acta Math. Sinica.* **22**, no. 6, 1847-1862 (2006).

[24] D. E. Edmunds, V. Kokilashvili and A. Meskhi, On the boundedness and compactness of the weighted Hardy operators in $L^{p(x)}$ spaces, *Georgian Math. J.* **12**, no. 1, 27–44 (2005).

[25] D. E. Edmunds, V. Kokilashvili and A. Meskhi, *Bounded and compact integral operators*, Kluwer Academic Publishers, Dordrecht, (2002).

[26] D. E. Edmunds and A. Meskhi, Potential-type operators in $L^{p(x)}$ spaces, *Z. Anal. Anwend.* **21**, 681-690 (2002).

[27] J. F. Fournier and J. Stewart, Amalgams of L^p and l^q, *Bull. Amer. Math. Soc. (N.S.).* **13**, no. 1, 1-21 (1985).

[28] T. Futamura and Y. Mizuta, Continuity properties of Riesz potentials for functions in $L^{p(\cdot)}$ of variable exponent, *Math. Inequal. Appl.* **8**, no. 4, 619–631 (2005).

[29] P. Harjulehto, P. Hästö, V. Latvala and O. Toivanen, Critical variable exponent functionals in image restoration, *Appl. Math. Lett.* **26** , no. 1, 56–60, (2013).

[30] P. Harjulehto, P. Hästö and M. Pere, Variable exponent Lebesgue spaces on metric spaces the Hardy-Littlewood maximal operator, *Real Anal. Exchange.* **30**, 87–103 (2004-2005).

[31] H. P. Heinig and A. Kufner, Weighted Friedrichs inequalities in amalgams, *Czechoslovak Math. J.* **43(118)**, no. 2, 285–308 (1993).

[32] F. Holland, Harmonic analysis on amalgams of L^p and l^q, *J. London Math. Soc.* **(2)** 10, 295–305 (1975).

[33] A. Jakimovski and D. Russell, Interpolation by functions with m–th derivative in pre–assigned spaces In: Cheney, E.W. (ed) Approximation Theory III , Conference Proc. Texas, pp. 531-536. Academic Press, New York. (1980).

[34] L. Kantorovich and G. Akilov, *Functional Analysis*, Pergamon, Oxford (1982).

[35] G. Köthe, *Topological vector spaces*, Springer-Verlag, (1969).

[36] V. Kokilashvili, A. Meskhi and M. A. Zaighum, Weighted kernel operators in variable exponent amalgam spaces, *J. Inequal. Appl.*, **2013**: 173, 27 pp, (2013).

[37] V. Kokilashvili, A. Meskhi and M. A. Zaighum, Positive kernel operators in $L^{p(x)}$ spaces, *Positivity.*, **17**, no. 4, 1123–1140 (2013).

[38] V. Kokilashvili, A. Meskhi and M. Sarwar, Potential operators in variable exponent Lebesgue spaces Two-weight estimates, *J. Inequal. Appl.* (2010), DOI: 10.1155/2010/329571.

[39] V. Kokilashvili and A. Meskhi, Two–weight inequalities for fractional maximal functions and singular integrals in $L^{p(\cdot)}$ spaces, *J. Math. Sci. (N.Y.)*, **173**, no. 6, 656–673 (2011).

[40] V. Kokilashvili, A. Meskhi and M. Sarwar, One and two weight estimates for one-sided operators in $L^{p(\cdot)}$ spaces, *Eurasian Math. J.* **1**, no.1, 73–110 (2010).

[41] V. Kokilashvili, A. Meskhi and L. E. Persson, *Weighted norm inequalities for integral transforms with product kernels*, Nova Science Publishers, New York, 2010.

[42] V. Kokilashvili and A. Meskhi, Weighted criteria for generalized fractional maximal functions and potentials in Lebesgue spaces with variable exponent, *Integr. Trans. Spec. Func.* **18**, no.9, 609–628 (2007).

[43] V. Kokilashvili and A. Meskhi, On one–sided potentials with multiple kernels, *Integr. Transf. Spec. Funct.* **16**, no. 8, 669–683 (2005).

[44] V. Kokilashvili, On a progress in the theory of integral operators in weighted Banach Function Spaces. In "Function Spaces, Differential Operators and Nonlinear Analysis", Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28–June 2, Math. Inst. Acad. Sci. of Czech Republic, Prague, (2004).

[45] V. Kokilashvili and S. Samko, Maximal and fractional operators in weighted $L^{p(x)}$ spaces. *Rev. Mat. Iberoamericana,* **20**, no. 2, 493-515 (2004).

[46] V. Kokilashvili and A. Meskhi, On a trace inequality for one-sided potentials with multiple kernels, *Frac. Calc. Appl. Anal.* **6**, no.4, 461–472 (2003).

[47] V. Kokilashvili and S. Samko, On Sobolev theorem for Riesz-type potentials in Lebesgue spaces with variable exponent, *Z. Anal. Anwendungen.* **22**, no. 4, 899-910 (2003).

[48] V. Kokilashvili, On Hardy's inequalities in weighted spaces, *Soobsch. Akad. Nauk Gruz. SSR.* **96**,no. 1, 37–40 (Russian)(1979).

[49] T. S. Kopaliani, A characterization of some weighted norm inequalities for maximal operators, *Z. Anal. Anwend.* **29**, no. 4, 401–412 (2010).

[50] T. S. Kopaliani, On some structural properties of Banach function spaces and boundedness of certain integral operators, *Czechoslovak Math. J.* **54**(129), no. 3, 791–805 (2004).

[51] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math. J.* **41**(116), no. 4, 592–618 (1991).

[52] A. Kufner, A. Maligranda and L. E. Persson, The Hardy inequality-about its history and some related results, *Vydavatelsky Servis Publishing House*, Pilsen, 2007.

[53] A. Kufner and L. E. Persson, Weighted inequalities of Hardy type, *World Scientific Publishing Co., Inc.*, River Edge, NJ, (2003).

[54] F. Mamedov and Y. Zeren, On a two–weighted estimation of maximal operator in the Lebesgue space with variable exponent, *Ann. Mat. Pura Appl.* (4)**190**, no. 2, 263–275 (2011).

[55] V. G. Maz'ya and I. E. Verbitsky, Capacitary inequalities for fractional integrals with applications to partial differential equations and Sobolev multipliers, *Ark. Mat.* **33**, no. 1, 81–115 (1995).

[56] V. G. Maz'ya, *Sobolev Spaces*, Springer, Berlin, (1985).

[57] Y. Mizuta, E. Nakai, Y. Sawano, and T. Shimomura, Littlewood–Paley theory for variable exponent Lebesgue spaces and Gagliardo–Nirenberg inequality for Riesz potentials, *J. Math. Soc. Japan*, **65**, no.2, 633–670 (2013).

[58] A. Meskhi and M. A. Zaighum, On the boundedness of product kernel operators with measures, *Georgian Math. J.* **19**, no. 3, 533–557 (2012).

[59] A. Meskhi and M. A. Zaighum, On the Boundedness of maximal and potential operators in variable exponent amalgam spaces. *J. Math. Inequal.*, (to appear).

[60] A. Meskhi and M. A. Zaighum, Positive kernel operators in $L^{p(x)}$ spaces with the decay condition at infinity, *Research Preprint Series at ASSMS*, Preprint no. 512, http://www.sms.edu.pk/journals/preprint/pre_512.pdf.

[61] A. Meskhi, *Measure of non-compactness for integral operators in weighted Lebesgue spaces*, Nova Science Publishers, New York, (2009).

[62] A. Meskhi, A note on two-weight inequalities for multiple Hardy-type operators, *J. Funct. Spaces Appl.* **3**, no. 3, 223–237 (2005).

[63] A. Meskhi, Criteria for the boundedness and compactness of integral transforms with positive kernels. *Proc. Edin. Math. Soc.* **44**, no. 2, 267–284 (2001).

[64] A. Meskhi, Solution of some weight problems for the Riemann-Liouville and Weyl operators, *Georgian Math. J.* **5**, no. 6, 565–574 (1998).

[65] B. Muckenhoupt, Hardy's inequality with weights, *Studia Math.* **44**, 31–38 (1972).

[66] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Math. **1034**, Berlin, (1983).

[67] J. Musielak and W. Orlicz, On modular spaces, *Studia Math.* **18**, 49-65 (1959).

[68] H. Nakano, *Topology of linear topological spaces*, Moruzen Co. Ltd, Tokyo, (1981).

[69] C. Okpoti, L.-E. Persson and A. Wedstig, Scales of weight characterizations for discrete Hardy and Carleman type inequalities, *Function Spaces Differential Operators and Nonlinear Analysis, Proceedings of the Conference held in Milovy,*

Bohemian- Moravian Uplands, Czech Republic, May 28-June 2, 2004, Mathematical Institute of the Academy of Sciences of the Czech Republic, Praha, (2005), pp. 236–258.

[70] Y. Rakotondratsimba, Fractional maximal and integral operators on weighted amalgam spaces, *J. Korean Math. Soc.* **36**, no. 5, 855-890 (1999).

[71] M. Ružička, Electrorheological fluids: modeling and mathematical theory, Lecture Notes in Mathematics, Vol. **1748**, Springer-Verlag, Berlin, (2000).

[72] P. O. Salvador, C. Ramírez Torreblanca, Hardy operators on weighted amalgams, *Proc. Roy. Soc. Edinburgh Sect. A.* **140**, no. 1, 175-188 (2010).

[73] S. Samko, Variable exponent Herz spaces, *Mediterr. J. Math.* DOI 10.1007/s00009-013-0285-x, (2013).

[74] S. Samko, Convolution type operators in $L^{p(x)}$, *Integral Transforms Spec. Funct.* **7**, no. 1-2, 123-144 (1998).

[75] S. Samko, Convolution type operators in $L^{p(x)}(R^n)$, *Integral Transforms Spec. Funct.* **7**, no. 3-4, 261–284 (1998).

[76] E. Sawyer and R. L. Wheeden, Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces, *Amer. J. Math.* **114**, no. 4, 813–874 (1992).

[77] E. Sawyer, Weighted inequalities for the two-dimensional Hardy operator, *Studia Math.* **82**, no. 1, 1–16 (1985).

[78] E. Sawyer, Two–weight norm inequalities for certain maximal and integral operators, in: Harmonic analysis, Minneapolis, Minn. 1981, pp. 102–127, Lecture Notes in Math. Vol. 908, Springer, Berlin, New York, 1982.

[79] G. Sinnamon, Spaces defined by the level function and their duals, *Studia Math.* **111**, no. 1, 19–52 (1994).

[80] G. Sinnamon, Operator on Lebesgue spaces with general measures, *Doctoral Thesis,* 1987.

[81] I. I. Sharapudinov, The topology of the space $\mathcal{L}^{p(t)}([0, 1])$, *Mat. Zametki.* **26**, no. 4, 613–632, 655 (Russian) (1979).

[82] J. Stewart and S. Watson, Irregular Amalgams, *Internat. J. Math. Math. Sci.* **9**, no. 2, 331–340 (1986).

[83] I. E. Verbitsky, Weighted norm inequalities for maximal operators and Pisier's Theorem on factorization through $L^{p,\infty}$, *Integr. Equ. Oper. Theory.* **15**, 124–153 (1992).

[84] N. Wiener, Tauberian theorem, *Ann. of Math.* **33**, 1–100 (1932).

[85] N. Wiener, On the representation of functions by trigonometrical integrals, *Math.Z.* **24**, no. 1, 575–616 (1926).

[86] A. Wedestig, Weighted inequalities for Hardy-type and their limiting inequalities, *doctoral Thesis,* 2003.

[87] V. V. Zhikov, On some variational problems, *Russian J. Math. Phys.* **5**, 105–116 (1997).

[88] V. V. Zhikov, On Lavrentievs phenomenon, *Russian J. Math. Phys.* **3**, 249–269 (1995).

[89] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Izv. Akad. Nauk SSSR Ser. Mat.* **50**, no. 4, 675–710 (Russian) (1987).