Variational Improvement to Near-Minimal Surfaces and Comparison with Numerical Outputs of Exact Expressions

By
Daud Ahmad

Supervised By
Prof. Dr. Shaban Ali Bhatti
Dr. Bilal Masud (Associate Professor)

UNIVERSITY OF THE PUNJAB
LAHORE- PAKISTAN
September, 2013
Variational Improvement to Near-Minimal Surfaces and Comparison with Numerical Outputs of Exact Expressions

By
Daud Ahmad

A THESIS
SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

AT

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF THE PUNJAB
LAHORE- Pakistan
September, 2013
Declaration

I, Daud Ahmad s/o Mr. Ahmad Din hereby declare that the dissertation has been written by me and is my original work. The work in this dissertation does not contain any material that has been submitted for the award of any other degree nor has it been submitted as part of the partial requirements for a degree in this university or any other university, except the information sources and literature that has been fully acknowledged and due reference is made in the text, to the best of my knowledge.

Daud Ahmad  (Dated: 17-09-2013)
Certificate

The undersigned hereby certify that the work presented in this thesis is original work of Mr. Daud Ahmad s/o Mr. Ahmad Din and is carried out under their supervision and endorse evaluation of the thesis titled "Variational Improvement to Near-Minimal Surfaces and Comparison with Numerical Outputs of Exact Expressions" by Daud Ahmad for the award of Ph. D. degree, through the official procedure of the University of the Punjab.

Research Supervisors:

________________________
Prof. Dr. Shaban Ali Bhatti

________________________
Dr. Bilal Masud
(Associate Professor)
Dedicated to

My Parents, My Wife Shazia,

Children Maheen, Waleed and Hassan
# Table of Contents

Table of Contents v

List of Tables vii

List of Figures viii

Abstract x

Acknowledgements xii

## 1 Introduction

1.1 Background to the Work ........................................... 1

1.2 Preliminaries ...................................................... 8

1.2.1 Regular curves ................................................. 8

1.2.2 Regular Surfaces ............................................... 12

1.3 Minimal Surfaces .................................................. 19

1.4 The Plateau Problem .............................................. 20

1.5 The Weierstarass-Enneper Integral Representations ............... 23

1.6 Related Developments ............................................ 27

1.6.1 Bilinear Interpolation ....................................... 28

1.6.2 Mean Curvature Flow and its Alternatives ................... 29

1.6.3 Harmonic Approximation ..................................... 29

1.6.4 Use of Differentiable Form of Step Function ................. 29

## 2 Coons patch

2.1 Multi-linear Interpolation ....................................... 31

2.1.1 Linear Interpolation ......................................... 33

2.2 Blending Functions ............................................... 34

2.3 Blended and Ruled Surfaces ...................................... 38

2.4 Bilinear Interpolation ............................................ 40

2.5 Coons Patches Spanned by Fixed Boundary Curves ............... 41

## 3 Generating a Coons Patch

3.1 Unit Step Function and its Relation with Dirac Delta Symbol .... 47
3.2 Comparison of Differentiable Step Function Interpolation with Bezier and Spline Interpolation ........................................... 53
3.3 Analytical Representations of the Step Function ........................................... 54
3.4 Constructing Four Continuous Curves From a Given Set of Finite Number of Boundary Curves ........................................... 57

4 Variational Improvement ......................................................... 61
4.1 A Technique For Variational Improvement ........................................... 62
4.2 The Technique Applied to Hemiellipsoid, a Hump Like Surface and a surface spanned by Four Arbitrary Lines ........................................... 71
4.2.1 Hemiellipsoid-A Surface with Corresponding Known Minimal Surface ...... 72
4.2.2 A Hump Like Surface Spanned by Four Boundary Straight Lines ............ 74
4.2.3 Surface spanned by four arbitrary boundary lines ................................... 85
4.3 The Technique Applied to a Surface Spanned by Five Arbitrary Lines .......... 89

5 Conclusion and Possible Applications ........................................... 101
5.1 Surfaces Spanned by Fixed Boundary ........................................... 102
5.2 Surfaces Spanned by Finite Number \( N \) of Boundary Curves for \( N > 4 \) ........ 102
5.3 Conclusion ................................................................. 103
5.4 Possible Extensions: Comparison with Numerical Output Expressions .......... 106

Appendix ................................................................. 108

Bibliography ............................................................. 109
List of Tables

4.1 Symmetric behaviour of percent decrease in area \( q_{01} \) for selected output values of \( x_i \) for \( 0 < x_i < 20 \). .......................................................... 100
List of Figures

2.1 A ruled surface obtained by fitting a surface between two arbitrary curves \( c_1(u) \) and \( c_2(u) \) through linear interpolation. ........................................... 39

2.2 Bilinear interpolation: a hyperbolic paraboloid is defined by four points \( x(i,j) \) ................................................................. 42

2.3 The ruled surface \( (r = 1, d = 2) \) with \( x, y \) as the horizontal plane and height along \( z \)-axis. ................................................................. 45

2.4 The ruled surface \( (r = 1, d = 2) \) with \( x, y \) as the horizontal plane and height along \( z \)-axis. ................................................................. 46

3.1 Unit step function \( S(u - u_0) \) ................................................................. 48

3.2 A piece wise function as given above, height rises as \( n \to \infty \) ................................................................. 50

3.3 A sequence function \( \frac{n}{\pi(1 + \pi^2 u^2)} \) peak rises as \( n \to \infty \) ................................................................. 51

3.4 A sequence function \( \frac{\sin(nu)}{\pi u} \) peak rises as \( n \to \infty \) ................................................................. 52

3.5 A sequence function \( \frac{n}{\sqrt{\pi} e^{-n^2 u^2}} \), peak rises as \( n \to \infty \) ................................................................. 53

3.6 Step function representations \( S(u - u_i), S^*(u - u_i) \) and \( S^{**}(u - u_i) \) for \( i = 1, l = 20, u_i = 0.25, \epsilon = 0.01 \) and \( k = 5 \). ................................................................. 55

3.7 Straight lines \( L_1(u) \) and \( L_2(u) \) joined by first, second and third step function representations respectively for \( l = 20, u_1 = 0.25, \epsilon = 0.01 \) and \( k = 5 \). ................................................................. 56

3.8 \( c_1(u) \) joining \( L_1(u), L_2(u) \) and \( L_3(u) \) for \( N = 9 \). ................................................................. 59

3.9 \( c_1(u) \) joining \( L_1(u), L_2(u), L_3(u) \) and \( L_4(u) \) for \( N = 14 \). ................................................................. 60

4.1 Graph of variational improvement to arc length of a curve of given length, comparing the results of eight iterations, where \( \gamma \) is the initial curve along with the curves \( X(\xi) \) of reduced lengths. ................................................................. 71

4.2 A typical hemiellipsoid, initial non-minimal surface of which boundary is an ellipse in the \( xy \)-plane. ................................................................. 73

4.3 Values of the variation in the parameter \( t(b, c) \) for the first iteration as a function of the semi-major and semi-minor axes \( b \) and \( c \) of the ellipse bounding the hemiellipsoid. 74
4.4 The percentage decrease $q_{01}$ in area $A(b, c)$ of hemiellipsoid as a function of semi-major and semi-minor axes $b$ and $c$ of the ellipse bounding the hemiellipsoid.

4.5 A surface spanned by four straight lines with the parametrization $x_0(u, v) = (u, v, 16uv(1-u)(1-v))$.

4.6 Graph of numerator of initial mean curvature denoted by $H_0$.

4.7 Mean square of mean curvature of the surface $x_1(u,v)$ as a function of $t$.

4.8 The surface $x_1(u,v)$ for $t = t_{min} = 0.0440856$.

4.9 Mean curvature of the surface $x_1(u,v)$ for $t_{min} = 0.0440856$.

4.10 Surface $x_2(u,v,t)$ for $t_{min} = 0.0440856$.

4.11 Mean curvature of the surface $x_2(u,v)$ for $t_{min} = 0.0440856$.

4.12 $m_2(u,v)$ for the surface $x_3(u,v)$.

4.13 The variation in parameter $t(r)$ as a function of the real scalar $r$. For the skew quadrilaterals ruled bounded by four arbitrary straight lines connecting four corners $x(0,0), x(0,1), x(1,0)$ and $x(1,1)$.

4.14 The dimensionless decrease in area $A(r)$ as a function of $r$ enclosed by a skew quadrilateral ruled bounded by four arbitrary straight lines connecting four corners $x(0,0), x(0,1), x(1,0)$ and $x(1,1)$.

4.15 Interpolating curve representing the dimensionless decrease in area $A(r)$ as a function of $r$ enclosed by a skew quadrilateral ruled bounded by four arbitrary straight lines connecting four corners $x(0,0), x(0,1), x(1,0)$ and $x(1,1)$.

4.16 An initial string $c_2(u) = L_4(u)$ with $d_1(v) = L_3(v)$ and $d_2(v) = L_5(v)$ modeling the time evolution of its ends and the combination $L(u)$ contains the two final strings $L_1(u)$ and $L_2(u)$ providing $c_1(u)$.

4.17 Cosine of the angle between $N$ and $k$ is shown to be positive for Coons patch in particular for $x_i = 4$. Thus angle between $N$ and $k$ remains within the range $0 \leq \theta \leq \pi/2$.

4.18 A representative graph for Coons patch for $x_i = 10$ that angle $\theta$ between $N$ and $k$ on Coons patch for $x_i = 10$ is smaller.

4.19 Coons patch for $x_i = 2, 6, 10$.

4.20 Spline curve passing through the data points $x_i = 2, 4, 5, 6, 8, 10, 12, 14, 16, 18$ along the horizontal axis for percentage decrease in area shown along the vertical axis.
Abstract

An algorithm using a suggested ansatz is presented to reduce the area of a surface spanned by a finite number of boundary curves by doing a variational improvement in the initial surface of which area is to be reduced. The ansatz we consider, consists of original surface plus a variational parameter multiplying the unit normal to the surface, numerator part of its mean curvature function and a function of its parameters chosen such that its variation at boundary points is zero. We minimize of its $\text{rms}$ mean curvature and for the same boundary decrease the area of the surface we generate. We do a complete numerical implementation for the boundary of surfaces, a) when the minimal surface is known, namely a hemiellipsoid spanned by an elliptic curve (in this case the area is reduced for the elliptic boundary by as much as 23 percent of original surface), and b) a hump like surface spanned by four straight lines in the same plane- in this case the area is reduced by about 37.9141 percent of original surface along with the case when the corresponding minimal surface is unknown, namely a bilinearly interpolating surface spanned by four bounding straight lines lying in different planes. (The four boundary lines of the bilinear interpolation can model the initial and final configurations of re-arranging strings). This is a special case of Coons patch, a surface frequently encountered in surface modelling- Area reduced for the bilinear interpolation is 0.8 percent of original surface, with no further decrease possible at least for the ansatz we used, suggesting that it is already a near-minimal surface. As a Coons patch is defined only for a boundary composed of four analytical curves, we extend the range of applicability of a Coons patch by telling how to write it for a boundary composed of an arbitrary number of boundary curves. We partition the curves in a clear and natural way into four groups and then join all the curves in each group into one analytic curve by using representations of the unit step function including a fully analytic suggested by us. Having a well parameterized Coons patch spanning a boundary composed of an arbitrary number of curves, we do calculations on it that are motivated by variational calculus that give a better optimized and possibly more smooth surface. A complete numerical implementation for a boundary composed of five straight lines is provided (that can model a string breaking) and get about 0.82 percent decrease of the area in this case as well. Given the demonstrated ability of our optimization algorithm to reduce area by as much as 37.9141 percent for a spanning surface not close to being a minimal
surface, this much smaller fractional decrease suggests that the Coons patch for five line boundary
we have been able to write is also close to being a minimal surface. That is it is a near-minimal
surface. This work compares the reduction in area for near-minimal surfaces (bilinear interpolation
spanned by four boundary lines and a Coons patch whose boundary is rewritten for a boundary
composed of five lines) with the surfaces whose minimal surfaces are already known (a hemiellipsoid
spanned by an elliptic disc and a hump like surface spanned by four straight lines lying in the
same plane) and we have been able to calculate numerically worked out differential geometry related
quantities like the metric, unit normal, root mean square of mean curvature and root mean square
of Gaussian curvature for the surface obtained through calculus of variations with reduced area.
Acknowledgements

I would like to express my deepest gratitude to my supervisors, Prof. Dr. Shaban Ali Bhatti and Dr. Bilal Masud (Associate Professor), for their continuous support, motivation, engagement and contribution in stimulating suggestions and encouragement during my research work. I would also like to express my deepest appreciation to my teachers Prof. Dr. Bashir Sadiq, Prof. Dr. Shaban Ali Bhatti and Dr. Bilal Masud, who taught me during my Ph. D. course work and continually and convincingly conveyed the spirit of learning the courses with motivation and excitement. I would like to express my sincere thanks to Prof. Dr. Kh. Haris Rashid, Director (CHEP) and Dean of Science, for having a sympathetic ear to share my problems and for his generous support and encouragement during the entire process. I am thankful to Prof. Dr. Shahid Saeed Siddiqui, Chairman (Retd.) and Prof. Dr. Muhammad Sharif (Chairman), Department of Mathematics, for providing departmental facilitates related to computer and softwares needed for the work. I would like to express my thanks as well to Dr. Munawar Iqbal, Dr. Muhammad Ayub Faridi and Dr. Faisal Akram (CHEP), my colleagues Dr. Muhammad Aslam, Dr. Muhammad Akram, Mr. Muhammad Riaz, Mr. Hafiz Muhammad Khalid (Department of Mathematics) for the discussions I have been involved in at one or the other occasion.

I am also thankful to University of the Punjab for partially funding, providing stipend and relaxation in workload during my PhD work. The Stipend was awarded to me by the University of the Punjab, Lahore for the period 2007–2010.

Finally I would like to thank my family for their patience and love, without their continuous support I might not have been able to complete this work.

Department of Mathematics, University of the Punjab, Lahore, September 17, 2013.
Chapter 1

Introduction

In this chapter we present historical overview and introduce the differential geometry related basic concepts including regular curves and surfaces, minimal surfaces, Plateau Problem, Weierstrass-Enneper Integral representations of the surfaces and related developments in the field.

1.1 Background to the Work

Variational methods are one of the active research areas of the optimization theory [1, 2]. A variational method tries to find the best values of the parameters in a trial function that optimize, subject to some algebraic, integral or differential constraints, a quantity dependant on the ansatz. A simple example of such a problem may be to find the curve of shortest length connecting two points. The solution is a straight line between the points in case of no constraints, with constraints, possibly many solutions may exist depending on the nature of constraints, called geodesics. One of the related problems is finding the path of stationary optical length connecting two points, as the Fermat’s principle of least time (1657) says that “Of all the possible paths that a light ray might take between two fixed points, the actual path is the one that minimises the travel time of the ray”. He showed that the laws of reflection and refraction could be derived from his principle. This is one of the example that shows that variational methods are sought for their simplicity, elegance and the general applications. One of the variational principles, Hamilton’s principle, also called the principle of least action, is a far reaching principle in classical mechanics that led to development of Lagrangian and Hamiltonian formalism. This formalism has significant applications in the theory
of relativity, quantum mechanics and non-mechanical systems e.g. quantum field theory. It is stated in the form “Of all the kinematically possible motions that take a mechanical system from one given configuration to another within a given time interval, the actual motion is the one that minimises the time integral of the Lagrangian of the system”.

Another related problem is a Plateau problem [3, 4] which is finding the surface with minimal area constrained by a given boundary curve (called a minimal surface- a surface of zero mean curvature at all points [5, 6]), having many applications in different branches of science like Mathematics, Computer Science, Physics, Chemistry and Biology. This problem is named after the blind Belgian physicist Joseph Plateau, who demonstrated in 1849 that a minimal surface can be obtained by immersing a wire frame, representing the boundaries, into soapy water. Since then many mathematicians contributed to the theory of minimal surfaces like Schwarz [7] (who discovered D (diamond), P (primitive), H (hexagonal), T (tetragonal) and CLP (crossed layers of parallels) triply periodic surfaces), Riemann [3], and Weierstrass [3]. Although mathematical solutions for specific boundaries had been obtained for years, but existence of a minimal solution for a given simple closed curve was independently proved in 1931 by the American mathematician Jesse Douglas [8] and the Hungarian Tibor Radó [9]. They approached to the solution through different methods. Douglas [8] minimized a quantity now named as Douglas Integral and his results held for arbitrary simple closed curve, while Radó [9] minimized the energy. The work of Radó was built on the previous work of R. Garnier [10] and held only for rectifiable simple closed curves. Achievements of L. Tonelli [11], R. Courant [12] [13], C. B. Morrey [14] [15], E. M. McShane [16], M. Shiffman [17], M. Morse [18], T. Tompkins [18], Osserman [19], Gulliver [20] and Karcher [21] and others contributed many revolutionary results in the subsequent years. The ref. [22] gives the related results of differential geometry and measure theory to solve the problems which have variational aspects. The ref. [23] gives developments in the theory of Plateau’s problem for parametric minimal surfaces and surfaces of prescribed constant mean curvature and its analytical framework, relying on the variational techniques. In the ref. [24], the author treats the Plateau problem as a “test problem” using theory of differential chains.

For a closed boundary composed of four straight lines efficient methods [25, 26, 27, 28], of generating a spanning surface are known and it is also well known how to find different differential geometry related properties for these surfaces. If needed such a surface can be replaced by other
slightly deformed surfaces with the differential geometry properties being closer to some desired values. Specifically we can mention the Coons patch \cite{29, 30} prescription (eq. (2.5.3) below), for generating the surface and our variational method \cite{31} (given by eq. (4.1.11)) based changes in it that generate a slightly changed surface from it that has lesser $rms$ mean curvature and area as compared to the original surface and hence is closer to being a minimal surface.

The anzatz we consider consists of original surface plus a variational parameter multiplying the unit normal to the surface or its substitute, numerator part of its mean curvature function and a function of the surface parameters chosen such that its variation at boundary points is zero. We minimize its $rms$ mean curvature and for the same boundary decrease the area of the surface we generate. Explicitly, the anzatz of eq. (4.1.11) consists of original surface $x_0(u,v)$ plus a variational parameter $t$ multiplying the unit normal $N_0(u,v)$ to the surface, numerator part $H_0$ of its mean curvature function eq. (4.1.13) and a function $uv(1-u)(1-v)$ of its parameters chosen such that its variation at boundary points is zero. To reduce the area, we minimize its $rms$ mean curvature function and for the same boundary decrease the area of the surface. For this, a variational area reduction scheme is outlined in section 4.1. This technique is applied to a hemiellipsoid, a hump like surface and bilinearly interpolating surface (these three surfaces are discussed in section 4.2) spanned by four bounding straight lines. (The four boundary lines of the bilinear interpolation can model the initial and final configurations of re-arranging strings.) As a demonstration of the effectiveness of the technique, the area of the hemiellipsoid is reduced for the same boundary by as much as 23 percent of the original area. For hump like surface the area of the surface is reduced by as much as 37.9141 percent of the original surface. For bilinear interpolation the decrease remains less than 0.8 percent of the original area, which may suggest that it is already a near minimal surface. For a surface spanned by more than four boundary curves, methods have been used for $N$-sides surface generation from arbitrary boundary edges. But these are 1) discrete surfaces \cite{32, 33}, 2) the relation of such a surface with Coons patch is at least not clear, 3) such surfaces are not studied from the view of differential geometry as we have done \cite{31} for the Coons patch, or 4) it is not told how to replace such a surface with a deformed one closer to being, say, a minimal surface. In the present thesis we present a prescription to avoid all these possibly unwanted features; we tell how to generalize both the surface generation of eq. (2.5.3) and the variational improvement of eq. (4.1.11).
to closed boundaries composed of an arbitrary finite number of curves. Presently, we are able to fully produce variationally improved surfaces only for a boundary composed of five straight lines, but our algorithms both for the variational improvement given by eq. (4.1.11) and, before it for the surface generation are general. In surface modeling a surface frequently encountered is a Coons patch eq. (2.5.3) satisfying conditions eq. (2.5.1) and eq. (2.5.2), that is defined only for a boundary composed of four analytical curves. We not only applied the technique to a Coons patch bounded by four lines given by eq. (2.5.3) satisfying conditions eqs. (2.5.1) and (2.5.2) but we attempt to extend the range of applicability of a Coons patch by telling how to write it for a boundary composed of an arbitrary number of boundary curves (section 3.4). We partition the curves in a clear and natural way into four groups and then join all the curves in each group into one analytic curve of eq. (3.4.4) by using representations of the unit step function given by eqs. (3.3.1) to (3.3.3) including one that is fully analytic. Having a well parameterized surface, we do some calculations on it that are motivated by calculus of variation and give a better optimized and possibly more smooth surface. For this, we use the ansatz of eq. (4.1.12) as mentioned above, minimize w.r.t. it the rms mean curvature and decrease the area of the surface we generate. We do a complete numerical implementation for a boundary composed of five straight lines, that can model a string breaking, and get about 0.82 percent decrease of the area. Given the demonstrated ability of our optimization algorithm to reduce area by as much as 23 percent for a hemiellipsoid (ref [31]), a surface spanned by an ellipse, not close of being a minimal surface and 37.9141 percent for a hump like surface, this much smaller fractional decrease suggests that the Coons patch we have been able to write is already close to being a minimal surface.

An emerging use of minimal surfaces in physics is that in string theories. A classical particle travels a geodesic with least distance whereas a classical string is an entity which traverses a minimal area. Amongst the string theories used in physics, two are worth mentioning. One is the theory of quantum chromodynamics (QCD) strings that models the gluonic field confining a quark and an antiquark within a meson. (The gluonic field connecting three quarks, within a proton or neutron, is modeled through Y-shaped strings. For a system composed of more than three quarks, minimization of the total length of a string network with only Y-shaped junctions may be a non-trivial Steiner-Tree Problem [34]). In the other string theory (or theories) string vibrations are supposed to generate
different elementary particles of the present high energy physics. Quite often string theories need a surface spanning the boundary composed of curves either connecting particles or describing the time evolution of particles. An important case can be a fixed boundary composed of four external curves. A common application of this boundary can be the time evolution of a string parameterized by $\sigma$ or $\beta$ variable; the time evolution itself is parameterized by the symbol $\tau$, the proper time of relativity. In this case two bounding curves parameterized by the respective $\sigma$ or $\beta$ represent the initial and final configurations of a string, and the other two curves (parameterized by the respective $\tau$ variables) describe the time evolution of the two ends of a string. String theories take action to be proportional to area. Combining this with the classical mechanics demand of the least action, minimal surfaces spanning the corresponding fixed boundaries get their importance. For example, see eq. 13 of ref. [35] for the Nambu-Goto ansatz for the minimal surface area and compare it with eqs. (2.5.6) and (2.5.7) below, along with ref. [36] for Nambu-Goto strings. Also relevant is the use in ref. [37] of Wilson minimal area law (MAL) to derive the quark antiquark potential in a certain approximation. A surface spanned by such a boundary is in space-time of relativity. An ordinary 3-dimensional spatial surface can span a boundary composed of two 3-dimensional curves connecting four particles and two other curves connecting the same four particles in a re-arranged (or exchanged) clustering; see for example Fig. 2 of ref. [38] and Fig. 5 of ref. [39]. An explicit expression of such a spanning surface can be found in eqs. 3, 4 of ref. [40] and eq. 22 of ref. [41]. This is a bilinear interpolation in ordinary 3-dimensional space and is similar to the linear interpolations in above mentioned eq. 13 of ref. [35], eq. 4.7 of ref. [42] and eq. 3.4 of ref. [37]. Ref. [42] clarifies that such a surface is used as a replacement to the exact minimal surfaces for the corresponding boundaries; see section 1.2 below for a minimal surface in the differential geometry. Even non-minimal surfaces have some usage in the mathematical modeling of quantum strings because 1) in contrast to classical strings, quantum strings can have any action and hence area as described by the path integral version of the quantum mechanics (see eq. 1 of [43] ) and 2) any surface spanning a boundary composed of quark lines (or quark connecting lines) corresponds to a physically allowed (gauge invariant) configuration of the gluonic field between these quarks; compare the non-minimal surface of Fig. 10.5 of ref. [44] with the minimal surface for the same boundary in Fig. 10.1 of the same ref. [44]. But it cannot be denied that minimal surfaces are the most important of the
spanning surfaces even in quantum theories. For example, the relation in eq. (1.14) of ref. [45] between an area and an important quantity (termed Wilson loop) related to the potential between a quark and antiquark connected by a QCD (gluonic) string is valid only if the area is of the minimal surface. (Though above mentioned eq. 1 of ref. [43] relates the Wilson loop to a “sum over all surfaces of the topology of rectangle bounded by the loop” implying that each spanning surface has some contribution in the Wilson loop, the minimal surface must contribute most.) Thus it is worth pointing out that the non-minimal linearly or bilinearly interpolating surfaces can replace minimal surfaces, can be effectively used as minimal surfaces or share some features in common with minimal surfaces; text just before eq. (1.15) of the above mentioned ref. [45] relates them, up to non-relativistic 1/(mass square) order, to the minimal surfaces. The purpose of our work reported in ref. [31] is explore further this “effective usability” or “sharing common features with minimal surfaces” of linearly or bilinearly interpolating surfaces. Before starting a description of our work, we want to 1) state the common feature we have chosen. This is the fractional reduction possible in the area for a fixed boundary; for an exact minimal surface this quantity is zero (at least for a small neighbourhood). For reducing area we use the variational area reduction, outlined in sect. 4.1, to our specific bilinear interpolation described in sec. 2.5. Moreover, we 2) point out that the bilinear interpolations used in string-theories-related works of physics are also used in the emerging discipline of the computer aided geometric design (CAGD) and hence the usefulness of our work reported in ref. [31] extends to CAGD along with physics and the differential geometry; as much as bilinear interpolations are near or related to minimal surfaces their study sheds some light on the above mentioned Plateau problem of the differential geometry itself.

Computer aided geometric design (CAGD) [29], [30] arose when mathematical descriptions of shapes facilitated the use of computers to process data and analyze related information. In the 1960s, it became possible to use computer control for basic and detailed design enabling utilization of a mathematical model stored in a computer instead of the conventional design based on drawings. The term geometric modeling is used to characterize the methods used in describing the geometry of an object. Over the years, various schemes were developed with a view to achieve this abstraction. S. A. Coons [29], [30] introduced the Coons patch in 1964. The Coons patch approach is based on the premise that a patch can be described in terms of four distinct boundary curves. Thus a
Coons patch can be a worth analyzing surface spanning a fixed set of boundary curves, at least when the number of curves is four. It is an active area of research and has seen enormous development during recent years that includes the work of G. E. Farin and D. Hansford [25], M. Hugentobler [46], Wang and Tang [27], Szilvási and Szabó [47, 26], S. Sarkar and P. Dey [28]. But most, if not all, of the work on it has been limited to its geometric descriptions and visualization and to interactive mathematical experiments with it; it has not been analyzed from the view of differential geometry and that is also what we aim to do in this dissertation. For us, Coons patch (see eq. (2.5.3) below) is relevant because the above mentioned bilinear interpolations (see also eq. (2.5.5) below) we basically study are a special case of Coons patch [29].

The dissertation is organized as follows. In the remaining chapter, basic differential geometry related concepts are presented in section 1.2, minimal surfaces in section 1.3, the Plateau problem in section 1.4, the Weierstarass-Enneper integral representations of minimal surfaces in section 1.5 and finally the related developments to achieve a minimal surface using approximation techniques in the section 1.6. In the next chapter 2, multilinear interpolation with emphasis on linear, bilinear interpolation and blended and ruled surfaces in sections through 2.1 to 2.4 and in section 2.5 basic definitions and constructions related to surfaces spanned by fixed boundary curves are presented and mean curvature of a bilinear interpolating surface is computed which is zero only for two coordinate curves (eqs. (2.5.13) and (2.5.14) below). In chapter 3, unit step function and its relation with Dirac delta as a distribution sequence is given in section 3.1. A comparison of differentiable step function interpolation, Beziér interpolation and spline Interpolation is provided in section 3.2. In section 3.3, we present, through eqs. (3.3.1) to (3.3.2), three analytical representations of the unit step function (shown in Figure 3.6) and give, through Figure 3.7, geometric description of boundary curves generated by these step function representations. An arbitrary boundary can be written as a limit of a collection of finite number of arbitrary curves and for a boundary composed of finite number of curves our plan is to first reduce it to only four bounding curves so that we can then apply our Coons patch based analysis. For this, techniques are needed to group a finite number of curves into four sets and then within each set combine all the curves as one continuous curve. In this way, one set would become, in the common notation for a Coons patch, $c_1(u) = x(u,0)$ and the other three as $c_2(u) = x(u,1)$, $d_1(v) = x(0,v)$ and $d_2(v) = x(1,v)$ with $0 \leq u,v \leq 1$. We
introduce our algorithms for grouping in the first paragraph of section 3.4. For the combinations, in the remaining portion of section 3.4 we present an iterative scheme that uses suitable step function representations to combine an arbitrary number of curves into one continuous curve. In the next chapter 4, we present an algorithm to reduce the area of a surface spanned by a finite number of boundary curves by introducing a variational improvement in a surface in section 4.1. Then in section 4.2 we apply this technique to reduce the area of 1) a non-minimal surface spanning a boundary for which the minimal surface is known - namely hemiellipsoid eq. (4.2.2) and a hump like surface eq. (4.2.7), to make sure the efficiency of the algorithm given by eq. (4.1.11) and 2) above mentioned bilinear interpolation spanned by four bounding lines eq. (4.2.35) (for which we do not need step function based construction) and a surface eq. (4.3.4) spanned by five arbitrary straight lines (a surface reduced through step function to the standard Coons patch prescription in section 4.3). Based on this comparison, we comment on the possible status of bilinear interpolation and a surface spanned by five lines as near-minimal surface. The last chapter 5 presents results, final remarks and mentions possible future developments.

1.2 Preliminaries

In this section we shall state few definitions related to local and global properties of curves and surfaces. Local properties of curves and surfaces depend on the behaviour of the curve or surface in the neighbourhood of a point. The techniques which are suitable to study these properties are the methods of differential calculus. Thus it is more convenient to define a curve or a surface in differential geometry by the functions that are differentiable for a certain number of times. The global properties of curves and surfaces study the behaviour of the curve or a surface as a whole, with the help of local properties of the entire curve or a surface.

1.2.1 Regular curves

A real function of a real variable is said to be differentiable (smooth) if it has derivatives of all orders at each point of its domain. We define the curve as follows,
Definition 1.2.1. A parameterized differentiable curve is a differentiable map \( x : I \rightarrow \mathbb{R}^3 \) of an open interval \( I = (\alpha, \beta) \) of the real line \( \mathbb{R} \) into \( \mathbb{R}^3 \).

This means that \( x \) is a function that maps each \( \xi \in I \), called the parameter of the curve, into a point \( x = (x(\xi), y(\xi), z(\xi)) \in \mathbb{R}^3 \) such that the functions \( x(\xi), y(\xi), z(\xi) \) are differentiable. The interval \( I \) may be entire real line. For each \( \xi \in I \) where \( x'(\xi) \neq 0 \) there is a well-defined straight line, which contains the point \( x(\xi) \) and the vector \( x'(\xi) \). This line is called the tangent line to \( x(\xi) \) at \( \xi \). For the study of the differential geometry of a curve it is essential that there exists such a tangent line at every point. Therefore, we call any point \( \xi \) where \( x'(\xi) = 0 \), a singular point of \( x(\xi) \) and restrict our attention to curves without singular points.

Definition 1.2.2. A parameterized differentiable curve \( x : I \rightarrow \mathbb{R}^3 \) is said to be regular if \( x' \neq 0 \) for all \( \xi \in I \).

A regular parameterized curve (regular curve) \( x = x(\xi) \) on \( I \) may have multiple points \( i.e., x(\xi_1) = x(\xi_2), \) for \( \xi_1, \xi_2 \in I, \xi_1 \neq \xi_2. \) But locally a regular parameterized curve behaves different. It may be shown [5] that for a regular parametric representation of a curve \( x(\xi) \) on \( I \), there exists a neighbourhood of each \( \xi_0 \in I \), in which \( x(\xi) \) is one-to-one.

The length of an arc may be approximated by taking polygonal arcs on the curve joined in a sequence. Let \( \gamma \) be an arc given by a parameterized differentiable curve \( x(\xi) \) for \( \alpha \leq \xi \leq \beta. \) A subdivision \( \alpha = \xi_0 < \xi_1 < \ldots < \xi_n = \beta \) of the interval \([\alpha, \beta]\) gives a sequence of points \( x(\xi_0), x(\xi_1), \ldots, x(\xi_m) \) on the given arc, which are joined to form an approximating polygonal arc. The total length \( p_m \) of these approximating \( m \) polygonal arcs approximates the length of the arc of given curve and is given by

\[
p_m = \sum_{k=1}^{m} |x(\xi_k) - x(\xi_{k-1})|. \tag{1.2.1}
\]

Here \( |x(\xi_k) - x(\xi_{k-1})| \) denotes the length of the chord joining the two adjacent points \( x(\xi_{k-1}), x(\xi_k). \) However, by introducing a better approximating polynomial arc by taking additional points on the arc and denoting this total new length of the polygonal arcs by \( p_n \), thus it can be seen that \( p_m < p_n \) for \( m < n \). Hence, we can define the length of the arc \( \gamma \) as the greatest sum of the lengths of the polygonal arcs. Let \( p \equiv \{p_m, p_n, \ldots\} \) be the set of lengths of polygonal arcs approximating the arc \( \gamma \) for all possible subdivisions. Now if this set is bounded above, the arc \( \gamma \) given by \( x(\xi) \) is called rectifiable. In this case, length of the given arc is defined as the supremum of the set \( p \).
Definition 1.2.3. It is known [5] that a regular arc \( \gamma \), given by \( x(\xi) \) is rectifiable and its length is given by the following integral.

\[
\begin{align*}
    s &= \int_\alpha^\beta \left| \frac{dx}{d\xi} \right| d\xi \\
    &= \int_\alpha^\beta \sqrt{\left( \frac{dx}{d\xi} \right)^2 + \left( \frac{dy}{d\xi} \right)^2 + \left( \frac{dz}{d\xi} \right)^2} d\xi.
\end{align*}
\] (1.2.2)

For a regular parameterized curve \( x : I \to \mathbb{R}^3 \) denoted by \( x = x(\xi) \) for \( \xi \in I \), consider the function

\[
s(\xi) = \int_{\xi_0}^{\xi} \left| \frac{dx}{d\xi} \right| d\xi,
\] (1.2.3)

where \( |x'(\xi)| = \left| \frac{dx}{d\xi} \right| \) is the length of the vector \( x'(\xi) \).

Note that \( s \geq 0 \) for \( \xi \geq \xi_0 \) and \( s < 0 \) for \( \xi < \xi_0 \), \( s(\xi) \) denotes the arc length of the curve \( x_0(\xi) \) between \( x(\xi_0) \) and \( x(\xi) \). For a rectangular coordinate system, if \( \gamma \) reduces to a straight segment of length \( \eta \), then the arc length formula (1.2.3) reduces to \( \eta_1^2 + \eta_2^2 + \eta_3^2 \), where \( \eta_1, \eta_2, \eta_3 \) denote the lengths of the orthogonal projections of the segment upon the respective coordinate axes.

It follows from the Fundamental Theorem of Calculus that eq. (1.2.3) has a continuous non-vanishing derivative and is of the same class of differentiability as that of \( x(\xi) \). This means that the arc length \( s(\xi) \) may be introduced as a parameter instead of the parameter \( \xi \), which is sometimes more convenient in computing the differential geometry related quantities. The derivative is given by

\[
\frac{ds}{d\xi} = \frac{d}{d\xi} \left( \int_{\xi_0}^{\xi} \left| \frac{dx}{d\xi} \right| du \right) = \left| \frac{dx}{d\xi} \right|.
\] (1.2.4)

The representation of the curve with arc length as parameter is not unique, as it depends on the choice of the initial point \( \xi_0 \), where \( s \) is taken to be zero. The representation \( x = x(s) \) is said to be a natural representation if

\[
|dx/ds| = 1.
\] (1.2.5)

A regular curve is uniquely determined by two scalar quantities namely the curvature and torsion, as function of the natural parameter. For a regular curve \( x = x(s) \), the derivative denoted by

\[
t = dx/ds = x^*
\] (1.2.6)

defines the unit tangent to the curve \( \gamma \).

Definition 1.2.4. Let \( x : I \to \mathbb{R}^3 \) be a curve parameterized by arc length \( s \in I \). The number \( |x^{**}(s)| = \kappa(s) \) is called the curvature of \( x \) at \( s \).
Alternatively, derivative of the unit tangent vector
\[ \kappa = \frac{dt}{ds} = \mathbf{t}' \] (1.2.7)
gives us the curvature vector on curve \( \gamma \). The reciprocal of magnitude of this curvature vector is called the radius of curvature at \( x(s) \). Thus the curvature vector \( \kappa \) is defined as the rate of change of the unit tangent vector \( t \) w.r.t the arc length \( s \). A point on the curve \( \gamma \) where the curvature vector \( \kappa = 0 \) is called an inflection point. If there are no inflection points on the curve \( \gamma \) i.e. \( \kappa(s) \neq 0 \ \forall s \), the unit vector defined by \( \mathbf{n}(s) = \frac{\kappa(s)}{|\kappa(s)|} \) is called principal normal unit vector to the curve \( \gamma \) at \( s \), \( \kappa(s) = x''(s) \) is normal to \( t = x'(s) \), as is obvious from eq. (1.2.5) that differentiating the eq.(1.2.6) gives \( x'(s) \cdot x''(s) = 0 \), so that \( t \cdot n = 0 \) and thus \( t \perp n \).

The plane determined by the unit tangent vector \( t(s) \) and the normal vector \( n(s) \) is called the osculating plane at \( s \). At points where \( \kappa(s) = 0 \), the normal vector and therefore the osculating plane is not defined. For local analysis of curves we need the osculating plane. The unit vector \( b(s) = t(s) \times n(s) \) is called the unit binormal vector at \( s \). The length \( |b'(s)| \) measures the rate of change of the neighboring osculating planes with the osculating plane at \( s \). That is \( b'(s) \) measures how rapidly the curve pulls away from the osculating plane at \( s \), in a neighbourhood of \( s \).

**Definition 1.2.5.** Let \( x : I \rightarrow \mathbb{R}^3 \) be a curve parameterized by arc length \( s \) such that \( x''(s) \neq 0, s \in I \). The number \( \tau(s) \) defined by \( b'(s) = -\tau(s)n(s) \) is called the torsion of \( x(s) \) at \( s \).

Along a curve \( x(s) \), the unit vectors \( t(s), n(s), b(s) \) as defined above satisfy the following set \([5]\) of eqs. (1.2.8),
\[ t'(s) = \kappa n(s), \quad n'(s) = -\kappa t(s) + \tau b, \quad \text{and} \quad b'(s) = -\tau n(s), \] (1.2.8)
are called the Frenet-Serret equations of a curve, where the dot ‘\( ' \) denote the derivative \( w.r.t \) the arc length \( s \). The plane determined by the unit tangent vector \( t(s) \) and the binormal vector \( b(s) \) is called the rectifying plane at \( s \) and that of unit normal vector \( n(s) \) and the binormal vector \( b(s) \) is called the normal plane at \( s \). Physically, we can think of a curve in \( \mathbb{R}^3 \) as being obtained from a straight line by its bending (curvature) and twisting (torsion). After reflecting on this construction, we are led to the following statement of “Fundamental existence and uniqueness theorem for space curves”, which shows that \( \kappa(s) \) and \( \tau(s) \) describe completely the local behavior of the curve.
Theorem 1.2.1. Let $\kappa(s)$ and $\tau(s)$ be arbitrary continuous functions on $s_1 \leq s \leq s_2$. Then there exists, except for position in space, one and only one space curve $\gamma$ for which $\kappa(s)$ is the curvature, $\tau(s)$ is the torsion and $s$ is a natural parameter along $\gamma$.

Thus, for given arbitrary continuous functions $\kappa(s)$ and $\tau(s)$, there exist solutions $t(s), n(s), b(s)$ of Frenet-Serret equations, such that $x = \int tds + C$ with the given curvature and torsion. In the following section we give few definitions related to the regular surfaces. These are also basically taken from the ref. [5].

1.2.2 Regular Surfaces

A regular surface may be thought of a surface obtained by gluing together pieces of a plane deformed and arranged in such a way that the resulting surface has no sharp points, edges, or self-intersections. Then, it makes sense to imagine a tangent plane at points of the surface to apply the usual calculus methods on the surface. We define below a surface in its parametric form that is more convenient to compute curvature and arc length of curves on a surface from the given parametrization of the surface. Likewise we can compute surface area and invariant quantities like the two fundamental forms, Gaussian, mean and principle curvatures on a surface. We state:

Definition 1.2.6. A subset $S \subset \mathbb{R}^3$ is a regular surface if, for each $P \in S$, there exists a neighborhood $V$ in $\mathbb{R}^3$ and a map $x : U \rightarrow V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

1. $x$ is differentiable, that means for $x(u,v) = (x(u,v), y(u,v), z(u,v))$, the functions $x(u,v)$, $y(u,v)$ and $z(u,v)$ have continuous partial derivatives of all orders in $U$, $\forall (u,v) \in U$.

2. $x$ is a homeomorphism. This means that $x$ has inverse $x^{-1} : V \cap S \rightarrow U$ which is continuous as it can be seen from the above condition that $x$ is continuous.

3. $x$ satisfies regularity condition i.e. for each $Q \in U$, the differential $dx_Q : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is one-to-one.

The variables $u$ and $v$ are called parameters of the surface. The mapping $x$ is called a regular parametric representation of the surface $S$, a system of local coordinates in a neighbourhood of $P$. The neighbourhood $V \cap S$ of $P$ in $S$ is called coordinate neighbourhood. The image of a coordinate
line \( v = v_0 \) in \( U \) is a curve \( \mathbf{x}(u, v_0) \) on \( S \) with \( u \) as its parameter. It is called the \( u \)-parameter curve. In the similar way we can define the \( u \)-parameter curve as the image denoted by \( \mathbf{x}(u_0, v) \), of the coordinate line \( u = u_0 \) in \( U \), with \( u \) as its parameter. Thus parametrization of a surface covers the surface \( S \) with two families of curves namely the \( u \)-parameter curves and \( v \)-parameter curves.

Let us denote \( \mathbf{x}_u(u_o, v_o) \) and \( \mathbf{x}_v(u_o, v_o) \) as the derivatives of \( \mathbf{x} \) at \((u_o, v_o)\) along \( u \) and \( v \) axes respectively. Thus \( \mathbf{x}_u(u_o, v_o) \) and \( \mathbf{x}_v(u_o, v_o) \) are vectors tangent to the \( u \) and \( v \)-parameter curves at \( \mathbf{x}(u_o, v_o) \) along the directions of increasing \( u \) and \( v \) respectively. The regularity condition of the surface requires that \(|\mathbf{x}_u \times \mathbf{x}_v| \neq 0\), i.e. at least one of the Jacobian determinants

\[
\frac{\partial (x, y)}{\partial (u, v)} \cdot \frac{\partial (y, z)}{\partial (u, v)} \cdot \frac{\partial (x, z)}{\partial (u, v)},
\]

is different from zero at \((u, v) \in U\). Thus, the vectors \( \mathbf{x}_u(u, v), \mathbf{x}_v(u, v) \) are linearly independent at each point of a regular surface. A plane \( T_P(S) \) passing through a point \( P(x(u, v), y(u, v), z(u, v)) \) containing \( \mathbf{x}_u(u, v) \) and \( \mathbf{x}_v(u, v) \), is called the tangent plane to the surface \( S \) at \( P \). At each point on the surface, we can define two unit vectors that are orthogonal to the tangent plane and they are of opposite sense. We choose the one that makes a right handed system with \( \mathbf{x}_u(u, v) \) and \( \mathbf{x}_v(u, v) \) at the point \( P \) of the surface as

\[
\mathbf{N}(u, v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|},
\]

The unit vector \( \mathbf{N}(u, v) \) varies continuously throughout the surface as \( \mathbf{x}_u \) and \( \mathbf{x}_v \) are differentiable and is perpendicular to both vectors \( \mathbf{x}_u \) and \( \mathbf{x}_v \) and the tangent plane at \( P \). The unit vector \( \mathbf{N}(u, v) \) is called a unit normal to the surface. We define now further geometric structures on a surface, the invariant quantities namely the first fundamental form, second fundamental form, mean curvature and Gaussian curvature. For the mapping \( \mathbf{x}(u, v) \), the differential

\[
d\mathbf{x}(u, v) = \mathbf{x}_u du + \mathbf{x}_v dv,
\]

is a one-to-one linear mapping of vectors \((du, dv)\) in the \( uv \)-plane onto the vectors \( d\mathbf{x}(u, v) = \mathbf{x}_u du + \mathbf{x}_v dv \), which lie in the tangent plane. One of the invariant quantities, namely, first fundamental form of the surface is defined as follows:

**Definition 1.2.7.** The quadratic form

\[
I(du, dv) = d\mathbf{x}(u, v) \cdot d\mathbf{x}(u, v),
\]
which because of eq. (1.2.11) reduces to

\[ I(du, dv) = Edu^2 + 2Fdu dv + Gdv^2, \]  

(1.2.13)
is called the first fundamental form of the surface \( \mathbf{x}(u, v) \). Here

\[ E(u, v) = \mathbf{x}_u \cdot \mathbf{x}_u, \quad F(u, v) = \mathbf{x}_u \cdot \mathbf{x}_v, \quad G(u, v) = \mathbf{x}_v \cdot \mathbf{x}_v, \]  

(1.2.14)

are called coefficients of first fundamental form in the basis \( \{ \mathbf{x}_u, \mathbf{x}_v \} \) of the tangent plane \( T_P(S) \).

First fundamental form plays an important role in computation of arc length of a parameterized curve on the surface and areas of regions without reference to the surrounding space \( \mathbb{R}^3 \) in which the surface lies.

**Definition 1.2.8.** Let \( \mathbf{x}(\xi) = \mathbf{x}(u(\xi), v(\xi)), (\alpha \leq \xi \leq \beta) \) be a regular arc contained in a coordinate neighbourhood corresponding to the given parametrization \( \mathbf{x}(u, v) \). Thus the length of a parameterized curve \( \mathbf{x} : I \to \mathbb{R}^3 \) is given by the following integral

\[ s = \int_{\alpha}^{\beta} \left| \frac{d\mathbf{x}(\xi)}{d\xi} \right| d\xi. \]  

(1.2.15)

This may be written as,

\[ s = \int_{\alpha}^{\beta} \left( \frac{d\mathbf{x}(\xi)}{d\xi} \cdot \frac{d\mathbf{x}(\xi)}{d\xi} \right)^{1/2} d\xi. \]  

(1.2.16)

Using eq. (1.2.11) for \( \mathbf{x} = \mathbf{x}(\xi) \), above eq. (1.2.16) becomes

\[ s = \int_{\alpha}^{\beta} \left( \left( \mathbf{x}_u \frac{du}{d\xi} + \mathbf{x}_v \frac{dv}{d\xi} \right) \cdot \left( \mathbf{x}_u \frac{du}{d\xi} + \mathbf{x}_v \frac{dv}{d\xi} \right) \right)^{1/2} d\xi. \]  

(1.2.17)

In terms of coefficients of first fundamental form above equation takes the form

\[ s = \int_{\alpha}^{\beta} \left( E \left( \frac{du}{d\xi} \right)^2 + 2F \left( \frac{du}{d\xi} \right) \left( \frac{dv}{d\xi} \right) + G \left( \frac{dv}{d\xi} \right)^2 \right)^{1/2} d\xi. \]  

(1.2.18)

Last eq. (1.2.18) helps us to calculate element of arc length, \( ds \) on \( S \), through

\[ \left( \frac{ds}{d\xi} \right)^2 = E \left( \frac{du}{d\xi} \right)^2 + 2F \left( \frac{du}{d\xi} \right) \left( \frac{dv}{d\xi} \right) + G \left( \frac{dv}{d\xi} \right)^2. \]  

(1.2.19)
**Definition 1.2.9.** Let \( R \cap S \) be a bounded region of a regular surface contained in the coordinate neighbourhood of the parametrization \( \mathbf{x} : U \cap \mathbb{R}^2 \to S \). The positive number

\[
A(R) = \int \int_D |\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)| \, du \, dv,
\]

is called the area of the region \( R \) and \( D = \mathbf{x}^{-1}(R) \).

The function \( |\mathbf{x}_u \times \mathbf{x}_v| \), defined in \( U \), measures the area of the parallelogram generated by the vectors \( \mathbf{x}_u \) and \( \mathbf{x}_v \). The integral given above to determine the area of a bounded region of a surface does not depend on the parameterizations of the surface \( \mathbf{x}(u,v) \). Using the following identity

\[
|\mathbf{x}_u \times \mathbf{x}_v|^2 = (\mathbf{x}_u \cdot \mathbf{x}_u)(\mathbf{x}_v \cdot \mathbf{x}_v) - (\mathbf{x}_u \cdot \mathbf{x}_v)^2,
\]

the integral \( A(R) \) may be written as

\[
A(R) = \int \int_D \sqrt{EG - F^2} \, du \, dv.
\]

Another invariant quantity, called the second fundamental form denoted by \( II(du, dv) \), is defined as follows:

**Definition 1.2.10.** The quadratic form \( II(du, dv) \) defined in \( T_P(S) \) is given by the quantity

\[
II(du, dv) = -d\mathbf{N} \cdot d\mathbf{x},
\]

where \( \mathbf{N} = \mathbf{N}(u,v) \), the unit normal given by eq (1.2.10) to the surface \( \mathbf{x}(u,v) \) is a function of \( u \) and \( v \). The differential \( d\mathbf{N}(u,v) \) is given by

\[
d\mathbf{N} = \mathbf{N}_u du + \mathbf{N}_v dv.
\]

Note that \( \mathbf{N} \cdot \mathbf{N} = 1 \). Therefore \( \mathbf{N} \cdot d\mathbf{N} = 0 \). Thus \( d\mathbf{N} \) is orthogonal to \( \mathbf{N} \) and hence \( d\mathbf{N} \) lies in the tangent plane \( T_P(S) \). Eq. (1.2.23), along with (1.2.11) and (1.2.24), gives

\[
II(du, dv) = e \, du^2 + 2f \, du \, dv + g \, dv^2,
\]

where

\[
e = \mathbf{N} \cdot \mathbf{x}_{uu}, \quad f = \mathbf{N} \cdot \mathbf{x}_{uv}, \quad \text{and} \quad g = \mathbf{N} \cdot \mathbf{x}_{vv}
\]

are called coefficients of second fundamental form. Second fundamental form \( II(du, dv) \) is a homogeneous function of second degree in \( du \) and \( dv \).
An important geometric entity, the curvature of a curve $\gamma$ is the rate of change of the tangent line to the curve $\gamma$. The idea may be extended for a regular surface that how rapidly a surface $S$ pulls away from the tangent plane $T_P(S)$ in a neighborhood of a point $P \in S$ or equivalently to measure the rate of change at $P$ of a unit normal vector field $N$ in the neighborhood of $P$. For a regular parameterized surface $\mathbf{x} : U \subset \mathbb{R}^2 \to S$, a unit normal at a point $P \in S$ may be given by the eq. (1.2.10) at each point $Q$ in $\mathbf{x}(U)$. Thus, we have a differentiable map $N : \mathbf{x}(U) \to \mathbb{R}^3$, that associates to each $Q \in \mathbf{x}(U)$, a unit normal vector $N(Q)$, called a differentiable field of unit normal vectors on $\mathbf{x}(U)$. Not all the surfaces admit such a differentiable field of unit normal vectors e.g. for a Mobius strip one can not define such a field, as it can be seen intuitively that after a turn around the Mobius strip, the normal would point out in a direction opposite to $N$. A surface is called an orientable surface if it admits a differentiable field of unit normal vectors defined on the entire surface and choice of such a field $N$ is called an orientation of $S$.

**Definition 1.2.11.** Let $S \subset \mathbb{R}^3$ be a surface with an orientation $N$. The map $N : S \to \mathbb{R}^3$ takes its values in the unit sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3\}. \quad (1.2.27)$$

The map $N : S \to S^2$, thus defined is called the Gauss Map of $S$.

**Definition 1.2.12.** Let $\gamma$ be a regular curve $\mathbf{x}(u(\xi), v(\xi))$ in $S$ given by $\mathbf{x}(u,v)$ passing through $P \in S$, $\kappa$ the curvature of $\gamma$ at $P$, and $\cos \theta = \mathbf{n} \cdot N$, where $\mathbf{n}$ is the normal vector to $\gamma$ and $N$ is the normal vector to $S$ at $P$. The number $\kappa_n = \kappa \cos \theta$ is then called the normal curvature of $\gamma \subset S$ at $P$.

Thus $\kappa_n$ is the projection of the vector $\kappa$ over the normal to the surface at $P$, with a sign given by the orientation $N$ of $S$ at $P$ and hence $\kappa_n = \kappa_n N = (\kappa \cdot N) N$. Recalling the definitions for the unit tangent vector given by eq. (1.2.6) and the curvature vector given by eq. (1.2.7), it can be easily shown that

$$\kappa_n = \frac{II(du, dv)}{I(du, dv)}, \quad (1.2.28)$$

where $I(du, dv)$ and $II(du, dv)$ are given by the eqs. (1.2.12) and (1.2.25) respectively. For $u = u(\xi)$ and $v = v(\xi)$, eq. (1.2.28) may be rewritten as

$$\kappa_n = \frac{e (du/d\xi)^2 + 2f (du/d\xi)(dv/d\xi) + g (dv/d\xi)^2}{E (du/d\xi)^2 + 2F (du/d\xi)(dv/d\xi) + G (dv/d\xi)^2}. \quad (1.2.29)$$
Note that $\kappa_n$ depends only on the ratio $(du/d\xi)/(dv/d\xi)$ when considered as a function of $(du/d\xi)$ and $(dv/d\xi)$ i.e. the direction of the tangent line to $\gamma$ at the point $p$, otherwise it is a function of the coefficients of first and second fundamental forms. This leads to the following result by Meusnier:

**Theorem 1.2.2.** All curves lying on a surface $S$ and having at a given point $P \in S$ the same tangent line have at this point the same normal curvatures.

It may be seen [5] that for each $P \in S$, there exists an orthonormal basis $\{e_1, e_2\}$ such that the linear map $dN_P(e_1) = -\kappa_1 e_1$ and $dN_P(e_2) = -\kappa_2 e_2$ and $\kappa_1$ and $\kappa_2$ ($\kappa_1 \geq \kappa_2$) are the maximum and minimum of the second fundamental form $II_P(du, dv)$ given by eq. (1.2.25) restricted to the unit circle of $T_P(S)$. That is, they are the extreme values of the normal curvature at $P$.

**Definition 1.2.13.** The maximum normal curvature $\kappa_1$ and the minimum normal curvature $\kappa_2$ are called the principal curvatures at $P$. The corresponding directions, that is, the directions given by the eigenvectors $e_1, e_2$ are called principal directions at $P$.

**Definition 1.2.14.** Let $P \in S$ and let $dN_P : T_P(S) \to T_P(S)$ be the differential of the Gauss map. The determinant of $dN_P$ is the Gaussian curvature $K$ of $S$ at $P$. The negative of half of the trace of $dN_P$ is called the mean curvature $H$ of $S$ at $P$.

It may be seen that for a locally parameterized surface $x = x(u,v)$, mean curvature $H$ and Gaussian curvature $K$ in terms of principal curvatures $\kappa_1$ and $\kappa_2$ may be given by the following expressions:

$$H = \frac{1}{2} (\kappa_1 + \kappa_2) = \frac{Ge - 2Ff + Eg}{2(EG - F^2)},$$

$$K = \kappa_1 \kappa_2 = \frac{eg - f^2}{EG - F^2},$$

where $E(u,v), F(u,v), G(u,v), e(u,v), f(u,v)$ and $g(u,v)$ are the coefficients of first fundamental form and second fundamental form given by eqs. (1.2.14) and (1.2.26), respectively and $N(u,v)$ is the unit normal to the surface $x(u,v)$ as given by eq. (1.2.10). Here, the quantities $EG - F^2$ and $eg - f^2$ are determinants of first fundamental form and second fundamental form respectively and $Ge - 2Ff + Eg$ is the trace of the second fundamental form. Thus Gaussian curvature of a surface in $R^3$ is the ratio of the determinants of the second and first fundamental forms, and mean curvature is the half of the trace of the second fundamental form. One of the curvatures is the extrinsic property
of the surface and other is the intrinsic property of the surface. The mean curvature $H$ of a surface $S$

is the extrinsic measure of the curvature. It describes locally the curvature of an embedded surface in

another space. Gaussian curvature at a point $P \in S$ turns out to be an intrinsic property of the surface. A property of a surface is said to be an intrinsic property if and only if it depends only on the coefficients of the first fundamental form [5]. The value of the Gaussian curvature depends only on the Coefficients of the first fundamental form and its derivatives which is the content of famous theorem called Gauss's Theorema Egregium (that the Gaussian curvature is an intrinsic invariant of a surface). It may be noted that $EG - F^2 \geq 0$, thus sign of the Gaussian curvature agrees with sign of $eg - f^2$. A point on a surface is said to be elliptic if $K > 0$, hyperbolic if $K < 0$, parabolic or planar if $K = 0$. At an elliptic point both principal curvatures are of same sign and thus all curves passing through an elliptic point have their normal vectors pointing toward the same side of the tangent plane. At a hyperbolic point, the principal curvatures are of opposite signs, and therefore there are curves through $P$ whose normal vectors at $P$ point toward any of the sides of the tangent plane at $P$. At a parabolic point, one of the principal curvature is not zero but at a planar point, all the principal curvatures are zero. Below, we define the root mean square root of the mean curvature and Gaussian curvature and a dimensionless quantity of their ratios, useful in comparison related to the nature of points on the variationally improved surfaces.

**Definition 1.2.15.** The root mean square root of the quantities $H$ and $K$, for $0 \leq u \leq 1$ and $0 \leq v \leq 1$ denoted by $\mu$ and $\nu$ are given by the following expressions,

\[
\mu = \left( \int_0^1 \int_0^1 H^2 \; dudv \right)^{1/2}, \quad (1.2.32)
\]
\[
\nu = \left( \int_0^1 \int_0^1 K^2 \; dudv \right)^{1/2}. \quad (1.2.33)
\]

Mean curvature has the dimension of $1/\text{length}$ and Gaussian curvature has the dimension of $1/(\text{length})^2$. A dimensionless quantity from eqs. (1.2.32) and (1.2.33) is

\[
\frac{\mu^2}{\nu} = \frac{\int_0^1 \int_0^1 H^2 \; dudv}{\left( \int_0^1 \int_0^1 K^2 \; dudv \right)^{1/2}}, \quad (1.2.34)
\]
1.3 Minimal Surfaces

A regular parameterized surface is called minimal [5] if its mean curvature vanishes. A regular surface \( S \subset \mathbb{R}^3 \) is minimal if each of its parametrization is minimal. Let \( x : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \) be a regular parameterized surface. For a bounded domain \( D \subset U \) and a differentiable function \( h : \bar{D} \to \mathbb{R} \), where \( \bar{D} \) is the union of the domain \( D \) with its boundary \( \partial D \), the normal variation of \( x(\bar{D}) \), determined by \( h \), is the map given by \( \phi : \bar{D} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3 \) defined by

\[
\phi(u,v,t) = x(u,v) + th(u,v)N(u,v), \quad (u,v) \in \bar{D}, \ t \in (-\varepsilon, \varepsilon).
\]  

(1.3.1)

For each fixed \( t \in (-\varepsilon, \varepsilon) \), the map \( x^t : D \to \mathbb{R}^3 \) given by \( x^t(u,v) = \phi(u,v,t) \) is a parameterized surface. Let \( E,F,G \) and \( E^t,F^t,G^t \) denote first fundamental magnitudes of \( x(u,v) \) and \( x^t(u,v,t) \) respectively, \( e,f,g \) second fundamental magnitudes of \( x(u,v) \), \( N = N(u,v) \) unit normal given by eq.(1.2.10) to the surface \( x(u,v) \) and \( H \), the mean curvature given by eq. (1.2.30) of the surface \( x(u,v) \) at \( (u,v) \). Thus it can be shown [5]

\[
E^tG^t - (F^t)^2 = (EG - F^2)(1 - 4thH) + R,
\]  

(1.3.2)

where \( \lim_{t \to 0} (R/t) = 0 \). It follows that \( x^t \) is a regular parameterized surface for sufficiently small \( \varepsilon \).

The area \( A(t) \) of \( x^t(\bar{D}) \) is

\[
A(t) = \int_{\bar{D}} \sqrt{E^tG^t - (F^t)^2} \, dudv.
\]  

(1.3.3)

Substituting values of \( E^t, F^t, G^t \) in the above expression, we find

\[
A(t) = \int_{\bar{D}} \sqrt{1 - 4thH + \bar{R}} \sqrt{EG - F^2} \, dudv,
\]  

(1.3.4)

where \( \bar{R} = R/(EG - F^2) \). Thus for sufficiently small \( \varepsilon \), \( A(t) \) is differentiable function and its derivative at \( t = 0 \) is

\[
A'(0) = -\int_{\bar{D}} 2hH \sqrt{EG - F^2} \, dudv.
\]  

(1.3.5)

\( x(u,v) \) is minimal if and only if \( A'(0) = 0 \) for all such \( D \), i.e. normal variations of \( x(\bar{D}) \) that \( H \) must vanish. Using (1.3.5) this is possible if

\[
H = \frac{1}{2} \frac{Eg - 2Ff + Ge}{EG - F^2} = 0.
\]  

(1.3.6)
1.4 The Plateau Problem

Minimal surfaces are usually associated with soap films that can be obtained by immersing a wire frame into a soapy water, representing the boundaries, and withdrawing it back carefully from the soapy water. The related problem of existence and regularity of soap bubbles and the geometry of soap bubbles can be seen in the refs. [48, 49]. It can be shown [5] by physical considerations that the film will assume a position where at its regular points, principal curvatures (maximum bending) are equal in magnitude and opposite in direction i.e. the average or the mean curvature (1.2.30) is zero. This can generate minimal surfaces of different shapes depending on the wire frames. Soap films can also be formed that have singularities along the lines of the frames e.g. a cube frame has singularities [50] along its lines. The Plateau’s problem was motivated by this connection between minimal surfaces and soap films, which can be described as that for each closed curve \( \gamma \subset \mathbb{R}^3 \), there exists a surface \( S \) of minimum area with \( \gamma \) as its boundary. For convenience we choose the mean curvature as a vector defined by \( \mathbf{H} = H \mathbf{N} \) for an arbitrarily parameterized surface. The eq. (1.3.5) gives geometrical meaning to this vector quantity \( \mathbf{H} \). Let us choose \( h = H \), we have, for this particular variation,

\[
A'(0) = -\int_D 2(\mathbf{H} \cdot \mathbf{H}) \sqrt{EG - F^2} \, dudv < 0. \tag{1.4.1}
\]

This simply says that the initial area decreases in the direction of the vector \( \mathbf{H} \) by the deformation of \( \mathbf{x}(D) \). Following definitions are useful to understand the Weierstrass-Enneper representation of a minimal surface (section 1.5).

**Definition 1.4.1.** A regular parameterized surface \( \mathbf{x}(u, v) \) is isothermal if \( E = \mathbf{x}_u \cdot \mathbf{x}_u = x_u^2 = G = \lambda^2(u,v) \) and \( F = \mathbf{x}_u \cdot \mathbf{x}_v = 0 \). In other words local coordinates \( (u,v) \) on a surface \( M \) are called isothermal (conformal) if the metric \( ds^2 \) is given by \( ds^2 = \lambda^2 \left( du^2 + dv^2 \right) \) on \( M \), where \( \lambda^2 \) is a function of isothermal coordinates \( (u,v) \), called the conformal factor. Thus, in isothermal coordinates the tangential coordinate vectors at each point are perpendicular and have the same length.

An isothermal parametrization maps the \( uv \)-plane onto the given surface in a conformal, or angle preserving way [3, 4]. Typically, \( u \) and \( v \) are called isothermal parameters. In terms of the
matrix representation of the first fundamental form, we have \( E = G \) and \( F = 0 \), a symmetric form. Isothermal coordinates exist on any regular surface and on a minimal surface. Following result is about the existence of the isothermal coordinates on a surface. For a related detailed account of the general case, consult Chern [51] (or ref [52] page (54) which derives the explicit formulas for the isothermal coordinates for a a minimal surface).

**Theorem 1.4.1.** Let \( M \) be an arbitrary regular (of class \( C^2 \)) surface in \( \mathbb{R}^3 \). Then for any point \( P \) of \( M \) there is a neighborhood \( U, P \in U \), and local coordinates \((u,v)\) in this neighborhood \( U \) such that the induced metric \( ds^2 \) on \( M \), written in the \((u,v)\) coordinates, is given by \( ds^2 = \lambda^2(u,v)(du^2 + dv^2) \).

**Definition 1.4.2.** The radius vector \( x(u,v) \) of a surface \( M \) specified in the \((u,v)\)−coordinates that satisfies the condition \( \nabla^2 x(u,v) = 0 \), where \( \nabla^2 = \partial^2 / \partial u^2 + \partial^2 / \partial v^2 \), is called a harmonic radius vector. A generalization of a harmonic radius vector is a harmonic mapping. In isothermal coordinates the minimality of a surface is equivalent to the harmonicity of the radius vector specifying it, that is what we have described in the following result [5].

**Proposition 1.4.2.** Let \( x = x(u,v) \) be a regular parameterized surface and assume that \( x \) is isothermal. Then

\[
x_{uu} + x_{uv} = 2\lambda^2 H, \tag{1.4.2}
\]

where \( \lambda^2 = x_u \cdot x_u = x_v \cdot x_v \) and \( H = Hn \) is the mean curvature vector.

The Laplacian of a differentiable function \( h : U \subset \mathbb{R}^2 \to \mathbb{R} \) is defined by \( \Delta h = \partial^2 h / \partial u^2 + \partial^2 h / \partial v^2, \) \((u,v) \in U \). For \( H = 0 \), \( \Delta h = 0 \) and hence \( h \) is harmonic in \( U \). A direct consequence of the above theorem is the following result.

**Corollary 1.4.3.** Let \( x(u,v) = (x(u,v), y(u,v), z(u,v)) \) be a parameterized surface and assume that \( x \) is isothermal. Then \( x \) is minimal if and only if its coordinate functions \( x, y, z \) are harmonic.

**Remark 1.4.1.** For a parameterized surface \( x = x(u,v) \), for which mean curvature is given by the eq. (1.2.30), the vanishing condition of mean curvature is a partial differential equation. For example, for a Monge’s form of surface \( x = (u,v,h(u,v)) \), a locally parameterized surface where \( x : U \to \mathbb{R}^3 \) with \( U \) an open set in \( \mathbb{R}^2 \), the fundamental magnitudes are given by

\[
E = 1 + h_u^2, \quad F = h_u h_v, \quad G = 1 + h_v^2, \tag{1.4.3}
\]
Mean curvature for a Monge’s form of surface is given by substituting above values in the eq. (1.2.30), that results in the following expression

\[ H = \frac{(1 + h_v^2) h_{uu} - 2h_u h_v h_{uv} + (1 + h_u^2) h_{vv}}{2 (1 + h_u^2 + h_v^2)}. \]  

(1.4.5)

The vanishing condition of mean curvature for a minimal surface becomes

\[ (1 + h_v^2) h_{uu} - 2h_u h_v h_{uv} + (1 + h_u^2) h_{vv} = 0. \]  

(1.4.6)

Using eq. (1.2.31), we can find Gaussian curvature for Monge’s form of surface as

\[ K = \frac{h_{uu} h_{vv} - h_{uv}^2}{1 + h_u^2 + h_v^2}. \]  

(1.4.7)

In the optimization problem we aim for here, we eventually try to find a surface of a known boundary that has a least value of area. Area is evaluated by the area functional eq. (1.2.20), where \( D \subset \mathbb{R}^2 \) is a domain over which the surface \( x(u,v) \) is defined as a map, with the boundary condition \( x(\partial D) = \Gamma \), where \( \Gamma \) is the boundary curve for \( 0 \leq u \leq 1 \) and \( 0 \leq v \leq 1 \), \( x_u(u,v) \) and \( x_v(u,v) \) being partial derivatives of \( x(u,v) \) with respect to \( u \) and \( v \). It is known [5] that the first variation of \( A(x) \) vanishes everywhere if and only if the mean curvature \( H \) (eq. (1.2.30)) of \( x(u,v) \) is zero everywhere in it. Thus a surface of least area is also a surface of least \((zero) \) rms mean curvature spanning the given boundary. This means we can aim for the same surface using the condition of the least means square mean curvature in place of the condition of the least area. This is helpful as, unlike area, the \( ms \) mean curvature \( \mu^2 \) has not a square root in its integrand. For a minimal surface [5], [6] the mean curvature eq. (1.2.30) is identically zero. For minimization we use only the numerator part of mean curvature \( H \) given by eq. (1.2.30), as done in ref. [53] following ref. [3] who writes that “for a locally parameterized surface, the mean curvature vanishes when the numerator part of the mean curvature is equal to zero”. We use this numerator part \( H_0 \) to compute the \( rms \) curvature of the initial surface \( x_0(u,v) \) to be used in the ansatz eq. (4.1.11) to get first order variationally improved surface \( x_1(u,v) \) of lesser area. This process could be continued as an iterative process until a minimal surface is achieved. Once a minimal surface is achieved, this may help us to construct a conformal (isothermal) mapping from the complex \( w \)-plane to the boundary composed of arbitrary number of curves (utilizing the recursive relation to reduce this boundary
to four discussed in chapter 3). This will generate a geometrically intuitive local representation of minimal surface by a pair of complex-valued functions \( p(w) \) and \( q(w) \) (where \( w = \xi + i\eta \) being the corresponding complex coordinate), the so called Weierstrass representation of a minimal surface. Geometric characteristics of minimal surfaces, like the metric, the Gaussian curvature, the Gaussian mapping, and so on may be expressed in terms of the holomorphic functions \( p(z) \) and \( q(z) \), thus defining a complex structure on the surface that may provide deep insight into the surface for example by knowing the analyticity and the harmonicity properties of these complex valued functions. Thus, a minimal surface achieved through the iterative numerical processes may be compared with the exact expressions obtained using Weierstrass integral representation for the meaningful results. Using differential forms generalizes Wierstrass representation for constant mean curvature surfaces that yields Hamiltonian systems. The near-minimal surfaces obtained for the boundary composed of four and five lines in the chapter 4 may initially be the promising candidates for such an analysis to obtain exact expressions in terms of complex-valued functions \( p(z) \) and \( q(z) \). In the following section we recall some facts related to minimal surfaces and the complex differential geometry, and present the Weierstrass representation of a minimal surface in terms of complex functions of a complex variable.

1.5 The Weierstrass-Enneper Integral Representations

Theory of minimal surfaces may be developed as well with the help of complex analysis. One of such connections is the Weierstrass-Enneper Integral Representation of Minimal Surfaces which tells that any minimal surface may be represented by a pair of analytic functions in an appropriate conformal parametrization, and conversely [13, 52]. By requiring the parametrization of a minimal surface to be isothermal (see section 1.4), we can begin using complex analysis to better understand geometric characteristics of minimal surfaces by expressing them in terms of the functions \( (p, q) \) of the Weierstrass representation, namely the metric, the Gaussian curvature, the Gaussian mapping,
and so on. The representation gives the possibility of constructing a large number of new examples of minimal surfaces and is an essential tool in the theory of two dimensional minimal surfaces in $\mathbb{R}^3$.

Let us recall some related results from complex differential geometry.

Let $f(u,v)$ be a complex-valued function in a planar domain $U \subset \mathbb{R}^2$ with coordinates $(u,v)$. Let $z = u + iv$ be the corresponding complex coordinate and $\bar{z}$ its complex conjugate. The complex differential operators are given by

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

(1.5.1)

Complex notation is often convenient for example for $f(u,v) = \phi(u,v) + i\psi(u,v)$, it is easy to verify [52] that,

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) (\varphi + i\psi) = \frac{1}{2} \left( \frac{\partial \phi}{\partial u} + \frac{\partial \psi}{\partial v} + i \left( \frac{\partial \psi}{\partial u} - \frac{\partial \phi}{\partial v} \right) \right),$$

(1.5.2)

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) (\varphi + i\psi) = \frac{1}{2} \left( \frac{\partial \phi}{\partial u} - \frac{\partial \psi}{\partial v} + i \left( \frac{\partial \psi}{\partial u} + \frac{\partial \phi}{\partial v} \right) \right),$$

(1.5.3)

and

$$f_{uu} + f_{vv} = 4 \left( \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial \bar{z}} \right) \right).$$

(1.5.4)

It can be seen easily that Cauchy-Riemann conditions for a function $f(u,v)$ to be analytic are equivalent to

$$\frac{\partial f}{\partial \bar{z}} = 0.$$ 

(1.5.5)

Therefore, if above condition is satisfied by a function $f(u,v)$, then $f$ is analytic and its Taylor series does not contain $\bar{z}$ and we say that $f$ does not depend on $\bar{z}$.

**Definition 1.5.1.** Let $M$ be a minimal surface with an isothermal parametrization $\mathbf{x} = \mathbf{x}(u,v)$. Let $z = u + iv$ be the corresponding complex coordinate. Formally, we can solve for $u,v$ in terms of $z,\bar{z}$ to get $u = (z + \bar{z})/2$ and $v = (z - \bar{z})/(2i)$. Then the parametrization of $M$ can be written as:

$$\mathbf{x}(u,v) = (x_1(z, \bar{z}), x_2(z, \bar{z}), x_3(z, \bar{z})), $$

(1.5.6)

where $(x_1, x_2, x_3)$ are the standard Euclidean coordinates in $\mathbb{R}^3$. Let us regard the functions $x_k(z, \bar{z})$ to be complex-valued for $k = 1, 2, 3$. We can define using eq. (1.5.1) that

$$\frac{\partial x_k}{\partial z} = \frac{1}{2} \left( \frac{\partial x_k}{\partial u} - i \frac{x_k}{\partial v} \right).$$

(1.5.7)
Let
\[ \phi = \frac{\partial \mathbf{x}}{\partial z} = \left( \frac{\partial x_1}{\partial z}, \frac{\partial x_2}{\partial z}, \frac{\partial x_3}{\partial z} \right) = (\phi_1, \phi_2, \phi_3). \] (1.5.8)

Then
\[ (\phi)^2 = \sum_{k=1}^{3} (\phi_k)^2 = \sum_{k=1}^{3} \left( \frac{\partial x_k}{\partial z} \right)^2. \] (1.5.9)

A simple computation shows that
\[ (\phi)^2 = \frac{1}{4} \left( \left| \frac{\partial \mathbf{x}}{\partial u} \right|^2 - \left| \frac{\partial \mathbf{x}}{\partial v} \right|^2 - 2i \left( \frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial \mathbf{x}}{\partial v} \right) \right) = \frac{1}{4} (E - G - 2iF). \] (1.5.10)

Thus, we obtain the following result from the above eq. (1.5.10), as \( E = G \) and \( F = 0 \) for the isothermal coordinates \((u, v)\) on the surface \( M \) that
\[ (\phi)^2 = 0, \] (1.5.11)

which is the condition for the coordinates \((u, v)\) to be isothermal. Also note that
\[ |\phi|^2 = \sum_{k=1}^{3} |\phi_k|^2 = \sum_{k=1}^{3} \left| \frac{\partial x_k}{\partial z} \right|^2. \] (1.5.12)

Using eq. (1.5.7) in above eq. (1.5.12), reduces it to (for \( E = G \) for isothermal coordinates),
\[ |\phi|^2 = \frac{1}{2} \left( \left| \frac{\partial \mathbf{x}}{\partial u} \right|^2 + \left| \frac{\partial \mathbf{x}}{\partial v} \right|^2 \right) = \frac{1}{2} (E + G) = \lambda^2 \neq 0. \] (1.5.13)

Above eq. (1.5.13) specifies the condition for the surface \( M \) to be regular, that is, for \( \mathbf{x}_u \) and \( \mathbf{x}_v \) to be linearly independent. This is the regularity condition for the coordinates to be isothermal.

Finally, using eq. (1.5.1) and harmonicity condition for the minimal surfaces, we find that
\[ \frac{\partial \phi}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} \left( \frac{\partial \mathbf{x}}{\partial z} \right) = \frac{1}{4} \left( \frac{\partial^2 \mathbf{x}}{\partial u^2} + \frac{\partial^2 \mathbf{x}}{\partial v^2} \right) = \frac{1}{4} \nabla^2 \mathbf{x} = 0. \] (1.5.14)

Therefore \( \phi \) is holomorphic (all the coordinates of the function are holomorphic) and hence all of its components. In isothermal coordinates the minimality of a surface is equivalent to the harmonicity of the vector function \( \mathbf{x}(u, v) \) that specifies this surface. Thus,

**Corollary 1.5.1.** Let \( M \) be a parameterized surface in \( \mathbb{R}^3 \) specified locally by a vector function \( \mathbf{x}(u, v) \). Let \( z = u + iv \) be the corresponding complex coordinate and \( \phi = \partial \mathbf{x}/\partial z \) such that \((\phi)^2 = 0\). Then the fact that \( M \) is minimal is equivalent to \( \phi \) being holomorphic.
By Corollary 1.5.1 all minimal surfaces can be described locally (in a simply connected domain) by holomorphic functions \((\varphi^1, \varphi^2, \varphi^3)\) that satisfy the conditions given by eqs. (1.5.10) and (1.5.13). It is obvious that integrating the holomorphic functions \(\varphi_k\) restores the functions \(x^k(z, \bar{z})\) that gives a minimal surface \(x(u, v)\). It is summarized in the following result.

**Theorem 1.5.2.** Let \(M\) be a surface with parametrization \(x = (x_1, x_2, x_3)\) and let \(\varphi = (\varphi_1, \varphi_2, \varphi_3)\), where \(\varphi_k = \partial x_k / \partial z\). Then the given parameterized surface \(x\) is isothermal (i.e. \(E = G, F = 0\)) if and only if \(\varphi_2 = (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^3 = 0\). If \(x\) is isothermal then \(M\) is minimal if and only if each \(\varphi_k\) is analytic.

**Definition 1.5.2.** A pair of complex-valued functions \((p, q)\), defined on a simply-connected domain \(U\) of the complex plane \(\mathbb{C}\), such that \(p\) is analytic, \(q\) is meromorphic, and the product \(pq\) is analytic, is called a Weierstrass representation.

It follows from theorem 1.5.2 that \(x_k(z, \bar{z}) = c_k + 2 \text{Re} \int \varphi_k dz\) and hence we can find a formula for constructing minimal surfaces by determining analytic functions \(\varphi_k\) \((k = 1, 2, 3)\) such that \(\varphi^2 = 0\) and \(|\varphi|^2 \neq 0\). Let \(p(z)\) and \(q(z)\) be complex differentiable functions and consider

\[
\varphi_1 = p \left( 1 + q^2 \right), \quad \varphi_2 = -ip \left( 1 - q^2 \right), \quad \varphi_3 = -2ipq. \tag{1.5.15}
\]

Substitute \(\varphi_1, \varphi_2\) and \(\varphi_3\) from eq. (1.5.15) in

\[
\varphi^2 = (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2, \tag{1.5.16}
\]

to get,

\[
\varphi^2 = \left( p \left( 1 + q^2 \right) \right)^2 + \left( -ip \left( 1 - q^2 \right) \right)^2 + \left( -2ipq \right)^2, \tag{1.5.17}
\]

which reduces to

\[
\varphi^2 = \left( p^2 + 2p^2q^2 + p^2q^4 \right) - \left( p^2 - 2p^2q^2 + p^2q^4 \right) - \left( 4p^2q^2 \right) = 0. \tag{1.5.18}
\]

Now substitute \(\varphi_1, \varphi_2\) and \(\varphi_3\) from eq. (1.5.15) in

\[
|\varphi|^2 = |\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2, \tag{1.5.19}
\]

to get

\[
|\varphi|^2 = \varphi_1 \left( \bar{\varphi}_1 \right) + \varphi_2 \left( \bar{\varphi}_2 \right) + \varphi_3 \left( \bar{\varphi}_3 \right), \tag{1.5.20}
\]
which may be simplified to get

$$|\varphi|^2 = |p|^2 ((1 + q^2) (1 + \bar{q}^2) + (1 - q^2) (1 - \bar{q}^2) + 4pq),$$

(1.5.21)

that reduces finally to

$$|\varphi|^2 = 4|p|^2 \left(1 + |q|^2\right)^2 \neq 0.$$

(1.5.22)

It is to be noted that if \( p = 0 \) then \( \varphi_k = 0 \) for all \( k \).

Keeping in mind above terminology for the minimal surfaces and complex variable functions and their relation with minimal surfaces, we now state in the following result [52], Weierstrass Integral Representation for the minimal surfaces.

**Theorem 1.5.3.** Let \( p(w) \) be an analytic function and \( q(w) \) a meromorphic function in some domain \( \Omega \in \mathbb{C} \) (where \( w \) is the corresponding complex coordinate), having the property that at each point where \( q(w) \) has a pole of order \( m \), \( p(w) \) has a zero of order at least \( 2m \). Then every regular minimal surface has a local isothermal parametric representation of the form, for \( \mathbf{x} = (x_1(z), x_2(z), x_3(z)) \),

\[
\begin{align*}
  x_1(z) &= \text{Re} \int_0^z p(1 + q^2) \, d\omega, \\
  x_2(z) &= \text{Re} \int_0^z -ip(1 - q^2) \, d\omega, \\
  x_3(z) &= \text{Re} \int_0^z -2ipq \, d\omega.
\end{align*}
\]

(1.5.23)

**1.6 Related Developments**

It is known [5] that the first variation of \( A(x) \) vanishes if and only if the mean curvature \( H \) of \( \mathbf{x}(u,v) \) is zero everywhere in it. Thus to get a minimal (or, more precisely, a stationary) surface (section 1.3), we have to solve the differential equation obtained by setting the mean curvature \( H = 0 \) for each value of the two parameters, say, \( u \) and \( v \) parameterizing a surface spanning the fixed boundary (for example for Monge’s form of surface setting \( H = 0 \) reduces to the partial differential equation eq. (1.4.6)). In a numerical work, the problem has to be discretized by choosing a selection of the numerical values of the two parameters and finding the minimal surface position.
for each pair of the values. If the given boundary is a four-sided figure whose projection on a plane is a rectangle, the surface positions become simply the heights above the plane. Such numerical heights and resulting numerical minimal surfaces have been computed in ref. [53] for a variety of closed curve boundaries. A modification to the above algorithm was reported [54] to their algorithm that used linear combinations of Chebyshev polynomials as heights at the discretized positions. In this way, they replaced in the algorithm arbitrary heights by linear combinations of convenient polynomials with arbitrary coefficients. The immediate advantage of this use of polynomials was a reduction in the discretization error and a better convergence because polynomials are (smooth) analytic functions having simply calculable derivatives.

1.6.1 **Bilinear Interpolation**

In ref. [54] Furui and Masud have calculated the surface of least area bounded by four-sided figures whose projection on a plane is a rectangle, starting with the bilinear interpolation and using, for smoothness, the Chebyshev polynomial expansion in their discretized numerical algorithm to get closer to satisfying the zero mean curvature condition. They also have reported values for both the bilinear and improved areas, suggesting a quantitative evaluation of the bilinear interpolation. An analytical expression of the Schwarz minimal surface with polygonal boundaries and its 3-dimensional plot is also given. They [54] carried further their efforts to find analytic surfaces that can be taken as approximate minimal surfaces:

1. They read initial heights from a ruled analytic surface, namely the bilinear interpolation, spanning fixed boundary.

2. For knowing how much heights change on the exact minimal surface through numerical minimization they compared the areas generated by some numerically found points with those of the bilinear interpolation for each of the selected boundaries.

Bilinear interpolation has applications in mathematical modeling of the gluonic field in a two quarks, two anti-quarks system [41, 55].
1.6.2 Mean Curvature Flow and its Alternatives

Mean curvature flow is one of the numerical techniques to yield a surface. A surface is said to be minimal if its mean curvature given by eq. (1.2.30) is zero. A surface that evolves with mean curvature of the surface is governed by a heat type equation called the mean curvature flow equation. Minimal surfaces are the critical points of the mean curvature flow. In ref. [50], the minimal surfaces in connection with mean curvature flow are discussed along with the classification and structure of embedded minimal surfaces and the stable singularities of mean curvature flow. Ref. [56] presents an algorithm which allows to split the calculation of the mean curvature flow of surfaces with or without boundary into a series of Poisson problems on a series of surfaces. This gives a method to solve Plateau’s problem for constant mean curvature surfaces. Ref. [53] and [54] are alternative to Mean Curvature flow. They use the symbolic capabilities of Mathematica to formulate a discretized form of the problem and solve it numerically. Numerical technique mean curvature flow suffers from the problem of not giving any analytical expression in terms of parameters but yield grid points and plot approximately the surface.

1.6.3 Harmonic Approximation

Ref. [57] is related to harmonic approximation because it is shown there that minimizing area means (in the approximation used there) solving a Laplace (harmonic) differential equation. This is done using complex analysis concepts with the help of Schwarz-Christoffel transformation.

1.6.4 Use of Differentiable Form of Step Function

In chapter 3 of this dissertation we use different forms of the step function to join the curves in each of the four groups (see section 3.3) into a single analytic curve. This joining lets us use eq. (2.5.3) to write the Coons patch which is used in trying to find a minimal surface for a fixed boundary, this is the Plateau problem. In refs. [27, 58], a differentiable form of the step function is used for effectively deleting, through its vanishing value, a non-desired piece of a surface and keep only a desired piece of a surface for a corresponding range of parameters; practically this amounts to changing the boundary of the piece of the surface that is not deleted (in contrast to our problem in chapter 4 in which we find a minimal surface for a fixed boundary instead of shrinking the boundary).
[27], an example is given for a problem related to generating an approximately kink-free (non-self-overlapping structured quadrilateral grid in a given four sided 2D) region in $uv$–plane that generates by a Hermite Interpolation Mapping (HIM) a non-self-overlapping surface in 3D. The authors of this paper achieve this aim by shrinking the region with inverted normal; this results in a reduced area of the region. This area is reduced through a reduction in the region with inverted normal, as only the portion with inverted normal gives a non-zero output of the Heaviside function $H(t)$ into the functional minimization integral $\min E$ (section 4 page 274 of ref. [27]), where $E$ counts exactly the number of points with negative Jacobian in the $\xi, \eta$ square ($0 \leq \xi, \eta \leq 1$), by minimizing it this minimizes the degree of self-overlapping, and a zero $E$ indicates a mapping with no self-overlapping. The resulting vanishing output is what reduces the area that contributes to non-self-overlapping area $A$ of eq. (6) of reference [58] or $E$ of section 4 in reference [27] and thus the origin of the reduction of area in both papers is an elimination of the region with non-inverted normal from the integrand of $A$ of eq. (6) of reference [58] or $E$ of section 4 in reference [27]. In these references ([27] and [58]) area of the planar region is reduced by modifying/shrinking the boundary of the region with inverted normal that contributes to area without addressing the local (differential) geometry characteristics like mean curvature. Hence their problem is practically different to, for example, reducing the area for a boundary that is fixed like the one we address in chapter 4. In our case, the reduction of area is achieved solely through an improvement in the local characteristics of the surface like mean curvature; this is stated in section 3.3. As an initial surface for this minimization, we suggest a way to produce, for a given set of finite number of consecutive curves, a set of 4 boundary curves that can generate a Coons patch. In this way we have also extended the range of applicability of the Coons patch in computer aided geometric design or other research areas. The sketch of this dissertation has been already written at the end of section 1.1.
Chapter 2

Coons patch

In this chapter we introduce briefly multi-linear interpolation, in particular linear and bilinear interpolation, and the related concepts like blended functions, ruled surfaces and blending based Coons patch. Coons patch is used as initial surface in the variational improvement of chapter 4.

2.1 Multi-linear Interpolation

Discrete data set or a set of values for a function of one or more variables resulting from experimental observations or numerical computations may be used to study a mathematical model to predict its related properties. This is usually the requirement to find the value of the function at a point that lies in between two points by finding an interpolant that passes through these points. In interpolation we construct new data points within the range of a discrete set of known data points and then construct a function passing exactly through these data points. Different interpolation techniques are available depending on the nature of the problem, number of the data points, efficiency and accuracy desired in an algorithm used. These are namely linear interpolation, polynomial interpolation, spline interpolation and interpolation making use of rational or trigonometric functions etc. Linear interpolation is a method of curve fitting using linear polynomials that is heavily employed in mathematics particularly in numerical analysis, and in numerous applications including computer graphics. Linear interpolation between two known points is a straight line between these two points. Thus linear interpolation on a given set of data points in a plane is defined as the connection of linear interpolants between each pair of data points. This results in a continuous curve, with a
discontinuous derivative, thus of differentiability class $C_0$. A linear interpolant is a linear function and serves to generalize the problem to polynomial interpolation when the linear interpolant is replaced by a polynomial of higher degree. The examples of polynomial interpolation are Whittaker–Shannon \[59\] interpolation formula for a set of infinite number of data points, Hermite Interpolation \[60\] for finding a function when the data points as well as derivatives at those data points are known, Newton Form of Interpolation polynomials, Gauss interpolation formulae etc. Interpolation of functions of many variables is called multivariate interpolation. For example bilinear interpolation is used to interpolate functions of two variables say $u$ and $v$ on a regular 2-dimensional grid. The notion behind the bilinear interpolation is to perform linear interpolation in one dimension and then to perform linear interpolation in the second dimension. Trilinear interpolation is used to interpolate a function linearly in three dimensions. For the given data points on the lattice, this type of interpolation tries to locate approximately the position of an intermediate point $(x,y,z)$ within the local axial rectangular grid linearly and is quite often used in computer graphics and data analysis in numerical analysis. For example, for given eight lattice points $\psi(0,0,0), \psi(1,0,0), \psi(0,1,0), \psi(0,0,1), \psi(1,0,1), \psi(0,1,1), \psi(1,1,0)$, and $\psi(1,1,1)$ on a three dimensional rectangular grid, a trilinear interpolation may be given by,

\[
\psi(u,v,w) = \psi(0,0,0) (1-u) (1-v) (1-w) + \psi(1,0,0)u (1-v) (1-w) \\
+ \psi(0,1,0) (1-u) v (1-w) + \psi(0,0,1) (1-u) (1-v) w \\
+ \psi(1,0,1)u (1-v) w + \psi(0,1,1) (1-u) vw \\
+ \psi(1,1,0)uw (1-w) + \psi(1,1,1)uvw.
\]

(2.1.1)

For the linear interpolation in higher dimensions, we are unable to visualize geometry of the shapes; however same intuition may lead to higher dimensional interpolation. Rest of the chapter is devoted to the discussion of linear interpolation, blended functions, blended surfaces, ruled surfaces, bilinear interpolation and Coons patch, a bilinear starting initial surface spanned by fixed boundary curves to be worked out for its reduction in area in section 4.2.
2.1.1 Linear Interpolation

For two distinct points \((u_0, v_0)\) and \((u_1, v_1)\) in \(R^2\), a linear interpolant to find the unknown function \(h(u)\) for \(u_0 \leq u \leq u_1\) satisfies the following equation of a straight line

\[
\frac{h(u) - h(u_0)}{h(u_1) - h(u_0)} = \frac{u - u_0}{u_1 - u_0}.
\]  

(2.1.2)

Equivalently, we can write it in the form

\[
h(u) = \frac{u_1 - u}{u_1 - u_0} h(u_0) + \frac{u - u_0}{u_1 - u_0} h(u_1),
\]  

(2.1.3)

which is the linear interpolant function in the direction of \(u\). However, in order to find linear interpolant function for the two distinct points \(a\) and \(b\) in \(R^3\), it is convenient to use its parametric form. For the two distinct points \(a\) and \(b\) in \(R^3\), the set of all points \(x \in R^3\) satisfying

\[
x = x(t) = (1 - t)a + tb; \quad t \in R,
\]  

(2.1.4)

is the straight line that passes through the points \(a\) and \(b\). The straight line passes through \(a\) for \(t = 0\) and it passes through \(b\) for \(t = 1\). The point \(x\) lies in \(a\) and \(b\) for \(0 \leq t \leq 1\), otherwise it lies outside for all other values of \(t\). Equation (2.1.4) represents \(x\) as a barycentric combination (location of a point given in terms of center of mass e.g. when masses placed at its vertices) of two points in \(R^3\). For \(t \in R\), note that \(t = (1 - t) \cdot 0 + t \cdot 1\) implies that \(t\) is related to 0 and 1 by the same barycentric combination in \(R\) that relates \(x\) to \(a\) and \(b\) in \(R^3\). By the definition of affine maps, thus three points \(a, x, b\) are an affine map of the three one dimensional points 0,\(t, 1\). Hence, linear interpolation is an affine map of the real line onto a straight line in \(R^3\). Thus the straight line segment \([a, b]\) may be defined as the affine image of the unit interval \([0, 1]\). Also the straight line segment may be viewed as the affine image of any interval \([a, b]\). The interval \([a, b]\) may itself be obtained by an affine map from the interval \([0, 1]\) or vice versa. A mapping for \(t \in [0, 1]\) and \(u \in [a, b]\) is given by \(t = (u - a)/(b - a)\). The interpolated point on the straight line may now be given by

\[
x(t) = (1 - t)a + t b, \quad t \in R.
\]  

(2.1.5)

and

\[
x(u) = \frac{b - u}{b - a} a + \frac{u - a}{b - a} b.
\]  

(2.1.6)
It is well known [29] that linear interpolation is invariant under affine domain transformations, as \(a, u, b\) and \(0, t, 1\) are in the same ratio as the triple \(a, x, b\), where affine domain transformation is simply an affine map of the real line onto itself. The parameter \(t\) is sometimes called a local parameter of the interval \([a, b]\). For more general application, we can assume that a barycentric combination of three domain points \(r, s, t\) (not necessarily involving any interval endpoints)\n
\[s = (1 - \alpha) r + \alpha t\]  \hspace{1cm} (2.1.7)\n
maps to the corresponding range points given by\n
\[x(s) = (1 - \alpha) x(r) + \alpha x(t)\]  \hspace{1cm} (2.1.8)\n
\section{2.2 Blending Functions} \n
We state here few definitions related to blending functions, latter to be used to define blended surfaces in context to bilinear interpolation and Coons patch.

**Definition 2.2.1.** A function space [29] is a collection of functions of a given kind from one set to another set and is a vector space for a given basis (a linearly independent spanning set). A continuous function in a function space may be expressed as a linear combination of the basis functions. These basis functions are called blending functions in the literature of numerical analysis and approximation theory. The blending functions find their use in interpolation and they are used to define curves and surfaces. For an interval \([a, b]\) on the real line, we assume that \(C[a, b]\) denotes the set of all real-valued continuous functions. Let us assume that

\[(\alpha f + \beta g)(t) = \alpha f(t) + \beta g(t), \text{ for all } t \in [a, b], f, g \in C[a, b].\]  \hspace{1cm} (2.2.1)\n
Above relation defines the addition and multiplication of all elements of \(C[a, b]\) by a constant. We can then show that \(C[a, b]\) is a linear space over the real numbers. This is also true about the sets \(C^k[a, b]\) of real valued functions defined over \([a, b]\), which are \(k\)-times continuously differentiable functions. In addition to that \(C^{k+1}\) is a subspace of \(C^k\) for all \(k\). Furthermore, \(n\) functions \(f_1, f_2, ..., f_n \in C[a, b]\) are said to be linearly independent if
\[ \sum c_i f_i = 0, \forall t \in [a, b] \Rightarrow c_1 = \ldots = c_n = 0. \] (2.2.2)

The spaces of all polynomials of degree \( n \) denoted by \( \mathbb{P}^n \) given by \( p^n = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n \), \( \forall t \in [a, b] \) form subspaces of \( C[a, b] \). For fixed \( n \) the dimension of \( \mathbb{P}^n \) is \( n + 1 \): each \( p^n \in \mathbb{P}^n \) is determined uniquely by the \( n + 1 \) coefficients \( a_0, a_1, \ldots, a_n \). These can be interpreted as a vector in \( n + 1 \) dimensional linear space \( \mathbb{R}^{n+1} \), which has dimension \( n + 1 \). The monomials \( 1, t, t^2, \ldots, t^n \) are \( n + 1 \) linearly independent functions and thus form a basis for \( \mathbb{P}^n \). Another class of subspaces of \( C[a,b] \) may be defined by piecewise linear functions. For this assume that \( a = t_0 < t_1 < \ldots, t_n = b \) be a partition of the interval \( [a, b] \). A piecewise linear function is defined as a continuous function that is linear on each subinterval \( [t_i, t_{i+1}] \). Then the piecewise linear functions form a linear function space on a fixed partition of interval \( [a, b] \). A basis for this space may be given by the hat functions \( H_i(t) \) which are piecewise linear functions satisfying \( H_i(t_i) = 1 \) and \( H_i(t_j) = 0 \) for \( i \neq j \). A piecewise linear function \( f \) with \( f(t_j) = f_j \) can always be written as
\[ f(t) = \sum_{j=0}^{n} f_j H_j(t). \] (2.2.3)

Let us consider linear operators that assign a function \( Af \) to a given function \( f \). An operator \( \mathcal{A} : C[a, b] \mapsto C[a, b] \) is called linear if it leaves linear combinations invariant:
\[ \mathcal{A}(\alpha f + \beta g) = \alpha \mathcal{A}f + \beta \mathcal{A}g; \quad \alpha, \beta \in \mathbb{R}. \] (2.2.4)

For example, the derivative operator that assigns the derivative \( f' \) to a given function \( f : \mathcal{A}f = f' \) is a linear operator.

The blending functions are assumed to satisfy certain conditions [61] to generate a desired curve or a surface; otherwise arbitrary set of blending functions change a curve or a surface arbitrarily. The blending functions are chosen such that a curve has any or all of the following properties:

**Definition 2.2.2.** Let \( P_i \) be the set of points used to generate a curve. Then these points are called control points of a curve. Similarly, the set of points \( P_{ij} \) for \( i, j = 0, \ldots, n \) used to define a surface are called control points of the surface. Let us denote the blending functions \( f_i(u) \) to define a parameterized curve \( \mathbf{x}(u) \) by taking
\[ \mathbf{x}(u) = \sum_{i=0}^{n} f_i(u) P_i. \] (2.2.5)
In the similar way the blending function \( f_{ij}(u, v) \) may be used to define a parameterized surface \( x(u, v) \) by defining

\[
x(u, v) = \sum_{i,j=0}^{n} f_{ij}(u, v) P_{ij},
\]

(2.2.6)

where \( 0 \leq u \leq 1 \) for the curve and \( 0 \leq u, v \leq 1 \). The blending functions cannot be arbitrary as the shape of the curve or the surface defined by its parametric from corresponds to its shape defined by the control points. In order to have control over the shape of the surface, the blending functions must satisfy certain appropriate characteristics. For example, a parametric curve or a surface must depend on control points. It must be independent of particular coordinate system. To achieve this, the blending functions are assumed to form a partition of unity that means that the sum of blending function values at \( u \) identically sums to one. For a curve, this simply means that

\[
\sum_{i=0}^{n} f_{i}(u) = 1,
\]

(2.2.7)

and for a surface, the blending functions must satisfy the following condition

\[
\sum_{i,j=0}^{n} f_{ij}(u, v) = 1.
\]

(2.2.8)

This property is called invariance under affine transformation. (An affine transformation is the composition of two functions namely translation represented by vector addition and a linear transformation represented by matrix multiplication.)

**Definition 2.2.3.** A convex set is defined as the set in which the line joining the two points of the set is entirely contained within the set itself. Formally, this means that, a subset \( U \) of \( \mathbb{R} \) is convex if \( \forall a, b \in U \), the points of the set \( U \) satisfy the following condition that

\[
t a + (1 - t) b \in U,
\]

where \( t \in [0, 1] \) and \( t a + (1 - t) b \) is called the convex combination of the points \( a \) and \( b \). The concept of convex set may easily be extended to higher dimensional spaces that for a subset \( U \) of \( \mathbb{R}^n \), the convex combination \( t a + (1 - t) b \in U, \forall a, b \in U \). It can be seen that for a subset \( U \) of a vector space \( W \), the condition is equivalent to the requirement that \( \sum_{j=1}^{n} t_j P_j \in U \), the convex combination of the points \( P_1, P_2, \ldots, P_n \in U \) for \( \sum_{j=1}^{n} t_j = 1 \) where \( t_1, t_2, \ldots, t_n \geq 0 \). A convex combination of points is always inside those points, which leads to the definition of the convex hull of a point set. It is to
be noted that for any two convex sets $U$ and $V$ in vector space $W$, the intersection of two sets $U \cap V$ is itself a convex set. The intersection of all the convex sets containing the subset $U$ of a vector space $W$ is the smallest convex set that contains $U$, is called the convex hull of the set $U$. Thus a convex hull is the set that is formed by all convex combinations of a given point set. For example for a two dimensional set, imagine a string that is loosely circumscribed around the set, with nails driven through the points in the set. The string pulled tight serves as the boundary of the convex hull.

**Definition 2.2.4.** In order to keep the curve or the surface in the region of its control points, it is essential to confine the curve or the surface in its convex hull. Convex hull property exists in curves which are invariant under an affine transformation (satisfy the eq. (2.2.7)) or (2.2.8) and for which blending functions are all non-negative, this means that for $t \in [0, 1],
\begin{equation}
f_i(u) \geq 0, \quad 0 \leq u \leq 1, \quad i = 0, 1, \ldots n,
\end{equation}

or
\begin{equation}
f_{ij}(u, v) \geq 0, \quad 0 \leq u, v \leq 1, \quad i, j = 0, 1, \ldots n.
\end{equation}

**Definition 2.2.5.** For a symmetric curve it does not matter if the control points $P_0, P_1, \ldots P_n$ are arranged in reverse order $P_n, P_{n-1}, \ldots, P_0$. The curves that correspond to two different orderings look the same, differ only in the direction in which they are traversed. In order to find the symmetry condition that the parameterized curve does not depend on the control points if they are taken in reverse order, the parametric form of the curve when the curve is drawn taking the points in the order $P_0, P_1, \ldots P_n$ for $0 \leq u \leq 1$
\begin{equation}
x(u) = \sum_{i=0}^{n} f_i(u)P_i.
\end{equation}

Let us rewrite the parametric form of the curve, when the points are arranged in the reverse order $P_n, P_{n-1}, \ldots, P_0$ for $0 \leq v \leq 1$
\begin{equation}
x(v) = \sum_{i=0}^{n} f_i(v)P_{n-i}.
\end{equation}

Eq. (2.2.12) may be equivalently rewritten as
\begin{equation}
x(v) = \sum_{i=0}^{n} f_{n-i}(v)P_i.
\end{equation}
Two eqs. (2.2.11) and (2.2.13) represent the same curve and this is possible only when

$$f_i(u) = f_{n-i}(1 - u), \text{ for } v = 1 - u.$$  \hspace{1cm} (2.2.14)

**Definition 2.2.6.** A set of blending functions $f_i(t)$ is said to be linearly independent if there does not exist a set of constants $\alpha_i, i = 1, 2, ..n$, not all zero for which,

$$\sum_{i=0}^{n} \alpha_i f_i(t) = 0.$$  \hspace{1cm} (2.2.15)

If the blending functions are not linearly independence, then a blending function may be expressed in terms of the other. One of the disadvantage of linearly dependent blending functions is that a given curve can be expressed in terms of infinitely many different control point positions and this also means that, for certain control point arrangements, the curve may collapse to a single point, even though the control points are not all zero at that point.

### 2.3 Blended and Ruled Surfaces

A blending surface is defined by a collection of space curves that serve as the boundary of the surface and possess the information about the surface. The surface blends together the boundary curves for given blending functions with information of the surface at the blending boundaries. A blending surface may be represented by a parametric form and its shape is controlled by blending functions.

Let $c_1(u) = x(u, 0)$ and $c_2(u) = x(u, 1)$ be two parameterized curves. A surface joining the two opposite boundary curves $c_1(u)$ and $c_2(u)$ may be constructed in infinitely many ways, defined over the same parameter $u \in [0, 1]$. A simple parametric representation of a surface that linearly interpolates the curves $c_1(u)$ and $c_2(u)$ is of the form

$$x(u,v) = (1-v)c_1(u) + vc_2(u).$$  \hspace{1cm} (2.3.1)

Specifically, to determine $x(u,v)$, if we denote $c_1(u) = x(u, 0)$ and $c_2(u) = x(u, 1)$, the above equation may be rewritten in the form

$$x(u,v) = (1-v)x(u, 0) + vx(u, 1).$$  \hspace{1cm} (2.3.2)
This is one of the elementary blended surfaces, called a ruled surface, with linear blending functions
\( g_1(v) = 1 - v \) and \( g_2(v) = v \). It interpolates to \( c_1(u) \) for \( v = 0 \) and to \( c_1(u) \) for \( v = 1 \), as illustrated in the Figure 2.1. A ruled surface has its own importance and it is useful as well in the study

![Figure 2.1: A ruled surface obtained by fitting a surface between two arbitrary curves \( c_1(u) \) and \( c_2(u) \) through linear interpolation.](image)

of bilinearly blended surface, called the Coons patch. A ruled surface may be defined as a surface generated by line segments between corresponding points on the two given parameterized curves. The line segment on the ruled surface joining the two curves is called ruling of the surface at a particular value of the parameter. A ruled surface is called a cylindrical surface if all the rulings are parallel, conical surface if all the rulings intersect in a single point. Cylindrical and conical surfaces are special cases of a broader class of surfaces with zero Gaussian curvature, called developable surfaces. A developable surface can be flattened onto a plane by stretching or compressing or it is a surface that can be constructed by transforming a plane. It is to be noted that in a three-dimensional space \( R^3 \), a developable surface is a ruled surface. Cylinders and cones are the examples of developable surface in three-dimensional space.

Blending functions as mentioned above in the construction of a ruled surface may be linear or non-linear, however we work out the surfaces blended by linear functions. Above form eq. (2.3.2) does not impose any restriction on the input curves \( x(u, 0) \) and \( x(u, 1) \) other than that they must be defined on the same parameter \( u \in [0, 1] \), though we can also choose the parameter interval \([a, b]\) in which case we may use eq. (2.1.5). For example \( x(u, 0) \) may be chosen to be a cubic polynomial
curve and \( x(u, 1) \) a spline curve.

### 2.4 Bilinear Interpolation

Let us assume that \((u_0, v_0), (u_1, v_0), (u_0, v_1)\) and \((u_1, v_1)\) be four distinct points in \( R^2 \). A bilinear interpolant for the unknown function \( h(u, v) \) at the point \((u, v)\) may be found by using linear interpolation given by eq. (2.1.3). This can be done by interpolating the function \( h(u, v) \) (say) first along \( u \)-axis and then along the \( v \)-axis. For \( v = v_0 \), linear interpolation along \( u \)-axis is given by the following expression

\[
h(u, v_0) = \frac{u_1 - u}{u_1 - u_0} h(u_0, v_0) + \frac{u - u_0}{u_1 - u_0} h(u_1, v_0),
\] (2.4.1)

and for \( v = v_1 \)

\[
h(u, v_1) = \frac{u_1 - u}{u_1 - u_0} h(u_0, v_1) + \frac{u - u_0}{u_1 - u_0} h(u_1, v_1).
\] (2.4.2)

Taking \( u \) as fixed, interpolation along \( v \)-axis is given by

\[
h(u, v) = \frac{v_1 - v}{v_1 - v_0} h(u, v_0) + \frac{v - v_0}{v_1 - v_0} h(u, v_1)
\] (2.4.3)

Thus eqs. (2.4.1) and (2.4.2) in eq. (2.4.3) reduces it to the bilinear interpolant function given by

\[
h(u, v) = ((u_1 - u)(v_1 - v)h(u_0, v_0) + (u - u_0)(v_1 - v)h(u_1, v_0) + (u_1 - u)(v - v_0)h(u_0, v_1) + (u - u_0)(v_1 - v)h(u_1, v_1))/(u_1 - u_0)(v_1 - v_0).
\] (2.4.4)

In particular for the given points \((u_0, v_0) = (0, 0), (u_1, v_0) = (1, 0), (u_0, v_1) = (0, 1)\) and \((u_1, v_1) = (1, 1)\), the above formula for bilinear interpolation in \((u, v)\) may be rewritten as:

\[
h(u, v) = (1 - u)(1 - v) h(0, 0) + u(1 - v) h(1, 0) + (1 - u)v h(0, 1) + uv h(1, 1).
\] (2.4.5)

This may be represented in matrix form:

\[
h(u, v) = \begin{bmatrix} 1 - u & u \end{bmatrix} \begin{bmatrix} h(0, 0) & h(0, 1) \\ h(1, 0) & h(1, 1) \end{bmatrix} \begin{bmatrix} 1 - v \\ v \end{bmatrix}.
\] (2.4.6)

It is to be noted that bilinear interpolant is not linear rather is product of two linear functions in the variables \( u \) and \( v \) and is linear in each variable when one of the variables is held fixed. Let us denote the above linear functions by the functions \( f_1(u) = 1 - u, f_2(u) = u, g_1(v) = 1 - v \) and \( g_2(v) = v \).
They are called linear blending functions, hence the name bilinear interpolation. The set of all points \( x \in \mathbb{R}^3 \) given by a bilinear interpolant function \( x(u, v) \), interpolating the four distinct points \( x(0, 0), x(0, 1), x(1, 0), x(1, 1) \) in \( \mathbb{R}^3 \) for the given blending functions \( f_1(u), f_2(u), g_1(v), g_2(v) \) is of the form

\[
x(u, v) = \sum_{i=0}^{1} \sum_{j=0}^{1} x(i, j) f_{i+1}(u) g_{j+1}(v).
\] (2.4.7)

This may be represented in matrix form:

\[
x(u, v) = \begin{bmatrix} f_1(u) & f_2(u) \\
                   x(0, 0) & x(0, 1) \\
                   x(1, 0) & x(1, 1) \end{bmatrix} \begin{bmatrix} g_1(v) \\
                   g_2(v) \end{bmatrix}.
\] (2.4.8)

Equivalently we can write it in the form

\[
x(u, v) = f_1(u) g_1(v) x(0, 0) + f_1(u) g_2(v) x(0, 1) + f_2(u) g_1(v) x(1, 0) + f_2(u) g_2(v) x(1, 1). \] (2.4.9)

In particular the bilinear function is called a hyperbolic paraboloid given by eq. (2.4.5) through \( h(i, j) \) or given by eq. (2.4.7) through the four points \( x(i, j) \) for blending functions \( f_1(u) = 1 - u \), \( f_2(u) = u \), \( g_1(v) = 1 - v \) and \( g_2(v) = v \). The surface \( x(u, v) \) given by eq. (2.4.9) is linear in both \( u \) and \( v \) and it interpolates to the input points \( x(i, j) \). The bilinear interpolant can be viewed as a map of the unit square \( 0 \leq u, v \leq 1 \) in the \( u, v \)-plane. We say that the unit square is the domain of the interpolant, while the surface \( x(u, v) \) is its range. A line parallel to one of the axes in the domain corresponds to a curve in the range; it is called an isoparametric curve. Every isoparametric curve of the hyperbolic paraboloid of the last equation is a straight line; thus hyperbolic paraboloids are ruled surfaces. In particular, the isoparametric line \( u = 0 \) is mapped onto the straight line through \( x(0, 0) \) and \( x(0, 1) \); analogous statements hold for the other three boundary curves.

2.5 Coons Patches Spanned by Fixed Boundary Curves

For a minimal (or, more precisely, a stationary) surface, we have to solve the differential equation obtained by setting the mean curvature \( H \) given by eq. (1.2.30) equal to zero for each value of the two parameters, say, \( u \) and \( v \) parameterizing a surface spanning the fixed boundary. In this section, our purpose is to describe a starting surface bounded by the skew quadrilateral which is composed of four arbitrary straight lines connecting four corners \( x(0, 0), x(0, 1), x(1, 0) \) and \( x(1, 1) \); that is used in
Figure 2.2: Bilinear interpolation: a hyperbolic paraboloid is defined by four points \(\mathbf{x}(i,j)\)

Chapter 4 for the variational improvement. A preliminary effort to variationally improve the surface bounded by four straight lines towards being a minimal surface is in ref. [54]. The algorithm used for this variational improvement applies to a wider class of surfaces. Accordingly, now we point out a class of surfaces, namely Coons patch, that includes surfaces bounded by four straight lines.

A ruled surface discussed in section 2.3 interpolates to two boundary curves. However, it can be seen that it has four boundary curves, a simplest example of one of the Coons patches [29, 62]. For a boundary \(\Gamma\) composed of four continuous curves \(\mathbf{c}_1(u) = \mathbf{x}(u, 0), \mathbf{c}_2(u) = \mathbf{x}(u, 1), \mathbf{d}_1(v) = \mathbf{x}(0, v)\) and \(\mathbf{d}_2(v) = \mathbf{x}(1, v)\) over the parameters \(u, v \in [0, 1]\), a surface \(\mathbf{x}(u, v)\) spanning it can be the Coons patch [29]. (As mentioned above we choose this to be our initial surface for the variational process we report in chapter 4.) Using blending functions \(f_1(u), f_2(u), g_1(v)\) and \(g_2(v)\) satisfying the conditions that

\[
\sum_{i=1}^{2} f_i(u) = \sum_{i=1}^{2} g_i(v) = 1, \quad (2.5.1)
\]

i.e. \(f_1(u) + f_2(u) = 1, g_1(v) + g_2(v) = 1\) for non-barycentric combination of points and for \(j = 0, 1\)

\[
f_i(j) = g_i(j) = \delta_{i-1,j}, \quad (2.5.2)
\]

(i.e. \(f_1(0) = g_1(0) = 1, f_1(1) = g_1(1) = 0\) etc) in order to actually interpolate \(\mathbf{x}(0, 0), \mathbf{x}(0, 1), \mathbf{x}(1, 0)\)
and \( x(1, 1) \), the following equation defines Coons patch:

\[
x(u, v) = \left[ \begin{array}{cc} f_1(u) & f_2(u) \\ x(1, v) & x(0, v) \end{array} \right] + \left[ \begin{array}{cc} x(u, 0) & x(u, 1) \\ x(0, 0) & x(0, 1) \end{array} \right] \left[ \begin{array}{c} g_1(v) \\ g_2(v) \end{array} \right] - \left[ \begin{array}{cc} f_1(u) & f_2(u) \\ x(1, 0) & x(1, v) \end{array} \right] \left[ \begin{array}{c} g_1(v) \\ g_2(v) \end{array} \right].
\]

(2.5.3)

For instance, linear blending functions satisfying above conditions may be given by

\[
f_1(u) = 1 - u, \quad f_2(u) = u, \quad g_1(v) = 1 - v, \quad g_2(v) = v.
\]

(2.5.4)

Using these choices of blending functions in eqs. (2.5.3) we get our initial surface \( x_0(u, v) \). In the present form this prescription is apparently limited to a boundary composed of four straight lines or at most four continuous curves. For a more general boundary, we have to reduce, as mentioned above, it to four continuous curves. The next chapter 3 describes the algorithm we suggest for achieving this aim.

As a special case of the above, we consider a Coons patch for which all the three terms in eq. (2.5.3) are equal, so that this equation reduces to the following form:

\[
x(u, v) = \left[ \begin{array}{cc} f_1(u) & f_2(u) \\ x(1, v) & x(0, v) \end{array} \right] \left[ \begin{array}{c} x(0, 0) \\ x(0, 1) \\ x(0, 0) \\ x(1, 1) \end{array} \right] \left[ \begin{array}{c} g_1(v) \\ g_2(v) \end{array} \right].
\]

(2.5.5)

The boundary spanned by lines connecting the points \( x(0, 0) \), \( x(0, 1) \), \( x(1, 0) \) and \( x(1, 1) \) with linear blending functions given by eqs. (2.5.4) in eq. (2.5.5) can represent a time evolution of a string or, alternatively, a re-arrangement of a one set of two strings to the only possible other re-arranged set (of two strings) connecting the same two particles and two antiparticles. (If a string connects only a particle with antiparticle, this constraint allows only two string arrangements for a system composed of two particles and two antiparticles.) The surface \( x(u, v) \) of eq. (2.5.5) spanning this boundary is a surface that is needed in many models of string re-arrangements from one of these configurations to the other with the particle positions \( x(0, 0) \equiv 1 \) and \( x(1, 1) \equiv 2 \) and anti-particle positions \( x(1, 0) \equiv 3 \) and \( x(0, 1) \equiv 4 \). In this thesis we reduce the the area of a quadrilateral, using above linear blending functions and particle positions. This gives

\[
x_u(u, v) = (1 - v) r_{13} - v r_{24}
\]

(2.5.6)

and

\[
x_v(u, v) = (1 - u) r_{14} - u r_{23}
\]

(2.5.7)
as partial derivatives \( w.r.t. \ u \) and \( v \) with the following corners:

\[
x(0, 0) = \mathbf{r}_1, \quad x(1, 1) = \mathbf{r}_2, \quad x(1, 0) = \mathbf{r}_3, \quad x(0, 1) = \mathbf{r}_4.
\] (2.5.8)

\((\mathbf{x}(u, v))\) is our initial surface spanning four straight lines for chapter 4.) For real scalars \( r \) and \( d \), we consider two types of configurations of the four corners: \( \text{ruled}_1 \) and \( \text{ruled}_2 \). For \( \text{ruled}_1 \) we choose

\[
\mathbf{r}_1 = (0, 0, 0), \quad \mathbf{r}_2 = (r, d, 0), \quad \mathbf{r}_3 = (0, d, d), \quad \mathbf{r}_4 = (r, 0, d).
\] (2.5.9)

The mapping from \((u, v)\) to \((x, y, z)\) in this case is

\[
x(u, v) = r(u + v - 2uv), \quad y(u, v) = vd, \quad z(u, v) = ud.
\] (2.5.10)

For \( \text{ruled}_2 \) we choose

\[
x(0, 0) = \mathbf{r}_1, \quad x(1, 1) = \mathbf{r}_2, \quad x(1, 0) = \mathbf{r}_3, \quad x(0, 1) = \mathbf{r}_4.
\] (2.5.11)

The mapping from \((u, v)\) to \((x, y, z)\) in this case is

\[
x(u, v) = ur, \quad y(u, v) = vd, \quad z(u, v) = ud + vd(1 - 2u).
\] (2.5.12)

These definitions are such that for \( r = d \) the four position vectors \( \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \) and \( \mathbf{r}_4 \) lie at the corners of a regular tetrahedron. Figure 2.3 and Figure 2.4 below are 3 dimensional graphs of surfaces called the hyperbolic paraboloids for a choice of corners given by eq. (2.5.9) and eq. (2.5.11) respectively.

The expression for the mean curvatures \( H_{\text{ruled}_1} \) and \( H_{\text{ruled}_2} \), calculated using eq. (1.2.30), of our bilinear interpolations is

\[
H_{\text{ruled}_1} = -\frac{4r^3(2u - 1)(2v - 1)}{d(d^2 + 2r^2(2u - 1)u + 2(u - 1)v + 1))^{3/2}},
\] (2.5.13)

for the \( \text{ruled}_1 \) and

\[
H_{\text{ruled}_2} = \frac{4dr(2u - 1)(2v - 1)}{(d^2(1 - 2v)^2 + 2r^2(2u - 1)u + 1))^{3/2}},
\] (2.5.14)

for the \( \text{ruled}_2 \). The mean curvature for our starting surface is zero only for the coordinate lines \( u = \frac{1}{2} \) and \( v = \frac{1}{2} \), whereas for a minimal surface this should be zero for all values of \( u \) and \( v \). In section 4.2, we describe our effort to improve our surface towards being minimal.
Figure 2.3: The \textit{ruled}_1 surface ($r = 1, d = 2$) with $x, y$ as the horizontal plane and height along $z$-axis.
Figure 2.4: The \textit{ruled}_2 surface ($r = 1, d = 2$) with $x, y$ as the horizontal plane and height along $z$-axis.
Chapter 3

Generating a Coons Patch

In the first section 3.1 of the present chapter, unit step function is defined and its relation with Dirac delta is established. In section 3.2 a comparison of differentiable step function interpolation with Bezier interpolation and spline Interpolation is given. In section 3.3, we present, through eqs. (3.3.1) to (3.3.2), three analytical representations of the unit step function (shown in Figure 3.6) and give, through Figure 3.7, geometric description of boundary curves generated by these step function representations. In final section 3.4, an algorithm is presented to group a finite number of curves into four sets and then within each set combine all the curves as one continuous curve.

3.1 Unit Step Function and its Relation with Dirac Delta

Symbol

A unit step function also called Heaviside step function is a function which assumes a unit value for its argument \( u \geq u_0 \), otherwise it is zero. It is denoted by \( S(u - u_0) \) and is expressed by the following relation,

\[
S(u - u_0) = \begin{cases} 
0 & \text{if } u < u_0, \\
1 & \text{if } u > u_0,
\end{cases} \tag{3.1.1}
\]

as shown in the Figure 3.1. In particular for \( u_0 = 0 \), the above eq. (3.1.1) may be expressed as,

\[
S(u) = \begin{cases} 
0 & \text{if } u < 0, \\
1 & \text{if } u > 0,
\end{cases} \tag{3.1.2}
\]
The function $S(u - u_0)$ is undefined at $u = u_0$ and hence discontinuous at $u = u_0$. However a step function may be approximated by smooth functions as is discussed in detail in the section 3.3 below.

In the present section we discuss the related properties of the step function and its relation with Dirac delta, which comes out to be the derivative of step function. Unit step function may serve as a switch that turns on the given function at $u = u_0$, otherwise off. This can be seen by multiplying a function $h(u)$ by the unit step function $S(u - u_0)$, meaning

$$S(u - u_0)h(u) = \begin{cases} 0 & \text{if } u < u_0, \\ h(u) & \text{if } u > u_0. \end{cases} \quad (3.1.3)$$

A function $h(u)$ will assume its value in a specific interval (say) $u_0 \leq u \leq u_1$ otherwise zero, if we multiply the function $h(u)$ by the difference $S(u - u_0) - S(u - u_1)$, where $u_0 \leq u_1$, i.e.

$$[S(u - u_0) - S(u - u_1)]h(u) = \begin{cases} 0 & \text{if } u < u_0, \\ h(u) & \text{if } u_0 < u < u_1, \\ 0 & \text{if } u > u_1. \end{cases} \quad (3.1.4)$$

It is to be noted that the unit step function $S(u)$ can be viewed as the integral of Dirac Delta [63] denoted by $\delta(u)$ or the unit impulse, which is defined as the limit of a class of delta sequences $\delta_n(u)$. It is sometimes expressed as

$$S(u) = \int_{-\infty}^{u} \delta(u) du, \quad (3.1.5)$$
or if we shift the origin to $u = u_0$, this may be expressed as

$$S(u - u_0) = \int_{-\infty}^{u_0} \delta(u - u_0) \, du,$$  \hspace{1cm} (3.1.6)

where $\delta(u)$ (or $\delta(u - u_0)$) is called the Dirac delta function. Dirac delta function is not a function in the usual sense of definition of a function but is defined as the limit of integrals in a sequence $\delta_n(u)$ as follows

$$\int_{-\infty}^{\infty} \delta(u) h(u) \, du = \lim_{n \to \infty} \int_{-\infty}^{\infty} \delta_n(u) h(u) \, du,$$  \hspace{1cm} (3.1.7)

where $\delta_n(u)$ are functions with $\int_{-\infty}^{\infty} \delta_n(u) \, du = 1$ strongly peaked around $u = 0$. Dirac delta function is used to describe the phenomenon of a impulsive force, its nature such as the action of very large forces or voltages over very short intervals of time. For example, when a tennis ball is hit, a system is given a blow by hammer or when an aeroplane makes a hard landing etc. Dirac, an English physicist introduced the delta function in Quantum Mechanics. But it can be used outside the Quantum Mechanics as in above mentioned applications. The Dirac delta function practically has the following properties:

$$\delta(u) = \begin{cases} 0 & \text{if } u \neq 0, \\ \infty & \text{if } u = 0, \end{cases}$$  \hspace{1cm} (3.1.8)

and

$$\int_{-\infty}^{\infty} \delta(u) h(u) \, du = h(0),$$  \hspace{1cm} (3.1.9)

where $h(u)$ is any well-behaved function and the integration includes the origin. As a special case, in particular for $h(u) = 1$ and $h(0) = 1$, eq. (3.1.9) reduces to

$$\int_{-\infty}^{\infty} \delta(u) \, du = 1,$$  \hspace{1cm} (3.1.10)

which simply states that the area under the curve defined by the the Dirac delta or the impulse function is unity. Equation (3.1.8) implies that $\delta(u)$ must be infinitely high and infinitely thin spike at $u = 0$, as for example in the description of an impulsive force. The problem is that no such function does exists. However, the crucial property can be developed as the limit of a sequence of functions called the distribution (also called a generalized function, a function whose derivative does not exist in classical sense but it makes possible to differentiate a function represented by it), that
we may treat \( \delta(u) \) in the form given by eq. (3.1.7). Here \( \delta(u) \) is labelled as a distribution (not a function) defined by the sequences \( \delta_n(u) \) that satisfies the following relation

\[
\int_{-\infty}^{\infty} \delta_n(u)du = 1.
\] (3.1.11)

The delta function \( \delta(u) \) may be approximated by the sequences of functions in \( n \) as \( n \to \infty \), for example

\[
\delta_n(u) = \begin{cases} 
0 & \text{if } u < -\frac{1}{2n}, \\
 n & \text{if } -\frac{1}{2n} < u < \frac{1}{2n}, \\
0 & \text{if } u > \frac{1}{2n},
\end{cases}
\] (3.1.12)

\[
\delta_n(u) = \frac{n}{\pi (1 + n^2 u^2)},
\] (3.1.13)

\[
\delta_n(u) = \frac{\sin(nu)}{\pi u},
\] (3.1.14)

and

\[
\delta_n(u) = \frac{n}{\sqrt{\pi}} e^{-n^2 u^2}.
\] (3.1.15)

shown in the figures (3.2) to (3.5).

Figure 3.2: A piece wise function as given above, height rises as \( n \to \infty \).

It is assumed that \( \delta(u) \) is integrable and it offers no problem for larger values of \( u \). However the problem is that the limit \( \lim_{n \to \infty} \delta_n(u) \) does not exist. One of the way out to handle this troublesome situation lies in the theory of distributions, that \( \delta(u) \) may be interpreted as sequences of normalized functions and eqs. (3.1.7) and (3.1.11) could be established (though the limit \( \lim_{n \to \infty} \delta_n(u) \) does not
Figure 3.3: A sequence function \( \frac{n}{\pi(1+n^2 u^2)} \) peak rises as \( n \to \infty \).

exist as we can see in the examples for \( \delta_n(u); \) each diverges at \( u = 0 \). The integral property of eq. (3.1.9) is also useful in cases in which the argument of the delta function is itself a function with simple zeros on the real axis. Following are the few properties of the delta function that can be established easily \([64, 65]\). Dirac delta function is even in \( u \), i.e.

\[
\delta(-u) = \delta(u), \tag{3.1.16}
\]

\[
\delta(mu) = \frac{1}{m} \delta(u), \quad m > 0, \tag{3.1.17}
\]

\[
\delta(p(u)) = \sum_a \frac{\delta(u - a)}{|p'(a)|}, \quad p(a) = 0, \quad p'(a) \neq 0, \tag{3.1.18}
\]

where the function \( p(u) \) has simple zeros at \( u = 0 \) on the real axis. The first two derivatives of the Dirac delta function may be found to be

\[
\int_{-\infty}^{\infty} h(u) \delta'(u - u_0) du = -h'(u_0), \tag{3.1.19}
\]

\[
\int_{-\infty}^{\infty} h(u) \delta''(u - u_0) du = h''(u_0), \tag{3.1.20}
\]

that helps to generalize the result for the \( m \)-th order derivative, given by the following relation

\[
\int_{-\infty}^{\infty} h(u) \delta^m(u - u_0) du = (-1)^m h^n(u_0), \tag{3.1.21}
\]
Figure 3.4: A sequence function \( \frac{\sin(nu)}{\pi u} \) peak rises as \( n \to \infty \).

It is to be noted that Dirac delta function is significant as a part of an integrand, often regarded as a linear operator \( \delta(u - u_0) \) which when operates on \( h(u) \) yields \( h(u_0) \). Shifting the singularity to \( u = u_0 \), the Dirac Delta function may be denoted by \( \delta(u - u_0) \) and thus the eq. (3.1.9) may be rewritten as

\[
\int_{-\infty}^{\infty} \delta(u - u_0)h(u)du = h(u_0). \tag{3.1.22}
\]

Equation (3.1.8) for the singularity at \( u = u_0 \), results in the following relation, obtained from eq. (3.1.11) by a shift of variable,

\[
\int_{-\infty}^{\infty} \delta(u - u_0)du = 1. \tag{3.1.23}
\]

This may be generalized to higher dimensions. For example for two dimensional distributions, two dimensional Dirac delta \( \delta(u - u_0, v - v_0) = \delta(u - u_0)\delta(v - v_0) \) satisfies the relation,

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(u - u_0)\delta(v - v_0)du dv = 1, \tag{3.1.24}
\]

and

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(u - u_0)\delta(v - v_0)h(u, v)du dv = h(u_0, v_0). \tag{3.1.25}
\]

For three dimensional distributions, three dimensional Dirac delta \( \delta(u - u_0, v - v_0, w - w_0) = \)
Figure 3.5: A sequence function $\frac{n}{\sqrt{\pi}} e^{-n^2 u^2}$, peak rises as $n \to \infty$

\[ \delta(u - u_0)\delta(v - v_0)\delta(w - w_0) \text{ satisfies,} \]
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(u - u_0)\delta(v - v_0)\delta(w - w_0) \, du \, dv \, dw = 1 \quad (3.1.26) \]

and
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(u - u_0)\delta(v - v_0)\delta(w - w_0) \, h(u, v, w) \, du \, dv \, dw = h(u_0, v_0, w_0). \quad (3.1.27) \]

The above expressions may be found as well for other coordinate systems simply by knowing the transformation equations for the desired coordinate system in a given region.

### 3.2 Comparison of Differentiable Step Function Interpolation with Bezier and Spline Interpolation

A function is said to be smooth if it is infinitely differentiable, (but may not be analytic- A function is said to analytic if it is given by a locally convergent power series. However an analytic function of a real variable may be shown to be smooth. A complex valued function possessing derivatives of
all orders called holomorphic is analytic as well). For example,

\[ h(u) = \begin{cases} 
e^{-1/u} & \text{if } u > 0, \\ 0 & \text{if } u \leq 0, \end{cases} \]  

(3.2.1)

is smooth, \textit{i.e.} possess derivatives of all orders but is not analytic, as it can be seen that the derivatives of this function are all zero at \( u = 0 \) and so the Taylor series of the given function at \( u = 0 \) converges to \textit{zero} for all values of \( u \) and does not represent the function itself. It is to be noted [29] that Bzier interpolation introduces parametric curve or surfaces based on control points. Bzier interpolation is well behaved and suited to shape control, but it does not necessarily pass through all the control points. Another technique for interpolating a curve is spline interpolation [29, 66] for which the interpolant function is a special type of piecewise polynomial called a spline. To generate a smooth curve passing through the points, one can discretize the given piecewise boundary curves and use spline interpolation that passes through these points, but that would not assure continuity of third and higher derivatives. However, if extrapolated, the resulting curve simply agrees to the extrapolations of the respective constituent curves and thus does not develop any large fluctuations that would result from using an interpolating polynomial of high enough degree in place of splines but these high degree polynomials are computationally expensive and give more interpolation error. We intend to use the differentiable form as given by eq. (3.3.1) to eq. (3.3.2) of the unit step function eq.(3.4.2). This is explicated in detail in section 4.3 that how to group five lines (Figure 4.16) resulting into an initial parametric form of the surface, an initial surface \textit{e.g.} Coons patch for given blending functions eq. (2.5.3) satisfying conditions given by eqs. (2.5.1) and (2.5.2), composed of five lines is differentiable at each of its points. Our results are not affected by how this grouping is done in order to obtain an initial surface (Coons patch), as it can be seen in detail in the section 3.4.

### 3.3 Analytical Representations of the Step Function

In eq. (3.4.2), we use the step function for letting the parameter of our \textit{one} curve map into different given constituent curves as it increases. For this we have to let the step function multiply one curve only till it reaches a value corresponding to a junction; for higher values of the parameter the same
step function now multiplies a new curve, and so on till each of the curves gets multiplied by 1 for some range of values of the parameter. In the existing literature [27, 58] a step function is used for another purpose, namely for effectively deleting through its vanishing value a non-desired piece of a surface and keeping only a desired piece of a surface for a corresponding range of parameters; practically this amounts to changing the boundary of the piece of the surface that is not deleted. In ref. [67] we use different forms of the step function to join the curves in each of the four groups into a single analytic curve. (This joining lets us use eq. (2.5.3) to write the Coons patch which is used in trying to find a minimal surface for a fixed boundary in contrast to an effectively shrinking boundary of refs. [27, 58]; this minimization is called the Plateau problem and has its own importance in the mathematical literature.) We tried three step function representations \( S(u - u_i) \), \( S^*(u - u_i) \) and \( S^{**}(u - u_i) \) written below in eqs. (3.3.1) to (3.3.3). It is to be noted that, in addition to being analytic for \( \epsilon \neq 0 \) as that in ref. [58], our first and third forms, sketched in Figure 3.6, are also analytic even for \( \epsilon = 0 \) provided \( u \neq u_i \) and the second form is simply analytic throughout. In writing these, a real scalar \( l \) is used to replace unit interval \( 0 \leq u \leq 1 \) to an interval \( 0 \leq lu \leq l \) of arbitrary length \( l \). In the second form another variable \( k \) is introduced; larger \( k \) results in sharp transition at \( u = u_i \).

\[
S(u - u_i) = \frac{1}{2} \left( 1 + \frac{(u - u_i)l}{\sqrt{\epsilon + ((u - u_i)l)^2}} \right),
\]

(3.3.1)

\[
S^*(u - u_i) = \frac{1}{2} \left( 1 + \tanh k ((u - u_i)l) \right),
\]

(3.3.2)

\[
S^{**}(u - u_i) = \frac{1}{2} \left( 1 + \frac{(u - u_i)l}{|u - u_i|l + \epsilon} \right).
\]

(3.3.3)

Figure 3.6: Step function representations \( S(u - u_i) \), \( S^*(u - u_i) \) and \( S^{**}(u - u_i) \) for \( i = 1 \), \( l = 20 \), \( u_i = 0.25 \), \( \epsilon = 0.01 \) and \( k = 5 \).

As a special case let us assume that \( L_1(u) \) and \( L_2(u) \) be two successive analytical smooth curves
joined together using a differentiable step-function representation eqs. (3.3.1) to (3.3.3) to give us a smooth curve $L(u)$ that exactly interpolates the constituent curves $L_1(u)$ and $L_2(u)$ such that

$$
L(u) = \begin{cases} 
L_1(u) & \text{if } u < u_1, \\
L_2(u) & \text{if } u > u_1.
\end{cases}
$$

(3.3.4)

In particular for a line $L_1(u)$ obtained by joining $(0,0)$ to $(x_1, y_1) = (lu, y_1)$

$$
L_1(u) = \frac{y_1}{u_1} u,
$$

(3.3.5)

and line $L_2(u)$ obtained by joining the points $(x_1, y_1) = (lu, y_1)$ to $(l,0)$

$$
L_2(u) = \frac{y_1}{u_1 - 1} (u - 1),
$$

(3.3.6)

Using eq. (3.4.1) $L(u)$ may be written as a smooth curve joined through a step function representation given by eqs. (3.3.1) to (3.3.3) that is

$$
L(u) = L^1_1(u) = \frac{y_1}{u_1} u + S(u - u_1) \left( \frac{y_1}{u_1 - 1} (u - 1) - \frac{y_1}{u_1} u \right).
$$

(3.3.7)

The graphs of these combinations of $L_1(u)$ and $L_2(u)$ for the three step function representations $S(u-u_1), S^*(u-u_1)$ and $S^{**}(u-u_1)$ are sketched below Figure 3.7. Each of these is an analytical function guaranteed to pass through both the curves it combines.

![Figure 3.7: Straight lines $L_1(u)$ and $L_2(u)$ joined by first, second and third step function representations respectively for $l = 20$, $u_1 = 0.25$, $\epsilon = 0.01$ and $k = 5$.](image-url)
3.4 Constructing Four Continuous Curves From a Given Set of Finite Number of Boundary Curves

An arbitrary boundary can be written as a limit of a collection of finite number of arbitrary curves. And for a boundary composed of finite number of curves, we need to reduce it to only four bounding curves so as to be able to write a Coons patch for such a boundary. For this, techniques are needed to group a finite number of curves into four sets and then within each set combine all the curves as one continuous curve. We introduce our algorithms for grouping in this paragraph of present section. For the combinations, in the remaining portion of this section we present an iterative scheme that uses suitable step function representations to combine an arbitrary number of curves into one continuous curve. The step function representations are given in 3.3 through eqs. (3.3.1) to (3.3.3) that are analytical representations of the unit step function and have given, through Figure 3.7, geometric description of boundary curves generated by these step function representations. For reducing a boundary composed of finite number of curves to four bounding curves, first we need to group a finite number of curves into four sets, in the common notation for a Coons patch (introduced in the section 2.5) $c_1(u) = x(u, 0)$, and other three as $c_2(u) = x(u, 1), d_1(v) = x(0, v)$ and $d_2(v) = x(1, v)$ with $0 \leq u, v \leq 1$. Let us call the $N$ bounding curves as $L_i$ for $i = 1, 2, ..., N$, the greatest integer $\leq N/4$ as $m$ and the residue $s = N - 4m$ which can take values as 0, 1, 2 or 3. We put the first $m + c_1$ curves in the first group $c_{i=1}^{m+c_1}$ that eventually becomes $c_1(u)$ when we join all its elements into one continuous curve by using step functions of the section 3.3. Here $c_1 = min(s, 1)$, meaning that for four equal groups the first group is just the first quarter and for non-equal groups we put one of the $s$ extra curves in the first group. The next group, to become $d_1(v)$ after the joining(s), starts with the very next $m + c_1 + 1st$ curve and continues till $i = 2m + d_1$ with $d_1 = min(s, 2)$. The number of curves in this second group is $m + min(2, s) - min(1, s)$, meaning that for $s$ larger than one this group gets one of the $s$ curves. We call this group as $c_{i=m+c_1+1}^{2m+d_1}$. The third group, to become $c_2(u)$ after the joining(s), starts with the $2m + d_1 + 1st$ curve and continues till $i = 3m + s$. The number of curves in this second group is $m + s - min(2, s)$ meaning that only for $s = 3$ this group gets one of the $s$ curves. We call it $c_{i=2m+d_1+1}^{3m+s}$. The fourth and last group, to become $d_2(v)$, starts with the next $i = 3m + s + 1$ till at the end when $i = 4m + s = N$. The number of curves in
this last group is always \( m \) and thus it never gets any of the extra \( s \) curves. The label we use for this group is \( C^N_{i=3m+s+1} \).

For example for a boundary composed of 9-curves, \( N = 9 \), \( m = 2 \), \( s = 9 - 8 = 1 \), \( c_1 = \min(s, 1) = 1 \), \( d_1 = \min(s, 2) = 1 \) and this boundary is grouped into four sets of curves namely \( C_1^3 \), \( C_4^2 \), \( C_6^7 \) and \( C_9^8 \) forming the bounding curves \( c_1(u), d_1(v), c_2(u) \) and \( d_2(v) \) respectively, where \( c_1(u) \) comprises 3 curves and each of \( d_1(v), c_2(u) \) and \( d_2(v) \) has two curves. For another example, for \( N = 14 \), \( m = 3 \), \( s = 14 - 12 = 2 \). This corresponds to four sets of curves namely \( C_4^4 \), \( C_8^5 \), \( C_{11}^9 \) and \( C_{14}^{12} \) as the bounding curves \( c_1(u), d_1(v), c_2(u) \) and \( d_2(v) \) respectively for Coons patch in this case. Each of \( C_4^4 \) and \( C_8^5 \) includes 4-curves, and each of \( C_{11}^9 \) and \( C_{14}^{12} \) contains 3 curves.

The next task is to join \( m + c_1 \) curves in the first group \( C_{i=1}^{m+c_1} \) into one continuous curve \( c_1(u) \); the same process is to be done later for other three groups. Let us consider two consecutive curves \( L_i(u) \) and \( L_{i+1}(u) \), for \( i = 1, \ldots, m + c_1 - 1 \), in this group. These can be combined into a continuous curve

\[
L_i^1(u) = L_i^0(u) + S(u - u_i) \left( L_{i+1}^0(u) - L_i^0(u) \right),
\]

(3.4.1)

at their junction \( u_i \) for a smooth approximation (eq. (3.3.1) to eq. (3.3.3)) of the step function

\[
S(u - u_i) = \begin{cases} 
0 & \text{if } u < u_i, \\
1 & \text{if } u > u_i,
\end{cases}
\]

(3.4.2)

ensuring that

\[
L_i^1(u) = \begin{cases} 
L_i^0(u) & \text{if } u < u_i, \\
L_{i+1}^0(u) & \text{if } u > u_i.
\end{cases}
\]

(3.4.3)

We have added a superscript \( j \) to \( L_i^j \). This denotes the number of junctions in the joined curve and hence \( L_i^0 = L_i \) are not supposed to have junctions in them. In \( L_i \), \( i = 1, 2, \ldots, m + c_1 \) of the first group; similarly for the other three groups. To complete the task we have to be able to increase the superscript to the number of junctions of the respective group. We suggest a recursion relation to increase the superscript to any value, namely

\[
L_i^j(u) = L_i^0(u) + S(u - u_i) \left[ L_{i+1}^{j-1}(u) - L_i^0(u) \right].
\]

(3.4.4)

For \( j = 1 \) this equation becomes eq.(3.4.1) and this is the least value of the superscript for which eq. (3.4.4) should be used. Continuing the iterative process in above equation would express \( L_{i+1}^{j-1}(u) \)
in terms of the one with superscript further decreased. That is,

\[ L_{i+1}^{j-1}(u) = L_{i+1}^0(u) + S(u - u_{i+1}) \left[ L_{i+2}^{j-2}(u) - L_{i+1}^0(u) \right], \]  

(3.4.5)

and so on. This iteration allows us to extend our algorithm to merge any finite number of curves that is (one more than) the value we assign to the superscript \( j \). We illustrate below a merging of three and four curves by assigning the superscript \( j \) values of 2 and 3, which are the corresponding number of junctions. In our full scheme these combinations are needed when we partition 9 and 14 total number of curves respectively into our usual four groups; when the above mentioned set of curves \( C_3^4 \) is joined together to make \( c_1(u) \) for a boundary composed of 9-curves, it takes the following form

\[ c_1(u) = L_1^3(u) = L_1^0(u) + S(u - u_1) \left[ L_2^1(u) - L_1^0(u) \right], \]

(3.4.6)

where

\[ L_1^1(u) = L_2^0(u) + S(u - u_2) \left[ L_3^0(u) - L_2^0(u) \right], \]

(3.4.7)

as shown in Figure 3.8.

\[ c_1(u) = L_3^0(u) + S(u - u_1) \left[ L_2^0(u) - L_1^0(u) \right] \]

\[ L_1^1(u) = L_1^0(u) \]

\[ L_3^0(u) = L_3^0(u) \]

\[ L_2^0(u) = L_2^0(u) \]

\[ L_2^0(u) = L_2^0(u) + S(u - u_2) \left[ L_3^0(u) - L_2^0(u) \right] \]

Figure 3.8: \( c_1(u) \) joining \( L_1(u) \), \( L_2(u) \) and \( L_3(u) \) for \( N = 9 \).

Likewise, when the above mentioned set of curves \( C_4^4 \) is joined together to make \( c_1(u) \) for a boundary composed of 14-curves, it takes the following form

\[ c_1(u) = L_1^3(u) = L_1^0(u) + S(u - u_1) \left[ L_2^2(u) - L_1^0(u) \right], \]

(3.4.8)

where

\[ L_2^2(u) = L_2^0(u) + S(u - u_2) \left[ L_3^0(u) - L_2^0(u) \right], \]

(3.4.9)
and in turn
\[ L_3^1(u) = L_3^0(u) + S(u - u_3) \left[ L_3^0(u) - L_3^1(u) \right]. \] (3.4.10)

Substituting eqs. (3.4.9) and (3.4.10) in eq. (3.4.8) results in
\[ c_1(u) = L_1^0(u) + S(u - u_1) \left[ L_2^0(u) + S(u - u_2) \left[ L_3^0(u) + S(u - u_3) \left[ L_4^0(u) - L_3^0(u) \right] - L_2^0(u) \right] - L_1^0(u) \right], \] (3.4.11)
as shown in the Figure 3.9. In the similar way other constituent parts of Coons patch namely

Figure 3.9: \( c_1(u) \) joining \( L_1(u), L_2(u), L_3(u) \) and \( L_4(u) \) for \( N = 14 \).

\[ d_1(v), c_2(u) \text{ and } d_2(v) \] can be constructed using eq. (3.4.4).
Chapter 4

Variational Improvement

In this chapter we discuss a variational technique to reduce the area of a non-minimal surface \( \mathbf{x}(u, v) \), expecting that the reduced value of \( ms \) of mean curvature numerator, denoted by \( \mu^2 \), in turn reduces the area \( A \) of the surface \( \mathbf{x}(u, v) \). The related concepts like variation of a parametric surface with the help of a variational parameter, area reduction, ratio of \( rms \) of Gaussian curvature to the mean square of mean curvature are discussed in detail in the following section 4.1.

In the section 4.2 we apply the technique introduced in the section 4.1 to reduce the area of the non-minimal surfaces namely 1) a hemiellipsoid surface of which corresponding minimal surface is known as a flat surface boundary as an ellipse lying in the plane, the minimal surface in this case being an elliptic disc, 2) a hump like surface spanned by four boundary lines in the plane for which the corresponding minimal surface is known, namely a square disc and 3) a bilinearly interpolating surface spanned by four boundary lines lying in different planes for which the corresponding minimal surface is not known and that is the subject matter of chapter 2. For the hemiellipsoid eq. (4.2.2) with semi-major and semi-minor axes \( b \) and \( c \) (where \( 0 \leq b, c \leq 2 \), \( 0 \leq u \leq \frac{\pi}{2} \), \( 0 \leq v \leq 2\pi \) and step size 0.2) of the ellipse bounding the hemiellipsoid, the minimum value of variational parameter \( t(b, c) \) as the function of \( b \) and \( c \) is shown in Figure 4.3 and the corresponding dimensionless decrease in area \( A(b, c) \) of hemiellipsoid as a function of \( b \) and \( c \) is shown in the Figure 4.4. The reduction in area in this case is about 23 percent of the original surface. For the hump like surface eq. (4.2.7), first variational surface eq. (4.2.31) is shown in Figure 4.8, its mean curvature shown in Figure 4.9 as a function of the surface parameters \( u \) and \( v \) for minimum value of variational parameter \( t \). Mean curvature of second variational Figure 4.10 is shown in the Figure 4.11. In order to see the pattern
of reduction in area from \( \mathbf{x}_1(u, v) \) to \( \mathbf{x}_2(u, v) \), the graph of the quantity \( m_n(u, v) \) eq. (4.1.2) for \( n = 2 \) is shown in Figure 4.12 in the region for \( 0 \leq u, v \leq 1 \), so that we may guess the reduction in area for the higher iterations which we are unable to implement in the present computer algebra system that becomes irresponsive for higher iteration. The percentage decrease in area in this case is about 37.9141 percent of the original surface. For the bilinear interpolation eq. (4.2.35)(resulting from a bilinear interpolant eq. (2.5.5) as a special case of Coons patch eq. (2.5.3) with linear blending functions given by eqs.(2.5.4)), the dimensionless decrease in area as a function of scalar \( r \) (for \( 0 \leq r \leq 2 \) with step size \( h = 0.001 \)) is given in Figure (4.14) or (4.15). Bilinear interpolation may be used to model the initial and final configurations of rearranging strings. The reduction in area in this case remains less than 0.82 percent of the original area.

Then in the section 4.3, we apply our step function analytical form, standard Coons patch and our variational technique to reduce the area of a surface spanned by five lines. The resulting percentage decrease in area for few cases is reported in table 4.1 of initial surface Coons patch. The table 4.1 indicates a symmetric behaviour of decrease in area in the region considered and these values agree up to four decimal places. The data points of the table 4.1 are plotted in the Figure 4.13 along with spline curve giving us percentage decrease in area as numerical function of data points. In this Figure 4.13, the dots give computed values of decrease in area and the smooth graph passing through these points is the spline curve interpolating these points for better predictability that how the decrease in area in the Coons patch is associated with the given range of points. The Figure 4.13, as a string breaking model gives the relationship between reduction in area and the string breaking point. It indicates that string breaking generally in the middle may be closer to being a minimal surface than the string breaking significantly away from the middle.

### 4.1 A Technique For Variational Improvement

Reducing an arbitrary curve (or a finite collection of many continuous curves) to four continuous curves we may write \( e.g. \) Coons patch for a surface bounded by it. That is what we achieved in chapter 3. But that would not tell us anything about its characteristics in the differential geometry. For example, there is no guarantee that such a surface would be a minimal surface spanning its
boundary. We can calculate the differential geometry related functions of the two parameters \( u \) and \( v \) of the surface and then do integrations with respect to these parameters to get numbers characterizing the surface. If the \( \text{rms} \) mean curvature (see section 1.2) of the generated surface is non-zero, the surface is not a minimal surface and a challenge is to modify it so that it either becomes a minimal surface or, if that is not possible, gets closer to being a minimal surface. For this we can write an ansatz for a modification in the surface including a variational parameter (or parameters) and then solve the optimization problem of selecting the value(s) of the parameter(s) so as to minimize either its area directly or its \( \text{rms} \) mean curvature or \( \text{ms} \) of the numerator of the mean curvature function. To reduce area of a non-minimal surface \( x(u,v) \) using the expectation that reduced value of \( \text{ms} \) mean curvature, denoted by \( \mu^2 \), in turn reduces the area \( A \) of the surface \( x(u,v) \). Our scheme is to reduce the area of a given surface \( x(u,v) \) having fixed boundary, 1) spanned by an elliptic curve (section 4.2.1), namely the hemiellipsoid, 2) a hump like surface spanned by a four lines (section 4.2.2) for the surfaces, and 3) a surface spanned by finite number of curves for which the minimal surface is unknown (section 4.2.3 and 4.3). In each case we do this by obtaining a variationally selected surface \( x_1(u,v) \) eq. (4.1.11) and \( x_2(u,v) \) eq. (4.1.24) of lesser area. As these equations tell, in contrast to the first two examples with a known minimal surface, for the variational improvement in the surfaces we use an ansatz essentially consisting of the non-minimal surface denoted by the \( x_0(u,v) \), the initial surface plus a variational parameter \( t \) multiplying the unit normal \( N_0 \) to the initial surface and a function \( m_0(u,v) \) (eq. (4.1.12)) of surface parameters chosen such that its variation at boundary points is zero, otherwise the variation is a multiple of numerator \( H_0 \) of mean curvature function \( H \). The ansatz we suggest as an iterative scheme eq.(4.1.1) to minimize the area of a non-minimal surface \( x_n(u,v) \) \( (n = 0, 1, 2, ...) \) is

\[
x_{n+1}(u,v,t) = x_n(u,v) + t m_n(u,v) N_n,
\]

(4.1.1)

where \( t \) is our variational parameter and

\[
m_n(u,v) = uv(1-u)(1-v) H_n
\]

(4.1.2)

is chosen so that the variation at the boundary curves \( u = 0, u = 1, v = 0 \) and \( v = 1 \) is zero. Here \( H_n \) \( (n = 0, 1, 2, ...) \) denotes the numerator part of the mean curvature function eq. (1.2.30) of the
non-minimal surface \( \mathbf{x}(u,v) \) and is given by

\[
H_n = e_n G_n - 2F_n f_n + g_n E_n. \tag{4.1.3}
\]

For \( \mathbf{x}_n(u,v) \), we denote \( E_n, F_n, G_n, e_n, f_n \) and \( g_n \) as the fundamental magnitudes given by eqs. (1.2.14) and (1.2.26) and \( \mathbf{N}_n(u,v) \), the numerator part of unit normal given by eq. (1.2.10) to the surface \( \mathbf{x}(u,v) \). The subscript \( n \) is used not only to denote numerator part of the quantities but also to denote the \( n \text{th} \) iteration. For \( n = 0 \), for the initial surface \( \mathbf{x}_0(u,v) \), thus \( E_0(u,v), F_0(u,v), G_0(u,v), e_0(u,v), f_0(u,v) \) and \( g_0(u,v) \) denote the fundamental magnitudes, \( H_0(u,v) \) the numerator part of the mean curvature, \( m_0(u,v) = u v (1 - u) (1 - v) H_0 \), a function of \( u, v \) with no variation at its boundary curves and \( \mathbf{N}_0(u,v) \) the unit normal. For \( n \neq 0 \), \( E_n(u,v,t), F_n(u,v,t), G_n(u,v,t), e_n(u,v,t), f_n(u,v,t) \) and \( g_n(u,v,t) \) depend not only the surface parameters but on the variational parameter as well. For \( \mathbf{x}_n(u,v) \) eq. (4.1.1), for \( n \neq 0 \), the numerator \( H_n \) eq. (4.1.3) of mean curvature of \( \mathbf{x}(u,v) \), is a polynomial in \( t \). It would have the following familiar expression

\[
H_n(u,v,t) = E_n g_n - 2F_n f_n + G_n e_n = \sum_{i=0}^{6} (p_i(u,v)) t^i. \tag{4.1.4}
\]

As \( H^2_n(u,v,t) \) is a polynomial in \( t \), with real coefficients \( q_j(u,v) \) of \( t^j \) for \( j = 0, 1, 2, ..., 10 \), we rewrite eq. (4.1.4) in the form

\[
H^2_n(u,v,t) = \sum_{j=0}^{n} (q_j(u,v)) t^j. \tag{4.1.5}
\]

Here \( n \) turns out to be 10; there being no higher powers of \( t \) in the polynomials as it can be seen from the expression for \( E_n(u,v,t), F_n(u,v,t) \) and \( G_n(u,v,t) \) which are quadratic in \( t \) and \( e_n(u,v,t), f_n(u,v,t) \) and \( g_n(u,v,t) \) which are cubic in \( t \). Integrating (numerically if desired) these coefficients w.r.t. \( u \) and \( v \) in the range \( 0 \leq u, v \leq 1 \) we get the following integral for root mean square \( \text{rms} \)

\[
\mu_n(t) = \left( \int_0^1 \int_0^1 H^2_n(u,v,t) \, du \, dv \right)^{\frac{1}{2}} = \left( t^j \int_0^1 \int_0^1 \sum_{i=0}^{n} (q_j(u,v)) \, du \, dv \right)^{\frac{1}{2}}. \tag{4.1.6}
\]

The expression in the parentheses on right hand side of above equation turns out to be a polynomial in \( t \) of degree \( n \), which can be minimized w.r.t. \( t \) to find \( t_{\text{min}} \). The resulting value of \( t \) completely specify new surface \( \mathbf{x}_{n+1}(u,v) \). For this value of \( t = t_{\text{min}} \), new surface \( \mathbf{x}_{n+1}(u,v) \) is expected to have lesser area than that of surface \( \mathbf{x}_n(u,v) \). Thus the root mean square \( \text{(rms)} \) eq. (1.2.32) of mean curvature eq. (1.2.30) of this \( H_n \), for \( 0 \leq u \leq 1 \) and \( 0 \leq v \leq 1 \) and \( t = t_{\text{min}} \) can be calculated. That
We denote the root mean square (rms) eq. (1.2.33) of Gaussian curvature eq. (1.2.31) for \(0 \leq u \leq 1\) and \(0 \leq v \leq 1\), as \(\nu_n\) given by the following expression,

\[
\nu_n(t) = \left( \int_0^1 \int_0^1 K_n^2(u, v, t = t_{\text{min}}) \ dudv \right)^{\frac{1}{2}}, \tag{4.1.8}
\]

where \(K_n\) is the numerator part of \(K\) of eq. (1.2.31). Using eqs. (4.1.6) and (4.1.8), ratio of the mean square of mean curvature to the \(\text{rms}\) of Gaussian curvature eq. (1.2.34) takes the following form,

\[
\frac{\mu_n^2}{\nu_n} = \frac{\int_0^1 \int_0^1 H_n^2(u, v, t = t_{\text{min}}) \ dudv}{\left( \int_0^1 \int_0^1 K_n^2(u, v, t = t_{\text{min}}) \ dudv \right)^{\frac{1}{2}}}. \tag{4.1.9}
\]

Now for \(x_n(u, v)\), we compute \(E_n(u, v, t = t_{\text{min}})\), \(F_n(u, v, t = t_{\text{min}})\) and \(G_n(u, v, t = t_{\text{min}})\) and denote the area integral eq. (1.2.22), as \(A_n\) and it is given by the following expression

\[
A_n = \int_0^1 \int_0^1 \sqrt{E_n G_n - F_n^2} \ dudv. \tag{4.1.10}
\]

The ansatz eq. (4.1.1), for the first order reduction (eq.(4.1.11)) in the area of a non-minimal surface \(x_0(u, v)\) is

\[
x_1(u, v, t) = x_0(u, v) + t m_0(u, v) N_0, \tag{4.1.11}
\]

where \(t\) is our variational parameter and

\[
m_0(u, v) = uv(1 - u)(1 - v)H_0, \tag{4.1.12}
\]

is chosen so that the variation at the boundary curves \(u = 0, u = 1, v = 0\) and \(v = 1\) is zero. \(N_0\) as mentioned before, is unit normal eq. (1.2.10) to the non-minimal surface \(x_0(u, v)\). \(H_0\), given by the following eq. (4.1.13),

\[
H_0(u, v, t) = E_0 y_0 - 2F_0 f_0 + G_0 \epsilon_0, \tag{4.1.13}
\]

is numerator of the mean curvature of the surface \(x(u, v)\) for each stage of iteration of the algorithm. We call the root mean square (\(\text{rms}\)) of this \(H_0\), for \(0 \leq u \leq 1\) and \(0 \leq v \leq 1\), as \(\mu_0\). That is,

\[
\mu_0 = \left( \int_0^1 \int_0^1 H_0^2 \ dudv \right)^{1/2}. \tag{4.1.14}
\]
We denote the root mean square (rms) of Gaussian curvature for \(0 \leq u \leq 1\) and \(0 \leq v \leq 1\), as \(\nu_0\) for \(x_0(u,v)\) given by the following expression,

\[
\nu_0 = \left( \int_0^1 \int_0^1 K_0^2 \, dudv \right)^{1/2},
\]  

(4.1.15)

where \(K_0\) is the numerator part of \(K\), eq. (1.2.31). Using eqs. (4.1.14) and (4.1.15), eq. (1.2.34) takes the following form,

\[
\nu_0 \frac{\mu_0}{\nu_0} = \left( \int_0^1 \int_0^1 K_0^2 \, dudv \right)^{1/2}.
\]  

(4.1.16)

With the above notation eq. (1.2.20) becomes, for \(x_n(u,v)\),

\[
A_0 = \int_0^1 \int_0^1 \sqrt{E_0G_0 - F_0^2} \, dudv.
\]  

(4.1.17)

Calling the fundamental magnitudes for \(x_1(u,v)\) as \(E_1(u,v,t)\), \(F_1(u,v,t)\), \(G_1(u,v,t)\), \(e_1(u,v,t)\), \(f_1(u,v,t)\) and \(g_1(u,v,t)\), we denote the numerator of mean curvature for \(x_1(u,v)\) eq. (4.1.11) as \(H_1(u,v,t)\). We find following expressions for \(H_1(u,v,t)\) and \(H_1^2(u,v,t)\) using eqs. (4.1.4) and (4.1.5) for \(n = 1\)

\[
H_1(u,v,t) = E_1 g_1 - 2 F_1 f_1 + G_1 e_1 = \sum_{i=0}^{6} (p_i(u,v)) \, t^i,
\]  

(4.1.18)

and

\[
H_1^2(u,v,t) = \sum_{j=0}^{n} (q_j(u,v)) \, t^i.
\]  

(4.1.19)

As discussed before, \(H_1^2(u,v,t)\) is a polynomial in \(t\), with real coefficients \(q_j(u,v)\) of \(t^j\) for \(j = 0, 1, 2, 3...10\), here \(n\) turns out to be \(10\); there being no higher powers of \(t\) in the polynomials as it can be seen from the expression for \(E_1(u,v,t)\), \(F_1(u,v,t)\) and \(G_1(u,v,t)\) which are quadratic in \(t\) and \(e_1(u,v,t)\), \(f_1(u,v,t)\) and \(g_1(u,v,t)\) which are cubic in \(t\). Integrating (numerically if desired) these coefficients w.r.t. \(u\) and \(v\) in the range \(0 \leq u,v \leq 1\) we find the integral for root mean square rms denoted by \(\mu_1(t)\), as given below

\[
\mu_1(t) = \left( \int_0^1 \int_0^1 H_1^2(u,v,t) \, dudv \right)^{\frac{1}{2}} = \left( t^i \int_0^1 \int_0^1 \sum_{i=0}^{n} (q_i(u,v)) \, dudv \right)^{\frac{1}{2}}.
\]  

(4.1.20)

The expression in the parentheses on right hand side of above equation is a polynomial in \(t\) of degree \(n\), which can be minimized w.r.t. \(t\) to find \(t_{min}\) for the first order reduction in area of \(x_0\). The resulting value of \(t\) completely specify new surface \(x_1(u,v)\). New surface \(x_1(u,v)\) is expected to
have lesser area than that of original surface $x_0(u, v)$. Root mean square root of Gaussian curvature in this case turns out to be

$$\nu_1(t) = \left( \int_0^1 \int_0^1 K_1^2(u, v, t = t_{\text{min}}) \ dudv \right)^{\frac{1}{2}}. \quad (4.1.21)$$

Ratio of this $rms$ of Gaussian curvature to the mean square of mean curvature is

$$\frac{\mu_1^2}{\nu_1} = \frac{\int_0^1 \int_0^1 H_1^2 \ dudv}{\left( \int_0^1 \int_0^1 K_1^2 \ dudv \right)^{1/2}}. \quad (4.1.22)$$

The area $A_1$ of the surface $x_1(u, v, t)$ for $0 \leq u \leq 1$ and $0 \leq v \leq 1$ is given by

$$A_1 = \int_0^1 \int_0^1 \sqrt{E_1 G_1 - F_1^2} \ dudv, \quad (4.1.23)$$

where $E_1 = E_1(u, v, t)$, $F_1 = F_1(u, v, t)$, $G_1 = G_1(u, v, t)$ for $t = t_{\text{min}}$.

The second order reduction in area is given by forming surface $x_2(u, v, t)$

$$x_2(u, v, t) = x_1(u, v) + t m_1(u, v) N_1(u, v), \quad (4.1.24)$$

where $t$ is again variational parameter which would take a new value now. $N_1(u, v)$ is the unit normal to the first variational surface $x_1(u, v)$ and

$$m_1(u, v) = uv(1 - u)(1 - v)H_1, \quad (4.1.25)$$

is chosen so that the variation at the boundary curves $u = 0$, $u = 1$, $v = 0$ and $v = 1$ is zero. We denote the numerator of mean curvature for $x_2(u, v)$ eq. (4.1.24) as $H_2(u, v, t)$. It would have the following familiar expression

$$H_2(u, v, t) = E_2 g_2 - 2F_2 f_2 + G_2 e_2. \quad (4.1.26)$$

As $H_2^2(u, v, t)$ is a polynomial in $t$, with real coefficients $q_i(u, v)$, we rewrite eq. (4.1.26) in the form

$$H_2^2(u, v, t) = \sum_{i=0}^{n} (q_i(u, v)) \ t^i. \quad (4.1.27)$$

As before, $n$ is 10; there being no higher powers of $t$ in the polynomials as it can be seen from the expression for $E_2(u, v, t)$, $F_2(u, v, t)$ and $G_2(u, v, t)$ which are quadratic in $t$ and $e_2(u, v, t)$, $f_2(u, v, t)$ and $g_2(u, v, t)$ which are cubic in $t$. Integrating (numerically if desired) these coefficients $w.r.t.$ $u$ and $v$ in the range $0 \leq u, v \leq 1$ we get the following integral for root mean square $rms$

$$\mu_2(t) = \left( \int_0^1 \int_0^1 H_2^2(u, v, t) \ dudv \right)^{\frac{1}{2}} = \left( t^i \int_0^1 \int_0^1 \sum_{i=0}^{n} (q_i(u, v)) \ dudv \right)^{\frac{1}{2}}. \quad (4.1.28)$$
The expression in the parentheses on right hand side of above equation is a polynomial in \( t \) of degree \( n \), which can be minimized \( w.r.t. \ t \) to find \( t_{\text{min}} \). The resulting value of \( t \) completely specify new surface \( x_2(u, v) \). New surface \( x_2(u, v) \) is expected to have lesser area than that of original surface \( x_0(u, v) \) and \( x_1(u, v) \). Root mean square root of Gaussian curvature in this case turns out to be

\[
\nu_2(t) = \left( \int_0^1 \int_0^1 K_2^2(u, v, t = t_{\text{min}}) \ dudv \right)^{\frac{1}{2}}.
\] (4.1.29)

Ratio of this mean square (ms) of mean curvature to the \( \text{rms} \) of Gaussian curvature, denoted by \( \mu_2^2/\nu_2 \) is given by,

\[
\frac{\mu_2^2}{\nu_2} = \frac{\int_0^1 \int_0^1 H_2^2 \ dudv}{\left( \int_0^1 \int_0^1 K_2^2 \ dudv \right)^{1/2}}.
\] (4.1.30)

The area \( A_2 \) of the surface \( x_2(u, v, t) \) for \( 0 \leq u \leq 1 \) and \( 0 \leq v \leq 1 \) and \( t = t_{\text{min}} \) is given by

\[
A_2 = \int_0^1 \int_0^1 \sqrt{E_2 G_2 - F_2^2} \ dudv,
\] (4.1.31)

where \( E_2 = E_2(u, v, t), \ F_2 = F_2(u, v, t), \ G_2 = G_2(u, v, t) \) are the fundamental magnitudes of the surface \( x_2(u, v) \) for \( t = t_{\text{min}} \). Complexity in computer algebra system calculations did not allow us to compute third order reduction in area. For completeness, we include the prescription for the third order reduction. The expression for the surface \( x_3(u, v, t) \) is

\[
x_3(u, v, t) = x_2(u, v) + t m_2(u, v)N_2(u, v),
\] (4.1.32)

where \( t \) is a variational parameter, \( N_2(u, v) \), unit normal to the surface \( x_2(u, v) \) and

\[
m_2(u, v) = uv(1 - u)(1 - v)H_2,
\] (4.1.33)

chosen so that the variation at the boundary curves \( u = 0, u = 1, v = 0 \) and \( v = 1 \) is zero, \( H_2 \) given by eq. (4.1.26). In order to see a geometrically meaningful (relative) change in area we calculate the dimension less area. Let \( A_0 \) be the initial area and \( A_f \) the area of the minimal surface (the subscript \( f \) denotes here that a minimal surface is flat). We can define maximum possible change to be achieved as \( \Delta A_{\text{max}} = A_0 - A_f \). Let \( A_i \) and \( A_j \) for \( i < j \) where \( i, j = 0, 1, 2, ... \) be the areas of the surfaces \( x_i(u, v) \) and \( x_j(u, v) \) obtained through \( ith \) and \( jth \) iterations respectively. Then the difference of the areas is denoted by \( \Delta A_{ij} \). When we know the minimal surface, the percentage decrease \( p_{ij} \) in area can be computed in such cases by multiplying quotient of \( \Delta A_{ij} \) and \( \Delta A_{\text{max}} \) by
100. Thus we have

\[ p_{ij} = 100 \frac{\Delta A_{ij}}{\Delta A_{\text{max}}} = 100 \frac{A_i - A_j}{A_0 - A_f} \text{ for } i < j \text{ and } i, j = 0, 1, 2, ... \]  \hspace{1cm} (4.1.34)

When we do not know the minimal surface (as e.g. in case of bilinear interpolation and Coons patch bounded by five straight lines), the percentage decrease \( q_{ij} \) in area can be computed by using

\[ q_{ij} = 100 \frac{A_i - A_j}{A_0} \text{ for } i < j \text{ and } i, j = 0, 1, 2, ... \]  \hspace{1cm} (4.1.35)

It is to be noted that for a two dimensional variational problem, the ansatz of eq. (4.1.1) is meant for determining and investigating a surface of given boundary, to find its smallest possible area and differential geometry related quantities motivated by Variational calculus. In order to foresee that how much this ansatz may be effective, we write it for a one dimensional variational problem of a known minimal straight line. This might determine the smallest possible length of an arc of a curve bounded by two given end points. The ansatz given below (eq. (4.1.36)) for the one dimensional variational problem to find the smallest possible length of an arc of a given non-minimal curve \( x_n(\xi) \) \( n = 0, 1, 2, ... \) joining the same two ends of the straight line, may be written in the form

\[ x_{n+1}(\xi, t) = x_n(\xi) + t m_n(\xi) N_n(\xi), \]  \hspace{1cm} (4.1.36)

where \( t \) is our variational parameter and

\[ m_n(\xi) = \xi (1 - \xi) H_n \]  \hspace{1cm} (4.1.37)

is chosen so that the variation at the boundary curves \( \xi = 0 \) and \( \xi = 1 \) is zero. Here \( N_n(\xi) \) is in the direction of normal to the curve given simply by 2nd order derivative of \( x_n(\xi) \) \( w.r.t. \) the curve parameter \( \xi \), serving as the the curvature multiple of the unit vector in the direction normal to the curve, thus \( H_n = 1 \) is the convenient choice in this case. We give below the variational curves and their reduced lengths skipping the details as they are better illustrated for the non-minimal surfaces in the following sections 4.2 and 4.3.

Let us apply the ansatz to the curve given by

\[ x_0(\xi) = (\xi, \xi - \xi^8), \quad 0 \leq \xi \leq 1. \]  \hspace{1cm} (4.1.38)
The ansatz eq. (4.1.36) along with eq. (4.1.37) gives following expressions for variationally improved curves

\[ \mathbf{x}_1(\xi) = (\xi, \xi - 7.42857\xi^7 + 6.42857\xi^8), \quad (4.1.39) \]

\[ \mathbf{x}_2(\xi) = (\xi, \xi - 22.131\xi^6 + 40.2381\xi^7 - 19.1071\xi^8), \quad (4.1.40) \]

\[ \mathbf{x}_3(\xi) = (\xi, \xi - 60.6973\xi^5 + 122.159\xi^6 - 128.943\xi^7 + 46.4814\xi^8), \quad (4.1.41) \]

\[ \mathbf{x}_4(\xi) = (\xi, \xi - 39.6743\xi^4 + 177.61\xi^5 - 320.449\xi^6 + 261.908\xi^7 - 80.3952\xi^8), \quad (4.1.42) \]

\[ \mathbf{x}_5(\xi) = (\xi, \xi - 21.394\xi^3 + 141.344\xi^4 - 414.012\xi^5 + 605.86\xi^6 - 434.714\xi^7 + 121.916\xi^8), \quad (4.1.43) \]

\[ \mathbf{x}_6(\xi) = (\xi, \xi - 5.02685\xi^2 + 50.057\xi^3 - 249.339\xi^4 + 622.028\xi^5 - 820.916\xi^6 + 547.646\xi^7 - 145.446\xi^8), \quad (4.1.44) \]

\[ \mathbf{x}_7(\xi) = (\xi, 0.623914\xi + 6.58385\xi^2 - 73.106\xi^3 + 327.961\xi^4 - 764.604\xi^5 + 960.763\xi^6 - 617.46\xi^7 + 159.24\xi^8), \quad (4.1.45) \]

and

\[ \mathbf{x}_8(\xi) = (\xi, 1.07777\xi - 8.98864\xi^2 + 77.659\xi^3 - 334.761\xi^4 + 755.916\xi^5 - 926.532\xi^6 + 583.748\xi^7 - 148.119\xi^8). \quad (4.1.46) \]

Corresponding lengths of the curves are

\[ \rho_0 = 1.7329, \quad \rho_1 = 1.46525, \quad \rho_2 = 1.30988, \quad \rho_3 = 1.24103, \quad \rho_4 = 1.20133, \quad (4.1.47) \]

\[ \rho_5 = 1.16958, \quad \rho_6 = 1.1459, \quad \rho_7 = 1.12682 \quad \text{and} \quad \rho_8 = 1.11081. \]

Percentage decrease in length w.r.t. maximum possible change in length may be obtained by using

\[ \rho_{\alpha\beta} = 100\frac{\rho_{\alpha} - \rho_{\beta}}{\rho_{\alpha} - 1}, \quad \text{where} \quad \alpha < \beta \quad \text{and} \quad \alpha, \beta = 0, 1, 2, \ldots. \quad (4.1.48) \]
Using values from eq. (4.1.47) in eq. (4.1.48), we get

\[ \rho_{01} = 36.5206, \quad \rho_{12} = 21.1996, \quad \rho_{23} = 9.3933, \quad \rho_{34} = 5.41714, \quad \rho_{45} = 4.33218, \]
\[ \rho_{56} = 3.2307, \quad \rho_{07} = 2.60332, \quad \rho_{78} = 2.18471 \quad \text{or} \quad \rho_{08} = 84.8815. \]

(4.1.49)

We have been able to manage eight iterations that yields the following graph for the comparison with the reduction in the initial length of the given curve \( \gamma \), whose initial length is computed to be \( \rho_0 = 1.73291 \) as shown in the Fig. 4.1.

![Graph of variational improvement to arc length of a curve of given length, comparing the results of eight iterations, where \( \gamma \) is the initial curve along with the curves \( X(\xi) \) of reduced lengths.]

Figure 4.1: Graph of variational improvement to arc length of a curve of given length, comparing the results of eight iterations, where \( \gamma \) is the initial curve along with the curves \( X(\xi) \) of reduced lengths.

### 4.2 The Technique Applied to Hemiellipsoid, a Hump Like Surface and a surface spanned by Four Arbitrary Lines

In this section we apply the technique introduced in the above section 4.1 to reduce the area of three different surfaces. In first instance we apply this technique to reduce the area of a non-minimal surface spanning a boundary for which the minimal surface is known namely hemiellipsoid given below (eq. (4.2.2)) whose boundary is an ellipse lying in a plane and thus minimal area in this case is that of the elliptic disc. In this case we obtained a reduction \( q_{01} \) in area as about 23 percent of original surface. In the second example we worked on a hump like surface spanned by four boundary
lines in the same plane; this is subject matter of section 4.2.2. In this case the reduction \( p_{02} \) in area is about 37.9141 percent of maximum possible change in initial area of the surface. These two examples in which we already know the area of the minimal surfaces, demonstrate the efficiency of the algorithm given by eq. (4.1.11). In the third example we apply this technique to reduce the area of a bilinearly interpolating surface spanned by four boundary lines lying in different planes for which the corresponding minimal surface is not known. Area reduced in this case is 0.8 percent of original surface, with no further decrease possible at least in this case, suggesting that this surface is already a near-minimal surface.

4.2.1 Hemiellipsoid-A Surface with Corresponding Known Minimal Surface

We apply the technique introduced in the section 4.1 to the following surface \( \mathbf{x}(u, v) \) namely hemiellipsoid given by eq. (4.2.2) below along with linear blending functions eq. (2.5.4), whose boundary is an ellipse. Simpler alternative of the above mentioned unit normal \( \mathbf{N}(u, v) \), having always a positive projection on it and thus not changing the signs of variable surface change for each portion of it, in case of hemiellipsoid of eq. (4.2.2) is found to be

\[
\mathbf{k} = (0, 0, 1). \tag{4.2.1}
\]

A hemiellipsoid

\[
\mathbf{x}_0 \ (u, v) = (\sin u \cos v, b \sin u \sin v, c \cos u). \tag{4.2.2}
\]

with \( b \) and \( c \) being constants and \( 0 \leq u \leq \frac{\pi}{2} \) and \( 0 \leq v \leq 2\pi \), is a non-minimal surface with its bounding curve an ellipse in the \( xy \)-plane; see Figure 4.2. Near \( z = 0 \) of the hemiellipsoid of Fig. 4.2, \( \mathbf{N}(u, v) \) and \( \mathbf{k} \) are orthogonal. In this case we shall treat hemiellipsoid as the initial non-minimal surface and the elliptical disc as the minimal surface for the given boundary, namely the ellipse. Thus, eq. (4.1.13) along with eqs. (1.2.14) and (1.2.26) gives

\[
H_0 = -bc \sin^3 u \left( \sin^2 u \left( b^2 \cos^2 v + c^2 + \sin^2 v \right) + (b^2 + 1) \cos^2 u \right). \tag{4.2.3}
\]

The root mean square (rms) of beginning curvature numerator given by eq. (4.1.14) takes the form

\[
\mu_0 = \left( \int_0^1 \int_0^1 b^2 c^2 \sin^6 u \left( \sin^2 u \left( b^2 \cos^2 v + c^2 + \sin^2 v \right) + (b^2 + 1) \cos^2 u \right)^2 \ dudv \right)^{1/2}. \tag{4.2.4}
\]
The beginning or initial area of the Coons patch given by eq. (4.1.17) takes the form in this case

$$A_0 = \int_0^1 \int_0^1 \sqrt{\sin^2 u \left( c^2 \sin^2 u \left( b^2 \cos^2 v + \sin^2 v \right) + b^2 \cos^2 u \right)} \, dudv. \quad (4.2.5)$$

Substituting $H_0$ from eq. (4.2.3) in eq. (4.1.12) gives us expression for $m_0(u,v)$. Thus in this case eq. (4.1.11) becomes

$$x_1(u,v) = (\sin u \cos v, b \sin u \sin v, c \cos u - \frac{1}{4} b c t(u-1)u(v-1)v \sin^3 u)$$

$$2(\frac{b^2}{2} - 2c^2 + 1) \cos(2u) + (\frac{b^2}{2} - 1) \sin^2 u \cos(2v) + 3b^2 + 2c^2 + 3). \quad (4.2.6)$$

Finding the fundamental magnitudes $E_1(u,v,t)$, $F_1(u,v,t)$, $G_1(u,v,t)$, $e_1(u,v,t)$, $f_1(u,v,t)$ and $g_1(u,v,t)$ for the above surface $x_1(u,v)$ of eq. (4.2.6), we can obtain the area $A_1$ using eq. (4.1.23) and $H_1(u,v,t)$ using eq. (4.1.18) and after performing the integrations mentioned in eq. (4.1.20), the mean square curvature $\mu_1^2(t)$ for $x_1(u,v)$ can be calculated. These are the similar details (like mean curvature, Gaussian curvature, dimension less quantities e.g. reduction in area and ratios of the integrals of the ms of mean curvature to the rms of Gaussian curvature etc.) as given below for the non-minimal surfaces namely a hump like surface (section 4.2.2) and a surface spanned by 4 non-coplanar lines (section 4.2.3). They have not been included for this “first instance” but rather
included for the “second and the third examples” because the formalism is well illustrated by these “second and third examples”. Also, that these expressions for the first example are too lengthy to be presented. For chosen values of \( b \) and \( c \) we can generate a table of their values within the range \( 0 \leq u \leq \frac{\pi}{2} \) and \( 0 \leq v \leq 2\pi \). For our purpose we took \( 0 \leq b, c \leq 2 \) with a step size 0.2 and \( 0 \leq u \leq \frac{\pi}{2} \) and \( 0 \leq v \leq 2\pi \), yielding a table of values. Interpolation surface for the corresponding minimum values \( t(b, c) \) for the first iteration as a function of \( b \) and \( c \) is given by Figure 4.3.

![Figure 4.3: Values of the variation in the parameter \( t(b, c) \) for the first iteration as a function of the semi-major and semi-minor axes \( b \) and \( c \) of the ellipse bounding the hemiellipsoid.](image)

In this case the dimensionless decrease \( p \) in area for different values of \( b \) and \( c \) is \( 0 \leq p \leq 23 \) that may be seen from the Figure 4.4.

### 4.2.2 A Hump Like Surface Spanned by Four Boundary Straight Lines

In this section we shall reduce the area of a hump like surface spanning four straight lines on a square with parametrization

\[
x_0(u, v) = (u, v, 16uv(1 - u)(1 - v)) \tag{4.2.7}
\]

where \( 0 \leq u, v \leq 1 \) as shown in Figure 4.5. In this case a plane bounded by four straight lines is a minimal surface. Spanning the variation of boundary composed of four straight lines we apply the eq. (4.1.11) along with eq. (4.1.12) mentioned in the section 4.1 to the surface \( x(u, v) \) given by eq. (4.2.7) bounded by four straight lines \( 0 \leq u, v \leq 1 \). A convenient possible choice of above mentioned
alternative to unit normal \( \mathbf{N}_0(u, v) \), having always a positive projection on it in case of surface given by eq. (4.2.7) is:

\[
\mathbf{k} = (0, 0, 1). \tag{4.2.8}
\]

It can be seen easily that component of the unit normal \( \mathbf{N}_0 \) along \( \mathbf{k} \) remains positive for \( 0 \leq u, v \leq 1 \). Thus angle \( \vartheta \) between \( \mathbf{N}_0 \) and \( \mathbf{k} \) remains within the interval \( 0 \leq \vartheta \leq \pi/2 \); this assures us that signs of changes along \( \mathbf{N}_0 \) and \( \mathbf{k} \) are same, so a local increase (decrease) in the area integrand while moving along \( \mathbf{N}_0 \) means an increase (decrease) in moving along \( \mathbf{k} \). Computationally it is comparatively easier to move along \( \mathbf{k} \) than along \( \mathbf{N}_0 \). We get a net decrease in area with an optimal numerical value of the coefficient \( t \) even when it multiplies \( \mathbf{k} \) not \( \mathbf{N}_0 \) as in eq. (4.1.11); this indicates that the choice \( \mathbf{k} \) instead of \( \mathbf{N}_0 \) has been useful. The surface given by eq. (4.2.7) is a non-minimal surface and the \( xy \)-plane bounded by \( 0 \leq u, v \leq 1 \) is a minimal surface in this case. Fundamental magnitudes for this initial surface are

\[
E_0 = 1 + (16(1-u)(1-v)v - 16u(1-v)v)^2 \tag{4.2.9}
\]

\[
F_0 = (16(1-u)u(1-v) - 16(1-u)uv)(16(1-u)(1-v)v - 16u(1-v)v) \tag{4.2.10}
\]

\[
G_0 = 1 + (16(1-u)u(1-v) - 16(1-u)uv)^2 \tag{4.2.11}
\]

\[
e_0 = -32v + 32v^2 \tag{4.2.12}
\]
Figure 4.5: A surface spanned by four straight lines with the parametrization $\mathbf{x}_0(u, v) = (u, v, 16uv(1-u)(1-v))$.

\[ f_0 = 16 - 32u - 32v + 64uv \]  \hspace{1cm} (4.2.13)

\[ g_0 = -32u + 32u^2. \]  \hspace{1cm} (4.2.14)

Thus, eq. (4.1.13) along with eqs. (1.2.14) and (1.2.26) gives

\[ H_0 = 32((1 + 256(-1 + u)^2u^2(1 - 2v)^2)(-1 + v)v - 256(1 - 2u)^2(-1 + u)u(1 - 2v)^2 \]

\[ (-1 + v)v + (-1 + u)u(1 + 256(1 - 2u)^2(-1 + v)^2v^2)). \]  \hspace{1cm} (4.2.15)

The initial surface is shown in the Figure 4.5 and the initial mean curvature $H_0$ eq. (4.2.15) is shown in the Figure 4.6 in the region for $0 \leq u, v \leq 1$. In this figure for the region around $u = 0.25$ and $v = 0.5$ (or for the region in which $u, v$ interchanged), there is almost a saddle shape and the mean curvature seems to be reducing in this region. But in the middle region around $u = v = 0.5$, mean curvature is more than the other region, it seems that in this region both the principal curvatures have the same sign. The root mean square ($rms$) of initial curvature given by eq. (4.1.14) comes out to be

\[ \mu_0 = 40.468 \]  \hspace{1cm} (4.2.16)
Ratio (1.2.34) of \( \text{rms} \) Gaussian curvature to the initial mean square of mean curvature of the initial surface \( x_0(u, v) \)

\[
\frac{\mu_0^2}{\nu_0} = 28.9318 \quad (4.2.17)
\]

Substituting value of \( H_0 \) from eq. (4.2.15) in eq. (4.1.12) and then eq. (4.1.11) gives us following expression for

\[
x_1(u, v, t) = (u, v, 16(1 - u)u(1 - v)v + t(1 - u)u(1 - v)v(-2(16 - 32u - 32v + 64uv)
\]

\[
(16(1 - u)u(1 - v) - 16(1 - u)uv)(16(1 - u)(1 - v)v - 16u(1 - v)v) + \]

\[
(-32v + 32v^2)(1 + (16(1 - u)u(1 - v) - 16(1 - u)uv)^2) + (-32u + 32u^2)
\]

\[
(1 + (16(1 - u)(1 - v)v - 16u(1 - v)v^2)).
\]

Fundamental magnitudes for this variational surface, \( E_1(u, v, t) \), \( F_1(u, v, t) \), \( G_1(u, v, t) \), \( e_1(u, v, t) \), \( f_1(u, v, t) \), \( g_1(u, v, t) \) are as follow:

\[
E_1(u, v, t) = 1 + 256(1 - 2u)^2(-1 + v)^2v^2(1 + 2t((-1 + v)v + 1536u^3(-1 + v)v(3 + 8
\]

\[
(-1 + v)v) - 768u^4(-1 + v)v(3 + 8(-1 + v)v) + 2u(-1 + 256(-1 + v)
\]

\[
v(1 + 3(-1 + v)v)) + 2u^2(1 - 128(-1 + v)v(11 + 30(-1 + v)v))\) ^2,
\]
Inserting these values of fundamental magnitudes in eq. (4.1.18) we find the expression for $H_1(u, v, t)$,
mean curvature of $x_1(u,v,t)$

$$H_1(u,v,t) = -32(-1 + v)v(-1 + 2t(-1 + 6u - 6u^2 - 15(17 + 256(-1 + u)u(1 + 3(-1 + u)u)v + 3(341 + 256(-1 + u)u(19 + 55(-1 + u)u)v^2 - 1536(1 - 2u)^2(1 + 10(-1 + u)v - 1536(-1 + u)^2u^2)$$

$$+ (1 - 2v)^2(1 + 2t(-1 + u) - 2v + 512(-1 + u)u(1 + 3(-1 + u)u)v + 2)$$

$$- 128(-1 + u)u(11 + 30(-1 + u)u)v^2 + 1536(-1 + u)u(3 + 8(-1 + u)u$$

$$v^3 - 768(-1 + u)u(3 + 8(-1 + u)u)v^4)) + 8192(1 - 2u)^2(-1 + u)u(1 - 2v)^2$$

$$(-1 + v)v(-1 + 2t(-2(-1 + u) + v - 256(-1 + u)u(-2 + 3u(-1 + 4 + 3u)v +$$

$$(-1 + 256(-1 + u)u(8 + 33(-1 + u)u)v^2 - 3072(1 - 2u)^2(-1 + u)uv^3 +$$

$$1536(-1 - 2u)^2(-1 + u)uv^4))(-1 + 4t(-1 + u)u + v - 256(-1 + u)u$$

$$(-2 + 3u(-1 + 3u) + u(-1 + 256(-1 + u)u(11 + 45(-1 + u)u)v^2 - 4608(1 - 2u)^2(-1 + u)uv^3 +$$

$$2304(1 - 2u)^2(-1 + u)uv^4))(-1 + 2t(-2(-1 + v)$$

$$v - 3072u^3(1 - 2v)^2(-1 + v)v + 1536u^4(1 - 2v)^2(-1 + v)v + u(-1 - 256(-1 + v)$$

$$v(-2 + 3v(-1 + 3v)) + u^2(-1 + 256(-1 + v)u(8 + 33(-1 + v)u)v(1 - 256(-1 + v)$$

$$u(-1 + 2t(-1 - 6(-1 + v)v - 1536u^3(1 - 2v)^2(1 + 10(-1 + v) + 768u^4$$

$$+ (1 - 2v)^2(1 + 10(-1 + v)v) - 15u(17 + 256(-1 + v)u(1 + 3(-1 + v) + 3u(341 + 256(-1 + v)u(19 + 55(-1 + v)u$$

$$v) + (1 + 256(-1 + u)u(1 + 3(-1 + u)u)v^2 - 512(-1 + u)u(3 + 8(-1 + u)u$$

$$v^3 + 256(-1 + u)u(3 + 8(-1 + u)u)^2),$$

$$(4.2.25)$$

The coefficients $p_i(u,v)$ of $t^i$ for $i = 0, 1, 2, 3$ in the expansion of $H_1(u,v,t)$ are

$$p_0(u,v) = -32(-1 + u)u + v - 256(-1 + u)u(1 + 3(-1 + u)u)v + (-1 + 256(-1 + u)u(1 + 11(-1 + u)u)$$

$$v^2 - 512(-1 + u)u(3 + 8(-1 + u)u$$

$$v^3 + 256(-1 + u)u(3 + 8(-1 + u)u)^2),$$

$$(4.2.26)$$
\[ p_1(u, v) = 64(-786432u^7(-1 + v)^2v^2(21 + 4(-1 + v)v(31 + 48(-1 + v)v) + 196608u^8(-1 + v)^2v^2(21 + 4(-1 + v)v(31 + 48(-1 + v)v)) - (-1 + v)v(-1 + 3(-1 + v)v(85 + 256(-1 + v)v)) + u(-1 + 4(-1 + v)v(-3 + 64(-1 + v)v(19 + 54(-1 + v)v)) - 768u^5(-3 + 2(-1 + v)v(-27 + 2(-1 + v)v(7835 + 256(-1 + v)v(184 + 285(-1 + v)v))) + 256u^6(-3 + 2(-1 + v)v(-27 + 2(-1 + v)v(26651 + 768(-1 + v)v(206 + 319(-1 + v)v)))) + 2u^2(-127 + 2(-1 + v)v(-1213 + 64(-1 + v)v(1405 + 2(-1 + v)v(4891 + 8064(-1 + v)v))) - 2u^3(-639 + 512(-1 + v)v(-23 + (-1 + v)v(3171 + 2(-1 + v)v(10139 + 15936(-1 + v)v)))) + u^4(-2559 + 256(-1 + v)v(-181 + 4(-1 + v)v(11765 + (-1 + v)v(72347 + 112320(-1 + v)v))))), \]

\[ (4.2.27) \]

\[ p_2(u, v) = -32768(-1 + u)u(-1 + v)v((-1 + v)^2v^2(6 + 19(-1 + v)v) - 1179648u^9(-1 + v)^2v^2(14 + (-1 + v)v(119 + 32(-1 + v)v(11 + 12(-1 + v)v))) + 2359296u^{10}(-1 + v)^2v^2(14 + (-1 + v)v(119 + 32(-1 + v)v(11 + 12(-1 + v)v))) + u(-1 + v)v(-9 + 2(-1 + v)v(2029 + 2(-1 + v)v(7093 + 12672(-1 + v)v(149 + 164(-1 + v)v))) + 1536u^8(-1 + v)v(-33 + 16(-1 + v)v(14533 + 6(-1 + v)v(20663 + 128(-1 + v)v(479 + 524(-1 + v)v)))) - u^3(31 + 8(-1 + v)v(-4561 + 8(-1 + v)v(143089 + 32(-1 + v)v(43313 + 48(-1 + v)v(2981 + 3528(-1 + v)v))))) + u^2(6 + (-1 + v)v(-4049 + 2(-1 + v)v(367651 + 2(-1 + v)v(1912651 + 384(-1 + v)v(17455 + 21504(-1 + v)v)))))) - 3u^5(19 + 4(-1 + v)v(-23989 + 256(-1 + v)v(53227 + 12(-1 + v)v(39553 + 4(-1 + v)v(30469 + 34384(-1 + v)v))))) + u^6(19 + 4(-1 + v)v(-83125 + 256(-1 + v)v(326171 + 12(-1 + v)v(237065 + 12(-1 + v)v(59735 + 66416(-1 + v)v))))) + u^4(63 + 2(-1 + v)v(-69931 + 16(-1 + v)v(1582433 + 32(-1 + v)v(456095 + 24(-1 + v)v(60259 + 69408(-1 + v)v))))), \]

\[ (4.2.28) \]
\[ p_3(u, v) = 65536(-1 + u)u(-1 + v)v(-1 + v)^3v^3(3 + 10(-1 + v)v) - 25367150592u^{13}(1 - 2v)^2(-1 + v)^3v^3(3 + 8(-1 + v)v)(2 + (-1 + v)v(9 + 16(-1 + v)v)) + 3623878656u^{14}(1 - 2v)^2(-1 + v)^3v^3(3 + 8(-1 + v)v)(2 + (-1 + v)v(9 + 16(-1 + v)v)) - u(-1 + v)^2v^2(-6 + (-1 + v)v(2793 + 8(-1 + v)v(2525 + 4704(-1 + v)v)) + 2359296u^{12}(-1 + v)^2v^2(-21 + (-1 + v)v(203161 + 64(-1 + v)v(35519 + 768(-1 + v)v)(421 + 355(-1 + v)v)))) - 14155776u^{11}(-1 + v)^2v^2(-21 + (-1 + v)v(63385 + 64(-1 + v)v(11131 + (-1 + v)v(48765 + 128(-1 + v)v(797 + 674(-1 + v)v)))))) - u^2(-1 + v)v(6 + (-1 + v)v(-5090 + (-1 + v)v(999971 + 8(-1 + v)v(1355491 + 96(-1 + v)v(51427 + 64512(-1 + v)v)))) - 768u^3(-1 + v)v(245 + 4(-1 + v)v(-386905 + 8(-1 + v)v(38922251 + 256(-1 + v)v(1740737 + 24(-1 + v)v(322558 + (-1 + v)v(683777 + 584928(-1 + v)v)))))) + 768u^4(-1 + v)v(49 + 4(-1 + v)v(-254789 + 8(-1 + v)v(45198319 + 256(-1 + v)v(1999045 + 24(-1 + v)v(367093 + (-1 + v)v(772045 + 655968(-1 + v)v))))))) + u^5(3(-1 + v)v(2781 + 4(-1 + v)v(-248143 + 2(-1 + v)v(14160189 + 128(-1 + v)v(1627565 + 6(-1 + v)v(1477367 + 384(-1 + v)v(9583 + 9216(-1 + v)v))))))) - 8u^7(-5 + 4(-1 + v)v(14285 + 128(-1 + v)v(-179703 + 4(-1 + v)v(15643685 + 32(-1 + v)v(5816093 + 48(-1 + v)v(556153 + 36(-1 + v)v(33637 + 29392(-1 + v)v)))))) + u^4(-19 + (-1 + v)v(28573 + 4(-1 + v)v(-3466831 + (-1 + v)v(508303293 + 128(-1 + v)v(53765501 + 18(-1 + v)v(15240289 + 512(-1 + v)v(70459 + 65280(-1 + v)v))))))) + 2u^6(-5 + 4(-1 + v)v(49565 + 64(-1 + v)v(-2244879 + 64(-1 + v)v(17885581 + 4(-1 + v)v(52018853 + 48(-1 + v)v(4883029 + 72(-1 + v)v(145395 + 125528(-1 + v)v)))))))) + u^5(49 + (-1 + v)v(126811 + 4(-1 + v)v(-21499593 + 256(-1 + v)v(15874307 + 8(-1 + v)v(25388918 + 3(-1 + v)v(41251309 + 288(-1 + v)v(327257 + 295936(-1 + v)v))))))) + u^6(-63 + (-1 + v)v(312153 + 4(-1 + v)v(-78350147 + 256(-1 + v)v(76888369 + 8(-1 + v)v(117926386 + 3(-1 + v)v(185099599 + 288(-1 + v)v(1428851 + 1267712(-1 + v)v))))))))).

(4.2.29)

\[ H_1(u, v, t) \] is a polynomial in \( t \) and thus \( H_1^2(u, v, t) \) is polynomial in \( t \) as well. We find the non-zero coefficients \( q_i(u, v) \) of \( t^4 \) for \( i = 0, 1, 2, 3, 4, 5, 6 \) and integrate these coefficients for \( 0 \leq u, v \leq 1 \) as mentioned in eq. (4.1.19) to get expression for mean square of mean curvature (4.1.20) as a
polynomial in $t$, given by

$$
\mu_1^2(t) = 1637.65 - 20425t + 195725t^2 - 898809t^3 + 2.98414 \times 10^6t^4 - 5.10679 \times 10^6t^5 + 4.1912 \times 10^6t^6,
$$

(4.2.30)

shown in Figure 4.7. Minimizing above polynomial of eq. (4.2.30) for $t$ gives us $t_{\text{min}} = 0.0889327$.

Figure 4.7: Mean square of mean curvature of the surface $x_1(u,v)$ as a function of $t$

We find the variationally improved surface $x_1(u,v)$ of eq. (4.1.11) for this minimum value of $t$, that is,

$$
x_1(u,v) = (u, v, (1 + u)u(1 + v)v(16 + v(-2.84585 + 2.84585v) + u^4v(2185.61 + v(-8013.91 + (11656.6 - 5828.3v)v)) + u^3v(-4371.22 + v(16027.8 + v(-23313.2 + 11656.6v))) + u^2(2.84585 + v(2914.15 + v(-10928.1 + (16027.8 - 8013.91v)v)) + u(-2.84585 + v(-728.537 + v(2914.15 + v(-4371.22 + 2185.61v)))))
$$

(4.2.31)

shown in Figure 4.8. For this $t_{\text{min}}$ mean curvature of $x_1(u,v)$ has a lengthy expression. We show its numerical values, it is a function of $u,v$ in Figure 4.9. Initial area of surface $x_0(u,v)$ (using eq. (4.1.17)) is 2.4945189 units and that of surface given by eq. (4.2.31) comes out to be 2.11589 units as given by eq. (4.1.23) for $t_{\text{min}} = 0.0889327$. Percentage decrease in original area in this case comes out to be 15.1784. The ratio of root mean square Gaussian to the mean square of mean curvature in this case is

$$
\frac{\mu_1^2}{\nu_1} = 19.2038.
$$

(4.2.32)

Substituting $H_1(u,v)$ (shown in Figure 4.9) in eqs. (4.1.25) and (4.1.24) results in expression for
Figure 4.8: The surface $x_1(u, v)$ for $t = t_{\text{min}} = 0.0440856$

variational surface $x_2(u, v, t)$. We find fundamental coefficients $E_2(u, v, t)$, $F_2(u, v, t)$, $G_2(u, v, t)$, $e_2(u, v, t)$, $f_2(u, v, t)$, $g_2(u, v, t)$ for this variational surface $x_2(u, v, t)$ and insert these fundamental coefficients in eq. (4.1.26) to get the expression for $H_2(u, v, t)$, mean curvature of $x_2(u, v)$, and thus mean square of mean curvature $H_2(u, v, t)$ using eq. (4.1.28) gives the following expression,

$$\mu_2^2(t) = 897.323 - 14022.3t + 207068t^2 - 1.07708 \times 10^6t^3 + 6.45464 \times 10^6t^4 - 9.9155 \times 10^6t^5 + 1.99266 \times 10^7t^6.$$  

(4.2.33)

Minimizing this expression results in $t_{\text{min}} = 0.0440856$. For this $t_{\text{min}} = 0.0440856$ we find variationally improved surface $x_2(u, v)$ as shown in Figure (4.10). The mean curvature numerator $H_2(u, v)$ of $x_2(u, v)$, again a lengthy expression, is shown in Figure. 4.11 through its values. The graph of quantity $m_2(u, v)$ is shown in Figure 4.12 in the region for $0 \leq u, v \leq 1$ in order to see the pattern of reduction in area from $x_1(u, v)$ to $x_2(u, v)$. It is to be noted that in this figure for the region around $u = 0.2$ and $v = 0.5$ (or for the region in which $u, v$ interchanged), there is almost a saddle shape. This means that lesser mean curvature $H$, and thus the quantity $m(u, v)$ seems to be reducing and hence for the same $t$-values, the reduction in the area is smaller than that for $x_0(u, v)$ to $x_1(u, v)$ is lesser as well. But in the middle region around $u = v = 0.5$, it seems like both principal curvatures have the same sign and hence mean curvature in this region is more than the other region. This is
helpful in particular to guess the reduction in area for the higher iterations which we are unable to be implemented in the present computer algebra system that becomes irresponsive for higher iteration. This means that this understanding of the quantity \( m(u, v) \) (standing for the quantities \( m_1(u, v) \) and \( m_2(u, v) \) according to the context) may predict the behaviour of the area reduction pattern for the surface \( \mathbf{x}(u, v) \) that for less (more) mean curvature the quantity \( m(u, v) \) is less (more). Thus lesser \( m(u, v) \) means lesser change from \( \mathbf{x}_1(u, v) \) to \( \mathbf{x}_2(u, v) \) or than from \( \mathbf{x}_0(u, v) \) to \( \mathbf{x}_1(u, v) \). This lesser change is expected in \( u = v = 0.5 \) region even if we have not been able to actually managed the next iteration concretely. Related quantities like root mean square of mean curvature of \( \mathbf{x}_2(u, v) \) and root mean square of Gaussian Curvature are \( \mu_2 = 24.74225 \), and \( \nu_2 = 57.842 \). Ratio (1.2.34) of \( ms \) mean curvature to the \( rms \) Gaussian curvature of the initial surface \( \mathbf{x}_0(u, v) \) is \( \frac{\mu_2^2}{\nu_0} = 28.9318 \), that of \( \mathbf{x}_1(u, v) \) is \( \frac{\mu_2^2}{\nu_1} = 19.2038 \) and for \( \mathbf{x}_2(u, v) \) is \( \frac{\mu_2^2}{\nu_2} = 10.5836 \). This is only the two dimensional variational problem in which we have been able to calculate second variation of the surface. In other cases, only first variational surface is calculated. However in case of one dimensional variational problem of a curve of known minimal length, the ansatz eq. (4.1.36) along with eq. (4.1.37) was manageable to work through eight iterations giving us reduction in length up to 84.8815 percent of maximum change in length (see Figure 4.1). For the surface discussed in the present section, area of the initial surface \( \mathbf{x}_0(u, v) \) comes out to be \( A_0 = 2.49451 \), that of \( \mathbf{x}_1(u, v) \) is \( A_1 = 2.11589 \) and area of the variationally improved surface \( \mathbf{x}_2(u, v) \) comes out to be \( A_2 = 1.927876 \). The percentage
Figure 4.10: Surface $\mathbf{x}_2(u, v, t)$ for $t = t_{\text{min}} = 0.0440856$

decrease in area from $\mathbf{x}_0(u, v)$ to $\mathbf{x}_1(u, v)$ is $p_{01} = 25.3341$, from $\mathbf{x}_1(u, v)$ to $\mathbf{x}_2(u, v)$ is $p_{12} = 12.58$
and that from $\mathbf{x}_0(u, v)$ to $\mathbf{x}_2(u, v)$ is $p_{02} = 37.9141$.

4.2.3 Surface spanned by four arbitrary boundary lines

Now we apply the technique introduced in the section 4.1 to the eq. (2.5.5) along with linear blending functions eq. (2.5.4) for a surface $\mathbf{x}(u, v)$ whose boundary is composed of four straight lines connecting four arbitrary corner points $\mathbf{x}(0, 0), \mathbf{x}(0, 1), \mathbf{x}(1, 0)$ and $\mathbf{x}(1, 1)$. For its corners we choose the configuration eq. (2.5.9), for a selection of integer values of $r$ and $d$. The results for the configuration (2.5.11) have not been included as they are similar to those for the configuration (2.5.9). We found that the above mentioned simpler alternative of the unit normal $\mathbf{N}(u, v)$, having always a positive projection on it, in case of configuration eq. (2.5.9) is

$$\mathbf{k} = (-1, 0, 0).$$

(4.2.34)

Inserting values of blending functions and boundary points in the eq. (2.5.5) we find

$$\mathbf{x}_0(u, v) = (r(u + v - 2uv), v, ud),$$

(4.2.35)
with fundamental magnitudes having the expressions as

\[ E_0 = d^2 + r^2(1-2u)^2, \quad F_0 = r^2(1-2u)(1-2v), \quad G_0 = d^2 + r^2(1-2u)^2, \quad (4.2.36) \]

\[ e_0 = 0, \quad f_0 = 2d^2r, \quad \text{and} \quad g_0 = 0. \quad (4.2.37) \]

Thus, eq. (4.1.13) gives

\[ H_0 = -4d^2r^3(-1+2u)(-1+2v). \quad (4.2.38) \]

The root mean square (rms) of beginning curvature given by eq. (4.1.14) takes the form

\[ \mu_0 = \frac{4d^2r^3}{3}. \quad (4.2.39) \]

The beginning or initial area of the Coons patch given by eq. (4.1.17) takes the form in this case

\[ A_0 = \int_0^1 \int_0^1 d\sqrt{d^2 + 2r^2(2u^2 - 2u + 2v^2 - 2v + 1)} \; dudv. \quad (4.2.40) \]

The scalars \( r \) and \( d \) can arbitrarily be chosen. Geometrical properties depend only on ratios of lengths, without changing the ratio itself and thus without loss of generality \( d = 1 \), so that the eq. (4.3.9) takes the form

\[ A_0 = \int_0^1 \int_0^1 \sqrt{4r^2u^2 - 4r^2u + 4r^2v^2 - 4r^2v + 2r^2 + 1} \; dudv. \quad (4.2.41) \]

Substituting \( H_0 \) from eq. (4.2.38) in eq. (4.1.12), we have

\[ m_0(u, v) = 16r^3u^3v^3 - 24r^3u^3v^2 + 8r^3u^3v - 24r^3u^2v^3 + 36r^3u^2v^2 \\
- 12r^3u^2v + 8r^3uv^3 - 12r^3uw^2 + 4r^3uv. \quad (4.2.42) \]
Using (4.2.42), variationally improved surface eq. (4.1.11) takes following form

\[ x_1(u, v, t) = (16r^3tu^3v^3 - 24r^3tu^3v^2 + 8r^3tu^3v - 24r^3tu^2v^3 + 36r^3tu^2v^2 - 12r^3tu^2v^2 + 8r^3tuv^3 - 12r^3tuv^2 + 4r^3tuv^2 - ruv + ru + rv, v, u). \]  

(4.2.43)

Fundamental magnitudes for this variationally improved surface are as follows:

\[ E_1(u, v, t) = (-t(8r^2(1 - u)u(1 - v)v(r(1 - v) - rv) - 4r(1 - u)(1 - v)v(r(1 - u) - ru)(r(1 - v) - rv) + 4ru(1 - v)v(r(1 - u) - ru)(r(1 - v) - rv)) + r(1 - v) - rv)^2 + 1, \]  

(4.2.44)

\[ F_1(u, v, t) = (-t(8r^2(1 - u)u(1 - v)v(r(1 - u) - ru) - 4r(1 - u)u(1 - v)v(r(1 - u) - ru)(r(1 - v) - rv)) + r(1 - u) - rv)(r(1 - v) - rv) + 4r(1 - u)uv(r(1 - u) - ru)(r(1 - v) - rv) + ru(r(1 - u) - rv) + 4ru(1 - v)v(r(1 - u) - ru) + (r(1 - u) - ru) + r(1 - v) - rv), \]  

(4.2.45)

\[ G_1(u, v, t) = (-t(8r^2(1 - u)u(1 - v)v(r(1 - u) - ru) - 4r(1 - u)u(1 - v)v(r(1 - u) - ru)(r(1 - v) - rv)) + r(1 - u) - ru)^2 + 1, \]  

(4.2.46)
After performing the integrations mentioned in eq. (4.1.20), the mean square curvature $\mu^2_H$ of surface (4.2.43) as
\[ e_1(u, v, t) = t(-96r^3uv^3 + 144r^3uv^2 - 48r^3uv + 48r^3v^3 - 72r^3v^2 + 24r^3), \]  
(4.2.47)
\[ f_1(u, v, t) = t(-144r^3u^2v^2 + 144r^3u^2v - 24r^3u^2 + 144r^3uv^2 - 144r^3uv + 24r^3) \]
\[ u - 24r^3v^2 + 24r^3v - 4r^3) + 2r, \]
(4.2.48)
and
\[ g_1(u, v, t) = t(-96r^3u^3v + 48r^3u^3v + 144r^3u^2v - 72r^3u^2 - 48r^3uv + 24r^3u). \]
(4.2.49)
Inserting these values of fundamental magnitudes in eq. (4.1.18) we find the expression for $H_1(u, v, t)$ of surface (4.2.43) as
\[ H_1(u, v, t) = [-4r^3(2u - 1)(2v - 1)] + [8r^3(2u - 1)(2v - 1)(r^2(6v - 5)(6v - 1) + u(-36(v - 1) v - 5) + 5(v - 1)v + 1 - 3(u^2 + v^2) + 3(u + v))]\]
\[ + [-32r^7(2u - 1)(2v - 1)(6u^4(2v - 1)v(18(v - 1)v + 5) + 1 - 12u^3(2v - 1)v(18(v - 1)v + 5) + 1) + u^2(2v - 1)v(138(v - 1)v + 37) + 7) + u(-2v - 1)v(30(v - 1)v + 7) - 1) + (v - 1)v(6v - 1)v + 1))\]
\[ + [u(2u - 1)(v - 1)v(2v - 1)(12u^4(3(v - 1)v(12(v - 1)v + 5) + 2) - 24u^3(3(v - 1)v(12(v - 1)v + 5) + 2) + 3u^2(12(v - 1)v(17(v - 1)v + 7) + 11) - 9u\]
\[ (4(v - 1)v(5(v - 1)v + 2) + 1) + 3(v - 1)v(8(v - 1)v + 3) + 1)]\]
(4.2.50)
After performing the integrations mentioned in eq. (4.1.20), the mean square curvature $\mu^2_H(t)$ for $x_1(u, v)$ becomes
\[ \mu^2_1(t) = \left( \frac{2048r^{18}}{2277275} \right) t^6 + \left( \frac{190464r^{16}}{25050025} \right) t^5 + \left( \frac{512r^{12}}{444675} \right) \left( \frac{153r^2 + 77}{3675} \right) t^4 + \left( \frac{256r^{10}}{3675} \right) \left( \frac{7r^2 + 3}{9} \right) t^3 \]
\[ + \left( \frac{32r^6}{1225} \right) \left( 29r^4 + 98r^2 + 119 \right) t^2 + \left( \frac{64}{75} \right) \left( 3r^2 + 5 \right) t + \left( \frac{16r^8}{9} \right), \]
(4.2.51)
which may be minimized for $t$ for every fixed value of $r$. Figure 4.13 represents this minimizing value of $t_{min}$ as the numerical function of $r$.

We find the variationally improved surface $x_1(u, v)$ eq. (4.1.11) and its area as given by eq. (4.1.23) for each $t_{min}$ for the corresponding $r$. For a selection of $r$ values for $0 \leq r \leq 2$ with step size 0.001, the dimension less decrease in area of surface $x(u, v)$ of eq. (2.5.5) can be seen in the Figure 4.14 and interpolating curve of the same is provided in Figure 4.15.
4.3 The Technique Applied to a Surface Spanned by Five Arbitrary Lines

In this section we use the ansatz of eq. (4.1.11) to reduce the area of a surface spanned by five arbitrary lines \( L_i \), \( i = 1, 2, 3, 4, 5 \) lying in different planes. We need to reduce this boundary of five lines to a boundary of four lines using technique developed in the section 3.4. Thus in the notation of section 3.4, for \( N = 5 \) bounding curves, \( m = 1 \), residue \( s = 1 \) that gives a boundary of four curves in which one curve is obtained by joining two curves and the remaining three curves. For \( 0 \leq u_i \leq 1 \), the step function \( S(u - u_i) \) satisfying the eq. (3.4.2) represented by eq. (3.3.1) is used to join the curves \( L_i \) and \( L_{i+1} \), using the technique discussed in section 3.3; other two step function representations \( S^*(u - u_i) \) and \( S^{**}(u - u_i) \) demand more CPU time, involve complicated trigonometric expressions and pose issues related to programming. As a special case for \( i = 1 \), let us assume that \( L_1(u) \) of eq. (3.3.5) and \( L_2(u) \) of eq. (3.3.6) be two successive analytical smooth curves joined together to give us smooth curve eq. (3.3.7) through the step function \( S(u - u_i) \) of eq. (3.3.1) so that eq (3.3.7) takes the following form:

\[
L(u) = L_1^1(u) = \frac{y_1}{u_1} u + \frac{1}{2} \left( 1 + \frac{l(u - u_1)}{\sqrt{\epsilon + (l(u - u_1))^2}} \right) \left( \frac{y_1}{u_1} (u - 1) - \frac{y_1}{u_1} u \right),
\]

(4.3.1)
Figure 4.14: The dimensionless decrease in area $A(r)$ as a function of $r$ enclosed by a skew quadrilateral ruled, bounded by four arbitrary straight lines connecting four corners $x(0,0), x(0,1), x(1,0)$ and $x(1,1)$.

which is continuous and differentiable at every point in the domain $0 \leq u \leq 1$. For using this work for modeling a string breaking, one can take $c_2(u) = L_4(u)$ as the initial string, $d_1(v) = L_3(v)$ and $d_2(v) = L_5(v)$ modeling the time evolution of its ends, the above $L(u)$ as the combination that contains the two final strings as shown below in the Figure 4.16 would become $c_1(u)$. For the four curves required in a general Coons patch eq. (2.5.3), we construct them from the boundary composed of five straight lines $L_1(u), L_2(u), L_3(u), L_4(u)$ and $L_5(u)$ connecting five arbitrary corner points. For this, two lines joining three corners are joined into one curve namely $c_1(u) = L_1^4(u)$ of eq. (4.3.1) and the remaining three boundary lines $d_1(v) = L_3(v), c_2(u) = L_4(u)$ and $d_2(v) = L_5(v)$. For linear blending functions $f_1 = 1 - u$, $f_2 = u$, $g_1 = 1 - v$ and $g_2 = v$, we have been able to reduce the area spanning pentagons. In case of a pentagon, when we convert it to a Coons patch, for the corners we choose:

$$r_1 = (0,0,0), \quad r_2 = (l,a,0), \quad r_3 = (0,a,0), \quad r_4 = (l,0,0). \quad (4.3.2)$$

We use a selection of integer values of $l$ and $a$. The four corners are labeled by the following Coons convention:

$$x(0,0) = r_1, \quad x(1,0) = r_4, \quad x(0,1) = r_3, \quad x(1,1) = r_2. \quad (4.3.3)$$

The $z$ component of our surface variable vector $x_0(u,v)$ is a single valued for all values of its $x$ and $y$ components and hence we can replace complicated $N_0$ in eq. (4.1.11) by a unit vector $k$ along
the $z$-axis to facilitate computations. We checked that this has a positive projection on a small enough angle with the original normal $N_0$ for the whole range of $u,v$ values; it can be seen in the Figure 4.17 that component of the unit normal $N_0$ along $k$ remains positive for $0 \leq u, v \leq 1$. Thus angle $\vartheta$ between $N$ and $k$ remains within the interval $0 \leq \vartheta \leq \pi/2$; this guarantees that signs of changes along $N_0$ and $k$ are same, so a local increase (decrease) in the area integrand while moving along $N$ means an increase (decrease) in moving along $k$. But moving along $k$ is much easier computationally. We get a net decrease in area with an optimal numerical value of the coefficient $t$ even when it multiplies $k$ not $N_0$ as in eq. (4.1.11); this indicates that the choice $k$ instead of $N_0$ has been useful. This choice of unit vector is graphically depicted in the Figure 4.18.

Inserting values of blending functions and boundary points in the eq. (2.5.3), we find

$$x_0(u,v) = \{ lu, av, vL(u) \},$$

(4.3.4)

whereas fundamental magnitudes have the following expressions

$$E_0(u,v) = l^2 + v^2 (L'(u))^2, \quad F_0(u,v) = vL(u) L'(u), \quad G_0(u,v) = a^2 L^2(u),$$

(4.3.5)

$$e_0(u,v) = alvL''(u), \quad f_0(u,v) = aL'(u) \quad \text{and} \quad g_0(u,v) = 0,$$

(4.3.6)
Figure 4.16: An initial string $c_2(u) = L_4(u)$ with $d_1(v) = L_3(v)$ and $d_2(v) = L_5(v)$ modeling the time evolution of its ends and the combination $L(u)$ contains the two final strings $L_1(u)$ and $L_2(u)$ providing $c_1(u)$.

and eq. (4.1.13) gives

$$H_0 = -2avL(u)L'(u)^2 + av\left(a^2 + L(u)^2\right)L''(u). \quad (4.3.7)$$

The root mean square (rms) of beginning curvature given by eq. (4.1.14) takes the form

$$\mu_0 = \left(\int_0^1 \int_0^1 \left(-2avL(u)L'(u)^2 + av\left(a^2 + L(u)^2\right)L''(u)\right)^2 \, dudv\right)^{1/2}. \quad (4.3.8)$$

The beginning or initial area of the Coons patch given by eq. (4.1.17) takes the form in this case

$$A_0 = \int_0^1 \int_0^1 \sqrt{-v^2L(u)^2L'(u)^2 + \left(a^2 + L(u)^2\right)^2 \left(t^2 + v^2L'(u)^2\right)} \, dudv. \quad (4.3.9)$$

Substituting $H_0$ from eq. (4.3.7) in eq. (4.1.12), we have

$$m_0(u,v) = (1-u)u(1-v)v\left(-2avL(u)L'^2(u) + av\left(a^2 + L^2(u)\right)L''(u)\right). \quad (4.3.10)$$

Variationally improved surface (4.1.11) takes following form

$$x_1(u,v,t) = \{lu, av, vL(u) + t(1-u)u(1-v)v\left(-2avL(u)L'^2(u) + av\left(a^2 + L^2(u)\right)L''(u)\right)\}. \quad (4.3.11)$$
Fundamental Magnitudes for this variationally improved surface are $E_1(u,v,t)$, $F_1(u,v,t)$, $G_1(u,v,t)$, $e_1(u,v,t)$, $f_1(u,v,t)$, $g_1(u,v,t)$ given by

$$E_1(u,v,t) = l^2 + (vL'(u) + t((1-u)(1-v)v(-2alvl(u)L'(u)^2 + alvl(a^2 + L(u)^2)L''(u)) - u(1-v)v(-2alvl(u)L'(u)^2 + alvl(a^2 + L(u)^2)L''(u)) + (1-u)u(1-v)v((2alvl(u)^3 - 2alvl(u)L'(u)L''(u) + alvl(a^2 + L(u)^2)L^{(3)}(u))))),$$

(4.3.12)

$$F_1(u,v,t) = (L(u) + t(1-u)u(1-v)v(-2alvl(u)L'(u)^2 + alvl(a^2 + L(u)^2)L''(u)) + (1-u)u(1-v)v(-2alvl(u)L'(u)^2 + alvl(a^2 + L(u)^2)L''(u)) - (1-u)uv(-2alvl(u)L'(u)^2 + alvl(a^2 + L(u)^2)L''(u))))(vL'(u) + t((1-u)

(4.3.13)
Figure 4.18: A representative graph for Coons patch for $x_i = 10$ that angle $\theta$ between $N$ and $k$ on Coons patch for $x_i = 10$ is smaller.

\[
G_1(u, v, t) = a^2 + (L(u) + t)((1 - u)v(-2alL(u)L'(u)^2 + al(a^2 + L(u)^2)L''(u)) + \\
(1 - u)(1 - v)(-2alvL(u)L'(u)^2 + alv(a^2 + L(u)^2)L''(u)) - (1 - u)uv \quad (4.3.14)
\]

\[
(-2alvL(u)L'(u)^2 + alv(a^2 + L(u)^2)L''(u)))^2.
\]

\[
e_1(u, v, t) = al(vL''(u)) + t(-2(1 - v)v(-2alvL(u)L'(u)^2 + alv(a^2 + L(u)^2)L''(u)) + \\
2(1 - u)(1 - v)v(-2alvL'(u)^3 - 2alvL(u)L'(u)L''(u) + alv(a^2 + L(u)^2) \\
L^{(3)}(u)) - 2u(1 - v)v(-2alvL'(u)^3 - 2alvL(u)L'(u)L''(u) + alv(a^2 + \\
L(u)^2)L^{(3)}(u)) + (1 - u)u(1 - v)v(-8alvL'(u)^2 L''(u) - 2alvL(u) \\
L''(u)^2 + alv(a^2 + L(u)^2)L^{(4)}(u)))),
\]
Inserting these values of fundamental magnitudes in eq. (4.1.18) we find the expression for $f_1(u, v, t)$:

$$f_1(u, v, t) = aL'(u) - 4a^2t^2vL(u)L'(u)^2 + 8a^2t^2uvL(u)L'(u)^2 + 6a^2t^2v^2L(u)L'(u)^2 - 12a^2t^2uv^2L(u)L'(u)^2 - 4a^2t^2uvL(u)^3 + 4a^2t^2uvL(u)^3 + 6a^2t^2uvL(u)^3 - 6a^2t^2uvL(u)^3 + 2a^4t^2vL''(u) - 4a^4t^2uvL''(u) - 3a^4t^2uvL''(u) + 6a^4t^2uvL''(u) - 4a^2t^2uvL(u)L''(u) + 4a^2t^2uvL(u)L''(u) + 6a^2t^2uvL(u)L''(u) - 6a^2t^2uvL(u)L''(u) - 2a^4t^2uvL(3)(u) - 2a^4t^2uvL(3)(u) - 3a^4t^2uvL(3)(u) + 3a^4t^2uvL(3)(u) + 2a^2t^2uvL(u)L''(u) - 2a^2t^2uvL(u)L''(u) - 3a^2t^2uvL(u)L''(u) + 3a^2t^2uvL(u)L''(u),$$

and

$$g_1(u, v, t) = alt(2(1 - u)v(1 - v)(-2aL(u)L'(u)^2 + al(a^2 + L(u)^2)L''(u)) - 2(1 - u)$$

$$uv(-2aL(u)L'(u)^2 + al(a^2 + L(u)^2)L''(u)) - 2(1 - u)u(-2avL(u))$$

$$L'(u)^2 + alv(a^2 + L(u)^2)L''(u)).$$

Inserting these values of fundamental magnitudes in eq. (4.1.18) we find the expression for $H_1(u, v, t)$.
as

\[ H_1(u, v, t) = alt(-2vL(u) + alt(-1 + u)uv(-2 + 3v)(-2L(u)L''(u) + a^2L''(u) + L^2(u)L''(u)))(2altv

(-2 + u(4 - 6v) + 3v)L(u)L''(u) - 2alt(-1 + u)uv(-2 + 3v)L'(u)^3 + L'(u)(1 - 2alt

(-1 + u)uv(-2 + 3v)L(u)L''(u)) + altv(-2 + 3v)(a^2 + L^2(u))((-1 + 2u)L''(u) + (-1 + u)u

L^{(3)}(u))(L'(u) + alt(1 - v)uv(2(-1 + u)L'(u)^3 + 2L(u)L'(u)((-1 + 2u)L'(u) + (-1 + u)u

L''(u)) + a^2((-1 - 2u)L''(u) - (-1 + u)uL^{(3)}(u)) + L^2(u)((-1 - 2u)L''(u) - (-1 + u)uL^{(3)}(u))))

+ 2alt(-1 + u)uv(-2 + 3v)(-2L(u)L''(u) + a^2L''(u) + L^2(u)L''(u))(t^2 + v^2(L'(u) + alt(1 - v)v

(2(-1 + u)L'(u)^3 + 2L(u)L'(u)((-1 + 2u)L'(u) + (-1 + u)uL''(u)) + a^2((-1 - 2u)L''(u)

- (-1 + u)uL^{(3)}(u)) + L^2(u)((-1 - 2u)L''(u) - (-1 + u)uL^{(3)}(u))))^2 + v(a^2 + (L(u)

+ alt(-1 + u)uv(-2 + 3v)(-2L(u)L''(u) + a^2L''(u) + L^2(u)L''(u)))^2)(L''(u) + alt(1 - v)v

((-4 + 8u)L'(u)^3 + 8(-1 + u)L^2(u)L''(u) + 2L(u)(2L^2(u) + 2(-1 + 2u)L'(u)L''(u) +

(-1 + u)L''^2(u)) + a^2(-2L''(u) + (2 - 4u)L^{(3)}(u) - (-1 + u)uL^{(4)}(u)) + L^2(u)(-2L''(u) +

(2 - 4u)L^{(3)}(u) - (-1 + u)uL^{(4)}(u))))).

(4.3.18)

Note that \( H^2_1(u, v, t) \) is a polynomial in \( t \), with real coefficients. Thus we rewrite above expression

using eq. (4.1.19), which reduces to the following form (the coefficients \( h_7(u, v), h_8(u, v), h_9(u, v) \) and

\( h_{10}(u, v) \) of \( t^7, t^9, t^{10} \) do not exist as it can be seen from above expression for \( H_1(u, v, t) \) in which

the surviving coefficients are that of \( t^0, t^1, t^2 \) and \( t^3 \) respectively).

\[ H^2_1(u, v, t) = h_0(u, v) + h_1(u, v) t + h_2(u, v) t^2 + h_3(u, v) t^3 + h_4(u, v) t^4 + \]

\[ h_5(u, v) t^5 + h_6(u, v) t^6, \]

where

\[ h_0(u, v) = (-2advL(u)L''(u) + a^3lvL''(u) + alvL^2(u)L''(u))^2, \]

(4.3.20)

\[ h_1(u, v) = 2(-2advL(u)L''(u) + a^3lvL''(u) + alvL^2(u)L''(u))(-2advL''(u)m_v(u, v) + 2adv

L(u)L''(u)m_v(u, v) + al^3m_{vv}(u, v) + alv^2L^2(u)m_{vv}(u, v) - 2advL(u)L'(u)

m_u(u, v) - 2advL(u)L'(u)m_{uv}(u, v) + a^3lm_{uu}(u, v) + alL^2(u)m_{uu}(u, v)). \]

(4.3.21)
Similarly we can find the remaining coefficients $h_2(u, v), h_3(u, v), h_4(u, v), h_5(u, v)$ and $h_6(u, v)$ as

\[
h_2(u, v) = (-2alvL''(u)m_v(u, v) + 2alvL(u)L''(u)m_v(u, v) + al^3m_{vv}(u, v) + alv^2L^2(u)m_{vv}(u, v)
- 2alL(u)L'(u)m_v(u, v) - 2alvL(u)L'(u)m_{uv}(u, v) + a^3lm_{uu}(u, v) + alL^2(u)m_{uu}(u, v))^2
+ 2(-2alvL(u)L''(u) + a^3tvL''(u) + alvL^2(u)L''(u))(alvL''(u)m_v^2(u, v) - 2alL'(u)m_v(u, v)
\]
\[
- 2alL(u)m_u(u, v)m_{uv}(u, v) + 2alL(u)m_v(u, v)m_{uv}(u, v) - 2alL'(u)m_v(u, v)m_{uv}(u, v)
\]
\[
(4.3.22)
\]

\[
h_3(u, v) = 2(-2alvL''(u)m_v(u, v) + 2alvL(u)L''(u)m_v(u, v) + al^3m_{vv}(u, v) + alv^2L^2(u)m_{vv}(u, v)
- 2alL(u)L'(u)m_v(u, v) - 2alvL(u)L'(u)m_{uv}(u, v) + a^3lm_{uu}(u, v) + alL^2(u)m_{uu}(u, v))
\]
\[
(alvL''(u)m_v^2(u, v) - 2alL'(u)m_v(u, v)m_v(u, v) + 2alvL'(u)m_{vv}(u, v)m_v(u, v)
- 2alvL'(u)m_v(u, v)m_{uv}(u, v) - 2alL(u)m_v(u, v)m_{uv}(u, v) + 2alL(u)m_v(u, v)m_{uv}(u, v))
\]
\[
+ 2(-2alvL(u)L''(u) + a^3tvL''(u) + alvL^2(u)L''(u))(alvL''(u)m_v^2(u, v) - 2alL'(u)m_v(u, v)m_{uv}(u, v)
\]
\[
- 2alm_v(u, v)m_{vu}(u, v)m_{uv}(u, v) + alm_v^2(u, v)m_{uv}(u, v)),
\]
\[
(4.3.23)
\]

\[
h_4(u, v) = (alvL''(u)m_v^2(u, v) - 2alL'(u)m_v(u, v)m_v(u, v) + 2alvL'(u)m_{vv}(u, v)m_{uu}(u, v)
\]
\[
- 2alvL'(u)m_v(u, v)m_{uv}(u, v) - 2alL(u)m_v(u, v)m_{uv}(u, v) + 2alL(u)m_v(u, v)m_{uv}(u, v)
\]
\[
m_{uu}(u, v))^2 + 2(-2alvL''(u)m_v(u, v) + 2alvL(u)L''(u)m_v(u, v) + al^3
\]
\[
m_{uv}(u, v) + alv^2L^2(u)m_{vv}(u, v) - 2alL(u)L'(u)m_v(u, v) - 2alvL(u)L'(u)
\]
\[
m_{uv}(u, v) + a^3lm_{uu}(u, v) + alL^2(u)m_{uu}(u, v))(alm_{vu}(u, v)m_v^2(u, v)
\]
\[
- 2alm_v(u, v)m_{vu}(u, v)m_{uv}(u, v) + alm_v^2(u, v)m_{uv}(u, v),
\]
\[
(4.3.24)
\]

\[
h_5(u, v) = 2(alvL''(u)m_v^2(u, v) - 2alL'(u)m_v(u, v)m_v(u, v) + 2alvL'(u)m_{vv}(u, v)m_{uu}(u, v) -
\]
\[
2alvL'(u)m_v(u, v)m_{uv}(u, v) - 2alL(u)m_v(u, v)m_{uv}(u, v) + 2alL(u)m_v(u, v)m_{uv}(u, v)
\]
\[
m_{uu}(u, v))(alm_{vv}(u, v)m_v^2(u, v) - 2alm_v(u, v)m_v(u, v)m_{uv}(u, v) +
\]
\[
alm_v^2(u, v)m_{uv}(u, v)),
\]
\[
(4.3.25)
\]
and

\[ h_6(u, v) = (almv(u, v)m^2_u(u, v) - 2almv(u, v)m_u(u, v)m_v(u, v) + alm^2_v(u, v)m_{uu}(u, v))^2. \]  (4.3.26)

Thus eq. (4.1.20) gives in this case after performing the integrations mentioned in it, the mean square curvature \( \mu^2_1 (t) \) for \( x_1(u, v) \) as

\[
\mu^2_1 = \int_0^1 \int_0^1 h_0(u, v) \, du \, dv + t \int_0^1 \int_0^1 h_1(u, v) \, du \, dv + t^2 \int_0^1 \int_0^1 h_2(u, v) \, du \, dv + \]
\[
t^3 \int_0^1 \int_0^1 h_3(u, v) \, du \, dv + t^4 \int_0^1 \int_0^1 h_4(u, v) \, du \, dv + t^5 \int_0^1 \int_0^1 h_5(u, v) \, du \, dv +
\]
\[
t^6 \int_0^1 \int_0^1 h_6(u, v) \, du \, dv,
\]  (4.3.27)

where the coefficients \( h_0(u, v) \) up to \( h_6(u, v) \) are given by the equations (4.3.20) to (4.3.26). The mean square curvature \( \mu^2_1 \) may be minimized for \( t \) for fixed values of \( a, l \) and \( y_k \). For this minimum value \( t_{\text{min}} \) we find the variationally improved surface \( x_1(u, v) \) and its area using eq. (4.1.23). In order to see a geometrically meaningful (relative) change in area we calculate the dimensionless area by dividing the difference of the (original) area of the Coons patch and the variationally decreased area by the original area. In particular for \( y_i = 10, a = 100 \) and \( l = 20 \), following table 4.1 provides the percentage decrease in area of the initial surface Coons patch eq. (2.5.3) for few cases that includes \( x_i = lu_i = 2, 4, 5, 6, 8, 10, 12, 14, 16 \) and \( 18 \). The table 4.1 indicates a symmetric behaviour of decrease in area for \( 0 \leq x_i \leq 20 \), as it can be seen that the decreases in area e.g. for \( x_i = 2 \) and \( x_i = 18 \), \( x_i = 4 \) and \( x_i = 16 \), \( x_i = 6 \) and \( x_i = 14 \), \( x_i = 8 \) and \( x_i = 12 \) etc. agree up to four decimal places. Figure 4.19 indicates that the relatively flat region staring from \( av = 0 \) expands inside the surface. This expansion is more widely spread when the ratio \( l/a \) tends to zero for larger values of \( a \) \((a \rightarrow \infty)\). The large \( a \) limit of the five line boundary has importance in the mathematical modeling of a string breaking into two, where the \( av = 0 \) straight line models the original string breaking into two final strings (straight lines) visible at \( av = 100 \). With our surface as the corresponding (relativistic) space-time world sheet of (of string), large length \( a \) means a large time evolution of the string ends at \( u = 0 \) and \( u = 1 \). Combining this with the usual quantum mechanical exponential dependence of the transition amplitudes on both time and energy [68] and Wick’s rotation [69] to imaginary time justified by a Contour integration [70] the transition amplitudes for larger energies get damped away for this large time evolution. Thus, in this limit, the string breaking probabilities
become specialized to the physically more interesting problem of the ground states of both the initial and final strings. For the gluonic strings connecting a quark and an antiquark, all five lines can be seen for example in Ref. [57]; other problems in any bosonic string theory have the same mathematical structure. (In a related application, for any value of time evolution or $a$ the area of string is proportional to its action and thus reducing area takes us closer to the non-quantum or classical minimal action. This area reduction is what we have done in the section 4.3.) Figure 4.20 represents the data points of table 4.1 along with spline curve giving us percentage decrease in area $A$ as numerical function of $x_i$, that shows the outcome of the ansatz used to calculate the decrease in area for the Coons patch for $0 \leq x_i \leq 20$. The dots give computed values of decrease in area and the smooth graph passing through these points is the spline curve interpolating these points for better predictability that how the decrease in area in the Coons patch is associated with the range of points $0 \leq x_i \leq 20$. Figure 4.20 indicates that reduction in area is smaller for the string breaking point $x_i$ generally in the middle and thus such a string breaking world sheet or symmetrical surface may be closer to being a minimal surface than the asymmetrical surfaces for the $x_i$ point significantly away from the middle where we have larger area reductions.

Figure 4.19: Coons patch for $x_i = 2, 6, 10$. 
Table 4.1: Symmetric behaviour of percent decrease in area $q_{01}$ for selected output values of $x_i$ for $0 < x_i < 20$.

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>Percentage decrease in area A</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.08214</td>
</tr>
<tr>
<td>4</td>
<td>0.05076</td>
</tr>
<tr>
<td>5</td>
<td>0.04301</td>
</tr>
<tr>
<td>6</td>
<td>0.03781</td>
</tr>
<tr>
<td>8</td>
<td>0.03208</td>
</tr>
<tr>
<td>10</td>
<td>0.04983</td>
</tr>
<tr>
<td>12</td>
<td>0.03208</td>
</tr>
<tr>
<td>14</td>
<td>0.03781</td>
</tr>
<tr>
<td>15</td>
<td>0.04301</td>
</tr>
<tr>
<td>16</td>
<td>0.05077</td>
</tr>
<tr>
<td>18</td>
<td>0.08215</td>
</tr>
</tbody>
</table>

Figure 4.20: Spline curve passing through the data points $x_i = 2, 4, 5, 6, 8, 10, 12, 14, 16, 18$ along the horizontal axis for percentage decrease in area shown along the vertical axis.
Chapter 5

Conclusion and Possible Applications

This dissertation investigated a possibility to reduce the area of a non-minimal surface for which standard techniques are not known. We studied some surfaces spanned by fixed boundary curves and did the numerical implementation of the ansatz eq. (4.1.11). The surfaces actually studied are for 1) the hemiellipsoid spanned by an ellipse, 2) a hump like surface eq.(4.5) and 3) bilinear interpolation, a special case of Coons patch. In the first two instances we know the corresponding minimal surface, whereas in third case the minimal surface is unknown. We also extended the range of applicability of 3), by studying a surface spanned by a finite number of boundary curves for the case when this $N > 4$. In this case as well, the corresponding minimal surface is not known. In order to reduce the area of a surface spanned by finite number of boundary curves, we first work a Coons patch for it. For this we introduce a grouping scheme to reduce it to four analytic curves using a differentiable form of a step function. We then wrote our ansatz to this boundary reduced to four as in the standard Coons patch. We used variational method to reduce the area of this surface by optimizing a variational parameter in our ansatz. We summarise the main points of this series of works and conclude the chapter with possible applications of the work.
5.1 Surfaces Spanned by Fixed Boundary

We have discussed a technique to reduce the area of a surface \( x(u, v) \) eq. (2.5.3) of obtaining variationally improved surface \( x_1(u, v) \) of eq. (4.1.11). We start with the application of this algorithm to a non-minimal surface spanning a boundary for which minimal surface is known namely hemiellipsoid eq. (4.2.2) (mentioned in ref. [31]). The algorithm is applied to another surface namely a hump like surface eq. (4.2.7) (section 4.2.2 see the Figure 4.5), spanned by four straight lines in the same plane for which the corresponding minimal surface is also known as a square disk. In this latter case the area is reduced by about 25.3341 percent of the maximum possible change \( \Delta A_{\text{max}} \) by the first variation of the surface and it is reduced by about 12.58 by the second variation of the surface. The dimensionless decrease \( p_{01} \) in the area of the hemiellipsoid eq. (4.2.2) for different values of \( b \) and \( c \) is \( 0 \leq p_{01} \leq 23 \) (see Figure 4.4) depending upon how much it is far from the minimal surface, namely the elliptic disk and that of a hump like surface this decrease in area is about 37.9141. This shows that our algorithm eq. (4.1.11) can significantly reduce area of surface that is far from being minimal. After noting this effectiveness, we applied this technique to reduce the area of a surface of \( x(u, v) \) given by eq. (2.5.5) bilinearly spanned by four non-planar boundary lines, a special case of Coons patch eq. (2.5.3), along with the configuration eq. (2.5.9), for a selection of \( r \) values for \( 0 \leq r \leq 2 \) with step size 0.001. This gave us a much lesser (in the range 0 to 0.82) dimensionless decrease for the surface \( x(u, v) \) spanned by four line boundary given by eq. (2.5.5), as seen in the Figure 4.14 or Figure 4.15. This suggests that this bilinear surface generated by four lines is already a near minimal surface.

5.2 Surfaces Spanned by Finite Number \( N \) of Boundary Curves for \( N > 4 \)

In this thesis, we have developed an algorithm eq. (3.4.4) to combine \( N \)-number of curves with the help of step functions shown in eqs. (3.3.1) to (3.3.3). We have applied our above mentioned technique to reduce the area of such a surface \( x(u, v) \) after converting this to the form of eq. (2.5.3) obtaining variationally improved surface \( x_1(u, v) \) of eq. (4.1.11) in this case as well. The algorithm
is applied to reduce the area of a surface of $x(u,v)$ eq. (2.5.3) spanned by five non-planar boundary lines using algorithm (3.4.4). Our resulting Coons patch of eq. (2.5.3) satisfied the conditions (2.5.1) and (2.5.2) along with the linear blending functions eq. (2.5.4), for a selection of values as given in table 4.1. This again gave us a much lesser (in the range 0 to 0.82) dimensionless decrease in less area of surface $x(u,v)$ of eq. (2.5.3), as seen in the Figure 4.13. This work is mentioned in ref [67]. It is to be noted that our variational technique reduces area by 23 percent for a surface mentioned in ref. [31] and 37.9141 for a hump like surface mentioned in above section. A much lesser decrease in this case suggests that $x(u,v)$ eq. (2.5.3) is also already a near minimal surface. We develop an algorithm (3.4.4) to combine $N$ number of curves with the help of differentiable form of step functions eqs. (3.3.1) to (3.3.3). This boundary composed of finite number of curves that is now reduced to four is used to construct Coons patch. Coons patch boundary is a combination of four analytical curves. For a Coons patch bounded by a finite number of boundary curves not necessarily in a plane we used an analytic representation of the unit step function (eqs. (3.3.1) to (3.3.3)) that lets join a collection of piecewise smooth curves into a differentiable curve that passes through all the data points, to reduce an arbitrary number of boundary curves to four analytic curves with each being an analytical function of the parameter of the curve. These four joined curves serve to form the boundary of a Coons patch eq. (2.5.3). Having the boundary, we are able to introduce a variational improvement in the resulting Coons patch using the ansatz of eq. (4.1.1) discussed in section 4.1.

### 5.3 Conclusion

Theoretically, a minimal surface may be represented by a pair of analytic functions in an appropriate conformal parameterization, and conversely [13](section 1.5). However, as it can be seen from eq. (1.5.23), that it may be difficult to calculate integrals involving complex variable functions. But this does not tell how to find a surface of smallest possible area with a prescribed fixed boundary (for $N \geq 4$) and if not possible then at least how to reduce the area of a non-minimal surface. In this dissertation we have tried to suggest a wider framework in which we report some efforts to reduce the area of a non-minimal surface suggesting an ansatz for a fixed boundary surface spanned
by \( N \geq 4 \) number of curves by giving an algorithm as well to reduce the boundary composed of \( N > 4 \) curves to *four* and then suggesting an ansatz to reduce its area and find differential geometry related quantities motivated by variational calculus. We do this by testing in certain cases where we already know a minimal surface for a boundary and we can guess the complete form of a surface of which we want to reduce the area. For ansatz the essential requirement is 1) whatever the vector function of surface parameters is added to the original surface should vanish at the boundary and 2) ansatz should have at least one variational parameter surviving the integrations with respect to the two usual parameters of a surface. The convenient actual choice depends on a variety of factors. In certain cases, we do know a minimal surface for a fixed boundary. If a closed boundary is fully in a plane, it divides the plane into two parts, a flat segment of the plane inside the boundary and one not inside. Being simply flat, the inside portion is obviously a minimal surface. We have considered such boundary only to test our algorithm if, starting from some deliberately chosen non-minimal surfaces, it actually ends up at these already known minimal flat surfaces. For the target surface being not only minimal but also flat, our query may be more than a testing of our algorithm. This is because, at least according to the traditional definition (see for example [5]) of a minimal surface as a surface with zero mean curvature, for a fixed boundary we may have more than one minimal surface because vanishing mean curvature is solution of an equation obtained by setting the derivative of the surface with respect to a parameter introduced in its temporarily modified expression. This means that a surface satisfying the resulting condition would be locally minimal in the set of all the surfaces generated by this prescription and hence is not proved to be unique. Given all this, in our testing of algorithm we have addressed three questions, namely 1) if our algorithm significantly reduces the area, 2) in how many iterations it regains the flat surface and 3) if it gets stuck in some other (locally) "minimal" surface before it reaches the flat surface. An alternative (and more practical) way of asking the first two questions is if what fraction of the total possible decrease (that is difference of non-minimal area and minimal area) our variational algorithm achieves in the maximum number of iterations we could implement within the available computational resources. In cases when we know the minimal surface and just want to check the efficiency of our algorithm, the simple ansatz eq. (4.1.11) of taking the variational surface to be non-minimal, would simply return the minimal surface if the variational parameter is taken to be
one. Obviously such an algorithm cannot be written for a case when we do not know any minimal surface for our boundary. So, in order to be able to apply our variational technique for such cases of actual interest, we deliberately complicated our algorithm even for the case of known minimal surface in such a way that it can be used more than once. Obviously to be re-iterated or re-used, the modified algorithm should perform less than perfectly in one iteration and is what we saw in section 4.2. In this dissertation, we have mentioned the “maximum number of iterations we could implement within the available computational resources”. This turned out to be 3 for a surface simpler than hemiellipsoid. In order to see that that how much the algorithm is efficient in reducing the area of a non-minimal surface, we tried our ansatz of eq. (4.1.36) for the case of a known minimal straight line and a deliberately chosen non-minimal curve joining the same two ends of the straight line. It is interesting to note that the non-minimal surface (bilinear interpolation) without a known corresponding minimal surface (ref. [31]) and Coons patch for five line boundary (ref. [67]) becomes a straight line in the one dimensional case. Thus in one dimension there is no need to judge it for being minimal or not and hence our work without a known minimal surface is not tried for the one-dimensional case. In the one-dimensional case, all the features of mean curvature of two dimensional are shown through simple second derivative, and we completed 8 iterations (section 4.1, the ansatz eq. (4.1.36) along with eq. (4.1.37) applied to the curve of eq. (4.1.38) giving us variationally improved curves of eqs. (4.1.39) to (4.1.46), percentage decreases $\rho_{\alpha\beta}$ for $\alpha, \beta = 0, 1, 2,...$ given by eq. (4.1.49) and a comparison between variationally improved curves provided in Figure 4.1). In the two dimensional case, our fractional decrease in area suddenly drops in 2nd iteration from 37.9141 percent to about 12 percent of the maximum possible decrease (namely the difference of the non-minimal area and the minimal area). In the one-dimensional case as well we did obtain a sudden decrease. But decrease continued in all the 8 iterations and hence we can not say our algorithm gets stuck in some other minimal surface than the flat one. For the two dimensional case, there still remains the possibility that the sudden decrease indicates that our algorithm getting stuck in some other minimal surface other than flat, but this possibility is not supported by our one dimensional experiments.
5.4 Possible Extensions: Comparison with Numerical Output Expressions

The problems investigated so far provide a basis for future work in several areas like in string rearrangement and computer aided geometric design. Following problems can be considered for useful studies.

1. The four boundary lines of the bilinear interpolation model the initial and final configurations of re-arranging strings [57], described in the sections 1.1 and 2.5. This might be of interest to see its future implications in string theory related works.

2. Bilinear interpolations used in string-theories-related works of physics are also used in the emerging discipline of the computer aided geometric design (CAGD) and hence the usefulness of the present work extends to CAGD along with physics and the differential geometry; as much as bilinear interpolations are near or related to minimal surfaces their study sheds some light on the above mentioned Plateau problem of the differential geometry itself.

3. A complete numerical implementation for a boundary composed of five straight lines is provided (that can model a string breaking) in the section 4.3 (see Figure 4.16). For our fully worked out boundary, we chose one that was initially composed of five straight lines that can model initial one and final two strings in a string breaking, along with the time evolution of its ends. The five-line boundary we have worked out has a variety of applications. For example, while modeling the string breaking amplitude (needed for a meson decay or other application in the bosonic string theory) we can take the initial string as one curve, the time evolution of its ends as the two adjacent curves and the two final strings modeled by the remaining two curves. The areas we calculate can be helpful in giving the required actions, reducing area taking us closer to the classical minimal action.

4. In surface modeling a surface frequently encountered is a Coons patch shown in (2.5.3) satisfying conditions (2.5.1) and (2.5.2), that is defined only for a boundary composed of four analytical curves. We not only applied the technique to a Coons patch bounded by four lines but we attempt to extend the range of applicability of a Coons patch by telling how to write
it for a boundary composed of an arbitrary number of boundary curves (section 3.4). We partition the curves in a clear and natural way into four groups and then join all the curves in each group into one analytic curve of eq. (3.4.4) by using representations of the unit step function shown in eqs. (3.3.1) to (3.3.3) including one that is fully analytic, giving a better optimized and possibly more smooth surfaces that may help generating surfaces for a variety of boundaries in computer graphics and related fields.

5. The exact minimal surface using Weierstrass-Enneper representation can be visualized. We are working to develop [40] a technique to perform integrations in (1.5.23) for discrete $z$ and then interpolate to generate the surface. It would be interesting to compare the surfaces of least area we have found with these exact minimal surfaces for areas etc.

6. In cases when we know the minimal surface and just want to check the efficiency of our algorithm, the simple ansatz eq. (4.1.11) of taking the variational surface to be non-minimal, would simply return the minimal surface if the variational parameter is taken to be one. It may be of interest to try other modifications of the ansatz simpler than that in ref. [31, 67] obtained by replacing the numerator of the mean curvature function there by one, in first variational surface or in all the variational surfaces.
Appendix

Publications

Following articles attached herewith are the outcome of this thesis. These articles have been published.


Following article resulted from the work is submitted as mentioned below.

Bibliography


110


Variational minimization on string-rearrangement surfaces, illustrated by an analysis of the bilinear interpolation

Daud Ahmad a,*, Bilal Masud b

a Department of Mathematics, University of the Punjab, Lahore, Pakistan
b Center for High Energy Physics, University of the Punjab, Lahore, Pakistan

A R T I C L E   I N F O

Keywords:
Coons patch
String-rearrangement
Bilinear interpolation
Variational improvement
Minimal surfaces

A B S T R A C T

In this paper we present an algorithm to reduce the area of a surface spanned by a finite number of boundary curves by initiating a variational improvement in the surface. The ansatz we suggest consists of original surface plus a variational parameter \( t \) multiplying the numerator \( H_0 \) of mean curvature function defined over the surface. We point out that the integral of the square of the mean curvature with respect to the surface parameter becomes a polynomial in this variational parameter. Finding a zero, if there is any, of this polynomial would end up at the same (minimal) surface as obtained by minimizing more complicated area functional itself. We have instead minimized this polynomial. Moreover, our minimization is restricted to a search in the class of all surfaces allowed by our ansatz. All in all, we have not yet obtained the exact minimal but we do reduce the area for the same fixed boundary. This reduction is significant for a surface (hemiellipsoid) for which we know the exact minimal surface. But for the bilinear interpolation spanned by four bounding straight lines, which can model the initial and final configurations of re-arranging strings, the decrease remains less than 0.8 percent of the original area. This may suggest that bilinear interpolation is already a near minimal surface.

© 2014 Elsevier Inc. All rights reserved.

1. Introduction

Variational methods are one of the active research areas of the optimization theory [1]. A variational method tries to find the best values of the parameters in a trial function that optimize, subject to some algebraic, integral or differential constraints, a quantity dependant on the ansatz. A simple example of such a problem may be to find the curve of shortest length connecting two points. The solution is a straight line between the points in case of no constraints and simplest metric, otherwise possibly many solutions may exist depending on the nature of constraints. Such solutions are called geodesics [2–4]. One of the related problems is finding the path of stationary optical length connecting two points, as the Fermat’s principle says that the rays of light traverse such a path. Another related problem is a Plateau problem [5,6] which is finding the surface with minimal area enclosed by a given curve. This problem is named after the blind Belgian physicist Joseph Plateau, who demonstrated in 1849 that a minimal surface can be obtained by immersing a wire frame, representing the boundaries, into soapy water. The Plateau problem attracted mathematicians like Schwarz [7] (who discovered D (diamond), P (primitive), H (hexagonal), T (tetragonal) and CLP (crossed layers of parallels) triply periodic surfaces), Riemann [5], and Weierstrass [5].

* Corresponding author.
E-mail addresses: daudahmadpu@yahoo.com (D. Ahmad), bilalmasud.chep@pu.edu.pk (B. Masud).

http://dx.doi.org/10.1016/j.amc.2014.01.172
0096-3003/© 2014 Elsevier Inc. All rights reserved.
Although mathematical solutions for specific boundaries had been obtained for years, but it was not until 1931 that the American mathematician Jesse Douglas [8] and the Hungarian Tibor Radó [9] independently proved the existence of a minimal solution for a given simple closed curve. Their methods were quite different. Douglas [8] minimized a functional now named as Douglas–Dirichlet Integral. This is easier to manage but has the same extremals in an unrestricted search [10] as the area functional. Douglas results held for arbitrary simple closed curve, while Radó [9] minimized the energy. The work of Radó was built on the previous work of Garnier [11] and held only for rectifiable simple closed curves. Many results were obtained in subsequent years, including revolutionary achievements of Tonelli [12], Courant [13,14], Morrey [15,16], McShane [17], Shiffman [18], Morse [19], Tompkins [19], Osserman [20], Gulliver [21] and Karcher [22] and others.

In addition to finding (above mentioned) alternative functionals, the search can be limited to a certain class of surfaces. A widely used such restriction is to search among all Bézier surfaces with the given boundary. Bézier models are widely used in computer aided geometric design (CAGD) because of their suitable geometric properties. For a control net \( P_j \) of a two dimensional parametric Bézier surface is given by

\[
X(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} B_i^n(u) B_j^m(v) P_{ij},
\]

where \( u, v \) are the parameters, \( B_i^n(u) = \binom{n}{i}u^i(1-u)^{n-i} \), the Bernstein polynomials of degree \( n \) and \( \binom{n}{i} = \frac{n!}{i!(n-i)!} \), binomial coefficients and \( D = [0, 1] \times [0, 1] \). The minimal Bézier surfaces as an example of the extremal of discrete version of Dirichlet functional may be found in the Monderde work [10], a restricted Plateau-Bézier problem defined as the surface of minimal area among all Bézier surfaces with the given boundary. A use of Dirichlet method and the extended bending energy method to obtain an approximate solution of Plateau-Bézier problem may be seen in work by Chen et al. [23]. Another restriction may be to find a surface in the parametric polynomial form as it can be seen in the Ref. [24] that finds a class of quintic parametric polynomial minimal surfaces. Bézier surfaces exactly deal with the case that the prescribed borders are polynomial curves. A more general case of borders is taken in Ref. [25] that study the Plateau-quasi-Bézier problem which includes the case when the boundary curves are catenaries and circular arcs. The Plateau-quasi-Bézier problem is related to the quasi-Bézier surface with minimal area among all the quasi-Bézier surfaces with prescribed border. They minimize the Dirichlet functional in place of original area functional.

An emerging use of minimal surfaces in physics is that in string theories. A classical particle travels a geodesic with least distance whereas a classical string is an entity which traverses a minimal area. Amongst the string theories used in physics, two are worth mentioning. One is the theory of quantum chromodynamics (QCD) strings that model the gluonic field confining a quark and an antiquark within a meson. (The gluonic field connecting three quarks, within a proton or neutron, is modeled through Y-shaped strings. For a system composed of more than three quarks, minimization of the total length of a string network with only Y-shaped junctions may be a non-trivial Steiner-Tree Problem [26]). In the other string theory (or theories) string vibrations are supposed to generate different elementary particles of the present high energy physics. Quite often string theories need a surface spanning the boundary composed of curves either connecting particles or describing the time evolution of particles. An important case can be a fixed boundary composed of four external curves. A common application of this boundary can be the time evolution of a string parameterized by \( \sigma \) or \( \beta \) [27] variable; the time evolution itself is parameterized by the symbol \( \tau \), the proper time of relativity. In this case two bounding curves parameterized by the respective \( \sigma \) or \( \beta \) represent the initial and final configurations of a string, and the other two curves (parameterized by the respective \( \tau \) variables) describe the time evolution of the two ends of a string.

String theories take action to be proportional to area. Combining this with the classical mechanics demand of the least action, minimal surfaces spanning the corresponding fixed boundaries get their importance. For example, see Eq. (13) of Ref. [27] for the Nambu-Goto ansatz for the minimal surface area and compare it with Eqs. (14) and (15) below, along with Ref. [28] for Nambu-Goto strings. Also relevant is the use in Ref. [29] of Wilson minimal area law (MAL) to derive the quark antiquark potential in a certain approximation. A surface spanned by such a boundary is in space–time of relativity. An ordinary 3-dimensional spatial surface can span a boundary composed of two 3-dimensional curves connecting four particles and two other curves connecting the same four particles in a re-arranged (or exchanged) clustering; see for example Figs. 2 of Ref. [30] and Fig. 5 of Ref. [31]. An explicit expression of such a spanning surface can be found in Eqs. (3) and (4) of Ref. [32] and Eq. (22) of Ref. [33]. This is a bilinear interpolation in ordinary 3-dimensional space and is similar to the linear interpolations in above mentioned Eq. (13) of Ref. [27], Eq. 4.7 of Ref. [34] and Eq. 3.4 of Ref. [29]. Ref. [34] clarifies that such a surface is used as a replacement to the exact minimal surfaces for the corresponding boundaries; see Section 2 below for a minimal surface in the differential geometry. Even non-minimal surfaces have some usage in the mathematical modeling of quantum strings because (1) in contrast to classical strings, quantum strings can have any action and hence area as described by the path integral version of the quantum mechanics (see Eq. (1) of [35]) and (2) any surface spanning a boundary composed of quark lines (or quark connecting lines) corresponds to a physically allowed (gauge invariant) configuration of the gluonic field between these quarks; compare the non-minimal surface of Fig. 10.5 of Ref. [36] with the minimal surface for the same boundary in Fig. 10.1 of the same Ref. [36]. But it cannot be denied that minimal surfaces are the most important of the spanning surfaces even in quantum theories. For example, the relation in Eq. 1.14 of Ref. [37] between an area and an important quantity (termed Wilson loop) related to the potential between a quark and antiquark connected by a QCD (gluonic) string is valid only if the area is of the minimal surface. (Though above mentioned Eq. (1) of Ref. [35] relates the Wilson
loop to a “sum over all surfaces of the topology of rectangle bounded by the loop” implying that each spanning surfaces has some contribution in the Wilson loop, the minimal surface must contribute most.) Thus it is worth pointing out that the non-minimal linearly or bilinearly interpolating surfaces can replace minimal surfaces, can be effectively used as minimal surfaces or share some features in common with minimal surfaces; text just before Eq. (1.15) of the above mentioned Ref. [37] relates them, up to non-relativistic 1/(mass square) order, to the minimal surfaces. The purpose of the present paper is explore further this “effective usability” or “sharing common features with minimal surfaces” of linearly or bilinearly interpolating surfaces. Before starting a description of our work, we want to (1) state the common feature we have chosen. This is the fractional reduction possible in the area for a fixed boundary; for an exact minimal surface this quantity is zero (at least for a small neighbourhood). For reducing area we use the variational area reduction, outlined in Section 4, to our specific bilinear interpolation described in Section 3. Moreover, we (2) point out that the bilinear interpolations used in string-theories-related works of physics are also used in the emerging discipline of the computer aided geometric design (CAGD) and hence the usefulness of the present paper extends to above mentioned CAGD along with physics and the differential geometry; as much as bilinear interpolations are near or related to minimal surfaces their study sheds some light on the above mentioned Plateau problem of the differential geometry itself.

Computer aided geometric design (CAGD) [38,39], mentioned above, arose when mathematical descriptions of shapes facilitated the use of computers to process data and analyze related information. In the 1960s, it became possible to use computer control for basic and detailed design enabling utilization of a mathematical model stored in a computer instead of the conventional design based on drawings. The term geometric modeling is used to characterize the methods used in describing the geometry of an object. Over the years, various schemes were developed with a view to achieve this abstraction. S. A. Coons [38,39] introduced the Coons patch in 1964. The Coons patch approach is based on the premise that a patch can be described in terms of four distinct boundary curves. Thus a Coons patch can be a worth analyzing surface spanning a fixed set of boundary curves. This is simple when the number of bounding curves is four. For a surface spanned by an arbitrary \( N \)-number of curves, it is still possible to find a Coons patch that is spanned by a boundary of four analytical curves by combining, as for example the way we did in Ref. [40], these \( N \)-number of curves into four groups and then joining these curves in each of four groups into a single analytic curve. This joining let us use Eq. (11) to write the Coons patch spanned by \( N > 4 \)-number of curves which may then be used to find the associated minimal surface by the ansatz Eq. (25). Using that formalism our technique can be applied to any number of curves, we have implemented it in full though numerical implementation has been limited to five straight lines. Ref. [41] points out that Coons patch can be considered a special case of the above mentioned Bézier surface. For us, Coons patch (see Eq. (11) below) is relevant because the above mentioned bilinear interpolations (see also Eq. (12) below) we basically study in this paper are a special case of Coons patch [38]. Coons patch analysis is an active area of research and has seen enormous development during recent years. But most, if not all, of the work on it has been limited to its geometric descriptions and visualization and to interactive mathematical experiments with it; it has not been analyzed from the view of differential geometry and that is also what we aim to do in this paper though we actually study only its special case of a bilinear interpolation. In trying to judge how close it is to being a minimal surface, we see how much its area can be reduced through our variational minimization. To carry out his minimization, we also had to reduce the area of a non-minimal surface spanning a boundary for which the minimal surface is known – finite number of boundary curves by introducing a variational improvement in a surface. Then in Section 5 we apply this technique to reduce the area of a non-minimal surface spanning a boundary for which the minimal surface is known – namely hemiellipsoid Eq. (32), to make sure the efficiency of the algorithm given by Eq. (25) and above mentioned bilinear interpolation spanned by four bounding lines for which the corresponding minimal surface is not known. Based on this comparison, we comment on the possible status of bilinear interpolation as an approximate minimal surface. The last Section presents results, final remarks and mentions possible future developments.

2. Differential geometry of minimal surfaces

In the optimization problem we aim for here, we eventually try to find a surface of a known boundary that has a least value of area. Area is evaluated by the area functional:

\[
A(x) = \int \int_D |\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)| du dv,
\]

where \( D \subset \mathbb{R}^2 \) is a domain over which the surface \( \mathbf{x}(u, v) \) is defined as a map, with the boundary condition \( \mathbf{x}(\partial D) = \Gamma \) for \( 0 \leq u \leq 1 \) and \( 0 \leq v \leq 1 \), \( \mathbf{x}_u(u, v) \) and \( \mathbf{x}_v(u, v) \) being partial derivatives of \( \mathbf{x}(u, v) \) with respect to \( u \) and \( v \). It is known [42] that the first variation of \( A(x) \) vanishes everywhere if and only if the mean curvature \( H \) of \( \mathbf{x}(u, v) \) is zero everywhere in it. Thus a surface of least area is also a surface of least (zero) \( \text{rms} \) mean curvature spanning the given boundary. This means we can aim for the same surface using the condition of the least means square mean curvature in place of the condition of the least area. This is helpful as, unlike area, the \( \text{ms} \) mean curvature has not a square root in its integrand. For a locally parameterized surface \( \mathbf{x} = \mathbf{x}(u, v) \), the mean curvature \( H \) may be given by
\[ H = \frac{G e - 2 F f + E g}{E G - F^2}, \]

where
\[ E = \langle x_u, x_u \rangle, \quad F = \langle x_u, x_v \rangle, \quad G = \langle x_v, x_v \rangle, \]
are the first fundamental coefficients and
\[ e = \langle N, x_u \rangle, \quad f = \langle N, x_v \rangle, \quad g = \langle N, x_v \rangle, \]
are the second fundamental coefficients with
\[ N(u, v) = \frac{x_u \times x_v}{|x_u \times x_v|}, \]
being the unit normal to the surface \( x(u, v) \). The root mean square root of the mean curvature \( H(u, v) \), for \( 0 \leq u \leq 1 \) and \( 0 \leq v \leq 1 \) denoted by \( \mu \) is given by the following expression,
\[ \mu = \left( \int_0^1 \int_0^1 H^2 \, du \, dv \right)^{1/2}. \]

For a minimal surface [42,43] the mean curvature (3) is identically zero. For minimization we use only the numerator part of the mean curvature \( H \) given by (3), as done in Ref. [44] following Ref. [5] who writes that “for a locally parameterized surface, the mean curvature vanishes when the numerator part of the mean curvature is equal to zero”. We call this numerator part \( H_0 \) as the rms curvature of the initial surface \( x_0(u, v) \) to be used in the ansatz Eq. (25) to get first order variationally improved surface \( x_1(u, v) \) of lesser area. This process could be continued as an iterative process until a minimal surface is achieved. Due to complexity of the calculations required for obtaining the second order improvement \( x_2(u, v) \), we have been able to calculate the first order \( x_1(u, v) \) only. The numerator part \( H_0 \) is denoted by
\[ H_0 = e_0 g_0 - 2 f_0 f_0 + g_0 e_0, \]
where \( E_0, F_0, G_0, e_0, f_0 \) and \( g_0 \) denote the fundamental magnitudes given by Eqs. (4) and (5), with \( N_0(u, v) \) being the unit normal given by Eq. (6) to the initial surface \( x_0(u, v) \). We call the root mean square (rms) of this \( H_0 \), for \( 0 \leq u \leq 1 \) and \( 0 \leq v \leq 1 \), as \( \mu_0 \). That is,
\[ \mu_0 = \left( \int_0^1 \int_0^1 H_0^2 \, du \, dv \right)^{1/2}. \]

In the notation of Eqs. (3)–(5) Eq. (2) becomes, for \( x_0(u, v) \),
\[ A_0 = \int_0^1 \int_0^1 \sqrt{E_0 G_0 - F_0^2} \, du \, dv. \]

### 3. Bilinear starting surface spanned by fixed boundary curves

For a minimal (or, more precisely, a stationary) surface, we have to solve the differential equation obtained by setting the mean curvature \( H \) given by Eq. (3) equal to zero for each value of the two parameters, say, \( u \) and \( v \) parameterizing a surface spanning the fixed boundary. In this section, our purpose is to describe a starting surface bounded by the skew quadrilateral which is composed of four arbitrary straight lines connecting four corners \( x(0, 0), x(0, 1), x(1, 0) \) and \( x(1, 1) \); in the next section we report the variational improvement to this start aimed towards minimizing the surface evolving from the starting surface of this section. Ref. [45] also includes a preliminary effort to variationally improve the surface bounded by four straight lines towards being a minimal surface. The algorithm used for this variational improvement applies to a wider class of surfaces. Accordingly, now we point out a class of surfaces, namely Coons patch, that includes surfaces bounded by four straight lines: Let \( c_1(u) \), \( c_2(u) \), \( d_1(v) \) and \( d_2(v) \) be four given arbitrary curves defined over the parameters \( u, v \in [0, 1] \). For \( c_1(u) = x(u, 0), c_2(u) = x(u, 1), d_1(v) = x(0, v) \) and \( d_2(v) = x(1, v) \), blending functions \( f_1(u), f_2(u), g_1(v) \) and \( g_2(v) \) satisfying the conditions that \( f_1(u) + f_2(u) = 1, g_1(v) + g_2(v) = 1 \) for non-barycentric combination of points and \( f_1(0) = g_1(0) = 1, f_1(1) = g_1(1) = 0 \) in order to actually interpolate \( x(0, 0), x(0, 1), x(1, 0) \) and \( x(1, 1) \), the following equation defines Coons patch:
\[ x(u, v) = [f_1(u) f_2(u)] \begin{bmatrix} x(0, v) \\ x(1, v) \end{bmatrix} + [x(u, 0) x(u, 1)] \begin{bmatrix} g_1(v) \\ g_2(v) \end{bmatrix} - [f_1(u) f_2(u)] \begin{bmatrix} x(0, 0) \\ x(0, 1) \end{bmatrix} \begin{bmatrix} g_1(v) \\ g_2(v) \end{bmatrix}. \]

As a special case of the above, we consider a Coons patch for which all the three terms in Eq. (11) are equal, so that this equation reduces to the following form:
\[ x(u, v) = [f_1(u) f_2(u)] \begin{bmatrix} x(0, 0) \\ x(0, 1) \end{bmatrix} \begin{bmatrix} g_1(v) \\ g_2(v) \end{bmatrix}. \]
The boundary spanned by lines connecting the points \( x(0, 0), x(0, 1), x(1, 0) \) and \( x(1, 1) \) with linear blending functions
\[
 f_1 = 1 - u, \quad f_2 = u, \quad g_1 = 1 - v, \quad g_2 = v, \quad (13)
\]
in Eq. (12) can represent a time evolution of a string or, alternatively, a re-arrangement of a one set of two strings to the only possible other re-arranged set (of two strings) connecting the same two particles and two antiparticles. (It is to be noted that a string connects only a particle with antiparticle. This constraint allows only two string arrangements for a system composed of two particles and two antiparticles.) Above \( x(u, v) \) Eq. (12) spanning this boundary is a surface that is needed in many models of string re-arrangements from one of these configurations to the other with the particle positions \( x(0, 0) = 1 \) and \( x(1, 1) = 2 \) and anti-particle positions \( x(1, 0) = 3 \) and \( x(0, 1) = 4 \). In this paper we reduce the area of a quadrilateral, using above linear blending functions and particle positions. This gives
\[
x_d(u, v) = (1 - v)r_{13} - vr_{24},
\]
(14) and
\[
x_v(u, v) = (1 - u)r_{14} - ur_{23},
\]
as partial derivatives w.r.t. \( u \) and \( v \) with the following corners:
\[
x(0, 0) = r_1, \quad x(1, 1) = r_2, \quad x(1, 0) = r_3, \quad x(0, 1) = r_4,
\]
(16) \( x(u, v) \) is our starting surface spanning four straight lines.) For real scalars \( r \) and \( d \), we consider two types of configurations of the four corners: ruled, and ruled. For ruled, we choose
\[
r_1 = (0, 0, 0), \quad r_2 = (r, d, 0), \quad r_3 = (0, d, d), \quad r_4 = (r, 0, d). \quad (17)
\]
The mapping from \( (u, v) \) to \( (x, y, z) \) in this case is
\[
x(u, v) = r(u + v - 2uv), \quad y(u, v) = vd, \quad z(u, v) = ud.
\]
For ruled, we choose
\[
x(0, 0) = r_1, \quad x(1, 1) = r_2, \quad x(0, 1) = r_3, \quad x(1, 0) = r_4.
\]
(19) The mapping from \( (u, v) \) to \( (x, y, z) \) in this case is
\[
x(u, v) = ur, \quad y(u, v) = vd, \quad z(u, v) = ud + vd(1 - 2u).
\]
These definitions are such that for \( r = d \) the four position vectors \( r_1, r_2, r_3 \) and \( r_4 \) lie at the corners of a regular tetrahedron. Figs. 1 and 2 below are 3D graphs of surfaces called the hyperbolic paraboloids for a choice of corners given by (17) and (19). The expression for the mean curvature, calculated using Eq. (3), of our bilinear interpolations is
\[
-\frac{4r^2(2u - 1)(2v - 1)}{d\left(d^2 + 2r^2(2u - 1)u + 2(v - 1)v + 1\right)^{3/2}}.
\]
(21) for the ruled, and
\[
\frac{4dr(2u - 1)(2v - 1)}{\left(d^2(1 - 2v)^2 + 2r^2(2u - 1)u + 1\right)^{3/2}},
\]
(22) for the ruled. The mean curvature for our starting surface is zero only for the \( u = \frac{1}{4} \) line and the \( v = \frac{1}{4} \) line, whereas for a minimal surface this should be zero for all values of \( u \) and \( v \). Below we describe our effort to improve our surface towards being minimal.

4. A technique for variational improvement

The area functional given by Eq. (2) is highly non-linear and is difficult to minimize due to its high non-linearity. Douglas replaced it by the extremal-sharing Dirichlet functional
\[
D(x) = \frac{1}{2} \int_D \left( \sum_i x_i^2 + \sum_j x_j^2 \right) du dv = \frac{1}{2} \int_D (E + G) du dv,
\]
(23) to give his famous solution to Plateau problem. A list of other possibilities of such functionals can be found in Ref. [46,23]. The Dirichlet Integral is related to the area functional Eq. (2) by the following relation
\[
(EG - F^2)^{\frac{1}{2}} \leq (EG)^{\frac{1}{2}} \leq \frac{E + G}{2}.
\]
(24) Thus, for a surface \( x(u, v) \), \( A(x) \leq D(x) \). The equality of the two integrals holds only for an isothermal patch i.e. for which \( E = G \) and \( F = 0 \). Both the functionals are defined as the integrals of positive functions, thus they are bounded below and both
the functions have a minimum for a compact domain. Thus finding a surface of minimal area is equivalent to solving vari-
atonal problem of finding a surface with appropriate boundary conditions for which the integrals are minimum. Douglas
suggested minimizing the Dirichlet integral that has the same extremal as the area functional. We suggest another functional
that has the same extremal as the area integral. This is based on observation that for the extremal (minimal surface) of the
area functional, the mean curvature is zero and hence an integral of the square of mean curvature would be least for this
area. (This is because this integral Eq. (7) is non-negative by construction and hence zero is its least value.) Thus a minimal
surface is also an extremal of the $\mu^2$ (Eq. (7)) along with being an extremal of the area functional. Now, as with Dirichlet
integral, $\mu^2$ has no square root unlike the area integral. Others [10,47] have converted Dirichlet integrals to a system of linear
equations for inner control points in terms of known boundary control points. We can convert our $\mu^2$ Eq. (30) to polynomial
in a variational parameter $t$ introduced through the ansatz Eq. (25). In Monterde work [10], a surface may be spanned by
given control points as is the case with Bézier surface [48]. We are considering a surface that is spanned by the fixed bound-
dary, though our straight line boundaries are in turn dictated by corner points. The Coons patch we are basing on is, according
to Ref. [41], is a special case of the Bézier surface Eq. (1).

In contrast to the work mentioned in above references, we choose the minimization of $ms$ mean curvature to reduce the
area of a non-minimal surface $x(u, v)$ in order to get a smooth variationally improved surface instead of Dirichlet integral.
This $ms$ mean curvature functional is positive as the integrand is positive and is zero only for a minimal surface. Thus we
try to find that value of variational parameter that makes this $ms$ mean curvature zero or the neighbouring value for which
the resulting variational surface is minimal or has reduced area. These surfaces are spanned by a fixed boundary curve, as is
the case with the hemiellipsoid Eq. (32) or the surfaces (Eq. (38)) spanned by four boundary curves. The area reduction in
the surface bounded by a skew quadrilateral composed by four straight lines (see below Eq. (38)) is included in the section (see
Section 5), whereas for a surface spanned by $N > 4$-number of curves, we have developed (see Ref. [40, eq. 16]) a formalism
that groups these curves into four and then in each group these curves are joined using step-function representation (Ref.
[40, eqs. 24-26]) into an analytic curve. Using that formalism we are able to write Coons patch out of it which can be used to
find a variationally improved surface of reduced area by the ansatz Eq. (25). The reduction scheme follows in the remain-
ing part of the present section.

Fig. 1. The ruled surface $(r = 1, d = 2)$ with $x, y$ as the horizontal plane and height along $z$–axis.
We want to reduce area of a non-minimal surface \( x(u, v) \) using the expectation that reduced value of \( m \) mean curvature, denoted by \( \mu^2 \), in turn reduces the area \( A \) of the surface \( x(u, v) \). As mentioned above, the \( m \) mean curvature \( \mu^2(t) \) reduces to a polynomial in the variational parameter \( t \) and can be solved for its minimum value as discussed above. Our scheme is to reduce the area of a surface \( x(u, v) \) given by Eq.(12)- a special case of Eq.(11), spanned by a fixed boundary, by obtaining a variationally selected surface \( x_1(u, v) \) of lesser area. For the variational improvement in surface (11), we suggest an ansatz essentially consisting of the original surface \( x_0(u, v) \) of Eq.(12) plus a variational parameter multiplying the numerator of its mean curvature. In our notation it becomes

\[
x_1(u, v, t) = x_0(u, v) + tm(u, v)k,
\]

where \( t \) is our variational parameter and

\[
m(u, v) = uv(1-u)(1-v)H_0.
\]

is chosen so that the variation at the boundary curves \( u = 0 \), \( u = 1 \), \( v = 0 \) and \( v = 1 \) is zero. \( k \) is a unit vector chosen such that it makes a small angle with the normal to the original surface and \( H_0 \), given by (8), is numerator of the initial mean curvature of the starting surface \( x_0(u, v) \). Calling the fundamental magnitudes for \( x_1(u, v) \) as \( E_1(u, v, t) \), \( F_1(u, v, t) \), \( G_1(u, v, t) \), \( e_1(u, v, t) \), \( f_1(u, v, t) \) and \( g_1(u, v, t) \), the area \( A_1 \) of the surface \( x_1(u, v, t) \) for \( 0 \leq u \leq 1 \) and \( 0 \leq v \leq 1 \) is given by

\[
A_1 = \int_0^1 \int_0^1 \sqrt{E_1G_1 - F_1^2}dudv.
\]

We denote the numerator of mean curvature for \( x_1(u, v) \) Eq. (25) as \( H_1(u, v, t) \). It would have the following familiar expression

\[
H_1(u, v, t) = E_1g_1 - 2F_1f_1 + G_1e_1.
\]
As \( H^2_1(u, v, t) \) is a polynomial in \( t \), with real coefficients \( h_i(u, v) \), we rewrite Eq. (28) in the form
\[
H^2_1(u, v, t) = \sum_{i=0}^{n} (h_i(u, v)) t^i.
\] (29)

Here \( n \) turns out to be 10; there being no higher powers of \( t \) in the polynomials as it can be seen from the expression for \( E_1(u, v, t) \), \( F_1(u, v, t) \) and \( G_1(u, v, t) \) which are quadratic in \( t \) and \( e_1(u, v, t) \), \( f_1(u, v, t) \) and \( g_1(u, v, t) \) which are cubic in \( t \). Integrating (numerically if needed) these coefficients \( w.r.t. \) \( u \) and \( v \) in the range \( 0 \leq u, v \leq 1 \) we get the following integral for the mean square curvature
\[
\mu^2_1(t) = \int_0^1 \int_0^1 H^2_1(u, v, t) dudv = t' \int_0^1 \int_0^1 n_{i=0} (h_i(u, v)) dudv.
\] (30)

The expression in the parentheses on right hand side of above equation turns out to be a polynomial in \( t \) of degree \( n \), which can be minimized \( w.r.t. \) \( t \) to find \( t_{\text{min}} \). The resulting value of \( t \) completely specify new surface \( x_i(u, v) \). New surface \( x_i(u, v) \) is expected to have lesser area than that of original surface \( x_o(u, v) \).

In order to see a geometrically meaningful (relative) change in area we calculate the dimension less area by dividing the difference of the (original) area of the Coons patch and the variationally decreased area by the original area.

5. The technique applied to hemiellipsoid and a surface spanned by four arbitrary lines

In this section we apply the technique introduced in the above Section 4 to reduce the area of two types of surfaces. In first instance we apply this technique to reduce the area of a non-minimal surface spanning a boundary for which the minimal surface is known namely hemiellipsoid Eq. (32) whose boundary is an ellipse lying in a plane and thus minimal area in this case is that of the elliptic disc. The reduction in area in this case makes sure the efficiency of the algorithm given by Eq. (25). In the second example we apply this technique to reduce the area of a bilinearly interpolating surface spanned by four boundary lines lying in different planes for which the corresponding minimal surface is not known.

5.1. Hemiellipsoid-a surface with corresponding known minimal surface

We apply the technique introduced in the Section 4 to the following surface \( x(u, v) \) namely hemiellipsoid given by Eq. (32) below along with linear blending functions Eq. (13), whose boundary is an ellipse. Simpler alternative of the above mentioned unit normal \( N(u, v) \), making a small angle with it, in case of hemiellipsoid Eq. (32) is found to be
\[
k = (0, 0, 1).
\] (31)

A hemiellipsoid
\[
x_o(u, v) = (\sin u \cos v, b \sin u \sin v, c \cos u).
\] (32)

with \( b \) and \( c \) being constants and \( 0 \leq u \leq \pi/2 \) and \( 0 \leq v \leq 2\pi \), is a non-minimal surface with its bounding curve an ellipse in the \( xy \)-plane; see Fig. 3. In this case we shall treat hemiellipsoid as the initial non-minimal surface and the elliptical disc as the minimal surface for the given boundary, namely the ellipse. Thus, Eq. (8) along with Eqs. (4) and (5) gives
\[
H_0 = -bc \sin^3 u \left( \sin^2 u \left( b^2 \cos^2 v + c^2 + \sin^2 v \right) + \left( b^2 + 1 \right) \cos^2 u \right).
\] (33)

The mean square curvature of beginning curvature given by Eq. (9) takes the form
\[
\mu^2_0 = \int_0^1 \int_0^1 b^2 \sin^6 u \left( \sin^2 u \left( b^2 \cos^2 v + c^2 + \sin^2 v \right) + \left( b^2 + 1 \right) \cos^2 u \right)^2 dudv.
\] (34)

The beginning or initial area of the Coons patch given by Eq. (10) takes the form in this case
\[
A_0 = \int_0^1 \int_0^1 \sin^2 u \left( b^2 \sin^2 u \left( b^2 \cos^2 v + \sin^2 v \right) + b^2 \cos^2 u \right) dudv.
\] (35)

For Hemiellipsoid, \( m(u, v) = (\pi/2 - u)H_0(0 \leq u \leq \pi/2) \) is the function that is zero at the boundary of the hemiellipsoid given by \( u = \pi/2 \). For \( H_0 \) from Eq. (33) gives us expression for \( m(u, v) \), thus in this case Eq. (25) becomes
\[
x_i(u, v, t) = (\sin u \cos v, b \sin u \sin v, c \cos u - \frac{1}{16} bct(\pi - 2u) \sin^3 u (2(b^2 - 2c^2 + 1) \cos(2u) - b^2 \cos(2u + v))
\]
\[-(b^2 - 1) \cos(2(u + v)) + 2b^2 \cos(2v) + 6b^2 + 4c^2 + \cos(2(u + v)) - 2 \cos(2v) + 6).\] (36)

Finding the fundamental magnitudes \( E_1(u, v, t) \), \( F_1(u, v, t) \), \( G_1(u, v, t) \), \( e_1(u, v, t) \), \( f_1(u, v, t) \) and \( g_1(u, v, t) \) for the above surface \( x_i(u, v) \) Eq. (36), we can obtain the area \( A_i \) using Eq. (27) and \( H^2_i(u, v, t) \) using Eq. (28) and after performing the integrations mentioned in Eq. (30), the mean square curvature \( \mu^2_i(t) \) for \( x_i(u, v) \) can be calculated. These are the similar details as
given below for the non-minimal surface spanned by 4—non-coplanar lines. They have not been included for this “first instance” but rather included for the “second example” because the formalism is well illustrated by this “second example”. Also, that these expressions are too lengthy to be presented. For chosen values of $b$ and $c$ we can generate a table of their values within the range $0 \leq u \leq \frac{\pi}{2}$ and $0 \leq v \leq 2\pi$. For our purpose we took $0 \leq b, c \leq 2$ with a step size 0.2 and $0 \leq u \leq \frac{\pi}{2}$ and $0 \leq v \leq 2\pi$, yielding a table of values. Interpolation surface for the corresponding minimum values $t(b, c)$ as a function of $b$ and $c$ is given by Fig. 4.

In this case the dimensionless decrease $p$ in area for different values of $b$ and $c$ is $0 \leq p \leq 15$ that may be seen from the Fig. 5.

5.2. Surface spanned by four arbitrary boundary lines

Now we apply the technique introduced in the Section 4 to Eq. (12) along with linear blending functions Eq. (13) for a surface $x(u, v)$ whose boundary is composed of four straight lines connecting four arbitrary corner points.
from Eq. (41), we have found that the above mentioned simpler alternative of the unit normal \( \mathbf{N}(u, v) \), making a small angle with it, in case of configuration (17) is
\[
\mathbf{k} = (-1, 0, 0).
\] (37)

Inserting values of blending functions and boundary points in Eq. (12) we find
\[
x_0(u, v) = (r(u + v - 2uv), vd, ud),
\] (38)
with fundamental magnitudes having the expressions as
\[
E_0 = d^2 + r^2 (1 - 2v)^2, F_0 = r^2 (1 - 2u)(1 - 2v), G_0 = d^2 + r^2 (1 - 2u)^2;
\] (39)
\[
e_0 = 0, f_0 = 2d^2r, \quad \text{and} \quad g_0 = 0.
\] (40)

Thus, Eq. (8) gives
\[
H_0 = -4d^2r^3(-1 + 2u)(-1 + 2v).
\] (41)

The root mean square (rms) of beginning curvature given by Eq. (9) takes the form
\[
\mu_0 = \frac{4d^2r^3}{3}.
\] (42)

The beginning or initial area of the Coons patch given by Eq. (10) takes the form in this case
\[
A_0 = \int_0^1 \int_0^1 d\sqrt{d^2 + 2r^2(2u^2 - 2u + 2v^2 - 2v + 1)dudv}.
\] (43)

The scalars \( r \) and \( d \) can arbitrarily be chosen. Geometrical properties depend only on ratios of lengths, without changing the ratio itself and thus without loss of generality \( d = 1 \), so that Eq. (43) takes the form
\[
A_0 = \int_0^1 \int_0^1 \sqrt{4r^2u^2 - 4r^2u + 4r^2v^2 - 4r^2v + 2r^2 + 1dudv}.
\] (44)

Substituting \( H_0 \) from Eq. (41) in Eq. (26), we have
\[
m(u, v) = 16r^3u^3v^3 - 24r^3u^2v^2 + 8r^3u^2v - 24r^3u^2v^2 + 36r^3u^2v^3 - 12r^3u^2v + 8r^3uv^3 - 12r^3uv^2 + 4r^3u.
\] (45)

Using (45), variationally improved surface Eq. (25) takes following form
\[
x_{f}(u, v, t) = (16r^3t^2u^3v^3 - 24r^3t^2u^2v^2 + 8r^3t^2u^2v - 24r^3t^2u^2v^2 + 36r^3t^2u^2v^3 - 12r^3t^2u^2v + 8r^3tuv^3 - 12r^3tuv^2 + 4r^3tu^2v - 2ruv + ru + rv, v, u).
\] (46)
Fundamental magnitudes for this variationally improved surface (46) are as follows:

\[
E_1(u, v, t) = (-t(8r^2(1-u)u(1-v)v(r(1-u) - rv) - 4r(1-u)(1-v)v(r(1-u) - ru)(r(1-v) - rv)) \\
+ 4ru(1-v)v(r(1-u) - ru)(r(1-v) - rv)) + r(1-v) - rv)^2 + 1,
\]

\[
F_1(u, v, t) = (-t(8r^2(1-u)u(1-v)v(r(1-u) - ru) - 4r(1-u)u(1-v)(r(1-u) - ru)(r(1-v) - rv)) \\
+ 4r(1-u)uv(r(1-u) - ru)(r(1-v) - rv) + r(1-u) - ru)(-t(8r^2(1-u)u(1-v)v(r(1-u) - rv)) \\
- 4r(1-u)(1-v)v(r(1-u) - ru)(r(1-v) - rv) + 4ru(1-v)v(r(1-u) - ru)(r(1-v) - rv)) \\
+ r(1-v) - rv),
\]

\[
G_1(u, v, t) = (-t(8r^2(1-u)u(1-v)v(r(1-u) - ru) - 4r(1-u)u(1-v)(r(1-u) - ru)(r(1-v) - rv)) \\
+ 4r(1-u)uv(r(1-u) - ru)(r(1-v) - rv) + r(1-u) - ru)^2 + 1,
\]

\[
e_1(u, v, t) = t(-96r^3u^3 + 144r^3u^2v - 48r^3uv^2 + 48r^3v^3 - 72r^3u^2 + 24r^3v),
\]

\[
f_1(u, v, t) = t(-144r^3u^2v^2 + 144r^3u^2v - 24r^3u^2 + 144r^3u^2v^2 - 144r^2uv^2 + 24r^3u - 24r^3v^2 + 24r^3v - 4r^3) + 2r.
\]

and

\[
g_1(u, v, t) = t(-96r^3u^2v + 48r^3u^2 + 144r^3u^2v - 72r^3u^2 - 48r^3uv + 24r^3u).
\]

Inserting these values of fundamental magnitudes in Eq. (28) we find the expression for \(H_1(u, v, t)\) of surface (46) as

\[
H_1(u, v, t) = [-4r^3(2u - 1)(2v - 1)] + [8r^3(2u - 1)(2v - 1)(r^2(u^2(6v - 1)(6u - 1) + u(-36u - 1)v - 5) \\
+ 5u - 1)v + 1 - 3u^2 + v^2 + 3(u + v))(u + [-32r^2(2u - 1)(2v - 1)(6u^2(2v - 1)v(18v - 1)v + 5) \\
+ 1 - 12u^2((2v - 1)v(18v - 1)v + 5) + 1 + u^2(2v - 1)v(138v - 1)v + 37) + 7) \\
+ u(2v - 1)v(30v - 1)v + 7) - 1 + (u - 1)v(6v - 1)v + 1))u^2 \\
+ u(2u - 1)v(2v - 1)(12u^4(3v - 1)v(12v - 1)v + 5) + 2) \\
- 24u^3(3v - 1)v(12v - 1)v + 5) + 2 + 3u^2(12v - 1)v(17v - 1)v + 7) \\
+ 11 + 9u(4v - 1)v(5v - 1)v + 2) + 1 + 3v - 1)v(8v - 1)v + 3) + 1)]u^2.
\]

After performing the integrations mentioned in Eq. (30), the mean square curvature \(\mu_i(t)\) for \(x_i(u, v)\) becomes

\[
\mu_i^2(t) = \left(\frac{2048r^{18}}{2277275} \right)^2 + \left(\frac{190464r^{16}}{25050025} \right)^4 \left(\frac{512r^{12}(153r^2 + 77)}{444675} \right)^2 + \left(\frac{256r^{10}(7r^2 + 3)}{3675} \right)
\]

\[
+ \left(\frac{32r^8(29r^4 + 98r^2 + 119)}{1225} \right)^2 + \left(\frac{64}{75} \right) \left(\frac{16r^2}{9} \right),
\]

which may be minimized for \(t\) for every fixed value of \(r\). Fig. 6 represents this minimizing value of \(t_{min}\) as the numerical function of \(r\).

\[\text{Fig. 6. The variation in parameter } t(r) \text{ depends on the variation of real scalar } r \text{ for the skew quadrilaterals ruled, bounded by four arbitrary straight lines connecting four corners } x(0, 0), x(0, 1), x(1, 0) \text{ and } x(1, 1).\]
We find the variationally improved surface $x(1; u, v)$ Eq. (25) and its area as given by Eq. (27) for each $t_{\text{min}}$ for the corresponding $r$. For a selection of $r$ values for $0 \leq r \leq 2$ with step size 0.001, the dimension less decrease in area of surface $x(u, v)$ of Eq. (12) can be seen in the Fig. 7 and interpolating curve of the same is provided in Fig. 8.

6. Conclusions

We have discussed a technique to reduce the area of a surface $x(u, v)$ Eq. (11) obtaining variationally improved surface $x_1(u, v)$ of Eq. (25). This algorithm is first applied to a non-minimal surface spanning a boundary for which minimal surface is known namely hemiellipsoid Eq. (32). The dimensionless decrease $p$ in the area of the hemiellipsoid Eq. (32) for different values of $b$ and $c$ is $0 \leq p \leq 15$ (see Fig. 5) depending upon how much it is far from the minimal surface, namely the elliptic disk. This shows our algorithm Eq. (25) can significantly reduce area of surface that is far from being minimal. After noting this effectiveness, we applied this technique to reduce the area of a surface of $x(u, v)$ Eq. (12) bilinearly spanned by four non-planar boundary lines, a special case of Coons patch Eq. (11), along with the configuration Eq. (17), for a selection of $r$ values for $0 \leq r \leq 2$ with step size 0.001. This gave us a much lesser (in the range 0 to 0.80) dimensionless decrease in less area of surface $x(u, v)$ of Eq. (12), as seen in the Fig. 7 or Fig. 8. This suggests that ruled_1 is already a near minimal surface.

References

Research Article

A Coons Patch Spanning a Finite Number of Curves Tested for Variationally Minimizing Its Area

Daud Ahmad¹ and Bilal Masud²

¹ Department of Mathematics, University of the Punjab, Lahore 54590, Pakistan
² Center for High Energy Physics, University of the Punjab, Lahore 54590, Pakistan

Correspondence should be addressed to Daud Ahmad; daud.math@pu.edu.pk

Received 15 September 2012; Revised 9 December 2012; Accepted 13 December 2012

Academic Editor: Yansheng Liu

Copyright © 2013 D. Ahmad and B. Masud. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In surface modeling a surface frequently encountered is a Coons patch that is defined only for a boundary composed of four analytical curves. In this paper we extend the range of applicability of a Coons patch by telling how to write it for a boundary composed of an arbitrary number of boundary curves. We partition the curves in a clear and natural way into four groups and then join all the curves in each group into one analytic curve by using representations of the unit step function including one that is fully analytic. Having a well-parameterized surface, we do some calculations on it that are motivated by differential geometry but give a better optimized and possibly more smooth surface. For this, we use an ansatz consisting of the original surface plus a variational parameter multiplying the numerator part of its mean curvature function and minimize with the respect to it the rms mean curvature and decrease the area of the surface we generate. We do a complete numerical implementation for a boundary composed of five straight lines, that can model a string breaking, and get about 0.82 percent decrease of the area. Given the demonstrated ability of our optimization algorithm to reduce area by as much as 23 percent for a spanning surface not close of being a minimal surface, this much smaller fractional decrease suggests that the Coons patch we have been able to write is already close of being a minimal surface.

1. Introduction

For a closed boundary composed of four curves efficient methods [1–5] of generating a spanning surface are known, and it is also well known how to find different differential geometry-related properties [6] for these surfaces. If needed such a surface can be replaced by other slightly deformed surfaces with the differential geometry properties be closer to some desired values. Specifically we can mention the Coons patch [7, 8] prescription for generating the surface and our variational method [9] (given by (31)) based changes in it that generate a slightly changed surface from it that has lesser rms mean curvature and hence is closer to be a minimal surface. S. A. Coons [7, 8] introduced the Coons patch in 1964. This approach is based on the premise that a patch can be described in terms of four distinct boundary curves. So when the number of boundary curves is four, a Coons patch is clearly a worth analyzing surface spanning it. It is an active area of research and has seen enormous development during recent years that includes the work of Farin and Hansford [1], Hugentobler and Schneider [10], Wang and Tang [4], Szilvási-Nagy and Szabó [2, 11], and Sarkar and Dey [5]. As for the variational methods, generally these try to find the best values of the parameters in a trial function that optimize, subject to some algebraic, integral, or differential constraints, a quantity dependant on the ansatz. Variational methods are one of the active research areas of the optimization theory [12, 13]. For example, work has been done on finding the path of stationary optical length connecting two points, as the Fermat’s principle says that the rays of light traverse such a path. In our previous work [9] we tried to find the best values of a parameter in the trial expression for a bounding surface spanning our boundary.

When the number of boundary curves is more than four, methods have been used for N sides surface generation from arbitrary boundary edges. But these are (1) discrete surfaces
The relation of such a surface with Coons patch is at least not clear, (3) such a surface is not studied from the view of differential geometry as we have done [9] for the Coons patch, or (4) it is not told how to replace such a surface with a deformed one closer to being, say, a minimal surface. In the present paper we present a prescription to avoid all these possibly unwanted features; we tell how to generalize both the surface generation equation (11) and the variational improvement equation (31) to closed boundaries composed of an arbitrary finite number of curves. Presently, we are able to fully produce variationally improved surfaces only for a boundary composed of five straight lines, but our algorithms both for the variational improvement equation (31) and, before it, for the surface generation are general. Our previous and present work falls in the category of plane problem [16, 17] which is finding the surface with minimal area constrained by a given boundary curve, named after the blind Belgian physicist Joseph Antoine Ferdinand Plateau. He showed in 1849 that a minimal surface can be realized in the form of soap films, stretched over various shapes of wire frames. Since then many mathematicians contributed to the theory of minimal surfaces like Schwarz [18] who discovered some triply periodic surfaces, Riemann and Weierstrass [16]. Mathematical solutions for specific boundaries were known for years, but existence of a minimal solution for a given simple closed curve was independently proved in 1931 by Douglas [19] and Radó [20]. They approached to the solution through different methods. Douglas [19] minimized a quantity now called as Douglas Integral for a minimal solution to an arbitrary simple closed curve, while Radó [20] minimized the energy. The work of Radó was built on the previous work of Garnier [21] for minimal solution only for rectifiable simple closed curves. Achievements of Tonelli [22], Courant [23, 24], Morrey [25, 26], McShane [27], Shiffman [28], Morse and Tompkins [29], Osserman [30], Gulliver [31], and Karcher [32] and others contributed with many revolutionary results in the subsequent years.

An arbitrary boundary can be written as a limit of a collection of finite number of arbitrary curves and for a boundary composed of finite number of our plan is to first reduce it to only four bounding curves so that we can then apply our Coons-patch-based analysis. For this reason, we are needed to group a finite number of curves into four sets and then within each set combine all the curves as one continuous curve. In this way, one set would become, in the common notation for a Coons patch, $c_2(u) = x(u, 0)$ and the other three as $c_1(u) = x(u, 1)$, $d_1(v) = x(0, v)$, and $d_2(v) = x(1, v)$ with $0 \leq u, v \leq 1$. We introduce our algorithm for grouping in the first paragraph of Section 3. For the combinations, in the remaining portion of Section 3 we present an iterative scheme that uses suitable step function representations to combine an arbitrary number of curves into one continuous curve. In Section 4 we present, in (24) to (26), three analytical representations of the unit step function and give, through Figure 4, geometric description of boundary curves generated by these step function representations. Before these two sections, in Section 2 we recall briefly some basic stuff related to minimalization of area and mean curvature and some simple facts about blending-based Coons patch as our initial surface. Having written the expression for the Coons patch spanning an arbitrary discredited boundary, in the Section 5 we describe our variational technique to reduce its rms mean curvature resulting in a decrease of its area as well. So far, our description of algorithms remains general. Then in next Section 6 we apply our step function analytical form, standard Coons patch and our variational techniques to reduce the area of a nonminimal surface spanned by five arbitrary straight lines. The resulting percentage decrease in area for few cases is reported in Table 1, and same data points are plotted in Figure 9 along with the curve that gives us percentage decrease in area as numerical function. Section 7 presents results, final remarks and mentions possible future developments.

## 2. Coons Patch as Initial Surface Composed of Four Continuous Curves and Related Quantities

Considering a surface $x(u, v)$ for $0 \leq u \leq 1$ and $0 \leq v \leq 1$ defined as a map over a domain $D \subset \mathbb{R}^2$ spanning a fixed boundary $x(\partial D) = \Gamma$, we can evaluate some quantities for that are of interest of differential geometry. We choose the area and rms mean curvature and in the later part of the paper report efforts to decrease both. Area is evaluated by the area functional:

$$A(x) = \int_D \|x_u(u, v) \times x_v(u, v)\| du dv,$$  

with $x_u(u, v)$ and $x_v(u, v)$ being partial derivatives of $x(u, v)$ with respect to $u$ and $v$. It is known [6] that the first variation of $A(x)$ vanishes everywhere if and only if the mean curvature $H$ of $x(u, v)$ is zero everywhere in it. Thus a surface of the least area is also a surface of the least (zero) rms mean curvature spanning the given boundary. Amongst the two, the mean square mean curvature (termed $\mu^2$ later) has not a square root in its integrand, and its minimization is easier than that of the area directly. For a locally parameterized surface $x = x(u, v)$, the mean curvature $H$ may be given by

$$H = \frac{Ge - 2Ff + Eg}{EG - F^2},$$
where

\[ E = \langle x_u, x_u \rangle, \quad F = \langle x_u, x_v \rangle, \quad G = \langle x_v, x_v \rangle \quad (3) \]

are the first fundamental coefficients, and

\[ e = \langle N, x_u \rangle, \quad f = \langle N, x_v \rangle, \quad g = \langle N, x_v \rangle \quad (4) \]

are the second fundamental coefficients with

\[ N(u, v) = \frac{x_u \times x_v}{|x_u \times x_v|} \quad (5) \]

being the unit normal to the surface \( x(u, v) \). For a minimal surface \([6, 33]\) the mean curvature \((2)\) is identically zero. For minimization we use only the numerator part of mean curvature \( H \) given by \((2)\), as done in \([34]\) following \([16]\) which explicitly mentions that “for a locally parameterized surface, the mean curvature vanishes when the numerator part of the mean curvature is equal to zero.” We call the numerator part of \((2)\) as \( H_0 \) corresponding to the initial surface \( x_0(u, v) \) that is used in the ansatz equation \((31)\) to get first-order variationally improved surface \( x_1(u, v) \) of lesser area; we chose our initial surface to be a Coons patch described later. This process could be continued as an iterative process until a minimal surface is achieved. But due to complexity of the calculations required for obtaining the second-order improvement \( x_2(u, v) \), we have been able to calculate the first-order surface \( x_1(u, v) \) only. The numerator part \( H_0 \) is denoted by

\[ H_0 = e_0 G_0 - 2 F_0 f_0 + g_0 E_0 \quad (6) \]

where \( E_0, F_0, G_0, e_0, f_0 \) and \( g_0 \) denote the fundamental magnitudes, given by \((3)\) and \((4)\), for the initial surface \( x_0(u, v) \), and \( N_0(u, v) \) is the unit normal, given by \((5)\), to this initial surface. We call the root mean square (rms) of this \( H_0 \), for \( 0 \leq u \leq 1 \) and \( 0 \leq v \leq 1 \), as \( \mu_0 \). That is,

\[ \mu_0 = \left( \int_0^1 \int_0^1 H_0^2 du dv \right)^{1/2}. \quad (7) \]

In the notation of \((2)\) to \((4)\), equation \((1)\) becomes, for \( x_0(u, v) \),

\[ A_0 = \int_0^1 \int_0^1 \sqrt{E_0 G_0 - F_0^2} \ du \ dv. \quad (8) \]

If the boundary \( \Gamma \) is composed of four continuous curves \( c_1(u) = x(u, 0), \ c_2(u) = x(u, 1), \ d_1(v) = x(0, v) \) and \( d_2(v) = x(1, v) \), a surface \( x(u, v) \) spanning it can be the Coons patch \([7]\). (As mentioned previously we choose this to be our initial surface for the variational process we report.) Using blending functions \( f_1(u), f_2(u), g_1(v) \) and \( g_2(v) \) satisfying the conditions that

\[ \sum_{i=1}^2 f_i(u) = 2, \quad \sum_{i=1}^2 g_i(v) = 1, \quad (9) \]

that is, \( f_1(u) + f_2(u) = 1, \ g_1(v) + g_2(v) = 1 \) for nonbarycentric combination of points and for \( j = 0, 1 \)

\[ f_i(j) = g_i(j) = \delta_{j-1,j} \quad (10) \]

in order to actually interpolate \( x(0, 0), x(0, 1), x(1, 0), \) and \( x(1, 1) \) the following equation defines Coons patch:

\[
\begin{align*}
x(u, v) &= \begin{bmatrix} f_1(u) & f_2(u) \end{bmatrix} \begin{bmatrix} x(0, 0) \ x(1, 0) \ x(0, 1) \ x(1, 1) \end{bmatrix} \begin{bmatrix} g_1(v) \\ g_2(v) \end{bmatrix} \\
&+ \begin{bmatrix} x(u, 0) \ x(u, 1) \end{bmatrix} \begin{bmatrix} g_1(v) \\ g_2(v) \end{bmatrix} \\
&- \begin{bmatrix} f_1(u) \ f_2(u) \end{bmatrix} \begin{bmatrix} x(0, 0) \ x(0, 1) \ x(1, 0) \ x(1, 1) \end{bmatrix} \begin{bmatrix} g_1(v) \\ g_2(v) \end{bmatrix}.
\end{align*}
\]

(11)

For instance, linear blending functions satisfying the previous conditions may be given by

\[
\begin{align*}
f_1(u) &= 1 - u, \quad f_2(u) = u, \\
g_1(v) &= 1 - v, \quad g_2(v) = v.
\end{align*}
\]

(12)

Using these choices of blending functions in \((11)\) we get our initial surface \( x_0(u, v) \). In the present form this prescription is apparently limited to a boundary composed of four straight lines or at most four continuous curves. For a more general boundary, we have to reduce, as mentioned previously, it to four continuous curves. The next section describes the algorithm we suggest for achieving this aim.

### 3. Construction of the Four Continuous Boundary Curves

As said previously, an arbitrary boundary can be written as a limit of a collection of finite number of arbitrary curves. For reducing it to four bounding curves, first we need to group a finite number of curves into four sets (in the notation introduced in the previous section) \( c_1(u) = x(u, 0), \ c_2(u) = x(u, 1), \ d_1(v) = x(0, v) \) and \( d_2(v) = x(1, v) \) with \( 0 \leq u, v \leq 1 \). Let us call the \( N \) bounding curves as \( L_i \) for \( i = 1, 2, \ldots, N \), the greatest integer \( \leq N/4 \) as \( m \) and the residue \( s = N - 4m \) which can take values as \( 0, 1, 2, \) or \( 3 \). We put the first \( m + c_1 \) curves in the first group \( G_{m+c_1} \) that essentially becomes \( c_1(u) \) when we join all its elements into one continuous curve by using step functions of the next section. Here \( c_1 = \min(s, 1) \), meaning that for four equal groups the first group is just the first quarter and for nonequal groups we put one of the \( s \) extra curves in the first group. The next group, to become \( d_1(v) \) after the joining \((s)\), starts with the very next \( m + c_1 + 1 \) curve and continues till \( i = 2m + d_1 \) with \( d_1 = \min(s, 2) \). The number of curves in this second group is \( m + \min(2, s) - \min(1, s) \) meaning that for \( s \) larger than one this group gets one of the \( s \) curves. We call this group \( G_{2m+d_1} \). The third group, to become \( c_2(u) \) after the joining \((s)\), starts with the \( 2m + d_1 + 1 \)st curve and continues till \( i = 3m + s \) with \( d_1 = \min(s, 2) \). The number of curves in this second group is \( m + s - \min(2, s) \) meaning that only for \( s = 3 \) this group gets one of the \( s \) curves. We call it \( G_{3m+s} \). The fourth and last group, to become \( d_2(v) \), starts with the next \( i = 3m + s + 1 \) till the end when \( i = 4m + s = N \). The number of curves in this last group is always \( m \), and thus it never gets any of the extra \( s \) curves. The label we use for this
group is $C^N_{m+1}$ for example for a boundary composed of 9 curves, $N = 9, m = 2, s = 9 = 8 = 1$, $c_1 = \min(s, 1) = 1$, $d_1 = \min(s, 2) = 1$ correspond to four sets of curves, namely, $C^i_1, C^i_2, C^i_3, C^i_4$ as the bounding curves $c_1(u), d_1(v), c_2(u)$, and $d_2(v)$, respectively, where $c_1(u)$ comprises 3 curves and each of $d_1(v), c_2(u)$, and $d_2(v)$ has two curves. For another example note that $N = 14, m = 3, s = 14 = 12 = 2$ correspond to four sets of curves, namely, $C^i_1, C^i_2, C^i_3, C^i_4$ as the bounding curves $c_1(u), d_1(v), c_2(u)$, and $d_2(v)$, respectively, for Coons patch in this case. Each of $C^i_1$ and $C^i_3$ includes 4 curves, and each of $C^i_2$ and $C^i_4$ contains 3 curves.

The next task is to join $m + c_1$ curves in the first group $C^m_{i+1}$ into one continuous curve $c_1(u)$; the same process is to be done later for other three groups. Let us consider two consecutive curves $L_i(u)$ and $L_{i-1}(u)$, for $i = 1, \ldots, m + c_1 - 1$, in this group. These can be combined into a continuous curve

$$L_i^1(u) = L_i^0(u) + S(u - u_i) \left( L_{i+1}^0(u) - L_i^0(u) \right),$$

(13)

at their junction $u_i$ for a smooth approximation ((24) to (26)) of the step function

$$S(u - u_i) = \begin{cases} 
0 & \text{if } u < u_i, \\
1 & \text{if } u > u_i, 
\end{cases}$$

(14)

ensuring that

$$L_i^1 = \begin{cases} 
L_i^0(u) & \text{if } u < u_i, \\
L_{i+1}^0(u) & \text{if } u > u_i.
\end{cases}$$

(15)

We have added a superscript $j$ to $L_i^j$. This denotes the number of junctions in the joined curve and hence $L_i^0 = L_i$ are not supposed to have junctions in them. In $L_i$, $i = 1, 2, \ldots, m + c_1$ of the first group, similarly for the other three groups. To complete the task we have to be able to increase the superscript to the number of junctions of the respective group. We suggest a recursion relation to increase the superscript to any value, namely,

$$L_i^1(u) = L_i^0(u) + S(u - u_i) \left[ L_{i+1}^{j-1}(u) - L_i^0(u) \right].$$

(16)

For $j = 1$ this equation becomes (13), and this is the least value of the superscript for which (16) should be used. Continuing the iterative process in the previous equation would express $L_{i+1}^{j-1}(u)$ in terms of the one with superscript further decreased. That is,

$$L_{i+1}^{j-1}(u) = L_{i+1}^0(u) + S(u - u_{i+1}) \left[ L_{i+2}^{j-2}(u) - L_{i+1}^0(u) \right],$$

(17)

and so on. This iteration allows us to extend our algorithm to merge any finite number of curves, that is, (one more than) the value we assign to the superscript $j$. We illustrated later a merging of three and four curves by assigning the superscript $j$ values of 2 and 3, which are the corresponding number of junctions. In our full scheme these combinations are needed when we partition 9 and 14 total number of curves, respectively, into our usual four groups; when the previously mentioned set of curves $C^i_1$ is joined together to make $c_1(u)$ for a boundary composed of 9 curves, it takes the following form:

$$c_1(u) = L_1^0(u) = L_1^1(u) + S(u - u_1) \left[ L_1^2(u) - L_1^0(u) \right],$$

(18)

where

$$L_2^1(u) = L_2^0(u) + S(u - u_2) \left[ L_2^2(u) - L_2^0(u) \right],$$

(19)

as shown in Figure 1.

Likewise, when the previously mentioned set of curves $C^i_1$ is joined together to make $c_1(u)$ for a boundary composed of 14 curves, it takes the following form:

$$c_1(u) = L_1^0(u) = L_1^1(u) + S(u - u_1) \left[ L_1^2(u) - L_1^0(u) \right],$$

(20)

where

$$L_2^1(u) = L_2^0(u) + S(u - u_2) \left[ L_2^0(u) - L_2^0(u) \right],$$

(21)

and in turn

$$L_3^1(u) = L_3^0(u) + S(u - u_3) \left[ L_3^0(u) - L_3^0(u) \right].$$

(22)

Substituting (21) and (22) in (20) results in

$$c_1(u) = L_1^0(u) + S(u - u_1) \left[ L_2^0(u) + S(u - u_2) \right] \left[ L_3^0(u) + S(u - u_3) \right] - L_2^0(u) - L_3^0(u),$$

(23)

which is shown in Figure 2. In the similar way other constituent parts of Coons patch, namely, $d_1(v), c_2(u)$, and $d_2(v)$ can be constructed using (16).

4. Analytical Representations of the Step Function

In (14), we have used the step function for letting the parameter of our one curve map into different given constituent curves as it increases. For this we have to let the step function multiply one curve only till it reaches a value corresponding to a junction; for higher values of the parameter the same step function now multiplies a new curve, and so on till each of the curves gets multiplied by 1 for some range of values of the parameter. In the existing literature [3, 4], a step function is used for another purpose, namely, for effectively deleting through its vanishing value a nondesired piece of a surface and keeping only a desired piece of a surface for a corresponding range of parameters; practically this amounts
A differentiable step-function representation equations (24)–(26) to give us smooth curve $L(u)$ that exactly interpolates the constituent curves $L_1(u)$ and $L_2(u)$ such that

$$L(u) = \begin{cases} L_1(u) & \text{if } u < u_1, \\ L_2(u) & \text{if } u > u_1. \end{cases}$$

In particular for a line $L_1(u)$ obtained by joining $(0,0)$ to $(x_1, y_1) = (lu_1)$

$$L_1(u) = \frac{y_1}{lu_1} (lu), \quad (28)$$

and line $L_2(u)$ obtained by joining the points $(x_1, y_1) = (lu_1, y_1)$ to $(1,0)$

$$L_2(u) = \frac{y_1}{lu_1 - 1} (lu - l). \quad (29)$$

Using (13), $L(u)$ may be written as a smooth curve joined through a step function representation given by (24)–(26)

$$L(u) = L_1^*(u) = \frac{y_1}{u_1} u + S(u-u_1) \left( \frac{y_1}{u_1 - 1} (u - 1) - \frac{y_1}{u_1} u \right). \quad (30)$$

The graphs of these combinations of $L_1(u)$ and $L_2(u)$ for the three-step function representations $S(u-u_1), S^*(u-u_1)$ and $S^{**}(u-u_1)$ are sketched in Figure 4. Each of these is an analytical function guaranteed to pass through both the curves it combines. One can discretize both the curves and use Splines to generate a smooth curve passing through the resulting points, but that would not assure continuity of third and higher derivatives. If extrapolated, $L(u)$ simply agrees to the respective constituent curves $L_1(u)$ and $L_2(u)$ and thus does not develop any large fluctuations that would result from using an interpolating polynomial of high enough degree in place of Splines.

5. A Technique for Variational Improvement

Reducing an arbitrary curve (or a finite collection of many continuous curves) to four continuous curves let us write Coons patch for a surface bounded by it. But that would not tell us anything about its characteristics in the differential geometry. For example, there is no guarantee that such a surface would be a minimal surface spanning its boundary. We can calculate the differential-geometry-related functions of the two parameters $u$ and $v$ of the surface and then do integrations with respect to these parameters to get numbers characterizing the surface. If the rms mean curvature (see Section 2) of the generated surface is nonzero, the surface is not a minimal surface, and a challenge is to modify it so that it either becomes a minimal surface or, if that is not possible, gets closer to being a minimal surface. For this we can write an ansatz for a modification in the surface including a variational parameter (or parameters) and then solve the optimization problem of selecting the value(s) of the parameter(s) so as to minimize either its area directly or its
rms mean curvature. The ansatz we suggest [9] to minimize the area of a nonminimal surface $x_0(u, v)$ is

$$x_1(u, v, t) = x_0(u, v) + tm(u, v) N_0,$$  

(31)

where $t$ is our variational parameter; the rest of the modification,

$$m(u, v) = uv(1 - u)(1 - v) H_0,$$  

(32)

is chosen so that the variation at the boundary curves $u = 0, u = 1, v = 0, v = 1$ is zero. (Other choices of $m(u, v)$, for example $m(u, v) = u^2 v^2 (1 - u^2)(1 - v^2) H_0$, that vanish at the boundary points are possible, but they would take more CPU time.) In (31), $N_0$ is unit normal to the nonminimal surface $x_0(u, v)$, and $H_0$, given by (6), is numerator of the initial mean curvature of the starting surface $x_0(u, v)$. Calling the fundamental magnitudes for $x_1(u, v)$ as $E_1(u, v, t), F_1(u, v, t), G_1(u, v, t), e_1(u, v, t), f_1(u, v, t)$, and $g_1(u, v, t)$, the area $A_1(t)$ of the surface $x_1(u, v, t)$ for $0 \leq u \leq 1$ and $0 \leq v \leq 1$ is given by

$$A_1(t) = \int_0^1 \int_0^1 \sqrt{E_1 G_1 - F_1^2} \, du \, dv.$$  

(33)

We denote, the numerator of mean curvature for $x_1(u, v)$ of (31) by $H_1(u, v, t)$. It would have the following familiar expression:

$$H_1(u, v, t) = E_1 g_1 - 2 F_1 f_1 + G_1 e_1.$$  

(34)

As $H_1^2(u, v, t)$ is a polynomial in $t$, with real coefficients $h_i(u, v)$, we rewrite (34) in the form

$$H_1^2(u, v, t) = \sum_{i=0}^n (h_i(u, v)) t^i.$$  

(35)

Here $n$ turns out to be 10; there being no higher powers of $t$ in the polynomials as it can be seen from the expression for $E_1(u, v, t), F_1(u, v, t), g_1(u, v, t), e_1(u, v, t), f_1(u, v, t)$, and $g_1(u, v)$ which are cubic in $t$. Integrating (numerically if needed) these coefficients with respect to $u$ and $v$ in the range $0 \leq u, v \leq 1$ we get $\mu$ of mean curvature $H_1(u, v, t)$ of the new surface $x_1(u, v, t)$

$$\mu_1(t) = \left( \int_0^1 \int_0^1 H_1^2(u, v, t) \, du \, dv \right)^{1/2}$$

$$= \left( t^l \int_0^1 \int_0^1 \sum_{i=0}^n (h_i(u, v)) \, du \, dv \right)^{1/2}.$$  

(36)

The expression in the parentheses on right-hand side of previous equation turns out to be a polynomial in $t$ of degree $n$, which can be minimized with respect to $t$ to find $t_{\text{min}}$. The resulting value of $t$ completely specifies new surface $x_1(u, v)$. New surface $x_1(u, v)$ is expected to have lesser area than that of original surface $x_0(u, v)$.

Figure 3: Step function representations $S(u - u_i), S'(u - u_i)$ and $S''(u - u_i)$ for $i = 1, l = 20, u_i = 0.25, e = 0.01$ and $k = 5$. 
6. The Technique Applied to a Surface Spanned by Five Arbitrary Lines

In this section we use the ansatz of (31) to reduce the area of a surface spanned by five arbitrary lines lying in different planes. For $0 \leq u_i \leq 1$ with $l$ any real scalar, the step function $S(u - u_i)$ satisfying (14) represented by (24) is used to join the curves $L_i$ and $L_{i+1}$, using the technique discussed in Section 4; other two step function representations $S'(u - u_i)$ and $S''(u - u_i)$ demand more CPU time, involve complicated trigonometric expressions, and pose issues related to programming. As a special case let us assume that $L_1(u)$ equation (28) and $L_2(u)$ equation (29) be two successive analytical smooth curves joined together to give us smooth curve equation (30) through the step function $S(u - u_i)$ of (24) so that (30) takes the following form:

$$L(u) = L_1(u) = \frac{y_1}{u_1} u + \frac{1}{2} \left( 1 + \frac{1}{\sqrt{\epsilon + l(u - u_1)^2}} \right) \left( \frac{y_1}{u_1} (u - 1) - \frac{y_1}{u_1} u \right),$$

which is continuous and differentiable at every point in the domain $0 \leq u \leq 1$. For using this work for modeling a string breaking, one can take $c_2(u) = L_4(u)$ as the initial string, $d_1(v) = L_3(v)$, and $d_2(v) = L_5(v)$ modeling the time evolution of its ends, the later $L(u)$ as the combination that contains the two final strings as shown below in Figure 5.

For the four curves required in a general Coons patch equation (11), we construct them from the boundary composed of five straight lines $L_1(u)$, $L_2(u)$, $L_3(u)$, $L_4(u)$, and $L_5(u)$ connecting five arbitrary corner points. For this, two lines joining three corners are joined into one curve, namely, $c_1(u) = L_1(u)$ equation (37) and the remaining three boundary lines $d_1(v) = L_3(v), c_2(u) = L_4(u)$ and $d_2(v) = L_5(v)$. For linear blending functions $f_1 = 1 - u, f_2 = u, g_1 = 1 - v$, and $g_2 = v$, we have been able to reduce the area spanning pentagons. In case of a pentagon, when we convert it to a Coons patch, for the corners we choose

$$r_1 = (0, 0, 0), \quad r_2 = (l, a, 0),$$

$$r_3 = (0, a, 0), \quad r_4 = (l, 0, 0).$$

Figure 4: Straight lines $L_i(u)$ and $L_{i+1}(u)$ joined by first-, second, and third-step function representations, respectively, for $i = 1, l = 20, u_i = 0.25, \epsilon = 0.01,$ and $k = 5.$
\( L(u) = L_1(u) \)
\( L(\gamma) = d_1(\gamma) \)
\( L(\varphi) = d_2(\varphi) \)

**Figure 5:** An initial string \( c_2(u) = L_4(u) \) with \( d_1(V) = L_3(V) \) and \( d_2(V) = L_5(V) \) modeling the time evolution of its ends and the combination \( L(u) \) contains the two final strings \( L_1(u) \) and \( L_2(u) \).

We use a selection of integer values of \( l \) and \( a \). The four corners are labeled by the following Coons convention:

\[
\begin{align*}
  x(0,0) &= r_1, & x(1,0) &= r_4, \\
  x(0,1) &= r_3, & x(1,1) &= r_2.
\end{align*}
\]

The \( z \) component of our surface variable vector \( x_0(u,v) \) is a single valued function for all values of its \( x \) and \( y \) components, and hence we can replace complicated \( N_0 \) in (31) by a unit vector \( k \) along the \( z \)-axis to facilitate computations. We checked that this makes a small enough angle with the original normal \( N_0 \); it can be seen in Figure 6 that component of the unit normal \( N_0 \) along \( k \) remains positive for \( 0 \leq u, v \leq 1 \). Thus angle \( \theta \) between \( N \) and \( k \) remains within the interval \( 0 \leq \theta \leq \pi/2 \); this guarantees that signs of changes along \( N_0 \) and \( k \) are the same, so a local increase (decrease) in the area integrand while moving along \( N \) means an increase (decrease) in moving along \( k \). But moving along \( k \) is much easier computationally. We get a net decrease in area with an optimal numerical value of the coefficient \( t \) even when it multiplies \( k \) not \( N_0 \) as in (31); this indicates that the choice \( k \) instead of \( N_0 \) has been useful. This choice of unit vector is graphically depicted in the Figure 7. Inserting values of blending functions and boundary points in (11), we find

\[
x(u,v) = [l(u,av,vL(u))],
\]

whereas fundamental magnitudes have the following expressions:

\[
\begin{align*}
  E(u,v) &= l^2 + v^2(L'(u))^2, \\
  F(u,v) &= vL(u)L'(u), \\
  G(u,v) &= a^2L^2(u), \\
  e(u,v) &= avL''(u), \\
  f(u,v) &= alL'(u), \\
  g(u,v) &= 0,
\end{align*}
\]

and (6) gives

\[
H_0 = -2alvL(u)L'(u)^2 + av(a^2 + L(u)^2)L''(u).
\]
The root mean square (rms) of beginning curvature given by (7) takes the form:

\[
\mu_0 = \left( \int_0^1 \int_0^1 (-2alv(u)L'(u))^2 + alv(a^2 + L(u)^2) L''(u))^2 \, du \, dv \right)^{1/2}.
\]

(43)

The beginning or initial area of the Coons patch given by (8) takes the form in this case

\[
A_0 = \int_0^1 \int_0^1 \sqrt{-v^2L(u)^2 L'(u)^2 + (a^2 + L(u)^2)(u^2 + v^2L'(u)^2)} \, du \, dv.
\]

(44)

Substituting \(H_0\) from (42) in (32), we have

\[
m(u, v) = (1 - u)u (1 - v) v \times \left( -2alv(u)L'(u)^2 + alv(a^2 + L(u)^2) L''(u) \right).
\]

(45)

Variationally improved surface (31) takes following form

\[
x_i(u, v, t) = \left[ lu, av, vL(u) + t \left( 1 - u \right) u (1 - v) v \times \left( -2alv(L(u)L'^2(u) + alv(a^2 + L^2(u)) L''(u) \right) \right].
\]

(46)

Fundamental magnitudes for this variationally improved surface are \(E_i(u, v, t), F_i(u, v, t), G_i(u, v, t), e_i(u, v, t), f_i(u, v, t), \) and \(g_i(u, v, t)\) included in Appendix A. Inserting these values of fundamental magnitudes in (34) we find the expression for \(H_1(u, v, t)\) as

\[
H_1(u, v, t) = al \left( -2v \left( L(u) + alt (-1 + u)uv(-2 + 3v) \right. \right.
\]

\[
\left. \times \left( -2L(u)L'^2(u) + a^2 L''(u) \right) + L^2(u)L''(u) \right) \right.
\]

\[
\times \left( 2alt(-2 + u(4 - 6v) + 3v)L(u)L'^2(u) - 2alt(-1 + u)uv(-2 + 3v)L'(u)^3 \right.
\]

\[
+ \left. L'(u) \left( -1 + 2alt(-1 + u) \right) \times uv(-2 + 3v)L(u)L''(u) \right)
\]

\[
+ \left. alt(-2 + 3v) \left( a^2 + L^2(u) \right) \times \left( (-1 + 2u)L''(u) \left( -1 + u \right) u L^{(3)}(u) \right) \right)
\]

\[
\times \left. \left( L'(u) + alt(1 - v) v \right. \right.
\]

\[
\left. \times \left( 2(-1 + u)uL'(u)^3 + 2L(u)L'(u) \left( (-1 + 2u)L'(u) \right. \right. \right.
\]

\[
\left. + \left. (-1 + u)uL''(u) \right) + a^2 \left( (1 - 2u)L''(u) \right. \right.
\]

\[
\left. - \left. (-1 + u)uL^{(3)}(u) \right) \right) \right)
\]

\[
+ \left. 2alt(-1 + u)u(-1 + 3v) \right. \times \left. \left( -2L(u)L'^2(u) \right) \right).
\]
\[ + a^2 L''(u) + L^2(u) L''(u) \]
\[ \times \left( I^2 + v^2 \left( L'(u) + alt(1 - v) v \times \frac{a^2}{2} \right) \times (L(u) + alt(-1 + u) u L^{(4)}(u)) \right) + 2L(u) L'(u) \times ( -1 + u u L^{(2)}(u) ) + a^2 \left( -2L''(u) \right) \]
\[ + (2 - 4u) L^2(u) \]
\[ + (-1 + u u L^{(4)}(u)) \].

Note that \( H_1^2(u,v,t) \) is a polynomial in \( t^i \) with real coefficients. For this purpose we rewrite the previous expression using (35), which reduces to the following form (the coefficients \( h_1(u,v), h_2(u,v), h_3(u,v), h_4(u,v), \) and \( h_5(u,v) \) of \( t^i \)'s, \( t^i \)'s do not exist as it can be seen from the previous expression for

\[ H_1^2(u,v,t) = h_0(u,v) + h_1(u,v) t \]
\[ + h_2(u,v) t^2 + h_3(u,v) t^3 \]
\[ + h_4(u,v) t^4 \]
\[ + h_5(u,v) t^5 + h_6(u,v) t^6, \]

where

\[ h_0(u,v) = \left( -2advL(u) L^{2}(u) \right)^2, \]
\[ h_1(u,v) = 2 \left( -2advL(u) L^{2}(u) \right) \]
\[ + a^3 \left( aL''(u) + alvl^2(u) L''(u) \right)^2, \]
\[ -2avlL^2(u) m_u(u,v) \]
\[ + 2advL(u) L''(u) m_u(u,v) \]
\[ + al^3 m_{uv}(u,v) \]
\[ + al^2 L(2(u) m_{uv}(u,v) \]
\[ - 2advL(u) L'(u) m_u(u,v) \]
\[ - 2advL(u) L'(u) m_{uv}(u,v) \]
\[ + a^3 m_{uu}(u,v) + alL^2(u) m_{uu}(u,v) \],

Similarly we can find the remaining coefficients \( h_2(u,v), h_3(u,v), h_4(u,v), h_5(u,v), \) and \( h_6(u,v) \), and they have been included in Appendix B. Thus (36) in this case after performing the integrations mentioned in (36), the mean square curvature \( \mu_1^2(t) \) for \( x_1(u,v) \) becomes

\[ \mu_1^2 = \int_0^1 \int_0^1 h_0(u,v) du dv \]
\[ + t \int_0^1 \int_0^1 h_1(u,v) du dv \]
\[ + t^2 \int_0^1 \int_0^1 h_2(u,v) du dv + t^3 \int_0^1 \int_0^1 h_3(u,v) du dv \]
\[ + t^4 \int_0^1 \int_0^1 h_4(u,v) du dv + t^5 \int_0^1 \int_0^1 h_5(u,v) du dv \]
\[ + t^6 \int_0^1 \int_0^1 h_6(u,v) du dv, \]
square curvature $\mu_2^2$ may be minimized for $t$ for fixed values of $a$, $l$, and $y_k$. For this minimum value $t_{\text{min}}$ we find the variationally improved surface $x(u, v)$ and its area using (33). In order to see a geometrically meaningful (relative) change in area we calculate the dimension less area by dividing the difference of the (original) area of the Coons patch and the variationally decreased area by the original area. In particular for $y_i = 10$, $a = 100$, and $l = 20$, following Table 1 provides the percentage decrease in area of the initial surface Coons patch equation (11) for few cases that includes $x_i = l u_i = 2, 4, 5, 6, 8, 10, 12, 14, 16$, and $18$. The Table 1, indicates a symmetric behaviour of decrease in area for example for $0 \leq x_i \leq 20$, as it can be seen that the decreases in area for example for $x_i = 2$ and $x_i = 18$, $x_i = 4$ and $x_i = 16$, $x_i = 6$ and $x_i = 14$, $x_i = 8$ and $x_i = 12$, and so forth, agree up to four decimal places. Table 1 and Figure 8 indicate that the relatively flat region starting from $av = 0$ expands inside the surface as the ratio $l/a$ tends to zero for larger values of $a$ ($a \to \infty$). The large $a$ limit of the five line boundary has importance in the mathematical modeling of a string breaking into two, where the $av = 0$ straight line models the original string breaking into two final strings (straight lines) visible at $av = 100$. With our surface as the corresponding string space-time world sheet of relativity, large length $a$ means a large time evolution of the string ends at $u = 0$ and $u = 1$. Combining this with the usual quantum mechanical exponential dependence of the transition amplitudes on both time and energy [35] and Wick’s rotation [36] to imaginary time justified by a Contour integration [37] the transition amplitudes for larger energies get damped away for this large time evolution. Thus, in this limit, the string breaking probabilities become specialized to the physically more interesting problem of the ground states of both the initial and final strings. For the gluonic strings connecting a quark and an antiquark, all five lines can be seen for example in [38]; other problems in any bosonic string theory have the same mathematical structure. (In a related application, for any value of time evolution or $a$ the area of string is proportional to its action, and thus reducing area takes us closer to the nonquantum or classical minimal action. This area reduction is what we have done in the second part of our work). Figure 9 represents the data points of Table 1 along with spline curve giving us percentage decrease in area $A$ as numerical function of $x_i$ that show the outcome of the ansatz used to calculate the decrease in area for the Coons patch for $0 \leq x_i \leq 20$. The dots give computed values of decrease in area, and the smooth graph passing through these points is the spline curve interpolating these points for better predictability that how the decrease in area in the Coons patch is associated with the range of points $0 \leq x_i \leq 20$. Figure 9 indicates that reduction in area is smaller for the string breaking point $x_i$ generally in the middle, and thus a string breaking world sheet or symmetrical surface may...
be closer to being a minimal surface than the asymmetrical surfaces for the \( x_i \) point significantly away from the middle where we have larger area reductions.

7. Conclusions

We have developed an algorithm (16) to combine \( N \) number of curves with the help of step functions equations (24) to (26). We have discussed a technique to reduce the area of a surface \( x(u, v) \) equation (11) obtaining variationally improved surface \( x_i(u, v) \) of (31). The algorithm is applied to reduce the area of a surface of \( x(u, v) \) equation (11) spanned by five nonplanar boundary lines with the help of algorithm (16), extended boundary Coons patch equation (11), satisfying the conditions (9) and (10) along with the linear blending functions equation (12), for a selection of values as given in the previous table. This gave us a much lesser (in the range 0 to 0.82) dimensionless decrease in less area of surface \( x(u, v) \) of equation (11), as seen in the Figure 9. It is to be noted that our variational technique reduces area by 23 percent for a surface mentioned in [9]. A much lesser decrease in this case suggests that \( x(u, v) \) equation (11) is already a near minimal surface. The five-line boundary we have worked out has a variety of applications including the string theory one mentioned in the previous paragraph.

Appendices

A. Fundamental Magnitudes of Variationally Improved Surface

The expressions for \( E_1(u, v, t), F_1(u, v, t), G_1(u, v, t), e_1(u, v, t), \) \( f_1(u, v, t), g_1(u, v, t) \) are given below:

\[
E_1(u, v, t) = I^2 + \left(vL'(u) + \right.
\begin{align*}
& + t \left((1 - u)(1 - v) v \right.
n + & \left((-2alvL(u)L'(u))^2 \times \right.
+ & \left(alv(a^2 + L(u)^2)L''(u) \right)
- & \left(u(1 - v)v(-2alvL(u)L'(u))^2 \times \right.
+ & \left(alv(a^2 + L(u)^2)L''(u) \right)
\times & \left(L''(u) \right)
+ & \left((1 - u)u(1 - v) \times \right.
- & \left(2alvL(u)L'(u)^3 \times \right.
+ & \left(alv(a^2 + L(u)^2)L''(u) \right)
\right)\right),
\end{align*}
\]

\[
F_1(u, v, t) = \left(L(u) + t \left((1 - u)u(1 - v) \times \right.
\begin{align*}
& + 2alvL(u)L'(u)^2 + \right.
+ & \left(alv(a^2 + L(u)^2)L''(u) \right)
+ & \left((1 - u)v \times \right.
- & \left(2alvL(u)L'(u)^3 \times \right.
+ & \left(alv(a^2 + L(u)^2)L''(u) \right)
\times & \left(L''(u) \right)
- & \left(2alvL(u)L'(u)^3 \times \right.
+ & \left(alv(a^2 + L(u)^2)L''(u) \right)
\right)\right),
\end{align*}
\]

\[
e_1(u, v, t) = al \left(vL''(u) + t \left(-2(1 - v) v \right.
\begin{align*}
& \times \left(-2alvL(u)L'(u)^2 + \right.
+ & \left(alv(a^2 + L(u)^2)L''(u) \right)
+ & \left((1 - u)v \times \right.
- & \left(2alvL(u)L'(u)^3 \times \right.
+ & \left(alv(a^2 + L(u)^2)L''(u) \right)
\right)\right),
\end{align*}
\]

Abstract and Applied Analysis
\[ f_1(u, v, t) = aL'(u) - 4a^2L^2vL(u) L'(u)^2 + 8a^2L^2tuvL(u) L''(u)^2 + 6a^2L^2t^2L(u) L'(u)^2 - 12a^2L^2tuL'(u)^3 + 4a^2L^2tuL(u)^2L'(u) - 4a^2L^2tuvL'(u)^3 + 6a^2L^2t^2L'(u)^3 - 6a^2L^2tu^2vL'(u)^3 + 2a^2L^2tvL''(u) - 4a^2L^2tuL''(u) - 3a^2L^2tvL'(u)^3 + 6a^2L^2tuL'(u)^3L''(u) - 4a^2L^2tuvL(u)^2L''(u) - 3a^2L^2tvL(u)^2L''(u) + 6a^2L^2tuL(u)^2L''(u)k - 4a^2L^2tuvL(u) L'(u) L''(u) + 4a^2L^2tuL(u) L'(u) L''(u) + 6a^2L^2tuL(u)^2L'(u) L''(u) - 6a^2L^2tuL(u) L''(u) + 2a^2L^2tuL(u)^2L''(u) + 2a^2L^2tuL(u)^3L''(u) - 3a^2L^2tuL(u)^3L''(u) + 3a^2L^2tuL(u)^2L''(u)L''(u) - 2a^2L^2tuL(u)^2L''(u)L''(u) + 2aL^2tuL(u)^2L''(u) + 2aL^2tuL(u)^2L''(u)L''(u) + 2aL^2tuL(u)^2L''(u) + 2aL^2tuL(u)^2L''(u)L''(u) + aL^2tuL(u)^2L''(u) + 2aL^2tuL(u)^2L''(u)L''(u) + aL^2tuL(u)^2L''(u)L''(u) + 2aL^2tuL(u)^2L''(u)L''(u) + \dots \]
\[ h_4 (u, v) = (alv^2 (u) m^2_v (u, v) \]
\[ + 2 \left( -2avl (u) L^2 (u) + a^3 vL'' (u) \right) \]
\[ + alvL^2 (u) L'' (u) \]
\[ \times \left( alm_{vv} (u, v) m^2_v (u, v) \right) \]
\[ - 2alm_v (u, v) m_u (u, v) m_{uv} (u, v) \]
\[ + alm^2_v (u, v) m_{uv} (u, v) \right), \quad \text{(B.2)} \]

\[ h_5 (u, v) = (alv^2 (u) m^2_v (u, v) \]
\[ - 2avl (u) m_v (u, v) m_{uv} (u, v) \]
\[ + 2avl (u) m_{vv} (u, v) m_u (u, v) \]
\[ - 2avl (u) L (u) m_v (u, v) m_{uv} (u, v) \]
\[ - 2al (u) L (u) m_u (u, v) m_{uv} (u, v) \]
\[ + 2al (u) m_v (u, v) m_{uv} (u, v) \]
\[ + 2al (u) (m_v (u, v) m_{uv} (u, v))^2 \]
\[ + 2 \left( -2avl^2 (u) m_v (u, v) \right) \]
\[ + 2avl (u) L'' (u) m_v (u, v) \]
\[ + al^3 m_{vv} (u, v) + al^2 vL^2 (u) m_{vv} (u, v) \]
\[ - 2al (u) L (u) L' (u) m_v (u, v) \]
\[ + 2al (u) L (u) L' (u) m_{uv} (u, v) \]
\[ + a^3 l m_{uv} (u, v) + al^2 (u) m_{uv} (u, v) \]
\[ \times \left( alm_{vv} (u, v) m^2_v (u, v) \right) \]
\[ - 2alm_v (u, v) m_u (u, v) m_{uv} (u, v) \]
\[ + alm^2_v (u, v) m_{uv} (u, v) \right), \quad \text{(B.3)} \]

\[ h_6 (u, v) = (alm_{vv} (u, v) m^2_v (u, v) \]
\[ - 2alm_v (u, v) m_u (u, v) m_{uv} (u, v) \]
\[ + alm^2_v (u, v) m_{uv} (u, v) \right), \quad \text{(B.4)} \]

\[ h_6 (u, v) = (alm_{vv} (u, v) m^2_v (u, v) \]
\[ - 2alm_v (u, v) m_u (u, v) m_{uv} (u, v) \]
\[ + alm^2_v (u, v) m_{uv} (u, v) \right)^2. \quad \text{(B.5)} \]