General Inequalities for Generalized Convex Functions

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DECLARATION

I, Mr. **Asif Raza Khan** Registration No. **100-GCU-PHD-SMS-09**, student at **Abdus Salam School of Mathematical Sciences GC University Lahore** in the subject of **Mathematics,** year of admission **2009**, hereby declare that the matter printed in thesis titled

"General Inequalities for Generalized Convex Functions"

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RESEARCH COMPLETION CERTIFICATE

Certified that the research work contained in this thesis titled

"General Inequalities for Generalized Convex Functions"

has been carried out and completed by **Mr. Asif Raza Khan** Registration No. **100-GCU-PHD-SMS-09** under my supervision.

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To my parents, to my wife and to the memory of
Khala Jan

Table of Contents

Abstract

It is a fact that, the theory of inequalities, priding on a history of more than two centuries, plays a significant role in almost all fields of mathematics and in major areas of science. In the present dissertation, we will study the general inequalities, namely integral inequalities and discrete inequalities for generalized convex functions. Therefore, we will introduce some generalized convex functions which include functions with nondecreasing increments, $\Delta-$ and $\nabla-$convex functions, and $n-$convex functions of higher orders. By using these functions, we will provide a generalization of the Brunk's theorem, of the Levinson-type inequalities, of the Burkill-Mirsky-Pečarić's result and of the result related to arithmetic integral mean. We will also discuss the Popoviciu-type characterization of positivity of sums and integrals for higher order convex functions of n variables and we will give some related results. Our dissertation also provides generalizations of some of the celebrated and fundamental identities and inequalities including Montgomery's identities, Ostrowski-, Grüss-, Čebyšev- and Fan-type inequalities. Moreover, we will also apply an elegant method of producing $n-$exponentially and logarithmically convex functions for positive linear functionals constructed with the help of majorization-type results, Favard-, Berwald- and Jensen-type inequalities. The generalization and the following refinements of Jensen-Mercer's inequalities are also provided with some applications. The Lagrange- and Cauchy-type mean value theorems are also proved and shown to be useful in studying Stolarsky-type means defined for the positive linear functionals.

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Notations and Terminologies

Following is the table of notations and terminologies to be used frequently in the dissertation.

$\mathbb{R} = (-\infty, \infty)$	Set of real numbers
$\mathbb{R}_+ = (0, \infty)$	Set of positive real numbers
$\mathbb{R}_* = [0, \infty)$	Set of nonnegative real numbers
$\mathbb{R}^k = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{k-times}$	Set of real numbers of dimension k
$\mathbb{R}^k_+ = \underbrace{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+}_{k-times}$	Set of positive real numbers of dimension k
$\mathbb{R}^k_* = \underbrace{\mathbb{R}_* \times \cdots \times \mathbb{R}_*}_{k-times}$	Set of nonnegative real numbers of dimension k
$C(\mathbf{I})$	Space of real-valued continuous functions defined on $\mathbf{I} \subset \mathbb{R}^k$, $k \geq 1$
$C^{(n_1,\ldots,n_k)}(\mathbf{I})$	Space of real-valued continuously differentiable functions of order $(n_1, \ldots, n_k)$ defined on $\mathbf{I} \subset \mathbb{R}^k$, $k \geq 1$
$BV(I)$	Space of real-valued functions of bounded variations defined on $I \subset \mathbb{R}$
$L_p(\mu)$	Space of pth power $\mu-$integrable functions
$\Delta^{(n_1,\ldots,n_k)}_{h_1,\ldots,h_k}$	Finite difference operator of order $(n_1, \ldots, n_k)$ of step length $h_1, \ldots, h_k$, $k \geq 1$
$\Delta_{(n_1,\ldots,n_k)}$	Divided difference operator of order $(n_1, \ldots, n_k)$, $k \geq 1$
$\binom{n}{r}$	Binomial coefficient (n choose r)
χ_I	Characteristic function of the set I

Introduction

> "Mathematics has been called the science of tautology; that is to say, mathematicians have been accused of spending their time proving that things are equal to themselves. This statement (appropriately by a philosopher) is rather inaccurate on two counts. In the first place, mathematics, although the language of science, is not a science. Rather, it is a creative art. Secondly, the fundamental results of mathematics are often *inequalities* rather than *equalities*."

– Edwin F. Beckenbach and Richard Bellman

Why do we study inequalities? The answer to this question was given by Bellman in a very concrete and elegant fashion [60]: "There are three reasons for the study of inequalities: practical, theoretical and aesthetic. In many practical investigations, it is necessary to bound one quantity by another. The classical inequalities are very useful for this purpose. From the theoretical point of view, very simple questions give rise to entire theories. For example, we may ask when the nonnegativity of one quantity implies that to another. This simple question leads to the theory of positive operators and theory of differential inequalities. ... Another question which gives rise to much interesting research is that of finding equalities associated with inequalities. We use the principle that every inequality should come from an equality which makes the inequality obvious. Along these lines, we may also look for representation which make inequalities obvious. Often, these representations are maxima or minima of certain quantities. ... Finally, let us turn to aesthetic aspects. As has been pointed out, beauty is in the eyes of the beholder. However, it is generally agreed that certain pieces of music, art or mathematics are beautiful. There is an elegance to inequalities that makes them very attractive."

Whether it is the study of fractional calculus, game theory, spectral theory, control theory, operations research, quantum mechanics, engineering, economics or even biology, inequalities are encountered everywhere. Inequalities are among the most significant tools in many fields of mathematics, including optimization theory, interpolation theory, functional analysis, harmonic analysis, geometry and calculus of variation etc. Inequalities also play an important role in the theory of partial/ordinary differential and integral equations, as many authors use integral inequalities in the study of existence, uniqueness, boundedness, stability and asymptotic behaviour of solutions of differential equations. Due to various applications of inequalities in different areas of mathematics and further domains of sciences, new types of interesting inequalities are discovered every year and efforts are made to extend and to improve the classical ones.

Among the many types of inequalities, integral inequalities are of supreme importance because over the last few decades this field has proven to be an extensively applicable field. The integral inequalities of various types have been widely studied in most subjects involving mathematical analysis. These inequalities are particularly useful in approximation theory and in numerical analysis where estimates of approximation errors are involved. Integral inequalities that establish bounds on the physical quantities are of great significance in the sense that these types of inequalities are not only useful in nonlinear analysis, numerical integration, approximation theory, probability theory, stochastic analysis, statistics, information theory and integral operator theory but also have applications in the area of physics, technology and biological sciences. The theory of integral inequalities has gained increasing significance over the last century as is apparent from the large number of publications on the subject. With the growing range of applications, the theory of integral inequalities enjoys a rapid increase of interest and widespread recognition as an important area of mathematical analysis. The following lines from [21] well justify our statement "It was noted in the preface of the book *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, 1991, by D. S. Mitrinović, J. E. Pečarić and A. M. Fink; since the writing of the classical book by Hardy, Littlewood and Pŏlya (1934), the subject of differential and integral inequalities has grown by about 800%. Ten years on, we can confidently assert that this growth will increase even more significantly. Twenty pages of Chapter XV in the above mentioned book are devoted to integral inequalities involving functions with bounded derivatives, or, Ostrowski type inequalities. This is now itself a special domain of the Theory of Inequalities with many powerful results and a large number of applications in Numerical Integration, Probability Theory and Statistics, Information Theory and Integral Operator Theory." For more details on the subject cited above we refer to e.g. [21], [60], [65] and [92] and the references wherein.

In the present dissertation, several general inequalities are provided for certain generalized convex functions. To be more specific, the present dissertation mostly deals with integral inequalities for generalized convex functions. Nevertheless, in some subchapters we will also study discrete inequalities. To our eye, the term general inequalities proves subjective to some extent. To our understanding, a general inequality is an inequality which is not confined to a specific function, but it is valid for a class of functions. Moreover, it may have the ability to give birth to many different inequalities by substituting suitable functions and conditions in it.

0.1 Brief Historical and Literature Review

The origins of the theory of inequalities reach back into the 19$^{\text{th}}$ century, to eminent mathematicians such as Gauss, Cauchy and Čebyšev [57]. In the years thereafter, the charm and importance of this field attracted many distinguished mathematicians including Poincaré, Lyapunov, Hölder, Hadamard and Jensen. Around the turn of the century, a large number of inequalities were introduced, some of which became classical, while most of the inequalities remained isolated and unconnected. The systematic development of the theory of inequalities began with the ground breaking work "Inequalities" of Hardy et al., published in 1934, which transformed the field of inequalities from a collection of isolated formulae into a systematic discipline. From 1934 to 1960, a number of papers devoted to inequalities were published: some of them introduced new inequalities, others sharpened or extended classical inequalities. Furthermore, various inequalities were linked by finding their common source, while other papers gave a large number of miscellaneous applications. The book "Inequalities" by Beckenbach and Bellman, which appeared in 1961, contains an account of some results on inequalities obtained over the span of time 1934-1960 [57]. The history of the theory of inequalities would be incomplete without the reference to Mitrinović's "Analytic inequalities" (1970). This book brought the field one step ahead and filled the gaps in the field by including most of the topics in this book which were not included in the aforementioned books.

0.2 Objectives of the Dissertation

As mentioned earlier, integral inequalities represent an intensely researched domain, due to their high applicability in several fields of mathematics and other areas of sciences. The main objective of our dissertation is to provide various integral and discrete identities and inequalities involving generalized convex functions. For this purpose, we introduce some generalized convex functions such as functions with nondecreasing increments, $\Delta-$ and $\nabla-$convex functions, and $n-$convex functions of higher orders. By using these generalized convex functions, our aim is to give generalizations of some important results and well-known identities and inequalities such as Montgomery's identities, Ostrowski-, Grüss-, Čebyšev- and Fan-type inequalities (see for reference [75] and [81]). We are interested in the generalization of Brunk's theorem [13], Levinson-type inequalities [49], Burkill-Mirsky-Pečarić's result [69] and arithmetic integral mean of functions defined on some interval [45]. We plan to discuss the Popoviciu-type characterization of positivity of sums and integrals for higher order convex functions of n variables. We also intend to discuss the $n-$exponential and logarithmic convexity for the Favard- and Berwald-type inequalities and the majorization-type results (see [47, 48] and [50]). The Jensen-type and the reverse Jensen-type inequalities [77], presented as discrete and continuous versions will also come under our consideration, in constructing $n-$exponentially and logarithmically convex functions. We wish to arrive at generalizations of Jensen-Mercer's inequalities with some refinements (see [54] and [62]). Finally, we will study $n-$exponential and logarithmic convexity by the aid of some examples.

0.3 Dissertation Overview

The present dissertation is organized in the following way:

Chapter 1 presents certain basic definitions, notations and preliminary results related to convex functions, generalized convex functions and convex functions of other types. We subdivide the chapter into two main sections. The first section entitled Convex Functions provides the historical background, the motivation, the applications and the definitions of convex functions. The second section deals with definitions and remarks related to generalized convex functions and other type of convex functions, such as logarithmically and $n-$exponentially convex functions. In this same chapter, we provide an introductory material which is repeatedly used throughout our dissertation.

Chapter 2 extends the idea of functions with nondecreasing increments by defining a new class of functions with nondecreasing increments of higher order and provides the results related to this new class. The chapter encompasses four main sections: the first section consists of some important preliminary results related to functions with nondecreasing increments, namely Brunk's theorem, the Jensen-Steffensen-type inequality and the Burkill-Mirsky-Pečarić's result. In the second section, we introduce the functions with nondecreasing increments of order n and we provide the generalization of Brunk's theorem, after presenting some constructions. In the third section, we considered the functions with nondecreasing increments of order three, in order to obtain the Levinson-type inequalities and generalizations of Burkill-Mirsky-Pečarić's result. In the last section, we provide a result for an arithmetic integral mean of a function with nondecreasing increments of higher order.

Chapter 3 discusses the Popoviciu-type characterization of positivity of sums and integrals for higher order convex functions of n variables. This chapter consists of five parts. After a brief introduction, the first section gives two identities for sum $\sum_{i=1}^{N} p_i f(x_i)$ which involve divided differences $\Delta_{(n)} f$ and $\nabla_{(n)} f$, respectively. These identities are the basic tools for obtaining the necessary and sufficient conditions that the inequality $\sum_{i=1}^{N} p_i f(x_i) \geq 0$ holds for every $n-$convex function or $\nabla-$convex function of higher order. In the second and in the third sections respectively, we obtain one discrete identity for the sum $\sum \ldots \sum P_{k_1 \ldots k_n} f(x_{1k_1}, \ldots, x_{nk_n})$ and one integral identity for $\Lambda(f) = \int \ldots \int P(x_1, \ldots, x_n) f(x_1, \ldots, x_n) dx_1 \ldots dx_n$ of Popoviciu-type. These results are in fact generalizations of the results given in [44] and [75]. In both the sections, we also obtain necessary and sufficient conditions under which these identities are nonnegative for higher order convex functions of $n-$variables. The forth section presents the mean value theorems, while in the last section we apply the functional $\Lambda(f)$ on the family of certain exponentially convex functions and we discuss some of its major properties.

Chapter 4 uses the results obtained in Chapter 3 and it provides a generalization of Čebyšev's and Fan's integral identities and inequalities for functions of two variables by using higher order derivatives. The chapter consists of four sections. The first section gives us some preliminary results related to the discrete and continuous Čebyšev's identity and inequality. In this section, some notations are also defined in order to use them in the main results of Chapter 4. The second and the third sections provide the generalization of discrete and continuous Čebyšev's identity and inequality, while the forth section deals with the generalization of integral Fan's identity and inequality.

Chapter 5 again applies the results from Chapter 3 in order to provide the weighted Montgomery's identities for higher order differentiable functions of two variables. These identities help us in arriving at the generalized Ostrowski- and Grüss-type inequalities for double weighted integrals of higher order differentiable functions. Chapter 5 has four sections. In the first section, some preliminary results related to Montgomery's identities are given whereas in the second section, some generalizations of the results, which were established in the first section, are provided. The third and the forth sections respectively consist of Ostrowski-type and Grüss-type inequalities for double weighted integrals for higher order differentiable functions.

Chapter 6 provides us with new research methods to generalize results connected to majorization, to Favard- and to Berwald-type inequalities, by means of second-order divided difference. This is in fact a generalization of the results given in [47, 48] and [50]. There are four main sections of this chapter. The first section is devoted to some basic definitions and preliminary results related to majorization and to Favard- and to Berwald-type inequalities. The second section provides the mean value theorems. The third section generates the $n-$exponential and logarithmic convexity for the majorization-type results and the Favard- and Berwald-type inequalities by using the class of continuous functions in linear functionals constructed in the first section. In the third section, we also construct positive-semidefinite matrices for majorization-type results, Favard- and Berwald-type inequalities. In the last section, we vary on a choice of a family of functions in order to construct different examples of exponentially convex functions and to construct some Stolarsky means. In the end, we also prove the monotonicity property.

Chapter 7 is devoted to the Jensen-type inequalities. The chapter is based on seven sections. The first section introduces the well-known Jensen's inequality and states its different variants. The second section recalls basic results given in [77], where the uniform treatment of the Jensen-type inequalities and of the reverse Jensen-type inequalities is discussed. Its first and second subsections discusses respectively discrete and continuous versions of the aforementioned inequalities. By using these inequalities, we construct some nonnegative linear functionals. Following a brief introduction to Neizgoda's extension of Mercer's inequality, the third section provides certain generalization of Niezgoda's result. The following subsections are fully based on the refinements and on the applications respectively. By applying isotonic linear functionals, the forth section provides some more generalized results as compared to the results from third section. Our results in the third and in the forth sections help us to construct some nonnegative linear functionals, which we may use in the upcoming sections. This way, the fifth section provides the generalized mean value theorems. In the sixth section, we produce the $n-$exponential and the logarithmic convexity of the nonnegative linear functionals constructed in the second, in the third and in the forth sections. Finally, in the last section, we give examples of the families of functions for which the obtained results can be applied. These examples are served as applications to generalized Lagrange- and Cauchy-type means.

It is worth mentioning that the most of the contents of the dissertation are parts of the following articles [38–44]. Some of them were presented at the 5^{th} International Conference on 21^{st} Century Mathematics, Lahore, 2011, at the 13^{th} International Pure Mathematics Conference, Islamabad, 2012 and at the 6^{th} International Conference on 21^{st} Century Mathematics, Lahore, 2013.

Chapter 1

Preliminaries

> "It seems to me that the notion of convex function is just as fundamental as positive function or increasing function. If I am not mistaken in this, the notion ought to find its place in elementary exposition of the theory of real functions."

–J. Jensen

The present chapter is intended to describe convex functions of different type as well as its generalizations. It also covers some related material which would be used in the sequel.

1.1 Convex Functions

Although the systematic study of convex functions was commenced by Jensen [34, 35] and one may find its roots in the works of Hermite [31], Hölder [32], Stolz [90] and Hadamard [29], some authors still believe that it may be traced back to Gibbs [88, p. 287]. The theory of convex functions has a unique place in mathematics due to several reasons: firstly, it is among the most important theories per se, which touches almost all fields of mathematics such as optimization theory, control theory, operations research, geometry, differential equations, functional analysis, operator theory, probability theory, numerical integration, information theory, integral operator theory etc. The theory of convex functions plays an important role in other branches of sciences as well, such as physics, statistics, mechanics, economics, finance, engineering and management sciences. Due to its wide range of applications it has attracted many economists, engineers along with pure mathematicians to be more interested in convex analysis [61]: “Convexity appear like an octopus, tentacles reaching far and wide, its shape and color changing as it roams from one area to the next. It is quite clear that research opportunity abounds”.

Secondly, convex functions are closely related to the theory of inequalities and many famous inequalities are consequences of the applications of convex functions. It is no exaggeration to argue that the theory of convex functions has become a special domain of inequality theory with a number of powerful results and numerous applications in many branches of mathematics.

Thirdly, it has high geometric and intuitive content. The subject of convex geometry is well developed, as the geometric approach to convex functions is the one many prefer [85].

Fourthly, the comparison of means stands at the core of the notion of convexity. In fact, nowadays the study of convex functions has evolved into a larger theory about functions, which are adapted to other geometries of the domain and/or they obey other laws of comparison of means [61].

Fifthly, the theory of convex functions permits an easy generalization to an abstract setting.

Finally, the class of convex functions may be characterized in a variety of way as we see in present dissertation.

Due to all the aforementioned reasons, many books have been written on the topic of “convex functions”. Here we mention some remarkable works such as “Convex analysis” by Rockafellar [86], “Convex functions” by Robert and Verberg [85] and “Convex functions, partial ordering and statistical applications” by Pečarić et al. [80]. Moreover, many classical books of inequalities like “Inequalities” by Hardy et al. [30], “Inequalities” by Beckenbach and Bellman [7], and “Analytic inequalities” by Mitrinović [57] have treated the topic of convex functions extensively. For a detailed discussion on the books related to the topic we refer the reader to [88].

Now, we recall some useful definitions and significant results about convex functions extracted from [80]. Throughout the section I stands for an interval in $\mathbb{R}$.

Definition 1.1.1. A function $f\colon I \to \mathbb{R}$ is called *convex in the $J-$sense* or *$J-$convex* or *mid-convex* if

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{f(x_1)+f(x_2)}{2}$$

holds for each $x_1, x_2 \in I$.

Definition 1.1.2. A function $f : I \to \mathbb{R}$ is called *convex* if the inequality

$$f\left(\lambda x_1 + (1-\lambda)\,x_2\right) \;\le\; \lambda\, f(x_1) \;+\; (1-\lambda)\, f(x_2) \tag{1.1.1}$$

holds for each $x_1, x_2 \in I$ and $\lambda \in [0,1]$.

Remark 1.1.1. (a) If inequality (1.1.1) is strict for each $x_1 \neq x_2$ and $\lambda \in (0,1)$, then f is called *strictly convex.*

(b) If the inequality in (1.1.1) is reversed, then f is called *concave.* If it is strict for each $x_1 \neq x_2$ and $\lambda \in (0,1)$, then f is called *strictly concave.*

(c) A $J-$convex function is convex if it is continuous as well. □

The following proposition gives us an alternate definition of convex functions [80, p. 2].

Proposition 1.1.1. *A function $f : I \to \mathbb{R}$ is convex if the inequality*

$$(x_3 - x_2)\, f(x_1) + (x_1 - x_3)\, f(x_2) + (x_2 - x_1)\, f(x_3) \ge 0$$

holds for each $x_1,\ x_2,\ x_3 \in I$ such that $x_1 < x_2 < x_3$.

The following result can be deduce from Proposition 1.1.1.

Proposition 1.1.2. *If a function $f : I \to \mathbb{R}$ is convex, then the inequality*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(y_2) - f(y_1)}{y_2 - y_1}$$

holds for each $x_1, x_2, y_1, y_2 \in I$ such that $x_1 \le y_1,\ x_2 \le y_2,\ x_1 \neq x_2,\ y_1 \neq y_2$.

We introduce here generalized convex functions and give certain definitions of different types of convex functions.

1.2 Generalized Convex Functions and Other Types

The core of mathematics is to generalize concepts and results. Therefore, in the present section we define some generalized convex functions in order to generalize certain important identities and inequalities.

1.2.1 Functions with Nondecreasing Increments

The notion of functions with nondecreasing increments was introduced by Brunk [13] in 1964. Let us introduce some notations to recall the definition of functions with nondecreasing increments as follows:

Let $\mathbb{R}^k$ denote the $k-$dimensional vector lattice of points $\mathbf{x} = (x_1, \ldots, x_k)$, x_i be real for $i \in \{1, \ldots, k\}$, with the partial ordering $\mathbf{x} = (x_1, \ldots, x_k) \leq \mathbf{y} = (y_1, \ldots, y_k)$ if and only if $x_i \leq y_i$ for each $i \in \{1, \ldots, k\}$. We denote

$$a\mathbf{x} + b\mathbf{y} = (ax_1 + by_1, \ldots, ax_k + by_k),$$

where $a, b \in \mathbb{R}$ and $k-$tuple $(0, \ldots, 0)$ is denoted by $\mathbf{0}$. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$, $\mathbf{a} \leq \mathbf{b}$, a set $\{\mathbf{x} \in \mathbb{R}^k : \mathbf{a} \leq \mathbf{x} \leq \mathbf{b}\}$ is called an *interval* $[\mathbf{a}, \mathbf{b}]$.

Brunk gave the following definition:

Definition 1.2.1. A function $f : \mathbf{I} \to \mathbb{R}$ is said to have *nondecreasing increments* if

$$f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) \leq f(\mathbf{b} + \mathbf{h}) - f(\mathbf{b}) \tag{1.2.1}$$

holds for each $\mathbf{a} \in \mathbf{I}$, $\mathbf{b} + \mathbf{h} \in \mathbf{I}$, $\mathbf{h} \in \mathbb{R}^k_*$, $\mathbf{a} \leq \mathbf{b}$, where $\mathbf{I}$ is an interval in $\mathbb{R}^k$.

Brunk observed that even if $k = 1$, inequality (1.2.1) does not imply continuity. It is of interest to note that such a function is convex along positively oriented lines, i.e., lines whose direction cosines are nonnegative, with equations of the form $\mathbf{x} = \mathbf{a}t + \mathbf{b}$ where $\mathbf{0} \leq \mathbf{a}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$. Moreover, in one dimension case these functions are known as *Wright-convex*. The class of Wright-convex functions is properly contained in class of $J-$convex functions and it properly contains the class of convex functions.

1.2.2 $\Delta-$ and $\nabla-$ Convex Functions

Now we define further generalized convex functions which can be found in (see e.g.) [39], [44] and [80].

Definition 1.2.2. The *nth order divided difference* of a function $f : I \to \mathbb{R}$ at distinct points $x_i, x_{i+1}, \ldots, x_{i+n} \in I = [a, b] \subset \mathbb{R}$ for some $i \in \mathbb{N}$ is defined recursively by:

$$\begin{aligned} [x_j; f] &= f(x_j), \quad j \in \{i, \ldots, i+n\} \\ [x_i, \ldots, x_{i+n}; f] &= \frac{[x_{i+1}, \ldots, x_{i+n}; f] - [x_i, \ldots, x_{i+n-1}; f]}{x_{i+n} - x_i}. \end{aligned}$$

It may easily be verified that

$$[x_i, \dots, x_{i+n}; f] = \sum_{k=0}^{n} \frac{f(x_{i+k})}{\prod_{j=i, j\neq i+k}^{i+n} (x_{i+k} - x_j)}.$$

Remark 1.2.1. Let us denote $[x_i, \dots, x_{i+n}; f]$ by $\Delta_{(n)} f(x_i)$. The value $[x_i, \dots, x_{i+n}; f]$ is independent of the order of the points $x_i, x_{i+1}, \dots, x_{i+n}$. We can extend this definition by including the cases in which two or more points coincide by taking respective limits. □

Definition 1.2.3. A function $f : I \to \mathbb{R}$ is called *convex of order* n or $n-$*convex* if for all choices of $(n+1)$ distinct points $x_i, \dots, x_{i+n}$ we have $\Delta_{(n)} f(x_i) \geq 0$. Further, we say that if nth order derivative $f_{(n)}$ exists, then f is $n-$*convex* if and only if $f_{(n)} \geq 0$.

Definition 1.2.4. A function $f : I \to \mathbb{R}$ is called $\nabla-$*convex of order* n if for all choices of $(n+1)$ distinct points $x_i, \dots, x_{i+n}$ we have $\nabla_{(n)} f(x_i) = (-1)^n \Delta_{(n)} f(x_i) \geq 0$.

Remark 1.2.2. For $n = 2$ and $i = 0$, we get the *second order divided difference* of a function $f : I \to \mathbb{R}$ which is defined recursively by

$$\begin{aligned}
[x_j; f] &= f(x_j) \qquad , \quad j \in \{0, 1, 2\}, \\
[x_j, x_{j+1}; f] &= \frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j}, \quad j \in \{0, 1\}, \\
[x_0, x_1, x_2; f] &= \frac{[x_1, x_2; f] - [x_0, x_1; f]}{x_2 - x_0},
\end{aligned} \tag{1.2.2}$$

for arbitrary points $x_0, x_1, x_2 \in I$. Now, we discuss some limiting cases as follows: taking the limit as $x_1 \to x_0$ in (1.2.2), we get

$$\lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_2; f] = \frac{f(x_2) - f(x_0) - f'(x_0)(x_2 - x_0)}{(x_2 - x_0)^2}, \quad x_2 \neq x_0,$$

provided that $f'(x_0)$ exists. Furthermore, taking the limits as $x_i \to x_0$, $i \in \{1, 2\}$ in (1.2.2), we obtain

$$\lim_{\substack{x_1 \to x_0 \\ x_2 \to x_0}} [x_0, x_1, x_2; f] = [x_0, x_0, x_0; f] = \frac{f''(x_0)}{2},$$

provided that $f''(x_0)$ exists. □

We can extend the definition of divided difference up to order $(n_1, \dots, n_k)$. For that let us denote $I_1 \times \cdots \times I_k = [a_1, b_1] \times \cdots \times [a_k, b_k] \subset \mathbb{R}^k$.

Definition 1.2.5. Let $f : I_1 \times \cdots \times I_k \to \mathbb{R}$ be a function. Then, the *divided difference of order* $(n_1, \ldots, n_k)$ of the function f at distinct points $x_{ji_j}, \ldots, x_{j(i_j+n_j)} \in I_j$, for $j \in \{1, \ldots, k\}$ is given as

$$\Delta_{(n_1,\ldots,n_k)} f(x_{1i_1}, \ldots, x_{ki_k}) = \begin{bmatrix} x_{1i_1}, & \ldots & , x_{1(i_1+n_1)} \\ \vdots & \vdots & \vdots \\ x_{ki_k}, & \ldots & , x_{k(i_k+n_k)} \end{bmatrix} f$$
$$= \left[x_{1i_1}, \ldots, x_{1(i_1+n_1)}; \left[x_{2i_2}, \ldots, x_{2(i_2+n_2)}; \left[\ldots; [x_{ki_k}, \ldots, x_{k(i_k+n_k)}; f]\right]\right]\right].$$

Definition 1.2.6. We say that a function $f : I_1 \times \cdots \times I_k \to \mathbb{R}$ is *convex of order* $(n_1, \ldots, n_k)$ or $(n_1, \ldots, n_k)-$*convex* if $\Delta_{(n_1,\ldots,n_k)} f(x_{1i_1}, \ldots, x_{ki_k}) \geq 0$, where $x_{ji_j}, \ldots, x_{j(i_j+n_j)} \in I_j$, for $j \in \{1, \ldots, k\}$. Further, we say that if all partial derivatives $\frac{\partial^{n_1+\cdots+n_k} f}{\partial x_1^{n_1} \cdots \partial x_k^{n_k}}$ (denoted by $f_{(n_1,\ldots,n_k)}$) exist, then f is $(n_1, \ldots, n_k)-$*convex* if and only if $f_{(n_1,\ldots,n_k)} \geq 0$.

For other results about convex functions of higher order we refer to the book [80].

Definition 1.2.7. We also define (n, m) *order finite difference* of the function f for $x \in I = [a, b]$, $y \in J = [c, d]$ and $h, k \in \mathbb{R}$ as follows

$$\begin{aligned} \Delta_{h,k}^{(n,m)} f(x, y) &= \Delta_h^{(n)}(\Delta_k^{(m)} f(x, y)) = \Delta_k^{(m)}(\Delta_h^{(n)} f(x, y)) \\ &= \sum_{i=0}^{n} \sum_{j=0}^{m} (-1)^{n+m-i-j} \binom{n}{i} \binom{m}{j} f(x + ih, y + jk). \end{aligned}$$

provided $x + ih \in I$ for $i \in \{0, \ldots, n\}$ and $y + jk \in J$ for $j \in \{0, \ldots, m\}$. Moreover, we say that a function $f : I \times J \to \mathbb{R}$ is *convex of order* (n, m) *or* $(n, m)-$*convex* if $\Delta_{h,k}^{(n,m)} f(x, y) \geq 0$ holds for each $x \in I$, $y \in J$ and $h, k \in \mathbb{R}$.

Definition 1.2.8. *Divided and finite differences of order* (n, m) *of a sequence* (a_{ij}) $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$ are defined as $\Delta_{(n,m)} a_{ij} = \Delta_{(n,m)} f(x_i, y_j)$ and $\Delta^{(n,m)} a_{ij} = \Delta_{1,1}^{(n,m)} f(x_i, y_j)$ respectively, where $x_i = i$, $y_j = j$ and $f : \{1, \ldots, n\} \times \{1, \ldots, m\} \to \mathbb{R}$ is the function $f(i, j) = a_{ij}$. Moreover, we say that a sequence (a_{ij}) is *convex of order* (n, m) *or* $(n, m)-$*convex* if $\Delta^{(n,m)} a_{ij} \geq 0$ holds for $n, m \geq 0$ and $i, j \in \mathbb{N}$.

1.2.3 Logarithmically Convex Functions

A number of important inequalities arise from the logarithmic convexity of some functions as one can see in [51]. Logarithmic convexity plays an important role in various fields namely reliability theory and survival analysis, economics, statistics, social sciences, information theory and optimization etc. (see for reference [12]). Its applications can also be found in applied mathematics as well.

Now, we recall some definitions. The following definition is originally given by Jensen in 1906 [35]. Here I is an interval in $\mathbb{R}$.

Definition 1.2.9. A function $f : I \to \mathbb{R}_+$ is called $\log -$*convex in* $J-$*sense* if the inequality

$$f^2\left(\frac{x_1+x_2}{2}\right) \leq f\left(x_1\right) f\left(x_2\right)$$

holds for each $x_1, x_2 \in I$.

Definition 1.2.10. [80, p. 7] A function $f : I \to \mathbb{R}_+$ is called $\log -$*convex* if the inequality

$$f(\lambda x_1 + (1-\lambda)x_2) \leq [f(x_1)]^{\lambda}[f(x_2)]^{(1-\lambda)}$$

holds for each $x_1, x_2 \in I$ and $\lambda \in [0, 1]$.

Remark 1.2.3. A function log-convex in the $J-$sense is log-convex if it is continuous as well. □

1.2.4 $n-$Exponentially Convex Functions

Bernstein [8] and Widder [91] independently introduced an important sub-class of convex functions, which is called class of exponentially convex functions on a given open interval and studied some properties of this newly defined class. Exponentially convex functions have many nice properties, e.g., these functions are analytic on their domain. These functions also provide us positive-semidefinite matrices. Moreover, they play an important role in studying the properties of Stolarsky and Cauchy means, such as monotonicity of these means etc. For further study of the class of exponentially convex functions we refer [2], [33] and [59].

Pečarić and Perić in [79] introduced the notion of $n-$exponentially convex functions which is in fact generalization of the concept of exponentially convex functions. In the present subsection, we discus the same notion of $n-$exponential convexity by describing related definitions and some important results with some remarks from [79].

Definition 1.2.11. A function $f : I \to \mathbb{R}$ is *$n-$exponentially convex in the $J-$sense* if the inequality

$$\sum_{i,j=1}^{n} u_i u_j f\left(\frac{t_i + t_j}{2}\right) \geq 0$$

holds for each $t_i \in I$ and $u_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$.

Definition 1.2.12. A function $f : I \to \mathbb{R}$ is *$n-$exponentially convex* if it is $n-$exponentially convex in the $J-$sense and continuous on I.

Remark 1.2.4. We can see from the definition that $1-$exponentially convex functions in the $J-$sense are in fact nonnegative functions. Also, $n-$exponentially convex functions in the $J-$sense are $k-$exponentially convex in the $J-$sense for every $k \in \mathbb{N}$ such that $k \leq n$. □

Definition 1.2.13. A function $f : I \to \mathbb{R}$ is *exponentially convex in the $J-$sense*, if it is $n-$exponentially convex in the $J-$sense for each $n \in \mathbb{N}$.

Remark 1.2.5. A function $f : I \to \mathbb{R}$ is *exponentially convex* if it is $n-$exponentially convex in the $J-$sense and continuous on I. □

Proposition 1.2.1. *If function $f : I \to \mathbb{R}$ is $n-$exponentially convex in the $J-$sense, then the matrix*

$$\left[f\left(\frac{t_i + t_j}{2}\right)\right]_{i,j=1}^{m}$$

is positive-semidefinite. Particularly

$$\det\left[f\left(\frac{t_i + t_j}{2}\right)\right]_{i,j=1}^{m} \geq 0$$

for each $m \in \mathbb{N}, m \leq n$ and $t_i \in I$ for $i \in \{1, \ldots, m\}$.

Corollary 1.2.1. *If function $f : I \to \mathbb{R}$ is exponentially convex, then the matrix*

$$\left[f\left(\frac{t_i + t_j}{2}\right)\right]_{i,j=1}^{m}$$

is positive-semidefinite. Particularly

$$det\left[f\left(\frac{t_i + t_j}{2}\right)\right]_{i,j=1}^{m} \geq 0$$

for each $m \in \mathbb{N}$ and $t_i \in I$ for $i \in \{1, \ldots, m\}$.

Corollary 1.2.2. *If function $f : I \to \mathbb{R}_+$ is exponentially convex, then f is* $\log -$*convex.*

Remark 1.2.6. A function $f : I \to \mathbb{R}_+$ is $\log -$convex in $J-$sense if and only if the inequality

$$u_1^2 f(t_1) + 2u_1u_2 f\left(\frac{t_1+t_2}{2}\right) + u_1^2 f(t_2) \geq 0$$

holds for each $t_1, t_2 \in I$ and $u_1, u_2 \in \mathbb{R}$. It follows that a positive function is log-convex in the $J-$sense if and only if it is $2-$exponentially convex in the $J-$sense. Also, using basic convexity theory it follows that a positive function is log-convex if and only if it is $2-$exponentially convex. □

The next chapter is devoted to the class of functions with nondecreasing increments of higher order. In next chapter, we are going to give generalization of Brunk's theorem, Levinson-type inequalities, Burkill-Mirsky-Pečarić's results and a result related to arithmetic integral means.

Chapter 2

Functions with Nondecreasing Increments of Higher Order

> "One reason why mathematics enjoys special esteem, above all other sciences, is that its laws are absolutely certain and indisputable, while those of other sciences are to some extent debatable and in constant danger of being overthrown by newly discovered facts."

–Albert Einstein

In the present chapter, we introduce and investigate a class of functions with nondecreasing increments of higher order. We provide a generalization of Brunk's theorem for this class of functions. Also, we consider functions with nondecreasing increments of order three and for this class we obtain the Levinson-type inequalities and a generalizations of Burkill-Mirsky-Pečarić's result. We also give a result for the integral mean of functions with nondecreasing increments of higher order.

The contents of the present chapter was published in Journal of Inequalities and Applications in year 2013 [43].

2.1 Preliminaries

Regarding the topic functions with nondecreasing increments, Mitrinović et al. [59] pointed out that weights in Jensen-Steffensen's inequality for convex functions and in Popoviciu's generalization of Čebyšev's inequality satisfy the same conditions. There arises a natural question "Can we find a generalization which contains both these results?". They gave the remark that the answer to this question is given by Brunk in [13] by introducing the class of functions with nondecreasing increments. In that article, Brunk also discussed some properties of the class of functions with nondecreasing increments and gave some results. The most remarkable result is the following Brunk theorem (see also [80, p. 266] and [43]). Throughout the chapter $\mathbf{I}$ and $[\mathbf{a},\mathbf{b}]$ are intervals in $\mathbb{R}^k$.

Proposition 2.1.1. *Let $X : [a,b] \to \mathbf{I}$ be a nondecreasing continuous map and let H be a function of bounded variation and continuous from the left on $[a,b)$ with $H(a) = 0$. Then*

$$\int_a^b f\left(\mathbf{X}(t)\right)\, dH(t) \geq 0$$

holds for every continuous function $f : \mathbf{I} \to \mathbb{R}$ with nondecreasing increments if and only if

$$\begin{aligned} H(b) &= 0, \\ \int_{[a,b)} H(u) d\mathbf{X}(u) &= 0, \end{aligned}$$

and

$$\int_{[a,t\}} H(u) d\mathbf{X}(u) \geq 0 \quad \text{for} \quad [a,t\} \subset [a,b),$$

where $\int H\, d\mathbf{X} = \left(\int H\, dX_1, \ldots, \int H\, dX_k\right)$ and the symbol $[a,t\}$ refer to either of the left intervals $[a,t]$ or $[a,t)$.

More results about functions with nondecreasing increments can be found in [69, 71]. The following theorem is the Jensen-Steffensen-type inequality for a function with nondecreasing increments which is extracted from [71].

Proposition 2.1.2. *Let $\mathbf{X} : [a,b] \to \mathbf{I}$ be a nondecreasing continuous map and let $H \in BV[a,b]$ such that*

$$H(a) \leq H(x) \leq H(b), \quad H(a) < H(b). \tag{2.1.1}$$

If $f : \mathbf{I} \to \mathbb{R}$ is a continuous function with nondecreasing increments, then

$$f\left(\frac{\int_a^b \mathbf{X}(t)\, dH(t)}{\int_a^b dH(t)}\right) \leq \frac{\int_a^b f\left(\mathbf{X}(t)\right)\, dH(t)}{\int_a^b dH(t)}$$

holds, where $\int_a^b \mathbf{X}\, dH = \left(\int_a^b X_1\, dH, \ldots, \int_a^b X_k\, dH\right)$.

Now, let us describe monotonicity in means which we will use in next proposition. Throughout this chapter, we will use the notation $\mathbf{X_i} = (x_{i1}, \ldots, x_{ik})$ for $i \in \{1, \ldots, n\}$ where $\mathbf{X_i} \in \mathbb{R}^k$.

Definition 2.1.1. [69] A finite sequence $(\mathbf{X_1}, \ldots, \mathbf{X_n}) \in \mathbf{I}^n$ is said to be *nondecreasing in means* with respect to weights $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n_+$ if the inequalities

$$\mathbf{X_1} \leq A_2(\mathbf{X}; \mathbf{w}) \leq \cdots \leq A_n(\mathbf{X}; \mathbf{w}) \tag{2.1.2}$$

hold, where

$$A_j(\mathbf{X}; \mathbf{w}) = \frac{1}{W_j} \sum_{i=1}^{j} w_i \mathbf{X_i}, \quad W_j = \sum_{i=1}^{j} w_i.$$

If the directions of inequalities are reversed in (2.1.2), then the sequence $(\mathbf{X_1}, \ldots, \mathbf{X_n})$ is said to be *nonincreasing in means*.

The following proposition gives us a Jensen-type inequality for functions with nondecreasing increments when the finite sequence of $k-$tuples $(\mathbf{X_1}, \ldots, \mathbf{X_n})$ is monotone in means. The following is a Pečarić's generalization of Burkill-Mirsky's result which we refer as to Burkill-Mirsky-Pečarić's result (see [69]).

Proposition 2.1.3. *Let $f : \mathbf{I} \to \mathbb{R}$ be a continuous function with nondecreasing increments and $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n_+$. If $(\mathbf{X_1}, \ldots, \mathbf{X_n}) \in \mathbf{I}^n$ is nondecreasing or nonincreasing in means with respect to weights w_i for $i \in \{1, \ldots, n\}$, then the inequality*

$$f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i \mathbf{X_i}\right) \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i f(\mathbf{X_i})$$

holds.

2.2 Functions with Nondecreasing Increments of Order n

Let us write $\Delta_{\mathbf{h_1}} f(\mathbf{x}) = f(\mathbf{x} + \mathbf{h_1}) - f(\mathbf{x})$ and inductively,

$$\Delta_{\mathbf{h_1}} \Delta_{\mathbf{h_2}} \cdots \Delta_{\mathbf{h_n}} f(\mathbf{x}) = \Delta_{\mathbf{h_1}} (\Delta_{\mathbf{h_2}} \cdots \Delta_{\mathbf{h_n}} f(\mathbf{x})) \quad \text{for} \quad n \geq 2,$$

where $\mathbf{x}, \mathbf{x} + \mathbf{h_1} + \cdots + \mathbf{h_n} \in \mathbf{I}$, $\mathbf{h_i} \in \mathbb{R}^k_*$ for $i \in \{1, \ldots, n\}$. Using this notation with $\mathbf{h} = \mathbf{h_1}$, $\mathbf{s} = \mathbf{h_2}$, $\mathbf{b} = \mathbf{a} + \mathbf{s}$, condition (1.2.1) becomes

$$\Delta_{\mathbf{h_1}} \Delta_{\mathbf{h_2}} f(\mathbf{a}) \geq 0.$$

Let us extend Definition 1.2.1 to the following.

Definition 2.2.1. $f : \mathbf{I} \to \mathbb{R}$ is said to be a *function with nondecreasing increments of order* n if

$$\Delta_{\mathbf{h_1}} \cdots \Delta_{\mathbf{h_n}} f(\mathbf{x}) \geq 0$$

holds whenever $\mathbf{x}, \mathbf{x} + \mathbf{h_1} + \cdots + \mathbf{h_n} \in \mathbf{I}$, $\mathbf{h_i} \in \mathbb{R}^k_*$ for $i \in \{1, \ldots, n\}$.

Remark 2.2.1. Every solution of Cauchy equation $f(\mathbf{x_1} + \mathbf{x_2}) = f(\mathbf{x_1}) + f(\mathbf{x_2})$ is a function with nondecreasing increments of order n with null increments, i.e., $\Delta_{\mathbf{h_1}} \cdots \Delta_{\mathbf{h_n}} f(\mathbf{x}) = 0$. If the nth partial derivatives $f_{i_1 \cdots i_n}(\mathbf{x}) = \frac{\partial^n}{\partial x_{i_1} \cdots x_{i_n}} f(\mathbf{x})$ exist, they are nonnegative. If f is a continuous function with nondecreasing increments of order n, it may be approximated uniformly on $\mathbf{I}$ by polynomials having nonnegative nth partial derivatives. To see this, we set, for convenience, $\mathbf{I} = [\mathbf{0}, \mathbf{1}]$ where $\mathbf{1} = (1, \ldots, 1)$. It is known that the Bernstein polynomials

$$\sum_{i_1=0}^{n_1} \cdots \sum_{i_k=0}^{n_k} f\left(\frac{i_1}{n_1}, \ldots, \frac{i_k}{n_k}\right) \prod_{j=1}^{k} \binom{n_j}{i_j} x_j^{i_j} (1 - x_j)^{n_j - i_j}$$

converge uniformly to f on $\mathbf{I}$ as $n_1 \to \infty, \ldots, n_k \to \infty$, if f is continuous. Furthermore, if f is a function with nondecreasing increments of order n, these polynomials have nonnegative nth partial derivatives, as may be shown by repeated application of the formula (see [13] and [43])

$$\frac{d}{dx} \sum_{i=0}^{n} \binom{n}{i} a_i x^i (1 - x)^{n-i} = n \sum_{i=0}^{n-1} \binom{n-1}{i} (a_{i+1} - a_i) x^i (1 - x)^{n-1-i}.$$

□

The aim of the rest of the present section is to give generalization of Proposition 2.1.1. Let us introduce some further notations.

Let $p_1, \ldots, p_r$ be positive integers such that $p_1 + \cdots + p_r = w$. Let $(i_1^{p_1} \cdots i_r^{p_r})_p$ be a set of all permutations with repetitions whose elements are from the multiset

$$S = \{\underbrace{i_1, \ldots, i_1}_{p_1-times}, \underbrace{i_2, \ldots, i_2}_{p_2-times}, \ldots, \underbrace{i_r, \ldots, i_r}_{p_r-times}\}, \quad i_1 < \cdots < i_r, \quad i_1, \ldots, i_r \in \{1, \ldots, k\}.$$

There are $\frac{w!}{p_1! p_2! \cdots p_r!}$ elements in the class $(i_1^{p_1} \cdots i_r^{p_r})_p$.

For $0 < p_1 \leq p_2 \leq \cdots \leq p_r,\ p_1 + \cdots + p_r = w$, let $(p_1 \cdots p_r)_c$ be a set whose elements are described in the following way. We say that permutation $j_1 \cdots j_w$ belongs to the set $(p_1 \cdots p_r)_c$ if and only if there exist $i_1, i_2, \ldots, i_r \in \{1, \ldots, k\}$, $i_1 < i_2 < \cdots < i_r$ and permutation σ of the multiset $\{p_1 \cdots p_r\}$ such that $j_1 \cdots j_w \in (i_1^{\sigma(p_1)} \cdots i_r^{\sigma(p_r)})_p$. Family of all classes $(p_1 \cdots p_r)_c$ is denoted with C_w^k.

For illustration, we describe the above notation on one example. Let $k = 5$ and $w = 4$. Classes $(p_1 \cdots p_r)_c$ are the following: $(1,1,1,1)_c$, $(1,1,2)_c$, $(1,3)_c$, $(2,2)_c$ and $(4)_c$. Let us describe the elements of the set $(1,1,2)_c$. There are three different permutations of the multiset $\{1,1,2\}$. These are

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}.$$

So, $(i_1^{\sigma(p_1)} \cdots i_r^{\sigma(p_r)})_p$ are $(i_1, i_2, i_3, i_3)_p$, $(i_1, i_2, i_2, i_3)_p$, $(i_1, i_1, i_2, i_3)_p$, where $i_1 < i_2 < i_3$ and $i_1, i_2, i_3 \in \{1,2,3,4,5\}$. If, for example, $(i_1, i_2, i_3, i_3)_p = (2,3,5,5)_p$, then it contains all permutations with repetitions of elements $2,3,5,5$, i.e., $(2,3,5,5)_p = \{2355, 2535, 2533, \ldots, 5532\}$ and it has $\frac{4!}{2!} = 12$ elements.

In the following text, $H \in BV[a,b]$ with $H(a) = 0$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$. Let $K_{i_1 \cdots i_n}^n$ be a function such that

$$\begin{aligned} K_{i_1 \cdots i_n}^n(t) &= \int_a^t K_{i_1 \cdots i_{n-1}}^{n-1}(x_n)\, dX_{i_n}(x_n), \qquad n \geq 2 \\ \text{and} \qquad K_{i_1}^1(t) &= \int_a^t H(x_1)\, dX_{i_1}(x_1). \end{aligned}$$

Further, we write

$$\begin{aligned} \prod(S)(x) &= \prod_{j \in S}(X_j(t) - X_j(x)), \\ \text{and} \qquad \prod(\phi)(x) &= 1, \end{aligned}$$

where S is a multiset with elements from $\{1, \ldots, k\}$. Clearly

$$\begin{aligned} d\left\{\prod(S)(x)\right\} &= -\sum_{j \in S} dX_j(x) \prod(S \setminus \{j\})(x) \\ \text{and} \qquad dK_{i_1 \cdots i_n}^n(t) &= K_{i_1 \cdots i_{n-1}}^{n-1}(t)\, dX_{i_n}(t). \end{aligned}$$

Now, the following result holds.

Lemma 2.2.1. *Let w be a fixed positive integer. Then*

$$\int_a^t \prod\left(\{i_1, \ldots, i_w\}\right)(x) dH(x)$$

$$= \sum_{j_1=1}^{w} \sum_{\substack{j_2 = 1 \\ j_2 \neq j_1}}^{w} \cdots \sum_{\substack{j_m = 1 \\ j_m \neq j_k \\ k < m}}^{w} \int_a^t \prod \left(\{i_1, \dots, i_w\} \setminus \{i_{j_1}, \dots, i_{j_m}\}\right)(x)\, dK^m_{i_{j_1} \cdots i_{j_m}}(x)$$

holds for each $m \in \{1, \dots, w\}$.

Proof. We prove it using induction on m. For $m = 1$, using integration by parts, we have

$$\begin{aligned}
\int_a^t \prod \left(\{i_1, \dots, i_w\}\right)(x)\, dH(x) &= -\int_a^t H(x)\, d\left(\prod \left(\{i_1, \dots, i_w\}\right)(x)\right) \\
&= \int_a^t H(x) \sum_{j_1=1}^{w} dX_{j_1}(x) \prod \left(\{i_1, \dots, i_m\} \setminus \{i_{j_1}\}\right)(x) \\
&= \sum_{j_1=1}^{w} \int_a^t \prod \left(\{i_1, \dots, i_w\} \setminus \{i_{j_1}\}\right)(x)\, dK^1_{i_{j_1}}(x).
\end{aligned}$$

Let us suppose that the statement holds for $m - 1$ and let us apply integration by parts on the right-hand side of the formula.

$$\begin{aligned}
&\int_a^t \prod \left(\{i_1, \dots, i_w\}\right)(x)\, dH(x) \\
&= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_{m-1} = 1 \\ j_{m-1} \neq j_k \\ k < m-1}}^{w} \int_a^t \prod \left(\{i_1, \dots, i_w\} \setminus \{i_{j_1}, \dots, i_{j_{m-1}}\}\right)(x)\, dK^{m-1}_{i_{j_1} \cdots i_{j_{m-1}}}(x) \\
&= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_{m-1} = 1 \\ j_{m-1} \neq j_k \\ k < m-1}}^{w} (-1) \int_a^t K^{m-1}_{i_{j_1} \cdots i_{j_{m-1}}}(x) \times \\
&\qquad \times d\left(\prod \left(\{i_1, \dots, i_w\} \setminus \{i_{j_1}, \dots, i_{j_{m-1}}\}\right)(x)\right)
\end{aligned}$$

$$= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_{m-1}=1 \\ j_{m-1} \neq j_k \\ k < m-1}}^{w} (-1) \int_a^t K^{m-1}_{i_{j_1} \cdots i_{j_{m-1}}}(x) \times$$

$$\times (-1) \sum_{\substack{j_m=1 \\ j_m \neq j_k \\ k < m}}^{w} dX_{i_{j_m}}(x) \prod \left(\{i_1, \ldots, i_w\} \setminus \{i_{j_1}, \ldots, i_{j_m}\}\right)(x)$$

$$= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_m=1 \\ j_m \neq j_k \\ k < m}}^{w} \int_a^t \prod \left(\{i_1, \ldots, i_w\} \setminus \{i_{j_1}, \ldots, i_{j_m}\}\right)(x)\, K^{m-1}_{i_{j_1} \cdots i_{j_{m-1}}}(x)\, dX_{i_{j_m}}(x)$$

$$= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_m=1 \\ j_m \neq j_k \\ k < m}}^{w} \int_a^t \prod \left(\{i_1, \ldots, i_w\} \setminus \{i_{j_1}, \ldots, i_{j_m}\}\right)(x)\, dK^{m}_{i_{j_1} \cdots i_{j_m}}(x).$$

■

Especially for $m = w$, we have

$$\begin{aligned} \int_a^t \prod \left(\{i_1, \ldots, i_w\}\right)(x) dH(x) &= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_w=1 \\ j_w \neq j_k \\ k < w}}^{w} \int_a^t dK^{w}_{i_{j_1} \cdots i_{j_w}}(x) \\ &= \sum_{j_1=1}^{w} \cdots \sum_{\substack{j_w=1 \\ j_w \neq j_k \\ k < w}}^{w} K^{w}_{i_{j_1} \cdots i_{j_w}}(t) \end{aligned}$$

$$= \quad p_1! \cdots p_r! \sum_{i_{j_1} \cdots i_{j_w} \in (i_1^{p_1} \cdots i_r^{p_r})_p} K^w_{i_{j_1} \cdots i_{j_w}}(t) \tag{2.2.1}$$

where $\{i_{j_1}, \ldots, i_{j_w}\} = \{\underbrace{i_1, \ldots, i_1}_{p_1-times}, \ldots, \underbrace{i_r, \ldots, i_r}_{p_r-times}\}, \quad i_1 < i_2 < \cdots < i_r, \quad i_1, i_2, \ldots, i_r \in \{1, \ldots, k\}, \quad p_1 + \cdots + p_r = w.$

Example 2.2.1. *If* $w = 3,\ i_1 = i_2 = 1,\ i_3 = 2,$ *then*

$$\begin{aligned} \int_a^t \prod (\{1,1,2\})(x) dH(x) &= \sum_{j_1=1}^{3} \sum_{\substack{j_2 = 1 \\ j_2 \neq j_1}}^{3} \sum_{\substack{j_3 = 1 \\ j_3 \neq j_1, j_2}}^{3} K^3_{i_{j_1} i_{j_2} i_{j_3}}(t) \\ &= 2!\ 1!\ (K^3_{112} + K^3_{121} + K^3_{211}). \end{aligned}$$

□

Furthermore, if we suppose

$$\int_a^b X_{j_1}(u) \cdots X_{j_s}(u)\, dH(u) = 0 \quad \text{for} \quad j_1, \ldots, j_s \in \{1, \ldots, k\},\ s \in \{0, \ldots, w\},$$

then

$$\begin{aligned} p_1! \cdots p_r! \sum K^w_{i_{j_1} \cdots i_{j_w}}(b) &= \int_a^b \prod (\{i_1, \ldots, i_w\})(x)\, dH(x) \\ = \sum (-1)^s \int_a^b X_{j_1}(x) \cdots X_{j_s}(x) X_{j_{s+1}}(b) \cdots X_{j_w}(b)\, dH(x) &= 0. \end{aligned} \tag{2.2.2}$$

Now, we state our main theorems of this section:

Theorem 2.2.1. *Let* $\mathbf{X} : [a, b] \to \mathbf{I}$ *be a continuous function and let* $H \in BV[a, b]$ *with* $H(a) = H(b) = 0$. *Further, assume that* f *has continuous* $(n-1)th$ *partial derivatives for* $n \geq 2$. *If*

$$\int_a^b X_{i_1}(u) \cdots X_{i_m}(u)\, dH(u) \quad = \quad 0 \quad \text{for} \quad i_1, \ldots, i_m \in \{1, \ldots, k\},\ m \in \{1, \ldots, n-1\},$$

then

$$\begin{aligned} \int_a^b f(\mathbf{X}(t))\, dH(t) = (-1)^{n-1} \sum_{(p_1 \cdots p_r)_c \in C^k_{n-1}} \frac{1}{p_1! \cdots p_r!} \sum_{(i_1^{p_1} \cdots i_r^{p_r})_p \subset (p_1 \cdots p_r)_c} \times \\ \times \int_a^b f_{\underbrace{i_1 \cdots i_1}_{p_1-times} \cdots \underbrace{i_r \cdots i_r}_{p_r-times}}(\mathbf{X}(t)) d\left(\int_a^t \prod (\{i_1^{p_1}, \ldots, i_r^{p_r}\})(x)\, dH(x)\right). \end{aligned} \tag{2.2.3}$$

Proof. The proof follows from induction on n. Let $n = 2$,

$$\begin{aligned}
\int_a^b f(\mathbf{X}(t))\, dH(t) &= -\sum_{i=1}^k \int_a^b f_i(\mathbf{X}(t))\, H(t) dX_i(t) \\
&= -\sum_{i=1}^k \int_a^b f_i(\mathbf{X}(t))\, dK_i^1(t) \\
&= -\sum_{i=1}^k \int_a^b f_i(\mathbf{X}(t))\, d\left(\int_a^t H(x) dX_i(x)\right) \\
&= -\sum_{i=1}^k \int_a^b f_i(\mathbf{X}(t))\, d\left(\int_a^t H(x) d(X_i(x) - X_i(t))\right) \\
&= \sum_{i=1}^k \int_a^b f_i(\mathbf{X}(t))\, d\left(\int_a^t H(x) d(X_i(t) - X_i(x))\right) \\
&= -\sum_{i=1}^k \int_a^b f_i(\mathbf{X}(t))\, d\left(\int_a^t (X_i(t) - X_i(x)) dH(x)\right) \\
&= -\sum_{i=1}^k \int_a^b f_i(\mathbf{X}(t))\, d\left(\int_a^t \prod(\{i\})(x)\, dH(x)\right).
\end{aligned}$$

If we have $\int_a^b X_{i_1}(u) \cdots X_{i_m}(u)\, dH(u) = 0$ for $i_1, \ldots, i_m \in \{1, \ldots, k\}$, $m \in \{1, \ldots, n-2\}$ and if we suppose that (2.2.3) holds for $(n-1)$, then

$$\begin{aligned}
&\int_a^b f(\mathbf{X}(t))\, dH(t) \\
&= (-1)^{n-2} \sum_{(p_1 \cdots p_r)_c \in C_{n-2}^k} \frac{1}{p_1! \cdots p_r!} \sum_{(i_1^{p_1} \cdots i_r^{p_r})_p \subset (p_1 \cdots p_r)_c} \int_a^b f_{i_1^{p_1} \cdots i_r^{p_r}}(\mathbf{X}(t)) \times \\
&\qquad \times d\left(\int_a^t \prod\left(\{i_1^{p_1}, \ldots, i_r^{p_r}\}\right)(x)\, dH(x)\right) \\
&= (-1)^{n-2} \sum_{(p_1 \cdots p_r)_c \in C_{n-2}^k} \frac{1}{p_1! \cdots p_r!} \sum_{(i_1^{p_1} \cdots i_r^{p_r})_p} \int_a^b f_{i_1^{p_1} \cdots i_r^{p_r}}(\mathbf{X}(t)) \times \\
&\qquad \times d\left(p_1! \cdots p_r! \sum_{i_{j_1} \cdots i_{j_{n-2}} \in (i_1^{p_1} \cdots i_r^{p_r})_p} K_{i_{j_1} \cdots i_{j_{n-2}}}^{n-2}(t)\right)
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{n-1} \sum_{(p_1\cdots p_r)_c\in C^k_{n-2}} \sum_{(i_1^{p_1}\cdots i_r^{p_r})_p} \int_a^b df_{i_1^{p_1}\cdots i_r^{p_r}}(\mathbf{X}(t))\times \\
&\qquad \times \sum_{i_{j_1}\cdots i_{j_{n-2}}\in(i_1^{p_1}\cdots i_r^{p_r})_p} K^{n-2}_{i_{j_1}\cdots i_{j_{n-2}}}(t) \\
&= (-1)^{n-1} \sum_{(p_1\cdots p_r)_c\in C^k_{n-2}} \sum_{(i_1^{p_1}\cdots i_r^{p_r})_p} \int_a^b \sum_{i_{n-1}=1}^{k} f_{i_1^{p_1}\cdots i_r^{p_r} i_{n-1}}(\mathbf{X}(t))\times \\
&\qquad \times dX_{i_{n-1}}(t) \left(\sum_{i_{j_1}\cdots i_{j_{n-2}}} K^{n-2}_{i_{j_1}\cdots i_{j_{n-2}}}(t) \right) \\
&= (-1)^{n-1} \sum_{\substack{(s_1\cdots s_g)_c \in C^k_{n-1} \\ s_1+\cdots+s_g = n-1}} \sum_{(i_1^{s_1}\cdots i_g^{s_g})_p\subset(s_1\cdots s_g)_c} \int_a^b f_{i_1^{s_1}\cdots i_g^{s_g}}(\mathbf{X}(t))\times \\
&\qquad \times \left(\sum_{l_1\cdots l_{n-1}\in(i_1^{s_1}\cdots i_g^{s_g})_p} K^{n-2}_{l_1\cdots l_{n-2}}(t)\, dX_{l_{n-1}}(t) \right) \\
&= (-1)^{n-1} \sum_{(s_1\cdots s_g)_c\in C^k_{n-1}} \sum_{(i_1^{s_1}\cdots i_g^{s_g})_p} \int_a^b f_{i_1^{s_1}\cdots i_g^{s_g}}(\mathbf{X}(t))\, d\left(\sum_{l_1\cdots l_{n-1}} K^{n-1}_{l_1\cdots l_{n-1}}(t) \right) \\
&= (-1)^{n-1} \sum_{(s_1\cdots s_g)_c\in C^k_{n-1}} \sum_{(i_1^{s_1}\cdots i_g^{s_g})_p} \int_a^b f_{i_1^{s_1}\cdots i_g^{s_g}}(\mathbf{X}(t))\times \\
&\qquad \times d\left(\frac{1}{s_1!\cdots s_g!} \int_a^t \prod \left(\{i_1^{s_1}\cdots i_g^{s_g}\}\right)(x) dH(x) \right)
\end{aligned}
$$

by (2.2.1) and (2.2.2). Hence we have (2.2.3). ■

Theorem 2.2.2. *Let $X : [a,b] \to \mathbf{I}$ be a nondecreasing continuous map and let $H \in BV[a,b]$ with $H(a) = 0$. Then*

$$\int_a^b f\left(\mathbf{X}(t)\right)\, dH(t) \geq 0 \tag{2.2.4}$$

holds for every continuous function f with nondecreasing increments of order n on $\mathbf{I}$ if and only if

$$H(b) \;=\; 0, \tag{2.2.5}$$

$$\int_a^b X_{i_1}(t)\cdots X_{i_m}(t)\,dH(t) \;=\; 0, \tag{2.2.6}$$

for $i_1,\ldots,i_m \in \{1,\ldots,k\},\ m\in\{1,\ldots,n-1\}$ *and*

$$(-1)^n \int_a^t \prod(\{i_1,\ldots,i_{n-1}\})(u)\,dH(u) \geq 0 \tag{2.2.7}$$

for each $t\in[a,b],\ i_1,\ldots,i_{n-1}\in\{1,\ldots,k\}$.

Proof. Necessity: The validity of (2.2.4) for constant functions $f^1\equiv 1$ and $f^2\equiv -1$ implies (2.2.5). From (2.2.4) for $f^3(\mathbf{x}) = x_{i_1}\cdots x_{i_s}$ and $f^4(\mathbf{x}) = -x_{i_1}\cdots x_{i_s}$, for $i_1,\ldots,i_s\in\{1,\ldots,k\},\ s\in\{1,\ldots,n-1\}$, we have (2.2.6).

Inequality (2.2.7) is obtained from (2.2.4) on setting, for fixed $t\in[a,b]$ and fixed $i_1,\ldots,i_{n-1}\in\{1,\ldots,k\}$,

$$f^5(x) = -[x_{i_1}-X_{i_1}(t)]^-\cdots[x_{i_{n-1}}-X_{i_{n-1}}(t)]^- \quad \text{where} \quad c^- = \min\{c,0\},\ c\in\mathbb{R}.$$

Sufficiency: Since f may be approximated uniformly on $\mathbf{I}$ by functions with continuous and nonnegative nth partial derivatives, we may assume that the nth partials $f_{i_1\cdots i_n}$ exist and are continuous and nonnegative. By Theorem 2.2.1 and (2.2.6), we have

$$\begin{aligned}
&\int_a^b f\left(\mathbf{X}(t)\right)\,dH(t)\\
&= (-1)^n \sum_{(p_1\cdots p_r)_c\in C^k_{n-1}} \frac{1}{p_1!\cdots p_r!} \sum_{(i_1^{p_1}\cdots i_r^{p_r})_p\subset(p_1\cdots p_r)_c} \sum_{i_n=1}^{k} \int_a^b f_{i_1^{p_1}\cdots i_r^{p_r} i_n}\left(\mathbf{X}(t)\right)\times\\
&\qquad \times dX_{i_n}(t)\int_a^t \prod\left(\{i_1^{p_1}\cdots i_r^{p_r}\}\right)(x)\,dH(x).
\end{aligned}$$

By (2.2.7), each term in the sum is nonnegative so that (2.2.4) is verified. ■

2.3 Functions with Nondecreasing Increments of Order three

2.3.1 On Inequalities of Levinson-type

Levinson in [49] proved that:

Proposition 2.3.1. *If a real-valued function f defined on $[0,2a] \subset \mathbb{R}$ has a nonnegative third derivative, then*

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(y_i) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i y_i\right) \quad (2.3.1)$$

holds for $0 < x_i < a$, $y_i = 2a - x_i$ and $w_i > 0$, $i \in \{1,\ldots,n\}$ such that $W_n = \sum_{i=1}^{n} w_i$.

Remark 2.3.1. If $a = \frac{1}{2}$, $w_1 = \cdots = w_n = 1$ and $f(x) = \ln(x)$, then Levinson's inequality (2.3.1) becomes the renowned Fan's inequality

$$\frac{G_n}{G'_n} \leq \frac{A_n}{A'_n},$$

where

$$A_n = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad A'_n = \frac{1}{n}\sum_{i=1}^{n}(1 - x_i)$$

and

$$G_n = \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \quad G'_n = \left(\prod_{i=1}^{n}(1 - x_i)\right)^{1/n}.$$

□

In [68], Pečarić showed that instead of variables with sum equal to $2a$, we can use variables with constant difference and that result becomes a source of some further generalizations [80, pp. 74,75]. In fact, he proved that if f is a real-valued 3-convex function on $[a,b]$ and x_i, y_i for $i \in \{1,\ldots,n\}$ are $2n$ points in $[a,b]$ such that $y_1 - x_1 = y_2 - x_2 = \cdots = y_n - x_n > 0$ and $w_i > 0$, $i \in \{1,\ldots,n\}$, then (2.3.1) is valid.

The following theorem is a generalization of the Levinson's inequality.

Theorem 2.3.1. *Let $H \in BV[a,b]$ such that (2.1.1) holds and let $\mathbf{X} : [a,b] \to [\mathbf{0},\mathbf{d}]$, $(\mathbf{d} > \mathbf{0})$ be a nondecreasing continuous map. If f is a continuous function with nondecreasing increments of order three on $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$, then the inequality*

$$\frac{\int_a^b f\left(\mathbf{X}(t)\right)\, dH(t)}{\int_a^b dH(t)} - f\left(\frac{\int_a^b \mathbf{X}(t)\, dH(t)}{\int_a^b dH(t)}\right) \leq \frac{\int_a^b f\left(2\mathbf{d} - \mathbf{X}(t)\right)\, dH(t)}{\int_a^b dH(t)} - f\left(\frac{\int_a^b (2\mathbf{d} - \mathbf{X}(t))\, dH(t)}{\int_a^b dH(t)}\right)$$

holds.

Proof. If f is a function with nondecreasing increments of order three on $\mathbf{J}$, then the following inequality holds

$$\Delta_{\mathbf{h_1}}\Delta_{\mathbf{h_2}}\Delta_{\mathbf{h_3}} f(\mathbf{x}) \geq 0 \quad \text{for} \quad \mathbf{x}, \mathbf{x}+\mathbf{h_1}+\mathbf{h_2}+\mathbf{h_3} \in \mathbf{J}, \quad \mathbf{h_1}, \mathbf{h_2}, \mathbf{h_3} \in \mathbb{R}^k_*,$$

i.e.,

$$\Delta_{\mathbf{h_1}}\Delta_{\mathbf{h_2}} \left(f(\mathbf{x}+\mathbf{h_3}) - f(\mathbf{x}) \right) \geq 0. \tag{2.3.2}$$

If $\mathbf{x} \in \mathbf{I}$ and $\mathbf{h_3} = 2\mathbf{d} - 2\mathbf{x}$, we have

$$\Delta_{\mathbf{h_1}}\Delta_{\mathbf{h_2}} \left(f(2\mathbf{d} - \mathbf{x}) - f(\mathbf{x}) \right) \geq 0,$$

i.e., the function $\mathbf{x} \mapsto f(2\mathbf{d}-\mathbf{x}) - f(\mathbf{x})$ is a function with nondecreasing increments of order two, i.e., it is a function with nondecreasing increments. Now, using Proposition 2.1.2, we obtain Theorem 2.3.1. ■

Theorem 2.3.2. *Let $H \in BV[a,b]$ such that* (2.1.1) *holds and let f be a continuous function with nondecreasing increments of order three on $[\mathbf{c}, \mathbf{d}] \subset \mathbb{R}^k$. Let $\mathbf{0} < \mathbf{a} < \mathbf{d} - \mathbf{c}$. If $\mathbf{X} : [a,b] \to [\mathbf{c}, \mathbf{d} - \mathbf{a}]$ is a nondecreasing continuous map, then the inequality*

$$\frac{\int_a^b f(\mathbf{X}(t))\, dH(t)}{\int_a^b dH(t)} - f\left(\frac{\int_a^b \mathbf{X}(t)\, dH(t)}{\int_a^b dH(t)} \right) \leq \frac{\int_a^b f(\mathbf{a}+\mathbf{X}(t))\, dH(t)}{\int_a^b dH(t)} - f\left(\frac{\int_a^b (\mathbf{a}+\mathbf{X}(t))\, dH(t)}{\int_a^b dH(t)} \right)$$

holds.

Proof. Using (2.3.2) for $\mathbf{h_3} = \mathbf{a} = \text{constant} \in \mathbb{R}^k$, we have that $\mathbf{x} \mapsto f(\mathbf{a}+\mathbf{x}) - f(\mathbf{x})$ is a function with nondecreasing increments, so from Proposition 2.1.2, we obtain Theorem 2.3.2. ■

Remark 2.3.2. For $k = 1$, Theorem 2.3.2 gives us a result from [68]. □

Corollary 2.3.3. (*a*) *Let $\mathbf{X}$ satisfy the assumptions of Theorem* 2.3.1. *Then the inequalities*

$$0 \leq \left(\int_a^b dH(t) \right)^{k-1} \int_a^b \prod_{i=1}^k X_i(t)\, dH(t) - \prod_{i=1}^k \int_a^b X_i(t)\, dH(t)$$
$$\leq \left(\int_a^b dH(t) \right)^{k-1} \int_a^b \prod_{i=1}^k \left(2d_i - X_i(t)\right) dH(t) - \prod_{i=1}^k \int_a^b \left(2d_i - X_i(t)\right) dH(t)$$

hold.

(b) *If* $\mathbf{X}$ *satisfies the assumptions of Theorem 2.3.2, then the inequalities*

$$0 \le \left(\int_a^b dH(t)\right)^{k-1} \int_a^b \prod_{i=1}^k X_i(t)\, dH(t) - \prod_{i=1}^k \int_a^b X_i(t)\, dH(t)$$
$$\le \left(\int_a^b dH(t)\right)^{k-1} \int_a^b \prod_{i=1}^k (a_i + X_i(t))\ dH(t) - \prod_{i=1}^k \int_a^b (a_i + X_i(t))\ dH(t)$$

hold, where all components of $\mathbf{X}$ *are nonnegative.*

Proof. The function $f(\mathbf{x}) = x_1 \cdots x_k$ is a function with nondecreasing increments of orders two and three for $\mathbf{x} \in \mathbb{R}_*^k$. So, using Proposition 2.1.2, Theorems 2.3.1 and 2.3.2, we obtain Corollary 2.3.3. ■

2.3.2 Generalizations of Burkill-Mirsky-Pečarić's Result

In the current subsection, we consider a sequence of $k-$tuples $(\mathbf{X_1}, \ldots, \mathbf{X_n})$ which is monotone in means.

Theorem 2.3.4. *Let* $(\mathbf{X_1}, \ldots, \mathbf{X_n}) \in [\mathbf{0}, \mathbf{d}]^n$, $(\mathbf{d} > \mathbf{0})$ *be nondecreasing or nonincreasing in means with respect to positive weights* w_i *for* $i \in \{1, \ldots, n\}$. *If* f *is a continuous function with nondecreasing increments of order three on* $\mathbf{J} = [\mathbf{0}, 2\mathbf{d}]$,*then the inequality*

$$\frac{1}{W_n}\sum_{i=1}^n w_i f(\mathbf{X_i}) - f\left(\frac{1}{W_n}\sum_{i=1}^n w_i \mathbf{X_i}\right)$$
$$\le \frac{1}{W_n}\sum_{i=1}^n w_i f\left(2\mathbf{d} - \mathbf{X_i}\right) - f\left(\frac{1}{W_n}\sum_{i=1}^n w_i \left(2\mathbf{d} - \mathbf{X_i}\right)\right)$$

holds.

Proof. By following the proof of Theorem 2.3.1, we obtain Theorem 2.3.4 by simply replacing "Proposition 2.1.2" by "Proposition 2.1.3". ■

Theorem 2.3.5. *Let* $(\mathbf{X_1}, \ldots, \mathbf{X_n}) \in [\mathbf{c}, \mathbf{d} - \mathbf{a}]^n$, $(\mathbf{0} < \mathbf{a} < \mathbf{d} - \mathbf{c})$ *be nondecreasing or nonincreasing in means with respect to positive weights* w_i *for* $i \in \{1, \ldots, n\}$. *If* f *is a continuous function with nondecreasing increments of order three on* $\mathbf{J} = [\mathbf{c}, \mathbf{d}]$, *then the following inequality holds*

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i f(\mathbf{X_i}) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \mathbf{X_i}\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(\mathbf{a}+\mathbf{X_i}) - f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i (\mathbf{a}+\mathbf{X_i})\right)$$

Proof. By following the proof of Theorem 2.3.2, we obtain Theorem 2.3.5 by simply replacing "Proposition 2.1.2" by "Proposition 2.1.3". ■

Corollary 2.3.6. (*a*) *Let* $\mathbf{X}$ *satisfy the assumptions of Theorem* 2.3.4. *Then the inequalities*

$$0 \leq W_n^{k-1}\sum_{i=1}^{n} w_i^k \left(\prod_{j=1}^{k} x_{ij}\right) - \prod_{j=1}^{k}\left(\sum_{i=1}^{n} w_i x_{ij}\right) \leq W_n^{k-1}\sum_{i=1}^{n} w_i^k \left(\prod_{j=1}^{k} (2d_j - x_{ij})\right) - \prod_{j=1}^{k}\left(\sum_{i=1}^{n} w_i (2d_j - x_{ij})\right)$$

hold.

(*b*) *If* $\mathbf{X}$ *satisfies the assumptions of Theorem* 2.3.5. *Then the inequalities*

$$0 \leq W_n^{k-1}\sum_{i=1}^{n} w_i^k \left(\prod_{j=1}^{k} x_{ij}\right) - \prod_{j=1}^{k}\left(\sum_{i=1}^{n} w_i x_{ij}\right) \leq W_n^{k-1}\sum_{i=1}^{n} w_i^k \left(\prod_{j=1}^{k} (a_j + x_{ij})\right) - \prod_{j=1}^{k}\left(\sum_{i=1}^{n} w_i (a_j + x_{ij})\right)$$

hold, where all components of $\mathbf{X}$ *are nonnegative.*

Proof. We consider again the function $f(\mathbf{x}) = x_1 \cdots x_k$ which is a function with nondecreasing increments of orders two and three for $\mathbf{x} \in \mathbb{R}_*^k$. So, using Proposition 2.1.3, Theorems 2.3.4 and 2.3.5, we obtain Corollary 2.3.6. ■

2.4 Arithmetic Integral Mean

It is known that if $f : [0, a] \to \mathbb{R}$, $a > 0$, is a nonnegative and nondecreasing function, then the function F, defined as

$$F(x) = \frac{1}{x}\int_0^x f(t)dt$$

is also a nondecreasing function on $[0, a]$. Let us observe that F is an arithmetic integral mean of a function f on an interval $[0, a]$. This result was generalized in [45] by considering a real-valued function f for which $\Delta_h^n f(x) \geq 0$ holds for any $h > 0$, where Δ_h^n is defined as follows:

$$\Delta_h^0 f(x) = f(x), \qquad \Delta_h^n f(x) = \Delta_h^{n-1} f(x+h) - \Delta_h^{n-1} f(x).$$

Here, we extend the above-mentioned result to functions with nondecreasing increments of higher order.

Theorem 2.4.1. *Let the function $f : [\mathbf{a}, \mathbf{b}] \to \mathbb{R}$ be continuous and with nondecreasing increments of order n. Then the function F, defined as*

$$F(\mathbf{x}) = \left(\prod_{i=1}^{k}(x_i - a_i)\right)^{-1} \int_{a_1}^{x_1} \cdots \int_{a_k}^{x_k} f(\mathbf{u})\mathbf{du},$$

is a function with nondecreasing increments of order n on $[\mathbf{a}, \mathbf{b}]$, where $\mathbf{u} = (u_1, \ldots ,u_k)$ and $\mathbf{du} = du_1 \cdots du_k$.

Proof. Let $\mathbf{x} > \mathbf{a} = (a_1, \ldots, a_k)$. Then

$$F(\mathbf{x}) = \int_0^1 \cdots \int_0^1 f\left(\mathbf{a} + \mathbf{s}\left(\mathbf{x} - \mathbf{a}\right)\right) \mathbf{ds},$$

where we used the substitutions $u_i = a_i + s_i(x_i - a_i)$, $i \in \{1, \ldots, k\}$, $0 \leq s_i \leq 1$, where $\mathbf{a} + \mathbf{s}(\mathbf{x} - \mathbf{a}) = (a_1 + s_1(x_1 - a_1), \ldots, a_k + s_k(x_i - a_k))$ and $\mathbf{ds} = ds_1 \cdots ds_k$. Now, we have

$$\begin{aligned} \Delta_{\mathbf{h_1}} \cdots \Delta_{\mathbf{h_n}} F(\mathbf{x}) &= \Delta_{\mathbf{h_1}} \cdots \Delta_{\mathbf{h_n}} \int_0^1 \cdots \int_0^1 f(\mathbf{a} + \mathbf{s}(\mathbf{x} - \mathbf{a}))\mathbf{ds} \\ &= \int_0^1 \cdots \int_0^1 \Delta_{\mathbf{h_1}} \cdots \Delta_{\mathbf{h_n}} f(\mathbf{a} + \mathbf{s}(\mathbf{x} - \mathbf{a}))\mathbf{ds} \geq 0 \end{aligned}$$

because if $f(\mathbf{x})$ is a function with nondecreasing increments of order n, then the function $f(\mathbf{a} + \mathbf{s}(\mathbf{x} - \mathbf{a}))$ is also a function with nondecreasing increments of order n. ∎

In the next chapter, we are going to deduce some very general identities of Popoviciu-type for sums $\sum p_k f(x_k)$, $\sum \cdots \sum P_{k_1 \cdots k_n} f(x_{1k_1}, \ldots, x_{nk_n})$ and integral $\int \cdots \int P(x_1, \ldots, x_n) f(x_1, \ldots, x_n) dx_1 \cdots dx_n$. Using the obtained identities, positivity of these expressions are characterized for convex functions of higher order. An application in terms of exponential convexity will also be given.

Chapter 3

Popoviciu-Type Characterization of Positivity of Sums and Integrals for Higher Order Convex Functions

"Behind every theorem lies an inequality."

–A. N. Kolmogorov

In the present chapter, we will provide several identities for sum $\sum p_k f(x_k)$, one discrete identity for $\sum \cdots \sum p_{k_1 \cdots k_n} f(x_{1k_1}, \ldots, x_{nk_n})$ and one integral identity for $\Lambda(f) = \int \cdots \int p(x_1, \ldots, x_n) f(x_1, \ldots, x_n) dx_1 \cdots dx_n$. Using these identities and some special convex and $\nabla-$convex functions of higher orders, we obtain necessary and sufficient conditions under which the above-mentioned sums and integral are nonnegative. Finally, we will discuss some properties of the functional $\Lambda(f)$.

Some of the contents of the present chapter was published in Journal of Mathematical Inequalities in year 2013 [44].

Throughout this chapter we will use the following notations:
$I = [a, b] \subset \mathbb{R}$, $J = [c, d] \subset \mathbb{R}$, $I_j = [a_j, b_j] \subset \mathbb{R}$ for $j \in \{1, \ldots, n\}$.
For some fixed integer a and $m \in \mathbb{N}$:

$$a^{(m)} = a(a-1)\cdots(a-m+1), \quad a^{(0)} = 1.$$

For some real sequence (a_n), $n \in \mathbb{N}$ and $m \in \{2, 3, \ldots\}$:

$$\Delta^{(1)} a_n = \Delta a_n = a_{n+1} - a_n, \quad \Delta^{(m)} a_n = \Delta(\Delta^{(m-1)} a_n)$$

and

$$\nabla^{(1)}a_n = \nabla a_n = a_n - a_{n+1}, \quad \nabla^{(m)}a_n = \nabla(\nabla^{(m-1)}a_n)$$

Also for n distinct real numbers x_i, $i \in \{1,\ldots,n\}$ and $m \geq 0$:

$$(x_k - x_i)^{(m+1)} = (x_k - x_i)(x_k - x_{i+1})\cdots(x_k - x_{i+m}), \quad (x_k - x_i)^{(0)} = 1$$

and

$$(x_k - x_i)^{\{m+1\}} = (x_k - x_i)(x_{k-1} - x_i)\cdots(x_{k-m} - x_i), \quad (x_k - x_i)^{\{0\}} = 1.$$

We also recall from Chapter 1:

$$\nabla_{(n)}f(x_i) = (-1)^n \Delta_{(n)}f(x_i)$$

where $\Delta_{(n)}f$ represents divided difference of order n of function f whereas $\Delta_{(i_1,\ldots,i_n)}f$ represents divided difference of order $(i_1,\ldots,i_n)$ of function f of n variables.

3.1 Discrete Identity and Inequalities for Functions of One Variable

Let us state two results from [56] and [67] respectively as follows.

Proposition 3.1.1. *Let $p_k \in \mathbb{R}$ for $k \in \{1,\ldots,N\}$. Then for any real sequence (a_k), $k \in \{1,\ldots,N\}$ the identity*

$$\begin{aligned} \sum_{k=1}^{N} p_k a_k &= \sum_{i=0}^{m-1}\sum_{k=i+1}^{N} p_k (k-1)^{(i)} \frac{\Delta^{(i)}a_1}{i!} \\ &+ \sum_{i=m+1}^{N}\left(\sum_{k=i}^{N} p_k (k-i+m-1)^{(m-1)}\right)\frac{\Delta^{(m)}a_{i-m}}{(m-1)!} \end{aligned} \qquad (3.1.1)$$

holds.

Proposition 3.1.2. *Let $p_k \in \mathbb{R}$ for $k \in \{1,\ldots,N\}$. Then for any real sequence (a_k), $k \in \{1,\ldots,N\}$ the identity*

$$\begin{aligned} \sum_{k=1}^{N} p_k a_k &= \sum_{i=0}^{m-1}\sum_{k=1}^{N-i} p_k (N-i)^{(i)} \frac{\nabla^{(i)}a_{N-i}}{i!} \\ &+ \sum_{i=1}^{N-m}\left(\sum_{k=1}^{i} p_k (i-k+m-1)^{(m-1)}\right)\frac{\nabla^{(m)}a_i}{(m-1)!} \end{aligned} \qquad (3.1.2)$$

holds.

A result analogous to (3.1.1) for real functions was proved by Popoviciu [83] which may be stated as:

Proposition 3.1.3. *Let $p_k \in \mathbb{R}$ for $k \in \{1, \dots, N\}$. If $f : I \to \mathbb{R}$ is a function and $x_k,\ k \in \{1, \dots, N\}$ be mutually distinct point from I, then the identity*

$$\begin{aligned}\sum_{k=1}^{N} p_k f(x_k) &= \sum_{i=0}^{m-1} \Big(\sum_{k=i+1}^{N} p_k (x_k - x_1)^{(i)} \Big) \Delta_{(i)} f(x_1) \\ &+ \sum_{i=m+1}^{N} \Big(\sum_{k=i}^{N} p_k (x_k - x_{i-m+1})^{(m-1)} \Big) \Delta_{(m)} f(x_{i-m})(x_i - x_{i-m}) \qquad (3.1.3)\end{aligned}$$

holds.

Now, let us prove an identity which is a generalization of formula (3.1.2). In fact, it is a formula which is similar to Popoviciu's result (3.1.3) but involving the operator ∇.

Theorem 3.1.1. *Let $p_k \in \mathbb{R}$ for $k \in \{1, \dots, N\}$. If $f : I \to \mathbb{R}$ be a function and $x_k,\ k \in \{1, \dots, N\}$ be mutually distinct point from I, then the identity*

$$\begin{aligned}\sum_{k=1}^{N} p_k f(x_k) &= \sum_{i=0}^{m-1} \Big(\sum_{k=1}^{N-i} p_k (x_N - x_k)^{\{i\}} \Big) \nabla_{(i)} f(x_{N-i}) \\ &+ \sum_{i=1}^{N-m} \Big(\sum_{k=1}^{i} p_k (x_{i+m-1} - x_k)^{\{m-1\}} \Big) \nabla_{(m)} f(x_i)(x_{i+m} - x_i) \qquad (3.1.4)\end{aligned}$$

holds.

Proof. Let us prove it by induction on m. For $m = 1$, we have

$$\sum_{k=1}^{N} p_k f(x_k) = \sum_{k=1}^{N} p_k f(x_N) + \sum_{i=1}^{N-1} \Big(\sum_{k=1}^{i} p_k \Big) \Big(f(x_i) - f(x_{i+1}) \Big)$$

which is true.

Suppose that (3.1.4) is valid. Then

$$\begin{aligned}&\sum_{i=0}^{m} \Big(\sum_{k=1}^{N-i} p_k (x_N - x_k)^{\{i\}} \Big) \nabla_{(i)} f(x_{N-i}) \\ &+ \sum_{i=1}^{N-m-1} \Big(\sum_{k=1}^{i} p_k (x_{i+m} - x_k)^{\{m\}} \Big) \nabla_{(m+1)} f(x_i)(x_{i+m+1} - x_i)\end{aligned}$$

$$
\begin{aligned}
&= A + \sum_{k=1}^{N-m} p_k (x_N - x_k)^{\{m\}} \nabla_{(m)} f(x_{N-m}) \\
&+ \sum_{i=1}^{N-m-1} B(-1)^{m+1} \Big(\Big[x_{i+1}, \dots, x_{i+m+1}; f \Big] - \Big[x_i, \dots, x_{i+m}; f \Big] \Big) \\
&= A + \sum_{k=1}^{N-m} p_k (x_N - x_k)^{\{m\}} \nabla_{(m)} f(x_{N-m}) \\
&+ \sum_{k=1}^{N-m-1} p_k (x_{N-1} - x_k)^{\{m\}} (-1)^{m+1} [x_{N-m}, \dots, x_N; f] \\
&+ \sum_{i=1}^{N-m-2} B(-1)^{m+1} [x_{i+1}, \dots, x_{i+m+1}; f] - \sum_{i=2}^{N-m-1} B(-1)^{m+1} [x_i, \dots, x_{i+m}; f] \\
&- p_1 (x_{m+1} - x_1)^{\{m\}} (-1)^{m+1} [x_1, \dots, x_{m+1}; f] \\
&= A + \sum_{k=1}^{N-m} p_k (x_{N-1} - x_k)^{\{m-1\}} \nabla_{(m)} f(x_{N-m})(x_N - x_{N-m}) \\
&+ \sum_{i=2}^{N-m-1} (-1)^m [x_i, \dots, x_{i+m}; f] \Big(\sum_{k=1}^{i} p_k (x_{i+m} - x_k)^{\{m\}} \\
&- \sum_{k=1}^{i-1} p_k (x_{i+m-1} - x_k)^{\{m\}} \Big) + p_1 (x_{m+1} - x_1)^{\{m\}} \nabla_{(m)} f(x_1) \\
&= A + \sum_{k=1}^{N-m} p_k (x_{N-1} - x_k)^{\{m-1\}} \nabla_{(m)} f(x_{N-m})(x_N - x_{N-m}) \\
&+ \sum_{i=2}^{N-m-1} \Big(\sum_{k=1}^{i} p_k (x_{i+m-1} - x_k)^{\{m-1\}} \Big) \nabla_{(m)} f(x_i)(x_{i+m} - x_i) \\
&+ p_1 (x_m - x_1)^{\{m-1\}} \nabla_{(m)} f(x_1)(x_{m+1} - x_1) \\
&= A + \sum_{i=1}^{N-m} \Big(\sum_{k=1}^{i} p_k (x_{i+m-1} - x_k)^{\{m-1\}} \Big) \nabla_{(m)} f(x_i)(x_{i+m} - x_i) = \sum_{k=1}^{N} p_k f(x_k).
\end{aligned}
$$

where $A = \sum_{i=0}^{m-1} \left(\sum_{k=1}^{N-i} p_k (x_N - x_k)^{\{i\}} \right) \nabla_{(i)} f(x_{N-i})$, $B = \sum_{k=1}^{i} p_k (x_{i+m} - x_k)^{\{m\}}$. Thus, identity (3.1.4) is proved. ∎

From identity (3.1.4) we can obtain the following result about necessary and sufficient conditions that inequality $\sum_{k=1}^{N} p_k f(x_k) \geq 0$ holds for every $\nabla-$convex function of order m.

Theorem 3.1.2. *Let the assumptions of Theorem* 3.1.1 *be valid and let* $x_1 < x_2 < \cdots < x_N$*. Then the inequality*

$$\sum_{k=1}^{N} p_k f(x_k) \geq 0 \tag{3.1.5}$$

holds for every $\nabla-$*convex function* f *of order* m *if and only if*

$$\sum_{k=1}^{N-i} p_k (x_N - x_k)^{\{i\}} = 0, \qquad i \in \{0, \ldots, m-1\}, \tag{3.1.6}$$

$$\sum_{k=1}^{i} p_k (x_{i+m-1} - x_k)^{\{m-1\}} \geq 0, \qquad i \in \{1, \ldots, N-m\}. \tag{3.1.7}$$

Proof. If the inequalities (3.1.6) and (3.1.7) are satisfied, then the first sum in identity (3.1.4) is equal to 0, the second sum is nonnegative and hence the inequality (3.1.5) holds.

Conversely, if for each $\nabla-$convex functions of order m inequality (3.1.5) holds, then we consider the functions $h^1(x) = x^r$ and $h^2(x) = -x^r$, $r \leq m-1$. Functions h^1 and h^2 are $\nabla-$convex functions of order m and for $r \leq m-1$, we have

$$\sum_{k=1}^{N} p_k x_k^r = 0.$$

From this equality we obtain (3.1.6). For each $i \in \{1, \ldots, N-m\}$, $m > 1$, the function

$$h^3(x) = \begin{cases} (x_{i+1} - x) \cdots (x_{i+m-1} - x) & , \ x < x_{i+1} \\ 0 & , \ x \geq x_{i+1} \end{cases}$$

is $\nabla-$convex of order m and using these facts we obtain (3.1.7). ∎

The next theorem is a generalization of the result from [70, pp. 121-122].

Theorem 3.1.3. *Let the assumptions of Theorem* 3.1.1 *be valid and let* $x_1 < x_2 < \cdots < x_N$*. Then the following statements are valid:*

(*a*) *The inequality*

$$\sum_{k=1}^{N} p_k f(x_k) \geq 0$$

holds for every convex function f of order $r, r+1, \ldots, m$ for $r \in \{0, \ldots, m\}$ if and only if

$$\sum_{k=i+1}^{N} p_k (x_k - x_1)^{(i)} = 0, \qquad i \in \{0, \ldots, r-1\}, \tag{3.1.8}$$

$$\sum_{k=i+1}^{N} p_k (x_k - x_1)^{(i)} \geq 0, \qquad i \in \{r, \ldots, m-1\}, \tag{3.1.9}$$

$$\sum_{k=i}^{N} p_k (x_k - x_{i-m+1})^{(m-1)} \geq 0, \qquad i \in \{m+1, \ldots, N\}.$$

For $r = 0$ (or $r = m$), condition (3.1.8) (or (3.1.9)) can be omitted.

(*b*) *The inequality*

$$\sum_{k=1}^{N} p_k f(x_k) \geq 0$$

holds for every $\nabla-$convex function f of order $r, r+1, \ldots, m$ for $r \in \{0, \ldots, m\}$ if and only if

$$\sum_{k=1}^{N-i} p_k (x_N - x_k)^{\{i\}} = 0, \qquad i \in \{0, \ldots, r-1\}, \tag{3.1.10}$$

$$\sum_{k=1}^{N-i} p_k (x_N - x_k)^{\{i\}} \geq 0, \qquad i \in \{r, \ldots, m-1\}, \tag{3.1.11}$$

$$\sum_{k=1}^{i} p_k (x_{i+m-1} - x_k)^{\{m-1\}} \geq 0, \qquad i \in \{1, \ldots, N-m\}.$$

For $r = 0$ (or $r = m$), condition (3.1.10) (or (3.1.11)) can be omitted.

Remark 3.1.1. Since the proof of this theorem is similar to the proof of Theorem 3.1.2 so we omit the details. The result for the special case $f(x_k) = a_k$ can be found in [82], see also [80, p. 257]. □

3.2 Discrete Identity and Inequality for Functions of n Variables

For our main theorems of this section we define some notations to be used as follows.

Let for $r \in \{0,\dots,n\}$, $j \in \{1,\dots,n\}$, ${}^nC_r(i_j,m_j)$ be the set of all $n-$tuples in which on the kth place we put m_k or i_k and r places are filled with constants from the set $\{m_1,\dots,m_n\}$ while on the other $n-r$ places we put variables from the set $\{i_1,\dots,i_n\}$. For example:
${}^nC_1(i_j,m_j) = \{(m_1,i_2,\dots,i_n),(i_1,m_2,\dots,i_n),\dots,(i_1,i_2,\dots,i_{n-1},m_n)\}$,
${}^nC_2(i_j,m_j) = \{(m_1,m_2,i_3,\dots,i_n),(m_1,i_2,m_3,i_4,\dots,i_n),\dots,(m_1,i_2,\dots,i_{n-1},m_n),$
$(i_1,m_2,m_3,i_4,\dots,i_n),\ \dots,(i_1,m_2,i_3,\dots,i_{n-1},m_n),\dots,(i_1,i_2,\dots,i_{n-2},m_{n-1},m_n)\}$.

Note that the number of elements of the class ${}^nC_r(i_j,m_j)$ are equal to the binomial coefficient $\binom{n}{r}$. We introduce $\overline{\Delta}$ involving variables $i_1,\dots,i_n$ and constants $m_1,\dots,m_n$ as follows. For $(i_1,\dots,i_n)\in {}^nC_0(i_j,m_j)$, we have

$$\begin{aligned}\overline{\Delta}(i_1,\dots,i_n) &= \sum_{i_n=0}^{m_n-1}\cdots\sum_{i_1=0}^{m_1-1}\sum_{k_1=i_1+1}^{N_1}\cdots\sum_{k_n=i_n+1}^{N_n} p_{k_1\cdots k_n}\prod_{j=1}^{n}(x_{jk_j}-x_{j1})^{(i_j)}\times\\ &\times\ \Delta_{(i_1,\dots,i_n)}f(x_{11},\dots,x_{n1}),\end{aligned}$$

For $(i_1,\dots,i_{t-1},m_t,i_{t+1},\dots,i_n)\in {}^nC_1(i_j,m_j)$, we have

$$\begin{aligned}&\overline{\Delta}(i_1,\dots,i_{t-1},m_t,i_{t+1},\dots,i_n)\\ &=\sum_{i_n=0}^{m_n-1}\cdots\sum_{i_{t+1}=0}^{m_{t+1}-1}\sum_{i_t=m_t+1}^{N_t}\sum_{i_{t-1}=0}^{m_{t-1}-1}\cdots\sum_{i_1=0}^{m_1-1}\sum_{k_1=i_1+1}^{N_1}\cdots\sum_{k_{t-1}=i_{t-1}+1}^{N_{t-1}}\sum_{k_t=i_t}^{N_t}\sum_{k_{t+1}=i_{t+1}+1}^{N_{t+1}}\cdots\times\\ &\times\sum_{k_n=i_n+1}^{N_n}p_{k_1\cdots k_n}\Big(\prod_{j=1,j\neq t}^{n}(x_{jk_j}-x_{j1})^{(i_j)}\Big)(x_{tk_t}-x_{t(i_t-m_t+1)})^{(m_t-1)}\times\\ &\times(x_{ti_t}-x_{t(i_t-m_t)})\Delta_{(i_1,\dots,i_{t-1},m_t,i_{t+1},\dots,i_n)}f(x_{11},\dots,x_{(t-1)1},x_{t(i_t-m_t)},x_{(t+1)1},\dots,x_{n1}).\end{aligned}$$

In general, for $(i_1,\dots,i_{s-1},m_s,i_{s+1},\dots,i_{t-1},m_t,i_{t+1},\dots,i_n)\in {}^nC_r(i_j,m_j)$, we have

$$\begin{aligned}&\overline{\Delta}(i_1,\dots,i_{s-1},m_s,i_{s+1},\dots,i_{t-1},m_t,i_{t+1},\dots,i_n)\\ &=\sum_{i_n=0}^{m_n-1}\cdots\sum_{i_{t+1}=0}^{m_{t+1}-1}\sum_{i_t=m_t+1}^{N_t}\sum_{i_{t-1}=0}^{m_{t-1}-1}\cdots\sum_{i_{s+1}=0}^{m_{s+1}-1}\sum_{i_t=m_s+1}^{N_s}\sum_{i_{s-1}=0}^{m_{s-1}-1}\cdots\sum_{i_1=0}^{m_1-1}\times\\ &\times\sum_{k_1=i_1+1}^{N_1}\cdots\sum_{k_{s-1}=i_{s-1}+1}^{N_{s-1}}\sum_{k_s=i_s}^{N_s}\sum_{k_{s+1}=i_{s+1}+1}^{N_{s+1}}\cdots\sum_{k_{t-1}=i_{t-1}+1}^{N_{t-1}}\sum_{k_t=i_t}^{N_t}\sum_{k_{t+1}=i_{t+1}+1}^{N_{t+1}}\cdots\sum_{k_n=i_n+1}^{N_n}\times\\ &\times p_{k_1\cdots k_n}\prod_{j=1,j\notin I_r}^{n}(x_{jk_j}-x_{j1})^{(i_j)}\prod_{j\in I_r}(x_{jk_j}-x_{j(i_j-m_j+1)})^{(m_j-1)}(x_{ji_j}-x_{j(i_j-m_j)})\times\\ &\times\Delta_{(i_1,\dots,i_{s-1},m_s,i_{s+1},\dots,i_{t-1},m_t,i_{t+1},\dots,i_n)}\times\\ &\times f(x_{11},\dots,x_{(s-1)1},x_{s(i_s-m_s)},x_{(s+1)1},\dots,x_{(t-1)1},x_{t(i_t-m_t)},x_{(t+1)1},\dots,x_{n1})\end{aligned}$$

where I_r is a set of all r indices $s, \ldots, t$ of used constants $m_s, \ldots, m_t$.

Finally, for $(m_1, \ldots, m_n) \in {}^nC_n(i_j, m_j)$, we have

$$\begin{aligned}&\overline{\Delta}(m_1, \ldots, m_n)\\&= \sum_{i_n=m_n+1}^{N_n} \cdots \sum_{i_1=m_1+1}^{N_1} \sum_{k_1=i_1}^{N_1} \cdots \sum_{k_n=i_n}^{N_n} p_{k_1\cdots k_n} \Delta_{(m_1,\ldots,m_n)} f\big(x_{1(k_1-m_1)}, \ldots, x_{n(k_n-m_n)}\big) \times\\&\times \prod_{j=1}^{n} \Big((x_{jk_j} - x_{j(i_j-m_j+1)})^{(m_j-1)} (x_{ji_j} - x_{j(i_j-m_j)}) \Big)\end{aligned}$$

The following theorem gives an identity for sum $\sum \cdots \sum p_{k_1\cdots k_n} f(x_{1k_1}, \ldots, x_{nk_n})$ involving n variables.

Theorem 3.2.1. *Let $f : I_1 \times \cdots \times I_n \to \mathbb{R}$ be a function. Let $p_{k_1\ldots k_n} \in \mathbb{R}$ and let $x_{jk_j} \in I_j$ be distinct real numbers for $k_j \in \{1, \ldots, N_j\}$, $j \in \{1, \ldots, n\}$, where $I_j = [a_j, b_j] \subset \mathbb{R}$. Then, we have*

$$\sum_{k_1=1}^{N_1} \cdots \sum_{k_n=1}^{N_n} p_{k_1\cdots k_n} f(x_{1k_1}, \ldots, x_{nk_n}) = \sum_{r=0}^{n} \sum_{(p_1,\ldots,p_n)\in {}^nC_r(i_j,m_j)} \overline{\Delta}(p_1, \ldots, p_n). \quad (3.2.1)$$

Proof. We start with considering

$$\sum_{k_1=1}^{N_1} \cdots \sum_{k_n=1}^{N_n} p_{k_1\cdots k_n} f(x_{1k_1}, \ldots, x_{nk_n}) = \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-1}=1}^{N_{n-1}} \Big[\sum_{k_n=1}^{N_n} Q_{k_n}^{(1,1)} F_{x_{nk_n}}^{(1,1)}(x_{nk_n}) \Big]$$

where $Q_{k_n}^{(1,1)} = p_{k_1\cdots k_n}$ and $F_{x_{nk_n}}^{(1,1)}(x_{nk_n}) = f(x_{1k_1}, \ldots, x_{nk_n})$ where $Q_{k_n}^{(1,1)}$ represents that this function only depends on k_n and independent of other $n-1$ variables. Similarly $F_{x_{nk_n}}^{(1,1)}$ represents that this is only function of variable x_{nk_n} and independent of other $n-1$ variables. So using Proposition 3.1.3 we get,

$$\begin{aligned}&\sum_{k_1=1}^{N_1} \cdots \sum_{k_n=1}^{N_n} p_{k_1\cdots k_n} f(x_{1k_1}, \ldots, x_{nk_n})\\&= \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-1}=1}^{N_{n-1}} \Big[\sum_{i_n=0}^{m_n-1} \Big(\sum_{k_n=i_n+1}^{N_n} Q_{k_n}^{(1,1)} (x_{nk_n} - x_{n1})^{(i_n)} \Delta_{(i_n)} F_{x_{nk_n}}^{(1,1)}(x_{n1})\\&+ \sum_{i_n=m_n+1}^{N_n} \sum_{k_n=i_n}^{N_n} Q_{k_n}^{(1,1)} (x_{nk_n} - x_{n(i_n-m_n+1)})^{(m_n-1)} \Big) \times\\&\times \Delta_{(m_n)} F_{x_{nk_n}}^{(1,1)}(x_{n(i_n-m_n)}) (x_{ni_n} - x_{n(i_n-m_n)}) \Big]\end{aligned}$$

$$= \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{i_n=0}^{m_n-1} \Big[\sum_{k_{n-1}=1}^{N_{n-1}} \Big(\sum_{k_n=i_n+1}^{N_n} p_{k_1\cdots k_n}(x_{nk_n} - x_{n1})^{(i_n)} \Big) \times$$
$$\times \Delta_{(i_n)} f(x_{1k_1}, \ldots, x_{(n-1)k_{n-1}}, x_{n1}) \Big]$$
$$+ \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{i_n=m_n+1}^{N_n} \Big[\sum_{k_{n-1}=1}^{N_{n-1}} \Big(\sum_{k_n=i_n}^{N_n} p_{k_1\cdots k_n}(x_{nk_n} - x_{n(i_n-m_n+1)})^{(m_n-1)} \times$$
$$\times (x_{ni_n} - x_{n(i_n-m_n)}) \Big) \Delta_{(m_n)} f(x_{1k_1}, \ldots, x_{(n-1)k_{n-1}}, x_{n(i_n-m_n)}) \Big]$$
$$= \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{i_n=0}^{m_n-1} \Big[\sum_{k_{n-1}=1}^{N_{n-1}} Q^{(2,1)}_{k_{n-1}} F^{(2,1)}_{x_{(n-1)k_{n-1}}} \Big]$$
$$+ \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{i_n=m_n+1}^{N_n} \Big[\sum_{k_{n-1}=1}^{N_{n-1}} Q^{(2,2)}_{k_{n-1}} F^{(2,2)}_{x_{n-1k_{n-1}}} \Big]$$

where

$$\begin{aligned} Q^{(2,1)}_{k_{n-1}} &= \sum_{k_n=i_n+1}^{N_n} p_{k_1\cdots k_n}(x_{nk_n} - x_{n1})^{(i_n)}, \\ Q^{(2,2)}_{k_{n-1}} &= \sum_{k_n=i_n}^{N_n} p_{k_1\cdots k_n}(x_{nk_n} - x_{n(i_n-m_n+1)})^{(m_n-1)}(x_{ni_n} - x_{n(i_n-m_n)}), \\ F^{(2,1)}_{x_{(n-1)k_{n-1}}}(x_{(n-1)k_{n-1}}) &= \Delta_{(i_n)} f(x_{1k_1}, \ldots, x_{(n-1)k_{n-1}}, x_{n1}), \\ F^{(2,2)}_{x_{(n-1)k_{n-1}}}(x_{(n-1)k_{n-1}}) &= \Delta_{(m_n)} f(x_{1k_1}, \ldots, x_{(n-1)k_{n-1}}, x_{n(i_n-m_n)}). \end{aligned}$$

Note that, this time we assume $Q^{(2,1)}_{k_{n-1}}$ to be only dependent on k_{n-1}, whereas $F^{(2,1)}_{x_{(n-1)k_{n-1}}}$ is considered to be a function of variable $x_{(n-1)k_{n-1}}$ as far as $Q^{(2,2)}_{k_{n-1}}$ is concerned, it only depends on k_{n-1} and $F^{(2,2)}_{x_{(n-1)k_{n-1}}}$ is function of single variable $x_{(n-1)k_{n-1}}$.

So, again applying Proposition 3.1.3, we have

$$\sum_{k_1=1}^{N_1} \cdots \sum_{k_n=1}^{N_n} p_{k_1\cdots k_n} f(x_{1k_1}, \ldots, x_{nk_n})$$
$$= \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{i_n=0}^{m_n-1} \Big[\sum_{i_{n-1}=0}^{m_{n-1}-1} \sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}} Q^{(2,1)}_{k_{n-1}}(x_{(n-1)k_{n-1}} - x_{(n-1)1})^{(i_{n-1})} \times$$
$$\times \Delta_{(i_{n-1})} F^{(2,1)}_{x_{(n-1)k_{n-1}}}(x_{(n-1)1})$$

$$+\sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}}\sum_{k_{n-1}=i_{n-1}}^{N_{n-1}} Q^{(2,1)}_{k_{n-1}}\big(x_{(n-1)k_{n-1}}-x_{(n-1)(i_{n-1}-m_{n-1}+1)}\big)^{(m_{n-1}-1)}\times$$

$$\times\,\Delta_{(i_{n-1})}F^{(2,1)}_{x_{(n-1)k_{n-1}}}\big(x_{(n-1)(i_{n-1}-m_{n-1})}\big)\big(x_{(n-1)(i_{n-1})}-x_{(n-1)(i_{n-1}-m_{n-1})}\big)\Big]$$

$$+\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{i_n=m_n+1}^{N_n}\Big[\sum_{i_{n-1}=0}^{m_{n-1}-1}\sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}} Q^{(2,2)}_{k_{n-1}}\big(x_{(n-1)k_{n-1}}-x_{(n-1)1}\big)^{(i_{n-1})}\times$$

$$\times\,\Delta_{(i_{n-1})}F^{(2,2)}_{x_{(n-1)k_{n-1}}}\big(x_{(n-1)1}\big)$$

$$+\sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}}\sum_{k_{n-1}=i_{n-1}}^{N_{n-1}} Q^{(2,2)}_{k_{n-1}}\big(x_{(n-1)k_{n-1}}-x_{(n-1)(i_{n-1}-m_{n-1}+1)}\big)^{(m_{n-1}-1)}\times$$

$$\times\,\Delta_{(m_{n-1})}F^{(2,2)}_{x_{(n-1)k_{n-1}}}\big(x_{(n-1)(i_{n-1}-m_{n-1})}\big)\big(x_{(n-1)(i_{n-1})}-x_{(n-1)(i_{n-1}-m_{n-1})}\big)\Big]$$

$$=\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-3}=1}^{N_{n-3}}\sum_{i_n=0}^{m_n-1}\sum_{i_{n-1}=0}^{m_{n-1}-1}\Big[\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}} Q^{(2,1)}_{k_{n-1}}\big(x_{(n-1)k_{n-1}}-x_{(n-1)1}\big)^{(i_{n-1})}\times$$

$$\times\,\Delta_{(i_{n-1})}F^{(2,1)}_{x_{(n-1)k_{n-1}}}\big(x_{(n-1)1}\big)\Big]+\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-3}=1}^{N_{n-3}}\sum_{i_n=0}^{m_n-1}\sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}}\times$$

$$\times\Big[\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{k_{n-1}=i_{n-1}}^{N_{n-1}} Q^{(2,1)}_{k_{n-1}}\big(x_{(n-1)k_{n-1}}-x_{(n-1)(i_{n-1}-m_{n-1}+1)}\big)^{(m_{n-1}-1)}\times$$

$$\times\,\Delta_{(m_{n-1})}F^{(2,1)}_{x_{(n-1)k_{n-1}}}\big(x_{(n-1)(i_{n-1}-m_{n-1})}\big)\big(x_{(n-1)(i_{n-1})}-x_{(n-1)(i_{n-1}-m_{n-1})}\big)\Big]$$

$$+\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-3}=1}^{N_{n-3}}\sum_{i_n=m_n+1}^{N_n}\sum_{i_{n-1}=0}^{m_{n-1}-1}\Big[\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}} Q^{(2,2)}_{k_{n-1}}\big(x_{(n-1)k_{n-1}}-x_{(n-1)1}\big)^{(i_{n-1})}\times$$

$$\times\,\Delta_{(i_{n-1})}F^{(2,2)}_{x_{(n-1)k_{n-1}}}\big(x_{(n-1)1}\big)\Big]+\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-3}=1}^{N_{n-3}}\sum_{i_n=m_n+1}^{N_n}\sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}}\times$$

$$\times\Big[\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{k_{n-1}=i_{n-1}}^{N_{n-1}} Q^{(2,2)}_{k_{n-1}}\big(x_{(n-1)k_{n-1}}-x_{(n-1)(i_{n-1}-m_{n-1}+1)}\big)^{(m_{n-1}-1)}\times$$

$$\times\,\Delta_{(m_{n-1})}F^{(2,2)}_{x_{(n-1)k_{n-1}}}\big(x_{(n-1)(i_{n-1}-m_{n-1})}\big)\big(x_{(n-1)(i_{n-1})}-x_{(n-1)(i_{n-1}-m_{n-1})}\big)\Big]$$

$$=\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-3}=1}^{N_{n-3}}\sum_{i_n=0}^{m_n-1}\sum_{i_{n-1}=0}^{m_{n-1}-1}\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}}\sum_{k_n=i_n+1}^{N_n} p_{k_1\cdots k_n}\big(x_{nk_n}-x_{n1}\big)^{(i_n)}\times$$

$$\times\,\big(x_{(n-1)k_{n-1}}-x_{(n-1)1}\big)^{(i_{n-1})}\Delta_{(i_{n-1},i_n)}f\big(x_{1k_1},\ldots,x_{(n-2)k_{n-2}},x_{(n-1)1},x_{n1}\big)$$

$$+\sum_{k_1=1}^{N_1}\cdots\sum_{k_{n-3}=1}^{N_{n-3}}\sum_{i_n=0}^{m_n-1}\sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}}\sum_{k_{n-2}=1}^{N_{n-2}}\sum_{k_{n-1}=i_{n-1}}^{N_{n-1}}\sum_{k_n=i_n+1}^{N_n} p_{k_1\cdots k_n}\big(x_{nk_n}-x_{n1}\big)^{(i_n)}\times$$

$$\times \left(x_{(n-1)k_{n-1}} - x_{(n-1)(i_{n-1}-m_{n-1}+1)}\right)^{(m_{n-1}-1)} \left(x_{(n-1)(i_{n-1})} - x_{(n-1)(i_{n-1}-m_{n-1})}\right)\times$$
$$\times \Delta_{(m_{n-1},i_n)} f\left(x_{1k_1}, \dots, x_{(n-2)k_{n-2}}, x_{(n-1)(i_{n-1}-m_{n-1})}, x_{n1}\right)$$
$$+ \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-3}=1}^{N_{n-3}} \sum_{i_n=m_n+1}^{N_n} \sum_{i_{n-1}=0}^{m_{n-1}-1} \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{k_{n-1}=i_{n-1}+1}^{N_{n-1}} \sum_{k_n=i_n}^{N_n} p_{k_1\cdots k_n}\times$$
$$\times \left(x_{nk_n} - x_{n(i_n-m_n+1)}\right)^{(m_n-1)} \left(x_{ni_n} - x_{n(i_n-m_n)}\right)\left(x_{(n-1)k_{n-1}} - x_{(n-1)1}\right)^{(i_{n-1})}\times$$
$$\times \Delta_{(i_{n-1},m_n)} f\left(x_{1k_1}, \dots, x_{(n-2)k_{n-2}}, x_{(n-1)1}, x_{n(i_n-m_n)}\right)$$
$$+ \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n-3}=1}^{N_{n-3}} \sum_{i_n=m_n+1}^{N_n} \sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}} \sum_{k_{n-2}=1}^{N_{n-2}} \sum_{k_{n-1}=i_{n-1}}^{N_{n-1}} \sum_{k_n=i_n}^{N_n} p_{k_1\cdots k_n}\times$$
$$\times \left(x_{nk_n} - x_{n(i_n-m_n+1)}\right)^{(m_n-1)} \left(x_{(n-1)k_{n-1}} - x_{(n-1)(i_{n-1}-m_{n-1}+1)}\right)^{(m_{n-1}-1)}\times$$
$$\times \Delta_{(m_{n-1},m_n)} f\left(x_{1k_1}, \dots, x_{(n-2)k_{n-2}}, x_{(n-1)(i_{n-1}-m_{n-1})}, x_{n(i_n-m_n)}\right)\times$$
$$\times \left(x_{ni_n} - x_{n(i_n-m_n)}\right)\left(x_{(n-1)(i_{n-1})} - x_{(n-1)(i_{n-1}-m_{n-1})}\right).$$

Continuing in the similar fashion we finally get identity (3.2.1). ■

Remark 3.2.1. If we set $n = 2$ in previous theorem then we get following corollary, which we will use frequently in other results. □

Corollary 3.2.2. *Let $f : I \times J \to \mathbb{R}$ be a function and let $p_{ij} \in \mathbb{R}$ for $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$. Then the following identity holds*

$$\sum_{i=1}^{N}\sum_{j=1}^{M} p_{ij} f(x_i, y_j)$$
$$= \sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\sum_{s=t+1}^{N}\sum_{r=k+1}^{M} p_{sr}(x_s - x_1)^{(t)}(y_r - y_1)^{(k)}\Delta_{(t,k)} f(x_1, y_1)$$
$$+ \sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\sum_{s=t}^{N}\sum_{r=k+1}^{M} p_{sr}(x_s - x_{t-n+1})^{(n-1)}(y_r - y_1)^{(k)}\Delta_{(n,k)} f(x_{t-n}, y_1)(x_t - x_{t-n})$$
$$+ \sum_{k=m+1}^{M}\sum_{t=0}^{n-1}\sum_{s=t+1}^{N}\sum_{r=k}^{M} p_{sr}(x_s - x_1)^{(t)}(y_r - y_{k-m+1})^{(m-1)}\Delta_{(t,m)} f(x_1, y_{k-m})(y_k - y_{k-m})$$
$$+ \sum_{k=m+1}^{M}\sum_{t=n+1}^{N}\sum_{s=t}^{N}\sum_{r=k}^{M} p_{sr}(x_s - x_{t-n+1})^{(n-1)}(y_r - y_{k-m+1})^{(m-1)}\times$$
$$\times \Delta_{(n,m)} f(x_{t-n}, y_{k-m})(x_t - x_{t-n})(y_k - y_{k-m}) \tag{3.2.2}$$

where $(x_i, y_j) \in I \times J$ *are distinct points for* $i \in \{1, \ldots, N\}$ *and* $j \in \{1, \ldots, M\}$.

Remark 3.2.2. In [82] some results for two sequences were proved. If in Corollary 3.2.2 we simply put $f(x_i, y_j) = f(x_i)g(y_i)$, then we obtain the similar statement for two functions f and g as follows. □

Corollary 3.2.3. *Let* $f : I \to \mathbb{R}$ *and* $g : J \to \mathbb{R}$ *be two functions and let* $p_{ij} \in \mathbb{R}$ *for* $i \in \{1, \ldots, N\}$ *and* $j \in \{1, \ldots, M\}$. *Then the following identity holds*

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{M} p_{ij} f(x_i, y_j) \\
&= \sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\sum_{s=t+1}^{N}\sum_{r=k+1}^{M} p_{sr}(x_s - x_1)^{(t)}\Delta_{(t)} f(x_1)(y_r - y_1)^{(k)}\Delta_{(k)} g(y_1) \\
&+ \sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\sum_{s=t}^{N}\sum_{r=k+1}^{M} p_{sr}(x_s - x_{t-n+1})^{(n-1)}\Delta_{(n)} f(x_{t-n})(x_t - x_{t-n})(y_r - y_1)^{(k)}\Delta_{(k)} g(y_1) \\
&+ \sum_{k=m+1}^{M}\sum_{t=0}^{n-1}\sum_{s=t+1}^{N}\sum_{r=k}^{M} p_{sr}(x_s - x_1)^{(t)} \times \\
&\times \Delta_{(t)} f(x_1)(y_r - y_{k-m+1})^{(m-1)}\Delta_{(m)} g(y_{k-m})(y_k - y_{k-m}) \\
&+ \sum_{k=m+1}^{M}\sum_{t=n+1}^{N}\sum_{s=t}^{N}\sum_{r=k}^{M} p_{sr}(x_s - x_{t-n+1})^{(n-1)}\Delta_{(n)} f(x_{t-n})(x_t - x_{t-n}) \times \\
&\times (y_r - y_{k-m+1})^{(m-1)}\Delta_{(m)} g(y_{k-m})(y_k - y_{k-m})
\end{aligned}$$

where $(x_i, y_j) \in I \times J$ *are distinct points for* $i \in \{1, \ldots, N\}$ *and* $j \in \{1, \ldots, M\}$.

Theorem 3.2.4. *Let the assumptions of Theorem* 3.2.1 *be valid. Then the inequality*

$$\sum_{k_1=1}^{N_1} \cdots \sum_{k_n=1}^{N_n} p_{k_1 \cdots k_n} f(x_{1k_1}, \ldots, x_{nk_n}) \geq 0 \tag{3.2.3}$$

holds for every convex function f *of order* $(m_1, \ldots, m_n)$ *if and only if*

$$\sum_{k_1=i_1+1}^{N_1} \cdots \sum_{k_n=i_n+1}^{N_n} p_{k_1 \cdots k_n} \prod_{j=1}^{n} (x_{jk_j} - x_{j1})^{(i_j)} = 0, \tag{3.2.4}$$

$\forall\, i_1 \in \{0, \ldots, m_1 - 1\},\ \ldots,\ i_n \in \{0, \ldots, m_n - 1\}$,

$$\sum_{k_1=i_1+1}^{N_1} \cdots \sum_{k_n=i_n+1}^{N_n} p_{k_1 \cdots k_n} (x_{1k_1} - x_{1(i_1-m_1+1)})^{(m_1-1)} \prod_{j=2}^{n} (x_{jk_j} - x_{j1})^{(i_j)} = 0, \tag{3.2.5}$$

$\forall\ i_1 \in \{m_1+1,\dots,N_1\},\ i_2 \in \{0,\dots,m_2-1\},\ \dots,\ i_n \in \{0,\dots,m_n-1\},$

$$\vdots$$

$$\sum_{i_1=m_1+1}^{N_1} \cdots \sum_{i_{n-1}=m_{n-1}+1}^{N_{n-1}} \sum_{k_n=i_n}^{N_n} p_{k_1\cdots k_n} \prod_{j=1}^{n-1} (x_{jk_j} - x_{j1})^{(i_j)} (x_{nk_n} - x_{n(i_n-m_n+1)})^{(m_n-1)} = 0, \tag{3.2.6}$$

$\forall\ i_1 \in \{0,\dots,m_1-1\},\ \dots,\ i_{n-1} \in \{0,\dots,m_{n-1}-1\},\ i_n \in \{m_n+1,\dots,N_n\},$

$$\vdots$$

$$\sum_{k_1=i_1+1}^{N_1} \cdots \sum_{k_n=i_n+1}^{N_n} p_{k_1\cdots k_n} \prod_{j=1}^{n} (x_{jk_j} - x_{j(i_j-m_j+1)})^{(m_j-1)} \geq 0, \tag{3.2.7}$$

$\forall\ i_1 \in \{m_1+1,\dots,N_1\},\ \dots,\ i_n \in \{m_n+1,\dots,N_n\}.$

Proof. If (3.2.4), (3.2.5), ..., (3.2.6) hold then all these sums are zero in (3.2.1) and the required inequality (3.2.3) holds by using (3.2.7).

Conversely, let (3.2.3) hold for every convex function f of order $(m_1,\dots,m_n)$. Let us consider the following functions

$$f^1(x_{1k_1},\dots,x_{nk_n}) \;=\; \prod_{j=1}^{n} (x_{jk_j} - x_{j1})^{(i_j)} \quad \text{and} \quad f^2 = -f^1,$$

for $i_1 \in \{0,\dots,m_1-1\},\ \dots,\ i_n \in \{0,\dots,m_n-1\}$. Since these functions are convex of order $(m_1,\dots,m_n)$, so by (3.2.3) the inequalities

$$\sum_{k_1=1}^{N_1} \cdots \sum_{k_n=1}^{N_n} p_{k_1\cdots k_n} f^k(x_{1k_1},\dots,x_{nk_n}) \geq 0 \quad \text{for} \quad k \in \{1,2\}$$

hold and we get required inequality (3.2.4). In the same way if we consider the following functions for $i_1 \in \{m_1+1,\dots,N_1\},\ i_2 \in \{0,\dots,m_2-1\},\ \dots,\ i_n \in \{0,\dots,m_n-1\}$

$$f^3(x_{1k_1},\dots,x_{nk_n}) = \begin{cases} (x_{1k_1} - x_{1(i_1-m_1+1)})^{(m_1-1)} \prod_{j=2}^{n} (x_{jk_j} - x_{j1})^{(i_j)} & , \quad x_{1(i_1-1)} < x_{1k_1}, \\ 0 & , \quad x_{1(i_1-1)} \geq x_{1k_1}, \end{cases}$$

and
$$f^4 = -f^3$$

such that $\Delta_{(m_1,\dots,m_n)} f^k \geq 0$ for $k \in \{3,4\}$, then we get the required equality (3.2.5).

Similarly, if we consider in (3.2.3) the following functions for $i_1 \in \{0,\dots,m_1 -$

$1\}, \ \ldots, \ i_{n-1} \in \{0, \ldots, m_{n-1}-1\}, \ i_n \in \{m_n+1, \ldots, N_n\}$

$$f^5(x_{1k_1}, \ldots, x_{nk_n}) = \begin{cases} (x_{nk_n} - x_{n(i_n-m_n+1)})^{(m_n-1)} \prod_{j=1}^{n-1}(x_{jk_j} - x_{j1})^{(i_j)} & , \quad x_{n(i_n-1)} < x_{nk_n}, \\ 0 & , \quad x_{n(i_n-1)} \geq x_{nk_n}, \end{cases}$$

and
$$f^6 = -f^5$$

such that $\Delta_{(m_1,\ldots,m_n)} f_k \geq 0 \quad \text{for} \quad k \in \{5,6\}$, then we get the required equality (3.2.6) and so on.

The last inequality (3.2.7) is followed by considering the following function in (3.2.3) for $i_1 \in \{m_1+1, \ldots, N_1\}, \ \ldots, \ i_n \in \{m_n+1, \ldots, N_n\}$.

$$f^7(x_{1k_1}, \ldots, x_{nk_n}) = \begin{cases} \prod_{j=1}^{n}(x_{jk_j} - x_{j(i_j-m_j+1)})^{(m_j-1)} & , \quad x_{1(i_1-1)} < x_{1k_1}, \ldots, x_{n(i_n-1)} < x_{nk_n}, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

■

Corollary 3.2.5. *Let $f : I \times J \to \mathbb{R}$ be a function and let $p_{ij} \in \mathbb{R}$ for $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, M\}$. For real numbers $x_1 < x_2 < \ldots < x_N$, $x_i \in I$, $y_1 < y_2 < \ldots < y_M$, $y_j \in J$, the inequality*

$$\sum_{i=1}^{N} \sum_{j=1}^{M} p_{ij} f(x_i, y_j) \geq 0$$

holds for every convex function f of order (n,m) if and only if

$$\begin{aligned}
\sum_{s=t+1}^{N} \sum_{r=k+1}^{M} p_{sr}(x_s - x_1)^{(t)}(y_r - y_1)^{(k)} &= 0, & & k \in \{0, \ldots, m-1\} \\ & & & t \in \{0, \ldots, n-1\} \\
\sum_{s=t}^{N} \sum_{r=k+1}^{M} p_{sr}(x_s - x_{t-n+1})^{(n-1)}(y_r - y_1)^{(k)} &= 0, & & k \in \{0, \ldots, m-1\} \\ & & & t \in \{n+1, \ldots, N\} \\
\sum_{s=t+1}^{N} \sum_{r=k}^{M} p_{sr}(x_s - x_1)^{(t)}(y_r - y_{k-m+1})^{(m-1)} &= 0, & & k \in \{m+1, \ldots, M\} \\ & & & t \in \{0, \ldots, n-1\} \\
\sum_{s=t}^{N} \sum_{r=k}^{M} p_{sr}(x_s - x_{t-n+1})^{(n-1)}(y_r - y_{k-m+1})^{(m-1)} &\geq 0, & & k \in \{m+1, \ldots, M\} \\ & & & t \in \{n+1, \ldots, N\}.
\end{aligned}$$

Remark 3.2.3. The case when $f(x_i, y_j) = a_{ij}$ for $i \in \{1, \ldots, N\}$, $j \in \{1, \ldots, M\}$ and $m = n = 1$ was considered in [75]. The case when $f(x_i, y_j) = a_i b_j$, where (a_i) for $i \in \{1, \ldots, N\}$ is an $n-$convex sequence and (b_j) for $j \in \{1, \ldots, M\}$ is an $m-$convex sequence was researched in [82]. Also the case $f(x_i, y_j) = a_i b_j$ for monotonic $n-$tuples **a** and **b** was considered by Popoviciu in [84].

□

3.3 Integral Identity and Inequality for Higher Order Differentiable Functions of n Variables

As we done in previous section, for the present section also we introduce some notations to simplify the statement of our main theorems as follows.

For variables $i_1, \ldots, i_n$ and constants $m_1+1, \ldots, m_n+1$ we define $\tilde{\Delta}$ in the following way:

$$\tilde{\Delta}(i_1, \ldots, i_n) = \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} p(x_1, \ldots, x_n) f_{(i_1, \ldots, i_n)}(a_1, \ldots, a_n) \times$$
$$\times \prod_{j=1}^{n} \frac{(y_j - a_j)^{i_j}}{i_j!} dy_n \cdots dy_1,$$

and

$$\tilde{\Delta}(i_1, \ldots, i_{k-1}, m_k, i_{k+1}, \ldots, i_n) =$$
$$\sum_{i_1=0}^{m_1} \cdots \sum_{i_{k-1}=0}^{m_{k-1}} \sum_{i_{k+1}=0}^{m_{k+1}} \cdots \sum_{i_n=0}^{m_n} \int_{a_k}^{b_k} \int_{a_1}^{b_1} \cdots \int_{a_{k-1}}^{b_{k-1}} \int_{x_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} \cdots \int_{a_n}^{b_n} p(x_1, \ldots, x_n) \times$$
$$\times f_{(i_1, \ldots, i_{k-1}, m_k+1, i_{k+1}, \ldots, i_n)} \frac{(y_k - x_k)^{m_k}}{m_k!} \prod_{j=1,\, j\neq k}^{n} \frac{(y_j - a_j)^{i_j}}{i_j!} dy_n \cdots dy_1 dx_k.$$

Similarly, we can define $\tilde{\Delta}$ for any $n-$tuple from ${}^nC_r(i_j, m_j)$ (where ${}^nC_r(i_j, m_j)$ was introduced in the start of previous section) for some $j \in \{1, \ldots, n\}$ and finally we define

$$\begin{aligned} \tilde{\Delta}(m_1, \ldots, m_n) &= \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} p(x_1, \ldots, x_n) \times \\ &\times \; f_{(m_1+1, \ldots, m_n+1)}(x_1, \ldots, x_n) \prod_{j=1}^{n} \frac{(y_j - x_j)^{m_j}}{m_j!} dy_n \cdots dy_1 dx_n \cdots dx_1. \end{aligned}$$

Now we are ready to state our main theorems of this section.

Theorem 3.3.1. *Let $p, f : I_1 \times \cdots \times I_n \to \mathbb{R}$ be integrable functions and let $f \in C^{(m_1+1,\ldots,m_n+1)}(I_1 \times \cdots \times I_n)$. Then the identity*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} p(x_1, \ldots, x_n) f(x_1, \ldots, x_n) dx_n \cdots dx_1 = \sum_{r=0}^{n} \sum_{(p_1,\ldots,p_n) \in {}^n C_r(i_j, m_j+1)} \tilde{\Delta}(p_1, \ldots, p_n) \tag{3.3.1}$$

holds.

Proof. We consider the Taylor expansion:

$$\begin{aligned} f(x_1, \ldots, x_n) &= \sum_{i_n=0}^{m_n} f_{(0,\ldots,0,i_n)}(x_1, \ldots, x_{n-1}, a_n) \frac{(x_n - a_n)^{i_n}}{i_n!} \\ &+ \int_{a_n}^{x_n} f_{(0,\ldots,0,m_n+1)}(x_1, \ldots, x_{n-1}, y_n) \frac{(x_n - y_n)^{m_n}}{m_n!} dy_n. \end{aligned}$$

Multiply the above formula with $p(x_1, \ldots, x_n)$ and integrate it over $[a_n, b_n]$ by variable x_n. Then we have

$$\begin{aligned} &\int_{a_n}^{b_n} p(x_1, \ldots, x_n) f(x_1, \ldots, x_n) dx_n \\ &= \sum_{i_n=0}^{m_n} f_{(0,\ldots,0,i_n)}(x_1, \ldots, x_{n-1}, a_n) \int_{a_n}^{b_n} p(x_1, \ldots, x_n) \frac{(x_n - a_n)^{i_n}}{i_n!} dx_n \\ &+ \int_{a_n}^{b_n} \Big(\int_{a_n}^{x_n} p(x_1, \ldots, x_n) f_{(0,\ldots,0,m_n+1)}(x_1, \ldots, x_{n-1}, y_n) \frac{(x_n - y_n)^{m_n}}{m_n!} dy_n \Big) dx_n. \end{aligned} \tag{3.3.2}$$

Let us use the following Taylor expansions:

$$\begin{aligned} &f_{(0,\ldots,0,i_n)}(x_1, \ldots, x_{n-1}, a_n) \\ &= \sum_{i_{n-1}=0}^{m_{n-1}} f_{(0,\ldots,0,i_{n-1},i_n)}(x_1, \ldots, x_{n-2}, a_{n-1}, a_n) \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \\ &+ \int_{a_{n-1}}^{x_{n-1}} f_{(0,\ldots,0,m_{n-1}+1,i_n)}(x_1, \ldots, x_{n-2}, y_{n-1}, a_n) \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} dy_{n-1}, \end{aligned}$$

$$
\begin{aligned}
&f_{(0,\ldots,0,m_n+1)}(x_1,\ldots,x_{n-1},y_n)\\
&=\sum_{i_{n-1}=0}^{m_{n-1}} f_{(0,\ldots,0,i_{n-1},m_n+1)}(x_1,\ldots,x_{n-2},a_{n-1},y_n)\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\\
&+\int_{a_{n-1}}^{x_{n-1}} f_{(0,\ldots,0,m_{n-1}+1,m_n+1)}(x_1,\ldots,x_{n-2},y_{n-1},y_n)\times\\
&\times\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}dy_{n-1}.
\end{aligned}
$$

Putting these two formulae in (3.3.2) and integrate over $[a_{n-1},b_{n-1}]$ by variable x_{n-1}. Then, we have

$$
\begin{aligned}
&\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(x_1,\ldots,x_n)f(x_1,\ldots,x_n)dx_n dx_{n-1}\\
&=\int_{a_{n-1}}^{b_{n-1}}\Big[\sum_{i_n=0}^{m_n}\sum_{i_{n-1}=0}^{m_{n-1}} f_{(0,\ldots,0,i_{n-1},i_n)}(x_1,\ldots,x_{n-2},a_{n-1},a_n)\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\times\\
&\times\int_{a_n}^{b_n} p(x_1,\ldots,x_n)\frac{(x_n-a_n)^{i_n}}{i_n!}dx_n\Big]dx_{n-1}\\
&+\int_{a_{n-1}}^{b_{n-1}}\Big[\sum_{i_n=0}^{m_n}\int_{a_{n-1}}^{x_{n-1}} f_{(0,\ldots,0,m_{n-1}+1,i_n)}(x_1,\ldots,x_{n-2},y_{n-1},a_n)\times\\
&\times\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}dy_{n-1}\int_{a_n}^{b_n} p(x_1,\ldots,x_n)\frac{(x_n-a_n)^{i_n}}{i_n!}dx_n\Big]dx_{n-1}\\
&+\int_{a_{n-1}}^{b_{n-1}}\Big[\int_{a_n}^{b_n}\int_{a_n}^{x_n} p(x_1,\ldots,x_n)\sum_{i_{n-1}=0}^{m_{n-1}} f_{(0,\ldots,0,i_{n-1},m_n+1)}(x_1,\ldots,x_{n-2},a_{n-1},y_n)\times\\
&\times\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dy_n dx_n\Big]dx_{n-1}\\
&+\int_{a_{n-1}}^{b_{n-1}}\Big[\int_{a_n}^{b_n}\int_{a_n}^{x_n} p(x_1,\ldots,x_n)\times\\
&\times\int_{a_{n-1}}^{x_{n-1}} f_{(0,\ldots,0,m_{n-1}+1,m_n+1)}(x_1,\ldots,x_{n-2},y_{n-1},y_n)\times\\
&\times\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dy_{n-1}dy_n dx_n\Big]dx_{n-1}.
\end{aligned}
$$

In the first summand we change the order of summation, use linearity of integral and get

$$\sum_{i_n=0}^{m_n}\sum_{i_{n-1}=0}^{m_{n-1}}\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(x_1,\dots,x_n)f_{(0,\dots,0,i_{n-1},i_n)}(x_1,\dots,x_{n-2},a_{n-1},a_n)\times$$
$$\times\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-a_n)^{i_n}}{i_n!}dx_ndx_{n-1}.$$

The second summand is rewritten as

$$\int_{a_{n-1}}^{b_{n-1}}\Big[\sum_{i_n=0}^{m_n}\int_{a_{n-1}}^{x_{n-1}} f_{(0,\dots,0,m_{n-1}+1,i_n)}(x_1,\dots,x_{n-2},y_{n-1},a_n)\times$$
$$\times\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}dy_{n-1}\int_{a_n}^{b_n} p(x_1,\dots,x_n)\frac{(x_n-a_n)^{i_n}}{i_n!}dx_n\Big]dx_{n-1}$$
$$=\int_{a_{n-1}}^{b_{n-1}}\Big[\sum_{i_n=0}^{m_n}\int_{a_{n-1}}^{x_{n-1}}\int_{a_n}^{b_n} p(x_1,\dots,x_n)\frac{(x_n-a_n)^{i_n}}{i_n!}\times$$
$$\times f_{(0,\dots,0,m_{n-1}+1,i_n)}(x_1,\dots,x_{n-2},y_{n-1},a_n)\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}dx_ndy_{n-1}\Big]dx_{n-1}$$
$$=\sum_{i_n=0}^{m_n}\int_{a_{n-1}}^{b_{n-1}}\int_{a_{n-1}}^{x_{n-1}}\int_{a_n}^{b_n} p(x_1,\dots,x_n)f_{(0,\dots,0,m_{n-1}+1,i_n)}(x_1,\dots,x_{n-2},y_{n-1},a_n)\times$$
$$\times\frac{(x_n-a_n)^{i_n}}{i_n!}\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}dx_ndy_{n-1}dx_{n-1}$$
$$=\sum_{i_n=0}^{m_n}\int_{a_{n-1}}^{b_{n-1}}\int_{y_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(x_1,\dots,x_n)f_{(0,\dots,0,m_{n-1}+1,i_n)}(x_1,\dots,x_{n-2},y_{n-1},a_n)\times$$
$$\times\frac{(x_n-a_n)^{i_n}}{i_n!}\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}dx_ndx_{n-1}dy_{n-1}$$

where in the last equation we used the Fubini theorem for variables y_{n-1} and x_{n-1}. Let us point out that firstly, the variable x_{n-1} is changed from a_{n-1} to b_{n-1} while the variable y_{n-1} is changed from a_{n-1} to x_{n-1}. After changing the order of integration we have that variable y_{n-1} is changed from a_{n-1} to b_{n-1} while the variable x_{n-1} is changed from y_{n-1} to b_{n-1}.

Similarly, the third summand is rewritten as:

$$\int_{a_{n-1}}^{b_{n-1}}\Big[\int_{a_n}^{b_n}\int_{a_n}^{x_n} p(x_1,\dots,x_n)\sum_{i_{n-1}=0}^{m_{n-1}} f_{(0,\dots,0,i_{n-1},m_n+1)}(x_1,\dots,x_{n-2},a_{n-1},y_n)\times$$
$$\times\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dy_ndx_n\Big]dx_{n-1}$$

$$= \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \int_{a_n}^{x_n} p(x_1, \ldots, x_n) f_{(0,\ldots,0,i_{n-1},m_n+1)}(x_1, \ldots, x_{n-2}, a_{n-1}, y_n) \times$$
$$\times \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dy_n dx_n dx_{n-1}$$
$$= \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \int_{y_n}^{b_n} p(x_1, \ldots, x_n) f_{(0,\ldots,0,i_{n-1},m_n+1)}(x_1, \ldots, x_{n-2}, a_{n-1}, y_n) \times$$
$$\times \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dx_n dy_n dx_{n-1}$$
$$= \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_n}^{b_n} \int_{a_{n-1}}^{b_{n-1}} \int_{y_n}^{b_n} p(x_1, \ldots, x_n) f_{(0,\ldots,0,i_{n-1},m_n+1)}(x_1, \ldots, x_{n-2}, a_{n-1}, y_n) \times$$
$$\times \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dx_n dx_{n-1} dy_n$$

where we use the Fubini theorem twice, firstly for changing y_n and x_n and then for y_n and x_{n-1}.
The fourth summand is rewritten as

$$\int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \int_{a_n}^{x_n} \int_{a_{n-1}}^{x_{n-1}} p(x_1, \ldots, x_n) f_{(0,\ldots,0,m_{n-1}+1,m_n+1)}(x_1, \ldots, x_{n-2}, y_{n-1}, y_n) \times$$
$$\times \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dy_{n-1} dy_n dx_n dx_{n-1}$$
$$= \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} \int_{y_{n-1}}^{b_{n-1}} \int_{y_n}^{b} p(x_1, \ldots, x_n) f_{(0,\ldots,0,m_{n-1}+1,m_n+1)}(x_1, \ldots, x_{n-2}, y_{n-1}, y_n) \times$$
$$\times \frac{(x_{n-1} - y_{n-1})^{m_{n-1}}}{m_{n-1}!} \frac{(x_n - y_n)^{m_n}}{m_n!} dx_n dx_{n-1} dy_n dy_{n-1},$$

where we use the Fubini theorem several times. Firstly, we change y_n and x_n, then x_n and y_{n-1}, then y_{n-1} and y_n, then y_{n-1} and x_{n-1}, then y_n and x_{n-1}. Using all these results we get

$$\int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(x_1, \ldots, x_n) f(x_1, \ldots, x_n) dx_n dx_{n-1}$$
$$= \sum_{i_n=0}^{m_n} \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_{n-1}}^{b_{n-1}} \int_{a_n}^{b_n} p(x_1, \ldots, x_n) f_{(0,\ldots,0,i_{n-1},i_n)}(x_1, \ldots, x_{n-2}, a_{n-1}, a_n) \times$$
$$\times \frac{(x_{n-1} - a_{n-1})^{i_{n-1}}}{i_{n-1}!} \frac{(x_n - a_n)^{i_n}}{i_n!} dx_n dx_{n-1}$$

$$+\sum_{i_n=0}^{m_n}\int_{a_{n-1}}^{b_{n-1}}\int_{y_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(x_1,\ldots,x_n)f_{(0,\ldots,0,m_{n-1}+1,i_n)}(x_1,\ldots,x_{n-2},y_{n-1},a_n)\times$$
$$\times\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}\frac{(x_n-a_n)^{i_n}}{i_n!}dx_ndx_{n-1}dy_{n-1}$$
$$+\sum_{i_{n-1}=0}^{m_{n-1}}\int_{a_n}^{b_n}\int_{a_{n-1}}^{b_{n-1}}\int_{y_n}^{b_n} p(x_1,\ldots,x_n)f_{(0,\ldots,0,i_{n-1},m_n+1)}(x_1,\ldots,x_{n-2},a_{n-1},y_n)\times$$
$$\times\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dx_ndx_{n-1}dy_n$$
$$+\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}\int_{y_{n-1}}^{b_{n-1}}\int_{y_{n-1}}^{b_{n-1}}\int_{y_n}^{b_n} p(x_1,\ldots,x_n)\times$$
$$\times f_{(0,\ldots,0,m_{n-1}+1,m_n+1)}(x_1,\ldots,x_{n-2},y_{n-1},y_n)\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}\times$$
$$\times\frac{(x_n-y_n)^{m_n}}{m_n!}dx_ndx_{n-1}dy_ndy_{n-1}.$$

Now, using Taylor expansion again and integrate over $[a_{n-2},b_{n-2}]$ by variable x_{n-2}. If we proceed in the similar fashion as we done before, then we finally get:

$$\int_{a_{n-2}}^{b_{n-2}}\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(x_1,\ldots,x_n)f(x_1,\ldots,x_n)dx_ndx_{n-1}dx_{n-2}$$
$$=\sum_{i_n=0}^{m_n}\sum_{i_{n-1}=0}^{m_{n-1}}\sum_{i_{n-2}=0}^{m_{n-2}}\int_{a_{n-2}}^{b_{n-2}}\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(x_1,\ldots,x_n)\times$$
$$\times f_{(0,\ldots,0,i_{n-2},i_{n-1},i_n)}(x_1,\ldots,x_{n-3},a_{n-2},a_{n-1},a_n)\times$$
$$\times\frac{(x_{n-2}-a_{n-2})^{i_{n-2}}}{i_{n-2}!}\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-a_n)^{i_n}}{i_n!}dx_ndx_{n-1}dx_{n-2}$$
$$+\sum_{i_n=0}^{m_n}\sum_{i_{n-1}=0}^{m_{n-1}}\int_{a_{n-2}}^{b_{n-2}}\int_{y_{n-2}}^{b_{n-2}}\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(x_1,\ldots,x_n)\times$$
$$\times f_{(0,\ldots,0,m_{n-2}+1,i_{n-1}+1,i_n)}(x_1,\ldots,x_{n-3},y_{n-2},a_{n-1},a_n)\times$$
$$\times\frac{(x_{n-2}-y_{n-2})^{m_{n-2}}}{m_{n-2}!}\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-a_n)^{i_n}}{i_n!}dx_ndx_{n-1}dx_{n-2}dy_{n-2}$$
$$+\sum_{i_n=0}^{m_n}\sum_{i_{n-2}=0}^{m_{n-2}}\int_{a_{n-1}}^{b_{n-1}}\int_{a_{n-2}}^{b_{n-2}}\int_{y_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(x_1,\ldots,x_n)\times$$
$$\times f_{(0,\ldots,0,i_{n-2},m_{n-1}+1,i_n)}(x_1,\ldots,x_{n-3},a_{n-2},y_{n-1},a_n)\times$$
$$\times\frac{(x_{n-2}-a_{n-2})^{i_{n-2}}}{i_{n-2}!}\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}\frac{(x_n-a_n)^{i_n}}{i_n!}dx_ndx_{n-1}dx_{n-2}dy_{n-1}$$

$$+\sum_{i_n=0}^{m_n}\int_{a_{n-2}}^{b_{n-2}}\int_{a_{n-1}}^{b_{n-1}}\int_{y_{n-2}}^{b_{n-2}}\int_{y_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n} p(x_1,\ldots,x_n)\times$$
$$\times f_{(0,\ldots,0,m_{n-2}+1,m_{n-1}+1,i_n)}(x_1,\ldots,x_{n-3},y_{n-2},y_{n-1},a_n)\times$$
$$\times\frac{(x_{n-2}-y_{n-2})^{m_{n-2}}}{m_{n-2}!}\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}\frac{(x_n-a_n)^{i_n}}{i_n!}dx_n dx_{n-1}dx_{n-2}dy_{n-1}dy_{n-2}$$
$$+\sum_{i_{n-1}=0}^{m_{n-1}}\sum_{i_{n-2}=0}^{m_{n-2}}\int_{a_n}^{b_n}\int_{a_{n-2}}^{b_{n-2}}\int_{a_{n-1}}^{b_{n-1}}\int_{y_n}^{b_n} p(x_1,\ldots,x_n)\times$$
$$\times f_{(0,\ldots,0,i_{n-2},i_{n-1},m_n+1)}(x_1,\ldots,x_{n-3},a_{n-2},a_{n-1},y_n)\times$$
$$\times\frac{(x_{n-2}-a_{n-2})^{i_{n-2}}}{i_{n-2}!}\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dx_n dx_{n-1}dx_{n-2}dy_n$$
$$+\sum_{i_{n-1}=0}^{m_{n-1}}\int_{a_{n-2}}^{b_{n-2}}\int_{a_n}^{b_n}\int_{y_{n-2}}^{b_{n-2}}\int_{a_{n-1}}^{b_{n-1}}\int_{y_n}^{b_n} p(x_1,\ldots,x_n)\times$$
$$\times f_{(0,\ldots,0,m_{n-2}+1,i_{n-1},m_n+1)}(x_1,\ldots,x_{n-3},y_{n-2},a_{n-1},y_n)\times$$
$$\times\frac{(x_{n-2}-y_{n-2})^{m_{n-2}}}{m_{n-2}!}\frac{(x_{n-1}-a_{n-1})^{i_{n-1}}}{i_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dx_n dx_{n-1}dx_{n-2}dy_n dy_{n-2}$$
$$+\sum_{i_{n-2}=0}^{m_{n-2}}\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}\int_{a_{n-2}}^{b_{n-2}}\int_{y_{n-1}}^{b_{n-1}}\int_{y_n}^{b_n} p(x_1,\ldots,x_n)\times$$
$$\times f_{(0,\ldots,0,i_{n-2},m_{n-1}+1,m_n+1)}(x_1,\ldots,x_{n-3},a_{n-2},y_{n-1},y_n)\times$$
$$\times\frac{(x_{n-2}-a_{n-2})^{i_{n-2}}}{i_{n-2}!}\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dx_n dx_{n-1}dx_{n-2}dy_n dy_{n-1}$$
$$+\int_{a_{n-2}}^{b_{n-2}}\int_{a_{n-1}}^{b_{n-1}}\int_{a_n}^{b_n}\int_{y_{n-2}}^{b_{n-2}}\int_{y_{n-1}}^{b_{n-1}}\int_{y_n}^{b_n} p(x_1,\ldots,x_n)\times$$
$$\times f_{(0,\ldots,0,m_{n-2}+1,m_{n-1}+1,m_n+1)}(x_1,\ldots,x_{n-3},y_{n-2},y_{n-1},y_n)\times$$
$$\times\frac{(x_{n-2}-y_{n-2})^{m_{n-2}}}{m_{n-2}!}\frac{(x_{n-1}-y_{n-1})^{m_{n-1}}}{m_{n-1}!}\frac{(x_n-y_n)^{m_n}}{m_n!}dx_n dx_{n-1}dx_{n-2}dy_n dy_{n-1}dy_{n-2}.$$

Then we use the Taylor expansion again and we integrate the result over $[a_{n-3}, b_{n-3}]$ by variable x_{n-3}. If we continue this process, we get required identity. ■

Remark 3.3.1. If we set $n = 2$ in the previous theorem, we get the following corollary, which we will frequently use in other results. □

Corollary 3.3.2. *Let $p, f : I\times J \to \mathbb{R}$ be integrable functions and let $f \in C^{(n+1,m+1)}(I\times J)$. Then the identity*

$$\int_a^b\int_a^b P(x,y)f(x,y)dy\,dx$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{m}\int_a^b\int_a^b P(s,t)f_{(i,j)}(a,a)\frac{(s-a)^i}{i!}\frac{(t-a)^j}{j!}dt\,ds$$

$$+ \sum_{j=0}^{m}\int_a^b\int_x^b\int_a^b P(s,t)f_{(n+1,j)}(x,a)\frac{(s-x)^n}{n!}\frac{(t-a)^j}{j!}dt\,ds\,dx$$

$$+ \sum_{i=0}^{n}\int_a^b\int_a^b\int_y^b P(s,t)f_{(i,m+1)}(a,y)\frac{(s-a)^i}{i!}\frac{(t-y)^m}{m!}dt\,ds\,dy$$

$$+ \int_a^b\int_a^b\int_x^b\int_y^b P(s,t)f_{(n+1,m+1)}(x,y)\frac{(s-x)^n}{n!}\frac{(t-y)^m}{m!}dt\,ds\,dy\,dx$$

holds.

Remark 3.3.2. If in Corollary 3.3.2 we simply put $n = m = 0$, then we get the following corollary. In fact the following identity was considered by Pečarić in Theorem 10 of [75]. □

Corollary 3.3.3. *Let $P, f : I^2 \to R$ be integrable functions and if f has the continuous partial derivatives $f_{(1,0)}$, $f_{(0,1)}$ and $f_{(1,1)}$on I^2 then*

$$\int_a^b\int_a^b P(x,y)f(x,y)dxdy = f(a,a)P_1(a,a)+\int_a^b P_1(x,a)f_{(1,0)}(x,a)dx$$
$$+\int_a^b P_1(a,y)f_{(0,1)}(a,y)dy + \int_a^b\int_a^b P_1(x,y)f_{(1,1)}(x,y)dxdy$$

where

$$P_1(x,y) = \int_x^b\int_y^b P(s,t)dtds,$$

$$f_{(1,0)} = \frac{\partial f}{\partial x},\quad f_{(0,1)} = \frac{\partial f}{\partial y}\quad \text{and}\quad f_{(1,1)} = \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}.$$

Remark 3.3.3. If in Corollary 3.3.2 we replace $f(x,y)$ by $f(x)g(y)$, then we get the following result. □

Corollary 3.3.4. *Let $f \in C^{(n+1)}(I)$ and $g \in C^{(m+1)}(J)$ be two functions. Further let $p : I \times J \to \mathbb{R}$ be an integrable function. Then the identity*

$$\int_a^b\int_a^b P(x,y)f(x)g(y)dy\,dx$$
$$= \sum_{i=0}^{n}\sum_{j=0}^{m}\int_a^b\int_a^b P(s,t)f_{(i)}(a)g_{(j)}(a)\frac{(s-a)^i}{i!}\frac{(t-a)^j}{j!}dt\,ds$$
$$+\sum_{j=0}^{m}\int_a^b\int_x^b\int_a^b P(s,t)f_{(n+1)}(x)g_{(j)}(a)\frac{(s-x)^n}{n!}\frac{(t-a)^j}{j!}dt\,ds\,dx$$

$$+ \sum_{i=0}^{n} \int_a^b \int_a^b \int_y^b P(s,t) f_{(i)}(a) g_{(m+1)}(y) \frac{(s-a)^i}{i!} \frac{(t-y)^m}{m!} dt\, ds\, dy$$
$$+ \int_a^b \int_a^b \int_x^b \int_y^b P(s,t) f_{(n+1)}(x) g_{(m+1)}(y) \frac{(s-x)^n}{n!} \frac{(t-y)^m}{m!} dt\, ds\, dy\, dx$$

holds.

Corollary 3.3.5. *Let the assumptions of Theorem* 3.3.1 *be valid and if* $p \equiv 1$. *Then the following identity holds*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n) dx_n \cdots dx_1$$
$$= \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} \prod_{j=1}^{n} \frac{(b_j - a_j)^{i_j+1}}{(i_i+1)!} f_{(i_1,\ldots,i_n)}(a_1, \ldots, a_n)$$
$$+ \sum_{i_2=0}^{m_2} \cdots \sum_{i_n=0}^{m_n} \int_{a_1}^{b_1} \frac{(b_1-y_1)^{m_1+1}}{(m_1+1)!} \prod_{j=2}^{n} \frac{(b_j-a_j)^{i_j+1}}{(i_j+1)!} f_{(m_1+1,i_2,\ldots,i_n)}(y_1, a_2, \ldots, a_n) dy_1$$
$$+ \cdots +$$
$$+ \sum_{i_1=0}^{m_1} \cdots \sum_{i_{n-1}=0}^{m_{n-1}} \int_{a_n}^{b_n} \frac{(b_n-y_n)^{m_n+1}}{(m_n+1)!} \prod_{j=1}^{n-1} \frac{(b_j-a_j)^{i_j+1}}{(i_j+1)!} \times$$
$$\times f_{(i_1,\ldots,i_{n-1},m_n+1)}(a_1, \ldots, a_{n-1}, y_n) dy_n$$
$$+ \cdots +$$
$$+ \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \prod_{j=1}^{n} \frac{(b_j-y_j)^{m_j+1}}{(m_j+1)!} f_{(m_1+1,\ldots,m_n+1)}(y_1, \ldots, y_n) dy_n \cdots dy_1.$$

Remark 3.3.4. For $n = 2$ in the above corollary we get Theorem 6.16 in the book [21] by simply putting $x = a$ and $y = c$.
Now we state our next main theorem: □

Theorem 3.3.6. *Let the assumptions of Theorem* 3.3.1 *be valid. Then the inequality*

$$\Lambda(f) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} p(x_1, \ldots, x_n) f(x_1, \ldots, x_n) dx_n \cdots dx_1 \geq 0 \tag{3.3.3}$$

holds for every $(m_1+1, \ldots, m_n+1)-$*convex function* f *on* $I_1 \times \cdots \times I_n$ *if and only if*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} p(x_1, \ldots, x_n) \prod_{j=1}^{n} \frac{(y_j - a_j)^{i_j}}{i_j!} dy_n \cdots dy_1 = 0, \tag{3.3.4}$$

$$i_1 \in \{0,1,\ldots,m_1\},\ \ldots,\ i_n \in \{0,1,\ldots,m_n\},$$

$$\int_{a_1}^{b_1}\int_{x_1}^{b_1}\cdots\int_{a_n}^{b_n} p(x_1,\ldots,x_n)\frac{(y_1-x_1)^{m_1}}{m_1!}\prod_{j=2}^{n}\frac{(y_j-a_j)^{i_j}}{i_j!}\,dy_n\cdots dy_1dx_1 = 0, \tag{3.3.5}$$

$$i_2 \in \{0,1,\ldots,m_2\},\ldots,i_n \in \{0,1,\ldots,m_n\}, \forall x_1 \in [a_1,b_1],$$

$$\vdots$$

$$\int_{a_n}^{b_n}\int_{a_1}^{b_1}\cdots\int_{a_{n-1}}^{b_{n-1}}\int_{y_n}^{b_n} p(x_1,\ldots,x_n)\prod_{j=1}^{n-1}\frac{(y_j-a_j)^{i_j}}{i_j!}\frac{(y_n-x_n)^{m_n}}{m_n!}\,dy_n\cdots dy_1dx_n = 0, \tag{3.3.6}$$

$$i_1 \in \{0,1,\ldots,m_1\},\ldots,i_{n-1} \in \{0,1,\ldots,m_{n-1}\}, \forall x_n \in [a_n,b_n],$$

$$\vdots$$

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\int_{x_1}^{b_1}\cdots\int_{x_n}^{b_n} p(x_1,\ldots,x_n)\prod_{j=1}^{n}\frac{(y_j-x_j)^{m_j}}{m_j!}\,dy_n\cdots dy_1dx_n\cdots dx_1 \geq 0, \tag{3.3.7}$$

$$\forall x_1 \in [a_1,b_1],\ldots,\ x_n \in [a_n,b_n].$$

Proof. If (3.3.4), (3.3.5), $\ldots$, (3.3.6) hold, then all these sums are zero in (3.3.1) and the required inequality (3.3.3) holds by using (3.3.7).

Conversely, if we consider in (3.3.3) the following functions

$$g^1(y_1,\ldots,y_n) \;=\; \prod_{j=1}^{n}\frac{(y_j-a_j)^{i_j}}{i_j!} \quad \text{and} \quad g^2 = -g^1$$

for $i_1 \in \{0,1,\ldots,m_1\},\ \ldots,\ i_n \in \{0,1,\ldots,m_n\}$ such that $g^k_{(m_1+1,\ldots,m_n+1)} \geq 0, \quad k \in \{1,2\}$, then we get the required equality (3.3.4).

In the same way, if we consider in (3.3.3) the following functions for $i_2 \in \{0,1,\ldots,m_2\}$, $\ldots, i_n \in \{0,1,\ldots,m_n\}, \forall x_1 \in [a_1,b_1]$

$$g^3(y_1,\ldots,y_n) = \begin{cases} \frac{(y_1-x_1)^{m_1}}{m_1!}\prod_{j=2}^{n}\frac{(y_j-a_j)^{i_j}}{i_j!} & , \quad x_1 < y_1, \\ 0 & , \quad x_1 \geq y_n, \end{cases} \quad \text{and} \quad g^4 = -g^3$$

such that $g^k_{(m_1+1,\dots,m_n+1)} \geq 0, \quad k \in \{3,4\}$, then we get the required equality (3.3.5). Similarly, if we consider in (3.3.3) the following functions for $i_1 \in \{0,1,\dots,m_1\},\dots,$ $i_{n-1} \in \{0,1,\dots,m_{n-1}\}, \forall x_n \in [a_n,b_n]$

$$g^5(y_1,\dots,y_n) = \begin{cases} \prod_{j=1}^{n-1} \frac{(y_j-a_j)^{i_j}}{i_j!} \frac{(y_n-x_n)^{m_n}}{m_n!} & , \quad x_n < y_n, \\ 0 & , \quad x_n \geq y_n, \end{cases} \quad \text{and} \quad g^6 = -g^5$$

such that $g^k_{(m_1+1,\dots,m_n+1)} \geq 0, \quad k \in \{5,6\}$, then we get the required equality (3.3.6) and so on.

The last inequality (3.3.7) is followed by considering the following function in (3.3.3) $\forall x_1 \in [a_1,b_1],\ \dots,\ x_n \in [a_n,b_n]$,

$$g^7(y_1,\dots,y_n) = \begin{cases} \prod_{j=1}^{n} \frac{(y_j-x_j)^{m_j}}{m_j!} & , \quad x_1 < y_1,\dots,x_n < y_n, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

■

3.4 Mean Value Theorems

It is a well known fact that many results of classical real analysis are a consequence of the mean value theorem. Lagrange's and Cauchy's mean value theorems are among the most important theorems of differential calculus. For detailed discussion on the topic we refer to [87]. Here we state some generalized mean value theorems of Lagrange- and of Cauchy-type.

Theorem 3.4.1. *Let $\Lambda : C^{(m_1+1,\dots,m_n+1)}(I_1 \times \cdots \times I_n) \to \mathbb{R}$ be the linear functional defined in* (3.3.3) *and let $p : I_1 \times \cdots \times I_n \to \mathbb{R}$ be an integrable function such that the conditions* (3.3.4), (3.3.5), …, (3.3.6), …, (3.3.7) *of Theorem* 3.3.6 *be satisfied. Then there exists $(\xi_1,\dots,\xi_n) \in I_1 \times \cdots \times I_n$ such that*

$$\Lambda(f) = f_{(m_1+1,\dots,m_n+1)}(\xi_1,\dots,\xi_n)\Lambda(f_0) \tag{3.4.1}$$

where $f_0(x_1,\dots,x_n) = \prod_{j=1}^{n} \frac{x_j^{m_j+1}}{(m_j+1)!}$.

Proof. Since $f_{(m_1+1,\dots,m_n+1)}$ is continuous on $(I_1 \times \cdots \times I_n)$, so it attains its maximum and minimum value on $(I_1 \times \cdots \times I_n)$. Let

$$L = \min_{(x_1,\dots,x_n)\in I_1\times\cdots\times I_n} f_{(m_1+1,\dots,m_n+1)}(x_1,\dots,x_n)$$

and

$$U = \max_{(x_1,\dots,x_n)\in I_1\times\cdots\times I_n} f_{(m_1+1,\dots,m_n+1)}(x_1,\dots,x_n).$$

Then the function

$$G(x_1,\dots,x_n) = Uf_0(x_1,\dots,x_n) - f(x_1,\dots,x_n)$$

gives us

$$G_{(m_1+1,\dots,m_n+1)}(x_1,\dots,x_n) = U - f_{(m_1+1,\dots,m_n+1)}(x_1,\dots,x_n) \geq 0,$$

i.e., G is $(m_1+1,\dots,m_n+1)-$convex function. Hence $\Lambda(G) \geq 0$ by Theorem 3.3.6 and we conclude that

$$\Lambda(f) \leq U\Lambda(f_0).$$

Similarly, we have

$$L\Lambda(f_0) \leq \Lambda(f).$$

Combining the two inequalities we get

$$L\Lambda(f_0) \leq \Lambda(f) \leq U\Lambda(f_0)$$

which gives us (3.4.1). ■

Theorem 3.4.2. *Let all the assumptions of Theorem* 3.4.1 *be valid. Then there exists* $(\xi_1,\dots,\xi_n) \in I_1\times\cdots\times I_n$ *such that*

$$\frac{\Lambda(f)}{\Lambda(g)} = \frac{f_{(m_1+1,\dots,m_n+1)}(\xi_1,\dots,\xi_n)}{g_{(m_1+1,\dots,m_n+1)}(\xi_1,\dots,\xi_n)}$$

provided that the denominator of the left-hand side is nonzero.

Proof. Let $h \in C^{(m_1+1,\dots,m_n+1)}(I_1\times\cdots\times I_n)$ be defined as

$$h = \Lambda(g)f - \Lambda(f)g.$$

Using Theorem 3.4.1 there exists $(\xi_1,\dots,\xi_n)$ such that

$$0 = \Lambda(h) = h_{(m_1+1,\dots,m_n+1)}(\xi_1,\dots,\xi_n)\Lambda(f_0)$$

or

$$\Big[\Lambda(g)f_{(m_1+1,\dots,m_n+1)}(\xi_1,\dots,\xi_n) - \Lambda(f)g_{(m_1+1,\dots,m_n+1)}(\xi_1,\dots,\xi_n)\Big]\Lambda(f_0) = 0$$

which gives us required result. ■

Corollary 3.4.3. *Let all the assumptions of Theorem* 3.4.2 *be satisfied with* $m = m_1 = m_2 = \ldots = m_n$. *Then there exists* $(\xi_1, \ldots, \xi_n) \in I_1 \times \cdots \times I_n$ *such that*

$$(\xi_1 \cdots \xi_n)^{q-q'} = \frac{[(q'+1)q' \cdots (q'-n+1)]^n \Lambda((x_1 \cdots x_n)^{q+1})}{[(q+1)q \cdots (q-n+1)]^n \Lambda((x_1 \cdots x_n)^{q'+1})}$$

for $-\infty < q \neq q' < +\infty$ *and* $q, q' \notin \{-1, 0, 1, \ldots, n-1\}$.

Proof. If we put $f(x_1, \ldots, x_n) = \frac{(x_1 \cdots x_n)^{q+1}}{[(q+1)!]^n}$ and $g(x_1, \ldots, x_n) = \frac{(x_1 \cdots x_n)^{q'+1}}{[(q'+1)!]^n}$ in Theorem 3.4.2, then we get the required result. ■

Remark 3.4.1. Special cases of Theorems 3.4.1, 3.4.2 and Corollary 3.4.3 for $n = 2$ can be found in [44]. □

For our next theorem we recall the Hölder's Inequality for functional

$$A(F) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\, dx_n \cdots dx_1$$

as follows:

$$A(FG) \leq A(F^q)^{1/q} A(G^{q'})^{1/q'}$$

where $1/q + 1/q' = 1, \quad q, q' > 1$.

Let us introduce some notations for simplifications of statements as follows:

$$\begin{aligned} \overline{\Lambda}(f) &= \Lambda(f) - \sum_{r=0}^{n-1} \sum_{(p_1,\ldots,p_n) \in {}^n C_r(i_j, m_j+1)} \tilde{\Delta}(p_1, \ldots, p_n) \\ &= \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} p(x_1, \ldots, x_n) f_{(m_1+1,\ldots,m_n+1)}(x_1, \ldots, x_n) \times \\ &\quad \times \frac{(y_1 - x_1)^{m_1}}{m_1!} \cdots \frac{(y_n - x_n)^{m_n}}{m_n!} dy_n \cdots dy_1 dx_n \cdots dx_1. \end{aligned}$$

Theorem 3.4.4. *Let* $p : I_1 \times \cdots \times I_n \to \mathbb{R}$ *be an integrable function and let* $f \in C^{(m_1+1,\ldots,m_n+1)}(I_1 \times \cdots \times I_n)$. *If* $\left|f_{(m_1+1,\ldots,m_n+1)}\right|^q$ *is an integrable function such that*

$$\| f_{(m_1+1,\ldots,m_n+1)} \|_q = \left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \left|f_{(m_1+1,\ldots,m_n+1)}(x_1, \ldots, x_n)\right|^q dx_n \cdots dx_1 \right)^{1/q} < \infty,$$

then the following inequality holds

$$\begin{aligned} \left|\overline{\Lambda}(f)\right| \leq \| f_{(m_1+1,\ldots,m_n+1)} \|_q \Bigg(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \Bigg| \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} p(x_1, \ldots, x_n) \times \\ \times \frac{(y_1 - x_1)^{m_1}}{m_1!} \cdots \frac{(y_n - x_n)^{m_n}}{m_n!} dy_n \cdots dy_1 \Bigg|^{q} dx_n \cdots dx_1 \Bigg)^{1/q'} \end{aligned}$$

where $1/q + 1/q' = 1$, $q, q' > 1$.

Remark 3.4.2. The proof of the theorem is easily followed by applying the Hölders inequality. Moreover, when we consider the case $q \to 1$ then $r \to \infty$, we get the following corollary. □

Corollary 3.4.5. *Let all the assumptions of the Theorem* 3.4.4 *be valid. Then the inequality*

$$\left|\overline{\Lambda}(f)\right| \leq M \prod_{i=1}^{n}(b_i - a_i) \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} f_{(m_1+1,\ldots,m_n+1)}(x_1,\ldots,x_n)\, dx_n \cdots dx_1$$

holds, where

$$M = ess\ sup\left(\int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} p(x_1,\ldots,x_n)\frac{(y_1-x_1)^{m_1}}{m_1!} \cdots \frac{(y_n-x_n)^{m_n}}{m_n!} dy_n \cdots dy_1\right).$$

Remark 3.4.3. For the case $p \equiv 1$, we get the following corollary. □

Corollary 3.4.6. *Let all the assumptions of the Theorem* 3.4.4 *be valid and if* $p \equiv 1$. *Then we have*

$$\left|\overline{\Lambda}(f)\right| \leq \prod_{i=1}^{n} \frac{(b_i - a_i)^{m_i+2}}{(m_i+2)!} \parallel f_{(m_1+1,\ldots,m_n+1)} \parallel_q .$$

3.5 Exponential Convexity

Let $I = [a,b] \subset \mathbb{R}_+$ and $\Omega = \{\varphi^{(t)} : I^n \to \mathbb{R} : t \in \mathbb{R}\}$ be a family of functions defined as:

$$\varphi^{(t)}(x_1,\ldots,x_n) = \begin{cases} \frac{(x_1\cdots x_n)^t}{[t(t-1)\cdots(t-m)]^n} & , \quad t \notin \{0,\ldots,m\}, \\ \frac{(x_1\cdots x_n)^t[\ln(x_1\cdots x_n)]^n}{(-1)^{m-t}n![t!(m-t)!]^n} & , \quad t \in \{0,\ldots,m\}. \end{cases}$$

Clearly $\varphi^{(t)}_{(m+1,\ldots,m+1)}(x_1,\ldots,x_n) = (x_1\cdots x_n)^{t-m-1} = e^{(t-m-1)\ln(x_1\cdots x_n)}$ for $(x_1,\ldots,x_n) \in I^n$ so $\varphi^{(t)}$ is $(m+1,\ldots,m+1)-$convex function and $t \mapsto \varphi^{(t)}_{(m+1,\ldots,m+1)}(x_1,\ldots,x_n)$ is exponentially convex function on $\mathbb{R}$. From Corollary 1.2.2 we know that every positive function which is exponentially convex is log $-$convex. So, we state our next theorem.

Theorem 3.5.1. *Let* $\Lambda : C^{(m+1,\ldots,m+1)}(I^n) \to \mathbb{R}$ *be a linear functional as defined in* (3.3.3) *and let the conditions* (3.3.4), (3.3.5), …, (3.3.6), …, (3.3.7) *of Theorem* 3.3.6 *for function* p *be satisfied and* $\varphi^{(t)}$ *be a function defined above. Then the following statements hold:*

(a) *The function* $t \mapsto \Lambda(\varphi^{(t)})$ *is continuous on* $\mathbb{R}$.

(b) *The function* $t \mapsto \Lambda(\varphi^{(t)})$ *is exponentially convex on* $\mathbb{R}$.

(c) *If the function* $t \mapsto \Lambda(\varphi^{(t)})$ *is positive on* $\mathbb{R}$, *then* $t \mapsto \Lambda(\varphi^{(t)})$ *is log-convex on* $\mathbb{R}$. *Moreover, the following Lyapunov's inequality holds for* $r < s < t$; $r,\ s,\ t\ \in I$

$$(\Lambda_k(f_s))^{t-r} \ \leq\ (\Lambda_k(f_r))^{t-s}\,(\Lambda_k(f_t))^{s-r}. \tag{3.5.1}$$

(d) *The matrix* $\left[\Lambda(\varphi^{(\frac{t_i+t_j}{2})})\right]_{i,j=1}^{m}$ *is positive-semidefinite. Particularly*

$$\det\left[\Lambda(\varphi^{(\frac{t_i+t_j}{2})})\right]_{i,j=1}^{m} \ \geq\ 0$$

for each $t_i \in\ \mathbb{R}$ *and* $m \in \mathbb{N}$ *for* $i \in \{1, \ldots, m\}$.

(e) *If the function* $t \mapsto \Lambda(\varphi^{(t)})$ *is differentiable on* $\mathbb{R}$. *Then for every* $s, t, u, v \in \mathbb{R}$ *such that* $s \leq u$ *and* $t \leq v$, *we have*

$$\mu_{s,t}(\Lambda, \Omega) \leq \mu_{u,v}(\Lambda, \Omega) \tag{3.5.2}$$

where

$$\mu_{s,t}(\Lambda, \Omega) = \begin{cases} \left(\frac{\Lambda(\varphi^{(s)})}{\Lambda(\varphi^{(t)})}\right)^{\frac{1}{s-t}} & , \quad s \neq t, \\ \exp\left(\frac{\frac{d}{ds}\Lambda(\varphi^{(s)})}{\Lambda(\varphi^{(s)})}\right) & , \quad s = t. \end{cases} \tag{3.5.3}$$

Proof.

(a) For fixed $n \in \mathbb{N} \cup \{0\}$, using Ĺ Hôpital rule $n-$times and applying limit, we get

$$\begin{aligned}
\lim_{t \to 0} \Lambda(\varphi^{(t)}) &= \lim_{t \to 0} \frac{\int_a^b \cdots \int_a^b p(x_1, \ldots, x_n)(x_1 \cdots x_n)^t dx_n \cdots dx_1}{[t(t-1)\cdots(t-m)]^n} \\
&= \frac{\int_a^b \cdots \int_a^b p(x_1, \ldots, x_n)\Big(\ln(x_1 \cdots x_n)\Big)^n dx_n \cdots dx_1}{(-1)^m n!\Big(m!\Big)^n} \\
&= \Lambda(\varphi^{(0)}).
\end{aligned}$$

In the similar fashion we can get

$$\lim_{t \to k} \Lambda(\varphi^{(t)}) = \Lambda(\varphi^{(k)}), \quad k \in \{1, \ldots, m\}.$$

So we conclude that the function $t \mapsto \Lambda(\varphi^{(t)})$ is continuous on $\mathbb{R}$.

(b) Let us define the function

$$\omega = \sum_{i,j=1}^{k} u_i u_j \varphi^{(\frac{t_i+t_j}{2})},$$

where $t_i, u_i \in \mathbb{R}$, $i \in \{1, \ldots, k\}$.
Since the function $t \mapsto \varphi^{(t)}_{(m+1,\ldots,m+1)}$ is exponentially convex, we have

$$\omega_{(m+1,\ldots,m+1)} = \sum_{i,j=1}^{k} u_i u_j \varphi^{(\frac{t_i+t_j}{2})}_{(m+1,\ldots,m+1)} \geq 0,$$

which implies that ω is $(m+1,\ldots,m+1)-$convex function on I^n and therefore we have $\Lambda(\omega) \geq 0$. Hence $\sum_{i,j=1}^{k} u_i u_j \Lambda(\varphi^{(\frac{t_i+t_j}{2})}) \geq 0$. We conclude that the function $t \mapsto \Lambda(\varphi^{(t)})$ is exponentially convex on $\mathbb{R}$.

(c) It is a direct consequence of (b) by using Corollary 1.2.2. As the function $t \mapsto \Lambda(f_t)$ is log-convex, i.e., $\ln(\Lambda(f_t))$ is convex, so by using Proposition 1.1.1, we have

$$\ln(\Lambda(f_s))^{t-r} \leq \ln(\Lambda(f_r))^{t-s} + \ln(\Lambda(f_t))^{s-r},$$

which gives us (3.5.1).

(d) This is a consequence of Corollary 1.2.1.

(e) From Proposition 1.1.2, the inequality

$$\frac{\phi(s) - \phi(t)}{s - t} \leq \frac{\phi(u) - \phi(v)}{u - v} \tag{3.5.4}$$

holds $\forall\, s, t, u, v \in I \subset \mathbb{R}$ such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$.
Since by (c), $\Lambda(\varphi^{(t)})$ is log $-$convex, so set $\phi(x) = \ln(\Lambda(\varphi^{(x)}))$ in (3.5.4) we have

$$\frac{\ln(\Lambda(\varphi^{(s)})) - \ln(\Lambda(\varphi^{(t)}))}{s - t} \leq \frac{\ln(\Lambda(\varphi^{(u)})) - \ln(\Lambda(\varphi^{(v)}))}{u - v} \tag{3.5.5}$$

for $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, which is equivalent to (3.5.2). The cases for $s = t$ and / or $u = v$ are easily followed from (3.5.5) by taking respective limits. ■

Remark 3.5.1. Here we notice that Theorem 3.5.1 generalizes Theorem 5.6 of [44]. □

The next two chapters are related to this chapter through Corollaries 3.2.2 and 3.3.2, because we will use these corollaries as lemmas in proof of the main theorems of next two chapters. In the upcoming chapter, we will discuss the generalization of Čebyšev's and Fan's identities and inequalities.

Chapter 4

Generalized Čebyšev's and Fan's Identities and Inequalities

"Mathematics is the supreme judge; from its decisions there is no appeal."

– Tobias Dantzig

The present chapter is devoted to the generalization of Čhebyšev's and Fan's identities and inequalities. The chapter is based on four sections. The first section introduces Čebyšev's inequality and gives some related results. Some notations are also defined. In the second section we discuss the generalization of discrete Čhebyšev's identity and inequality. In the third and in the forth sections the generalization of Čebyšev's and Fan's identities and inequalities are given respectively.

4.1 Introduction and Preliminaries

A classic result due to Čebyšev [16, 17] may be stated as (see also [80, p. 197]). Throughout this chapter $I = [a, b] \subset \mathbb{R}$.

Proposition 4.1.1. *Let $f, g : I \to \mathbb{R}$ and $p : I \to \mathbb{R}_+$ be integrable functions. If f and g are monotonic in the same sense, then the inequality*

$$\int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \geq 0 \qquad (4.1.1)$$

holds provided that the integrals exist. If f and g are monotonic in the opposite sense, then the reverse of the inequality in (4.1.1) *is valid. In both cases, equality in* (4.1.1) *holds if and only if either f or g is constant almost everywhere.*

A discrete analogue of the previous proposition may also be given as follows (see [80, p. 197]).

Proposition 4.1.2. *Let* **a** *and* **b** *be two real* $n-$*tuples monotonic in the same sense and* **p** *be a nonnegative* $n-$*tuple. Then the inequality*

$$\sum_{i=1}^{N} p_i \sum_{i=1}^{N} p_i a_i b_i - \sum_{i=1}^{N} p_i a_i \sum_{j=1}^{N} p_i b_i \geq 0 \tag{4.1.2}$$

holds. If **a** *and* **b** *are monotonic in the opposite sense, then the reverse of the inequality in* (4.1.2) *holds. In both cases equality in* (4.1.2) *holds if and only if either* $a_1 = a_2 = \ldots = a_n$ *or* $b_1 = b_2 = \ldots = b_n$.

For detailed discussion on the topic of Čhebyšev's inequality we refer to the books [59] and [80].

Ostrowski [63] gave the following result related to Čebyšev's inequality:

Proposition 4.1.3. *Let* $f, g \in C^{(1)}(I)$ *be two monotonic functions and let* $p : I \to \mathbb{R}_+$ *be an integrable function. Then there exist* ξ, $\eta \in I$ *such that*

$$T(f, g, p) = f'(\xi)g'(\eta)T(x - a, x - a, p), \tag{4.1.3}$$

where

$$T(f, g, p) = \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx. \tag{4.1.4}$$

For other generalizations of Proposition 4.1.3, [73] can be seen. In [75], Pečarić gave the following generalization of Proposition 4.1.3 by using the functional

$$C(f, p) = \int_a^b \int_a^b p(x, y)f(x, x)dy\, dx - \int_a^b \int_a^b p(x, y)f(x, y)dy\, dx, \tag{4.1.5}$$

where p and f are integrable functions.

Proposition 4.1.4. *Let* $p : I^2 \to \mathbb{R}$ *be an integrable function such that*

$$X(x, x) = \overline{X}(x, x) \quad \forall\, x \in I$$

and let either

$$X(x, y) \geq 0, \quad a \leq y \leq x \leq b, \qquad \overline{X}(x, y) \geq 0, \quad a \leq x \leq y \leq b$$

or its reverse inequalities be valid, where

$$X(x,y)=\int_x^b\int_a^y p(s,t)\,dt\,ds \quad \text{and} \quad \overline{X}(x,y)=\int_a^x\int_y^b p(s,t)\,dt\,ds.$$

If $f\in C^{(2)}(I^2)$, then there exists $(\xi,\eta)\in I^2$ such that

$$C(f,p)=f_{(1,1)}(\xi,\eta)C\left((x-a)(y-a),p\right).$$

Now let us state the discrete analogous of Proposition 4.1.4 from [75] using the functional $C_\Delta(a,p)$ defined as

$$C_\Delta(a,p)=\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}a_{ii}-\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}a_{ij},$$

where $a_{ij},\ p_{ij}\in\mathbb{R}$ for $i,j\in\{1,\dots,N\}$.

Proposition 4.1.5. *The inequality*

$$C_\Delta(a,p)\geq 0 \tag{4.1.6}$$

holds for each real number a_{ij} for $i,j\in\{1,\dots,N\}$ such that $\Delta^{(1,1)}\ a_{ij}\geq 0$ for $i,j\in\{1,\dots,N-1\}$ if and only if

$$X_{j+1,j}=\overline{X}_{j,j+1},\quad j\in\{1,\dots,N-1\}$$

and

$$\begin{aligned} X_{ij} &\geq 0, \quad i\in\{j+1,\dots,n\} \quad \text{for} \quad j\in\{1,\dots,N-1\}\\ \overline{X}_{ij} &\geq 0, \quad i\in\{1,\dots,j-1\} \quad \text{for} \quad j\in\{2,\dots,N\}\end{aligned}$$

hold. If $\Delta^{(1,1)}a_{ij}\leq 0$ for $i,j\in\{1,\dots,N-1\}$, then the reverse inequality in (4.1.6) *is valid, where*

$$X_{ij}=\sum_{r=i}^{N}\sum_{s=1}^{j}p_{rs} \quad \text{and} \quad \overline{X}_{ij}=\sum_{r=1}^{i}\sum_{s=j}^{N}p_{rs}.$$

In 1952, Fan [24] proposed as a problem the following result (see also [58]):

Proposition 4.1.6. *Let $(x,y)\mapsto w(x,y)$ be a nonnegative Lebesgue integrable function over the square $\{(x,y): a\leq x\leq b \ \text{ and } \ a\leq y\leq b\}$. Suppose that B is a positive constant such that $\int_a^b w(x,y)dy\leq B$ for almost all $x\in[a,b]$ and also*

$\int_a^b w(x,y)dx \le B$ *for almost all* $y \in [a,b]$. *If two finite-valued functions* f *and* g *are both nonnegative and nonincreasing on* $[a,b]$*, then the following inequality holds*

$$\int_a^b \int_a^b w(x,y)f(x)g(y)dx\,dy \le B\int_a^b f(x)g(x)dx. \tag{4.1.7}$$

Remark 4.1.1. If $w(x,y) = const$, then (4.1.7) becomes special case of (4.1.1). □

For generalization of Fan's result, Pečarić in [75] considered the following expression for integrable functions f, p and q,

$$K(f,p,q) = \int_a^b q(x)f(x,x)dx - \int_a^b \int_a^b p(x,y)f(x,y)dx\,dy \tag{4.1.8}$$

and gave the following result.

Proposition 4.1.7. *Let* $p : I^2 \to \mathbb{R}$ *and* $q : I \to \mathbb{R}$ *be two integrable functions such that* $P(x,a) = Q(x), \quad P(a,y) = Q(y), \quad P(x,y) \le Q(max\{x,y\}), \quad \forall\, x,\, y \in I,$

$$\textit{where} \quad Q(x) = \int_x^b q(t)dt \quad \textit{and} \quad P(x,y) = \int_x^b \int_y^b p(s,t)dt\,ds.$$

If $f \in C^{(2)}(I^2)$*, then there exists* $(\xi,\eta) \in I^2$ *such that*

$$K(f,p,q) = f_{(1,1)}(\xi,\eta)K\left((x-a)(y-a),p,q\right).$$

Under the assumptions of Proposition 4.1.7, we introduce the following notations for simplification of statements of the up coming theorems:

$$P^{(i,j)}(x,y) = \int_x^b \int_y^b p(s,t)\frac{(s-x)^i}{i!}\frac{(t-y)^j}{j!}dtds, \tag{4.1.9}$$

$$\overline{P}^{(i,j)}(x,y) = \int_x^b \int_y^b p(s,t)\frac{(s-x)^i}{i!}\frac{(s-y)^j}{j!}dtds, \tag{4.1.10}$$

$$Q^{(i,j)}(x) = \int_x^b q(s)\frac{(s-x)^i}{i!}\frac{(s-a)^j}{j!}ds, \tag{4.1.11}$$

$$\begin{aligned} R(x,y) = &\int_{max\{x,y\}}^b \int_a^b p(s,t)\frac{(s-x)^N}{N!}\frac{(s-y)^M}{M!}dt\,ds \\ &- \int_x^b \int_y^b p(s,t)\frac{(s-x)^N}{N!}\frac{(t-y)^M}{M!}dt\,ds, \end{aligned} \tag{4.1.12}$$

$$\begin{aligned} \overline{R}(x,y) = &\int_{max\{x,y\}}^b q(s)\frac{(s-x)^N}{N!}\frac{(s-y)^M}{M!}\,ds \\ &- \int_x^b \int_y^b p(s,t)\frac{(s-x)^N}{N!}\frac{(t-y)^M}{M!}dt\,ds, \end{aligned} \tag{4.1.13}$$

$$f_0(x,y) = \frac{(x-a)^{N+1}(y-a)^{M+1}}{(N+1)!(M+1)!}. \tag{4.1.14}$$

Let $f, p : I^2 \to \mathbb{R}$ and $q : I \to \mathbb{R}$ be three functions such that p, q are integrable and $f_{(N,M)}$ exists and is absolutely continuous (in the sense of Carathéodory [89]). Then, $\overline{C}$ and $\overline{K}$ given below are well defined:

$$\begin{aligned}\overline{C}(f,p) &= C(f,p) - \sum_{i=0}^{N}\sum_{j=0}^{M} f_{(i,j)}(a,a)\left[\overline{P}^{(i,j)}(a,a) - P^{(i,j)}(a,a)\right] \\ &- \sum_{j=0}^{M}\int_a^b f_{(N+1,j)}(x,a)\left[\overline{P}^{(N,j)}(x,a) - P^{(N,j)}(x,a)\right] dx \\ &- \sum_{i=0}^{N}\int_a^b f_{(i,M+1)}(a,y)\left[\overline{P}^{(i,M)}(a,y) - P^{(i,M)}(a,y)\right] dy, \end{aligned} \tag{4.1.15}$$

where C is defined in (4.1.5).

$$\begin{aligned}\overline{K}(f,p,q) &= K(f,p,q) - \sum_{j=0}^{M}\sum_{i=0}^{N} f_{(i,j)}(a,a)\left[Q^{(i,j)}(a) - P^{(i,j)}(a,a)\right] \\ &- \sum_{j=0}^{M}\int_a^b f_{(N+1,j)}(x,a)\left[Q^{(N,j)}(x) - P^{(N,j)}(x,a)\right] dx \\ &- \sum_{i=0}^{N}\int_a^b f_{(i,M+1)}(a,y)\left[Q^{(M,i)}(y) - P^{(i,M)}(a,y)\right] dy, \end{aligned} \tag{4.1.16}$$

where K is defined in (4.1.8).

4.2 Generalized Discrete Čebyšev's Identity and Inequality

Now, we state main theorems of this section as follows:

Theorem 4.2.1. *Let $(x_i, y_j) \in I^2$ for $i, j \in \{1, \ldots, N\}$ be mutually distinct points and let $f : I^2 \to \mathbb{R}$ be a function and $p_{ij} \in \mathbb{R}$ for $i, j \in \{1, \ldots, N\}$. Then,*

$$C_\Delta(f,p) = \sum_{i=1}^{N}\sum_{j=1}^{N} p_{ij} f(x_i, y_i) - \sum_{i=1}^{N}\sum_{j=1}^{N} p_{ij} f(x_i, y_j)$$

$$= \sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\Delta_{(t,k)}f(x_1,y_1)\left[\sum_{s=max\{t,k\}+1}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_1)^{(k)}\right.$$

$$\left.-\sum_{s=t+1}^{N}\sum_{r=k+1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_r-y_1)^{(k)}\right]$$

$$+\sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\Delta_{(n,k)}f(x_{t-n},y_1)(x_t-x_{t-n})\times$$

$$\times\left[\sum_{s=max\{t,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_1)^{(k)}\right.$$

$$\left.-\sum_{s=t}^{N}\sum_{r=k+1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_r-y_1)^{(k)}\right]$$

$$+\sum_{k=m+1}^{N}\sum_{t=0}^{n-1}\Delta_{(t,m)}f(x_1,y_{k-m})(y_k-y_{k-m})\times$$

$$\times\left[\sum_{s=max\{t+1,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_{k-m+1})^{(m-1)}\right.$$

$$\left.-\sum_{s=t+1}^{N}\sum_{r=k}^{N}p_{sr}(x_s-x_1)^{(t)}(y_r-y_{k-m+1})^{(m-1)}\right]$$

$$+\sum_{k=m+1}^{N}\sum_{t=n+1}^{N}\Delta_{(n,m)}f(x_{t-n},y_{k-m})(x_t-x_{t-n})(y_k-y_{k-m})\times$$

$$\times\left[\sum_{s=max\{t,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_{k-m+1})^{(m-1)}\right.$$

$$\left.-\sum_{s=t}^{N}\sum_{r=k}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_r-y_{k-m+1})^{(m-1)}\right] \quad (4.2.1)$$

holds, where $a^{(k)} = a(a-1)\ldots(a-k+1)$ *and* $a^{(0)} = 1$.

Proof. We start the proof by considering the expression

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{p}_{ij}f(x_i,y_i)$$

where $\tilde{p}_{ij}$ is defined as

$$\tilde{p}_{ij} = \begin{cases} \sum_{r=1}^{N} p_{ir} & , \quad i = j, \\ 0 & , \quad i \neq j. \end{cases}$$

By Corollary 3.2.2, we get

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{p}_{ij}f(x_i,y_i) = \sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}f(x_i,y_i)\\
&= \sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\Delta_{(t,k)}f(x_1,y_1) \sum_{s=max\{t+1,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_1)^{(k)}\\
&+ \sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\Delta_{(n,k)}f(x_{t-n},y_1)(x_t-x_{t-n})\times\\
&\times \sum_{s=max\{t,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_1)^{(k)}\\
&+ \sum_{k=m+1}^{N}\sum_{t=0}^{n-1}\Delta_{(t,m)}f(x_1,y_{k-m})(y_k-y_{k-m})\times\\
&\times \sum_{s=max\{t+1,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_{k-m+1})^{(m-1)}\\
&+ \sum_{k=m+1}^{N}\sum_{t=n+1}^{N}\Delta_{(n,m)}f(x_{t-n},y_{k-m})(x_t-x_{t-n})(y_k-y_{k-m})\times\\
&\times \sum_{s=max\{t,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_{k-m+1})^{(m-1)}.
\end{aligned}$$

So, we get our required result by putting the expressions $\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}f(x_i,y_i)$ and $\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}f(x_i,y_j)$ in $C_\Delta(f,p) = \sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}f(x_i,y_i) - \sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}f(x_i,y_j)$. ■

Remark 4.2.1. If we put $x_i = i$, $y_j = j$ and $f(x_i,y_j) = f(i,j) = a_{ij}$ in Theorem 4.2.1, then we get the following corollary. □

Corollary 4.2.2. *Let $p_{ij},\ a_{ij} \in \mathbb{R}$ for $i,j \in \{1,\ldots,N\}$. Then, the identity*

$$C_\Delta(a,p) = \sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}a_{ii} - \sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}a_{ij}$$

$$= \sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\Delta^{(t,k)}a_{11}\left[\sum_{s=max\{t,k\}+1}^{N}\sum_{r=1}^{N}p_{sr}\binom{s-1}{t}\binom{s-1}{k}\right.$$
$$\left.-\sum_{s=t+1}^{N}\sum_{r=k+1}^{N}p_{sr}\binom{s-1}{t}\binom{r-1}{k}\right]+\sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\Delta^{(n,k)}a_{(t-n)1}\times$$
$$\times\left[\sum_{s=max\{t,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}\binom{s-t+n-1}{n-1}\binom{s-1}{k}\right.$$
$$\left.-\sum_{s=t}^{N}\sum_{r=k+1}^{N}p_{sr}\binom{s-t+n-1}{n-1}\binom{r-1}{k}\right]+\sum_{k=m+1}^{N}\sum_{t=0}^{n-1}\Delta^{(t,m)}a_{1(k-m)}\times$$
$$\times\left[\sum_{s=max\{t+1,k\}}^{N}\sum_{r=1}^{N}p_{sr}\binom{s-1}{t}\binom{s-k+m-1}{m-1}\right.$$
$$\left.-\sum_{s=t+1}^{N}\sum_{r=k}^{N}p_{sr}\binom{s-1}{t}\binom{r-k+m-1}{m-1}\right]$$
$$+\sum_{k=m+1}^{N}\sum_{t=n+1}^{N}\Delta^{(n,m)}a_{(t-n)(k-m)}\left[\sum_{s=max\{t,k\}}^{N}\sum_{r=1}^{N}p_{sr}\binom{s-t+n-1}{n-1}\binom{s-k+m-1}{m-1}\right.$$
$$\left.-\sum_{s=t}^{N}\sum_{r=k}^{N}p_{sr}\binom{s-t+n-1}{n-1}\binom{r-k+m-1}{m-1}\right],$$

holds, where $\Delta^{(t,k)}a_{ij}$ represents finite difference of order (t,k) of the sequence (a_{ij}).

Remark 4.2.2. If we put $n = m = 1$ in Corollary 4.2.2, then we get Theorem 3 of [63]. □

Before we state our next theorem, under the assumptions of Theorem 4.2.1 we introduce some notations as follows:

$$\overline{C}_{\Delta}(f,p) = C_{\Delta}(f,p) - \sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\Delta_{(t,k)}f(x_1,y_1)\times$$
$$\times\left[\sum_{s=max\{t+1,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_1)^{(k)}\right.$$
$$\left.-\sum_{s=t+1}^{N}\sum_{r=k+1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_r-y_1)^{(k)}\right]$$

$$
\begin{aligned}
&-\sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\Delta_{(n,k)}f(x_{t-n},y_1)(x_t-x_{t-n})\times\\
&\times\Bigg[\sum_{s=max\{t,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_1)^{(k)}\\
&-\sum_{s=t}^{N}\sum_{r=k+1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_r-y_1)^{(k)}\Bigg]\\
&-\sum_{k=m+1}^{N}\sum_{t=0}^{n-1}\Delta_{(t,m)}f(x_1,y_{k-m})(y_k-y_{k-m})\times\\
&\times\Bigg[\sum_{s=max\{t+1,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_{k-m+1})^{(m-1)}\\
&-\sum_{s=t+1}^{N}\sum_{r=k}^{N}p_{sr}(x_s-x_1)^{(t)}(y_r-y_{k-m+1})^{(m-1)}\Bigg],
\end{aligned}
\tag{4.2.2}
$$

$$
\begin{aligned}
R_\Delta(t,k) &= \Bigg[\sum_{s=max\{t,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_{k-m+1})^{(m-1)}\\
&- \sum_{s=t}^{N}\sum_{r=k}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_r-y_{k-m+1})^{(m-1)}\Bigg].
\end{aligned}
\tag{4.2.3}
$$

Theorem 4.2.3. *let $p_{ij}\in\mathbb{R}$ for $i,j\in\{1,\ldots,N\}$ and let (x_i) and (y_j) for $i,j\in\{1,\ldots,N\}$ be two real sequences that are monotonic in the same sense. We also assume that f is an $(n,m)-$convex function. If*

$$R_\Delta(t,k)\geq 0,\quad t\in\{n+1,\ldots,N\},\quad k\in\{m+1,\ldots,N\},$$

then

$$\overline{C}_\Delta(f,p)\geq 0,$$

where $\overline{C}_\Delta$ and R_Δ are defined in (4.2.2) and (4.2.3) respectively.

Proof. The result follows easily by using identity (4.2.1). ■

Remark 4.2.3. If we put $x_i=i$, $y_j=j$ and $f(x_i,y_j)=f(i,j)=a_{ij}$ in Theorem 4.2.3 for $n=m=1$, then we get Theorem 3 of paper [75] and hence in this theorem for $a_{ij}=f(a_i,b_j)$ we get Corollary 2 of paper [75]. □

Theorem 4.2.4. *Let $p_{ij} \in \mathbb{R}$ and let $(x_i, y_j) \in I^2$ be the distinct points, where $i, j \in \{1, \ldots, N\}$. If $f, g : I^2 \to \mathbb{R}$ are two functions such that the inequalities*

$$R_\Delta(t,k) \geq 0, \quad t \in \{n+1, \ldots, N\}, \quad k \in \{m+1, \ldots, N\} \tag{4.2.4}$$

and

$$L\Delta_{(n,m)} g(x_i, y_j) \leq \Delta_{(n,m)} f(x_i, y_j) \leq U\Delta_{(n,m)} g(x_i, y_j) \tag{4.2.5}$$

hold, then the following inequalities are valid

$$L\overline{C}_\Delta(g,p) \leq \overline{C}_\Delta(f,p) \leq U\overline{C}_\Delta(g,p), \tag{4.2.6}$$

where R_Δ is defined in (4.2.3) *and L and U are some real constants.*

Proof. Let $F_1(x_i, y_j) = f(x_i, y_j) - Lg(x_i, y_j)$ and $F_2(x_i, y_j) = Ug(x_i, y_j) - f(x_i, y_j)$. Then $\Delta_{(n,m)} F_1(x_i, y_j) \geq 0$ and $\Delta_{(n,m)} F_2(x_i, y_j) \geq 0$. So, from Theorem 4.2.3 we easily obtain Theorem 4.2.4. ■

Remark 4.2.4. If reverse inequalities hold in (4.2.4) and (4.2.5), then the inequalities in (4.2.6) still hold. Moreover, if the reverse inequality holds in (4.2.4), then the reverse inequalities in (4.2.6) are valid. □

Remark 4.2.5. If we put $x_i = i$, $y_j = j$ and $f(x_i, y_j) = f(i,j) = a_{ij}$ and $g(i,j) = b_{ij}$ in previous theorem then we get Theorem 4 of paper [75]. □

4.3 Generalized Integral Čebyšev's Identity and Inequality

The first main result of this section is as follows:

Theorem 4.3.1. *Let $p, f : I^2 \to \mathbb{R}$ be two functions such that p is integrable, $f_{(N+1,M)}$ and $f_{(N,M+1)}$ exist and are absolutely continuous. Then, we have*

$$\begin{aligned} C(f,p) &= \int_a^b \int_a^b p(x,y) f(x,x) dy\, dx - \int_a^b \int_a^b p(x,y) f(x,y) dy\, dx \\ &= \sum_{i=0}^{N} \sum_{j=0}^{M} f_{(i,j)}(a,a) \left[\overline{P}^{(i,j)}(a,a) - P^{(i,j)}(a,a) \right] \\ &+ \sum_{j=0}^{M} \int_a^b f_{(N+1,j)}(x,a) \left[\overline{P}^{(N,j)}(x,a) - P^{(N,j)}(x,a) \right] dx \end{aligned}$$

$$+ \sum_{i=0}^{N} \int_a^b f_{(i,M+1)}(a,y) \left[\overline{P}^{(i,M)}(a,y) - P^{(i,M)}(a,y) \right] dy$$
$$+ \int_a^b \int_a^b f_{(N+1,M+1)}(x,y) R(x,y) dy\, dx$$

where $\overline{P}^{(i,j)}$, $P^{(i,j)}$ *and* R *are defined in* (4.1.10), (4.1.9) *and* (4.1.12) *respectively.*

Proof. To prove this identity, first we find an expression for $\int_a^b \int_a^b p(x,y) f(x,x) dy\, dx$ as follows. First we expand $f(x,x)$ in Taylor expansion of two variables and multiply it with $p(x,y)$ and integrate it over I^2 by variables x and y to get

$$\int_a^b \int_a^b p(x,y) f(x,x) dy\, dx$$
$$= \int_a^b \left[\sum_{j=0}^{M} \left(\sum_{i=0}^{N} f_{(i,j)}(a,a) \frac{(x-a)^i}{i!} \right) \int_a^b p(x,y) \frac{(x-a)^j}{j!} dy \right] dx$$
$$+ \int_a^b \left[\sum_{j=0}^{M} \left(\int_a^x f_{(N+1,j)}(s,a) \frac{(x-s)^N}{N!} ds \right) \int_a^b p(x,y) \frac{(x-a)^j}{j!} dy \right] dx$$
$$+ \int_a^b \left[\int_a^b \int_a^x p(x,y) \left(\sum_{i=0}^{N} f_{(i,M+1)}(a,t) \frac{(x-a)^i}{i!} \right) \frac{(x-t)^M}{M!} dt\, dy \right] dx$$
$$+ \int_a^b \left[\int_a^b \int_a^x \left(\int_a^x p(x,y) f_{(N+1,M+1)}(s,t) \frac{(x-s)^N}{N!} ds \right) \frac{(x-t)^M}{M!} dt\, dy \right] dx$$

In the first summand, we change the order of summation, use linearity of integral to obtain

$$\sum_{i=0}^{N} \sum_{j=0}^{M} \int_a^b \int_a^b p(x,y) f_{(i,j)}(a,a) \frac{(x-a)^i}{i!} \frac{(x-a)^j}{j!} dy dx.$$

By using Fubini's theorem, the second summand is rewritten as:

$$\int_a^b \left[\sum_{j=0}^{M} \left(\int_a^x f_{(N+1,j)}(s,a) \frac{(x-s)^N}{N!} ds \right) \int_a^b p(x,y) \frac{(x-a)^j}{j!} dy \right] dx$$
$$= \int_a^b \left[\sum_{j=0}^{M} \left(\int_a^x \int_a^b p(x,y) \frac{(x-a)^j}{j!} f_{(N+1,j)}(s,a) \frac{(x-s)^N}{N!} dy\, ds \right) \right] dx$$
$$= \sum_{j=0}^{M} \int_a^b \int_a^x \int_a^b p(x,y) f_{(N+1,j)}(s,a) \frac{(x-s)^N}{N!} \frac{(x-a)^j}{j!} dy\, ds\, dx$$

$$= \sum_{j=0}^{M} \int_a^b \int_s^b \int_a^b p(x,y) f_{(N+1,j)}(s,a) \frac{(x-s)^N}{N!} \frac{(x-a)^j}{j!} dy\, dx\, ds,$$

Similarly, the third summand is rewritten as:

$$\int_a^b \left[\int_a^b \int_a^x p(x,y) \left(\sum_{i=0}^{N} f_{(i,M+1)}(a,t) \frac{(x-a)^i}{i!} \right) \frac{(x-t)^M}{M!} dt\, dy \right] dx$$
$$= \sum_{i=0}^{N} \int_a^b \int_a^b \int_a^x p(x,y) f_{(i,M+1)}(a,t) \frac{(x-a)^i}{i!} \frac{(x-t)^M}{M!} dt\, dy\, dx$$
$$= \sum_{i=0}^{N} \int_a^b \int_a^b \int_t^b p(x,y) f_{(i,M+1)}(a,t) \frac{(x-a)^i}{i!} \frac{(x-t)^M}{M!} dy\, dx\, dt,$$

Finally, the fourth summand is rewritten as:

$$\int_a^b \left[\int_a^b \int_a^x \left(\int_a^x p(x,y) f_{(N+1,M+1)}(s,t) \frac{(x-s)^N}{N!} ds \right) \frac{(x-t)^M}{M!} dt\, dy \right] dx$$
$$= \int_a^b \int_a^b \int_a^x \int_a^x p(x,y) f_{(N+1,M+1)}(s,t) \frac{(x-s)^N}{N!} \frac{(x-t)^M}{M!} ds\, dt\, dy\, dx$$
$$= \int_a^b \int_a^b \int_{max\{s,t\}}^b \int_a^b p(x,y) f_{(N+1,M+1)}(s,t) \frac{(x-s)^N}{N!} \frac{(x-t)^M}{M!} dy\, dx\, dt\, ds.$$

Now, we add up all these results to get

$$\int_a^b \int_a^b p(x,y) f(x,x) dy\, dx$$
$$= \sum_{i=0}^{N} \sum_{j=0}^{M} \int_a^b \int_a^b p(x,y) f_{(i,j)}(a,a) \frac{(x-a)^i}{i!} \frac{(x-a)^j}{j!} dy\, dx$$
$$= \sum_{j=0}^{M} \int_a^b \int_s^b \int_a^b p(x,y) f_{(N+1,j)}(s,a) \frac{(x-s)^N}{N!} \frac{(x-a)^j}{j!} dy\, dx\, ds$$
$$= \sum_{i=0}^{N} \int_a^b \int_a^b \int_t^b p(x,y) f_{(i,M+1)}(a,t) \frac{(x-a)^i}{i!} \frac{(x-t)^M}{M!} dy\, dx\, dt$$
$$= \int_a^b \int_a^b \int_{max\{s,t\}}^b \int_a^b p(x,y) f_{(N+1,M+1)}(s,t) \frac{(x-s)^N}{N!} \frac{(x-t)^M}{M!} dy\, dx\, dt\, ds,$$

when we change the names of variables on the right-hand side $x \leftrightarrow s$, $y \leftrightarrow t$, then we have,

$$\int_a^b \int_a^b p(x,y)f(x,x)dy\,dx$$
$$= \sum_{i=0}^{N}\sum_{j=0}^{M}\int_a^b \int_a^b p(s,t)f_{(i,j)}(a,a)\frac{(s-a)^{i+j}}{i!j!}dt\,ds$$
$$+\sum_{j=0}^{M}\int_a^b \int_x^b \int_a^b p(s,t)f_{(N+1,j)}(x,a)\frac{(s-x)^N}{N!}\frac{(s-a)^j}{j!}dt\,ds\,dx$$
$$+\sum_{i=0}^{N}\int_a^b \int_a^b \int_x^b p(s,t)f_{(i,M+1)}(a,y)\frac{(s-a)^i}{i!}\frac{(s-y)^M}{M!}dt\,ds\,dy$$
$$+\int_a^b \int_a^b \int_{max\{x,y\}}^b \int_a^b p(s,t)f_{(N+1,M+1)}(x,y)\frac{(s-x)^N}{N!}\frac{(s-y)^M}{M!}\,dt\,ds\,dy\,dx,$$

by using defined notations we finally obtain

$$\int_a^b \int_a^b p(x,y)f(x,x)dy\,dx = \sum_{i=0}^{N}\sum_{j=0}^{M} f_{(i,j)}(a,a)\overline{P}^{(i,j)}(a,a)$$
$$+\sum_{j=0}^{M}\int_a^b f_{(N+1,j)}(x,a)\overline{P}^{(N,j)}(x,a)\,dx + \sum_{i=0}^{N}\int_a^b f_{(i,M+1)}(a,y)\overline{P}^{(i,M)}(a,y)\,dy$$
$$+\int_a^b \int_a^b f_{(N+1,M+1)}(x,y)\int_{max\{x,y\}}^b \int_a^b p(s,t)\frac{(s-x)^N}{N!}\frac{(s-y)^M}{M!}\,dt\,ds\,dy\,dx,$$

where $\overline{P}^{(i,j)}$ is defined in (4.1.9).
Using the above expression for $\int_a^b \int_a^b p(x,y)f(x,x)dy\,dx$ and Corollary 3.3.2 in

$$C(f,p) = \int_a^b \int_a^b p(x,y)f(x,x)dy\,dx - \int_a^b \int_a^b p(x,y)f(x,y)dy\,dx,$$

we get our required identity. ■

Remark 4.3.1. If in Theorem 4.3.1 we put $f(x,y) = f(x)g(y)$ and $p(x,y) = p(x)p(y)$, then we may state the following corollary. □

Corollary 4.3.2. *Let $p, f, g : I \to \mathbb{R}$ be three functions such that p is integrable and*

$f_{(N)}$ and $g_{(M)}$ exist and are absolutely continuous. Then, we have

$$\begin{aligned}
&T(f,g,p) = T(P_N(f),P_M(g),p) + T(R_N(f),P_M(g),p) + T(P_N(f),R_M(g),p)\\
&+\int_a^b p(x)\,dx \int_a^b\int_a^b\int_{max\{x,y\}}^b \frac{f_{(N+1)}(x)(s-x)^N}{N!}\frac{g_{(M+1)}(y)(s-y)^M}{M!}\,p(s)\,ds\,dy\,dx\\
&-\int_a^b R_N(f)(x)p(x)\,dx\int_a^b R_M(g)(x)p(x)\,dx \qquad (4.3.1)
\end{aligned}$$

where $P_k(h)(x) = \sum_{i=0}^{k}\frac{h^{(i)}(a)(x-a)^i}{i!}$, $R_k(h)(x) = \int_a^x \frac{h^{(N+1)}(s)(x-s)^N}{N!}\,ds$, $k \in \mathbb{N}$ for a function h and T is defined in (4.1.4).

Proof. We can get (4.3.1) directly by using Taylor formula for functions f and g. ■

Corollary 4.3.3. *Let the assumptions of Theorem* 4.3.1 *be valid. Then for $s,t>1$ such that $1/s+1/t=1$, we have*

$$\left|\overline{C}(f,p)\right| \le \left(\int_a^b\int_a^b \left|f_{(N+1,M+1)}(x,y)\right|^s\,dy\,dx\right)^{1/s}\left(\int_a^b\int_a^b |R(x,y)|^t\,dy\,dx\right)^{1/t} \qquad (4.3.2)$$

where $\overline{C}$ and R are defined in (4.1.15) *and* (4.1.12) *respectively.*

Proof. We can get (4.3.2) by using Hölder's inequality for integrals in Theorem 4.3.1.
■

Theorem 4.3.4. *Let the assumptions of Theorem* 4.3.1 *be valid. If the inequality*

$$R(x,y)\ge 0$$

holds for every $(x,\,y)\in I^2$, then there exist $(\xi,\,\eta)\in I^2$ such that

$$\overline{C}(f,p) = f_{(N+1,M+1)}(\xi,\eta)\,C\left(f_0,p\right),$$

where R, f_0 and $\overline{C}$ are defined in (4.1.12), (4.1.14) *and* (4.1.15), *respectively.*

Proof. We have

$$\overline{C}(f,p) = \int_a^b\int_a^b f_{(N+1,M+1)}(x,y)R(x,y)dy\,dx, \qquad (4.3.3)$$

using the mean value theorem for double integrals we get

$$\overline{C}(f,p) = f_{(N+1,M+1)}(\xi,\eta)\int_a^b\int_a^b R(x,y)dy\,dx.$$

If we put $f = f_0$ in above expression, then we obtain

$$\overline{C}\,(f_0, p) = C\,(f_0, p) = \int_a^b \int_a^b R(x, y)\, dy\, dx$$

and hence we get what we wanted. ■

Remark 4.3.2. (a) We can also proof Theorem 4.3.4 by following the proof of Theorem 3.4.1.

(b) For $N = M = 0$, Theorem 4.3.4 is equivalent to Proposition 4.1.4.

(c) If we choose $f(x, y) = f(x)g(y)$ and $p(x, y) = p(x)p(y)$ in Theorem 4.3.4 with $N = M = 0$, then we get (4.1.3).

□

Theorem 4.3.5. *Let the assumptions of Theorem* 4.3.1 *be valid and let* $g_{(N+1,M+1)} \neq 0$ *on* I^2 *where* $g \in C^{(N+1,M+1)}(I^2)$*. If the inequality*

$$R(x, y) \geq 0$$

holds for every $(x,\, y) \in I^2$*, then there exist* $(\xi,\, \eta) \in I^2$ *such that*

$$\overline{C}(f, p) = \frac{f_{(N+1,M+1)}(\xi, \eta)}{g_{(N+1,M+1)}(\xi, \eta)}\, \overline{C}(g, p),$$

where R *and* $\overline{C}$ *are defined in* (4.1.12) *and* (4.1.15) *respectively.*

Proof. Using (4.3.3) and the integral mean value theorem we have

$$\begin{aligned}
\overline{C}(f, p) &= \int_a^b \int_a^b \frac{f_{(N+1,M+1)}(x, y)}{g_{(N+1,M+1)}(x, y)}\, g_{(N+1,M+1)}(x, y) R(x, y) dy\, dx \\
&= \frac{f_{(N+1,M+1)}(\xi, \eta)}{g_{(N+1,M+1)}(\xi, \eta)} \int_a^b \int_a^b g_{(N+1,M+1)}(x, y) R(x, y) dy\, dx \\
&= \frac{f_{(N+1,M+1)}(\xi, \eta)}{g_{(N+1,M+1)}(\xi, \eta)}\, \overline{C}(g, p).
\end{aligned}$$

■

Remark 4.3.3. (a) We can also proof Theorem 4.3.5 by following the proof of Theorem 3.4.2.

(b) For $N = M = 0$, Theorem 4.3.5 becomes Theorem 2 of [75].

□

Theorem 4.3.6. *Let $p, f : I^2 \to \mathbb{R}$ be two functions such that p is integrable and f is $(N+1, M+1)-$convex. If the inequality*

$$R(x, y) \geq 0$$

holds for every $(x,\, y) \in I^2$, then the following inequality is valid

$$\overline{C}(f, p) \geq 0,$$

where R and $\overline{C}$ are defined in (4.1.12) *and* (4.1.15) *respectively.*

Proof. If f is $(N+1, M+1)-$convex function it may be approximated uniformly on I^2 by polynomials having nonnegative partial derivatives of order $(N+1, M+1)$. It is known that the Bernstein polynomials $B^{n,m}$ defined as

$$B^{n,m}(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} f(a_i, b_j)(x-a)^i (b-x)^{n-i} (y-a)^j (b-y)^{m-j}$$

(where $a_i = a + i\frac{b-a}{n}$, $b_j = a + j\frac{b-a}{m}$) converge uniformly to f on I^2 as $n, m \to \infty$ provided that f is continuous. Further, if f is $(N+1, M+1)-$convex function these polynomials have nonnegative partial derivatives of order $(N+1, M+1)$, i.e., $B^{n,m}_{(N+1,M+1)} \geq 0$ which can be prove by induction by using the following formula:

$$\begin{aligned} & B^{n,m}_{(N+1,M+1)}(x, y) = (N+1)!(M+1)! \binom{n}{N+1} \binom{m}{M+1} \times \\ & \times \sum_{i=0}^{n-N-1} \sum_{j=0}^{m-M-1} \binom{n-N-1}{i} \binom{m-M-1}{j} \times \\ & \times \Delta^{(N+1,M+1)} f(a_i, b_j)(x-a)^i (b-x)^{n-N-1-i} (y-a)^j (b-y)^{m-M-1-j}. \end{aligned}$$

As (a_i) and (b_j) are increasing sequences and f is $(N+1, M+1)-$convex function, so we have $\Delta^{(N+1,M+1)} f(a_i, b_j) \geq 0$. Since R is continuous and $B^{n,m}_{(N+1,M+1)} \geq 0$ on I^2 so by (4.1.15) we obtain

$$\begin{aligned} \overline{C}(B^{n,m}, p) \;&=\; \int_a^b \int_a^b B^{n,m}_{(N+1,M+1)}(x, y) \left[\int_{max\{s,t\}}^b \int_a^b p(s,t) \frac{(x-s)^N}{N!} \frac{(x-t)^M}{M!} dt\, ds \right. \\ &\quad - \left. \int_x^b \int_y^b p(s,t) \frac{(s-x)^N}{N!} \frac{(t-y)^M}{M!} dt\, ds \right] dy\, dx \;\geq\; 0, \end{aligned}$$

or we can write $\overline{C}(B^{n,m},p)$ as

$$\overline{C}(B^{n,m},p)=\int_a^b\int_a^b B^{n,m}_{(N+1,M+1)}(x,y)R(x,y)dy\,dx. \tag{4.3.4}$$

Now by letting $n,m\to\infty$ through an appropriate sequence, the uniform convergence of $B^{n,m}_{(N+1,M+1)}$ to $f_{(N+1,M+1)}$ provides our desired result. ■

Theorem 4.3.7. *Let the assumptions of Theorem* 4.3.6 *be valid. Then there exist* $(\xi,\eta)\in I^2$ *such that*

$$\overline{C}(f,p)=R(\xi,\eta)\left(f_{(N,M)}(b,b)-f_{(N,M)}(a,b)-f_{(N,M)}(b,a)+f_{(N,M)}(a,a)\right),$$

where R *and* $\overline{C}$ *are defined in* (4.1.12) *and* (4.1.15) *respectively.*

Proof. Since R is continuous and $B^{n,m}_{(N+1,M+1)}\geq 0$ on I^2, where $B^{n,m}$ is Bernstien polynomial, by same arguments used in proof of the Theorem 4.3.4, starting from (4.3.4), we obtain

$$\begin{aligned}
&\overline{C}(B^{n,m},p)\\
&=\int_a^b\int_a^b R(x,y)B^{n,m}_{(N+1,M+1)}(x,y)\,dy\,dx\\
&=R(\xi_{n,m},\eta_{n,m})\int_a^b\int_a^b B^{n,m}_{(N+1,M+1)}(x,y)\,dy\,dx\\
&=R(\xi_{n,m},\eta_{n,m})\left(B^{n,m}_{(N,M)}(b,b)-B^{n,m}_{(N,M)}(a,b)-B^{n,m}_{(N,M)}(b,a)+B^{n,m}_{(N,M)}(a,a)\right).
\end{aligned}$$

The points $x_{n,m}=(\xi_{n,m},\eta_{n,m})$ have a limit point (ξ,η) in I^2 as $n,m\to\infty$, so letting $n,m\to\infty$ through an appropriate sequence, the uniform convergence of $B^{n,m}_{(N,M)}$ to $f_{(N,M)}$ provides our desired result. ■

Remark 4.3.4. For $N=M=0$, Theorem 4.3.7 becomes Theorem 6 of [75]. □

4.4 Generalized Integral Fan's Identity and Inequality

Theorem 4.4.1. *Let the assumptions of Theorem* 4.3.1 *be valid and let* $q:I\to\mathbb{R}$ *be an integrable function. Then the following identity holds*

$$K(f,p,q) \;=\; \sum_{j=0}^{M}\sum_{i=0}^{N} f_{(i,j)}(a,a)\left[Q^{(i,j)}(a)-P^{(i,j)}(a,a)\right]$$

$$
\begin{aligned}
&+ \sum_{j=0}^{M} \int_a^b f_{(N+1,j)}(x,a)\left[Q^{(N,j)}(x) - P^{(N,j)}(x,a)\right] dx \\
&+ \sum_{i=0}^{N} \int_a^b f_{(i,M+1)}(a,y)\left[Q^{(M,i)}(y) - P^{(i,M)}(a,y)\right] dy \\
&+ \int_a^b \int_a^b f_{(N+1,M+1)}(x,y)\overline{R}(x,y)\, dy\, dx,
\end{aligned}
$$

where $P^{(i,j)}$, $Q^{(i,j)}$ and $\overline{R}$ are defined in (4.1.9), (4.1.11) *and* (4.1.13) *respectively.*

Proof. The proof of this theorem is analogous to proof of Theorem 4.3.1. We only need the following substitution $\int_a^b p(x,y)dy = q(x)$. ■

Remark 4.4.1. If in Theorem 4.4.1 we put $f(x,y) = f(x)g(y)$ and $p(x,y) = \frac{q(x)q(y)}{\int_a^b q(t)\,dt}$ where q is an integrable function such that $\int_a^b q(t)\,dt \neq 0$, then we state the following corollary. □

Corollary 4.4.2. *Let the assumptions of Corollary* 4.3.2 *be valid for functions f and g and let $q : I \to \mathbb{R}$ be an integrable function such that $\int_a^b q(t)\,dt \neq 0$. Then the identity*

$$
\begin{aligned}
T(f,g,q) &= T(P_N(f), P_M(g), q) + T(R_N(f), P_M(g), q) + T(P_N(f), R_M(g), q) \\
&+ \int_a^b \int_a^b \int_{max\{x,y\}}^b \frac{f_{(N+1)}(x)(s-x)^N}{N!} \frac{g_{(M+1)}(y)(s-y)^M}{M!} q(s)\, ds\, dy\, dx \\
&- \int_a^b R_N(f)(x)q(x)\, dx \int_a^b R_M(g)(x)q(x)\, dx
\end{aligned}
$$

holds, where $P_k(h)(x) = \sum_{i=0}^{k} \frac{h^{(i)}(a)(x-a)^i}{i!}$, $R_k(h)(x) = \int_a^x \frac{h^{(N+1)}(s)(x-s)^N}{N!} ds$, $k \in \mathbb{N}$ for a function h and T is defined in (4.1.4).

Corollary 4.4.3. *Let the assumptions of Theorem* 4.4.1 *be valid. Then for $s, t > 1$ such that $1/s + 1/t = 1$, we have*

$$
|\,\overline{K}(f,p,q)\,| \leq \left(\int_a^b \int_a^b |f_{(N+1,M+1)}(x,y)|^s \, dy\, dx \right)^{1/s} \left(\int_a^b \int_a^b \left|\overline{R}(x,y)\right|^t \, dy\, dx \right)^{1/t},
$$

where $\overline{R}$ and $\overline{K}$ are defined in (4.1.13) *and* (4.1.16) *respectively.*

Theorem 4.4.4. *Let the assumptions of Theorem* 4.3.6 *be valid. If the inequality*

$$
\overline{R}(x,y) \geq 0
$$

holds for every $(x, y) \in I^2$, *then there exist* $(\xi, \eta) \in I^2$ *such that*

$$\overline{K}(f, p, q) = f_{(N+1,M+1)}(\xi, \eta) K\left(f_0, p, q\right),$$

where $\overline{R}$, f_0 *and* $\overline{K}$ *are defined in* (4.1.13), (4.1.14) *and* (4.1.16) *respectively.*

Remark 4.4.2. The proof of this theorem may also be given in two different ways analogous to the proof of Theorem 4.3.4 and this theorem gives us Proposition 4.1.7 for $N = M = 0$. □

Theorem 4.4.5. *Let the assumptions of Theorem* 4.4.1 *be valid. If the inequality*

$$\overline{R}(x, y) \geq 0$$

holds for every $(x, y) \in I^2$, *then there exist* $(\xi, \eta) \in I^2$ *such that*

$$\overline{K}(f, p, q) = \frac{f_{(N+1,M+1)}(\xi, \eta)}{g_{(N+1,M+1)}(\xi, \eta)} \overline{K}(g, p, q),$$

where $\overline{R}$, f_0 *and* $\overline{K}$ *are defined in* (4.1.13), (4.1.14) *and* (4.1.16) *respectively.*

Remark 4.4.3. The proof of Theorem 4.4.5 may also be given in two different ways analogous to the proof of Theorem 4.3.5. Also for $N = M = 0$ we get Theorem 16 of [75]. □

Theorem 4.4.6. *Let the assumptions of Theorem* 4.3.6 *be valid for function* f *and let* $q : I \to \mathbb{R}$ *be an integrable function. If the inequality*

$$\overline{R}(x, y) \geq 0$$

holds for every $(x, y) \in I^2$, *then the following inequality holds*

$$\overline{K}(f, p, q) \geq 0,$$

where $\overline{R}$ *and* $\overline{K}$ *are defined in* (4.1.13) *and* (4.1.16) *respectively.*

Proof. The proof is analogous to proof of Theorem 4.3.6 so we omit the details. ■

The next chapter deals with the generalizations of Montgomery identities for higher order differentiable functions of two variables. We will also discuss the generalization of other important inequalities including the Ostrowski- and Grüss-type inequalities which are in fact a direct consequence of the Montgomery identities.

Chapter 5

Montgomery's Identities for Double Weighted Integrals of Higher Order Differentiable Functions

> "Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show."

–Bertrand Russell

The present chapter deals with the generalizations of weighted Montgomery's identities of two variable functions and its consequences. On one hand, the Montgomery identities have many applications in various fields, on the other hand these identities capture other well known and important identities and inequalities which include Ostrowski-, Čebyšev- and Grüss-type inequalities.

The Ostrowski-type inequalities have many applications in the field of numerical integrations and in probability theory. We can also obtain special means with the help of such inequalities. The celebrated Čebyšev's inequality is also a special case of the Ostrowski-type inequalities. As far as we are concerned with the Grüss-type inequalities, these inequalities play a paramount role in numerical integrations and in other fields. For detailed discussion on the topic we refer to the following monographs [6], [21] and [92].

In recent years a rapid advancement in generalizations and improvements of these type of inequalities has been observed. In present chapter we have also proposed certain generalizations of Montgomery's identities and hence generalizations of Ostrowski- and Grüs-type inequalities by using higher order differentiable functions. These identities and inequalities generalize many results given in [5], [22], [23], [28], [64] and [81] etc.

The contents of the present chapter belong to the article [41].

5.1 Preliminaries

The following identity is known as Montgomery's identity in literature [60] (see also [81]).

Proposition 5.1.1. *Let $f \in C^{(1)}[a,b]$. Then the identity*

$$(b-a)f(x) = \int_a^b f(s)ds + \int_a^b p(x,s)f'(s)ds, \tag{5.1.1}$$

holds for Peano kernel p defined as

$$p(x,s) = \begin{cases} s-a & , \quad a \le s \le x, \\ s-b & , \quad x < s \le b. \end{cases}$$

In [72], the following generalization of (5.1.1) can be found.

Proposition 5.1.2. *Let $f \in C^{(1)}[a,b]$. Then the identity*

$$f(x) = \int_a^b w(s)f(s)ds + \int_a^b p_w(x,s)f'(s)ds,$$

holds for weighted Peano kernel p_w defined as

$$p_w(x,s) = \begin{cases} W(s) & , \quad a \le s \le x, \\ W(s)-1 & , \quad x < s \le b, \end{cases}$$

where $w : [a,b] \to \mathbb{R}_*$ is such that $\int_a^b w(s)ds = 1$ and

$$W(s) = \begin{cases} 0 & , \quad s < a, \\ \int_a^s w(\xi)d\xi & , \quad s \in [a,b], \\ 1 & , \quad s > b. \end{cases}$$

For functions of two variables the following generalized identities were obtained by authors in [5] and [22].

Proposition 5.1.3. *Let $f \in C^{(1,1)}([a,b]\times[c,d])$. Then identities*

$$(b-a)(d-c)f(x,y) = -\int_a^b\int_c^d f(s,t)\,dt\,ds + (d-c)\int_a^b f(s,y)\,ds$$
$$+(b-a)\int_c^d f(x,t)\,dt + \int_a^b\int_c^d p(x,s)q(y,t)f_{(1,1)}(s,t)\,dt\,ds,$$

and

$$(b-a)(d-c)f(x,y) = \int_a^b\int_c^d f(s,t)\,dt\,ds + \int_a^b\int_c^d p(x,s)f_{(1,0)}(s,t)\,dt\,ds$$
$$+\int_a^b\int_c^d q(y,t)f_{(0,1)}(s,t)\,dt\,ds + \int_a^b\int_c^d p(x,s)\;q(y,t)f_{(1,1)}(s,t)\,dt\,ds,$$

hold, where p and q are the Peano kernals.

Pečarić and Vukelić in [81] gave the following weighted Montgomery's identities for functions of two variables.

Proposition 5.1.4. *Let $p:[a,b]\times[c,d]\to\mathbb{R}$ be an integrable function and P is defined as*

$$P(x,y) = \int_x^b\int_y^d p(\xi,\eta)\,d\eta\,d\xi. \tag{5.1.2}$$

If $f\in C^{(1,1)}([a,b]\times[c,d])$, then the following identity holds

$$P(a,c)f(x,y) = \int_a^b\int_c^d p(s,t)f(s,t)dtds + \int_a^b \hat{P}(x,s)f_{(1,0)}(s,y)\,ds$$
$$+\int_c^d \tilde{P}(y,t)f_{(0,1)}(x,t)dt - \int_a^b\int_c^d \bar{P}^{(N,M)}(x,s,y,t)f_{(1,1)}(s,t)dtds, \tag{5.1.3}$$

$$\text{where}\quad \hat{P}(x,s) = \begin{cases} \int_a^s\int_c^d p(\xi,\eta)d\eta\,d\xi &, \quad a\le s\le x,\\ -P(s,c) &, \quad x<s\le b,\end{cases}$$

$$\tilde{P}^{(i,M)}(x,y,t) = \begin{cases} \int_a^b\int_c^t p(\xi,\eta)d\eta\,d\xi &, \quad c\le t\le y,\\ -P(a,t) &, \quad y<t\le d,\end{cases}$$

$$\text{and}\quad \bar{P}(x,s,y,t) = \begin{cases} \int_a^s\int_c^t p(\xi,\eta)d\eta d\xi &, \quad a\le s\le x \quad , \quad c\le t\le y,\\ -\int_s^b\int_c^t p(\xi,\eta)d\eta d\xi &, \quad x<s\le b \quad , \quad c\le t\le y,\\ -\int_a^s\int_t^d p(\xi,\eta)d\eta d\xi &, \quad a\le s\le x \quad , \quad y<t\le d,\\ P(s,t) &, \quad x<s\le b, \quad y<t\le d.\end{cases}$$

Proposition 5.1.5. *Let the assumptions of Proposition* 5.1.4 *be valid. Then the identity*

$$P(a,c)f(x,y) = -\int_a^b\int_c^d p(s,t)f(s,t)\,dt\,ds + \int_a^b\int_c^d p(s,t)f(s,y)\,dt\,ds + \int_a^b\int_c^d p(s,t)f(x,t)\,dt\,ds + \int_a^b\int_c^d \bar{P}(x,s,y,t)f_{(1,1)}(s,t)\,dt\,ds, \quad (5.1.4)$$

holds, where $\bar{P}$ *is as defined in Proposition* 5.1.4.

Proposition 5.1.6. *Let the assumptions of Proposition* 5.1.4 *be valid. Then the identity*

$$\begin{aligned}[P(a,c)]^2 f(x,y) &= P(a,c)\int_a^b\int_c^d p(s,t)f(s,t)\,dt\,ds \\ &+ \int_a^b\left(\int_a^b\int_c^d p(\xi,t)\hat{P}(x,s)f_{(1,0)}(s,t)\,dt\,ds\right)d\xi \\ &+ \int_c^d\left(\int_a^b\int_c^d p(s,\eta)\tilde{P}(y,t)f_{(0,1)}(s,t)\,dt\,ds\right)d\eta \\ &+ \int_a^b\int_c^d \check{P}(x,s,y,t)f_{(1,1)}(s,t)\,dt\,ds,\end{aligned}$$

holds, where $\hat{P}$, $\tilde{P}$ *and* $\bar{P}$ *are defined in Proposition* 5.1.4 *and*

$$\check{P}(x,s,y,t) = 2\hat{P}(x,s)\tilde{P}(y,t) - P(a,c)\bar{P}(x,s,y,t).$$

5.2 Montgomery's Identities for Double Weighted Integrals of Higher Order Differentiable Functions

In the start of this section, we introduce some notations to reduce our lengthy expressions as follows:

$$P^{(i,j)}_{(a,c)\to(b,d)}(x,y) = \int_a^b\int_c^d p(\xi,\eta)\frac{(\xi-x)^i}{i!}\frac{(\eta-y)^j}{j!}\,d\eta\,d\xi, \quad (5.2.1)$$

$$P^{(0,j)}_{(a,c)\to(b,d)}(y) = \int_a^b\int_c^d p(\xi,\eta)\frac{(\eta-y)^j}{j!}\,d\eta\,d\xi, \quad (5.2.2)$$

$$P^{(i,0)}_{(a,c)\to(b,d)}(x) = \int_a^b\int_c^d p(\xi,\eta)\frac{(\xi-x)^i}{i!}\,d\eta\,d\xi, \quad (5.2.3)$$

$$R(x,y;f) = -\sum_{i=1}^{N}\sum_{j=1}^{M} f_{(i,j)}(x,y)P^{(i,j)}_{(a,c)\to(b,d)}(x,y) - \sum_{j=1}^{M} f_{(0,j)}(x,y)P^{(0,j)}_{(a,c)\to(b,d)}(y)$$
$$- \sum_{i=1}^{N} f_{(i,0)}(x,y)P^{(i,0)}_{(a,c)\to(b,d)}(x). \quad (5.2.4)$$

For our next theorem we give a lemma by restating Corollary 3.3.2 using our notations as follows.

Lemma 5.2.1. *Let $p : [a,b]\times[c,d] \to \mathbb{R}$ be an integrable function and let $f \in C^{(N+1,M+1)}([a,b]\times[c,d])$. Then the following identity holds*

$$\int_a^b\int_c^d p(x,y)f(x,y)dy\,dx$$
$$= \sum_{i=0}^{N}\sum_{j=0}^{M} P^{(i,j)}_{(a,c)\to(b,d)}(a,c)f_{(i,j)}(a,c)$$
$$+ \sum_{j=0}^{M}\int_a^b P^{(N,j)}_{(x,c)\to(b,d)}(x,c)f_{(N+1,j)}(x,c)\,dx$$
$$+ \sum_{i=0}^{N}\int_c^d P^{(i,M)}_{(a,y)\to(b,d)}(a,y)f_{(i,M+1)}(a,y)\,dy$$
$$+ \int_a^b\int_c^d P^{(N,M)}_{(x,y)\to(b,d)}(x,y)f_{(N+1,M+1)}(x,y)\,dy\,dx,$$

Now we give generalizations of Propositions 5.1.4, 5.1.5 and 5.1.6 respectively as follows:

Theorem 5.2.1. *Let the assumptions of Lemma* 5.2.1 *be valid. Then the identity*

$$P(a,c)f(x,y) = R(x,y;f) + \int_a^b\int_c^d p(s,t)f(s,t)dt\,ds$$
$$+ \sum_{j=0}^{M}\int_a^b \hat{P}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,ds + \sum_{i=0}^{N}\int_c^d \tilde{P}^{(i,M)}(x,y,t)f_{(i,M+1)}(x,t)\,dt$$
$$- \int_a^b\int_c^d \bar{P}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,dt\,ds, \quad (5.2.5)$$

holds, where

$$\hat{P}^{(N,j)}(x,s,y) = \begin{cases} P^{(N,j)}_{(a,c)\to(s,d)}(s,y) & , \quad a \le s \le x, \\ -P^{(N,j)}_{(s,c)\to(b,d)}(s,y) & , \quad x < s \le b, \end{cases}$$

$$\tilde{P}^{(i,M)}(x,y,t) = \begin{cases} P^{(i,M)}_{(a,c)\to(b,t)}(x,t) & , \quad c \le t \le y, \\ -P^{(i,M)}_{(a,t)\to(b,d)}(x,t) & , \quad y < t \le d, \end{cases}$$

$$\text{and} \quad \bar{P}^{(N,M)}(x,s,y,t) = \begin{cases} P^{(N,M)}_{(a,c)\to(s,t)}(s,t) & , \quad a \le s \le x \quad , \quad c \le t \le y, \\ -P^{(N,M)}_{(s,c)\to(b,t)}(s,t) & , \quad x < s \le b \quad , \quad c \le t \le y, \\ -P^{(N,M)}_{(a,t)\to(s,d)}(s,t) & , \quad a \le s \le x \quad , \quad y < t \le d, \\ P^{(N,M)}_{(s,t)\to(b,d)}(s,t) & , \quad x < s \le b \quad , \quad y < t \le d, \end{cases}$$

where $P^{(i,j)}_{(.,.)\to(.,.)}$ *for* $i,j \in \{N,M\}$ *is defined in* (5.2.1), *and* P *and* R *are as defined in* (5.1.2) *and* (5.2.4) *respectively.*

Proof. Using Lemma 5.2.1 for $[a,x]\times[c,y]$, we get

$$\begin{aligned}
&\int_a^x \int_c^y p(s,t) f(s,t) dt\, ds = \int_x^a \int_y^c p(s,t) f(s,t) dt\, ds \\
&= \sum_{i=0}^{N}\sum_{j=0}^{M} P^{(i,j)}_{(x,y)\to(a,c)}(x,y) f_{(i,j)}(x,y) + \sum_{j=0}^{M}\int_x^a P^{(N,j)}_{(s,y)\to(a,c)}(s,y) f_{(N+1,j)}(s,y)\, ds \\
&+ \sum_{i=0}^{N}\int_y^c P^{(i,M)}_{(x,t)\to(a,c)}(x,t) f_{(i,M+1)}(x,t)\, dt \\
&+ \int_x^a \int_y^c P^{(N,M)}_{(s,t)\to(a,c)}(s,t) f_{(N+1,M+1)}(s,t)\, dt\, ds \\
&= \sum_{i=0}^{N}\sum_{j=0}^{M} \Big[P^{(i,j)}_{(x,y)\to(b,d)}(x,y) - P^{(i,j)}_{(x,c)\to(b,d)}(x,y) - P^{(i,j)}_{(a,y)\to(b,d)}(x,y) \\
&+ P^{(i,j)}_{(a,c)\to(b,d)}(x,y) \Big] f_{(i,j)}(x,y) \\
&- \sum_{j=0}^{M}\int_a^x \Big[P^{(N,j)}_{(s,y)\to(b,d)}(s,y) - P^{(N,j)}_{(s,c)\to(b,d)}(s,y) - P^{(N,j)}_{(a,y)\to(b,d)}(s,y) \\
&+ P^{(N,j)}_{(a,c)\to(b,d)}(s,y) \Big] f_{(N+1,j)}(s,y)\, ds \\
&- \sum_{i=0}^{N}\int_c^y \Big[P^{(i,M)}_{(x,t)\to(b,d)}(x,t) - P^{(i,M)}_{(x,c)\to(b,d)}(x,t) - P^{(i,M)}_{(a,t)\to(b,d)}(x,t) \\
&+ P^{(i,M)}_{(a,c)\to(b,d)}(x,t) \Big] \times f_{(i,M+1)}(x,t)\, dt \\
&+ \int_a^x \int_c^y \Big[P^{(N,M)}_{(s,t)\to(b,d)}(s,t) - P^{(N,M)}_{(s,c)\to(b,d)}(s,t) - P^{(N,M)}_{(a,t)\to(b,d)}(s,t) \\
&+ P^{(N,M)}_{(a,c)\to(b,d)}(s,t) \Big] f_{(N+1,M+1)}(s,t)\, dt\, ds.
\end{aligned}$$

Similarly for $[x,b]\times[c,y]$, we have

$$\int_x^b\int_c^y p(s,t)f(s,t)dt\,ds = -\int_x^b\int_y^c p(s,t)f(s,t)dt\,ds$$

$$= -\sum_{i=0}^{N}\sum_{j=0}^{M} f_{(i,j)}(x,y)\Big[P^{(i,j)}_{(x,y)\to(b,d)}(x,y) - P^{(i,j)}_{(x,c)\to(b,d)}(x,y)\Big]$$

$$-\sum_{j=0}^{M}\int_x^b f_{(N+1,j)}(s,y)\Big[P^{(N,j)}_{(s,y)\to(b,d)}(s,y) - P^{(N,j)}_{(s,c)\to(b,d)}(s,y)\Big]\,ds$$

$$+\sum_{i=0}^{N}\int_c^y f_{(i,M+1)}(x,t)\Big[P^{(i,M)}_{(x,t)\to(b,d)}(x,t) - P^{(i,M)}_{(x,c)\to(b,d)}(x,t)\Big]\,dt$$

$$+\int_x^b\int_c^y f_{(N+1,M+1)}(s,t)\Big[P^{(N,M)}_{(s,t)\to(b,d)}(s,t) - P^{(N,M)}_{(s,c)\to(b,d)}(s,t)\Big]\,dt\,ds.$$

For $[a,x]\times[y,d]$, we obtain

$$\int_a^x\int_y^d p(s,t)f(s,t)dt\,ds = -\int_x^a\int_y^d p(s,t)f(s,t)dt\,ds$$

$$= -\sum_{i=0}^{N}\sum_{j=0}^{M} f_{(i,j)}(x,y)\Big[P^{(i,j)}_{(x,y)\to(b,d)}(x,y) - P^{(i,j)}_{(a,y)\to(b,d)}(x,y)\Big]$$

$$+\sum_{j=0}^{M}\int_a^x f_{(N+1,j)}(s,y)\Big[P^{(N,j)}_{(s,y)\to(b,d)}(s,y) - P^{(N,j)}_{(a,y)\to(b,d)}(s,y)\Big]\,ds$$

$$-\sum_{i=0}^{N}\int_y^d f_{(i,M+1)}(x,t)\Big[P^{(i,M)}_{(x,t)\to(b,d)}(x,t) - P^{(i,M)}_{(a,t)\to(b,d)}(x,t)\Big]\,dt$$

$$+\int_a^x\int_y^d f_{(N+1,M+1)}(s,t)\Big[P^{(N,M)}_{(s,t)\to(b,d)}(s,t) - P^{(N,M)}_{(a,t)\to(b,d)}(s,t)\Big]\,dt\,ds.$$

Finally for $[x,b]\times[y,d]$, we have

$$\int_x^b\int_y^d p(s,t)f(s,t)dt\,ds = \sum_{i=0}^{N}\sum_{j=0}^{M} f_{(i,j)}(x,y)P^{(i,j)}_{(x,y)\to(b,d)}(x,y)$$

$$+\sum_{j=0}^{M}\int_x^b f_{(N+1,j)}(s,y)P^{(N,j)}_{(s,y)\to(b,d)}(s,y)\,ds + \sum_{i=0}^{N}\int_y^d f_{(i,M+1)}(x,t)P^{(i,M)}_{(x,t)\to(b,d)}(x,t)\,dt$$

$$+\int_x^b\int_y^d f_{(N+1,M+1)}(s,t)P^{(N,M)}_{(s,t)\to(b,d)}(s,t)\,dt\,ds.$$

Adding up the four expressions we get our required result. ■

Theorem 5.2.2. *Let the assumptions of Lemma* 5.2.1 *be valid. Then the identity*

$$P(a,c)f(x,y) = R(x,y;f) + \sum_{j=1}^{M}\int_a^b\int_c^d p(s,\eta)\frac{(\eta-y)^j}{j!}f_{(0,j)}(s,y)\,d\eta\,ds$$
$$+\sum_{i=1}^{N}\int_a^b\int_c^d p(\xi,t)\frac{(\xi-x)^i}{i!}f_{(i,0)}(x,t)\,dt\,d\xi - \int_a^b\int_c^d p(s,t)f(s,t)dt\,ds$$
$$+\int_a^b\int_c^d p(s,t)f(s,y)\,dt\,ds + \int_a^b\int_c^d p(s,t)f(x,t)\,dt\,ds$$
$$+\int_a^b\int_c^d \bar{P}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,dt\,ds, \tag{5.2.6}$$

holds, where $\bar{P}^{(N,M)}$ *is as in Theorem* 5.2.1, P *and* R *are defined in* (5.1.2) *and* (5.2.4) *respectively.*

Proof. First we find an expression for

$$\int_a^b \hat{P}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,ds$$

by using integration by parts as follows:

$$\int_a^b \hat{P}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,ds$$
$$= \int_a^x P^{(N,j)}_{(a,c)\to(s,d)}(s,y)f_{(N+1,j)}(s,y)\,ds - \int_x^b P^{(N,j)}_{(s,c)\to(b,d)}(s,y)f_{(N+1,j)}(s,y)\,ds$$
$$= \int_a^x P^{(N,j)}_{(a,c)\to(s,d)}(s,y)f_{(N+1,j)}(s,y)\,ds + \int_x^b P^{(N,j)}_{(b,c)\to(s,d)}(s,y)f_{(N+1,j)}(s,y)\,ds$$
$$= P^{(N,j)}_{(a,c)\to(x,d)}(x,y)f_{(N,j)}(x,y) + \int_a^x P^{(N-1,j)}_{(a,c)\to(s,d)}(s,y)f_{(N,j)}(s,y)\,ds$$
$$+P^{(N,j)}_{(x,c)\to(b,d)}(x,y)f_{(N,j)}(x,y) + \int_x^b P^{(N-1,j)}_{(b,c)\to(s,d)}(s,y)f_{(N,j)}(s,y)\,ds$$
$$= P^{(N,j)}_{(a,c)\to(b,d)}(x,y)f_{(N,j)}(x,y) + \int_a^x P^{(N-1,j)}_{(a,c)\to(s,d)}(s,y)f_{(N,j)}(s,y)\,ds$$
$$+\int_x^b P^{(N-1,j)}_{(b,c)\to(s,d)}(s,y)f_{(N,j)}(s,y)\,ds$$
$$= P^{(N,j)}_{(a,c)\to(b,d)}(x,y)f_{(N,j)}(x,y) + \int_a^b P^{(N-1,j)}_{(a,c)\to(s,d)}(s,y)f_{(N,j)}(s,y)\,ds,$$

continuing in similar fashion, we finally get

$$\int_a^b \hat{P}^{(N,j)}(x,s,y) f_{(N+1,j)}(s,y)\,ds$$
$$= \int_a^b \int_c^d p(\xi,\eta) \frac{(\eta-y)^j}{j!} \Big[\sum_{k=0}^{N} \frac{(\xi-x)^k}{k!} f_{(k,j)}(x,y) \Big]\, d\eta\, d\xi$$
$$- \int_a^b \int_c^d p(s,\eta) \frac{(\eta-y)^j}{j!} f_{(0,j)}(s,y)\, d\eta\, ds. \tag{5.2.7}$$

Similarly
$$\int_c^d \tilde{P}^{(i,M)}(x,y,t) f_{(i,M+1)}(x,t)\,dt$$
$$= \int_a^b \int_c^d p(\xi,\eta) \frac{(\xi-x)^i}{i!} \Big[\sum_{l=0}^{M} \frac{(\eta-y)^l}{l!} f_{(i,l)}(x,y) \Big]\, d\eta\, d\xi$$
$$- \int_a^b \int_c^d p(\xi,t) \frac{(\xi-x)^i}{i!} f_{(i,0)}(x,t)\, d\xi\, dt. \tag{5.2.8}$$

If we put all these values in (5.2.5), then after some cancelation and some rearrangements we get our required identity. ■

Theorem 5.2.3. *Let $f \in C^{(2N+1,\,2M+1)}([a,b]\times[c,d])$. Then the identity*

$$[P(a,c)]^2 f(x,y) = P(a,c)R(x,y;f) + P(a,c)\int_a^b \int_c^d p(s,t)f(s,t)dt\,ds$$
$$+ \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,y) R(s,y;f_{(N+1,j)})\,ds$$
$$+ \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(x,y,t) R(x,t;f_{(i,M+1)})\,dt$$
$$+ \sum_{i=0}^{N}\sum_{j=0}^{M} \int_a^b \int_a^b \int_c^d \hat{P}^{(N,j)}(x,s,y) p(\xi,t) \frac{(\xi-x)^i}{i!} f_{(N+1+i,j)}(s,t)\,dt\,ds\,d\xi$$
$$+ \sum_{i=0}^{N}\sum_{j=0}^{M} \int_c^d \int_a^b \int_c^d \tilde{P}^{(i,M)}(x,y,t) p(s,\eta) \frac{(\eta-y)^j}{j!} f_{(i,M+1+j)}(s,t)\,dt\,ds\,d\eta$$
$$+ \int_a^b \int_c^d \Big[2\sum_{i=0}^{N}\sum_{j=0}^{M} \hat{P}^{(N,j)}(x,s,y)\tilde{P}^{(i,M)}(x,y,t) f_{(N+1+i,M+1+j)}(s,t)$$
$$- \bar{P}^{(N,M)}(x,s,y,t) f_{(N+1,M+1)}(s,t) \Big]\, dt\, ds, \tag{5.2.9}$$

holds, where p, P, $\hat{P}^{(N,j)}$, $\tilde{P}^{(i,M)}$ are $\bar{P}^{(N,M)}$ are as in Theorem 5.2.1.

Proof. Summing (5.2.7) for $j \in \{0, \dots, M\}$ and (5.2.8) for $i \in \{0, \dots, N\}$, we get respectively for each $(x, y) \in [a, b] \times [c, d]$.

$$\begin{aligned} P(a,c)f(x,y) &= R(x,y;f) + \sum_{j=0}^{M} \int_a^b \int_c^d p(s,\eta) \frac{(\eta - y)^j}{j!} f_{(0,j)}(s,y)\, d\eta\, ds \\ &+ \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,y) f_{(N+1,j)}(s,y)\, ds, \end{aligned} \tag{5.2.10}$$

and

$$\begin{aligned} P(a,c)f(x,y) &= R(x,y;f) + \sum_{i=0}^{N} \int_a^b \int_c^d p(\xi,t) \frac{(\xi - x)^i}{i!} f_{(i,0)}(x,t)\, dt\, d\xi \\ &+ \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(x,y,t) f_{(i,M+1)}(x,t)\, dt. \end{aligned} \tag{5.2.11}$$

By using formula (5.2.10) for partial derivatives $f_{(i,M+1)}$ for $i \in \{0, \dots, N\}$, we obtain

$$\begin{aligned} &P(a,c)f_{(i,M+1)}(x,t) \\ &= R(x,t;f_{(i,M+1)}) + \sum_{j=0}^{M} \int_a^b \int_c^d p(s,\eta) \frac{(\eta - t)^j}{j!} f_{(i,M+1+j)}(s,t)\, d\eta\, ds \\ &+ \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,t) f_{(N+1+i,M+1+j)}(s,t)\, ds. \end{aligned} \tag{5.2.12}$$

Similarly, by using formula (5.2.11) for partial derivatives $f_{(N+1,j)}$ for $j \in \{0, \dots, M\}$, we have

$$\begin{aligned} &P(a,c)f_{(N+1,j)}(s,y) \\ &= R(s,y;f_{(N+1,j)}) + \sum_{i=0}^{N} \int_a^b \int_c^d p(\xi,t) \frac{(\xi - s)^i}{i!} f_{(N+1+i,j)}(s,t)\, dt\, d\xi \\ &+ \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(s,y,t) f_{(N+1+i,M+1+j)}(s,t)\, dt. \end{aligned} \tag{5.2.13}$$

Substituting (5.2.12) and (5.2.13) into (5.2.5), we get

$$\begin{aligned} P(a,c)f(x,y) &= R(x,y;f) + \int_a^b \int_c^d p(s,t) f(s,t) dt\, ds \\ &+ \frac{1}{P(a,c)} \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,y) \Big[R(s,y;f_{(N+1,j)}) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=0}^{N} \int_a^b \int_c^d p(\xi,t) \frac{(\xi-s)^i}{i!} f_{(N+1+i,j)}(s,t)\, dt\, d\xi \\
&+ \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(s,y,t) f_{(N+1+i,M+1+j)}(s,t)\, dt \Big]\, ds \\
&+ \frac{1}{P(a,c)} \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(x,y,t) \Big[R(x,t; f_{(i,M+1)}) \\
&+ \sum_{j=0}^{M} \int_a^b \int_c^d p(s,\eta) \frac{(\eta-t)^j}{j!} f_{(i,M+1+j)}(s,t)\, d\eta\, ds \\
&+ \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,t) f_{(N+1+i,M+1+j)}(s,t)\, ds \Big]\, dt \\
&- \int_a^b \int_c^d \bar{P}^{(N,M)}(x,s,y,t) f_{(N+1,M+1)}(s,t)\, dt\, ds.
\end{aligned}
$$

After some rearrangements and using Fubini's Theorem we obtain our required result. ■

Remark 5.2.1. For $N = M = 0$, Propositions 5.1.4, 5.1.5 and 5.1.6 become special cases of Theorems 5.2.1, 5.2.2 and 5.2.3 respectively. □

5.2.1 Special Cases

If $p(s,t) = q(s)r(t)$ in identities (5.2.5), (5.2.6) and (5.2.9), then we get respectively the following special cases:

$$
\begin{aligned}
& f(x,y) P_{a\to b}(q) P_{c\to d}(r) = Q(x,y;f) + \int_a^b \int_c^d q(s) r(t) f(s,t) dt\, ds \\
& + \sum_{j=0}^{M} \int_a^b \hat{Q}^{(N,j)}(x,s,y) f_{(N+1,j)}(s,y)\, ds \\
& + \sum_{i=0}^{N} \int_c^d \tilde{Q}^{(i,M)}(x,y,t) f_{(i,M+1)}(x,t)\, dt \\
& - \int_a^b \int_c^d \bar{Q}^{(N,M)}(x,s,y,t) f_{(N+1,M+1)}(s,t)\, dt\, ds,
\end{aligned}
$$

$$
f(x,y) P_{a\to b}(q) P_{c\to d}(r) = Q(x,y;f) + \sum_{j=1}^{M} \int_a^b q(s) f_{(0,j)}(s,y)\, ds\, Q^{(j)}_{c\to d}(r,y)
$$

$$+\sum_{i=1}^{N} Q^{(i)}_{a\to b}(q,x)\int_c^d r(t)f_{(i,0)}(x,t)\,dt - \int_a^b\int_c^d q(s)r(t)f(s,t)dt\,ds$$
$$+\int_a^b\int_c^d q(s)r(t)f(s,y)\,dt\,ds + \int_a^b\int_c^d q(s)r(t)f(x,t)\,dt\,ds$$
$$-\int_a^b\int_c^d \bar{Q}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,dt\,ds,$$
$$f(x,y)[P_{a\to b}(q)P_{c\to d}(r)]^2 = P_{a\to b}(q)P_{c\to d}(r)Q(x,y;f)$$
$$+\sum_{j=0}^{M}\int_a^b \hat{Q}^{(N,j)}(x,s,y)Q(s,y;f_{(N+1,j)})\,ds$$
$$+\sum_{i=0}^{N}\int_c^d \tilde{Q}^{(i,M)}(x,y,t)Q(x,t;f_{(i,M+1)})\,dt$$
$$+P_{a\to b}(q)P_{c\to d}(r)\int_a^b\int_c^d q(s)r(t)f(s,t)dt\,ds$$
$$+\sum_{i=0}^{N}\sum_{j=0}^{M} Q^{(i)}_{a\to b}(q,x)\int_a^b\int_c^d \hat{Q}^{(N,j)}(x,s,y)r(t)f_{(N+1+i,j)}(s,t)\,dt\,ds$$
$$+\sum_{i=0}^{N}\sum_{j=0}^{M} Q^{(j)}_{c\to d}(r,y)\int_a^b\int_c^d \tilde{Q}^{(i,M)}(x,y,t)q(s)\,f_{(i,M+1+j)}(s,t)\,dt\,ds$$
$$+\int_a^b\int_c^d\Big[2\sum_{i=0}^{N}\sum_{j=0}^{M}\hat{Q}^{(N,j)}(x,s,y)\tilde{Q}^{(i,M)}(x,y,t)f_{(N+1+i,M+1+j)}(s,t)$$
$$-\bar{Q}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\Big]\,dt\,ds,$$

where $\quad P_{a\to b}(q) = \int_a^b q(s)\,ds, \qquad\qquad Q^{(i)}_{a\to b}(q,x) = \int_a^b q(\xi)\frac{(\xi-x)^i}{i!}\,d\xi,$

$$\begin{aligned}
Q^{(i,j)}_{(a,c)\to(b,d)}(x,y) &= Q^{(i)}_{a\to b}(q,x)Q^{(j)}_{c\to d}(r,y),\\
Q^{(0,j)}_{(a,c)\to(b,d)}(y) &= P_{a\to b}(q)\;Q^{(j)}_{c\to d}(r,y),\\
Q^{(i,0)}_{(a,c)\to(b,d)}(x) &= Q^{(i)}_{a\to b}(q,x)\;P_{c\to d}(r),
\end{aligned}$$

$$\begin{aligned}
Q(x,y;f) &= -\sum_{i=1}^{N}\sum_{j=1}^{M} f_{(i,j)}(x,y)Q^{(i,j)}_{(a,c)\to(b,d)}(x,y)\\
&- \sum_{j=1}^{M} f_{(0,j)}(x,y)Q^{(0,j)}_{(a,c)\to(b,d)}(y) - \sum_{i=1}^{N} f_{(i,0)}(x,y)Q^{(i,0)}_{(a,c)\to(b,d)}(x),
\end{aligned}$$

$$\hat{Q}^{(N,j)}(x,s,y) = \begin{cases} Q^{(N,j)}_{(a,c)\to(s,d)}(s,y) & , \quad a \le s \le x, \\ -Q^{(N,j)}_{(s,c)\to(b,d)}(s,y) & , \quad x < s \le b, \end{cases}$$

$$\tilde{Q}^{(i,M)}(x,y,t) = \begin{cases} Q^{(i,M)}_{(a,c)\to(b,t)}(x,t) & , \quad c \le t \le y, \\ -Q^{(i,M)}_{(a,t)\to(b,d)}(x,t) & , \quad y < t \le d, \end{cases}$$

$$\text{and} \quad \bar{Q}^{(N,M)}(x,s,y,t) = \begin{cases} Q^{(N,M)}_{(a,c)\to(s,t)}(s,t) & , \quad a \le s \le x, \quad c \le t \le y, \\ -Q^{(N,M)}_{(s,c)\to(b,t)}(s,t) & , \quad x < s \le b, \quad c \le t \le y, \\ -Q^{(N,M)}_{(a,t)\to(s,d)}(s,t) & , \quad a \le s \le x, \quad y < t \le d, \\ Q^{(N,M)}_{(s,t)\to(b,d)}(s,t) & , \quad x < s \le b, \quad y < t \le d. \end{cases}$$

Particularly, if $p \equiv 1$ in identities (5.2.5), (5.2.6) and (5.2.9) then the expressions will look like

$$P_{a\to b} = b - a, \qquad Q^{(i)}_{a\to b}(x) = \frac{(b-x)^{i+1} - (a-x)^{i+1}}{(i+1)!},$$

$$\begin{aligned} Q(x,y;f) &= -\sum_{i=1}^{N}\sum_{j=1}^{M} \frac{(b-x)^{i+1} - (a-x)^{i+1}}{(i+1)!} \, \frac{(d-y)^{j+1} - (c-y)^{j+1}}{(j+1)!} \, f_{(i,j)}(x,y) \\ &- (b-a)\sum_{j=1}^{M} \frac{(d-y)^{j+1} - (c-y)^{j+1}}{(j+1)!} \, f_{(0,j)}(x,y) \\ &- (d-c)\sum_{i=1}^{N} \frac{(b-x)^{i+1} - (a-x)^{i+1}}{(i+1)!} \, f_{(i,0)}(x,y), \end{aligned}$$

$$\hat{Q}^{(N,j)}(x,s,y) = \begin{cases} -\frac{(a-s)^{N+1}}{(N+1)!} \, \frac{(d-y)^{j+1}-(c-y)^{j+1}}{(j+1)!} & , \quad a \le s \le x, \\ -\frac{(b-s)^{N+1}}{(N+1)!} \, \frac{(d-y)^{j+1}-(c-y)^{j+1}}{(j+1)!} & , \quad x < s \le b, \end{cases}$$

$$\tilde{Q}^{(i,M)}(x,y,t) = \begin{cases} -\frac{(c-t)^{M+1}}{(M+1)!} \, \frac{(b-x)^{i+1}-(a-x)^{i+1}}{(i+1)!} & , \quad c \le t \le y, \\ -\frac{(d-t)^{M+1}}{(M+1)!} \, \frac{(b-x)^{i+1}-(a-x)^{i+1}}{(i+1)!} & , \quad y < t \le d, \end{cases}$$

$$\text{and} \quad \bar{Q}^{(N,M)}(x,s,y,t) = \begin{cases} \frac{(a-s)^{N+1}}{(N+1)!} \, \frac{(c-t)^{M+1}}{(M+1)!} & , \quad a \le s \le x \quad , \quad c \le t \le y, \\ \frac{(b-s)^{N+1}}{(N+1)!} \, \frac{(c-t)^{M+1}}{(M+1)!} & , \quad x < s \le b \quad , \quad c \le t \le y, \\ \frac{(a-s)^{N+1}}{(N+1)!} \, \frac{(d-t)^{M+1}}{(M+1)!} & , \quad a \le s \le x \quad , \quad y < t \le d, \\ \frac{(b-s)^{N+1}}{(N+1)!} \, \frac{(d-t)^{M+1}}{(M+1)!} & , \quad x < s \le b \quad , \quad y < t \le d. \end{cases}$$

5.3 Ostrowski's Inequalities for Double Weighted Integrals of Higher Order Differentiable Functions

Let us recall an inequality which is known as Ostrowski's inequality in literature [64].

Proposition 5.3.1. *Let $f \in C^{(1)}[a,b]$ satisfy the condition $|f'(x)| \leq M$ for each $x \in [a,b] \subset \mathbb{R}$. Then the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq M\left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a), \quad x \in [a,b], \tag{5.3.1}$$

holds.

There were given many generalizations of this inequality. In [81], Pečarić and Vukelić also have given generalizations of this inequality by using identities (5.1.3) and (5.1.4). By using identities (5.2.5) and (5.2.6) we can give generalized results of Ostrowski-type for higher order differentiable functions of two independent variables as follows:

Theorem 5.3.1. *Let $f \in C^{(N+1,M+1)}([a,b]\times[c,d])$. Then the inequality*

$$\left| f(x,y) - \frac{1}{P(a,c)}\int_a^b\int_c^d p(s,t)f(s,t)\,dt\,ds \right| \leq D(x,y) + \sum_{j=0}^{M}\hat{D}^{(0,j)}(x,y) + \sum_{i=0}^{N}\tilde{D}^{(i,0)}(x,y) + \bar{D}(x,y),$$

holds for each $(x,y) \in [a,b]\times[c,d]$, where

$$D(x,y) = \frac{1}{|P(a,c)|}|R(x,y;f)|,$$

$$\hat{D}^{(0,j)}(x,y) = \frac{1}{|P(a,c)|}\left(\sum_{j=0}^{M}\int_a^b |\hat{P}^{(N,j)}(x,s,y)|^{\hat{q}_j}ds\right)^{1/\hat{q}_j} .\|f_{(N+1,j)}\|_{\hat{p}_i},$$

provided that $f_{(N+1,j)} \in L_{\hat{p}_j}([a,b]\times[c,d])$, $1/\hat{p}_j + 1/\hat{q}_j = 1$,

$$\tilde{D}^{(i,0)}(x,y) = \frac{1}{|P(a,c)|}\left(\sum_{i=0}^{N}\int_c^d |\tilde{P}^{(i,M)}(x,y,t)|^{\tilde{q}_i}dt\right)^{1/\tilde{q}_i} .\|f_{(i,M+1)}\|_{\tilde{p}_i},$$

provided that $f_{(i,M+1)} \in L_{\tilde{p}_i}([a,b]\times[c,d])$, $1/\tilde{p}_i + 1/\tilde{q}_i = 1$,

$$\bar{D}(x,y) = \frac{1}{|P(a,c)|}\left(\int_a^b\int_c^d |\bar{P}^{(N,M)}(x,s,y,t)|^{\bar{q}}dt\,ds\right)^{1/\bar{q}}.\|f_{(N+1,M+1)}\|_{\bar{p}},$$
provided that $f_{(N+1,M+1)} \in L_{\bar{p}}([a,b]\times[c,d]),\ 1/\bar{p}+1/\bar{q}=1,$

where p, P, $\hat{P}^{(N,j)}$, $\tilde{P}^{(i,M)}$ *and* $\bar{P}^{(N,M)}$ *are as in Theorem* 5.2.1 *whereas* R *is defined in* (5.2.4).

Proof. Identity (5.2.5) can be rewritten as

$$\begin{aligned}
& f(x,y) - \frac{1}{P(a,c)}\int_a^b\int_c^d p(s,t)f(s,t)\,dt\,ds \\
& = \frac{1}{P(a,c)}\Big[R(x,y;f) + \sum_{j=0}^{M}\int_a^b \hat{P}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,ds \\
& + \sum_{i=0}^{N}\int_c^d \tilde{P}^{(i,M)}(x,y,t)f_{(i,M+1)}(x,t)\,dt \\
& - \int_a^b\int_c^d \bar{P}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,dt\,ds\Big].
\end{aligned}$$

Now, taking absolute value and applying Hölder's inequality for double integrals, we easily obtain our required inequality. ■

Remark 5.3.1. For $N=M=0$, Theorem 4 of [81] becomes special case of Theorem 5.3.1 and we also retrieve results of [23] by simply putting $p\equiv 1$. □

Theorem 5.3.2. *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be a continuous function such that* $f\in C^{(N+1,M+1)}((a,b)\times(c,d))$ *and* $|f_{(N+1,M+1)}|^q$ *be an integrable function such that*

$$\left\|f_{(N+1,M+1)}\right\|_q := \left(\int_a^b\int_c^d |f_{(N+1,M+1)}(s,t)|^q\,dt\,ds\right)^{1/q} < \infty.$$

Then the inequality

$$\begin{aligned}
& \Big|\int_a^b\int_c^d p(s,t)f(x,t)\,dt\,ds - \Big[R(x,y;f) \\
& + \sum_{j=1}^{M}\int_a^b\int_c^d p(s,\eta)\frac{(\eta-y)^j}{j!}f_{(0,j)}(s,y)\,d\eta\,ds \\
& + \sum_{i=1}^{N}\int_a^b\int_c^d p(\xi,t)\frac{(\xi-x)^i}{i!}f_{(i,0)}(x,t)\,dt\,d\xi \\
& + \int_a^b\int_c^d p(s,t)f(x,t)\,dt\,ds
\end{aligned}$$

$$+\int_a^b\int_c^d p(s,t)f(s,y)dt\,ds - P(a,c)f(x,y)\Big]\Big|$$
$$\leq \left(\int_a^b\int_c^d |\bar{P}^{(N,M)}(x,s,y,t)|\,dt\,ds\right)^{1/q}\|f_{(N+1,M+1)}\|_{q'}.$$

holds for each $(x,y)\in[a,b]\times[c,d]$, *where* $1/q+1/q'=1$, $q,q'>1$ *and* P, $\bar{P}^{(N,M)}$ *are as in Theorem* 5.2.1.

Proof. Identity (5.2.6) may be rewritten as

$$\int_a^b\int_c^d p(s,t)f(s,t)\,dt\,ds - \Big[R(x,y;f)$$
$$+\int_a^b\int_c^d p(s,t)f(s,y)\,dt\,ds + \int_a^b\int_c^d p(s,t)f(x,t)dt\,ds$$
$$+\sum_{j=1}^{M}\int_a^b\int_c^d p(s,\eta)\frac{(\eta-y)^j}{j!}f_{(0,j)}(s,y)\,d\eta\,ds$$
$$+\sum_{i=1}^{N}\int_a^b\int_c^d p(\xi,t)\frac{(\xi-x)^i}{i!}f_{(i,0)}(x,t)\,dt\,d\xi - P(a,c)f(x,y)\Big]$$
$$=\int_a^b\int_c^d \bar{P}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,dt\,ds.$$

Now, taking absolute value and applying Hölder's inequality for double integrals, we easily obtain our required inequality. ■

Remark 5.3.2. For $N=M=0$, Theorem 5 of [81] becomes special case of Theorem 5.3.2 and we also retrieve results of [5] and [22] by simply putting $p\equiv 1$. □

5.4 Grüss' Inequalities for Double Weighted Integrals of Higher Order Differentiable Functions

In 1935, Grüss [27] gave the following celebrated integral inequality.

Proposition 5.4.1. *Let* $f,g:[a,b]\to\mathbb{R}$ *be two integrable functions satisfying the conditions*

$$m_1\leq f(x)\leq M_1,\quad m_2\leq g(x)\leq M_2,$$

for each $x \in [a,b]$ and some real constants m_1, m_2, M_1 and M_2. Then the following inequality holds

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \leq$$
$$\leq \frac{1}{4}(M_1-m_1)(M_2-m_2).$$

In [81], Pečarić and Vukelić gave new Grüss-type inequalities for double weighted integrals by using identities (5.1.3) and (5.1.4). Now, we give more generalized results by using higher order differentiable functions of two independent variables but in order to simplify the details of the presentations we define the following notations.

$$A^{(i,j)}(x,y) = p(x,y)[f_{(i,j)}(x,y)g(x,y) + g_{(i,j)}(x,y)f(x,y)]P^{(i,j)}_{(a,c)\to(b,d)}(x,y), \tag{5.4.1}$$

$$A(x,y) = p(x,y)\int_a^b\int_c^d p(s,t)[f(s,t)g(x,y) + g(s,t)f(x,y)]\,dt\,ds, \tag{5.4.2}$$

$$\hat{A}^{(N,j)}(x,y) = p(x,y)\int_a^b [f_{(N+1,j)}(s,y)g(x,y) + g_{(N+1,j)}(s,y)f(x,y)] \times \times \hat{P}^{(N,j)}(x,s,y)\,ds, \tag{5.4.3}$$

$$\tilde{A}^{(i,M)}(x,y) = p(x,y)\int_c^d [f_{(i,M+1)}(x,t)g(x,y) + g_{(i,M+1)}(x,t)f(x,y)] \times \times \tilde{P}^{(i,M)}(x,y,t)\,dt, \tag{5.4.4}$$

$$\bar{A}^{(N,M)}(x,y) = p(x,y)\int_a^b\int_c^d [f_{(N+1,M+1)}(s,t)g(x,y) + g_{(N+1,M+1)}(s,t)f(x,y)] \times \times \bar{P}^{(N,M)}(x,s,y,t)\,dt\,ds, \tag{5.4.5}$$

$$B^{(i,j)}(x,y) = |p(x,y)g(x,y)|\,\|f_{(i,j)}(x,y)\|_\infty + |p(x,y)f(x,y)|\,\|g_{(i,j)}(x,y)\|_\infty, \tag{5.4.6}$$

$$C^{(i,j)}(x,y) = \frac{(\max\{b-x,x-a\})^{i+1}}{(i+1)!}\,\frac{(\max\{d-y,y-c\})^{j+1}}{(j+1)!} \times \times \int_a^b\int_c^d |p(\xi,\eta)|\,d\eta\,d\xi, \tag{5.4.7}$$

$$C^{(0,j)}(y) = (b-a)\frac{(\max\{d-y,y-c\})^{j+1}}{(j+1)!}\int_a^b\int_c^d |p(\xi,\eta)|\,d\eta\,d\xi, \tag{5.4.8}$$

$$C^{(i,0)}(x) = (d-c)\frac{(\max\{b-x,x-a\})^{i+1}}{(i+1)!}\int_a^b\int_c^d |p(\xi,\eta)|\,d\eta\,d\xi, \tag{5.4.9}$$

$$\hat{C}^{(N,j)}(x,y) = \int_a^b |\hat{P}^{(N,j)}(x,s,y)|\,ds, \tag{5.4.10}$$

$$\tilde{C}^{(i,M)}(x,y) = \int_c^d |\tilde{P}^{(i,M)}(x,y,t)|\,dt, \tag{5.4.11}$$

$$\bar{C}^{(N,M)}(x,y) = \int_a^b \int_c^d |\bar{P}^{(N,M)}(x,s,y,t)|\,dt\,ds, \tag{5.4.12}$$

$$\begin{aligned} F(x,y) &= R(x,y;f) + \int_a^b \int_c^d p(s,t)f(s,y)\,dt\,ds + \int_a^b \int_c^d p(s,t)f(x,t)\,dt\,ds \\ &+ \sum_{j=1}^{M} \int_a^b \int_c^d p(s,\eta)\frac{(\eta-y)^j}{j!} f_{(0,j)}(s,y)\,d\eta\,ds \\ &+ \sum_{i=1}^{N} \int_a^b \int_c^d p(\xi,t)\frac{(\xi-x)^i}{i!} f_{(i,0)}(x,t)\,dt\,d\xi, \end{aligned} \tag{5.4.13}$$

$$\begin{aligned} G(x,y) &= R(x,y;g) + \int_a^b \int_c^d p(s,t)g(s,y)\,dt\,ds + \int_a^b \int_c^d p(s,t)g(x,t)\,dt\,ds \\ &+ \sum_{j=1}^{M} \int_a^b \int_c^d p(s,\eta)\frac{(\eta-y)^j}{j!} g_{(0,j)}(s,y)\,d\eta\,ds \\ &+ \sum_{i=1}^{N} \int_a^b \int_c^d p(\xi,t)\frac{(\xi-x)^i}{i!} g_{(i,0)}(x,t)\,dt\,d\xi, \end{aligned} \tag{5.4.14}$$

where $f, g \in C^{(N+1,M+1)}([a,b]\times[c,d])$ and p, P, $\hat{P}^{(N,j)}$, $\tilde{P}^{(i,M)}$ and $\bar{P}^{(N,M)}$ are as in Theorem 5.2.1 whereas R is defined in (5.2.4).

Now, we present our main results of this section by using notations introduced earlier in this section, which are as follows:

Theorem 5.4.1. *Let $p : [a,b]\times[c,d] \to \mathbb{R}$ be an integrable function and let $f,\, g \in C^{(N+1,M+1)}([a,b]\times[c,d])$. Then the inequality*

$$\begin{aligned} &\Bigg| \frac{1}{P(a,c)} \int_a^b \int_c^d p(x,y)f(x,y)g(x,y)\,dy\,dx \\ &- \left(\frac{1}{P(a,c)} \int_a^b \int_c^d p(x,y)f(x,y)\,dy\,dx\right)\left(\frac{1}{P(a,c)} \int_a^b \int_c^d p(x,y)g(x,y)\,dy\,dx\right)\Bigg| \leq \\ &\leq \frac{1}{2[P(a,c)]^2} \int_a^b \int_c^d \Big[\sum_{i=1}^{N}\sum_{j=1}^{M} B^{(i,j)}(x,y)C^{(i,j)}(x,y) \\ &+ \sum_{j=1}^{M} B^{(0,j)}(y)C^{(0,j)}(y) + \sum_{i=1}^{N} B^{(i,0)}(x)C^{(i,0)}(x) + B^{(N+1,j)}(x,y)\hat{C}^{(N,j)}(x,y) \\ &+ B^{(i,M+1)}(x,y)\tilde{C}^{(i,M)}(x,y) + B^{(N+1,M+1)}(x,y)\bar{C}^{(N,M)}(x,y)\Big]\,dy\,dx \end{aligned}$$

holds, where P *is defined in* (5.1.2).

Proof. From (5.2.5) for $(x,y) \in [a,b] \times [c,d]$, we have

$$\begin{aligned} P(a,c)f(x,y) &= R(x,y;f) + \int_a^b \int_c^d p(s,t)f(s,t)dt\,ds \\ &+ \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,y) f_{(N+1,j)}(s,y)\,ds \\ &+ \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(x,y,t) f_{(i,M+1)}(x,t)\,dt \\ &- \int_a^b \int_c^d \bar{P}^{(N,M)}(x,s,y,t) f_{(N+1,M+1)}(s,t)\,dt\,ds, \qquad (5.4.15) \end{aligned}$$

$$\begin{aligned} P(a,c)g(x,y) &= R(x,y;g) + \int_a^b \int_c^d p(s,t)g(s,t)dt\,ds \\ &+ \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,y) g_{(N+1,j)}(s,y)\,ds \\ &+ \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(x,y,t) g_{(i,M+1)}(x,t)\,dt \\ &- \int_a^b \int_c^d \bar{P}^{(N,M)}(x,s,y,t) g_{(N+1,M+1)}(s,t)\,dt\,ds. \qquad (5.4.16) \end{aligned}$$

Now, if we multiply (5.4.15) by $p(x,y)g(x,y)$ and (5.4.16) by $p(x,y)f(x,y)$ and add them, then we obtain

$$\begin{aligned} 2P(a,c)p(x,y)f(x,y)g(x,y) &= -\sum_{i=1}^{N}\sum_{j=1}^{M} A^{(i,j)}(x,y) - \sum_{j=1}^{M} A^{(0,j)}(y) \\ &- \sum_{i=1}^{N} A^{(i,0)}(x) + A(x,y) + \hat{A}^{(N,j)}(x,y) \\ &+ \tilde{A}^{(i,M)}(x,y) - \bar{A}^{(N,M)}(x,y). \qquad (5.4.17) \end{aligned}$$

If we integrate (5.4.17) over $[a,b] \times [c,d]$ and divide both sides by $2P(a,c)$, then we get

$$\begin{aligned} &\int_a^b \int_c^d p(x,y)f(x,y)g(x,y)\,dy\,dx \\ &= \frac{1}{2P(a,c)} \int_a^b \int_c^d \Big[-\sum_{i=1}^{N}\sum_{j=1}^{M} A^{(i,j)}(x,y) - \sum_{j=1}^{M} A^{(0,j)}(y) \end{aligned}$$

$$-\sum_{i=1}^{N} A^{(i,0)}(x) + A(x,y) + \hat{A}^{(N,j)}(x,y) + \tilde{A}^{(i,M)}(x,y) - \bar{A}^{(N,M)}(x,y)\Big]\, dy\, dx.$$

It can be rewritten as

$$\begin{aligned}
&\frac{1}{P(a,c)} \int_a^b \int_c^d p(x,y) f(x,y) g(x,y)\, dy\, dx \\
&- \left(\frac{1}{P(a,c)} \int_a^b \int_c^d p(x,y) f(x,y)\, dy\, dx\right) \left(\frac{1}{P(a,c)} \int_a^b \int_c^d p(x,y) g(x,y)\, dy\, dx\right) \\
&= \frac{1}{2[P(a,c)]^2} \int_a^b \int_c^d \Big[-\sum_{i=1}^{N}\sum_{j=1}^{M} A^{(i,j)}(x,y) - \sum_{j=1}^{M} A^{(0,j)}(y) \\
&- \sum_{i=1}^{N} A^{(i,0)}(x) + \hat{A}^{(N,j)}(x,y) + \tilde{A}^{(i,M)}(x,y) - \bar{A}^{(N,M)}(x,y)\Big]\, dy\, dx. \qquad (5.4.18)
\end{aligned}$$

Using (5.4.1),. . .,(5.4.12), we have the following inequalities for all $(x,y) \in [a,b]\times[c,d]$

$$\begin{aligned}
|A^{(i,j)}(x,y)| &\leq B^{(i,j)}(x,y)\; C^{(i,j)}(x,y), \\
|A^{(0,j)}(y)| &\leq B^{(0,j)}(y)\; C^{(0,j)}(y), \\
|A^{(i,0)}(x)| &\leq B^{(i,0)}(x)\; C^{(i,0)}(x), \\
|\hat{A}^{(N,j)}(x,y)| &\leq B^{(N+1,j)}(x,y)\; \hat{C}^{(N,j)}(x,y), \\
|\tilde{A}^{(i,M)}(x,y)| &\leq B^{(i,M+1)}(x,y)\; \tilde{C}^{(i,M)}(x,y), \\
|\bar{A}^{(N,M)}(x,y)| &\leq B^{(N+1,M+1)}(x,y)\; \bar{C}^{(N,M)}(x,y).
\end{aligned}$$

Taking absolute value on both sides in (5.4.18) and using all these inequalities in it, we get our required result. ■

Theorem 5.4.2. *Let the assumptions of Theorem* 5.4.1 *be valid. Then the inequality*

$$\begin{aligned}
&\Bigg|\frac{1}{P(a,c)} \int_a^b \int_c^d p(x,y) f(x,y) g(x,y)\, dy\, dx \\
&+ \left(\frac{1}{P(a,c)} \int_a^b \int_c^d p(x,y) f(x,y)\, dy\, dx\right) \left(\frac{1}{P(a,c)} \int_a^b \int_c^d p(x,y) g(x,y)\, dy\, dx\right) \\
&- \frac{1}{2[P(a,c)]^2} \int_a^b \int_c^d p(x,y)[g(x,y)F(x,y) + f(x,y)G(x,y)]\, dy\, dx\Bigg| \\
&\leq \frac{1}{2[P(a,c)]^2} \int_a^b \int_c^d B^{(N+1,M+1)}(x,y)\; \bar{C}^{(N,M)}(x,y)\, dy\, dx
\end{aligned}$$

holds, where P is defined in (5.1.2).

Proof. From (5.2.6) for $(x,y) \in [a,b] \times [c,d]$ we have

$$\begin{aligned} P(a,c)f(x,y) &= F(x,y) - \int_a^b \int_c^d p(s,t)f(s,t)dt\,ds \\ &+ \int_a^b \int_c^d \bar{P}^{(N,M)}(x,s,y,t) f_{(N+1,M+1)}(s,t)\,dt\,ds, \end{aligned} \tag{5.4.19}$$

$$\begin{aligned} P(a,c)g(x,y) &= G(x,y) - \int_a^b \int_c^d p(s,t)g(s,t)dt\,ds \\ &+ \int_a^b \int_c^d \bar{P}^{(N,M)}(x,s,y,t) g_{(N+1,M+1)}(s,t)\,dt\,ds. \end{aligned} \tag{5.4.20}$$

If we multiply (5.4.19) by $p(x,y)g(x,y)$ and (5.4.20) by $p(x,y)f(x,y)$ and add them, then we get

$$\begin{aligned} 2P(a,c)p(x,y)f(x,y)g(x,y) &= p(x,y)g(x,y)F(x,y) + p(x,y)f(x,y)G(x,y) \\ &- A(x,y) + \bar{A}^{(N,M)}(x,y). \end{aligned} \tag{5.4.21}$$

If we integrate (5.4.21) over $[a,b] \times [c,d]$ and divide both sides by $2P(a,c)$, then we get

$$\begin{aligned} &\int_a^b \int_c^d p(x,y)f(x,y)g(x,y)\,dy\,dx \\ &= \frac{1}{2P(a,c)} \int_a^b \int_c^d p(x,y)[g(x,y)F(x,y) + f(x,y)G(x,y)]\,dy\,dx \\ &- \frac{1}{P(a,c)} \left(\int_a^b \int_c^d p(x,y)f(x,y)\,dy\,dx \right) \left(\int_a^b \int_c^d p(x,y)g(x,y)\,dy\,dx \right) \\ &+ \frac{1}{2P(a,c)} \int_a^b \int_c^d \bar{A}^{(N,M)}(x,y)\,dy\,dx. \end{aligned} \tag{5.4.22}$$

Also we have

$$|\bar{A}^{(N,M)}(x,y)| \leq B^{(N+1,M+1)}(x,y)\, \bar{C}^{(N,M)}(x,y). \tag{5.4.23}$$

From (5.4.22) and (5.4.23), we obtain our required inequality. ■

Remark 5.4.1. For $N = M = 0$, Theorems 6 and 7 of [81] become special cases of Theorems 5.4.1 and 5.4.2 respectively and we also retrieve results of [64] by simply putting $p \equiv 1$. For $N = M = 0$, we can also find similar results as given in [28]. □

The main subject of the next two chapters is the $n-$exponential convexity. In the next chapter we discuss the $n-$exponential convexity for Favard's and Berwald's inequalities and the majorization-type results.

Chapter 6

$n-$Exponential Convexity for Majorization, Favard's and Berwald's Inequalities

> "All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove."
>
> –G. H. Hardy

The present chapter deals with the $n-$exponential convexity method. We will produce the $n-$exponentially and logarithmically convex functions for majorization, Favard's and Berwald's inequalities by using a class of different functions in linear functionals. We will also construct positive-semidefinite matrices for these functionals. We will vary on a choice of a family of functions in order to construct different examples of exponentially convex functions and in order to generate some means. We will also prove the monotonic property.

The results of the present chapter are extracted from [39, 40].

6.1 Majorization and Related Results

These lines are extracted form the book [51] "Although inequalities play a fundamental role in nearly all branches of mathematics, inequalities are usually obtained by ad hoc methods rather than as consequences of some underlying 'Theory of Inequalities'. For certain kinds of inequalities, the notion of majorization leads to such a theory that is sometimes extremely useful and powerful for deriving inequalities. Moreover, the derivation of an inequality by methods of majorization is often very helpful both for providing a deeper understanding and for suggesting natural generalizations."

In our construction for main theorems of this chapter, we recall the definitions of majorization and state some results related to majorization, Favard's and Berwald's inequalities as follows.

For fixed $n \geq 2$, $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ denote two $n-$tuples and $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$, $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ be their ordered components.

Definition 6.1.1. For $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$,

$$\mathbf{x} \prec \mathbf{y} \quad \text{if} \quad \begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \quad , & k \in \{1, \ldots, n-1\}, \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]} \quad , \end{cases}$$

when $\mathbf{x} \prec \mathbf{y}$, $\mathbf{x}$ *is said to be majorized by* $\mathbf{y}$ or $\mathbf{y}$ *majorizes* $\mathbf{x}$.

This notion and notation of majorization was introduced by Hardy et al. in [30]. Now, we state the well-known majorization theorem from the same book [30] as follows.

Proposition 6.1.1. *Let* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$. *The inequality*

$$\Lambda_1(\mathbf{x}, \mathbf{y}; f) = \sum_{i=1}^{n} f(y_i) - \sum_{i=1}^{n} f(x_i) \geq 0 \tag{A1}$$

holds for every continuous convex function $f : \mathbb{R} \to \mathbb{R}$ *if and only if* $\mathbf{x} \prec \mathbf{y}$. *Moreover, if* f *is a strictly convex function, then equality in* (A1) *is valid if and only if* $x_{[i]} = y_{[i]}$ *for each* $i \in \{1, \ldots, n\}$.

The following weighted version of majorization theorem was given by Fuchs in [26] (see also [51, p. 580] and [80, p. 323]).

Proposition 6.1.2. *Let* $\mathbf{w} \in \mathbb{R}^n$ *and let* $\mathbf{x}$, $\mathbf{y}$ *be two nonincreasing real* $n-$*tuples such that*

$$\sum_{i=1}^{k} w_i x_i \leq \sum_{i=1}^{k} w_i y_i, \quad k \in \{1, \ldots, n-1\}$$

$$\text{and} \quad \sum_{i=1}^{n} w_i x_i = \sum_{i=1}^{n} w_i y_i.$$

Then for every continuous convex function $f : \mathbb{R} \to \mathbb{R}$, *the following inequality holds*

$$\sum_{i=1}^{n} w_i f(x_i) \leq \sum_{i=1}^{n} w_i\, f(y_i). \tag{6.1.1}$$

Remark 6.1.1. In the Proposition 6.1.2, Fuchs used the real weights and two sequences monotonic in the same sense, here we state two results from [47] in which authors considered only one sequence to be monotonic but they compromised on weights by taking it positive. □

Proposition 6.1.3. *Let the following assumptions be valid:* $f : J \to \mathbb{R}$ (*where* J *is an interval in* $\mathbb{R}$) *is a convex function,* $\mathbf{w} \in \mathbb{R}_+^n$ *and* $\mathbf{x}$, $\mathbf{y} \in J^n$ *satisfying*

$$\begin{aligned} \sum_{i=1}^{k} w_i x_i \quad &\leq \quad \sum_{i=1}^{k} w_i y_i, \quad k \in \{1, \ldots, n-1\}, \\ \text{and} \quad \sum_{i=1}^{n} w_i x_i \quad &= \quad \sum_{i=1}^{n} w_i\, y_i. \end{aligned}$$

Then the followings statements are valid:

(*a*) *For every nonincreasing* $n-$*tuple* $\mathbf{x}$, *inequality* (6.1.1) *holds,*

(*b*) *For every nondecreasing* $n-$*tuple* $\mathbf{y}$, *the reverse inequality in* (6.1.1) *holds.*

Moreover, if f *is strictly convex and* $\mathbf{x} \neq \mathbf{y}$, *then* (6.1.1) *and reverse inequality in* (6.1.1), *whichever holds, is strict.*

Motivated by inequality (6.1.1), under the assumptions of Proposition 6.1.2 (or Proposition 6.1.3), we define the functional λ_1 by

$$\lambda_1(\mathbf{x}, \mathbf{y}, \mathbf{w}; f) = \sum_{i=1}^{n} w_i f\left(y_i\right) - \sum_{i=1}^{n} w_i f\left(x_i\right).$$

Now, we define the functional Λ_2 in terms of λ_1 by

$$\Lambda_2 = \begin{cases} \lambda_1 & , \quad \text{if inequality (6.1.1) holds,} \\ -\lambda_1 & , \quad \text{if reverse inequality in (6.1.1) holds.} \end{cases} \tag{A2}$$

Note that, whenever it is defined, Λ_2 is nonnegative.

Here, we give an important result from Anwar et al. [4] which is given in book of Marshall et al. [51, p. 666].

Proposition 6.1.4. *Let the assumptions of Proposition* 6.1.2 *be valid and let the function* $\varphi_t : \mathbb{R}_+ \to \mathbb{R}$ *be defined as*

$$\varphi_t(x) = \begin{cases} \frac{x^t}{t(t-1)} & , \quad t \notin \{0,1\}, \\ -\ln(x) & , \quad t = 0, \\ x\ln(x) & , \quad t = 1. \end{cases} \tag{6.1.2}$$

Then $\Lambda_2(\mathbf{x}, \mathbf{y}, \mathbf{w}; \varphi_t)$ *is* log *−convex in* t*, with* $\mathbf{x}$, $\mathbf{y}$ *and* $\mathbf{w}$ *fixed.*

Proposition 6.1.5. *Let* $\mathbf{w}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n_*$ *and let* $g, f : \mathbb{R}_* \to \mathbb{R}$ *be the functions such that* g *is increasing and* $f \circ g^{-1}$ *is convex. Further, suppose that*

$$\begin{aligned} \sum_{i=1}^{k} w_i g(x_i) &\leq \sum_{i=1}^{k} w_i g(y_i), \quad k \in \{1, \dots, n-1\}, \\ \text{and} \quad \sum_{i=1}^{n} w_i g(x_i) &= \sum_{i=1}^{n} w_i\, g(y_i) \end{aligned}$$

are valid, then, we have:

(*a*) *For every nonincreasing* $n-$*tuple* $\mathbf{x}$*, inequality* (6.1.1) *holds,*

(*b*) *For every nondecreasing* $n-$*tuple* $\mathbf{y}$*, the reverse inequality in* (6.1.1) *holds.*

Moreover, if $f \circ g^{-1}$ *is strictly convex and* $\mathbf{x} \neq \mathbf{y}$*, then* (6.1.1) *and reverse inequality in* (6.1.1)*, whichever holds, is strict.*

Under the assumptions of Proposition 6.1.5, let us define the functional Λ_3 in terms of λ_1 by

$$\Lambda_3 = \begin{cases} \lambda_1 & , \quad \text{if inequality (6.1.1) holds,} \\ -\lambda_1 & , \quad \text{if reverse inequality in (6.1.1) holds.} \end{cases} \tag{A3}$$

Note that, whenever it is defined, Λ_3 is nonnegative.

We also give some integral inequalities related to majorization. The following result is a consequence of Theorem 1 in [71] (see also [80, p. 328]).

Proposition 6.1.6. *Let* x, $y : [a,b] \to \mathbb{R}$ *be nondecreasing continuous functions and let* $H \in BV[a,b]$*. Further, suppose that*

$$\begin{aligned} \int_u^b x(t)\, dH(t) &\leq \int_u^b y(t)\, dH(t), \quad u \in (a,b), \\ \int_a^b x(t)\, dH(t) &= \int_a^b y(t)\, dH(t), \end{aligned}$$

are valid, then for every continuous convex function $f : \mathbb{R} \to \mathbb{R}$*,*

$$\Lambda_4(x, y, H; f) = \int_a^b f\,(y(t))\; dH(t) \;-\int_a^b f\,(x(t))\; dH(t) \geq 0 \tag{A4}$$

holds.

Here we give another important result from Anwar et. al. [4] which is also given in [51, p. 667].

Proposition 6.1.7. *Let* x*,* $y : [a, b] \to \mathbb{R}$ *be nonincreasing continuous functions and let* $H \in BV[a, b]$*. Further suppose that*

$$\begin{aligned} \int_a^u x(t)\, dH(t) \;&\leq\; \int_a^u y(t)\, dH(t), \quad u \in (a, b), \\ \int_a^b x(t)\, dH(t) \;&=\; \int_a^b y(t)\, dH(t), \end{aligned}$$

are valid, then

$$\Psi(x, y, H; \varphi_t) = \int_a^b \varphi_t(y(t))dH(t) - \int_a^b \varphi_t(x(t))dH(t)$$

is log-convex in t*, where* φ_t *is defined in* (6.1.2)*.*

We can find variety of applications of the Propositions 6.1.4 and 6.1.7 in article [46] but we quote here an application in statistics which is given in the book [51, p. 668].

Corollary 6.1.1. *If* W *is positive random variable for which expectation exist and* $\alpha \geq \beta$*, then the function*

$$\begin{aligned} g(t) &= \frac{EW^{\alpha t} - (EW^{\beta t})(EW^{\alpha}/EW^{\beta})^t}{t(t-1)}, & t \notin \{0, 1\}, \\ g(t) &= (\ln(EW^{\alpha}) - E\ln(W^{\alpha})) - (\ln(EW^{\beta}) - E\ln(W^{\beta})), & t = 0, \\ g(t) &= E(W^{\alpha}\ln(W^{\alpha})) - (EW^{\alpha})(\ln(EW^{\alpha})) - E(W^{\beta}\ln(W^{\beta})) & \\ &\quad - (EW^{\beta})(\ln(EW^{\beta}))(EW^{\alpha}/EW^{\beta}), & t = 1 \end{aligned}$$

is log-*convex.*

Remark 6.1.2. The Propositions 6.1.4 and 6.1.7 give us log-convexity but we can find more generalized results proved by Anwar et al. [4] which give us positive-semidefinite matrices and exponential convexity for positive $n-$tuples $\mathbf{x}$ and $\mathbf{y}$. Also in [46] we find similar results for nonnegative and for real $n-$tuples. Nevertheless, we give much more general results than results of [4] and [46] in new direction by using second-order divided differences. □

For our next theorem, we recall a definition with some notations from [80, p. 330] as follows. Let $F, G : \mathbb{R}_* \to \mathbb{R}$ be two nondecreasing continuous functions which pass through origin and define $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$.

Definition 6.1.2.

$$\overline{G} \prec \overline{F} \quad \text{if} \quad \begin{cases} \int_0^u \overline{G}(z)\,dz \le \int_0^u \overline{F}(z)dz & , \quad u \in \mathbb{R}_+, \\ \int_0^\infty \overline{G}(z)dz = \int_0^\infty \overline{F}(z)dz & , \end{cases}$$

provided that integrals exist, when $\overline{G} \prec \overline{F}$, $\overline{G}$ *is said to be majorized by* $\overline{F}$ or $\overline{F}$ *majorizes* $\overline{G}$.

Boland and Proschan in [11] gave the following result.

Proposition 6.1.8. *The inequality*

$$\Lambda_5(F, G; f) = \int_0^\infty f(t)\,dG(t) - \int_0^\infty f(t)\,dF(t) \ge 0, \tag{A5}$$

holds for every convex function f if and only if $\overline{G} \prec \overline{F}$, provided that integrals exist.

Following result from [47] is in fact slight extension of Theorem 2 in [76] and it also generalizes Lemma 2 in [50].

Proposition 6.1.9. *Let $w : [a, b] \to \mathbb{R}$ be a weight function and let $x, y : [a, b] \to \mathbb{R}_*$ be two functions. Suppose that the functions $f, g : \mathbb{R}_* \to \mathbb{R}_+$ are such that g is increasing and $f \circ g^{-1}$ is convex. Further, suppose that*

$$\begin{aligned} \int_a^u g(x(t))w(t)dt \quad &\le \quad \int_a^u g(y(t))w(t)dt, \quad u \in (a, b), \\ \textit{and} \quad \int_a^b g(x(t))w(t)dt \quad &= \quad \int_a^b y(t)w(t)dt. \end{aligned}$$

are valid, then we have:

(*a*) *For every nonincreasing function x defined on $[a,b]$, the following inequality holds*

$$\int_a^b f(x(t))w(t)dt \leq \int_a^b f(y(t))w(t)dt. \tag{6.1.3}$$

(*b*) *For every nondecreasing function y defined on $[a,b]$, the reverse inequality in* (6.1.3) *holds.*

Moreover, if $f \circ g^{-1}$ is strictly convex and $x \neq y$ (a.e.), then (6.1.3) *and reverse inequality in* (6.1.3)*, whichever holds, is strict.*

Motivated by inequality (6.1.3), under the assumptions of Proposition 6.1.9, we define the functional λ_2 by

$$\lambda_2(x,y,w;f) = \int_a^b f\left(y(t)\right)\, w(t)\, dt - \int_a^b f\left(x(t)\right)\, w(t)\, dt \geq 0.$$

Now, we define the functional Λ_6 in terms of λ_2 by

$$\Lambda_6 = \begin{cases} \lambda_2 & , \quad \text{if inequality (6.1.3) holds,} \\ -\lambda_2 & , \quad \text{if reverse inequality in (6.1.3) holds.} \end{cases} \tag{A6}$$

Note that, whenever it is defined, Λ_6 is nonnegative.

6.2 Favard's and Berwald's Inequalities

Now, we state some results related to Favard's and Berwald's inequalities. The following result is due to Favard [25] which is referred as Favard's inequality in literature.

Proposition 6.2.1. *Let $\varphi : [a,b] \to \mathbb{R}_*$ be a continuous concave function such that $\varphi \not\equiv 0$. Then for every convex function $f : [0, 2\alpha] \to \mathbb{R}$, the inequality*

$$\frac{1}{b-a}\int_a^b f\left(\varphi(t)\right) dt \leq \frac{1}{2\alpha}\int_0^{2\alpha} f(t)dt$$

holds, where

$$\alpha = \frac{1}{b-a}\int_a^b \varphi(t)dt.$$

Some generalized results related to Favard's inequality and its reverse inequality can be found in [36, pp. 412-413]. Moreover, Berwald in [9] gave the following generalization of Favard's inequality:

Proposition 6.2.2. *Let $\varphi : [a,b] \to \mathbb{R}_*$ be a continuous concave function such that $\varphi \not\equiv 0$ and let $f, g : [0, L] \to \mathbb{R}$ be two functions such that g is a strictly monotonic continuous function and $f \circ g^{-1}$ is convex where L is sufficiently large. If r is the unique positive root of the equation*

$$\frac{1}{b-a}\int_a^b g(\varphi(t))dt = \frac{1}{r}\int_0^r g(t)dt,$$

then the following inequality holds

$$\frac{1}{b-a}\int_a^b f(\varphi(t))dt \leq \frac{1}{r}\int_0^r f(t)dt.$$

The following two results are generalizations of discrete weighted Favard's and Berwald's inequalities respectively proved by Latif et al. in [47]:

Proposition 6.2.3. *Let $\mathbf{w}, \mathbf{x}, \mathbf{y} \in \mathbb{R}_*^n$ and let $f : \mathbb{R}_* \to \mathbb{R}$ be a convex function.*

(a) *Let $\mathbf{x}/\mathbf{y}$ be a nonincreasing $n-$tuple. If $\mathbf{x}$ is a nondecreasing $n-$tuple , then the inequality*

$$\sum_{i=1}^n w_i\, f\left(\frac{x_i}{\sum_{i=1}^n x_i w_i}\right) \leq \sum_{i=1}^n w_i\, f\left(\frac{y_i}{\sum_{i=1}^n y_i w_i}\right). \tag{6.2.1}$$

holds, where $\mathbf{x}/\mathbf{y} = (x_1/y_1, \ldots, x_n/y_n)$ provided that $y_i \neq 0$ for $i \in \{1, \ldots, n\}$.
If $\mathbf{y}$ is a nonincreasing $n-$tuple, then the reverse inequality in (6.2.1) *holds.*

(b) *Let $\mathbf{x}/\mathbf{y}$ be a nondecreasing $n-$tuple. If $\mathbf{x}$ is a nonincreasing $n-$tuple, then the inequality* (6.2.1) *is valid.*
If $\mathbf{y}$ is a nondecreasing $n-$tuple, then the reverse inequality in (6.2.1) *holds.*

Moreover, if f is strictly convex and $\mathbf{x} \neq \mathbf{y}$, then the inequality in (6.2.1) *and reverse inequality in* (6.2.1)*, whichever holds, is strict.*

Motivated by inequality (6.2.1), under the assumptions of Proposition 6.2.3, we define the functional λ_3 by

$$\lambda_3(\mathbf{x}, \mathbf{y}, \mathbf{w}; f) = \sum_{i=1}^n w_i\, f\left(\frac{y_i}{\sum_{i=1}^n y_i w_i}\right) - \sum_{i=1}^n w_i\, f\left(\frac{x_i}{\sum_{i=1}^n x_i w_i}\right) \geq 0.$$

Now, we define the functional Λ_7 in terms of λ_3

$$\Lambda_7 = \begin{cases} \lambda_3 & , \quad \text{if inequality (6.2.1) holds,} \\ -\lambda_3 & , \quad \text{if reverse inequality in (6.2.1) holds.} \end{cases} \tag{A7}$$

Note that, whenever it is defined, Λ_7 is nonnegative.

Proposition 6.2.4. *Let $\mathbf{w}, \mathbf{x}, \mathbf{y} \in \mathbb{R}_*^n$ and let the function $f, g : \mathbb{R}_* \to \mathbb{R}$ be such that g is increasing and continuous and $f \circ g^{-1}$ is convex. Suppose that λ is such that*

$$\sum_{i=1}^{n} w_i g\left(x_i\right) = \sum_{i=1}^{n} w_i g\left(\lambda\, y_i\right).$$

(a) *Let $\mathbf{x}/\mathbf{y}$ be a nonincreasing $n-$tuple. If $\mathbf{x}$ is an nondecreasing $n-$tuple, then the following inequality holds.*

$$\sum_{i=1}^{n} w_i\, f\left(x_i\right) \leq \sum_{i=1}^{n} w_i\, f\left(\lambda\, y_i\right). \tag{6.2.2}$$

If $\mathbf{y}$ is a nonincreasing $n-$tuple, then the reverse inequality in (6.2.2) *holds.*

(b) *Let $\mathbf{x}/\mathbf{y}$ be a nondecreasing $n-$tuple. If $\mathbf{x}$ is a nonincreasing $n-$tuple, then the inequality* (6.2.2) *is valid.*
If $\mathbf{y}$ is a nondecreasing $n-$tuple, then the reverse inequality in (6.2.2) *holds.*

Moreover, if f is strictly convex function and $\mathbf{x} \neq \mathbf{y}$, then the inequality in (6.2.2) *and reverse inequality in* (6.2.2)*, whichever holds, is strict.*

Motivated by inequality (6.2.2), under the assumptions of Proposition 6.2.4, we define the functional λ_4 by

$$\lambda_4(\mathbf{x}, \mathbf{y}, \mathbf{w}, \lambda; f) = \sum_{i=1}^{n} w_i\, f\left(\lambda\, y_i\right) - \sum_{i=1}^{n} w_i\, f\left(x_i\right) \geq 0.$$

Now, we define the functional Λ_8 in terms of λ_4 by

$$\Lambda_8 = \begin{cases} \lambda_4 & , \quad \text{if inequality (6.2.1) holds,} \\ -\lambda_4 & , \quad \text{if reverse inequality in (6.2.1) holds.} \end{cases} \tag{A8}$$

Note that, whenever it is defined, Λ_8 is nonnegative.

The following two results are generalized integral versions of weighted Favard's and Berwald's inequalities respectively, proved by Latif et al. in [47]:

Proposition 6.2.5. *Let $w : [a, b] \to \mathbb{R}$ be a weight function and let $x, y : [a, b] \to \mathbb{R}_*$ be two functions. Suppose that $f : \mathbb{R}_* \to \mathbb{R}$ is a convex function.*

(*a*) *Let* x/y *be a nonincreasing function on* $[a, b]$*. If* x *is a nondecreasing function on* $[a, b]$*, then the following inequality holds*

$$\int_a^b f\left(\frac{x(t)}{\int_a^b x(t)w(t)dt}\right) w(t)dt \leq \int_a^b f\left(\frac{y(t)}{\int_a^b y(t)w(t)dt}\right) w(t)dt. \tag{6.2.3}$$

If y *is a nonincreasing function on* $[a, b]$*, then the reverse inequality in* (6.2.3) *holds.*

(*b*) *Let* x/y *be a nondecreasing function on* $[a, b]$*. If* x *is a nonincreasing function on* $[a, b]$*, then the inequality* (6.2.3) *is valid.*
If y *is an nondecreasing function on* $[a, b]$*, then the reverse inequality in* (6.2.3) *holds.*

Moreover, if f *is strictly convex on* $[a, b]$ *and* $x \neq y$ *(a.e.), then the inequality in* (6.2.3) *and the reverse inequality in* (6.2.3)*, whichever holds, is strict.*

Motivated by inequality (6.2.3), under the assumptions of Proposition 6.2.5, we define the functional λ_5 by

$$\lambda_5(x, y, w; f) = \int_a^b f\left(\frac{y(t)}{\int_a^b y(t)w(t)dt}\right) w(t)dt - \int_a^b f\left(\frac{x(t)}{\int_a^b x(t)w(t)dt}\right) w(t)dt \geq 0.$$

Now, we define the functional Λ_9 in terms of λ_5 by

$$\Lambda_9 = \begin{cases} \lambda_5 & , \quad \text{if inequality (6.2.3) holds,} \\ -\lambda_5 & , \quad \text{if reverse inequality in (6.2.3) holds.} \end{cases} \tag{A9}$$

Note that, whenever it is defined, Λ_9 is nonnegative.

Proposition 6.2.6. *Let* $w : [a, b] \to \mathbb{R}$ *be a weight function and let* $x, y : [a, b] \to \mathbb{R}_+$ *be two functions. Suppose that the functions* $f, g : \mathbb{R}_* \to \mathbb{R}$ *are such that* g *is increasing and continuous and* $f \circ g^{-1}$ *is convex. Further suppose that* λ *is such that*

$$\int_a^b g\left(x(t)\right) w(t)dt = \int_a^b g\left(\lambda y(t)\right) w(t)dt.$$

(*a*) *Let* x/y *be a nonincreasing function on* $[a, b]$*. If* x *is a nondecreasing function on* $[a, b]$*, then the following inequality holds*

$$\int_a^b f\left(x(t)\right) w(t)dt \leq \int_a^b f\left(\lambda y(t)\right) w(t)dt. \tag{6.2.4}$$

If y *is a nonincreasing function on* $[a, b]$*, then the reverse inequality in* (6.2.4) *holds.*

(*b*) *Let* x/y *be a nondecreasing function on* $[a,b]$*. If* x *is a nonincreasing function on* $[a,b]$*, then the inequality* (6.2.4) *is valid.*
If y *is a nondecreasing function on* $[a,b]$*, then the reverse inequality in* (6.2.4) *holds.*

Moreover, if $f \circ g^{-1}$ *is strictly convex function on* $[a,b]$ *and* $x \neq y$ *(a.e.), then the inequality in* (6.2.4) *and the reverse inequality in* (6.2.4)*, whichever holds, is strict.*

Motivated by the inequality (6.2.4), under the assumptions of Proposition 6.2.6, we define the functional λ_6 by

$$\lambda_6(x,y,w,\lambda;f) = \int_a^b f\left(\lambda y(t)\right) w(t)dt - \int_a^b f\left(x(t)\right) w(t)dt \geq 0.$$

Now, we define the functional Λ_{10}, in terms of λ_6 by

$$\Lambda_{10} = \begin{cases} \lambda_6 & , \quad \text{if inequality (6.2.4) holds,} \\ -\lambda_6 & , \quad \text{if reverse inequality in (6.2.4) holds.} \end{cases} \tag{A10}$$

Note that, whenever it is defined, Λ_{10} is nonnegative.

6.3 Mean Value Theorems

For the sake of completion, we only state here two theorems which will be used in examples. For the idea of the proof see proof of Theorems 3.4.1 and 3.4.2.

Theorem 6.3.1. *Let* Λ_5 *be a linear functional as defined in* (*A*5) *under the assumptions of Proposition* 6.1.8 *and let* $f \in C^{(2)}(K)$*, where* K *is a compact interval in* $\mathbb{R}_*$*. Then there exists* $\xi \in K$ *such that*

$$\Lambda_5(F,G;f) = f''(\xi)\Lambda_5(F,G;f_0),$$

where $f_0(x) = \frac{x^2}{2}$*.*

Theorem 6.3.2. *Let* Λ_5 *be a linear functional as defined in* (*A*5) *under the assumptions of Proposition* 6.1.8 *and let* $f, g \in C^{(2)}(K)$*, where* K *is a compact interval in* $\mathbb{R}_*$*. Then there exists* $\xi \in K$ *such that*

$$\frac{\Lambda_5(F,G;f)}{\Lambda_5(F,G;g)} = \frac{f''(\xi)}{g''(\xi)}$$

provided that the denominator of the left-hand side is nonzero.

Remark 6.3.1. If the inverse of $\frac{f''}{g''}$ exists, then from the above mean value theorem we get the following generalized mean

$$\xi = \left(\frac{f''}{g''}\right)^{-1}\left(\frac{\Lambda_5(F,G;f)}{\Lambda_5(F,G;g)}\right). \tag{6.3.1}$$

□

Remark 6.3.2. For the functionals Λ_k, $k \in \{1,2,3,4,6,7,8,9,10\}$ (as defined in $(A1)$, $(A2)$, $(A3)$, $(A4)$,$(A6)$, $(A7)$, $(A8)$, $(A9)$,(A10)) the results similar to Theorems 6.3.1 and 6.3.2 can be found in [4] and [47, 48]. In the similar way, we can use Remark 6.3.1 for these functionals as well. □

6.4 $n-$Exponential Convexity for Majorization, Favard's and Berwald's Inequalities

As mentioned earlier, this important sub-class of $n-$exponentially convex functions is recently introduced by Pečarić and Perić. In this section, we use the same notion of $n-$exponential convexity and prove it for some important results mentioned in introduction section of present chapter. It is worth mentioning that throughout this section Λ_k for $k \in \{1,\dots,10\}$ refer to functionals as defined in $(A1),\dots,(A10)$. Also in the remaining part of the chapter for the sake of brevity we write $\Lambda_k(.,.,.,.;f) = \Lambda_k(f)$ for $k \in \{1,\dots,10\}$. Let us denote domain of f_t by $Dom(f_t)$ where $Dom(f_t)$ varies from functional to functional. Throughout this section I is an interval in $\mathbb{R}$.
By using the idea of [33], we produce $n-$exponentially convex functions and hence as a consequence we produce exponentially and logarithmically convex functions by applying the functionals Λ_k, $k \in \{1,\dots,10\}$ on a given family of functions with the same property.

Theorem 6.4.1. *Let $D_1 = \{f_t : t \in I\}$ be a class of continuous functions such that the function $t \mapsto [z_0, z_1, z_2; f_t]$ is $n-$exponentially convex in the $J-$sense on I for any three mutually distinct points $z_0, z_1, z_2 \in Dom(f_t)$. Let Λ_k be the linear functionals for $k \in \{1,2,4,5,7,9\}$. Then the following statements are valid:*

(a) *The function $t \mapsto \Lambda_k(f_t)$ is $n-$exponentially convex function in the $J-$sense on I.*

(b) *If the function $t \mapsto \Lambda_k(f_t)$ is continuous on I, then the function $t \mapsto \Lambda_k(f_t)$ is $n-$exponentially convex on I.*

Proof.

(*a*) Fix $k \in \{1, 2, 4, 5, 7, 9\}$. Let us define the function ω for $t_i \in I$, $u_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$ as follows

$$\omega = \sum_{i,j=1}^{n} u_i u_j f_{\frac{t_i+t_j}{2}},$$

Since the function $t \mapsto [z_0, z_1, z_2; f_t]$ is $n-$exponentially convex in the $J-$sense, so

$$[z_0, z_1, z_2; \omega] = \sum_{i,j=1}^{n} u_i u_j [z_0, z_1, z_2; f_{\frac{t_i+t_j}{2}}] \geq 0$$

which implies that ω is convex function on $Dom(f_t)$ and therefore $\Lambda_k(\omega) \geq 0$. Hence

$$\sum_{i,j=1}^{n} u_i u_j \Lambda_k(f_{\frac{t_i+t_j}{2}}) \geq 0.$$

We conclude that the function $t \mapsto \Lambda_k(f_t)$ is an $n-$exponentially convex function on I in $J-$sense.

(*b*) This part is easily follows from definition of $n-$exponentially convex function.

■

As a consequence of the above theorem we give the following corollaries:

Corollary 6.4.2. *Let $D_2 = \{f_t : t \in I\}$ be a class of continuous functions such that the function $t \mapsto [z_0, z_1, z_2; f_t]$ is an exponentially convex in the $J-$sense on I for any three mutually distinct points $z_0, z_1, z_2 \in Dom(f_t)$. Let Λ_k be the linear functionals for $k \in \{1, 2, 4, 5, 7, 9\}$. Then the following statements are valid:*

(*a*) *The function $t \mapsto \Lambda_k(f_t)$ is exponentially convex in the $J-$sense on I.*

(*b*) *If the function $t \mapsto \Lambda_k(f_t)$ is continuous on I, then the function $t \mapsto \Lambda_k(f_t)$ is exponentially convex on I.*

(*c*) *The matrix $\left[\Lambda_k\left(f_{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^{m}$ is positive-semidefinite. Particularly,*

$$\det\left[\Lambda_k\left(f_{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^{m} \geq 0$$

for each $m \in \mathbb{N}$ and $t_i \in I$ where $i \in \{1, \ldots, m\}$.

Proof. Proof follows directly from Theorem 6.4.1 by using definition of exponential convexity and Corollary 1.2.1. ■

Corollary 6.4.3. *Let $D_3 = \{f_t : t \in I\}$ be a class of continuous functions such that the function $t \mapsto [z_0, z_1, z_2; f_t]$ is 2−exponentially convex in the J−sense on I for any three mutually distinct points $z_0, z_1, z_2 \in Dom(f_t)$. Let Λ_k be the linear functionals for $k \in \{1, 2, 4, 5, 7, 9\}$. Then the following statements are valid:*

(a) *If the function $t \mapsto \Lambda_k(f_t)$ is continuous on I, then it is 2−exponentially convex on I. If the function $t \mapsto \Lambda_k(f_t)$ is additionally positive, then it is also log-convex on I. Moreover, the following Lyapunov's inequality holds for $r < s < t$; $r,\ s,\ t\ \in I$*

$$(\Lambda_k(f_s))^{t-r} \leq (\Lambda_k(f_r))^{t-s} (\Lambda_k(f_t))^{s-r}. \tag{6.4.1}$$

(b) *If the function $t \mapsto \Lambda_k(f_t)$ is positive and differentiable on I, then for every $s, t, u, v \in I$ such that $s \leq u$ and $t \leq v$, we have*

$$\mu_{s,t}(\Lambda_k, D_3) \leq \mu_{u,v}(\Lambda_k, D_3)$$

where $\mu_{s,t}$ is defined in (3.5.3)

Proof.

(a) It follows directly form Theorem 6.4.1 and Remark 1.2.6. For remaining part follow proof of Theroem 3.5.1 part (c).

(b) Follow proof of Theorem 3.5.1 part (e). ■

Remark 6.4.1. The results from Theorem 6.4.1 and Corollaries 6.4.2 and 6.4.3 still hold when any two (three) points $z_0, z_1, z_2 \in [a, b]$ coincide for a family of differentiable (twice differentiable) functions f_t such that the function $t \mapsto [z_0, z_1, z_2; f_t]$ is $n-$exponentially convex, exponentially convex and 2−expoenetially convex in the $J-$sense respectively. □

Theorem 6.4.4. *Let $D_4 = \{f_t : t \in I\}$ be a class of continuous functions such that the function $t \mapsto [z_0, z_1, z_2; f_t \circ g^{-1}]$ is $n-$exponentially convex in the $J-$sense on I for any three mutually distinct points $z_0, z_1, z_2 \in \mathbb{R}_*$ where the function g is increasing (and continuous also for functionals Λ_k, $k \in \{8, 10\}$). Let Λ_k be linear functionals for $k \in \{3, 6, 8, 10\}$. Then the following statements are valid:*

(a) *The function $t \mapsto \Lambda_k(f_t)$ is $n-$exponentially convex in the $J-$sense on I.*

(b) *If the function $t \mapsto \Lambda_k(f_t)$ is continuous on I, then the function $t \mapsto \Lambda_k(f_t)$ is $n-$exponentially convex on I.*

Proof.

(a) Fix $k \in \{3, 6, 8, 10\}$. Let us define the function ω for $t_i \in I$, $u_i \in \mathbb{R}$, $i \in \{1, \dots, n\}$ as follows

$$\omega = \sum_{i,j=1}^{n} u_i u_j f_{\frac{t_i+t_j}{2}},$$

which implies that

$$\omega \circ g^{-1} = \sum_{i,j=1}^{n} u_i u_j f_{\frac{t_i+t_j}{2}} \circ g^{-1}.$$

Since the function $t \mapsto [z_0, z_1, z_2; f_t \circ g^{-1}]$ is $n-$exponentially convex in the $J-$sense, we have

$$[z_0, z_1, z_2; \omega \circ g^{-1}] = \sum_{i,j=1}^{n} u_i u_j [z_0, z_1, z_2; f_{\frac{t_i+t_j}{2}} \circ g^{-1}] \geq 0,$$

which implies that $\omega \circ g^{-1}$ is convex function on $\mathbb{R}_*$ and therefore $\Lambda_k(\omega \circ g^{-1}) \geq 0$. Hence

$$\sum_{i,j=1}^{n} u_i u_j \Lambda_k(f_{\frac{t_i+t_j}{2}} \circ g^{-1}) \geq 0.$$

We conclude that the function $t \mapsto \Lambda_k(f_t)$ is an $n-$exponentially convex function on I in $J-$sense.

(b) This part is easily followed by definition of $n-$exponentially convex functions.

■

As a consequence of the above theorem we give the following corollaries:

Corollary 6.4.5. *Let $D_5 = \{f_t : t \in I\}$ be a class of continuous functions such that the function $t \mapsto [z_0, z_1, z_2; f_t \circ g^{-1}]$ is an exponentially convex in the $J-$sense on I for any three mutually distinct points $z_0, z_1, z_2 \in \mathbb{R}_*$ where the function g is increasing (and continuous also for functionals Λ_k, $k \in \{8, 10\}$). Let Λ_k be linear functionals for $k \in \{3, 6, 8, 10\}$. Then the following statements are valid:*

(a) *The function $t \mapsto \Lambda_k(f_t)$ is exponentially convex in the $J-$sense on I.*

(b) *If the function* $t \mapsto \Lambda_k(f_t)$ *is continuous on* J, *then the function* $t \mapsto \Lambda_k(f_t)$ *is exponentially convex on* I.

(c) *The matrix* $\left[\Lambda_k\left(f_{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^m$ *is positive-semidefinite. Particularly,*

$$\det\left[\Lambda_k\left(f_{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^m \geq 0$$

for each $m \in \mathbb{N}$ *and* $t_i \in I$ *where* $i \in \{1, \ldots, m\}$.

Corollary 6.4.6. *Let* $D_6 = \{f_t : t \in I\}$ *be a class of continuous functions such that the function* $t \mapsto [z_0, z_1, z_2; f_t \circ g^{-1}]$ *is* $2-$*exponentially convex in the* $J-$*sense on* I *for any three mutually distinct points* $z_0, z_1, z_2 \in \mathbb{R}_*$ *where the function* g *is increasing (and continuous also for functionals* Λ_k, $k \in \{8, 10\}$ *). Let* Λ_k *be linear functionals for* $k \in \{3, 6, 8, 10\}$. *Then the following statements are valid:*

(a) *If the function* $t \mapsto \Lambda_k(f_t)$ *is continuous on I, then it is* $2-$*exponentially convex function on* I. *If the function* $t \mapsto \Lambda_k(f_t)$ *is additionally positive, then it is also log-convex on* I. *Moreover, the following inequality holds for* $r < s < t$; $r,\ s,\ t \in I$

$$(\Lambda_k(f_s))^{t-r} \leq (\Lambda_k(f_r))^{t-s}\,(\Lambda_k(f_t))^{s-r}.$$

(b) *If the function* $t \mapsto \Lambda_k(f_t)$ *is positive and differentiable on* I, *then for every* $s, t, u, v \in I$ *such that* $s \leq u$ *and* $t \leq v$, *we have*

$$\mu_{s,t}(\Lambda_k, D_6) \leq \mu_{u,v}(\Lambda_k, D_6) \tag{6.4.2}$$

where $\mu_{s,t}$ *is defined in* (3.5.3).

Remark 6.4.2. The proofs of Corollaries 6.4.5 and 6.4.6 are similar to the proofs of Corollaries 6.4.2 and 6.4.3 respectively, so we omit the details. Moreover, the results from Theorem 6.4.4 and Corollaries 6.4.5 and 6.4.6 still hold when any two (three) points $z_0, z_1, z_2 \in [a, b]$ coincide for a family of differentiable (twice differentiable) functions f_t such that the function $t \mapsto [z_0, z_1, z_2; f_t]$ is $n-$exponentially convex, exponentially convex and $2-$expoenetially convex in the $J-$sense respectively. □

Now, we give two important remarks and one useful corollary from [33], which we will use in some examples in next section.

Remark 6.4.3. For $\mu_{s,t}(\Lambda_k, \Omega)$ defined with (3.5.3) we will refer as mean if

$$a \leq \mu_{s,t}(\Lambda_k, \Omega) \leq b$$

for $s, t \in I$ and $k \in \{1, \ldots, 10\}$ where $\Omega = \{f_t : t \in I\}$ be a family of functions and $[a, b] \subset Dom(f_t)$. □

Theorems 6.4.1 and 6.4.4 give us the following corollary.

Corollary 6.4.7. *Let $a, b \in \mathbb{R}$ and Λ_k be linear functionals for $k \in \{1, \ldots, 10\}$. Let $\Omega = \{f_t : t \in I\}$ be a family of functions in $C^{(2)}[a,b]$ ($[a,b] \subset Dom(f_t)$). If*

$$a \leq \left(\frac{\frac{d^2 f_s}{dx^2}}{\frac{d^2 f_t}{dx^2}}\right)^{\frac{1}{s-t}} (\xi) \leq b,$$

for $\xi \in [a,b]$, $s, t \in I$, then $\mu_{s,t}(\Lambda_k, \Omega)$ is a mean for $k \in \{1, \ldots, 10\}$.

Remark 6.4.4. In some examples, we will get means of this type:

$$\left(\frac{\frac{d^2 f_s}{dx^2}}{\frac{d^2 f_t}{dx^2}}\right)^{\frac{1}{s-t}} (\xi) = \xi, \quad \xi \in [a,b], \quad s \neq t.$$

□

6.5 Examples with Applications

In this section, we use various classes of functions $\Omega = \{f_t : t \in I\}$ for any open interval $I \subset \mathbb{R}$ to construct different examples of exponentially convex functions and applications to Stolarsky-type means. Let us consider some examples:

Example 6.5.1. *Let $\Omega_1 = \{\psi_t : \mathbb{R} \to \mathbb{R}_* : t \in \mathbb{R}\}$ be a family of functions defined by*

$$\psi_t(x) = \begin{cases} \frac{1}{t^2} e^{tx} & , \quad t \neq 0, \\ \frac{1}{2} x^2 & , \quad t = 0. \end{cases}$$

Here we observe that ψ_t is convex with respect to $g(x) = x$ which is increasing and continuous. Since, $\psi_t(x)$ is a convex function on $\mathbb{R}$ and $t \mapsto \frac{d^2}{dx^2}\psi_t(x)$ is exponentially convex function [33]. Using analogous arguing as in the proof of Theorems 6.4.1 *and* 6.4.4*, we have that $t \mapsto [z_0, z_1, z_2; \psi_t]$ is exponentially convex (and so exponentially convex in the $J-$sense). Using Corollary* 6.4.2 *and* 6.4.5 *we conclude that $t \mapsto \Lambda_k(\psi_t)$, $k \in \{1, \ldots, 10\}$ are exponentially convex in the $J-$sense. It is easy to see that these mappings are continuous, so they are exponentially convex.*

Assume that $t \mapsto \Lambda_k(\psi_t) > 0$ for $k \in \{1, \ldots, 10\}$. By introducing convex functions ψ_t in (7.5.1)*, we obtain the following means: for $k \in \{1, \ldots, 10\}$*

$$\mathfrak{M}_{s,t}(\Lambda_k, \Omega_1) = \begin{cases} \frac{1}{s-t} \ln\left(\frac{\Lambda_k(\psi_s)}{\Lambda_k(\psi_t)}\right) & , \quad s \neq t, \\ \frac{\Lambda_k(id.\psi_s)}{\Lambda_k(\psi_s)} - \frac{2}{s} & , \quad s = t \neq 0, \\ \frac{\Lambda_k(id.\psi_0)}{3\Lambda_k(\psi_0)} & , \quad s = t = 0. \end{cases}$$

where id stands for identity function on $\mathbb{R}$. *Here* $\mathfrak{M}_{s,t}(\Lambda_k, \Omega_1) = \ln(\mu_{s,t}(\Lambda_k, \Omega_1))$, $k \in \{1, \ldots, 10\}$ *are in fact means.* □

Remark 6.5.1. If we choose this class of function in Theorem 6.4.1 then for Λ_1 we get Theorem 32 of [46] and similarly for Λ_k, $k \in \{2, 3, 4\}$, the Theorems 34, 40, 41 and 42 of [46] all become special cases of Theorem 6.4.1.

We observe here that $\left(\frac{\frac{d^2\psi_s}{dx^2}}{\frac{d^2\psi_t}{dx^2}}\right)^{\frac{1}{s-t}} (\ln(\xi)) = \xi$ is a mean for $\xi \in [a, b]$ where $a, b \in \mathbb{R}_+$. We also note that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$ after the substitution $(x_i, y_i) \mapsto (\ln(x_i), \ln(y_i))$, $i \in \{1, \ldots, n\}$ in Λ_k, $k \in \{1, 2\}$ in Theorem 6.4.1 we get the Corollaries 36 and 38 of [46]. We can also obtain similar results for Λ_3 and Λ_4 by this substitution in Theorem 6.4.1. □

Example 6.5.2. *Let* $\Omega_2 = \{\varphi_t : \mathbb{R}_+ \to \mathbb{R} : t \in \mathbb{R}\}$ *be a family of functions as defined in* (6.1.2). *Since* $\varphi_t(x)$ *is a convex function for* $x \in \mathbb{R}_+$ *and* $t \mapsto \frac{d^2}{dx^2}\varphi_t(x)$ *is exponentially convex, so by the same arguments given in previous example we conclude that* $\Lambda_k(\varphi_t)$, $k \in \{1, \ldots, 10\}$ *are exponentially convex.*

We assume that $\Lambda_k(\varphi_t) > 0$ *for* $k \in \{1, \ldots, 10\}$. *For this family of convex functions we obtain the following means: for* $k \in \{1, \ldots, 10\}$

$$\mathfrak{M}_{s,t}(\Lambda_k, \Omega_2) = \begin{cases} \left(\frac{\Lambda_k(\varphi_s)}{\Lambda_k(\varphi_t)}\right)^{\frac{1}{s-t}} & , \quad s \neq t, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Lambda_k(\varphi_0\varphi_s)}{\Lambda_k(\varphi_s)}\right) & , \quad s = t \notin \{0, 1\}, \\ \exp\left(1 - \frac{\Lambda_k(\varphi_0{}^2)}{2\Lambda_k(\varphi_0)}\right) & , \quad s = t = 0, \\ \exp\left(-1 - \frac{\Lambda_k(\varphi_0\varphi_1)}{2\Lambda_k(\varphi_1)}\right) & , \quad s = t = 1. \end{cases}$$

Here $\mathfrak{M}_{s,t}(\Lambda_k, \Omega_2) = \mu_{s,t}(\Lambda_k, \Omega_2)$, $k \in \{1, \ldots, 10\}$ *are in fact means.* □

Remark 6.5.2. If we choose the class of functions given in (6.1.2) in Theorem 6.4.1, then for Λ_1 we get Theorems 2.4 and 2.5 of [4] and similarly for Λ_k, $k \in \{2, 3\}$, the Theorems 2.6, 2.7, 4.4, 4.5, 4.11 and 4.12 of [4] all become special cases of Theorem 6.4.1. We can also obtain similar results for Λ_4 as in [4]. Further, in this choice of family Ω_2, we have

$$\left(\frac{\frac{d^2\varphi_s}{dx^2}}{\frac{d^2\varphi_t}{dx^2}}\right)^{\frac{1}{s-t}} (\xi) = \xi, \quad \xi \in [a, b], \ s \neq t, \text{ where } a, b \in \mathbb{R}_+.$$

So, using Remark 6.4.4 we have an important conclusion that $\mu_{s,t}(\Lambda_k, \Omega_2)$ is in fact mean for $k \in \{1, \ldots, 4\}$. We can extend these means in other cases as given in [4]. □

Example 6.5.3. *Let* $\overline{\Omega}_2 = \{\overline{\varphi}_t : \mathbb{R}_* \to \mathbb{R} : t \in \mathbb{R}_+\}$ *be a family of functions defined as*

$$\overline{\varphi}_t(x) = \begin{cases} \frac{x^t}{t(t-1)} & , \quad t \neq 1, \\ x\ln(x) & , \quad t = 1, \end{cases} \tag{6.5.1}$$

here we use the convention $0\,ln(0) = 0$*. Since,* $\frac{d^2}{dx^2}\overline{\varphi}_t(x) = x^{t-2} = e^{(t-2)\ln(x)}$ *for* $x > 0$ *by same argument given in Example* 6.5.1 *we conclude that* $\Lambda_k(\overline{\varphi}_t),\ k \in \{1, \dots, 10\}$ *are exponentially convex functions with respect to* t*.* □

Remark 6.5.3. If we choose class of functions given in (6.5.1) in Theorem 6.4.1 then we get Theorems 18 and 19 of [46] for Λ_1 and similarly for Λ_k, $k \in \{2, 3, 4\}$, Theorems 20, 26, 27, 28 and 29 of [46] become special cases of Theorem 6.4.1. Further, in this choice of family $\overline{\Omega}_2$ we have

$$\left(\frac{\frac{d^2\overline{\varphi}_s}{dx^2}}{\frac{d^2\overline{\varphi}_t}{dx^2}}\right)^{\frac{1}{s-t}}(\xi) = \xi, \quad \xi \in [a, b],\ s \neq t,\ \text{where } a, b \in \mathbb{R}_+.$$

So, using Remark 6.4.4 we conclude that $\mu_{s,t}(\Lambda_k, \overline{D})$ is mean for $k \in \{1, \dots, 10\}$. We can also extend these means in other cases as given in [46]. All the means $\mu_{s,t}(\Lambda_k, \overline{D})$ are calculated for two parameters s and t, now we move towards three parameters namely r, s, t. For $r > 0$ by substituting $x_i = x_i^r$, $y_i = y_i^r$, $t = t/r$, $s = s/r$ in $\mu_{s,t}(\Lambda_k, \overline{D})$, we get similar results as given in [46]. □

Example 6.5.4. *Let* $\Omega_3 = \{\theta_t : \mathbb{R}_+ \to \mathbb{R}_+ : t \in \mathbb{R}_+\}$ *be a family of functions defined by*

$$\theta_t(x) = \frac{e^{-x\sqrt{t}}}{t}.$$

Since $t \mapsto \frac{d^2}{dx^2}\theta_t(x) = e^{-x\sqrt{t}}$ *is exponentially convex for* $x > 0$*, being the Laplace transform of a nonnegative function [33]. So, by same argument given in Example* 6.5.1 *we conclude that* $\Lambda_k(\theta_t),\ k \in \{1, \dots, 10\}$ *are exponentially convex.*

We assume that $\Lambda_k(\theta_t) > 0$ *for* $k \in \{1, \dots, 10\}$*. For this family of functions we have the following possible cases of* $\mu_{s,t}(\Lambda_k, \Omega_3)$*: for* $k \in \{1, \dots, 10\}$

$$\mu_{s,t}(\Lambda_k, \Omega_3) = \begin{cases} \left(\frac{\Lambda_k(\theta_s)}{\Lambda_k(\theta_t)}\right)^{\frac{1}{s-t}} & , \quad s \neq t, \\ \exp\left(-\frac{\Lambda_k(id.\theta_s)}{2\sqrt{s}\,\Lambda_k(\theta_s)} - \frac{1}{s}\right) & , \quad s = t. \end{cases}$$

By (6.3.1)*,* $\mathfrak{M}_{s,t}(\Lambda_k, \Omega_3) = -(\sqrt{s} + \sqrt{t})\ln(\mu_{s,t}(\Lambda_k, \Omega_3)), \quad k \in \{1, \dots, 10\}$ *defines a class of means.* □

Example 6.5.5. *Let $\Omega_4 = \{\phi_t : \mathbb{R}_+ \to \mathbb{R}_+ : t \in \mathbb{R}_+\}$ be a family of functions defined by*

$$\phi_t(x) = \begin{cases} \frac{t^{-x}}{(\ln(t))^2} & , \quad t \neq 1, \\ \frac{x^2}{2} & , \quad t = 1. \end{cases}$$

Since $\frac{d^2}{dx^2}\phi_t(x) = t^{-x} = e^{-x\ln(t)} > 0$ for $x > 0$, so by same argument given in Example 6.5.1 *we conclude that $t \mapsto \Lambda_k(\phi_t),\ k \in \{1, \ldots, 10\}$ are exponentially convex.*

We assume that $\Lambda_k(\phi_t) > 0$ for $k \in \{1, \ldots, 10\}$. For this family of functions we have the following possible cases of $\mu_{s,t}(\Lambda_k, \Omega_4)$: for $k \in \{1, \ldots, 10\}$

$$\mu_{s,t}(\Lambda_k, \Omega_4) = \begin{cases} \left(\frac{\Lambda_k(\phi_s)}{\Lambda_k(\phi_t)}\right)^{\frac{1}{s-t}} & , \quad s \neq t, \\ \exp\left(-\frac{\Lambda_k(id.\phi_s)}{s\Lambda_k(\phi_s)} - \frac{2}{s\ln(s)}\right) & , \quad s = t \neq 1, \\ \exp\left(-\frac{1}{3}\frac{\Lambda_k(id.\phi_1)}{\Lambda_k(\phi_1)}\right) & , \quad s = t = 1. \end{cases}$$

By (6.3.1)*, $\mathfrak{M}_{s,t}(\Lambda_k, \Omega_4) = -L(s,t)\ln(\mu_{s,t}, (\Lambda_k, \Omega_4)), \quad k \in \{1, \ldots, 10\}$ defines a class of means, where $L(s,t)$ is Logarithmic mean defined as:*

$$L(s,t) = \begin{cases} \frac{s-t}{\ln(s)-\ln(t)} & , \quad s \neq t, \\ s & , \quad s=t. \end{cases} \tag{6.5.2}$$

□

Remark 6.5.4. Monotonicity of $\mu_{s,t}(\Lambda_k, \Omega_j)$ follow form (6.4.2) for $k \in \{1, \ldots, 10\}$, $j \in \{1, \ldots, 4\}$. The special cases of $\mu_{s,t}(\Lambda_5, \Omega_j)$ for $j \in \{1, \ldots, 4\}$ in terms of weighted power means are given in [40]. □

The last chapter of our dissertation is based on the results related to the Jensen-type inequalities. The main subject of the next chapter consists of the generalizations and refinements of the Jensen-type, of the reverse Jensen-type and of the Jensen-Mercer's inequalities.

Chapter 7

Jensen-type Inequalities

"Every mathematician loves an inequality."

–A. M. Fink

The present chapter gives generalizations of the Jensen-type and of the reverse Jensen-type inequalities. These inequalities are dealt with in two different cases, discrete and continuous. Generalizations and refinements of the Jensen-Mercer's inequities are also given with some applications. Mean value theorems and $n-$exponential convexity for functionals constructed from the stated inequalities are also be discussed. The chapter ends with some applications to the Cauchy means.

The contents of the present chapter are given in [38] and [42].

7.1 Introductions and Preliminaries

The well known Jensen's inequality for convex functions is one of the most celebrated inequalities in mathematics and statistics. Jensen's inequality asserts a remarkable relation between the mean and the mean of function values. Any generalization or refinements of Jensen's inequality is a source to enrichment of monotone property of mixed means. Applications of Jensen's inequality in statistics and probability related to the expectation of a convex function of a random variable are of great significance. Moreover, many other important inequalities may be obtained from it such as Hölder's and Minkowski's inequalities. Thus, some generalizations and refinements of inequalities of Jensen-type are discussed in the present chapter with some applications. For detailed discussion and history of the topic we refer to [57] and [80] (see also [37]).

Let us start with Jensen's inequality for convex functions [80]. Throughout this chapter $[a, b]$ and $[c, d]$ are intervals in $\mathbb{R}$, also m_1 and m_2 are defined as $m_1 = \min\limits_{1\leq i\leq n}\{x_i\}$ and $m_2 = \max\limits_{1\leq i\leq n}\{x_i\}$ where $x_i \in [a, b]$ for $i \in \{1, \ldots, n\}$.

Proposition 7.1.1. *Let $x_i \in [a, b]$ and $w_i \in \mathbb{R}_*$ for $i \in \{1, \ldots, n\}$ such that $W_n = \sum_{i=1}^n w_i = 1$. If $\varphi : [a, b] \to \mathbb{R}$ is a convex function, then the inequality*

$$\varphi\left(\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} w_i \varphi(x_i) \tag{7.1.1}$$

holds.

In paper [54], Mercer proved the following variant of Jensen's inequality, which we will refer to as Mercer's inequality.

Proposition 7.1.2. *Let $x_i \in [a, b]$ and $w_i \in \mathbb{R}_*$ for $i \in \{1, \ldots, n\}$ such that $W_n = \sum_{i=1}^n w_i = 1$. If $\varphi : [a, b] \to \mathbb{R}$ is a convex function, then the inequality*

$$\varphi\left(m_1 + m_2 - \sum_{i=1}^{n} w_i x_i\right) \leq \varphi(m_1) + \varphi(m_2) - \sum_{i=1}^{n} w_i \varphi(x_i) \tag{7.1.2}$$

holds, where $m_1 = \min\limits_{1\leq i\leq n}\{x_i\}$ and $m_2 = \max\limits_{1\leq i\leq n}\{x_i\}$

There are many versions, variants and generalizations of Proposition 7.1.1 and Proposition 7.1.2, see e.g. [1], [19], [62] and [66]. Here we state few integral versions of Jensen's inequality from [80, pp. 58–59] which will be needed in the main theorems of the present chapter.

Proposition 7.1.3. *Let $f : [c, d] \to [a, b]$ be a continuous function. If the function $H : [c, d] \to \mathbb{R}$ is nondecreasing, bounded and $H(c) \neq H(d)$, then for every continuous convex function $\varphi : [a, b] \to \mathbb{R}$ the inequality*

$$\varphi\left(\frac{\int_c^d f(t)dH(t)}{\int_c^d dH(t)}\right) \leq \frac{\int_c^d \varphi(f(t))dH(t)}{\int_c^d dH(t)} \tag{7.1.3}$$

holds.

Inequality (7.1.3) can hold under different set of assumptions. For example, for a monotonic f, assumptions on H can be relaxed. The following proposition gives Jensen-Steffensen's inequality.

Proposition 7.1.4. *If $f : [c,d] \to [a,b]$ is continuous and monotonic (either nonincreasing or nondecreasing) and $H : [c,d] \to \mathbb{R}$ is either continuous or of bounded variation satisfying $H(c) \leq H(t) \leq H(d)$ for all $t \in [c,d]$, $H(c) < H(d)$, then* (7.1.3) *holds.*

If we replace the assumption of monotonicity of f over the whole interval $[c,d]$ in Proposition 7.1.4 with monotonicity over subintervals, we obtain the following, Jensen-Boas' inequalities.

Proposition 7.1.5. *If $H : [c,d] \to \mathbb{R}$ is continuous or of bounded variation satisfying*

$$H(c) \leq H(x_1) \leq H(y_1) \leq H(x_2) \leq \cdots \leq H(y_{k-1}) \leq H(x_k) \leq H(d)$$

for all $x_i \in (y_{i-1}, y_i)$ ($y_0 = c$, $y_k = d$) and $H(c) < H(d)$, and if f is continuous and monotonic (either nonincreasing or nondecreasing) in each of the k intervals (y_{i-1}, y_i), then inequality (7.1.3) *holds.*

7.2 Jensen-type and Reverse Jensen-type Inequalities

Starting from the discrete Jensen's inequality, Mercer in [53, 55] gave two mean value theorems of the Lagrange- and of the Cauchy-type. Having in mind the integral Jensen's inequality, the authors in [78] gave similar results in integral form. The generalization of these results for the real Stieltjes measure and several other interesting results concerning Jensen-type and reverse Jensen-type inequalities are given in [77] using the Green function $G : [a,b] \times [a,b] \to \mathbb{R}$ defined by

$$G(t,s) = \begin{cases} \frac{(s-a)(t-b)}{b-a} & , \quad a \leq s \leq t, \\ \frac{(t-a)(s-b)}{b-a} & , \quad t \leq s \leq b, \end{cases} \tag{7.2.1}$$

where G is continuous convex function with respect to both s and t.

Here, we introduce some notations to be used in up coming subsections as follows:

$$W_j = \sum_{i=1}^{j} w_i, \quad \overline{W_j} = W_n - W_{j-1} \quad \text{for} \quad j \in \{1, \ldots, n\}$$

and

$$\overline{x} = \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i,$$

where $w_i, x_i \in I \subset \mathbb{R}$ for $i \in \{1, \dots, n\}$ and $W_n = \sum_{i=1}^{n} w_i \neq 0$. Also

$$\overline{g} = \frac{\int_c^d g(x)d\lambda(x)}{\int_c^d \lambda(x)} \in [a, b]$$

where $g, \lambda : [c, d] \to \mathbb{R}$ are two functions such that image of g is a subset of $[a, b]$ and $\lambda(c) \neq \lambda(d)$.

7.2.1 Discrete Jensen-type and Reverse Jensen-type Inequalities

The discrete Jensen's inequality (7.1.1) asserts that for convex function φ on interval $I \subset \mathbb{R}$ the inequality

$$\varphi(\bar{x}) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i \varphi(x_i) \tag{7.2.2}$$

holds, where $w_i \in \mathbb{R}_+$ and $x_i \in I$ for $i \in \{1, \dots, n\}$.

In [77], the generalization of above result is given, allowing that w_i may also be negative, with the sum different from 0, but with a supplementary demand on w_i, x_i given using the Green function G defined in (7.2.1). In [77], the following result is derived.

Proposition 7.2.1. *Let the following assumptions be valid: $x_i \in [c, d] \subset [a, b]$, $w_i \in \mathbb{R}$, $i \in \{1, \dots, n\}$ be such that $W_n \neq 0$ and $\overline{x} \in [a, b]$. Then the following statements are equivalent:*

(a) *For every continuous convex function $\varphi : [a, b] \to \mathbb{R}$, the following inequality holds*

$$\varphi(\overline{x}) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i \varphi(x_i). \tag{7.2.3}$$

(b) *For each $s \in [a, b]$, the inequality*

$$G(\overline{x}, s) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i G(x_i, s) \tag{7.2.4}$$

holds, where the function G is defined in (7.2.1).

Moreover, if the sign of inequality is changed in both (7.2.3) *and* (7.2.4)*, then the statements* (*a*) *and* (*b*) *are also equivalent.*

Remark 7.2.1. Note that in the case when all $w_i > 0$, $i \in \{1, \dots, n\}$ (or when all $w_i \geq 0$, $i \in \{1, \dots, n\}$ and $W_n > 0$) inequality (7.2.3) becomes discrete Jensen's inequality (7.2.2) and by this inequality we have that for each $s \in [a, b]$ the inequality (7.2.4) holds. If $\mathbf{x} = (x_1, \dots, x_n)$ is monotonous $n-$tuple (i.e., either $x_1 \leq x_2 \leq \dots \leq x_n$ or $x_1 \geq x_2 \geq \dots \geq x_n$) and $0 \leq W_k \leq W_n$ for $k \in \{1, \dots, n-1\}$ and $W_n > 0$, then by the discrete Jensen-Steffensen's inequality (see [80, p. 57]) we also have that for each $s \in [a, b]$ inequality (7.2.4) holds.

On the other hand, if $\mathbf{w} = (w_1, \dots, w_n)$ is such that $w_1 > 0$, $w_2, \dots, w_n \leq 0$ and $W_n > 0$, then by the reverse Jensen's inequality (see [15, p. 45]) we have that for each $s \in [a, b]$ the reverse inequality in (7.2.4) holds. If $\mathbf{x} = (x_1, \dots, x_n)$ is monotonous $n-$tuple and $\mathbf{w} = (w_1, \dots, w_n)$ is such that there exists $m \in \{1, \dots, n\}$ so that $W_k \leq 0$ for $k < m$ and $\overline{W_k} \leq 0$ for $k > m$, and $W_n > 0$, then by the reverse Jensen-Steffensen's inequality (see [80, p. 83]) we have that for each $s \in [a, b]$ the reverse inequality in (7.2.4) holds. □

Motivated by the inequality (7.2.3), under the assumptions of Proposition 7.2.1, we define the functional λ_7 by

$$\lambda_7(\mathbf{x}, \mathbf{w}; \varphi) = \frac{1}{W_n} \sum_{i=1}^{n} w_i \varphi(x_i) - \varphi(\overline{x}).$$

Now, we define the functional Λ_{11} in terms of λ_7

$$\Lambda_{11} = \begin{cases} \lambda_7 \quad , & \text{if for each } s \in [a, b] \text{ inequality (7.2.4) holds,} \\ -\lambda_7 \quad , & \text{if for each } s \in [a, b] \text{ the reverse inequality in (7.2.4) holds.} \end{cases} \tag{A11}$$

Note that, whenever it is defined, Λ_{11} is nonnegative.

The similar results may also be derived for the reverse Jensen's inequality in discrete case. We state the following result from [77].

Proposition 7.2.2. *Let the assumptions of Proposition* 7.2.1 *be valid with the condition that* $c \neq d$*. Then the following statements are equivalent:*

(*a*) *For every continuous convex function* $\varphi : [a, b] \to \mathbb{R}$*, the following inequality holds*

$$\frac{1}{W_n} \sum_{i=1}^{n} w_i \varphi(x_i) \leq \frac{d - \overline{x}}{d - c} \varphi(c) + \frac{\overline{x} - c}{d - c} \varphi(d). \tag{7.2.5}$$

(*b*) *For each* $s \in [a, b]$, *the inequality*

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i G(x_i, s) \leq \frac{d-\overline{x}}{d-c}G(c, s) + \frac{\overline{x}-c}{d-c}G(d, s) \tag{7.2.6}$$

holds, where the function G *is defined in* (7.2.1).

Moreover, if the sign of inequality is changed in both (7.2.5) *and* (7.2.6), *then the statements* (*a*) *and* (*b*) *are also equivalent.*

Remark 7.2.2. If we set that all $w_i \in \mathbb{R}_+$ for $i \in \{1, \ldots, n\}$, then (7.2.5) becomes classical reverse Jensen's inequality (see [70, p. 48]) and by this inequality we have that for each $s \in [a, b]$ inequality (7.2.6) holds. □

Remark 7.2.3. If we set $c = a$ and $d = b$ in Proposition 7.2.2, then inequality (7.2.6) becomes (see [77])

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i G(x_i, s) \leq 0.$$

□

Motivated by inequality (7.2.5), under the assumptions of Proposition 7.2.2, we define the functional λ_8 by

$$\lambda_8(\mathbf{x}, \mathbf{w}; \varphi) = \frac{d-\overline{x}}{d-c}\varphi(c) + \frac{\overline{x}-c}{d-c}\varphi(d) - \frac{1}{W_n}\sum_{i=1}^{n} w_i \varphi(x_i).$$

Now, we define the functional Λ_{12} in terms of λ_8

$$\Lambda_{12} = \begin{cases} \lambda_8 \quad , & \text{if for each } s \in [a, b] \text{ inequality (7.2.6) holds,} \\ -\lambda_8 \quad , & \text{if for each } s \in [a, b] \text{ the reverse inequality in (7.2.6) holds.} \end{cases} \tag{A12}$$

Note that, whenever it is defined, Λ_{12} is nonnegative.

7.2.2 Integral Jensen-type and Reverse Jensen-type Inequalities

The following theorem, extracted from [77], gave the conditions on the real Stieltjes measure $d\lambda$ (not necessarily positive), such that $\lambda(c) \neq \lambda(d)$, for which the Jensen's inequality holds for every continuous convex function φ.

Proposition 7.2.3. *Let the following assumptions be valid:* $g : [c, d] \to [a, b]$ *be a continuous function,* $\lambda \in C[c, d]$ *or* $\lambda \in BV[c, d]$ *such that* $\lambda(c) \neq \lambda(d)$ *and* $\overline{g} \in [a, b]$*. Then the following statements are equivalent:*

(a) *For every continuous convex function* $\varphi : [a, b] \to \mathbb{R}$*, the following inequality holds*

$$\varphi\left(\overline{g}\right) \leq \frac{\int_c^d \varphi\left(g(x)\right) d\lambda(x)}{\int_c^d \lambda(x)}. \tag{7.2.7}$$

(b) *For each* $s \in [a, b]$*, the inequality*

$$G\left(\overline{g}, s\right) \leq \frac{\int_c^d G\left(g(x), s\right) d\lambda(x)}{\int_c^d \lambda(x)} \tag{7.2.8}$$

holds, where the function G *is defined in* (7.2.1).

Moreover, if the sign of inequality is changed in both (7.2.7) *and* (7.2.8)*, then the statements* (a) *and* (b) *are also equivalent.*

Remark 7.2.4. For the case of positive measure $d\lambda$, we get the well known results. If the function λ is nondecreasing and bounded with $\lambda(c) \neq \lambda(d)$, then inequality (7.2.7) becomes integral Jensen's inequality and by this inequality, we have that for each $s \in [a, b]$ the inequality (7.2.8) holds. On the other hand, if the function g is continuous and monotonic, and either $\lambda \in C[c, d]$ or $\lambda \in BV[c, d]$, satisfying $\lambda(c) \leq \lambda(x) \leq \lambda(d)$ for each $x \in [c, d]$ and $\lambda(c) < \lambda(d)$, then inequality (7.2.7) becomes integral Jensen-Steffensen's inequality given by Boas in [10] (see also [80, p. 59]) and by this inequality, we have that for each $s \in [a, b]$ the inequality (7.2.8) holds.

Next, if g is continuous function and λ is the function of bounded variation, nonincreasing on the intervals $[c, \gamma]$ and $(\gamma, d]$ such that $\lambda(d) > \lambda(c)$, then by the reverse Jensen's inequality (see [74] or [80, p. 84]), we have that for each $s \in [a, b]$ the reverse inequality in (7.2.8) holds. For more discussion on such type of inequalities we refer [77] and [80]. □

Motivated by the inequality (7.2.7), under the assumptions of Proposition 7.2.3, we define the functional λ_9 by

$$\lambda_9(g, \lambda; \varphi) = \frac{\int_c^d \varphi\left(g(x)\right) d\lambda(x)}{\int_c^d \lambda(x)} - \varphi\left(\overline{g}\right).$$

Now, we define the functional Λ_{13} in terms of λ_9

$$\Lambda_{13} = \begin{cases} \lambda_9 & , \quad \text{if for each } s \in [a,b] \text{ inequality (7.2.8) holds,} \\ -\lambda_9 & , \quad \text{if for each } s \in [a,b] \text{ the reverse inequality in (7.2.8) holds.} \end{cases} \tag{A13}$$

Note that, whenever it is defined, Λ_{13} is nonnegative.

The similar results may also be derived for the reverse Jensen's inequality in integral case. We state the following result from [77].

Proposition 7.2.4. *Let the assumptions of Proposition* 7.2.3 *be valid. In addition, we assume that* $m, M \in [a,b]$ $(m \neq M)$ *such that* $m \leq g(t) \leq M$ *for each* $t \in [c,d]$*. Then the following statements are equivalent:*

(*a*) *For every continuous convex function* $\varphi : [a,b] \to \mathbb{R}$*, the following inequality holds*

$$\frac{\int_c^d \varphi\left(g(x)\right) d\lambda(x)}{\int_c^d \lambda(x)} \leq \frac{M-\overline{g}}{M-m}\varphi(m) + \frac{\overline{g}-m}{M-m}\varphi(M). \tag{7.2.9}$$

(*b*) *For each* $s \in [a,b]$*, the inequality*

$$\frac{\int_c^d G\left(g(x), s\right) d\lambda(x)}{\int_c^d \lambda(x)} \leq \frac{M-\overline{g}}{M-m}G(m,s) + \frac{\overline{g}-m}{M-m}G(M,s) \tag{7.2.10}$$

holds, where the function G *is defined in* (7.2.1)*.*

Moreover, if the sign of inequality is changed in both (7.2.9) *and* (7.2.10)*, then the statements* (*a*) *and* (*b*) *are also equivalent.*

Remark 7.2.5. If we set $m = a$ and $M = b$ in Proposition 7.2.4, then inequality (7.2.10) becomes (see [77])

$$\frac{\int_c^d G\left(g(x), s\right) d\lambda(x)}{\int_c^d \lambda(x)} \leq 0.$$

□

Motivated by the inequality (7.2.9), under the assumptions of Proposition 7.2.4, we define the functional λ_{10} by

$$\lambda_{10}(g, \lambda; \varphi) = \frac{M-\overline{g}}{M-m}\varphi(m) + \frac{\overline{g}-m}{M-m}\varphi(M) - \frac{\int_c^d \varphi\left(g(x)\right) d\lambda(x)}{\int_c^d d\lambda(x)}.$$

Now, we define the functional Λ_{14} in terms of λ_{10}

$$\Lambda_{14} = \begin{cases} \lambda_{10} & , \quad \text{if for each } s \in [a,b] \text{ inequality (7.2.10) holds,} \\ -\lambda_{10} & , \quad \text{if for each } s \in [a,b] \text{ the reverse inequality in (7.2.10) holds.} \end{cases} \tag{A14}$$

Note that, whenever it is defined, Λ_{14} is nonnegative.

7.3 Generalizations and Refinements of Jensen-Mercer's Inequality

The following extension of (7.1.2) is given by Niezgoda in [62] which we will refer to as Niezgoda's inequality.

Proposition 7.3.1. *Let the following assumptions be valid:* $\mathbf{a} = (a_1, \ldots, a_m) \in [a,b]^m$, $\mathbf{X} = (x_{ij})$ *is an* $n \times m$ *matrix such that* $x_{ij} \in [a,b]$ *for each* $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$ *and* $\mathbf{a}$ *majorizes each row of* $\mathbf{X}$, *that is*

$$\mathbf{x}_{i.} = (x_{i1}, \ldots, x_{im}) \prec (a_1, \ldots, a_m) = \mathbf{a} \textit{ for each } i \in \{1, \ldots, n\}.$$

Then for every continuous convex function $\varphi : [a,b] \to \mathbb{R}$, *the inequality*

$$\varphi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}\right) \leq \sum_{j=1}^{m} \varphi(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i \varphi(x_{ij}), \tag{7.3.1}$$

holds, where $\sum_{i=1}^{n} w_i = 1$ *with* $w_i \geq 0$.

Here we state some results needed in the main theorems of this section. The following proposition is a consequence of Theorem 1 in [71] (see also [80, p. 328]) represents an integral majorization result.

Proposition 7.3.2. *Let* f, $g : [c,d] \to [a,b]$ *be two nonincreasing continuous functions and let* $H : [c,d] \to \mathbb{R}$ *be a function of bounded variation. If*

$$\begin{aligned} \int_c^u f(t)\, dH(t) \quad &\leq \quad \int_c^u g(t)\, dH(t), \quad \textit{for each} \quad u \in (c,d), \\ \textit{and} \quad \int_c^d f(t)\, dH(t) \quad &= \quad \int_c^d g(t)\, dH(t), \end{aligned}$$

hold, then for every continuous convex function $\varphi : [a,b] \to \mathbb{R}$ the following inequality holds

$$\int_c^d \varphi(f(t))\,dH(t) \le \int_c^d \varphi(g(t))\,dH(t). \tag{7.3.2}$$

Remark 7.3.1. If $f,\ g : [c,d] \to [a,b]$ are two nondecreasing continuous functions such that

$$\begin{aligned} \int_u^d f(t)\,dH(t) &\le \int_u^d g(t)\,dH(t), \quad \text{for each} \quad u \in (c,d), \\ \text{and} \quad \int_c^d f(t)\,dH(t) &= \int_c^d g(t)\,dH(t), \end{aligned}$$

where $H \in BV[c,d]$, then again inequality (7.3.2) holds. In the present section, we will state our results for nonincreasing f and g satisfying the assumption of Proposition 7.3.2, but they are still valid for nondecreasing f and g satisfying the above condition see for example [51, p. 584]. □

The following lemma shows that the subintervals in the Jensen-Boas' inequality (see Proposition 7.1.5) can be disjoint for the inequality of type (7.1.3) to hold.

Lemma 7.3.1. *Let $c = d_0 \le c_1 < d_1 < c_2 < d_2 < \ldots < c_k < d_k \le c_{k+1} = d$ be the partition of the interval $[c,d]$ and let $I = \bigcup_{l=1}^k (c_l, d_l)$. Further, let $H : [c,d] \to \mathbb{R}$ be continuous or a function of bounded variation such that $L = \int_I dH(t) > 0$ and*

$$H(c_l) \le H(t) \le H(d_l) \quad \textit{for all } t \in (c_i, d_i) \textit{ and } 1 \le l \le k. \tag{7.3.3}$$

Then for every function $f : [c,d] \to [a,b]$ which is continuous and monotonic (either nonincreasing or nondecreasing) in each of the k intervals (c_i, d_i) and every convex and continuous function $\varphi : [a,b] \to \mathbb{R}$, the following inequality holds

$$\varphi\left(\frac{1}{L}\int_I f(t)\,dH(t)\right) \le \frac{1}{L}\int_I \varphi(f(t))\,dH(t).$$

Proof. Denote $w_i = \int_{c_i}^{d_i} dH(t)$. Due to (7.3.3), if $H(c_i) = H(d_i)$ then dH is a null-measure on $[c_i, d_i]$ and $w_i = 0$, while otherwise $w_i > 0$. Denote $S = \{i : w_i > 0\}$ and

$$x_i = \frac{1}{w_i}\int_{c_i}^{d_i} f(t)\,dH(t), \quad \text{for } i \in S.$$

Notice that

$$L = \int_I dH(t) = \sum_{i\in S} w_i > 0, \quad \int_I \varphi(f(t))\,dH(t) = \sum_{i \in S}\int_{c_i}^{d_i} \varphi(f(t))\,dH(t)$$

and, due to Proposition 7.1.4,

$$w_i\varphi(x_i) \leq \int_{c_i}^{d_i} \varphi(f(t))\, dH(t), \qquad \text{for } i \in S.$$

Therefore, taking into account the discrete Jensen's inequality, we finally get

$$\varphi\left(\frac{1}{L}\int_I f(t)\, dH(t)\right) = \varphi\left(\frac{1}{L}\sum_{i\in S} w_i x_i\right) \leq \frac{1}{L}\sum_{i\in S} w_i\varphi(x_i) \leq$$
$$\leq \frac{1}{L}\sum_{i\in S}\int_{a_i}^{b_i} \varphi(f(t))\, dH(t) = \frac{1}{L}\int_I \varphi(f(t))\, dH(t).$$

■

The following theorem is our main result of this section and it gives a generalization of the Proposition 7.3.1.

Theorem 7.3.1. *Let the following assumptions be valid:* $c = d_0 \leq c_1 < d_1 < c_2 < d_2 < \ldots < c_k < d_k \leq c_{k+1} = d$ *is partition of* $[c,d]$, $I = \bigcup_{l=1}^{k}(c_l, d_l)$, $I^c = [c,d]\backslash I = \bigcup_{l=1}^{k+1}[d_{l-1}, c_l]$ *and* $H : [c,d] \to \mathbb{R}$ *be a function of bounded variation such that* $L = \int_{I^c} dH(t) > 0$ *and*

$$H(c_{l-1}) \leq H(t) \leq H(d_l) \quad \textit{for all } t \in (c_{l-1}, d_l) \textit{ and } 1 \leq l \leq k.$$

Furthermore, let (X, Σ, μ) *be a measure space with positive finite measure* μ, *let* $g : [c,d] \to [a,b]$ *be a nonincreasing continuous function and let* $f : X \times [c,d] \to [a,b]$ *be a measurable function such that the mapping* $t \mapsto f(s,t)$ *is nonincreasing and continuous for each* $s \in X$ *and*

$$\int_c^u f(s,t)\, dH(t) \leq \int_c^u g(t) dH(t), \qquad u \in (c,d),$$
$$\int_c^d f(s,t)\, dH(t) = \int_c^d g(t)\, dH(t). \tag{7.3.4}$$

Then for every continuous convex function $\varphi : [a,b] \to \mathbb{R}$, *the inequality*

$$\varphi\left(\frac{1}{L}\left(\int_c^d g(t)dH(t) - \frac{1}{\mu(X)}\int_I\int_X f(s,t)d\mu(s)dH(t)\right)\right) \leq$$
$$\leq \frac{1}{L}\left(\int_c^d \varphi(g(t))dH(t) - \frac{1}{\mu(X)}\int_I\int_X \varphi(f(s,t))d\mu(s)dH(t)\right) \tag{7.3.5}$$

holds.

Proof. Using Fubini's theorem, equality (7.3.4) and the integral Jensen's inequality (7.1.3) we get

$$\varphi\left(\frac{1}{L}\left(\int_c^d g(t)dH(t)-\frac{1}{\mu(X)}\int_I\int_X f(s,t)d\mu(s)dH(t)\right)\right)=$$
$$\varphi\left(\frac{1}{\mu(X)}\int_X\left[\frac{1}{L}\int_{I^c} f(s,t)dH(t)\right]d\mu(s)\right)\leq\frac{1}{\mu(X)}\int_X\varphi\left(\frac{1}{L}\int_{I^c} f(s,t)dH(t)\right)d\mu(s). \tag{7.3.6}$$

Applying Lemma 7.3.1 and Proposition 7.3.2, respectively, we have

$$\begin{aligned}\varphi\left(\frac{1}{L}\int_{I^c} f(s,t)dH(t)\right)&\leq\frac{1}{L}\int_{I^c}\varphi(f(s,t))dH(t)\\&\leq\frac{1}{L}\left(\int_c^d\varphi(g(t))dH(t)-\int_I\varphi(f(s,t))dH(t)\right).\end{aligned}\tag{7.3.7}$$

Finally, combining (7.3.6) and (7.3.7) we obtain inequality (7.3.5). ■

The following result is a direct consequence of Theorem 7.3.1.

Corollary 7.3.2. *Let the following assumptions be valid:* $\mathbf{a}=(a_1,\ldots,a_m)\in[a,b]^m$, $\mathbf{X}=(x_{ij})$ *is an* $n\times m$ *matrix such that* $x_{ij}\in[a,b]$ *for each* $i\in\{1,\ldots,n\}$, $j\in\{1,\ldots,m\}$ *and* $\mathbf{a}$ *majorizes each row of* $\mathbf{X}$. *Further, let* $c_l,d_l\in\mathbb{N}$ *for* $l\in\{1,\ldots,k\}$ *be such that* $1=d_0\leq c_1<d_1<c_2<d_2<\ldots<c_k<d_k\leq c_{k+1}=m+1$ *and denote* $L=\sum_{l=1}^{k+1}(c_l-d_{l-1})$. *Then for every continuous convex function* $\varphi:[a,b]\to\mathbb{R}$, *the inequality*

$$\varphi\left(\frac{1}{L}\left(\sum_{j=1}^m a_j-\frac{1}{W_n}\sum_{l=1}^k\sum_{j=c_l}^{d_l-1}\sum_{i=1}^n w_i x_{ij}\right)\right)\leq$$
$$\leq\frac{1}{L}\left(\sum_{j=1}^m\varphi(a_j)-\frac{1}{W_n}\sum_{l=1}^k\sum_{j=c_l}^{d_l-1}\sum_{i=1}^n w_i\varphi(x_{ij})\right)$$

holds, where $W_n=\sum_{i=1}^n w_i>0$ *with* $w_i\geq 0$.

Proof. The proof of the corollary follows from Theorem 7.3.1 by taking step functions. More concretely, for $c=d_0=1$, $d=c_{k+1}=m+1$, $g(t)=\sum_{j=1}^m a_j\,\chi_{[j,j+1)}(t)$, $f(s,t)=\sum_{i=1}^n\sum_{j=1}^m x_{ij}\,\chi_{[i,i+1)}(s)\chi_{[j,j+1)}(t)$, $X=[1,n+1)$, $d\mu(s)=\sum_{i=1}^n w_i\chi_{[i,i+1)}(s)d\lambda(s)$ and $H(t)=t$. ■

Remark 7.3.2. If in Corollary 7.3.2 we simply take $k=1$, $c_1=1$ and $d_1=m$ and assume that $W_n=\sum_{i=1}^n w_i=1$, then we get Niezgoda's inequality (7.3.1). □

7.3.1 Refinements

In the present subsection, we give some refinements and for that purpose we need some construction: we assume that $\Omega \subset X$ with $\mu(\Omega), \mu(\Omega^c) > 0$ and we define the following notations

$$W_\Omega = \frac{\mu(\Omega)}{\mu(X)}, \quad W_{\Omega^c} = \frac{\mu(\Omega^c)}{\mu(X)} = 1 - W_\Omega.$$

Using these notations, under assumptions of Theorem 7.3.1, we define the following functional

$$\begin{aligned} F_1(f,g,\varphi;\Omega) &= W_\Omega \varphi\left(\frac{1}{L}\left(\int_c^d g(t)dH(t) - \frac{1}{\mu(\Omega)}\int_I\int_\Omega f(s,t)d\mu(s)dH(t)\right)\right) \\ &+ W_{\Omega^c}\varphi\left(\frac{1}{L}\left(\int_c^d g(t)dH(t) - \frac{1}{\mu(\Omega^c)}\int_I\int_{\Omega^c} f(s,t)d\mu(s)dH(t)\right)\right). \end{aligned} \tag{7.3.8}$$

We also assume that $I \subset \{1,\ldots,n\}$ with $I \neq \emptyset$ and $I \neq \{1,\ldots,n\}$. We define $W_I = \sum_{i\in I} w_i > 0$ and $W_{\bar{I}} = 1 - \sum_{i\in I} w_i > 0$, where w_i's are nonnegative weights. Under the assumptions of Corollary 7.3.2 and Propositions 7.1.2 and 7.3.1, we define the following functionals respectively

$$\begin{aligned} F_2(\mathbf{X},\mathbf{w},\varphi;I) &= W_I\varphi\left(\frac{1}{L}\left(\sum_{j=1}^m a_j - \frac{1}{W_I}\sum_{l=1}^k\sum_{j=c_l}^{d_l-1}\sum_{i\in I} w_i x_{ij}\right)\right) \\ &+ W_{\bar{I}}\frac{1}{L}\left(\sum_{j=1}^m \varphi(a_j) - \frac{1}{W_{\bar{I}}}\sum_{l=1}^k\sum_{j=c_l}^{d_l-1}\sum_{i\in \bar{I}} w_i \varphi(x_{ij})\right), \end{aligned} \tag{7.3.9}$$

$$\begin{aligned} F_3(\mathbf{X},\mathbf{w},\varphi;I) &= W_I\varphi\left(\sum_{j=1}^m a_j - \frac{1}{W_I}\sum_{j=1}^{m-1}\sum_{i\in I} w_i x_{ij}\right) \\ &+ W_{\bar{I}}\varphi\left(\sum_{j=1}^m a_j - \frac{1}{W_{\bar{I}}}\sum_{j=1}^{m-1}\sum_{i\in \bar{I}} w_i x_{ij}\right), \end{aligned} \tag{7.3.10}$$

$$\begin{aligned} F_4(\mathbf{x},\mathbf{w},\varphi;I) &= W_I\varphi\left(m_1 + m_2 - \frac{1}{W_I}\sum_{i\in I} w_i x_i\right) \\ &+ W_{\bar{I}}\varphi\left(m_1 + m_2 - \frac{1}{W_{\bar{I}}}\sum_{i\in \bar{I}} w_i x_i\right). \end{aligned} \tag{7.3.11}$$

The following refinement of (7.3.5) is valid.

Theorem 7.3.3. *Let the assumptions of Theorem* 7.3.1 *be valid. Then for any nonempty* $\Omega \subset X$*, the inequality*

$$\varphi\left(\frac{1}{L}\left(\int_c^d g(t)dH(t) - \frac{1}{\mu(X)}\int_I\int_X f(s,t)d\mu(s)dH(t)\right)\right) \leq F_1(f,g,\varphi;\Omega) \leq$$
$$\leq \frac{1}{L}\left(\int_c^d \varphi(g(t))dH(t) - \frac{1}{\mu(X)}\int_I\int_X \varphi(f(s,t))d\mu(s)dH(t)\right) \quad (7.3.12)$$

hold, where F_1 *is defined in* (7.3.8).

Proof. By using convexity of the function φ, we have

$$\begin{aligned}
&\varphi\left(\frac{1}{L}\left(\int_c^d g(t)dH(t) - \frac{1}{\mu(X)}\int_I\int_X f(s,t)d\mu(s)dH(t)\right)\right)\\
&= \varphi\left(W_\Omega\left[\frac{1}{L}\left(\int_c^d g(t)dH(t) - \frac{1}{\mu(\Omega)}\int_\Omega\int_I f(s,t)dH(t)\right)d\mu(s)\right]\right.\\
&\left.+ W_{\Omega^c}\left[\frac{1}{L}\left(\int_c^d g(t)dH(t) - \frac{1}{\mu(\Omega^c)}\int_{\Omega^c}\int_I f(s,t)dH(t)\right)d\mu(s)\right]\right)\\
&\leq W_\Omega\varphi\left(\frac{1}{L}\left(\int_c^d g(t)dH(t) - \frac{1}{\mu(\Omega)}\int_\Omega\int_I f(s,t)dH(t)\right)d\mu(s)\right)\\
&+W_{\Omega^c}\varphi\left(\frac{1}{L}\left(\int_c^d g(t)dH(t) - \frac{1}{\mu(\Omega^c)}\int_{\Omega^c}\int_I f(s,t)dH(t)\right)d\mu(s)\right)\\
&= F_1(f,g,\varphi;\Omega)
\end{aligned}$$

for any Ω, which proves the first inequality in (7.3.12).

By inequality (7.3.5) we also have

$$\begin{aligned}
F_1(f,g,\varphi;\Omega) &= W_\Omega\varphi\left(\frac{1}{L}\left(\int_c^d g(t)dH(t) - \frac{1}{\mu(\Omega)}\int_I\int_\Omega f(s,t)d\mu(s)dH(t)\right)\right)\\
&+ W_{\Omega^c}\varphi\left(\frac{1}{L}\left(\int_c^d g(t)dH(t) - \frac{1}{\mu(\Omega^c)}\int_I\int_{\Omega^c} f(s,t)d\mu(s)dH(t)\right)\right)\\
&\leq W_\Omega\left[\frac{1}{L}\left(\int_c^d \varphi(g(t))dH(t) - \frac{1}{\mu(\Omega)}\int_I\int_\Omega \varphi(f(s,t))d\mu(s)dH(t)\right)\right]\\
&+ W_{\Omega^c}\left[\frac{1}{L}\left(\int_c^d \varphi(g(t))dH(t) - \frac{1}{\mu(\Omega^c)}\int_I\int_{\Omega^c} \varphi(f(s,t))d\mu(s)dH(t)\right)\right]\\
&= \frac{1}{L}\left(\int_c^d \varphi(g(t))dH(t) - \frac{1}{\mu(X)}\int_I\int_X \varphi(f(s,t))d\mu(s)dH(t)\right)
\end{aligned}$$

for any Ω, which proves the second inequality in (7.3.12). ■

Remark 7.3.3. Following are the direct consequences of the previous theorem. Under the assumptions of Theorem 7.3.3, following inequalities hold

$$\varphi\left(\frac{1}{L}\left(\int_c^d g(t)dH(t) - \frac{1}{\mu(X)}\int_I\int_X f(s,t)d\mu(s)dH(t)\right)\right) \leq$$
$$\leq \inf_{\{\Omega:0<\mu(\Omega)<\mu(X)\}} F_1(f,g,\varphi;\Omega)$$

and

$$\sup_{\{\Omega:0<\mu(\Omega)<\mu(X)\}} F_1(f,g,\varphi;\Omega) \leq$$
$$\leq \frac{1}{L}\left(\int_c^d \varphi(g(t))dH(t) - \frac{1}{\mu(X)}\int_I\int_X \varphi(f(s,t))d\mu(s)dH(t)\right).$$

□

Corollary 7.3.4. *Let the assumptions of Corollary* 7.3.2 *be valid. Then for any nonempty subset I of $\{1,\ldots,n\}$, the inequalities*

$$\varphi\left(\frac{1}{L}\left(\sum_{j=1}^m a_j - \frac{1}{W_n}\sum_{l=1}^k\sum_{j=c_l}^{d_l-1}\sum_{i=1}^n w_i x_{ij}\right)\right) \leq F_2(\mathbf{X},\mathbf{w},\varphi;I) \leq$$
$$\leq \frac{1}{L}\left(\sum_{j=1}^m \varphi(a_j) - \frac{1}{W_n}\sum_{l=1}^k\sum_{j=c_l}^{d_l-1}\sum_{i=1}^n w_i\varphi(x_{ij})\right),$$

hold, where F_2 is defined in (7.3.9).

Remark 7.3.4. If in Corollary 7.3.2 we simply put $k=1$, $c_1=1$ and $d_1=m$, then we get the following result which is in fact refinement of Niezgoda's inequality (7.3.1). □

Corollary 7.3.5. *Let the assumptions of Proposition* 7.3.1 *be valid. Then for any nonempty subset I of $\{1,\ldots,n\}$,*

$$\varphi\left(\sum_{j=1}^m a_j - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i x_{ij}\right) \leq F_3(\mathbf{X},\mathbf{w},\varphi;I) \leq \sum_{j=1}^m \varphi(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^n w_i\varphi(x_{ij}), \tag{7.3.13}$$

holds, where F_3 is defined in (7.3.10).

For next corollary we recall the following definition from [51, p. 20].

Definition 7.3.1. An $m \times m$ matrix $\mathbf{A} = (a_{jk})$ is said to be doubly stochastic, if $a_{jk} \geq 0$ and $\sum_{j=1}^{m} a_{jk} = \sum_{k=1}^{m} a_{jk} = 1$ for all $j, k \in \{1, \ldots, m\}$.

Remark 7.3.5. It is a well-known fact that if $\mathbf{A}$ is an $m \times m$ doubly stochastic matrix, then

$$\mathbf{aA} \prec \mathbf{a} \text{ for each real } m\text{−tuple } \mathbf{a} = (a_1, \ldots, a_m). \tag{7.3.14}$$

□

By applying Corollary 7.3.5 and (7.3.14), we obtain:

Corollary 7.3.6. *Let the following assumptions be valid:* $\mathbf{a} = (a_1, \ldots, a_m) \in [a, b]^m$ *and* $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$ *are* $m \times m$ *doubly stochastic matrices. Set*

$$\mathbf{X} = (x_{ij}) = \begin{pmatrix} \boldsymbol{a}\mathbf{A}_1 \\ \vdots \\ \boldsymbol{a}\mathbf{A}_n \end{pmatrix}.$$

Then for every continuous convex function $\varphi : [a, b] \to \mathbb{R}$ *inequality* (7.3.13) *holds.*

Remark 7.3.6. Related refinements for Jensen's inequality is given by Dragomir in [20]. □

Corollary 7.3.7. *Let* $x_i \in [a, b]$ *and* $w_i \in \mathbb{R}_+$ *for* $i \in \{1, \ldots, n\}$ *such that* $\sum_{i=1}^{n} w_i = 1$. *If* $\varphi : [a, b] \to \mathbb{R}$ *is a convex function, then for any nonempty subset* I *of* $\{1, \ldots, n\}$, *the inequalities*

$$\varphi\left(m_1 + m_2 - \sum_{i=1}^{n} w_i x_i\right) \leq F_4(\mathbf{x}, \mathbf{w}, \varphi; I) \leq \varphi(m_1) + \varphi(m_2) - \sum_{i=1}^{n} w_i \varphi(x_i) \tag{7.3.15}$$

hold, where F_4 *is defined in* (7.3.11).

Proof. If in (7.3.13) we set $m = 2, a_1 = m_1, a_2 = m_2$ and $x_{i1} = x_i$ for $i \in \{1, \ldots, n\}$ we get (7.3.15). ■

Remark 7.3.7. Corollary 7.3.7 in fact provides us refinement of Jensen-Mercer's inequality (7.1.2). Moreover, In [52] from the proof of Theorem 2.3 we have left inequality of (7.3.15). □

Remark 7.3.8. We observe that the inequality (7.3.15) can be written in an equivalent form as

$$\varphi\left(m_1 + m_2 - \sum_{i=1}^{n} w_i x_i\right) \leq \min_{I} F_4(\mathbf{x}, \mathbf{w}, \varphi; I)$$

and

$$\max_I F_4(\mathbf{x}, \mathbf{w}, \varphi; I) \leq \varphi(m_1) + \varphi(m_2) - \sum_{i=1}^{n} w_i \varphi(x_i).$$

For other related results one may see [38]. □

7.3.2 Applications

For results of the current subsection, we give some construction here and call the construction by **H** whose details are as under.

H: For $\emptyset \neq I \subset \{1, \ldots, n\}$, let A_I, G_I, H_I and $M_I^{[r]}$ be the arithmetic, geometric, harmonic means and power mean of order $r \in \mathbb{R}$ respectively of $x_i \in [a, b]$ $(0 < a < b)$, formed with the nonnegative weights w_i, $i \in I$. For $I = \{1, \ldots, n\}$ we denote the arithmetic, geometric, harmonic and power means by A_n, G_n, H_n, and $M_n^{[r]}$ respectively.

Here we introduce some notations as follows:

$$\begin{aligned}
\tilde{A}_I &= m_1 + m_2 - \frac{1}{W_I} \sum_{i \in I} w_i x_i = m_1 + m_2 - A_I, \\
\tilde{G}_I &= \frac{m_1 m_2}{\left(\prod_{i \in I} x_i^{w_i} \right)^{\frac{1}{W_I}}} = \frac{m_1 m_2}{G_I}, \\
\tilde{H}_I &= \left({m_1}^{-1} + {m_2}^{-1} - \frac{1}{W_I} \sum_{i \in I} w_i x_i^{-1} \right)^{-1} = \left({m_1}^{-1} + {m_2}^{-1} - H_I^{-1} \right)^{-1}, \\
\tilde{M}_I^{[r]} &= \begin{cases} \left({m_1}^r + {m_2}^r - \left(M_I^{[r]} \right)^r \right)^{\frac{1}{r}} & , \quad r \neq 0, \\ \tilde{G}_I & , \quad r = 0, \end{cases}
\end{aligned}$$

where

$$M_I^{[r]} = \begin{cases} \left(\frac{1}{W_I} \sum_{i \in I} w_i x_i^r \right)^{\frac{1}{r}} & , \quad r \neq 0, \\ \left(\prod_{i \in I} x_i^{w_i} \right)^{\frac{1}{W_I}} & , \quad r = 0. \end{cases}$$

Note that

$$\begin{aligned}
M_I^{[1]} = A_I, &\quad \tilde{M}_I^{[1]} = \tilde{A}_I \\
M_I^{[0]} = G_I, &\quad \tilde{M}_I^{[0]} = \tilde{G}_I \\
M_I^{[-1]} = H_I, &\quad \tilde{M}_I^{[-1]} = \tilde{H}_I.
\end{aligned}$$

Theorem 7.3.8. *Let the assumptions given in* **H** *be valid. Then the following inequalities hold.*

(*a*) $\tilde{G}_n \leq \min_I \tilde{A}_I^{W_I} \tilde{A}_{\bar{I}}^{W_{\bar{I}}}$ *and* $\tilde{A}_n \geq \max_I \tilde{A}_I^{W_I} \tilde{A}_{\bar{I}}^{W_{\bar{I}}}$.

(*b*) $\tilde{G}_n \leq \min_I \left[W_I \tilde{G}_I + W_{\bar{I}} \tilde{G}_{\bar{I}}\right]$ *and* $\tilde{A}_n \geq \max_I \left[W_I \tilde{G}_I + W_{\bar{I}} \tilde{G}_{\bar{I}}\right]$.

Proof.

(*a*) Applying Corollary 7.3.7 to the convex function $\varphi(x) = -\ln(x)$, we obtain

$$-\ln(\tilde{A}_n) \leq -W_I \ln(\tilde{A}_I) - W_{\bar{I}} \ln(\tilde{A}_{\bar{I}}) \leq -\ln(\tilde{G}_n). \tag{7.3.16}$$

Now required results follow from Remark 7.3.8 and (7.3.16).

(*b*) Applying Corollary 7.3.7 to the convex function $\varphi(x) = \exp(x)$ and replacing m_1, m_2 and x_i with $\ln(m_1)$, $\ln(m_2)$ and $\ln(x_i)$ respectively and using Remark 7.3.8, we obtain what we wanted. ■

The following particular case of Theorem 7.3.8 is of interest.

Corollary 7.3.9. *Under the assumptions of Theorem* 7.3.8*, the following inequalities hold.*

(*a*) $\frac{1}{\tilde{G}_n} \leq \min_I \frac{1}{\tilde{H}_I^{W_I} \tilde{H}_{\bar{I}}^{W_{\bar{I}}}}$ *and* $\frac{1}{\tilde{H}_n} \geq \max_I \frac{1}{\tilde{H}_I^{W_I} \tilde{H}_{\bar{I}}^{W_{\bar{I}}}}$.

(*b*) $\frac{1}{\tilde{G}_n} \leq \min_I \left[\frac{W_I}{\tilde{G}_I} + \frac{W_{\bar{I}}}{\tilde{G}_{\bar{I}}}\right]$ *and* $\frac{1}{\tilde{H}_n} \geq \max_I \left[\frac{W_I}{\tilde{G}_I} + \frac{W_{\bar{I}}}{\tilde{G}_{\bar{I}}}\right]$.

Proof. Proof follows directly from Theorem 7.3.8 by the substitutions $m_1 \to \frac{1}{m_1}$, $m_2 \to \frac{1}{m_2}$, and $x_i \to \frac{1}{x_i}$. ■

Theorem 7.3.10. *Let the assumptions given in* **H** *be valid. For* $r \leq 1$*, the following inequalities hold*

$$\tilde{M}_n^{[r]} \leq \min_I \left[W_I \tilde{M}_I^{r]} + W_{\bar{I}} \tilde{M}_{\bar{I}}^{[r]}\right] \quad \text{and} \quad \tilde{A}_n \geq \max_I \left[W_I \tilde{M}_I^{[r]} + W_{\bar{I}} \tilde{M}_{\bar{I}}^{[r]}\right]. \tag{7.3.17}$$

For $r \geq 1$*, the inequalities in* (7.3.17) *are reversed.*

Proof. For $r \leq 1$, $r \neq 0$, use Corollary 7.3.7 for the convex function $\varphi(x) = x^{\frac{1}{r}}$ and replacing m_1, m_2 and x_i with ${m_1}^r$, ${m_2}^r$ and x_i^r respectively and for $r = 0$ use Corollary 7.3.7 for the convex function $\varphi(x) = \exp(x)$, replacing m_1, m_2 and x_i with $\ln(m_1)$, $\ln(m_2)$ and $\ln(x_i)$ respectively, we obtain (7.3.17) by Remark 7.3.8.

If $r \geq 1$, then the function $\varphi(x) = x^{\frac{1}{r}}$ is concave, so the inequalities in (7.3.17) are reversed. ■

Remark 7.3.9. Clearly, part (b) of Theorem 7.3.8 is a direct consequences of Theorem 7.3.10. Moreover, for $r=-1$, we get the following special case of Theorem 7.3.10. □

Corollary 7.3.11. *Under the assumptions of Theorem* 7.3.10*, the following inequalities hold*

$$\tilde{H}_n \leq \min_I \left[W_I \tilde{H}_I + W_{\bar{I}} \tilde{H}_{\bar{I}}\right] \quad \textit{and} \quad \tilde{A}_n \geq \max_I \left[W_I \tilde{H}_I + W_{\bar{I}} \tilde{H}_{\bar{I}}\right].$$

Theorem 7.3.12. *Let the assumptions given in* **H** *be valid and let* $r, s \in \mathbb{R}$, $r \leq s$.

(a) *For* $s \geq 0$*, the following inequalities hold*

$$\left(\tilde{M}_n^{[r]}\right)^s \leq \min_I \left[W_I \left(\tilde{M}_I^{[r]}\right)^s + W_{\bar{I}} \left(\tilde{M}_{\bar{I}}^{[r]}\right)^s\right], \tag{7.3.18}$$

$$\left(\tilde{M}_n^{[r]}\right)^s \geq \max_I \left[W_I \left(\tilde{M}_I^{[r]}\right)^s + W_{\bar{I}} \left(\tilde{M}_{\bar{I}}^{[r]}\right)^s\right]. \tag{7.3.19}$$

(b) *For* $s < 0$*, the inequalities* (7.3.18) *and* (7.3.19) *are reversed.*

Proof.

(a) Let $s \geq 0$. Using Corollary 7.3.7 and Remark 7.3.8 to the convex function $\varphi(x) = x^{\frac{s}{r}}$ and replacing m_1, m_2 and x_i with ${m_1}^r$, ${m_2}^r$ and x_i^r respectively, we obtain (7.3.18) and (7.3.19).

(b) If $s < 0$, then the function $\varphi(x) = x^{\frac{s}{r}}$ is concave so inequalities in (7.3.18) and (7.3.19) are reversed. ■

Let us state a definition from [57, p. 215].

Definition 7.3.2. Let $\varphi : [a,b] \to \mathbb{R}$ be a strictly monotonic continuous function. Then for a given $n-$tuple $\mathbf{x} = (x_1, \ldots, x_n) \in [a,b]^n$ and nonnegative $n-$tuple $\mathbf{w} = (w_1, \ldots, w_n)$ with $\sum_{i=1}^n w_i = 1$, the value

$$M_\varphi^{[n]} = \varphi^{-1}\left(\sum_{i=1}^n w_i \varphi(x_i)\right)$$

is well-defined and is called *quasi-arithmetic mean* of $\mathbf{x}$ with weight $\mathbf{w}$.

If we define

$$\tilde{M}_\varphi^{[n]} = \varphi^{-1}\left(\varphi(m_1) + \varphi(m_2) - \sum_{i=1}^n w_i \varphi(x_i)\right),$$

then we have the following results.

Theorem 7.3.13. *Let $\varphi, \psi : [a,b] \to \mathbb{R}$ be two strictly monotonic continuous functions. If $\psi \circ \varphi^{-1}$ is convex on $[a,b]$, then the following inequalities hold*

$$\psi\left(\tilde{M}_{\varphi}^{[n]}\right) \leq \min_{I}\left[W_I \psi\left(\tilde{M}_{\varphi}^{[I]}\right) + W_{\bar{I}} \psi\left(\tilde{M}_{\varphi}^{[\bar{I}]}\right)\right], \tag{7.3.20}$$

$$\psi\left(\tilde{M}_{\varphi}^{[n]}\right) \geq \max_{I}\left[W_I \psi\left(\tilde{M}_{\varphi}^{[I]}\right) + W_{\bar{I}} \psi\left(\tilde{M}_{\varphi}^{[\bar{I}]}\right)\right], \tag{7.3.21}$$

$$\textit{where} \quad \tilde{M}_{\varphi}^{[J]} = \varphi^{-1}\Big(\varphi(m_1) + \varphi(m_2) - \frac{1}{W_J}\sum_{i\in J} w_i \varphi(x_i)\Big).$$

Proof. Applying Corollary 7.3.7 to the convex function $f = \psi \circ \varphi^{-1}$ and replacing m_1, m_2, and x_i with $\varphi(m_1)$, $\varphi(m_2)$ and $\varphi(x_i)$ respectively and then using Remark 7.3.8, we obtain (7.3.20) and (7.3.21). ■

Remark 7.3.10. Theorems 7.3.8, 7.3.10 and 7.3.12 follow from Theorem 7.3.13, by choosing adequate functions φ, ψ and appropriate substitutions. □

7.4 Generalization and Refinement of Jensen-Mercer's Inequality Using Isotonic Linear Functionals

At the start of this section, we recall some useful definitions from [80] and some assumptions which we will use in next two theorems.

Let $\mathfrak{A}$ be an algebra of subsets of $E \neq \emptyset$ and let L be a linear class of functions $f : E \to \mathbb{R}$ having the properties:

L1 : $f, g \in L \Rightarrow (af + bg) \in L$ for each $a, b \in \mathbb{R}$,

L2 : $\mathbf{1} \in L$, i.e., if $f(t) = 1$ for each $t \in E$, then $f \in L$,

L3 : $f \in L$, $E_1 \in \mathfrak{A} \Rightarrow f.\chi_{E_1} \in L$,

where χ_{E_1} is the characteristic function of E_1. It follows from L_2 and L_3 that $\chi_{E_1} \in L$ for every $E_1 \in \mathfrak{A}$.

Definition 7.4.1. An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying the following properties:

A1 : $A(af + bg) = aA(f) + bA(g)$ for $f, g \in L, a, b \in \mathbb{R}$, A2 : $f \in L,\ f(t) \geq 0$ on $E \Rightarrow A(f) \geq 0$.

Remark 7.4.1. It follows from L_3 that for every $E_1 \in \mathfrak{A}$ such that $A(\chi_{E_1}) > 0$, the functional A_1 defined for each $f \in L$ as $A_1(f) = \frac{A(f.\chi_{E_1})}{A(\chi_{E_1})}$ is an isotonic linear functional with $A(\mathbf{1}) = 1$. Moreover, we observe that

$$A(\chi_{E_1}) + A(\chi_{E\setminus E_1}) = 1,$$

$$A(f) = A(f.\chi_{E_1}) + A(f.\chi_{E\setminus E_1}).$$

□

In [19] Cheung et al. gave the following variant of the Jessen's inequality.

Proposition 7.4.1. *Let L satisfy properties $L1$ and $L2$ on a nonempty set E and let $\varphi : [a, b] \to \mathbb{R}$ be a continuous convex function. If A is an isotonic linear functional on L with $A(\mathbf{1}) = 1$, then for all $f \in L$ such that $\varphi(f),\ \varphi(m_1 + m_2 - A(f)) \in L$ (so that $a \leq f(t) \leq b$ for all $t \in E$), we have*

$$\varphi(m_1 + m_2 - A(f)) \leq \varphi(m_1) + \varphi(m_2) - A(\varphi(f)). \tag{7.4.1}$$

If φ is concave then the inequality (7.4.1) *is reversed.*

The following refinement of (7.4.1) holds.

Theorem 7.4.1. *Let the assumptions of Proposition* 7.4.1 *be valid. If φ is convex, then the inequality*

$$\varphi(m_1 + m_2 - A(f)) \leq F_5(A, f, \varphi; E_1) \leq \varphi(m_1) + \varphi(m_2) - A(\varphi(f)), \tag{7.4.2}$$

holds for each $E_1 \in \mathfrak{A}$ such that $0 < A(\chi_{E_1}) < 1$, where

$$\begin{aligned} F_5(A, f, \varphi; E_1) &= A(\chi_{E_1})\varphi\left(m_1 + m_2 - \frac{A(f.\chi_{E_1})}{A(\chi_{E_1})}\right) \\ &+ A(\chi_{E\setminus E_1})\varphi\left(m_1 + m_2 - \frac{A(f.\chi_{E\setminus E_1})}{A(\chi_{E\setminus E_1})}\right) \end{aligned} \tag{7.4.3}$$

Proof. The first inequality follows by using definition of convex function and the second follows by using (7.4.1) for $A_1(f)$ instead of $A(f)$. ■

Remark 7.4.2. In [52] from the proof of Theorem 4.1 we have left inequality of (7.4.3).
□

Remark 7.4.3. We observe that the inequality (7.4.2) may be written in an equivalent form as

$$\begin{aligned}\varphi(m_1+m_2-A(f)) &\leq \min_{E_1\in\mathfrak{A}} F_5(A,f,\varphi;E_1),\\ \varphi(m_1)+\varphi(m_2)-A(\varphi(f)) &\geq \max_{E_1\in\mathfrak{A}} F_5(A,f,\varphi;E_1).\end{aligned}$$

□

The following particular case of Theorem 7.4.1 is of interest:

Corollary 7.4.2. *Let (Ω, P, μ) be a probability measure space and let $f:\Omega\to[a,b]$ be a measurable function. Then for every continuous convex function $\varphi:[a,b]\to\mathbb{R}$ and for any set E_1 in P with $\mu(E_1),\mu(\Omega\backslash E_1)>0$ the following inequalities hold*

$$\begin{aligned}\varphi\left(m_1+m_2-\int_\Omega f d\mu\right) &\leq \min_{E_1\in P}\left[\mu(E_1)\varphi\left(m_1+m_2-\frac{1}{\mu(E_1)}\int_{E_1} f d\mu\right)\right.\\ &+ \left.\mu(\Omega\setminus E_1)\varphi\left(m_1+m_2-\frac{1}{\mu(\Omega\setminus E_1)}\int_{\Omega\setminus E_1} f d\mu\right)\right],\\ \varphi(m_1)+\varphi(m_2)-\int_\Omega \varphi(f)d\mu &\geq \max_{E_1\in P}\left[\mu(E_1)\varphi\left(m_1+m_2-\frac{1}{\mu(E_1)}\int_{E_1} f d\mu\right)\right.\\ &+ \left.\mu(\Omega\setminus E_1)\varphi\left(m_1+m_2-\frac{1}{\mu(\Omega\setminus E_1)}\int_{\Omega\setminus E_1} f d\mu\right)\right].\end{aligned}$$

Proof. It is a special case of Theorem 7.4.1 for the functional A defined on the class $L_1(\mu)$ as $A(f)=\int_\Omega f d\mu$. ■

Remark 7.4.4. We may also obtain similar results as in Theorem 7.3.13 for the generalized quasi-arithmetic means of Mercers type defined in [19] as

$$\tilde{M}_\varphi(f,A)=\varphi^{-1}(\varphi(m_1)+\varphi(m_2)-A(\varphi(f))).$$

□

For our next two sections, we give here some constructions as follows. Under the assumptions of Theorem 7.3.3 using (7.3.12) we define the following functionals:

$$\begin{aligned}\Lambda_{15}(f,g;\varphi) &= \varphi\Big(\frac{1}{L}\Big(\int_c^d g(t)dH(t)-\frac{1}{\mu(X)}\int_I\int_X f(s,t)d\mu(s)dH(t)\Big)\Big)\\ &\quad - F_1(f,g,\varphi;\Omega)\geq 0, \qquad \text{(A15)}\end{aligned}$$

$$\Lambda_{16}(f,g;\varphi) = F_1(f,g,\varphi;\Omega) - \frac{1}{L}\Big(\int_c^d \varphi(g(t))dH(t) - \frac{1}{\mu(X)}\int_I\int_X \varphi(f(s,t))\times$$
$$\times\, d\mu(s)dH(t)\Big) \geq 0, \tag{A16}$$
$$\Lambda_{17}(f,g;\varphi) = \varphi\Big(\frac{1}{L}\Big(\int_c^d g(t)dH(t) - \frac{1}{\mu(X)}\int_I\int_X f(s,t)d\mu(s)dH(t)\Big)\Big)$$
$$- \frac{1}{L}\Big(\int_c^d \varphi(g(t))dH(t) - \frac{1}{\mu(X)}\int_I\int_X \varphi(f(s,t))d\mu(s)dH(t)\Big) \geq 0. \tag{A17}$$

Also, under the assumptions of Theorem 7.4.1 using (7.4.2) we define the following functionals:

$$\Lambda_{18}(A,f;\varphi) = F_5(A,f,\varphi;E_1) - \varphi(m_1+m_2-A(f)) \geq 0, \tag{A18}$$
$$\Lambda_{19}(A,f;\varphi) = \varphi(m_1)+\varphi(m_2) - A(\varphi(f)) - F_5(A,f,\varphi;E_1) \geq 0, \tag{A19}$$
$$\Lambda_{20}(A,f;\varphi) = \varphi(m_1)+\varphi(m_2) - A(\varphi(f)) - \varphi(m_1+m_2-A(f)) \geq 0. \tag{A20}$$

Remark 7.4.5. For the sake of brevity, in next two sections at some places we will use the notations $\Lambda_k(.,.;\varphi) = \Lambda_k(\varphi)$ for $k \in \{11,\ldots,20\}$. □

7.5 Mean Value Theorems

Now we give mean value theorems for Λ_k, $k \in \{15,\ldots,20\}$. Here $\varphi_0(x) = \frac{x^2}{2}$. For the idea of the proof, see proof of Theorems 3.4.1 and 3.4.2.

Theorem 7.5.1. *Let Λ_k be linear functionals for $k \in \{15,\ldots,20\}$ as defined in (A15), ..., (A20) and let $\varphi \in C^{(2)}[a,b]$. Then there exists $\xi_k \in [a,b]$ such that*

$$\Lambda_k(\varphi) = \varphi''(\xi_k)\Lambda_k(\varphi_0), \quad k \in \{15,\ldots,20\}.$$

Theorem 7.5.2. *Let Λ_k be linear functionals for $k \in \{15,\ldots,20\}$ as defined in (A15), ..., (A20) and let $\varphi,\psi \in C^{(2)}[a,b]$. Then there exists $\xi_k \in [a,b]$ such that*

$$\frac{\Lambda_k(\varphi)}{\Lambda_k(\psi)} = \frac{\varphi''(\xi_k)}{\psi''(\xi_k)}, \qquad k \in \{15,\ldots,20\},$$

provided that the denominator of the left-hand side is nonzero.

Remark 7.5.1. If the inverse of $\frac{\varphi''}{\psi''}$ exists, then from the above mean value theorems we can give generalized means

$$\xi_k = \left(\frac{\varphi''}{\psi''}\right)^{-1}\left(\frac{\Lambda_k(\varphi)}{\Lambda_k(\psi)}\right), \qquad k \in \{15, \ldots, 20\}. \tag{7.5.1}$$

□

Remark 7.5.2. Similar mean value theorems for functionals Λ_k for $k \in \{11, \ldots, 14\}$ as defined in $(A11), \ldots, (A14)$ can be found in [77]. □

7.6 $n-$Exponential Convexity for Jensen-type Inequalities

Concluding as before, we get our results concerning the $n-$exponential convexity and exponential convexity for our functionals Λ_k, $k \in \{11, \ldots, 20\}$ as defined in $(A11), \ldots, (A20)$. Here we have $[a, b] \subset Dom(f_t)$ where $Dom(f_t)$ being the domain of function f_t. Throughout the section I is an interval in $\mathbb{R}$.

Theorem 7.6.1. *Let $D_7 = \{f_t \in C[a, b] : t \in I\}$ be a family of functions such that the function $t \mapsto [z_0, z_1, z_2; f_t]$ is $n-$exponentially convex in the $J-$sense on I for any three mutually distinct points $z_0, z_1, z_2 \in [a, b]$. Let $\Lambda_k(f_t)$, $k \in \{11, \ldots, 20\}$ be linear functionals. Then $t \mapsto \Lambda_k(f_t)$ is an $n-$exponentially convex function in the $J-$sense on I. If the function $t \mapsto \Lambda_k(f_t)$ is continuous on I, then it is $n-$exponentially convex on I.*

As a consequence of the above theorem we can state the following two corollaries.

Corollary 7.6.2. *Let $D_8 = \{f_t \in C[a, b] : t \in I\}$ be a family of functions such that the function $t \mapsto [z_0, z_1, z_2; f_t]$ is exponentially convex in the $J-$sense on I for any three mutually distinct points $z_0, z_1, z_2 \in [a, b]$. Let $\Lambda_k(f_t)$, $k \in \{11, \ldots, 20\}$ be linear functionals. Then*

(a) *The function $t \mapsto \Lambda_k(f_t)$ is an exponentially convex function in the $J-$sense on I. If the function $t \mapsto \Lambda_k(f_t)$ is continuous on I, then it is exponentially convex on I.*

(b) *The matrix $\left[\Lambda_k\left(f_{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^{m}$ is a positive-semidefinite. Particularly,*

$$\det\left[\Lambda_k\left(f_{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^{m} \geq 0$$

for each $m \in \mathbb{N}$ *and* $t_i \in I$ *where* $i \in \{1, \ldots, m\}$.

Corollary 7.6.3. *Let* $D_9 = \{f_t \in C[a,b] : t \in I\}$ *be a family of functions such that the function* $t \mapsto [z_0, z_1, z_2; f_t]$ *is* $2-$*exponentially convex in the* $J-$*sense on* I *for any three mutually distinct points* $z_0, z_1, z_2 \in [a,b]$. *Let* $\Lambda_k(f_t)$, $k \in \{11, \ldots, 20\}$ *be linear functionals. Then the following statements hold:*

(a) *If the function* $t \mapsto \Lambda_k(f_t)$ *is continuous on I, then it is* $2-$*exponentially convex function on* I. *If the function* $t \mapsto \Lambda_k(f_t)$ *is additionally positive, then it is also log-convex on* I. *Moreover, the following inequality holds for* $r < s < t$; $r, s, t \in I$

$$(\Lambda_k(f_s))^{t-r} \leq (\Lambda_k(f_r))^{t-s} (\Lambda_k(f_t))^{s-r}.$$

(b) *If the function* $t \mapsto \Lambda_k(f_t)$ *is positive and differentiable on* I, *then for every* $s, t, u, v \in I$ *such that* $s \leq u$ *and* $t \leq v$, *we have*

$$\mu_{s,t}(\Lambda_k, D) \leq \mu_{u,v}(\Lambda_k, D) \tag{7.6.1}$$

where $\mu_{s,t}$ *is defined in* (3.5.3).

Remark 7.6.1. The proofs of the Theorem 7.6.1 and Corollaries 7.6.2 and 7.6.3 are similar to the proofs of Theorem 6.4.1 and Corollaries 6.4.2 and 6.4.3 respectively, so we omit the details. We also note that the results from Theorem 7.6.1, Corollaries 7.6.2 and 7.6.3 still hold when any two (three) points $z_0, z_1, z_2 \in [a,b]$ coincide for a family of differentiable (twice differentiable) functions f_t such that the function $t \mapsto [z_0, z_1, z_2; f_t]$ is $n-$exponentially convex, exponentially convex and $2-$expoenetially convex in the $J-$sense, respectively. □

Remark 7.6.2. Results for the Jensen-Steffensen's inequality regarding exponential convexity, which are special case of some of the results given here, were given in [3]. □

7.7 Examples with Applications

In present section, we give the same examples as given in the previous chapter and give some means and mean-type results.

Under the assumptions of Theorem 7.4.1, for the present section, we consider the following conditions to be valid for all $t,\ t_o \in I$:

$$\begin{aligned}\lim_{t\to t_0} A(f_t) &= A\left(\lim_{t\to t_0} f_t\right) = A\left(f_{t_0}\right)\\ \lim_{t\to t_0} \frac{A(f_t)-A(f_{t_0})}{t-t_0} &= A\left(\lim_{t\to t_0} \frac{f_t-f_{t_0}}{t-t_0}\right) = A\left(f'_{t_0}\right)\end{aligned}$$

where I is an interval in $\mathbb{R}$.

To avoid repetition, from Examples 6.5.1 − 6.5.5 we note that all the mappings $t \mapsto \Lambda_k(f_t)$ for $k \in \{11, \ldots, 20\}$ are exponentially convex for $f_t \in \Omega_j$ for $j \in \{1, \ldots, 4\}$ in the following examples.

Example 7.7.1. *Let $\Omega_1 = \{\psi_t : \mathbb{R} \to \mathbb{R}_* : t \in \mathbb{R}\}$ be the family of functions defined by*

$$\psi_t(x) = \begin{cases} \frac{1}{t^2}e^{tx} & , \quad t \neq 0, \\ \frac{1}{2}x^2 & , \quad t = 0. \end{cases}$$

By introducing this family of convex functions in (7.5.1) for $k \in \{11, \ldots, 20\}$, we obtain the following means:

$$\mathfrak{M}_{s,t}(\Lambda_k, \Omega_1) = \begin{cases} \frac{1}{s-t}\ln\left(\frac{\Lambda_k(\psi_s)}{\Lambda_k(\psi_t)}\right) & , \quad s \neq t, \\ \frac{\Lambda_k(id.\psi_s)}{\Lambda_k(\psi_s)} - \frac{2}{s} & , \quad s = t \neq 0, \\ \frac{\Lambda_k(id.\psi_0)}{3\Lambda_k(\psi_0)} & , \quad s = t = 0. \end{cases}$$

where id stands for identity function on $\mathbb{R}$.
Since, $\mathfrak{M}_{s,t}(\Lambda_k, \Omega_1) = \ln(\mu_{s,t}(\Lambda_k, \Omega_1))$ for $k \in \{11, \ldots, 20\}$, so by (7.6.1) these means are monotonic, where $\mu_{s,t}$ is defined in (3.5.3). □

Example 7.7.2. *Let $\Omega_2 = \{\varphi_t : \mathbb{R}_+ \to \mathbb{R} : t \in \mathbb{R}\}$ be the family of functions defined by*

$$\varphi_t(x) = \begin{cases} \frac{x^t}{t(t-1)} & , \quad t \notin \{0, 1\}, \\ -\ln(x) & , \quad t = 0, \\ x\ln(x) & , \quad t = 1. \end{cases} \tag{7.7.1}$$

We assume that $[a, b] \subset \mathbb{R}_+$ and $\Lambda_k(\varphi_t) > 0$ for $k \in \{11, \ldots, 20\}$. By introducing this family of convex functions in (7.5.1) for $k \in \{11, \ldots, 20\}$ we have the following *means:*

$$\mu_{s,t}(\Lambda_k,\Omega_2)=\begin{cases}\left(\frac{\Lambda_k(\varphi_s)}{\Lambda_k(\varphi_t)}\right)^{\frac{1}{s-t}} & , \quad s\neq t,\\ \exp\left(\frac{1-2s}{s(s-1)}-\frac{\Lambda_k(\varphi_0\varphi_s)}{\Lambda_k(\varphi_s)}\right) & , \quad s=t\neq 0,1,\\ \exp\left(1-\frac{\Lambda_k({\varphi_0}^2)}{2\Lambda_k(\varphi_0)}\right) & , \quad s=t=0,\\ \exp\left(-1-\frac{\Lambda_k(\varphi_0\varphi_1)}{2\Lambda_k(\varphi_1)}\right) & , \quad s=t=1.\end{cases}$$

Monotonicity of these means follows directly from (7.6.1). □

Remark 7.7.1. If Λ_k for $k\in\{11,\ldots,20\}$ are positive, then using Theorem 7.5.2 with $\varphi=\varphi_s\in\Omega_2$ and $\psi=\varphi_t\in\Omega_2$ yield that there exist some $\xi_k\in[a,b]\,,\quad k\in\{11,\ldots,20\}$, such that

$$\xi_k^{s-t}=\frac{\Lambda_k(\varphi_s)}{\Lambda_k(\varphi_t)},\quad k\in\{11,\ldots,20\}.$$

Since the function $\xi_k\mapsto\xi_k^{s-t}$ is invertible for $s\neq t$, we then have

$$L_k\leq\left(\frac{\Lambda_k(\varphi_s)}{\Lambda_k(\varphi_t)}\right)^{\frac{1}{s-t}}\leq U_k,\quad k\in\{11,\ldots,20\}, \tag{7.7.2}$$

where $L_k,U_k\in[a,b]$ for $k\in\{11,\ldots,20\}$, which shows that in this case $\mu_{s,t}(\Lambda_k,\Omega_2)$ for $k\in\{11,\ldots,20\}$ are means.

Now, we impose one additional parameter r. For $r\neq 0$ by substituting $s\to\frac{s}{r}$, $t\to\frac{t}{r}$ and $\mathbf{x}\to\mathbf{x}^r$ for $k\in\{11,12\}$, $g\to g^r$ for $k\in\{13,14\}$, $f\to f^r$ and $g\to g^r$ for $k\in\{15,16,17\}$ and $f\to f^r$ for $k\in\{18,19,20\}$ in (7.7.2), we get

$$\begin{aligned}L_k &\leq \left(\frac{\Lambda_k(\mathbf{x}^r,\mathbf{w};\varphi_s)}{\Lambda_k(\mathbf{x}^r,\mathbf{w};\varphi_t)}\right)^{\frac{r}{s-t}}\leq U_k \quad \text{for } k\in\{11,12\},\\ L_k &\leq \left(\frac{\Lambda_k(g^r,\lambda;\varphi_s)}{\Lambda_k(g^r,\lambda;\varphi_t)}\right)^{\frac{r}{s-t}}\leq U_k \quad \text{for } k\in\{13,14\},\\ L_k &\leq \left(\frac{\Lambda_k(f^r,g^r;\varphi_s)}{\Lambda_k(f^r,g^r;\varphi_t)}\right)^{\frac{r}{s-t}}\leq U_k \quad \text{for } k\in\{15,16,17\},\\ L_k &\leq \left(\frac{\Lambda_k(A,f^r;\varphi_s)}{\Lambda_k(A,f^r;\varphi_t)}\right)^{\frac{r}{s-t}}\leq U_k \quad \text{for } k\in\{18,19,20\},\end{aligned}$$

where $\mathbf{x}^r=(x_1^r,\ldots,x_n^r)$.

Here we define new generalized means as follows.

- for $k \in \{11, 12\}$:

$$\mu_{s,t;r}(\Lambda_k(\mathbf{x}^r, \mathbf{w}; \varphi_t), \Omega_2) = \begin{cases} \left(\mu_{\frac{s}{r},\frac{t}{r}}(\Lambda_k(\mathbf{x}^r, \mathbf{w}; \varphi_t), \Omega_2)\right)^{\frac{1}{r}} & , \quad r \neq 0, \\ \mu_{s,t}(\Lambda_k(\ln(\mathbf{x}), \mathbf{w}; \varphi_t), \Omega_1) & , \quad r = 0, \end{cases}$$

- for $k \in \{13, 14\}$:

$$\mu_{s,t;r}(\Lambda_k(g^r, \lambda; \varphi_t), \Omega_2) = \begin{cases} \left(\mu_{\frac{s}{r},\frac{t}{r}}(\Lambda_k(g^r, \lambda; \varphi_t), \Omega_2)\right)^{\frac{1}{r}} & , \quad r \neq 0, \\ \mu_{s,t}(\Lambda_k(\ln(g), \lambda; \varphi_t), \Omega_1) & , \quad r = 0, \end{cases}$$

- for $k \in \{15, 16, 17\}$:

$$\mu_{s,t;r}(\Lambda_k(f^r, g^r; \varphi_t), \Omega_2) = \begin{cases} \left(\mu_{\frac{s}{r},\frac{t}{r}}(\Lambda_k(f^r, g^r; \varphi_t), \Omega_2)\right)^{\frac{1}{r}} & , \quad r \neq 0, \\ \mu_{s,t}(\Lambda_k(\ln(f), \ln(g); \varphi_t), \Omega_1) & , \quad r = 0, \end{cases}$$

- for $k \in \{18, 19, 20\}$:

$$\mu_{s,t;r}(\Lambda_k(A, f^r; \varphi_t), \Omega_2) = \begin{cases} \left(\mu_{\frac{s}{r},\frac{t}{r}}(\Lambda_k(A, f^r; \varphi_t), \Omega_2)\right)^{\frac{1}{r}} & , \quad r \neq 0, \\ \mu_{s,t}(\Lambda_k(A, \ln(f); \varphi_t), \Omega_1) & , \quad r = 0, \end{cases}$$

where $\ln(\mathbf{x}) = (\ln(x_1), \ldots, \ln(x_n))$. These new generalized means are monotonic. If $s,\ t,\ u,\ v \in \mathbb{R},\ r \neq 0$ such that $s \leq u,\ t \leq v$, then we have

$$\mu_{s,t;r}(\Lambda_k, \Omega_2) \leq \mu_{u,v;r}(\Lambda_k, \Omega_2), \quad k \in \{11, \ldots, 20\}.$$

The above inequalities are easily followed by using the fact that $\mu_{s,t}(\Lambda_k, \Omega_2)$ for $k \in \{11, \ldots, 20\}$ are monotonic in both parameters and using the inequalities given below for $k \in \{11, \ldots, 20\}$

$$\mu_{\frac{s}{r},\frac{t}{r}}(\Lambda_k, \Omega_2) = \left(\frac{\Lambda_k(\varphi_{\frac{s}{r}})}{\Lambda_k(\varphi_{\frac{t}{r}})}\right)^{\frac{r}{s-t}} \leq \left(\frac{\Lambda_k(\varphi_{\frac{u}{r}})}{\Lambda_k(\varphi_{\frac{v}{r}})}\right)^{\frac{r}{u-v}} = \mu_{\frac{u}{r},\frac{v}{r}}(\Lambda_k, \Omega_2),$$

where $r, s, t, u, v \in \mathbb{R},\ r \neq 0$ such that $\frac{s}{r} \leq \frac{u}{r}, \frac{t}{r} \leq \frac{v}{r}$. For $r = 0$, we obtain the required result by taking the limit $r \to 0$. □

Example 7.7.3. *Let* $\Omega_3 = \{\theta_t : \mathbb{R}_+ \to \mathbb{R}_+ : t \in \mathbb{R}_+\}$ *be the family of functions defined by*

$$\theta_t(x) = \frac{e^{-x\sqrt{t}}}{t}.$$

We assume that $[a,b] \subset \mathbb{R}_+$ *and* $\Lambda_k(\theta_t) > 0,\ k \in \{11,\dots,20\}$*. For this family of convex functions* $\mu_{s,t}(\Lambda_k,\Omega_3)$*,* $k \in \{11,\dots,20\}$ *from* (7.6.1) *become*

$$\mu_{s,t}(\Lambda_k,\Omega_3) \;=\; \begin{cases} \left(\frac{\Lambda_k(\theta_s)}{\Lambda_k(\theta_t)}\right)^{\frac{1}{s-t}} & , \quad s \neq t, \\ \exp\left(-\frac{\Lambda_k(id.\theta_s)}{2\sqrt{s}(\Lambda_k(\theta_s))} - \frac{1}{s}\right) & , \quad s = t. \end{cases}$$

Monotonicity of $\mu_{s,t}(\Lambda_k,\Omega_3)$ *for* $k \in \{11,\dots,20\}$ *follow from* (7.6.1)*. By* (7.5.1) $\mathfrak{M}_{s,t}(\Lambda_k,\Omega_3) = -(\sqrt{s}+\sqrt{t})\ln(\mu_{s,t}(\Lambda_k,\Omega_3)),\quad k \in \{11,\dots,20\}$ *defines a class of means.* □

Example 7.7.4. *Let* $\Omega_4 = \{\phi_t : \mathbb{R}_+ \to \mathbb{R}_+ : t \in \mathbb{R}_+\}$ *be the family of functions defined by*

$$\phi_t(x) = \begin{cases} \frac{t^{-x}}{(\ln(t))^2} & , \quad t \neq 1, \\ \frac{x^2}{2} & , \quad t = 1. \end{cases}$$

We assume that $[a,b] \subset \mathbb{R}_+$ *and* $\Lambda_k(\phi_t) > 0,\quad k \in \{11,\dots,20\}$*. For this family of convex functions* $\mu_{s,t}(\Lambda_k,\Omega_4)$*,* $k \in \{11,\dots,20\}$ *from* (7.6.1) *become*

$$\mu_{s,t}(\Lambda_k,\Omega_4) = \begin{cases} \left(\frac{\Lambda_k(\phi_s)}{\Lambda_k(\phi_t)}\right)^{\frac{1}{s-t}} & , \quad s \neq t, \\ \exp\left(-\frac{\Lambda_k(id.\phi_s)}{s\Lambda_k(\phi_s)} - \frac{2}{s\ln(s)}\right) & , \quad s = t \neq 1, \\ \exp\left(\frac{1}{3}\frac{\Lambda_k(id.\phi_1)}{\Lambda_k(\phi_1)}\right) & , \quad s = t = 1. \end{cases}$$

Monotonicity of $\mu_{s,t}(\Lambda_k,\Omega_4)$ *for* $k \in \{11,\dots,20\}$ *follow from* (7.6.1)*. By* (7.5.1)

$$\mathfrak{M}_{s,t}(\Lambda_k,\Omega_4) = -L(s,t)\ln(\mu^j_{s,t}(\Omega_4)),\quad k \in \{11,\dots,20\}$$

defines a class of means, where $L(s,t)$ *is Logarithmic mean defined in* (6.5.2)*.* □

As a conclusion, Paul Erdös argued that

> "Every human activity, good or bad, except mathematics, must come to an end."

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List of Publications

[1] **(ISI) A. R. Khan, N. Latif and J. E. Pečarić, Exponential convexity for majorization, *J. Inequal. Appl.*, 2012 (2012): 105, 1–13.**

[2] **(ISI) A. R. Khan, J. E. Pečarić and S. Varošanec, On some inequalities for functions with nondecreasing increments of higher order, *J. Inequal. Appl.*, 2013 (2013): 8, 1–14.**

[3] M. A. Khan, A. R. Khan and J. E. Pečarić, On the refinements of Jensen-Mercer's inequality, *Rev. Anal. Numér. Théor. Approx.* **41** (1) (2012), 62–81.

[4] **(ISI) A. R. Khan, J. E. Pečarić and S. Varošanec, Popoviciu type characterization of positivity of sums and integrals for convex functions of higher order, *J. Math. Inequal.*, 7 (2) (2013), 195–212.**

[5] **(ISI) A. R. Khan, J. E. Pečarić and M. R. Lipanović, n−Exponential convexity for Jensen-type inequalities, *J. Math. Inequal.*, 7 (3) (2013), 313–335.**

[6] A. R. Khan, J. E. Pečarić and M. Praljak, Weighted Montgomery inequalities for higher order differentiable functions of two variables, *Rev. Anal. Numer. Theor. Approx.*, **42** (1) (2013), 49–71.

[7] A. R. Khan, N. Latif and J. E. Pečarić, $n-$Exponential convexity for majorization, Favard's and Berwald's inequalities, *Advances in Inequal.*, to appear.

Note: Other three papers are submitted in reputed journals.