HERMITIAN GEOMETRY OF TWISTOR SPACES

Name : **Danish Ali**

Year of Admission : **2009**

Registration No. : **111-GCU-PHD-SMS-09**

Abdus Salam School of Mathematical Sciences

GC University Lahore, Pakistan

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Name	**:**	**Danish Ali**
Year of Admission	**:**	**2009**
Registration No.	**:**	**111-GCU-PHD-SMS-09**

Abdus Salam School of Mathematical Sciences

GC University Lahore, Pakistan

DECLARATION

I, **Mr. Danish Ali** Registration No. **111-GCU-PHD-SMS-09** student at **Abdus Salam School of Mathematical Sciences GC University** in the subject of **Mathematics** year of admission **2009**, hereby declare that the matter printed in this thesis titled

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(ii) No direct major work had already been done by me or anybody else on this topic; I worked on, for the Ph. D. degree.

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RESEARCH COMPLETION CERTIFICATE

Certified that the research work contained in this thesis titled

"HERMITIAN GEOMETRY OF TWISTOR SPACES"

has been carried out and completed by **Mr. Danish Ali** Registration No. **111-GCU-PHD-SMS-09** under my supervision.

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Date **Oleg Mushkarov**

Supervisor

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Date **Johann Davidov**

Co-Supervisor

Submitted Through

Prof. Dr. A. D. Raza Choudary
Director General
Abdus Salam School of Mathematical Sciences
GC University Lahore
Pakistan.

Controller of Examination
GC University Lahore
Pakistan.

Dedicated

to

Ami and Abu

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Abstract

In the present thesis we investigate the almost Hermitian geometry of the twistor spaces of oriented Riemannian 4-manifolds.

Holomorphic and orthogonal bisectional curvatures have been intensively explored on Kähler manifolds and a lot of important results have been obtained in this case. But in the non-Kähler case these curvatures are not very well studied and it seems that the main reason for that is the lack of interesting examples. The first part of the thesis is devoted to the study of the curvature properties of Atiyah-Hitchin-Singer and Eells-Salamon almost Hermitian structures. This is used to provide some interesting examples of almost Hermitian 6-manifolds of constant or strictly positive holomorphic, Hermitian and orthogonal bisectional curvatures.

In the second part of the thesis we determine the Gray-Hervella classes of the so-called compatible almost Hermitian structures on the twistor spaces, recently introduced by G. Deschamps . The interest in determining these classes is motivated by the fact that the Gray-Hervella classification is a very useful tool in studying almost complex manifolds. Our results in this direction generalize the well known integrability theorems by Atiyah-Hitchin-Singer, Eells-Salamon and Deschamps and show that there is a close relation between the properties of the spectrum of the anti-self-dual Weyl tensor of an almost Kähler 4-manifold and the almost Hermitian geometry of its twistor space.

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Introduction

The main purpose of the present thesis is to study certain aspects of the geometry of twistor spaces of oriented Riemannian 4-manifolds as almost Hermitian manifolds.

An almost Hermitian manifold is an almost complex manifold endowed with a Riemannian metric such that the almost complex structure is an isometry; in this case the metric and the almost complex structure are said to be compatible. The investigation of almost Hermitian manifolds is one of the central topics of the contemporary differential geometry.[6, 11, 27, 29, 37, 52]

From mathematical point of view, the basic idea of the twistor theory created by the English physicist R. Penrose [46, 47, 48] is that the geometry of a conformal manifold M can be encoded in holomorphic terms of the so-called twistor space associated to M. Penrose's idea has been developed in the context of Riemannian geometry by Atiyah, Hitchin and Singer [8] in the case of manifolds of dimension four. In particular, they have defined the (negative) twistor space of an oriented Riemannian 4-manifold (M, g) as the two-sphere bundle $\pi : \mathcal{Z} \to M$, whose fibre at a point p of M consists of all complex structures on the tangent space T_pM compatible with the metric and the opposite orientation of M.The positive twistor space is defined in a similar way-it parameterizes the complex structures on the tangent spaces of M compatible with the metric and the given orientation of M. Changing the orientation of M interchanges the role of the negative and positive twistor spaces.

The smooth manifold $\mathcal{Z}$ admits two natural almost-complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$, introduced by Atiyah-Hitchin-Singer [8] and Eells-Salamon [23] respectively. It is a result of Atiyah-Hitchin-Singer [ibid.] that $\mathcal{J}_1$ is integrable, i.e. it comes from a complex structure, if and only if the base manifold is self-dual (a restriction on its curvature). Eells and Salamon [ibid.] have shown that, although $\mathcal{J}_2$ is never integrable, it can be used to construct harmonic maps. The twistor space $\mathcal{Z}$ also admits a one-parameter family of natural Riemannian metrics h_t compatible with the almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$. The almost Hermitian structures $(h_t, \mathcal{J}_1)$ and $(h_t, \mathcal{J}_2,)$ on the twistor space $\mathcal{Z}$ have been studied from differential-geometric point of view by several authors.[1, 8, 17, 18, 19, 20, 21, 22, 24, 25, 32, 34, 43, 44, 45, 54]

In Chapter 1 we collect some basic facts about oriented Riemannian 4-manifolds and their twistor spaces, such as Singer-Thorpe decomposition[49] of curvature tensors, definition of the twistor space of a 4-manifold, the Atiyah-Hitchin-Singer and Eells-Salamon almost Hermitian structures $(\mathcal{J}_1, h_t)$, $(\mathcal{J}_2, h_t)$ and explicit formulas for the various curvatures of a twistor space. Examples of twistor spaces of certain concrete four dimensional manifolds are also given in this chapter.

In Chapter 2 we study the twistor spaces of oriented Riemannian four-manifolds as a source of almost Hermitian six-manifolds of constant or strictly positive holomorphic, Hermitian and orthogonal bisectional curvatures.

Given an almost Hermitian manifold (M, g, J) one can define various types of curvatures related to the almost Hermitian structure (g, J). The most important ones are the holomorphic sectional curvature [37], the holomorphic bisectional curvature, Hermitian bisectional curvature, and the orthogonal (totally real) bisectional curvature

[28], [9], [12]. These curvatures have been studied intensively on Kähler manifolds and a lot of important results have been obtained. For example, the well-known uniformization theorem for complete Kähler manifolds of constant holomorphic sectional curvature states that any such manifold is either the complex projective space $\mathbb{CP}^n$ with the Fubini-Study metric, a quotient of $\mathbb{C}^n$ with the flat metric or a quotient of the unit ball in $\mathbb{C}^n$ with the hyperbolic metric [37]. Moreover, due to the solution of the Frankel conjecture given by Mori [42] and Siu-Yau [50], we know that the complex projective spaces are the only compact complex manifolds admitting Kähler metrics of positive holomorphic bisectional curvature. Note also that Mok [41] proved the so-called generalized Frankel conjecture stating that any compact simply-connected Kähler manifold with nonnegative holomorphic bisectional curvature is biholomorphic to a compact Hermitian symmetric space. We refer to [38], [15], [31] for analogous results under some weaker conditions on the holomorphic bisectional curvature. The case of negative holomorphic bisectional curvature is not so rigid. For example, recently To and Yeung [51] have constructed such Kähler metrics on any Kodaira surface.

In the non-Kähler case the holomorphic curvatures mentioned above are not so well studied and it seems that the main reason for that is the lack of interesting examples. Complete results are obtained only for complex dimension 2 in which case it has been proved that every compact Hermitian surface of constant holomorphic or Hermitian sectional curvature is a complex space form [5]. In higher dimensions it is still an open question posed by Balas and Gauduchon [9], [10] whether there are compact non-Kähler Hermitian manifolds of non-zero constant holomorphic sectional curvature of the Hermitian connection (we call it Hermitian sectional curvature in short).

In Section 2.1 we obtain an explicit formula for the holomorphic bisectional curvature of the twistor spaces $(\mathcal{Z}, h_t, \mathcal{J}_n), n = 1, 2, t > 0$, when the base manifold (M, g) is Einstein and self-dual (Proposition 2.1.1). We use this formula to show that the holomorphic bisectional curvature of a twistor space is never constant (Theorem 2.1.4) and, when the base manifold is a real space form, we determine all t for which this curvature is strictly positive (Theorem 2.1,5). In particular, it follows that the so-called "squashed" metric on $\mathbb{CP}^3$ ([11], Example 9.83) is a non-Kähler Hermitian-Einstein metric of positive holomorphic bisectional curvature. This shows that a recent result of Kalafat and Koca [35] in dimension four cannot be extended to higher dimensions.

It is well known [33] that the orthogonal bisectional curvature of a Kähler manifold is constant if and only if it is a complex space form. In Section 2.2 we prove that the orthogonal bisectional curvature of the twistor space $(\mathcal{Z}, h_t, \mathcal{J}_1)$ of a self-dual Einstein 4-manifold (M, g) is constant if and only if $\mathcal{Z}$ is Kähler and (M, g) is of constant sectional curvature $1/t$ (Theorem 2.2.1).

In Section 2.3 we prove that the Hermitian bisectional curvature of a non-Kähler Hermitian manifold is never a non-zero constant (Theorem 2.3.2) which gives a partial negative answer to the question of Balas and Gauduchon mentioned above. We also study the problem when the holomorphic and Hermitian bisectional curvatures of an almost Hermitian manifold coincide. This is motivated by a result of Vezzoni [53] that an almost Kähler manifold has that property if and only if it is Kähler. In Theorem 2.3.1 we extend this result to a more general class of almost Hermitian manifolds containing the Hermitian, almost Kähler and nearly Kähler manifolds. Finally, in Theorem 2.3.3 we show that the holomorphic and Hermitian sectional curvatures of a twistor space $(\mathcal{Z}, h_t, \mathcal{J}_n)$ coincide if and only if the base manifold (M, g) is Einstein and self-dual with $ts = 12$ if $n = 1$ and $ts = 6$ if $n = 2$. Then, as is well known [32], $M = S^4$ or $M = \mathbb{CP}^2$, hence $\mathcal{Z} = \mathbb{CP}^3$ or $\mathcal{Z} = SU(3)/S(U(1) \times U(1) \times U(1))$. Note that $(h_t, \mathcal{J}_1)$ for $t = 12/s$ is the standard Kähler-Einstein structure on $M = \mathbb{CP}^2$. For $t = 6/s$, $(\mathcal{Z}, h_t)$ is a Riemannian 3-symmetric space (in the sense of Wolf and Gray [56]) and $\mathcal{J}_2$ is its canonical almost-complex structure. In this case $(\mathcal{Z}, h_t, \mathcal{J}_2)$ is a nearly Kähler manifold. Note also that for $M = S^4$ and $t = 6/s$, h_t is the "squashed" Einstein metric on $\mathbb{CP}^3$ mentioned above.

The main purpose of Chapter 3 is to determine the Gray-Hervella classes of the so-called compatible almost Hermitian structures on the twistor space of an oriented four-dimensional Riemannian manifold (M, g) considered by G. Deschamps in [22].

In 1980s A. Gray and L.M. Hervella [29] proposed a natural way to classify the almost Hermitian manifolds by studying a representation of the unitary group on the space of tensors satisfying the same identities as the covariant derivative of the Kähler form of an almost Hermitian manifold. This representation has four irreducible components, which determine sixteen classes of almost Hermitian manifolds playing an important role in Hermitian geometry.

G. Deschamps observed in [22] that one can obtain almost complex structures on $\mathcal{Z}$ compatible with the metrics h_t by means of a fibre-preserving map $f : \mathcal{Z} \to \mathcal{Z}$. Given such a map, the corresponding almost complex structure $\mathcal{J}_f$ is defined as follows. The Levi-Civita connection of (M, g) gives rise to a splitting $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of $\mathcal{Z}$ into horizontal and vertical subbundles. The structure $\mathcal{J}_f$ on a horizontal space $\mathcal{H}_\sigma$, $\sigma \in \mathcal{Z}$ is defined as the horizontal lift of the complex structure $f(\sigma)$ on $T_\pi(\sigma)$; on $\mathcal{V}_\sigma$ it is set to be equal to $\mathcal{J}_1$. Thus, if $f = id$ we obtain the almost complex structure $\mathcal{J}_1$ and if f is the antipodal map $\sigma \mapsto -\sigma$ we get $-\mathcal{J}_2$. In fact, LeBrun [39] was the first to use self-maps of a twistor space to construct complex structures with interesting properties. He considered the twistor space $\mathcal{Z}$ of a $K3$-surface M with a hyper-Kähler metric. In this case $\mathcal{Z}$ is diffeomorphic to $M \times S^2$ and LeBrun used the complex structures on $\mathcal{Z}$ defined by means of the maps $\mathbb{CP} \ni [u,v] \mapsto [u^{m-1}, v^{m-1}]$, $m = 1, 2, ...$, to show that the Chern numbers c_1^3 and c_1c_2 of a compact complex 3-manifold are not determined by the topology of the underlying smooth manifold.

In Section 3.1 we derive coordinate-free formulas for the covariant derivative of the Kähler 2-form of the almost Hermitian structure $(h_t, \mathcal{J}_f)$ corresponding to an arbitrary fibre-preserving map $f : \mathcal{Z} \to \mathcal{Z}$. We use these formulas to determine the Gray-Hervella classes of $(h_t, \mathcal{J}_f)$ for some particular maps f. More precisely, let (M, g, J) be an almost Hermitian manifold of real dimension four and λ be a complex number. Consider M with the orientation induced by the almost complex structure J. Then J is a section of the (positive) twistor space $\mathcal{Z}$ of (M, g). At any point $p \in M$, the complex structure $J_p : T_pM \to T_pM$ is a point of the fibre $\mathcal{Z}_p$ at p of $\mathcal{Z}$. Since $\mathcal{Z}_p$ is the unit 2-sphere, we can compose the stereographic projection of $\mathcal{Z}_p$ from the point J_p with the linear map $z \mapsto \lambda z$ of the complex plane, then go back to the sphere by the inverse stereographic projection. This way we obtain a fibre-preserving map $f_\lambda^+ : \mathcal{Z} \to \mathcal{Z}$ whose restriction to every fibre is a holomorphic map. If we use in a similar way the stereographic projection from the point $-J_p$, we get a map f_λ^- such that the points $f_\lambda^+(\sigma)$ and $f_\lambda^-(\sigma)$ are symmetric with respect to the plane $(\mathbb{R}J_p)^\perp$; thus the restrictions to the fibres of $\mathcal{Z}$ are anti-holomorphic maps. In particular, $f_1^+(\sigma) = \sigma$, so $\mathcal{J}_{f_1^+}$ is the Atiyah-Hitchin-Singer almost complex structure $\mathcal{J}_1$, whereas $f^-_{-1}(\sigma) = -\sigma$ and $\mathcal{J}_{f^-_{-1}}$ is the conjugate structure of the Eells-Salamon almost complex structure $\mathcal{J}_2$. For $\lambda = 0$, we have $f^{\pm}_0 \equiv \mp J \circ \pi$; note that the structures J and $-J$ induce the same orientation, since $dim\, M = 4$ and belong to the same Gray-Hervella classes. In case $\lambda = 0$, the integrability condition for the corresponding almost complex structure $\mathcal{J}_{f^{\pm}_0}$ has been given in [22], where this structure is denoted by $\mathbb{J}_\infty$.

In Section 3.2, Theorem 3.2.1 we establish all possible Gray-Hervella classes of the almost Hermitian structure $(h_t, \mathcal{J}_{f^{\pm}_0})$ on the twistor space $\mathcal{Z}$ and found the geometric conditions on the base manifold M under which this structure belongs to each of these classes. In case $\lambda \neq 0, 1$ and the base manifold M is Kähler, the integrability condition for the almost complex structure $\mathcal{J}_{f^+_\lambda}$ has been obtained in [22] (the structure being denoted there by $\mathbb{J}_{\lambda Id}$).

Under the assumptions that $\lambda \neq 0$ and M is Kähler, in Section 3.3, Theorem 3.3.2 we determine the Gray-Hervella classes of the almost Hermitian structure $(h_t, \mathcal{J}_{f^{\pm}_\lambda})$. At the end of this section, we also discuss the case when $|\lambda| = 1$ without the Kähler assumption on the base manifold M. In order to maintain the reasonable length of the chapter , we only discuss some of the basic Gray-Hervella classes.

The results of Chapter 2 are published in [2, 4] and those of Chapter 3 in [3].

Chapter 1

Twistor Spaces of Oriented Riemannian 4-Manifolds

In this chapter we recall the definitions of Atiyah, Hitchin, Singer [8] and Eells, Salamon [23] almost Hermitian structures on the twistor spaces of oriented Riemannian 4-manifolds as well as some explicit formulas for their curvatures.

1.1 Curvature Decomposition in Dimension 4

The Riemannian geometry in 4 dimensions has some special features most of which may be derived from the action of the Hodge star operator. Using it one may split 2-forms into self-dual and anti self-dual forms. This can be applied in particular to the curvature form of any bundle with connection over an oriented 4-manifold.

Let M be a (connected) oriented Riemannian 4-manifold with metric g. Then g induces a metric on the bundle $\Lambda^2 TM$ of 2-vectors by the formula

$$g(X_1 \wedge X_2, X_3 \wedge X_4) = \frac{1}{2}\det(g(X_i, X_j)).$$

The Levi-Civita connection of M determines a connection on the vector bundle

$\Lambda^2 TM$ (both denoted by ∇) and the respective curvatures are related by

$$R(X,Y)(Z\wedge T) = R(X,Y)Z\wedge T + Z\wedge R(X,Y)T$$

for $X,Y,Z,T\in\chi(M)$; $\chi(M)$ stands for the Lie algebra of smooth vector fields on M. (For the curvature tensor R we adopt the following definition $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X,\nabla_Y]$). The curvature operator $\mathcal{R}$ is the self-adjoint endomorphism of $\Lambda^2 TM$ defined by

$$g(\mathcal{R}(X\wedge Y), Z\wedge T) = g(R(X,Y)Z,T)$$

for all $X,Y,Z,T\in\chi(M)$.

Recall that [11] the Hodge star operator $*$ is define as follows: If α,β are exterior p-forms and ω_g is the volume form of an oriented Riemannian manifold (M,g), then

$$\alpha\wedge *\beta = (\alpha,\beta)\omega_g$$

In dimension 4 the Hodge star operator defines an endomorphism $*$ of $\Lambda^2 TM$ with $*^2 = Id$. Hence we have the orthogonal decomposition

$$\Lambda^2 TM = \Lambda^2_+ TM \oplus \Lambda^2_- TM$$

where $\Lambda^2_\pm TM$ are the subbundles of $\Lambda^2 TM$ corresponding to the (± 1)-eigenvalues of $*$. Let (E_1,E_2,E_3,E_4) be a local oriented orthonormal frame of TM. Set

$$\begin{aligned} s_1 &= E_1\wedge E_2 + E_3\wedge E_4 & \bar{s}_1 &= E_1\wedge E_2 - E_3\wedge E_4 \\ s_2 &= E_1\wedge E_3 + E_4\wedge E_2 & \bar{s}_2 &= E_1\wedge E_3 - E_4\wedge E_2 \\ s_3 &= E_1\wedge E_4 + E_2\wedge E_3 & \bar{s}_3 &= E_1\wedge E_4 - E_2\wedge E_3 \end{aligned} \tag{1.1.1}$$

Then (s_1,s_2,s_3) (resp.$(\bar{s}_1,\bar{s}_2,\bar{s}_3)$) is a local orthonormal frame of $\Lambda^2_+ TM$ (resp.$\Lambda^2_- TM$) determining an orientation of $\Lambda^2_+ TM$ (resp.$\Lambda^2_- TM$) which does not depend on the choice of E_1, E_2, E_3, E_4. Changing the orientation of M interchanges the role of $\Lambda^2_+ TM$ and $\Lambda^2_- TM$ (respectively (s_1, s_2, s_3) and $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$)

For every $p \in M$, the group $SO(4)$ acts in a natural way on the space of 4-tensors on T_pM having the same symmetries as the Riemannian curvature tensor. The irreducible decomposition of this space under the action of $SO(4)$ has been found by Singer and Thorpe [49].

The matrix of $\mathcal{R}$ with respect to the frame $(s_i, \bar{s}_i)$ of $\Lambda^2 TM$ has the form

$$\mathcal{R} = \begin{bmatrix} A & B \\ {}^tB & C \end{bmatrix}$$

where the 3×3-matrices A and C are symmetric and have equal traces. Let $\mathcal{B}$, $\mathcal{W}_+$ and $\mathcal{W}_-$ be the endomorphisms of $\Lambda^2 TM$ with matrices:

$$\mathcal{B} = \begin{bmatrix} 0 & B \\ {}^tB & 0 \end{bmatrix}, \quad \mathcal{W}_+ = \begin{bmatrix} A - \frac{s}{6}I & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{W}_- = \begin{bmatrix} 0 & 0 \\ 0 & C - \frac{s}{6}I \end{bmatrix}$$

where s is the scalar curvature and I is the unit 3×3-matrix. Then

$$\mathcal{R} = \frac{s}{6} Id + \mathcal{B} + \mathcal{W}_+ + \mathcal{W}_- \tag{1.1.2}$$

is the irreducible decomposition of $\mathcal{R}$ under the action of $SO(4)$ found by Singer & Thorpe. Note that

$$\mathcal{W}_+ : \Lambda^2_+ TM \longrightarrow \Lambda^2_+ TM$$

$$\mathcal{W}_- : \Lambda^2_- TM \longrightarrow \Lambda^2_- TM$$

$$\mathcal{B} : \Lambda^2_\pm TM \longrightarrow \Lambda^2_\mp TM$$

Here $\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_-$ and $\mathcal{B}$ represent the Weyl conformal tensor and traceless Ricci tensor, respectively. The manifold M is called self-dual (anti-self-dual) if $\mathcal{W}_- = 0$ $(\mathcal{W}_+ = 0)$. It is Einstein exactly when $\mathcal{B} = 0$.

1.2 Almost Hermitian Structures on Twistor Spaces

For every $a \in \Lambda^2 TM$, define a skew-symmetric endomorphism of $T_{\pi(a)}M$ by

$$g(K_a X, Y) = 2g(a, X \wedge Y), \quad X, Y \in T_pM. \tag{1.2.1}$$

If $\sigma \in \Lambda^2_- TM$ is a unit vector, then K_σ is a complex structure on the vector space $T_{\pi(\sigma)}M$ compatible with the metric and the opposite orientation of M. Conversely, the 2-vector σ dual to one half of the Kähler 2-form of such a complex structure is a unit vector in $\Lambda^2_- TM$. Thus the unit sphere subbundle $\pi : \mathcal{Z} \to M$ of $\pi : \Lambda^2_- TM \to M$ parameterizes the complex structures on the tangent spaces of M compatible with its metric and the opposite orientation. Its total space $\mathcal{Z}$ is called the (negative) twistor space of M.

Similarly the unit sphere subbundle of $\pi : \Lambda^2_+ TM \to M$ parameterizes the complex structures on the tangent spaces of M compatible with its metric and the given orientation of M. Its total space called the positive twistor space of M.

Remark.

If we endow $\Lambda^2 TM$ with the metric $2g$, as many authors do, then the twistor bundle of M is the sphere subbundle of $\Lambda^2_- TM$ of radius $\sqrt{2}$. Moreover the curvature operator acting on $\Lambda^2 TM$ is one half of the operator used here.

For every $\sigma \in \mathcal{Z}$, the tangent space $T_{\pi(\sigma)}M$ has an oriented orthonormal frame of the form $E', K_\sigma E', E'', K_\sigma E''$. So setting $E_1 = E', E_2 = K_\sigma E', E_3 = E'', E_4 = -K_\sigma E''$ and defining $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ by means of (1.1.1) we obtain an oriented orthonormal basis of $\Lambda^2_- T_{\pi(\sigma)}M$ for which $\sigma = \bar{s}_1$; for $E_1 = E', E_2 = E'', E_3 = K_\sigma E', E_4 = K_\sigma E''$, we have $\sigma = \bar{s}_2$ and if $E_1 = E', E_2 = E'', E_3 = -K_\sigma E'', E_4 = K_\sigma E'$, we have $\sigma = \bar{s}_3$.

The Levi-Civita connection ∇ of M gives rise to a splitting $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of $\mathcal{Z}$ into horizontal and vertical components. More precisely, let $\pi : \Lambda^2_- TM \to M$ be the natural projection. By definition, the vertical space at $\sigma \in \mathcal{Z}$ is $\mathcal{V}_\sigma$ =Ker$(\pi|\mathcal{Z})_{*\sigma}$ ($T_\sigma\mathcal{Z}$ is always considered as a subspace of $T_\sigma(\Lambda^2_- TM)$). Note that $\mathcal{V}_\sigma$ consists of those vectors of $T_\sigma\mathcal{Z}$ which are tangent to the fibre $\mathcal{Z}_p = \pi^{-1}(p) \cap \mathcal{Z}$, $p = \pi(\sigma)$, of $\mathcal{Z}$ through the point σ. Since $\mathcal{Z}_p$ is the unit sphere in the vector space $\Lambda^2_- T_pM$, $\mathcal{V}_\sigma$ is the orthogonal complement of σ in $\Lambda^2_- T_pM$. Let s be a local section of $\mathcal{Z}$ such that $s(p) = \sigma$. Since s has a constant length, $\nabla_X s \in \mathcal{V}_\sigma$ for all $X \in T_pM$. Given $X \in T_pM$, the vector $X^h_\sigma = s_*X - \nabla_X s \in T_\sigma\mathcal{Z}$ depends only on p and σ. By definition, the horizontal space at σ is $\mathcal{H}_\sigma = \{X^h_\sigma : X \in T_pM\}$. The map $X \to X^h_\sigma$ is an isomorphism between T_pM and $\mathcal{H}_\sigma$ with inverse map $\pi_* \mid \mathcal{H}_\sigma$.

Following [8] and [23] define two almost–complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ on $\mathcal{Z}$ by

$$\mathcal{J}_n V = (-1)^n \sigma \times V \text{ for } V \in \mathcal{V}_\sigma$$

$$\mathcal{J}_n X^h_\sigma = (K_\sigma X)^h_\sigma \text{ for } X \in T_pM, p = \pi(\sigma).$$

It is well-known [8] that $\mathcal{J}_1$ is integrable (i.e. comes from a complex structure) if and only if M is self-dual. Unlike $\mathcal{J}_1$, the almost-complex structure $\mathcal{J}_2$ is never integrable [23].

Let h_t be the Riemannian metric on $\mathcal{Z}$ given by

$$h_t = \pi^* g + t g^v$$

where $t > 0$, g is the metric of M and g^v is the restriction of the metric of $\Lambda^2 TM$ to the vertical distribution $\mathcal{V}$.

The action of $SO(4)$ on $\Lambda^2\mathbb{R}^4$ preserves the decomposition $\Lambda^2\mathbb{R}^4 = \Lambda^2_+\mathbb{R}^4 \oplus \Lambda^2_-\mathbb{R}^4$. Thus, considering S^2 as the unit sphere in $\Lambda^2_-\mathbb{R}^4$, we have an action of the group $SO(4)$ on S^2. Then, if $SO(M)$ denotes the principal bundle of the oriented orthonormal frames on M, the twistor bundle of M is the associated bundle $SO(M) \times_{SO(4)} S^2$. It follows from the Vilms theorem (see, for example, [11, Theorem 9.59]) that the projection map $\pi : (\mathcal{Z}, h_t) \to (M, g)$ is a Riemannian submersion with totally geodesic fibres (this can also be proved by a direct computation). Note also that the almost-complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ are compatible with the metrics h_t.

1.3 Examples of Twistor Spaces

In this section we consider some well known examples of twistor spaces.

Example 1. Suppose that $\Lambda^2_- TM$ admits a global frame $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$. Then the bundle $\Lambda^2_- TM$ is isomorphic to the trivial bundle $M \times \mathbb{R}^3 \to M$, an isomorphism being given by

$$F(\sigma) = (\pi(\sigma), (y_1, y_2, y_3)) \text{ where } y_k = g(\sigma, \bar{s}_k(\pi(\sigma)), \ k = 1, 2, 3.$$

Clearly, the restriction $f = F|\mathcal{Z}$ is a diffeomorphism of $\mathcal{Z}$ onto $M \times S^2$ and $f^{-1} = F^{-1}|M \times S^2$. If $x_1, ..., x_4$ are local coordinates of M, then $(\widetilde{x}_i = x_i \circ \pi, y_k)$, $i = 1, ..., 5$, $k = 1, 2, 3$, are local coordinates of $\Lambda^2_- TM$. If ξ_1, ξ_2, ξ_3 are the standard coordinates of $\mathbb{R}^3$, then (x_i, ξ_k) are local coordinates of $M \times \mathbb{R}^3$. Let $V = \sum_{k=1}^3 v_k \dfrac{\partial}{\partial y_k}(\sigma)$ be a vertical vector at a point $\sigma = \sum_{i=1}^3 y_i \bar{s}_i(\pi(\sigma))$ of $\mathcal{Z}$. Then,

$$f_*(V) = \sum_{k=1}^3 v_k \frac{\partial}{\partial \xi_k}(y), \quad y = (y_1, y_2, y_3).$$

The right-hand side of this identity is a vector tangent to S^2 at the point y . Thus f sends vertical spaces of $\mathcal{Z}$ onto vertical spaces of the trivial bundle $M \times S^2 \to M$.

Every smooth section φ of $M \times \mathbb{R}^3 \to M$ has the form (Id, Φ) where $\Phi : M \to \mathbb{R}^3$ is a smooth function. The trivial connection D on $M \times \mathbb{R}^3 \to M$ is defined by $D_X\varphi = (p, X(\Phi))$ for $X \in T_pM$. The horizontal space of $M \times \mathbb{R}^3 \to M$ with respect to D is tangent to $M \times S^2$ since the latter is the unit sphere subbundle of of the bundle $M \times \mathbb{R}^3 \to M$ endowed with the product metric.

Now suppose in addition that the frame $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ is parallel, $\nabla \bar{s}_k = 0$, $k = 1, 2, 3$. Take $\sigma \in \mathcal{Z}$ and set $p = \pi(\sigma)$. Let s be a (local) section of $\mathcal{Z}(M)$ such that $s(p) = \sigma$ and $\nabla s|_p = 0$. Then for every $X \in T_pM$, we have $f_*(X^h_\sigma) = f_*(s_*(X)) = (f \circ s)_*(X)$. Setting $\varphi = f \circ s$ we get a section of $M \times S^2$ for which $\varphi = (Id, (g(s, \bar{s}_1), g(s, \bar{s}_2), g(s, \bar{s}_3)))$. For $X \in T_pM$, we have $Xg(s, \bar{s}_k) = g(\nabla_X s, \bar{s}_k(p)) + g(s(p), \nabla_X \bar{s}_k) = 0$ since $\nabla s|_p = \nabla \bar{s}_k|_p = 0$. Thus, considering φ as a section of $M \times \mathbb{R}^3$, $D_X \varphi = 0$, hence $\varphi_*(X) = X^h_{f(\sigma)}$ where the right-hand side is the lift of X to the horizontal spaces of $M \times S^2$. Therefore $f_*(X^h_\sigma) = \varphi_*(X) = X^h_{f(\sigma)}$. In other words, f_* sends horizontal spaces onto horizontal spaces.

We can define two almost complex structures I_1 and I_2 on the manifold $M \times S^2$ in a way similar to that we used on the twistor space. For $(p, y) \in M \times S^2$ and $X \in T_pM$ we set $I_n X^h_{(p,y)} = (K_\sigma X)^h_{(p,y)}$ where $\sigma = f^{-1}(p, y) = \sum_{k=1}^3 y_k \bar{s}_k(p)$, $n = 1, 2$. On the vertical subbundle TS^2 the structure I_n is defined as $(-1)^{n+1}$ (*the standard complex structure of* S^2). It is easy to check that the map f is biholomorphic with respect to the structures $\mathcal{J}_n$ on $\mathcal{Z}(M)$ and I_n on $M \times S^2$.

The following examples are given in [8]

a). Take $M = \mathbb{R}^4$ with the standard metric. Set $E_i = \dfrac{\partial}{\partial x_i}$, $i = 1, .., 4$ where x_i are the standard coordinates of $\mathbb{R}^4$. Define $\bar{s}_1, \bar{s}_2, \bar{s}_3$ by means of $E_1, ..., E_4$. Then we have $\nabla \bar{s}_k = 0$, $k = 1, 2, 3$, since $\nabla E_i = 0$. Thus the twistor space $(\mathcal{Z}, \mathcal{J}_n)$ is biholomorphic to $(\mathbb{R}^4 \times S^2, I_n)$. The space $\mathbb{R}^4$ is flat, so I_1 is integrable. Note that if we identify $\mathbb{R}^4$ with $\mathbb{C}^2$, the structure I_1 is not the product structure on $\mathbb{C}^2 \times S^2$.

b). Let $M = \mathbb{R}^4/\Gamma$ be a torus (Γ being a lattice). Consider the torus M with the flat metric induced by the metric of $\mathbb{R}^4$. The vector fields E_i descend to global frame on the torus We denote this frame again by E_i and define $\bar{s}_1, \bar{s}_2, \bar{s}_3$ by means of it. Again $\nabla \bar{s}_k = 0$, hence $(\mathcal{Z}, \mathcal{J}_n)$ is biholomorphic to $((\mathbb{R}^4/\Gamma) \times S^2, I_n)$. Note that the structure I_1 on the torus has been considered in [13, Example de fibration au sense de la proposition I.2.2]

Example 2. The twistor space of the unit 4-sphere S^4 endowed with its standard metric and orientation can be identified with the complex projective space $\mathbb{CP}^3$. More precisely, if the sphere S^4 is identified with the quaternion projective line $\mathbb{HP}^1$, the twistor bundle over S^4 has the form $\pi : \mathbb{CP}^3 \longrightarrow \mathbb{HP}^1$, where the projection is given by the tautological formula

$$\pi : [z_1, z_2, z_3, z_4] \longmapsto [z_1 + z_2 j, z_3 + z_4 j]$$

We refer to [55] for a proof of this fact.

Example 3. Consider the complex projective space $\mathbb{CP}^2$ with the Fubini-Study metric and the orientation induced by the complex structure. Then the twistor space $\mathcal{Z}$, is identified with the flag manifold $F = F(1, 1, 1; \mathbb{C}^3)$ (see [11]). We consider the points of F as triples (E_0, E_1, E_2) of mutually orthogonal one-dimensional complex subspaces in $\mathbb{C}^3$. Then the manifold F evidently coincides with the homogeneous space $F = SU(3)/S(U(1) \times U(1) \times U(1))$.

1.4 The Riemannian Curvature of Twistor Spaces

The sectional curvature of Levi-Civita connection of the metric h_t on the twistor space $\mathcal{Z}$ of an oriented Riemannian 4-manifold (M, g) has been computed in [18] by means of the O'Neill formulas [11]. In this section we present (without proofs) some explicit formulas obtained in [18] for various curvatures of h_t which we will use later.

Denote by $R, \mathcal{R}, s$ the curvature tensor, the curvature operator and the scalar curvature of (M, g) respectively.

1.4.1 The Sectional Curvature of Twistor Spaces

Let $R_{\mathcal{Z}}$ be the curvature tensor of $(\mathcal{Z}, h_t)$.

Proposition 1.4.1. *Let $E, F \in T_\sigma\mathcal{Z}$ and $X = \pi_* E$, $Y = \pi_* F$, $V = \mathcal{V}E$, $W = \mathcal{V}F$. Then the sectional curvature of $(\mathcal{Z}, h_t)$ is given by*

$$\begin{aligned} h_t(R_{\mathcal{Z}}(E \wedge F)E, F) &= g(R(X \wedge Y)X, Y) - tg((\nabla_X \mathcal{R})(X \wedge Y), \sigma \times W) \\ &+ tg((\nabla_Y \mathcal{R})(X \wedge Y), \sigma \times V) - 3tg(\mathcal{R}(\sigma), X \wedge Y)g(\sigma \times V, W) \\ &- t^2 g(R(\sigma \times V)X, R(\sigma \times W)Y) + \frac{t^2}{4} \parallel R(\sigma \times W)X + R(\sigma \times V)Y \parallel^2 \\ &- \frac{3t}{4} \parallel R(X \wedge Y)\sigma \parallel^2 + t(\parallel V \parallel^2 \parallel W \parallel^2 - g(V, W)^2). \end{aligned}$$

In the case when M is self dual and Einstein this formula simplifies significantly:

Corollary 1.4.2. *Let M be a self-dual Einstein manifold with scalar curvature s. Then*

$$\begin{aligned} h_t(R_{\mathcal{Z}}(E \wedge F)E, F) &= g(R(X \wedge Y)X, Y) - \frac{ts}{2} g(\sigma, X \wedge Y)g(\sigma \times V, W) \\ &- \frac{1}{2}(\frac{ts}{12})^2 g(X, Y)g(V, W) + 3(\frac{ts}{12})^2 g(X \wedge Y, V \times W) \\ &+ (\frac{ts}{24})^2(\parallel X \parallel^2 \parallel W \parallel^2 + \parallel Y \parallel^2 \parallel V \parallel^2) \\ &- 6t(\frac{s}{24})^2(\parallel X \wedge Y \parallel^2 - 2g(\sigma, X \wedge Y)^2) \\ &+ t(\parallel V \parallel^2 \parallel W \parallel^2 - g(V, W)^2). \end{aligned}$$

1.4.2 The Ricci Curvature of Twistor Spaces

Denote by $c_{\mathcal{Z}}$ the Ricci tensor of the twistor space $(\mathcal{Z}, h_t)$.

Proposition 1.4.3. *Let $E \in T_\sigma\mathcal{Z}$, $X = \pi_* E$ and $V = \mathcal{V}E$,. Then*

$$\begin{aligned} c_{\mathcal{Z}}(E, E) &= c_M(X, X) + tTrace(A \longrightarrow (\nabla_A R)(\sigma \times V, X)) \\ &+ (\frac{t^2}{4}) \parallel \mathcal{R}(\sigma \times V) \parallel^2 - (\frac{t}{2}) \parallel i_X \circ \mathcal{R}_- \parallel_p^2 \\ &+ (\frac{t}{2}) \parallel i_X \circ \mathcal{R} \parallel^2 + \parallel V \parallel^2 \end{aligned}$$

where $i_X : \Lambda^2 TM \longrightarrow TM$ *is the interior product and* $\mathcal{R}_-$ *is the restriction of* $\mathcal{R}$ *on* $\Lambda^2_- TM$.

Corollary 1.4.4. *The scalar curvature* $s_{\mathcal{Z}}$ *of twistor space* $(\mathcal{Z}, h_t)$ *is given by*

$$s_{\mathcal{Z}}(\sigma) = s_M(p) + (\frac{t}{4})(\| \mathcal{R}(\sigma) \|^2 - \| \mathcal{R}_- \|_p^2) + \frac{2}{t}$$

where $p = \pi(\sigma)$ *and* s_M *is scalar curvature of* M.

Corollary 1.4.5. *Let* M *be a self dual Einstein 4-manifold with scalar curvature* s. *Then the Ricci tensor* $c_{\mathcal{Z}}$ *and the scalar curvature* $s_{\mathcal{Z}}$ *of* $(\mathcal{Z}, h_t)$ *are given by*

$$\begin{aligned} c_{\mathcal{Z}}(E,E) &= (\frac{s}{4} - t(\frac{s}{12})^2) \| X \|^2 + (1 + (\frac{ts}{12})^2) \| V \|^2, \\ s_{\mathcal{Z}} &= \frac{2}{t} + s - (\frac{t}{72}) s^2 \end{aligned}$$

where $X = \pi_* E$ *and* $V = \mathcal{V}E$

Using the above formulas one can prove the following well-known result of Friedrich and Grunewald [24]:

Proposition 1.4.6. *The metric* h_t *is Einstein if and only if* (M, g) *is a self-dual Einstein manifold with scalar curvature* $s = \frac{6}{t}$ *or* $\frac{12}{t}$.

1.4.3 The Holomorphic Sectional Curvature of Twistor Spaces

Proposition 1.4.7. *Let* M *be a self dual Einstein manifold with sectional curvature* S *and scalar curvature* s. *Let* $E \in T_\sigma \mathcal{Z}$ *be an* h_t*-unit vector and* K_σ *the complex structure on* T_pM, $p = \pi(\sigma)$, *defined by* σ. *Then*

$$\begin{aligned} H_n(E) &= S(X, K_\sigma X) \| X \|^4 + t \| V \|^4 \\ &+ (2(\frac{st}{24})^2 (3(-1)^{n+1}) + (-1)^n + 1(\frac{st}{4})) \| X \|^2 \| V \|^2 \end{aligned}$$

where $X = \pi_* E$ *and* $V = \mathcal{V}E$

Using this proposition the following statement has been proved in [18].

Proposition 1.4.8. *The almost Hermitian manifold* $(\mathcal{Z}, h_t, \mathcal{J}_1)$ *has a constant holomorphic sectional curvature* χ *if and only if* M *is of constant sectional curvature* $\chi = \frac{1}{t}$
The holomorphic sectional curvature of $(\mathcal{Z}, h_t, \mathcal{J}_2)$ *is never constant.*

Chapter 2

Holomorphic Curvatures of Twistor Spaces

In this chapter we study the twistor spaces of oriented Riemannian four-manifolds as a source of almost Hermitian six-manifolds of constant or strictly positive holomorphic, Hermitian and orthogonal bisectional curvatures. In particular, we obtain explicit formulas for these curvatures in the case when the base manifold is Einstein and self-dual, and observe that the "squashed" metric on $\mathbb{CP}^3$ is a non-Kähler Hermitian-Einstein metric of positive holomorphic bisectional curvature. This shows that a recent result of Kalafat and Koca [35] in dimension four cannot be extended to higher dimensions. We also prove that the Hermitian bisectional curvature of a non-Kähler Hermitian manifold is never a non-zero constant which gives a partial negative answer to a question of Balas and Gauduchon [10].

Finally, motivated by an integrability result of Vezzoni [53] for almost Kähler manifolds, we study the problem when the holomorphic and the Hermitian bisectional curvatures of an almost Hermitian manifold coincide. We extend the result of Vezzoni to a more general class of almost Hermitian manifolds and describe the twistor spaces having this curvature property.

2.1 Holomorphic bisectional curvature of a twistor space

Let (M, g) be an oriented Riemannian four-manifold. Denote by $\mathcal{Z}$ the negative twistor space of M. Moreover denote by $\times$ the usual vector cross product on the oriented 3-dimensional vector space $\Lambda^2_- T_pM$, $p \in M$, endowed with the metric g. Then it is easy to check that

$$g(R(a)b, c) = -g(\mathcal{R}(b \times c), a) \tag{2.1.1}$$

for $a \in \Lambda^2 T_pM$, $b, c \in \Lambda^2_+ T_pM$, and

$$g(\sigma \times V, X \wedge K_\sigma Y) = g(\sigma \times V, K_\sigma X \wedge Y) = -g(V, X \wedge Y) \tag{2.1.2}$$

for $V \in \mathcal{V}_\sigma$, $X, Y \in T_pM$.

It is also easy to show that for every $a, b \in \Lambda^2_- T_pM$

$$K_a \circ K_b = -g(a, b)Id - K_{a \times b}. \tag{2.1.3}$$

Now let (N, h, J) be an almost Hermitian manifold. Then the holomorphic sectional curvature and the holomorphic bisectional curvature are defined respectively by

$$H(X) = R(X, JX, X, JX), \| X \| = 1;$$

$$H(X, Y) = R(X, JX, Y, JY), \| X \| = \| Y \| = 1$$

where R is the Riemannian curvature of (N, h).

First we give an explicit formula for the holomorphic bisectional curvature $H_{t,n}$ of the twistor space $(\mathcal{Z}, h_t, \mathcal{J}_n), n = 1, 2, t > 0$, when the base manifold (M, g) is self-dual and Einstein.

Proposition 2.1.1. *Let (M, g) be a self-dual Einstein manifold with scalar curvature s and let $E, F \in T_\sigma \mathcal{Z}$ be arbitrary h_t-unit tangent vectors with $X = \pi_* E$, $Y = \pi_* F$, $V = \mathcal{V}E$, $W = \mathcal{V}F$. Then*

$$\begin{aligned} H_{t,n}(E, F) &= R(X, K_\sigma X, Y, K_\sigma Y) + t \parallel V \parallel^2 \parallel W \parallel^2 \\ &+ 2t(\frac{s}{24})^2 (\parallel X \parallel^2 \parallel Y \parallel^2 - g(X, Y)^2 - g(K_\sigma X, Y)^2) \\ &+ (-1)^n (2(\frac{ts}{24})^2 - \frac{ts}{12})(\parallel X \parallel^2 \parallel W \parallel^2 + \parallel Y \parallel^2 \parallel V \parallel^2) \\ &+ (2(\frac{ts}{24})^2 (1 + (-1)^n) - \frac{ts}{12})(g(K_\sigma X, Y) g(\sigma \times V, W) \\ &+ (-1)^n g(X, Y) g(V, W)) \end{aligned} \tag{2.1.4}$$

where K_σ is the complex structure on T_pM, $p = \pi(\sigma)$, determined by σ via (1.2.1).

Proof. Note that

$$g(V, \sigma \times W)^2 + g(V, W)^2 = \parallel V \parallel^2 \parallel W \parallel^2 \tag{2.1.5}$$

since if $W \neq 0$, $W, \sigma \times W$ is a basis of $\mathcal{V}_\sigma$. Note also that

$$\parallel (-1)^n \sigma \times V \parallel^2 = \parallel V \parallel^2, \tag{2.1.6}$$

and

$$(\sigma \times V) \times W = -\sigma g(V, W) \tag{2.1.7}$$

Now we show that

$$g(K_\sigma Y \wedge K_\sigma X, V \times W) = -\frac{1}{2} g(K_\sigma X, Y) g(\sigma \times V, W) \tag{2.1.8}$$

We may assume that $\parallel V \times W \parallel = 1$. Then

$$g(K_\sigma Y \wedge K_\sigma X, V \times W) = \frac{1}{2} g(K_{V \times W} K_\sigma Y, K_\sigma X)$$

We have

$$K_{V \times W} \circ K_\sigma Y = -g(V \times W, \sigma) Y - K_{(V \times W) \times \sigma} Y$$

by (2.1.3), where $(V \times W) \times \sigma = 0$, since $V \perp \sigma$, $W \perp \sigma$

Hence

$$\begin{aligned} g(K_\sigma Y \wedge K_\sigma X, V \times W) &= -\frac{1}{2} g(K_\sigma X, Y) g(\sigma, V \times W) \\ &= -\frac{1}{2} g(K_\sigma X, Y) g(\sigma \times V, W) \end{aligned}$$

The well-known expression of the Riemannian curvature tensor by means of sectional curvatures (cf. e.g. [37]) gives:

$$\begin{aligned} 6h_t(R_{\mathcal{Z}}(E, \mathcal{J}_n E)F, \mathcal{J}_n F) &= S(\mathcal{J}_n E + \mathcal{J}_n F, E + F) - S(\mathcal{J}_n E + \mathcal{J}_n F, E) \\ &- S(\mathcal{J}_n E + \mathcal{J}_n F, F) - S(\mathcal{J}_n E, E + F) - S(\mathcal{J}_n F, E + F) \\ &+ S(\mathcal{J}_n E, F) + S(\mathcal{J}_n F, E) - S(E + \mathcal{J}_n F, \mathcal{J}_n E + F) \\ &+ S(E + \mathcal{J}_n F, \mathcal{J}_n E) + S(E + \mathcal{J}_n F, F) + S(E, F + \mathcal{J}_n E) \\ &+ S(\mathcal{J}_n F, F + \mathcal{J}_n E) - S(E, F) - S(\mathcal{J}_n F, \mathcal{J}_n E) \qquad (2.1.9) \end{aligned}$$

where S denotes the sectional curvature. Now using Proposition 1.4.1 and identities (2.1.2) – (2.1.6) one can obtain after a simple but long computation the desired formula for the holomorphic bisectional curvature $H_{t,n}(E, F)$. □

Note that in the case when the base manifold (M, g) is of constant sectional curvature we have

$$g(R(X, K_\sigma X)Y, K_\sigma Y) = \frac{s}{6} g(X \wedge K_\sigma X, Y \wedge K_\sigma Y) = \frac{s}{12}(g(X, Y)^2 + g(K_\sigma X, Y)^2)$$

and the proposition above implies the following

Corollary 2.1.2. *Let (M,g) be a 4-manifold of constant sectional curvature and scalar curvature s. Then*

$$\begin{aligned} H_{t,n}(E,F) &= \frac{s}{12}(g(X,Y)^2+g(K_\sigma X,Y)^2)+t\parallel V\parallel^2\parallel W\parallel^2 \\ &+ 2t(\frac{s}{24})^2(\parallel X\parallel^2\parallel Y\parallel^2 - g(X,Y)^2-g(K_\sigma X,Y)) \\ &+ (-1)^n(2(\frac{ts}{24})^2-\frac{ts}{12})(\parallel X\parallel^2\parallel W\parallel^2+\parallel Y\parallel^2\parallel V\parallel^2) \\ &+ (2(\frac{ts}{24})^2(1+(-1)^n)-\frac{ts}{12})(g(K_\sigma X,Y)g(\sigma\times V,W) \\ &+ (-1)^n g(X,Y)g(V,W)). \end{aligned} \tag{2.1.10}$$

Setting $E=F$ in (2.1.9) we obtain the following.

Corollary 2.1.3. *Let (M,g) be a self-dual Einstein manifold with sectional curvature K and scalar curvature s, and let $E,F\in T_\sigma\mathcal{Z}$ be arbitrary h_t-unit tangent vectors with $X=\pi_*E$,$Y=\pi_*F$,$V=\mathcal{V}E$, $W=\mathcal{V}F$. The holomorphic sectional curvature of $(\mathcal{Z},h_t,\mathcal{J}_n)$ is given by*

$$H_{t,n}(E)=K(X,K_\sigma X)\|X\|^4+t\|V\|^4+(2(\frac{st}{24})^2(3(-1)^n+1)+(-1)^{n+1}(\frac{st}{24}))\|X\|^2\|V\|^2$$

Next we apply Corollary 2.1.2 to prove the following

Theorem 2.1.4. *The holomorphic bisectional curvature of the twistor space $(\mathcal{Z},h_t,\mathcal{J}_n)$ of an oriented Riemannian 4-manifold (M,g) is never constant.*

Proof. Suppose on the contrary that $(\mathcal{Z},h_t,\mathcal{J}_1)$ has constant holomorphic bisectional curvature. Then $(\mathcal{Z},h_t,\mathcal{J}_1)$ has constant holomorphic sectional curvature. According to [18, Proposition 5.2], (M,g) has constant sectional curvature and $st=12$.

Setting $st = 12$ and $n = 1$ in (2.1.10) gives

$$\begin{aligned} H_{t,1}(E,F) &= \frac{1}{t}[g(X,Y)^2 + g(K_\sigma X,Y)^2] + t\|V\|^2\|W\|^2 \\ &+ \frac{1}{2t}[\|X\|^2\|Y\|^2 - g(X,Y)^2 - g(K_\sigma X,Y)^2] + \frac{1}{2}[\|X\|^2\|W\|^2 + \\ &+ |Y\|^2\|V\|^2] - g(K_\sigma X,Y)g(\sigma\times V,W) + g(X,Y)g(V,W) \\ &= \frac{1}{2t}[(g(X,Y)+tg(V,W))^2 + g(X,K_\sigma Y) - tg(V,\sigma\times W)^2(g(X,X) \\ &+ tg(V,V))(g(Y,Y)+tg(W,W))] \\ &= \frac{1}{2t}[h_t(E,F)^2 + h_t(E,\mathcal{J}_1F)^2 + h_t(E,E)h_t(F,F)] \end{aligned}$$

The right hand side of the above identity is not constant because

$$h_t(E,F)^2 + h_t(E,\mathcal{J}_1F)^2$$

is not constant, which contradicts to our assumption.

According to [18, Proposition 5.2], the holomorphic sectional curvature of $(\mathcal{Z},h_t,\mathcal{J}_2)$ is never constant, therefore the holomorphic bisectional curvature of $(\mathcal{Z},h_t,\mathcal{J}_2)$ is never constant too. □

As another application of Proposition 2.1.1, we prove the following:

Theorem 2.1.5. *Let (M,g) be an oriented Riemannian 4-manifold of constant sectional curvature.*

(i) The holomorphic bisectional curvature of $(\mathcal{Z},h_t,\mathcal{J}_1)$ is positive if and only if $0 < ts < 24$.

(ii) If (M,g) is a flat manifold, the holomorphic bisectional curvature of $(\mathcal{Z},h_t,\mathcal{J}_n)$ is non-negative.

Proof. To prove (i), we first note that

$$\| X \|^2 . \| Y \|^2 - g(X,Y)^2 - g(K_\sigma X,Y)^2 = \| X \|^2 . \| Z \|^2 \geq 0,$$

where Z is the projection of Y on the orthogonal complement of the vector space $Span(X,K_\sigma X)$. Now suppose that $0 < ts < 24$. Then using the Cauchy-Schwarz inequality we get

$$\frac{1}{2}(\frac{s}{6}g(X,Y)^2 + t \parallel V \parallel^2 . \parallel W \parallel^2) \geq \sqrt{\frac{ts}{6}}|g(X,Y)||g(V,W)|$$

and

$$\frac{1}{2}(\frac{s}{6}g(K_\sigma X,Y)^2 + t \parallel V \parallel^2 . \parallel W \parallel^2) \geq \sqrt{\frac{ts}{6}}|g(K_\sigma X,Y)||g(\sigma \times V,W)|.$$

We also have

$$\frac{1}{2}(\parallel X \parallel^2 . \parallel W \parallel^2 + \parallel Y \parallel^2 . \parallel V \parallel^2) \geq |g(X,Y)||g(V,W)|$$

and

$$\frac{1}{2}(\parallel X \parallel^2 . \parallel W \parallel^2 + \parallel Y \parallel^2 . \parallel V \parallel^2) \geq |g(K_\sigma X,Y)||g(\sigma \times V,W)|.$$

Since $0 < ts < 24$ it follows from (2.1.10) and the inequalities above that

$$H_{t,1}(E,F) \geq (\sqrt{\frac{ts}{6}} - \frac{1}{2}(\frac{ts}{12})^2)(|g(K_\sigma X,Y)||g(\sigma \times V,W)| + |g(X,Y)||g(V,W)|) \geq 0.$$

It is easy to see that the inequality $H_{t,1}(E,F) \geq 0$ is strict since otherwise either $E = 0$ or $F = 0$.

Finally, observe that

$$H_{t,1}(X^h,W) = (\frac{ts}{12} - \frac{1}{2}(\frac{ts}{12})^2) \parallel X \parallel^2 \parallel W \parallel^2 \leq 0$$

if $ts \leq 0$ or $ts \geq 24$.

Part (ii) follows from (2.1.10) since if the metric g is flat we have

$$H_{t,n}(E,F) = t \parallel V \parallel^2 . \parallel W \parallel^2 \geq 0.$$

□

Now note that the metric h_t on the twistor space Z of a self-dual Einstein manifold M with positive scalar curvature s for $t = \frac{s}{6}$ is Einsten [24] (see also Proposition 1.4.6) and is not Kähler with respect to the complex structure $\mathcal{J}_1$ [25]. If we take $M = S^4$ with the round metric, then $(Z, \mathcal{J}_1) = \mathbb{CP}^3$ and h_t is the so-called "squashed" metric ([11, Example 9.83]). Thus, by Theorem 2.1.5, $\mathbb{CP}^3$ with the "squashed" metric and the standard complex structure is an Einstein Hermitian manifold of positive holomorphic bisectional curvature which is not isometric to $\mathbb{CP}^3$ with a rescaling of the Fubini-Study metric. This shows that the result of Kalafat and Koca mentioned above cannot be extented in real dimension six.

2.2 Orthogonal bisectional curvature of a twistor space

Given an almost Hermitian manifold (M, g, J) the orthogonal (totally real) bisectional curvature $B(X, Y)$ is defined by [12]:

$$B(X, Y) = g(R(X, JX)Y, JY),$$

where $X \perp Y, JY$ and $||X|| = ||Y|| = 1$.

Note that the orthogonal bisectional curvature is also called totally real bisectional curvature [10], [33].

It is well known [33] that the orthogonal bisectional curvature of a Kähler manifold of complex dimension ≥ 3 is constant if and only if it is a complex space form. So, it is natural to ask if the same holds for other classes of almost Hermitian manifolds. Motivated by this question we prove the following

Theorem 2.2.1. *Let (M, g) be a self-dual Einstein 4-manifold. Then $(\mathcal{Z}, h_t, \mathcal{J}_n)$ has constant orthogonal bisectional curvature if and only if $n = 1$ and (M, g) is of constant sectional curvature $\chi = \frac{1}{t}$.*

Proof. Denote by $B_{t,n}(E,F)$ the orthogonal bisectional curvature of the twistor space $(\mathcal{Z},h_t,\mathcal{J}_n)$ and let $B_{t,n}(E,F)\equiv c$. We first show that (M,g) has constant sectional curvature. Indeed, it follows from formula (2.1.1) that

$$g(\mathcal{R}(X\wedge K_\sigma X),Y\wedge K_\sigma Y)=c-2t(\frac{s}{24})^2 \qquad (2.2.1)$$

for all unit tangent vectors X,Y such that $X\perp\{Y,K_\sigma Y\}$. Set $d=c-2t(\frac{s}{24})^2$. Then for any orthonormal frame (E_1,E_2,E_3,E_4) we have

$$\begin{gathered} g(\mathcal{R}(E_1\wedge E_2),E_4\wedge E_3)=g(\mathcal{R}(E_1\wedge E_3),E_2\wedge E_4)= \\ g(\mathcal{R}(E_1\wedge E_4),E_3\wedge E_2)=d \end{gathered} \qquad (2.2.2)$$

which implies

$$g(\mathcal{R}(\bar{s}_i),\bar{s}_i)=\ g(\mathcal{R}(s_i),s_i)+4d \quad ,i=1,2,3.$$

Having in mind that

$$\mathcal{R}=\frac{s}{6}Id+W_+$$

we get

$$g(W_+(s_i),s_i)=-4d,i=1,2,3.$$

Since W_+ is trace free, it follows that $d=0$. Hence

$$g(W_+(\tau),\tau)=0$$

for all $\tau\in\Lambda^2_+TM$ since for any unit τ we can find an orthonormal frame (E_1,E_2,E_3,E_4) such that $\tau=s_1$. The above identity implies $W_+=0$ since W_+ is a symmetric operator. Hence $\mathcal{R}=\frac{s}{6}Id$, i.e. (M,g) is of constant sectional curvature $\chi=\frac{s}{12}$.

Now let E,F be arbitrary h_t-unit tangent vectors such that $E\perp\{F,\mathcal{J}_nF\}$. Set $X=\pi_*E$, $Y=\pi_*F$, $V=\mathcal{V}E$, $W=\mathcal{V}F$. Then

$$\parallel X\parallel^2+t\parallel V\parallel^2=1,\quad \parallel Y\parallel^2+t\parallel W\parallel^2=1,$$

$$g(X,Y)+tg(V,W)=0, \quad g(K_\sigma X,Y)+(-1)^n tg(\sigma\times V,W)=0.$$

Using (2.1.10) and the identity

$$g(V,W)^2+g(\sigma\times V,W)^2=\parallel V\parallel^2\parallel W\parallel^2$$

we get

$$B_{t,1}((E,F)=(\frac{ts}{12}-1)^2 t\parallel V\parallel^2\parallel W\parallel^2+\frac{ts}{12}(1-\frac{ts}{12})(\parallel V\parallel^2+\parallel W\parallel^2)+\frac{t}{2}(\frac{s}{12})^2$$

and

$$B_{t,2}((E,F)=(2(\frac{ts}{12})^2-\frac{ts}{3}+1)t\parallel V\parallel^2\parallel W\parallel^2-\frac{ts}{12}(\parallel V\parallel^2+\parallel W\parallel^2)+\frac{t}{2}(\frac{s}{12})^2.$$

Hence $B_{t,1}(E,F)\equiv const$ iff $ts=12$, whereas $B_{t,2}$ is never constant. □

Remarks.

1. If (M,g) is of constant sectional curvature χ and $t=\frac{1}{\chi}$, then $t=\frac{12}{s}$ and the twistor space $(\mathcal{Z},h_t,\mathcal{J}_1)$ is Kähler [25, 43]

2. If (M,g) has a constant sectional curvature, then the orthogonal bisectional curvature of the twistor space $(\mathcal{Z},h_t,\mathcal{J}_1)$ is strictly positive if and only if $0<ts<24$. This follows from Theorem 2.1.5 (or the identity for $B_{t,1}$ above) and the fact that

$$B_{t,1}(X^h,W)=\frac{s}{12}(1-\frac{ts}{24})\leq 0$$

for $ts\leq 0$ or $ts\geq 24$.

2.3 Hermitian bisectional curvature of a twistor space

It is well-known [40, 27] that every almost Hermitian manifold admits a unique connection for which the almost complex structure and the metric are parallel, and the

$(1,1)$-part of the torsion vanishes. It is usually called the Hermitian (or Chern) connection because, in the integrable case, it coincides with the Hermitian connection [16] of the tangent bundle considered as a Hermitian holomorphic bundle. This connection plays an important role in (almost) complex geometry since, by the Chern–Weil theory, the Chern classes of an almost Hermitian manifold are directly related to its curvature.

Given an almost Hermitian manifold (M, g, J), denote by ∇ its Riemannian connection. Then the Hermitian connection $\tilde{\nabla}$ is defined by (cf. e.g. [30, Th.6.1]):

$$\begin{aligned} g(\tilde{\nabla}_X Y, Z) &= g(\nabla_X Y, Z) + \frac{1}{2} g((\nabla_X J)(JY), Z) + \frac{1}{4} g((\nabla_Z J)(JY) \\ &- (\nabla_Y J)(JZ) - (\nabla_{JZ} J)(Y) + (\nabla_{JY} J)(Z), X) \end{aligned} \tag{2.3.1}$$

We denote by $\tilde{R}$ the curvature tensor of $\tilde{\nabla}$ and set $\tilde{H}(X,Y) = (\tilde{R}(X,JX)Y, JY)$,the holomorphic bisectional curvature of $\tilde{\nabla}$. For short we call $\tilde{H}(X,Y)$ the Hermitian bisectional curvature and $\tilde{H}(X) = \tilde{H}(X,X)$ the Hermitian sectional curvature of (M, g, J).

As we have already mentioned above, it is an open question posed by Ballas and Gauduchon [9],[10] whether there are compact non-Kähler Hermitian manifolds of non-zero constant Hermitian sectional curvature. It was pointed out to us by S. Kobayashi [36] that the answer to this question is negative if the Hermitian bisectional curvature is constant.

Theorem 2.3.1. *The Hermitian bisectional curvature of a Hermitian manifold of complex dimension ≥ 2 is never a non-zero constant.*

Proof. Let (M, g, J) be a Hermitian manifold of constant Hermitian bisectional curvature. Denote by γ the first Chern form of (M, g, J). Then

$$\gamma(X, JX) = \sum_{i=1}^{n} g(\tilde{R}(X, JX)E_i, JE_i) = nc$$

for any unit tangent vector X, where $n = dim_{\mathbb{C}} M$ and $c \neq 0$ is a constant. Hence $\gamma = ncF$ for the Kähler 2-form F of (M, g, J). Since γ is a closed 2-form we get $dF = 0$ and it follows that (M, g, J) is a Kähler manifold. Hence $\nabla = \widetilde{\nabla}$ and therefore the holomorphic bisectional curvature $H(X, Y)$ of (M, g, J) is constant. In particular, (M, g, J) is a complex space form and it is well-known [37] that its curvature tensor R is given by

$$\begin{aligned} R(X,Y,Z,T) &= \frac{c}{4}(g(X,Z)g(Y,T) - g(X,T)g(Y,Z) \\ &+ g(X,JZ)g(Y,JT) - g(X,JT)g(Y,JZ) + 2g(X,JY)g(Y,JT)). \end{aligned}$$

Hence for all unit tangent vectors X, Y we have

$$c = R(X, JX, Y, JY) = \frac{c}{2}(1 + g(X,Y)^2 + g(X,JY)^2).$$

Since $dim_{\mathbb{C}} M \geq 2$, there are unit X, Y such that $X \perp Y, JY$. It follows that $c = 0$, a contradiction. □

Remark.

Note that the same reasoning as above shows that if an almost Hermitian manifold has non-zero constant Hermitian bisectional curvature, then it is an almost Kähler manifold. Hence it is natural to ask the following questions:

- Are there compact non-Kähler and non-flat Hermitian manifolds of complex dimension ≥ 3 with vanishing Hermitian bisectional curvature?
- Are there compact non-Kähler almost Kähler manifolds of constant Hermitian bisectional curvature?

By a result of Vezzoni [53, Theorem 4.8], if (M, g, J) is an almost Kähler manifold (i.e its Kähler form is closed) whose holomorphic and Hermitian bisectional curvatures coincide, then it is a Kähler manifold. In the next theorem we extend this result to a more general class of almost Hermitian manifolds.

Theorem 2.3.2. *Let (M, g, J) be an almost Hermitian manifold such that*

$$(\nabla_X J(X) = \varepsilon(\nabla_{JX} J)(JX), \tag{2.3.2}$$

where $\varepsilon = \pm 1$. Then its holomorphic and Hermitian bisectional curvatures coincide if and only if (M, g, J) is a Kähler manifold.

Proof. We shall use the following relation between the holomorphic and Hermitian sectional curvatures $H(X)$ and $\tilde{H}(X)$ of an almost Hermitiam manifold (M, g, J) (c.f [21, identity (27)]):

$$\begin{aligned} \tilde{H}(X) &= H(X) + \frac{1}{8}(\|(\nabla_X J)(X)\|^2 + \|(\nabla_{JX} J)(JX)\|^2) \\ &+ \frac{3}{4} g((\nabla_X J)(X), (\nabla_{JX} J)(JX)). \end{aligned} \tag{2.3.3}$$

If the holomorphic and Hermitian bisectional curvatures of (M, g, J) coincide then $H(X) = \tilde{H}(X)$ for all $X \in TM$. Hence using the identity (2.3.3), we get

$$\|(\nabla_X J)(X)\|^2 + \|(\nabla_{JX} J)(JX)\|^2 + 6\ g((\nabla_X J)(X), (\nabla_{JX} J)(JX)) = 0 \tag{2.3.4}$$

for all X $\in TM$. Now (2.3.2) and (2.3.4) imply that $(\nabla_X J)(X) = 0$, i.e. (M, g, J) is a nearly Kähler manifold. In this case it follows from (2.3.1) that the Hermitian connection $\tilde{\nabla}$ is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}(\nabla_X J)(JY)$$

and using [21, identity (19)] we get

$$\begin{aligned} 4\tilde{H}_n(X, Y) &= 4\widetilde{R}(X, JX, Y, JY) = 4R(X, JX, Y, JY) + \\ &+ g((\nabla_X J)(Y), (\nabla_{JX} J)(JY)) - g((\nabla_X J)(JY), (\nabla_{JX} J)(Y)) \\ &= 4H(X, Y) - 2\|(\nabla_X J)(Y)\|^2 \end{aligned}$$

since for nearly Kähler manifolds we have

$$(\nabla_{JX}J)(JY) = -(\nabla_X J)(Y)$$

and

$$(\nabla_{JX}J)(Y) = (\nabla_X J)(JY) = -J(\nabla_X J)(Y)$$

Thus $\tilde{H}(X,Y) = H(X,Y)$ if and only if $(\nabla_X J)(Y) = 0$, i.e. when (M,g,J) is a Kähler manifold. □

Remarks.

1. According to the Gray-Hervella terminology [29] the almost Hermitian manifolds satisfying (2.3.2) with $\epsilon = 1$ are called G_1-spaces. This class contains the Hermitian and nearly Kähler manifolds. The identity (2.3.2) with $\epsilon = -1$ holds for almost Kähler and quasi Kähler manifolds (recall that the quasi Kähler condition is $(\nabla_X J)(Y) + (\nabla_{JX}J)(JY) = 0$).

2. The proof of Theorem 2.3.2 shows that the result of Vezzoni [53, Theorem 4.8] for almost Kähler manifolds holds also true under the weaker condition that the holomorphic and Hermitian sectional curvatures coincide.

We now describe the twistor spaces whose holomorphic and Hermitian sectional curvatures coincide.

Theorem 2.3.3. *Let (M,g) be an oriented 4-manifold with twistor space $(\mathcal{Z}, h_t, \mathcal{J}_n), n = 1,2, t > 0$. The holomorphic and Hermitian sectional curvatures of $(\mathcal{Z}, h_t, \mathcal{J}_n)$ coincide if and only if (M,g) is a self-dual Einstein manifold with $ts = 12$ for $n = 1$ and $ts = 6$ for $n = 2$.*

Proof. It follows from [18, identity (5.1)] and [21, identity (28)], that

$$H_{t,n}(X^h_\sigma) = R(X, K_\sigma X, X, K_\sigma X) - \frac{3t}{4}\|R(X \wedge K_\sigma X)\sigma\|^2_g$$

and

$$\tilde{H}_{t,n}(X^h_\sigma) = R(X, K_\sigma X, X, K_\sigma X) - \frac{t}{2}\|R(X \wedge K_\sigma X)\sigma\|^2_g.$$

Suppose $\widetilde{H}_n(E) \equiv H_n(E)$. Then $\widetilde{H}_n(X^h_\sigma) = H_n(X^h_\sigma)$ and the above identities imply that

$$R(X \wedge K_\sigma X)\sigma = 0 \tag{2.3.5}$$

for all $\sigma \in \mathcal{Z}$, $X \in T_pM$, $p = \pi(\sigma)$, where K_σ is the complex structure on T_pM defined by σ. Since $R(X \wedge K_\sigma X)\sigma$ is a vertical vector at σ, the identity (2.3.5) can be rewritten as $g(R(X \wedge K_\sigma X)\sigma, V)) = 0$ for all $V \in V_\sigma$. Now (2.1.1) implies that

$$g(\mathcal{R}(\sigma \times V), X \wedge K_\sigma X)) = 0$$

for all $\sigma \in \mathcal{Z}$, $V \in V_\sigma$ and $X \in T_pM$. Setting $\sigma = \bar{s}_1$ and $V = \bar{s}_2$, we get

$$g(\mathcal{R}(\bar{s}_3), X \wedge K_{\bar{s}_1}X) = 0 \tag{2.3.6}$$

for all $X \in T_pM$. Note that $K_{\bar{s}_1}E_1 = E_2$, $K_{\bar{s}_1}E_3 = -E_4$. Then setting $X = E_1$ and $X = E_3$ in (2.3.6), we obtain

$$g(\mathcal{R}(\bar{s}_3), E_1 \wedge E_2) = g(\mathcal{R}(\bar{s}_3), E_3 \wedge E_4) = 0$$

which is equivalent to

$$g(\mathcal{R}(\bar{s}_3), \bar{s}_1) = g(\mathcal{R}(\bar{s}_3), s_1) = 0.$$

Now set $X = E_1 + E_3$ and $X = E_1 + E_4$ in (2.3.6) to get

$$g(\mathcal{R}(\bar{s}_3), s_3) = g(\mathcal{R}(\bar{s}_3), s_2) = 0.$$

Similarly, we can prove that

$$g(\mathcal{R}(\bar{s}_i), \bar{s}_j) = 0 \text{ for } i \neq j$$

and

$$g(\mathcal{R}(\bar{s}_i), s_j) = 0 \text{ for } \text{ all } 1 \leq i, j \leq 3.$$

This identities imply that $\mathcal{R}(\sigma) = \lambda\sigma$ for some constant λ which means that $\mathcal{B} = \mathcal{W}_- = 0$, i.e. (M, g) is self-dual and Einstein. Hence the almost complex structure $\mathcal{J}_1$ is integrable, whereas $\mathcal{J}_2$ is quasi Kähler [43]. In particular, if D is the Riemannian connection of $(\mathcal{Z}, h_t)$ we have

$$(D_E\mathcal{J}_n)(E) = (-1)^{n+1}(D_{\mathcal{J}_nE}\mathcal{J}_n)(\mathcal{J}_nE), n = 1, 2. \tag{2.3.7}$$

On the other hand we know from (2.3.3) that $\widetilde{H}_n(E) = H_n(E)$ if and only if

$$\|(D_E\mathcal{J}_n)(E)\|_t^2 + \|(D_{\mathcal{J}_nE}\mathcal{J}_n)(\mathcal{J}_nE)\|_t^2 + 6h_t((D_E\mathcal{J}_n)(E), (D_{\mathcal{J}_nE}\mathcal{J}_n)(\mathcal{J}_nE)) = 0 \tag{2.3.8}$$

and using (2.3.7) and (2.3.8), we get $(D_E\mathcal{J}_n)(E) = 0$ for all $E \in T_\sigma\mathcal{Z}$. Thus $(\mathcal{Z}, h_t, \mathcal{J}_n)$ is a nearly K̈hler manifold which implies that [43]:

- $(\mathcal{Z}, h_t, \mathcal{J}_1)$ is Kähler, i.e. (M, g) is self-dual Einstein and $st = 12$.
- $(\mathcal{Z}, h_t, \mathcal{J}_2)$ is nearly Kähler, i.e. (M, g) is self-dual Einstein and $st = 6$.

This completes the proof of the theorem since by (2.3.3) in both cases the holomorphic and Hermitian sectional curvatures coincide. □

Chapter 3

Compatible Almost Complex Structures on Twistor Spaces and their Gray-Hervella Classes

In this chapter we determine the Gray-Hervella classes of the compatible almost complex structures on the twistor spaces of oriented Riemannian four-manifolds considered by G. Deschamps [22].

3.1 Compatible almost complex structures on twistor spaces

Let (M, g) be an oriented Riemannian four-manifold. In this chapter we denote by $\mathcal{Z}$ the unit sphere subbundle of $\wedge^2_+ TM$, i.e. the positive twistor space of M.

Let $f : \mathcal{Z} \to \mathcal{Z}$ be a morphism of the bundle $\mathcal{Z}$, i.e. a smooth map with $\pi \circ f = \pi$. Following [22] we define an almost complex structure $\mathcal{J}_f$ on the 6-manifold $\mathcal{Z}$ setting

$$\mathcal{J}_f V = \sigma \times V \quad for\ V \in \mathcal{V}_\sigma$$

$$\mathcal{J}_f X^h_\sigma = (K_{f(\sigma)} X)^h_\sigma \quad for\ X \in T_{\pi(\sigma)} M.$$

Note that, since the fibres of $\mathcal{Z}$ are spheres, the restriction of $\mathcal{J}_f$ to any fibre is the standard complex structure of the unit sphere.

In the case when $f = Id$, the almost complex structure $\mathcal{J}_f$ coincides with that defined by Atiyah-Hitchin-Singer [8]. In this case the almost complex structure $\mathcal{J}_f$ is integrable if and only if the base manifold M is anti-self-dual [8]. If f is the antipodal map $f(\sigma) = -\sigma$, $\mathcal{J}_f$ is the conjugate structure of the almost complex structure defined by Eells-Salamon [23]. This structure is never integrable [23]. The almost complex structure $\mathcal{J}_f$ is compatible with the Riemannian metrics $h_t, t > 0$ defined above.

Now we recall some well known formulas [18].

Denote by $\times$ the usual vector cross product on the oriented 3-dimensional vector space $\Lambda^2_+ T_pM$, $p \in M$, endowed with the metric g. Then it is easy to check that

$$g(R(a)b, c) = g(\mathcal{R}(b \times c), a) \tag{3.1.1}$$

for $a \in \Lambda^2 T_pM$, $b, c \in \Lambda^2_+ T_pM$, and

$$g(\sigma \times V, X \wedge K_\sigma Y) = g(\sigma \times V, K_\sigma X \wedge Y) = g(V, X \wedge Y) \tag{3.1.2}$$

for $V \in \mathcal{V}_\sigma$, $X, Y \in T_pM$.

It is also easy to show that for every $a, b \in \Lambda^2_+ T_pM$

$$K_a \circ K_b = -g(a, b)Id + K_{a\times b}. \tag{3.1.3}$$

Recall that $\mathcal{Z}$ admits a 1-parameter family h_t of Riemannian metric defined by

$$h_t(X^h_\sigma + V, Y_\sigma)^h + W) = g(X, Y) + tg(V, W)$$

for $X, Y \in T_\pi(\sigma)$, $V, W \in \mathcal{V}_\sigma$ $\sigma \in \mathcal{Z}$.

Then the projection $\pi : \mathcal{Z} \to M$ is a Riemannian submersion with totally geodesic fibres.

Let $(G, x_1, ..., x_4)$ be a local coordinate system of M and let $(E_1, ..., E_4)$ be an oriented orthonormal frame of TM on G. If (s_1, s_2, s_3) is the local frame of $\Lambda^2_+ TM$ defined by (1.1.1), then $\widetilde{x}_\alpha = x_\alpha \circ \pi$, $y_j(\sigma) = g(\sigma, (s_j \circ \pi)(\sigma))$, $1 \leq \alpha \leq 4$, $1 \leq j \leq 3$, are local coordinates of $\Lambda^2_+ TM$ on $\pi^{-1}(G)$.

The horizontal lift X^h on $\pi^{-1}(G)$ of a vector field

$$X = \sum_{\alpha=1}^{4} X^\alpha \frac{\partial}{\partial x_\alpha}$$

is given by

$$X^h = \sum_{\alpha=1}^{4} (X^\alpha \circ \pi) \frac{\partial}{\partial \widetilde{x}_\alpha} - \sum_{j,k=1}^{3} y_j (g(\nabla_X s_j, s_k) \circ \pi) \frac{\partial}{\partial y_k}. \qquad (3.1.4)$$

Hence

$$[X^h, Y^h] = [X, Y]^h + \sum_{j,k=1}^{3} y_j (g(R(X \wedge Y) s_j, s_k) \circ \pi) \frac{\partial}{\partial y_k}. \qquad (3.1.5)$$

for every vector fields X, Y on G. Let $\sigma \in \mathcal{Z}$. Using the standard identification $T_\sigma(\Lambda^2_+ T_p M) \cong \Lambda^2_+ T_p M$ formula (3.1.5) can be rewritten as

$$[X^h, Y^h]_\sigma = [X, Y]^h_\sigma + R_p(X \wedge Y)\sigma, \quad p = \pi(\sigma). \qquad (3.1.6)$$

Denote by D the Levi-Civita connection of $(\mathcal{Z}, h_t)$. Then we have the following:

Lemma 3.1.1. *([18]) If X, Y are vector fields on M and V is a vertical vector field on $\mathcal{Z}$, then*

$$(D_{X^h} Y^h)_\sigma = (\nabla_X Y)^h_\sigma + \frac{1}{2} R_p(X \wedge Y)\sigma, \qquad (3.1.7)$$

$$(D_V X^h)_\sigma = \mathcal{H}(D_{X^h} V)_\sigma = -\frac{t}{2} (R_p(\sigma \times V) X)^h_\sigma \qquad (3.1.8)$$

where $\sigma \in \mathcal{Z}$, $p = \pi(\sigma)$ and $\mathcal{H}$ means "the horizontal component".

Proof. Identity (3.1.7) follows from the Koszul formula for the Levi-Civita connection and (3.1.6).

Let W be a vertical vector field on $\mathcal{Z}$. Then

$$h_t(D_V X^h, W) = -h_t(X^h, D_V W) = 0$$

since the fibres are totally geodesic submanifolds, so $D_V W$ is a vertical vector field. Therefore $D_V X^h$ is a horizontal vector field. Moreover, $[V, X^h]$ is a vertical vector field, hence $D_V X^h = \mathcal{H} D_{X^h} V$. Then

$$h_t(D_V X^h, Y^h) = h_t(D_{X^h} V, Y^h) = -h_t(V, D_{X^h} Y^h).$$

Now (3.1.8) follows from (3.1.7) and (3.1.1). □

Let $\Omega(A, B) = h_t(\mathcal{J}_f A, B)$ be the Kähler 2-form of the almost Hermitian manifold $(\mathcal{Z}, h_t, \mathcal{J}_f)$. We now compute the covariant derivative of Ω.

Proposition 3.1.2. *Let $\sigma \in \mathcal{Z}$, $X, Y, Z \in T_p M$, $p = \pi(\sigma)$, $U, V, W \in \mathcal{V}_\sigma$. Then*

$$(D_{X^h_\sigma}\Omega)(Y^h_\sigma, Z^h_\sigma) = 2g(\mathcal{V} f_*(X^h_\sigma), Y \wedge Z); \tag{3.1.9}$$

$$(D_{X^h_\sigma}\Omega)(Y^h_\sigma, U) = -\frac{t}{2} g(\mathcal{R}(U), X \wedge Y) + \frac{t}{2} g(\mathcal{R}(\sigma \times U), X \wedge K_{f(\sigma)} Y); \tag{3.1.10}$$

$$(D_U\Omega)(Y^h_\sigma, Z^h_\sigma) = -\frac{t}{2} g(\mathcal{R}(\sigma \times U), Y \wedge K_{f(\sigma)} Z + K_{f(\sigma)} Y \wedge Z) + 2g(f_* U, Y \wedge Z); \tag{3.1.11}$$

$$(D_{X^h_\sigma}\Omega)(U, V) = 0, \quad (D_U\Omega)(Y^h_\sigma, V) = 0, \quad (D_U\Omega)(V, W) = 0. \tag{3.1.12}$$

Proof. Extend the tangent vectors Y, Z to vector fields in a neighbourhood of p such that $\nabla Y|_p = \nabla Z|_p = 0$.

To prove the first formula, we note that

$$(D_{X^h_\sigma}\Omega)(Y^h_\sigma, Z^h_\sigma) = X^h_\sigma(h_t(\mathcal{J}_f Y^h, Z^h)) - h_t(\mathcal{J}_f D_{X^h_\sigma} Y^h, Z^h) + h_t(Y^h, \mathcal{J}_f D_{X^h_\sigma} Z^h).$$

The vectors $\mathcal{J}_f D_{X^h_\sigma} Y^h$ and $\mathcal{J}_f D_{X^h_\sigma} Z^h$ are vertical in view of (3.1.7). Hence

$$(D_{X^h_\sigma}\Omega)(Y^h_\sigma, Z^h_\sigma) = X^h_\sigma(h_t(\mathcal{J}_f Y^h, Z^h)).$$

Let s be a section of the bundle $\mathcal{Z}$ around p such that $s(p) = \sigma$ and $\nabla s|_p = 0$. Then

$$\begin{aligned} X^h_\sigma(h_t(\mathcal{J}_f Y^h, Z^h)) &= X(h_t(\mathcal{J}_f Y^h, Z^h) \circ s) \\ &= X(g(K_{f(s)}Y, Z)) \\ &= 2X(g(f \circ s, Y \wedge Z)) \\ &= 2g(\nabla_X(f \circ s), Y \wedge Z). \end{aligned}$$

The map $\widetilde{s} = f \circ s$ is a section of $\mathcal{Z}$ with $\widetilde{s}(p) = f(\sigma)$. Then

$$X^h_{f(\sigma)} + \nabla_X(f \circ s) = \widetilde{s}_*(X) = f_{*\sigma}(s_{*p}(X)) = f_*(X^h_\sigma).$$

Therefore

$$\nabla_X(f \circ s) = \mathcal{V} f_*(X^h_\sigma).$$

Thus

$$(D_{X^h_\sigma}\Omega)(Y^h_\sigma, Z^h_\sigma) = 2g(\mathcal{V} f_*(X^h_\sigma), Y \wedge Z).$$

Extend U to a vertical vector field in a neighbourhood of σ. Identities (3.1.7) and (3.1.8) imply that

$$\begin{aligned} (D_{X^h_\sigma}\Omega)(Y^h_\sigma, U) &= -h_t(\mathcal{J}_f D_{X^h_\sigma} Y^h, U) - h_t(\mathcal{J}_f Y^h_\sigma, D_{X^h_\sigma} U) \\ &= -\frac{t}{2} g(\sigma \times R(X,Y)\sigma, U) + \frac{t}{2} g(K_{f(\sigma)}Y, R(\sigma \times U)X) \end{aligned}$$

This gives the second formula of the lemma since, in view of (3.1.1),

$$g(\sigma \times R(X,Y)\sigma, U) = -g(R(X,Y)\sigma, \sigma \times U) = g(\mathcal{R}(U), X \wedge Y).$$

Next, we have

$$(D_U\Omega)(Y^h_\sigma, Z^h_\sigma) = U(h_t(\mathcal{J}_f Y^h, Z^h)) + h_t(D_U Y^h, \mathcal{J}_f Z^h) - h_t(\mathcal{J}_f Y^h, D_U Z^h). \quad (3.1.13)$$

Moreover, $f = \sum_{i=1}^{3}(y_i \circ f)(s_i \circ \pi)$, therefore

$$U(h_t(\mathcal{J}_f Y^h, Z^h)) = 2\sum_{i=1}^{3} U((y_i \circ f)g(s_i, Y \wedge Z) \circ \pi) = 2\sum_{i=1}^{3} U(y_i \circ f)g(s_i, Y \wedge Z)_p.$$

The map f sends fibres to fibres, hence f_* sends vertical vectors to vertical vectors. In particular,

$$f_* U = \sum_{i=1}^{3} U(y_i \circ f)(\frac{\partial}{\partial y_i})_{f(\sigma)}.$$

It follows that

$$U(h_t(\mathcal{J}_f Y^h, Z^h)) = 2g(f_* U, Y \wedge Z)$$

and the third formula of the lemma follows from (3.1.13) and (3.1.8).

To prove the remaining formulas fix a point $\sigma \in \mathcal{Z}$ and set $p = \pi(\sigma)$. Take an oriented orthonormal frame $(E_1, ..., E_4)$ of M around the point p such that $\nabla E_\alpha|_p = 0$, $\alpha = 1, ..., 4$, and define an oriented orthonormal frame (s_1, s_2, s_3) of $\Lambda^2_+ TM$ by means of (1.1.1). We have $\nabla s_i|_p = 0$, $i = 1, 2, 3$, for the latter frame. Choose also a local coordinate system $(x_1, ..., x_4)$ of M near p, then define local coordinates $(\widetilde{x}_\alpha, y_i)$, $\alpha = 1, ..., 4$, $i = 1, 2, 3$, on $\mathcal{Z}$ as above.

Every section a of $\Lambda^2_+ TM$ on an open set G gives rise to a vertical vector field $\widetilde{a}$ on $\pi^{-1}(G)$ defined by

$$\widetilde{a}_\tau = a \circ \pi(\tau) - g(a \circ \pi(\tau), \tau)\tau, \quad \tau \in \pi^{-1}(G).$$

Note that, around every point of $\mathcal{Z}$, there exists a frame of vertical vector fields of this type. Further on, we shall use this notation without explicitly saying so.

Now take sections a and b of $\mathcal{Z}$ defined in a neighbourhood of $p = \pi(\sigma)$ and such that

$$a(p) = U, b(p) = V, \nabla a|_p = \nabla b|_p = 0.$$

Let $\widetilde{a}$ and $\widetilde{b}$ be the vertical vector fields associated to a and b. Then

$$\widetilde{a}_\sigma = U, \widetilde{b}_\sigma = V$$

and

$$(D_{X^h_\sigma}\Omega)(U,V) = X^h_\sigma(h_t(\mathcal{J}_f\widetilde{a},\widetilde{b})) - h_t(\mathcal{J}_f D_{X^h_\sigma}\widetilde{a}, V) + h_t(U, \mathcal{J}_f D_{X^h_\sigma}\widetilde{b}). \tag{3.1.14}$$

Set

$$\widetilde{a} = \sum_{i=1}^{3} \widetilde{a}_i \frac{\partial}{\partial y_i}, \quad \widetilde{b} = \sum_{i=1}^{3} \widetilde{b}_i \frac{\partial}{\partial y_i}.$$

Then

$$\widetilde{a}_i = \sum_{j=1}^{3} (\delta_{ij} - y_i y_j)(g(a, s_j) \circ \pi), \tag{3.1.15}$$

and similar for $\widetilde{b}_i$. Moreover,

$$\mathcal{J}_f\widetilde{a} = (y_2\widetilde{a}_3 - y_3\widetilde{a}_2)\frac{\partial}{\partial y_1} + (y_3\widetilde{a}_1 - y_1\widetilde{a}_3)\frac{\partial}{\partial y_2} + (y_1\widetilde{a}_2 - y_2\widetilde{a}_1)\frac{\partial}{\partial y_3}. \tag{3.1.16}$$

Hence

$$h_t(\mathcal{J}_f\widetilde{a},\widetilde{b}) = (y_2\widetilde{a}_3 - y_3\widetilde{a}_2)\widetilde{b}_1 + (y_3\widetilde{a}_1 - y_1\widetilde{a}_3)\widetilde{b}_2 + (y_1\widetilde{a}_2 - y_2\widetilde{a}_1)\widetilde{b}_3.$$

If

$$X = \sum_{\alpha=1}^{4} X^\alpha (\frac{\partial}{\partial x_\alpha})_p,$$

we have

$$X^h_\sigma = \sum_{\alpha=1}^{4} X^\alpha (\frac{\partial}{\partial \widetilde{x}_\alpha})_\sigma,$$

hence

$$X^h_\sigma(\widetilde{a}_i) = \sum_{j=1}^{3} (\delta_{ij} - y_i y_j) X(g(a, s_j)) = 0$$

since $\nabla_X a = \nabla_X s_j = 0$. Similarly,

$$X^h_\sigma(\widetilde{b}_i) = 0, i = 1, 2, 3.$$

It follows that

$$X^h_\sigma(h_t(\mathcal{J}_f\widetilde{a},\widetilde{b}))=0.$$

Using (3.1.4) and (3.1.15), one obtains by a straightforward computation that

$$[X^h,\widetilde{a}]_\sigma=(\widetilde{\nabla_X a})_\sigma=0.$$

Hence

$$D_{X^h_\sigma}\widetilde{a}=-D_{\widetilde{a}_\sigma}X^h\in\mathcal{H}_\sigma$$

in view of (3.1.8). Then

$$h_t(\mathcal{J}_f D_{X^h_\sigma}\widetilde{a},V)=0.$$

Similarly

$$h_t(U,\mathcal{J}_f D_{X^h_\sigma}\widetilde{b})=0.$$

Thus,

$$(D_{X^h_\sigma}\Omega)(U,V)=0$$

by (3.1.14). Also

$$(D_U\Omega)(Y^h_\sigma,V)=U(h_t(\mathcal{J}_fY^h,\widetilde{b}))-h_t(\mathcal{J}_fD_UY^h,V)-h_t(\mathcal{J}_fY^h,D_U\widetilde{b})=0$$

since $\mathcal{J}_fY^h$, $\mathcal{J}_fD_UY^h$ are horizontal vectors and $D_U\widetilde{b}$ is vertical.

Finally, the identity $(D_U\Omega)(V,W)=0$ is a consequence of the fact that the fibres of $\mathcal{Z}$ are totally geodesic submanifolds and $\mathcal{J}_f$ preserves the vertical distribution. □

Proposition 3.1.2 and the well known formula (c.f. [37])

$$d\Omega(A,B,C)=\underset{A,B,C}{\mathfrak{S}}(D_A\Omega)(B,C)$$

give the following

Corollary 3.1.3. *Let* $\sigma \in \mathcal{Z}$*,* $X, Y, Z \in T_pM$*,* $p = \pi(\sigma)$*,* $U, V, W \in \mathcal{V}_\sigma$*. Then*

$$d\Omega(X^h, Y^h, Z^h) = 2g(\mathcal{V}f_*(X^h), Y \wedge Z) + 2g(\mathcal{V}f_*(Y^h), Z \wedge X) + 2g(\mathcal{V}f_*(Z^h), X \wedge Y)$$

$$d\Omega(X^h, Y^h, U) = g(2f_*U - t\mathcal{R}(U), X \wedge Y)$$

$$d\Omega(X^h, U, V) = 0, \quad d\Omega(U, V, W) = 0.$$

Corollary 3.1.4. *Let* $\sigma \in \mathcal{Z}$*,* $X \in T_pM$*,* $p = \pi(\sigma)$*,* $U \in \mathcal{V}_\sigma$*. Then*

$$\begin{aligned} (\delta\Omega)(X^h_\sigma) &= Trace\{T_pM \ni A \to 2g(\mathcal{V}f_*(A^h_\sigma), X \wedge A)\} \\ &= Trace\{\mathcal{V}_\sigma \ni \tau \to g(\mathcal{V}f_*((K_\tau X)^h_\sigma), \tau)\}. \end{aligned} \tag{3.1.17}$$

$$\delta\Omega(U) = -tg(\mathcal{R}(\sigma \times U), f(\sigma)). \tag{3.1.18}$$

Proof. Let $E_1, ..., E_4$ be an orthonormal basis of T_pM, $p = \pi(\sigma)$ and τ_1, τ_2 a g-orthonormal basis of $\mathcal{V}_\sigma$. Then, by (3.1.9) and (3.1.12),

$$\begin{aligned} \delta\Omega(X^h_\sigma) &= -\sum_{i=1}^{4}(D_{(E_i)^h_\sigma}\Omega)((E_i)^h_\sigma, X^h_\sigma) - \sum_{m=1}^{2}(D_{\tau_m}\Omega)(\tau_m, X^h_\sigma) \\ &= -2\sum_{i=1}^{4} g(\mathcal{V}f_*((E_i)^h_\sigma), E_i \wedge X) \\ &= -2\sum_{i=1}^{4}\sum_{m=1}^{2} g(f_*((E_i)^h_\sigma), \tau_m)g(\tau_m, E_i \wedge X) \\ &= \sum_{i=1}^{2}\sum_{m=1}^{2} g(f_*((E_i)^h_\sigma), \tau_m)g(K_{\tau_m}X, E_i) \\ &= \sum_{m=1}^{2} g(f_*((K_{\tau_m}X)^h_\sigma), \tau_m). \end{aligned}$$

This proves (3.1.17). In view of (3.1.10) and (3.1.12), we have

$$\delta\Omega(U) = -\frac{t}{2}g(\mathcal{R}(\sigma \times U), \sum_{i=1}^{4} E_i \wedge K_{f(\sigma)}E_i)$$

Moreover, for $Y, Z \in T_pM$,

$$\begin{aligned} \sum_{i=1}^{4} g(E_i \wedge K_{f(\sigma)}E_i, Y \wedge Z) &= \frac{1}{2}\sum_{i=1}^{4}[-g(Y, E_i)g(K_{f(\sigma)}Z, E_i) + g(Z, E_i)g(K_{f(\sigma)}Y, E_i)] \\ &= g(K_{f(\sigma)}Y, Z) \\ &= 2g(f(\sigma), Y \wedge Z). \end{aligned}$$

Thus

$$\sum_{i=1}^{4} E_i \wedge K_{f(\sigma)} E_i = 2f(\sigma)$$

and the second formula of the corollary is proved. $\square$

Denote the Nijenhuis tensor of $\mathcal{J}_f$ by N. The next statement follows from Proposition 3.1.2, identity (3.1.2) and the well-known formula

$$h_t(N(A,B),C) = (D_A\Omega)(\mathcal{J}_f B, C) - (D_{\mathcal{J}_f B}\Omega)(A,C) - (D_B\Omega)(\mathcal{J}_f A, C) + (D_{\mathcal{J}_f A}\Omega)(B,C).$$

Corollary 3.1.5. *Let $\sigma \in \mathcal{Z}$, $X, Y, Z \in T_pM$, $p = \pi(\sigma)$, $U, V \in \mathcal{V}_\sigma$. Then*

$$\begin{aligned} h_t(N(X^h_\sigma, Y^h_\sigma), Z^h_\sigma) &= 2g(\mathcal{V}f_*(X^h_\sigma), K_{f(\sigma)}Y \wedge Z) - 2g(\mathcal{V}f_*(Y^h_\sigma), K_{f(\sigma)}X \wedge Z) \\ &+ 2g(\mathcal{V}f_*((K_{f(\sigma)}X)^h_\sigma), Y \wedge Z) - 2g(\mathcal{V}f_*((K_{f(\sigma)}Y)^h_\sigma), X \wedge Z) \end{aligned}$$

$$\begin{aligned} h_t(N(X^h_\sigma, Y^h_\sigma), U) &= -tg(\mathcal{R}(X \wedge K_{f(\sigma)}Y + K_{f(\sigma)}X \wedge Y), U) \\ &- tg(\mathcal{R}(X \wedge Y - K_{f(\sigma)}X \wedge K_{f(\sigma)}Y), \sigma \times U) \end{aligned}$$

$$h_t(N(X^h_\sigma, U), Z^h_\sigma) = -2g(f(\sigma) \times f_*(U), X \wedge Z) + 2g(f_*(\sigma \times U), X \wedge Z)$$

$$h_t(N(X^h_\sigma, U), V) = 0 \quad N(U,V) = 0.$$

Since

$$h_t(N(X^h, U), Z^h) = -2g(\mathcal{J}_f f_*(U), X \wedge Z) + 2g(f_*(\mathcal{J}_f U), X \wedge Z),$$

we have the following.

Corollary 3.1.6. *([22]) $\mathcal{H}(N(X^h, U)) = 0$ if and only if the restriction of f to every fibre is a holomorphic map.*

3.2 Gray-Hervella classes of the almost complex structures $\mathcal{J}_\omega$

In what follows we use the same notation for the Gray-Hervella classes as in [29]. For example, $\mathcal{K}$ is the class of Kähler manifolds, $\mathcal{W}_1$ is the class of nearly Kähler manifolds, $\mathcal{W}_2$ is the class of almost Kähler manifolds, $\mathcal{W}_3 \oplus \mathcal{W}_4$ is the class of Hermitian manifolds, $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ is the class of semi-Kähler or balanced manifolds, etc.

Let (g, J) be an almost Hermitian structure on a four-manifold M. Define a section ω of $\Lambda^2 TM$ by

$$g(\omega, X \wedge Y) = \frac{1}{2} g(JX, Y), \ \ X, Y \in TM.$$

Clearly, at any point, ω is the dual 2-vector of one half of the Kähler 2-form F of the almost Hermitian manifold (M, g, J). Consider M with the orientation yielded by the almost complex structure J. Then ω is a section of the twistor bundle $\mathcal{Z}$. As in [22], define a bundle map $f : \mathcal{Z} \to \mathcal{Z}$ setting $f = \omega \circ \pi$. Since the restriction of f to any fibre is a constant map, $f_*|\mathcal{V} = 0$. We also have

$$f_*(X^h_\sigma) = X^h_{\omega(p)} + \nabla_X \omega, \tag{3.2.1}$$

where $p = \pi(\sigma)$ and $X \in T_pM$. Note that

$$2g(\nabla_X \omega, Y \wedge Z) = (\nabla_X F)(Y, Z).$$

Denote by $\mathcal{J}_\omega$ the almost complex structure on $\mathcal{Z}$ determined by the map f defined by ω (in the notation of Introduction $\mathcal{J}_\omega = \mathcal{J}_{f_0^-}$)

In the next theorem we determine the Gray-Hervella classes of the almost Hermitian manifolds $(\mathcal{Z}, h_t, \mathcal{J}_\omega)$.

Theorem 3.2.1. *Let (M, g, J) be an almost Hermitian 4-manifold with Kähler 2-vector ω, self-dual Weyl tensor W_+ and scalar curvature s. The possible Gray-Hervella classes of its twistor space $(\mathcal{Z}, h_t, \mathcal{J}_\omega)$ are $\mathcal{W}$, $\mathcal{K}$, $\mathcal{W}_3$, $\mathcal{H} = \mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{S}K = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$, $\mathcal{G}_1 = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ and $\mathcal{G}_2 = \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$. Moreover*

(i) $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{K}$ if and only if (M, g, J) is Kähler and Ricci flat.

(ii) $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{S}K \cap \mathcal{H} = \mathcal{W}_3$ if and only if (M, g, J) is Kähler and scalar flat.

(iii) $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{H} = \mathcal{W}_3 \oplus \mathcal{W}_4$ if and only if (M, g, J) is Hermitian and

$$\mathcal{W}_+(\sigma) = \frac{s}{2} g(\sigma, \omega)\omega - \frac{s}{6}\sigma$$

for all $\sigma \in \Lambda^2_+ TM$.

(iv) $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{S}K = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ if and only if (M, g, J) is almost Kähler and

$$\mathcal{W}_+(\omega) = -\frac{s}{6}\omega.$$

(v) $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{G}_1 = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ if and only if (M, g, J) is Hermitian.

(vi) $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{G}_2 = \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ if and only if

$$\mathcal{W}_+(\sigma) = \frac{s}{2} g(\sigma, \omega)\omega - \frac{s}{6}\sigma$$

for all $\sigma \in \Lambda^2_+ TM$.

Proof. To determine the possible Gray-Hervella classes of the twistor space $(\mathcal{Z}, h_t, \mathcal{J}_\omega)$ we shall need several technical lemmas.

Given a point $\sigma \in \mathcal{Z}$, we take a basis $E_1, E_2 = JE_1, E_3, E_4 = JE_3$ of $T_{\pi(\sigma)}M$. Such a basis induces the orientation of M we have chosen and we define s_1, s_2, s_3 and $\bar{s}_1, \bar{s}_2, \bar{s}_3$ via (1.1.1) (so $\omega = s_1$). This notation will be used in the proofs of the next statements.

Lemma 3.2.2. *$(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{K}$ if and only if (M, g, J) is Kähler and Ricci flat.*

Proof. It follows from Proposition 3.1.2 and (3.2.1) that $(\mathcal{Z}, h_t, \mathcal{J}_\omega)$ is Kähler if and only if (M, g, J) is Kähler and for every $\sigma \in \mathcal{Z}$, $U \in \mathcal{V}_\sigma$ and $X, Y \in T_{\pi(\sigma)}M$

(i) $-g(\mathcal{R}(U), X \wedge Y) + g(\mathcal{R}(\sigma \times U), X \wedge JY) = 0,$

(ii) $g(\mathcal{R}(U), X \wedge JY + JX \wedge Y) = 0.$

The latter identity implies

$$g(\mathcal{R}(U), s_2) = g(\mathcal{R}(U), s_3) = 0. \tag{3.2.2}$$

It follows from identity (i) that

$$g(\mathcal{R}(U), E_1 \wedge E_2) = g(\mathcal{R}(U), E_3 \wedge E_4) = 0, \tag{3.2.3}$$

$$\begin{aligned} g(\mathcal{R}(U), E_1 \wedge E_3) &= g(\mathcal{R}(\sigma \times U), E_1 \wedge E_4), \\ g(\mathcal{R}(U), E_3 \wedge E_1) &= g(\mathcal{R}(\sigma \times U), E_3 \wedge E_2). \end{aligned} \tag{3.2.4}$$

We obtain from (3.2.3) that

$$g(\mathcal{R}(U), \bar{s}_1) = g(\mathcal{R}(U), s_1) = 0.$$

Thus $g(\mathcal{R}(U), s_i) = 0$ for $i = 1, 2, 3$ and every $U \in \Lambda^2_+ T_p M$.

It follows from (3.2.4) that

$$g(\mathcal{R}(\sigma \times U), \bar{s}_3) = 0.$$

Moreover, identities (3.2.2) and (3.2.4) imply

$$\begin{aligned} g(\mathcal{R}(U), \bar{s}_2) &= 2g(\mathcal{R}(U), E_1 \wedge E_3) \\ &= 2g(\mathcal{R}(\sigma \times U), E_1 \wedge E_4) \\ &= g(\mathcal{R}(\sigma \times U), \bar{s}_3). \end{aligned}$$

Therefore

$$g(\mathcal{R}(U), \bar{s}_2) = g(\mathcal{R}(U), \bar{s}_3) = 0,$$

thus

$$g(\mathcal{R}(U), \bar{s}_i) = 0, i = 1, 2, 3.$$

Hence $\mathcal{R}(U) = 0$ for every $U \in \Lambda^2_+ T_pM$. This shows that if $(\mathcal{Z}, h_t, \mathcal{J}_\omega)$ is Kähler, then (M, g, J) is a Kähler and Ricci flat manifold.

Conversely, suppose that (M, g, J) is Kähler and Ricci flat. Using the curvature decomposition (1.1.2), the Kähler curvature identities and the first Bianchi identity, one can see that

$$g(\mathcal{R}(s_1), s_1) = \frac{s}{3} s_1, \quad \mathcal{R}(s_2) = \mathcal{R}(s_3) = 0.$$

This implies the well-known fact (which can be traced back to [26]) that the eigenvalues of the operator $\mathcal{W}_+$ on a Kähler surface are $\frac{s}{3}, -\frac{s}{6}, -\frac{s}{6}$. It follows that $\mathcal{R}(U) = 0$ for every $U \in \Lambda^2_+ T_pM$, thus identities (i) and (ii) obviously are satisfied. □

Lemma 3.2.3. $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ *if and only if* $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{K}$.

Proof. The condition for $(\mathcal{Z}, h_t, \mathcal{J}_\omega)$ to be in the class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ is

$$\begin{gathered}(D_A\Omega)(B, C) + (D_{\mathcal{J}_\omega A}\Omega)(\mathcal{J}_\omega B, C) = \\ -\tfrac{1}{2}\{h_t(A, B)\delta\Omega(C) - h_t(A, C)\delta\Omega(B) - h_t(A, \mathcal{J}_\omega B)\delta\Omega(\mathcal{J}_\omega C) \\ +h_t(A, \mathcal{J}_\omega C)\delta\Omega(\mathcal{J}_\omega B)\}\end{gathered} \tag{3.2.5}$$

for every $A, B, C \in T\mathcal{Z}$. Proposition 3.1.2 and (3.2.1) imply that this condition is satisfied if and only if for every $\sigma \in \mathcal{Z}$, $X, Y, Z \in T_{\pi(\sigma)}M$ and $U, V, W \in \mathcal{V}_\sigma$ we have

(i) $$\begin{gathered}(\nabla_X F)(Y, Z) + (\nabla_{JX} F)(JY, Z) = \\ -\tfrac{1}{2}\{g(X, Y)\delta F(Z) - g(X, Z)\delta F(Y) - g(X, JY)\delta F(JZ) + g(X, JZ)\delta F(JY)\},\end{gathered}$$

(ii) $$\begin{gathered}-g(\mathcal{R}(U), X \wedge Y + JX \wedge JY) + g(\mathcal{R}(\sigma \times U), X \wedge JY - JX \wedge Y) = \\ g(X, Y)g(\mathcal{R}(\sigma \times U), \omega) + g(X, JY)g(\mathcal{R}(U), \omega),\end{gathered}$$

(iii) $g(\mathcal{R}(U), X \wedge Y - JX \wedge JY) + g(\mathcal{R}(\sigma \times U), X \wedge JY + JX \wedge Y) = 0,$

(iv) $g(U, V)\delta F(X) - g(U, \sigma \times V)\delta F(JX) = 0,$

(v) $$\begin{gathered}g(U, V)g(\mathcal{R}(\sigma \times W), \omega) - g(U, W)g(\mathcal{R}(\sigma \times V), \omega) + \\ g(U, \sigma \times V)g(\mathcal{R}(W), \omega) - g(U, \sigma \times W)g(\mathcal{R}(V), \omega) = 0.\end{gathered}$$

Clearly, identity (v), obtained from (3.2.5) for vertical vectors $A = U$, $B = V$, $C = W$, holds when $U = 0$. If $U \neq 0$, then $U, \sigma \times U$ is a basis of $\mathcal{V}_\sigma$ and it is easy to check that this identity is also satisfied. Thus identity (v) does not impose any restriction on the base manifold M. Identity (iv) implies that $\delta F = 0$. Then it follows from (i) that $(\nabla_X F)(Y, Z) + (\nabla_{JX} F)(JY, Z) = 0$. It is well-known (and easy to see) that, in dimension 4, the latter identity is equivalent to $dF = 0$. Take a point $p \in M$ and let $X \in T_pM$ be a unit vector. For every point $\sigma \in \mathcal{Z}$ with $\pi(\sigma) = p$ and every $U \in \mathcal{V}_\sigma$, identity (ii) with $Y = JX$ gives $2g(\mathcal{R}(U), X \wedge JX) = g(\mathcal{R}(U), s_1)$. Thus we have $2g(\mathcal{R}(U), E_1 \wedge E_2) = g(\mathcal{R}(U), s_1)$ and $2g(\mathcal{R}(U), E_3 \wedge E_4) = g(\mathcal{R}(U), s_1)$. This implies

$$g(\mathcal{R}(U), E_1 \wedge E_2) = g(\mathcal{R}(U), E_3 \wedge E_4) = g(\mathcal{R}(U), s_1) = 0$$

since $s_1 = E_1 \wedge E_2 + E_3 \wedge E_4$. It follows that

$$g(\mathcal{R}(U), \bar{s}_1) = 0, \quad g(\mathcal{R}(s_1), s_1) = g(\mathcal{R}(s_2), s_1) = g(\mathcal{R}(s_3), s_1) = 0. \tag{3.2.6}$$

Identity (iii) with $X = E_1$, $Y = E_3$ becomes

$$g(\mathcal{R}(U), s_2) + g(\mathcal{R}(\sigma \times U), s_3) = 0, \quad U \in \mathcal{V}_\sigma.$$

Applying the latter identity for $\sigma = s_2$ and $\sigma = s_3$ and taking into account (3.2.6) we see that

$$g(\mathcal{R}(s_2), s_2) = g(\mathcal{R}(s_2), s_3) = g(\mathcal{R}(s_3), s_3) = 0.$$

It follows that $g(\mathcal{R}(s_i), s_j) = 0$, $i, j = 1, 2, 3$. This means that (M, g) is anti-self-dual with zero scalar curvature.

Since $g(\mathcal{R}(U), \omega) = 0$ for every vertical vector U, identity (ii) takes the form

$$g(\mathcal{R}(U), X \wedge Y + JX \wedge JY) - g(\mathcal{R}(\sigma \times U), X \wedge JY - JX \wedge Y) = 0.$$

Setting in this identity $(X, Y) = (E_1, E_3)$ and $(X, Y) = (E_3, E_1)$ we obtain

$$g(\mathcal{R}(U), \bar{s}_2) - g(\mathcal{R}(\sigma \times U), \bar{s}_3) = 0, \quad g(\mathcal{R}(U), \bar{s}_2) + g(\mathcal{R}(\sigma \times U), \bar{s}_3) = 0.$$

This, together with (3.2.6), implies $g(\mathcal{R}(U), s_j^-) = 0$, $j = 1, 2, 3$. Thus

$$g(\mathcal{R}(s_i), \bar{s}_j) = 0, \quad i, j = 1, 2, 3$$

which means that $\mathcal{B} = 0$. We note also that, since $dim\, M = 4$, $dF = \theta \wedge F$, where $\theta = \delta F \circ J$, so the identity $\delta F = 0$ is equivalent to $dF = 0$, i.e. to (M, g, J) being almost Kähler. It follows that if $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in W_1 \oplus W_2 \oplus W_4$, then (M, g, J) is almost Kähler, anti-self-dual and Ricci flat manifold. According to [6, Propostion 1] these conditions are equivalent to the base manifold being Kähler and Ricci flat. For such a manifold we have $\nabla F = \delta F = 0$ and $\mathcal{R}(U) = 0$ for every vertical vector U, thus conditions (i) – (iv) are clearly satisfied. Now the proof of lemma follows from Lemma 3.2.2. □

Lemma 3.2.4. $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{S}K = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ *if and only if* (M, g, J) *is almost Kähler and* $\mathcal{W}_+(\omega) = -\frac{s}{6}\omega$.

Proof. The defining condition for the class of semi-Kähler manifolds is $\delta\Omega = 0$. According to Corollary 3.1.4 and (3.2.1), the twistor space is semi-Kähler if and only if $g(\delta\omega, X) = 0$, i.e. $\delta F(X) = 0$ and $g(\mathcal{R}(U), \omega \circ \pi(\sigma)) = 0$ for every $\sigma \in \mathcal{Z}$, $U \in \mathcal{V}_\sigma$, $X \in T_{\pi(\sigma)}M$. As we have mentioned the identity $\delta F = 0$ is equivalent to $dF = 0$ since $dim\, M = 4$. The identity $g(\mathcal{R}(U), \omega \circ \pi(\sigma)) = 0$ for all $U \in \mathcal{V}_\sigma$ holds if and only if $g(\mathcal{R}(\omega), s_i) = 0, i = 1, 2, 3$. This is equivalent to $\frac{s}{6}\omega + \mathcal{W}_+(\omega) = 0$. □

Lemma 3.2.5. $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{G}_1 = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ *if and only if the almost complex structure* J *is integrable.*

Proof. $(\mathcal{Z}, h_t, \mathcal{J}_\omega)$ belongs to the class $\mathcal{G}_1$ when

$$(D_A\Omega)(A,B) - (D_{\mathcal{J}_\omega A})(\mathcal{J}_\omega A, B) = 0, \quad A, B \in T\mathcal{Z}. \tag{3.2.7}$$

It follows from Proposition 3.1.2 and (3.2.1) that this condition holds if and only if for every $X, Y \in TM$

$$(\nabla_X F)(X,Y) - (\nabla_{JX} F)(JX, Y) = 0.$$

In dimension 4, the latter identity is equivalent to J being integrable. □

Lemma 3.2.6. $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{G}_2 = \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ *if and only if*

$$\mathcal{W}_+(\sigma) = \frac{s}{2} g(\sigma, \omega)\omega - \frac{s}{6}\sigma$$

for all $\sigma \in \Lambda^2_+ TM$.

Proof. The condition for $(\mathcal{Z}, h_t, \mathcal{J}_\omega)$ to be in the class $\mathcal{G}_2$ is

$$\underset{A,B,C}{\mathfrak{S}} \{(D_A\Omega)(B,C) - (D_{\mathcal{J}_\omega A}\Omega)(\mathcal{J}_\omega B, C)\} = 0. \tag{3.2.8}$$

By Proposition 3.1.2 and (3.2.1) this is equivalent to the following identities

(i) $\underset{X,Y,Z}{\mathfrak{S}} \{(\nabla_X F)(Y,Z) - (\nabla_{JX} F)(JY, Z)\} = 0, \quad X, Y, Z \in TM.$

(ii) $g(\mathcal{R}(U), X \wedge Y - JX \wedge JY) - g(\mathcal{R}(\sigma \times U), X \wedge JY + JX \wedge Y) = 0,$

for every $\sigma \in \mathcal{Z}$, $U \in \mathcal{V}_\sigma$, $X, Y \in T_{\pi(\sigma)}M$. Identity (i) is always satisfied in dimension four. Identity (ii) gives

$$g(\mathcal{R}(U), s_2) - g(\mathcal{R}(\sigma \times U), s_3) = 0.$$

Applying the latter identity for $\sigma = s_1, s_2, s_3$, it is easy to see that

$$g(\mathcal{R}(s_i), s_j) = 0 \text{ for } (i,j) \neq (1,1).$$

The curvature decomposition and the fact that $Trace\,\mathcal{W}_+ = 0$ then imply

$$\frac{s}{6} + g(\mathcal{W}_+(\omega), \omega) = g(\mathcal{R}(\omega), \omega) = \frac{s}{2}.$$

Thus the matrix of $\mathcal{W}_+$ with respect to the basis $s_1 = \omega, s_2, s_3$ is diagonal with diagonal entries $\frac{s}{3}, -\frac{s}{6}, -\frac{s}{6}$. Therefore

$$\mathcal{W}_+(\sigma) = \frac{s}{2} g(\sigma, \omega)\omega - \frac{s}{6}\sigma.$$

Conversely, suppose that this identity is fulfilled. Then

$$\mathcal{R}(\sigma) = \frac{s}{2} g(\sigma, \omega)\omega + \mathcal{B}(\sigma).$$

It is easy to check that if $\sigma \in \Lambda^2_+ T_pM$ and $\tau \in \Lambda^2_- T_pM$, the endomorphisms K_σ and K_τ of T_pM commute, $K_\sigma \circ K_\tau = K_\tau \circ K_\sigma$. This implies that, for every $X, Y \in T_pM$, the 2-vector $X \wedge Y - K_\sigma X \wedge K_\sigma Y$ is orthogonal to $\Lambda^2_- T_pM$, so it lies in $\Lambda^2_+ T_pM$. In particular,

$$g(\mathcal{B}(\sigma), X \wedge Y - JX \wedge JY) = 0.$$

We also have

$$g(\omega, X \wedge Y - JX \wedge JY) = 0.$$

Thus

$$g(\mathcal{R}(\sigma), X \wedge Y - JX \wedge JY) = 0$$

for every $\sigma \in \mathcal{Z}$, $X, Y \in T_{\pi(\sigma)}M$. It follows that condition (ii) is satisfied, hence $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{G}_2$. □

Lemma 3.2.7. *$(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{W}_1 \oplus \mathcal{W}_3$ if and only if (M, g, J) is Kähler and scalar flat.*

Proof. Note that

$$\mathcal{W}_1 \oplus \mathcal{W}_3 = (\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3) \cap (\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4).$$

Hence it follows from Lemmas 3.2.4 and 3.2.5 that $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{W}_1 \oplus \mathcal{W}_3$ if and only if (M, g, J) is Kähler and $\mathcal{W}_+(\omega) = -\frac{s}{6}\omega$. But, as we have already mentioned, it is well-known that for Kähler manifolds $\mathcal{W}_+(\omega) = \frac{s}{3}\omega$ and the above identity implies that $s = 0$. The converse statement follows from the fact that a Kähler manifold is scalar flat if and only if it is anti-self-dual. □

Lemma 3.2.8. $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{W}_2 \oplus \mathcal{W}_3$ *if and only if* (M, g, J) *is Kähler and scalar flat.*

Proof. It follows from Lemmas 3.2.4 and 3.2.6 that $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{W}_2 \oplus \mathcal{W}_3$ if and only if (M, g, J) is almost Kähler, anti-self-dual and scalar flat. Now the lemma follows from Proposition 1 in [6] according to which these conditions are equivalent to the base manifold being Kähler and scalar flat. □

Lemma 3.2.9. $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{W}_3$ *if and only if* (M, g, J) *is Kähler and scalar flat.*

Proof. The lemma follows from Lemmas 3.2.7 and 3.2.8. □

Lemma 3.2.10. $(\mathcal{Z}, h_t, \mathcal{J}_\omega) \in \mathcal{H} = \mathcal{W}_3 \oplus \mathcal{W}_4$ *if and only if the almost complex structure* J *is integrable and*

$$\mathcal{W}_+(\sigma) = \frac{s}{2} g(\sigma, \omega)\omega - \frac{s}{6}\sigma$$

for all $\sigma \in \Lambda^2_+ TM$.

Proof. The proof follows from Lemmas 3.2.5 and 3.2.6. □

We are now ready to prove Theorem 3.2.1.

Proof of Theorem 3.2.1.

It follows from Lemmas 3.2.2 and 3.2.3 that

$$\mathcal{K} = \mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{W}_1 \oplus \mathcal{W}_4 = \mathcal{W}_2 \oplus \mathcal{W}_4 = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4.$$

Lemmas 3.2.7, 3.2.8 and 3.2.9 imply that

$$\mathcal{W}_3 = \mathcal{W}_1 \oplus \mathcal{W}_3 = \mathcal{W}_2 \oplus \mathcal{W}_3.$$

Hence the first part of the theorem follows from Lemmas 3.2.4, 3.2.5, 3.2.6 and 3.2.10.

The statements (i)–(vi) follow respectively from Lemmas 3.2.2, 3.2.9 , 3.2.10, 3.2.4, 3.2.5 and 3.2.6 . □

Remark.

Concerning the geometric conditions in Theorem 3.2.1 we note that:

1. Any compact Kähler and Ricci flat surface is either a complex torus, a hyper-elliptic surface with a flat metric, a $K3$-surface with a Calabi-Yau metric or its $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$ quotient.[11]

2. The spectrum of the anti-self-dual Weyl tensor $\mathcal{W}_+$ of a Hermitian surface M is equal to $(\frac{k}{3}, -\frac{k}{6}, -\frac{k}{6})$, where k is the conformal scalar curvature [7]. Hence the curvature condition in Theorem 3.2.1, (iii) implies that $k = s$, i.e. $\delta\theta = \|\theta\|^2$, where θ is the Lee form of M [7]. If M is compact, then integrating this identity and using Stokes' formula, we see that $\theta = 0$, i.e. the surface M is Kähler.

3. We do not know non-Kähler examples of compact almost Kähler 4-manifolds

whose anti-self-dual Weyl tensor $\mathcal{W}_+$ satisfies the condition of Theorem 3.2.1, (iv).

3.3 Gray-Hervella classes of the almost complex structures $\mathcal{J}_\lambda^\pm$

As in [22], in order to define a fibre-preserving map $f : \mathcal{Z} \to \mathcal{Z}$, we shall use the stereographic projection of every fibre $\mathcal{Z}_{\pi(\sigma)}$ from the point $\omega_{\pi(\sigma)}$ onto the plane $(\mathbb{R}\omega_{\pi(\sigma)})^\perp$, the orthogonal complement being taken in $\Lambda^2_+ T_{\pi(\sigma)}M$. This stereographic projection Φ_σ and its inverse Φ_σ^{-1} are given by

$$\Phi_\sigma(\tau) = \frac{\tau - g(\tau, \omega_{\pi(\sigma)})\omega_{\pi(\sigma)}}{1 - g(\tau, \omega_{\pi(\sigma)})}, \quad \tau \in \mathcal{Z}_{\pi(\sigma)} \setminus \{\omega_{\pi(\sigma)}\},$$
$$\Phi_\sigma^{-1}(\zeta) = \frac{2\zeta + ||\zeta||^2 - 1}{||\zeta||^2 + 1}, \quad \zeta \in (\mathbb{R}\omega_{\pi(\sigma)})^\perp$$

The map Φ_σ is holomorphic with respect to the standard complex structure of $\mathcal{Z}_{\pi(\sigma)}$ and the complex structure of $(\mathbb{R}\omega_{\pi(\sigma)})^\perp$ given by $\zeta \to \omega_{\pi(\sigma)} \times \zeta$ (the latter structure is compatible with the metric g of $\Lambda^2_+ T_{\pi(\sigma)}M$). As usual, we also set $\Phi_\sigma(\omega_{\pi(\sigma)}) = \infty$, the "ideal" element of the plane $(\mathbb{R}\omega_{\pi(\sigma)})^\perp$.

Let $\lambda = a + ib$ be a fixed complex number and set $F_\lambda(\zeta) = \lambda\zeta$ for $\zeta \in (\mathbb{R}\omega_{\pi(\sigma)})^\perp$. Then

$$f_\lambda^+(\sigma) = \Phi_\sigma^{-1} \circ F_\lambda \circ \Phi_\sigma(\sigma)$$

is a self-map of $\mathcal{Z}$ whose restriction to any fibre is holomorphic. Similarly, denote by Ψ_σ the stereographic projection of $\mathcal{Z}_{\pi(\sigma)}$ from the point $-\omega_{\pi(\sigma)}$ onto the plane $(\mathbb{R}\omega_{\pi(\sigma)})^\perp$. Set $f_\lambda^-(\sigma) = \Phi_\sigma^{-1} \circ F_\lambda \circ \Psi_\sigma(\sigma)$. In this way we obtain another self-map of $\mathcal{Z}$ whose restriction to any fibre is anti-holomorphic. Clearly, the points $f_\lambda^-(\sigma)$ and $f_\lambda^+(\sigma)$ are symmetric with respect to the plane $(\mathbb{R}\omega_{\pi(\sigma)})^\perp$.

The maps $f_\lambda^\pm : \mathcal{Z} \to \mathcal{Z}$ are given by the following explicit formulas:

$$f_\lambda^\pm(\sigma) = [(a^2+b^2+1)+(a^2+b^2-1)g(\sigma,\omega_{\pi(\sigma)})]^{-1}$$

$$\{2a\sigma - 2b\sigma\times\omega_{\pi(\sigma)} - 2ag(\sigma,\omega_{\pi(\sigma)})\omega_{\pi(\sigma)}$$

$$\pm[(a^2+b^2-1)+(a^2+b^2+1)g(\sigma,\omega_{\pi(\sigma)})]\omega_{\pi(\sigma)}\}.$$

Denote by $\mathcal{J}_\lambda^\pm$ the almost complex structure on $\mathcal{Z}$ defined by means of the map $f_\lambda^\pm$. Note that $f_0^\pm = \mp\omega$ and $\mathcal{J}_0^\pm$ is the almost complex structure on $\mathcal{Z}$ yielded by the almost complex structure $\mp J$ on M and discussed in the preceding section. The structure $\mathcal{J}_\lambda^+$ is denoted by $J_{\lambda Id}$ in [22] where the integrability condition for this structure is found when the base manifold (M,g,J) is Kähler. Note also that $f_1^+(\sigma)=\sigma$ and $\mathcal{J}_1^+$ is the Atiyah-Hitchin-Singer almost complex structure, whereas $f_{-1}^-(\sigma)=-\sigma$ and $\mathcal{J}_{-1}^-$ is the conjugate structure of the Eells-Salamon almost complex structure. The Gray-Hervella classes of these structures have been determined in [43].

Since the restrictions to the fibres of the map $f_\lambda^-(\sigma)$ are not holomorphic, Corollary 3.1.6 implies the following.

Corollary 3.3.1. *The almost complex structure $\mathcal{J}_\lambda^-$ is never integrable.*

In this section we shall discuss the possible Gray-Hervella classes of the almost Hermitian manifolds $(Z,h_t,\mathcal{J}_\lambda^\pm)$. To do this we need to compute $\mathcal{V}(f_\lambda^\pm)_*(X_\sigma^h)$, $X\in T_{\pi(\sigma)}M$. Taking a section s of Λ_+^2TM around the point $p=\pi(\sigma)$ such that $s(p)=\sigma$ and $\nabla s|_p=0$, we have $\mathcal{V}(f_\lambda^\pm)_*(X_\sigma^h)=\nabla_X(f_\lambda^\pm\circ s)$. Using this formula, we can get an explicit expression for $\mathcal{V}(f_\lambda^\pm)_*(X_\sigma^h)$ which simplifies considerably in the case when (M,g,J) is a Kähler manifold or when $|\lambda|=1$. In fact,

$$\mathcal{V}(f_\lambda^\pm)_*(X_\sigma^h)=0$$

in the first case and

$$\mathcal{V}(f_\lambda^\pm)_*(X_\sigma^h) = -b\sigma\times\nabla_X\omega + (\pm1-a)[g(\sigma,\nabla_X\omega)\omega_{\pi(\sigma)} + g(\sigma,\omega)\nabla_X\omega] \qquad (3.3.1)$$

in the case when $|\lambda| = 1$. Let us note that if $|\lambda| = 1$, say $\lambda = e^{i\theta}$, the point $f_\lambda^+(\sigma)$ is obtained by rotating σ around the line $\mathbb{R}\omega_{\pi(\sigma)}$ at angle θ.

We are now ready to prove the following

Theorem 3.3.2. *Let (M, g, J) be a Kähler manifold and $\lambda \neq 0$ a complex number.*
(i) The possible Gray-Hervella classes of the twistor space $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^-)$ are $\mathcal{W}$, $\mathcal{Q}K = \mathcal{W}_1 \oplus \mathcal{W}_2$ and $\mathcal{S}K = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$. Moreover
(i_1) $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^-) \in \mathcal{S}K = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ if and only if (M, g, J) is scalar flat.
(i_2) $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^-) \in \mathcal{Q}K = \mathcal{W}_1 \oplus \mathcal{W}_2$ if and only if (M, g, J) is Ricci flat.
(ii) The possible Gray-Hervella classes of the twistor space $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^+)$ are $\mathcal{W}$ and $\mathcal{W}_3 = \mathcal{S}K \cap \mathcal{H}$. The latter case occurs if and only if (M, g, J) is scalar flat.

Proof. Given a point $p \in M$, we choose an orthonormal frame of vector fields $A_1, ..., A_4$ around p such that $A_3 = JA_2$, $A_4 = JA_1$ and use this frame to define sections s_1, s_2, s_3 of $\Lambda^2_+ TM$ via (1.1.1). Then $\omega = s_3$ and

$$
\begin{aligned}
s_1 &= A_1 \wedge A_2 - JA_1 \wedge JA_2,\\
s_2 &= A_1 \wedge JA_2 + JA_1 \wedge A_2,\\
s_3 &= A_1 \wedge JA_1 + A_2 \wedge JA_2.
\end{aligned}
$$

Suppose that (M, g, J) is a Kähler 4-manifold. Then, as we have mentioned,

$$
\mathcal{R}(s_1) = \mathcal{R}(s_2) = 0, \quad g(\mathcal{R}(s_3), s_3) = \frac{s}{3} s_3. \tag{3.3.2}
$$

In particular the Kähler metric g is anti-self-dual if and only if it is scalar flat.

To determine the possible Gray-Hervella classes of the twistor space $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^\pm)$ of an almost Hermitian manifold (M, g, J) we shall need several technical lemmas. Next we shall always assume that (M, g, J) is a Kähler 4-manifold with Kähler 2-vector ω and scalar curvature s and that $\lambda \neq 0$ is an arbitrary complex number.

Lemma 3.3.3. $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^\pm) \in \mathcal{S}K = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ *if and only if* $s = 0$.

Proof. We know that $\mathcal{V}(f_\lambda^\pm)_*(X_\sigma^h) = 0$ for all $X \in T_{\pi(\sigma)}M$, hence, by Corollary 3.1.4, the fundamental 2-form of $(h_t, \mathcal{J}_\lambda^\pm)$ is co-closed if and only if

$$g(\mathcal{R}(U), f_\lambda^\pm(\sigma)) = 0. \tag{3.3.3}$$

for every $\sigma \in \mathcal{Z}$ and $U \in \mathcal{V}_\sigma$. Setting in this identity $\sigma = s_1(p)$, $U = s_3(p)$ for $p \in M$ and taking into account (3.3.2), we obtain

$$(a^2 + b^2 - 1)g(\mathcal{R}(s_3), s_3) = 0.$$

Hence $g(\mathcal{R}(s_3), s_3) = 0$ if $a^2 + b^2 \neq 1$. If $a^2 + b^2 = 1$ we set $\sigma = \frac{1}{\sqrt{2}}(s_1 + s_3)$ and $U = s_1 - s_3$. Then $\sqrt{2} f_\lambda^\pm(\sigma) = as_1 + bs_2 \pm s_3$ and identity (3.3.3) gives again $g(\mathcal{R}(s_3), s_3) = 0$. It follows that $s = 0$.

Conversely, if $s = 0$, we have $g(\mathcal{R}(s_i), s_j) = 0$, $i, j = 1, 2, 3$, so $g(\mathcal{R}(\sigma), \tau) = 0$ for every $\sigma, \tau \in \Lambda_+^2 TM$. In particular, identity (3.3.3) is fulfilled, hence $(h_t, \mathcal{J}_\lambda^\pm)$ is semi-Kähler. □

Lemma 3.3.4. *(i)* $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^-) \in \mathcal{Q}K = \mathcal{W}_1 \oplus \mathcal{W}_2$ *if and only if* (M, g, J) *is Ricci flat.*
(ii) $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^+)$ *never belongs to the class* $\mathcal{W}_1 \oplus \mathcal{W}_2$*.*

Proof. Suppose that $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^\pm) \in \mathcal{W}_1 \oplus \mathcal{W}_2$. By the defining condition for the class of quasi Kähler manifolds

$$(D_A\Omega)(B, C) + (D_{\mathcal{J}_\lambda^\pm A}\Omega)(\mathcal{J}_\lambda^\pm B, C) = 0, \quad A, B \in T\mathcal{Z}. \tag{3.3.4}$$

Hence, according to the second formula of Proposition 3.1.2,

$$\begin{aligned} &g(\mathcal{R}(U), X \wedge Y + K_{f_\lambda^\pm(\sigma)}X \wedge K_{f_\lambda^\pm(\sigma)}Y) \\ &-g(\mathcal{R}(\sigma \times U), X \wedge K_{f_\lambda^\pm(\sigma)}Y - K_{f_\lambda^\pm(\sigma)}X \wedge Y) = 0. \end{aligned} \tag{3.3.5}$$

for every $\sigma \in \mathcal{Z}$, $X, Y \in T_{\pi(\sigma)}M$, $U \in \mathcal{V}_\sigma$. Setting $Y = K_{f^{\pm}_{\lambda}(\sigma)}X$ we get

$$g(\mathcal{R}(U), X \wedge K_{f^{\pm}_{\lambda}(\sigma)}X) = 0, \tag{3.3.6}$$

or, equivalently,

$$g(\mathcal{R}(U), X \wedge K_{f^{\pm}_{\lambda}(\sigma)}Y - K_{f^{\pm}_{\lambda}(\sigma)}X \wedge Y) = 0. \tag{3.3.7}$$

It is easy to check by means of (3.1.3) that for any $\tau \in \Lambda^2_+ T_pM$ and $X, Y \in T_pM$ with $X \perp Y$, the 2-vector $X \wedge K_\tau Y - K_\tau X \wedge Y$ is orthogonal to $\Lambda^2_+ T_pM$, hence it lies in $\Lambda^2_- T_pM$. Moreover, for every $\tau \in \Lambda^2_+ T_pM$, every vector of $\Lambda^2_- T_pM$ is a linear combination of vectors of the form $X \wedge K_\tau Y - K_\tau X \wedge Y$ with $X \perp Y$ and vectors of the form $Z \wedge K_\tau Z$, $\quad X, Y, Z \in T_pM$. Indeed, if $a_1, ..., a_4$ is an orthonormal basis of T_pM such that $a_3 = K_\tau a_2$, $a_4 = K_\tau a_1$, then it is positively oriented and

$$\begin{aligned}
a_1 \wedge a_2 - a_3 \wedge a_4 \quad &= -(a_1 \wedge K_\tau a_3 - K_\tau a_1 \wedge a_3), \\
a_1 \wedge a_3 - a_4 \wedge a_2 \quad &= a_1 \wedge K_\tau a_2 - K_\tau a_1 \wedge a_2, \\
a_1 \wedge a_4 - a_2 \wedge a_3 \quad &= a_1 \wedge K_\tau a_1 - a_2 \wedge K_\tau a_2.
\end{aligned}$$

Thus it follows from (3.3.6) and (3.3.7) that

$$g(\mathcal{R}(U), s^-) = 0$$

for every $\sigma \in \mathcal{Z}$, $U \in \mathcal{V}_\sigma$, $s^- \in \Lambda^2_- T_{\pi(\sigma)}M$. In particular,

$$g(\mathcal{R}(s_3), s^-) = 0,$$

hence in view of (3.3.2),

$$\mathcal{R}(s_3) = \frac{s}{3} s_3.$$

Now, setting $\sigma = s_1(p)$ and $U = s_3(p)$, $p \in M$, in (3.3.7), we obtain

$$sg(s_3, X \wedge Y + K_{f_\lambda^\pm(s_1)} X \wedge K_{f_\lambda^\pm(s_1)} Y) = 0. \tag{3.3.8}$$

This identity for $(X,Y) = (A_1, A_2)$ and $(X,Y) = (A_1, A_3)$ gives

$$sa(a^2 + b^2 - 1) = 0, \quad sb(a^2 + b^2 - 1) = 0.$$

Hence $s = 0$ if $a^2 + b^2 \neq 1$.

If $a^2 + b^2 = 1$, we set $\sigma = \frac{1}{\sqrt{2}}(s_1 + s_3)$ and $U = s_1 - s_3$. We have

$$\sqrt{2} f_\lambda^\pm(\sigma) = as_1 + bs_2 \pm s_3$$

and identity (3.3.8) with $(X,Y) = (A_1, A_2)$ and (A_1, A_3) gives $as = 0$ and $bs = 0$. Therefore $s = 0$. It follows that $\mathcal{R}(\tau) = 0$ for every $\tau \in \Lambda^2_+ TM$. Since (M, g, J) is a Kähler manifold this is equivalent to the metric g being Ricci flat. Moreover, in view of the third formula of Proposition 3.1.2, we get from (3.3.4) that

$$g((f_\lambda^\pm)_*(U), Y \wedge Z) + g((f_\lambda^\pm)_*(\mathcal{J}_\lambda^\pm U), K_{f_\lambda^\pm(\sigma)} Y \wedge Z) = 0 \tag{3.3.9}$$

for $Y, Z \in T_{\pi(\sigma)}M$ and $U \in \mathcal{V}_\sigma$. The restriction of f_λ^+ to any fibre of $\mathcal{Z}$ is holomorphic, hence

$$(f_\lambda^+)_*(\mathcal{J}_\lambda^+ U) = \mathcal{J}_\lambda^+ (f_\lambda^+)_*(U) = f_\lambda^+(\sigma) \times (f_\lambda^+)_*(U).$$

Then, by (3.1.2)

$$g((f_\lambda^+)_*(\mathcal{J}_\lambda^+ U), K_{f_\lambda^+(\sigma)} Y \wedge Z) = g((f_\lambda^+)_*(U), Y \wedge Z).$$

This and (3.3.9) imply $(f_\lambda^+)_*(U) = 0$. Therefore the restriction of f_λ^+ to every fibre of $\mathcal{Z}$ is a constant map which is a contradiction.

Now suppose that the Kähler manifold M is Ricci flat. In this case $\mathcal{R}(\tau) = 0$ for every $\tau \in \Lambda^2_+ TM$. Then, in view of Proposition 3.1.2, in order to prove that $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^-) \in \mathcal{W}_1 \oplus \mathcal{W}_2$ it is enough to show that, for every $U \in \mathcal{V}_\sigma$ and $Y, Z \in T_{\pi(\sigma)}M$, we have

$$(D_U\Omega)(Y^h_\sigma, Z^h_\sigma) + (D_{\mathcal{J}_\lambda^- U}\Omega)((K_{f_\lambda^-(\sigma)}Y)^h_\sigma, Z^h_\sigma) = 0$$

This is equivalent to identity (3.3.9) for f_λ^-. The restriction of the map f_λ^- to any fibre of $\mathcal{Z}$ is anti-holomorphic, hence

$$(f_\lambda^-)_*(\mathcal{J}_\lambda^- U) = -\mathcal{J}_\lambda^-(f_\lambda^-)_*(U) = -f_\lambda^-(\sigma) \times (f_\lambda^-)_*(U).$$

This, in view of (3.1.2), implies that identity (3.3.9) for f_λ^- is fulfilled. Therefore $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^-) \in \mathcal{W}_1 \oplus \mathcal{W}_2$. □

Lemma 3.3.5. *(i)* $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^-) \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ *if and only if* (M, g, J) *is Ricci flat.*
(ii) $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^+)$ *never belongs to the class* $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$.

Proof. Suppose that $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^\pm)$ is of class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$. Then

$$(D_{X^h_\sigma}\Omega)(X^h_\sigma, U) + (D_{\mathcal{J}_\lambda^\pm X^h_\sigma}\Omega)(\mathcal{J}_\lambda^\pm X^h_\sigma, U) = -\frac{1}{2}||X||^2 \delta\Omega(U)$$

for $X \in T_{\pi(\sigma)}M$, $U \in \mathcal{V}_\sigma$. By Proposition 3.1.2 and Corollary 3.1.4, this is equivalent to

$$2g(\mathcal{R}(U), X \wedge K_{f_\lambda^\pm(\sigma)}X) = ||X||^2 g(\mathcal{R}(U), f_\lambda^\pm(\sigma)) \tag{3.3.10}$$

or

$$g(\mathcal{R}(U), X \wedge K_{f_\lambda^\pm(\sigma)}Y - K_{f_\lambda^\pm(\sigma)}X \wedge Y) = g(X,Y)g(\mathcal{R}(U), f_\lambda^\pm(\sigma)). \tag{3.3.11}$$

for $X, Y \in T_{\pi(\sigma)}M$, $U \in \mathcal{V}_\sigma$. Take an orthonormal basis $E_1, ..., E_4$ of $T_{\pi(\sigma)}M$ such that $E_3 = K_{f_\lambda^\pm(\sigma)}E_2$, $E_4 = K_{f_\lambda^\pm(\sigma)}E_1$. Then identity (3.3.10) for $X = E_1$ and $X = E_2$

gives

$$2g(\mathcal{R}(U), E_1 \wedge E_4) = g(\mathcal{R}(U), f_\lambda^\pm(\sigma)), \quad 2g(\mathcal{R}(U), E_2 \wedge E_3) = g(\mathcal{R}(U), f_\lambda^\pm(\sigma)).$$

These identities imply $g(\mathcal{R}(U), E_1 \wedge E_4 - E_2 \wedge E_3) = 0$. Moreover, setting in (3.3.11) $(X, Y) = (E_1, E_3)$ and $(X, Y) = (E_1, E_2)$ we get

$$g(\mathcal{R}(U), E_1 \wedge E_2 - E_3 \wedge E_4) = 0, \quad g(\mathcal{R}(U), E_1 \wedge E_3 - E_4 \wedge E_2) = 0.$$

It follows that $g(\mathcal{R}(U), s^-) = 0$ for every $s^- \in \Lambda^2_- T_{\pi(\sigma)}M$, hence $g(\mathcal{R}(\sigma), s^-) = 0$ for $\sigma \in \mathcal{Z}$ and $s^- \in \Lambda^2_- T_{\pi(\sigma)}M$. This and (3.3.2) imply

$$\mathcal{R}(\sigma) = \frac{s}{3} g(\sigma, \omega)\omega, \quad \sigma \in \Lambda^2_+ TM.$$

Then, by Corollary 3.1.4,

$$\delta\Omega(s_3) = -tg(\mathcal{R}(s_1 \times s_3), f_\lambda^\pm(s_1)) = t\frac{s}{3} g(s_2, s_3) g(s_3, f_\lambda^\pm(s_1)) = 0.$$

Moreover, $\delta\Omega(s_1) = tg(\mathcal{R}(s_2), f_\lambda^\pm(s_3)) = 0$ and $\delta\Omega(s_2) = -tg(\mathcal{R}(s_1), f_\lambda^\pm(s_3)) = 0$. It follows that $\delta\Omega = 0$, hence $s = 0$ by Lemma 3.3.3. Finally note that an almost Hermitian manifold with $\delta\Omega = 0$ belongs to the class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ if and only if it belongs to the class $\mathcal{W}_1 \oplus \mathcal{W}_2$. Hence the lemma follows from Lemmas 3.3.4 and 3.3.3. □

Lemma 3.3.6. *([22])* $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^+) \in \mathcal{H} = \mathcal{W}_3 \oplus \mathcal{W}_4$ *if and only if* (M, g, J) *is scalar flat.*

Proof. By Corollaries 3.1.6 and 3.1.5, the almost complex structure $\mathcal{J}_\lambda^+$ is integrable if and only if

$$\begin{aligned} &g(\mathcal{R}(X \wedge K_{f_\lambda^+(\sigma)} Y + K_{f_\lambda^+(\sigma)} X \wedge Y), U) \\ &+ g(\mathcal{R}(X \wedge Y - K_{f_\lambda^+(\sigma)} X \wedge K_{f_\lambda^+(\sigma)} Y), \sigma \times U) = 0 \end{aligned} \tag{3.3.12}$$

It is easy to check that for every $\tau \in \Lambda^2_+ T_pM$ and $X, Y \in T_pM$, the 2-vector $X \wedge K_\tau Y + K_\tau X \wedge Y \in \Lambda^2_+ T_pM$ (and is orthogonal to τ). Therefore, in view of (3.3.2), $\mathcal{J}^+_\lambda$ is integrable if and only if

$$\begin{aligned} &g(X \wedge K_{f^+_\lambda(\sigma)}Y + K_{f^+_\lambda(\sigma)}X \wedge Y, s_3)g(\mathcal{R}(s_3), U) \\ &+g(X \wedge Y - K_{f^+_\lambda(\sigma)}X \wedge K_{f^+_\lambda(\sigma)}Y, s_3)g(\mathcal{R}(s_3), \sigma \times U) = 0 \end{aligned} \tag{3.3.13}$$

for $X, Y \in T_{\pi(\sigma)}M$ and $U \in \mathcal{V}_\sigma$. Set $\sigma = s_1$ and $U = s_3$. Then, since $\mathcal{R}(s_2) = 0$, identity (3.3.13) becomes

$$g(X \wedge K_{f^+_\lambda(\sigma)}Y + K_{f^+_\lambda(\sigma)}X \wedge Y, s_3)g(\mathcal{R}(s_3), s_3) = 0. \tag{3.3.14}$$

For $(X, Y) = (A_1, A_2)$ and $(X, Y) = (A_1, A_3)$, the vector $X \wedge K_{f^+_\lambda(\sigma)}Y + K_{f^+_\lambda(\sigma)}X \wedge Y$ is collinear to $-2bs_3 + (a^2 + b^2 - 1)s_2$ and $2as_3 - (a^2 + b^2 - 1)s_1$, respectively. Then identity (3.3.14) gives

$$bg(\mathcal{R}(s_3), s_3) = 0, \quad ag(\mathcal{R}(s_3), s_3) = 0.$$

Therefore $g(\mathcal{R}(s_3), s_3) = 0$, thus $s = 0$. This shows that if $\mathcal{J}^+_\lambda$ is integrable, then (M, g, J) is scalar flat.

Conversely, suppose that (M, g, J) is Kähler and scalar flat. Then

$$\mathcal{V}(f^+_\lambda)_*(X^h_\sigma) = 0$$

for every $\sigma \in \mathcal{Z}$ and $X \in T_{\pi(\sigma)}M$. Hence, by Corollary 3.1.5,

$$\mathcal{H}N(X^h, Y^h) = 0$$

for every X, Y. Since $s = 0$, we also have

$$g(\mathcal{R}(s_i), s_j) = 0$$

for $i, j = 1, 2, 3$. Thus

$$g(\mathcal{R}(\sigma), \tau) = 0$$

for every $\sigma, \tau \in \Lambda^2_+ TM$. Recall that for every $\tau \in \Lambda^2_+ TM$ and $X, Y \in T_{\pi(\tau)}M$, the 2-vector $X \wedge K_\tau Y + K_\tau X \wedge Y$ lies in $\Lambda^2_+ TM$. Then by Corollary 3.1.5 we get

$$\mathcal{V}N(X^h, Y^h) = 0.$$

Finally, the map f^+_λ is holomorphic, hence, by Corollary 3.1.6 we have

$$\mathcal{H}(N(X^h_\sigma, U)) = 0$$

for every $U \in \mathcal{V}_\sigma$. Now Corollary 3.1.5 implies that $N = 0$. □

Lemma 3.3.7. *(i)* $(\mathcal{Z}, h_t, \mathcal{J}^-_\lambda)$ *never belongs to the class* $\mathcal{G}_1 = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$.
(ii) $(\mathcal{Z}, h_t, \mathcal{J}^+_\lambda) \in \mathcal{G}_1 = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ *if and only if* (M, g, J) *is scalar flat.*

Proof. By the definition of the class $\mathcal{G}_1$ ([29]), $(\mathcal{Z}, h_t, \mathcal{J}^\pm_\lambda) \in \mathcal{G}_1$ if and only if

$$h_t(N(A, B), C) + h_t(N(C, B), A) = 0$$

for all $A, B, C \in T\mathcal{Z}$. By Corollary 3.1.5 this is equivalent to the identity

$$\begin{aligned} & tg(\mathcal{R}(U), X \wedge K_{f^\pm_\lambda(\sigma)} Y + K_{f^\pm_\lambda(\sigma)} X \wedge Y) \\ & + tg(\mathcal{R}(\sigma \times U), X \wedge Y - K_{f^\pm_\lambda(\sigma)} X \wedge K_{f^\pm_\lambda(\sigma)} Y) \\ & + 2g((f^\pm_\lambda)_*(\sigma \times U) - f^\pm_\lambda(\sigma) \times (f^\pm_\lambda)_*(U), X \wedge Y) = 0. \end{aligned} \tag{3.3.15}$$

To prove *(i)* note that the restriction of f^-_λ on the fibre is anti-holomorphic, thus

$$f^-_\lambda(\sigma) \times (f^-_\lambda)_*(U) = -(f^-_\lambda)_*(\sigma \times U).$$

Hence, if $(\mathcal{Z}, h_t, \mathcal{J}^-_\lambda) \in \mathcal{G}_1$, then, setting $\sigma = s_3(p)$, $U = s_1(p)$, and taking into account that $\mathcal{R}(s_1) = \mathcal{R}(s_2) = 0$, we obtain from (3.3.15)

$$(f^-_\lambda)_{*\, s_3}(s_1) = 0.$$

But a straightforward computation shows that $(f_\lambda^-)_{*\,s_3}(s_1) \neq 0$, a contradiction.

To prove (ii) notice that the restriction of f_λ^+ on the fibre is holomorphic and identity (3.3.15) takes the form (3.3.12). Hence $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^+)$ is of class $\mathcal{G}_1$ if and only if it is of class $\mathcal{H}$, the later condition being equivalent to $s = 0$ by Lemma 3.3.6. □

Lemma 3.3.8. (i) $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^-)$ *never belongs to the class* $\mathcal{G}_2 = \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$.
(ii) $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^+) \in \mathcal{G}_2 = \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ *if and only if* (M, g, J) *is scalar flat.*

Proof. The structure $(h_t, \mathcal{J}_\lambda^\pm)$ is of class $\mathcal{G}_2$ if and only if [29]

$$\underset{A,B,C}{\mathfrak{S}} h_t(N(A,B), \mathcal{J}_\lambda^\pm C) = 0, \quad A, B, C \in T\mathcal{Z}.$$

For $A = X_\sigma^h$, $B = U \in \mathcal{V}_\sigma$, $C = Y_\sigma^h$ this identity and Corollary 3.1.5 imply

$$\begin{aligned} &2g(\mathcal{J}_\lambda^\pm (f_\lambda^\pm)_*(U) - (f_\lambda^\pm)_*(\mathcal{J}_\lambda^\pm U), X \wedge K_{f_\lambda^\pm(\sigma)} Y + K_{f_\lambda^\pm(\sigma)} X \wedge Y) \\ &\quad -tg(\mathcal{R}(X \wedge K_{f_\lambda^\pm(\sigma)} Y + K_{f_\lambda^\pm(\sigma)} X \wedge Y), \sigma \times U) \\ &\quad +tg(\mathcal{R}(X \wedge Y - K_{f_\lambda^\pm(\sigma)} X \wedge K_{f_\lambda^\pm(\sigma)} Y), U) = 0. \end{aligned} \tag{3.3.16}$$

Set $\sigma = \omega(p)$, $p \in M$. We have $f_\lambda^\pm(\omega) = \pm\omega$, so $K_{f_\lambda^\pm(\omega)} = \pm J$. Then, setting $(X, Y) = (A_1, A_3)$, $(X, Y) = (A_1, A_2)$ and taking into account (3.3.2), we obtain from (3.3.16) that

$$g(\mathcal{J}_\lambda^\pm (f_\lambda^\pm)_*(U) - (f_\lambda^\pm)_*(\mathcal{J}_\lambda^\pm U), s_i) = 0, \quad i = 1, 2.$$

The map f_λ^- is anti-holomorphic on the fibres of $\mathcal{Z}$, so the latter identity gives

$$g((f_\lambda^-)_{*,\,s_3}(\mathcal{J}_\lambda^- U), s_i) = 0, \quad i = 1, 2, \quad U \in \mathcal{V}_{s_3}.$$

We set $U = s_2(p)$ and $U = s_1(p)$ and compute

$$f^-_{*,\,s_3}(s_1) = \frac{2as_1 + 2bs_2 - (a^2 + b^2 - 1)s_3}{2(a^2 + b^2)}, f^-_{*,\,s_3}(s_2) = \frac{2as_2 - 2bs_1 - (a^2 + b^2 - 1)s_3}{2(a^2 + b^2)}.$$

It follows that $a = 0$, $b = 0$, which contradicts to the assumption $\lambda \neq 0$. This proves statement (i).

Now suppose that $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^+)$ is of class $\mathcal{G}_2$. Then identity (3.3.16) becomes

$$\begin{aligned} &g(\mathcal{R}(X \wedge K_{f_\lambda^+(\sigma)}Y + K_{f_\lambda^+(\sigma)}X \wedge Y), \sigma \times U) \\ &-g(\mathcal{R}(X \wedge Y - K_{f_\lambda^+(\sigma)}X \wedge K_{f_\lambda^+(\sigma)}Y), U) = 0. \end{aligned} \tag{3.3.17}$$

We have $f_\lambda^+(s_1) = (a^2 + b^2 + 1)^{-1}(2as_1 + 2bs_2 + cs_3)$ where $c = a^2 + b^2 - 1$. Then

$$A_1 \wedge K_{f_\lambda^+(s_1)}A_2 + K_{f_\lambda^+(s_1)}A_1 \wedge A_2 = (a^2 + b^2 + 1)^{-1}(-2bs_3 + cs_2).$$

Thus, setting $\sigma = s_1$, $(X, Y) = (A_1, A_2)$, $U = s_2$ in (3.3.17) and taking into account that $\mathcal{R}(s_1) = \mathcal{R}(s_2) = 0$, we obtain $bg(\mathcal{R}(s_3), s_3) = 0$. Similarly, since

$$f_\lambda^+(s_2) = (a^2 + b^2 + 1)^{-1}(-2bs_1 + 2as_2 + cs_3),$$

setting $\sigma = s_2$, $(X, Y) = (A_1, A_2)$, $U = s_1$ we get

$$ag(\mathcal{R}(s_3), s_3) = 0.$$

It follows $g(\mathcal{R}(s_3), s_3) = 0$, hence $s = 0$. By Lemma 3.3.6 , $s = 0$ if and only if the almost complex structure $\mathcal{J}_\lambda^+$ is integrable. In particular, if $s = 0$, $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^+)$ is of class $\mathcal{G}_2$ and (ii) is proved. □

We are now ready to prove Theorem 3.3.2.

Proof of Theorem 3.3.2.

(i). It follows from statements (i) of Lemmas 3.3.7 and 3.3.8, and [29, Table I] that the possible nontrivial Gray-Hervella classes of $(\mathcal{Z}, h_t, \mathcal{J}_\lambda^-)$ are subclasses of $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ or $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$. Moreover statements (i) of Lemmas 3.3.4 and 3.3.5 imply that

$$\mathcal{W}_1 \oplus \mathcal{W}_2 = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3.$$

Hence the first part of the theorem follows from statements (i) of Lemmas 3.3.3, 3.3.7 and 3.3.8.

(ii). Using statements (ii) of Lemmas 3.3.3-3.3.8 we prove the second part of the theorem in a similar way. □

Now we shall discuss the case when $|\lambda| = 1$. In this case we have the simple formula (3.3.1) for $\mathcal{V}(f_\lambda^\pm)_*(X_\sigma^h)$, $\sigma \in \mathcal{Z}$, $X \in T_{\pi(\sigma)}M$. This simplifies the computations that should be done in order to determine the possible Gray-Hervella types of the almost Hermitian manifold $(h_t, \mathcal{J}_\lambda^\pm)$. Here we shall address only a few of the basic classes.

Proposition 3.3.9. *Suppose that* $|\lambda| = 1$ *and* $Re(\lambda) \neq \pm 1$ *for* $\mathcal{J}_\lambda^\pm$ *.Then:*
(i) *The almost Hermitian structure* $(h_t, \mathcal{J}_\lambda^-)$ *on the twistor space* $\mathcal{Z}$ *is (non-integrable) quasi Kähler if and only if* M *is Ricci flat.*
(ii) *The structure* $(h_t, \mathcal{J}_\lambda^+)$ *is never quasi Kähler.*
(iii) *The structures* $(h_t, \mathcal{J}_\lambda^\pm)$ *are not nearly Kähler or almost Kähler.*

Proof. It is convenient to prove first the following

Lemma 3.3.10. *If* $(h_t, \mathcal{J}_\lambda^\pm)$ *is quasi Kähler, then* (M, g, J) *is Kähler and Ricci flat.*

Proof of the lemma. Let $p \in M$, $X, Y, Z \in T_pM$. We have $f_\lambda^\pm(\omega_p) = \pm\omega_p$, hence by Proposition 3.1.2 and (3.3.1)

$$\begin{aligned} 0 &= \frac{1}{2}[(D_{X^h_{\omega(p)}}\Omega)(Y^h_{\omega(p)}, Z^h_{\omega(p)}) + (D_{(JX)^h_{\omega(p)}}\Omega)((JY)^h_{\omega(p)}, Z^h_{\omega(p)})] \\ &= -bg(\omega \times \nabla_X\omega, Y \wedge Z) + (\pm 1 - a)g(\nabla_X\omega, Y \wedge Z) \\ &- bg(\omega \times \nabla_{JX}\omega, JY \wedge Z) + (\pm 1 - a)g(\nabla_{JX}\omega, JY \wedge Z) \end{aligned}$$

This and identity (3.1.3) give

$$\begin{aligned} 0 &= -bg(\nabla_X J)(JY), Z) + (\pm 1 - a)g((\nabla_X J)(Y), Z) \\ &+ bg(\nabla_{JX} J)(Y), Z) + (\pm 1 - a)g((\nabla_{JX} J)(JY), Z). \end{aligned}$$

Thus

$$bJ[(\nabla_X J)(Y) - J(\nabla_{JX} J)(Y)] + (\pm 1 - a)[(\nabla_X J)(Y) - J(\nabla_{JX} J)(Y)] = 0.$$

By assumption $a \neq 1$ when considering $\mathcal{J}_\lambda^+$ and $a \neq -1$ for $\mathcal{J}_\lambda^-$. Thus

$$b^2 + (\pm 1 - a)^2 \neq 0$$

and the latter equation implies

$$(\nabla_X J)(Y) - J(\nabla_{JX} J)(Y) = 0.$$

This means that the almost Hermitian structure (g, J) on M is quasi Kähler. It follows that it is Kähler since $dim\, M = 4$. The assumption that $\mathcal{J}_\lambda^\pm$ is quasi Kähler implies identity (3.3.5) and, as in the proof of Lemma 3.3.4, we see that $s = 0$. Thus M is Kähler and Ricci flat. □

Now we are ready to prove Proposition 3.3.9

(i). If $(h_t, \mathcal{J}_\lambda^-)$ is quasi Kähler, M is Kähler and Ricci flat by the Lemma 3.3.10. Conversely, if M is such a manifold, $(h_t, \mathcal{J}_\lambda^-)$ is quasi Kähler by Lemma 3.3.4.

(ii). This statement follows from the Lemmas 3.3.10 and 3.3.4.

(iii). If $(h_t, \mathcal{J}_\lambda^\pm)$ is nearly Kähler or almost Kähler, it is quasi Kähler, hence M is Kähler by the lemma. Then, according to Lemma 3.3.5 (ii), $(h_t, \mathcal{J}_\lambda^+)$ does not belong to the class $\mathcal{N}K = \mathcal{W}_1$ or to the class $\mathcal{A}K = \mathcal{W}_2$. Also $(h_t, \mathcal{J}_\lambda^-)$ is not of class $\mathcal{N}K$ or $\mathcal{A}K$ by Lemmas 3.3.7 (i) and 3.3.8 (i). □

Proposition 3.3.11. *Let $|\lambda| = 1$ and $Re(\lambda) \neq 1$. Then the almost complex structure $\mathcal{J}_\lambda^+$ is integrable if and only if (M, g, J) is Kähler and scalar flat. In this case $(h_t, \mathcal{J}_\lambda^+) \in \mathcal{W}_3 = \mathcal{S}K \cap \mathcal{H}$.*

Proof. Suppose that the almost complex structure $\mathcal{J}_\lambda^+$ is integrable.

Let$p \in M$, $\sigma = s_1(p)$ and $X \in T_pM$. Then we have

$$\begin{aligned} \sigma \times \nabla_X \omega &= g(s_2, s_1 \times \nabla_X s_3)s_2 + g(s_3, s_1 \times \nabla_X s_3)s_3 \\ &= -g(s_3, \nabla_X s_3)s_2 + g(s_2, \nabla_X s_3)s_3 \\ &= g(s_2, \nabla_X s_3)s_3. \end{aligned}$$

Thus by (3.3.1)

$$\mathcal{V}(f_\lambda^+)_*(X^h_{s_1(p)}) = [-bg(\nabla_X s_3, s_2) + (1-a)g(\nabla_X s_3, s_1)]s_3.$$

It is convenient to set

$$\phi_i = -bg(\nabla_{A_i} s_3, s_2) + (1-a)g(\nabla_{A_i} s_3, s_1)_p, i = 1, ..., 4$$

We have

$$K_{f_\lambda^+(s_1)} = aK_{s_1} + bK_{s_2}$$

and, using Corollary 3.1.5, it is easy to see that

$$h_t(N(A_1^h, A_2^h), A_1^h)_{s_1(p)} = 2b\phi_1, \quad h_t(N(A_1^h, A_3^h), A_1^h)_{s_1(p)} = -2a\phi_1.$$

It follows that $\phi_1 = 0$ since $N = 0$ and $a^2 + b^2 \neq 0$. We also have

$$h_t(N(A_1^h, A_2^h), A_2^h)_{s_1(p)} = 2b\phi_2, \quad h_t(N(A_2^h, A_4^h), A_2^h)_{s_1(p)} = 2a\phi_2,$$

$$h_t(N(A_1^h, A_3^h), A_3^h)_{s_1(p)} = 2a\phi_3, \quad h_t(N(A_3^h, A_4^h), A_3^h)_{s_1(p)} = 2b\phi_3$$

$$h_t(N(A_2^h, A_4^h), E_2^h)_{s_1(p)} = 2a\phi_4, \quad h_t(N(A_3^h, A_4^h), A_4^h)_{s_1(p)} = 2b\phi_4.$$

It follows that $\phi_2 = \phi_3 = \phi_4 = 0$. Thus

$$-bg(\nabla_{A_i} s_3, s_2) + (1-a)g(\nabla_{A_i} s_3, s_1) = 0, \quad i = 1, ..., 4. \tag{3.3.18}$$

Now set $\sigma = s_2(p)$. We have

$$\mathcal{V}f_*(X^h_{s_2(p)}) = bg(\nabla_X s_3, s_1) + (1-a)g(\nabla_X s_3, s_2)s_3$$

and

$$K_{f_\lambda^+(s_2)} = -bK_{s_1} + aK_{s_2}.$$

Then a similar computation as above gives

$$bg(\nabla_{A_i} s_3, s_1) + (1-a)g(\nabla_{A_i} s_3, s_2) = 0, \quad i = 1, ..., 4. \tag{3.3.19}$$

It follows from (3.3.18) and (3.3.19) that

$$g(\nabla_{A_i} s_3, s_1) = g(\nabla_{A_i} s_3, s_2) = 0, \quad i = 1, ..., 4, \tag{3.3.20}$$

Also we have $g(\nabla_{A_i} s_3, s_3) = 0$ since s_3 is of constant length, hence, by (3.3.20), the almost complex structure J is Kählerian. By Corollary 3.1.5, the identity

$$\mathcal{V}N(X^h_\sigma, Y^h_\sigma) = 0$$

is equivalent to

$$g(\mathcal{R}(X \wedge K_{f^+_\lambda(\sigma)} Y + K_{f^+_\lambda(\sigma)} X \wedge Y), U)$$
$$+ g(\mathcal{R}(X \wedge Y - K_{f^+_\lambda(\sigma)} X \wedge K_{f^+_\lambda(\sigma)} Y), \sigma \times U) = 0$$

for $X, Y \in T_{\pi(\sigma)}M$ and $U \in \mathcal{V}_\sigma$. Setting in the latter identity $\sigma = s_1(p)$, $U = s_3(p)$, $(X, Y) = (A_1, A_2)$ and $(X, Y) = (A_1, A_3)$, and taking into account (3.3.2), we get

$$bg(\mathcal{R}(s_3), s_3) = 0, \quad ag(\mathcal{R}(s_3), s_3) = 0.$$

Therefore $g(\mathcal{R}(s_3), s_3) = 0$, thus $s = 0$. This proves that if $\mathcal{J}^+_\lambda$ is integrable, then (M, g, J) is Kähler and scalar flat. The converse follows from Theorem 3.3.2 (ii). $\square$

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