FRACTIONAL DIFFERENTIAL EQUATIONS IN NONREFLEXIVE BANACH SPACES

Since 1864

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DECLARATION

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has been carried out and completed by Mr. **Ghaus ur Rahman** Registration No. **102-GCU-PHD-SMS-09** under my supervision.

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Abstract

This thesis is devoted to fractional calculus in nonreflexive Banach spaces and existence results for the fractional differential equations. Using fractional Pettis integral and fractional pseudo-derivative, we discussed fractional calculus and fractional differential equations in nonreflexive Banach spaces, equipped with weak topology. We obtained some results on existence of solution of fractional differential equations. Furthermore, applying fractional Pettis integral and fractional pseudo-derivative we discussed the existence of solution of multi-term fractional differential equation, in nonreflexive Banach spaces, equipped with weak topology. Finally, assuming the concept of Riemann-Pettis integral, we introduced and studied the notions of fractional Riemann-Pettis integral and fractional Caputo weak derivative. Using these tool we obtained an existence result for weak solution of fractional differential equations in a nonreflexive Banach space equipped with the weak topology.

Chapter 1

Preliminaries and Introduction

1.1 Introduction

The mathematical field that deals with derivatives of any real order is called fractional calculus. For a long time, it was only considered as a pure mathematical branch. Nevertheless, during the last two decades, fractional calculus has attracted the attention of many researchers and it has been successfully applied in various areas like computational biology, computational fluid dynamics and economics *etc* [80].

The study of first order ordinary differential equations in Banach spaces (reflexive or not) equipped with the weak topology was initiated in the 1950's. Let E be Banach space and let $f(\cdot,\cdot):[a,b]\times E\to E$ be continuous. It is well known that if E is finite dimensional, then for each $(t_0,y_0)\in[a,b)\times E$, there exists a continuous differentiable function $y(\cdot)$ which is a solution of the Cauchy problem

$$y'(t)=f(t,y(t)),\quad y(t_0)=y_0, \tag{1.1.1}$$

on some open interval which contains t_0. In 1950, Dieudonné [32] showed that in the case $E=\mathbf{c}_0$ the Cauchy problem (1.1.1) has no solutions for every continuous function $f(\cdot,\cdot)$.

The notion of the measure of non-compactness was introduced by Kuratowski [51] in 1930. Ambrosetti [8] was first one who had an idea to use the Kuratowski measure of non-compactness and Darbo fixed point theorem to prove an existence result for (1.1.1) in infinite dimensional Banach spaces. Szep [86] was the first author related to the existence of weak solutions for (1.1.1), where $f(\cdot,\cdot):[a,b]\times E\to E$ is a weakly continuous and E is a reflexive Banach space. Chow and Schuur [24] treated the case where E is separable and reflexive $f(\cdot,\cdot)$ is a weakly continuous function with bounded range. Kato in [46] shown that if $f(\cdot,\cdot) : [a,b] \times B_E[y_0,r] \to E$ is weakly continuous, then all we needed to assure the existence of solutions to (1.1.1) is the relatively weak compactness of $f([a,b] \times B_E[y_0,r])$. Pianigiani [69] shown that in every nonreflexive rectractive Banach space there exists a weakly continuous function $f(\cdot,\cdot)$ such that (1.1.1) does not have a weak solution, and Perri [66] showed that this property is true in every nonreflexive Banach space.

The measure of weak non-compactness was introduced by De Blasi [27], and it was used by Cramer, Lakshmikantham and Mitchell [23] to obtain an existence result for weak solutions of (1.1.1) in nonreflexive Banach spaces. Using the measure of weak non-compactness, Cichoń [17], Cichoń and Kubiaczyk [19], Dutkiewicz and Szufla [34], Gomaa [40], O'Regan [62], [64], have improved and generalized previous results. For a review of this topic we refer to Cichoń [18], Deimling [28], Hashem [43] and Teixeira [89].

The existence of weak solution of differential equations in nonreflexive Banach spaces, equipped with weak topology was studied for the first time by Cramer et.al [23]. The authors imposed weak compactness type conditions in terms of the measure of weak non-compactness. Moreover, for existence and uniqueness of solution, the authors imposed weak dissipative type conditions. Using these existence and partial ordering induced by cones, existence of extremal solutions and comparison results to the weak topology are also proved in this article.

In recent years, fractional differential equations in Banach spaces has been studied intensively. The general literature on fractional differential equations in finite or infinite dimensional Banach spaces is extensive and considers different topics on the existence and qualitative properties of solutions are considered. Concerning the literature on fractional differential equations we refer to the books [55], [80] and the references cited therein. Only a few papers consider fractional differential equations in reflexive Banach spaces equipped with the weak topology e.g [3], [13], [14], [43], [74], [76].

In [74], Salem et.al defined the fractional order Pettis-integral operator in reflexive Banach spaces and also investigated the properties of such operator. Furthermore, the authors used O'Regan fixed point theorem (see [64]), to establish an existence result for nonlinear Pettis-fractional order integral equations of the type

$$x(t) = g(t) + \lambda I^q f(\cdot, x(\cdot)),\ \ t \in [0,1],\ \ 0 < q < 1.$$

Moreover, the authors exhibit the existence of solution of Cauchy problem

$$\frac{dx}{dt} = f(t, D^q x(t)),\ \ t \in [0,1], \quad x(0) = x_0.$$

In [76], the author used O'Regan fixed point theorem (see [64]) to establish an existence of solution for the fractional order integral equation

$$x(t) = g(t) + \lambda I^q f(\cdot, x(\cdot)), \quad t \in [0,1], \quad q > 0,$$

where f is nonlinear weakly-weakly continuous. Moreover, the authors exhibit the solution of Cauchy problem

$$\frac{dx}{dt} = f(t, x(t)), \quad t \in [0,1], \quad x(0) = x_0.$$

Some of the recent progress in this direction are [13], [14], [73]. For a recent review of this topic we refer to [43].

So for, we discussed those differential equations which contain only one differential operator. Nevertheless, in certain cases we need to solve fractional differential equations containing more than one differential operators. This type of fractional differential equation is called multi-term fractional differential equation. Multi-term fractional differential equations also have numerous applications in physical sciences and other branches of sciences [29]. The existence of the solutions of multi-term fractional differential equations was studied by many authors [11, 25, 26, 75, 77, 78]. The main tool used in [11, 25, 26], is the Krasnoselskii's fixed point theorem on a cone while the main tool used in [77], is the technique associated with the measure of non-compactness and fixed point theorem. In [77], the author exhibits the existence of monotonic solution for the multi-term fractional differential equations in Banach spaces, using Riemann-Liouville fractional derivative and no compactness condition is assumed on the nonlinearity of the function f.

In [75], the author studied the existence of weak solution of the Cauchy problem in reflexive Banach spaces equipped with weak topology, the author imposed weak-weak continuity assumption on f . Similarly in [78], the author exhibits the existence of global monotonic solution for the Cauchy problem, the author assumed f to be Carathéodory which has linear growth.

This dissertation is devoted to fractional calculus in nonreflexive Banach spaces and existence results for the fractional differential equations. Using Pettis integral and fractional pseudo derivative we present more general results for the existence of solutions to fractional differential equations in nonreflexive Banach spaces, equipped with weak topology. Assume the following fractional differential equation

$$\begin{cases} D^q y(t) = f(t, y(t)),\ t \in T, \\ y(0) = y_0 \end{cases} \tag{1.1.2}$$

where D^q is a fractional pseudo-derivative, f is given function, and T is a bounded interval of real numbers such that $0 \in T$ and E is a nonreflexive Banach space. Furthermore, we also establish an existence result for multi-term fractional differential equation,

$$\left(D^{\alpha_m} - \sum_{i=1}^{m-1} a_i D^{\alpha_i} \right) u(t) = f(t,\, u(t)) \ \ for\ t \in [0,1],\ u(0) = 0, \tag{1.1.3}$$

where D^{α_m} and D^{α_i} are fractional pseudo-derivative of order α_m and α_i, $i = 1, 2, \ldots,$ $m-1$, respectively. The function $f(t, \cdot) : [0,1] \times E \to E$ is weakly-weakly sequentially continuous for every $t \in [0,1]$ and $f(\cdot, u(\cdot))$ is Pettis integrable for every weakly absolutely continuous function $u(\cdot) : [0,1] \to E$, where E is nonreflexive Banach space, $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_m < 1$ and $a_1, a_2 \ldots a_{m-1}$ are real numbers such that $a := \sum_{i=1}^{m-1} \frac{|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} < 1$, and $\Gamma(\cdot)$ is the Euler's gamma function.

Finally, we exhibit an existence result for weak solution of fractional differential equation (1.1.2), where D^q is a fractional Caputo-derivative, f is a given function, T is a bounded interval of real numbers and E is a nonreflexive Banach space.

Personal Contributions:

The results of this thesis are based on the following articles.

[3] R.P. Agarwal, V. Lupulescu, D. O'Regan, Ghaus ur Rahman, Fractional Calculus and Fractional Differential Equation in nonreflexive Banach spaces, *Communications in Nonlinear Science and Numerical Simulation* (to appear).

[4] R.P. Agarwal, V. Lupulescu, D. O'Regan, Ghaus ur Rahman, Multi-term Fractional Differential Equation in nonreflexive Banach spaces, *Advances in Difference Equation, 2013, 2013:302.*

[5] R.P. Agarwal, V. Lupulescu, D. O'Regan, Ghaus ur Rahman, Weak Solution for Fractional Differential Equations in nonreflexive Banach spaces via Riemann Pettis integral, Mathematische Nachrichten, (to appear)

In the paper [3], using fractional Pettis integral and fractional pseudo-derivative, we discussed fractional calculus and fractional differential equations in nonreflexive Banach spaces, equipped with weak topology. Our contribution about properties of fractional Pettis integral and fractional Pseudo-derivative can be found in chapter 2. We gave some results on existence of solution of fractional differential equations and equivalence between integral equation and Cauchy equation. The results of this paper can be found in chapter 2.

In the paper [4], using fractional Pettis integral and fractional pseudo-derivative we discussed the existence of solution of multi-term fractional differential equation, in nonreflexive Banach spaces, equipped with weak topology .

In the paper [5], using the concept of Riemann-Pettis integral, we introduced and studied the notions of fractional Riemann-Pettis integral and fractional Caputo weak derivative. Using these tool we obtain an existence result for fractional differential equations in a nonreflexive Banach space equipped with the weak topology.

1.2 Preliminaries

Let E be a Banach space with the norm $\|\cdot\|$ and let E^* be the topological dual of E. If $x^* \in E^*$, then its value on an element $x \in E$ will be denoted by $\langle x^*, x\rangle$. The space E endowed with the weak topology $\sigma(E, E^*)$ will be denoted by E_w. Consider an interval $T = [0, b]$ of $\mathbb{R}$, the set of real numbers, endowed with the Lebesgue σ-algebra $\mathcal{L}(T)$ and the Lebesgue measure λ. A function $x(\cdot) : T \to E$ is said to be *almost separable valued* if there exists a null set $N \in \mathcal{L}(T)$ such that $x(T \setminus N)$ is a separable set in E (equivalently, $x(T \setminus N)$ is contained in a separable closed subspace of E.

Definition 1.2.1. A function $x(\cdot) : T \to E$ is said to be *strongly measurable* on T if there exists a sequence of simple functions $x_n(\cdot) : T \to E$ such that $\lim\limits_{n\to\infty} x_n(t) = x(t)$ for a.e. $t \in T$.

Definition 1.2.2. A function $x(\cdot) : T \to E$ is said to be *weakly measurable* (or *scalarly measurable*) on T if for every $x^* \in E^*$ the real valued function $t \mapsto \langle x^*, x(t)\rangle$ is Lebesgue measurable on T.

Definition 1.2.3. Let $x(t)$ be a function from T into E. Then $x(t)$ is weakly continuous at t_0 if for every $x^* \in E^*$, the scalar function $\langle x^*, x(t)\rangle$ is continuous at t_0.

***Remark* 1.2.1.** (Pettis theorem) It is well known that a weakly measurable and almost separable valued function $x(\cdot) : T \to E$ is strongly measurable ([67, Theorem 1.1]).

Further, we also assume the set of all measurable functions from T to $\mathbb{R}$ and (T, Σ, μ) be a measure space. We denote by $L^p(T)$ the space of all real measurable functions $f : T \to \mathbb{R}$, whose absolute value raised to the $p - th$ power has finite integral, or equivalently, that

$$\|f\|_p \equiv \left(\int_T |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} < \infty,$$

where $1 \le p < \infty$. Moreover by $L^\infty(T)$ we denote, the space of all measurable and essentially bounded real functions defined on T. Let $C(T, E)$ denote the space of all strongly continuous functions $y(\cdot) : T \to E$, endowed with the supremum norm $||y(\cdot)||_c = sup_{t\in T} \|y(t)\|$. Also, we consider the space $C(T, E)$ with its weak topology $\sigma(C(T, E), C(T, E)^*)$. It is known that (see [33, 82])

$$C(T, E)^* = M(T,\ E^*),$$

where $M(T,\ E^*)$ is the space of all bounded regular vector measures from $\mathcal{B}(T)$ into E^* which are of bounded variation. Here, $\mathcal{B}(T)$ denotes the σ-algebra of Borel measurable subsets of T. Therefore, a sequence $\{y_n(\cdot)\}_{n\geq 1}$ converges weakly to $y(\cdot)$ in $C(T_0, E))$ if and only if

$$\langle m(\cdot), y_n(\cdot) - y(\cdot)\rangle \to 0 \ \text{ as } n \to \infty, \tag{1.2.1}$$

for all $m(\cdot) \in M(T_0, E^*)$. In [59, Lemma 9] it is shown that a sequence $\{y_n(\cdot)\}_{n\geq 1}$ converges weakly to $y(\cdot)$ in $C(T_0, E))$ if and only if $y_n(t)$ tends weakly to $y(t)$ for each $t \in T$.

By $C_w(T, E)$ we will denote the space of all weakly continuous functions from T into E_w endowed with the topology of weak uniform convergence. A set $N \in \mathcal{L}(T)$ is called a null set if $\lambda(N) = 0$.

Definition 1.2.4. A function $x(\cdot) : T \to E$ is said to be *absolutely continuous* on T (AC, for short) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \sum_{k=1}^{m} [x(b_k) - x(a_k)] \right\| < \varepsilon$$

for every finite disjoint family $\{(a_k, b_k)\,; 1 \leq k \leq m\}$ of subintervals of T such that $\sum\limits_{k=1}^{m} (b_k - a_k) < \delta$. If the last inequality is replaced by $\sum\limits_{k=1}^{m} ||x(b_k) - x(a_k)|| < \varepsilon$, then we say that $x(\cdot)$ is a *strongly absolutely continuous* (sAC) function.

Definition 1.2.5. A function $x(\cdot) : T \to E$ is said to be *weakly absolutely continuous* (wAC) on T if for every $x^* \in E^*$ the real valued function $t \mapsto \langle x^*, x(t)\rangle$ is AC on T.

***Remark* 1.2.2.** Each sAC function is an AC function, and each AC function is a wAC function. If E is a weakly sequentially complete space, then every wAC function is an AC function ([48]).

1.3 Differentiability in Banach Spaces

In this section we will recall some notion about differentiability in Banach spaces.

Definition 1.3.1. A function $x(\cdot) : T \to E$ is said to be *strongly differentiable at* $t_0 \in T$ if there exists an element $x'_s(t_0) \in E$ such that

$$\lim_{h \to 0} \left\| \frac{x(t_0 + h) - x(t_0)}{h} - x'_s(t_0) \right\| = 0.$$

The element $x'_s(t_0)$ will be also denoted by $\frac{d_s}{dt} x(t_0)$ and it is called the *strong derivative* of $x(\cdot)$ at $t_0 \in T$.

Definition 1.3.2. A function $x(\cdot) : T \to E$ is said to be *weakly differentiable at* $t_0 \in T$ if there exists an element $x'_w(t_0) \in E$ such that

$$\lim_{h \to 0} \left\langle x^*, \frac{x(t_0 + h) - x(t_0)}{h} \right\rangle = \langle x^*, x'_w(t_0) \rangle$$

for every $x^* \in E^*$. The element $x'_w(t_0)$ will be also denoted by $\frac{d_w}{dt} x(t_0)$ and it is called the *weak derivative* of $x(\cdot)$ at $t_0 \in T$.

It is easy to see that if the elements $x'_s(t_0)$ and $x'_w(t_0)$ exist, then they are uniquely determined. If a function $x(\cdot) : T \to E$ is strongly differentiable (weakly differentiable) at each point $t \in T$, then we say that $x(\cdot)$ is strongly differentiable (weakly differentiable) on T. In this case, the vector valued function $t \mapsto x'_s(t)$ is called the strong derivative (weak derivative) of $x(\cdot)$. Obviously, a strongly differentiable function $x(\cdot) : T \to E$ is also weakly differentiable, but the converse is not true ([81, Example 7.3.6]). Also, it is obvious that if $x(\cdot) : T \to E$ is a function weakly differentiable on T, then the real function $t \mapsto \langle x^*, x(t) \rangle$ is differentiable on T. Moreover, in this case we have that

$$\frac{d}{dt} \langle x^*, x(t) \rangle = \langle x^*, x'_w(t) \rangle, \ t \in T, \tag{1.3.1}$$

for every $x^* \in E^*$. For the converse the following result is known.

Proposition 1.3.1. *([81, Theorem 7.3.3]) If* E *is a weakly sequentially complete space and* $x(\cdot) : T \to E$ *is a function such that for every* $x^* \in E^*$ *the real function* $t \mapsto \langle x^*, x(t) \rangle$ *is differentiable on*T*, then* $x(\cdot)$ *is weakly differentiable on* T

Proposition 1.3.2. *([67, Theorem 1.2]) If $x(\cdot) : T \to E$ is a.e. weakly differentiable on T, then its weak derivative $x'_w(\cdot)$ is strongly measurable on T.*

Definition 1.3.3. A function $x(\cdot) : T \to E$ is said to be *pseudo-differentiable* on T to a function $y(\cdot) : T \to E$ if for every $x^* \in E^*$ there exists a null set $N(x^*) \in \mathcal{L}(T)$ such that the real function $t \mapsto \langle x^*, x(t)\rangle$ is differentiable on $T \setminus N(x^*)$ and

$$\frac{d}{dt}\langle x^*, x(t)\rangle = \langle x^*, y(t)\rangle\,,\ t \in T \setminus N(x^*). \tag{1.3.2}$$

The function $y(\cdot)$ is called a *pseudo-derivative* of $x(\cdot)$ and it will be denoted by $x'_p(\cdot)$ or by $\frac{d_p}{dt}x(\cdot)$.

We end this section with some remarks.

- Clearly, if $x(\cdot) : T \to E$ is a function a.e. weakly differentiable on T, then $x(\cdot)$ is pseudo-differentiable on T and $x'_p(\cdot) = x'_w(\cdot)$ a.e. on T

- A pseudo-derivative $x'_p(\cdot)$ of a pseudo-differentiable function $x(\cdot) : T \to E$ need not be strongly measurable [83]. However, in [84] it was shown that $x'_p(\cdot)$ is weakly measurable on T.

- In general, a pseudo-derivative of a pseudo-differentiable function $x(\cdot) : T \to E$ is not unique. Moreover, two pseudo-derivatives of $x(\cdot)$ need not be a.e. equal [83]. However, if E has a *countable determining set*, that is, a countable set $W^* \subset E^*$ such that $\|x\| = \sup_{x^* \in W^*} |\langle x^*, x\rangle|$ for every $x \in E$, then any two pseudo-derivative of $x(\cdot)$ are a.e. equal [83].

- Even if E^* is separable and $x(\cdot) : T \to E$ is a Lipschitz function, we cannot guarantee that $x'_p(\cdot)$ exists on T; in fact, $x'_p(\cdot)$ need not exist on any subset of T of positive Lebesgue measure [84].

1.4 Integration in Banach Spaces

The concept of Bochner integral and Pettis integral are well known [30, 61, 81]. Nevertheless, we recall the definition of Bochner integral, Pettis integral and Riemann integral.

Definition 1.4.1. Let $(T; \Sigma; \mu)$ be a measure space and $f : T \to E$ be a function. The function f is called Bochner-integrable if there exists a sequence of simple functions $f_n : T \to E$ such that $\int_T \|f - f_n\| d\mu \to 0$, as $n \to \infty$. The Bochner integral of f is then defined as $\int_T f d\mu = \lim_{n\to\infty} \int_T f_n d\mu$, where the integral for simple functions is defined in the obvious way: if $f_n = \sum x_i \chi_{A_i}$, then $\int_T f_n d\mu = \lim_{n\to\infty} \sum_i^n x_i \mu(A_i)$. It is easy to see that such a limit indeed exists and does not depend on the choice of a sequence f_n approximating the given function f.

Definition 1.4.2. A weakly measurable function $x(\cdot) : T \to E$ is said to be *Pettis integrable* on T if
(a) $x(\cdot)$ is *scalarly integrable*; that is, for every $x^* \in E^*$ the real function $t \mapsto \langle x^*, x(t)\rangle$ is Lebesgue integrable on T;
(b) for every set $A \in \mathcal{L}(T)$ there exists an element $x_A \in E$ such that

$$\langle x^*, x_A\rangle = \int_A \langle x^*, x(s)\rangle\, ds \tag{1.4.1}$$

for every $x^* \in E^*$. The element $x_A \in E$ is called the *Pettis integral* on A and it will be denoted by $\int_A x(s)ds$.

It is easy to show that a Bochner integrable function $x(\cdot) : T \to E$ is Pettis integrable and both integrals of $x(\cdot)$ are equal on each Lebesgue measurable subset A of T ([81, Proposition 2.3.1]).

***Remark* 1.4.1.** It is known that if $x(\cdot) : T \to E$ is Bochner integrable on T, then the function $y(\cdot) : T \to E$, given by

$$y(t) = (B)\int_0^t x(s)ds,\ t \in T,$$

is *AC* and a.e. differentiable on T, and $y'_s(t) = x(t)$ for a.e. $t \in T$. Also, if a function $x(\cdot) : T \to E$ is *AC* and a.e. strongly differentiable on T, then $x'_s(\cdot)$ is Bochner integrable on T and

$$x(t) = x(0) + (B)\int_0^t x'_s(s)ds,\ t \in T.$$

***Remark* 1.4.2.** In the case of the Pettis integral, in [60, 67] it was shown that if $x(\cdot) : T \to E$ is *AC* and a.e. weakly differentiable on T, then $x'_w(\cdot)$ is Pettis integrable on T and

$$x(t) = x(0) + \int_0^t x'_w(s)ds,\ t \in T.$$

In 1994 Kadets [45] prove that there exists a strongly measurable and Pettis integrable function $x(\cdot) : T \to E$ such that the indefinite Pettis integral

$$y(t) = \int_0^t x(s)ds,\ t \in T, \tag{1.4.2}$$

is not weakly differentiable on a set of positive Lebesgue measure (see also [60, 68]). In 1995 Dilworth and Girardi [31] showed that there always exists a strongly measurable and Pettis integrable function $x(\cdot) : T \to E$ such that the indefinite Pettis integral (1.4.2) is nowhere weakly differentiable. The best result for a descriptive definition of the Pettis integral is that given by Pettis in [67].

Proposition 1.4.1. *[67] Let $x(\cdot) : T \to E$ be a weakly measurable function.*
(a) If $x(\cdot)$ is Pettis integrable on T, then the indefinite Pettis integral (1.4.2) is AC on T and $x(\cdot)$ is a pseudo-derivative of $y(\cdot)$.
(b) If $y(\cdot) : T \to E$ is an AC function on T and it has a pseudo-derivative $x(\cdot)$ on T, then $x(\cdot)$ is Pettis integrable on T and

$$y(t) = y(0) + \int_0^t x(s)ds, t \in T.$$

It is known that the Pettis integrals of two strongly measurable functions $x(\cdot) : T \to E$ and $y(\cdot) : T \to E$ coincide over every Lebesgue measurable set in T if and only if $x(\cdot) = y(\cdot)$ a.e. on T [67, Theorem 5.2]. Since a pseudo-derivative of a pseudo-differentiable function $x(\cdot) : T \to E$ is not unique and two pseudo-derivatives of $x(\cdot)$ need not be a.e. equal, then the concept of weakly equivalence plays an important role in the following.

Definition 1.4.3. Two weakly measurable functions $x(\cdot) : T \to E$ and $y(\cdot) : T \to E$ are said to be *weakly equivalent* on T if for every $x^* \in E^*$ we have that $\langle x^*, x(t)\rangle = \langle x^*, y(t)\rangle$ for a.e. $t \in T$.

***Remark* 1.4.3.** From an integral point of view, the weak measurable functions with the same indefinite Pettis integral are weakly equivalent. Obviously, every two pseudo-derivatives of a pseudo-differentiable function are weakly equivalent, if there exist. However, the functions that are weakly equivalent to strongly measurable functions need not themselves be strongly measurable ([37, Example 2.3]). In fact, to say that a weakly measurable function $x(\cdot) : T \to E$ is weakly equivalent to a strongly measurable function is the same as saying that the indefinite integral of $x(\cdot)$ is given by a Bochner integrable function (see [85]). An example of a weakly measurable function that is not strongly measurable but is weakly equivalent to a strongly measurable function can be found in [61, Example 3.1].

If two weakly measurable functions $x(\cdot) : T \to E$ and $y(\cdot) : T \to E$ are weakly equivalent on T, then we will write $x(\cdot) \approx y(\cdot)\ t \in T$.

Proposition 1.4.2. *[67] A weakly measurable function $x(\cdot) : T \to E$ is Pettis integrable on T and $\langle x^*, x(\cdot)\rangle \in L^\infty(T)$ for every $x^* \in E^*$, if and only if the function $t \mapsto \varphi(t)x(t)$ is Pettis integrable on T for every $\varphi(\cdot) \in L^1(T)$.*

We will now define the fractional integral of a vector-valued function using the Riemann-Pettis integral. Also, we will establish some properties of them. First, we recall the notion of Riemann integral for vector-valued functions.

The notion of Riemann integral for vector-valued functions was introduced by Graves [42].

Definition 1.4.4. A vector-valued function $x(\cdot) : T \to E$ is said to be *Riemann integrable* (or *R-integrable*, for short) on T if for any partition $\{t_0, ..., t_n\}$ of T and any choice of points $\xi_i \in [t_{i-1}, t_i]$, $i = 1, ..., n$, the sum given by

$$\sum_{i=1}^{n}(t_i - t_{i-1})x(\xi_i) \tag{1.4.3}$$

converge strongly to some $x_T \in E$ provided $\max_{1\le i\le n} |t_i - t_{i-1}| \to 0$ as $n \to \infty$.

The element x_T is called the Riemann-Graves integral of $x(\cdot)$ and it will be denoted by $(R)\int_0^b x(t)dt$.

Let us recall that a function $x(\cdot) : T \to E$ is said to be *scalarly Riemann integrable* if for every $x^* \in E^*$ the real function $t \mapsto \langle x^*, x(t)\rangle$ is Riemann integrable on T. In the following, let us recall some properties of Riemann integral. First, let us denote by $P^\infty(T, E)$ the space of all weakly measurable and Pettis integrable functions $x(\cdot) : T \to E$ with the property that $\langle x^*, x(\cdot)\rangle \in L^\infty(T)$ for every $x^* \in E^*$.

An R-integrable function $x(\cdot) : T \to E$ is also scalarly Riemann integrable and

$$\int_a^b \langle x^*, x(t)\rangle\, dt = \left\langle x^*, (R)\int_a^b x(t)dt \right\rangle$$

for every $x^* \in E^*$ (see [41, Theorem 7]). Using the Uniform Boundedness Principle it is easy to see that every scalarly Riemann integrable function is bounded. Therefore, if $x(\cdot) : T \to E$ is R-integrable on T, then for every $x^* \in E^*$ the real function $t \mapsto \langle x^*, x(t) \rangle$ is bounded and a.e. continuous on T. It follows that every R-integrable function $x(\cdot)$ is Pettis integrable (in fact, $x(\cdot) \in P^\infty(T, E)$) and every strongly measurable and R-integrable function $x(\cdot)$ is Bochner integrable (see [41, Theorem 15]).

Graves ([42, Theorem 1]) shows that any function which is discontinuous on a set of Lebesgue measure zero is R-integrable. Also, Graves [42] gives an example of a discontinuous everywhere function which is R-integrable. We also remark that there exist strongly measurable and R-integrable functions which are not a.e. continuous [70], and R-integrable functions $x(\cdot) : T \to E$ such that $\|x(\cdot)\|$ is not measurable and, hence neither Riemann integrable nor Lebesgue integrable [67]. Therefore, a R-integrable function is not necessarily Bochner integrable. More details on the properties of R-integral can be found in the works [9], [41], [42] and [71].

1.5 Weak Measure of Non-Compactness and Weak variant of Arcoli-Arzela Theorem

Let us denote by $P_{wk}(E)$ the set of all weakly compact subset of E. The *weak measure of non-compactness* [27] is the set function $\beta : P_{wk}(E) \to [0, \infty)$ defined by

$$\beta(A) = \inf\{r > 0;\ \text{there exist } K \in P_{wk}(E) \text{ such } A \subset K + rB_1\},$$

where B_1 is the closed unit ball in E. The properties of weak non-compactness measure are analogous to the properties of measure of non-compactness:

(i) $A \subseteq B$ implies that $\beta(A) \leq \beta(B)$;

(ii) $\beta(A) = \beta(cl_w A)$, where $cl_w A$ denotes the weak closure of A;

(iii) $\beta(A) = 0$ if and only if $cl_w A$ is weakly compact;

(iv) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$;

(v) $\beta(A) = \beta(conv(A))$;

(vi) $\beta(A + B) \leq \beta(A) + \beta(B)$;

(vii) $\beta(x + A) = \beta(A)$, for all $x \in E$;

(viii) $\beta(\lambda A) = |\lambda|\beta(A)$, for all $\lambda \in \mathbb{R}$;
(ix) $\beta(\cup_{0\leq r\leq r_0} rA) = r_0\beta(A)$;
(x) $\beta(A) \leq 2diam(A)$.

Lemma 1.5.1. *([8, 23]) Let $H \subset C(T, E)$ be bounded and equicontinuous. Then*
(i) the function $t \to \beta(H(t))$ is continuous on T,
(ii) $\beta_c(H) = \sup_{t\in T} \beta(H(t)) = \beta(H(t))$,
where $\beta_c(\cdot)$ denote the weak non-compactness measure on $C(T, E)$ and $H(t) = \{u(t),$ $u \in H\}, \; t \in T$.

Definition 1.5.1.

- Let $\{x_n(t)\}$ be a sequence of functions from T into E, then $\{x_n(t)\}$ converges weakly uniformly to $x(t)$, where $x : T \to E$, if for $\varepsilon > 0$, $x^* \in E^*$ there exists $N = N(x^*, \varepsilon)$ such that $n > N$ implies

$$\Big|\langle x^*, x_n(t) - x(t)\rangle\Big| < \varepsilon \; for \; all \, t \in T.$$

- The family $\{x_n(t)\}$ is said to be weakly equicontinuous if given $\varepsilon > 0$, $x^* \in E^*$, there exists a $\delta = \delta(x^*, \varepsilon)$ such that

$$\Big|\langle x^*, x_n(t) - x_n(s)\rangle\Big| < \varepsilon$$

whenever $|t - s| < \delta$ and for any $n \in \mathbb{N}$.

- if $\{x_n\}$ is a sequence in E, then $\{x_n\}$ is weakly Cauchy if given $\varepsilon > 0$, $x^* \in E^*$, there exists $N = N(x^*, \varepsilon)$ such that $n, m \geq N$ implies that

$$\Big|\langle x^*, (x_n - x_m)\rangle\Big| < \varepsilon;$$

- The Banach space E is weakly sequentially complete if every weakly Cauchy sequence is weakly convergent in E.

Theorem 1.5.2. *[54] Let E be a metrizable locally convex topological vector space and let K be a closed convex subset of E, and let Q be a weakly sequentially continuous map of K into itself. If for some $y \in K$ the implication*

$$\overline{V} = \overline{conv}(Q(V) \cup \{y\}) \Rightarrow V \text{ is relatively weakly compact,}$$

holds for every subset V of K, then Q has a fixed point.

Chapter 2

Pseudo Solution for Fractional Differential Equations

In chapter 1 we recalled some basic definitions and results about differentiation and integration in Banach spaces. In the present chapter we establish an existence result for the fractional differential equations in nonreflexive Banach spaces.

$$\begin{cases} D_p^\alpha y(t) = f(t, y(t)) \\ y(0) = y_0 \end{cases}$$

where $D_p^\alpha y(\cdot)$ is a fractional pseudo-derivative of a weakly absolutely continuous and pseudo-differentiable function $y(\cdot) : T \to E$, the function $f(t, \cdot) : T \times E \to E$ is weakly-weakly sequentially continuous for every $t \in T$ and $f(\cdot, y(\cdot))$ is Pettis integrable for every weakly absolutely continuous function $y(\cdot) : T \to E$, T is a bounded interval of real numbers containing zero and E is a nonreflexive Banach space.

2.1 Fractional Pettis integral and Abel integral equation

Let us denote by $P^\infty(T, E)$ the space of all weakly measurable and Pettis integrable functions $x(\cdot) : T \to E$ with the property that $\langle x^*, x(\cdot)\rangle \in L^\infty(T)$ for every $x^* \in E^*$. Since for each $t \in T$ the real valued function $s \mapsto (t-s)^{\alpha-1}$ is Lebesgue integrable on $[0, t]$ for every $\alpha > 0$ then, by Proposition 1.4.2, the *fractional Pettis integral*

$$I^\alpha x(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, \; t \in T,$$

exists for every function $x(\cdot) \in P^{\infty}(T, E)$ as a function from T into E. Moreover, we have that

$$\langle x^*, I^{\alpha}x(t) \rangle = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \langle x^*, x(s) \rangle \, ds, \; t \in T,$$

for every $x^* \in E^*$, and the real function $t \mapsto \langle x^*, I^{\alpha}x(t) \rangle$ is continuous (in fact, bounded and uniformly continuous on T if $T = \mathbb{R}$) on T for every $x^* \in E^*$ ([9, Proposition 1.3.2]).

Example 2.1. Let T be the interval $[0,1]$ and define $f : T \to L^{\infty}(T)$ by $f(t) = \chi_{[0,t]}$. Then this function is weakly measurable and Pettis integrable, but not strongly measurable (see [37], [85]). To compute the fractional Pettis integral of f, we consider $\psi \in L^1(T)$, and let x^* be the element of $L^{\infty}(T)^*$ corresponding to ψ. Since

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \langle x^*, f(s) \rangle \, ds = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\langle x^*, \chi_{[0,s]} \right\rangle ds =$$

$$= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 \psi(\tau)\chi_{[0,s]}(\tau) d\tau ds = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \psi(\tau) d\tau ds$$

$$= \int_0^t \int_{\tau}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \psi(\tau) ds d\tau = \int_0^t \frac{(t-\tau)^{\alpha}}{\Gamma(1+\alpha)} \psi(\tau) d\tau$$

$$= \int_0^1 \frac{(t-\tau)^{\alpha}}{\Gamma(1+\alpha)} \chi_{[0,t]}(\tau)\psi(\tau) d\tau = \left\langle x^*, \frac{(t-\tau)^{\alpha}}{\Gamma(1+\alpha)} \chi_{[0,t]} \right\rangle,$$

we infer that

$$\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds \right)(\tau) = \frac{(t-\tau)^{\alpha}}{\Gamma(1+\alpha)} \chi_{[0,t]} \in L^{\infty}(T), \;\; t \in [0,1].$$

Lemma 2.1.1. *If $x(\cdot), y(\cdot) \in P^{\infty}(T, E)$ are weakly equivalent on T, then $I^{\alpha}x(t) = I^{\alpha}y(t)$ on T.*

Proof. If $x(\cdot), y(\cdot) \in P^{\infty}(T, E)$ are weakly equivalent on T, then for every $x^* \in E^*$ there exists a null set $N(x^*) \in \mathcal{L}(T)$ such that for every $t \in T$ we have that $\langle x^*, x(s) \rangle = \langle x^*, y(s) \rangle$ for $s \in [0,t] \setminus N(x^*)$. It follows that $\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \langle x^*, x(s) \rangle = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \langle x^*, y(s) \rangle$ for $s \in [0,t] \setminus N(x^*)$ and $t \in T$, and so $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \langle x^*, x(s) \rangle \, ds = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \langle x^*, y(s) \rangle \, ds$ for $t \in T$. Therefore, we have that $\langle x^*, I^{\alpha}x(t) \rangle = \langle x^*, I^{\alpha}y(t) \rangle$ for every $x^* \in E^*$ and every $t \in T$, and thus $I^{\alpha}x(t) = I^{\alpha}y(t)$ on T. $\square$

Lemma 2.1.2. *The fractional Pettis integral is a linear operator from $P^{\infty}(T, E)$ into $P^{\infty}(T, E)$. Moreover, if $x(\cdot) \in P^{\infty}(T, E)$, then for $\alpha, \beta \in (0,1)$ we have* *(a)* $I^{\alpha}I^{\beta}x(t) = I^{\alpha+\beta}x(t)$, $t \in T$,

(b) $\lim_{\alpha\to 1} I^{\alpha}x(t) = I^{1}x(t)$ *weakly uniformly on* T *if only these integrals exist on* T.

Proof. Let $x(\cdot) \in P^{\infty}(T,E)$ and $\alpha > 0$. Since for each $t \in T$ the real valued function $s \mapsto (t-s)^{\alpha-1}$ is Lebesgue integrable on $[0,t]$ and for every $x^* \in E^*$ the real function $t \mapsto \langle x^*, x(t)\rangle$ is essentially bounded on T, then for every $x^* \in E^*$ the real function $t \mapsto \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \langle x^*, x(s)\rangle\, ds$ is continuous on T. Moreover, for every $x^* \in E^*$ we have

$$\int_T \langle x^*, I^{\alpha}x(t)\rangle\, dt = \int_0^b \left\langle x^*, \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \right\rangle dt =$$

$$\int_0^b \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \langle x^*, x(s)ds\rangle\, dt = \int_0^b \int_s^b \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \langle x^*, x(s)dt\rangle\, ds =$$

$$\int_0^b \frac{(b-s)^{\alpha}}{\Gamma(\alpha+1)} \langle x^*, x(s)\rangle\, ds = \left\langle x^*, \int_0^b \frac{(b-s)^{\alpha}}{\Gamma(\alpha+1)} x(s)ds \right\rangle = \langle x^*, x_T\rangle,$$

where $x_T = \int_0^b \frac{(b-s)^{\alpha}}{\Gamma(\alpha+1)} x(s)ds$. It follows that the function $t \mapsto I^{\alpha}x(t)$ is weakly measurable, $\langle x^*, x(\cdot)\rangle \in L^{\infty}(T)$ for every $x^* \in E^*$ and there exists a $x_T \in E$ such that $\langle x^*, x_T\rangle = \int_T \langle x^*, I^{\alpha}x(t)\rangle\, dt$; that is, $I^{\alpha}x(\cdot) \in P^{\infty}(T,E)$. Obviously, I^{α} is a linear operator. Next, since for every $x^* \in E^*$ the real function $t \mapsto \langle x^*, x(t)\rangle$ is essentially bounded on T, then from the semigroup property of fractional integration (see [38, Theorem 2.2]) it follows that for every $x^* \in E^*$ we have

$$I^{\alpha}I^{\beta}\langle x^*, x(t)\rangle = I^{\alpha+\beta}\langle x^*, x(t)\rangle$$

for $t \in T$. Therefore, for every $x^* \in E^*$ we have

$$\left\langle x^*, I^{\alpha}I^{\beta}x(t)\right\rangle = \left\langle x^*, I^{\alpha+\beta}x(t)\right\rangle$$

for $t \in T$ and so $I^{\alpha}I^{\beta}x(t) = I^{\alpha+\beta}x(t)$ for $t \in T$. For (b) see [74]. □

In the following, consider $\alpha \in (0,1)$ and for a given function $x(\cdot) \in P^{\infty}(T,E)$ we also denote by $x_{1-\alpha}(t)$ the fractional Pettis integral

$$I^{1-\alpha}x(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds,\ t \in T.$$

Consider the following *Abel integral equation*

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds = y(t),\ t \in T, \qquad (2.1.1)$$

where $y(\cdot) \in P^{\infty}(T, E)$ is a given function. Our aim is to find the conditions under which the integral equation (2.1.1) has a solution $x(\cdot) \in P^{\infty}(T, E)$. For this, suppose that $x(\cdot) \in P^{\infty}(T, E)$ is a solution of (2.1.1). Then from (2.1.1) it follows that

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \langle x^*, x(s) \rangle \, ds = \langle x^*, y(t) \rangle \, , \, t \in T, \tag{2.1.2}$$

for every $x^* \in E^*$. Since $\langle x^*, x(\cdot) \rangle \in L^1(T)$ and $\langle x^*, y(\cdot) \rangle \in L^1(T)$ for every $x^* \in E^*$, then using the same reasoning as in [80, Section 1.2] it follows that

$$\int_0^t \langle x^*, x(s) \rangle \, ds = \langle x^*, y_{1-\alpha}(t) \rangle \, , \, t \in T,$$

for every $x^* \in E^*$. Therefore, for every $x^* \in E^*$ the real function $t \mapsto \langle x^*, y_{1-\alpha}(t) \rangle$ is an AC function on T, and so $y_{1-\alpha}(\cdot) : T \to E$ is wAC on T. Moreover, for every $x^* \in E^*$ the real function $t \mapsto \langle x^*, y_{1-\alpha}(t) \rangle$ is a.e. differentiable on T and $\frac{d}{dt} \langle x^*, y_{1-\alpha}(\cdot) \rangle \in L^1(T)$. Then for every $x^* \in E^*$ there exists a null set $N(x^*) \in \mathcal{L}(T)$ such that the real function $t \mapsto \langle x^*, y_{1-\alpha}(t) \rangle$ is differentiable on $T \setminus N(x^*)$ and

$$\frac{d}{dt} \langle x^*, y_{1-\alpha}(t) \rangle = \langle x^*, x(t) \rangle \text{ for } t \in T \setminus N(x^*), \tag{2.1.3}$$

and thus $x(\cdot)$ is a pseudo-derivative of $y_{1-\alpha}(\cdot)$. Consequently, if $x(\cdot) \in P^{\infty}(T, E)$ is a solution of (2.1.1), then $y_{1-\alpha}(\cdot)$ is wAC on T, $x(\cdot) \in P^{\infty}(T, E)$ is a pseudo-derivative of $y_{1-\alpha}(\cdot)$, and $y_{1-\alpha}(0) = 0$.

Moreover, if $\widetilde{x}(\cdot) \in P^{\infty}(T, E)$ is another solution of (2.1.1), then it is not difficult to see that $x(\cdot)$ and $\widetilde{x}(\cdot)$ are weakly equivalent.

Conversely, suppose that the function $y_{1-\alpha}(\cdot)$ is wAC on T, has a pseudo-derivative $z(\cdot) \in P^{\infty}(T, E)$ and $y_{1-\alpha}(0) = 0$. Since $y_{1-\alpha}(\cdot)$ is wAC on T, then for every $x^* \in E^*$ the real function $t \mapsto \langle x^*, y_{1-\alpha}(t) \rangle$ is a.e. differentiable on T, and so for every $x^* \in E^*$ there exists a null set $N(x^*) \in \mathcal{L}(T)$ such that the real function $t \mapsto \langle x^*, y_{1-\alpha}(t) \rangle$ is differentiable on $T \setminus N(x^*)$ and $\langle x^*, y_{1-\alpha}(\cdot) \rangle \in L^1(T)$. For every $x^* \in E^*$, we consider a function $g(x^*)(\cdot) \in L^1(T)$ such that $g(x^*)(t) = \frac{d}{dt} \langle x^*, y_{1-\alpha}(t) \rangle$, $t \in T \setminus N(x^*)$. Further, since $y_{1-\alpha}(\cdot)$ has the pseudo-derivative $z(\cdot) \in P^{\infty}(T, E)$, then for every $x^* \in E^*$ there exists a null set $M(x^*) \in \mathcal{L}(T)$ such that the real function $t \mapsto \langle x^*, y_{1-\alpha}(t) \rangle$ is differentiable on $T \setminus M(x^*)$ and

$$\frac{d}{dt} \langle x^*, y_{1-\alpha}(t) \rangle = \langle x^*, z(t) \rangle \, , \, t \in T \setminus M(x^*).$$

It follows that for every $x^* \in E^*$ we can choose the null set $A(x^*) := N(x^*) \cup M(x^*)$ such that $t \mapsto \langle x^*, y_{1-\alpha}(t)\rangle$ is differentiable on $T \setminus A(x^*)$ and

$$g(x^*)(t) = \frac{d}{dt}\langle x^*, y_{1-\alpha}(t)\rangle = \langle x^*, z(t)\rangle, \, t \in T \setminus A(x^*). \tag{2.1.4}$$

Moreover, we observe that $g(x^*)(\cdot) \in L^\infty(T)$. We will now show that for every $x^* \in E^*$ the real function $g(x^*)(\cdot)$ is a solution of (2.1.2). For this purpose we replace $\langle x^*, x(\cdot)\rangle$ by $g(x^*)(\cdot)$ in the left-hand side of (2.1.2) and denote the result by $f(x^*)(\cdot)$, that is,

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(x^*)(s)ds = f(x^*)(t), \, t \in T. \tag{2.1.5}$$

If we show that for every $x^* \in E^*$ the functions $f(x^*)(\cdot)$ and $\langle x^*, y(\cdot)\rangle$ are equal on T, then our assertion is proved. Note using (2.1.5), the equality $f(x^*)(\cdot) = \langle x^*, y(\cdot)\rangle$ on T is shown in the proof of Theorem 2.1 from [80]. Consequently, since for every $x^* \in E^*$ the real function $g(x^*)(\cdot)$ is a solution of (2.1.2), it follows that the pseudo-derivative $z(\cdot)$ of $y_{1-\alpha}(\cdot)$ is weakly equivalent with a solution of (2.1.1). Finally, we remark that if $\alpha \in (0,1)$ is replaced by $\beta := 1-\alpha \in (0,1)$, then the previous reasoning remain valid.

Summarizing the above we obtain.

Theorem 2.1.3. *If $y(\cdot) \in P^\infty(T,E)$, then the Abel integral equation (2.1.1) has a solution in $P^\infty(T,E)$ if and only if the function $y_{1-\alpha}(\cdot)$ has the following properties:*

(a) $y_{1-\alpha}(\cdot)$ is wAC on T,

(b) $y_{1-\alpha}(\cdot)$ has a pseudo-derivative belonging to $P^\infty(T,E)$;

(c) $y_{1-\alpha}(0) = 0$.

If these conditions are satisfied, then any solution $x(\cdot) \in P^\infty(T,E)$ of (2.1.1) is weakly equivalent to a pseudo-derivative of $y_{1-\alpha}(\cdot)$; that is, $x(t) \approx \frac{d_p}{dt}\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} y(s)ds$ on T. If E is a weakly sequentially complete space, then condition (a) is replaced by

(a′) $y_{1-\alpha}(\cdot)$ is AC on T.

Corollary 2.1.4. *If $y(\cdot) \in P^\infty(T,E)$, then the Abel integral equation*

$$\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds = y(t), \, t \in T, \tag{2.1.6}$$

has a solution in $P^\infty(T,E)$ if and only if the function $y_\alpha(\cdot)$ has the following properties:

(a) $y_\alpha(\cdot)$ is wAC on T,

(b) $y_\alpha(\cdot)$ has a pseudo-derivative belonging to $P^\infty(T,E)$;

(c) $y_\alpha(0)=0$.

If these conditions are satisfied, then any solution $x(\cdot)\in P^\infty(T,E)$ of (2.1.6) is weakly equivalent to a pseudo-derivative of $y_\alpha(\cdot)$; that is, $x(t)\approx \frac{d_p}{dt}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds$ on T. If E is a weakly sequentially complete space, then condition (a) is replaced by

(a′) $y_\alpha(\cdot)$ is AC on T.

Corollary 2.1.5. *If $f(\cdot,\cdot):T\times E\to E$ is a function such that $f(\cdot,y(\cdot))\in P^\infty(T,E)$ for every wAC function $y(\cdot):T\to E$, then the function $f_\alpha(\cdot)$ given by*

$$f_\alpha(t)=\int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(\alpha)}f(s,y(s))ds,\ t\in T, \tag{2.1.7}$$

has the following properties:

(a) $f_\alpha(\cdot)$ is wAC on T,

(b) $f(\cdot,y(\cdot))$ is a pseudo-derivative of $f_\alpha(\cdot)$;

(c) $f_\alpha(0)=0$.

If E is a weakly sequentially complete space, then wAC is replaced by AC.

2.2 Fractional Pseudo-Derivative

In the following, consider $\alpha\in(0,1)$. If $y(\cdot):T\to E$ is a pseudo-differentiable function with a pseudo-derivative $x(\cdot)\in P^\infty(T,E)$ on T, then the following fractional Pettis integral

$$I^{1-\alpha}x(t)=\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}x(s)ds$$

exists on T. The fractional Pettis integral $I^{1-\alpha}x(\cdot)$ is called a *fractional pseudo-derivative* of $y(\cdot)$ on T and it will be denoted by $D_p^\alpha y(\cdot)$; that is,

$$D_p^\alpha y(t)=I^{1-\alpha}x(t),\ \ t\in T. \tag{2.2.1}$$

If $y(\cdot) : T \to E$ is an a.e. weakly differentiable function with the weak derivative $y'_w(\cdot) \in P^\infty(T, E)$ on T, then

$$D_w^\alpha y(t) := I^{1-\alpha} y'_w(t), \ t \in T, \tag{2.2.2}$$

is called the *fractional weak derivative* of $y(\cdot)$ on T.

Example 2.2. Let $\mathbf{c}_0$ be the space of all real sequences converging to zero and $T = [0, 1]$. Then $\mathbf{c}_0$ is a Banach space with respect to the norm $\|x\|_{\mathbf{c}_0} := \max\limits_{n\geq 1} |x_n|$, where $x = \{x_n\}_{n\geq 1}$. Also, we recall that $\mathbf{c}_0$ is not weakly sequentially complete. Let $x(\cdot) : T \to \mathbf{c}_0$ be defined by

$$x(t)(n) = x_n(t) = \begin{cases} n, & \text{if } 0 \leq t \leq \frac{1}{2n} \\ -n, & \text{if } \frac{1}{2n} < t \leq \frac{1}{n} \\ 0, & \text{if } \frac{1}{n} < t \leq 1. \end{cases}$$

Also, consider the function $y(\cdot) : T \to \mathbf{c}_0$ be defined by

$$y(t)(n) = y_n(t) = \begin{cases} nt, & \text{if } 0 \leq t \leq \frac{1}{2n} \\ 1 - nt, & \text{if } \frac{1}{2n} < t \leq \frac{1}{n} \\ 0, & \text{if } \frac{1}{n} < t \leq 1. \end{cases}$$

Then it is easy to see that $y_n(t) = \int_0^t x_n(s)ds$, $0 \leq t \leq 1$. Let $x^* \in \mathbf{c}_0^*$. Then there exists a sequence of real numbers $\{\lambda_n(x^*)\}_{n\geq 1}$ such that the series $\sum_{n=1}^\infty \lambda_n(x^*)$ is absolutely convergent and

$$\langle x^*, y(t)\rangle = \sum_{n=1}^\infty \lambda_n(x^*) y_n(t) = \sum_{n\geq 1} \lambda_n(x^*) \int_0^t x_n(s)ds.$$

Since

$$|\langle x^*, x(t)\rangle| = \left|\sum_{n=1}^\infty \lambda_n(x^*) x_n(t)\right| \leq \sum_{n=1}^\infty |\lambda_n(x^*)| < \infty$$

by the Fubini theorem we have

$$\langle x^*, y(t)\rangle = \int_0^t \left[\sum_{n=1}^\infty \lambda_n(x^*) x_n(s)\right] ds,$$

and so the function $t \mapsto \langle x^*, y(t)\rangle$ is absolutely continuous for every $x^* \in \mathbf{c}_0^*$ (see also [48]). Thus, for every $x^* \in E^*$ there exists a null set $N(x^*) \in \mathcal{L}(T)$ such that the real function $t \mapsto \langle x^*, x(t)\rangle$ is differentiable on $T \setminus N(x^*)$ and $\frac{d}{dt}\langle x^*, y(t)\rangle = \langle x^*, x(t)\rangle$, $t \in T \setminus N(x^*)$. It follows that the function $x(\cdot)$ is a pseudo-derivative of $y(\cdot)$ such that $x(\cdot) \in P^\infty(T)$. To compute the fraction pseudo-derivative of $x(\cdot)$, let $x^* \in \mathbf{c}_0^*$ be given and let $\sum_{n=1}^\infty \lambda_n(x^*)$ be its corresponding absolutely convergent series. We have

$$\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, x(s)\rangle \, ds = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \left[\sum_{n=1}^\infty \lambda_n(x^*) x_n(s)\right] ds$$

$$= \sum_{n=1}^\infty n\lambda_n(x^*) \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds = \sum_{n=1}^\infty \lambda_n(x^*) \frac{nt^{1-\alpha}}{\Gamma(2-\alpha)}$$

if $0 \le t \le \frac{1}{2n}$,

$$\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, x(s)\rangle \, ds = \sum_{n=1}^\infty n\lambda_n(x^*) \left[\int_0^{1/2n} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds - \int_{1/2n}^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds\right]$$

$$= \sum_{n=1}^\infty \lambda_n(x^*) \frac{n}{\Gamma(2-\alpha)} \left[t^{1-\alpha} - 2\left(t - \frac{1}{2n}\right)^{1-\alpha}\right]$$

if $\frac{1}{2n} < t \le \frac{1}{n}$, and

$$\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, x(s)\rangle \, ds = \sum_{n=1}^\infty n\lambda_n(x^*) \left[\int_0^{1/2n} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds - \int_{1/2n}^{1/n} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds\right]$$

$$= \sum_{n=1}^\infty \lambda_n(x^*) \frac{n}{\Gamma(2-\alpha)} \left[t^{1-\alpha} - 2\left(t - \frac{1}{2n}\right)^{1-\alpha} + \left(t - \frac{1}{n}\right)^{1-\alpha}\right]$$

if $\frac{1}{n} < t \le 1$. It follows that

$$D_p^\alpha x(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s) ds = z(t),\ 0 \le t \le 1,$$

where $z(\cdot) \in \mathbf{c}_0$ is defined by

$$z(t)(n) = \begin{cases} \frac{nt^{1-\alpha}}{\Gamma(2-\alpha)}, & \text{if } 0 \le t \le \frac{1}{2n} \\ \frac{n}{\Gamma(2-\alpha)} \left[t^{1-\alpha} - 2\left(t - \frac{1}{2n}\right)^{1-\alpha}\right], & \text{if } \frac{1}{2n} < t \le \frac{1}{n} \\ \frac{n}{\Gamma(2-\alpha)} \left[t^{1-\alpha} - 2\left(t - \frac{1}{2n}\right)^{1-\alpha} + \left(t - \frac{1}{n}\right)^{1-\alpha}\right], & \text{if } \frac{1}{n} < t \le 1. \end{cases}$$

***Remark* 2.2.1.** If $x(\cdot), \widetilde{x}(\cdot) \in P^\infty(T, E)$ are two pseudo-derivatives of $y(\cdot) : T \to E$, then $x(\cdot) \approx \widetilde{x}(\cdot)$ on T. Thus, Lemma 2.1.1 implies that $I^{1-\alpha}x(t) = I^{1-\alpha}\widetilde{x}(t)$ on T, and so $D_p^\alpha y(\cdot)$ does not depend on the pseudo-derivatives of $y(\cdot)$. If $y(\cdot) : T \to E$ is a.e. weakly differentiable on T, then its weak derivative $y'_w(\cdot)$ is a pseudo-derivative of $y(\cdot)$, and thus $D_p^\alpha y(t) = D_w^\alpha y(t)$ for $t \in T$.

Lemma 2.2.1. *If $y(\cdot) : T \to E$ is a pseudo-differentiable function with a pseudo-derivative $x(\cdot) \in P^\infty(T, E)$, then the function*

$$y_{1-\alpha}(t) := \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} y(s)ds, \ t \in T,$$

is wAC and it has a pseudo-derivative $\frac{d_p}{dt} y_{1-\alpha}(\cdot) \in P^\infty(T, E)$ such that

$$\frac{d_p}{dt} y_{1-\alpha}(t) \eqsim \frac{t^{-\alpha}}{\Gamma(1-\alpha)} y(0) + I^{1-\alpha} x(t) \text{ on } T. \tag{2.2.3}$$

Proof. Since $x(\cdot) \in P^\infty(T, E)$ is a pseudo-derivative of $y(\cdot)$, then for every $x^* \in E^*$ there exists a null set $N(x^*) \in \mathcal{L}(T)$ such that the real function $t \mapsto \langle x^*, y(t) \rangle$ is differentiable on $T \setminus N(x^*)$ and $\frac{d}{dt} \langle x^*, y(t) \rangle = \langle x^*, x(t) \rangle$ for $t \in T \setminus N(x^*)$. From the last equality we infer that

$$\int_0^s \langle x^*, x(\tau) \rangle \, d\tau = \langle x^*, y(s) \rangle - \langle x^*, y(0) \rangle, \ s \in T, \tag{2.2.4}$$

for every $x^* \in E^*$. From (2.2.4) we obtain that

$$\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \int_0^s \langle x^*, x(\tau) \rangle \, d\tau ds =$$

$$\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, y(s) \rangle \, ds - \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, y(0) \rangle \, ds, \ t \in T,$$

for every $x^* \in E^*$. It follows that

$$\begin{aligned} &\int_0^t \int_0^s \frac{(s-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, x(\tau) \rangle \, d\tau ds \\ &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, y(s) \rangle \, ds - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \langle x^*, y(0) \rangle, \end{aligned}$$

and so

$$\langle x^*, y_{1-\alpha}(t) \rangle = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \langle x^*, y(0) \rangle + \int_0^t z(s)ds, \ t \in T, \tag{2.2.5}$$

for every $x^* \in E^*$, where $z(s) = \int_0^s \frac{(s-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, x(\tau) \rangle \, d\tau, \ s \in [0, t]$. Since $x(\cdot) \in P^\infty(T, E)$, then from (2.2.5) we infer that the function $t \mapsto \langle x^*, y_{1-\alpha}(t) \rangle$ is AC on T for every $x^* \in E^*$. Thus, from (2.2.5) it follows that for every $x^* \in E^*$ we have

$$\frac{d}{dt} \langle x^*, y_{1-\alpha}(t) \rangle = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, y(0) \rangle + \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, x(\tau) \rangle \, d\tau, \ t \in T,$$

for a.e. $t \in T$. Therefore, the function $y_{1-\alpha}(\cdot)$ is wAC and has a pseudo-derivative weakly equivalent to $\frac{t^{-\alpha}}{\Gamma(1-\alpha)} y(0) + I^{1-\alpha} x(t)$ on T. □

***Remark* 2.2.2.** Relation (2.2.3) can be written as

$$D_p^{\alpha}x(t) \approx \frac{t^{-\alpha}}{\Gamma(1-\alpha)}y(0) - \frac{d_p}{dt}y_{1-\alpha}(t) \ on\ T. \tag{2.2.6}$$

In this case $\frac{d_p}{dt}y_{1-\alpha}(t)$ will be denoted by ${}^{RL}D_p^{\alpha}x(t)$ and it is called a *Riemann-Liouville pseudo-derivative* of $x(\cdot)$. The formula (2.2.6) suggests us that we can extend the definition of the fractional pseudo-derivative for functions $x(\cdot) \in P^{\infty}(T, E)$. Therefore, if $x(\cdot) \in P^{\infty}(T, E)$, then a fractional pseudo-derivative of $x(\cdot)$ is defined by (2.2.6). It follows that a fractional pseudo-derivative $D_p^{\alpha}x(t)$ are also defined for functions $x(\cdot)$ for which a Riemann-Liouville fractional pseudo-derivative ${}^{RL}D_p^{\alpha}x(t)$ exists.

Corollary 2.2.2. *Let E be a weakly sequentially complete space. If $y(\cdot) : T \to E$ is a.e. weakly differentiable, then the function $y_{1-\alpha}(\cdot)$ is AC on T.*

***Remark* 2.2.3.** Let $y(\cdot) : T \to E$ be a pseudo-differentiable function with a pseudo-derivative $x(\cdot) \in P^{\infty}(T, E)$. Then as in the proof of Lemma 2.2.1 we can show that the function

$$y_{\alpha}(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds, \ t \in T,$$

is wAC and has a pseudo-derivative $\frac{d_p}{dt}y_{\alpha}(t)$ such that

$$\frac{d_p}{dt}y_{\alpha}(t) \approx \frac{t^{\alpha-1}}{\Gamma(\alpha)}y(0) + I^{\alpha}x(t) \text{ on } T.$$

Also, if E is a weakly sequentially complete space and $y(\cdot) : T \to E$ is a.e. weakly differentiable, then the function $y_{\alpha}(\cdot)$ is AC on T.

Lemma 2.2.3. *If $y(\cdot) : T \to E$ is a pseudo-differentiable function with a pseudo-derivative $x(\cdot) \in P^{\infty}(T, E)$ and $\alpha, \beta \in (0, 1)$, then*

(a) $I^{\alpha}D_p^{\alpha}y(t) = y(t) - y(0)$ on T;

(b) $D_p^{\alpha}I^{\alpha}y(t) = y(t)$ on T.

Proof. Indeed, using (2.2.1), Lemma 2.1.2 and Proposition 1.4.1, we have

$$\begin{aligned} I^{\alpha}D_p^{\alpha}y(t) &= I^{\alpha}I^{1-\alpha}x(t) = I^1x(t) = \int_0^t x(s)ds \\ &= y(t) - y(0), \ t \in T. \end{aligned}$$

Also, using Lemma 2.1.1, Remark 2.2.3 and Proposition 1.4.1 we have

$$\begin{aligned} D_p^\alpha I^\alpha y(t) &= I^{1-\alpha}\frac{d_p}{dt}I^\alpha y(t) = I^{1-\alpha}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}y(0) + I^\alpha x(t)\right] \\ &= y(0) + \int_0^t x(s)ds = y(0) + y(t) - y(0) = y(t) \end{aligned}$$

on T. □

2.3 Differential Equations With Fractional Pseudo-Derivatives

In this section we establish an existence result for the following fractional differential equation

$$\begin{cases} D_p^\alpha y(t) = f(t,\, y(t)) \\ y(0) = y_0 \end{cases} \tag{2.3.1}$$

where $D_p^\alpha y(\cdot)$ is a fractional pseudo-derivative of the function $y(\cdot) : T \to E$ and $f(\cdot,\cdot) : T \times E \to E$ is a given function. Along with the Cauchy problem (2.3.1) consider the following integral equation

$$y(t) = y_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds, \quad t \in T, \tag{2.3.2}$$

where the integral is in the sense of Pettis.

A continuous function $y(\cdot) : T \to E$ is said to be a *solution* of (2.3.1) if $y(\cdot)$ is pseudo-differentiable, has a pseudo-derivative belonging to $P^\infty(T,E)$, $D_p^\alpha y(t) \approx f(t,\, y(t))$ for $t \in T$ and $y(0) = y_0$.
To prove a result of the existence of solutions for (2.3.1) we need some preliminary results.

Lemma 2.3.1. *Let $f(\cdot,\cdot) : T \times E \to E$ be a function such that $f(\cdot, y(\cdot)) \in P^\infty(T,E)$ for every continuous function $y(\cdot) : T \to E$. Then a continuous function $y(\cdot) : T \to E$ is a solution of (2.3.1) if and only if it satisfies the integral equation (2.3.2).*

Proof. Indeed, if a continuous function $y(\cdot) : T \to E$ is a solution of (2.3.1), then from Lemma 2.2.3(a) it follows that $y(t) - y(0) = I^\alpha f(t, y(t))$ on T; that is, $y(\cdot)$ satisfies the integral equation (2.3.2). Conversely, suppose that a continuous function $y(\cdot) : T \to E$ satisfies the integral equation (2.3.2). Then the function $z(\cdot) := f(\cdot, y(\cdot)) \in P^\infty(T, E)$ satisfies the Abel equation

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} z(s) ds = v(s), \quad t \in T,$$

where $v(t) := y(t) - y_0$, $t \in T$. From Theorem 2.1.3 and Remark 2.2.2 it follows that $v_{1-\alpha}(\cdot)$ has a pseudo-derivative on T and

$$z(t) \approx \frac{d_p}{dt} v_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} y_0 - \frac{d_p}{dt} y_{1-\alpha}(t) \text{ for } t \in T.$$

Then by (2.2.6) we have that $z(t) \approx D_p^\alpha y(t)$ for $t \in T$; that is, $D_p^\alpha y(t) \approx f(t, y(t))$ on T.

2.4 An Existence Results of Fractional Differential Equation

In this section we shell discuss the existence of solutions of fractionl differential equations in nonreflexive Banach spaces. We recall that a function $f(\cdot) : E \to E$ is said to be *sequentially continuous* from E_w *into* E_w (or *weakly-weakly sequentially continuous*) if for every weakly convergent sequence $\{x_n\}_{n \geq 1} \subset E$, the sequence $\{f(x_n)\}_{n \geq 1}$ is weakly convergent in E.

By a *Gripenberg function* we mean a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(\cdot)$ is continuous, nondecreasing with $g(0) = 0$ and $u \equiv 0$ is the only continous solution of

$$u(t) \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(u(s)) ds, \;\; u(0) = 0. \tag{2.4.1}$$

The problem of uniqueness of the null solution of (2.4.1) was studied by Gripenberg in [39].

Theorem 2.4.1. *Assume $f(\cdot, \cdot) : T \times E \to E$ be a function such that:*
(h1) $f(t, \cdot)$ is weakly-weakly sequentially continuous for every $t \in T$;
(h2) $f(\cdot, y(\cdot)) \in P^\infty(T, E)$ for every continuous function $y(\cdot) : T \to E$;
(h3) $||f(t, y)|| \leq M$ for all $(t, y) \in T \times E$;
(h4) for every bounded set $A \subseteq E$ we have

$$\beta(f(T \times A)) \leq g(\beta(A)),$$

where $g(\cdot)$ *is a Gripenberg function. Then (2.3.1) admits a solution* $y(\cdot)$ *on an interval* $T_0 = T$ *with* $a = \min\left\{b, \left(\frac{r\Gamma(\alpha+1)}{M}\right)^{1/\alpha}\right\}$.

Proof. In our proof we shall use some ideas from the papers of Cichoń [22] and Salem & El–Sayed [74] (see also Cichoń & all. [22], Salem & Cichoń [79]). We define the nonlinear operator $Q(\cdot) : C(T_0, E) \to C(T_0, E)$ by

$$(Qy)(t) = y_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds, \quad t \in T_0.$$

If $y(\cdot) \in C(T_0, E)$, then by (h2) we have that $f(\cdot, y(\cdot)) \in P^\infty(T, E)$ and so the operator Q makes sense. To show that Q is well defined, let $t_1, t_2 \in T_0$ with $t_2 > t_1$. Without loss of generality, assume that $(Qy)(t_2) - (Qy)(t_1) \neq 0$. Then by the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and $\|(Qy)(t_2) - (Qy)(t_1)\| = |\langle y^*, (Qy)(t_2) - (Qy)(t_1)\rangle|$. Then

$$\begin{aligned}
&\|(Qy)(t_2) - (Qy)(t_1)\| = |\langle y^*, (Qy)(t_2) - (Qy)(t_1)\rangle| \\
=\; & \left| \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \langle y^*, f(s, y(s))\rangle\, ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \langle y^*, f(s, y(s))\rangle\, ds \right| \\
\leq\; & \int_0^{t_1} \left(\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \right) |\langle y^*, f(s, y(s))\rangle|\, ds + \qquad (2.4.2) \\
& \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |\langle y^*, f(s, y(s))\rangle|\, ds \\
\leq\; & \frac{M}{\Gamma(1+\alpha)} \left[t_1^\alpha - t_2^\alpha + 2(t_2-t_1)^\alpha\right] \leq \frac{2M}{\Gamma(1+\alpha)} (t_2-t_1)^\alpha,
\end{aligned}$$

so Q maps $C(T_0, E)$ into itself. Let $r > 0$ and let $\widetilde{B}$ be the convex, closed and equicontinous set defined by

$$\begin{aligned}
\widetilde{B} &= \{y(\cdot) \in C(T_0, E); \|y(\cdot)\|_c \leq \|y_0\| + r,\ \|y(t_2) - y(t_1)\| \\
&\leq \frac{2M}{\Gamma(1+\alpha)} (t_2-t_1)^\alpha \text{ for all } t_1, t_2 \in T_0\}.
\end{aligned}$$

We will show that Q maps $\widetilde{B}$ into itself and Q restricted to the set $\widetilde{B}$ is weakly-weakly sequentially continuous. To show that $Q : \widetilde{B} \to \widetilde{B}$, let $y(\cdot) \in \widetilde{B}$ and $t \in T_0$. Again, without loss of generality, assume that $(Qy)(t) \neq 0$. By the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and $\|(Qy)(t)\| = |\langle y^*, (Qy)(t)\rangle|$.

Then by (h3), we have

$$\begin{aligned}\|(Qy)(t)\| &\leq \|y_0\| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\langle y^*, f(s,y(s))\rangle|\, ds \\ &\leq \|y_0\| + \frac{Ma^\alpha}{\Gamma(\alpha+1)} \leq \|y_0\| + r,\end{aligned}$$

and using (2.4.2) it follows that Q maps $\widetilde{B}$ into $\widetilde{B}$. Next, we show that Q is weakly-weakly sequentially continuous. First, we recall that the weak convergence in $\widetilde{B} \subset C(T_0, E)$ is exactly the weak pointwise convergence. Let $\{y_n(\cdot)\}_{n\geq 1}$ be a sequence in $\widetilde{B}$ such that $y_n(\cdot)$ converges weakly to $y(\cdot)$ in $\widetilde{B}$. Then $y_n(t)$ converges weakly to $y(t)$ in E for each $t \in T_0$. Since $\widetilde{B}$ is a closed convex set, by Mazur's lemma we have $y(\cdot) \in \widetilde{B}$. Further, by (h1) it follows that $f(t, y_n(t))$ converges weakly to $f(t, y(t))$ for each $t \in T_0$. Then the Lebesgue dominated convergence theorem for the Pettis integral (see [88]) yields $I^\alpha y_n(t)$ converging weakly to $I^\alpha y(t)$ in E for each $t \in T_0$. Since $\widetilde{B}$ is equicontinuous subset of $C(T_0, E)$ it follows that $Q(\cdot)$ is weakly-weakly sequentially continuous.

Suppose that $V \subset \widetilde{B}$ such that $V = \overline{co}(Q(V) \cup \{y(\cdot)\})$ for some $y(\cdot) \in \widetilde{B}$. We will show that V is relatively weakly compact in $C(T_0, E)$. Let

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, V(s))ds = \left\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds; y(\cdot) \in V\right\}$$

and $(QV)(t) = y_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, V(s))ds$. Let $t \in T_0$ and $\varepsilon > 0$. If we choose $\eta > 0$ such that $\eta < \left(\frac{\varepsilon\Gamma(\alpha+1)}{M}\right)^{1/\alpha}$ and $\int_{t-\eta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds \neq 0$ then, by the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and

$$\left\|\int_{t-\eta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds\right\| = \left|\left\langle y^*, \int_{t-\eta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds\right\rangle\right|.$$

It follows that

$$\left\|\int_{t-\eta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds\right\| \leq \int_{t-\eta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\langle y^*, f(s, y(s))\rangle|\, ds \leq \varepsilon,$$

and thus using property (x) measure of the non-compactness we infer

$$\beta\left(\int_{t-\eta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, V(s))ds\right) \leq 2\varepsilon. \tag{2.4.3}$$

Since by Lemma 1.5.1 the function $s \to v(s) := \beta(V(s))$ is continuous on $[0, t-\eta]$ it follows that $s \to (t-s)^{\alpha-1} g(v(s))$ is continuous on $[0, t-\eta]$. Hence, there exists $\delta > 0$ such that

$$\left\|(t-\tau)^{\alpha-1} g(v(\tau)) - (t-s)^{\alpha-1} g(v(s))\right\| < \frac{\varepsilon}{2}$$

and

$$\|g(v(\xi)) - g(v(\tau))\| < \frac{\varepsilon}{2\eta^{\alpha-1}}.$$

If $|\tau - s| < \delta$ and $|\tau - \xi| < \delta$ with $\tau, s, \xi \in [0, t-\eta]$, then it follows that

$$\begin{aligned}&|(t-\tau)^{\alpha-1} g(v(\xi)) - (t-s)^{\alpha-1} g(v(s))| \leq \\ &|(t-\tau)^{\alpha-1} g(v(\tau)) - (t-s)^{\alpha-1} g(v(s))| + (t-\tau)^{\alpha-1} |g(v(\xi)) - g(v(\tau))| < \varepsilon,\end{aligned}$$

that is

$$|(t-\tau)^{\alpha-1} g(v(\xi)) - (t-s)^{\alpha-1} g(v(s))| < \varepsilon, \tag{2.4.4}$$

for all $\tau, s, \xi \in [0, t-\eta]$ with $|\tau - s| < \delta$ and $|\tau - \xi| < \delta$. Consider the following partition of the interval $[0, t-\eta]$ into n parts $0 = t_0 < t_1 \ldots < t_n = t - \eta$ such that $t_i - t_{i-1} < \delta$ $(i = 1, 2, \ldots, n)$. By Lemma 1.5.1 for each i there exists $s_i \in [t_{i-1}, t_i]$ such that $\beta(V([t_{i-1}, t_i])) = v(s_i)$, $i = 1, 2, \ldots, n$. Then we have (see [37, Theorem 2.2])

$$\begin{aligned}&\int_0^{t-n} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, V(s)) ds \subset \\ &\subset \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (t-s)^{\alpha-1} f(s, V(s)) ds \\ &\subset \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1}) \overline{conv} \{(t-s)^{\alpha-1} f(s, y(s)); s \in [t_{i-1}, t_i],\ y \in V\},\end{aligned}$$

and so

$$\begin{aligned}&\beta\left(\int_0^{t-\eta} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, V(s)) ds\right) \leq \\ &\leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1}) \beta\left(\overline{conv} \{(t-s)^{\alpha-1} f(s, y(s)); s \in [t_{i-1}, t_i],\ y \in V\}\right) \\ &= \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1}) \beta\left(\{(t-s)^{\alpha-1} f(s, y(s)); s \in [t_{i-1}, t_i],\ y \in V\}\right)\end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1})(t - t_i)^{\alpha-1} \beta \left(f(T_0 \times V[t_{i-1}, t_i]) \right)$$

$$\leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1})(t - t_i)^{\alpha-1} g(\beta(V[t_{i-1}, t_i])) \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1})(t - t_i)^{\alpha-1} g(v(s_i)).$$

Using (2.4.4) we have that

$$|(t - t_i)^{\alpha-1} g(v(s_i)) - (t - s)^{\alpha-1} g(v(s))| < \varepsilon.$$

This implies that

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1})(t - t_i)^{\alpha-1} g(v(s_i)) \leq \int_0^{t-\eta} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(v(s))ds + \varepsilon(t - \eta).$$

Thus we obtain

$$\beta \left(\int_0^{t-\eta} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, V(s))ds \right) \leq \int_0^{t-\eta} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(v(s))ds + \varepsilon(t - \eta). \quad (2.4.5)$$

Because

$$(QV)(t) \subset \frac{1}{\Gamma(\alpha)} \int_0^{t-\eta} (t-s)^{\alpha-1} f(s, V(s))ds + \frac{1}{\Gamma(\alpha)} \int_{t-\eta}^{t} (t-s)^{\alpha-1} f(s, V(s))ds,$$

then by virtue of (2.4.3) and (2.4.5) we have

$$\begin{aligned} \beta((QV)(t)) &\leq \int_0^{t-\eta} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(v(s))ds + \varepsilon(t - \eta) + 2\varepsilon \\ &\leq \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(v(s))ds + \varepsilon(t + \tau). \end{aligned}$$

As the last inequality is true for every $\varepsilon > 0$, we infer

$$\beta((QV)(t)) \leq \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(v(s))ds.$$

Because $V = \overline{co}(Q(V) \cup \{y(\cdot)\})$ then

$$\beta(V(t)) = \beta \left(\overline{co}(Q(V(t)) \cup \{y(t)\}) \right) \leq \beta((QV(t)))$$

and thus

$$v(t) \leq \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(v(s))ds \text{ for } t \in T_0.$$

Since $g(\cdot)$ is a Gripenberg function, it follows that $v(t) = 0$ for $t \in T_0$. Since V as a subset of $\widetilde{B}$ is equicontinuous, by Lemma 1.5.1 we infer

$$\beta_c(V(T_0)) = \sup_{t \in T_0} \beta(V(t)) = 0.$$

Thus, by Arzelá-Ascoli's theorem we obtain that V is weakly relatively compact in $C(T_0, E)$. Using Theorem 1.5.2 there exists a fixed point of the operator Q which is a solution of (2.3.1). □

If E is reflexive and $f(\cdot, \cdot) : T \times E \to E$ is bounded, then (h4) is automatically satisfied since a subset of a reflexive Banach space is weakly compact iff it is closed in the weak topology and bounded in the norm topology.

We end this section with some remarks. If for $\alpha = 1$ we put $D_p^1 y(\cdot) = y_p'(\cdot)$, then from Theorem 2.4.1 we obtain the following result (see [17], [49]).

Corollary 2.4.2. *If $f(\cdot, \cdot) : T \times E \to E$ is a function that satisfies the conditions (h1)-(h4) in Theorem 2.4.1, then the differential equation*

$$\begin{cases} y_p'(t) = f(t, y(t)) \\ y(0) = y_0 \end{cases} \tag{2.4.6}$$

has a solution on $[0, a]$ with $a = \min\{b, r/M\}$.

2.5 Conclusions

In this chapter, we developed the fractional calculus for functions with values in a nonreflexive Banach space equipped with the weak topology. Involving the concept of Pettis integral, we introduced and studied the notions of fractional Pettis integral and pseudo-fractional derivative. Using these tools we obtain an existence result for fractional differential equations in a nonreflexive Banach space equipped with the weak topology.

Chapter 3

Multi-term Fractional Differential Equation

3.1 Multi-term Fractional Differential Equations

In chapter 2, we introduced pseudo solution of fractional differential equations in nonreflexive Banach spaces. We also exhibited the existence of solution of fractional differential equations in nonreflexive Banach spaces. Furthermore, in last chapter we considered those differential equations which has one differential operator but in certain cases we deal with differential equations which contain more than one differential operator. This type of differential equations are known as multi-term differential equations. In this chapter we will establish an existence result for the multi-term fractional differential equations in nonreflexive Banach spaces. Consider the following multi-term fractional differential equation,

$$\left(D^{\alpha_m} - \sum_{i=1}^{m-1} a_i D^{\alpha_i}\right) u(t) = f(t,\, u(t)) \quad for\ t \in [0,1],\ u(0) = 0. \tag{3.1.1}$$

where $D^{\alpha_m}u(\cdot)$ and $D^{\alpha_i}u(\cdot)$ are fractional pseudo-derivative of a weakly absolutely continuous and pseudo-differentiable function $u(\cdot) : [0,1] \to E$ of order α_m and α_i, $i = 1, 2, \ldots, m-1$, respectively, the function $f(t,\cdot) : [0,1] \times E \to E$ is weakly-weakly sequentially continuous for every $t \in [0,1]$ and $f(\cdot, y(\cdot))$ is Pettis integrable for every weakly absolutely continuous function $y(\cdot) : [0,1] \to E$, E is nonreflexive Banach space, $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_m < 1$ and $a_1, a_2 \ldots a_{m-1}$ are real numbers such that $a := \sum_{i=1}^{m-1} \frac{|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} < 1$.

Along with the Cauchy problem (3.1.1) consider the following integral equation,

$$u(t) = \sum_{i=1}^{m-1} \int_0^t \frac{a_i(t-s)^{(\alpha_m - \alpha_i - 1)}}{\Gamma(\alpha_m - \alpha_i)} u(s)ds + \int_0^t \frac{(t-s)^{\alpha_m - 1}}{\Gamma(\alpha_m)} f(s, u(s))ds, \qquad (3.1.2)$$

$t \in T = [0, 1]$, where the integral is in the sense of Pettis.

Definition 3.1.1. A continuous function $u(\cdot) : T \to E$ is said to be a solution of (3.1.1) if,

(i) $u(\cdot)$ has pseudo derivative of order α_i, $i = 1, 2, \dots m$,

(ii) the pseudo derivative of $u(\cdot)$ of order α_i, $i = 1, 2, \dots m$, belong to $P^\infty(T, E)$,

(iii) $\left(D^{\alpha_m} - \sum\limits_{i=1}^{m-1} a_i D^{\alpha_i} \right) u(t) \approx f(t, u(t))$ for all $t \in T$,

(iv) $u(0) = 0$.

Reasoning as in the proof of the Lemma 2.3.1 we can easily prove the following result.

Lemma 3.1.1. *Let $f(\cdot, \cdot) : T \times E \to E$ be a function such that $f(\cdot, u(\cdot)) \in P^\infty(T, E)$ for every continuous function $u(\cdot) : T \to E$. Then a continuous function $u(\cdot) : T \to E$ is a solution of (3.1.1) if and only if it satisfies the integral equation (3.1.2).*

We recall that a function $f(\cdot) : E \to E$ is said to be *sequentially continuous* from *E_w into E_w* (or *weakly-weakly sequentially continuous*) if for every weakly convergent sequence $\{x_n\}_{n \geq 1} \subset E$, the sequence $\{f(x_n)\}_{n \geq 1}$ is weakly convergent in E.

3.2 Existence of Solution

In this section we establish an existence result for the multi-term fractional differential equation (3.1.1) in nonreflexive Banach spaces.

Theorem 3.2.1. *Let $r > 0$. Assume $f(\cdot, \cdot) : T \times E \to E$ be a function such that:*

(h1) $f(t, \cdot)$ is weakly-weakly sequentially continuous for every $t \in T$;

(h2) $f(\cdot, u(\cdot))$ is Pettis integrable for every continuous function $u(\cdot) : T \to E$;

(h3) $||f(t, y)|| \leq M$ for all $(t, y) \in T \times E$;

(h4) for every bounded set $A \subseteq E$ we have

$$\beta(f(T \times A)) \leq g(\beta(A)),$$

where $g(\cdot)$ is a Gripenberg function. Then (3.1.1) admits a solution $u(\cdot)$ on an interval $T_0 = [0, a_0]$ with

$$a_0 = \min \left\{ 1, \left[\frac{r(1-a)\Gamma(\alpha_m + 1)}{M} \right]^{1/\alpha_m} \right\}.$$

Proof: We define the nonlinear operator $Q(\cdot) : C(T_0, E) \to C(T_0, E)$ by

$$(Qu)(t) = \sum_{i=1}^{m-1} \int_0^t \frac{a_i(t-s)^{\alpha_m - \alpha_i - 1}}{\Gamma(\alpha_m - \alpha_i)} u(s)ds + \int_0^t \frac{(t-s)^{\alpha_m - 1}}{\Gamma(\alpha_m)} f(s, u(s))ds, \qquad ,$$

for all $t \in [0, a_0]$. If $y(\cdot) \in C(T_0, E)$, then by (h2) we have that $f(\cdot, y(\cdot)) \in P^\infty(T_0, E)$ and so the operator Q makes sense. To show that Q is well defined, let $t, s \in T_0$ with $t > s$. Without loss of generality, assume that $(Qy)(t) - (Qy)(s) \neq 0$. Then by the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and $\|(Qy)(t) - (Qy)(s)\| = |\langle y^*, (Qy)(t) - (Qy)(s) \rangle|$. Then

$$\begin{aligned} \|(Qy)(t) - (Qy)(s)\| &= |\langle y^*, (Qy)(t) - (Qy)(s) \rangle| \leq \\ &\leq \sum_{i=1}^{m-1} \frac{|a_i|}{\Gamma(\alpha_m - \alpha_i)} \int_0^s \left[(s-\tau)^{\alpha_m - \alpha_i - 1} - (t-\tau)^{\alpha_m - \alpha_i - 1} \right] |\langle y^*, u(\tau) \rangle| \, d\tau \\ &\quad + \int_s^t (t-\tau)^{\alpha_m - \alpha_i - 1} |\langle y^*, f(\tau, u(\tau)) \rangle| \, d\tau + \\ &\quad + \frac{1}{\Gamma(\alpha_m)} \int_0^s \left[(s-\tau)^{\alpha_m - 1} - (t-\tau)^{\alpha_m - 1} \right] |\langle y^*, u(\tau) \rangle| \, d\tau \\ &\quad + \int_s^t (t-\tau)^{\alpha_m - 1} |\langle y^*, f(\tau, u(\tau)) \rangle| \, d\tau \\ &\leq 2 \left[\sum_{i=1}^{m-1} \frac{r|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} + \frac{M}{\Gamma(\alpha_m + 1)} \right] (t-s)^{\alpha_m}. \end{aligned} \tag{3.2.1}$$

so Q maps $C(T_0, E)$ into itself. Let $r > 0$ and let $\widetilde{B}$ be the convex, closed and equicontinous set defined by

$$\begin{aligned} \widetilde{B} &= \{ y(\cdot) \in C(T_0, E); \|y(\cdot)\|_c \leq r, \ \|y(t) - y(s)\| \leq \\ &\leq 2 \left[\sum_{i=1}^{m-1} \frac{r|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} + \frac{M}{\Gamma(\alpha_m + 1)} \right] (t-s)^{\alpha_m} \text{ for all } t, s \in T_0 \}. \end{aligned}$$

We will show that Q maps $\widetilde{B}$ into itself and Q restricted to the set $\widetilde{B}$ is weakly-weakly sequentially continuous. To show that $Q : \widetilde{B} \to \widetilde{B}$, let $y(\cdot) \in \widetilde{B}$ and $t \in T_0$.

Again, without loss of generality, assume that $(Qy)(t) \neq 0$. By the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and $\|(Qy)(t)\| = |\langle y^*, (Qy)(t)\rangle|$. Then by (h3), we have

$$\begin{aligned}
&\|(Qu)(t)\| = |\langle y^*, (Qy)(t)\rangle| \leq \\
\leq\ & \sum_{i=1}^{m-1} \frac{|a_i|}{\Gamma(\alpha_m - \alpha_i)} \int_0^t (t-s)^{\alpha_m - \alpha_i - 1} \, |\langle y^*, u(\tau)\rangle| \, ds + \\
& \frac{1}{\Gamma(\alpha_m)} \int_0^t (t-s)^{\alpha_m - 1} \, |\langle y^*, f(\tau, u(\tau))\rangle| \, ds \leq \\
\leq\ & \sum_{i=1}^{m-1} \frac{r|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} + \frac{Mt^{\alpha_m}}{\Gamma(\alpha_m + 1)} \leq ra + (1-a)r = r
\end{aligned}$$

and using (3.2.1) it follows that Q maps $\widetilde{B}$ into $\widetilde{B}$. Next, we show that Q is weakly-weakly sequentially continuous. First, we recall that the weak convergence in $\widetilde{B} \subset C(T_0, E)$ is exactly the weak pointwise convergence. Let $\{u_n(\cdot)\}_{n\geq 1}$ be a sequence in $\widetilde{B}$ such that $u_n(\cdot)$ converges weakly to $u(\cdot)$ in $\widetilde{B}$. Then $u_n(t)$ converges weakly to $u(t)$ in E for each $t \in T_0$. Since $\widetilde{B}$ is a closed convex set, by Mazur's lemma we have $u(\cdot) \in \widetilde{B}$. Further, by (h1) it follows that $f(t, u_n(t))$ converges weakly to $f(t, u(t))$ for each $t \in T_0$. Then the Lebesgue dominated convergence theorem for the Pettis integral (see [88]) yields $I^\alpha u_n(t)$ converging weakly to $I^\alpha u(t)$ in E for each $t \in T_0$. Since $\widetilde{B}$ is equicontinuous subset of $C(T_0, E)$ it follows that $Q(\cdot)$ is weakly-weakly sequentially continuous.

Suppose that $V \subset \widetilde{B}$ such that $V = \overline{co}(Q(V) \cup \{y(\cdot)\})$ for some $y(\cdot) \in \widetilde{B}$. We will show that V is relatively weakly compact in $C(T_0, E)$. Let $t \in T_0$ and $\varepsilon > 0$. If we choose $\eta > 0$ such that $\eta < \left(\frac{\varepsilon\Gamma(\alpha_m+1)}{M + r\Gamma(\alpha_m+1)}\right)^{1/\alpha_m}$ and

$$\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^t (t-s)^{\alpha_m - \alpha_i - 1} u(s) ds + \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^t (t-s)^{\alpha_m - 1} f(s, u(s)) ds \neq 0$$

then, by the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and

$$\begin{aligned}
&\left\| \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - \alpha_i - 1} u(s) ds + \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - 1} f(s, u(s)) ds \right\| \\
= & \left| \left\langle y^*, \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - \alpha_i - 1} u(s) ds + \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - 1} f(s, u(s)) ds \right\rangle \right| \\
\leq & \sum_{i=1}^{m-1} \frac{|a_i|}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - \alpha_i - 1} \left| \langle y^*, u(\tau) \rangle \right| ds + \\
& + \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - 1} \left| \langle y^*, f(\tau, u(\tau)) \rangle \right| ds \\
\leq & \sum_{i=1}^{m-1} \frac{r|a_i|\eta^{\alpha_m - \alpha_i}}{\Gamma(\alpha_m - \alpha_i + 1)} + \frac{M\eta^{\alpha_m}}{\Gamma(\alpha_m + 1)} \leq r\eta^{\alpha_m} + \frac{M\eta^{\alpha_m}}{\Gamma(\alpha_m + 1)} \\
\leq & \frac{M + r\Gamma(\alpha_m + 1)}{\Gamma(\alpha_m + 1)} \eta^{\alpha_m} < \varepsilon.
\end{aligned}$$

and thus using property (x) measure of the non-compactness we infer

$$\begin{aligned}
&\beta \left(\left\{ \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - \alpha_i - 1} u(s) ds + \right. \right. \\
&\left. \left. \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - 1} f(s, u(s)) ds, \; u \in V \right\} \right) \leq 2\varepsilon.
\end{aligned} \tag{3.2.2}$$

Since by Lemma 1.5.1 the function $t \to v(t) := \beta(V(t))$ is continuous on $[0, t-\eta]$ it follows that $s \to (t-s)^{\alpha_m - 1} g(v(s))$ is continuous on $[0, t-\eta]$. Hence, there exists $\delta > 0$ such that

$$\left\| (t-\tau)^{\alpha_m - 1} g(v(\tau)) - (t-s)^{\alpha_m - 1} g(v(s)) \right\| < \frac{\varepsilon}{2}$$

and

$$\| g(v(\xi)) - g(v(\tau)) \| < \frac{\varepsilon}{2\eta^{\alpha_m - 1}}.$$

If $|\tau - s| < \delta$ and $|\tau - \xi| < \delta$ with τ, s, $\xi \in [0, t-\eta]$, then it follows that

$$\begin{aligned}
|(t-\tau)^{\alpha_m - 1} g(v(\xi)) - (t-s)^{\alpha_m - 1} g(v(s))| &\leq |(t-\tau)^{\alpha_m - 1} g(v(\tau)) - (t-s)^{\alpha_m - 1} g(v(s))| \\
&+ (t-\tau)^{\alpha_m - 1} |g(v(\xi)) - g(v(\tau))| \\
&< \varepsilon,
\end{aligned}$$

that is

$$|(t-\tau)^{\alpha_m - 1} g(v(\xi)) - (t-s)^{\alpha_m - 1} g(v(s))| < \varepsilon, \tag{3.2.3}$$

for all τ, s, $\xi \in [0, t-\eta]$ with $|\tau - s| < \delta$ and $|\tau - \xi| < \delta$. Consider the following partition of the interval $[0, t-\eta]$ into n parts $0 = t_0 < t_1 \ldots < t_n = t - \eta$ such that $t_i - t_{i-1} < \delta$ $(i = 1, 2, \ldots, n)$. By Lemma 1.5.1 for each i there exists $s_i \in [t_{i-1}, t_i]$ such that $\beta(V([t_{i-1}, t_i])) = v(s_i)$, $i = 1, 2, \ldots, n$. Then we have (see [37], Theorem 2.2)

$$\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m - \alpha_i - 1} V(s) ds + \frac{1}{\Gamma(\alpha_m)} \int_0^{t-\eta} (t-s)^{\alpha_m - 1} f(s, V(s)) ds \subset$$

$$\subset \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - \alpha_i - 1} V(s) ds + \sum_{j=1}^{n} \frac{1}{\Gamma(\alpha_m)} \int_{t_{i-1}}^{t_i} (t-s)^{\alpha_m - 1} f(s, V(s)) ds \subset$$

$$\subset \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - \alpha_i - 1} V(s) ds +$$

$$+ \frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i - t_{i-1}) \overline{conv} \{(t-s)^{\alpha_m - 1} f(s, u(s)); s \in [t_{i-1}, t_i],\ u \in V\},$$

and so

$$\beta \left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m - \alpha_i - 1} V(s) ds + \frac{1}{\Gamma(\alpha_m)} \int_0^{t-\eta} (t-s)^{\alpha_m - 1} f(s, V(s)) ds \right) \leq$$

$$\leq \beta \left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m - \alpha_i - 1} V(s) ds \right) +$$

$$+ \frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i - t_{i-1}) \beta \left(\overline{conv} \{(t-s)^{\alpha_m - 1} f(s, u(s)); s \in [t_{i-1}, t_i],\ u \in V\} \right)$$

$$= \beta \left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m - \alpha_i - 1} V(s) ds \right) +$$

$$+ \frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i - t_{i-1}) \beta \left(\{(t-s)^{\alpha_m - 1} f(s, u(s)); s \in [t_{i-1}, t_i],\ u \in V\} \right) \leq$$

$$\leq \beta \left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m - \alpha_i - 1} V(s) ds \right) +$$

$$+ \frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i - t_{i-1}) (t - t_i)^{\alpha_m - 1} \beta \left(f([0, a_0] \times V[t_{i-1}, t_i]) \right)$$

$$\leq \beta\left(\sum_{i=1}^{m-1}\frac{a_i}{\Gamma(\alpha_m-\alpha_i)}\int_0^{t-\eta}(t-s)^{\alpha_m-\alpha_i-1}V(s)ds\right)+$$

$$+\frac{1}{\Gamma(\alpha_m)}\sum_{i=1}^{n}(t_i-t_{i-1})(t-t_i)^{\alpha_m-1}g(V[t_{i-1},t_i])\leq$$

$$\leq \beta\left(\sum_{i=1}^{m-1}\frac{a_i}{\Gamma(\alpha_m-\alpha_i)}\int_0^{t-\eta}(t-s)^{\alpha_m-\alpha_i-1}V(s)ds\right)+$$

$$+\frac{1}{\Gamma(\alpha_m)}\sum_{i=1}^{n}(t_i-t_{i-1})(t-t_i)^{\alpha_m-1}g(v(s_i)).$$

Using (3.2.3) we have that

$$|(t-t_i)^{\alpha_m-1}g(v(s_i))-(t-s)^{\alpha_m-1}g(v(s))|<\varepsilon.$$

This implies that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha_m)}\sum_{j=1}^{n}(t_i-t_{i-1})(t-t_i)^{\alpha_m-1}g(v(s_i)) \\ \leq\ &\frac{1}{\Gamma(\alpha_m)}\int_0^{t-\eta}(t-s)^{\alpha_m-1}g(v(s))ds+\varepsilon(t-\eta)/\Gamma(\alpha_m).\end{aligned} \tag{3.2.4}$$

By using (3.2.2) we claim that

$$\beta\left(\sum_{i=1}^{m-1}\frac{a_i}{\Gamma(\alpha_m-\alpha_i)}\int_{t-\eta}^{t}(t-s)^{\alpha_m-\alpha_i-1}V(s)ds\right)\leq 2\varepsilon. \tag{3.2.5}$$

Because if we let that

$$A(t)=\frac{1}{\Gamma(\alpha_m)}\int_{t-\eta}^{t}(t-s)^{\alpha_m-1}f(s,V(s))ds,$$

$$B(t)=\sum_{i=1}^{m-1}\frac{a_i}{\Gamma(\alpha_m-\alpha_i)}\int_{t-\eta}^{t}(t-s)^{\alpha_m-\alpha_i-1}V(s)ds,$$

then $a+B(t)\subset A(t)+B(t)$ for $a\in A(t)$, implies that $\beta(B(t))\leq\beta(A(t)+B(t))<2\varepsilon$. From relations (3.2.4) and (3.2.5) we obtain

$$\beta\left(\sum_{i=1}^{m-1}\frac{a_i}{\Gamma(\alpha_m-\alpha_i)}\int_0^{t-\eta}(t-s)^{\alpha_m-\alpha_i-1}V(s)ds+\frac{1}{\Gamma(\alpha_m)}\int_0^{t-\eta}(t-s)^{\alpha_m-1}f(s,V(s))ds\right)\leq \tag{3.2.6}$$

$$\leq 2\varepsilon+\frac{1}{\Gamma(\alpha_m)}\int_0^{t-\eta}(t-s)^{\alpha_m-1}g(V(s))ds+\varepsilon(t-\eta)/\Gamma(\alpha_m).$$

Since

$$(QV)(t) \subset \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m - \alpha_i - 1} V(s) ds +$$
$$\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - \alpha_i - 1} V(s) ds +$$
$$+ \frac{1}{\Gamma(\alpha_m)} \int_0^{t-\eta} (t-s)^{\alpha_m - 1} f(s, V(s)) ds + \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - 1} f(s, V(s)) ds,$$

then by virtue of (3.2.2) and (3.2.6) we have

$$\begin{aligned} \beta((QV)(t)) &\leq \frac{1}{\Gamma(\alpha_m)} \int_0^{t-\eta} (t-s)^{\alpha_m - 1} g(v(s)) ds + \varepsilon(t-\eta)/\Gamma(\alpha_m) + 4\varepsilon \\ &\leq \frac{1}{\Gamma(\alpha_m)} \int_0^{t} (t-s)^{\alpha_m - 1} g(v(s)) ds + \varepsilon((t+4)/\Gamma(\alpha_m)). \end{aligned}$$

As the last inequality is true for every $\varepsilon > 0$, we infer

$$\beta((QV)(t)) \leq \frac{1}{\Gamma(\alpha_m)} \int_0^{t} (t-s)^{\alpha_m - 1} g(v(s)) ds, \ t \in [0, a_0],$$

Because $V = \overline{co}(Q(V) \cup \{y(\cdot)\})$ then

$$\beta(V(t)) = \beta\left(\overline{co}(Q(V(t)) \cup \{y(t)\})\right) \leq \beta((QV(t)))$$

and thus

$$v(t) \leq \frac{1}{\Gamma(\alpha_m)} \int_0^{t} (t-s)^{\alpha_m - 1} g(v(s)) ds \ for \ t \in T_0.$$

Since $g(\cdot)$ is a Gripenberg function, it follows that $v(t) = 0$ for $t \in T_0$. Since V as a subset of $\widetilde{B}$ is equicontinuous, by Lemma 1.5.1 we infer

$$\beta_c(V(T_0)) = \sup_{t \in T_0} \beta(V(t)) = 0.$$

Thus, by Arzelá-Ascoli's theorem we obtain that V is weakly relatively compact in $C(T_0, E)$. Using Theorem 1.5.2 there exists a fixed point of the operator Q which is a solution of (3.1.1).

3.3 Conclusion

In this chapter, involving the concept of Pettis integral, we developed the existence of solution of multi-term fractional differential equations in nonreflexive Banach spaces, equipped with weak topology.

Chapter 4

Weak Solution For Fractional Differential Equation

In chapter 2, 3 we discussed the existence of solution of fractional differential equations, with one differential operator and multi-term fractional equations respectively, in nonreflexive Banach spaces. In the present chapter we will establish an existence of weak solution for the following fractional differential equation

$$\begin{cases} D_w^\alpha y(t) = f(t, y(t)) \\ y(0) = y_0 \end{cases} \tag{4.0.1}$$

where $D_w^\alpha y(\cdot)$ is fractional Caputo weak derivative of the function $y(\cdot) : T \to E$ and $f(\cdot, \cdot) : T \times E \to E$ is a given function, T is bounded interval of real numbers containing 0 and E is nonreflexive Banach space.

4.1 Preliminaries

Let $x(\cdot) : T \to E$ be a given function and $\alpha > 0$. As it is well known that, the *fractional Riemann-Liouville* integral of order $\alpha > 0$ of $x(\cdot)$ is defined by

$$I^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, \ \ t \in [0, b], \tag{4.1.1}$$

provided that the right side is point-wise defined on T. Also, the *fractional Caputo derivative* of order $\alpha \in (0, 1]$ of $x(\cdot)$ is defined by

$$D^\alpha x(t) := I^{1-\alpha} x'(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x'(s)ds, \ \ t \in T, \tag{4.1.2}$$

provided that the right side is point-wise defined on T. In the above definitions the integral sign "$\int$" can be one of the following integrals: Riemann integral, Riemann-Pettis integral, weak Riemann integral, Bochner integral, Pettis integral or other kind of integral. Also, the derivative can be one of the following: strong derivative, weak derivative or a Pseudo-derivative. The definitions, properties and applications to fractional differential equations of fractional calculus using Bochner integral or Pettis integral can be found in the papers [3], [12], [74].

4.2 Vector-Valued Fractional Integral And Abel Integral Equation

In this section we set out to state vector-valued fractional integral and derivatives and prove results from [5], that will be used in the remainder of this dissertation. We will discuss fractional Pettis integral, fractional Bochner integral and Riemann-Pettis integral.

Proposition 4.2.1. *If $x(\cdot) : T \to E$ is R-integrable on T, then*

$$I^\alpha x(t) = (P)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, \quad t \in T, \tag{4.2.1}$$

that is, $I^\alpha x(t)$ exists on T as a fractional Pettis integral.

Proof. Indeed, since $x(\cdot)$ is R-integrable on T then, by Theorem 15 from [41], it follows that $x(\cdot)$ is Pettis integrable on T. Moreover, by Theorem 7 from [41], $x(\cdot)$ is also scalarly Riemann integrable on T, and so it is weakly measurable and bounded on T. Thus, $\langle x^*, x(\cdot)\rangle \in L^\infty(T)$ for every $x^* \in E^*$. Next, since the real function $s \mapsto \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$ belongs to $L^1([0,t])$ for each $t \in (0,b]$ then, by Corollary 3.41 from [67], it follows that the function $s \mapsto \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)$ is Pettis integrable on $[0,t]$ for every $t \in (0,b]$. □

***Remark* 4.2.1.** If $x(\cdot) : T \to E$ is strongly measurable and R-integrable on T, then fractional integral $I^\alpha x(t)$ exists a.e. on T as a fractional Bochner integral. This result is a direct consequence of Theorem 15 of [41] and Theorem 2.4 from [76].

Proposition 4.2.2. *Let $\alpha \in (0,1)$ and let $x(\cdot) : T \to E$ be a strongly differentiable function on T. If the strong derivative $x'(\cdot)$ of $x(\cdot)$ is R-integrable on T, then*

$$D^{\alpha}x(t) = (B)\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}x'(s)ds, \quad t \in T, \tag{4.2.2}$$

that is, $D^{\alpha}x(t)$ exists a.e. on T as a fractional Bochner integral.

Proof. Since $x(\cdot)$ is strongly differentiable on T, then $x(\cdot)$ is strongly continuous on T, and so $x'(\cdot)$ is strongly measurable on T. It follows that $x'(\cdot)$ is Bochner integrable and thus, by Theorem 2.4 from [76], $I^{1-\alpha}x(\cdot)$ exists a.e. on T as a fractional Bochner integral. □

The main properties of fractional Bochner or fractional Pettis integral together with their applications to fractional differential equations are well known, being the subject of several works.

In the following, we will focus on the study of Riemann-Pettis integrability and its applications to fractional calculus and fractional differential equations. Graves [42] gave an example of a R-integrable function $x(\cdot) : [0,1] \to E$ which is not weakly continuous for any $t \in [0,1]$. A similar example was given by Alexiewicz and Orlicz [6] in the case when E is not separable. Moreover, Alexiewicz and Orlicz [6] gave an example of weakly continuous function which is not R-integrable. Kerner [47] has considered another type of integral, corresponding to weak convergence.

A function $x(\cdot) : T \to E$ is said to be *Riemann-Pettis integrable* (or *RP-integrable*, for short) on T if $x(\cdot)$ is a scalarly Riemann integrable and, for each interval $I \subset T$, there exists an element $z_I \in E$ such that

$$\langle x^*, z_I \rangle = \int_I \langle x^*, x(s) \rangle \, ds \tag{4.2.3}$$

for every $x^* \in E^*$. The element z_I will be denoted by $(w)\int_I x(s)ds$ and it is called the *weak Riemann integral* of $x(\cdot)$ on I. Also, a RP-integrable function is sometime called *weakly Riemann integrable* function. In fact, RP-integrability on T is equivalent to the weak convergence of Riemann sums (1.4.3).

It is easy to see that every R-integrable function is RP-integrable, and every RP-integrable function is Pettis integrable (see [41]). Alexiewicz and Orlicz [6] give an example which proves that neither RP-integrability nor weakly continuity do imply

R-integrability. We shall denote by $RP(T, E)$ the set of all RP-integrable function from T into E.

Proposition 4.2.3. *(Kerner [47], Alexiewicz and Orlicz [6]). Every weakly continuous function from T into E is RP-integrable on T, that is, $C_w(T, E) \subset RP(T, E)$.*

The following properties are well known and easy to prove, they being direct consequences of the definition and of the properties of the weak differentiability.

Proposition 4.2.4. *If $x(\cdot) : T \to E$ is weakly continuous on T, then the function $y(\cdot) : T \to E$, given by*

$$y(t) = (w) \int_0^t x(s)ds, \quad t \in T, \tag{4.2.4}$$

is weakly differentiable on T and $y'_w(t) = x(t)$ for every $t \in T$.

Proposition 4.2.5. *If $x(\cdot) : T \to E$ is weakly differentiable on T and $x'_w(\cdot)$ is weakly continuous on T, then*

$$x(t) = x(0) + (w) \int_0^t x'_w(s)ds, \quad t \in T. \tag{4.2.5}$$

Proposition 4.2.6. *If $x(\cdot) : T \to E$ is RP-integrable on T, then*

$$I_w^\alpha x(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, \quad t \in T, \tag{4.2.6}$$

exists on T as a fractional Pettis integral. Moreover, I_w^α is a linear operator from $RP(T, E)$ into $P^\infty(T, E)$, and for α, $\beta \in (0, 1)$ we have

$$I_w^\alpha I_w^\beta x(t) = I_w^{\alpha+\beta} x(t), \; t \in T. \tag{4.2.7}$$

Proof. Since each RP-integrable function $x(\cdot) : T \to E$ is Pettis integrable, then as in the proof of Proposition 4.2.1 we infer that $I_w^\alpha x(t)$ exists on T as a fractional Pettis integral. The linearity of I_w^α is obviously, and (4.2.7) can be obtained in the same manner as in Lemma 3.2 from [2]. $\square$

We remark that if $x(\cdot) : T \to E$ is R-integrable on T and measurable then $I_w^\alpha x(t)$ exists on T as a fractional Bochner integral and $I_w^\alpha x(t) = I^\alpha x(t)$ for $t \in T$.

***Remark* 4.2.2.** Also, if $x(\cdot) \in C_w(T,E)$, then $x(\cdot)$ is bounded on T. Then, if we put $M := \sup_{t\in T}||x(t)||$ and fix $x^* \in E^*$, it is easy to check that for $t, s \in T$, $s \leq t$, we have

$$|\langle x^*, I_w^\alpha x(t)\rangle - \langle x^*, I_w^\alpha x(s)\rangle| \leq \frac{2M||x^*||}{\Gamma(1+\alpha)}(t-s)^\alpha.$$

It follows that the real-valued function $t \mapsto \langle x^*, I_w^\alpha x(t)\rangle$ is continuous on T for every $x^* \in E^*$, an so I_w^α is a linear operator from $C_w(T,E)$ into $C_w(T,E)$.

Since the weak derivative $x'_w(\cdot)$ of a weakly differentiable function $x(\cdot) : T \to E$ is strongly measurable, then with the same proof as in Proposition 4.2.2, we obtain the following result.

Proposition 4.2.7. *Let $\alpha \in (0,1)$ and let $x(\cdot) : T \to E$ be a weakly differentiable function on T. If the weak derivative $x'_w(\cdot)$ of $x(\cdot)$ is RP-integrable on T, then*

$$D_w^\alpha x(t) := I_w^{1-\alpha} x'(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x'_w(s)ds, \ \ t \in T, \tag{4.2.8}$$

exists a.e. on T as a fractional Bochner integral (and so as a fractional Pettis integral).

Clearly, if $x(\cdot) : T \to E$ is R-integrable on T, then $D_w^\alpha x(t)$ exists on T as a fractional Bochner integral and $D_w^\alpha x(t) = D^\alpha x(t)$ for $t \in T$.

Example 1. Let $\mathbf{c}$ be the space of all real converging sequences. Consider $y(\cdot) : [0, 2\pi] \to \mathbf{c}$ given by

$$y(t)(n) = \frac{\sin nt}{n}, \ \ t \in [0, 2\pi], \ \ n = 1, 2, \ldots.$$

Since $y(\cdot)$ is bounded and scalarly integrable on $[0, 2\pi]$, then $y(\cdot)$ is RP-integrable on $[0, 2\pi]$. Also, for each $x^* \in \mathbf{c}^* = \mathbf{l}^1$ there exists a unique $a = (a_1, a_2, \ldots) \in \mathbf{l}^1$ such that

$$\langle x^*, y(t)\rangle = \sum_{n=1}^{\infty} a_n \frac{\sin nt}{n}, \ \ t \in [0, 2\pi].$$

Since $\sum_{n=1}^{\infty} a_n \frac{\sin nt}{n}$ is uniformly and absolutely convergent on $[0, 2\pi]$, then for each $x^* \in \mathbf{c}^*$ we have (see [58, page 355])

$$\int_0^t \tfrac{(t-s)^{1/2-1}}{\Gamma(1/2)} \langle x^*, y(s)\rangle\, ds = \int_0^t \tfrac{(t-s)^{-1/2}}{\Gamma(1/2)} \left(\sum_{n=1}^{\infty} a_n \tfrac{\sin ns}{n} \right) ds$$

$$= \sum_{n=1}^{\infty} \tfrac{a_n}{n} \int_0^t \tfrac{(t-s)^{-1/2}}{\Gamma(1/2)} \sin ns ds = \sum_{n=1}^{\infty} \tfrac{a_n}{n} z(t)(n) = \langle x^*, z(t)\rangle\,,$$

where

$$z(t)(n) = \frac{1}{n}\sqrt{\frac{2}{n}}\left[C\left(\sqrt{\frac{2nt}{\pi}}\right)\sin nt - S\left(\sqrt{\frac{2nt}{\pi}}\right)\cos nt\right],$$

and $C(x)$, $S(x)$ are Fresnel integrals

$$C(x) = \int_0^x \cos\frac{\pi s^2}{2}ds \text{ and } S(x) = \int_0^x \sin\frac{\pi s^2}{2}ds,$$

respectively. Obviously, $z(\cdot) \in \mathbf{c}$, and thus

$$I_w^{1/2}y(t)(n) = z(t)(n),\ t \in [0, 2\pi],\ n = 1, 2, ...$$

Also, we remark that $y(\cdot)$ is not weakly differentiable on $[0, 2\pi]$ (see [81]).
If we consider $x(\cdot) : [0, 2\pi] \to \mathbf{c}$ given by

$$x(t)(n) = \frac{\sin nt}{n^2},\ t \in [0, 2\pi],\ n = 1, 2, ...,$$

then $x(\cdot)$ is weakly differentiable and it is easy to see that

$$x'_w(t)(n) = \frac{\cos nt}{n},\ t \in [0, 2\pi],\ n = 1, 2, ...$$

Since $\sum\limits_{n=1}^{\infty} a_n \frac{\cos nt}{n}$ is uniformly and absolutely convergent on $[0, 2\pi]$, then for each $x^* \in \mathbf{c}^*$ we have (see [58, page 354])

$$\int_0^t \tfrac{(t-s)^{-1/2}}{\Gamma(1-1/2)} \langle x^*, x'_w(s)\rangle\, ds = \int_0^t \tfrac{(t-s)^{-1/2}}{\Gamma(1/2)} \left(\sum_{n=1}^{\infty} a_n \tfrac{\cos ns}{n}\right) ds$$

$$= \sum_{n=1}^{\infty} \tfrac{a_n}{n} \int_0^t \tfrac{(t-s)^{-1/2}}{\Gamma(1/2)} \cos ns ds = \sum_{n=1}^{\infty} a_n u(t)(n) = \langle x^*, u(t)\rangle\,,$$

where

$$u(t)(n) = \frac{1}{n}\sqrt{\frac{2}{n}}\left[C\left(\sqrt{\frac{2nt}{\pi}}\right)\cos nt + S\left(\sqrt{\frac{2nt}{\pi}}\right)\sin nt\right].$$

Obviously, $u(\cdot) \in \mathbf{c}$, and thus

$$D_w^{1/2}x(t)(n) = u(t)(n),\ t \in [0, 2\pi],\ n = 1, 2, ...$$

Lemma 4.2.8. *If $x(\cdot) : T \to E$ is a weakly differentiable function such that $x'_w(\cdot)$ is RP-integrable on T, then the function*

$$x_{1-\alpha}(t) := \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds,\ t \in T,$$

is wAC and weakly differentiable on T. Moreover, $(x_{1-\alpha})'_w(\cdot)$ *is RP-integrable and*

$$(x_{1-\alpha})'_w(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} y(0) + I^{1-\alpha} x'_w(t) \quad a.e. \ on \ T. \tag{4.2.9}$$

Proof. Since $x(\cdot)$ is weakly differentiable on T, then for every $x^* \in E^*$ the real function $t \mapsto \langle x^*, y(t)\rangle$ is differentiable on T and $\frac{d}{dt}\langle x^*, x(t)\rangle = \langle x^*, x'_w(t)\rangle$ for $t \in T$. The last equality implies

$$\int_0^s \langle x^*, x'_w(\tau)\rangle \, d\tau = \langle x^*, x(s)\rangle - \langle x^*, x(0)\rangle, \ s \in T, \tag{4.2.10}$$

for every $x^* \in E^*$. From (4.2.10) we obtain that

$$\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \int_0^s \langle x^*, x'_w(\tau)\rangle \, d\tau ds =$$

$$\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, x(s)\rangle \, ds - \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, x(0)\rangle \, ds, \ t \in T,$$

for every $x^* \in E^*$. It follows that

$$\int_0^t \int_0^s \frac{(s-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, x'_w(\tau)\rangle \, d\tau ds$$

$$= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, x(s)\rangle \, ds - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \langle x^*, x(0)\rangle,$$

and so

$$\langle x^*, x_{1-\alpha}(t)\rangle = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \langle x^*, x(0)\rangle + \int_0^t y(s)ds, \ t \in T, \tag{4.2.11}$$

for every $x^* \in E^*$, where $y(s) = \int_0^s \frac{(s-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, x'_w(\tau)\rangle \, d\tau, \ s \in [0,t]$. Since $y(\cdot)$ is Lebesgue integrable on $[0,t]$ for every $t \in (0,b]$, then from (4.2.11) it follows that the real function $t \mapsto \langle x^*, x_{1-\alpha}(t)\rangle$ is AC on T for every $x^* \in E^*$. Hence, from (4.2.11) we infer that for every $x^* \in E^*$ we have

$$\frac{d}{dt}\langle x^*, x_{1-\alpha}(t)\rangle = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, x(0)\rangle + \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \langle x^*, x'_w(\tau)\rangle \, d\tau,$$

for a.e. $t \in T$. Therefore, the function $x_{1-\alpha}(\cdot)$ is wAC and weakly differentiable on T, and (4.2.9) holds. □

***Remark* 4.2.3.** Relation (4.2.9) can be written as

$$D^\alpha_w x(t) = (x_{1-\alpha})'_w(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} y(0) \quad a.e. \ on \ T. \tag{4.2.12}$$

In this case $(x_{1-\alpha})'_w(t)$ will be denoted by ${}^{RL}D_w^\alpha x(t)$ and it is called the *weak Riemann-Liouville derivative* of $x(\cdot)$. The formula (4.2.12) suggests us that we can extend the definition of the weak Caputo fractional derivative for RP-integrable functions. Therefore, if $x(\cdot) \in RP(T,E)$, then its weak Caputo derivative is defined by (4.2.12). It follows that the weak Caputo fractional derivatives $D_w^\alpha x(t)$ are also defined for functions $x(\cdot)$ for which the weak Riemann-Liouville fractional derivatives ${}^{RL}D_w^\alpha x(t)$ exist.

***Remark* 4.2.4.** If $x(\cdot)$ is not weakly differentiable, then $x_{1-\alpha}(\cdot)$ will not be weakly differentiable.

Example 2. Consider $y(\cdot) : [0, 2\pi] \to \mathbf{c}$ given by

$$y(t)(n) = \frac{\sin nt}{\sqrt{2n}}, \ t \in [0, 2\pi], \ n = 1, 2, ...,$$

where $\mathbf{c}$ is the space of all real convergent sequences. As in Example 1 we obtain that

$$y_{1-1/2}(t)(n) = I_w^{1/2} y(t)(n) = z(t)(n), \ t \in [0, 2\pi], \ n = 1, 2, ...,$$

where $z(\cdot) \in \mathbf{c}$ is given by

$$z(t)(n) = \frac{1}{n}\left[C\left(\sqrt{\frac{2nt}{\pi}}\right)\sin nt - S\left(\sqrt{\frac{2nt}{\pi}}\right)\cos nt\right].$$

Now we show that $y_{1-1/2}(\cdot)$ is not weakly differentiable on $[0, 2\pi]$. Since $\mathbf{c}^* = \mathbf{l}^1$ for each $x^* \in \mathbf{c}^* = \mathbf{l}^1$ there exists a unique $a = (a_1, a_2, ...) \in \mathbf{l}^1$ such that

$$\langle x^*, z(t)\rangle = \sum_{n=1}^{\infty} a_n z(t)(n), \ t \in [0, 2\pi].$$

Since $\sum\limits_{n=1}^{\infty} a_n \frac{\cos nt}{\sqrt{n}}$ and $\sum\limits_{n=1}^{\infty} a_n \frac{\sin t}{\sqrt{n}}$ are uniformly and absolutely convergent on $[0, 2\pi]$ and the sequences $C\left(\sqrt{\frac{2nt}{\pi}}\right)$ and $C\left(\sqrt{\frac{2nt}{\pi}}\right)$ are uniformly bounded on $[0, 2\pi]$ it follows that

$$\frac{d}{dt}\langle x^*, z(t)\rangle = \sum_{n=1}^{\infty} a_n z'(t)(n) = C\left(\sqrt{\frac{2nt}{\pi}}\right)\cos nt + S\left(\sqrt{\frac{2nt}{\pi}}\right)\sin t$$

for $t \in [0, 2\pi]$. We suppose that the weak derivative of $z(\cdot)$ at a point $t_0 \in [0, 2\pi]$ exists and $z'_w(t_0) = (b_1^0, b_2^0, ...) \in \mathbf{c}$. Then we have

$$\sum_{n=1}^{\infty} a_n z'(t_0)(n) = \frac{d}{dt}\langle x^*, z(\cdot)\rangle(t_0) = \langle x^*, z'_w(t_0)\rangle = \sum_{n=1}^{\infty} a_n b_n^0$$

for all $x^* \in \mathbf{c}^*$. From the above formula we obtain that $b_n^0 = C\left(\sqrt{\frac{2nt_0}{\pi}}\right)\cos nt_0 + S\left(\sqrt{\frac{2nt_0}{\pi}}\right)\sin t_0$, $n = 1, 2, ...$ Note that $\lim_{n\to\infty} b_n^0$ does not exist. This contradicts the hypothesis $(b_1^0, b_2^0, ...) \in \mathbf{c}$. Hence, $y_{1-1/2}(\cdot)$ is not weakly differentiate at any point of $[0, 2\pi]$..

***Remark* 4.2.5.** Let $x(\cdot) : T \to E$ be a weakly differentiable function function such that $x'_w(\cdot)$ is RP-integrable. Then as in the proof of Lemma 4.2.8 we can show that the function

$$x_\alpha(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds,\ t \in T,$$

is wAC and weakly differentiable and

$$(x_\alpha)'_w(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} x(0) + I_w^\alpha x'_w(t) \text{ on } T.$$

Also, if E is a weakly sequentially complete space and $x(\cdot) : T \to E$ is weakly differentiable, then the function $x_\alpha(\cdot)$ is AC on T.

Corollary 4.2.9. *Let E be a weakly sequentially complete space. If $y(\cdot) : T \to E$ is weakly differentiable , then the function $y_{1-\alpha}(\cdot)$ is AC on T.*

Proposition 4.2.10. *If $x(\cdot) : T \to E$ is a weakly differentiable function such that $x'_w(\cdot)$ is RP-integrable on T and $\alpha, \beta \in (0, 1)$, then*
(a) $I_w^\alpha D_w^\alpha x(t) = x(t) - x(0)$ on T;
(b) $D_w^\alpha I_w^\alpha x(t) = x(t)$ on T.

Proof. Indeed, using (4.2.8), Proposition 4.2.5 and Proposition 4.2.6, we have

$$\begin{aligned} I_w^\alpha D_w^\alpha x(t) &= I_w^\alpha I_w^{1-\alpha} x'_w(t) = I_w^1 x'_w(t) = (w)\int_0^t x'_w(s)ds \\ &= x(t) - x(0),\ t \in T. \end{aligned}$$

Also, using Remark 4.2.5 and Proposition 4.2.5 we have

$$\begin{aligned} D_w^\alpha I_w^\alpha x(t) &= I_w^{1-\alpha}(I_w^\alpha x)'_w(t) = I_w^{1-\alpha}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} x(0) + I_w^\alpha x(t)\right] \\ &= x(0) + (w)\int_0^t x'_w(s)ds = x(0) + x(t) - x(0) = x(t) \end{aligned}$$

on T. □

Theorem 4.2.11. *If $y(\cdot) \in RP(T, E)$, then the Abel integral equation*

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds = y(t),\ t \in T = T, \tag{4.2.13}$$

has a solution in $x(\cdot) \in RP(T, E)$ if and only if the function $y_{1-\alpha}(\cdot)$ has the following properties:

(a) $y_{1-\alpha}(\cdot)$ is wAC on T,

(b) $y_{1-\alpha}(\cdot)$ is weakly differentiable a.e. on T and

$$x(t) = (y_{1-\alpha})'_w(t),\ \text{for a.e. } t \in T, \tag{4.2.14}$$

(c) $y_{1-\alpha}(0) = 0$.

Proof. Suppose that $x(\cdot) \in RP(T, E)$ is a solution of (4.2.13). Then from (4.2.13) it follows that

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \langle x^*, x(s)\rangle\, ds = \langle x^*, y(t)\rangle\,,\ t \in T, \tag{4.2.15}$$

for every $x^* \in E^*$. Since $\langle x^*, x(\cdot)\rangle$ and $\langle x^*, y(\cdot)\rangle$ are Riemann integrable (in particular, Lebesgue integrable) on T for every $x^* \in E^*$, using the same reasoning as in [80, Section 1.2] it follows that

$$\int_0^t \langle x^*, x(s)\rangle\, ds = \langle x^*, y_{1-\alpha}(t)\rangle\,,\ t \in T, \tag{4.2.16}$$

for every $x^* \in E^*$. Since for every $x^* \in E^*$ the real-valued function $t \mapsto \langle x^*, x(t)\rangle$ is Lebesgue integrable on T, then it follows that for every $x^* \in E^*$ the real-valued function $t \mapsto \langle x^*, y_{1-\alpha}(t)\rangle$ is differentiable a.e. on T,

$$\frac{d}{dt}\langle x^*, y_{1-\alpha}(t)\rangle = \langle x^*, x(t)\rangle \ \text{ for a.e. } t \in T,$$

and $t \mapsto \frac{d}{dt}\langle x^*, y_{1-\alpha}(t)\rangle$ is a bounded function on T. Therefore, for every $x^* \in E^*$ the function $t \mapsto \langle x^*, y_{1-\alpha}(t)\rangle$ is AC and weakly differentiable a.e. on T, and we are finished.

Conversely, suppose that the function $y_{1-\alpha}(\cdot)$ is wAC and weakly differentiable a.e. on T, and $y_{1-\alpha}(0) = 0$. Since $y_{1-\alpha}(\cdot)$ is weakly differentiable a.e. on T, then for every $x^* \in E^*$ the real function $t \mapsto \langle x^*, y_{1-\alpha}(t)\rangle$ is differentiable a.e. on T and $\frac{d}{dt}\langle x^*, y_{1-\alpha}(t)\rangle = \langle x^*, (y_{1-\alpha})'_w(t)\rangle$ for a.e. $t \in T$. If for every $x^* \in E^*$, we put

$g(x^*)(t) := \frac{d}{dt}\langle x^*, y_{1-\alpha}(t)\rangle$, for a.e. $t \in T$, then $g(x^*)(\cdot) \in L^1(T)$. We will now show that for every $x^* \in E^*$ the real function $g(x^*)(\cdot)$ is a solution of (4.2.15). For this purpose we replace $\langle x^*, x(\cdot)\rangle$ by $g(x^*)(\cdot)$ in the left-hand side of (4.2.15) and denote the result by $f(x^*)(\cdot)$, that is,

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(x^*)(s)ds = f(x^*)(t),\ t \in T. \tag{4.2.17}$$

If we show that for every $x^* \in E^*$ the functions $f(x^*)(\cdot)$ and $\langle x^*, y(\cdot)\rangle$ are equal on T, then our assertion is proved. Note using (4.2.17), the equality $f(x^*)(\cdot) = \langle x^*, y(\cdot)\rangle$ on T is shown in the proof of Theorem 2.1 from [80]. □

***Remark* 4.2.6.** If $\alpha \in (0,1)$ is replaced by $\beta := 1-\alpha \in (0,1)$, then the results of Theorem 4.2.11 remain valid, that is, if $y(\cdot) \in RP(T,E)$, then the integral equation

$$\int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} x(s)ds = y(t),\ t \in T,$$

has a solution in $x(\cdot) \in RP(T,E)$ if and only if the function $y_\beta(\cdot)$ has the following properties:

(a) $y_\beta(\cdot)$ is wAC on T,

(b) $y_\beta(\cdot)$ is weakly differentiable a.e. on T and

$$x(t) = (y_\beta)'_w(t),\ \text{ for a.e. } t \in T,$$

(c) $y_\beta(0) = 0$.

4.3 Differential Equation With Caputo Weak Derivatives

In this section we establish an existence of weak solution for the following fractional differential equation, in nonreflexive Banach spaces.

$$\begin{cases} D_w^\alpha y(t) = f(t, y(t)) \\ y(0) = y_0 \end{cases} \tag{4.3.1}$$

where $D_w^\alpha y(\cdot)$ is fractional Caputo weak derivative of the function $y(\cdot) : T \to E$ and $f(\cdot,\cdot) : T \times E \to E$ is a given function.

A continuous function $y(\cdot) : T \to E$ is said to be a *weak solution* of (4.3.1) if $y(\cdot)$ is weakly differentiable, $y'_w(\cdot)$ is RP-integrable, $D^\alpha_w y(t) = f(t, y(t))$ for a.e. $t \in T$ and $y(0) = y_0$.

Lemma 4.3.1. *Let $f(\cdot, \cdot) : T \times E \to E$ be a function such that $f(\cdot, y(\cdot))$ is weakly continuous for every continuous function $y(\cdot) : T \to E$. Then a continuous function $y(\cdot) : T \to E$ is a weak solution of (4.3.1) if and only if it satisfies the integral equation*

$$y(t) = y_0 + (P)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds, \quad t \in T. \tag{4.3.2}$$

Proof. Indeed, if the continuous function $y(\cdot) : T \to E$ is a weak solution of (4.3.1), then from Proposition 4.2.10 it follows that $y(t) - y(0) = I^\alpha_w f(t, y(t))$ on T; that is, $y(\cdot)$ satisfies the integral equation (4.3.2). Conversely, suppose that the continuous function $y(\cdot) : T \to E$ satisfies the integral equation (4.3.2). Then the weakly continuous function $z(\cdot) := f(\cdot, y(\cdot))$ satisfies the Abel equation

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} z(s)ds = v(s), \quad t \in T,$$

where $v(t) := y(t) - y_0$, $t \in T$. From Theorem 4.2.11 it follows that $v_{1-\alpha}(\cdot)$ is weakly differentiable a.e. on T and

$$z(t) = (v_{1-\alpha})'_w(t) = (y_{1-\alpha})'_w(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} y_0 \text{ for a.e. } t \in T.$$

Then by (4.2.12) we have that $z(t) = D^\alpha_w y(t)$ for a.e. $t \in T$; that is, $D^\alpha_w y(t) = f(t, y(t))$ *a.e.* on T. ☐

In the follows, assume that the function $f(\cdot, \cdot) : T \times E \to E$ satisfies the following assumptions:

(h1) $f(\cdot, \cdot)$ is weakly-weakly continuous;

(h2) $f(\cdot, \cdot)$ is bounded, that is, there exists $M > 0$ such that $||f(t, y)|| \le M$ for all $(t, y) \in T \times E$;

(h3) $g : [0, \infty) \to [0, \infty)$ is a non-decreasing continuous function such that $g(0) = 0$ and $g(t) < t$ for all $t > 0$;

(h4) $\beta(f(T \times A)) \le g(\beta(A))$ for every bounded set $A \subset E$.

Theorem 4.3.2. *Suppose that $f(\cdot,\cdot) : T \times E \to E$ satisfies (h1)-(h4). Then there exist an interval $T_\delta = [0,\delta] \subset T$ such that the set of solutions of (4.3.1) defined on T_δ is non-empty and compact in the space $C_w(T_\delta, E)$.*

Proof. In our proof we shall use some ideas from the papers [7, 21]. Let $\delta \in (0, b]$ be such that $\frac{\delta^\alpha}{\Gamma(\alpha+1)} < 1$. Consider the nonlinear operator $Q : C_w(T_\delta, E) \to C_w(T_\delta, E)$ defined by

$$(Qy)(t) = y_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds, \quad t \in T_\delta.$$

We remark that for $y(\cdot) \in C_w(T_\delta, E)$ we have that, by Remark 4.2.1, the operator Q is well defined. Let $t \in T_\delta$. By the Hahn-Banach theorem, there exists $x^* \in E^*$ such that $\|x^*\| = 1$ and $\|(Qy)(t)\| = |\langle x^*, (Qy)(t)\rangle|$. Then, using the assumption (h2), we have

$$\begin{aligned}\|(Qy)(t)\| &\le \|y_0\| + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\langle y^*, f(s, y(s))\rangle| \, ds \\ &\le \|y_0\| + \frac{M\delta^\alpha}{\Gamma(\alpha+1)} < \|y_0\| + M.\end{aligned}$$

Let $r := \|y_0\| + M$ and

$$\widetilde{B} := \{y(\cdot) \in C_w(T_\delta, E); \|y(\cdot)\|_c \le r\}.$$

We shall consider $\widetilde{B}$ as a topological subspace of $C_w(T_\delta, E)$. It is easy to see that $\widetilde{B}$ is convex and closed, and $Q(\widetilde{B}) \subset \widetilde{B}$. Next we show that $Q(\widetilde{B})$ is an equicontinuous set. Let $t, s \in T_\delta$. We suppose without loss of generality that $s < t$ and $(Qy)(t) \neq (Qy)(s)$. By the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and

$\|(Qy)(t)-(Qy)(s)\| = |\langle y^*, (Qy)(t)-(Qy)(s)\rangle|$. Then

$$
\begin{aligned}
&\|(Qy)(t)-(Qy)(s)\| = |\langle y^*, (Qy)(t)-(Qy)(s)\rangle| \\
= \;& \left| \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \langle y^*, f(\tau, y(\tau))\rangle \, d\tau - \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \langle y^*, f(\tau, y(\tau))\rangle \, d\tau \right| \\
\le \;& \int_0^s \left(\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \right) |\langle y^*, f(\tau, y(\tau))\rangle| \, d\tau + \\
& \int_s^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} |\langle y^*, f(\tau, y(\tau))\rangle| \, d\tau \\
\le \;& \frac{M}{\Gamma(1+\alpha)} \left[s^\alpha - t^\alpha + 2\,(t-s)^\alpha \right] \le \frac{2M}{\Gamma(1+\alpha)} (t-s)^\alpha \,,
\end{aligned}
$$

and so $Q(\widetilde{B})$ is an equicontinuous set. Next we will prove that Q restricted to $\widetilde{B}$ is a continuous operator. For this, fix $y(\cdot) \in \widetilde{B}$, $\varepsilon > 0$ and $y^* \in E^*$ with $\|y^*\| \le 1$. Since $f(\cdot,\cdot)$ is weakly-weakly continuous we have, by a Krasnoselskii type Lemma (see [87]), that there exists a weak neighborhood W of 0 in E such that $|\langle y^*, f(s, y(s)) - f(s, z(s))\rangle| \le \frac{\varepsilon \Gamma(1+\alpha)}{\delta^\alpha}$ for $s \in T_\delta$ and $z(\cdot) \in \widetilde{B}$ with $y(s) - z(s) \in W$. Then it follows that

$$
\begin{aligned}
|\langle y^*, (Qy)(t) - (Qz)(s)\rangle| &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\langle y^*, f(s, y(s)) - f(s, z(s))\rangle| \, ds \\
&\le \frac{\varepsilon \delta^\alpha}{\Gamma(1+\alpha)} \le \varepsilon,
\end{aligned}
$$

and thus Q restricted to $\widetilde{B}$ is a continuous operator.

Let $K := \overline{conv} Q(\widetilde{B})$. Since $Q(\widetilde{B})$ is bounded and equicontinuous in $C(T_\delta, E)$ it follows that K is also bounded and equicontinuous. Let V be a subset of K such that $\beta_c(V) \neq 0$, $V(t) := \{y(t); y(\cdot) \in V\}$ and $(QV)(t) := \{(Qy)(t); y(\cdot) \in V\}$. Let $t \in T_\delta$ and $\varepsilon > 0$. If we choose $\eta > 0$ such that $\eta < \left(\frac{\varepsilon\Gamma(\alpha+1)}{M}\right)^{1/\alpha}$ and $\int_{t-\eta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) ds \neq 0$ then, by the Hahn-Banach theorem, there exists a $y^* \in E^*$ with $\|y^*\| = 1$ and

$$
\left\| \int_{t-\eta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) ds \right\| = \left| \left\langle y^*, \int_{t-\eta}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s)) ds \right\rangle \right|.
$$

It follows that

$$\left\|\int_{t-\eta}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds\right\| \leq \int_{t-\eta}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|\langle y^{*},f(s,y(s))\rangle\right|ds \leq \varepsilon,$$

and thus using property (x) of the measure of noncompactness we infer

$$\beta\left(\left\{\int_{t-\eta}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,V(s))ds\right\}\right) \leq 2\varepsilon. \qquad (4.3.3)$$

Next, since $s \to (t-s)^{\alpha-1}$ is continuous on $[0,t-\eta]$ it follows that there exists $\gamma > 0$ such that

$$\left|(t-\tau)^{\alpha-1}-(t-s)^{\alpha-1}\right| < \varepsilon$$

for all $\tau, s \in [0,t-\eta]$ with $|\tau - s| < \gamma$. Consider the following partition of the interval $[0,t-\eta]$ into n parts $0 = t_0 < t_1 \ldots < t_n = t-\eta$ such that $t_i - t_{i-1} < \delta$ $(i = 1,2,\ldots,n)$ and put $T_i = [t_{i-1},t_i]$. Then we have (see [21])

$$\int_{0}^{t-n}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds =$$

$$= \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{n}(w)\int_{t_{i-1}}^{t_i}(t-s)^{\alpha-1}f(s,y(s))ds$$

$$\in \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{n}(t_i-t_{i-1})\overline{conv}\{(t-s)^{\alpha-1}f(s,z); s\in T_i,\ z\in Z\},$$

where $Z := \{y(s); y(\cdot)\in V\}$, so

$$\beta\left(\int_{0}^{t-\eta}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,V(s))ds\right) \leq$$

$$\leq \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{n}(t_i-t_{i-1})\beta\left(\overline{conv}\{(t-s)^{\alpha-1}f(s,z); s\in T_i,\ z\in Z\}\right),$$

where

$$\int_{0}^{t-\eta}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,V(s))ds := \left\{\int_{0}^{t-\eta}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds; y(\cdot)\in V\right\}.$$

From above and by the properties of β we have

$$\beta\left(\int_{0}^{t-\eta}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,V(s))ds\right) \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1}) \beta \left(\{ (t-s)^{\alpha-1} f(s,z); s \in T_i, \ z \in Z \} \right)$$

$$\leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1})(t - t_i)^{\alpha-1} \beta \left(f(T_i \times Z) \right)$$

$$\leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1})(t - t_i)^{\alpha-1} g(\beta(Z)).$$

Using the continuity on $[0, t-\eta]$ for the real valued function $s \to (t-s)^{\alpha-1}$, we have that

$$\frac{1}{\Gamma(\alpha)}(t_i - t_{i-1})(t-t_i)^{\alpha-1} g(\beta(Z)) \leq g(\beta(Z)) \int_{t_{i-1}}^{t_i} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \frac{\varepsilon\,(t_i - t_{i-1})}{\Gamma(\alpha)} \beta(Z)$$

and so

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1})(t-t_i)^{\alpha-1} g(\beta(Z)) \leq g(\beta(Z)) \int_0^{t-\eta} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \frac{\varepsilon(t-\eta)}{\Gamma(\alpha)} \beta(Z).$$

Thus we obtain

$$\begin{aligned} \beta\left(\int_0^{t-\eta} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, V(s)) ds \right) &\leq \frac{\delta^{\alpha}}{\Gamma(\alpha+1)} g(\beta(Z)) + \frac{\varepsilon(t-\eta)}{\Gamma(\alpha)} \beta(Z) \\ &\leq g(\beta(Z)) + \frac{\varepsilon\delta}{\Gamma(\alpha)} \beta(Z). \end{aligned} \tag{4.3.4}$$

Since

$$(QV)(t) \subset \frac{1}{\Gamma(\alpha)} \int_0^{t-\eta} (t-s)^{\alpha-1} f(s, V(s)) ds + \frac{1}{\Gamma(\alpha)} \int_{t-\eta}^{t} (t-s)^{\alpha-1} f(s, V(s)) ds,$$

then by virtue of (4.3.3) and (4.3.4) we have

$$\beta((QV)(t)) \leq g(\beta(Z)) + \frac{\varepsilon\delta}{\Gamma(\alpha)} \beta(Z) + 2\varepsilon, \ t \in T_\delta.$$

As the last inequality is true for every $\varepsilon > 0$, we have

$$\beta((QV)(t)) \leq g(\beta(Z)), \ t \in T_\delta.$$

Using Lemma 1.5.1 and (h3) we have

$$\beta((QV)(t)) \leq g(\beta(V(t))) \leq g(\beta_c(V)), \ t \in T_\delta$$

and thus $\beta_c((QV)) \leq g(\beta_c(V)) \leq \beta_c(V)$. Using similar arguments as in the middle of the proof of Theorem 2.3 in [7], we can show that K is a closed convex subset of $C_w(T_\delta, E)$. Therefore, by a Schauder-Tichonov type theorem [7, Cor 2.1] it follows that the set of the fixed points of Q in $\widetilde{B}$ is non-empty and compact, so the set of solutions of the problem (4.3.1) on T_δ is non-empty and compact in $C_w(T_\delta, E)$. $\square$

We end this paper with a remark. If for $\alpha = 1$ we put $D^1 y(\cdot) = y'(\cdot)$, then from Theorem 4.3.2 we obtain the following generalization of some known results (see [23, 53, 86, 87]).

Corollary 4.3.1. *If $f(\cdot,\cdot) : T \times E \to E$ is a function such that all conditions from Theorem 4.3.2 hold, then the differential equation*

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(0) = y_0 \end{cases} \tag{4.3.5}$$

has a weak solution on $[0, \delta]$.

4.4 Conclusions

In this chapter, we introduced and studied the notions of the fractional Riemann-Pettis integral and the fractional Caputo weak derivative. Using these tools we obtain an existence result for fractional differential equations in a nonreflexive Banach space equipped with the weak topology. It is well known that there exist functions $x(\cdot) : T \to E$ that are strongly mesurable and scalarly Lebesgue integrable on T, but they are neither Pettis integrable, nor Bochner integrable on T, and so $x(\cdot)$ is also neither Riemann integrable, nor Riemann-Pettis integrable on T. An example is given by the function $x(\cdot) : [0.1] \to \mathbf{c}_0$ defined by

$$x(t) = \left(n\chi_{(0,\frac{1}{n}]}(t)\right)_{n\geq 1}, \quad t \in [0, 1].$$

We note that this function is Dunford integrable on T. Therefore, to define the notion of a fractional integral for the function $x(\cdot)$ it will be necessary to use other kinds of integral such as the Dunford integral, the Henstock–Kurzweil integral or the Henstock–Kurzweil–Pettis integral.

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