SOME CONTRIBUTIONS TO THE DYNAMICAL SYSTEMS ON TIME SCALES

Name : **AWAIS YOUNUS**

Year of Admission : **2008**

Registration No. : **118-GCU-PHD-SMS-08**

Abdus Salam School of Mathematical Sciences

GC University Lahore, Pakistan

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By

Name	**:**	**AWAIS YOUNUS**
Year of Admission	**:**	**2008**
Registration No.	**:**	**118-GCU-PHD-SMS-08**

Abdus Salam School of Mathematical Sciences
GC University Lahore, Pakistan

<u>DECLARATION</u>

I, **<u>Mr. Awais Younus</u>** Registration No. **<u>118-GCU-PHD-SMS-08</u>** student at **Abdus Salam School of Mathematical Sciences GC University** in the subject of **Mathematics** year of admission **2008**, hereby declare that the matter printed in this thesis titled

"Some Contributions to the Dynamical Systems on Time Scales"

is my own work and that

(i) I am not registered for the similar degree elsewhere contemporaneously.

(ii) No direct major work had already been done by me or anybody else on this topic; I worked on, for the Ph. D. degree.

(iii) The work, I am submitting for the Ph. D. degree has not already been submitted elsewhere and shall not in future be submitted by me for obtaining similar degree from any other institution.

Dated: ------------------------ ----------------------------------

Signature

<u>RESEARCH COMPLETION CERTIFICATE</u>

Certified that the research work contained in this thesis titled

“Some Contributions to the Dynamical Systems on Time Scales”

has been carried out and completed by **Mr. Awais Younus** Registration No. **118-GCU-PHD-SMS-08** under my supervision.

Date

Vasile Lupulescu

Supervisor

Submitted Through

Prof. Dr. A. D. Raza Choudary

Director General

Abdus Salam School of Mathematical Sciences

GC University Lahore

Pakistan.

Controller of Examination

GC University Lahore

Pakistan.

To my family and some

very dear friends

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TABLE OF CONTENTS

1.0 INTRODUCTION

The theory of time scale was introduced in 1988 by S. Hilger [37] in order to create a theory which unify continuous and discrete analysis. In these three decades there has been significant growth in the theory of dynamic systems on time scale, covering a variety of different qualitative aspects. We refer to the books [16, 17, 45] and the papers [1, 4, 29, 31, 41, 43, 51, 63, 65, 67, 68, 70, 71].

Differential equations with impulses have a considerable importance in varied applications as physics, engineering, biology, medicine, economics, neuronal networks, social sciences, and so on. Many investigations have been carried out concerning the existence, uniqueness, and asymptotic properties of solutions. We refer to the monographs [7, 11, 44, 62] and the paper [59, 63]. It is well known that the study of controllability plays an important role in the control theory. In recent years, some research dealing with the study of controllability for impulsive systems [13, 20, 36, 47, 49, 66, 69, 72].

The most of the dynamical systems are analyzed in either the continuous or discrete time domain. The population dynamical models in continuous time are usually appropriate for organism that have overlapping generations. On other hand, many biological populations are more accurately described by non-overlapping gernations. The dynamics of these populations often are more appropriately expressed by difference equations. A hybrid model, so-called sequential-continuous dynamical models, was developed by Busenberg and Cooke [25] for models of vertically transmitted diseases (see also [26]). The sequential-continuous systems are characterized by the fact that they, during certain periods of time, are governed by continuous equations and during the other periods, are governed by sequential equations. A such sequential-continuous model can be formulated by the help of dynamical systems on time scales. For more details and results in this area see [14], [15], [22] and [70].

Some authors studied impulsive dynamic systems on time scale [6, 11, 12, 27, 40, 46, 48, 50]. The study of stability, controllability and observability for dynamical systems on time scale has been studied in few works [8, 9, 30, 32, 33, 38, 57, 58], but there has been no result about the controllability and observability of piecewise linear time-varying impulsive control systems. The main purpose of chapter 3 is to derive necessary and sufficient criteria for controllability and observability of a class of such systems on time scale.

Basic qualitative results about Volterra integral and integro-differential equations have been studied by many authors (see e.g. [18, 24]). Notable exceptions that have dispensed with the stability condition on the coefficient matrix have been the works of Burton [18, 19], Corduneanu [22], Choi and Koo [23], Mahfoud [54], Medina [55], Rao and Srinivas [61], among others. In [18], the author investigates the stability and boundedness of the solution involving the anti-derivatives of the kernel. Sufficient conditions for uniformly bounded solution are developed in [54]. In [61], the asymptotic behavior of the solution of a Volterra integro-differential equation in which the coefficient matrix is not necessarily stable is discussed . The resolvent of a Volterra integro-differential equation was first investigated by Grossman and Miller in [35]. In the discrete case resolvent equation was obtained by Elaydi in [28].

Volterra type equations (both integral and intgro-dynamic) on time scales become a new field of interest. In [42], Kulik and Tisdell obtained basic qualitative and quantitative results for Volterra integral equations. Furthermore, in [39] Karpuz studied the existence and the uniqueness of solutions to generalized Volterra integral equations.

Chapter 4 is devoted to the Volterra intgrodynamical systems (abbreviated as VIDS) on time scales. First section deals with the study of the relation between principal matrix and resolvent of linear VIDS. In section 2 we investigate the asymptotic behavior of the solution of the VIDS. The main aim in this section is to develop an equivalence system of the given VIDS having a potential to give the sufficient conditions for asymptotic stability. In Section 3 we first discuss the uniform boundedness of the solutions of VIDS by constructing a Lyapunov functional. Further results for boundedness, the uniform boundedness and stability of the solution are developed by an equivalence system of VIDS, which is constructed by using the antiderivative of the kernel. For the discrete time scale $\mathbb{T}^{r}_{(q,h)}$ we give the related results as corollaries.

Personal Contribution

The results of this dissertation are taken from the following articles.

[51] V. Lupulescu, A. Younus, On controllability and observability for a class of linear impulsive dynamic systems on time scales, *Mathematical and Computer Modelling*, **54 (2011)** 1300-1310.

[52] V. Lupulescu, A. Younus, Controllability and observability for a class of time-varying impulsive systems on time scales, *Electron. J. Qual. Theory Differ. Equ.* **95 (2011)** 1-30.

[53] V. Lupulescu, S. K. Ntouyas, A. Younus, Qualitative aspects of a Volterra integro-dynamic system on time scales, *Electron. J. Qual. Theory Differ. Equ.* **5 (2013)** 1-35.

2.0 PRELIMINARIES

In this chapter, we recall some basic definitions and results in the calculus on time scale analysis before reading the new results obtained in the remaining chapter. We refer to [16, 17], and also to the paper [4, 5], for more information on analysis on time scales. A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$, and the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T}; s > t\}$ (supplemented by $\inf \emptyset = \sup \mathbb{T}$), the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T}; s < t\}$ (supplemented by $\sup \emptyset = \inf \mathbb{T}$), while the *graininess* $\mu : \mathbb{T} \to \mathbb{R}_+$ is given by $\mu(t) := \sigma(t) - t$. The point $t \in \mathbb{T}$ is *left-dense*, *left-scattered*, *right-dense*, *right-scattered* if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. A time scale $\mathbb{T}$ is said to be *discrete* if t is left-scattered and right-scattered for each $t \in \mathbb{T}$. The notations $[a, b]$, $[a, b)$, and so on, will denote time scale intervals such as $[a, b] := \{t \in \mathbb{T}; a \leq t \leq b\}$, where $a, b \in \mathbb{T}$. Let $\mathbb{R}^n$ be the space of n-dimensional column vectors $x = \text{col}(x_1, x_2, ...x_n)$ with a norm $||\cdot||$. Also, by the same symbol $||\cdot||$ we will denote the corresponding matrix norm in the space $M_n(\mathbb{R})$ of $n \times n$ matrices. We recall that $||A|| := \sup\{||Ax||; ||x|| \leq 1\}$ and the following inequality $||Ax|| \leq ||A||||x||$ holds for all $A \in M_n(\mathbb{R})$ and $x \in \mathbb{R}^n$.

Definition 2.0.1. *([16, Definition 1.58])A function $w : \mathbb{T} \to \mathbb{R}^n$ is said to be rd-contionous if*

(i) *w is continuous at every right-dense point $t \in \mathbb{T}$,*

(ii) *$w(t^-) := \lim_{s \to t^-} w(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.*

The set of all rd-continuous functions $w : \mathbb{T} \to \mathbb{R}^n$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R}^n)$.

Definition 2.0.2. *A function $w : \mathbb{T} \to \mathbb{R}$ is said to be regressive (respectively positively regressive) if $1 + \mu(t)w(t) \neq 0$ (respectively $1 + \mu(t)w(t) > 0$) for all $t \in \mathbb{T}^k$.*

The set $\mathcal{R}$ (respectively $\mathcal{R}^+$) of all regressive (respectively positively regressive) functions from $\mathbb{T}$ to $\mathbb{R}$ is an Abelian group with respect to the circle addition operation $\oplus$, given by

$$(w \oplus q)(t) := w(t) + q(t) + \mu(t)w(t)q(t).$$

The inverse element of $w \in \mathcal{R}$ is given by

$$(\ominus w)(t) = -\frac{w(t)}{1 + \mu(t)w(t)}$$

and so, the circle subtraction operation $\ominus$ is defined by

$$(w \ominus q)(t) = (w \oplus (\ominus q))(t) = \frac{w(t) - q(t)}{1 + \mu(t)q(t)}.$$

The space of all rd-continuous and regressive functions from $\mathbb{T}$ to $\mathbb{R}$ is denoted by $C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$. Also,

$$C_{rd}^{+}\mathcal{R}(\mathbb{T}, \mathbb{R}) := \{w \in C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R}); 1 + \mu(t)w(t) > 0 \text{ for all } t \in \mathbb{T}^k\}.$$

The set of rd-continuous (respectively rd-continuous and regressive) functions $A : \mathbb{T} \to M_n(\mathbb{R})$ is denoted by $C_{rd}(\mathbb{T}, M_n(\mathbb{R}))$ (respectively by $C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$). We recall that a matrix-valued function A is said to be regressive if $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^k$, where I is the $n \times n$ identity matrix. Moreover, the set $\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$ of all regressive matrix-valued functions is a group with respect to the addition operation $\oplus$ define

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t)$$

for all $t \in \mathbb{T}^k$. The inverse element of $A \in \mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$ is given by

$$(\ominus A)(t) = -[I + \mu(t)A(t)]^{-1}A(t) = -A(t)[I + \mu(t)A(t)]^{-1}$$

for all $t \in \mathbb{T}^k$.

Definition 2.0.3. *([16, Definition 1.10])A function $w:\mathbb{T}\to\mathbb{R}^n$ is said to be differentiable at $t\in\mathbb{T}^k$, with Δ-derivative $w^{\Delta}(t)\in\mathbb{R}^n$ if given $\varepsilon>0$ there exists a neighborhood U of t such that, for all $s\in\mathbb{T}\cap U$,*

$$||w^{\sigma}(t)-w(s)-w^{\Delta}(t)[\sigma(t)-s]||\leq\varepsilon|\sigma(t)-s|,$$

where $w^{\sigma}(t):=w(\sigma(t))$.

We denote by $C^1_{rd}(\mathbb{T},\mathbb{R}^n)$ the set of all functions $w:\mathbb{T}\to\mathbb{R}^n$ that are differentiable on $\mathbb{T}$ and its Δ-derivative $w^{\Delta}(t)\in C_{rd}(\mathbb{T},\mathbb{R}^n)$.

Theorem 2.0.4. *([4, 16]) Assume that $w:\mathbb{T}\to\mathbb{R}^n$ and let $t\in\mathbb{T}^k$*

(i) *If w is differentiable at t, then w is continuous at t.*

(ii) *If w is continuous at t and t is right-scattered, then w is differentiable at t with*

$$w^{\Delta}(t)=\frac{w^{\sigma}(t)-w(t)}{\sigma(t)-t}.$$

(iii) *If w is differentiable at t and t is right-dense, then*

$$w^{\Delta}(t)=\lim_{s\to t}\frac{w^{\sigma}(t)-w(s)}{t-s}.$$

(iv) *If w is differentiable at t, then $w^{\sigma}(t)=w(t)+\mu(t)w^{\Delta}(t)$.*

(v) *If $w,q:\mathbb{T}\to\mathbb{R}^n$ are both differentiable at t, then the product w^Tq is also differentiable at t and*

$$(w^Tq)^{\Delta}(t)=w^T(t)q^{\Delta}(t)+(w^T)^{\Delta}(t)q^{\sigma}(t).$$

Theorem 2.0.5. *([16]) If $M,N:\mathbb{T}\to M_n(\mathbb{R})$ are differentiable matrices, then*

(i) *$M^{\sigma}(t)=M(t)+\mu(t)M^{\Delta}(t)$ for all $t\in\mathbb{T}$;*

(ii) *$(M^T)^{\Delta}=(M^{\Delta})^T$;*

(iii) *$(M+N)^{\Delta}=M^{\Delta}+N^{\Delta}$, and $(MN)^{\Delta}=M^{\Delta}N^{\sigma}+MN^{\Delta}=M^{\sigma}N^{\Delta}+M^{\Delta}N$;*

(iv) *$(M^{-1})^{\Delta}=-(M^{\sigma})^{-1}M^{\Delta}M^{-1}=-M^{-1}M^{\Delta}(M^{\sigma})^{-1}$if MM^{σ} is invertible;*

(v) $(MN^{-1})^{\Delta} = (M^{\Delta} - MN^{-1}N^{\Delta})(N^{\sigma})^{-1} = [M^{\Delta} - (MN^{-1})^{\sigma}N^{\Delta}]N^{-1}$ *if* NN^{σ} *is invertible.*

Definition 2.0.6. *Let* $w \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$*. A function* $q : \mathbb{T} \to \mathbb{R}^n$ *is called the antiderivative of* w *on* $\mathbb{T}$ *if it is differentiable on* $\mathbb{T}$ *and satisfies* $q^{\Delta}(t) = w(t)$ *for all* $t \in \mathbb{T}^k$*. In this cases, we define*

$$\int_a^t w(s)\Delta s = q(t) - q(a), \quad a, t \in \mathbb{T}.$$

Theorem 2.0.7. *(Existence of Antiderivatives) Every function* $w \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$ *has an antiderivative. In particular if* $\tau \in \mathbb{T}$*, then the function* $W : \mathbb{T} \to \mathbb{R}^n$ *defined by*

$$W(t) := \int_{\tau}^t w(s)\Delta s \text{ for } t \in \mathbb{T}$$

is an antiderivative of w*.*

Theorem 2.0.8. *([16, Theorem 1.77]) If* $a, b, c \in \mathbb{T}$*,* $\alpha, \beta \in \mathbb{R}$*, and* $w, q \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$*, then*

(i) $\int_a^b (\alpha w + \beta q)(t)\Delta t = \alpha \int_a^b w(t)\Delta t + \beta \int_a^b q(t)\Delta t$;

(ii) $\int_a^b w(t)\Delta t = -\int_b^a w(t)\Delta t$;

(iii) $\int_a^{\sigma(a)} w(t)\Delta t = \mu(a)w(a)$;

(iv) $\int_a^b w(t)\Delta t = \int_a^c w(t)\Delta t + \int_c^b w(t)\Delta t$;

(v) $\int_a^b w^T(t)q^{\Delta}(t)\Delta t = (w^T q)(b) - (w^T q)(a) - \int_a^b (w^T)^{\Delta}(t)q(\sigma(t))\Delta t$;

(vi) $\left\| \int_a^b w(t)\Delta t \right\| \leq \int_a^b ||w(t)||\Delta t.$

Let $w \in C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}^k$. Then the *exponential function* on $\mathbb{T}$ is defined by

$$e_w(t, s) = \exp\left(\int_s^t \xi_{\mu(t)}(w(\tau))\Delta\tau\right) \text{ with } \xi_h(z) := \begin{cases} \frac{\ln(1+hz)}{h} & \text{if } h \neq 0 \\ z & \text{if } h = 0, \end{cases}$$

and it is the unique solution of the initial value problem $y^{\Delta} = w(t)y$, $y(s) = 1$.

Theorem 2.0.9. *([16, Theorem 2.36]) If* $w, q \in C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$ *then the following hold:*

(i) $e_0(t, s) = 1$ *and* $e_w(t, t) = 1$*;*

(ii) $e_w(\sigma(t), s) = [1 + \mu(t)w(t)]e_w(t, s)$*;*

(iii) $\frac{1}{e_w(t,s)} = e_w(s,t) = e_{\ominus w}(t,s)$*;*

(iv) $e_w(t,s)e_w(s,r) = e_w(t,r)$*;*

(v) $e_w(t,s)e_q(t,s) = e_{w\oplus q}(t,s)$ *and* $\frac{e_w(t,s)}{e_q(t,s)} = e_{w\ominus q}(t,s)$*;*

(vi) $\left(\frac{1}{e_w(\cdot,s)}\right)^{\Delta} = -\frac{w(t)}{e_w^{\sigma}(\cdot,s)}$*;*

(vii) *If* $w \in C_{rd}^{+}\mathcal{R}(\mathbb{T},\mathbb{R})$ *then* $e_w(t,s) > 0$ *for all* $t,s \in \mathbb{T}$*;*

(viii) $\int_a^b w(s)e_w(c,\sigma(s))\Delta s = e_w(c,a) - e_w(c,b)$*.*

Example 2.0.10. If we consider discrete time scale (see [21, 60])

$$\mathbb{T}^r_{(q,h)} = \{rq^k + [k]_q h : k \in \mathbb{Z}\} \cup \left\{\frac{h}{1-q}\right\},$$

where $r \in \mathbb{R}$, $q \geq 1$, $h \geq 0$, $q+h > 1$, $[k]_q = \frac{q^k-1}{q-1}$, $k \in \mathbb{R}$, $q \neq 1$ and $[k]_1 = k \left(= \lim\limits_{q\to 1} \frac{q^k-1}{q-1}\right)$. It is easy to see that $\mathbb{T}^r_{(q,h)} = \mathbb{T}^r_q = \{rq^k : k \in \mathbb{Z}\} \cup \{0\}$ provided $h = 0$ and $\mathbb{T}^r_{(q,h)} = \mathbb{T}^r_h$ $= \{r + kh : k \in \mathbb{Z}\}$ provided $q = 1$ (in this case we put $h/(1-q) = -\infty$). It is clear that, for $t \in \mathbb{T}^r_{(q,h)}$, we have

$$\sigma(t) = qt + h \text{ and } \mu(t) = (q-1)t + h.$$

Let $t \in \mathbb{T}^r_{(q,h)}$ and $w : \mathbb{T}^r_{(q,h)} \to \mathbb{R}$. Then the Δ- (q,h)-derivative of w at t is

$$\Delta_{(q,h)} w(t) := \frac{w(qt+h) - w(t)}{(q-1)t + h}$$

and the (q,h)-integral is

$$\int_a^b w(t)\Delta t = \sum_{t\in[a,b)} w(t)\mu(t).$$

For $z \neq -1/(q't+h)$, where $q' = q-1$ the exponential function $e_z(t,s)$ has the form

$$e_z(t,s) = \prod_{r\in[s,t)} (1 + \mu(r)z), \text{ for all } t,s \in \mathbb{T}^r_{(q,h)}$$

and

$$e_{\ominus z}(t,s) = \prod_{r\in[s,t)} \frac{1}{(1+\mu(r)z)}, \text{ for all } t,s \in \mathbb{T}^r_{(q,h)}.$$

Lemma 2.0.11. *([16, Theorem 2.38]) If* $w,\ q \in C_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$*. Then* $e^{\Delta}_{w\ominus q}(\cdot,t_0) = (w - q)\dfrac{e_w(\cdot,t_0)}{e^{\sigma}_q(\cdot,t_0)}$*.*

Proposition 2.0.12. *([16, Theorem 5.24]) If $A \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$ and $h \in C_{prd}(\mathbb{T}, \mathbb{R}^n)$, then for each $(\tau, \eta) \in \mathbb{T} \times \mathbb{R}^n$ the initial value problem*

$$x^{\Delta} = A(t)x + h(t), \quad x(\tau) = \eta$$

has a unique solution given by

$$x(t) = \Phi_A(t, \tau)\eta + \int_{\tau}^{t} \Phi_A(t, \sigma(s))h(s)\Delta s, \quad t \geq \tau.$$

Lemma 2.0.13. *([16, Theorem 6.2]) Let $\alpha \in \mathbb{R}$ with $\alpha \in C^{+}_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$. Then*

$$e_{\alpha}(t, s) \geq 1 + \alpha(t - s) \text{ for all } t \geq s.$$

Lemma 2.0.14. *([16, Corollary 6.7]) Let $y \in C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$, $w \in C^{+}_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$, $w \geq 0$ and $\alpha \in \mathbb{R}$. Then*

$$y(t) \leq \alpha + \int_{\tau}^{t} y(s)w(s)\Delta s \text{ for all } t \in \mathbb{T},$$

implies

$$y(t) \leq \alpha e_w(t, \tau) \text{for all } t \in \mathbb{T}.$$

The following theorem shows that we can express the matrix exponential as a finite sum of powers of the matrix A with infinitely rd-continuous delta differentiable functions as coefficients.

Proposition 2.0.15. *([29, Theorem 5.1]) For the system (3.3) with $A \in M_n(\mathbb{R})$ constant, there exist scalar functions $\gamma_0(t, \tau), ..., \gamma_{n-1}(t, \tau) \in C^{\infty}_{rd}(\mathbb{T}_+, \mathbb{R})$ such that the unique solution has representation*

$$e_A(t, \tau) = \sum_{i=0}^{n-1} \gamma_i(t, \tau)A^i. \tag{2.1}$$

Theorem 2.0.16. *([39, Theorem 7]) Let $a, b \in \mathbb{T}$ with $b > a$ and assume that $w : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is integrable on $\{(t, s) \in \mathbb{T} \times \mathbb{T} : b > t > s \geq a\}$. Then*

$$\int_a^b \int_a^{\eta} w(\eta, \xi)\Delta\xi\Delta\eta = \int_a^b \int_{\sigma(\xi)}^{b} w(\eta, \xi)\Delta\eta\Delta\xi.$$

It is easy to verify that the above result holds for $w \in C_{rd}(\mathbb{T} \times \mathbb{T}, \mathbb{R}^n)$.

3.0 IMPULSIVE DYNAMIC EQUATIONS

This chapter deals with the study of variation of parameter formula, controllability and observability of the following linear impulsive dynamical system

$$\begin{cases} x^{\Delta} = A_k(t)x + B_k(t)u, \ \ t \in [t_{k-1}, t_k), \\ x(t_k^+) = (1 + c_k)x(t_k), \ \ k = 1, 2, ..., \\ y(t) = C_k(t)x + D_k(t)u, \\ x(t_0) = x_0, \end{cases} \tag{3.1}$$

where $[t_{k-1}, t_k) \subset \mathbb{T}_0 := [t_0, \infty) \cap \mathbb{T}$, $t_k \in \mathbb{T}_0$ are right-dense, $0 \leq t_0 < t_1 < t_2 < ... < t_k < ...$, such that $\lim_{k\to\infty} t_k = \infty$, $x(t_k^+) := \lim_{h\to 0^+} x(t_k + h)$, $x(t_k^-) := \lim_{h\to 0^+} x(t_k - h)$ and $c_k \in \mathbb{R}$ are constants. Also, we assume that $A_k \in C_{rd}\mathcal{R}(\mathbb{T}_0, M_n(\mathbb{R}))$, $B_k \in C_{rd}\mathcal{R}(\mathbb{T}_0, M_{n\times m}(\mathbb{R}))$, $C_k \in C_{rd}\mathcal{R}(\mathbb{T}_0, M_{p\times n}(\mathbb{R}))$, $D_k \in C_{rd}\mathcal{R}(\mathbb{T}_0, M_{p\times m}(\mathbb{R}))$, $x \in \mathbb{R}^n$ is the sate variable, $u \in \mathbb{R}^m$ is the control input, and $y \in \mathbb{R}^p$ is the output.

Corresponding to impulsive system (3.1), consider the following dynamic system on time scales

$$x^{\Delta} = A_k(t)x \tag{3.2}$$

where $k = 1, 2, ...,$ and $t \in [t_{k-1}, t_k)$.

A matrix $X_{A_k} \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$ is said to be a *matrix solution* of (3.2) if each column of X_{A_k} satisfies (3.2) for all $t \in [t_{k-1}, t_k)$. A *fundamental matrix* of (3.2) is a matrix solution X_{A_k} of (3.2) such that $\det X_{A_k}(t) \neq 0$ for all $t \in [t_{k-1}, t_k)$. A *transition matrix* of (3.2) at initial time $\tau \in [t_{k-1}, t_k)$ is a fundamental matrix such that $X_{A_k}(\tau) = I$. The transition matrix of (3.2) at initial time $\tau \in [t_{k-1}, t_k)$ will be denoted by $\Phi_{A_k}(t, \tau)$. Therefore, the transition matrix of (3.2) at initial time $\tau \in [t_{k-1}, t_k)$ is the unique solution of the following matrix initial value problem

$$X^{\Delta} = A_k(t)X, \quad X(\tau) = I \tag{3.3}$$

and $x(t) = \Phi_{A_k}(t,\tau)\eta$ for $\tau \in [t_{k-1}, t_k)$, is the unique solution of initial value problem

$$x^{\Delta} = A_k(t)x, \quad x(\tau) = \eta.$$

If $A_k(t) = A_k$ is a constant matrix, then we use the notation $e_{A_k}(t,\tau)$ instead of $\Phi_{A_k}(t,\tau)$.

If we take $A_k(t) = A(t)$, $B_k(t) = B(t)$, $C_k(t) = C(t)$ and $D_k(t) = D(t)$ for each $k = 1, 2, ...,$ in (3.1), then we obtain the following linear time-varying impulsive dinamical system (see [51])

$$\begin{cases} x^{\Delta} = A(t)x + B(t)u,\ t \in \mathbb{T}_0 \setminus \{t_k\}_{k=1}^{\infty}, \\ x(t_k^+) = (1 + c_k)x(t_k),\ k = 1, 2, ..., \\ y(t) = C(t)x + D(t)u, \\ x(t_0) = x_0. \end{cases} \tag{3.4}$$

Our first result establishes the Variation of Constants formula for linear impulsive dynamical system (3.1).

Lemma 3.0.17. *For any $t \in (t_{k-1}, t_k]$, $k = 1, 2, ...,$ the solution of the initial value problem (3.1) is given by*

$$\begin{aligned} x(t) \;=\; & \Phi_{A_k}(t, t_{k-1}) \prod_{i=1}^{k-1}(1 + c_i) \prod_{i=k-1}^{1} \Phi_{A_i}(t_i, t_{i-1})x_0 \qquad (3.5) \\ & + \int_{t_{k-1}}^{t} \Phi_{A_k}(t, \sigma(\tau))B_k(\tau)u(\tau)\Delta\tau + \sum_{i=1}^{k-1}[\prod_{j=i}^{k-1}(1 + c_j) \int_{t_{i-1}}^{t_i} \Phi_{A_k}(t, t_{k-1}) \\ & \times \prod_{r=k-1}^{i+1} \Phi_{A_r}(t_r, t_{r-1})\Phi_{A_i}(t_i, \sigma(\tau))B_i(\tau)u(\tau)\Delta\tau]. \end{aligned}$$

Proof. If $t \in [t_0, t_1]$, then by Proposition 2.0.12, the unique solution of (3.1) is given by

$$x(t) = \Phi_{A_1}(t, t_0)x_0 + \int_{t_0}^{t} \Phi_{A_1}(t, \sigma(\tau))B_1(\tau)u(\tau)\Delta\tau,\ t \in [t_0, t_1].$$

For $t \in (t_1, t_2]$ the initial value problem

$$\begin{cases} x^\Delta = A_2(t)x + B_2(t)u, \\ x(t_1^+) = (1 + c_1)x(t_1), \end{cases}$$

has the unique solution

$$x(t) = \Phi_{A_2}(t, t_1)x(t_1^+) + \int_{t_1}^{t} \Phi_{A_2}(t, \sigma(\tau))B_2(\tau)u(\tau)\Delta\tau.$$

Since

$$\begin{aligned} x(t_1^+) &= (1 + c_1)x(t_1) \\ &= (1 + c_1)\Phi_{A_1}(t_1, t_0)x_0 + (1 + c_1)\int_{t_0}^{t_1} \Phi_{A_1}(t_1, \sigma(\tau))B_1(\tau)u(\tau)\Delta\tau, \end{aligned}$$

it follows that

$$\begin{aligned} x(t) &= \Phi_{A_2}(t, t_1)(1 + c_1)\Phi_{A_1}(t_1, t_0)x_0 \\ &\quad +\Phi_{A_2}(t, t_1)(1 + c_1)\int_{t_0}^{t_1} \Phi_{A_1}(t_1, \sigma(\tau))B_1(\tau)u(\tau)\Delta\tau \\ &\quad + \int_{t_1}^{t} \Phi_{A_2}(t, \sigma(\tau))B_2(\tau)u(\tau)\Delta\tau \end{aligned}$$

and so that (3.5) is true for $k = 2$.

Next, suppose that (3.5) is true for $k = p$, that is, for $t \in (t_{p-1}, t_p]$, we have

$$\begin{aligned} x(t) &= \Phi_{A_p}(t, t_{p-1})\prod_{i=1}^{p-1}(1 + c_i)\prod_{i=p-1}^{1}\Phi_{A_i}(t_i, t_{i-1})x_0 \\ &\quad + \int_{t_{p-1}}^{t} \Phi_{A_p}(t, \sigma(\tau))B_p(\tau)u(\tau)\Delta\tau + \sum_{i=1}^{p-1}[\prod_{j=i}^{p-1}(1 + c_j)\int_{t_{i-1}}^{t_i} \Phi_{A_p}(t, t_{p-1}) \\ &\quad \times \prod_{r=p-1}^{i+1}\Phi_{A_r}(t_r, t_{r-1})\Phi_{A_i}(t_i, \sigma(\tau))B_i(\tau)u(\tau)\Delta\tau]. \end{aligned}$$

Then, for $t \in (t_p, t_{p+1}]$, the initial value problem

$$\begin{cases} x^\Delta = A_{p+1}(t)x + B_{p+1}(t)u, \\ x(t_p^+) = (1 + c_p)x(t_p), \end{cases}$$

has the unique solution

$$x(t) = \Phi_{A_{p+1}}(t, t_p)x(t_p^+) + \int_{t_p}^{t} \Phi_{A_{p+1}}(t, \sigma(\tau))B_{p+1}(\tau)u(\tau)\Delta\tau, \ \ t \in (t_p, t_{p+1}].$$

Since

$$\begin{aligned} x(t_p^+) &= (1+c_p)x(t_p) = \prod_{i=1}^{p}(1+c_i)\prod_{i=p}^{1}\Phi_{A_i}(t_i, t_{i-1})x_0 \\ &+ (1+c_p)\int_{t_{p-1}}^{t_p}\Phi_{A_p}(t_p, \sigma(\tau))B_p(\tau)u(\tau)\Delta\tau + \sum_{i=1}^{p-1}[\prod_{j=i}^{p}(1+c_j)\int_{t_{i-1}}^{t_i}\Phi_{A_p}(t_p, t_{p-1}) \\ &\times \prod_{r=p-1}^{i+1}\Phi_{A_r}(t_r, t_{r-1})\Phi_{A_i}(t_i, \sigma(\tau))B_i(\tau)u(\tau)\Delta\tau]. \end{aligned}$$

It follows that

$$\begin{aligned} x(t) &= \Phi_{A_{p+1}}(t, t_p)\prod_{i=1}^{p}(1+c_i)\prod_{i=p}^{1}\Phi_{A_i}(t_i, t_{i-1})x_0 \\ &+ \int_{t_p}^{t}\Phi_{A_{p+1}}(t, \sigma(\tau))B_{p+1}(\tau)u(\tau)\Delta\tau + \sum_{i=1}^{p}[\prod_{j=i}^{p}(1+c_j)\int_{t_{i-1}}^{t_i}\Phi_{A_{p+1}}(t, t_p) \\ &\times \prod_{r=p}^{i+1}\Phi_{A_r}(t_r, t_{r-1})\Phi_{A_i}(t_i, \sigma(\tau))B_i(\tau)u(\tau)\Delta\tau] \end{aligned}$$

and thus (3.5) is true for $k = p+1$. Therefore, by induction, (3.5) is proved. □

Corollary 3.0.18. *For any $t \in (t_k, t_{k+1}]$, $k = 1, 2, ...$, the solution of the initial value problem (3.4) is given by*

$$\begin{aligned} x(t) &= \prod_{j=1}^{k}(1+c_j)\Phi_A(t, t_0)x_0 + \sum_{i=1}^{k}\prod_{j=i}^{k}(1+c_j)\int_{t_{i-1}}^{t_i}\Phi_A(t, \sigma(\tau))B(\tau)u(\tau)\Delta\tau \\ &+ \int_{t_k}^{t}\Phi_A(t, \sigma(\tau))B(\tau)u(\tau)\Delta\tau. \end{aligned}$$

Remark 3.0.19. If $\mathbb{T} = \mathbb{R}$, then we obtain the result of Lemma 3.0.17 in [72] and the Corollary 3.0.18 in [36] . The version of non impulsive case on time scales ($c_i = -1$) can be found in [16, Theorem 5.24].

3.1 CONTROLLABILITY

The following section start with the definition of the controllable system.

Definition 3.1.1. *The impulsive system (3.1) (respectively (3.4)) is called controllable on $[t_0, t_f]$, with $t_f > t_0$, if given any initial state $x_0 \in \mathbb{R}^n$ there exists a piecewise rd-continuous input signal $u(\cdot) : [t_0, t_f] \to \mathbb{R}^m$ such that the corresponding solution of (3.1) (respectively (3.4)) satisfies $x(t_f) = 0$.*

We consider the following Gramian matrices:

$$G_i := G(t_0, t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} \Psi_i(t_0, \sigma(\tau)) B_i(\tau) B_i^T(\tau) \Psi_i^T(t_0, \sigma(\tau)) \Delta\tau, \tag{3.6}$$

for $i = 1, 2, ..., k-1$ and

$$G_k := G(t_0, t_{k-1}, t_f) = \int_{t_{k-1}}^{t_f} \Psi_k(t_0, \sigma(\tau)) B_k(\tau) B_k^T(\tau) \Psi_k^T(t_0, \sigma(\tau)) \Delta\tau, \tag{3.7}$$

where $\Psi_1(\tau) := \Psi_1(t_0, \sigma(\tau) = \Phi_{A_1}(t_0, \sigma(\tau))$, for $\tau \in (t_0, t_1]$ and

$$\Psi_i(\tau) := \Psi_i(t_0, \sigma(\tau)) = \prod_{j=1}^{i-1} \Phi_{A_j}(t_{j-1}, t_j) \Phi_{A_i}(t_{i-1}, \sigma(\tau)), \;\; \tau \in (t_{i-1}, t_i], \tag{3.8}$$

for $i = 2, 3, ..., k$.

If $A_k(t) = A_k$ and $B_k(t) = B_k$ are constant matrices then

$$G_i := G(t_0, t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} \Psi_i(t_0, \sigma(\tau)) B_i B_i^T \Psi_i^T(t_0, \sigma(\tau)) \Delta\tau, \tag{3.9}$$

for $i = 1, 2, ..., k-1$ and

$$G_k := G(t_0, t_{k-1}, t_f) = \int_{t_{k-1}}^{t_f} \Psi_k(t_0, \sigma(\tau)) B_k B_k^T \Psi_k^T(t_0, \sigma(\tau)) \Delta\tau, \tag{3.10}$$

where $\Psi_1(\tau) := \Psi_1(t_0, \sigma(\tau) = e_{A_1}(t_0, \sigma(\tau))$, for $\tau \in (t_0, t_1]$ and

$$\Psi_i(\tau) := \Psi_i(t_0, \sigma(\tau)) = \prod_{j=1}^{i-1} e_{A_j}(t_{j-1}, t_j) e_{A_i}(t_{i-1}, \sigma(\tau)), \;\; \tau \in (t_{i-1}, t_i], \tag{3.11}$$

for $i = 2, 3, ..., k$.

The Gramian matrix in the case of time scale was defined in [32]. The above definitions are adopted from [32] for impulsive case. Now we are formulating the results for controllability.

Theorem 3.1.2. (i) *If there exists at least* $l \in \{1, 2, ..., k\}$ *such that rank*$(G_l) = n$, *then the impulsive system (3.1) is controllable on* $[t_0, t_f]$ $(t_f \in (t_{k-1}, t_k])$.

(ii) *Assume that* $c_i \neq -1$, $i = 1, 2, ..., k-1$. *If the impulsive system (3.1) is controllable on* $[t_0, t_f]$ $(t_f \in (t_{k-1}, t_k])$, *then*

$$\operatorname{rank}(G_0\ G_1\ ...\ G_k) = n. \tag{3.12}$$

Proof. (i) Let $l \in \{1, 2, ..., k\}$ be such that rank$(G_l) = n$, that is, $G(t_0, t_{l-1}, t_l)$ is invertible. Then for a given $x_0 \in \mathbb{R}^n$, choose

$$u(t) = \begin{cases} a_l B_l^T(t)\Psi_l^T G_l^{-1} x_0 & \text{if } t \in (t_{l-1}, t_l] \\ 0 & \text{if } t \in [t_0, t_f] \setminus (t_{l-1}, t_l], \end{cases} \tag{3.13}$$

where a_l is a constant such that

$$\prod_{i=1}^{k-1}(1 + c_i) + a_l \prod_{j=l}^{k-1}(1 + c_j) = 0.$$

Obviously, the control input $u(\cdot)$ is piecewise rd-continuous on $[t_0, t_f]$. By Lemma 3.0.17, we have

$$\begin{aligned} x(t_f) &= \Phi_{A_k}(t_f, t_{k-1}) \prod_{i=1}^{k-1}(1 + c_i) \prod_{i=k-1}^{1} \Phi_{A_i}(t_i, t_{i-1}) x_0 + [\prod_{j=l}^{k-1}(1 + c_j) a_l \\ &\times \int_{t_{l-1}}^{t_l} \Phi_{A_k}(t_f, t_{k-1}) \prod_{r=k-1}^{l+1} \Phi_{A_r}(t_r, t_{r-1}) \Phi_{A_l}(t_l, \sigma(\tau)) B_l(\tau) B_l^T(\tau) \Psi_l^T(\tau) G_l^{-1} \Delta\tau] x_0. \end{aligned}$$

Since

$$\begin{aligned} \prod_{r=k-1}^{l+1} \Phi_{A_r}(t_r, t_{r-1}) \Phi_{A_l}(t_l, \sigma(\tau)) \Psi_l^{-1}(\tau) &= \Phi_{A_{k-1}}(t_{k-1}, t_{k-2}) ... \Phi_{A_l}(t_l, \sigma(\tau)) \\ \times \Phi_{A_l}(\sigma(\tau), t_{l-1}) \Phi_{A_{l-1}}(t_{l-1}, t_{l-2}) ... \Phi_{A_1}(t_1, t_0) &= \prod_{i=k-1}^{1} \Phi_{A_i}(t_i, t_{i-1}), \end{aligned}$$

it follows that

$$
\begin{aligned}
x(t_f) &= \Phi_{A_k}(t_f, t_{k-1}) \prod_{i=1}^{k-1}(1+c_i) \prod_{i=k-1}^{1} \Phi_{A_i}(t_i, t_{i-1}) x_0 \\
&\quad + \left[\prod_{j=l}^{k-1}(1+c_j) a_l \int_{t_{l-1}}^{t_l} \Phi_{A_k}(t_f, t_{k-1}) \prod_{r=k-1}^{l+1} \Phi_{A_r}(t_r, t_{r-1}) \Phi_{A_l}(t_l, \sigma(\tau)) \Psi_l^{-1}(\tau) \right. \\
&\quad \left. \times\, \Psi_l(\tau) B_l(\tau) B_l^T(\tau) \Psi_l^T(\tau) G_l^{-1} \Delta\tau \right] x_0.
\end{aligned}
$$

Therefore, we obtain

$$
x(t_f) = \left[\prod_{i=1}^{k-1}(1+c_i) + \prod_{j=l}^{k-1}(1+c_j) a_l \right] \Phi_{A_k}(t_f, t_{k-1}) \prod_{i=k-1}^{1} \Phi_{A_i}(t_i, t_{i-1}) = 0
$$

and so that the impulsive system (3.1) is controllable on $[t_0, t_f]$.

(ii) Suppose that (3.1) is controllable on $[t_0, t_f]$ and $\operatorname{rank}(G_0\ G_1\ ...\ G_k) < n$. Then there exists nonzero $x_\alpha \in \mathbb{R}^n$ such that

$$
0 = x_\alpha^T G(t_0, t_{i-1}, t_i) x_\alpha = \int_{t_{i-1}}^{t_i} x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i(\tau) B_i^T(\tau) \Psi_i^T(t_0, \sigma(\tau)) x_\alpha \Delta\tau,
$$

for $i = 1, 2, ..., k-1$ and

$$
0 = x_\alpha^T G(t_0, t_{k-1}, t_f) x_\alpha = \int_{t_{k-1}}^{t_f} x_\alpha^T \Psi_k(t_0, \sigma(\tau)) B_k(\tau) B_k^T(\tau) \Psi_k^T(t_0, \sigma(\tau)) x_\alpha \Delta\tau.
$$

Since $x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i(\tau)$ are rd-continuous functions and

$$
x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i(\tau) B_i^T(\tau) \Psi_i^T(t_0, \sigma(\tau)) x_\alpha = \left\| x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i(\tau) \right\|^2,
$$

for $\tau \in (t_{i-1}, t_i]$, $i = 1, 2, ..., k$, then from the last equalities we obtain

$$
x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i(\tau) = 0, \ \ \tau \in (t_{i-1}, t_i],\ i = 1, 2, ..., k. \tag{3.14}
$$

However, the impulsive system (3.1) is controllable on $[t_0, t_f]$, and so choosing $x_0 = x_\alpha$, there exists a piecewise rd-continuous input $u(\cdot)$ such that

$$\begin{aligned}
0 &= x(t_f) = \Phi_{A_k}(t_f, t_{k-1}) \prod_{i=1}^{k-1} (1 + c_i) \prod_{i=k-1}^{1} \Phi_{A_i}(t_i, t_{i-1}) x_\alpha \\
&+ \int_{t_{k-1}}^{t_f} \Phi_{A_k}(t_f, \sigma(\tau)) B_k(\tau) u(\tau) \Delta\tau + \sum_{i=1}^{k-1} [\prod_{j=i}^{k-1} (1 + c_j) \\
&\times \int_{t_{i-1}}^{t_i} \Phi_{A_k}(t_f, t_{k-1}) \prod_{r=k-1}^{i+1} \Phi_{A_r}(t_r, t_{r-1}) \Phi_{A_i}(t_i, \sigma(\tau)) B_i(\tau) u(\tau) \Delta\tau].
\end{aligned} \tag{3.15}$$

Multiplying by $\Phi_{A_1}(t_0, t_1)\Phi_{A_2}(t_1, t_2)...\Phi_{A_k}(t_{k-1}, t_f)$ in (3.15) we obtain

$$\begin{aligned}
\prod_{i=1}^{k-1} (1 + c_i) x_\alpha &= -\sum_{i=1}^{k-1} [\prod_{j=i}^{k-1} (1 + c_j) \Phi_{A_1}(t_0, t_1) \Phi_{A_2}(t_1, t_2) ... \Phi_{A_i}(t_{i-1}, t_i) \\
&\times \int_{t_{i-1}}^{t_i} \Phi_{A_i}(t_i, \sigma(\tau)) B_i(\tau) u(\tau) \Delta\tau] \\
&- \Phi_{A_1}(t_0, t_1) \Phi_{A_2}(t_1, t_2) ... \Phi_{A_k}(t_{k-1}, t_f) \int_{t_{k-1}}^{t_f} \Phi_{A_k}(t_f, \sigma(\tau)) B_k(\tau) u(\tau) \Delta\tau.
\end{aligned}$$

Now, using (3.14) and multiplying by x_α^T to the both side of the above equality, we obtain

$$\begin{aligned}
\prod_{i=1}^{k-1} (1 + c_i) x_\alpha^T x_\alpha &= -\sum_{i=1}^{k-1} \left[\prod_{j=i}^{k-1} (1 + c_j) \int_{t_{i-1}}^{t_i} x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i(\tau) u(\tau) \Delta\tau \right] \\
- \int_{t_{k-1}}^{t_f} x_\alpha^T \Psi_k(t_0, \sigma(\tau)) B_k(\tau) u(\tau) \Delta\tau &= 0.
\end{aligned}$$

Since $\prod_{j=1}^{k} (1 + c_j) \neq 0$, it follows that $x_\alpha x_\alpha^T = 0$. This contradicts $x_\alpha \neq 0$ and so we conclude that $\text{rank}(G_0 \ G_1 \ ... \ G_k) = n$. □

Example 3.1.3. Consider the following impulsive system on time scale

$$\begin{cases}
x^{\triangle}(t) = A_k(t) x(t) + B_k(t) u(t), \ t \in [t_{k-1}, t_k), \\
x(t_k^+) = \frac{1}{2} x(t_k), \ t = t_k \quad : \quad k = 1, 2, 3, \\
x(0) = x_0,
\end{cases} \tag{3.16}$$

where

$$A_1 = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, B_1 = \begin{pmatrix} e_3(\sigma(t),0) \\ 0 \end{pmatrix}, \tag{3.17}$$
$$A_2 = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ e_3(\sigma(t),\frac{1}{2}) \end{pmatrix},$$
$$A_3 = \begin{pmatrix} -3 & -2 \\ 3 & 4 \end{pmatrix}, B_3 = \begin{pmatrix} 0 \\ e_{-2}(\sigma(t),\frac{5}{2}) \end{pmatrix}.$$

Then the exponential matrices corresponding to A_1, A_2 and A_3 are given by

$$e_{A_1}(0,\sigma(t)) = \begin{pmatrix} -e_2(0,\sigma(t)) & 0 \\ e_3(0,\sigma(t)) & e_3(0,\sigma(t)) \end{pmatrix},$$
$$e_{A_2}(0,\sigma(t)) = \begin{pmatrix} e_1(0,\sigma(t)) & -e_1(0,\sigma(t)) \\ 0 & e_3(0,\sigma(t)) \end{pmatrix},$$
$$e_{A_3}(0,\sigma(t)) = \begin{pmatrix} \frac{3}{5}e_{-2}(0,\sigma(t)) & \frac{1}{5}e_{-2}(0,\sigma(t)) \\ -\frac{1}{5}e_3(0,\sigma(t)) & -\frac{2}{5}e_3(0,\sigma(t)) \end{pmatrix},$$

respectively. We have to compute the following matrices

$$G_i := G(0,t_{i-1},t_i) = \int_{t_{i-1}}^{t_i} \Psi_i(t_0,\sigma(\tau))B_i(\tau)B_i^T(\tau)\Psi_i^T(t_0,\sigma(\tau))\triangle\tau, \tag{3.18}$$

where

$$\Psi_1(0,\sigma(t)) = e_{A_1}(0,\sigma(t)) \quad t \in (0,t_1]$$

and

$$\Psi_i(0,\sigma(t)) = \prod_{j=1}^{i-1} e_{A_j}(t_{j-1},t_j)e_{A_i}(t_{i-1},\sigma(t)) \quad t \in (t_{i-1},t_i],\ i = 2,3.$$

If $\mathbb{T} = \mathbb{R}$ then $\sigma(t) = t$, $\mu(t) = 0$ and $e_w(t,\tau) = e^{w(t-\tau)}$. Next, if we choose $t_k = \frac{4k-3}{2}$, $k = 1,2,3$, then we have

$$\Psi_1(0,t)B_1(t)B_1^T(t)\Psi_1^T(0,t) = \begin{pmatrix} e^{2t} & -e^t \\ -e^t & 1 \end{pmatrix}, \tag{3.19}$$

$$\Psi_2(0,t)B_2(t)B_2^T(t)\Psi_2^T(0,t) = \begin{pmatrix} e^{4t-4} & e^{2t-7/2} - e^{4t-9/2} \\ e^{2t-7/2} - e^{4t-9/2} & e^{4t-5} - 2e^{2t-4} + e^{-3} \end{pmatrix} \tag{3.20}$$

and

$$\Psi_3(0,t)B_3(t)B_3^T(t)\Psi_3^T(0,t) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \tag{3.21}$$

where

$$\begin{aligned} a &= \frac{1}{25}(4e^{19-10t} + 4e^{13/2-5t} + e^{-6}), \\ b &= \frac{1}{25}(2e^{2-5t} - 4e^{6-5t} + 4e^{29/2-10t} - 4e^{37/2-10t} - e^{-13/2}), \\ c &= \frac{1}{25}(4e^{10-10t} - 8e^{14-10t} + 4e^{18-10t} - 4e^{3/2-5t} + 4e^{11/2-5t} + e^{-7}). \end{aligned}$$

Substituting (3.19), (3.20) and (3.21) in (3.18), we obtain

$$G_1 = \begin{pmatrix} \frac{1}{2}e - \frac{1}{2} & 1 - e^{1/2} \\ 1 - e^{1/2} & \frac{1}{2} \end{pmatrix},$$

$$G_2 = \begin{pmatrix} \frac{1}{4}e^{6} - \frac{1}{4}e^{-2} & \frac{1}{2}e^{3/2} - \frac{1}{4}e^{-5/2} - \frac{1}{4}e^{11/2} \\ \frac{1}{2}e^{3/2} - \frac{1}{4}e^{-5/2} - \frac{1}{4}e^{11/2} & \frac{11}{4}e^{-3} - e + \frac{1}{4}e^{5} \end{pmatrix}$$

and

$$G_3 = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

where

$$\begin{aligned} a &= \frac{2}{125}(8e^{-6} - 2e^{-16} - e^{-26}), \\ b &= \frac{2}{125}(2e^{-21/2} - 8e^{-13/2} + 2e^{-33/2} - e^{-41/2} + e^{-53/2} - e^{-61/2}), \\ c &= \frac{2}{125}(8e^{-7} - 4e^{-11} + e^{-15} - 2e^{-17} + 2e^{-21} - e^{-27} + 2e^{-31} - e^{-35}). \end{aligned}$$

Then we obtain

$$\begin{aligned} \det G_3 &\approx 1.371\,2 \times 10^{-12}, \\ \det G_2 &\approx 5.051\,8, \\ \det G_1 &\approx 8.7324 \times 10^{-3}. \end{aligned}$$

It follows that $\mathrm{rank}(G_i) = 2$, $i = 1, 2, 3$.

Further, if we choose $\mathbb{T} = \mathbb{P}_{1,1} = \bigcup_{k=0}^{\infty}[2k, 2k+1]$, then $e_w(t, t_0) = (1+w)^j e^{w(t-t_0)} e^{-wj}$ for $t_0 \in [2i, 2i+1)$, $t \in [2(i+j), 2(i+j)+1]$ with $j \geq 0$. In this case, $\mu(t) = 0$ if $t \in \bigcup_{k=0}^{\infty}[2k, 2k+1)$ and $\mu(t) = 1$ if $t \in \bigcup_{k=0}^{\infty}\{2k+1\}$.Then it follows that

$$\Psi_1(0,t)B_1(t)B_1^T(t)\Psi_1^T(0,t) = \begin{pmatrix} e^{2t} & -e^t \\ -e^t & 1 \end{pmatrix} t \in (0, \frac{1}{2}], \tag{3.22}$$

$$\Psi_2(0,t)B_2(t)B_2^T(t)\Psi_2^T(0,t)$$
$$= \begin{cases} \begin{pmatrix} e^{4t-4} & e^{2t-7/2} - e^{4t-9/2} \\ e^{2t-7/2} - e^{4t-9/2} & e^{4t-5} - 2e^{2t-4} + e^{-3} \end{pmatrix}, & t \in (\frac{1}{2}, 1] \\ \begin{pmatrix} 4e^{4t-8} & 2e^{2t-11/2} - 4e^{4t-17/2} \\ 2e^{2t-11/2} - 4e^{4t-17/2} & 4e^{4t-9} - 4e^{2t-6} + e^{-3} \end{pmatrix}, & t \in [2, \frac{5}{2}] \end{cases} \tag{3.23}$$

and

$$\Psi_3(0,t)B_3(t)B_3^T(t)\Psi_3^T(0,t) = \begin{cases} \begin{pmatrix} a & b \\ b & c \end{pmatrix}, & t \in (\frac{5}{2}, 3] \\ \begin{pmatrix} d & e \\ e & f \end{pmatrix}, & t \in [4, \frac{9}{2}], \end{cases} \tag{3.24}$$

where

$$\begin{aligned}
a &= \frac{1}{25}(e^{21-10t} - e^{17/2-5t} + \frac{1}{4}e^{-4}) \\
b &= \frac{1}{25}(\frac{1}{4}e^{6-5t} + e^{8-5t} - \frac{1}{2}e^{37/2-10t} - e^{41/2-10t} - \frac{1}{4}e^{-9/2}) \\
c &= \frac{1}{25}(\frac{1}{4}e^{16-10t} + e^{18-10t} + e^{20-10t} - \frac{1}{2}e^{11/2-5t} - e^{15/2-5t} + \frac{1}{4}e^{-5}) \\
d &= \frac{1}{100}(\frac{1}{9}e^{31-10t} - \frac{2}{3}e^{27/2-5t} + e^{-4}) \\
e &= \frac{1}{100}(\frac{1}{6}e^{11-5t} + \frac{2}{3}e^{13-5t} - \frac{1}{18}e^{57/2-10t} - \frac{1}{9}e^{61/2-10t} - e^{-9/2}) \\
f &= \frac{1}{100}(\frac{1}{36}e^{26-10t} + \frac{1}{9}e^{28-10t} + \frac{1}{9}e^{30-10t} - \frac{1}{3}e^{21/2-5t} - \frac{2}{3}e^{25/2-5t} + e^{-5}).
\end{aligned}$$

Substituting (3.22), (3.23) and (3.24) in (3.18) we obtain

$$G_1 = \begin{pmatrix} \frac{1}{2}e - \frac{1}{2} & 1 - e^{1/2} \\ 1 - e^{1/2} & \frac{1}{2} \end{pmatrix},$$

$$G_2 = \begin{pmatrix} e^2 - \frac{1}{4}e^{-2} - \frac{3}{4} & \frac{7}{4}e^{-1/2} - \frac{1}{2}e^{-3/2} - e^{3/2} - \frac{1}{4}e^{-5/2} \\ \frac{7}{4}e^{-1/2} - \frac{1}{2}e^{-3/2} - e^{3/2} - \frac{1}{4}e^{-5/2} & e - \frac{11}{4}e^{-1} + e^{-2} + \frac{7}{4}e^{-3} \end{pmatrix}$$

and

$$G_3 = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix},$$

where

$$\begin{aligned} \alpha &= -\frac{1}{9000}(-54e^{-4} + 23e^{-9} + e^{-14} - 60e^{-13/2}), \\ \beta &= -\frac{1}{18000}(120e^{-7} + 30e^{-9} + 108e^{-9/2} - 46e^{-19/2} - 29e^{-23/2} \\ &\quad -2e^{-29/2} - e^{-33/2}), \\ \gamma &= \frac{1}{36000}(216e^{-5} + 36e^{-9} - 92e^{-10} - 116e^{-12} - 35e^{-14} - 4e^{-15} - 4e^{-17} \\ &\quad -e^{-19} + 240e^{-15/2} + 120e^{-19/2}). \end{aligned}$$

Then

$$\begin{aligned} \det G_3 &\approx 1.4581 \times 10^{-11}, \\ \det G_2 &\approx 0.12274, \\ \det G_1 &\approx 8.7324 \times 10^{-3}. \end{aligned}$$

It follows that $\text{rank}(G_i) = 2$, $i = 1, 2, 3$. Therefore, by Theorem 3.1.2 the impulsive system (3.16) is controllable in the both cases.

If $A_k(t) = A(t)$, $B_k(t) = B(t)$ for each $k = `1, 2, ...$ and $t_0 = t_f$ then from (3.6) and (3.7), we obtain the following matrices:

$$H_{i-1}(t_{i-1}, t_i, t_f) := \int_{t_{i-1}}^{t_i} \Phi_A(t_f, \sigma(\tau))B(\tau)B^T(\tau)\Phi_A^T(t_f, \sigma(\tau))\Delta\tau,\ i = 1, 2, ..., k \tag{3.25}$$

and

$$H_k(t_k, t_f) := \int_{t_k}^{t_f} \Phi_A(t_f, \sigma(\tau))B(\tau)B^T(\tau)\Phi_A^T(t_f, \sigma(\tau))\Delta\tau. \tag{3.26}$$

Corollary 3.1.4. *(i) If there exists at least $l \in \{0, 1, ..., k\}$ such that*

$$rank(H_l(t_{l-1}, t_l, t_f)) = n,$$

then the impulsive system (3.4) is controllable on $[t_0, t_f]$ $(t_f \in (t_k, t_{k+1}])$.

(ii) Assume that $c_i \neq -1$, $i = 1, 2, ..., k$. If the impulsive system (3.4) is controllable on $[t_0, t_f]$ $(t_f \in (t_k, t_{k+1}])$, then

$$\text{rank}(H_0 \ H_1 \ \dots \ H_k) = n.$$

Example 3.1.5. Consider the following impulsive system on time scale

$$\begin{cases} x^{\triangle}(t) = A(t)x(t) + B(t)u(t), \ t \in [t_{k-1}, t_k), \\ x(t_k^+) = \frac{1}{2}x(t_k), \ t = t_k \quad : \quad k = 1, 2, 3, \\ x(0) = x_0, \end{cases} \tag{3.27}$$

where

$$A(t) = \begin{pmatrix} -3 & -2 \\ 3 & 4 \end{pmatrix} \text{ and } B(t) = \begin{pmatrix} 0 \\ e_{-2}(\sigma(t), 0) \end{pmatrix}.$$

Then the fundamental solution matrix of (3.27) is given by

$$e_A(t, 0) = \begin{pmatrix} 2e_{-2}(t, 0) & e_3(t, 0) \\ -e_{-2}(t, 0) & -3e_3(t, 0) \end{pmatrix}.$$

To compute $G(0, \sigma(t)) := e_A(0, \sigma(t))B(t)B^T(t)e_A^T(0, \sigma(t))$, we first observe that

$$e_A(0, \sigma(t)) = \begin{pmatrix} \frac{3}{5}e_{-2}(0, \sigma(t)) & \frac{1}{5}e_{-2}(0, \sigma(t)) \\ -\frac{1}{5}e_3(0, \sigma(t)) & -\frac{2}{5}e_3(0, \sigma(t)) \end{pmatrix}$$

and simple calculation give us

$$G(0, \sigma(t)) = \begin{pmatrix} \frac{1}{25} & -\frac{2}{25}e_{\frac{5}{1-2\mu(t)}}(0, \sigma(t)) \\ -\frac{2}{25}e_{\frac{5}{1-2\mu(t)}}(0, \sigma(t)) & \frac{4}{25}e^2_{\frac{5}{1-2\mu(t)}}(0, \sigma(t)) \end{pmatrix}.$$

If $\mathbb{T}=\mathbb{R}$ then $e_w(t,\tau)=e^{w(t-\tau)}$. Next, if we choose $t_k=\frac{4k-3}{2}$, $k=1,2,...$, then it follows

$$\begin{aligned} G_{k-1} &= \int_{t_{k-1}}^{t_k} G(0,\sigma(t))dt \\ &= \begin{pmatrix} \frac{2}{25} & -\frac{2}{125}e^{-5/2(4k-3)}(1-e^{10}) \\ -\frac{2}{125}e^{-5/2(4k-3)}(1-e^{10}) & \frac{2}{125}e^{15-20k}(e^{20}-1) \end{pmatrix}, \end{aligned}$$

for $k=2,3,...$ and

$$\begin{aligned} G_0 &= \int_0^{t_1} G(0,\sigma(t))dt \\ &= \begin{pmatrix} \frac{1}{50} & -\frac{2}{125}(1-e^{-5/2}) \\ -\frac{2}{125}(1-e^{-5/2}) & \frac{2}{125}(1-e^{-5}) \end{pmatrix}. \end{aligned}$$

Since

$$\det G_0 \approx 1.0215\times 10^{-4}$$

and

$$\det G_{k-1} \approx -(1.9265\times 10^5)e^{-5(4k-3)} \ \textit{for}\ k=2,3,...$$

it follows that $\operatorname{rank}(G_k)=2$ for all $k=0,1,....$

Further, if we choose $\mathbb{T}=\mathbb{P}_{1,1}=\bigcup_{k=0}^{\infty}[2k,2k+1]$ then it follows that

$$G(0,\sigma(t))=\begin{pmatrix} \frac{1}{25} & -\frac{2}{25}\left(\frac{1}{6}\right)^k e^{-5(t-k)} \\ -\frac{2}{25}\left(\frac{1}{6}\right)^k e^{-5(t-k)} & \frac{4}{25}\left(\frac{1}{6}\right)^{2k} e^{-10(t-k)} \end{pmatrix},$$

for $t\in[2k,2k+1]$, $k=0,1,2,....$If we choose $t_k=\frac{4k-3}{2}$, $k=1,2,...$, then it follows

$$\begin{aligned} G_0 &= \int_0^{t_1} G(0,\sigma(t))dt \\ &= \begin{pmatrix} \frac{1}{50} & -\frac{2}{125}(1-e^{-5/2}) \\ -\frac{2}{125}(1-e^{-5/2}) & \frac{2}{125}(1-e^{-5}) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} G_{k-1} &= \int_{t_{k-1}}^{t_k} G(0,\sigma(t))dt \\ &= \begin{pmatrix} \frac{1}{25} & \frac{-2\left(\frac{1}{6}\right)^k e^{-5k}}{125}(e^{10}-e^{15}-e^{\frac{15}{2}}+e^{\frac{35}{2}}) \\ \frac{-2\left(\frac{1}{6}\right)^k e^{-5k}}{125}(e^{10}-e^{15}-e^{\frac{15}{2}}+e^{\frac{35}{2}}) & \frac{2\left(\frac{1}{6}\right)^{2k} e^{-10k}}{125}(e^{15}-e^{20}+e^{30}-e^{45}) \end{pmatrix} \end{aligned}$$

for $k=2,3,....$ Since

$$\det G_0 \approx 1.0215\times 10^{-4}$$

and

$$\det G_{k-1} \approx -2.2358\times 10^{16}\times 6^{-2k}\times e^{-10k} \text{ for } k=2,3,...$$

it follows that $\operatorname{rank}(G_k)=2$ for all $k=0,1,....$ Therefore, by Corollary 3.1.4 the impulsive system (3.27) is controllable.

Remark 3.1.6. If $\mathbb{T}=\mathbb{R}$, then we obtain the result of Theorem 3.1.2 in [72] and Corollary 3.1.4 in [36]. The version of non impulsive case on time scales ($c_i=-1$) can be found in [32, Theorem 2.2] and [38].

Theorem 3.1.7. *Assume that $c_i\neq -1$, $i=1,2,...,k-1$ and $A_k(t)=A_k$, $B_k(t)=B_k$ are constant matrices. Then the impulsive system (3.1) is controllable on $[t_0,t_f](t_f\in(t_{k-1},t_k])$ if and only if*

$$rank(W_1\ W_2\ ...\ W_k)=n, \tag{3.28}$$

where $W_i=\Lambda_i(B_i\ A_iB_i\ ...\ A_i^{n-1}B_i)$ for $i=1,2,...,k-1$, $W_k=\Lambda_{k-1}e_{A_k}(t_{k-1},t_f)\ (B_k\ A_kB_k\ ...\ A_k^{n-1}B_k)$ and $\Lambda_i=e_{A_1}(t_0,t_1)e_{A_2}(t_1,t_2)...e_{A_i}(t_{i-1},t_i)$.

Proof. Suppose that the impulsive system (3.1) is controllable on $[t_0,t_f]$. If the rank condition (3.28) does not hold, then there exists nonzero $x_\alpha\in\mathbb{R}^n$ such that

$$x_\alpha^T\Lambda_iA_i^jB_i=0,$$

for $i=1,2,...,k,\ \ j=0,1,...,n-1$. Using (2.1), (3.9) and (3.10), we obtain that

$$\begin{aligned}
x_\alpha^TG(t_0,t_{i-1},t_i) &= \int_{t_{i-1}}^{t_i} x_\alpha^T\Psi_i(t_0,\sigma(\tau))B_iB_i^T\Psi_i(\tau)\Delta\tau\\
&= \int_{t_{i-1}}^{t_i} x_\alpha^T\Lambda_{i-1}e_{A_i}(t_{i-1},\sigma(\tau))B_iB_i^T\Psi_i(\tau)\Delta\tau\\
&= \int_{t_{i-1}}^{t_i} x_\alpha^T\Lambda_{i-1}e_{A_i}(t_{i-1},t_i)e_{A_i}(t_i,\sigma(\tau))B_iB_i^T\Psi_i(\tau)\Delta\tau\\
&= \int_{t_{i-1}}^{t_i} x_\alpha^T\Lambda_ie_{A_i}(t_i,\sigma(\tau))B_iB_i^T\Psi_i(\tau)\Delta\tau\\
&= \int_{t_{i-1}}^{t_i} \left[\sum_{j=0}^{n-1}\gamma_{ij}(t_i,\sigma(\tau))x_\alpha^T\Lambda_iA_i^jB_i\right]B_i^T\Psi_i(\tau)\Delta\tau=0,
\end{aligned}$$

for $i = 1, 2, ..., k-1$. Similarly, $x_\alpha^T G(t_0, t_{k-1}, t_f) = 0$. It follows that rank$(G_0\ G_1\ ...\ G_k) < n$. This contradicts the conclusion (ii) of Theorem 3.1.2 and therefore, we can conclude that the condition (3.28) is true.

Conversely, suppose that (3.28) holds. If the impulsive system (3.1) is not controllable on $[t_0, t_f]$ $(t_f \in (t_{k-1}, t_k])$, then it follows from conclusion (i) of Theorem 3.1.2 that the matrices $G(t_0, t_{i-1}, t_i)$ $(i = 1, 2, ..., k-1)$ and $G(t_0, t_{k-1}, t_f)$ are not invertible. Thus there exists nonzero $x_\alpha \in \mathbb{R}^n$ such that

$$0 = x_\alpha^T G(t_0, t_{i-1}, t_i) x_\alpha = \int_{t_{i-1}}^{t_i} x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i B_i^T \Psi_i^T(t_0, \sigma(\tau)) x_\alpha \Delta\tau,$$

for $i = 1, 2, ..., k-1$ and

$$0 = x_\alpha^T G(t_0, t_{k-1}, t_f) x_\alpha = \int_{t_{k-1}}^{t_f} x_\alpha^T \Psi_k(t_0, \sigma(\tau)) B_k B_k^T \Psi_k^T(t_0, \sigma(\tau)) x_\alpha \Delta\tau.$$

Exactly as in proof of Theorem 3.1.2, it follows that

$$0 = x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i = x_\alpha^T \Lambda_i e_{A_i}(t_i, \sigma(\tau)) B_i, \ \ \tau \in (t_{i-1}, t_i]$$

and

$$0 = x_\alpha^T \Psi_k(t_0, \sigma(\tau)) B_k = x_\alpha^T \Lambda_k e_{A_k}(t_f, \sigma(\tau)) B_k = 0, \ \ \tau \in (t_{k-1}, t_f].$$

By continuity of $e_{A_i}(t_i, \cdot)$ and density of $\sigma((t_{i-1}, t_i])$ in the interval $(\sigma(t_{i-1}), \sigma(t_i)] = (t_{i-1}, t_i]$ we obtain that

$$x_\alpha^T \Lambda_i e_{A_i}(t_i, \tau) B_i = 0 \ \text{ for all } \tau \in (t_{i-1}, t_i],\ i = 1, 2, ..., k-1. \tag{3.29}$$

Also, by continuity of $e_{A_k}(t_f, \cdot)$ and density of $\sigma((t_{k-1}, t_f])$ in the interval $(\sigma(t_{k-1}), \sigma(t_f)] = (t_{k-1}, t_f]$ we obtain that

$$x_\alpha^T \Lambda_k e_{A_k}(t_f, \tau) B_k = 0 \ \text{ for all } \tau \in (t_{k-1}, t_f]. \tag{3.30}$$

In particular, if we take $\tau = t_i$ in (3.29) and $\tau = t_f$ in (4.5), then it follows that $x_\alpha^T \Lambda_i B_i = 0$ for $i = 1, 2, ..., k$. Since $e_{A_i}(t_i, \cdot)$ is delta differentiable and $\Delta_\tau e_{A_i}(t_i, \tau) = -e_{A_i}(t_i, \sigma(\tau)) A_i$ (see [16, Theorem 5.23]), then subsequent derivatives and the density argument as above, gives

$$(-1)^j x_\alpha^T \Lambda_i e_{A_i}(t_i, \tau) A_i^j B_i = 0, \tau \in (t_{i-1}, t_i], \tag{3.31}$$

for $j = 0, 1, ..., n-1$ and $i = 1, 2, ..., k-1$. Similarly,

$$(-1)^j x_\alpha^T \Lambda_k e_{A_k}(t_f, \tau) A_k^j B_k = 0 \tau \in (t_{k-1}, t_f], \tag{3.32}$$

for $j = 0, 1, ..., n-1$. If we take $\tau = t_i$ in (3.31) and $\tau = t_f$ in (4.4), then it follows that $x_\alpha^T \Lambda_i A_i^j B_i = 0$ for $i = 1, 2, ..., k$ and $j = 0, 1, ..., n-1$. Therefore,

$$x_\alpha^T \Lambda_i (B_i \ A_i B_i \ ... \ A_i^{n-1} B_i) = 0,$$

which implies that the rank condition (3.28) fails. This contradiction proves that the impulsive system (3.1) is controllable on $[t_0, t_f]$ $(t_f \in (t_{k-1}, t_k])$. □

If $A(t) = A$, $B(t) = B$ are constant matrices then (3.25) and (3.26) yields that

$$H_{i-1}(t_{i-1}, t_i, t_f) := \int_{t_{i-1}}^{t_i} e_A(t_f, \sigma(\tau)) B B^T e_A^T(t_f, \sigma(\tau)) \Delta\tau, \ i = 1, 2, ..., k$$

and

$$H_k(t_k, t_f) := \int_{t_k}^{t_f} e_A(t_f, \sigma(\tau)) B B^T e_A^T(t_f, \sigma(\tau)) \Delta\tau.$$

Corollary 3.1.8. *Assume that $c_i \neq -1$, $i = 1, 2, ..., k$ and $A(t) = A$, $B(t) = B$ are constant matrices. Then the impulsive system (3.4) is controllable on $[t_0, t_f]$ $(t_f \in (t_k, t_{k+1}])$ if and only if*

$$rank(B \ AB \ ... \ A^{n-1} B) = n.$$

Remark 3.1.9. If $\mathbb{T} = \mathbb{R}$, then we obtain the result of Theorem 3.1.7 in [72] and the Corollary 3.1.8 in [36]. The version for non impulsive case $(c_i = -1)$ of the above theorem can be found in [8, Corollary 3], [32, Theorem 2.7] and [38].

3.2 OBSERVABILITY

Definition 3.2.1. *The impulsive system (3.1) (respectively (3.4)) is called state observable on $[t_0, t_f]$ $(t_f > t_0)$ if any initial state $x(t_0) = x_0 \in \mathbb{R}^n$ is uniquely determined by the corresponding system input $u(t)$ and system output $y(t)$ for $t \in [t_0, t_f]$.*

our first result on observability is as follows

Theorem 3.2.2. *Assume that $1 + c_i \geq 0$, $i = 1, 2, ..., k-1$. Then the impulsive system (3.1) is observable on $[t_0, t_f](t_f \in (t_{k-1}, t_k])$ if and only if the matrix*

$$M(t_0, t_f) := M(t_0, t_0, t_1) + \sum_{i=2}^{k-1} \prod_{j=1}^{i-1} (1 + c_j) M(t_0, t_{i-1}, t_i) + \prod_{j=1}^{k-1} (1 + c_j) M(t_0, t_{k-1}, t_f)$$

is invertible, where

$$\begin{aligned} M(t_0, t_{i-1}, t_i) &= \int_{t_{i-1}}^{t_i} \Omega_i^T(\tau, t_0) C_i^T(\tau) C_i(\tau) \Omega_i(\tau, t_0) \Delta\tau, \; i = 1, 2, ..., k-1, \\ M(t_0, t_{k-1}, t_f) &= \int_{t_{k-1}}^{t_f} \Omega_k^T(\tau, t_0) C_k^T(\tau) C_k(\tau) \Omega_k(\tau, t_0) \Delta\tau \end{aligned}$$

and

$$\Omega_i(\tau, t_0) = \Phi_{A_i}(\tau, t_{i-1}) \Phi_{A_{i-1}}(t_{i-1}, t_{i-2}) ... \Phi_{A_1}(t_1, t_0),$$

for $\tau \in (t_{i-1}, t_i]$ and $i = 1, 2, ..., k$.

Proof. Suppose that $M(t_0, t_f)$ is invertible. From (3.5) and (3.1) we obtain

$$y(t) = C_1(t) \Phi_{A_1}(t, t_0) x_0 + C_1(t) \int_{t_0}^{t} \Phi_{A_1}(t, \sigma(\tau)) B_1(\tau) u(\tau) \Delta\tau + D_1(t) u(t), \tag{3.33}$$

for $t \in [t_0, t_1]$ and

$$\begin{aligned} y(t) &= C_l(t) x(t) + D_l(t) u(t) \qquad (3.34) \\ &= C_l(t) \Phi_{A_l}(t, t_{l-1}) \prod_{i=1}^{l-1} (1 + c_i) \prod_{i=l-1}^{1} \Phi_{A_i}(t_i, t_{i-1}) x_0 \\ &\quad + C_l(t) \sum_{i=1}^{l-1} \left[\prod_{j=i}^{l-1} (1 + c_j) \int_{t_{i-1}}^{t_i} \Phi_{A_l}(t, \sigma(\tau)) \prod_{r=l-1}^{i+1} \Phi_{A_r}(t_r, t_{r-1}) \Phi_{A_i}(t_i, \sigma(\tau)) B_i(\tau) u(\tau) \Delta\tau \right] \\ &\quad + C_l(t) \int_{t_{l-1}}^{t} \Phi_{A_l}(t, \sigma(\tau)) B_l(\tau) u(\tau) \Delta\tau + D_l(t) u(t), \end{aligned}$$

for $t \in (t_{l-1}, t_l]$ and $l = 2, 3, ...k$. It is easy to see from the Definition 3.2.1 that the observability of system (*3.1*) is equivalent to the observability of $y(t)$ given by

$$y(t) = \begin{cases} C_1(t)\Phi_{A_1}(t, t_0)x_0, \ t \in [t_0, t_1] \\ \prod_{i=1}^{l-1}(1 + c_i)C_l(t)\Omega_l(t, t_0)x_0, \ t \in (t_{l-1}, t_l], l = 1, 2, ..., k, \end{cases} \tag{3.35}$$

as $u(t) = 0$. Now, multiplying $\Omega_l^T(t, t_0)C_l^T(t)$ to both sides of (3.35) and integrating with respect to t from t_0 to t_f, we have

$$\begin{aligned} \int_{t_0}^{t_f} \Omega_l^T(\tau, t_0)C_l^T(\tau)y(\tau)\Delta\tau \ &= \ \Bigg[\int_{t_0}^{t1} \Phi_{A_1}^T(\tau, t_0)C_1^T(\tau)C_1(\tau)\Phi_{A_1}(\tau, t_0)\Delta\tau \\ &+ \sum_{i=2}^{k-1}\prod_{j=1}^{i-1}(1 + c_j)\int_{t_{i-1}}^{t_i} \Omega_i^T(\tau, t_0)C_i^T(\tau)C_l(\tau) \ \Omega_i(\tau, t_0)\Delta\tau \\ &+ \prod_{j=1}^{k-1}(1 + c_j)\int_{t_{k-1}}^{t_f} \Omega_k^T(\tau, t_0)C_k^T(\tau)C_k(\tau)\Omega_k(\tau, t_0)\Delta\tau\Bigg] x_0 \end{aligned}$$

and so that

$$\int_{t_0}^{t_f} \Omega_l^T(\tau, t_0)C_l^T(\tau)y(\tau)\Delta\tau = [M(t_0, t_0, t_1) + \sum_{i=2}^{k-1}\prod_{j=1}^{i-1}(1 + c_j)M(t_0, t_{i-1}, t_i) \tag{3.36}$$
$$+ \prod_{j=1}^{k-1}(1 + c_j)M(t_0, t_{k-1}, t_f)]x_0.$$

Obviously, the left-hand side of (3.36) depend on $y(t), \ t \in [t_0, t_f]$. Since the matrix $M(t_0, t_f)$ is invertible, so from linear algebraic equation (3.36) we deduce that $x(t_0) = x_0$ is uniquely determined by the corresponding system output $y(t)$ for $t \in [t_0, t_f]$.

Conversely, if we suppose that the matrix $M(t_0, t_f)$ is not invertible, then there exist nonzero $x_\alpha \in \mathbb{R}^n$ such that $x_\alpha^T M(t_0, t_f)x_\alpha = 0$. Since $1 + c_i \geq 0$, $i = 1, 2....., k$, $M(t_0, t_{i-1}, t_i)$, $i = 1, 2, ...k - 1$, and $M(t_0, t_{k-1}, t_f)$ are positive semidefinite matrices, we have

$$\begin{aligned} x_\alpha^T M(t_0, t_{i-1}, t_i)x_\alpha \ &= \ 0, i = 0, 1, ..., k - 1, \\ x_\alpha^T M(t_0, t_{k-1}, t_f)x_\alpha \ &= \ 0. \end{aligned} \tag{3.37}$$

Choose $x_0 = x_\alpha$. Then, from (3.35) and (3.37), it follows that

$$\begin{aligned}\int_{t_0}^{t_f} y^T(\tau)y(\tau)\Delta\tau &= \sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} y^T(\tau)y(\tau)\Delta\tau + \int_{t_{k-1}}^{t_f} y^T(\tau)y(\tau)\Delta\tau \\ &= \int_{t_0}^{t_1} x_\alpha^T \Phi_{A_1}^T(\tau,t_0)C_1^T(\tau)C_1(\tau)\Phi_{A_1}(\tau,t_0)x_\alpha\Delta\tau \\ &\quad + \sum_{i=2}^{k-1}\left[\prod_{j=1}^{i-1}(1+c_j)\right]^2 \int_{t_{i-1}}^{t_i} x_\alpha^T\Omega_i^T(\tau,t_0)C_i^T(\tau)C_i(\tau)\Omega_i(\tau,t_0)x_\alpha\Delta\tau \\ &\quad + \left[\prod_{j=1}^{k-1}(1+c_j)\right]^2 \int_{t_{k-1}}^{t_f} x_\alpha^T\Omega_k^T(\tau,t_0)C_k^T(\tau)C_k(\tau)\Omega_k(\tau,t_0)x_\alpha\Delta\tau.\end{aligned}$$

Further, we have

$$\begin{aligned}\int_{t_0}^{t_f} y^T(\tau)y(\tau)\Delta\tau &= x_\alpha^T M(t_0,t_0,t_1)x_\alpha + \sum_{i=1}^{k-1}\left[\prod_{j=1}^{i}(1+c_j)\right]^2 x_\alpha^T M(t_0,t_{i-1},t_i)x_\alpha \\ &\quad + \left[\prod_{j=1}^{k-1}(1+c_j)\right]^2 x_\alpha^T M(t_0,t_{k-1},t_f)x_\alpha \\ &= 0\end{aligned}$$

and so that

$$\int_{t_0}^{t_f} \|y(\tau)\|^2 \Delta\tau = 0.$$

It follows that

$$0 = y(t) = \begin{cases} C_1(t)\Phi_{A_1}(t,t_0)x_0, t \in [t_0,t_1], \\ \prod_{j=1}^{l-1}(1+c_j)C_l(t)\Omega_l(t,t_0)x_\alpha,\ t\in(t_{l-1},t_l], l=1,2,...k-1, \\ \prod_{j=1}^{k-1}(1+c_j)C_k(t)\Omega_k(t,t_0)x_\alpha,\ t\in(t_{k-1},t_f]. \end{cases}$$

The last equality implies, by Definition 3.2.1, that the impulsive system is not observable on $[t_0, t_f]$ $(t_f \in (t_{k-1}, t_k])$. □

Example 3.2.3. Consider the following impulsive system on a time scale

$$\begin{cases} x^{\triangle}(t) = A_k(t)x(t) + B_k(t)u(t), \quad t \in [t_{k-1}, t_k), \\ x(t_k^+) = \frac{1}{2}x(t_k),\ k = 1,2,3, \\ y(t) = C_k(t)x(t) + D_k(t)u(t), \\ x(0) = x_0, \end{cases} \tag{3.38}$$

where

$$\begin{aligned} A_1 &= \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix},\ C_1 = \begin{pmatrix} 0 & e_{-3}(0,t) \end{pmatrix}, \\ A_2 &= \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix},\ C_2 = \begin{pmatrix} 0 & e_3(\frac{1}{2},t) \end{pmatrix}, \\ A_3 &= \begin{pmatrix} -3 & -2 \\ 3 & 4 \end{pmatrix},\ C_3 = \begin{pmatrix} 0 & e_3(\frac{5}{2},t) \end{pmatrix}. \end{aligned} \tag{3.39}$$

Then the exponential matrices corresponding to A_1, A_2 and A_3 are given by

$$\begin{aligned} e_{A_1}(t,t_0) &= \begin{pmatrix} -e_2(t,0) & 0 \\ e_2(t,0) & e_3(t,0) \end{pmatrix} \\ e_{A_2}(t,t_0) &= \begin{pmatrix} e_1(t,0) & e_3(t,0) \\ 0 & e_3(t,0) \end{pmatrix} \\ e_{A_3}(t,t_0) &= \begin{pmatrix} 2e_{-2}(t,0) & e_3(t,0) \\ -e_{-2}(t,0) & -3e_3(t,0) \end{pmatrix} \end{aligned}$$

respectively. We have to compute the following matrix

$$M(0,\frac{9}{2}) := M(0,0,\frac{1}{2}) + \frac{1}{2}M(0,\frac{1}{2},\frac{5}{2}) + \frac{1}{4}M(0,\frac{5}{2},\frac{9}{2}),$$

where

$$\begin{aligned} M(0,0,\frac{1}{2}) &= \int_0^{1/2} \Omega_1^T(\tau,0)C_1^T(\tau)C_1(\tau)\ \Omega_1(\tau,0)\Delta\tau, \\ M(0,\frac{1}{2},\frac{5}{2}) &= \int_{1/2}^{5/2} \Omega_2^T(\tau,0)C_2^T(\tau)C_2(\tau)\ \Omega_2(\tau,0)\Delta\tau, \\ M(0,\frac{5}{2},\frac{9}{2}) &= \int_{5/2}^{9/2} \Omega_3^T(\tau,0)C_3^T(\tau)C_3(\tau)\ \Omega_3(\tau,0)\Delta\tau \end{aligned} \tag{3.40}$$

and

$$\Omega_i(s,0) = \Phi_{A_i}(s,t_{i-1})\Phi_{A_{i-1}}(t_{i-1},t_{i-2})...\Phi_{A_1}(t_1,0),\ \ s\in(t_{i-1},t_i],\quad i=1,2,3.$$

If $\mathbb{T}=\mathbb{R}$ then

$$M(0,0,\tfrac{1}{2}) = \begin{pmatrix} -\frac{1}{10}\left(-e^5+1\right) & -\frac{1}{11}\left(-e^{11/2}+1\right) \\ -\frac{1}{11}\left(-e^{11/2}+1\right) & -\frac{1}{12}\left(-e^6+1\right) \end{pmatrix},$$

$$M(0,\tfrac{1}{2},\tfrac{5}{2}) = \begin{pmatrix} 2e^2 & 2e^{5/2} \\ 2e^{5/2} & 2e^3 \end{pmatrix}$$

and

$$M(0,\frac{5}{2},\frac{9}{2}) = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix},$$

where

$$\begin{aligned} a_1 &= -\frac{1}{10}\left(12e^4+e^{-6}-e^6-2e^{-10}+14e^{10}+e^{-14}-193e^{14}-12\right), \\ a_2 &= -\frac{1}{10}\left(-6e^{1/2}+12e^{9/2}+e^{-11/2}-e^{-19/2}+7e^{21/2}-193e^{29/2}\right), \\ a_3 &= \frac{1}{10}e^{-5}\left(-12e^{10}+193e^{20}-1\right). \end{aligned}$$

We obtain

$$\det M(0,\frac{9}{2}) \approx -1.7799\times 10^9.$$

Further, if $\mathbb{T}=\mathbb{P}_{1,1}=\bigcup\limits_{k=0}^{\infty}[2k,2k+1]$, then

$$\begin{aligned} M(0,0,\frac{1}{2}) &= \begin{pmatrix} -\frac{1}{10}\left(-e^5+1\right) & -\frac{1}{11}\left(-e^{11/2}+1\right) \\ -\frac{1}{11}\left(-e^{11/2}+1\right) & -\frac{1}{12}\left(-e^6+1\right) \end{pmatrix}, \\ M(0,\frac{1}{2},\frac{5}{2}) &= \begin{pmatrix} e^2 & e^{5/2} \\ e^{5/2} & e^3 \end{pmatrix} \end{aligned}$$

and

$$M(0,\frac{5}{2},\frac{9}{2}) = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix},$$

where

$$
\begin{aligned}
a_1 &= \frac{33}{10}e^3 - \frac{9}{10}e - \frac{1}{10}e^{-2} - \frac{3}{8}e^{-1} + \frac{1}{10}e^{-4} + \frac{2}{5}e^4 - \frac{1}{40}e^{-6} - \frac{56}{5}e^6 \\
&\quad + \frac{824}{5}e^8 + 12e^{7/2} - 24e^{11/2}, \\
a_2 &= 6e^4 - 24e^6 - \frac{1}{10}e^{-\frac{3}{2}} - \frac{9}{20}e^{\frac{3}{2}} + \frac{1}{20}e^{-7/2} + \frac{33}{10}e^{7/2} - \\
&\quad - \frac{28}{5}e^{13/2} + \frac{824}{5}e^{17/2}, \\
a_3 &= \frac{33}{10}e^4 - \frac{1}{10}e^{-1} + \frac{824}{5}e^9 - 24e^{13/2}.
\end{aligned}
$$

We obtain

$$\det M(0, \frac{9}{2}) \approx -9.4 \times 10^5.$$

Therefore, by Theorem 3.2.2 the system (3.38) is observable in the both cases.

Corollary 3.2.4. *Assume that $1 + c_i \geq 0, i = 1, 2, ..., k$. Then the impulsive system (3.4) is observable on $[t_0, t_f](t_f \in (t_k, t_{k+1}])$ if and only if the matrix*

$$N(t_0, t_f) := N(t_0, t_0, t_1) + \sum_{i=1}^{k-1} \prod_{j=1}^{i} (1 + c_j) N(t_0, t_i, t_{i+1}) + \prod_{j=1}^{k} (1 + c_j) N(t_0, t_k, t_f)$$

is invertible, where

$$N(t_0, t_i, t_{i+1}) = \int_{t_i}^{t_{i+1}} \Phi_A^T(\tau, t_0) C^T(\tau) C(\tau) \Phi_A(\tau, t_0) \Delta\tau \;\; i = 0, 1, 2, ..., k-1$$

and

$$N(t_0, t_k, t_f) = \int_{t_k}^{t_f} \Phi_A^T(\tau, t_0) C^T(\tau) C(\tau) \Phi_A(\tau, t_0) \Delta\tau.$$

Example 3.2.5. Consider the following impulsive system on time scales

$$
\begin{cases}
x^{\triangle}(t) = A(t)x(t) + B(t)u(t), \; t \in \mathbb{I}_0 \backslash \{t_k\}, \\
x(t_k^+) = \frac{1}{2}x(t_k), \; t = t_k, \; k = 1, 2, \\
y(t) = C(t)x(t) + D(t)u(t), \\
x(0) = x_0,
\end{cases}
\tag{3.41}
$$

where

$$A(t) = \begin{pmatrix} -3 & -2 \\ 3 & 4 \end{pmatrix}, \; C(t) = \begin{pmatrix} 0 & e_3(\frac{5}{2}, t) \end{pmatrix}, \; \mathbb{I}_0 = \mathbb{T}_0 \cap [0.3] \text{ and } t_f = \frac{11}{4}.$$

Then the fundamental solution matrix of (3.41) is given by

$$e_A(t,0) = \begin{pmatrix} 2e_{-2}(t,0) & e_3(t,0) \\ -e_{-2}(t,0) & -3e_3(t,0) \end{pmatrix}.$$

Then we have

$$\begin{aligned} T(t,0) \quad : & = e_A^T(t,0)C^T(t)C(t)e_A(t,0) \\ & = \begin{pmatrix} \frac{e_{-2}^2(t,0)}{e_3^2(t,0)} & 3\frac{e_{-2}(t,0)}{e_3(t,0)} \\ 3\frac{e_{-2}(t,0)}{e_3(t,0)} & 9 \end{pmatrix}. \end{aligned}$$

If $\mathbb{T}_0 = [0,\infty)$, then $\mathbb{I}_0 = [0,3]$ and $e_w(t,0) = e^{wt}$. Next, if we choose $t_1 = \frac{1}{2}$ and $t_2 = \frac{5}{2}$, then it follows that

$$\begin{aligned} N(0,0,\frac{1}{2}) &= \int_0^{1/2} T(t,0)dt = \begin{pmatrix} \frac{1}{10}\left(1-e^{-5}\right) & \frac{3}{5}\left(1-e^{-5/2}\right) \\ \frac{3}{5}\left(1-e^{-5/2}\right) & \frac{9}{2} \end{pmatrix}, \\ N(0,\frac{1}{2},\frac{5}{2}) &= \int_{1/2}^{5/2} T(t,0)dt = \begin{pmatrix} \frac{1}{10}e^{-5}\left(1-e^{-20}\right) & \frac{3}{5}e^{-5/2}\left(1-e^{-10}\right) \\ \frac{3}{5}e^{-5/2}\left(1-e^{-10}\right) & 18 \end{pmatrix} \end{aligned}$$

and

$$N(0,\frac{5}{2},\frac{11}{2}) = \int_{5/2}^{11/4} T(t,0)dt = \begin{pmatrix} \frac{1}{10}e^{-25}\left(1-e^{-5/2}\right) & \frac{3}{5}e^{-25/2}\left(1-e^{-5/4}\right) \\ \frac{3}{5}e^{-25/2}\left(1-e^{-5/4}\right) & \frac{9}{4} \end{pmatrix}.$$

Then we obtain

$$\det N(0,11/2) = \det\left[N(0,0,\frac{1}{2}) + \frac{1}{2}N(0,\frac{1}{2},\frac{5}{2}) + \frac{1}{4}N(0,\frac{5}{2},\frac{11}{2})\right] \approx 1.070\,5.$$

It follows that $\operatorname{rank}(G_k) = 2$ for all $k = 1,2,....$

Further, if we choose $\mathbb{T} = \mathbb{P}_{1,1} = \bigcup_{k=0}^{\infty}[2k, 2k+1]$, then $\mathbb{I}_0 = [0,1]\cup[2,3]$ and $e_w(t,0) = (1+w)^k e^{w(t-k)}$ for $t \in [2k, 2k+1]$. If we choose $t_1 = \frac{1}{2}$ and $t_2 = \frac{5}{2}$, then

$$\begin{aligned} N(0,0,\frac{1}{2}) &= \int_0^{1/2} T(t,0)dt = \begin{pmatrix} \frac{1}{10}\left(1-e^{-5}\right) & \frac{3}{5}\left(1-e^{-5/2}\right) \\ \frac{3}{5}\left(1-e^{-5/2}\right) & \frac{9}{2} \end{pmatrix}, \\ N(0,\frac{1}{2},\frac{5}{2}) &= \int_{1/2}^{5/2} T(t,0)dt = \int_{1/2}^{1} T(t,0)dt + \int_2^{5/2} T(t,0)dt \\ &= \begin{pmatrix} \frac{1}{160}\left(16e^{-5} - 15e^{-10} - e^{-15}\right) & \frac{3}{20}\left(-3e^{-5} + 4e^{-\frac{5}{2}} - e^{-\frac{15}{2}}\right) \\ \frac{3}{20}\left(-3e^{-5} + 4e^{-\frac{5}{2}} - e^{-\frac{15}{2}}\right) & 3^2 \end{pmatrix} \end{aligned}$$

and

$$N(0,\frac{5}{2},\frac{11}{2})=\int_{5/2}^{11/4}T(t,0)dt=\begin{pmatrix} \frac{1}{160}e^{-15}\left(1-e^{-5/2}\right) & -\frac{3}{20}e^{-5/2}\left(1-e^{-5/4}\right) \\ -\frac{3}{20}e^{-5/2}\left(1-e^{-5/4}\right) & \frac{9}{4} \end{pmatrix}.$$

Then we obtain

$$\det N(0,11/2)=\det\left[N(0,0,\frac{1}{2})+\frac{1}{2}N(0,\frac{1}{2},\frac{5}{2})+\frac{1}{4}N(0,\frac{5}{2},\frac{11}{2})\right]\approx 0.626\,26.$$

Therefore, by Corollary 3.2.4 the impulsive system (3.41) is observable.

Remark 3.2.6. If $\mathbb{T}=\mathbb{R}$, then we obtain the result of Theorem 3.2.2 in [72] and and Corollary 3.2.4 in [36]. The version of non impulsive case on time scales ($c_i=-1$) can be found in [32, Theorem 3.2] and [38].

In the following, we consider the sufficient and necessary criterion for time-invariant case. For impulsive system (*3.1*), we denote

$$S=\begin{pmatrix} V_1 \\ . \\ . \\ . \\ V_k \end{pmatrix} \text{ and } V_i=\begin{pmatrix} C_i \\ C_iA_i \\ . \\ . \\ . \\ C_iA_i^{n-1} \end{pmatrix}\Upsilon_i, \tag{3.42}$$

where $\Upsilon_i=e_{A_i}(t_i,t_{i-1})...e_{A_2}(t_2,t_1)e_{A_1}(t_1,t_0)$, for $i=1,2,...k$.

Theorem 3.2.7. *Assume that $1+c_i\geq 0$, $i=1,2,...,k$ and $A_k(t)=A_k$, $C_k(t)=C_k$ are constant matrices. Then impulsive system (3.1) is observable on $[t_0,t_f]$ ($t_f\in(t_{k-1},t_k]$) if and only if rank$(S)=n$.*

Proof. Suppose $rank(S) = n$ and we have to show that system (*3.1*) is observable on $[t_0, t_f](t_f \in (t_{k-1}, t_k])$. If otherwise, namely, system(*3.1*) is not observable, then by Theorem 3.2.2, it follows that the matrix $M(t_0, t_f)$ is not invertible. Hence there exists a nonzero vector x_α such that $x_\alpha^T M(t_0, t_f)x_\alpha = 0$. Similar to the proof of Theorem 3.2.2, we obtain

$$\begin{aligned} x_\alpha^T M(t_0, t_{i-1}, t_i)x_\alpha &= \int_{t_{i-1}}^{t_i} [x_\alpha^T \Omega_i^T(\tau, t_0)C_i^T][C_i\Omega_i(\tau, t_0)x_\alpha]\Delta\tau \\ &= \int_{t_{i-1}}^{t_i} [C_i\Omega_i(\tau, t_0)x_\alpha]^T[C_i\Omega_i(\tau, t_0)x_\alpha]\Delta\tau = 0, \ i = 1, 2, ..., k-1 \end{aligned}$$

and

$$x_\alpha^T M(t_0, t_{k-1}, t_f)x_\alpha = \int_{t_{k-1}}^{t_f} [C_i\Omega_k(\tau, t_0)x_\alpha]^T[C_i\Omega_k(\tau, t_0)x_\alpha]\Delta\tau = 0.$$

Since $\Omega_i(\tau, t_0) = e_{A_i}(\tau, t_{i-1})...e_{A_2}(t_2, t_1)e_{A_1}(t_1, t_0)$ for $i = 1, 2, ..., k$, it follows that

$$C_i e_{A_i}(\tau, t_{i-1})...e_{A_2}(t_2, t_1)e_{A_1}(t_1, t_0)x_\alpha = 0, \tag{3.43}$$

for $\tau \in (t_{i-1}, t_i]$, $i = 1, 2, ..., k-1$ and

$$C_k e_{A_k}(\tau, t_{k-1})...e_{A_2}(t_2, t_1)e_{A_1}(t_1, t_0)x_\alpha = 0, \tag{3.44}$$

for $\tau \in (t_{k-1}, t_f]$. Obviously, at $\tau = t_{i-1}$, we have $C_i\Upsilon_{i-1}x_\alpha = 0$, $i = 1, 2, ..., k$ and differentiating in (3.43) and (3.44) j times and evaluating the result at $\tau = t_{i-1}$, $i = 1, 2, ..., k$, we obtain

$$C_i A_i^j \Upsilon_{i-1}x_\alpha = 0, \ i = 1, 2, ..., k \text{ and } j = 0, 1, 2, ...n-1. \tag{3.45}$$

Therefore, by (3.42) and (3.45) it follows that $Sx_\alpha = 0$ and moreover, $x_\alpha \neq 0$ implies that $rank(S) < n$ which leads to a contradiction with the assumption that $rank(S) = n$. The proof of the sufficiency part is finished.

Conversely, we suppose that $rank(S) < n$. Then there exist $x_\alpha \neq 0$ such that $Sx_\alpha = 0$, which leads to (3.45). By (2.1) and (3.45) we have

$$M(t_0, t_{i-1}, t_i)x_\alpha = \int_{t_{i-1}}^{t_i} \sum_{j=0}^{n-1} \gamma_{ij}(\tau, t_{i-1})[C_i\Omega_i(\tau, t_0)]^T[C_i A_i^j \Upsilon_{i-1}]x_\alpha\Delta\tau = 0,$$

for $i = 1, 2, ..., k-1$ and

$$M(t_0, t_{k-1}, t_f)x_\alpha = \int_{t_{k-1}}^{t_f} \sum_{j=0}^{n-1} \gamma_{ij}(\tau, t_0)[C_k \Omega_k(\tau, t_0)]^T [C_k A_k^j \Upsilon_{k-1}] x_\alpha \Delta\tau = 0$$

and so that by (3.45), we obtain $M(t_0, t_f)x_\alpha = 0$. Since $x_\alpha \neq 0$ the matrix $M(t_0, t_f)x_\alpha$ is not invertible. Hence system (*3.1*) is not observable from Theorem 3.2.2 and it contradicts with the assumption of observability. The proof is completed. □

In the following, we consider the sufficient and necessary criterion for time-invariant case of impulsive system (*3.4*). Let us denote

$$R = \begin{pmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ . \\ CA^{n-1} \end{pmatrix}. \tag{3.46}$$

Corollary 3.2.8. *Assume that* $1 + c_i \geq 0$, $i = 1, 2, ..., k$ *and* $A(t) = A, C(t) = C$ *are constant matrices. Then impulsive system (3.4) is observable on* $[t_0, t_f]$ $(t_f \in (t_k, t_{k+1}])$ *if and only if* $rank(R) = n$.

Remark 3.2.9. If $\mathbb{T} = \mathbb{R}$, then we obtain the result of Theorem 4 in [72] and the Corollary 3.2.8. The version of non impulsive case on time scales ($c_i = -1$) can be found in [8, Theorem 4], [32, Theorem 3.5] and [38].

3.3 APPLICATIONS

Example 3.3.1. Consider the following application to population growth model with impulse

$$\begin{cases} N^{\triangle}(t) = r_k N(t) + c_k U(t), & t \neq t_k, \\ N(t_k^+) = (r_{k+1} - r_k) N(t_k), & t = t_k, \\ N(0) = N_0, \end{cases}$$

where $N(t)$ is the number of population at the time t, r_k is the rate of population growth between two consecutive impulsive points and $U(t)$ is a control input. Such model can be describe the evaluation of cicada magicicada septendecim. In this case it is need to consider the time scale $\mathbb{T} = \mathbb{P}_{1,1}$ (see [16, Example 1.39]) Using the Theorem 2 it is easy to see that the system is controllable.

Example 3.3.2. Next application is an impulsive model in Nonelectronic [69, Example 11.1.1], that is

$$\begin{cases} \theta^{\triangle}(t) = -\frac{\gamma}{\pi}\theta(t) + \gamma(a - b\cos t), & t \neq t_k, \\ \theta(t_k^+) = -3\pi, & t = t_k, \\ \theta(0) = \theta_0, \\ |\theta(0)| < \pi. \end{cases}$$

Using the Theorem 2, with $A = -\frac{\gamma}{\pi}$, $B = \gamma$ and $n = 1$, it is easy to see that the system is controllable if $\gamma \neq 0$ and $\gamma \neq \pi$. The controllability of this system is independent of the choice of the time scale $\mathbb{T}$.

3.4 CONCLUSION

In this chapter, the issue on the controllability and observability criteria for linear impulsive time-varying systems on time scales has been addressed. Several sufficient and necessary criteria for state controllability and observability of such systems have been established, respectively, by the variation of parameters for time-varying impulsive systems on time scales. In addition, several examples and applications have been presented to show the effectiveness of proposed results. As it has been shown that a larger class of systems are considered, the results generalize some known results in [8, 32, 36, 38, 51, 72].

4.0 VOLTERRA INTEGRODYNAMIC SYSTEMS

In a very recent paper [2], Adivar introduces the principal matrix solutions and variation of parameters for Volterra integro-dynamic equations. Motivated by the interesting nature of this problem, an attempt has been made to study some stability and boundedness properties of the following system

$$\begin{cases} x^{\Delta}(t) = A(t)x(t) + \displaystyle\int_{t_0}^{t} K(t,s)x(s)\Delta s + F(t), \;\; t \in \mathbb{T}_0 = [t_0, \infty) \\ x(t_0) = x_0, \end{cases} \tag{4.1}$$

where $0 \leq t_0 \in \mathbb{T}^k$ is fixed, A (not necessarily stable) is an $n \times n$ matrix function, F is an n-vector function, which are both continuous on $\mathbb{T}_0$ and K is an $n \times n$ matrix function, which is continuous on $\Omega := \{(t,s) \in \mathbb{T}_0 \times \mathbb{T}_0 : t_0 \leq s \leq t < \infty\}$.

Then, from (4.1), we obtain the following discrete variant for $\mathbb{T} = \mathbb{T}^r_{(q,h)}$

$$\begin{cases} \Delta_{(q,h)} x(t) = A(t)x(t) + \displaystyle\sum_{s \in [t_0, t)} K(t,s)x(s)\mu(s) + F(t), \\ x(t_0) = x_0, \end{cases} \tag{4.2}$$

where A is an $n \times n$ matrix function , F is an n-vector function on $\mathbb{T}^r_{(q,h)}$ and K is an $n \times n$ matrix function on $\Omega_{(q,h)} := \{(t,s) \in \mathbb{T}^r_{(q,h)} \times \mathbb{T}^r_{(q,h)} : t_0 \leq s \leq t < \infty\}$.

4.1 RESOLVENT

Lemma 4.1.1. *If $A(t)$ and $K(t,s)$ are the continuous functions given in equation (4.1), then*

$$\Delta_s R(t,s) = -R(t,\sigma(s))A(s) - \int_{\sigma(s)}^{t} R(t,\sigma(u))K(u,s)\Delta u \tag{4.3}$$

is equivalent to

$$R(t,s) = I + \int_{s}^{t} R(t,\sigma(u))W(u,s)\Delta u, \tag{4.4}$$

where

$$W(t,s) = A(t) + \int_{s}^{t} K(t,u)\Delta u. \tag{4.5}$$

Proof. Substituting (4.5) in (4.4) and using Theorem 2.0.16, we obtain

$$\begin{aligned} R(t,s) &= I + \int_s^t R(t,\sigma(u))\left[A(u) + \int_s^u K(u,v)\Delta v\right]\Delta u \\ &= I + \int_s^t R(t,\sigma(u))A(u)\Delta u + \int_s^t R(t,\sigma(u))\int_s^u K(u,v)\Delta v\Delta u \\ &= I + \int_s^t R(t,\sigma(u))A(u)\Delta u + \int_s^t \int_{\sigma(v)}^t R(t,\sigma(u))K(u,v)\Delta u\Delta v. \end{aligned}$$

Differentiating with respect to s, we obtain (4.3).

Conversely, we have to show that (4.3) implies (4.4). So, integrating (4.3) from s to t, we obtain

$$R(t,t) - R(t,s) = -\int_s^t R(t,\sigma(u))A(u)\Delta u - \int_s^t \int_{\sigma(v)}^t R(t,\sigma(u))K(u,v)\Delta u\Delta v,$$

which implies that

$$R(t,s) = I + \int_s^t R(t,\sigma(u))A(u)\Delta u + \int_s^t \int_{\sigma(v)}^t R(t,\sigma(u))K(u,v)\Delta u\Delta v.$$

Furthermore, using Theorem 2.0.16, we have

$$\begin{aligned} R(t,s) &= I + \int_s^t R(t,\sigma(u))A(u)\Delta u + \int_s^t R(t,\sigma(u))\int_s^u K(u,v)\Delta v\Delta u \\ &= I + \int_s^t R(t,\sigma(u))\left[A(u) + \int_s^u K(u,v)\Delta v\right]\Delta u \\ &= I + \int_s^t R(t,\sigma(u))W(u,s)\Delta u. \end{aligned}$$

Hence, (4.3) and (4.4) are equivalent systems, and the proof is completed. □

Theorem 4.1.2. *Assume A and K are continuous functions given in (4.1). Then the function $R(t,s)$, as defined in (4.4), exists on $t_0 \leq s \leq t$ and is continuous in (t,s). $\Delta_s R(t,s)$ exists, is continuous and satisfies the equation (4.3) on $t_0 \leq s \leq t$, for each $t > t_0$. Moreover, given any vector x_0 and any continuous function $F(t)$, equation (4.1) is equivalent to the system*

$$x(t) = R(t,t_0)x_0 + \int_{t_0}^{t} R(t,\sigma(s))F(s)\Delta s. \tag{4.6}$$

Proof. Since $W(t,s)$ is continuous in s for each fixed t, the existence of $R(t,s)$ on $t_0 \leq s \leq t$ is trivial (see [39, Theorem 1]). From the above calculations, it follows that for each fixed t, $\Delta_s R(t,s)$ exists and satisfies (4.3) by Lemma 4.1.1. Since K is continuous on $t_0 \leq s \leq t < \infty$, we have

$$|W(t,s)| \leq |A(t)| + \int_{t_0}^{t} |K(t,u)|\,\Delta u = w(t),$$

and w is continuous. Application of the Gronwall inequality (see [16, Theorem 6.4]) in (4.4), yields the estimate

$$\begin{aligned} |R(t,\sigma(s))| &= \left| I + \int_{\sigma(s)}^{t} R(t,\sigma(u))W(u,\sigma(s))\Delta u \right| \\ &\leq 1 + \int_{t_0}^{t} |R(t,\sigma(u))|\, w(u)\Delta u \\ &\leq 1 + \int_{t_0}^{t} e_w(t,\sigma(u))w(u)\Delta u \\ &= w_0(t), \end{aligned} \tag{4.7}$$

which implies that $R(t,\sigma(s))$ is continuous. Using this fact in (4.3) it is apparent that $\Delta_s R(t,s)$ is continuous, and that

$$\begin{aligned} |\Delta_s R(t,s)| &= \left| -R(t,\sigma(s))A(s) - \int_{\sigma(s)}^{t} R(t,\sigma(u))K(u,s)\Delta u \right| \\ &\leq w_0(t)\,|A(s)| + \int_{\sigma(s)}^{t} w_0(t)\,|K(u,s)|\,\Delta u \\ &\leq w_0(t)\left(A(s) + \int_{\sigma(s)}^{T} K(u,s)\Delta u \right) \end{aligned} \tag{4.8}$$

if $t_0 \leq s \leq t \leq T$. Using (4.7) and dominated convergence, it follows the continuity of $R(t,s)$ in t for a fixed s. From (4.8), it is clear that $R(t,s)$ is uniformly continuous for $t_0 \leq s \leq t \leq T$. Hence, by [34, Theorem 5, p. 102], $R(t,s)$ is continuous in the pair (t,s).

Now let $x(t)$ be a solution of (4.1) on an interval $t_0 \le t \le T$. If we take $p(s) = R(t,s)x(s)$, then we have

$$p^{\Delta}(s) = \Delta_s R(t,s)x(s) + R(t,\sigma(s))x^{\Delta}(s),$$

and by (4.1), it follows

$$\begin{aligned} p^{\Delta}(s) &= \Delta_s R(t,s)x(s) + R(t,\sigma(s))A(s)x(s) \\ &\quad + R(t,\sigma(s)) \int_{t_0}^{s} K(s,\tau)x(\tau)\Delta\tau + R(t,\sigma(s))F(s). \end{aligned}$$

Integrating from t_0 to t we have

$$\begin{aligned} p(t) - p(t_0) &= \int_{t_0}^{t} \Delta_s R(t,s)x(s)\Delta s + \int_{t_0}^{t} R(t,\sigma(s))A(s)x(s)\Delta s \\ &\quad + \int_{t_0}^{t} R(t,\sigma(s)) \int_{t_0}^{s} K(s,\tau)x(\tau)\Delta\tau\Delta s \\ &\quad + \int_{t_0}^{t} R(t,\sigma(s))F(s)\Delta s. \end{aligned}$$

Using Theorem 2.0.16, we obtain

$$\begin{aligned} & x(t) - R(t,t_0)x_0 \\ = & \int_{t_0}^{t} \Delta_s R(t,s)x(s)\Delta s + \int_{t_0}^{t} R(t,\sigma(s))A(s)x(s)\Delta s \\ & + \int_{t_0}^{t} \left(\int_{\sigma(s)}^{t} R(t,\sigma(\tau))K(\tau,s)\Delta\tau \right) x(s)\Delta s + \int_{t_0}^{t} R(t,\sigma(s))F(s)\Delta s \\ = & \int_{t_0}^{t} \left[\Delta_s R(t,s) + R(t,\sigma(s))A(s) + \int_{\sigma(s)}^{t} R(t,\sigma(\tau))K(\tau,s)\Delta\tau \right] x(s)\Delta s \\ & + \int_{t_0}^{t} R(t,\sigma(s))F(s)\Delta s. \end{aligned}$$

Furthermore, by using (4.3) we obtain (4.6). Moreover, if $x(t)$ solves (4.6) on an interval $t_0 \le t \le \tau$, then it is easy to see that $x(t)$ solves (4.1), which completes the proof. □

Consider the adjoint dynamical equation [16, Theorem 5.27],

$$y^{\Delta}(t) = -A^T(t)y^{\sigma}(t) - f(t) \tag{4.9}$$

where A^T is the transpose of A. Let us extend this definition to the integro-dynamic equation (4.1).

Definition 4.1.3. *For a fixed t the adjoint to (4.1) is*

$$\begin{cases} y^{\Delta}(s) = -A^T(s)y^{\sigma}(s) - \int_{\sigma(s)}^{t} K^T(u,s)y^{\sigma}(u)\Delta u - f(s) \\ y(t) = y_0 \end{cases} \tag{4.10}$$

where $s \in [t_0, t]$.

It is easy to see by Theorem 2.0.16 that (4.10) is equivalent to an integral equation

$$y(s) = y_0 + \int_s^t \left[A^T(u) + \int_s^u K(u,v)\Delta v\right] y^{\sigma}(u)\Delta u + \int_s^t f(u)\Delta u. \tag{4.11}$$

For the next result, we define, for a fixed t, the space of continuous function

$$C_{y_0}[t_0, t] := \{\varphi \in C[t_0, t] : \varphi(t) = y_0\}$$

and the metric

$$d_{\beta}^1(\varphi, \psi) := \sup\{|\varphi(s) - \psi(s)|\, e_{\beta}(s, t_0) : t_0 \le s \le t\}.$$

The metric space $(C_{y_0}[t_0, t], d_{\beta}^1)$ is complete by replacing β with $\ominus\beta$ in

$$d_{\beta}(\varphi, \psi) := \sup\left\{\frac{|\varphi(s) - \psi(s)|}{e_{\beta}(s, t_0)} : t_0 \le s \le t\right\},$$

where $\ominus\beta = -\beta/(1 + \mu(t)\beta)$ (see [42, Lemma 3.1]).

Theorem 4.1.4. *For a fixed $t \in \mathbb{T}_0$ such that $t > t_0$ and a given $y_0 \in \mathbb{R}^n$, there is a unique solution $y(s)$ of (4.11) on the interval $[t_0, t]$ satisfying the condition $y(t) = y_0$.*

Proof. We define the mapping

$$(P\varphi)(s) := y_0 + \int_s^t \left[A^T(u) + \int_s^u K(u,v)\Delta v\right] \varphi^{\sigma}(u)\Delta u + \int_s^t f(u)\Delta u$$

for all $\varphi \in C_{y_0}[t_0, t]$. For a given $\varphi \in C_{y_0}[t_0, t]$, it follows that $P\varphi$ is continuous on $[t_0, t]$ and that $(P\varphi)(t) = y_0$. Thus, $P : C_{y_0}[t_0, t] \to C_{y_0}[t_0, t]$. For an arbitrary pair of functions $\varphi, \psi \in C_{y_0}[t_0, t]$,

$$\begin{aligned} & |(P\phi)(s) - (P\psi)(s)| \\ = & \left|\int_s^t \left[A^T(u) + \int_s^u K^T(u,v)\Delta v\right] (\varphi^{\sigma}(u) - \psi^{\sigma}(u))\Delta u\right| \\ \le & \int_s^t \left[\left|A^T(u)\right| + \int_s^u \left|K^T(u,v)\right| \Delta v\right] \left|(\varphi^{\sigma}(u) - \psi^{\sigma}(u))\right| \Delta u. \end{aligned}$$

Since $A(u)$ and $K(u,v)$ are continuous for $t_0 \leq s \leq u \leq t$, then there is $\beta > 1$ such that

$$\left|A^T(u)\right| + \int_s^u \left|K^T(u,v)\right| \Delta v \leq \beta - 1.$$

Then we obtain the following estimation

$$|(P\phi)(s) - (P\psi)(s)| \leq \int_s^t (\beta - 1) \left|(\varphi^\sigma(u) - \psi^\sigma(u))\right| \Delta u. \tag{4.12}$$

Now, we have to show that P is a contraction on $C_{y_0}[t_0, t]$. Multiplying (4.12) by $e_\beta(s, t_0)$ we obtain

$$\begin{aligned}
& |(P\phi)(s) - (P\psi)(s)| \, e_\beta(s, t_0) \\
\leq \; & \int_s^t (\beta - 1) e_\beta(s, t_0) \left|(\varphi^\sigma(u) - \psi^\sigma(u))\right| \Delta u \\
\leq \; & \int_s^t (\beta - 1) e_\beta(s, \sigma(u)) \left|(\varphi^\sigma(u) - \psi^\sigma(u))\right| e_\beta(\sigma(u), t_0) \Delta u \\
\leq \; & d^1_\beta(\varphi, \psi) \int_s^t (\beta - 1) e_\beta(s, \sigma(u)) \Delta u \\
= \; & d^1_\beta(\varphi, \psi) \frac{(\beta - 1)}{-\beta} \int_s^t [e_\beta(s, u)]^\Delta \Delta u \\
= \; & d^1_\beta(\varphi, \psi) \frac{(\beta - 1)}{-\beta} [e_\beta(s, t) - 1)] \\
\leq \; & d^1_\beta(\varphi, \psi) \frac{(\beta - 1)}{\beta}.
\end{aligned}$$

Taking supremum over s, we have

$$d^1_\beta(P\phi, P\psi) \leq \frac{(\beta - 1)}{\beta} d^1_\beta(\varphi, \psi).$$

Therefore, by Banach fixed point theorem, P has a unique fixed point in $C_{y_0}[t_0, t]$. It follows that, (4.11) has a unique solution on the interval $[t_0, t]$. □

Definition 4.1.5. *The principal matrix solution of*

$$y^{\Delta}(s) = -A^T(s)y^{\sigma}(s) - \int_{\sigma(s)}^{t} K^T(u,s)y^{\sigma}(u)\Delta u \tag{4.13}$$

is the $n \times n$ matrix function

$$Z_1(t,s) := [y^1(t,s) \;\; y^2(t,s) \;\; \dots \;\; y^n(t,s)], \tag{4.14}$$

where $y^i(t,s)$ (t fixed) is the unique solution of (4.13) on $[t_0,t]$ that satisfies the condition $y^i(t,t) = e^i$.

By virtue of this definition, $Z_1(t,s)$ is the unique matrix solution of

$$\Delta_s Z_1(t,s) = -A^T(s)Z_1(t,\sigma(s)) - \int_{\sigma(s)}^{t} K^T(u,s)Z_1(t,\sigma(u))\Delta u, \tag{4.15}$$

such that $Z_1(t,t) = I$, on the interval $[t_0,t]$. Reasoning as in the proof of [2, Theorem 12], we conclude that for a given $y_0 \in \mathbb{R}^n$, the unique solution of (4.13) satisfying the condition $y(t) = y_0$ is

$$y(s) = Z_1(t,s)y_0 \tag{4.16}$$

for $t_0 \le s \le t$.

Taking the transpose of (4.13) we obtain

$$(y^T)^{\Delta}(s) = -(y^T)^{\sigma}(s)A(s) - \int_{\sigma(s)}^{t} (y^T)^{\sigma}(u)K(u,s)\Delta u. \tag{4.17}$$

The solution satisfying the condition $y^T(t) = y_0^T$ is the transpose of (4.16), namely,

$$y^T(s) = y_0^T Z_1^T(t,s), \tag{4.18}$$

where

$$R(t,s) := Z_1^T(t,s).$$

Consequently, $R(t,s)$ is the principal matrix solution of the transposed equation. As a result, Lemma 18 from [2], has the following adjoint variant.

Theorem 4.1.6. *The solution of (4.17) on* $[t_0, t]$ *satisfying the condition* $y^T(t) = y_0^T$ *is*

$$y^T(s) = y_0^T R(t,s) \tag{4.19}$$

where $R(t,s)$ *is the principal matrix solution of (4.17).*

It follows from (4.15) that $R(t,s)$ is the unique matrix solution of (4.3).

The principal matrix $Z(t,s)$ ([2, Theorem 12]) and the solution of the adjoint equation (4.17) are related via the expression

$$\Delta_u[y^T(u)Z(u,s)] = (y^T)^\Delta(u)Z(u,s) + (y^T)^\sigma(u)\Delta_u Z(u,s) \tag{4.20}$$

for $t_0 \le s \le u \le t$.

Theorem 4.1.7. $R(t,s) \equiv Z(t,s)$.

Proof. Select any $t > t_0$. For a given row n-vector, let $y^T(s)$ be the unique solution of (4.17) on $[t_0, t]$ such that $y^T(t) = y_0^T$. Integrating (4.20) from s to t we have

$$y^T(t)Z(t,s) - y^T(s)Z(s,s) = \int_s^t \left[(y^T)^\Delta(u)Z(u,s) + (y^T)^\sigma(u)\Delta_u Z(u,s)\right]\Delta u.$$

By the use of (4.17), we obtain

$$\begin{aligned} y^T(t)Z(t,s) - y^T(s) &= \int_s^t \Big[(y^T)^\sigma(u)\Delta_u Z(u,s) - (y^T)^\sigma(u)A(u)Z(u,s) \\ &\quad - \left(\int_{\sigma(u)}^t (y^T)^\sigma(v)K(v,u)\Delta v\right) Z(u,s)\Big]\Delta u. \end{aligned}$$

Interchanging the order of integration by using Theorem 2.0.16, we obtain

$$\begin{aligned} y_0^T Z(t,s) - y^T(s) &= \int_s^t (y^T)^\sigma(u)\Big[\Delta_u Z(u,s) - A(u)Z(u,s) \\ &\quad - \int_s^u K(u,v)Z(v,s)\Delta v\Big]\Delta u. \end{aligned}$$

By [2, Theorem 19], the integrand is zero. Hence,

$$y^T(s) = y_0^T Z(t,s).$$

On the other hand,

$$y^T(s) = y_0^T R(t,s).$$

Therefore, by uniqueness of the solution $y^T(s)$,

$$y_0^T Z(t,s) = y_0^T R(t,s). \tag{4.21}$$

Now let y_0^T be the transpose of the i-th basis vector e^i. Then (4.21) implies that the i-th column of $R(t,s)$ and $Z(t,s)$ are equal for $t_0 \leq s \leq t$. The conclusion follows as t is arbitrary. □

The continuous version ($\mathbb{T} = \mathbb{R}$) of the Theorem 4.1.6 can be found in [10, Theorem 2.7].

Now we are generalizing the idea of resolvent to discuss the asymptotic stability of (4.1) in the next section.

Some result of this section are simultaneously obtain by the Adivar et al.in [3].

4.2 ASYMPTOTIC STABILITY

Our first result in this section, is to present an equivalent system which involves an arbitrary function. A proper choice of this function has the potential to supply us a stable matrix B corresponding to A.

Theorem 4.2.1. *Let $L(t,s)$ be an $n \times n$ continuously differentiable matrix function on Ω. Then (4.1) is equivalent to the following system*

$$\begin{cases} y^\Delta(t) = B(t)y(t) + \int_{t_0}^t G(t,s)y(s)\Delta s + H(t), \ \ t \in \mathbb{T}_0, \\ y(t_0) = y_0, \end{cases} \tag{4.22}$$

where

$$\begin{aligned} B(t) &= A(t) - L(t,t), \\ H(t) &= F(t) + L(t,t_0)x_0 + \int_{t_0}^t L(t,\sigma(s))F(s)\Delta s \end{aligned} \tag{4.23}$$

and

$$G(t,s) = K(t,s) + \Delta_s L(t,s) + L(t,\sigma(s))A(s) + \int_{\sigma(s)}^{t} L(t,\sigma(\tau))K(\tau,s)\Delta\tau. \tag{4.24}$$

Proof. Let $x(t)$ be any solution of (4.1) on $\mathbb{T}_0$. If we take $p(s) = L(t,s)x(s)$, then we have

$$p^{\Delta}(s) = \Delta_s L(t,s)x(s) + L(t,\sigma(s))x^{\Delta}(s)$$

and by (4.1) it follows

$$\begin{aligned} p^{\Delta}(s) &= \Delta_s L(t,s)x(s) + L(t,\sigma(s))A(s)x(s) \\ &\quad + L(t,\sigma(s))\int_{t_0}^{s} K(s,\tau)x(\tau)\Delta\tau + L(t,\sigma(s))F(s). \end{aligned}$$

Integrating from t_0 to t we have

$$\begin{aligned} p(t) - p(t_0) &= \int_{t_0}^{t} \Delta_s L(t,s)x(s)\Delta s + \int_{t_0}^{t} L(t,\sigma(s))A(s)x(s)\Delta s \\ &\quad + \int_{t_0}^{t} L(t,\sigma(s))\int_{t_0}^{s} K(s,\tau)x(\tau)\Delta\tau\Delta s \\ &\quad + \int_{t_0}^{t} L(t,\sigma(s))F(s)\Delta s. \end{aligned}$$

Using Theorem 2.0.16, we obtain

$$\begin{aligned} p(t) - p(t_0) &= \int_{t_0}^{t} \Delta_s L(t,s)x(s)\Delta s + \int_{t_0}^{t} L(t,\sigma(s))A(s)x(s)\Delta s \\ &\quad + \int_{t_0}^{t} \left(\int_{\sigma(\tau)}^{t} L(t,\sigma(s))K(s,\tau)\Delta s\right) x(\tau)\Delta\tau \\ &\quad + \int_{t_0}^{t} L(t,\sigma(s))F(s)\Delta s. \end{aligned}$$

By change of variable, it follows

$$\begin{aligned} p(t) - p(t_0) &= \int_{t_0}^{t} \Bigg[\Delta_s L(t,s) + L(t,\sigma(s))A(s) \\ &\quad + \int_{\sigma(s)}^{t} L(t,\sigma(u))K(u,s)\Delta u\Bigg] x(s)\Delta s + \int_{t_0}^{t} L(t,\sigma(s))F(s)\Delta s. \end{aligned}$$

Further on, using (4.23) and (4.24), we obtain

$$(A(t)-B(t))x(t)=\int_{t_0}^{t}(G(t,s)-K(t,s))x(s)\Delta s+H(t)-F(t).$$

From (4.1) we have

$$x^{\Delta}(t)=B(t)x(t)+\int_{t_0}^{t}G(t,s)x(s)\Delta s+H(t)$$

for $t_0\le s\le t<\infty$. Hence, $x(t)$ is a solution of (4.22).

Conversely, let $y(t)$ be any solution of (4.22) on $\mathbb{T}_0$. We shall show that it satisfies (4.1). Consider

$$Z(t)=y^{\Delta}(t)-F(t)-A(t)y(t)-\int_{t_0}^{t}K(t,s)y(s)\Delta s.$$

Then by (4.22) and (4.23) we have

$$\begin{aligned}Z(t) &= -L(t,t)y(t)+L(t,t_0)x_0+\int_{t_0}^{t}G(t,s)y(s)\Delta s\\ &\quad+\int_{t_0}^{t}L(t,\sigma(s))F(s)\Delta s-\int_{t_0}^{t}K(t,s)y(s)\Delta s.\end{aligned}$$

Using (4.24), we obtain

$$\begin{aligned}Z(t) &= -L(t,t)y(t)+L(t,t_0)x_0+\int_{t_0}^{t}L(t,\sigma(s))F(s)\Delta s\\ &\quad-\int_{t_0}^{t}K(t,s)y(s)\Delta s+\int_{t_0}^{t}\Bigg[K(t,s)+\Delta_s L(t,s)+L(t,\sigma(s))A(s)\\ &\quad+\int_{\sigma(s)}^{t}L(t,\sigma(\tau))K(\tau,s)\Delta\tau\Bigg]y(s)\Delta s.\end{aligned}$$

Again by Theorem 2.0.16 and change of variable, it follows

$$\begin{aligned}Z(t) &= -L(t,t)y(t)+\int_{t_0}^{t}\left[\Delta_s L(t,s)+L(t,\sigma(s))A(s)\right]y(s)\Delta s\\ &\quad+\int_{t_0}^{t}L(t,\sigma(s))\left[\int_{t_0}^{s}K(s,\tau)y(\tau)\Delta\tau\right]\Delta s\\ &\quad+L(t,t_0)x_0+\int_{t_0}^{t}L(t,\sigma(s))F(s)\Delta s.\end{aligned}\tag{4.25}$$

Now, by setting $q(s)=L(t,s)y(s)$, we get

$$q^{\Delta}(s)=\Delta_s L(t,s)y(s)+L(t,\sigma(s))y^{\Delta}(s).\tag{4.26}$$

Integrating (4.26) from t_0 to t, it becomes

$$q(t) - q(t_0) = \int_{t_0}^{t} \left[\Delta_s L(t,s) y(s) + L(t,\sigma(s)) y^{\Delta}(s)\right] \Delta s$$

and therefore, we have

$$L(t,t)y(t) - L(t,t_0)x_0 = \int_{t_0}^{t} \left[\Delta_s L(t,s) y(s) + L(t,\sigma(s)) y^{\Delta}(s)\right] \Delta s. \tag{4.27}$$

Substituting (4.27) in (4.25) we obtain

$$\begin{aligned} Z(t) &= -\int_{t_0}^{t} L(t,\sigma(s)) y^{\Delta}(s) \Delta s + \int_{t_0}^{t} L(t,\sigma(s)) A(s) y(s) \Delta s \\ &\quad + \int_{t_0}^{t} L(t,\sigma(s)) \left[\int_{t_0}^{s} K(s,\tau) y(\tau) \Delta\tau\right] \Delta s + \int_{t_0}^{t} L(t,\sigma(s)) F(s) \Delta s \\ &= -\int_{t_0}^{t} L(t,\sigma(s)) Z(s) \Delta s, \end{aligned}$$

which implies $Z(t) \equiv 0$, by the uniqueness of the solution of Volterra integral equations [42]. Hence $y(t)$ is a solution of (4.1). □

As a straightforward consequence of Theorem 4.2.1 we obtain Lemma 2.1 of [61]. Also, it is to be noted that, if $L(t,s)$ is the differentiable resolvent corresponding to the kernel $K(t,s)$, then the equations (4.22), (4.23) and (4.24) give the usual variation of constants formula (4.6).

Corollary 4.2.2. *Let $L(t,s)$ be a $n \times n$ matrix function on $\Omega_{(q,h)}$. Then (4.2) is equivalent to the following system*

$$\begin{cases} \Delta_{(q,h)} y(t) = B(t) y(t) + \sum_{s \in [t_0, t)} G(t,s) y(s) \mu(s) + H(t), \\ y(t_0) = y_0, \end{cases} \tag{4.28}$$

where

$$\begin{aligned} B(t) &= A(t) - L(t,t), \\ H(t) &= F(t) + L(t,t_0)x_0 + \sum_{s \in [t_0,t)} L(t,\sigma(s)) F(s) \mu(s) \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} G(t,s) &= K(t,s) + L(t,\sigma(s))A(n) + \frac{L(t,\sigma(s)) - L(t,s)}{\mu(s)} \\ &\quad + \sum_{s \in [\sigma(s),t)} L(t,\sigma(\tau)) K(\tau,s) \sigma(\tau). \end{aligned}$$

Our next result is about the estimation of the solution of (4.1). For this result we assume that matrix B commutes with its integral, so B commutes with its matrix exponential, that is, $B(t)e_B(t,s) = e_B(t,s)B(t)$, [**?**, 31].

Theorem 4.2.3. *Let $B \in C(\mathbb{T}, M_n(\mathbb{R}))$ and M, $\alpha > 0$. Assume that matrix B commutes with its integral. If*

$$\|e_B(t,s)\| \le M e_\alpha(s,t), t, s \in \Omega, \tag{4.30}$$

then every solution $x(t)$ of (4.1) satisfies

$$\begin{aligned} \|x(t)\| &\le M \|x_0\| e_\alpha(t_0,t) + M \int_{t_0}^{t} e_\alpha(\sigma(s),t) \|H(s)\| \Delta s \\ &+ M \int_{t_0}^{t} \left[\int_{\sigma(s)}^{t} e_\alpha(\sigma(\tau),t) \|G(\tau,s)\| \Delta\tau \right] \|x(s)\| \Delta s. \end{aligned} \tag{4.31}$$

Proof. Let $x(t)$ be the solution of (4.22) and define $q(t) = e_B(t_0,t)x(t)$. Then

$$q^\Delta(t) = -B(t)e_B(t_0,\sigma(t))x(t) + e_B(t_0,\sigma(t))x^\Delta(t).$$

Substituting for $x^\Delta(t)$ from (4.22) and integrating from t_0 to t, we obtain

$$q(t) - q(t_0) = \int_{t_0}^{t} e_B(t_0,\sigma(s))H(s)\Delta s + \int_{t_0}^{t} e_B(t_0,\sigma(s)) \left[\int_{t_0}^{s} G(s,\tau)x(\tau)\Delta\tau \right] \Delta s.$$

Using Theorem 2.0.16, we obtain

$$\begin{aligned} x(t) &= e_B(t,t_0)x_0 + \int_{t_0}^{t} e_B(t,\sigma(s))H(s)\Delta s \\ &+ \int_{t_0}^{t} \left[\int_{\sigma(s)}^{t} e_B(t,\sigma(\tau))G(\tau,s)\Delta\tau \right] x(s)\Delta s. \end{aligned} \tag{4.32}$$

Hence, using (4.30) and applying norm on (4.32), we obtain (4.31), which completes the proof. □

The continuous version ($\mathbb{T} = \mathbb{R}$) of the Theorem 4.2.3 can be found in [61, Lemma 2.3].

Corollary 4.2.4. *Let* $B : \mathbb{T}^r_{(q,h)} \to M_n(\mathbb{R})$ *and* M, $\alpha > 0$. *If*

$$\left\| \prod_{r\in[s,t)} (I + \mu(r)B) \right\| \le \prod_{r\in[s,t)} \frac{M}{(1+\mu(r)\alpha)} \tag{4.33}$$

then every solution x *of* (4.2) *satisfies*

$$\begin{aligned} \|x(t)\| &\le \prod_{r\in[t_0,t)} \frac{M\|x_0\|}{(1+\mu(r)\alpha)} + M \sum_{s\in[t_0,t)} \prod_{r\in[\sigma(s),t)} \frac{\|H(t)\|\mu(s)}{(1+\mu(r)\alpha)} \\ &+M \sum_{s\in[t_0,t)} \left[\sum_{\tau\in[\sigma(s),t)} \prod_{r\in[\sigma(\tau),t)} \frac{\|G(\tau,s)\|\mu(\tau)}{(1+\mu(r)\alpha)} \right] \|x(s)\|\mu(s). \end{aligned} \tag{4.34}$$

In the next theorem we present sufficient conditions for asymptotic stability.

Theorem 4.2.5. *Let* $L(t,s)$ *be an* $n \times n$ *continuously differentiable matrix function on* Ω, *such that*

(a) *the assumptions of Theorem* 4.2.3 *holds,*
(b) $\|L(t,s)\| \le \dfrac{L_0 e_\gamma(s,t)}{(1+\mu(t)\alpha)(1+\mu(t)\gamma)}$,
(c) $\sup\limits_{t_0\le s\le t<\infty} \displaystyle\int_{\sigma(s)}^{t} e_\alpha(\sigma(\tau),t)\|G(\tau,s)\|\Delta\tau \le \alpha_0$,
(d) $F(t) \equiv 0$,

where L_0, $\gamma > \alpha$, α_0, *are positive real constants.*

If $\alpha \ominus M\alpha_0 > 0$, *then every solution* $x(t)$ *of* (4.1) *tends to zero exponentially as* $t \to +\infty$.

Proof. In view of Theorem 4.2.1 and the fact that $L(t,s)$ satisfies (a), it is enough to show that every solution of (4.22) tends to zero as $t \to +\infty$. From (a) and using (4.31) we obtain the following inequality

$$\begin{aligned} e_\alpha(t,0)\|x(t)\| &\le M\|x_0\| e_\alpha(t_0,0) + M\int_{t_0}^{t} e_\alpha(\sigma(s),0)\|H(s)\|\Delta s \\ &+M\int_{t_0}^{t}\left[\int_{\sigma(s)}^{t} e_\alpha(\sigma(\tau),0)\|G(\tau,s)\|\Delta\tau\right]\|x(s)\|\Delta s. \end{aligned} \tag{4.35}$$

Since

$$\int_{t_0}^{t} e_\alpha(\sigma(s),0)\|H(s)\|\Delta s \le L_0\|x_0\| e_\gamma(t_0,0)\int_{t_0}^{t} \frac{e_\alpha(\sigma(s),0)e_\gamma(0,s)}{(1+\mu(s)\alpha)(1+\mu(s)\gamma)}\Delta s,$$

then by Lemma 2.0.11 and the fact that $\gamma > \alpha$, we obtain

$$\int_{t_0}^{t} e_\alpha(\sigma(s),0)\,\|H(s)\|\,\Delta s \le \frac{L_0\,\|x_0\|\,e_\alpha(t_0,0)}{\gamma-\alpha}.$$

Using (4.35), (b), (c) and (d) we have

$$\begin{aligned} e_\alpha(t,0)\,\|x(t)\| \;&\le\; M\,\|x_0\|\,e_\alpha(t_0,0) + ML_0\,\|x_0\|\,\frac{e_\alpha(t_0,0)}{\gamma-\alpha} \\ &\quad + M\int_{t_0}^{t} \alpha_0 e_\alpha(s,0)\,\|x(s)\|\,\Delta s, \end{aligned}$$

which implies

$$\begin{aligned} e_\alpha(t,0)\,\|x(t)\| \;&\le M\,\|x_0\|\left(1+\frac{L_0}{\gamma-\alpha}\right)e_\alpha(t_0,0) \\ &\quad + M\int_{t_0}^{t} \alpha_0 e_\alpha(s,0)\,\|x(s)\|\,\Delta s. \end{aligned} \tag{4.36}$$

Lemma 2.0.14 yields that

$$e_\alpha(t,0)\,\|x(t)\| \le M\,\|x_0\|\left(1+\frac{L_0}{\gamma-\alpha}\right)e_\alpha(t_0,0)e_{M\alpha_0}(t,t_0).$$

Using [16, Theorem 2.36], we obtain

$$\|x(t)\| \le M\,\|x_0\|\left(1+\frac{L_0}{\gamma-\alpha}\right)e_{\alpha\ominus M\alpha_0}(t_0,0)e_{\alpha\ominus M\alpha_0}(0,t).$$

By Lemma 2.0.13 we have $e_{\alpha\ominus M\alpha_0}(0,t) \le \dfrac{1}{1+(\alpha\ominus M\alpha_0)t}$, so we obtain

$$\|x(t)\| \le \frac{M\,\|x_0\|\left(1+\frac{L_0}{\gamma-\alpha}\right)e_{\alpha\ominus M\alpha_0}(t_0,0)}{1+(\alpha\ominus M\alpha_0)t}.$$

Hence, in view of $\alpha\ominus M\alpha_0 > 0$, we obtain the required result. □

Theorem 4.2.5 generalizes the continuous version ($\mathbb{T}=\mathbb{R}$) of [61, Theorem 2.5].

Corollary 4.2.6. *Let $L(t,s)$ be a $n\times n$ matrix function on $\Omega_{(q,h)}$, such that*

(a) *all the assumptions of Corollary 3.4 holds,*

(b) $\|L(t,s)\| \le \displaystyle\prod_{r\in[s,t)} \frac{L_0}{(1+\alpha\mu(t))(1+\gamma\mu(t))(1+\mu(r)\gamma)}$,

(c) $\displaystyle\sup_{t_0\le s\le t<\infty}\sum_{\tau\in[\sigma(s),t)}\prod_{r\in[\sigma(\tau),t)}(1+\mu(r)\alpha)\,\|G(\tau,s)\|\,\mu(\tau) \le \alpha_0$,

(d) $F(n)\equiv 0$,

where L_0, $\gamma > \alpha$, α_0, are positive real constants. If $\alpha \ominus M\alpha_0 > 0$, then every solution $x(t)$ of (4.2) tends to zero exponentially as $t \to +\infty$.

Example 4.2.7. Let us consider the following Volterra integro-dynamic equation

$$\begin{cases} x^{\Delta}(t) = \ominus 2x(t) + \int_0^t e_{\ominus 2}(t,s)x(s)\Delta s, \\ x(0) = 1, \end{cases} \tag{4.37}$$

where $A(t) = \ominus 2$ and $K(t,s) = e_{\ominus 2}(t,s)$. Now consider $L(t,s) = 0$ then $B(t) = \ominus 2$. The matrix function $G(t,s)$ given in (4.24) becomes

$$G(t,s) = e_{\ominus 2}(t,s). \tag{4.38}$$

Let $\mathbb{T} = \mathbb{R}$. Then we have

$$|e_B(t,s)| = |e_{-2}(t,s)| = e^{2(s-t)} \leq Me^{2(s-t)},\ M = 2$$

and

$$0 = |L(t,s)| < L_0 e^{3(s-t)},\ L_0 = 1.$$

Here the constants are $\alpha = 2$ and $\gamma = 3$. From (4.38) it follows that

$$G(t,s) = e^{-2(t-s)} \tag{4.39}$$

Then from (4.39), we obtain that $G(t,s)$ is a positive function, and

$$\begin{aligned} \int_s^t e^{2(\tau-t)} |G(\tau,s)|\, d\tau &= \int_s^t e^{2(\tau-t)} e^{-2(\tau-s)} d\tau \\ &= e^{2(s-t)}(t-s) \\ &\leq \frac{(t-s)}{1+2(t-s)} \\ &< \frac{1}{2}, \end{aligned}$$

from which it follows that

$$\sup_{0 \leq s \leq t < \infty} \int_s^t e^{\frac{1}{2}(\tau-t)} |G(\tau,s)|\, d\tau \leq \frac{1}{2}.$$

Since $\alpha_0 = \frac{1}{2}$, then we that $\alpha - M\alpha_0 > 0$. Therefore, all the assumptions of Theorem 4.2.5 hold for the system (4.37), it follows that the solution of (4.37) tends to zero exponentially as $t \to +\infty$. Now we consider $\mathbb{T} = \mathbb{N}$. Then we have

$$|e_B(t,s)| = \left|e_{-\frac{2}{3}}(t,s)\right| = \left(\frac{1}{3}\right)^{t-s} \leq M\,(3)^{s-t}\,,\ M = 2$$

$$0 = |L(t,s)| < \frac{L_0\,(4)^{s-t}}{8},\quad L_0 = 1.$$

Here the constants are $\alpha = 2$ and $\gamma = 3$. From (4.38) it follows that

$$G(t,s) = \left(\frac{1}{3}\right)^{t-s}$$

Now we have to calculate

$$\begin{aligned}
& \sum_{\tau\in[s+1,t)} 3^{\tau+1-t}\,|G(\tau,s)| \\
= & \sum_{\tau\in[s+1,t)} 3^{\tau+1-t}\left(\frac{1}{3}\right)^{\tau-s} \\
= & \; 3^{s-t+1}(t-s-2) < 3^{s-t+1}(t-s-1) \\
< & \; \frac{1}{2},
\end{aligned}$$

from which it follows that

$$\sup_{0\leq s\leq t<\infty} \sum_{\tau\in[s+1,t)} 2^{\tau+1-t}\,|G(\tau,s)|\,d\tau \leq \frac{1}{2}.$$

Since $\alpha_0 = \frac{1}{2}$, then we that $\frac{\alpha - M\alpha_0}{1+M\alpha_0} > 0$. Therefore, all the assumptions of Theorem 4.2.5 hold for the system(4.37), it follows that the solution of (4.37) tends to zero exponentially as $t \to +\infty$.

Theorem 4.2.8. *Let $L \in C(\Omega, M_n(\mathbb{R}))$ such that $\Delta_s L(t,s) \in C(\Omega, M_n(\mathbb{R}))$ for $(t,s) \in \Omega$ and*

(i) the assumptions (a), (b) and (d) of Theorem 4.2.5 hold,

(ii) $\|\Delta_s L(t,s)\| \leq N_0 e_\delta(s,t)$ and $\|K(t,s)\| \leq K_0 e_\theta(s,t)$,

(iii) $\|A(t)\| \leq A_0$ for $t_0 \leq t < \infty$,

(iv) $\sup_{t_0 \le s \le t < \infty} \int_{\sigma(s)}^{t} \left[(K_0 + N_0)(1 + \mu(\tau)\alpha) + \frac{A_0 L_0 + (\tau - \sigma(s)) L_0 K_0}{\mu(\tau)\alpha} \right] \Delta\tau \le \alpha_0^\star$, *for some* $\alpha_0^\star > 0$,

where A_0, N_0, K_0, δ *and* θ *are positive real numbers such that* $\gamma > \delta > \alpha$, $\theta > \alpha$. *If* $\alpha \ominus M\alpha_0^\star > 0$, *then every solution* $x(t)$ *of (4.1) tends to zero exponentially as* $t \to +\infty$.

Proof. From (4.24) we obtain

$$\begin{aligned} \|G(t,s)\| &\le \|K(t,s)\| + \|\Delta_s L(t,s)\| + \|L(t,\sigma(s))\| \, \|A(s)\| \\ &\quad + \int_{\sigma(s)}^{t} \|L(t,\sigma(u))\| \, \|K(u,s)\| \, \Delta u, \end{aligned}$$

which implies

$$\begin{aligned} \|G(t,s)\| &\le K_0 e_\theta(s,t) + N_0 e_\delta(s,t) + \frac{L_0 e_\gamma(s,t)}{(1+\mu(t)\alpha)(1+\mu(t)\gamma)} A_0 \\ &\quad + \int_{\sigma(s)}^{t} \frac{L_0 K_0 e_\gamma(u,t) e_\theta(s,u)}{(1+\mu(t)\alpha)(1+\mu(t)\gamma)} \Delta u. \end{aligned} \tag{4.40}$$

Since $\lambda > \delta > \alpha$, $\theta > \alpha$, then from (i), (ii) and (iii), (4.40) becomes

$$\begin{aligned} \|G(t,s)\| &\le K_0 e_\alpha(s,t) + N_0 e_\alpha(s,t) \\ &\quad + \frac{L_0 e_\alpha(s,t)}{(1+\mu(t)\alpha)(1+\mu(t)\gamma)} A_0 + \frac{(\tau - \sigma(s)) L_0 K_0 e_\alpha(s,t)}{(1+\mu(t)\alpha)(1+\mu(t)\gamma)}, \end{aligned} \tag{4.41}$$

and

$$\begin{aligned} e_\alpha(\sigma(t),0) \|G(t,s)\| &\le \Bigg[(K_0 + N_0)(1 + \mu(\tau)\alpha) \\ &\quad + \frac{A_0 L_0 + (\tau - \sigma(s)) L_0 K_0}{\mu(\tau)\alpha} \Bigg] e_\alpha(s,0). \end{aligned}$$

Integrating the above inequality and using (iv), we obtain the following

$$\int_{\sigma(s)}^{t} e_\alpha(\sigma(\tau),0) \|G(\tau,s)\| \Delta\tau \le \alpha_0^\star e_\alpha(s,0). \tag{4.42}$$

Substituting (4.42) in (4.35) we obtain the following

$$\begin{aligned} e_\alpha(t,0) \|x(t)\| &\le M \|x_0\| \Big(1 + \frac{L_0}{\gamma - \alpha}\Big) e_\alpha(t_0,0) \\ &\quad + M \int_{t_0}^{t} \alpha_0^\star e_\alpha(s,0) \|x(s)\| \Delta s. \end{aligned}$$

Lemma 2.0.14 yields that

$$e_\alpha(t,0)\left\|x(t)\right\| \le M\left\|x_0\right\|\left(1+\frac{L_0}{\gamma-\alpha}\right)e_\alpha(t_0,0)e_{M\alpha_0^\star}(t,t_0).$$

Using [16, Theorem 2.36], we obtain

$$\left\|x(t)\right\| \le M\left\|x_0\right\|\left(1+\frac{L_0}{\gamma-\alpha}\right)e_{\alpha\ominus M\alpha_0^\star}(t_0,0)e_{\alpha\ominus M\alpha_0^\star}(0,t).$$

Then by Lemma 2.0.13, we have

$$\left\|x(t)\right\| \le \frac{M\left\|x_0\right\|\left(1+\frac{L_0}{\gamma-\alpha}\right)e_{\alpha\ominus M\alpha_0^\star}(t_0,0)}{1+(\alpha\ominus M\alpha_0^\star)t}.$$

Hence, in view of (i) and $\alpha\ominus M\alpha_0^\star > 0$, we obtain the required result. □

The continuous version ($\mathbb{T}=\mathbb{R}$) of the above theorem can be found in [61, Corollary 2.6].

Corollary 4.2.9. *Let $L(t,s)$ and $L(t,\sigma(s)) \in \Omega_{(q,h)}$, such that*

(i) *the assumptions (a), (d) of Corollary 3.6 hold,*

(ii) $\left\|L(t,\sigma(s))\right\| \le \prod_{\delta\in[s,t)} \frac{N_0}{(1+\mu(r)\delta)}$ *and* $\left\|K(n,m)\right\| \le \prod_{\theta\in[s,t)} \frac{K_0}{(1+\mu(r)\theta)}$,

(iii) $\left\|B(t)\right\| \le B_0$ *for $t_0 \le t < \infty$, where B_0, N_0, K_0, δ and θ are positive real numbers such that $\delta > \alpha$, $\theta > \alpha$,*

(iv) $\sup_{t_0\le s\le t<\infty} \sum_{\tau\in[\sigma(s),t)} (K_0+N_0)(1+\mu(\tau)\alpha)\mu(\tau) + A_0L_0 + (\tau-\sigma(s))L_0K_0 \le \alpha_0^\star,$ *for some $\alpha_0^\star > 0$.*

If $\alpha\ominus M\alpha_0^\star > 0$, then every solution $x(t)$ of (4.2) tends to zero exponentially as $t\to+\infty$.

4.3 BOUNDEDNESS

In the first result of this section, we give sufficient conditions to insure that (4.1) has bounded solutions. Our results are general and apply to (4.1) whether $A(t)$ is stable, identically zero, or completely unstable, and do not require $A(t)$ to be constant nor $K(t,s)$ to be a convolution kernel. Let $C(t)$ and $D(t,s)$ be continuous $n \times n$ matrices, $t_0 \le s \le t < \infty$. Let $s \in [t_0, \infty)$ and assume that $C(t)$ is an $n \times n$ regressive matrix. The unique matrix solution of initial valued problem

$$Y^{\Delta} = C(t)Y, \quad Y(s) = I, \tag{4.43}$$

is called the matrix exponential function (at s), and it is denoted by $e_C(t,s)$ (see [16, Definition 5.18]). Also, if $H(t,s)$ is an $n \times n$ regressive matrix, satisfying

$$\begin{cases} \Delta_t H(t,s) = C(t)H(t,s) + D(t,s), \\ \\ H(s,s) = A(s) - C(s) \end{cases} \tag{4.44}$$

then

$$H(t,s) = e_C(t,s)[A(s) - C(s)] + \int_s^t e_C(t,\sigma(\tau))D(\tau,s)\Delta\tau. \tag{4.45}$$

Theorem 4.3.1. *Let $e_C(t,s)$ be the solution of (4.43), and suppose there are positive constants N, J and M such that*

(i) $\|e_C(t,t_0)\| \le N,$

(ii) $\displaystyle\int_{t_0}^t \left\| e_C(t,s)[A(s) - C(s)] + \int_s^t e_C(t,\sigma(\tau))K(\tau,s)\Delta\tau \right\| \Delta s \le J < 1,$

(iii) $\displaystyle\left\| \int_{t_0}^t e_C(t,\sigma(u))[F(u) - G(t)x(t)]\Delta u \right\| \le M.$

Then all the solutions of (4.1) are uniformly bounded, and the zero solution of corresponding homogenous equation of (4.1) is uniformly stable with initial condition $x(t_0) = 0$.

Proof. Consider the following functional

$$V(t, x(.)) = x(t) - \int_{t_0}^t H(t,s)x(s)\Delta s. \tag{4.46}$$

The derivative of $V(t,x(.))$ along a solution $x(t)=x(t,t_0,x_0)$ of (4.1) satisfies

$$V^{\Delta}(t,x(.)) = x^{\Delta}(t) - \Delta_t \int_{t_0}^{t} H(t,s)x(s)\Delta s.$$

From Theorem 1.117 of [16], we obtain

$$\begin{aligned} V^{\Delta}(t,x(.)) &= x^{\Delta}(t) - H(\sigma(t),t)x(t) - \int_{t_0}^{t} \Delta_t H(t,s)x(s)\Delta s \\ &= A(t)x(t) - H(\sigma(t),t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\Delta s \\ &\quad - \int_{t_0}^{t} \Delta_t H(t,s)x(s)\Delta s + F(t), \end{aligned}$$

or

$$\begin{aligned} V^{\Delta}(t,x(.)) &= [A(t) - H(\sigma(t),t)]x(t) + F(t) \\ &\quad + \int_{t_0}^{t} \Big[K(t,s) - \Delta_t H(t,s)\Big] x(s)\Delta s. \end{aligned} \tag{4.47}$$

By using (4.45) and Theorems 1.75, 5.21 of [16] we have the following expression

$$\begin{aligned} H(\sigma(t),t) &= e_C(\sigma(t),t)[A(t)-C(t)] + \int_{t}^{\sigma(t)} e_C(\sigma(t),\sigma(\tau))D(\tau,t)\Delta\tau \\ &= (I+\mu(t)C(t))e_C(t,t)[A(t)-C(t)] + \mu(t)e_C(\sigma(t),\sigma(t))D(t,t) \\ &= (I+\mu(t)C(t))[A(t)-C(t)] + \mu(t)D(t,t) \\ &= [A(t)-C(t)] + \mu(t)[C(t)A(t) - C^2(t) + D(t,t)] \end{aligned}$$

which implies that

$$H(\sigma(t),t) = [A(t)-C(t)] + G(t), \tag{4.48}$$

where $G(t) = \mu(t)[C(t)A(t) - C^2(t) + D(t,t)]$. Substituting (4.48) in (4.47) it follows that

$$V^{\Delta}(t,x(.)) = C(t)x(t) - G(t)x(t) + \int_{t_0}^{t} \Big[K(t,s) - \Delta_t H(t,s)\Big] x(s)\Delta s + F(t).$$

By (4.44) and (4.46) we have

$$V^{\Delta}(t,x(.)) = C(t)V(t,x(.)) + \int_{t_0}^{t} [K(t,s) - D(t,s)]x(s)\Delta s + F(t) - G(t)x(t).$$

Thus

$$V(t,x(.)) = e_C(t,t_0)x_0 + \int_{t_0}^{t} e_C(t,\sigma(u))g(u,x(.))\Delta u, \tag{4.49}$$

where

$$g(t, x(.)) = \int_{t_0}^{t} [K(t,s) - D(t,s)]x(s)\Delta s + F(t) - G(t)x(t).$$

Let $D(t,s) = K(t,s)$. Then by (4.45), (ii) is precisely $\int_{t_0}^{t} \|H(t,s)\| \Delta s \leq J < 1$. By (4.49) and $(i) - (iii)$,

$$\begin{aligned} |V(t, x(.))| &= \left\| e_C(t,t_0)x_0 + \int_{t_0}^{t} e_C(t,\sigma(u))[F(u) - G(t)x(t)]\Delta u \right\| \\ &\leq \|e_C(t,t_0)\| \|x_0\| + \left\| \int_{t_0}^{t} e_C(t,\sigma(u))[F(u) - G(t)x(t)]\Delta u \right\| \\ &\leq N \|x_0\| + M. \end{aligned}$$

If $\|x_0\| < B_1$ for some constant, and if $Q = NB_1 + M$, then by (4.46) we obtain

$$\|x(t)\| - \int_{t_0}^{t} \|H(t,s)\| \|x(s)\| \Delta s \leq \|V(t, x(.))\| \leq Q. \tag{4.50}$$

Now, either there exists $B_2 > 0$ such that $\|x(t)\| < B_2$ for all $t \geq t_0$, and thus $x(t)$ is uniformly bounded, or there exists a monotone sequence $\{t_n\}$ tending to infinity such that $\|x(t_n)\| = \max_{t_0 \leq t \leq t_n} \|x(t)\|$ and $\|x(t_n)\| \to \infty$ as $t_n \to \infty$. By (ii) and (4.50) we have

$$\|x(t_n)\| (1 - J) \leq \|x(t_n)\| - \int_{t_0}^{t_n} \|H(t_n, s)\| \|x(s)\| \Delta s \leq Q,$$

a contradiction. This complete the proof. □

It is noted that the Theorem 4.3.1 generalizes the continuous version ($\mathbb{T} = \mathbb{R}$) of [54, Theorem 1].

Corollary 4.3.2. *Suppose that there are positive constants N, J and M such that*

(i) $\left\| \prod_{r \in [s,t)} (I + \mu(r)C(r)) \right\| \leq N,$

(ii) $\sum_{s \in [t_0,t)} \left\| \prod_{r \in [s,t)} (I + \mu(r)C(r))[A(s) - C(s)] + \sum_{\tau \in [s,t)} \prod_{r \in [\sigma(\tau),t)} (I + \mu(r)C(r)) \times K(\tau,s)\mu(\tau)\mu(s) \right\| \leq J < 1,$

(iii) $\left\| \sum_{u \in [t_0,t)} \prod_{r \in [\sigma(u),t)} (I + \mu(r)C(r))[F(u) + G(t)x(t)]\mu(u) \right\| \leq M.$

Then all solutions of (4.2) are uniformly bounded, and the zero solution of corresponding homogenous equation of (4.2) with initial condition $x(t_0) = 0$ is uniformly stable.

In the second part of this section, we consider the system (4.1) with $F(t)$ is bounded and suppose that

$$C(t,s) = -\int_t^\infty K(u,s)\Delta u \tag{4.51}$$

is defined and continuous on Ω. The matrix $E(t)$ on $[t_0, \infty)$ is defined by

$$E(t) = A(t) - C(\sigma(t), t). \tag{4.52}$$

Then (4.1) is equivalent to the following system

$$\begin{cases} x^\Delta(t) = E(t)x(t) + \Delta_t \displaystyle\int_{t_0}^t C(t,s)x(s)\Delta s + F(t),\ t \in \mathbb{T}_0, \\ x(t_0) = x_0. \end{cases} \tag{4.53}$$

Theorem 4.3.3. *Let $E \in C(\mathbb{T}, M_n(\mathbb{R}))$ and M, $\alpha > 0$. Assume that $E(t)$ commutes with its integral. If*

$$\|e_E(t,s)\| \leq M e_\alpha(s,t), \quad t, s \in \Omega, \tag{4.54}$$

then every solution $x(t)$ of (4.1) with $x(t_0) = x_0$ satisfies

$$\begin{aligned} \|x(t)\| &\leq M\|x_0\| e_\alpha(t_0,t) + M\int_{t_0}^t e_\alpha(\sigma(s),t)\|F(s)\|\Delta s \\ &+ M\int_{t_0}^t \|E(u)\| e_\alpha(\sigma(u),t)\left[\int_{t_0}^u \|C(u,s)\|\|x(s)\|\Delta s\right]\Delta u \\ &+ \int_{t_0}^t \|C(t,s)\|\|x(s)\|\Delta s. \end{aligned} \tag{4.55}$$

Proof. Let $x(t)$ be the solution of (4.1) and define $q(t) = e_E(t_0,t)x(t)$. Then

$$q^\Delta(t) = -E(t)e_E(t_0,\sigma(t))x(t) + e_E(t_0,\sigma(t))x^\Delta(t).$$

Substituting for $x^\Delta(t)$ from (4.53) and integrating from t_0 to t, we obtain

$$\begin{aligned} q(t) - q(t_0) &= \int_{t_0}^t e_E(t_0,\sigma(s))F(s)\Delta s \\ &+ \int_{t_0}^t e_E(t_0,\sigma(u))\left[\Delta_u \int_{t_0}^u C(u,s)x(s)\Delta s\right]\Delta u. \end{aligned}$$

Applying the integration by parts on the second term of the right hand side [16, Theorem 1.77], we obtain

$$\begin{aligned} x(t) &= e_E(t,t_0)x_0 + \int_{t_0}^t e_E(t,\sigma(s))H(s)\Delta s + \int_{t_0}^t C(t,s)x(s)\Delta s \\ &+ \int_{t_0}^t E(u)e_E(t,\sigma(s))\left[\int_{t_0}^u C(u,s)x(s)\Delta s\right]\Delta u. \end{aligned} \tag{4.56}$$

Hence, using (4.54) and applying norm on (4.56), we obtain (4.55), which completes the proof. □

The continuous version ($\mathbb{T}=\mathbb{R}$) of the Theorem 4.3.3 can be found in [18, Lemma 2] with $D \equiv 1$

Corollary 4.3.4. *Let* $E : \mathbb{T}^r_{(q,h)} \to M_n(\mathbb{R})$ *and* M, $\alpha > 0$. *If*

$$\left\| \prod_{r\in[s,t)} (I+\mu(r)E(r)) \right\| \le \prod_{r\in[s,t)} \frac{M}{(1+\mu(r)\alpha)}$$

then every solution $x(t)$ *of (4.2) satisfies*

$$\begin{aligned} \|x(t)\| &\le \prod_{r\in[t_0,t)} \frac{M\|x_0\|}{(1+\mu(r)\alpha)} + M\sum_{s\in[t_0,t)}\prod_{r\in[\sigma(s),t)} \frac{\|F(s)\|\mu(s)}{(1+\mu(r)\alpha)} \\ &+M\sum_{u\in[t_0,t)}\prod_{r\in[\sigma(u),t)} \frac{\|E(u)\|\mu(u)}{(1+\mu(r)\alpha)}\left[\sum_{s\in[t_0,u)}\|C(u,s)\|\,\|x(s)\|\,\mu(s)\right] \\ &+\sum_{s\in[t_0,t)}\|C(t,s)\|\,\|x(s)\|\,\mu(s). \end{aligned}$$

Our next result concerns the Boundedness of solutions of (4.1).

Theorem 4.3.5. *Let* $x(t)$ *be a solution of (4.1). If* $\|E(t)\| \le d$ *on* $[t_0,\infty)$ *for some* $d > 0$, $F(t)$ *is bounded and* $\sup_{t_0\le t<\infty}\int_{t_0}^t \|C(t,s)\|\Delta s \le \beta$, *with* β *sufficiently small, then* $x(t)$ *is bounded.*

Proof. For the given t_0 and bounded $F(t)$ there is $C_1 > 0$ with

$$M \|x_0\| e_\alpha(t_0, t) + M \sup_{t_0 \le t < \infty} \int_{t_0}^{t} e_\alpha(\sigma(s), t) \|F(s)\| \Delta s < C_1. \tag{4.57}$$

Substituting (4.57) in (4.55) we obtain

$$\begin{aligned} \|x(t)\| &\le C_1 + Md \int_{t_0}^{t} e_\alpha(\sigma(u), t) \left[\int_{t_0}^{u} \|C(u,s)\| \|x(s)\| \Delta s \right] \Delta u \\ &\quad + \int_{t_0}^{t} \|C(t,s)\| \|x(s)\| \Delta s, \\ &\le C_1 + \frac{Md}{\alpha} \beta \sup_{t_0 \le s < \infty} \|x(s)\| + \beta \sup_{t_0 \le s < \infty} \|x(s)\| \\ &= C_1 + \beta \left[1 + \frac{Md}{\alpha} \right] \sup_{t_0 \le s < \infty} \|x(s)\|. \end{aligned}$$

Let β be chosen so that $\beta \left[1 + \frac{Md}{\alpha} \right] = m < 1$. Then

$$\|x(t)\| \le C_1 + m \sup_{t_0 \le s < t} \|x(s)\|.$$

Let $C_2 > \|x_0\|$ and $C_1 + mC_2 < C_2$. If $\|x(t)\|$ is not bounded then there exists a first $t_1 > t_0$ with $\|x(t_1)\| = C_2$. Then

$$C_2 = \|x(t_1)\| \le C_1 + mC_2 < C_2,$$

a contradiction. This completes the proof. □

The Theorem 4.3.5 generalizes the continuous version ($\mathbb{T} = \mathbb{R}$) of [19, Theorem 2.6.3].

Corollary 4.3.6. *Let $x(t)$ be a solution of (4.2). If $\|E(t)\| \le d$ on $[t_0, \infty)$ for some $d > 0$, $F(t)$ is bounded and $\sup_{t_0 \le t < \infty} \sum_{s \in [t_0, t)} \|C(t,s)\| \mu(s) \le \beta$, for some sufficiently small β, then $x(t)$ is bounded.*

Example 4.3.7. Let us consider the following Volterra integro-dynamic equation

$$\begin{cases} x^{\Delta}(t) = \frac{\ominus a(1+a^2)}{a^2} x(t) + \int_{t_0}^{t} e_{\ominus a}(\sigma(t), s) x(s) \Delta s + F(t), \\ x(t_0) = 1, \end{cases} \tag{4.58}$$

where $A(t) = \frac{\ominus a(1+a^2)}{a^2}$, $K(t,s) = e_{\ominus a}(\sigma(t), s)$ with $a > 2$. Assume that $F(t)$ is bounded function.

Since we have that

$$\begin{aligned} \int_{t}^{\infty} e_{\ominus a}(\sigma(u), s) \Delta u &= \lim_{b \to \infty} -\frac{1}{a} \int_{t}^{b} \frac{-a}{e_a(\sigma(u), s)} \Delta u \\ &= \lim_{b \to \infty} -\frac{1}{a} \int_{t}^{b} \left(\frac{1}{e_a(u,s)} \right)^{\Delta} \Delta u \\ &= \lim_{b \to \infty} -\frac{1}{a} \left[\frac{1}{e_a(b,s)} - \frac{1}{e_a(t,s)} \right] \\ &= \frac{1}{a e_a(t,s)}. \end{aligned}$$

Using (4.52), we have

$$\begin{aligned} |E(t)| &= \left| A(t) + \int_{\sigma(t)}^{\infty} e_{\ominus a}(\sigma(u), t) \Delta u \right| \\ &= \left| \frac{\ominus a(1+a^2)}{a^2} + \frac{1}{a e_a(\sigma(t), t)} \right| \\ &= \left| \frac{-(1+a^2)}{a(1+\mu(t)a)} + \frac{1}{a(1+\mu(t)a)} \right| \\ &= |\ominus a| \leq a. \end{aligned}$$

Hence

$$|E(t)| \leq a. \tag{4.59}$$

Now, we have to approximate

$$\begin{aligned} \int_{t_0}^{t} |C(t,s)| \Delta s &\leq \int_{t_0}^{t} \frac{1}{a e_a(t,s)} \Delta s \\ &= \frac{1}{a} \left[\frac{1}{e_a(t,t)} - \frac{1}{e_a(t,t_0)} \right] \\ &= \frac{1}{a} \left[1 - \frac{1}{e_a(t,t_0)} \right] \\ &\leq \frac{1}{a}, \end{aligned}$$

therefore

$$\int_{t_0}^{t} |C(t,s)|\,\Delta s \le \frac{1}{a}, \quad t \ge t_0. \tag{4.60}$$

Finally, by taking the supremum over t in (4.60), over $[t_0,\infty)_{\mathbb{T}}$, we obtain

$$\sup_{t_0\le t<\infty} \int_{t_0}^{t} |C(t,s)|\,\Delta s \le \frac{1}{a}.$$

Obviously, in this case $d = \alpha = a$, $M \ge 1$ and $\beta = \frac{1}{a}$. If we choose $a > M+1$, then $\beta(1+\frac{Md}{\alpha}) < 1$. It follows that all the assumptions of Theorem 4.3.5 are satisfied, hence all solutions of (4.58) are bounded.

Theorem 4.3.8. *If $F(t) = 0$ in (4.1), $\|E(t)\| \le d$ on $[t_0,\infty)$ for some $d > 0$, and $\int_{t_0}^{t} \|C(t,s)\|\,\Delta s \le \beta$, for β sufficiently small, then the zero solution of (4.1) with initial condition $x(t_0) = 0$ is uniformly stable .*

Proof. Let $\epsilon > 0$ be given. We wish to find $\delta > 0$ such that $t_0 \ge 0$, $\|x_0\| < \delta$, and $t \ge t_0$ implies $\|x(t,x_0)\| < \epsilon$. Let $\delta < \epsilon$ with δ yet to be determined. If $\|x_0\| < \delta$, then $M\|x_0\| \le M\delta$. From (4.55) with $F(t) = 0$,

$$\begin{aligned} \|x(t)\| &\le M\delta + \frac{Md}{\alpha}\beta \sup_{t_0\le s<t} \|x(s)\| + \beta \sup_{t_0\le s<t} \|x(s)\| \\ &= M\delta + \beta\Big[1+\frac{Md}{\alpha}\Big] \sup_{t_0\le s<t} \|x(s)\|. \end{aligned}$$

First take β so that $\beta\Big[1+\dfrac{Md}{\alpha}\Big] \le \dfrac{3}{4}$ and δ so that $M\delta + \dfrac{3}{4}\epsilon < \epsilon$. If $\|x_0\| < \delta$ and if there exists $t_1 > t_0$ with $\|x(t_1)\| = \epsilon$, we have

$$\epsilon = \|x(t_1)\| < M\delta + \frac{3}{4}\epsilon < \epsilon,$$

a contradiction. Thus the zero solution is uniformly stable. The proof is complete. □

The continuous version ($\mathbb{T} = \mathbb{R}$) of the above theorem can be found in [19, Theorem 2.6.4].

Corollary 4.3.9. *If $F(t) = 0$ in (4.2), $\|E(t)\| \le d$ on $[t_0,\infty)$ for some $d > 0$, and $\sum_{s\in[t_0,t)} \|C(t,s)\|\,\mu(s) \le \beta$, sufficiently small, then the zero solution of (4.2) is uniformly stable with initial condition $x(t_0) = 0$.*

Example 4.3.10. *If we consider $F(t) = 0$ in Example 4.4, then by (4.59) and (4.60), Theorem 4.3.5 yields that the zero solution of (4.58) is uniformly stable.*

4.4 CONCLUSION

In this chapter we investigated the resolvent of a linear Volterra integro-dynamic system on time scales. Using the adjoint system we obtained a relation between the principle matrix and the resolvent. It follows from Theorem 4.1.7 that both (principle matrix and resolvent of the system (4.1)) are equivalent. In section 2 we obtained some sufficient results about asymptotic stability of (4.1) where the matrix $A(t)$ need not be stable. In the last section we obtained some results about the doundedness, uniform boundedness and stability of (4.1).

BIBLIOGRAPHY

[1] M. Adivar, Function bounds for solutions of Volterra integro dynamic equations on time scales, *Electron. J. Qual. Theory Differ. Equ.* **(7) (2010)** 1-22.

[2] M. Adivar, Principal matrix solutions and variation of parameters for Volterra integro-dynamic equations on time scales, *Glasg. Math. J.* **53 (3) (2011)** 463-480.

[3] M. Adivar,Y. N. Raffoul, Necessary and sufficient conditions for uniform stability of Volterra integro-dynamic equations using new resolvent equation, *An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat. (accepted).*

[4] R. P. Agarwal, M. Bohner, Basic calculus on time scales and some of its applications, *Results Math.* **35 (1999)** 3-22.

[5] R. P. Agarwal, M. Bohner, D. O'Regan, A. Peterson, Dynamic equations on time scales: A survey, *J. Comput. Appl. Math.* **141 (2002)** 1-26.

[6] M. U. Akhmet, M. Turan, The differential equations on time scales through impulsive differential equations, *Nonlinear Anal. Real World. Appl.* **65 (11) (2006)** 2043-2060.

[7] D. D. Bainov and S. I. Kostadinov, *Abstract Impulsive Differential Equations*, World Scientific, New Jersey, 1995.

[8] Z. Bartosiewicz, E. Pawłuszewicz, Realizations of linear control systems on time scales, *Control and Cybernetics* **35 (4) (2006)** 769-786.

[9] Z. Bartosiewicz, E. Pawłuszewicz, Realizations of nonlinear control systems on time scales, *IEEE Trans. Automat. Control* **53 (2008)** 571-575.

[10] L. C. Becker, Principal matrix solutions and variation of parameters for a Volterra integro-differential equation and its adjoint, *Electron. J. Qual. Theory Differ. Equ.* **(14) (2006)** 1-22.

[11] M. Benchohra, J. Henderson and S. K. Ntouyas, *Impulsive Differential Equations and Inclusions, Contemporary Mathematics and Its Applications 2*, Hindawi Publ. Corp., New York, 2006.

[12] M. Benchohra, J. Henderson, S. K. Ntouyas, A. Ouahab, On first order impulsive dynamic equations on time scales, *J. Differ. Equ. Appl.*, **10 (2004)** 541-548.

[13] H. Baek, Dynamic analysis of an impulsively controlled predator-prey system, *Electron. J. Qual. Theory Differ. Equ.* **19 (2010)** 1-14.

[14] M. Bohner, M. Fan, J. M. Zhang, Existence of periodic solutions in predator prey and competition dynamic systems, *Nonlinear Anal. Real World. Appl.* **7 (2006)** 1193-1204.

[15] M. Bohner, M. Fan, J. M. Zhang, Periodicity of scalar dynamic equations and applications to population models, *J. Math. Anal. Appl.* **330 (2007)** 1-9.

[16] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhäuser, Boston, 2001.

[17] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[18] T. A. Burton, An integrodifferential equation, *Proc. Amer. Math. Soc.* **79 (3) (1980)** 303-399.

[19] T. A. Burton, *Volterra Integral and Differential Equations*, second edition, Elsevier, 2005.

[20] Y. K. Chang, W. T. Li, J. J. Nieto, Controllability of evolution differential inclusions in Banach spaces, *Nonlinear Anal. Real World. Appl.* **67 (2007)** 623-632.

[21] J. Čermák, L. Nechvátal, On (q,h)-analogue of fractional calculus, *J. Nonlinear Math. Phys.* **17 (1) (2010)** 1-18.

[22] P. Chesson, Understanding the role of environmental variation in population and community dynamics, *Theor. Popul. Biol.* **64 (2003)** 253-254.

[23] S. K. Choi, N. J. Koo, Asymptotic property of linear Volterra difference system, *J. Math. Anal. Appl.* **321 (2006)** 260-272.

[24] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, Cambridge, 1991.

[25] C. V. Coffman, B. D. Coleman, On the growth of populations with narrow spread in reproductive age: III. Periodic variations in the environment. *J. Math. Biol.* **7 (1979)** 281-301.

[26] K. L. Cooke and M. Witten, One-dimensional linear and logistic harvesting models, *Mathematical Modelling* **7 (1986)** 301-340.

[27] M. Cichoń, B. Satco, A. Sikorska-Nowak, Impulsive nonlocal differential equations through differential equations on time scales, *Appl. Math. Comput.* **218 (6) (2011)** 2449-2458.

[28] S. Elaydi, Periodic and stability of linear Volterra difference system, *J. Math. Anal. Appl.* **181 (1994)** 483-492.

[29] J. J. DaCunha, Transition matrix and generalized matrix exponential via the Peano-Baker series, *J. Differ. Equ. Appl.* **11 (15) (2005)**, 1245-1264.

[30] J. J. DaCunha, Stability for time varying linear dynamic systems on time scales, *J. Comput. Appl. Math.* **176 (2005)** 381-410.

[31] J. J. DaCunha, J. M. Davis, A unified Floquet theory for discrete, continuous, and hybrid periodic linear systems, *J. Differential Equations* **251 (11) (2011)** 2987-3027.

[32] J. M. Davis, I. A. Gravagne, B. J. Jackson, R. J. Marks II, Controllability, Observability, Realizability, and Stability of Dynamical Linear Systems, Electron. *J. Differential Equations* **37 (2009)** 1-32.

[33] L. Fausett, K. Murty, Controllability, observability, and realizability criteria on time scale dynamical systems. *Nonlinear Stud.* **11 (2004)** 627-638.

[34] L. M. Graves, *The Theory of Functions of Real Variables*, McGraw-Hill, New York, 1946.

[35] S. I. Grossman, R. K. Miller, Perturbation theory for Volterra integrodifferential system, *J. Differential Equations* **8 (1970)** 457-474.

[36] Z. H. Guan, T. H. Qian, X. Yu, On controllability and observability for a class of impulsive systems, *Systems & Control Letters* **47 (2002)** 247-257.

[37] S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrmsmannigfaltingkeiten*, PhD thesis, Univarsi. Wurzburg, 1988.

[38] B. J. Jackson, *A General Linear Systems Theory on Time Scales: Transforms, Stability, and Control*, Ph.D. thesis, Baylor University, 2007.

[39] B. Karpuz, Basics of Volterra integral equations on time scales, *arXiv 1102.5588v1* [math.CA] 28 Feb 2011 (submitted).

[40] E. R. Kaufmann, N. Kosmatov, Y.N. Raffoul, Impulsive dynamic systems on a time scale, Electron. *J. Differential Equations* **67 (2008)** 1-9.

[41] E. R. Kaufmann, Y. N. Raffoul, Periodic solutions for a neutral nonlinear dynamical equation on a time scale, *J. Math. Anal. Appl.* **319 (2006)** 315-325.

[42] T. Kulik, C. C. Tisdell, Volterra integral equations on time scales: basic qualitative and quantitative results with applications to initial value problems on unbounded domains, *Int. J. Difference Equ. 3 (1) (2008)* 103-133.

[43] N. Kosmatov; Multi-point boundary value problems on time scales at resonance, *J. Math. Anal. Appl.* **323 (2006)** 253-266.

[44] V. Lakshmikantham, D. D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[45] V. Lakshmikantham, S. Sivasundaram and B. Kaymakcalan, *Dynamic Systems on Measure Chains*, Kluwer Academic Publishers, Dordrecht, 1996.

[46] V. Lakshmikantham, A. S. Vatsala, Hybrid systems on time scales, *J. Comput. Appl. Math.* **141 (2002)** 227-235.

[47] S. Leela, F. A. McRae, S. Sivasundaram, Controllability of impulsive differential equations, *J. Math. Anal. Appl.* **177 (1993)** 24-30.

[48] H. Liu, X. Xiang, A class of the first order impulsive dynamic equations on time scales, *Nonlinear Anal. Real World. Appl.* **69 (2008)** 2803-2811.

[49] X. Z. Liu, A. R. Willms, Impulsive controllability oflinear dynamical systems with applications to maneuvers of spacecraft, *Math. Problems Eng.* **2 (1996)** 277-299.

[50] V. Lupulescu, A. Zada, Linear impulsive dynamic systems on time scales, *Electron. J. Qual. Theory Differ. Equ.* **11 (2010)** 1-30.

[51] V. Lupulescu, A. Younus, On controllability and observability for a class of linear impulsive dynamic systems on time scales, *Mathematical and Computer Modelling*, **54 (2011)** 1300-1310.

[52] V. Lupulescu, A. Younus, Controllability and observability for a class of time-varying impulsive systems on time scales, *Electron. J. Qual. Theory Differ. Equ.* **95 (2011)** 1-30.

[53] V. Lupulescu, S. K. Ntouyas, A. Younus, Qualitative aspects of Volterra integro-dynamic system on time scales, *Electron. J. Qual. Theory Differ. Equ.* **5 (2013)** 1-35.

[54] W. E. Mahfoud, Boundedness properties in Volterra integro-differential systems, *Proc. Amer. Math. Soc.* **100 (1)** (1987) 37-45.

[55] R. Medina, Asymptotic behavior of Volterra difference equations, *Comput. Math. Appl.* **41 (5-6)** (2001) 679-687.

[56] A. Özbekler, A. Zafer, Picone type formula for non-selfadjoint impulsive differential equations with discontinuous solutions, *Electron. J. Qual. Theory Differ. Equ.* **35 (2010)** 1-12.

[57] E. Pawłuszewicz, D. F. M. Torres, Avoidance Control on Time Scales, *J. Optim. Theory Appl.* **145 (3) (2010)** 527-542.

[58] E. Pawłuszewicz, D. F. M. Torres, Backward Linear Control Systems on Time Scales, *International Journal of Control* **83 (8) (2010)** 1573-1580.

[59] L. Di Piazza, B. Satco, A new result on impulsive differential equations involving non-absolutely convergent integrals, *J. Math. Anal. Appl.* **352 (2) (2009)**, 954-963.

[60] M. R. S. Rahmat, The (q, h)-Laplace transform on discrete time scales, *Comput. Math. Appl.* **62 (1)** (2011) 272-281.

[61] M. R. M. Rao, P. Srinivas, Asymptotic behavior of solutions of Volterra integro-differential equations, *Proc. Amer. Math. Soc.* **94 (1) (1985)** 55-60.

[62] A. M. Samoilenko, N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[63] B. Satco, Solution tube method for impulsive periodic differential inclusions of first order, *Nonlinear Anal.* **75 (1) (2012)** 260-269.

[64] B. Satco, A Cauchy problem on time scales with applications, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat.*, **57 (1) (2011)** 221-234.

[65] B. Satco, Volterra integral equations governed by highly oscillatory functions on time scales. *An.Ştiinţ.Univ."Ovidius" ConstanţaSer.Mat.*, **17 (3) (2009)** 233-240.

[66] G. M. Xie, L. Wang, Controllability and observability of a class of linear impulsive systems, *J. Math. Anal. Appl.* **304 (2005)** 336-355.

[67] F. Xu, Multiple positive solutions of nonlinear singular m-point boundary value problem for second-order dynamic equations with sign changing coefficients on time scales, *Electron. J. Qual. Theory Differ. Equ.* **20 (2010)** 1-13.

[68] H. Xu, Existence and uniqueness of positive solutions for a third-order three-point problem on time scales, *Electron. J. Qual. Theory Differ. Equ.* **56 (2010)** 1-12.

[69] T. Yang, *Impulsive Control Theory,* Springer-Verlag Berlin Heidelberg, 2001.

[70] J. Zhanga, M. Fan, H. Zhu, Periodic solution of single population models on time scales, *Mathematical and Computer Modelling* **52 (2010)** 515-521.

[71] C. Zhai, R. Song, Existence and uniqueness of positive solutions for Neumann problems of second order impulsive differential equations, *Electron. J. Qual. Theory Differ. Equ.* **76 (2010)** 1-9.

[72] S. Zhao, J. Sun, Controllability and observability of a class of time-varying impulsive systems, *Nonlinear Anal. Real World. Appl.* **10 (2009)** 1370-1380.