ON GENERALIZATION OF INEQUALITIES FOR MONOTONE FUNCTIONS

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By

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DECLARATION

I, **Mr. Saad Ihsan Butt** Registration No. **107-GCU-PHD-SMS-09** student at **Abdus Salam School of Mathematical Sciences GC University** in the subject of **Mathematics** year of admission **2009**, hereby declare that the matter printed in this thesis titled

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is my own work and that

(i) I am not registered for the similar degree elsewhere contemporaneously.

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(iii) The work, I am submitting for the Ph. D. degree has not already been submitted elsewhere and shall not in future be submitted by me for obtaining similar degree from any other institution.

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RESEARCH COMPLETION CERTIFICATE

Certified that the research work contained in this thesis titled

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has been carried out and completed by **Mr. Saad Ihsan Butt** Registration No. **107-GCU-PHD-SMS-09** under my supervision.

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Date	**Josip Pečarić**
	Supervisor

Submitted Through

Prof. Dr. A. D. Raza Choudary	**-------------------------------**
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Dedicated

to

my family and friends

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Abstract

The thesis comprises of generalized inequalities for monotone functions from which we deduce important inequalities such as reversed Hardy type inequalities, generalized Hermite-Hadamard's inequalities etc by putting suitable functions. The present thesis is divided into three chapters.

The first chapter includes generalized inequalities given for C-monotone functions and multidimensional monotone functions. As a result of these inequalities, we deduce reversed Hardy inequalities for C-monotone functions and multidimensional reversed Hardy type inequalities with the optimal constant. Furthermore, we construct functionals from the differences of above inequalities and gives their n-exponential convexity and exponential convexity. By using log-convexity of these functionals we give refinements of these inequalities. Also we give mean-value theorems for these functionals and deduce Cauchy means for them.

The second chapter consists of inequalities valid for monotone functions of the form f/h and f'/h. These are also very interesting as by putting suitable functions we get one side of Hermite-Hadamard's inequality and generalized Hermite-Hadamard's inequality. Similarly as in the first chapter, we make functionals of these inequalities and gives results regarding n-exponential convexity and exponential convexity. Also we give mean value theorems of Lagrange and Cauchy type as well as we obtain non-symmetric Stolarsky means with and without parameter.

In the third and the last chapter we consider Petrović type functionals obtained from Petrović type inequalities and investigate their properties like superadditivity, subadditivity, monotonicity and n-exponential convexity.

Also at the end of each chapter we discuss examples in which we construct further exponential convex functions and their relative properties.

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Saad Ihsan Butt

Chapter 1

Introduction and Preliminaries

Without any doubt, inequalities are in the roots of almost all fields of mathematics as they play a key role in their prosperity. As G. H. Hardy says that "All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove."
Generalized results always have great importance in mathematics. Specially in the field of inequalities generalized inequalities lie at the heart of a great deal of mathematics as they give birth to many different inequalities by substituting effective functions and conditions. G. H. Hardy discovered the classical Hardy inequality while proving the Hilbert inequality. The classical Hardy inequality appeared in 1925 [17], which gives a relation between the L_p norm of the average value of the function and the L_p norm of the function. The Hardy inequality was then extensively studied and many general inequalities were developed by considering it as a role model. However P. F. Renaud [15] in (1986) gave the reversed Hardy inequality. Since there is not so much work on the reversed Hardy–type inequalities so in this thesis we give something interesting about reversed Hardy type inequalities and there refinements both in one and higher dimensions.
In modern era, convex functions has occupied significant commanding place in mathematics as it is involved in many research articles and books related to analysis. For convex function Hermite-Hadamard's inequality gives the natural geometrical interpretation of means of function and there domain and having a large number of applications in related inequalities. Later on G. H. Toader proved one side of the Hermite-Hadamard's inequality using starshaped functions. However, we prove and generalize one side of the Hermite-Hadamard's inequality for increasing functions of the form f/h and highlight many interesting aspects of these inequalities. Also we give important properties of Petrović type functionals obtained from Petrović type

inequalities (see Ch 5 [19]) also called as inequalities for star shaped functions.

1.1 Monotone Functions

The term "monotone" express the sense of no change, maintaining order or moving in the same direction. It is introduced to explore order theory. This fact is easy to understand in terms of inequalities. Throughout the section $I \subseteq \mathbb{R}$ be an interval in $\mathbb{R}$ unless stated otherwise and the function f be a real valued function on it.

Definition 1.1.1. A function $f : I \to \mathbb{R}$ is called increasing(decreasing) if for all distinct points $x_1 \geq x_2 \in I$ one has $f(x_1) \geq f(x_2)$ $\big(f(x_1) \leq f(x_2)\big)$. The function f is called strictly increasing or decreasing if the strict inequalities hold. A function f is called monotone function on I if it is either increasing or decreasing on I.

Remark 1.1.1. Mostly, we use the following criterion for monotonicity of the functions:

(a) If $f \in C^1(I)$, then it is monotone on I iff the sign of f' remains the same throughout I. In particular, if $f'(x) > 0(f'(x) < 0)$, except on a set of points of I which does not contain any interval of I then f is an strictly increasing(decreasing) function; if $f'(x) \geq 0(f'(x) \leq 0)$ then f is increasing(decreasing) respectively.

(b) If f and h are monotone in the same sense, then foh is increasing and if f and h are monotone in the opposite sense then foh is decreasing in their respective domains.

1.2 Convex Functions

Systematic study of convex functions started over the period 1905–1906 by thought provoking ideas and fascinating work of Jensen. However there also exists some literature about convex functions even before Jensen because one may find the existence of the roots of there definition in the work of O. Hölder(1989) and J. Hadamard(1893). The study of convex functions are used as a major tool to solve optimization problems in analysis. However the soil of inequalities involving convex functions is quite fertile as it increases the growth of many branches of mathematics with considerable high rate. That is why the study of such inequalities have been given great importance in literature. Historically the definition of convex functions originates as a real-valued function with real domain.

Definition 1.2.1 ([19, p.1])**.** A function $f : I \to \mathbb{R}$ is called convex if for all $x_1, x_2 \in I$ and $\lambda \in [0, 1]$, the inequality

$$f(\lambda x_1 + (1-\lambda) x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2) \tag{1.2.1}$$

holds.
Also f is said to be strictly convex if for $x_1 \neq x_2$ and $\lambda \in (0, 1)$ strict inequality holds in (1.2.1) . Moreover f is called concave function if the inequality in (1.2.1) is reversed and is strictly concave if for all $x_1 \neq x_2$ and $\lambda \in (0, 1)$ strict inequality holds in (1.2.1).

The following lemmas can be considered as equivalent definitions of convex functions.

Lemma 1.2.1 ([19, p. 2])**.** *A function $f : I \to \mathbb{R}$ is convex iff the inequality*

$$(x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \ge 0 \tag{1.2.2}$$

holds for every $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

Lemma 1.2.2 ([19, p. 2])**.** *If f is a convex function on an interval I and if $x_1 \le y_1$, $x_2 \le y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the inequality*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(y_2) - f(y_1)}{y_2 - y_1} \tag{1.2.3}$$

is valid. If the function f is concave then (1.2.3) is reversed.

In order to define convex function $f : I \to \mathbb{R}$ for higher order (see [19] page 14). A kth order divided difference of f at distinct points $x_o, x_1, \ldots, x_k$ in I may be defined recursively by

$$[x_i; f] = f(x_i), \qquad (i = 0, 1, \ldots, k)$$

and

$$[x_0, x_1, \ldots, x_k; f] = \frac{[x_1, x_2, \ldots, x_k; f] - [x_0, x_1, \ldots, x_{k-1}; f]}{x_k - x_0}. \tag{1.2.4}$$

Divided differences are found to be very handy and interesting when we have to operate with different functions having different degree of smoothness. The values of the divided difference are independent of the order of the points $x_0, x_1, \dots, x_k$ and may be extended to include the case when the points are equal, that is

$$[\underbrace{x, x, \dots, x}_{j+1 times}; f] = \frac{f^{(j)}(x)}{j!},$$

provided that $f^{(j)}$ exists. Note that (1.2.4) is equivalent to

$$[x_0, x_1, \dots, x_k; f] = \sum_{j=0}^{k} \frac{f(x_j)}{w'(x_k)}, \quad \text{where } w(x) = \prod_{j=0}^{k} (x - x_j).$$

Since for the results related to n-exponential convexity, we have to restrict ourselves to the definition of divided differences for $k = 1, 2$. Also we need the following remarks of our interest.

Remark 1.2.1. One can note that if for all $x_0, x_1 \in I$, $[x_0, x_1; f] \geq 0$ then f is increasing on I and if for all $x_0, x_1, x_2 \in I$, $[x_0, x_1, x_2; f] \geq 0$, then f is convex on I.

Definition 1.2.2 ([19, p.14])**.** A function $f : I \to \mathbb{R}$ is said to be n-convex, $n \geq 0$, on I iff for all choices of $(n + 1)$ distinct points in I,

$$[x_0, x_1, \dots, x_n; f] \geq 0 \tag{1.2.5}$$

holds.

1.3 n-Exponential Convexity and Exponential Convexity

First time exponentially convex functions were introduced by Bernstein [4]. Independently of Bernstein, but some what later Widder [29] introduced these functions, as a sub-class of convex functions in a given interval (a, b), and denoted this class by $W_{a,b}$. After the initial development, there is a big gap in time before applications and examples of interest were constructed. One of the reasons is that, aside from absolutely monotone functions and completely monotone functions, as special classes of exponentially convex functions, there is no operative criteria to recognize exponential convexity of functions. Later on J. Pečarić and I. Perić introduce the definition of n-exponential convex functions (see [25]). The following results for n-exponentially convex functions are cited from [25].

Definition 1.3.1. A function $f : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I, if

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and every $x_i \in I$, $i = 1, \ldots, n$.
A function $f : I \longrightarrow \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 1.3.1. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact non-negative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N},\ k \leq n$.

By using some linear algebra and the definition of a positive semi-definite matrix, we have the following proposition.

Proposition 1.3.1. *If f is n-exponentially convex in the Jensen sense, then for any $x_i \in I$, $i = 1, \ldots, n$, the matrix*

$$\left[f\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k}$$

is positive semi-definite for all $k \in \mathbb{N},\ k \leq n$. In particular,

$$\det\left[f\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k} \geq 0$$

for all $k \in \mathbb{N},\ k \leq n$.

Definition 1.3.2. A function $f : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I , if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$. Moreover, a function $f : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous on I.

Remark 1.3.2. A function $f : I \to \mathbb{R}$ is log-convex in the Jensen sense, i. e.

$$f\left(\frac{x_1 + x_2}{2}\right)^2 \leq f(x_1)f(x_2), \quad \text{for all } x_1, x_2 \in I, \tag{1.3.1}$$

if and only if

$$\xi_1^2 f(x_1) + 2\xi_1\xi_2 f\left(\frac{x_1 + x_2}{2}\right) + \xi_2^2 f(x_2) \geq 0$$

holds for every $\xi_1, \xi_2 \in \mathbb{R}$ and $x_1, x_2 \in I$, i. e, if and only if f is 2-exponentially convex in the Jensen sense. By induction from (1.3.1) we have

$$f\left(\frac{1}{2^k}x_1 + (1 - \frac{1}{2^k})x_2\right) \leq f(x_1)^{\frac{1}{2^k}} f(x_2)^{1-\frac{1}{2^k}}.$$

Therefore, if f is continuous and $f(x_1) = 0$ for some $x_1 \in I$, then from the last inequality and non-negativity of f (see Remark 1.3.1) we get $f(x_2) = \lim_{k\to\infty} f(\frac{1}{2^k}x_1 + (1 - \frac{1}{2^k})x_2) = 0$ for all $x_2 \in I$. Hence, a 2-exponentially convex function is either identically equal to zero or it is strictly positive and log-convex.

Chapter 2

Reversed Hardy Type Inequalities and their Refinements

We divide the present chapter in three main sections. In the first section, we will give refinements of some inequalities given in [20] for generalized monotone functions by using classical log-convexity method to some functionals. In the second section, we deduce reversed Hardy type inequalities for more general class of C-monotone functions by generalizing the inequalities given by J. Pečarić, I. Perić and L. E. Persson in [20]. In the last section, some multidimensional generalizations of reversed Hardy type inequalities for monotone functions is given. Also, we discuss n-exponential convexity, exponential convexity and related results for some functionals obtained from the differences of the respective inequalities. In each section, we will give mean-value theorems and Cauchy means for these functionals. The results of this chapter are given in [10], [13] and [12].

2.1 Introduction

The classical Hardy inequality (see [17]) for $h \geq 0$, $p > 1$ is given as

$$\left(\int_0^\infty \left(\frac{1}{x} \int_0^x h(t)dt \right)^p dx \right)^{1/p} \leq \frac{p}{p-1} \left(\int_0^\infty h^p(x)dx \right)^{1/p}. \tag{2.1.1}$$

When $h \geq 0$ is a decreasing function, then reversed Hardy inequality given by P. F. Renaud (1986) in [15],

$$\int_0^\infty \left(\frac{1}{x} \int_0^x h(t)dt \right)^p dx \geq \frac{p}{p-1} \int_0^\infty h^p(x)dx \tag{2.1.2}$$

holds.
Let us denote

$$\widetilde{H}_p(a,b,K,h,g) = \left(\int_a^b K(x) h^p(x) d\big[g^p(x) \big] \right)^{1/p},$$

$$\widetilde{H}_p(a,b,K,h,g) = \left(\int_a^b K(x) h^p(x) d\big[- g^p(x) \big] \right)^{1/p},$$

and

$$G_p(h,x) = \left(\int_x^\infty \big(t^{-\alpha} h(t) \big)^p \frac{dt}{t} \right)^{1/p} \text{ and } \widetilde{G}_p(h,x) = \left(\int_0^x \big(t^{-\alpha} h(t) \big)^p \frac{dt}{t} \right)^{1/p}.$$

We consider the following theorem of H. Heinig and L. Maligranda.

Theorem 2.1.1. *[16] Let $-\infty \le a < b \le \infty$ and let h and g be positive functions on (a,b), where g is continuous on (a,b).*

(a) Suppose that h is a decreasing function on (a,b) and g is an increasing function on (a,b), where $g(a+0) = 0$. Then, for any $p \in (0,1]$

$$H_1(a,b,1,h,g) \le H_p(a,b,1,h,g). \tag{2.1.3}$$

If $1 \le p < \infty$, then inequality (2.1.3) holds in the reversed direction.

(b) Suppose that h is an increasing function on (a,b) and g is a decreasing function on (a,b), where $g(b-0) = 0$. Then, for any $p \in (0,1]$

$$\widetilde{H}_1(a,b,1,h,g) \le \widetilde{H}_p(a,b,1,h,g). \tag{2.1.4}$$

If $1 \le p < \infty$, then inequality (2.1.4) holds in the reversed direction.

We consider positive real valued functions h and g defined on an interval (a,b), $-\infty \le a < b \le \infty$. We say that h is C-decreasing $\big(C$-increasing$\big)$ for $C \ge 1$, if $h(x) \le Ch(y)$ $\big(h(y) \le Ch(x)\big)$ whenever $y \le x$; $y, x \in (a,b)$.
In this chapter the terms positive, decreasing and increasing shall be interpreted as nonnegative, nonincreasing and nondecreasing, respectively. We shall consider positive real valued functions h and g defined on an interval (a,b), $-\infty \le a < b \le \infty$. Moreover, the function denoted by g will be monotone throughout the paper and we assume that the function denoted by h is integrable with respect to the measure generated by g, i. e., $\int_a^b h(x)dg(x) < \infty$ for increasing g and $\int_a^b h(x)d[-g(x)] < +\infty$ for decreasing g. Some extensions of Theorem 2.1.1 were obtained in [20] as:

Theorem 2.1.2. *Assume that* $0 < p < q < \infty$ *and* $-\infty \le a < b \le \infty$ *.*

(a) If h *is* C*-decreasing and* g *is increasing, differentiable such that* $g(a+0) = 0$*, then*

$$H_q(a, b, 1, h, g) \le C^{1-\frac{p}{q}} H_p(a, b, 1, h, g). \tag{2.1.5}$$

(b) If h *is* C*-increasing and* g *is increasing, differentiable such that* $g(a+0) = 0$*, then*

$$H_q(a, b, 1, h, g) \ge C^{\frac{p}{q}-1} H_p(a, b, 1, h, g). \tag{2.1.6}$$

(c) If h *is* C*-increasing and* g *is decreasing, differentiable such that* $g(b-0) = 0$*, then*

$$\widetilde{H}_q(a, b, 1, h, g) \le C^{1-\frac{p}{q}} \widetilde{H}_p(a, b, 1, h, g). \tag{2.1.7}$$

(d) If h *is* C*-decreasing and* g *is decreasing, differentiable such that* $g(b-0) = 0$*, then*

$$\widetilde{H}_q(a, b, 1, h, g) \ge C^{\frac{p}{q}-1} \widetilde{H}_p(a, b, 1, h, g). \tag{2.1.8}$$

As a special case, we consider C-monotone functions with respect to power functions.
For $C_1, C_2 \ge 1$, $-\infty < \alpha_1 \le \alpha_2 < \infty$, we say that $h \in Q^{\alpha_1}(C_1)$ if $h(x)x^{-\alpha_1}$ is C_1-increasing and $h \in Q_{\alpha_2}(C_2)$ if $h(x)x^{-\alpha_2}$ is C_2-decreasing.

Theorem 2.1.3. *[20] Let* $0 < p \le q < \infty$*.*

(a) If $h \in Q^{\alpha_1}(C)$, $\alpha > \alpha_1$, *then for any* $x \ge 0$

$$G_q(h, x) \le p^{1/p} q^{-1/q} \big(\alpha - \alpha_1\big)^{1/p-1/q} C^{1-p/q} G_p(h, x). \tag{2.1.9}$$

(b) If $h \in Q_{\alpha_2}(C)$, $\alpha_2 > \alpha$, *then for any* $x \ge 0$

$$\widetilde{G}_q(h, x) \le p^{1/p} q^{-1/q} \big(\alpha_2 - \alpha\big)^{1/p-1/q} C^{1-p/q} \widetilde{G}_p(h, x). \tag{2.1.10}$$

For multidimensional case we consider the following notions.
For $\mathbf{a} = (a_1, a_2, \ldots, a_m), \mathbf{b} = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$, the notation $\mathbf{a} < \mathbf{b}\,(\mathbf{a} \le \mathbf{b})$ means that $a_i < b_i\,(a_i \le b_i)$, $i = 1, 2, \ldots, m$ and $(\mathbf{a}, \mathbf{b}) = \{\mathbf{x} | a_i < x_i < b_i, i = 1, 2, \ldots, m\}$.
The functions h considered in this chapter are assumed to be measurable and positive on $(\mathbf{a}, \mathbf{b})$. We say that h is increasing (decreasing) if $h(\mathbf{x}) \le h(\mathbf{y}) (h(\mathbf{x}) \ge h(\mathbf{y})$ for

$\mathbf{a} < \mathbf{x} \leq \mathbf{y} < \mathbf{b}$. We also consider $\mathbf{g} = (g_1, \dots, g_m)$ where $g_i = g_i(x_i),\ i = 1, 2, \dots, m$ and use the notations

$$\int_{\mathbf{a}}^{\mathbf{b}} \dots \mathbf{d}[\mathbf{g}(\mathbf{x})] = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \dots \int_{a_m}^{b_m} \dots dg_1(x_1) dg_2(x_2)\ \dots\ dg_m(x_m),$$

$$\int_{\mathbf{a}}^{\mathbf{b}} \dots \mathbf{d}\big[-\mathbf{g}(\mathbf{x})\big] = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \dots \int_{a_m}^{b_m} \dots d\big[-g_1(x_1)\big] d\big[-g_2(x_2)\big]\ \dots\ d\big[-g_m(x_m)\big]$$

and $\mathbf{g}^p(\mathbf{x}) = (g_1^p(\mathbf{x}), g_2^p(\mathbf{x}), \dots, g_m^p(\mathbf{x}))$. Moreover, we say that $\mathbf{g}$ is increasing (decreasing, positive, or differentiable) if $g_i,\ i = 1, 2, \dots, m$ are increasing (decreasing, positive, or differentiable), respectively. We assume that, all of the integrals appearing in the chapter exist and are finite. The following two theorems involving multidimensional monotone functions are given in [20] and [21].

Theorem 2.1.4. *Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$, $\mathbf{a} < \mathbf{b}$, $m \in \mathbb{N}$, $h : (\mathbf{a}, \mathbf{b}) \to \mathbb{R}$ be positive, $\mathbf{g} = (g_1, \dots, g_m)$, $g_i : (a_i, b_i) \to \mathbb{R}$, and $\mathbf{g}$ is positive and differentiable. Moreover, let $f : [0, \infty) \to \mathbb{R}$ be a twice differentiable function such that $f(0) = 0$ and $\lim_{t \to 0} t f'(t) = 0$.*

(a) Suppose that h is decreasing and $\mathbf{g}$ is increasing, where $g_i(a_i + 0) = 0,\ i = 1, 2, \dots, m$. If f is convex, then

$$f\Big(\int_{\mathbf{a}}^{\mathbf{b}} h(\mathbf{x}) \mathbf{d}[\mathbf{g}(\mathbf{x})]\Big) \geq \int_{\mathbf{a}}^{\mathbf{b}} f'\Big(h(\mathbf{x}) \prod_{i=1}^{m} g_i(x_i)\Big) h(\mathbf{x}) \mathbf{d}[\mathbf{g}(\mathbf{x})]. \qquad (2.1.11)$$

If f is concave, then (2.1.11) holds in reversed direction.

(b) Suppose that h is increasing and $\mathbf{g}$ is decreasing, where $g_i(b_i - 0) = 0,\ i = 1, 2, \dots, m$. If f is convex, then

$$f\Big(\int_{\mathbf{a}}^{\mathbf{b}} h(\mathbf{x}) \mathbf{d}\big[-\mathbf{g}(\mathbf{x})\big]\Big) \geq \int_{\mathbf{a}}^{\mathbf{b}} f'\Big(h(\mathbf{x}) \prod_{i=1}^{m} g_i(x_i)\Big) h(\mathbf{x}) \mathbf{d}\big[-\mathbf{g}(\mathbf{x})\big]. \qquad (2.1.12)$$

If f is concave, then (2.1.12) holds in reversed direction.

Theorem 2.1.5. *Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$, $\mathbf{a} < \mathbf{b}$, $m \in \mathbb{N}$ $h : (\mathbf{a}, \mathbf{b}) \to \mathbb{R}$ be positive, $\mathbf{g} = (g_1, \dots, g_m)$, $g_i : (a_i, b_i) \to \mathbb{R}$, and $\mathbf{g}$ is positive and continuous.*

(a) Suppose that h is decreasing and $\mathbf{g}$ is increasing, where $g_i(a_i + 0) = 0,\ i = 1, 2, \dots, m$. Then, for any $p \geq 1$,

$$\int_{\mathbf{a}}^{\mathbf{b}} h(\mathbf{x}) \mathbf{d}[\mathbf{g}(\mathbf{x})] \geq \Big(\int_{\mathbf{a}}^{\mathbf{b}} h^p(\mathbf{x}) \mathbf{d}[\mathbf{g}^p(\mathbf{x})]\Big)^{1/p}. \qquad (2.1.13)$$

If $0 < p \leq 1$, then the inequality (2.1.13) holds in reversed direction.

(b) *Suppose that h is increasing and $\mathbf{g}$ is decreasing, where $g_i(b_i - 0) = 0$, $i = 1, 2, \ldots, m$. Then, for any $p \geq 1$,*

$$\int_{\mathbf{a}}^{\mathbf{b}} h(\mathbf{x})\mathbf{d}[-\mathbf{g}(\mathbf{x})] \geq \left(\int_{\mathbf{a}}^{\mathbf{b}} h^p(\mathbf{x})\mathbf{d}[-\mathbf{g}^p(\mathbf{x})] \right)^{1/p}. \tag{2.1.14}$$

If $0 < p \leq 1$, then the inequality (2.1.14) holds in reversed direction.

(c) *The inequalities in (2.1.13) and (2.1.14) are sharp.*

We use the following example of our interest in our functionals (that we later construct in this chapter) to prove there log-convexity, refinements and to deduce Cauchy means related to them.
Let a family of functions $f_p : [0, \infty) \to \mathbb{R},\ p > 0,$ be defined by

$$f_p(x) = \begin{cases} \frac{x^p}{p(p-1)}, & p > 0, p \neq 1, \\ x \log x, & p = 1, \end{cases} \tag{2.1.15}$$

with $0 \log 0 = 0$. Then $f_p''(x) = x^{p-2}$, that is f_p is convex for $x > 0$.

2.2 Refinements of Integral Inequalities for Monotone Functions

In the present section, we will prove some improvements and refinements of the inequalities given in Theorem 2.1.2 and 2.1.3 by using log-convexity method [2]. We consider the following theorem.

Theorem 2.2.1. *Let $f : [0, \infty) \to \mathbb{R}$ be a convex and differentiable function such that $f(0) = 0$ and let $-\infty \leq a < b \leq \infty$.*

(a) *If h is C-decreasing and g is increasing, differentiable such that $g(a + 0) = 0$, then*

$$f\left(C \int_a^b h(x)dg(x) \right) \geq C \int_a^b f'\big(h(x)g(x)\big)h(x)dg(x). \tag{2.2.1}$$

(b) *If h is C-increasing and g is increasing, differentiable such that $g(a + 0) = 0$, then*

$$f\left(\frac{1}{C} \int_a^b h(x)dg(x) \right) \leq \frac{1}{C} \int_a^b f'\big(h(x)g(x)\big)h(x)dg(x). \tag{2.2.2}$$

(c) If h is C-increasing and g is decreasing, differentiable such that $g(b-0)=0$, then

$$f\Big(C\int_a^b h(x)d[-g(x)]\Big) \geq C\int_a^b f'\big(h(x)g(x)\big)h(x)d[-g(x)]. \tag{2.2.3}$$

(d) If h is C-decreasing and g is decreasing, differentiable such that $g(b-0)=0$, then

$$f\Big(\frac{1}{C}\int_a^b h(x)d[-g(x)]\Big) \leq \frac{1}{C}\int_a^b f'\big(h(x)g(x)\big)h(x)d[-g(x)]. \tag{2.2.4}$$

(e) If the condition "f is convex" is replaced by "f is concave," then all the inequalities (2.2.1)-(2.2.4) *hold in reversed direction.*

Remark 2.2.1. It was given in [20] that f is a positive convex function but from the proof of Theorem 2.2.1 given there, it is clear that the results are still valid without the condition of positivity of f.

Remark 2.2.2. For the special case $f(x)=x^p$, $p>1$, the inequalities (2.2.1)-(2.2.4) becomes

$$H_1^p(a,b,1,h,g) \geq C^{1-p}H_p^p(a,b,1,h,g), \tag{2.2.5}$$

$$H_1^p(a,b,1,h,g) \leq C^{p-1}H_p^p(a,b,1,h,g), \tag{2.2.6}$$

$$\widetilde{H}_1^p(a,b,1,h,g) \geq C^{1-p}\widetilde{H}_p^p(a,b,1,h,g), \tag{2.2.7}$$

and

$$\widetilde{H}_1^p(a,b,1,h,g) \leq C^{p-1}\widetilde{H}_p^p(a,b,1,h,g). \tag{2.2.8}$$

If the condition $p>1$ is replaced by $0<p<1$, then all inequalities (2.2.5)-(2.2.8) hold in reversed direction.

We consider the following functionals as follows:

(M_1) Under the assumptions of Theorem 2.2.1(a), we define linear functional as

$$\mathcal{L}_1(f) = f\Big(C\int_a^b h(x)dg(x)\Big) - C\Big(\int_a^b f'\big(h(x)g(x)\big)h(x)dg(x)\Big).$$

(M_2) Under the assumptions of Theorem 2.2.1(b), we define linear functional as

$$\mathcal{L}_2(f) = \frac{1}{C}\Big(\int_a^b f'\big(h(x)g(x)\big)h(x)dg(x)\Big) - f\Big(\frac{1}{C}\int_a^b h(x)dg(x)\Big).$$

(M_3) Under the assumptions of Theorem 2.2.1(c), we define linear functional as

$$\mathcal{L}_3(f) = f\bigg(C\int_a^b h(x)d[-g(x)]\bigg) - C\bigg(\int_a^b f'\big(h(x)g(x)\big)h(x)d[-g(x)]\bigg).$$

(M_4) Under the assumptions of Theorem 2.2.1(d), we define linear functional as

$$\mathcal{L}_4(f) = \frac{1}{C}\bigg(\int_a^b f'\big(h(x)g(x)\big)h(x)d[-g(x)]\bigg) - f\bigg(\frac{1}{C}\int_a^b h(x)d[-g(x)]\bigg).$$

We will consider the classical method from [2] (see also book [1] and references given in it) to prove the log-convexity of the functionals defined as above by considering a family of convex function defined in (2.1.15). Let us denote

$$R_l^n(K,h,g) = \int_a^b K(x)\big(\frac{1}{l} + \log\big(h(x)g(x)\big)\big)^n h^l(x)d\big[g^l(x)\big]$$

and

$$\widetilde{R}_l^n(K,h,g) = \int_a^b K(x)\big(\frac{1}{l} + \log\big(h(x)g(x)\big)\big)^n h^l(x)d\big[-g^l(x)\big].$$

Using functions defined in (2.1.15), we get

$$\mathcal{L}_1(f_p) = \begin{cases} \frac{C^pH_1^p(a,b,1,h,g) - CH_p^p(a,b,1,h,g)}{p(p-1)}, & p>0, p\neq 1 \\ CH_1^1(a,b,1,h,g)\log\big(CH_1^1(a,b,1,h,g)\big) - CR_1^1(1,h,g), & p=1. \end{cases} \tag{2.2.9}$$

$$\mathcal{L}_2(f_p) = \begin{cases} \frac{\frac{1}{C}H_p^p(a,b,1,h,g) - \frac{1}{C^p}H_1^p(a,b,1,h,g)}{p(p-1)}, & p>0, p\neq 1 \\ \frac{1}{C}R_1^1(1,h,g) - \frac{1}{C}H_1^1(a,b,1,h,g)\log\big(\frac{1}{C}H_1^1(a,b,1,h,g)\big), & p=1. \end{cases} \tag{2.2.10}$$

$$\mathcal{L}_3(f_p) = \begin{cases} \frac{C^p\widetilde{H}_1^p(a,b,1,h,g) - C\widetilde{H}_p^p(a,b,1,h,g)}{p(p-1)}, & p>0, p\neq 1 \\ C\widetilde{H}_1^1(a,b,1,h,g)\log\big(C\widetilde{H}_1^1(a,b,1,h,g)\big) - C\widetilde{R}_1^1(1,h,g), & p=1. \end{cases} \tag{2.2.11}$$

$$\mathcal{L}_4(f_p) = \begin{cases} \frac{\frac{1}{C}\widetilde{H}_p^p(a,b,1,h,g) - \frac{1}{C^p}\widetilde{H}_1^p(a,b,1,h,g)}{p(p-1)}, & p>0, p\neq 1 \\ \frac{1}{C}\widetilde{R}_1^1(1,h,g) - \frac{1}{C}\widetilde{H}_1^1(a,b,1,h,g)\log\big(\widetilde{H}_1^1(a,b,1,h,g)\big), & p=1. \end{cases} \tag{2.2.12}$$

We will prove the log-convexity and related results for functionals $\mathcal{L}_k$, $k = 1,\dots,4$.

Theorem 2.2.2. *Let linear functionals* $\mathcal{L}_k$, $k = 1,\dots,4$ *be defined as above and* $\mathcal{L}_k(f_p)$ *be positive. Then for* $k = 1,\dots,4$

(a) for all $p, q > 0$

$$\mathcal{L}_k^2(f_{\frac{p+q}{2}}) \leq \mathcal{L}_k(f_p)\mathcal{L}_k(f_q), \tag{2.2.13}$$

that is $p \mapsto \mathcal{L}_k(f_p)$ *is log-convex in the Jensen sense,*

(b) Also $p \mapsto \mathcal{L}_k(f_p)$ *is log-convex, that is, for* $p < q < r$ $(p, q, r \in \mathbb{R}^+)$

$$\big(\mathcal{L}_k(f_q)\big)^{r-p} \leq \big(\mathcal{L}_k(f_p)\big)^{r-q}\big(\mathcal{L}_k(f_r)\big)^{q-p}. \tag{2.2.14}$$

Proof. (a) Suppose that $k = 1, \ldots, 4$ is arbitrary.
We shall use the idea from $\big[$[2], Theorem 4$\big]$. Let us consider the function defined by

$$\lambda(x) = u^2 f_p(x) + 2uwf_r(x) + w^2 f_q(x)$$

where $r = \frac{p+q}{2}$, $u, w \in \mathbb{R}$. We have

$$\lambda''(x) = u^2x^{p-2} + 2uwx^{r-2} + w^2x^{q-2} = \big(ux^{\frac{p}{2}-1} + wx^{\frac{q}{2}-1}\big)^2 \geq 0,\ x > 0.$$

Therefore λ is convex for $x > 0$. Hence, $\mathcal{L}_k(\lambda) \geq 0$, that is

$$u^2\mathcal{L}_k(f_p) + 2uw\mathcal{L}_k(f_r) + w^2\mathcal{L}_k(f_q) \geq 0$$

therefore we get (2.2.13).
(b) Since $\mathcal{L}_k$ is continuous, so it is log-convex. Therefore (2.2.14) is valid too.
Since k was taken to be arbitrary, so the above results hold for all $k = 1, \ldots, 4$. □

Corollary 2.2.3. *If* $s > 0$, $p < q < r$ $(p, q, r \in \mathbb{R}^+)$ *and* $p, q, r \neq s$, *then the following inequalities hold.*

$$\left[\frac{C^qH_s^q(a,b,1,h,g) - C^sH_q^q(a,b,1,h,g)}{q(q-s)}\right]^{r-p} \leq \left[\frac{C^pH_s^p(a,b,1,h,g) - C^sH_p^p(a,b,1,h,g)}{p(p-s)}\right]^{r-q} \left[\frac{C^rH_s^r(a,b,1,h,g) - C^sH_r^r(a,b,1,h,g)}{r(r-s)}\right]^{q-p}, \tag{2.2.15}$$

$$\left[\frac{\frac{1}{C^s}H_q^q(a,b,1,h,g) - \frac{1}{C^q}H_s^q(a,b,1,h,g)}{q(q-s)}\right]^{r-p} \leq \left[\frac{\frac{1}{C^s}H_p^p(a,b,1,h,g) - \frac{1}{C^p}H_s^p(a,b,1,h,g)}{p(p-s)}\right]^{r-q} \left[\frac{\frac{1}{C^s}H_r^r(a,b,1,h,g) - \frac{1}{C^r}H_s^r(a,b,1,h,g)}{r(r-s)}\right]^{q-p}, \tag{2.2.16}$$

$$\left[\frac{C^q\widetilde{H}_s^q(a,b,1,h,g)-C^s\widetilde{H}_q^q(a,b,1,h,g)}{q(q-s)}\right]^{r-p}\leq\left[\frac{C^p\widetilde{H}_s^p(a,b,1,h,g)-C^s\widetilde{H}_p^p(a,b,1,h,g)}{p(p-s)}\right]^{r-q}$$
$$\left[\frac{C^r\widetilde{H}_s^r(a,b,1,h,g)-C^s\widetilde{H}_r^r(a,b,1,h,g)}{r(r-s)}\right]^{q-p}, \qquad (2.2.17)$$

$$\left[\frac{\frac{1}{C^s}\widetilde{H}_q^q(a,b,1,h,g)-\frac{1}{C^q}\widetilde{H}_s^q(a,b,1,h,g)}{q(q-s)}\right]^{r-p}\leq\left[\frac{\frac{1}{C^s}\widetilde{H}_p^p(a,b,1,h,g)-\frac{1}{C^p}\widetilde{H}_s^p(a,b,1,h,g)}{p(p-s)}\right]^{r-q}$$
$$\left[\frac{\frac{1}{C^s}\widetilde{H}_r^r(a,b,1,h,g)-\frac{1}{C^r}\widetilde{H}_s^r(a,b,1,h,g)}{r(r-s)}\right]^{q-p}. \quad (2.2.18)$$

Proof. For $k=1$, we have,

$$\mathcal{L}_1(f_p)=\frac{C^p\left(\int_a^b h(x)dg(x)\right)^p-C\left(\int_a^b h^p(x)d\big[g^p(x)\big]\right)}{p(p-1)}.$$

Since $s>0$, so $p/s<q/s<r/s$. Also for h is C-decreasing, h^s is C^s-decreasing. We make substitutions $h\to h^s,\ g\to g^s,\ C\to C^s,\ p\to p/s,\ q\to q/s,$ and $r\to r/s$ in (2.2.14). We get

$$\left[\frac{C^qH_s^q(a,b,1,h,g)-C^sH_q^q(a,b,1,h,g)}{\frac{q(q-s)}{s^2}}\right]^{\frac{r-p}{s}}\leq\left[\frac{C^pH_s^p(a,b,1,h,g)-C^sH_p^p(a,b,1,h,g)}{\frac{p(p-s)}{s^2}}\right]^{\frac{r-q}{s}}$$
$$\left[\frac{C^rH_s^r(a,b,1,h,g)-C^sH_r^r(a,b,1,h,g)}{\frac{r(r-s)}{s^2}}\right]^{\frac{q-p}{s}}.$$

After simplification, we get (2.2.15). Similarly for $k=2,3,4$, we get (2.2.16)-(2.2.18) respectively. □

Remark 2.2.3. From the inequalities (2.2.15)-(2.2.18) for $(q<s)$, we get refinement for inequalities obtained from Theorem (2.1.2) and reversion when $(q>s)$. Similarly, we can get such refinement and reversions in all other cases for p,s and r,s.

Corollary 2.2.4. *For* $s>0$, $p<q<r$ $(p,q,r\in\mathbb{R}^+)$ *and* $p,q,r\neq s$.

(a) If $h \in Q^{\alpha_1}(C)$, $\alpha > \alpha_1$, then for any $x > 0$ the following inequality holds

$$\left[\frac{C^q\big[s(\alpha-\alpha_1)\big]^{q/s}G_s^q(h,x) - C^s\big[q(\alpha-\alpha_1)\big]G_q^q(h,x)}{q(q-s)}\right]^{r-p} \leq \left[\frac{C^p\big[s(\alpha-\alpha_1)\big]^{p/s}G_s^p(h,x) - C^s\big[p(\alpha-\alpha_1)\big]G_p^p(h,x)}{p(p-s)}\right]^{r-q} \left[\frac{C^r\big[s(\alpha-\alpha_1)\big]^{r/s}G_s^r(h,x) - C^s\big[r(\alpha-\alpha_1)\big]G_r^r(h,x)}{r(r-s)}\right]^{q-p}. \quad (2.2.19)$$

(b) If $h \in Q_{\alpha_2}(C)$, $\alpha_2 > \alpha$, then for any $x \geq 0$ the following inequality holds

$$\left[\frac{C^q\big[s(\alpha_2-\alpha)\big]^{q/s}\widetilde{G}_s^q(h,x) - C^s\big[q(\alpha_2-\alpha)\big]\widetilde{G}_q^q(h,x)}{q(q-s)}\right]^{r-p} \leq \left[\frac{C^p\big[s(\alpha_2-\alpha)\big]^{p/s}\widetilde{G}_s^p(h,x) - C^s\big[p(\alpha_2-\alpha)\big]\widetilde{G}_p^p(h,x)}{p(p-s)}\right]^{r-q} \left[\frac{C^r\big[s(\alpha_2-\alpha)\big]^{r/s}\widetilde{G}_s^r(h,x) - C^s\big[r(\alpha_2-\alpha)\big]\widetilde{G}_r^r(h,x)}{r(r-s)}\right]^{q-p}. \quad (2.2.20)$$

Proof. (a) It is a simple consequence of Corollary 2.2.3. Since $h \in Q^{\alpha_1}(C)$, by making substitutions $h \to h(t)t^{-\alpha_1}$ and $g \to t^{(\alpha_1-\alpha)}$ in (2.2.17), we get (2.2.19).
(b) Since $h \in Q^{\alpha_2}(C)$, by making substitutions $h \to h(t)t^{-\alpha_2}$ and $g \to t^{(\alpha_2-\alpha)}$ in (2.2.15), we get (2.2.20). □

2.3 Reversed Hardy Inequality for C-Monotone Functions

In the present section, we prove the reversed Hardy type inequalities for a more general class of C-monotone functions by obtaining some new inequalities from the inequalities given by J. Pečarić, I. Perić and L. E. Persson in [20]. Moreover, by constructing some linear functionals and n-exponential convex functions related to the obtained inequalities, we give refinements of the reversed Hardy type inequalities. Also we will give mean value theorems and deduce Cauchy means for these functionals. Now, we state our main result, which is a generalization of the Theorem 2.2.1.

Theorem 2.3.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a convex and differentiable function such that $f(0) = 0$ and let $-\infty \leq a < b \leq \infty$. Furthermore, let $k : (a,b) \to [0,\infty)$ be a positive integrable function, $K_1(x) = \int_x^b k(t)dt$ and $K_2(x) = \int_a^x k(t)dt$.*

(a) If h is C-decreasing and g is increasing, differentiable such that $g(a+0)=0$, then

$$\int_a^b k(x) f\Big(C \int_a^x h(t) dg(t)\Big) dx \geq C \int_a^b K_1(x) f'\big(h(x)g(x)\big) h(x) dg(x). \quad (2.3.1)$$

(b) If h is C-increasing and g is increasing, differentiable such that $g(a+0)=0$, then

$$\int_a^b k(x) f\Big(\frac{1}{C} \int_a^x h(t) dg(t)\Big) dx \leq \frac{1}{C} \int_a^b K_1(x) f'\big(h(x)g(x)\big) h(x) dg(x). \quad (2.3.2)$$

(c) If h is C-increasing and g is decreasing, differentiable such that $g(b-0)=0$, then

$$\int_a^b k(x) f\Big(C \int_x^b h(t) d[-g(t)]\Big) dx \geq C \int_a^b K_2(x) f'\big(h(x)g(x)\big) h(x) d[-g(x)]. \quad (2.3.3)$$

(d) If h is C-decreasing and g is decreasing, differentiable such that $g(b-0)=0$, then

$$\int_a^b k(x) f\Big(\frac{1}{C} \int_x^b h(t) d[-g(t)]\Big) dx \leq \frac{1}{C} \int_a^b K_2(x) f'\big(h(x)g(x)\big) h(x) d[-g(x)]. \quad (2.3.4)$$

(e) If the condition "f is convex" is replaced by "f is concave," then all the inequalities (2.3.1)-(2.3.4) *hold in reversed direction.*

Proof. (a) Under the given conditions, we have the following inequality by Theorem 2.2.1,

$$f\Big(C \int_a^t h(x) dg(x)\Big) \geq C \int_a^t f'\big(h(x)g(x)\big) h(x) dg(x).$$

Multiplying the above inequality with a positive function k and integrating from a to b and applying Fubini theorem on the integral on the R.H.S, we get

$$\int_a^b k(t) f\Big(C \int_a^t h(x) dg(x)\Big) dt \geq C \int_a^b k(t) \int_a^t f'\big(h(x)g(x)\big) h(x) dg(x) dt.$$

Denote $L = f'(0)$. We have

$$\int_a^b \int_a^t k(t) f'\big(h(x)g(x)\big) h(x) dg(x) dt =$$
$$\int_a^b \int_a^t k(t)\Big(f'\big(h(x)g(x)\big) - L\Big) h(x) dg(x) dt + L \int_a^b \int_a^t k(t) h(x) dg(x) dt \tag{2.3.5}$$

Since the functions under the integrals on the R.H.S. are positive, we can apply Fubini theorem. Furthermore, since the second integral on the R.H.S. is finite we can change the order of integration in the integral on the L.H.S.
Therefore,

$$\int_a^b k(t) f\Big(C \int_a^t h(x) dg(x)\Big) dt \geq$$
$$C \int_a^b f'\big(h(x)g(x)\big) h(x) \Big(\int_x^b k(t) dt\Big) dg(x). \tag{2.3.6}$$

Since $K_1(x) = \int_x^b k(t)dt$, we get (2.3.1).
Similarly, we can prove inequality given in (b).

(c) Under the given conditions, we have the following inequality by Theorem 2.2.1,

$$f\Big(C \int_t^b h(x) d[-g(x)]\Big) \geq C \int_t^b f'\big(h(x)g(x)\big) h(x) d[-g(x)].$$

Multiplying the above inequality with a positive function k, integrating from a to b and changing the order of integration in the integral on the R.H.S., we get

$$\int_a^b k(t) f\Big(C \int_t^b h(x) d[-g(x)]\Big) dt \geq C \int_a^b k(t) \int_t^b f'\big(h(x)g(x)\big) h(x) d[-g(x)] dt$$
$$= C \int_a^b \int_a^x k(t) f'\big(h(x)g(x)\big) h(x) dt d[-g(x)2. \tag{2.3.7}$$

Since $K_2(x) = \int_a^x k(t)dt$, we get (2.3.3).
Similarly, we can prove inequality given in (d).

□

Remark 2.3.1. If the function f in Theorem 2.3.1 is monotone (i. e., f' is of the same sign everywhere), then we can apply Fubini theorem directly to the integral on the L.H.S. of (2.3.5). In that case we do not need integrability of the function k and differentiability of ϕ at 0.

Corollary 2.3.2. *Let* $-\infty \leq a < b \leq \infty$ *and* $p > 1$.

(a) *If* h *is* C*-decreasing and* g *is increasing, differentiable such that* $g(a+0) = 0$, *then*

$$\int_a^b k(x)\Big(\int_a^x h(t)d[g(t)]\Big)^p dx \geq C^{1-p} \int_a^b K_1(x)h^p(x)d[g^p(x)]. \tag{2.3.8}$$

(b) *If* h *is* C*-increasing and* g *is increasing, differentiable such that* $g(a+0) = 0$, *then*

$$\int_a^b k(x)\Big(\int_a^x h(t)d[g(t)]\Big)^p dx \leq C^{p-1} \int_a^b K_1(x)h^p(x)d[g^p(x)]. \tag{2.3.9}$$

(c) *If* h *is* C*-increasing and* g *is decreasing, differentiable such that* $g(b-0) = 0$, *then*

$$\int_a^b k(x)\Big(\int_x^b h(t)d[-g(t)]\Big)^p dx \geq C^{1-p} \int_a^b K_2(x)h^p(x)d[-g^p(x)]. \tag{2.3.10}$$

(d) *If* h *is* C*-decreasing and* g *is decreasing, differentiable such that* $g(b-0) = 0$, *then*

$$\int_a^b k(x)\Big(\int_x^b h(t)d[-g(t)]\Big)^p dx \leq C^{p-1} \int_a^b K_2(x)h^p(x)d[-g^p(x)]. \tag{2.3.11}$$

(e) *If the condition "* $p > 1$*"is replaced by "* $0 < p < 1$*", then all the inequalities* (2.3.8)-(2.3.11) *hold in reversed direction.*

Proof. Consider $f(x) = x^p$ in the Theorem 2.3.1 (a)-(d), we get (2.3.8)-(2.3.11). □

Corollary 2.3.3. *Let* $-\infty < a < b \leq \infty$ *and* $p > 1$.

(a) *If* h *is* C*-decreasing, then*

$$\int_a^b \frac{1}{(x-a)^p}\Big(\int_a^x h(t)dt\Big)^p dx \geq \frac{pC^{1-p}}{p-1} \int_a^b \big[1 - (\frac{x-a}{b-a})^{p-1}\big]h^p(x)dx. \tag{2.3.12}$$

(b) *If* h *is* C*-increasing, then*

$$\int_a^b \frac{1}{(x-a)^p}\Big(\int_a^x h(t)dt\Big)^p dx \leq \frac{pC^{p-1}}{(p-1)} \int_a^b \big[1 - (\frac{x-a}{b-a})^{p-1}\big]h^p(x)dx. \tag{2.3.13}$$

(e) *If the condition "* $p > 1$*"is replaced by "* $0 < p < 1$*", then the inequalities* (2.3.12) *and* (2.3.13) *hold in reversed direction.*

Proof. Consider $k(t) = (t-a)^{-p}, t \in (a,b)$, and $g(x) = x - a$ in (2.3.8) and (2.3.9). By considering Remark 2.3.1, we get (2.3.12) and (2.3.13) respectively. □

Remark 2.3.2. Consider $(a = 0, b = \infty)$, and $C = 1$ in (2.3.12), we get reversed Hardy (2.1.2).

Theorem 2.3.4. *Let* $p > 1$*, then*

(a) *If* $h \in Q^{\alpha_1}(C)$*,* $\alpha > \alpha_1$*, then for any* $x \geq 0$ *the following inequality holds*

$$\int_a^\infty k(x)\left(\int_x^\infty h(t)t^{-\alpha}\frac{dt}{t}\right)^p dx \geq p\big[C(\alpha-\alpha_1)\big]^{p-1}\int_a^\infty K_2(x)h^p(x)x^{-p\alpha}\frac{dx}{x}. \tag{2.3.14}$$

(b) *If* $h \in Q_{\alpha_2}(C)$*,* $\alpha_2 > \alpha$*, then for any* $x \geq 0$ *the following inequality holds*

$$\int_0^b k(x)\left(\int_0^x h(t)t^{-\alpha}\frac{dt}{t}\right)^p dx \geq p\big[C(\alpha_2-\alpha)\big]^{p-1}\int_0^b K_1(x)h^p(x)x^{-p\alpha}\frac{dx}{x}. \tag{2.3.15}$$

Proof. (a) Since $h \in Q^{\alpha_1}(C)$, by making substitutions $h \to h(t)t^{-\alpha_1}$ and taking $g(t) = t^{(\alpha_1-\alpha)}$ in (2.3.10), we get (2.3.14).

(b) Since $h \in Q^{\alpha_2}(C)$, by making substitutions $h \to h(t)t^{-\alpha_2}$ and taking $g(t) = t^{(\alpha_2-\alpha)}$ in (2.3.8), we get (2.3.15).

□

We also construct the linear functionals as differences of inequalities from Theorem 2.3.1 as:

(M_5) Under the assumptions of Theorem 2.3.1, we define linear functional as

$$\mathcal{L}_5(f) = \int_a^b k(x)f\left(C\int_a^x h(t)dg(t)\right)dx$$
$$-C\int_a^b K_1(x)f'\big(h(x)g(x)\big)h(x)dg(x).$$

(M_6) Under the assumptions of Theorem 2.3.1, we define linear functional as

$$\mathcal{L}_6(f) = \frac{1}{C}\int_a^b K_1(x) f'\big(h(x)g(x)\big)h(x)dg(x) - \int_a^b k(x) f\left(\frac{1}{C}\int_a^x h(t)dg(t)\right)dx.$$

(M_7) Under the assumptions of Theorem 2.3.1, we define linear functional as

$$\mathcal{L}_7(f) = \int_a^b k(x) f\left(C\int_x^b h(t)d[-g(t)]\right)dx - C\int_a^b K_2(x) f'\big(h(x)g(x)\big)h(x)d[-g(x)].$$

(M_8) Under the assumptions of Theorem 2.3.1, we define linear functional as

$$\mathcal{L}_8(f) = \frac{1}{C}\int_a^b K_2(x) f'\big(h(x)g(x)\big)h(x)d[-g(x)] - \int_a^b k(x) f\left(\frac{1}{C}\int_x^b h(t)d[-g(t)]\right)dx.$$

Remark 2.3.3. Under the assumptions of Theorem 2.2.1 and Theorem 2.3.1 with f as a convex function the linear functionals $\mathcal{L}_k(f) \geq 0$ for $k = 1, \ldots, 8$.

2.3.1 n-Exponential Convexity and Refinements of Functionals Involving C-Monotone Functions

A brief introduction and related results about n-exponentially and exponentially convex functions are mentioned in the introductory chapter. In this section, we investigate the properties of the functionals $\mathcal{L}_k(f)$ for $k = 1, \ldots, 8$ regarding n-exponential and exponential convexity. In order to obtain our main results regarding the exponential convexity, we need to define different families of functions. Let $I \subseteq [0, \infty)$ and $J \subseteq \mathbb{R}$ be intervals. For distinct points $u_0, u_1, u_2 \in I$, let $\mathbf{E}_i$, $i = 1, 2, 3$, denote a family of functions with the following property:

$\mathbf{E}_1 = \{f_t : I \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1, u_2; f_t]$ is n-exponentially convex in the Jensen sense on $J\}$.

$\mathbf{E}_2 = \{f_t : I \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1, u_2; f_t]$ is exponentially convex in the Jensen sense on $J\}$.

$\mathbf{E}_3 = \{f_t : I \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1, u_2; f_t]$ is 2-exponentially convex in the Jensen sense on $J\}$.

Theorem 2.3.5. *Let $\mathcal{L}_k$ be the linear functionals defined as in (M_k) for $k = 1, \ldots, 8$ associated with a family $\mathbf{E}_1$. Then $t \mapsto \mathcal{L}_k(f_t)$ is an n-exponentially convex function in the Jensen sense on J. If the function $t \mapsto \mathcal{L}_k(f_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. We prove n-exponential convexity in the Jensen sense of the function $t \mapsto \mathcal{L}_k(f_t)$, for $k = 1, \ldots, 8$. To do this, consider the families of functions defined in $\mathbf{E}_1$, for $\xi_i \in \mathbb{R}$, $i = 1, \ldots, n$, and $t_i \in J$, $i = 1, \ldots, n$, define the function

$$\psi(u) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{t_i+t_j}{2}}(u).$$

We have

$$[u_0, u_1, u_2; \psi] = \sum_{i,j=1}^{n} \xi_i \xi_j \left[u_0, u_1, u_2; f_{\frac{t_i+t_j}{2}}\right].$$

Since the function $t \mapsto [u_0, u_1, u_2; f_t]$ is n-exponentially convex in the Jensen sense on J, the right-hand side of the above expression is non-negative which implies that $\psi(u)$ is convex on I (see Remark 1.2.1).
Hence, taking into account assumption M_k's with Remark 2.3.3, we have

$$\mathcal{L}_k(\psi) \geq 0, \text{ for } k = 1, \ldots, 8$$

that is,

$$\sum_{i,j=1}^{n} \xi_i \xi_j \mathcal{L}_k \left(f_{\frac{t_i+t_j}{2}}\right) \geq 0.$$

Therefore, we conclude that the functions $t \mapsto \mathcal{L}_k(f_t)$, $k = 1, \ldots, 8$ are n-exponentially convex in the Jensen sense on J .
If the function $t \mapsto \mathcal{L}_k(f_t)$ is also continuous on J then $t \mapsto \mathcal{L}_k(f_t)$ is n-exponentially convex by definition for $k = 1, \ldots, 8$. □

The following corollary is an immediate consequence of the above theorem.

Corollary 2.3.6. *Let $\mathcal{L}_k$ be the linear functionals defined as in (M_k) for $k = 1, \ldots, 8$ associated with a family $\mathbf{E}_2$. Then $t \mapsto \mathcal{L}_k(f_t)$ is an exponentially convex function in the Jensen sense on J. If $t \mapsto \mathcal{L}_k(f_t)$ is continuous on J, then it is exponentially convex on J.*

Proof. Follows from the previous theorem. □

Corollary 2.3.7. *Let $\mathcal{L}_k$ be the linear functionals defined as in (M_k) for $k = 1, \ldots, 8$ associated with a family $\mathbf{E}_3$. Then the following statements hold:*

(i) If the function $t \mapsto \mathcal{L}_k(f_t)$ is strictly positive and continuous on J, then it is 2-exponentially convex on J, and thus, log-convex. Also for $r, s, t \in J$ such that $r < s < t$, we have

$$(\mathcal{L}_k(f_s))^{t-r} \leq (\mathcal{L}_k(f_r))^{t-s}(\mathcal{L}_k(f_t))^{s-r}. \tag{2.3.16}$$

(ii) If the function $t \mapsto \mathcal{L}_k(f_t)$ is strictly positive and differentiable on J, then for all $t, r, u, v \in J$ such that $t \leq u$, $r \leq v$, we have

$$\mathfrak{B}(t, r; \mathcal{L}_k, \mathbf{E}_3) \leq \mathfrak{B}(u, v; \mathcal{L}_k, \mathbf{E}_3), \;\; k = 1, \ldots, 8,$$

where

$$\mathfrak{B}(t, r; \mathcal{L}_k, \mathbf{E}_3) = \begin{cases} \left(\frac{\mathcal{L}_k(f_t)}{\mathcal{L}_k(f_r)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{\frac{d}{d_t}(\mathcal{L}_k(f_t))}{\mathcal{L}_k(f_t)}\right), & t = r. \end{cases} \tag{2.3.17}$$

Proof. (i) This is an immediate consequence of Theorem 2.3.5 and Remark 1.3.2. Since $t \mapsto \mathcal{L}_k(f_t)$ is strictly positive, so for $r, s, t \in J$ such that $r < s < t$ with $h(t) = \log \mathcal{L}_k(f_t)$ in Lemma 1.2.1 gives

$$(t - s) \log \mathcal{L}_k(f_r) + (r - t) \log \mathcal{L}_k(f_s) + (s - r) \log \mathcal{L}_k(f_t) \geq 0.$$

This is equivalent to inequality (2.3.16).

(ii) By (i), the function $t \mapsto L_k(f_t)$ is log-convex on J, which means that the function $t \mapsto \log L_k(f_t)$ is convex on J. Hence, by using Lemma 1.2.2 with $t \leq u$, $r \leq v$, $t \neq r$, $u \neq v$, we obtain

$$\frac{\log \mathcal{L}_k(f_t) - \log \mathcal{L}_k(f_r)}{t - r} \leq \frac{\log \mathcal{L}_k(f_u)) - \log\log \mathcal{L}_k(f_v)}{u - v},$$

that is,

$$\mathfrak{B}(t, r; \mathcal{L}_k, \mathbf{E}_3) \leq \mathfrak{B}(u, v; \mathcal{L}_k, \mathbf{E}_3).$$

Finally, if $t = r \leq u$, by taking the limit $\lim_{r \longrightarrow t}$, we have

$$\mathfrak{B}(t, t; \mathcal{L}_k, \mathbf{E}_3) \leq \mathfrak{B}(u, v; \mathcal{L}_k, \mathbf{E}_3).$$

Other possible cases are treated similarly. □

Remark 2.3.4. The results given in Theorem 2.3.5, Corollary 2.3.6 and Corollary 2.3.7 hold when two of the points $u_0, u_1, u_2 \in I$ coincide, that is to say when families $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ are replaced, respectively, with a family $(f_t)_{t\in J}$ of differentiable functions f_t such that for every $u_0 \neq u_1$ the function $t \mapsto [u_0, u_0, u_1; f_t]$ is, respectively, n-exponentially convex in the Jensen sense, exponentially convex in the Jensen sense and 2-exponentially convex in the Jensen sense. Moreover, the above results also hold when all three points coincide, i.e. for a family of twice differentiable functions f_t such that the mapping $t \mapsto [u_0, u_0, u_0; f_t]$ satisfies analogous properties. These results can be proved easily as before by using the extension of the divided differences to the case when some or all of the points u_0, u_1, u_2 are equal.

Since we have proved the log-convexity of the functionals $\mathcal{L}_k$, $k = 1, \ldots, 8$. As we are interested to give the refinements of reversed Hardy inequality for C-monotone functions, we consider the family of functions given in (2.1.15) and apply it to the functionals $\mathcal{L}_k$, $k = 5, \ldots, 8$.

Corollary 2.3.8. *If $p < q < r$ $(p, q, r \in \mathbb{R}^+\backslash\{1\})$, then the following inequalities hold.*

$$\left[\frac{\int_a^b k(x)\left(C\int_a^x h(t)dg(t)\right)^q dx - C\int_a^b K_1(x)h^q(x)dg^q(x)}{q(q-1)}\right]^{r-p} \tag{2.3.18}$$

$$\leq \left[\frac{\int_a^b k(x)\left(C\int_a^x h(t)dg(t)\right)^p dx - C\int_a^b K_1(x)h^p(x)dg^p(x)}{p(p-1)}\right]^{r-q}$$

$$\left[\frac{\int_a^b k(x)\left(C\int_a^x h(t)dg(t)\right)^r dx - C\int_a^b K_1(x)h^r(x)dg^r(x)}{r(r-1)}\right]^{q-p},$$

$$\left[\frac{\frac{1}{C}\int_a^b K_1(x)h^q(x)dg^q(x) - \frac{1}{C^q}\int_a^b k(x)\left(C\int_a^x h(t)dg(t)\right)^q dx}{q(q-1)}\right]^{r-p} \tag{2.3.19}$$

$$\leq \left[\frac{\frac{1}{C}\int_a^b K_1(x)h^p(x)dg^p(x) - \frac{1}{C^p}\int_a^b k(x)\left(C\int_a^x h(t)dg(t)\right)^p dx}{p(p-1)}\right]^{r-q}$$

$$\left[\frac{\frac{1}{C}\int_a^b K_1(x)h^r(x)dg^r(x) - \frac{1}{C^r}\int_a^b k(x)\left(C\int_a^x h(t)dg(t)\right)^r dx}{r(r-1)}\right]^{q-p},$$

$$\left[\frac{\int_a^b k(x)\left(C\int_x^b h(t)d[-g(t)]\right)^q dx - C\int_a^b K_2(x)h^q(x)d[-g^q(x)]}{q(q-1)}\right]^{r-p} \tag{2.3.20}$$

$$\leq \left[\frac{\int_a^b k(x)\left(C\int_x^b h(t)d[-g(t)]\right)^p dx - C\int_a^b K_2(x)h^p(x)d[-g^p(x)]}{p(p-1)}\right]^{r-q}$$

$$\left[\frac{\int_a^b k(x)\left(C\int_x^b xh(t)d[-g(t)]\right)^r dx - C\int_a^b K_2(x)h^r(x)d[-g^r(x)]}{r(r-1)}\right]^{q-p},$$

$$\left[\frac{\frac{1}{C}\int_a^b K_2(x)h^q(x)d[-g^q(x)] - \frac{1}{C^q}\int_a^b k(x)\left(C\int_x^b h(t)d[-g(t)]\right)^q dx}{q(q-1)}\right]^{r-p} \tag{2.3.21}$$

$$\leq \left[\frac{\frac{1}{C}\int_a^b K_2(x)h^p(x)d[-g^p(x)] - \frac{1}{C^p}\int_a^b k(x)\left(C\int_x^b h(t)d[-g(t)]\right)^p dx}{p(p-1)}\right]^{r-q}$$

$$\left[\frac{\frac{1}{C}\int_a^b K_2(x)h^r(x)d[-g^r(x)] - \frac{1}{C^r}\int_a^b k(x)\left(C\int_a^x h(t)d[-g(t)]\right)^r dx}{r(r-1)}\right]^{q-p},$$

Proof. By Corollary 2.3.7(i) the functions $t \mapsto \mathcal{L}_k(f_t), k = 5, \ldots, 8$ are log-convex. Hence, for $p < q < r$ putting family of functions f_p in $(\mathcal{L}_5)$-$(\mathcal{L}_8)$ in (2.3.16), we get (2.3.18)-(2.3.21) respectively. □

Corollary 2.3.9. *For $p < q < r$ $(p, q, r \in \mathbb{R}^+\backslash\{1\})$, the following inequalities hold.*

(a) If $h \in Q^{\alpha_1}(C)$, $\alpha > \alpha_1$, then for any $x \geq 0$ the following inequality holds

$$\left[\frac{\left[C(\alpha-\alpha_1)\right]^q \int_a^\infty k(x)\left(\int_x^\infty h(t)t^{-\alpha}\frac{dt}{t}\right)^q dx - qC(\alpha-\alpha_1)\int_a^\infty K_2(x)h^q(x)x^{-q\alpha}\frac{dx}{x}}{q(q-1)}\right]^{r-p} \tag{2.3.22}$$

$$\leq \left[\frac{\left[C(\alpha-\alpha_1)\right]^p \int_a^\infty k(x)\left(\int_x^\infty h(t)t^{-\alpha}\frac{dt}{t}\right)^p dx - pC(\alpha-\alpha_1)\int_a^\infty K_2(x)h^p(x)x^{-p\alpha}\frac{dx}{x}}{p(p-1)}\right]^{r-q}$$

$$\left[\frac{\left[C(\alpha-\alpha_1)\right]^r \int_a^\infty k(x)\left(\int_x^\infty h(t)t^{-\alpha}\frac{dt}{t}\right)^r dx - rC(\alpha-\alpha_1)\int_a^\infty K_2(x)h^r(x)x^{-r\alpha}\frac{dx}{x}}{r(r-1)}\right]^{q-p},$$

(b) If $h \in Q_{\alpha_2}(C)$, $\alpha_2 > \alpha$, then for any $x \geq 0$ the following inequality holds

$$\left[\frac{\big[C(\alpha_2-\alpha)\big]^q \int_0^b k(x)\Big(\int_0^x h(t)t^{-\alpha}\frac{dt}{t}\Big)^q dx - qC(\alpha_2-\alpha)\int_0^b K_1(x)h^q(x)x^{-q\alpha}\frac{dx}{x}}{q(q-1)}\right]^{r-p} \tag{2.3.23}$$
$$\leq \left[\frac{\big[C(\alpha_2-\alpha)\big]^p \int_0^b k(x)\Big(\int_0^x h(t)t^{-\alpha}\frac{dt}{t}\Big)^p dx - pC(\alpha_2-\alpha)\int_0^b K_1(x)h^p(x)x^{-p\alpha}\frac{dx}{x}}{p(p-1)}\right]^{r-q}$$
$$\left[\frac{\big[C(\alpha_2-\alpha)\big]^r \int_0^b k(x)\Big(\int_0^x h(t)t^{-\alpha}\frac{dt}{t}\Big)^r dx - rC(\alpha_2-\alpha)\int_0^b K_1(x)h^r(x)x^{-r\alpha}\frac{dx}{x}}{r(r-1)}\right]^{q-p},$$

Proof. (a) It follows from Corollary 2.3.8 by making substitution $h \to h(t)t^{-\alpha_1}$ and taking $g(t) = t^{(\alpha_1-\alpha)}$ in (2.3.20).
(b) By making substitution $h \to h(t)t^{-\alpha_2}$ and taking $g(t) = t^{(\alpha_2-\alpha)}$ in (2.3.18), we get (2.3.23). □

2.3.2 Mean-Value Theorems of Functionals Involving C-Monotone Functions

Now we state and prove mean-value theorem of Lagrange and Cauchy type for the linear functionals $\mathcal{L}_k,\ k = 1, \dots, 8$ defined by $(M_1) - (M_8)$. Also, assume that $\mathcal{L}_k(f)$ are well defined for $f \in C^2[0, c]$

Theorem 2.3.10. *Let $\mathcal{L}_k,\ k = 1, \dots, 8$ be linear functionals defined by $(M_1) - (M_8)$ and $f \in C^2[0, c]$, $c > 0$, such that $f(0) = 0$. Then there exists $\xi_k \in [0, c]$ such that the identity*

$$\mathcal{L}_k(f) = \frac{f''(\xi_k)}{2}\mathcal{L}_k(x^2) \tag{2.3.24}$$

holds for $k = 1, \dots, 8$.

Proof. Fix $k = 1, \dots, 8$.
Since f'' is continuous on $[0, c]$, it attains its maximum and minimum value on $[0, c]$. Let

$$m = \min_{x\in[0,c]}\{f''(x)\} \qquad \text{and} \qquad M = \max_{x\in[0,c]}\{f''(x)\}.$$

Let us consider functions $F_1, F_2 : [0,c] \to \mathbb{R}$ defined by

$$F_1(x) = M\frac{x^2}{2} - f(x) \quad \text{and} \quad F_2(x) = f(x) - m\frac{x^2}{2}.$$

Clearly

$$F_1''(x) = M - f''(x) \geq 0,$$

and

$$F_2''(x) = f''(x) - m \geq 0,$$

so F_1, F_2 are convex functions. Also, $F_1(0) = 0 = F_2(0)$. Hence, from Theorem 2.2.1 and 2.3.1 for F_1 and F_2 respectively, it follows

$$\mathcal{L}_k(f) \leq \frac{M}{2}\mathcal{L}_k(x^2) \tag{2.3.25}$$

and

$$\mathcal{L}_k(f) \geq \frac{m}{2}\mathcal{L}_k(x^2). \tag{2.3.26}$$

Combining (2.3.25) and (2.3.26), we get

$$\frac{m}{2}\mathcal{L}_k(x^2) \leq \mathcal{L}_k(f) \leq \frac{M}{2}\mathcal{L}_k(x^2).$$

If $\mathcal{L}_k(x^2) = 0$ then $\mathcal{L}_k(f) = 0$ and (2.3.24) holds for all $\xi_k \in [0,c]$. Otherwise

$$m \leq \frac{2\mathcal{L}_k(f)}{\mathcal{L}_k(x^2)} \leq M.$$

Since f'' is continuous, there exists $\xi_k \in [0,c]$ such that (2.3.24) holds and the proof is complete. □

Theorem 2.3.11. *Let $\mathcal{L}_k,\ k = 1,\ldots,8$ be linear functionals defined by $(M_1) - (M_8)$ and $f_1, f_2 \in C^2[0,c]$, $c > 0$, such that $f_1(0) = 0 = f_2(0)$. Then there exists $\xi_k \in [0,c]$ such that the equality*

$$\frac{\mathcal{L}_k(f_1)}{\mathcal{L}_k(f_2)} = \frac{f_1''(\xi_k)}{f_2''(\xi_k)} \tag{2.3.27}$$

holds for $k = 1,\ldots,8$, provided that the denominators are non zero.

Proof. Fix $k = 1, \dots, 8$ and define $L \in C^2[0, c]$ in the way that

$$L = c_1 f_1 - c_2 f_2$$

where c_1 and c_2 are defined by $c_1 = \mathcal{L}_k(f_2)$ and $c_2 = \mathcal{L}_k(f_1)$. Now from Theorem 2.3.10 for the function L, it follows

$$\left(c_1 \frac{f_1''(\xi_k)}{2} - c_2 \frac{f_2''(\xi_k)}{2} \right) \mathcal{L}_k(x^2) = 0. \tag{2.3.28}$$

Since for (2.3.27) the denominators are nonzero, so we have $\mathcal{L}_k(x^2) \neq 0$ (because if it is zero, then $\mathcal{L}_k(f_2) = 0$ by Theorem 2.3.10), therefore (2.3.28) gives (2.3.27). □

Corollary 2.3.12. *Let $\mathcal{L}_k,\ k = 1, \dots, 8$ be linear functionals defined by $(M_1) - (M_8)$, for distinct positive real numbers l and r different from 1, there exists $\xi_k \in [0, c]$ such that*

$$\xi_k^{l-r} = \frac{r(r-1)\mathcal{L}_k(x^l)}{l(l-1)\mathcal{L}_k(x^r)} \tag{2.3.29}$$

holds for $k = 1, \dots, 8$.

Proof. Taking $f_1(x) = x^l$ and $f_2(x) = x^r$ in (2.3.27), for distinct positive real numbers l and r different from 1 we obtain (2.3.29). □

Remark 2.3.5. Since for fix $k = 1, \dots, 8$ the function $\xi_k \to \xi_k^{l-r}$, $l \neq r$ is invertible, then from (2.3.29) we get

$$m \leq \left(\frac{r(r-1)\mathcal{L}_k(x^l)}{l(l-1)\mathcal{L}_k(x^r)} \right)^{\frac{1}{l-r}} \leq M; \qquad r \neq l,\ r, l \neq 1.$$

2.3.3 Cauchy Means of Functionals Involving C-Monotone Functions

In this section, we deduce Cauchy means from Theorem 2.3.11. Suppose that f_1''/f_2'' has inverse. Then (2.3.27) gives

$$\xi_k = \left(\frac{f_1''}{f_2''}\right)^{-1} \left(\frac{\mathcal{L}_k(f_1)}{\mathcal{L}_k(f_2)} \right). \tag{2.3.30}$$

We conclude that the expression on the R. H. S. of the above equation is also a mean. For $r, l \in \mathbb{R}^+$, we define Cauchy means

$$M_{l,r}^k = \left(\frac{r(r-1)\mathcal{L}_k(x^l)}{l(l-1)\mathcal{L}_k(x^r)} \right)^{\frac{1}{l-r}},\ r \neq l,\ r, l \neq 1. \tag{2.3.31}$$

For instance, we give Cauchy Means explicitly in two and three parameters for $k = 1, \ldots, 4$. Also, we have continuous extensions of these means in other cases. Therefore by taking respective limits, we have the following

$$M^1_{r,r} = \begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{C^r H^r_1(a,b,1,h,g)\log\left(CH^1_1(a,b,1,h,g)\right) - CR^1_r(1,h,g)}{\left(C^r H^r_1(a,b,1,h,g) - CH^r_r(a,b,1,h,g)\right)}\right), & r \neq 1, \\ \exp\left(-1 + \frac{CH_1(a,b,1,h,g)\left(\log\left(CH^1_1(a,b,1,h,g)\right)\right)^2 + CH^1_1(a,b,1,h,g) - CR^2_1(1,h,g)}{2\left(CH^1_1(a,b,1,h,g)\log\left(CH^1_1(a,b,1,h,g)\right) - CR^1_1(1,h,g)\right)}\right), & r = 1. \end{cases} \tag{2.3.32}$$

$$M^2_{r,r} = \begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{\frac{1}{C}KR^1_r(1,h,g) - \frac{1}{C^r}H^r_1(a,b,1,h,g)\log\left(\frac{1}{C}H^1_1(a,b,1,h,g)\right)}{\left(\frac{1}{C}H^r_r(a,b,1,h,g) - \frac{1}{C^r}H^r_1(a,b,1,h,g)\right)}\right), & r \neq 1, \\ \exp\left(-1 + \frac{-\frac{1}{C}H^1_1(a,b,1,h,g) + \frac{1}{C}R^2_1(1,h,g) - \frac{1}{C}H^1_1(a,b,1,h,g)\left(\log\left(\frac{1}{C}H^1_1(a,b,1,h,g)\right)\right)^2}{2\left(\frac{1}{C}R^1_1(1,h,g) - \frac{1}{C}H^1_1(a,b,1,h,g)\log\left(\frac{1}{C}H^1_1(a,b,1,h,g)\right)\right)}\right), & r = 1. \end{cases} \tag{2.3.33}$$

$$M^3_{r,r} = \begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{C^r \widetilde{H}^r_1(a,b,1,h,g)\log\left(C\widetilde{H}^1_1(a,b,1,h,g)\right) - C\widetilde{R}^1_r(1,h,g)}{\left(C^r \widetilde{H}^r_1(a,b,1,h,g) - C\widetilde{H}^r_r(a,b,1,h,g)\right)}\right), & r \neq 1, \\ \exp\left(-1 + \frac{C\widetilde{H}_1(a,b,1,h,g)\left(\log\left(C\widetilde{H}^1_1(a,b,1,h,g)\right)\right)^2 + C\widetilde{H}^1_1(a,b,1,h,g) - C\widetilde{R}^2_1(1,h,g)}{2\left(C\widetilde{H}^1_1(a,b,1,h,g)\log\left(C\widetilde{H}^1_1(a,b,1,h,g)\right) - C\widetilde{R}^1_1(1,h,g)\right)}\right), & r = 1. \end{cases} \tag{2.3.34}$$

$$M^4_{r,r} = \begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{\frac{1}{C}\widetilde{R}^1_r(1,h,g) - \frac{1}{C^r}\widetilde{H}^r_1(a,b,1,h,g)\log\left(\frac{1}{C}\widetilde{H}^1_1(a,b,1,h,g)\right)}{\left(\frac{1}{C}\widetilde{H}^r_r(a,b,1,h,g) - \frac{1}{C^r}\widetilde{H}^r_1(a,b,1,h,g)\right)}\right), & r \neq 1, \\ \exp\left(-1 + \frac{-\frac{1}{C}\widetilde{H}^1_1(a,b,1,h,g) + \frac{1}{C}\widetilde{R}^2_1(1,h,g) - \frac{1}{C}\widetilde{H}^1_1(a,b,1,h,g)\left(\log\left(\frac{1}{C}\widetilde{H}^1_1(a,b,1,h,g)\right)\right)^2}{2\left(\frac{1}{C}\widetilde{R}^1_1(1,h,g) - \frac{1}{C}\widetilde{H}^1_1(a,b,1,h,g)\log\left(\frac{1}{C}\widetilde{H}^1_1(a,b,1,h,g)\right)\right)}\right), & r = 1. \end{cases} \tag{2.3.35}$$

Corollary 2.3.13. *For distinct positive real numbers l, r and s, there exist $\xi_k \in [0, c]$, $k = 1, \ldots, 4$ such that the following identities*

$$\xi_1^{l-r} = \frac{r(r-s)\Big(C^l H^l_s(a,b,1,h,g) - C^s H^l_l(a,b,1,h,g)\Big)}{l(l-s)\Big(C^r H^r_s(a,b,1,h,g) - C^s H^r_r(a,b,1,h,g)\Big)}, \tag{2.3.36}$$

$$\xi_2^{l-r} = \frac{r(r-s)\Big(\frac{1}{C^s}H^l_l(a,b,1,h,g) - \frac{1}{C^l}H^l_s(a,b,1,h,g)\Big)}{l(l-s)\Big(\frac{1}{C^s}H^r_r(a,b,1,h,g) - \frac{1}{C^r}H^r_s(a,b,1,h,g)\Big)}, \tag{2.3.37}$$

$$\xi_3^{l-r} = \frac{r(r-s)\Big(C^l \widetilde{H}_s^l(a,b,1,h,g) - C^s \widetilde{H}_l^l(a,b,1,h,g)\Big)}{l(l-s)\Big(C^r \widetilde{H}_s^r(a,b,1,h,g) - C^s \widetilde{H}_r^r(a,b,1,h,g)\Big)}, \tag{2.3.38}$$

$$\xi_4^{l-r} = \frac{r(r-s)\Big(\frac{1}{C^s} \widetilde{H}_l^l(a,b,1,h,g) - \frac{1}{C^l} \widetilde{H}_s^l(a,b,1,h,g)\Big)}{l(l-s)\Big(\frac{1}{C^s} \widetilde{H}_r^r(a,b,1,h,g) - \frac{1}{C^r} \widetilde{H}_s^r(a,b,1,h,g)\Big)}, \tag{2.3.39}$$

hold.

Proof. For $k=1$, making substitutions $h \to h^s,\ g \to g^s,\ C \to C^s,\ f_1(x) = x^{l/s},$ and $f_2(x) = x^{r/s}$ in (2.3.27), we get (2.3.36).
Similarly for $k = 2, 3, 4$, making substitutions as above in (2.3.27), we get (2.3.37), (2.3.38) and (2.3.39) respectively.

□

Remark 2.3.6. Since the function $\xi_k \to \xi_k^{l-r}$ is invertible for all $k = 1, \ldots, 4$, from (2.3.36)-(2.3.39), we can again formulate the corresponding Cauchy means for distinct positive real numbers l, r and s.

They are given as follows:

$$M^1_{l,r,s} = \begin{cases} \left(\dfrac{r(r-s)\Big(C^l H_s^l(a,b,1,h,g) - C^s H_l^l(a,b,1,h,g)\Big)}{l(l-s)\Big(C^r H_s^r(a,b,1,h,g) - C^s H_r^r(a,b,1,h,g)\Big)}\right)^{\frac{1}{l-r}}, & l \neq r \neq s, \\ \left(\dfrac{s\Big(C^l H_s^l(a,b,1,h,g) - C^s H_l^l(a,b,1,h,g)\Big)}{l(l-s)\Big(C^s H_s^s(a,b,1,h,g) \log\big(C^s H_s^s(a,b,1,h,g)\big) - sC^s R_s^1(1,h,g)\Big)}\right)^{\frac{1}{l-s}}, & l \neq r = s, \\ \exp\left(\dfrac{s-2r}{r(r-s)} + \dfrac{\big(C^s H_s^r(a,b,1,h,g)\big)^{r/s} \log\big(C^s H_s^s(a,b,1,h,g)\big) - sC^s R_r^1(1,h,g)}{s\Big(\big(C^s H_s^r(a,b,1,h,g)\big)^{r/s} - C^s H_r^r(a,b,1,h,g)\Big)}\right), & l = r \neq s, \\ \exp\left(\dfrac{-1}{s} + \dfrac{C^s H_s^s(a,b,1,h,g)\Big(\log\big(C^s H_s^s(a,b,1,h,g)\big)\Big)^2 + C^s H_s^s(a,b,1,h,g) - s^2 C^s R_s^2(1,h,g)}{2s\Big(C^s H_s^s(a,b,1,h,g) \log\big(C^s H_s^s(a,b,1,h,g)\big) - sC^s R_s^1(1,h,g)\Big)}\right), & l = r = s. \end{cases} \tag{2.3.40}$$

$$M^2_{l,r,s} = \begin{cases} \left(\dfrac{r(r-s)\left(\frac{1}{C^s} H^l_l(a,b,1,h,g) - \frac{1}{C^l} H^l_s(a,b,1,h,g)\right)}{l(l-s)\left(\frac{1}{C^s} H^r_r(a,b,1,h,g) - \frac{1}{C^r} H^r_s(a,b,1,h,g)\right)} \right)^{\frac{1}{l-r}}, & l \neq r \neq s, \\ \left(\dfrac{s\left(\frac{1}{C^s} H^l_l(a,b,1,h,g) - \frac{1}{C^l} H^l_s(a,b,1,h,g)\right)}{l(l-s)\left(\frac{s}{C^s} R^1_s(1,h,g) - \left(\frac{1}{C^s} H^s_s(a,b,1,h,g)\right)\log\left(\frac{1}{C^s} H^s_s(a,b,1,h,g)\right)\right)} \right)^{\frac{1}{l-s}}, & l \neq r = s, \\ \exp\left(\dfrac{s-2r}{r(r-s)} + \dfrac{\frac{s}{C^s} R^1_r(1,h,g) - \left(\frac{1}{C^s} H^r_s(a,b,1,h,g)\right)^{r/s} \log\left(\frac{1}{C^s} H^s_s(a,b,1,h,g)\right)}{s\left(\left(\frac{1}{C^s} H^r_s(a,b,1,h,g)\right)^{r/s} - C^s H^r_r(a,b,1,h,g)\right)} \right), & l = r \neq s, \\ \exp\left(\dfrac{-1}{s} + \dfrac{-\frac{1}{C^s} H^s_s(a,b,1,h,g) + \frac{s^2}{C^s} R^2_s(1,h,g) - \frac{1}{C^s} H^s_s(a,b,1,h,g)\left(\log\left(\frac{1}{C^s} H^s_s(a,b,1,h,g)\right)\right)^2}{2s\left(\frac{s}{C^s} R^1_s(1,h,g) - \left(\frac{1}{C^s} H^s_s(a,b,1,h,g)\right)\log\left(\frac{1}{C^s} H^s_s(a,b,1,h,g)\right)\right)} \right), & l = r = s. \end{cases} \tag{2.3.41}$$

$$M^3_{l,r,s} = \begin{cases} \left(\dfrac{r(r-s)\left(C^l \widetilde{H}^l_s(a,b,1,h,g) - C^s \widetilde{H}^l_l(a,b,1,h,g)\right)}{l(l-s)\left(C^r \widetilde{H}^r_s(a,b,1,h,g) - C^s \widetilde{H}^r_r(a,b,1,h,g)\right)} \right)^{\frac{1}{l-r}}, & r \neq l \neq s\,, \\ \left(\dfrac{s\left(C^l \widetilde{H}^l_s(a,b,1,h,g) - C^s \widetilde{H}^l_l(a,b,1,h,g)\right)}{l(l-s)\left(C^s \widetilde{H}^s_s(a,b,1,h,g)\log\left(C^s \widetilde{H}^s_s(a,b,1,h,g)\right) - sC^s \widetilde{R}^1_s(1,h,g)\right)} \right)^{\frac{1}{l-s}}, & l \neq r = s, \\ \exp\left(\dfrac{s-2r}{r(r-s)} + \dfrac{\left(C^s \widetilde{H}^r_s(a,b,1,h,g)\right)^{r/s} \log\left(C^s \widetilde{H}^s_s(a,b,1,h,g)\right) - sC^s \widetilde{R}^1_r(1,h,g)}{s\left(\left(C^s \widetilde{H}^r_s(a,b,1,h,g)\right)^{r/s} - C^s \widetilde{H}^r_r(a,b,1,h,g)\right)} \right), & l = r \neq s, \\ \exp\left(\dfrac{-1}{s} + \dfrac{C^s \widetilde{H}^s_s(a,b,1,h,g)\left(\log\left(C^s \widetilde{H}^s_s(a,b,1,h,g)\right)\right)^2 + C^s \widetilde{H}^s_s(a,b,1,h,g) - s^2 C^s \widetilde{R}^2_s(1,h,g)}{2s\left(C^s \widetilde{H}^s_s(a,b,1,h,g)\log\left(C^s \widetilde{H}^s_s(a,b,1,h,g)\right) - sC^s \widetilde{R}^1_s(1,h,g)\right)} \right), & l = r = s. \end{cases} \tag{2.3.42}$$

$$M^4_{l,r,s} = \begin{cases} \left(\dfrac{r(r-s)\left(\frac{1}{C^s}\widetilde{H}^l_l(a,b,1,h,g) - \frac{1}{C^l}\widetilde{H}^l_s(a,b,1,h,g) \right)}{l(ls)\left(\frac{1}{C^s}\widetilde{H}^r_r(a,b,1,h,g) - \frac{1}{C^r}\widetilde{H}^r_s(a,b,1,h,g) \right)} \right)^{\frac{1}{l-r}}, & r \neq l \neq s, \\ \left(\dfrac{s\left(\frac{1}{C^s}\widetilde{H}^l_l(a,b,1,h,g) - \frac{1}{C^l}\widetilde{H}^l_s(a,b,1,h,g) \right)}{l(l-s)\left(\frac{s}{C^s}\widetilde{R}^1_s(1,h,g) - \left(\frac{1}{C^s}\widetilde{H}^s_s(a,b,1,h,g) \right) \log\left(\frac{1}{C^s}\widetilde{H}^s_s(a,b,1,h,g) \right) \right)} \right)^{\frac{1}{l-s}}, & l \neq r = s, \\ \exp\left(\dfrac{s-2r}{r(r-s)} + \dfrac{\frac{s}{C^s}\widetilde{R}^1_r(1,h,g) - \left(\frac{1}{C^s}\widetilde{H}^r_s(a,b,1,h,g) \right)^{r/s} \log\left(\frac{1}{C^s}\widetilde{H}^s_s(a,b,1,h,g) \right)}{s\left(\left(\frac{1}{C^s}\widetilde{H}^r_s(a,b,1,h,g) \right)^{r/s} - C^s\widetilde{H}^r_r(a,b,1,h,g) \right)} \right), & l = r \neq s, \\ \exp\left(\dfrac{-1}{s} + \dfrac{-\frac{1}{C^s}\widetilde{H}^s_s(a,b,1,h,g) + \frac{s^2}{C^s}\widetilde{KR}^2_s(1,h,g) - \frac{1}{C^s}\widetilde{H}^s_s(a,b,1,h,g)\left(\log\left(\frac{1}{C^s}\widetilde{H}^s_s(a,b,1,h,g) \right) \right)^2}{2s\left(\frac{s}{C^s}\widetilde{R}^1_s(1,h,g) - \left(\frac{1}{C^s}\widetilde{H}^s_s(a,b,1,h,g) \right) \log\left(\frac{1}{C^s}\widetilde{H}^s_s(a,b,1,h,g) \right) \right)} \right), & l = r = s. \end{cases} \tag{2.3.43}$$

As an application to the above means, we consider the following corollary.

Corollary 2.3.14. *If $l, r, s \in \mathbb{R}^+$, then the following holds:*

(a) If $h \in Q^{\alpha_1}(C)$, $\alpha > \alpha_1$, then for any $x \geq 0$, we have the following means

$$\widetilde{M}^3_{l,r,s} = \begin{cases} \left(\dfrac{r(r-s)\left(\left[C^s sid_1\right]^{l/s} G^l_s(h,x) - C^s\left[lid_1\right] G^l_l(h,x) \right)}{l(l-s)\left[C^s sid_1\right]^{r/s} G^r_s(h,x) - C^s\left[rid_1\right] G^r_r(h,x)} \right)^{\frac{1}{l-r}}, & r \neq l \neq s, \\ \left(\dfrac{s\left(\left[C^s sid_1\right]^{l/s} G^l_s(h,x) - C^s\left[lid_1\right] G^l_l(h,x) \right)}{l(l-s)\left(C^s sid_1 G^s_s(h,x) \log\left(C^s sid_1 G^s_s(h,x) \right) - C^s sid_1 G^s_s(h,x) - C^s s^2 id_1 O^1_s(h,x) \right.} \right)^{\frac{1}{l-s}}, & l \neq r = s, \\ \exp\left(\dfrac{s-2r}{r(r-s)} + \dfrac{\left[C^s sid_1\right]^{r/s} G^r_s(h,x) \log\left(\left[C^s sid_1\right] G^s_s(h,x) \right) - C^s sid_1 G^r_r(h,x) - C^s rsid_1 O^1_r(h,x)}{s\left(\left[C^s sid_1\right]^{r/s} G^r_s(h,x) - C^s\left[rid_1\right] G^r_r(h,x) \right)} \right), & l = r \neq s, \\ \exp\left(\dfrac{-1}{s} + \dfrac{C^s sid_1 G^s_s(h,x)\left(\log\left(C^s sid_1 G^s_s(h,x) \right) \right)^2 - 2C^s s^2 id_1 O^1_s(h,x) - C^s s^3 id_1 O^2_s(h,x)}{2s\left(C^s sid_1 G^s_s(h,x) \log\left(C^s sid_1 G^s_s(h,x) \right) - C^s sid_1 G^s_s(h,x) - C^s s^2 id_1 O^1_s(h,x) \right)} \right), & l = r = s. \end{cases} \tag{2.3.44}$$

(b) If $h \in Q_{\alpha_2}(C)$, $\alpha_2 > \alpha$, then for any $x \geq 0$, we have the following means

$$\widetilde{M}^1_{l,r,s} = \begin{cases} \left(\dfrac{r(r-s)\left(\left[C^s sid_2\right]^{l/s} \widetilde{G}^l_s(h,x) - C^s\left[lid_2\right] \widetilde{G}^l_l(h,x) \right)}{l(l-s)\left[C^s sid_2\right]^{r/s} \widetilde{G}^r_s(h,x) - C^s\left[r(\alpha-\alpha_1)\right] \widetilde{G}^r_r(h,x)} \right)^{\frac{1}{l-r}}, & r \neq l \neq s, \\ \left(\dfrac{s\left(\left[C^s sid_2\right]^{l/s} \widetilde{G}^l_s(h,x) - C^s\left[lid_2\right] \widetilde{G}^l_l(h,x) \right)}{l(l-s)\left(C^s sid_2 \widetilde{G}^s_s(h,x) \log\left(C^s sid_2 \widetilde{G}^s_s(h,x) \right) - C^s sid_2 \widetilde{G}^s_s(h,x) - C^s s^2 id_2 \widetilde{O}^1_s(h,x) \right.} \right)^{\frac{1}{l-s}}, & l \neq r = s, \\ \exp\left(\dfrac{s-2r}{r(r-s)} + \dfrac{\left[C^s sid_2\right]^{r/s} \widetilde{G}^r_s(h,x) \log\left(\left[C^s sid_2\right] \widetilde{G}^s_s(h,x) \right) - C^s sid_2 \widetilde{G}^r_r(h,x) - C^s rsid_2 \widetilde{O}^1_r(h,x)}{s\left(\left[C^s sid_2\right]^{r/s} \widetilde{G}^r_s(h,x) - C^s\left[rid_2\right] \widetilde{G}^r_r(h,x) \right)} \right), & l = r \neq s, \\ \exp\left(\dfrac{-1}{s} + \dfrac{C^s sid_2 \widetilde{G}^s_s(h,x)\left(\log\left(C^s sid_2 \widetilde{G}^s_s(h,x) \right) \right)^2 - 2C^s s^2 id_2 \widetilde{O}^1_s(h,x) - C^s s^3 id_2 \widetilde{O}^2_s(h,x)}{2s\left(C^s sid_2 \widetilde{G}^s_s(h,x) \log\left(C^s sid_2 \widetilde{G}^s_s(h,x) \right) - C^s sid_2 \widetilde{G}^s_s(h,x) - C^s s^2 id_2 \widetilde{O}^1_s(h,x) \right)} \right), & l = r = s. \end{cases} \tag{2.3.45}$$

where

$$O^n_s(h,x) = \left(\int_x^\infty \left(t^{-\alpha} h(t) \right)^s \left(\log\left(t^{-\alpha} h(t) \right) \right)^n \frac{dt}{t} \right)^{1/p},$$

$$id_1 = (\alpha - \alpha_1) \text{ and } id_2 = (\alpha_2 - \alpha)$$

and

$$\widetilde{O}^n_s(h,x) = \left(\int_0^x \left(t^{-\alpha}h(t)\right)^s \left(\log\left(t^{-\alpha}h(t)\right)\right)^n \frac{dt}{t}\right)^{1/p}.$$

Proof. (a) It is a simple consequence of above means. Since $h \in Q^{\alpha_1}(C)$, so making substitutions $h \to h(t)t^{-\alpha_1}$ and $g \to t^{(\alpha_1-\alpha)}$ in (2.3.42). We get (2.3.44).
(b) Since $h \in Q_{\alpha_2}(C)$, so making substitutions $h \to h(t)t^{-\alpha_2}$ and $g \to t^{(\alpha_2-\alpha)}$ in (2.3.40). We get (2.3.45).

□

Before giving the Cauchy means for $k = 5,\dots,8$, we formulate $\mathcal{L}_k(f_p)$ for $k = 5,\dots,8$ for the class of functions defined in (2.1.15) as:
$\mathcal{L}_5(f_p)$

$$= \begin{cases} \frac{C^p}{p(p-1)} \int_a^b k(x)H_1^p(a,x,1,h,g)dx - CH_p^p(a,b,K_1,h,g), & p > 0, p \neq 1 \\ C\int_a^b k(x)H_1^1(a,x,1,h,g)\log\left[CH_1^1(a,x,1,h,g)\right]dx - CR_1^1(K_1,h,g), & p = 1. \end{cases}$$

$\mathcal{L}_6(f_p)$

$$= \begin{cases} \frac{\frac{1}{C}\int_a^b H_p^p(a,b,K_1,h,g) - \frac{1}{C^p}\int_a^b k(x)H_1^p(a,x,1,h,g)dx}{p(p-1)}, & p > 0, p \neq 1 \\ \frac{1}{C}R_1^1(K_1,h,g) - \frac{1}{C}\int_a^b \left(k(x)H_1^1(a,x,1,h,g)\log\left[CH_1^1(a,x,1,h,g)\right]\right)dx, & p = 1. \end{cases}$$

$\mathcal{L}_7(f_p)$

$$= \begin{cases} \frac{C^p\int_a^b k(x)\widetilde{H}_1^p(x,b,1,h,g)dx - C\widetilde{H}_p^p(a,b,K_2,h,g)}{p(p-1)}, & p > 0, p \neq 1 \\ C\int_a^b \left(k(x)\widetilde{H}_1^1(x,b,1,h,g)\log\left[C\widetilde{H}_1^1(x,b,1,h,g)\right]\right)dx - C\widetilde{R}_1^1(K_2,h,g), & p = 1. \end{cases}$$

$\mathcal{L}_8(f_p)$

$$= \begin{cases} \frac{\frac{1}{C}\int_a^b \widetilde{H}_p^p(a,b,K_2,h,g) - \frac{1}{C^p}\int_a^b k(x)\widetilde{H}_1^p(x,b,1,h,g)dx}{p(p-1)}, & p > 0, p \neq 1 \\ \frac{1}{C}\widetilde{R}_1^1(K_2,h,g) - \frac{1}{C}\int_a^b \left(k(x)\widetilde{H}_1^1(x,b,1,h,g)\log\left[C\widetilde{H}_1^1(x,b,1,h,g)\right]\right)dx, & p = 1. \end{cases}$$

These expressions for $\mathcal{L}_k(f_p)$ inserted in (2.3.31) give Cauchy means for $l \neq r$. Also, we have continuous extensions of Cauchy means in other cases. Therefore by taking respective limits, we have the following

$M^5_{r,r} =$

$$\begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{\int_a^b \left[k(x)C^r H_1^r(a,x,1,h,g)\log\left(CH_1^1(a,x,1,h,g)\right)\right]dx - CR_r^1(K_1,h,g)}{\left(\int_a^b k(x)C^r H_1^r(a,x,1,h,g)dx - CH_r^r(a,b,K_1,h,g)\right)}\right), & r \neq 1, \\ \exp\left(-1 + \frac{\int_a^b \left[k(x)CH_1^1(a,x,1,h,g)\left(\log\left(CH_1^1(a,x,1,h,g)\right)\right)^2\right]dx + CH_1^1(a,b,K_1,h,g) - CR_1^2(K_1,h,g)}{2\left(\int_a^b \left[CH_1^1(a,x,1,h,g)\log\left(CH_1^1(a,x,1,h,g)\right)\right]dx - CR_1^1(K_1,h,g)\right)}\right), & r = 1. \end{cases}$$

$M^6_{r,r} =$

$$\begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{\frac{1}{C}R_r^1(K_1,h,g) - \int_a^b \left[k(x)\frac{1}{C^r}H_1^r(a,x,1,h,g)\log\left(\frac{1}{C}H_1^1(a,x,1,h,g)\right)\right]dx}{\left(\frac{1}{C}H_r^r(a,b,K_1,h,g) - \int_a^b k(x)\frac{1}{C^r}H_1^r(a,x,1,h,g)dx\right)}\right), & r \neq 1, \\ \exp\left(-1 + \frac{-\frac{1}{C}H_1^1(a,b,K_1,h,g) + \frac{1}{C}R_1^2(K_1,h,g) - \int_a^b \left[k(x)\frac{1}{C}H_1^1(a,x,1,h,g)\left(\log\left(\frac{1}{C}H_1^1(a,x,1,h,g)\right)\right)^2\right]dx}{2\left(\frac{1}{C}R_1^1(K_1,h,g) - \int_a^b \left[\frac{1}{C}H_1^1(a,x,1,h,g)\log\left(\frac{1}{C}H_1^1(a,x,1,h,g)\right)\right]dx\right)}\right), & r = 1. \end{cases}$$

$M^7_{r,r} =$

$$\begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{\int_a^b \left[k(x)C^r \widetilde{H}_1^r(x,b,1,h,g)\log\left(C\widetilde{H}_1^1(x,b,1,h,g)\right)\right]dx - C\widetilde{R}_r^1(K_2,h,g)}{\left(\int_a^b k(x)C^r \widetilde{H}_1^r(x,b,1,h,g)dx - C\widetilde{H}_r^r(a,b,K_2,h,g)\right)}\right), & r \neq 1, \\ \exp\left(-1 + \frac{\int_a^b \left[k(x)C\widetilde{H}_1^1(x,b,1,h,g)\left(\log\left(C\widetilde{H}_1^1(x,b,1,h,g)\right)\right)^2\right]dx + C\widetilde{H}_1^1(a,b,K_2,h,g) - C\widetilde{R}_1^2(K_2,h,g)}{2\left(\int_a^b \left[C\widetilde{H}_1^1(x,b,1,h,g)\log\left(C\widetilde{H}_1^1(x,b,1,h,g)\right)\right]dx - C\widetilde{R}_1^1(K_2,h,g)\right)}\right), & r = 1. \end{cases}$$

$M^8_{r,r} =$

$$\begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{\frac{1}{C}\widetilde{R}_r^1(K_2,h,g) - \int_a^b \left[k(x)\frac{1}{C^r}\widetilde{H}_1^r(x,b,1,h,g)\log\left(\frac{1}{C}\widetilde{H}_1^1(x,b,1,h,g)\right)\right]dx}{\left(\frac{1}{C}\widetilde{H}_r^r(a,b,K_2,h,g) - \int_a^b k(x)\frac{1}{C^r}\widetilde{H}_1^r(x,b,1,h,g)dx\right)}\right), & r \neq 1, \\ \exp\left(-1 + \frac{-\frac{1}{C}\widetilde{H}_1^1(a,b,K_2,h,g) + \frac{1}{C}\widetilde{R}_1^2(K_2,h,g) - \int_a^b \left[k(x)\frac{1}{C}\widetilde{H}_1^1(x,b,1,h,g)\left(\log\left(\frac{1}{C}\widetilde{H}_1^1(x,b,1,h,g)\right)\right)^2\right]dx}{2\left(\frac{1}{C}\widetilde{R}_1^1(K_2,h,g) - \int_a^b \left[\frac{1}{C}\widetilde{H}_1^1(x,b,1,h,g)\log\left(\frac{1}{C}\widetilde{H}_1^1(x,b,1,h,g)\right)\right]dx\right)}\right), & r = 1. \end{cases}$$

Now we deduce the monotonicity of means defined by (2.3.31) in the form of Dresher's inequality as follows.

Theorem 2.3.15. *Let $M^k_{l,r}$ be given as in* (2.3.31) *and $r, l, u, v \in \mathbb{R}^+$ such that $r \leq u$, $l \leq v$. Then*

$$M^k_{l,r} \leq M^k_{v,u},\ k = 1, \dots 8. \tag{2.3.46}$$

Proof. By Corollary 2.3.7(i), $\mathcal{L}_k$ is log-convex. We set $f = \log \mathcal{L}_k$ and $x_1 = l$, $x_2 = r$, $y_1 = u$, $y_2 = v$ in Lemma 1.2.2 and get

$$\frac{\log \mathcal{L}_k(f_l) - \log \mathcal{L}_k(f_r)}{l - r} \leq \frac{\log \mathcal{L}_k(f_v) - \log \mathcal{L}_k(f_u)}{v - u}. \tag{2.3.47}$$

By using properties of log function, we immediately get (2.3.46). □

Corollary 2.3.16. *Let* $M^k_{l,r,s}$, $k = 1, \ldots, 4$ *be given as before and* $r, l, u, v; s \in \mathbb{R}^+$ *such that* $l \leq v$, $r \leq u$. *Then*

$$M^k_{l,r,s} \leq M^k_{v,u,s},\ k = 1, \ldots 4. \tag{2.3.48}$$

Proof. By Theorem 2.3.15,

$$M^k_{l,r} \leq M^k_{v,u},\ k = 1, \ldots 4.$$

For $s > 0$, we set $h \to h^s,\ g \to g^s,\ C \to C^s,\ l \to l/s,\ r \to r/s,\ u \to v/s$ and $r \to v/s$ in above inequality for means and get (2.3.48). □

2.3.4 Examples of Functionals Involving C-Monotone Functions

In the previous section, we applied Theorem 2.3.5 to the family of functions f_p given by (2.1.15) and constructed exponentially convex functions. By using properties of exponentially convex functions, we refined the reverse Hardy inequality and constructed Cauchy means. In this section, we apply Theorem 2.3.5 to other families of convex functions to get other exponentially convex functions and Cauchy means.

Example 2.3.1. *Consider a family of functions*

$$\Upsilon_1 = \{\lambda_l : \mathbb{R} \to \mathbb{R} : l \in \mathbb{R}\}$$

defined by

$$\lambda_l(x) = \begin{cases} \frac{e^{lx}-1}{l^2}, & l \neq 0\,, \\ \frac{x^2}{2}\,, & l = 0\,. \end{cases}$$

Since $\frac{d^2\lambda_l}{dx^2}(x) = e^{lx} > 0$, *then* $l \mapsto \frac{d^2\lambda_l}{dx^2}(x)$ *is exponentially convex by definition. Analogously as in the proof of Theorem 2.3.5 we conclude that* $l \mapsto [u_0, u_1, u_2; \lambda_l]$ *is exponentially convex (and so exponentially convex in the Jensen sense).*
Notice that $\lambda_l(0) = 0$ *so by Corollary 2.3.6 we have that* $l \mapsto \mathcal{L}_k(\lambda_l)$, $k = 1, \ldots, 8$ *is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous so it is exponentially convex.*
For this family of functions, $\mathfrak{B}(l, r; \mathcal{L}_k, \boldsymbol{\Upsilon}_1)$, $k = 1, \ldots, 8$ *from* (2.3.17) *is equal to*

$$\mathfrak{B}(l, r; \mathcal{L}_k, \boldsymbol{\Upsilon}_1) = \begin{cases} \left(\frac{\mathcal{L}_k(\lambda_l)}{\mathcal{L}_k(\lambda_r)}\right)^{\frac{1}{l-r}}, & l \neq r\,, \\ \exp\left(\frac{\mathcal{L}_k(id\cdot\lambda_l)}{\mathcal{L}_k(\lambda_l)} - \frac{2}{l}\right), & l = r \neq 0\,, \\ \exp\left(\frac{\mathcal{L}_k(id\cdot\lambda_0))}{3\mathcal{L}_k(\lambda_0))}\right), & l = r = 0\,, \end{cases}$$

where id is the identity function. Also by Corollary 2.3.7 it is monotonic in parameters l *and* r*. Applying Theorem 2.3.11 for* $f_1 = \lambda_l$ *and* $f_2 = \lambda_r$ *there exists* ξ_k, $k = 1, \ldots, 8$ *such that*

$$e^{(l-r)\xi_k} = \frac{\mathcal{L}_k(\lambda_l)}{\mathcal{L}_k(\lambda_r)}.$$

Therefore

$$M^k_{l,r}(\Upsilon_1) = \log \mathfrak{B}(l, r; \mathcal{L}_k, \boldsymbol{\Upsilon}_1)$$

is a Cauchy mean.

Example 2.3.2. *Consider a family of functions*

$$\Upsilon_2 = \{\omega_l : [0, \infty) \to \mathbb{R} : l > 0\}$$

defined by

$$\omega_l(x) = \begin{cases} \frac{l^{-x}-1}{\log^2 l}, & l \neq 1, \\ \frac{x^2}{2}, & l = 1. \end{cases}$$

Since $l \mapsto \frac{d^2\omega_l}{dx^2}(x) = l^{-x}$ *is the Laplace transform of a positive function (see [29]) that is* $l^{-x} = \dfrac{1}{\Gamma(x)} \displaystyle\int_0^\infty e^{-lt} t^{x-1}\, dt$ *, it is exponentially convex on* $(0, \infty)$ *(see [22]). Analogously as in the proof of Theorem 2.3.5 we conclude that* $l \mapsto [u_0, u_1, u_2; \omega_l]$ *is exponentially convex (and so exponentially convex in the Jensen sense).*
Notice that $\omega_l(0) = 0$*, so by Corollary 2.3.6 we have that* $l \mapsto \mathcal{L}_k(\omega_l)$*,* $k = 1, \ldots, 8$ *is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous so it is exponentially convex.*
For this family of functions, $\mathfrak{B}(l, r; \mathcal{L}_k, \boldsymbol{\Upsilon}_2)$*,* $k = 1, \ldots, 8$ *from* (2.3.17) *is equal to*

$$\mathfrak{B}(l, r; \mathcal{L}_k, \boldsymbol{\Upsilon}_2) = \begin{cases} \left(\frac{\mathcal{L}_k(\omega_l)}{\mathcal{L}_k(\omega_r)}\right)^{\frac{1}{l-r}}, & l \neq r, \\ \exp\left(-\frac{\mathcal{L}_k(id\cdot\omega_l)}{l\mathcal{L}_k(\omega_l)} - \frac{2}{l \log l}\right), & l = r \neq 1, \\ \exp\left(-\frac{\mathcal{L}_k(id\cdot\omega_1))}{3\mathcal{L}_k(\omega_1))}\right), & l = r = 1, \end{cases}$$

where id is the identity function. Also by Corollary 2.3.7 it is monotonic in parameters l *and* r*. Applying Theorem 2.3.11 for* $f_1 = \omega_l$ *and* $f_2 = \omega_r$ *there exists* ξ_k, $k = 1, \ldots, 8$ *such that*

$$\left(\frac{l}{r}\right)^{-\xi_k} = \frac{\mathcal{L}_k(\omega_l)}{\mathcal{L}_k(\omega_r)}.$$

Therefore

$$M^k_{l,r}(\Upsilon_2) = -L(l, r) \log \mathfrak{B}(l, r; \mathcal{L}_k, \boldsymbol{\Upsilon}_2)$$

is a Cauchy mean, where $L(l,r)$ is the logarithmic mean defined by

$$L(l,r) = \begin{cases} \frac{l-r}{\log l - \log r}\,, & l \neq r\,, \\ l\,, & l = r\,. \end{cases}$$

Example 2.3.3. *Consider a family of functions*

$$\Upsilon_3 = \{\mu_l : [0,\infty) \to l : l > 0\}$$

defined by

$$\mu_l(x) = \frac{e^{-x\sqrt{l}} - 1}{l}\,.$$

Also $l \mapsto \frac{d^2\mu_l}{dx^2}(x) = e^{-x\sqrt{l}}$ is the Laplace transform of a non-negative function (see [29]) that is $e^{-x\sqrt{l}} = \frac{l}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-lt}e^{-x^2/4t}}{t\sqrt{t}}\,dt$, it is exponentially convex on $(0,\infty)$ (see [22]).
Analogously as in the proof of Theorem 2.3.5 we conclude that $l \mapsto [u_0, u_1, u_2; \mu_l]$ is exponentially convex (and so exponentially convex in the Jensen sense).
Notice that $\mu_l(0) = 0$, so by Corollary 2.3.6 we have that $l \mapsto \mathcal{L}_k(\mu_l)$, $k = 1, \ldots, 8$ is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous so it is exponentially convex.
For this family of functions, $\mathfrak{B}(l, r; \mathcal{L}_k, \boldsymbol{\Upsilon}_3)$, $k = 1, \ldots, 8$ from (2.3.17) is equal to

$$\mathfrak{B}(l, r; \mathcal{L}_k, \boldsymbol{\Upsilon}_3) = \begin{cases} \left(\frac{\mathcal{L}_k(\mu_l)}{\mathcal{L}_k(\mu_r)}\right)^{\frac{1}{l-r}}, & l \neq r\,, \\ \exp\left(-\frac{\mathcal{L}_k(id\cdot\mu_l)}{2\sqrt{l}\mathcal{L}_k(\mu_l)} - \frac{1}{l}\right), & l = r\,, \end{cases}$$

where id is the identity function. Also by Corollary 2.3.7 it is monotonic in parameters l and r. Applying Theorem 2.3.11 for $f_1 = \mu_l$ and $f_2 = \mu_r$ there exists ξ_k, $k = 1, \ldots, 8$ such that

$$e^{-\xi_k(\sqrt{l}-\sqrt{r})} = \frac{\mathcal{L}_k(\mu_l)}{\mathcal{L}_k(\mu_r)}.$$

Therefore

$$M^k_{l,r}(\Upsilon_3) = -(\sqrt{l} + \sqrt{r}) \log \mathfrak{B}(l, r; \mathcal{L}_k, \boldsymbol{\Upsilon}_3)$$

is a Cauchy mean.

2.4 Multidimensional Generalization of Reversed Hardy Type Inequalities for Monotone Functions

In the present section, we will give results obtained from the inequalities given by Pečarić, Perić and Persson in [20] for multidimensional monotone functions. The obtained inequalities will be a generalization of the reversed Hardy inequalities. Also we include some interesting corollaries and remarks. We start the section by giving the following remark.

Remark 2.4.1. Theorems 2.1.4 and 2.1.5 give inequalities of the similar type but Theorem 2.1.4 gives the inequalities for the wider class of functions f, while in the particular case of $f(x) = x^p$ inequality in Theorem 2.1.5 gives the optimal constant.

Now, we will give some further results obtained from Theorems 2.1.4 and 2.1.5. For this we introduce the following further notions.
Let $\mathbf{A} \subseteq \mathbb{R}^n$ be measurable set, $\sigma : \mathbf{A} \to \mathbb{R}^m$ be a measurable function such that $\sigma(\mathbf{A}) \subseteq (\mathbf{a}, \mathbf{b})$ and $k : \mathbf{A} \to \mathbb{R}^+$ a measurable function. Denote $\sigma^{\leftarrow}(\mathbf{x}) = \{\mathbf{t} \in \mathbf{A} : \mathbf{x} < \sigma(\mathbf{t})\}$, $\sigma_{\leftarrow}(\mathbf{x}) = \{\mathbf{t} \in \mathbf{A} : \sigma(\mathbf{t}) < \mathbf{x}\}$, $K_1(\mathbf{x}) = \int_{\sigma^{\leftarrow}(\mathbf{x})} k(\mathbf{t})\mathbf{dt}$ and $K_2(\mathbf{x}) = \int_{\sigma_{\leftarrow}(\mathbf{x})} k(\mathbf{t})\mathbf{dt}$.

Theorem 2.4.1. *Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$, $\mathbf{a} < \mathbf{b}$, $m \in \mathbb{N}$, $h : (\mathbf{a}, \mathbf{b}) \to \mathbb{R}$ be positive, $\mathbf{g} = (g_1, \ldots, g_m)$, $g_i : (a_i, b_i) \to \mathbb{R}$, and $\mathbf{g}$ is positive and differentiable. Moreover, let $f : [0, \infty) \to \mathbb{R}$ be a two times differentiable function such that $f(0) = 0$ and $\lim_{t\to 0} tf'(t) = 0$. Let k, σ, K_1 and K_2 be the functions as given above.*

(a) Suppose that h is decreasing and $\mathbf{g}$ is increasing, where $g_i(a_i + 0) = 0$, $i = 1, 2, \ldots, m$. If f is convex, then

$$\int_{\mathbf{A}} k(\mathbf{x}) f\left(\int_{\mathbf{a}}^{\sigma(\mathbf{x})} h(\mathbf{t})\mathbf{d}[\mathbf{g}(\mathbf{t})]\right)\mathbf{dx} \geq \tag{2.4.1}$$
$$\int_{\mathbf{a}}^{\mathbf{b}} K_1(\mathbf{x}) f'\left(h(\mathbf{x}) \prod_{i=1}^{m} g_i(x_i)\right) h(\mathbf{x})\mathbf{d}[\mathbf{g}(\mathbf{x})].$$

If f is concave, then (2.4.1) holds in reversed direction.

(b) Suppose that h is increasing and $\mathbf{g}$ is decreasing, where $g_i(b_i - 0) = 0$, $i = 1, 2, \ldots, m$. If f is convex, then

$$\int_{\mathbf{A}} k(\mathbf{x}) f\left(\int_{\sigma(\mathbf{x})}^{\mathbf{b}} h(\mathbf{t})\mathbf{d}[-\mathbf{g}(\mathbf{t})]\right)\mathbf{dx} \geq \tag{2.4.2}$$

$$\int_{\mathbf{a}}^{\mathbf{b}} K_2(\mathbf{x}) f'\Big(h(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\Big) h(\mathbf{x})\mathbf{d}[-\mathbf{g}(\mathbf{x})].$$

If f is concave, then (2.4.2) *holds in reversed direction.*

Proof. (a) Under the given conditions, we have the following inequality by Theorem 2.1.4,

$$f\Big(\int_{\mathbf{a}}^{\sigma(\mathbf{t})} h(\mathbf{x})\mathbf{d}[\mathbf{g}(\mathbf{x})]\Big) \geq \int_{\mathbf{a}}^{\sigma(\mathbf{t})} f'\Big(h(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\Big) h(\mathbf{x})\mathbf{d}[\mathbf{g}(\mathbf{x})].$$

Multiplying the above inequality with a positive function k and integrating over $\mathbf{A}$, we get

$$\int_{\mathbf{A}} k(\mathbf{t}) f\Big(\int_{\mathbf{a}}^{\sigma(\mathbf{t})} h(\mathbf{x})\mathbf{d}[\mathbf{g}(\mathbf{x})]\Big)\mathbf{dt} \geq$$
$$\int_{\mathbf{A}} k(\mathbf{t}) \int_{\mathbf{a}}^{\sigma(\mathbf{t})} f'\Big(h(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\Big) h(\mathbf{x})\mathbf{d}[\mathbf{g}(\mathbf{x})]\mathbf{dt}.$$

Let χ_B denote the characteristic function of a set B and applying Fubini theorem on the integral on the R.H.S., we get

$$\int_{\mathbf{A}} k(\mathbf{t}) \int_{\mathbf{a}}^{\sigma(\mathbf{t})} f'\Big(h(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\Big) h(\mathbf{x})\mathbf{d}[\mathbf{g}(\mathbf{x})]\mathbf{dt}$$

$$= \int_{\mathbf{A}} k(\mathbf{t}) \int_{\mathbf{a}}^{\mathbf{b}} f'\Big(h(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\Big) h(\mathbf{x})\chi_{(\mathbf{a},\sigma(\mathbf{t}))}(\mathbf{x})\mathbf{d}[\mathbf{g}(\mathbf{x})]\mathbf{dt}$$
$$= \int_{\mathbf{a}}^{\mathbf{b}} f'\Big(h(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\Big) h(\mathbf{x})\int_{\mathbf{A}} k(\mathbf{t})\chi_{(\mathbf{a},\sigma(\mathbf{t}))}(\mathbf{x})\mathbf{dt}\mathbf{d}[\mathbf{g}(\mathbf{x})]$$
$$= \int_{\mathbf{a}}^{\mathbf{b}} f'\Big(h(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\Big) h(\mathbf{x})\int_{\mathbf{A}} k(\mathbf{t})\chi_{\sigma^{\leftarrow}(\mathbf{x})}(\mathbf{t})\mathbf{dt}\mathbf{d}[\mathbf{g}(\mathbf{x})].$$

Since $K_1(\mathbf{x}) = \int_{\mathbf{A}} k(\mathbf{t})\chi_{\sigma^{\leftarrow}(\mathbf{x})}(\mathbf{t})\mathbf{dt}$, we get (2.4.1).

(b) Under the given conditions, we have the following inequality by Theorem 2.1.4,

$$f\Big(\int_{\sigma(\mathbf{t})}^{\mathbf{b}} h(\mathbf{x})\mathbf{d}[-\mathbf{g}(\mathbf{x})]\Big) \geq \int_{\sigma(\mathbf{t})}^{\mathbf{b}} f'\Big(h(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\Big) h(\mathbf{x})\mathbf{d}[-\mathbf{g}(\mathbf{x})].$$

Multiplying the above inequality with a positive function k and integrating over $\mathbf{A}$, we get

$$\int_{\mathbf{A}} k(\mathbf{t}) f\Big(\int_{\sigma(\mathbf{t})}^{\mathbf{b}} h(\mathbf{x}) \mathbf{d}[-\mathbf{g}(\mathbf{x})]\Big) \mathbf{dt} \geq$$

$$\int_{\mathbf{A}} k(\mathbf{t}) \int_{\sigma(\mathbf{t})}^{\mathbf{b}} f'\Big(h(\mathbf{x}) \prod_{i=1}^{m} g_i(x_i)\Big) h(\mathbf{x}) \mathbf{d}[-\mathbf{g}(\mathbf{x})] \mathbf{dt}.$$

Changing of the order of integration as in (a), we get

$$\int_{\mathbf{A}} k(\mathbf{t}) \int_{\mathbf{a}}^{\mathbf{b}} f'\Big(h(\mathbf{x}) \prod_{i=1}^{m} g_i(x_i)\Big) h(\mathbf{x}) \chi_{(\sigma(\mathbf{t}),\mathbf{b})}(\mathbf{x}) \mathbf{d}[-\mathbf{g}(\mathbf{x})] \mathbf{dt}$$

$$= \int_{\mathbf{a}}^{\mathbf{b}} f'\Big(h(\mathbf{x}) \prod_{i=1}^{m} g_i(x_i)\Big) h(\mathbf{x}) \int_{\mathbf{A}} k(\mathbf{t}) \chi_{(\sigma(\mathbf{t}),\mathbf{b})}(\mathbf{x}) \mathbf{dt} \mathbf{d}[-\mathbf{g}(\mathbf{x})]$$

$$= \int_{\mathbf{a}}^{\mathbf{b}} f'\Big(h(\mathbf{x}) \prod_{i=1}^{m} g_i(x_i)\Big) h(\mathbf{x}) \int_{\mathbf{A}} k(\mathbf{t}) \chi_{\sigma_{\leftarrow}(\mathbf{x})}(\mathbf{t}) \mathbf{dt} \mathbf{d}[-\mathbf{g}(\mathbf{x})].$$

Since $K_2(\mathbf{x}) = \int_{\mathbf{A}} k(\mathbf{t}) \chi_{\sigma_{\leftarrow}(\mathbf{x})}(\mathbf{t}) \mathbf{dt}$, we get (2.4.2).

□

Theorem 2.4.2. *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$, $\mathbf{a} < \mathbf{b}$, $m \in \mathbb{N}$, $h : (\mathbf{a}, \mathbf{b}) \to \mathbb{R}$ *be positive,* $\mathbf{g} = (g_1, \ldots, g_m)$, $g_i : (a_i, b_i) \to \mathbb{R}$, *and* $\mathbf{g}$ *is positive and continuous. Let ,* k, σ, K_1 *and* K_2 *be the functions as defined in Theorem 2.4.1.*

(a) Suppose that h *is decreasing and* $\mathbf{g}$ *is increasing, where* $g_i(a_i + 0) = 0$, $i = 1, 2, \ldots, m$. *If* $p \geq 1$, *then*

$$\int_{\mathbf{A}} k(\mathbf{x}) \Big(\int_{\mathbf{a}}^{\sigma(\mathbf{x})} h(\mathbf{t}) \mathbf{d}[\mathbf{g}(\mathbf{t})]\Big)^p \mathbf{dx} \geq \int_{\mathbf{a}}^{\mathbf{b}} K_1(\mathbf{x}) h^p(\mathbf{x}) \mathbf{d}[\mathbf{g}^p(\mathbf{x})]. \tag{2.4.3}$$

If " $0 < p \leq 1$,*" then* (2.4.3) *holds in reversed direction.*

(b) Suppose that h *is increasing and* $\mathbf{g}$ *is decreasing, where* $g_i(b_i - 0) = 0$, $i = 1, 2, \ldots, m$. *If* $p \geq 1$, *then*

$$\int_{\mathbf{A}} k(\mathbf{x}) \Big(\int_{\sigma(\mathbf{x})}^{\mathbf{b}} h(\mathbf{t}) \mathbf{d}[-\mathbf{g}(\mathbf{t})]\Big)^p \mathbf{dx} \geq \int_{\mathbf{a}}^{\mathbf{b}} K_2(\mathbf{x}) h^p(\mathbf{x}) \mathbf{d}[-\mathbf{g}^p(\mathbf{x})]. \tag{2.4.4}$$

If " $0 < p \leq 1$,*" then* (2.4.4) *holds in reversed direction.*

(c) The inequalities in (a) *and* (b) *are sharp.*

Proof. The inequalities are derived from Theorem 2.1.5 in the same way as Theorem 2.4.1 is derived from Theorem 2.1.4.i. e. by replacing the upper (resp. lower) limit of integration $\mathbf{b}$ (resp. $\mathbf{a}$) in (2.1.13) (resp. (2.1.14)) with $\mathbf{h}(\mathbf{t})$, multiplying the obtained inequality by $k(\mathbf{t}) \geq 0$, integrating with respect to the variable $\mathbf{t}$ and applying Fubini's theorem on the R. H. S.

Regarding the sharpness statement in (c), equality in (a) is obtained for the functions of the form $h = C\prod_{i=1}^{m} \chi_{(a_i,e_i)}$ and in (b) for the functions of the form $h = C\prod_{i=1}^{m} \chi_{(e_i,b_i)}$, where C is a positive constant and $a_i \leq e_i \leq b_i$. □

After the suitable replacements of the functions h and $\mathbf{g}$, we get the following corollaries.

Corollary 2.4.3. *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$, $\mathbf{a} < \mathbf{b}$, $m \in \mathbb{N}$, $h : (\mathbf{a}, \mathbf{b}) \to \mathbb{R}$ *be positive,* $\mathbf{g} = (g_1, \ldots, g_m)$, $g_i : (a_i, b_i) \to \mathbb{R}$, *and* $\mathbf{g}$ *is strictly positive and differentiable. Let* f, k, σ, K_1 *and* K_2 *be as in Theorem 2.4.1.*

(a) Suppose that $\mathbf{x} \mapsto \frac{h(\mathbf{x})}{\prod_{i=1}^{m} g_i(x_i)}$ *is decreasing and* $\mathbf{g}$ *is increasing, where* $g_i(a_i+0) = 0,\ i = 1, 2, \ldots, m$. *If* f *is convex, then*

$$\int_{\mathbf{A}} k(\mathbf{x}) f\Big(\int_{\mathbf{a}}^{\sigma(\mathbf{x})} \frac{h(\mathbf{t})}{\prod_{i=1}^{m} g_i(t_i)} \mathbf{d}[\mathbf{g}(\mathbf{t})]\Big) \mathbf{dx} \geq \tag{2.4.5}$$

$$\int_{\mathbf{a}}^{\mathbf{b}} K_1(\mathbf{x}) f'\big(h(\mathbf{x})\big) \frac{h(\mathbf{x})}{\prod_{i=1}^{m} g_i(x_i)} \mathbf{d}[\mathbf{g}(\mathbf{x})].$$

If f *is concave, then* (2.4.5) *holds in reversed direction.*

(b) Suppose that $\mathbf{x} \mapsto \frac{h(\mathbf{x})}{\prod_{i=1}^{m} g_i(x_i)}$ *is increasing and* $\mathbf{g}$ *is decreasing, where* $g_i(b_i-0) = 0,\ i = 1, 2, \ldots, m$. *If* f *is convex, then*

$$\int_{\mathbf{A}} k(\mathbf{x}) f\Big(\int_{\sigma(\mathbf{x})}^{\mathbf{b}} \frac{h(\mathbf{t})}{\prod_{i=1}^{m} g_i(t_i)} \mathbf{d}[-\mathbf{g}(\mathbf{t})]\Big) \mathbf{dx} \geq \tag{2.4.6}$$

$$\int_{\mathbf{a}}^{\mathbf{b}} K_2(\mathbf{x}) f'\big(h(\mathbf{x})\big) \frac{h(\mathbf{x})}{\prod_{i=1}^{m} g_i(x_i)} \mathbf{d}[-\mathbf{g}(\mathbf{x})].$$

If f *is concave, then* (2.4.6) *holds in reversed direction.*

Proof. Replace $h(\mathbf{x}) \to \frac{h(\mathbf{x})}{\prod_{i=1}^{m} g_i(x_i)}$ in (2.4.1) and (2.4.2) respectively in the theorem above, we get (2.4.5) and (2.4.6) respectively. □

Corollary 2.4.4. *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$, $\mathbf{a} < \mathbf{b}$, $m \in \mathbb{N}$, $h : (\mathbf{a}, \mathbf{b}) \to \mathbb{R}$ *be positive,* $\mathbf{g} = (g_1, \ldots, g_m)$, $g_i : (a_i, b_i) \to \mathbb{R}$, *and* $\mathbf{g}$ *is positive and differentiable. Let* f, k, σ, K_1 *and* K_2 *be as in Theorem 2.4.1.*

(a) Suppose that $\mathbf{x} \mapsto h(\mathbf{x})/(\prod_{i=1}^m e^{g_i(x_i)})$ *is decreasing and* $\mathbf{g}$ *is increasing, where* $g_i(a_i + 0) = -\infty$, $i = 1, 2, \ldots, m$. *If* f *is convex, then*

$$\int_{\mathbf{A}} k(\mathbf{x}) f\Big(\int_{\mathbf{a}}^{\sigma(\mathbf{x})} h(\mathbf{t}) \mathbf{d}[\mathbf{g}(\mathbf{t})]\Big) \mathbf{dx} \geq \tag{2.4.7}$$

$$\int_{\mathbf{a}}^{\mathbf{b}} K_1(\mathbf{x}) f'\big(h(\mathbf{x})\big) h(\mathbf{x}) \mathbf{d}[\mathbf{g}(\mathbf{x})].$$

If f *is concave, then* (2.4.7) *holds in reversed direction.*

(b) Suppose that $\mathbf{x} \mapsto h(\mathbf{x})/(\prod_{i=1}^m e^{g_i(x_i)})$ *is increasing and* $\mathbf{g}$ *is decreasing, where* $g_i(b_i - 0) = -\infty$, $i = 1, 2, \ldots, m$. *If* f *is convex, then*

$$\int_{\mathbf{A}} k(\mathbf{x}) f\Big(\int_{\sigma(\mathbf{x})}^{\mathbf{b}} h(\mathbf{t}) \mathbf{d}[-\mathbf{g}(\mathbf{t})]\Big) \mathbf{dx} \geq \tag{2.4.8}$$

$$\int_{\mathbf{a}}^{\mathbf{b}} K_2(\mathbf{x}) f'\big(h(\mathbf{x})\big) h(\mathbf{x}) \mathbf{d}[-\mathbf{g}(\mathbf{x})].$$

If f *is concave, then* (2.4.8) *holds in reversed direction.*

Proof. Replace $g_i(x_i) \to e^{g_i(x_i)}$ in the Corollary 2.4.3 , we get (2.4.7) and (2.4.8) respectively. □

Now we consider different weight functions k in Theorem 2.4.2 to get reversed Hardy type inequalities.

Corollary 2.4.5. *Let* $p > 1$ *and* $h : (\mathbf{a}, \mathbf{b}) \to \mathbb{R}$ *be positive, then*
(a) If h *is decreasing, then*

$$\int_{\mathbf{a}}^{\mathbf{b}} \Bigg(\frac{1}{\prod_{i}^{m}(x_i - a_i)} \int_{\mathbf{a}}^{\mathbf{x}} h(\mathbf{t}) \mathbf{dt}\Bigg)^p \mathbf{dx} \tag{2.4.9}$$

$$\geq \frac{p^m}{(p-1)^m} \Bigg(\int_{\mathbf{a}}^{\mathbf{b}} \prod_{i}^{m} \left[1 - \frac{(x_i - a_i)^{p-1}}{(b_i - a_i)^{p-1}}\right] h^p(\mathbf{x}) \mathbf{dx}\Bigg).$$

(b) If h is increasing, then

$$\int_{\mathbf{a}}^{\mathbf{b}}\left(\frac{1}{\prod_{i}^{m}(b_i-x_i)}\int_{\mathbf{x}}^{\mathbf{b}} h(\mathbf{t})\mathbf{dt}\right)^p \mathbf{dx} \tag{2.4.10}$$

$$\geq \frac{p^m}{(p-1)^m}\left(\int_{\mathbf{a}}^{\mathbf{b}}\prod_{i}^{m}\left[1-\frac{(b_i-x_i)^{p-1}}{(b_i-a_i)^{p-1}}\right]h^p(\mathbf{x})\mathbf{dx}\right).$$

If " $0 < p < 1$," then (2.4.9) *and* (2.4.10) *hold in reversed direction.*

(c) The inequalities in (*a*) *and* (*b*) *are sharp.*

Proof. (*a*) Consider Theorem 2.4.2 with $k(\mathbf{t}) = \prod_{i}^{m}(x_i-a_i)^{-p}$, $\mathbf{t} \in \mathbf{A} = (\mathbf{a}, \mathbf{b})$, $g_i(x_i) = x_i - a_i$, $i = 1, 2, \ldots, m$ and $\sigma(\mathbf{t}) = \mathbf{t}$. Then $\sigma^{\leftarrow}(\mathbf{x}) = (\mathbf{x}, \mathbf{b})$, so (2.4.3) becomes (2.4.9).

(*b*) Consider Theorem 2.4.2 $k(\mathbf{t}) = \prod_{i}^{m}(b_i-x_i)^{-p}$, $\mathbf{t} \in \mathbf{A} = (\mathbf{a}, \mathbf{b})$, $g_i(x_i) = b_i - x_i$, $i = 1, 2, \ldots, m$ and $h(\mathbf{t}) = \mathbf{t}$. Then $\sigma_{\leftarrow}(\mathbf{x}) = (\mathbf{a}, \mathbf{x})$, so (2.4.4) becomes (2.4.10).

□

Remark 2.4.2. By making similar substitution as in Corollary 2.4.3, we conclude that if $\mathbf{x} \mapsto \frac{h^{1/p}(\mathbf{x})}{\prod_{i=1}^{m}(x_i-a_i)}$ is a decreasing function, then Corollary 2.4.5 gives the following power means type inequalities for $p > 1$

$$\int_{\mathbf{a}}^{\mathbf{b}}\left(\frac{1}{\prod_{i}^{m}(x_i-a_i)}\int_{\mathbf{a}}^{\mathbf{x}} \frac{h^{1/p}(\mathbf{t})}{\prod_{i}^{m}(t_i-a_i)}\mathbf{dt}\right)^p \mathbf{dx} \tag{2.4.11}$$

$$\geq \frac{p^m}{(p-1)^m}\left(\int_{\mathbf{a}}^{\mathbf{b}}\prod_{i}^{m}\left[\frac{1}{(x_i-a_i)^{p-1}}-\frac{1}{(b_i-a_i)^{p-1}}\right]\frac{h(\mathbf{x})}{\prod_{i}^{m}(x_i-a_i)}\mathbf{dx}\right).$$

If " $0 < p < 1$," then (2.4.11) holds in reversed direction.

Remark 2.4.3. If in the Corollary 2.4.5, we put $a_i = 0$ and b_i approach ∞ for all $i = 1, \ldots, m$ in the limit we get reversed Hardy inequality in m-dimension as

$$\int_{\mathbf{0}}^{\infty}\left(\frac{1}{\prod_{i}^{m} x_i}\int_{\mathbf{a}}^{\mathbf{x}} h(\mathbf{t})\mathbf{dt}\right)^p \mathbf{dx} \geq \frac{p^m}{(p-1)^m}\left(\int_{\mathbf{0}}^{\infty} h^p(\mathbf{x})\mathbf{dx}\right). \tag{2.4.12}$$

Corollary 2.4.6. *Let* $a, b \in \mathbb{R}$ *and denote* $\mathbf{a} = (a, a, \ldots, a), \mathbf{b} = (b, b, \ldots, b) \in \mathbb{R}^m$. *Let* $p > 1$ *and* $h : (\mathbf{a}, \mathbf{b}) \to \mathbb{R}$ *be positive, then*
(*a*) *If* h *is decreasing, then*

$$\int_a^b \left(\frac{1}{(x-a)^m} \int_a^x \cdots \int_a^x h(\mathbf{t}) \mathbf{dt} \right)^p dx \tag{2.4.13}$$

$$\geq \frac{p^m}{mp-1} \left(\int_{\mathbf{a}}^{\mathbf{b}} \prod_i^m (x_i - a)^{p-1} \left[\frac{1}{(\max\{x_i\} - a)^{mp-1}} - \frac{1}{(b-a)^{mp-1}} \right] h^p(\mathbf{x}) \mathbf{dx} \right)$$

(*b*) *If* h *is increasing, then*

$$\int_a^b \left(\frac{1}{(b-x)^m} \int_x^b \cdots \int_x^b h(\mathbf{t}) \mathbf{dt} \right)^p dx \tag{2.4.14}$$

$$\geq \frac{p^m}{mp-1} \left(\int_{\mathbf{a}}^{\mathbf{b}} \prod_i^m (b - x_i)^{p-1} \left[\frac{1}{(b - \min\{x_i\})^{mp-1}} - \frac{1}{(b-a)^{mp-1}} \right] h^p(\mathbf{x}) \mathbf{dx} \right).$$

If " $0 < p < 1$," $p \neq \frac{1}{m}$, *then* (2.4.13) *and* (2.4.14) *hold in reversed direction.*

Proof. (a) Consider Theorem 2.4.2 with $k(t) = (t-a)^{-mp}$, $t \in \mathbf{A} = (a, b)$, $g_i(x_i) = x_i - a$, $i = 1, 2, \ldots, m$ and $\sigma(t) = (t, t, \ldots, t)$. Then $\sigma^{\leftarrow}(\mathbf{x}) = (\max\{x_i\}, b)$, so (2.4.3) becomes (2.4.13).
(a) Consider Theorem 2.4.2 with $k(t) = (t-a)^{-mp}$, $t \in \mathbf{A} = (a, b)$, $g_i(x_i) = b - x_i$, $i = 1, 2, \ldots, m$ and $\sigma(t) = (t, t, \ldots, t)$. Then $\sigma_{\leftarrow}(\mathbf{x}) = (a, \min\{x_i\})$, so (2.4.4) becomes (2.4.14).

□

Remark 2.4.4. In the similar way as in Remark 2.4.2, if $\mathbf{x} \mapsto \frac{h^{1/p}(\mathbf{x})}{\prod_{i=1}^m (x_i - a)}$ is a decreasing function, then by Corollary 2.4.6, we get power means type inequality for $p > 1$

$$\int_a^b \left(\frac{1}{(x_i-a)^m} \int_{\mathbf{a}}^{\mathbf{x}} \frac{h^{1/p}(\mathbf{t})}{\prod_i^m (t_i - a)} d\mathbf{t} \right)^p dx \tag{2.4.15}$$

$$\geq \frac{p^m}{mp-1} \left(\int_{\mathbf{a}}^{\mathbf{b}} \left[\frac{1}{(\max\{x_i\} - a)^{mp-1}} - \frac{1}{(b-a)^{mp-1}} \right] \frac{h(\mathbf{x})}{\prod_i^m (x_i - a)} d\mathbf{x} \right).$$

If " $0 < p < 1$," $p \neq \frac{1}{m}$, then (2.4.15) holds in reversed direction.

Remark 2.4.5. From the one dimensional version of the inequality from Theorem 2.4.2, we can obtain reversed Hardy type inequality for radial functions on balls in $\mathbb{R}^m$. Let $B(\mathbf{x}, R)$ denotes the ball in $\mathbb{R}^m$ of radius R around $\mathbf{x}$ and

$$C_m = \int_{\phi_{m-1}=0}^{2\pi} \int_{\phi_{m-2}=0}^{\pi} \cdots \int_{\phi_1=0}^{\pi} \sin^{m-2}(\phi_1) \sin^{m-3}(\phi_3) \cdots \sin(\phi_{m-2}) \mathbf{d}\phi$$

be the surface area of the unit sphere S^{m-1}.

Consider Theorem 2.4.2 with $\mathbf{A} = [0, R]$, $\sigma : [0, R] \mapsto \mathbb{R}$ such that $\sigma(r) = r, g(r) = r^m$ and $k(r) = r^{-m(p-1)-1}$. For a decreasing $h : [0, R] \mapsto \mathbb{R}_+$ and $p > 1$, then (2.4.3) becomes

$$\int_0^R \left(\frac{m}{r^m} \int_0^r h(t) t^{m-1} dt \right)^p r^{m-1} dr \geq \frac{p}{p-1} \int_0^R \left[1 - \frac{r^{m(p-1)}}{R^{m(p-1)}} \right] h^p(r) r^{m-1} dr. \quad (2.4.16)$$

Multiplying the above inequality by C_m and dividing and multiplying the inner L. H. S. integral by C_m, we get

$$\int_0^R \int_{\mathbf{S^{m-1}}} \left(\frac{m}{C_m r^m} \int_0^r \int_{\mathbf{S^{m-1}}} h(t) J(t, \phi) dt \mathbf{d}\phi \right)^p J(r, \phi) dr \mathbf{d}\phi \quad (2.4.17)$$

$$\geq \frac{p}{p-1} \int_0^R \int_{\mathbf{S^{m-1}}} \left[1 - \frac{r^{m(p-1)}}{R^{m(p-1)}} \right] h^p(r) J(r, \phi) dr \mathbf{d}\phi.$$

where $J(r, \phi)$ is the Jacobian of the transformation from the Cartesian to the polar coordinates. Since the volume of $B_m(0, r)$ satisfies $V(B_m(0, r)) = \frac{C_m}{m} r^m$, after transformation to the Cartesian coordinates, we have

$$\int_{\mathbf{B_m(0,R)}} \left(\frac{1}{V(B_m(0, r))} \int_{\mathbf{B_m(0,r)}} h(\|\mathbf{x}\|) \mathbf{dx} \right)^p dr \quad (2.4.18)$$

$$\geq \frac{p}{p-1} \int_{\mathbf{B_m(0,R)}} \left[1 - \frac{\|\mathbf{x}\|^{m(p-1)}}{R^{m(p-1)}} \right] h^p(\|\mathbf{x}\|) \mathbf{dx}.$$

2.4.1 n-Exponential Convexity and Refinements of Functionals Involving Multidimensional Monotone Functions

In the present section, we will construct several linear functionals as differences of the L. H. S. and R. H. S. of some of the inequalities derived in the previous section. The obtained linear functionals will be used in construction of new families of exponentially convex functions and some related results will be derived.

For the sake of simplicity and to avoid many notions, we introduce the following definitions:
(M_9) Under the assumptions of Theorem 2.1.4, we define linear functional as

$$\mathcal{L}_9(f) = f\left(\int_{\mathbf{a}}^{\mathbf{b}} h(\mathbf{x})\mathbf{d}[\mathbf{g}(\mathbf{x})]\right) - \int_{\mathbf{a}}^{\mathbf{b}} f'\left(h(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\right)h(\mathbf{x})\mathbf{d}[\mathbf{g}(\mathbf{x})].$$

(M_{10}) Under the assumptions of Theorem 2.1.4, we define linear functional as

$$\mathcal{L}_{10}(f) = f\left(\int_{\mathbf{a}}^{\mathbf{b}} h(\mathbf{x})\mathbf{d}[-\mathbf{g}(\mathbf{x})]\right) - \int_{\mathbf{a}}^{\mathbf{b}} f'\left(h(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\right)h(\mathbf{x})\mathbf{d}[-\mathbf{g}(\mathbf{x})].$$

(M_{11}) Under the assumptions of Theorem 2.4.1, we define linear functional as

$$\mathcal{L}_{11}(f) = \int_{\mathbf{A}} k(\mathbf{x}) f\left(\int_{\mathbf{a}}^{\sigma(\mathbf{x})} h(\mathbf{t})\mathbf{d}[\mathbf{g}(\mathbf{t})]\right)\mathbf{dx}$$
$$-\int_{\mathbf{a}}^{\mathbf{b}} K_1(\mathbf{x}) f'\left(h(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\right)h(\mathbf{x})\mathbf{d}[\mathbf{g}(\mathbf{x})].$$

(M_{12}) Under the assumptions of Theorem 2.4.1, we define linear functional as

$$\mathcal{L}_{12}(f) = \int_{\mathbf{A}} k(\mathbf{x}) f\left(\int_{\sigma(\mathbf{x})}^{\mathbf{b}} h(\mathbf{t})\mathbf{d}[-\mathbf{g}(\mathbf{t})]\right)\mathbf{dx}$$
$$-\int_{\mathbf{a}}^{\mathbf{b}} K_2(\mathbf{x}) f'\left(h(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\right)h(\mathbf{x})\mathbf{d}[-\mathbf{g}(\mathbf{x})].$$

Remark 2.4.6. Under the assumptions of Theorem 2.1.4 and Theorem 2.4.1 with f as a convex function the linear functionals $\mathcal{L}_k(f) \geq 0$ for $k = 9, \ldots, 12$.

Now we are ready to investigate the properties of functionals as defined above, regarding n-exponential and exponential convexity. Let $\mathbf{E}_i$, $i = 1, 2, 3$ denote a family of functions defined earlier:

Theorem 2.4.7. *Let $\mathcal{L}_k$ be the linear functionals defined as in (M_k) for $k = 9, \ldots, 12$ associated with a family $\mathbf{E}_1$. Then $t \mapsto \mathcal{L}_k(f_t)$ is an n-exponentially convex function in the Jensen sense on J. If the function $t \mapsto \mathcal{L}_k(f_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. Follow similar steps as in Theorem 2.3.5. □

The following corollary is an immediate consequence of the above theorem.

Corollary 2.4.8. *Let $\mathcal{L}_k$ be the linear functionals defined as in (M_k) for $k = 9, \ldots, 12$ associated with a family $\mathbf{E}_2$. Then $t \mapsto \mathcal{L}_k(f_t)$ is an exponentially convex function in the Jensen sense on J. If $t \mapsto \mathcal{L}_k(f_t)$ is continuous on J, then it is exponentially convex on J.*

Proof. Follows from the previous theorem. □

Corollary 2.4.9. *Let $\mathcal{L}_k$ be the linear functionals defined as in (M_k) for $k = 9, \ldots, 12$ associated with a family $\mathbf{E}_3$. Then the following statements hold:*

(i) If the function $t \mapsto \mathcal{L}_k(f_t)$ is strictly positive and continuous on J, then it is 2-exponentially convex on J, and thus, log-convex. Also for $r, s, t \in J$ such that $r < s < t$, we have

$$(\mathcal{L}_k(f_s))^{t-r} \leq (\mathcal{L}_k(f_r))^{t-s}(\mathcal{L}_k(f_t))^{s-r}. \tag{2.4.19}$$

(ii) If the function $t \mapsto \mathcal{L}_k(f_t)$ is strictly positive and differentiable on J, then for all $t, r, u, v \in J$ such that $t \leq u$, $r \leq v$, we have

$$\mathfrak{B}(t, r; \mathcal{L}_k, \mathbf{E}_3) \leq \mathfrak{B}(u, v; \mathcal{L}_k, \mathbf{E}_3), \;\; k = 9, \ldots, 12,$$

where

$$\mathfrak{B}(t, r; \mathcal{L}_k, \mathbf{E}_3) = \begin{cases} \left(\frac{\mathcal{L}_k(f_t)}{\mathcal{L}_k(f_r)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{\frac{d}{dt}(\mathcal{L}_k(f_t))}{\mathcal{L}_k(f_t)}\right), & t = r. \end{cases} \tag{2.4.20}$$

Proof. Follow similar steps as in Corollary 2.3.7. □

Remark 2.4.7. Also, the comments given in Remark 2.3.4 are valid for the functionals $\mathcal{L}_k$ for $k = 9, \ldots, 12$.

The following Corollary is an immediate consequence of Corollary 2.4.9 by considering the family of functions given in (2.1.15) and applying it to the functionals $\mathcal{L}_k$ for $k = 9, \ldots, 12$.

Corollary 2.4.10. *If $p < q < r$ $(p, q, r \in \mathbb{R}^+\backslash\{1\})$, then the following inequalities hold.*

$$\left[\frac{q^{m-1}\left(\int_{\mathbf{a}}^{\mathbf{b}} h(\mathbf{x})\mathbf{d}[\mathbf{g}(\mathbf{x})]\right)^q - \int_{\mathbf{a}}^{\mathbf{b}} h^q(\mathbf{x})\mathbf{d}[\mathbf{g}^q(\mathbf{x})]}{q^m(q-1)}\right]^{r-p} \tag{2.4.21}$$

$$\leq \left[\frac{p^{m-1}\left(\int_{\mathbf{a}}^{\mathbf{b}} h(\mathbf{x})\mathbf{d}[\mathbf{g}(\mathbf{x})]\right)^{p} - \int_{\mathbf{a}}^{\mathbf{b}} h^{p}(\mathbf{x})\mathbf{d}[\mathbf{g}^{p}(\mathbf{x})]}{p^{m}(p-1)}\right]^{r-q}$$

$$\left[\frac{r^{m-1}\left(\int_{\mathbf{a}}^{\mathbf{b}} h(\mathbf{x})\mathbf{d}[\mathbf{g}(\mathbf{x})]\right)^{r} - \int_{\mathbf{a}}^{\mathbf{b}} h^{r}(\mathbf{x})\mathbf{d}[\mathbf{g}^{r}(\mathbf{x})]}{r^{m}(r-1)}\right]^{q-p},$$

$$\left[\frac{q^{m-1}\left(\int_{\mathbf{a}}^{\mathbf{b}} h(\mathbf{x})\mathbf{d}[-\mathbf{g}(\mathbf{x})]\right)^{q} - \int_{\mathbf{a}}^{\mathbf{b}} h^{q}(\mathbf{x})\mathbf{d}[-\mathbf{g}^{q}(\mathbf{x})]}{q^{m}(q-1)}\right]^{r-p} \tag{2.4.22}$$

$$\leq \left[\frac{p^{m-1}\left(\int_{\mathbf{a}}^{\mathbf{b}} h(\mathbf{x})\mathbf{d}[-\mathbf{g}(\mathbf{x})]\right)^{p} - \int_{\mathbf{a}}^{\mathbf{b}} h^{p}(\mathbf{x})\mathbf{d}[-\mathbf{g}^{p}(\mathbf{x})]}{p^{m}(p-1)}\right]^{r-q}$$

$$\left[\frac{r^{m-1}\left(\int_{\mathbf{a}}^{\mathbf{b}} h(\mathbf{x})\mathbf{d}[-\mathbf{g}(\mathbf{x})]\right)^{r} - \int_{\mathbf{a}}^{\mathbf{b}} h^{r}(\mathbf{x})\mathbf{d}[-\mathbf{g}^{r}(\mathbf{x})]}{r^{m}(r-1)}\right]^{q-p},$$

$$\left[\frac{q^{m-1}\int_{\mathbf{A}} k(\mathbf{x})\left(\int_{\mathbf{a}}^{\sigma(\mathbf{x})} h(\mathbf{t})\mathbf{d}[\mathbf{g}(\mathbf{t})]\right)^{q}\mathbf{dx} - \int_{\mathbf{a}}^{\mathbf{b}} K_1(\mathbf{x})h^{q}(\mathbf{x})\mathbf{d}[\mathbf{g}^{q}(\mathbf{x})]}{q^{m}(q-1)}\right]^{r-p} \tag{2.4.23}$$

$$\leq \left[\frac{p^{m-1}\int_{\mathbf{A}} k(\mathbf{x})\left(\int_{\mathbf{a}}^{\sigma(\mathbf{x})} h(\mathbf{t})\mathbf{d}[\mathbf{g}(\mathbf{t})]\right)^{p}\mathbf{dx} - \int_{\mathbf{a}}^{\mathbf{b}} K_1(\mathbf{x})h^{p}(\mathbf{x})\mathbf{d}[\mathbf{g}^{p}(\mathbf{x})]}{p^{m}(p-1)}\right]^{r-q}$$

$$\left[\frac{r^{m-1}\int_{\mathbf{A}} k(\mathbf{x})\left(\int_{\mathbf{a}}^{\sigma(\mathbf{x})} h(\mathbf{t})\mathbf{d}[\mathbf{g}(\mathbf{t})]\right)^{r}\mathbf{dx} - \int_{\mathbf{a}}^{\mathbf{b}} K_1(\mathbf{x})h^{r}(\mathbf{x})\mathbf{d}[\mathbf{g}^{r}(\mathbf{x})]}{r^{m}(r-1)}\right]^{q-p},$$

$$\left[\frac{q^{m-1}\int_{\mathbf{A}} k(\mathbf{x})\left(\int_{\sigma(\mathbf{x})}^{\mathbf{b}} h(\mathbf{t})\mathbf{d}[-\mathbf{g}(\mathbf{t})]\right)^{q}\mathbf{dx} - \int_{\mathbf{a}}^{\mathbf{b}} K_2(\mathbf{x})h^{q}(\mathbf{x})\mathbf{d}[-\mathbf{g}^{q}(\mathbf{x})]}{q^{m}(q-1)}\right]^{r-p} \tag{2.4.24}$$

$$\leq \left[\frac{p^{m-1}\int_{\mathbf{A}} k(\mathbf{x})\left(\int_{\sigma(\mathbf{x})}^{\mathbf{b}} h(\mathbf{t})\mathbf{d}[-\mathbf{g}(\mathbf{t})]\right)^{p}\mathbf{dx} - \int_{\mathbf{a}}^{\mathbf{b}} K_2(\mathbf{x})h^{p}(\mathbf{x})\mathbf{d}[-\mathbf{g}^{p}(\mathbf{x})]}{p^{m}(p-1)}\right]^{r-q}$$

$$\left[\frac{r^{m-1}\int_{\mathbf{A}} k(\mathbf{x})\left(\int_{\sigma(\mathbf{x})}^{\mathbf{b}} h(\mathbf{t})\mathbf{d}[-\mathbf{g}(\mathbf{t})]\right)^{r}\mathbf{dx} - \int_{\mathbf{a}}^{\mathbf{b}} K_2(\mathbf{x})h^{r}(\mathbf{x})\mathbf{d}[-\mathbf{g}^{r}(\mathbf{x})]}{r^{m}(r-1)}\right]^{q-p}.$$

Proof. For family of functions f_p the function $p \mapsto \mathcal{L}_k(f_p)$, $k = 9, \ldots, 12$ is log-convex by Corollary 2.4.9(i). Hence, for $p < q < r$, we get inequalities (2.4.31)-(2.4.24). $\square$

2.4.2 Mean-Value Theorems of Functionals Involving Multidimensional Monotone Functions

In this section, we will give Lagrange and Cauchy type mean-value theorems for the linear functionals $\mathcal{L}_k,\ k = 9, \ldots, 12$ defined by $(M_9) - (M_{12})$. Also, assume that $\mathcal{L}_k(f)$ are well defined for $f \in C^2[0, c]$.

Theorem 2.4.11. *Let $\mathcal{L}_k,\ k = 9, \ldots, 12$ be linear functionals defined by $(M_9)-(M_{12})$ and $f \in C^2[0, c]$, $c > 0$, such that $f(0) = 0$ and $\mathbf{lim}_{t\to 0} tf'(t) = 0$. Then there exists $\xi_k \in [0, c]$ such that the identity*

$$\mathcal{L}_k(f) = \frac{f''(\xi_k)}{2}\mathcal{L}_k(x^2) \tag{2.4.25}$$

holds for $k = 9, \ldots, 12$.

Proof. Follow similar steps as in Theorem 2.3.10. □

Theorem 2.4.12. *Let $\mathcal{L}_k,\ k = 9, \ldots, 12$ be linear functionals defined by $(M_9)-(M_{12})$ and $f_1, f_2 \in C^2[0, c]$, $c > 0$ such that $f_1(0) = 0 = f_2(0)$ and $\mathbf{lim}_{t\to 0} tf_1'(t) = 0 = \mathbf{lim}_{t\to 0} tf_2'(t)$. Then there exists $\xi_k \in [0, c]$ such that the equality*

$$\frac{\mathcal{L}_k(f_1}{\mathcal{L}_k(f_2)} = \frac{f_1''(\xi_k)}{f_2''(\xi_k)} \tag{2.4.26}$$

holds for $k = 9, \ldots, 12$, provided that the denominators are non zero.

Proof. Follow similar steps as in Theorem 2.3.11. □

Corollary 2.4.13. *Let $\mathcal{L}_k,\ k = 9, \ldots, 12$ be linear functionals defined by $(M_9)-(M_{12})$ and f_t functions given by (2.1.15). Then, for distinct positive real numbers l and r, there exists $\xi_k \in [0, c]$ such that*

$$\xi_k^{l-r} = \frac{\mathcal{L}_k(f_l)}{\mathcal{L}_k(f_r)} \tag{2.4.27}$$

holds for $k = 9, \ldots, 12$.

Proof. Taking $f_1 = f_l$ and $f_2 = f_r$ given by (2.1.15) in (2.4.26), for distinct positive real numbers l and r we obtain (2.4.27). □

Remark 2.4.8. Since for fix $k = 9, \ldots, 12$ the function $\xi_k \to \xi_k^{l-r}$, $l \neq r$ is invertible, then from (2.4.27) we get

$$m \leq \left(\frac{\mathcal{L}_k(f_l)}{\mathcal{L}_k(f_r)}\right)^{\frac{1}{l-r}} \leq M; \qquad r \neq l. \tag{2.4.28}$$

2.4.3 Cauchy Means of Functionals Involving Multidimensional Monotone Functions

In the present section, we deduce Cauchy means from Theorem 2.4.12. Suppose that f_1''/f_2'' has inverse. Then (2.4.26) gives

$$\xi_k = \Big(\frac{f_1''}{f_2''}\Big)^{-1}\left(\frac{\mathcal{L}_k(f_1)}{\mathcal{L}_k(f_2)}\right). \tag{2.4.29}$$

We conclude that the expression on the R. H. S. of the above equation is a Cauchy type mean of the interval $[0, c]$. Before giving Cauchy means, we introduce some notions for our convenience. Let us denote

$$\mathbf{H}_p(\mathbf{a}, \mathbf{b}, \mathbf{K}, h, \mathbf{g}) = \left(K(\mathbf{x})\int_{\mathbf{a}}^{\mathbf{b}} h^p(\mathbf{x})\mathbf{d}[\mathbf{g}^p(\mathbf{x})]\right)^{1/p},$$

$$\widetilde{\mathbf{H}}_p(\mathbf{a}, \mathbf{b}, \mathbf{K}, h, \mathbf{g}) = \left(K(\mathbf{x})\int_{\mathbf{a}}^{\mathbf{b}} h^p(\mathbf{x})\mathbf{d}[-\mathbf{g}^p(\mathbf{x})]\right)^{1/p},$$

$$\mathbf{R}_l^n(\mathbf{K}, h, \mathbf{g}) = \left(K(\mathbf{x})\int_{\mathbf{a}}^{\mathbf{b}} \Big(\frac{1}{l} + \log\big(h(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\big)\Big)^n h^l(\mathbf{x})\mathbf{d}\big[\mathbf{g}^l(\mathbf{x})\big]\right)$$

and

$$\widetilde{\mathbf{R}}_l^n(\mathbf{K}, h, \mathbf{g}) = \left(K(\mathbf{x})\int_{\mathbf{a}}^{\mathbf{b}} \Big(\frac{1}{l} + \log\big(h(\mathbf{x})\prod_{i=1}^{m} g_i(x_i)\big)\Big)^n h^l(\mathbf{x})\mathbf{d}\big[-\mathbf{g}^l(\mathbf{x})\big]\right).$$

For the family of functions f_p given by (2.1.15) and $r, l \in \mathbb{R}^+$, we denote the Cauchy means

$$M_{l,r}^k = \left(\frac{\mathcal{L}_k(f_l)}{\mathcal{L}_k(f_r)}\right)^{\frac{1}{l-r}}, \ r \neq l. \tag{2.4.30}$$

Also, we have continuous extensions of these means in other cases. Therefore by taking respective limits, we have the following $M_{r,r}^9 =$

$$\begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{\mathbf{H}_1^r(\mathbf{a},\mathbf{b},1,h,\mathbf{g})\log\left(\mathbf{H}_1^1(\mathbf{a},\mathbf{b},1,h,\mathbf{g})\right) - r^{1-m}\mathbf{R}_r^1(1,h,\mathbf{g})}{\mathbf{H}_1^r(\mathbf{a},\mathbf{b},1,h,\mathbf{g}) - r^{1-m}\mathbf{H}_r^r(\mathbf{a},\mathbf{b},1,h,\mathbf{g})}\right), & r \neq 1, \\ \exp\left(-1 + \frac{\mathbf{H}_1^1(\mathbf{a},\mathbf{b},1,h,\mathbf{g})\left(\log\left(\mathbf{H}_1^1(\mathbf{a},\mathbf{b},1,h,\mathbf{g})\right)\right)^2 + \mathbf{H}_1^1(\mathbf{a},\mathbf{b},1,h,\mathbf{g}) - \mathbf{R}_1^2(1,h,\mathbf{g})}{2\left(\mathbf{H}_1^1(\mathbf{a},\mathbf{b},1,h,\mathbf{g})\log\left(\mathbf{H}_1^1(\mathbf{a},\mathbf{b},1,h,\mathbf{g})\right) - \mathbf{R}_1^1(\mathbf{a},\mathbf{b},1,h,\mathbf{g})\right)}\right), & r = 1. \end{cases}$$

$M^{10}_{r,r} =$

$$\begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{\widetilde{\mathbf{H}}^r_1(\mathbf{a},\mathbf{b},1,h,\mathbf{g})\log\left(\widetilde{\mathbf{H}}^1_1(\mathbf{a},\mathbf{b},1,h,\mathbf{g})\right) - r^{1-m}\widetilde{\mathbf{R}}^1_r(1,h,\mathbf{g})}{\widetilde{\mathbf{H}}^r_1(\mathbf{a},\mathbf{b},1,h,\mathbf{g}) - r^{1-m}\widetilde{\mathbf{H}}^r_r(\mathbf{a},\mathbf{b},1,h,\mathbf{g})}\right), & r \neq 1, \\ \exp\left(-1 + \frac{\widetilde{\mathbf{H}}^1_1(\mathbf{a},\mathbf{b},1,h,\mathbf{g})\left(\log\left(\widetilde{\mathbf{H}}^1_1(\mathbf{a},\mathbf{b},1,h,\mathbf{g})\right)\right)^2 + \widetilde{\mathbf{H}}^1_1(\mathbf{a},\mathbf{b},1,h,\mathbf{g}) - \widetilde{\mathbf{R}}^2_1(1,h,\mathbf{g})}{2\left(\widetilde{\mathbf{H}}^1_1(\mathbf{a},\mathbf{b},1,h,\mathbf{g})\log\left(\widetilde{\mathbf{H}}^1_1(\mathbf{a},\mathbf{b},1,h,\mathbf{g})\right) - \widetilde{\mathbf{R}}^1_1(\mathbf{a},\mathbf{b},1,h,\mathbf{g})\right)}\right), & r = 1. \end{cases}$$

$M^{11}_{r,r} =$

$$\begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{\int_{\mathbf{A}}\left[k(\mathbf{x})\mathbf{H}^r_1(\mathbf{a},\sigma(\mathbf{x}),1,h,\mathbf{g})\log\left(\mathbf{H}^1_1(\mathbf{a},\sigma(\mathbf{x}),1,h,\mathbf{g})\right)\right]\mathbf{dx} - r^{1-m}\mathbf{R}^1_r(K_1,h,\mathbf{g})}{\int_{\mathbf{A}}k(x)\mathbf{H}^r_1(\mathbf{a},\sigma(\mathbf{x}),1,h,\mathbf{g})\mathbf{dx} - r^{1-m}\mathbf{H}^r_r(\mathbf{a},\mathbf{b},K_1,h,\mathbf{g})}\right), & r \neq 1, \\ \exp\left(-1 + \frac{\int_{\mathbf{A}}\left[k(\mathbf{x})\mathbf{H}^1_1(\mathbf{a},\sigma(\mathbf{x}),1,h,\mathbf{g})\left(\log\left(\mathbf{H}^1_1(\mathbf{a},\sigma(\mathbf{x}),1,h,\mathbf{g})\right)\right)^2\right]\mathbf{dx} + \mathbf{H}^1_1(\mathbf{a},\mathbf{b},K_1,h,\mathbf{g}) - \mathbf{R}^2_1(K_1,h,\mathbf{g})}{2\left(\int_{\mathbf{A}}k(\mathbf{x})\left[\mathbf{H}^1_1(\mathbf{a},\sigma(\mathbf{x}),1,h,\mathbf{g})\log\left(\mathbf{H}^1_1(\mathbf{a},\sigma(\mathbf{x}),1,h,\mathbf{g})\right)\right]\mathbf{dx} - \mathbf{R}^1_1(\mathbf{a},\mathbf{b},K_1,h,\mathbf{g})\right)}\right), & r = 1. \end{cases}$$

$M^{12}_{r,r} =$

$$\begin{cases} \exp\left(\frac{1-2r}{r(r-1)} + \frac{\int_{\mathbf{A}}\left[k(\mathbf{x})\widetilde{\mathbf{H}}^r_1(\sigma(\mathbf{x}),\mathbf{b},1,h,\mathbf{g})\log\left(\widetilde{\mathbf{H}}^1_1(\sigma(\mathbf{x}),\mathbf{b},1,h,\mathbf{g})\right)\right]\mathbf{dx} - r^{1-m}\widetilde{\mathbf{R}}^1_r(K_2,h,\mathbf{g})}{\int_{\mathbf{A}}k(x)\widetilde{\mathbf{H}}^r_1(\sigma(\mathbf{x}),\mathbf{b},1,h,\mathbf{g})\mathbf{dx} - r^{1-m}\widetilde{\mathbf{H}}^r_r(\mathbf{a},\mathbf{b},K_2,h,\mathbf{g})}\right), & r \neq 1, \\ \exp\left(-1 + \frac{\int_{\mathbf{A}}\left[k(\mathbf{x})\widetilde{\mathbf{H}}^1_1(\sigma(\mathbf{x}),\mathbf{b},1,h,\mathbf{g})\left(\log\left(\widetilde{\mathbf{H}}^1_1(\sigma(\mathbf{x}),\mathbf{b},1,h,\mathbf{g})\right)\right)^2\right]\mathbf{dx} + \widetilde{\mathbf{H}}^1_1(\mathbf{a},\mathbf{b},K_2,h,\mathbf{g}) - \widetilde{\mathbf{R}}^2_1(K_2,h,\mathbf{g})}{2\left(\int_{\mathbf{A}}k(\mathbf{x})\left[\widetilde{\mathbf{H}}^1_1(\sigma(\mathbf{x}),\mathbf{b},1,h,\mathbf{g})\log\left(\widetilde{\mathbf{H}}^1_1(\sigma(\mathbf{x}),\mathbf{b},1,h,\mathbf{g}),1,h,\mathbf{g})\right)\right]\mathbf{dx} - \widetilde{\mathbf{R}}^1_1(\mathbf{a},\mathbf{b},K_2,h,\mathbf{g})\right)}\right), & r = 1. \end{cases}$$

Now we deduce the monotonicity of means defined by (2.4.30) in the form of Dresher's inequality as follows.

Theorem 2.4.14. *Let $M^k_{l,r}$ be given as in (2.4.30) and $r, l, u, v \in \mathbb{R}^+$ such that $l \leq v$, $r \leq u$. Then*

$$M^k_{l,r} \leq M^k_{v,u},\ k = 9, \ldots, 12. \tag{2.4.31}$$

Proof. Similar as Theorem 2.3.15. □

2.4.4 Examples of Functionals Involving Multidimensional Monotone Functions

In the previous two sections we applied Theorem 2.4.7 to the family of functions f_p given by (2.1.15) and constructed exponentially convex functions. By using properties of exponentially convex functions, we refined the reverse Hardy inequality and constructed Cauchy means. In this section, we apply Theorem 2.4.7 to other families of convex functions to get other exponentially convex functions and Cauchy means.

Example 2.4.1. *Consider a family of functions*

$$\Upsilon_1 = \{\lambda_l : \mathbb{R} \to \mathbb{R} : l \in \mathbb{R}\}$$

defined by

$$\lambda_l(x) = \begin{cases} \frac{e^{lx}-1}{l^2}, & l \neq 0\,, \\ \frac{x^2}{2}\,, & l = 0\,. \end{cases}$$

Since $\frac{d^2\lambda_l}{dx^2}(x) = e^{lx} > 0$*, then* $l \mapsto \frac{d^2\lambda_l}{dx^2}(x)$ *is exponentially convex by definition. Analogously as in the proof of Theorem 2.4.7 we conclude that* $l \mapsto [u_0, u_1, u_2; \lambda_l]$ *is exponentially convex (and so exponentially convex in the Jensen sense).*
Notice that $\lambda_l(0) = 0$ *and* $\lim_{x\to 0} \lambda'(x)$ *is finite. By Corollary 2.4.8 we have that* $l \mapsto \mathcal{L}_k(\lambda_l)$*,* $k = 9, \ldots, 12$ *is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous so it is exponentially convex.*
For this family of functions, $\mathfrak{B}(l, r; \mathcal{L}_k, \boldsymbol{\Upsilon}_1)$*,* $k = 9, \ldots, 12$ *from* (2.4.20) *is equal to*

$$\mathfrak{B}(l, r; \mathcal{L}_k, \boldsymbol{\Upsilon}_1) = \begin{cases} \left(\frac{\mathcal{L}_k(\lambda_l)}{\mathcal{L}_k(\lambda_r)}\right)^{\frac{1}{l-r}}, & l \neq r\,, \\ \exp\left(\frac{\mathcal{L}_k(id\cdot\lambda_l)}{\mathcal{L}_k(\lambda_l)} - \frac{2}{l}\right), & l = r \neq 0\,, \\ \exp\left(\frac{\mathcal{L}_k(id\cdot\lambda_0))}{3\mathcal{L}_k(\lambda_0))}\right), & l = r = 0\,, \end{cases}$$

where id is the identity function. Also by Corollary 2.4.9 it is a monotonic in parameters l *and* r*. Applying Theorem 2.4.12 for* $f_1 = \lambda_l$ *and* $f_2 = \lambda_r$ *there exists* ξ_k*,* $k = 9, \ldots, 12$ *such that*

$$e^{(l-r)\xi_k} = \frac{\mathcal{L}_k(\lambda_l)}{\mathcal{L}_k(\lambda_r)}.$$

Therefore

$$M^k_{l,r}(\Upsilon_1) = \log \mathfrak{B}(l, r; \mathcal{L}_k, \boldsymbol{\Upsilon}_1)$$

is a Cauchy mean.

Example 2.4.2. *Consider a family of functions*

$$\Upsilon_2 = \{\omega_l : [0, \infty) \to \mathbb{R} : l > 0\}$$

defined by

$$\omega_l(x) = \begin{cases} \frac{l^{-x}-1}{\log^2 l}\,, & l \neq 1\,, \\ \frac{x^2}{2}\,, & l = 1\,. \end{cases}$$

Since $l \mapsto \frac{d^2\omega_l}{dx^2}(x) = l^{-x}$ *is the Laplace transform of a non-negative function (see [29]) that is* $l^{-x} = \dfrac{1}{\Gamma(x)} \displaystyle\int_0^{\infty} e^{-lt} t^{x-1}\, dt$ *, it is exponentially convex on* $(0, \infty)$ *(see [22]). Analogously as in the proof of Theorem 2.4.7 we conclude that* $l \mapsto [u_0, u_1, u_2; \omega_l]$ *is exponentially convex (and so exponentially convex in the Jensen sense).*
Notice that $\omega_l(0) = 0$ *and* $\lim_{x\to 0} \omega'(x)$ *is finite. By Corollary 2.4.8 we have that* $l \mapsto \mathcal{L}_k(\omega_l)$, $k = 9, \ldots, 12$ *is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous so it is exponentially convex.*
For this family of functions, $\mathfrak{B}(l, r; \mathcal{L}_k, \boldsymbol{\Upsilon}_2)$, $k = 9, \ldots, 12$ *from* (2.4.20) *is equal to*

$$\mathfrak{B}(l, r; \mathcal{L}_k, \boldsymbol{\Upsilon}_2) = \begin{cases} \left(\frac{\mathcal{L}_k(\omega_l)}{\mathcal{L}_k(\omega_r)}\right)^{\frac{1}{l-r}}, & l \neq r\,, \\ \exp\left(-\frac{\mathcal{L}_k(id\cdot\omega_l)}{l\mathcal{L}_k(\omega_l)} - \frac{2}{l\log l}\right), & l = r \neq 1\,, \\ \exp\left(-\frac{\mathcal{L}_k(id\cdot\omega_1))}{3\mathcal{L}_k(\omega_1))}\right), & l = r = 1\,, \end{cases}$$

where id is the identity function. Also by Corollary 2.4.9 it is a monotonic in parameters l *and* r. *Applying Theorem 2.4.12 for* $f_1 = \omega_l$ *and* $f_2 = \omega_r$ *there exists* ξ_k, $k = 9, \ldots, 12$ *such that*

$$\left(\frac{l}{r}\right)^{-\xi_k} = \frac{\mathcal{L}_k(\omega_l)}{\mathcal{L}_k(\omega_r)}.$$

Therefore

$$M^k_{l,r}(\Upsilon_2) = -L(l, r) \log \mathfrak{B}(l, r; \mathcal{L}_k, \boldsymbol{\Upsilon}_2)$$

is a Cauchy mean, where $L(l, r)$ *is the logarithmic mean defined by*

$$L(l, r) = \begin{cases} \frac{l-r}{\log l - \log r}\,, & l \neq r\,, \\ l\,, & l = r\,. \end{cases}$$

Example 2.4.3. *Consider a family of functions*

$$\Upsilon_3 = \{\mu_l : [0, \infty) \to l : l > 0\}$$

defined by

$$\mu_l(x) = \frac{e^{-x\sqrt{l}} - 1}{l}\,.$$

Also $l \mapsto \frac{d^2\mu_l}{dx^2}(x) = e^{-x\sqrt{l}}$ *is the Laplace transform of a non-negative function (see [29]) that is* $e^{-x\sqrt{l}} = \frac{l}{2\sqrt{\pi}} \int_0^{\infty} \frac{e^{-lt} e^{-x^2/4t}}{t\sqrt{t}}\, dt$ *, it is exponentially convex on* $(0, \infty)$ *(see [22]).*
Analogously as in the proof of Theorem 2.4.7 we conclude that $l \mapsto [u_0, u_1, u_2; \mu_l]$ *is*

exponentially convex (and so exponentially convex in the Jensen sense).
Notice that $\mu_l(0) = 0$ *and* $\lim_{x\to 0} x\mu'(x) = 0$. *By Corollary 2.4.8 we have that* $l \mapsto \mathcal{L}_k(\mu_l)$, $k = 9, \ldots, 12$ *is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous so it is exponentially convex.*
For this family of functions, $\mathfrak{B}(l, r; \mathcal{L}_k, \mathbf{\Upsilon}_3)$, $k = 9, \ldots, 12$ *from* (2.4.20) *is equal to*

$$\mathfrak{B}(l, r; \mathcal{L}_k, \mathbf{\Upsilon}_3) = \begin{cases} \left(\frac{\mathcal{L}_k(\mu_l)}{\mathcal{L}_k(\mu_r)}\right)^{\frac{1}{l-r}}, & l \neq r\,, \\ \exp\left(-\frac{\mathcal{L}_k(id\cdot\mu_l)}{2\sqrt{l}\mathcal{L}_k(\mu_l)} - \frac{1}{l}\right), & l = r\,, \end{cases}$$

where id is the identity function. Also by Corollary 2.4.9 it is a monotonic in parameters l *and* r. *Applying Theorem 2.4.12 for* $f_1 = \mu_l$ *and* $f_2 = \mu_r$ *there exists* ξ_k, $k = 9, \ldots, 12$ *such that*

$$e^{-\xi_k(\sqrt{l}-\sqrt{r})} = \frac{\mathcal{L}_k(\mu_l)}{\mathcal{L}_k(\mu_r)}.$$

Therefore

$$M^k_{l,r}(\Upsilon_3) = -(\sqrt{l} + \sqrt{r}) \log \mathfrak{B}(l, r; \mathcal{L}_k, \mathbf{\Upsilon}_3)$$

is a Cauchy mean.

Chapter 3

Hermite-Hadamard's Inequality for the Functions of the Form f/h and Related Mean-Value Theorems

Previously, we considered C-monotone and multidimensional functions in some functionals to get reversed Hardy type inequalities. In the present chapter, we consider starshaped and increasing functions of the form f/h to obtain one side of the Hermite-Hadamard's inequality and its generalization. The results of this chapter are given in [8], [9] and [11].

3.1 Introduction

Let us consider the sets of continuous, convex, starshaped and superadditive functions on $[a,b]$ respectively, given by

$$\begin{aligned}
C[a,b] &= \{f : [a,b] \to \mathbb{R} \text{ continuous}\},\\
K[a,b] &= \{f \in C[a,b]; f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y),\\
&\qquad\qquad\qquad\qquad \forall\ x, y \in [a,b], \forall\ \lambda \in [0,1]\},\\
St[a,b] &= \{f \in C[a,b]; (f(x) - f(a))/(x-a) \text{ is an increasing}\\
&\qquad\qquad\qquad\qquad \text{function } \forall\ x > a\},\\
S[a,b] &= \{f \in C[a,b]; f(x) + f(y) \leq f(x+y-a) + f(a),\\
&\qquad\qquad\qquad\qquad \forall\ x, y, x+y-a \in [a,b]\}.
\end{aligned}$$

Let us denote integral arithmetic mean of f on $[a, b]$ and arithmetic mean of a and b as follows:

$$A(f; a, b) = \frac{1}{b-a}\int_a^b f(x)dx \quad \text{and} \quad A(a, b) = \frac{a+b}{2}. \tag{3.1.1}$$

One of the most well-known inequalities in mathematics for convex functions is the Hermite-Hadamard's integral inequality given in [19](see also CH 5):

$$f\left(A(a, b)\right) \leq A(f; a, b) \leq A(f(a), f(b)). \tag{3.1.2}$$

If the function f is concave then (3.1.2) holds in the reversed direction. It gives an estimate from below and above of the mean value of a convex function. These inequalities for convex functions play an important role in nonlinear analysis.There is a large range of interesting applications of Hermite-Hadamard's inequality given in [19]. In [14] (see also[19]) L. Fejér established the following weighted generalization of (3.1.2).

Theorem 3.1.1. *Let $f \in K[a, b]$. Then the following inequality holds*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leq \int_a^b f(x)w(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)dx, \tag{3.1.3}$$

where $w : [a, b] \to \mathbb{R}$ is a nonnegative function which is integrable and symmetric about $\frac{a+b}{2}$.

G. Zabandan and A. Kilicman gave a different weighted version of the Hermite-Hadamard's inequality in [30] which is given below:

Theorem 3.1.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable convex function and $h : [a, b] \to [0, \infty]$ be a continuous function.*

(i) If h is decreasing on $[a, b]$, then

$$\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)h(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{3.1.4}$$

(ii) If h is increasing on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{\int_a^b h(x)dx}\int_a^b f(x)h(x)dx. \tag{3.1.5}$$

For $a = 0$ we denote by $C(b), K(b), St(b)$ and $S(b)$ the corresponding set of functions, submitted also to the condition $f(0) = 0$.
A. M. Bruckner and E. Ostrow have proved in [5] the strict inclusions:

$$K(b) \subset St(b) \subset S(b).$$

The following results are given in Gh. Toader [28]. For positive identity function, $p(x) = x$, the inequality

$$\int_0^x p(t) \left[\frac{f(x)}{x} - \frac{f(t)}{t} \right] dt \geq 0, \forall \ x \in [0, b] \tag{3.1.6}$$

holds for every $f \in S(b)$.

Remark 3.1.1. Clearly, for $f \in St(b)$ the inequality (3.1.6) is valid for all positive p.

Lemma 3.1.3. *For every $f \in S(b)$, the following inequality holds*

$$\int_0^x f(t)dt \leq \frac{xf(x)}{2}, \forall \ x \in [0, b]. \tag{3.1.7}$$

We can write (3.1.7) as:

$$\frac{1}{x} \int_0^x f(t)dt \leq \frac{f(x) + f(0)}{2}, \quad \text{where} \quad x \neq 0, \tag{3.1.8}$$

which is one of Hermite-Hadamard's inequalities. In (3.1.8) we see that the inequality

$$A(f; a, b) \leq A\left(f(a), f(b)\right),$$

holds for all $f \in S(b)$. Since $St(b) \subset S(b)$, the above inequality also holds for all $f \in St(b)$.

3.2 One of the Hermite-Hadamard's Inequality Involving starshaped Functions

Gh. Toader [28] considered the means defined in (3.1.1) and proved the following result.

Theorem 3.2.1. *Let $f \in St[a,b]$. Then the inequality*

$$A(f;a,b) \leq A\left(f(a), f(b)\right) \tag{3.2.1}$$

is valid, where $A(f;a,b)$ and $A\left(a,b\right)$ are defined in (3.1.1)*.*

Remark 3.2.1. If $\left(f(x) - f(a)\right)/(x-a)$ is strictly increasing for $x \in [a,b]$, then strict inequality holds in (3.2.1).

Now, we give mean value theorems for one side of the Hermite-Hadamard's inequality using starshaped function. Also, we give n-exponential convexity, related results and construct non-symmetric Stolarsky means from the functional that we obtained as the difference of one of the Hermite-Hadamard's inequality in the subsections.

3.2.1 Mean-Value Theorems for one of the Hermite-Hadamard's Inequality Involving starshaped Functions

To prove related mean-value theorems of Lagrange and Cauchy type, we consider the functions ϕ_1 and ϕ_2 defined in a following lemma.

Lemma 3.2.2. *Let $f \in C^1[a,b]$ and denote*

$$G_f(x) = \frac{f'(x)(x-a) - f(x) + f(a)}{(x-a)^2}, \quad \textit{where} \quad x \neq a. \tag{3.2.2}$$

Let $m, M \in \mathbb{R}$ be such that

$$m \leq G_f(x) \leq M \quad \textit{for all} \quad x \in [a,b]. \tag{3.2.3}$$

Let the functions ϕ_1 and ϕ_2 be defined by

$$\phi_1(x) = M(x-a)^2 - f(x) + f(a)$$

and

$$\phi_2(x) = f(x) - f(a) - m(x-a)^2.$$

Then $\phi_1, \phi_2 \in St[a,b]$.

Proof. Now

$$\left(\frac{\phi_1(x) - \phi_1(a)}{x-a}\right)' = \left(M(x-a) - \frac{f(x) - f(a)}{x-a}\right)' = M - G_f(x) \geq 0,$$

and
$$\left(\frac{\phi_2(x)-\phi_2(a)}{x-a}\right)' = \left(\frac{f(x)-f(a)}{x-a} - m(x-a)\right)' = G_f(x) - m \geq 0.$$
This gives that $\phi_1, \phi_2 \in St[a,b]$. □

Theorem 3.2.3. *Let $f \in C^1[a,b]$, $G_f \in C[a,b]$ as defined in Lemma 3.2.2, $A(f;a,b)$ the integral arithmetic mean of f on $[a,b]$ and $A(f(a),f(b))$ the arithmetic mean of $f(a)$ and $f(b)$. Then there exists $\xi \in [a,b]$ such that the identity*
$$A(f(a),f(b)) - A(f;a,b) = \frac{(b-a)^2}{6} G_f(\xi). \tag{3.2.4}$$
holds.

Proof. Since G_f is continuous on a compact set, it attains its maximum and minimum value on $[a,b]$. Let us consider
$$m = \min\{G_f(x)\}$$
and
$$M = \max\{G_f(x)\}.$$
Applying Theorem 3.2.1 on functions ϕ_1 and ϕ_2 defined in Lemma 3.2.2, we get the following inequalities:
$$A(f(a),f(b)) - A(f;a,b) \leq M\frac{(b-a)^2}{6}$$
$$A(f(a),f(b)) - A(f;a,b) \geq m\frac{(b-a)^2}{6}.$$
Combining both inequalities, we get
$$m\frac{(b-a)^2}{6} \leq A(f(a),f(b)) - A(f;a,b) \leq M\frac{(b-a)^2}{6}.$$
Since G_f is continuous on $[a,b]$, there exists $\xi \in [a,b]$ such that (3.2.4) holds and the proof is completed. □

Theorem 3.2.4. *Let $f, g \in C^1[a,b]$, $G_f, G_g \in C[a,b]$ as defined in Lemma 3.2.2, $A(f;a,b)$ the integral arithmetic mean of f on $[a,b]$ and $A(f(a),f(b))$ the arithmetic mean of $f(a)$ and $f(b)$. Then there exists $\xi \in [a,b]$ such that the equality*
$$\frac{A(f(a),f(b)) - A(f;a,b)}{A(g(a),g(b)) - A(g;a,b)} = \frac{f'(\xi)(\xi-a) - f(\xi) + f(a)}{g'(\xi)(\xi-a) - g(\xi) + g(a)},$$
holds, provided that the denominators are nonzero.

Proof. Similar as Theorem 2.3.11 □

3.2.2 n-Exponential Convexity of Functional Involving starshaped Functions

To start with the present section and own ward, we consider the following construction of our functional as:
(M_{13}) Under the assumptions of Theorem 3.2.1, we define liner functional as

$$\mathcal{L}_{13}(f) = A\left(f(a), f(b)\right) - A(f; a, b) \tag{3.2.5}$$

where $A(f; a, b)$ and $A\left(f(a), f(b)\right)$ are defined in (3.1.1).

Remark 3.2.2. Under the assumptions of Theorem 3.2.1, if f is a starshaped function on $[a, b]$ then $\mathcal{L}_{13}(f) \geq 0$.

Now we are ready to explore the properties of the functional $\mathcal{L}_{13}(f)$ regarding n-exponential and exponential convexity. In order to obtain our main results, we define different families of functions. Let $[a, b]$ and $J \subseteq \mathbb{R}$ be intervals. For distinct points $u_0, u_1 \in [a, b]$, let $\mathbf{E}_i$, $i = 4, 5, 6$, denote a family of functions with the following property:

$\mathbf{E}_4 = \{f_t : [a, b] \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1; F_t]$ is n-exponentially convex in the Jensen sense on J, where $F_t(u) = \frac{f_t(u)-f_t(a)}{u-a}\}$.

$\mathbf{E}_5 = \{f_t : [a, b] \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1; F_t]$ is exponentially convex in the Jensen sense on J, where $F_t(u) = \frac{f_t(u)-f_t(a)}{u-a}\}$.

$\mathbf{E}_6 = \{f_t : [a, b] \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1; F_t]$ is 2-exponentially convex in the Jensen sense on J, where $F_t(u) = \frac{f_t(u)-f_t(a)}{u-a}\}$.

Theorem 3.2.5. *Let $\mathcal{L}_{13}$ be the linear functional defined as in (M_{13}) associated with a family $\mathbf{E}_4$. Then $t \mapsto \mathcal{L}_{13}(f_t)$ is an n-exponentially convex function in the Jensen sense on J. If the function $t \mapsto \mathcal{L}_{13}(f_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. Follow similar steps as in Theorem 2.3.5 and using Remark 1.2.1 when divided difference of two points is nonnegative. □

Corollary 3.2.6. *Let $\mathcal{L}_{13}$ be the linear functional defined as in (M_{13}) associated with a family $\mathbf{E}_5$. Then $t \mapsto \mathcal{L}_{13}(f_t)$ is an exponentially convex function in the Jensen sense on J. If the function $t \mapsto \mathcal{L}_{13}(f_t)$ is continuous on J, then it is exponentially convex on J.*

Proof. Follows from the previous theorem. □

Corollary 3.2.7. *Let $\mathcal{L}_{13}$ be the linear functional defined as in (M_{13}) associated with a family $\mathbf{E}_6$. Then the following statements hold:*

(i) If the function $t \mapsto \mathcal{L}_{13}(f)$ is strictly positive and continuous on J, then it is 2-exponentially convex on J, and thus, log-convex. Also for $r, s, t \in J$ such that $r < s < t$, we have

$$(\mathcal{L}_{13}(f_s))^{t-r} \leq (\mathcal{L}_{13}(f_r))^{t-s}(\mathcal{L}_{13}(f_t))^{s-r}. \tag{3.2.6}$$

(ii) If the function $t \mapsto \mathcal{L}_{13}(f_t)$ is strictly positive and differentiable on J, then for all $t, r, u, v \in J$ such that $t \leq u$, $r \leq v$, we have

$$\mathfrak{B}(t, r; \mathcal{L}_{13}, \mathbf{E}_6) \leq \mathfrak{B}(u, v; \mathcal{L}_{13}, \mathbf{E}_6), \; k = 1, \ldots, 8,$$

where

$$\mathfrak{B}(t, r; \mathcal{L}_{13}, \mathbf{E}_6) = \begin{cases} \left(\frac{\mathcal{L}_{13}(f_t)}{\mathcal{L}_{13}(f_r)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{\frac{d}{d_t}(\mathcal{L}_{13}(f_t))}{\mathcal{L}_{13}(f_t)}\right), & t = r. \end{cases} \tag{3.2.7}$$

Proof. Follow similar steps as in Corollary 2.3.7. □

Remark 3.2.3. Also, the comments given in Remark 2.3.4 are valid for the functional $\mathcal{L}_{13}$.

3.2.3 Non-Symmetric Stolarsky Means of Functional Involving starshaped Functions

Let $r, s \in \mathbb{R}$ and let $a, b > 0$. The Stolarsky mean $E(a, b; r, s)$ of order (r, s) of a and b with $a \neq b$ are defined as

$$\begin{aligned} E(a, b; r, s) &= \left\{\frac{r(b^s - a^s)}{s(b^r - a^r)}\right\}^{\frac{1}{s-r}}, \text{ for } r \neq s, rs \neq 0; \\ E(a, b; r, 0) &= E(a, b; 0, r) = \left\{\frac{b^r - a^r}{r(\log b - \log a)}\right\}^{1/r}, \text{ for } r \neq 0; \\ E(a, b; r, r) &= e^{-\frac{1}{r}}\left(\frac{a^{a^r}}{b^{b^r}}\right)^{1/(a^r - b^r)}, \text{ for } r \neq 0; \\ E(a, b; 0, 0) &= \sqrt{ab}. \end{aligned}$$

Stolarsky[27] in 1975 (see also [19, page 120]) introduced these means. He also proved that the function $E(r, s; a, b)$ is increasing in both r and s. One can note that these

means are symmetric with respect to the variable a and b. In [23] and [25], new classes of symmetric means of Stolarsky type are introduced. In this section we consider a class of starshaped functions to introduce means of Stolarsky type with functional due to the difference of Hermite-Hadamard inequality.
Consider a family of functions

$$\Upsilon_4 = \{f_t : (0, \infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$f_t(x) = \begin{cases} \frac{(x-a)x^t}{t}, & t \neq 0; \\ (x-a)\log x, & t = 0. \end{cases} \tag{3.2.8}$$

Then $F_t(x) := (f_t(x) - f_t(a))/(x - a)$ is strictly increasing for $x \in (0, \infty)$ and for each $t \in \mathbb{R}$. One can note that $t \mapsto [u_0, u_0; F_t]$ is log-convex for all $t \in \mathbb{R}$ and hence $t \mapsto \mathcal{L}_{13}(f_t)$ is log-convex. Also for $r < s < t$, where $r, s, t \in \mathbb{R}$, we have

$$(\mathcal{L}_{13}(f_s))^{t-r} \leq (\mathcal{L}_{13}(f_r))^{t-s} (\mathcal{L}_{13}(f_t))^{s-r}. \tag{3.2.9}$$

From Corollary 3.2.7, we can define, for $t \neq r$ and $t, r \neq 0, -1, -2$,

$$\mathfrak{B}(t, r; \mathcal{L}_{13}, \Upsilon_4) = \left(\frac{r(r+1)(r+2)}{t(t+1)(t+2)} \frac{(B_1 t^2 + (B_2 - a^2)t + 2a^2)\, b^t - 2a^{t+2}}{(B_1 r^2 + (B_2 - a^2)r + 2a^2)\, b^r - 2a^{r+2}} \right)^{\frac{1}{t-r}} \tag{3.2.10}$$

where $B_1 = (b-a)^2$ and $B_2 = (b-2a)^2$, and for $t = r$ and $t \neq 0, -1, -2$,

$$\mathfrak{B}(t, r; \mathcal{L}_{13}, \Upsilon_4) = \exp\left(-\frac{3t^2+6t+2}{t(t+1)(t+2)} + \frac{\left(2B_1 t + B_2^2 - a^2\right) b^t + \left(B_1 t^2 + (B_2 - a^2)t + 2a^2\right) b^t \log b - 2a^{t+2}\log a}{(B_1 t^2 + (B_2 - a^2)t + 2a^2) b^t - 2a^{t+2}} \right).$$

However, to get the continuous extension of (3.2.10) in order to cover all choices of r and t, we consider the following.
For $t \neq 0, -1, -2$,

$$\mathfrak{B}(t, 0; \mathcal{L}_{13}, \Upsilon_4) = \left(\frac{2}{t(t+1)(t+2)} \frac{\left(B_1 t^2 + (B_2 - a^2)t + 2a^2\right) b^t - 2a^{t+2}}{B_2 - a^2(1 - 2\log b + 2\log a)} \right)^{\frac{1}{t}},$$

$$\mathfrak{B}(t, -1; \mathcal{L}_{13}, \Upsilon_4) = \left(\frac{-b}{t(t+1)(t+2)} \frac{\left(B_1 t^2 + (B_2 - a^2)t + 2a^2\right) b^t - 2a^{t+2}}{(\log b - 2)B_1 + (1 - \log b)B_2 - a^2(1 - 3\log b) - 2ab\log a} \right)^{\frac{1}{t+1}},$$

$$\mathfrak{B}(t, -2; \mathcal{L}_{13}, \Upsilon_4) = \left(\frac{2b^2}{t(t+1)(t+2)} \frac{\left(B_1 t^2 + (B_2 - a^2)t + 2a^2\right) b^t - 2a^{t+2}}{4(\log b - 1)B_1 + (1 - 2\log b)B_2 - a^2(1 - 4\log b) - 2b^2 \log a} \right)^{\frac{1}{t+2}}.$$

$$\mathfrak{B}(0, 0; \mathcal{L}_{13}, \Upsilon_4) = \exp\left(\frac{2B_1 + (2\log b - 3)B_2 + a^2\left(3 - 8\log b + 6\log a + 2(\log b)^2 - 2(\log a)^2\right)}{2(B_2 - a^2(1 + 2\log a - 2\log b))} \right).$$

$$\mathfrak{B}(-1, -1; \mathcal{L}_{13}, \Upsilon_4) = \exp\left(\frac{\left(2 - 4\log b + (\log b)^2\right) B_1 + (2 - \log b) B_2 \log b + a((3\log b - 2)a\log b - 2b(\log a)^2)}{2[(\log b - 2)B_1 + (1 - \log b)B_2 - a^2(1 - 3\log b) - 2ab\log a]} \right).$$

$$\mathfrak{B}\left(-2,-2;\mathcal{L}_{13},\Upsilon_4\right)$$
$$=\exp\left(\frac{2\left(-5+2\log b+2(\log b)^2\right)B_1+\left(3-4\log b-2(\log b)^2\right)B_2+a^2\left(-3+10\log b+4(\log b)^2\right)-2b^2(3+\log a)\log a}{2(4(\log b-1)B_1+(1-2\log b)B_2+a^2(-1+4\log b)-2b^2\log a)}\right).$$

Also note that if the function $t\mapsto\mathcal{L}_{13}(f_t)$ is positive and differentiable on $\mathbb{R}$ then for every $t,r,u,v\in\mathbb{R}$ such that $t\le u$, $r\le v$, we have

$$\mathfrak{B}(t,r;\mathcal{L}_{13},\Upsilon_4)\le\mathfrak{B}(u,v;\mathcal{L}_{13},\Upsilon_4).$$

If we apply Theorem 3.2.4 on functions $f=f_t$ and $g=f_r$, where $t\neq r$, then there exists some $\xi\in[a,b]$ such that

$$\frac{A(f_t(a),f_t(b))-A(f_t;a,b)}{A(f_r(a),f_r(b))-A(f_r;a,b)}=\xi^{t-r}.$$

Since the function $\xi\mapsto\xi^{t-r}$ is invertible for $t\neq r$, we then have

$$a\le\left(\frac{A(f_t(a),f_t(b))-A(f_t;a,b)}{A(f_r(a),f_r(b))-A(f_r;a,b)}\right)^{\frac{1}{t-r}}\le b,$$

that is

$$a\le\mathfrak{B}(t,r;\mathcal{L}_{13},\Upsilon_4)\le b,$$

which together with the fact that $\mathfrak{B}(t,r;\mathcal{L}_{13},\Upsilon_4)$ is continuous and monotonous with respect to its both arguments t and r, shows that shows that $\mathfrak{B}(t,r;\mathcal{L}_{13},\Upsilon_4)$ are means of a and b for all $t,r\in\mathbb{R}$. These means are non-symmetric with respect to its variable a and b.

3.3 Inequalities for Increasing Functions of the Form f/h

In the previous section, we considered one of the Hermite-Hadamard's inequality functional using starshaped functions and gave mean value theorems, n-exponential convexity and non-symmetric Stolarsky means. In this section we first give some interesting inequalities involving increasing functions of the form f/h and then construct some linear functionals from the differences of these inequalities to give some interesting properties. For this, we consider the following theorems.

Theorem 3.3.1. *Let h,g be positive integrable functions on $[a,b]$ and f be an integrable function such that f/h is an increasing function on $[a,b]$. Then the following results hold:*

(i) The inequality

$$\frac{\int_a^x g(t)f(t)dt}{\int_a^x g(t)h(t)dt} \le \frac{f(x)}{h(x)} \le \frac{\int_x^b g(t)f(t)dt}{\int_x^b g(t)h(t)dt} \tag{3.3.1}$$

is valid.

(ii) ϕ_1 *and* ϕ_2 *are increasing functions on* $[a, b]$, *where*

$$\phi_1(x) = \frac{\int_a^x g(t)f(t)dt}{\int_a^x g(t)h(t)dt} \qquad \textit{and} \qquad \phi_2(x) = \frac{\int_x^b g(t)f(t)dt}{\int_x^b g(t)h(t)dt}.$$

(iii) The following results for integrals are valid

$$\frac{\int_a^x g(t)f(t)dt}{\int_a^x g(t)h(t)dt} \le \frac{\int_a^b g(t)f(t)dt}{\int_a^b g(t)h(t)dt} \le \frac{\int_x^b g(t)f(t)dt}{\int_x^b g(t)h(t)dt}. \tag{3.3.2}$$

Proof. (i) Since f/h is an increasing function on $[a, b]$, therefore

$$\int_a^x h(t)g(t)\left[\frac{f(x)}{h(x)} - \frac{f(t)}{h(t)}\right]dt \ge 0$$

holds. Thus by rearranging, we get

$$\frac{f(x)}{h(x)}\int_a^x h(t)g(t)dt \ge \int_a^x g(t)f(t)dt,$$

hence, we have

$$\frac{f(x)}{h(x)} \ge \frac{\int_a^x g(t)f(t)dt}{\int_a^x g(t)h(t)dt}. \tag{3.3.3}$$

Similarly taking into account the difference

$$\int_x^b h(t)g(t)\left[\frac{f(t)}{h(t)} - \frac{f(x)}{h(x)}\right]dt \ge 0,$$

we have

$$\frac{f(x)}{h(x)} \le \frac{\int_x^b g(t)f(t)dt}{\int_x^b g(t)h(t)dt}. \tag{3.3.4}$$

Combining (3.3.3) and (3.3.4) we get (3.3.1). (ii) We have defined

$$\phi_1(x) = \frac{\int_a^x g(t)f(t)dt}{\int_a^x g(t)h(t)dt}$$

so by first derivative test we have that ϕ_1 is an increasing function on $[a,b]$ since

$$\phi_1'(x) = \frac{g(x)f(x)\int_a^x g(t)h(t)dt - g(x)h(x)\int_a^x g(t)f(t)dt}{\left(\int_a^x g(t)h(t)dt\right)^2} \geq 0$$

by (3.3.3). Similarly, by using (3.3.4) we can prove that ϕ_2 is an increasing function on $[a,b]$. (iii) As we have that ϕ_1 and ϕ_2 are increasing functions on $[a,b]$, we have

$$\phi_1(x) \leq \phi_1(b) \qquad \text{and} \qquad \phi_2(a) \leq \phi_2(x).$$

So, we get (3.3.2) by noting that $\phi_1(b) = \phi_2(a)$. □

Theorem 3.3.2. *Let h be any positive integrable function on $[a,b]$ and f be an integrable function such that f/h is an increasing function on $[a,b]$. Let g be an integrable function on $[a,b]$, such that*

$0 < Q(x) < Q(b)$ for all $x \in [a,b]$, where $Q(x) = \int_a^x g(t)h(t)dt$. Then (3.3.2) is valid.

Proof. For the sake of convenience, we will denote $F = f/h$ and $q = gh$ and consider the following integral

$$\int_a^x q(t)k(t)dt \quad \text{where} \quad \left(k(t) = \int_a^b q(s)[F(s) - F(t)]ds\right).$$

Using the weighted Montgomery's identity given in [24], that is

$$\int_a^b q(x)f(x)dx = f(c)Q(b) - \int_a^c Q(x)df(x) + \int_c^b \overline{Q}(x)df(x),$$

where $c \in [a,b]$, $\overline{Q}(x) = Q(b) - Q(x)$. We get

$$\begin{aligned}\int_a^x q(t)k(t)dt &= k(c)Q(x) - \int_a^c Q(t)dk(t)dt + \int_c^x \overline{Q}(t)dk(t)dt \\ &= k(x)\int_a^x q(t)dt - \int_a^x Q(t)dk(t), \quad (3.3.5)\end{aligned}$$

where

$$dk(t) = -Q(b)dF(t) \qquad \text{and} \qquad k(x) = -\int_a^x Q(t)dF(t) + \int_x^b \overline{Q}(t)dF(t).$$

Substituting the values of $dk(t)$ and $k(x)$ in (3.3.5), we get

$$\int_a^x q(t)k(t)dt = \overline{Q}(x)\int_a^x Q(t)dF(t) + Q(x)\int_x^b \overline{Q}(t)dF(t). \tag{3.3.6}$$

The right hand side of the above inequality is positive, since F is an increasing function on $[a,b]$, $0 < Q(x) < Q(b)$ and $\overline{Q}(x) > 0$. Hence , we have

$$\int_a^x q(t)k(t)dt = \int_a^x q(t)\Big(\int_a^b q(s)[F(s) - F(t)]ds\Big)dt \geq 0.$$

So by replacing the respective notations, we get

$$\int_a^x g(t)h(t)dt \int_a^b g(s)f(s)ds - \int_a^x g(t)f(t)dt \int_a^b g(s)h(s)ds \geq 0.$$

So be rearranging and using the property of definite integral, we have

$$\frac{\int_a^x g(t)f(t)dt}{\int_a^x g(t)h(t)dt} \leq \frac{\int_a^b g(t)f(t)dt}{\int_a^b g(t)h(t)dt}. \tag{3.3.7}$$

Now again using the property of definite integral, we can write

$$\int_a^x f(t)dt = \int_a^b f(t)dt - \int_x^b f(t)dt.$$

So by putting the value of $\int_a^x f(t)dt$ in (3.3.7), we get

$$\frac{\int_a^b g(t)f(t)dt - \int_x^b g(t)f(t)dt}{\int_a^b g(t)h(t)dt - \int_x^b g(t)h(t)dt} \leq \frac{\int_a^b g(t)f(t)dt}{\int_a^b g(t)h(t)dt}.$$

After simplification, we have

$$\frac{\int_a^b g(t)f(t)dt}{\int_a^b g(t)h(t)dt} \leq \frac{\int_x^b g(t)f(t)dt}{\int_x^b g(t)h(t)dt}. \tag{3.3.8}$$

Hence by combining (3.3.7) and (3.3.8), we get (3.3.2) □

Corollary 3.3.3. *Let f be an integrable function such that $\frac{f(t)-f(a)}{(t-a)^\alpha}$ is an increasing function on $[a,b]$, then generalized Hadamard's inequality*

$$\frac{1}{x-a}\int_a^x f(t)dt \leq \frac{f(x) + \alpha f(a)}{1+\alpha}. \tag{3.3.9}$$

is valid.

Proof. Substituting $g(t) = 1$, $h(t) = (t-a)^{\alpha}$, and $f(t) \to f(t) - f(a)$ in the first part of the inequality (3.3.1), we get

$$\frac{\int_a^x [f(t) - f(a)]dt}{\int_a^x (t-a)^{\alpha} dt} \leq \frac{f(x) - f(a)}{(x-a)^{\alpha}}.$$

After simplification we get (3.3.9). □

Remark 3.3.1. Also we can obtain generalized Hadamard's inequality

$$\frac{1}{b-x} \int_x^b f(t)dt \leq \frac{f(x) + \beta f(b)}{1+\beta} \tag{3.3.10}$$

by substituting $g(t) = 1$, $h(t) = (b-t)^{\beta}$, and $f(t) \to f(b) - f(t)$ in the second part of the inequality (3.3.1) and then simplifying the integrals we get (3.3.10).

Remark 3.3.2. Setting $\big(\alpha = 1,\ x = b\big)$ in (3.3.9) and $\big(\beta = 1,\ x = a\big)$ in (3.3.10), we get Hadamard's inequality (3.2.1).

Remark 3.3.3. We can also obtain (3.3.1) by using inequality of integrals for increasing functions on $[a, b]$. Since we have

$$\frac{\int_a^x g(t)f(t)dt}{\int_a^x g(t)dt} \leq f(x) \leq \frac{\int_x^b g(t)f(t)dt}{\int_x^b g(t)dt}, \tag{3.3.11}$$

by replacing $g \to gh$ and $f \to f/h$, where f/h is an increasing function on $[a, b]$ in (3.3.11), we get (3.3.1). Also, we can get (3.3.11) by just letting $h \equiv 1$ in (3.3.1).

Now we construct linear functionals of our interest from the differences of the above given inequalities to discuss their interesting properties.

(M_{14}) Under the assumptions of Theorem 3.3.1, we define linear functional as

$$\mathcal{L}_{14}(f) = \frac{f(x)}{h(x)} - \frac{\int_a^x g(t)f(t)dt}{\int_a^x g(t)h(t)dt}. \tag{3.3.12}$$

(M_{15}) Under the assumptions of Theorem 3.3.1, we define linear functional as

$$\mathcal{L}_{15}(f) = \frac{\int_x^b g(t)f(t)dt}{\int_x^b g(t)h(t)dt} - \frac{f(x)}{h(x)}. \tag{3.3.13}$$

(M_{16}) Under the assumptions of Corollary 3.3.3, we define linear functional as

$$\mathcal{L}_{16}(f) = \frac{f(x) + \alpha f(a)}{1+\alpha} - \frac{1}{x-a} \int_a^x f(t)dt. \tag{3.3.14}$$

Remark 3.3.4. Under the assumptions of Theorem 3.3.1 and Corollary 3.3.3 with f/h as an increasing function the linear functionals $\mathcal{L}_k(f) \geq 0$ for $k = 14, \ldots, 16$.

There is a very nice relationship between n-convex functions and increasing functions as follows:

Remark 3.3.5. Let $f : [a, b] \to \mathbb{R}$ be an n-convex function on $[a, b]$ for $n \geq 2$ with $f^{(k)}(a) = 0$, for $k = 1, \ldots, n-2$. Then $\frac{f(x)}{(x-a)^{n-1}}$ is an increasing function on $[a, b]$. (see [19, Theorem 1.43]).

Theorem 3.3.4. *Let $f : [a, b] \to \mathbb{R}$ be an n-convex function with $f^{(k)}(a) = 0$, for $k = 1, \ldots, n-2$. Then the following inequality holds:*

$$\frac{1}{x-a} \int_a^x f(t)dt \leq \frac{f(x) + (n-1)f(a)}{n}. \tag{3.3.15}$$

Proof. Setting $f(x) \to f(x) - f(a)$ in Remark 3.3.5, we get $\frac{f(x)-f(a)}{(x-a)^{n-1}}$, as an increasing function on $[a, b]$. Then (3.3.15) is an immediate consequence of Corollary 3.3.3, by taking $\alpha = n - 1$. □

Remark 3.3.6. As an application of the inequality

$$\frac{\int_a^x g(t)f(t)dt}{\int_a^x g(t)h(t)dt} \leq \frac{\int_a^b g(t)f(t)dt}{\int_a^b g(t)h(t)dt},$$

when $(f/h)(x) = \frac{f(x)-f(a)}{(x-a)^\alpha}$ is an increasing function, the following inequality holds

$$\frac{1}{x-a} \int_a^x f(t)dt \leq \frac{(x-a)^\alpha}{(b-a)^\alpha}\overline{f} + f(a)\big(1 - \frac{(x-a)^\alpha}{(b-a)^\alpha}\big),$$

where $\overline{f} = \frac{1}{b-a}\int_a^b f(t)dt$.
We get it by letting $g \equiv 1$, $f(t) \to f(t) - f(a)$ and $h(t) = (t-a)^\alpha$.

3.3.1 Mean-Value Theorems of Functionals Involving Increasing Functions of the Form f/h

In the present section, we state and prove the mean-value theorems of Lagrange and Cauchy type for the functionals defined in the previous section.

Theorem 3.3.5. *Let $\mathcal{L}_k$ be linear functionals defined in M_{14} and M_{15}. Consider $h(x), g(x) > 0$ for all $x \in [a, b]$, $h \in C^1[a, b]$, g is any real valued integrable function.*

If f is a function such that $f/h \in C^1[a,b]$, then there exist $\xi_k \in [a,b]$ such that the identity

$$\mathcal{L}_k(f) = X_f(\xi_k)\mathcal{L}_k(xh(x)) \ ; k = 14, 15 \tag{3.3.16}$$

holds, where $X_f(x) = \left(\frac{f}{h}\right)'(x) = \frac{f'(x)h(x)-f(x)h'(x)}{h^2(x)}$.

Proof. Fix $k = 14, 15$.
Since G_f is continuous on $[a,b]$, it attains its maximum and minimum value on $[a,b]$. Consider

$$m = \min\{X_f(x)\} \quad \text{and} \quad M = \max\{X_f(x)\}.$$

Let $F_1, F_2 : [a,b] \to \mathbb{R}$ be defined by

$$F_1(x) = Mxh(x) - f(x) \quad \text{and} \quad F_2(x) = f(x) - mxh(x).$$

Then

$$\frac{F_1(x)}{h(x)} = Mx - \frac{f(x)}{h(x)} \quad \text{and} \quad \frac{F_2(x)}{h(x)} = \frac{f(x)}{h(x)} - mx.$$

Furthermore,

$$\left(\frac{F_1(x)}{h(x)}\right)' = M - X_f(x) \geq 0,$$

so $\frac{F_1(x)}{h(x)}$ is an increasing function. Hence from Remark 3.3.4, we have

$$\mathcal{L}_k(x; F_1) \geq 0,$$

and by using the definition of the functionals we have

$$0 \leq M\mathcal{L}_k(xh(x)) - \mathcal{L}_k(f).$$

Hence, we have

$$\mathcal{L}_k(f) \leq M\mathcal{L}_k(xh(x)). \tag{3.3.17}$$

Also we have that $\frac{F_2(x)}{h(x)}$ is an increasing function. So from Remark 3.3.4, we have

$$\mathcal{L}_k(f) \geq m\mathcal{L}_k(xh(x)). \tag{3.3.18}$$

Combining (3.3.17) and (3.3.18), we get

$$m\mathcal{L}_k(xh(x)) \leq \mathcal{L}_k(f) \leq M\mathcal{L}_k(xh(x)).$$

If $\mathcal{L}_k(xh(x)) = 0$, then $\mathcal{L}_k(f) = 0$ and (3.3.16) holds for all $\xi_k \in [a, b]$ for $k = 14, 15$. Otherwise,

$$m \leq \frac{H_k(x; f)}{H_k(x; xh(x))} \leq M.$$

Since $(f(x)/h(x))'$ is continuous on $[a, b]$, there exists $\xi_k \in [a, b]$ for $k = 14, 15$ such that (3.3.16) holds. □

Theorem 3.3.6. *Let $\mathcal{L}_k$ be linear functionals defined in M_{14} and M_{15}. Consider $h(x), g(x) > 0$ for all $x \in [a, b]$, $h(x) \in C^1[a, b]$, g is any real valued integrable function. Let f, s be differentiable on (a, b) such that $X_f, X_s \in C[a, b]$ as defined in Theorem 3.3.5. Then there exists $\xi_k \in [a, b]$, such that the equality*

$$\frac{\mathcal{L}_k(f)}{\mathcal{L}_k(s)} = \frac{f'(\xi_k)h(\xi_k) - f(\xi_k)h'(\xi_k)}{s'(\xi_k)h(\xi_k) - s(\xi_k)h'(\xi_k)} \tag{3.3.19}$$

holds for $k = 14, 15$, provided that the denominators are non zero.

Proof. Follow similar steps as in Theorem 2.3.11. □

Since, we have established a theorem that relates increasing functions with n-convex functions very nicely. So we also consider the following mean value theorems to discover something more about the relationship between n-convex functions and increasing functions.

Theorem 3.3.7. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f \in C^n[a, b]$ with $f^k(a) = 0$, for $k = 1, \ldots, n-2$. Then there exists $\xi \in [a, b]$ such that*

$$\frac{f(x) + (n-1)f(a)}{n} - \frac{1}{x-a}\int_a^x f(t)dt = \frac{f^{(n)}(\xi)}{n!}\left[\frac{(x-a)^n}{n(n+1)}\right] \tag{3.3.20}$$

is valid.

Proof. Since $f^{(n)}(x)$ is continuous on $[a, b]$, it attains its maximum and minimum value on $[a, b]$. Let us consider

$$m = \min\{f^{(n)}(x)\} \quad \text{and} \quad M = \max\{f^{(n)}(x)\}.$$

Let us consider functions $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ defined by

$$\phi_1(x) = M\frac{(x-a)^n}{n!} - f(x) + f(a) \quad \text{and} \quad \phi_2(x) = f(x) - f(a) - m\frac{(x-a)^n}{n!}.$$

Then

$$\phi_1^{(n)}(x) = M - f^{(n)}(x) \geq 0,$$

and

$$\phi_2^{(n)}(x) = f^{(n)}(x) - m \geq 0.$$

So $\phi_i(x)$ $i = 1, 2$ are n-convex functions with $\phi_i^{(k)}(a) = 0$, for $k = 1, \ldots, n-2$. Hence from Theorem 3.3.4 with $\phi_1(x)$ we have

$$\begin{aligned} 0 &\leq \frac{\phi_1(x) + (n-1)\phi_1(a)}{n} - \frac{1}{x-a}\int_a^x \phi_1(t)dt \\ &= M\frac{(x-a)^n}{n(n!)} - \frac{f(x) - f(a)}{n} - \frac{1}{x-a}\int_a^x \left[M\frac{(t-a)^n}{n!} - f(t) + f(a)\right]dt \\ &= M\left[\frac{(x-a)^n}{n(n+1)(n!)}\right] - \frac{f(x) + (n-1)f(a)}{n} + \frac{1}{x-a}\int_a^x f(t)dt. \end{aligned}$$

Hence, we have

$$\frac{f(x) + (n-1)f(a)}{n} - \frac{1}{x-a}\int_a^x f(t)dt \leq M\left[\frac{(x-a)^n}{n(n+1)(n!)}\right]. \tag{3.3.21}$$

Also by considering $\phi_2(x)$ in Theorem 3.3.4, we have

$$\frac{f(x) + (n-1)f(a)}{n} - \frac{1}{x-a}\int_a^x f(t)dt \geq m\left[\frac{(x-a)^n}{n(n+1)(n!)}\right]. \tag{3.3.22}$$

Combining (3.3.21) and (3.3.22), we get

$$m\left[\frac{(x-a)^n}{n(n+1)(n!)}\right] \leq \frac{f(x) + (n-1)f(a)}{n} - \frac{1}{x-a}\int_a^x f(t)dt \leq M\left[\frac{(x-a)^n}{n(n+1)(n!)}\right].$$

If $(x = a)$, then $\frac{f(x)+(n-1)f(a)}{n} - \frac{1}{x-a}\int_a^x f(t)dt = 0$, that is

$$\frac{f(x) + (n-1)f(a)}{n} = \frac{1}{x-a}\int_a^x f(t)dt$$

and (3.3.20) holds for all $\xi \in (a, b)$. Otherwise,

$$m \leq \frac{\frac{f(x)+(n-1)f(a)}{n} - \frac{1}{x-a}\int_a^x f(t)dt}{\left[\frac{(x-a)^n}{n(n+1)(n!)}\right]} \leq M.$$

Since $f^{(n)}(x)$ is continuous on $[a, b]$, there exists $\xi \in [a, b]$ such that (3.3.20) holds and the proof is complete. □

Theorem 3.3.8. *Let $f_i : [a,b] \to \mathbb{R}$ be functions such that $f_i(x) \in C^n[a,b],\ i = 1,2$ with $f_i^{(k)}(a) = 0$, for $k = 1, \ldots, n-2$. Then there exists $\xi \in [a,b]$ such that the following equality*

$$\frac{\frac{f_1(x)+(n-1)f_1(a)}{n} - \frac{1}{x-a}\int_a^x f_1(t)dt}{\frac{f_2(x)+(n-1)f_2(a)}{n} - \frac{1}{x-a}\int_a^x f_2(t)dt} = \frac{f_1^{(n)}(\xi)}{f_2^{(n)}(\xi)} \tag{3.3.23}$$

holds, provided that the denominators are non zero.

Proof. Follow similar steps as in Theorem 2.3.11. □

3.3.2 n-Exponential Convexity of Functionals Involving Increasing Functions of the Form f/h

In the present section, we investigate the n exponential convexity and related properties of functionals defined in previous section. To move further, we define the following families of functions as in the previous sections. For distinct points $u_0, u_1 \in [a,b]$ and h be a positive integrable function on $[a,b]$. Let $\mathbf{E}_i$, $i = 7, \ldots, 12$, denote a family of functions with the following property:

$\mathbf{E}_7 = \{f_t : [a,b] \to \mathbb{R} \mid t \in J$ such that f_t is integrable and $t \mapsto [u_0, u_1; F_t]$ is n-exponentially convex in the Jensen sense on J, where $F_t(u) = \frac{f_t(u)}{h(u)}\}$.

$\mathbf{E}_8 = \{f_t : [a,b] \to \mathbb{R} \mid t \in J$ such that f_t is integrable and $t \mapsto [u_0, u_1; F_t]$ is exponentially convex in the Jensen sense on J, where $F_t(u) = \frac{f_t(u)}{h(u)}\}$.

$\mathbf{E}_9 = \{f_t : [a,b] \to \mathbb{R} \mid t \in J$ such that f_t is integrable and $t \mapsto [u_0, u_1; F_t]$ is 2-exponentially convex in the Jensen sense on J, where $F_t(u) = \frac{f_t(u)}{h(u)}\}$.

$\mathbf{E}_{10} = \{f_t : [a,b] \to \mathbb{R} \mid t \in J$ such that f_t is integrable and $t \mapsto [u_0, u_1; F_t]$ is n-exponentially convex in the Jensen sense on J, where $F_t(u) = \frac{f_t(u) - f_t(a)}{(u-a)^\alpha}\}$.

$\mathbf{E}_{11} = \{f_t : [a,b] \to \mathbb{R} \mid t \in J$ such that f_t is integrable and $t \mapsto [u_0, u_1; F_t]$ is exponentially convex in the Jensen sense on J, where$F_t(u) = \frac{f_t(u) - f_t(a)}{(u-a)^\alpha}\}$.

$\mathbf{E}_{12} = \{f_t : [a,b] \to \mathbb{R} \mid t \in J$ such that f_t is integrable and $t \mapsto [u_0, u_1; F_t]$ is 2-exponentially convex in the Jensen sense on J, where $F_t(u) = \frac{f_t(u) - f_t(a)}{(u-a)^\alpha}\}$.

Theorem 3.3.9. *Let $\mathcal{L}_k$ be the linear functionals defined as in (M_k) for $k = 14, 15$ associated with a family $\mathbf{E}_7$. Then $t \mapsto \mathcal{L}_k(f_t)$ is an n-exponentially convex function in the Jensen sense on J. If the function $t \mapsto \mathcal{L}_k(f_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. Follow similar steps as in Theorem 3.2.5. □

The following corollary is an immediate consequence of the above theorem.

Corollary 3.3.10. *Let $\mathcal{L}_k$ be the linear functionals defined as in (M_k) for $k = 14, 15$ associated with a family $\mathbf{E}_8$. Then $t \mapsto \mathcal{L}_k(f_t)$ is an exponentially convex function in the Jensen sense on J. If $t \mapsto \mathcal{L}_k(f_t)$ is continuous on J, then it is exponentially convex on J.*

Proof. Follows from the previous theorem. □

As an application the case when $h(x) = (x-a)^{\alpha}$ is of great importance for us. Since in the next section we will give explicitly non symmetric Stolarsky means with parameter α. So, we consider the following remark.

Remark 3.3.7. Theorem 3.3.9 and Corollary 3.3.10 also hold for the functional $\mathcal{L}_{16}$ defined as in (M_{16}) by considering f_t from $\mathbf{E}_{10}$ and $\mathbf{E}_{11}$ respectively.

Corollary 3.3.11. *Let $\mathcal{L}_k$ be the linear functionals defined as in (M_k) for $k = 14, 15$ associated with a family $\mathbf{E}_9$. Then the following statements hold:*

(i) If the function $t \mapsto \mathcal{L}_k(f_t)$ is strictly positive and continuous on J, then it is 2-exponentially convex on J, and thus, log-convex. Also for $r, s, t \in J$ such that $r < s < t$, we have

$$(\mathcal{L}_k(f_s))^{t-r} \leq (\mathcal{L}_k(f_r))^{t-s}(\mathcal{L}_k(f_t))^{s-r}. \tag{3.3.24}$$

(ii) If the function $t \mapsto \mathcal{L}_k(f_t)$ is strictly positive and differentiable on J, then for all $t, r, u, v \in J$ such that $t \leq u$, $r \leq v$, we have

$$\mathfrak{B}(t, r; \mathcal{L}_k, \mathbf{E}_9) \leq \mathfrak{B}(u, v; \mathcal{L}_k, \mathbf{E}_9), \ \ k = 14, 15,$$

where

$$\mathfrak{B}(t, r; \mathcal{L}_k, \mathbf{E}_9) = \begin{cases} \left(\frac{\mathcal{L}_k(f_t)}{\mathcal{L}_k(f_r)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{\frac{d}{dt}(\mathcal{L}_k(f_t))}{\mathcal{L}_k(f_t)}\right), & t = r. \end{cases} \tag{3.3.25}$$

Proof. Follow similar steps as in Corollary 2.3.7. □

Remark 3.3.8. Similar to Corollary 3.3.11, we can also give the following results for the linear functional $\mathcal{L}_{16}$;
Let $\mathcal{L}_{16}(f_t)$ be the linear functional defined as in (M_{16}). Consider $f_t \in \mathbf{E}_{12}$, then the following statements hold:

(i) If the function $t \mapsto \mathcal{L}_{16}(f_t)$ is strictly positive and continuous on J, then it is 2-exponentially convex on J, and thus, log-convex. Also for $r, s, t \in J$ such that $r < s < t$, we have

$$(\mathcal{L}_{16}(f_s))^{t-r} \le (\mathcal{L}_{16}(f_r))^{t-s}(\mathcal{L}_{16}(f_t))^{s-r}. \tag{3.3.26}$$

(ii) If the function $t \mapsto \mathcal{L}_{16}(f_t)$ is strictly positive and differentiable on J, then for all $t, r, u, v \in J$ such that $t \le u$, $r \le v$, we have

$$\mathfrak{B}(t, r; \mathcal{L}_{16}, \mathbf{E}_{12}) \le \mathfrak{B}(u, v; \mathcal{L}_{16}, \mathbf{E}_{12}),$$

where

$$\mathfrak{B}(t, r; \mathcal{L}_{16}, \mathbf{E}_{12}) = \begin{cases} \left(\frac{\mathcal{L}_{16}(f_t)}{\mathcal{L}_{16}(f_r)}\right)^{\frac{1}{t-r}}, & t \ne r, \\ \exp\left(\frac{\frac{d}{dt}(\mathcal{L}_{16}(f_t))}{\mathcal{L}_{16}(f_t)}\right), & t = r. \end{cases} \tag{3.3.27}$$

Remark 3.3.9. Also, the comments given in Remark 2.3.4 are valid for the functionals $\mathcal{L}_k$ for $k = 14, \ldots, 16$.

3.3.3 Non-Symmetric Stolarsky Means with Parameter α of Functional Involving Increasing Functions of the Form f/h

Consider a family of functions

$$\Upsilon_5 = \{f_t : (0, \infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$f_t(x) = \begin{cases} \frac{(x-a)^\alpha x^t}{t}, & t \ne 0; \\ (x-a)^\alpha \log x, & t = 0. \end{cases} \tag{3.3.28}$$

Then $F_t := \frac{f_t(x) - f_t(a)}{(x-a)^\alpha}$ is a strictly increasing function for $x \in (0, \infty)$ and for each $t \in \mathbb{R}$.

Moreover, since $\left(\frac{f_t(x)-f_t(a)}{(x-a)^\alpha}\right)' = x^{t-1}$, the mapping $t \mapsto \left(\frac{f_t(x)-f_t(a)}{(x-a)^\alpha}\right)'$ is exponentially convex (see [22]). Now, regarding Remark 3.3.7 and Remark 3.3.9, we get exponential convexity of the functional $\mathcal{L}_{16}(f_t)$. In addition, Remark 3.3.8 provides the log-convexity of the functional and for $r < s < t$, where $r, s, t \in \mathbb{R}$ (3.3.26) holds for the family of functions defined in (3.3.28). Also from Remark 3.3.8, we can define

$$\mathfrak{B}(t,r;\mathcal{L}_{16},\Upsilon_5)=\begin{cases}\left(\frac{r\left((x-a)^{\alpha+1}x^t-(\alpha+1)\int_a^x(p-a)^\alpha p^t\,dp\right)}{t\left((x-a)^{\alpha+1}x^r-(\alpha+1)\int_a^x(p-a)^\alpha p^r\,dp\right)}\right)^{\frac{1}{t-r}}, & t\neq r,\quad t,r\neq 0,\\ \left(\frac{\left((x-a)^{\alpha+1}x^t-(\alpha+1)\int_a^x(p-a)^\alpha p^t\,dp\right)}{t\left((x-a)^{\alpha+1}\ln x-(\alpha+1)\int_a^x(p-a)^\alpha \ln p\,dp\right)}\right)^{\frac{1}{t}}, & t\neq r=0,\\ \exp\left(\frac{-1}{t}+\frac{\left((x-a)^{\alpha+1}x^t\ln x-(\alpha+1)\int_a^x(p-a)^\alpha p^t\ln p\,dp\right)}{\left((x-a)^{\alpha+1}x^t-(\alpha+1)\int_a^x(p-a)^\alpha p^t\,dp\right)}\right), & t=r,\quad t,r\neq 0,\\ \exp\left(\frac{\left((x-a)^{\alpha+1}(\ln x)^2-(\alpha+1)\int_a^x(p-a)^\alpha(\ln p)^2\,dp\right)}{2\left((x-a)^{\alpha+1}\ln x-(\alpha+1)\int_a^x(p-a)^\alpha\ln p\,dp\right)}\right), & t=r=0,\end{cases}$$

where

$$\int_a^x (p-a)^\alpha p^t dp=\frac{a^t(x-a)^{\alpha+1}}{\alpha+1}F(-t,\alpha+1,\alpha+2;1-x/a)$$

and F denotes the hypergeometric function.

Also note that the function $t\mapsto\mathcal{L}_{16}(f_t)$ is strictly positive, so for every $t,r,v,w\in\mathbb{R}$ such that $t\leq v$, $r\leq w$, we have

$$\mathfrak{B}(t,r;\mathcal{L}_{16},\Upsilon_5)\leq\mathfrak{B}(v,w;\mathcal{L}_{16},\Upsilon_5),$$

where $t\mapsto\mathcal{L}_{16}(f_t)$ is differentiable when $t=r$.

As a special case when $(\alpha=1,x=b)$ in $\mathcal{L}_{16}(f)$ by Remark 3.3.2, we have classical Hadamard's functional. Also the above expressions in this case were given explicitly in [8].

3.4 Results for Increasing Functions of the Form f'/h

Unfortunately, inequalities (3.1.4) and (3.1.5) are not valid under the given assumptions. R. Jakšić, L. Kvesić and J. Pečarić found some errors in the results given by G. Zabandan and A. Kilicman. In their paper [18] they gave particular examples of functions that satisfy the conditions of Theorem 3.1.2, but for which the inequalities (3.1.4) and (3.1.5) are not valid. Moreover, in the same paper they also gave the best possible conditions under which the inequality (3.1.4) holds. The following result is given in [18].

Theorem 3.4.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function and let $h:[a,b]\to(0,\infty)$ be an integrable function.*

(i) *If the function* $\frac{f'}{h}$ *is increasing, then inequality (3.1.4) holds.*

(ii) *If the function* $\frac{f'}{h}$ *is decreasing, then inequality (3.1.4) holds in reversed direction.*

Remark 3.4.1. The class of functions for which $f'(x)/x$ is increasing is of special interest because it connects us with the superquadratic functions.
Suppose that the function $\phi : [0, \infty) \to \mathbb{R}$ is continuously differentiable and $\phi(0) \leq 0$. If $\phi'(x)/x$ is increasing then ϕ is superquadratic function (see [3]).

Corollary 3.4.2. *Let* $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ *be a differentiable function.*

(i) *If the function* $\frac{f'(x)}{x}$ *is increasing, then the following inequality*

$$\frac{2}{b^2 - a^2} \int_a^b x f(x) dx \leq \frac{f(a) + f(b)}{2} \tag{3.4.1}$$

holds.

Proof. If we put $h(x) = x$ in Theorem 3.4.1, we directly get (3.4.1). □

In the next sub-sections we will give some mean-value theorems and results related to them. Then, we will construct n-exponentially convex functions and exponentially convex functions by using the functional defined as the difference of the right and the left side of inequality (3.1.4) for different classes of functions. In the last section, we will give some interesting examples and construct Stolarsky means.

3.4.1 Mean-Value Theorems for One of Weighted Hermite-Hadamard's Inequality Involving Increasing Functions of the Form f'/h

To prove related mean-value theorems of Lagrange and Cauchy type, we need to consider functions ϕ_1 and ϕ_2 defined in the following lemma.

Lemma 3.4.3. *Let* $f : [a, b] \to \mathbb{R}$ *be a twice differentiable function and let* $h : [a, b] \to \mathbb{R}^+$ *be a differentiable integrable function. Denote*

$$Z_f(x) = \frac{f''(x)h(x) - f'(x)h'(x)}{h^2(x)}. \tag{3.4.2}$$

Let $m, M \in \mathbb{R}$ *be such that*

$$m \leq Z_f(x) \leq M \quad \textit{for all} \quad x \in [a, b]. \tag{3.4.3}$$

Let $\phi_1, \phi_2 \colon [a,b] \to \mathbb{R}$ be the functions defined by

$$\phi_1(x) = M \int_a^x th(t)dt - f(x) \tag{3.4.4}$$

and

$$\phi_2(x) = f(x) - m \int_a^x th(t)dt. \tag{3.4.5}$$

Then $\frac{\phi_1'}{h}$ and $\frac{\phi_2'}{h}$ are increasing functions.

Proof. We need to show that the first derivatives of $\frac{\phi_1'}{h}$ and $\frac{\phi_2'}{h}$ are positive functions. We have

$$\left(\frac{\phi_1'(x)}{h(x)}\right)' = \left(Mx - \frac{f'(x)}{h(x)}\right)' = M - Z_f(x) \geq 0,$$

and

$$\left(\frac{\phi_2'(x)}{h(x)}\right)' = \left(\frac{f'(x)}{g(x)} - mx\right)' = Z_f(x) - m \geq 0.$$

This shows us that $\frac{\phi_1'}{h}, \frac{\phi_2'}{h}$ are increasing functions. $\square$

Theorem 3.4.4. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function, $h : [a,b] \to \mathbb{R}^+$ a differentiable integrable function and let $Z_f \in C[a,b]$ be as defined in Lemma 3.4.3. Then there exists $\xi \in [a,b]$ such that the identity*

$$\frac{f(a)+f(b)}{2} - \frac{1}{\int_a^b h(x)dx} \int_a^b f(x)h(x)dx = \alpha Z_f(\xi) \tag{3.4.6}$$

holds, where

$$\alpha = \left(\frac{\int_a^b th(t)dt}{2} - \frac{\int_a^b h(x) \int_a^x th(t)dtdx}{\int_a^b h(x)dx}\right).$$

Proof. Since Z_f is continuous on a compact set, it attains its maximum and minimum value on it. Let us consider

$$m = \min\{Z_f(x)\}$$

and

$$M = \max\{Z_f(x)\}.$$

In Lemma 3.4.3 we have shown that $\frac{\phi_1'}{h}, \frac{\phi_2'}{h}$, where ϕ_1 and ϕ_2 are defined by (3.4.3) and (3.4.4) are increasing functions, so we can apply Theorem 3.4.1 to those functions and obtain the following inequalities:

$$\frac{f(a)+f(b)}{2} - \frac{1}{\int_a^b h(x)dx} \int_a^b f(x)h(x)dx \leq \alpha M,$$

$$\frac{f(a)+f(b)}{2}-\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)h(x)dx \geq \alpha m.$$

Combining both inequalities, we get

$$\alpha m \leq \frac{f(a)+f(b)}{2}-\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)h(x)dx \leq \alpha M.$$

If $\alpha = 0$ then $\frac{f(a)+f(b)}{2}-\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)h(x)dx = 0$ and (3.4.6) holds for all $\xi \in [a,b]$. Otherwise

$$m \leq \frac{\frac{f(a)+f(b)}{2}-\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)h(x)dx}{\alpha} \leq M.$$

Since Z_f is continuous on $[a,b]$, there exists $\xi \in [a,b]$ such that (3.4.6) holds and the proof is completed. □

Corollary 3.4.5. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a twice differentiable function and let $Z_f \in C[a,b]$ be as defined in Lemma 3.4.3. Then there exists $\xi \in [a,b]$ such that the identity*

$$\frac{f(a)+f(b)}{2}-\frac{2}{b^2-a^2}\int_a^b xf(x)dx = \alpha Z_f(\xi) \tag{3.4.7}$$

holds, where $\alpha = \frac{b^4+b^3a-4a^2b^2+ba^3+a^4}{30(b+a)}$ and $Z_f(\xi) = \frac{f''(\xi)}{\xi}-\frac{f'(\xi)}{\xi^2}$.

Proof. If we put $h(x) = x$ in Theorem 3.4.4, we get (3.4.7). □

Theorem 3.4.6. *Let $f_1, f_2 : [a,b] \to \mathbb{R}$ be twice differentiable functions, $h : [a,b] \to \mathbb{R}^+$ be a differentiable integrable functions and $Z_{f_1}, Z_{f_2} \in C[a,b]$ as defined in Lemma 3.4.3. Then there exists $\xi \in [a,b]$ such that the following equality is valid:*

$$\frac{\frac{f_1(a)+f_1(b)}{2}-\frac{1}{\int_a^b h(x)dx}\int_a^b f_1(x)h(x)dx}{\frac{f_2(a)+f_2(b)}{2}-\frac{1}{\int_a^b h(x)dx}\int_a^b f_2(x)h(x)dx} = \frac{Z_{f_1}(\xi)}{Z_{f_2}(\xi)} \tag{3.4.8}$$

provided that the denominators are nonzero.

Proof. Follow similar steps as in Theorem 2.3.11. □

Corollary 3.4.7. *Let $f_1, f_2 : [a,b] \subset (0,\infty) \to \mathbb{R}$ twice differentiable and $Z_{f_1}, Z_{f_2} \in C[a,b]$ as defined in Corollary 3.4.5. Then there exists $\xi \in [a,b]$ such that the following equality is valid:*

$$\frac{\frac{f_1(a)+f_1(b)}{2}-\frac{2}{b^2-a^2}\int_a^b xf_1(x)dx}{\frac{f_2(a)+f_2(b)}{2}-\frac{2}{b^2-a^2}\int_a^b xf_2(x))dx} = \frac{\xi f_1''(\xi)-f_1'(\xi)}{\xi f_2''(\xi)-f_2'(\xi)} \tag{3.4.9}$$

provided that the denominators are nonzero.

Proof. If we put $h(x) = x$ in Theorem 3.4.6, we directly get (3.4.9).

□

3.4.2 n-Exponential Convexity of Functional Involving Increasing Functions of the Form f'/h

In the present section, we construct our linear functional involving increasing functions of the form f'/h as:
(M_{17}) Under the assumptions of Theorem 3.4.1, we consider the following functional

$$\mathcal{L}_{17}(f) = \frac{f(a)+f(b)}{2} - \frac{1}{\int_a^b h(x)dx}\int_a^b f(x)h(x)dx. \tag{3.4.10}$$

Remark 3.4.2. Under the assumptions of Theorem 3.4.1, if f'/h is an increasing function on $[a, b]$ then $\mathcal{L}_{17}(f) \geq 0$.

Now we investigate the properties of functional as defined above, regarding n-exponential and exponential convexity by defining following families of functions.
Let $[a, b], J \subseteq \mathbb{R}$ be intervals and let h be a positive integrable function on $[a, b]$. For distinct points $u_0, u_1 \in [a, b]$, let $\mathbf{E}_i$, $i = 13, \ldots, 15$, denote a family of functions with the following property:

$\mathbf{E}_{13} = \{f_t : [a, b] \to \mathbb{R} \mid t \in J$ such that f_t is differentiable and $t \mapsto [u_0, u_1; F_t]$ is n-exponentially convex in the Jensen sense on J, where $F_t(u) = \frac{f_t'(u)}{h(u)}\}$.

$\mathbf{E}_{14} = \{f_t : [a, b] \to \mathbb{R} \mid t \in J$ such that f_t is differentiable and $t \mapsto [u_0, u_1; F_t]$ is exponentially convex in the Jensen sense on J, where $F_t(u) = \frac{f_t'(u)}{h(u)}\}$.

$\mathbf{E}_{15} = \{f_t : [a, b] \to \mathbb{R} \mid t \in J$ such that f_t is differentiable and $t \mapsto [u_0, u_1; F_t]$ is 2-exponentially convex in the Jensen sense on J, where $F_t(u) = \frac{f_t'(u)}{h(u)}\}$.

Theorem 3.4.8. *Let $\mathcal{L}_{17}$ be the linear functional defined as in (M_{17}) associated with a family $\mathbf{E}_{13}$. Then $t \mapsto \mathcal{L}_{17}(f_t)$ is n-exponentially convex function in the Jensen sense on J. If the function $t \mapsto \mathcal{L}_{17}(f_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. Follow similar steps as in Theorem 3.2.5. □

The following corollary is an immediate consequence of the above theorem.

Corollary 3.4.9. *Let $\mathcal{L}_{17}$ be the linear functional defined as in (M_{17}) associated with a family $\mathbf{E}_{14}$. Then $t \mapsto \mathcal{L}_{17}(f_t)$ is exponentially convex function in the Jensen sense on J. If $t \mapsto \mathcal{L}_{17}(f_t)$ is continuous on J then it is exponentially convex on J.*

Proof. Follows from the previous theorem. □

Corollary 3.4.10. *Let $\mathcal{L}_{17}$ be the linear functional defined as in (M_{17}) associated with a family $\mathbf{E}_{15}$. Then the following statements hold:*

(i) If the function $t \mapsto \mathcal{L}_{17}(f_t)$ is strictly positive and continuous on J, then it is 2-exponentially convex on J, and thus, log-convex. Also for $r, s, t \in J$ such that $r < s < t$, we have

$$(\mathcal{L}_{17}(f_s))^{t-r} \leq (\mathcal{L}_{17}(f_r))^{t-s}(\mathcal{L}_{17}(f_t))^{s-r}. \tag{3.4.11}$$

(ii) If the function $t \mapsto \mathcal{L}_{17}(f_t)$ is strictly positive and differentiable on J, then for all $t, r, u, v \in J$ such that $t \leq u$, $r \leq v$, we have

$$\mathfrak{B}(t, r; \mathcal{L}_{17}, \mathbf{E}_{15}) \leq \mathfrak{B}(u, v; \mathcal{L}_{17}, \mathbf{E}_{15}),$$

where

$$\mathfrak{B}(t, r; \mathcal{L}_{17}, \mathbf{E}_{15}) = \begin{cases} \left(\frac{\mathcal{L}_{17}(f_t)}{\mathcal{L}_{17}(f_r)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{\frac{d}{d_t}(\mathcal{L}_{17}(f_t))}{\mathcal{L}_{17}(f_t)}\right), & t = r. \end{cases} \tag{3.4.12}$$

Proof. Follow similar steps as in Corollary 2.3.7. □

Remark 3.4.3. Also, the comments given in Remark 2.3.4 are valid for the functional $\mathcal{L}_{17}$.

3.4.3 Stolarsky Means of Functional Involving Increasing Functions of the Form f'/h

In the present section, we vary on a choice of families of functions in order to construct different examples of log and exponentially convex functions and related results.

Example 3.4.1. *Consider a family of functions*

$$\Upsilon_6 = \{f_t : (0, \infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$f_t(u) = \begin{cases} \frac{1}{t}\int_a^u p^t h(p)dp, & t \neq 0, \\ \int_a^u \log p h(p)dp, & t = 0. \end{cases} \tag{3.4.13}$$

Since $\left(\frac{f_t'(u)}{h(u)}\right)' = u^{t-1}$, *the mapping* $t \mapsto \frac{f_t'(u)}{h(u)}$ *is exponentially convex function (see [22]).*
Analogously as in the proof of Theorem 3.4.8 we conclude that $t \mapsto [u_0, u_1; F_t]$ *where* $F_t = \frac{f_t'}{h}$ *is exponentially convex (and so it is also exponentially convex in the Jensen sense).*
Also by Corollary 3.4.9 we have that $t \mapsto \mathcal{L}_{17}(f_t)$ *is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous so it is exponentially convex. For this family of functions,* $\mathfrak{B}(t, r; \mathcal{L}_{17}, \Upsilon_6)$ *(defined in* (3.4.12)*), is equal to*

$$\mathfrak{B}(t,r;\mathcal{L}_{17},\Upsilon_6) = \begin{cases} \left(\dfrac{r\left(\int_a^b h(u)du\int_a^b p^t h(p)dp - 2\int_a^b h(u)\int_a^u p^t h(p)dpdu\right)}{t\left(\int_a^b h(u)du\int_a^b p^r h(p)dp - 2\int_a^b h(u)\int_a^u p^r h(p)dpdu\right)}\right)^{\frac{1}{t-r}}, & t \neq r,\ t,r \neq 0, \\ \left(\dfrac{\left(\int_a^b h(u)du\int_a^b p^t h(p)dp - 2\int_a^b h(u)\int_a^u p^t h(p)dpdu\right)}{t\left(\int_a^b h(u)du\int_a^b \log p h(p)dp - 2\int_a^b h(u)\int_a^u \log p h(p)dpdu\right)}\right)^{\frac{1}{t}}, & t \neq r = 0, \\ \exp\left(\dfrac{-1}{t} + \dfrac{\int_a^b h(u)du\int_a^b \log p p^t h(p)dp - 2\int_a^b h(u)\int_a^u \log p p^t h(p)dpdu}{\int_a^b h(u)du\int_a^b p^t h(p)dp - 2\int_a^b h(u)\int_a^u p^t h(p)dpdu}\right), & t = r,\quad t,r \neq 0, \\ \exp\left(\dfrac{\int_a^b h(u)du\int_a^b (\log p)^2 h(p)dp - 2\int_a^b h(u)\int_a^u (\log p)^2 h(p)dpdu}{2\left(\int_a^b h(u)du\int_a^b \log p h(p)dp - 2\int_a^b h(u)\int_a^u \log p h(p)dpdu\right)}\right), & t = r = 0, \end{cases}$$

Now if we put $h(x) = x$, then for $t \neq r$, $t, r \neq 0, -2, -4$ we have

$$\mathfrak{B}(t,r;\mathcal{L}_{17},\Upsilon_6) = \left(\frac{r(r+2)(r+4)}{t(t+2)(t+4)}\frac{(t+4)(b^2-a^2)\big(b^{t+2}+a^{t+2}\big) - 4\big(b^{t+4}-a^{t+4}\big)}{(r+4)(b^2-a^2)\big(b^{r+2}+a^{r+2}\big) - 4\big(b^{r+4}-a^{r+4}\big)}\right)^{\frac{1}{t-r}} \tag{3.4.14}$$

and for $t = r$ and $t \neq 0, -2, -4$,

$$\mathfrak{B}\left(t,t;\mathcal{L}_{17},\Upsilon_6\right) = \exp\left(\frac{-1}{t} + \frac{(b^2-a^2)(t+4)^2\mathbf{A} - 4(t+2)(t+4)\left(b^{t+4}\log b - a^{t+4}\log a\right) + 8(t+3)\left(b^{t+4}-a^{t+4}\right) + \mathbf{B}}{(t+2)(t+4)\left((t+4)(b^2-a^2)\left(b^{t+2}+a^{t+2}\right) - 4\left(b^{t+4}-a^{t+4}\right)\right)}\right),$$

where

$$A = \Big((t+2)\big(b^{t+2}\log b - a^{t+2}\log a\big) - \big(b^{t+2}-a^{t+2}\big)\Big)$$

and

$$B = 2(t+4)^2(t+2)(b^2-a^2)a^{t+2}\log a - 2(t+4)^2(b^2-a^2)a^{t+2}.$$

However, to get the continuous extension of (3.4.14) in order to cover all choices of r and t, we consider the following.
For $t \neq 0, -2, -4$,

$$\mathfrak{B}\left(t, 0; \mathcal{L}_{17}, \Upsilon_6\right) = \left(\frac{8}{t(t+2)(t+4)} \frac{(t+4)(b^2-a^2)\left(b^{t+2}+a^{t+2}\right)-4\left(b^{t+4}-a^{t+4}\right)}{b^4-a^4-4a^2b^2(\log b-\log a)}\right)^{\frac{1}{t}},$$

$$\mathfrak{B}\left(t, -2; \mathcal{L}_{17}, \Upsilon_6\right) = \left(\frac{-2}{t(t+2)(t+4)} \frac{(t+4)(b^2-a^2)\left(b^{t+2}+a^{t+2}\right)-4\left(b^{t+4}-a^{t+4}\right)}{(b^2+a^2)\left(\log a-\log b\right)+(b^2-a^2)}\right)^{\frac{1}{t+2}},$$

$$\mathfrak{B}\left(t, -4; \mathcal{L}_{17}, \Upsilon_6\right) = \left(\frac{8a^2b^2}{t(t+2)(t+4)} \frac{(t+4)(b^2-a^2)\left(b^{t+2}+a^{t+2}\right)-4\left(b^{t+4}-a^{t+4}\right)}{b^4-a^4-4a^2b^2\left(\log b-\log a\right)}\right)^{\frac{1}{t+4}},$$

$$\mathfrak{B}\left(0, 0; \mathcal{L}_{17}, \Upsilon_6\right) = \exp\left(\frac{b^4(4\log b-3)-a^4(4\log a-3)+8a^2b^2\left((\log a)^2-\log a+\log b-(\log b)^2\right)}{4\left(b^4-a^4-4a^2b^2\left(\log b-\log a\right)\right)}\right),$$

$$\mathfrak{B}\left(-2, -2; \mathcal{L}_{17}, \Upsilon_6\right) = \exp\left(\frac{a^2\left[\log b(1+\log b)+\log a(1-\log a)\right]+b^2\left[\log b(-1+\log b)-\log a(1+\log a)\right]}{(b^2-a^2)\left[-2-2(\log b+\log a)\right]+4b^2\log b-4a^2\log a}\right),$$

$$\mathfrak{B}\left(-4, -4; \mathcal{L}_{17}, \Upsilon_6\right) = \exp\left(\frac{b^4[3+4\log a]-a^4[3+4\log b]+8a^2b^2\left[-\log b(1+\log b)+\log a(1+\log a)\right]}{4\left[(b^4-a^4)+4a^2b^2(\log a-\log b)\right]}\right).$$

Also note that if the function $t \mapsto \mathcal{L}_{17}(f_t)$ is positive and differentiable on $\mathbb{R}$ then for every $t, r, u, v \in \mathbb{R}$ such that $t \leq u$, $r \leq v$, we have

$$\mathfrak{B}(t, r; \mathcal{L}_{17}, \Upsilon_6) \leq \mathfrak{B}(t, r; \mathcal{L}_{17}, \Upsilon_6). \tag{3.4.15}$$

If we apply Corollary 3.4.7 on functions $f_1 = f_t$ and $f_2 = f_r$, where $t \neq r$, then there exists some $\xi \in [a, b]$ such that

$$\frac{\frac{f_t(a)+f_t(b)}{2} - \frac{2}{b^2-a^2}\int_a^b x f_t(x)dx}{\frac{f_r(a)+f_r(b)}{2} - \frac{2}{b^2-a^2}\int_a^b x f_r(x)dx} = \xi^{t-r}.$$

Since the function $\xi \mapsto \xi^{t-r}$ is invertible for $t \neq r$, we then have

$$a \leq \left(\frac{\frac{f_t(a)+f_t(b)}{2} - \frac{2}{b^2-a^2}\int_a^b x f_t(x)dx}{\frac{f_r(a)+f_r(b)}{2} - \frac{2}{b^2-a^2}\int_a^b x f_r(x)dx}\right)^{\frac{1}{t-r}} \leq b,$$

that is

$$a \leq \mathfrak{B}(t, r; \mathcal{L}_{17}, \Upsilon_6) \leq b,$$

which together with the fact that $\mathfrak{B}(t, r; \mathcal{L}_{17}, \Upsilon_6)$ is continuous and monotonous with respect to its both arguments t and r, shows that $\mathfrak{B}(t, r; \mathcal{L}_{17}, \Upsilon_6)$ are means of a and b for all $t, r \in \mathbb{R}$.

Example 3.4.2. *Consider a family of functions*

$$\Upsilon_7 = \{f_t : (0,\infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$f_t(u) = \frac{-1}{\sqrt{t}} \int_a^u e^{-p\sqrt{t}} h(p)dp. \tag{3.4.16}$$

Since $\left(\frac{f_t'(u)}{h(u)}\right)' = e^{-u\sqrt{t}}$, *the mapping* $t \mapsto \frac{f_t'(u)}{h(u)}$ *is exponentially convex function (see [22]).*
Analogously as in the proof of Theorem 3.4.8 we conclude that $t \mapsto [u_0, u_1; F_t]$ *where* $F_t = \frac{f_t'}{h}$ *is exponentially convex (and so it is exponentially convex in the Jensen sense). Also by Corollary 3.4.9 we have that* $t \mapsto \mathcal{L}_{17}(f_t)$ *is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous so it is exponentially convex. For this family of functions,* $\mathfrak{B}(t, r; \mathcal{L}_{17}, \Upsilon_7)$ *from* (3.4.12) *is equal to*

$$\mathfrak{B}(t,r;\mathcal{L}_{17},\Upsilon_7) = \begin{cases} \left(\dfrac{\sqrt{r}\left(\int_a^b h(u)du \int_a^b e^{-p\sqrt{t}}h(p)dp - 2\int_a^b h(u) \int_a^u e^{-p\sqrt{t}}h(p)dpdu \right)}{\sqrt{t}\left(\int_a^b h(u)du \int_a^b e^{-p\sqrt{r}}h(p)dp - 2\int_a^b h(u) \int_a^u e^{-p\sqrt{r}}h(p)dpdu \right)} \right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\dfrac{-1}{2t} - \dfrac{\int_a^b h(u)du \int_a^b pe^{-p\sqrt{t}}h(p)dp - 2\int_a^b h(u) \int_a^u pe^{-p\sqrt{t}}h(p)dpdu }{2\sqrt{t}\left(\int_a^b h(u)du \int_a^b e^{-p\sqrt{t}}h(p)dp - 2\int_a^b h(u) \int_a^u e^{-p\sqrt{t}}h(p)dpdu \right)} \right), & t = r, \end{cases}$$

Example 3.4.3. *Consider a family of functions*

$$\Upsilon_8 = \{f_t : (0,\infty) \to \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$f_t(u) = \begin{cases} \frac{1}{t} \int_a^u e^{pt} h(p)dp, & t \neq 0, \\ \int_a^u p h(p)dp, & t = 0. \end{cases} \tag{3.4.17}$$

Since $\left(\frac{f_t'(u)}{h(u)}\right)' = e^{ut}$, *the mapping* $t \mapsto \frac{f_t'(u)}{h(u)}$ *is exponentially convex function (see [22]).*
Analogously as in the proof of Theorem 3.4.8 we conclude that $t \mapsto [u_0, u_1; F_t]$ *where* $F_t = \frac{f_t'}{h}$ *is exponentially convex (and so exponentially convex in the Jensen sense). Also by Corollary 3.4.9 we have that* $t \mapsto \mathcal{L}_{17}(f_t)$ *is exponentially convex in the Jensen*

sense. It is easy to verify that this mapping is continuous so it is exponentially convex. For this family of functions, $\mathfrak{B}(t,r;\mathcal{L}_{17},\Upsilon_8)$ *from* (3.4.12) *is equal to*

$$\mathfrak{B}(t,r;\mathcal{L}_{17},\Upsilon_8)=\begin{cases}\left(\dfrac{r\left(\int_a^b h(u)du\int_a^b e^{pt}h(p)dp-2\int_a^b h(u)\int_a^u e^{pt}h(p)dpdu\right)}{t\left(\int_a^b h(u)du\int_a^b e^{pr}h(p)dp-2\int_a^b h(u)\int_a^u e^{pr}h(p)dpdu\right)}\right)^{\frac{1}{t-r}}, & t\neq r,\ t,r\neq 0,\\ \left(\dfrac{\left(\int_a^b h(u)du\int_a^b e^{pt}h(p)dp-2\int_a^b h(u)\int_a^u e^{pt}h(p)dpdu\right)}{t\left(\int_a^b h(u)du\int_a^b ph(p)dp-2\int_a^b h(u)\int_a^u ph(p)dpdu\right)}\right)^{\frac{1}{t}}, & t\neq r=0,\\ \exp\left(\dfrac{-1}{t}+\dfrac{\int_a^b h(u)du\int_a^b pe^{pt}h(p)dp-2\int_a^b h(u)\int_a^u pe^{pt}h(p)dpdu}{\int_a^b h(u)du\int_a^b e^{pt}h(p)dp-2\int_a^b h(u)\int_a^u e^{pt}h(p)dpdu}\right), & t=r,\quad t,r\neq 0,\\ \exp\left(\dfrac{\int_a^b h(u)du\int_a^b p^2h(p)dp-2\int_a^b h(u)\int_a^u p^2h(p)dpdu}{2\left(\int_a^b h(u)du\int_a^b ph(p)dp-2\int_a^b h(u)\int_a^u ph(p)dpdu\right)}\right), & t=r=0,\end{cases}$$

Example 3.4.4. *Consider a family of functions*

$$\Upsilon_9=\{f_t:(0,\infty)\to\mathbb{R}:t\in\mathbb{R}\}$$

defined by

$$f_t(u)=\begin{cases}\frac{-1}{\log t}\int_a^u t^{-p}h(p)dp, & t\neq 1,\\ \int_a^u ph(p)dp, & t=1.\end{cases}\tag{3.4.18}$$

Since $\left(\frac{f_t'(u)}{h(u)}\right)'=t^{-u}$, *the mapping* $t\mapsto\frac{f_t'(u)}{h(u)}$ *is exponentially convex function (see [22]).*
Analogously as in the proof of Theorem 3.4.8 we conclude that $t\mapsto[u_0,u_1;F_t]$ *where* $F_t=\frac{f_t'}{h}$ *is exponentially convex (and so exponentially convex in the Jensen sense). Also by Corollary 3.4.9 we have that* $t\mapsto\mathcal{L}_{17}(f_t)$ *is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous so it is exponentially convex. For this family of functions,* $\mathfrak{B}(t,r;\mathcal{L}_{17},\Upsilon_9)$ *from* (3.4.12) *is equal to*

$$\mathfrak{B}(t,r;\mathcal{L}_{17},\Upsilon_9)=\begin{cases}\left(\dfrac{\log r\left(\int_a^b h(u)du\int_a^b t^{-p}h(p)dp-2\int_a^b h(u)\int_a^u t^{-p}h(p)dpdu\right)}{\log t\left(\int_a^b h(u)du\int_a^b r^{-p}h(p)dp-2\int_a^b h(u)\int_a^u r^{-p}h(p)dpdu\right)}\right)^{\frac{1}{t-r}}, & t\neq r,\ t,r\neq 1,\\ \left(\dfrac{\left(\int_a^b h(u)du\int_a^b t^{-p}h(p)dp-2\int_a^b h(u)\int_a^u t^{-p}h(p)dpdu\right)}{-\log t\left(\int_a^b h(u)du\int_a^b ph(p)dp-2\int_a^b h(u)\int_a^u ph(p)dpdu\right)}\right)^{\frac{1}{t-1}}, & t\neq r=1,\\ \exp\left(\dfrac{-1}{t\log t}-\dfrac{\int_a^b h(u)du\int_a^b pt^{-p}h(p)dp-2\int_a^b h(u)\int_a^u pt^{-p}h(p)dpdu}{t\left(\int_a^b h(u)du\int_a^b t^{-p}h(p)dp-2\int_a^b h(u)\int_a^u t^{-p}h(p)dpdu\right)}\right), & t=r,\quad t,r\neq 1,\\ \exp\left(\dfrac{\int_a^b h(u)du\int_a^b p^2h(p)dp-2\int_a^b h(u)\int_a^u p^2h(p)dpdu}{-2\left(\int_a^b h(u)du\int_a^b ph(p)dp-2\int_a^b h(u)\int_a^u ph(p)dpdu\right)}\right), & t=r=1,\end{cases}$$

Chapter 4

Superadditivity, Monotonicity and Exponential Convexity Of The Petrović type Functionals

In the present chapter we consider functionals derived from Petrović type inequalities and establish their superadditivity, subadditivity and monotonicity properties on the corresponding real n-tuples. By virtue of established results we also define some related functionals and investigate their properties regarding exponential convexity. Finally, the general results are then applied to some particular settings. The results given in this chapter are given in [7] and [6].

4.1 Introduction and Preliminary Results

In the present chapter we prove some interesting properties of the functionals derived by virtue of the Petrović and related inequalities (see [19], p.152–159). For the sake of simplicity these inequalities will be referred to as the Petrović type inequalities, while the corresponding functionals will be referred to as the Petrović type functionals.

Therefore, throughout this Introduction, we present the above mentioned Petrović type inequalities that will be the base in our research and also define the corresponding functionals that will be the subject of our study. We start with the following inequality:

Theorem 4.1.1. *Let* $I = (0, a] \subseteq \mathbb{R}_+$ *be an interval,* $(x_1, \ldots, x_n) \in I^n$ *and let*

$(p_1, \ldots, p_n) \in \mathbb{R}_+^n$ be a nonnegative real n-tuple such that

$$\sum_{i=1}^{n} p_i x_i \in I \quad \text{and} \quad \sum_{i=1}^{n} p_i x_i \geq x_j \ \text{ for } \ j = 1, \ldots, n. \tag{4.1.1}$$

If $f : I \to \mathbb{R}$ is such that the function $f(x)/x$ is decreasing on I, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i). \tag{4.1.2}$$

In addition, if $f(x)/x$ is increasing on I, then the sign of inequality in (4.1.2) is reversed.

Remark 4.1.1. It should be noticed here that if $f(x)/x$ is strictly increasing function on I, then the equality in (4.1.2) is valid if and only if we have equalities in (4.1.1) instead of inequalities, that is, if $x_1 = \ldots = x_n$ and $\sum_{i=1}^{n} p_i = 1$.

Motivated by the above theorem, we define the Petrović type functional as:
(M_{18}) Under the assumptions of Theorem 4.1.1, we consider the following functional

$$\mathcal{L}_{18}(\mathbf{x}, \mathbf{p}; f) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right), \tag{4.1.3}$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ and $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}_+^n$.

Remark 4.1.2. If (4.1.1) holds and $f(x)/x$ is decreasing on I, then

$$\mathcal{L}_{18}(\mathbf{x}, \mathbf{p}; f) \geq 0. \tag{4.1.4}$$

On the other hand, if (4.1.1) is valid and $f(x)/x$ is increasing on I, then

$$\mathcal{L}_{18}(\mathbf{x}, \mathbf{p}; f) \leq 0. \tag{4.1.5}$$

The above functional (4.1.3) will also be considered under slightly altered assumptions on real n-tuples $\mathbf{x}$ and $\mathbf{p}$. For that sake, the following result from [19] will be used in due course.

Theorem 4.1.2. *Suppose $I = (0, a] \subseteq \mathbb{R}_+$, $(x_1, \ldots, x_n) \in I^n$ is a real n-tuple such that $0 < x_1 \leq \ldots \leq x_n$ and let $(p_1, \ldots, p_n) \in \mathbb{R}_+^n$. Further, let $f : I \to \mathbb{R}$ be such that $f(x)/x$ is increasing on I.*

(i) If there exists m $(\leq n)$ such that

$$\bar{P}_1 \geq \bar{P}_2 \geq \ldots \geq \bar{P}_m \geq 1, \quad \bar{P}_{m+1} = \ldots = \bar{P}_n = 0, \tag{4.1.6}$$

where $P_k = \sum_{i=1}^{k} p_i$, $\bar{P}_k = P_n - P_{k-1}$, $k = 2, \ldots, n$, and $\bar{P}_1 = P_n$, then (4.1.2) *holds.*

(ii) If there exists m $(\leq n)$ such that

$$0 \leq \bar{P}_1 \leq \bar{P}_2 \leq \ldots \leq \bar{P}_m \leq 1, \quad \bar{P}_{m+1} = \ldots = \bar{P}_n = 0, \tag{4.1.7}$$

then the reverse inequality in (4.1.2) *holds.*

Remark 4.1.3. If $f(x)/x$ is increasing on I and (4.1.6) holds, then the Petrović-type functional $\mathcal{L}_{18}$ is nonnegative, i.e., inequality (4.1.4) is valid. Conversely, if $f(x)/x$ is increasing on I and conditions as in (4.1.7) are fulfilled, then the relation (4.1.5) holds.

In order to define another Petrović type functional, we cite the following Petrović type inequality involving a convex function.

Theorem 4.1.3. *Let $I = [0, a] \subseteq \mathbb{R}_+$, $(x_1, \ldots, x_n) \in I^n$ and let $(p_1, \ldots, p_n) \in \mathbb{R}_+^n$ fulfills conditions as in* (4.1.1). *If $f : I \to \mathbb{R}$ is a convex function, then*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \geq \sum_{i=1}^{n} p_i f(x_i) + \left(1 - \sum_{i=1}^{n} p_i\right) f(0). \tag{4.1.8}$$

Remark 4.1.4. If f is a concave function then $-f$ is convex, hence replacing f by $-f$ in Theorem 4.1.3, we obtain inequality

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i) + \left(1 - \sum_{i=1}^{n} p_i\right) f(0). \tag{4.1.9}$$

Remark 4.1.5. If the function f from Theorem 4.1.3 is strictly convex, then the inequality in (4.1.8) is strict, if all x_i's are not equal or $\sum_{i=1}^{n} p_i \neq 1$.

Now, regarding inequality (4.1.8) we define another Petrović type functional $\mathcal{L}_{19}$ as:

(M_{19}) Under the assumptions of Theorem 4.1.3, we consider the following functional

$$\mathcal{L}_{19}(\mathbf{x}, \mathbf{p}; f) = f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i) - \left(1 - \sum_{i=1}^{n} p_i\right) f(0), \tag{4.1.10}$$

provided that $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$, $I = [0, a]$, $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}_+^n$ and f is defined on I.

Remark 4.1.6. If (4.1.1) holds and $f : I \to \mathbb{R}$ is a convex function, then

$$\mathcal{L}_{19}(\mathbf{x}, \mathbf{p}; f) \geq 0. \tag{4.1.11}$$

If (4.1.1) holds and $f : I \to \mathbb{R}$ is a concave function, then

$$\mathcal{L}_{19}(\mathbf{x}, \mathbf{p}; f) \leq 0. \tag{4.1.12}$$

Finally, we shall also be concerned with an integral form of the Petrović type functional, based on the following integral Petrović type inequality.

Theorem 4.1.4. *Let $I \subseteq \mathbb{R}$ be an interval, $0 \in I$ and let $f : I \to \mathbb{R}$ be a convex function. Further, suppose $h : [a, b] \to I$ is continuous and monotone with $h(t_0) = 0$, where $t_0 \in [a, b]$ is fixed and g is a function of bounded variation with*

$$G(t) := \int_a^t dg(x), \quad \overline{G}(t) := \int_t^b dg(x).$$

(*a*) *If $\int_a^b h(t)dg(t) \in I$ and*

$$0 \leq G(t) \leq 1 \quad for \quad a \leq t \leq t_0, \;\; 0 \leq \overline{G}(t) \leq 1 \;\; for \;\; t_0 \leq t \leq b, \tag{4.1.13}$$

then

$$\int_a^b f(h(t))dg(t) \geq f\left(\int_a^b h(t)dg(t)\right) + \left(\int_a^b dg(t) - 1\right) f(0). \tag{4.1.14}$$

(*b*) *If $\int_a^b h(t)dg(t) \in I$ and either*

there exists an $s \leq t_0$ such that $G(t) \leq 0$ for $t < s$,

$$G(t) \geq 1 \;\; for \;\; s \leq t \leq t_0 \;\; and \;\; \overline{G}(t) \leq 0 \;\; for \;\; t > t_0 \tag{4.1.15}$$

or

there exists an $s \geq t_0$ such that $G(t) \leq 0$ for $t < t_0$,

$$\overline{G}(t) \geq 1 \; for \; t_0 < t < s \; and \; \overline{G}(t) \leq 0 \; for \; t \geq s, \tag{4.1.16}$$

then the reverse inequality in (4.1.14) *holds.*

In view of Theorem 4.1.4, we define the functional:
(M_{20}) Under the assumptions of Theorem 4.1.4, we consider the following functional

$$\begin{aligned}\mathcal{L}_{20}(h,g;f) &= \int_a^b f(h(t))dg(t) - f\left(\int_a^b h(t)dg(t)\right) \\ &\quad - \left(\int_a^b dg(t) - 1\right) f(0),\end{aligned} \tag{4.1.17}$$

which represents the integral form of the Petrović type functional.

Remark 4.1.7. If the functions f, g and h are defined as in the statement of Theorem 4.1.4 and (4.1.13) holds, then the functional $\mathcal{L}_{20}$ is nonnegative, i.e.,

$$\mathcal{L}_{20}(h,g;f) \geq 0. \tag{4.1.18}$$

Moreover, if either (4.1.15) or (4.1.16) hold, then

$$\mathcal{L}_{20}(h,g;f) \leq 0. \tag{4.1.19}$$

For a comprehensive inspection on the Petrović type inequalities including proofs and diverse applications, the reader is referred to [19].

The chapter is organized in the following way: After this Introduction, in Section 4.1.1 we prove superadditivity, subadditivity and monotonicity properties of functionals $\mathcal{L}_{18}, \mathcal{L}_{19}$ and $\mathcal{L}_{20}$. In addition, we also derive some bounds for the functional $\mathcal{L}_{18}$ via the non-weighted functional of the same type. By virtue of results from Section 4.1.1, in Section 4.1.2 we study some other classes of Petrović type functionals and investigate their properties regarding exponential convexity. Finally, in Section 4.1.3 we apply our general results to some particular settings.

Conventions. Throughout this chapter $\mathbb{R}_+$ denotes the set of nonnegative numbers (including zero). Further, bold letters $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{x}$ respectively denote real n-tuples $(p_1, p_2, \ldots, p_n)$, $(q_1, q_2, \ldots, q_n)$ and $(x_1, x_2, \ldots, x_n)$. Moreover, $\mathbf{p} \geq \mathbf{q}$ means that $p_i \geq q_i$ for all $i = 1, 2, \ldots n$.

4.1.1 Superadditivity, Subadditivity and Monotonicity of Petrović Type Functionals

In the present section, we derive some interesting properties of the Petrović type functionals $\mathcal{L}_{18}, \mathcal{L}_{19}$ and $\mathcal{L}_{20}$, defined in Introduction. More precisely, we establish the conditions under which the appropriate functional is superadditive (subadditive) and increasing (decreasing), with respect to the corresponding n-tuple of real numbers. Our first result refers to the Petrović type functional $\mathcal{L}_{18}$ defined by (4.1.3).

Theorem 4.1.5. *Let $I = (0, a] \subseteq \mathbb{R}_+$, $\mathbf{x} \in I^n$ and let nonnegative n-tuples $\mathbf{p}$, $\mathbf{q}$ fulfill conditions as in* (4.1.1). *If $f : I \to \mathbb{R}$ is such that the function $f(x)/x$ is decreasing on I, then the functional* (4.1.3) *possess the following properties:*

(i) $\mathcal{L}_{18}(\mathbf{x}, .; f)$ is superadditive on nonnegative n-tuples, i.e.,

$$\mathcal{L}_{18}(\mathbf{x}, \mathbf{p} + \mathbf{q}; f) \geq \mathcal{L}_{18}(\mathbf{x}, \mathbf{p}; f) + \mathcal{L}_{18}(\mathbf{x}, \mathbf{q}; f), \tag{4.1.20}$$

provided that $\sum_{i=1}^{n} (p_i + q_i)x_i \in I$.

(ii) If $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^n$ are such that $\mathbf{p} \geq \mathbf{q}$ and $\sum_{i=1}^{n} (p_i - q_i)x_i \geq x_j$, $j = 1, \ldots, n$, then

$$\mathcal{L}_{18}(\mathbf{x}, \mathbf{p}; f) \geq \mathcal{L}_{18}(\mathbf{x}, \mathbf{q}; f) \geq 0, \tag{4.1.21}$$

that is, $\mathcal{L}_{18}(\mathbf{x}, .; f)$ is increasing on nonnegative n-tuples.

(iii) If $f(x)/x$ is increasing on I, then the signs of inequalities in (4.1.20) *and* (4.1.21) *are reversed, i.e., $\mathcal{L}_{18}(\mathbf{x}, .; f)$ is subadditive and decreasing on nonnegative n-tuples.*

Proof. (i) Using definition (4.1.3) of the Petrović type functional $\mathcal{L}_{18}$ and utilizing the linearity of the sum, we have

$$\begin{aligned} \mathcal{L}_{18}(\mathbf{x}, \mathbf{p} + \mathbf{q}; f) &= \sum_{i=1}^{n} (p_i + q_i) f(x_i) - f\left(\sum_{i=1}^{n} (p_i + q_i)x_i\right) \\ &= \sum_{i=1}^{n} p_i f(x_i) + \sum_{i=1}^{n} q_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} q_i x_i\right). \end{aligned} \tag{4.1.22}$$

On the other hand, since $f(x)/x$ is decreasing function, Theorem 4.1.1 in the non-weighted case (for $n = 2$), yields inequality

$$f\left(\sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} q_i x_i\right) \leq f(\sum_{i=1}^{n} p_i x_i) + f(\sum_{i=1}^{n} q_i x_i). \tag{4.1.23}$$

Finally, combining relations (4.1.22) and (4.1.23), we obtain

$$\mathcal{L}_{18}(\mathbf{x},\mathbf{p}+\mathbf{q};f) \geq \sum_{i=1}^{n} p_i f(x_i) + \sum_{i=1}^{n} q_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i) - f(\sum_{i=1}^{n} q_i x_i).$$

Therefore we have

$$\mathcal{L}_{18}(\mathbf{x},\mathbf{p}+\mathbf{q};f) \geq \mathcal{L}_{18}(\mathbf{x},\mathbf{p};f) + \mathcal{L}_{18}(\mathbf{x},\mathbf{q};f),$$

as claimed.
(ii) Monotonicity follows easily from the superadditivity property. Since $\mathbf{p} \geq \mathbf{q} \geq 0$, we can represent $\mathbf{p}$ as the sum of two nonnegative n-tuples, namely $\mathbf{p} = (\mathbf{p}-\mathbf{q}) + \mathbf{q}$. Now, from relation (4.1.20) we get

$$\mathcal{L}_{18}(\mathbf{x},\mathbf{p};f) = \mathcal{L}_{18}(\mathbf{x},\mathbf{p}-\mathbf{q}+\mathbf{q};f) \geq \mathcal{L}_{18}(\mathbf{x},\mathbf{p}-\mathbf{q};f) + \mathcal{L}_{18}(\mathbf{x},\mathbf{q};f).$$

Finally, if the conditions as in (ii) are fulfilled, then, taking into account Theorem 4.1.1 we have that $\mathcal{L}_{18}(\mathbf{x},\mathbf{p}-\mathbf{q};f) \geq 0$, which implies that $\mathcal{L}_{18}(\mathbf{x},\mathbf{p};f) \geq \mathcal{L}_{18}(\mathbf{x},\mathbf{q};f)$.
(iii) The case of increasing function $f(x)/x$ is treated in the same way as in (i) and (ii), taking into account that the sign of the corresponding Petrović type inequality is reversed. □

By virtue of Theorem 4.1.2, the above properties of the functional $\mathcal{L}_{18}$ can also be derived in a slightly different setting.

Theorem 4.1.6. *Let* $I = (0,a] \subseteq \mathbb{R}_+$, $\mathbf{x} \in I^n$ *and let real* n*-tuples* $\mathbf{p}$, $\mathbf{q}$ *fulfill conditions as in* (4.1.6). *If* $f : I \to \mathbb{R}$ *is such that the function* $f(x)/x$ *is increasing on* I*, then the functional* $\mathcal{L}_{18}$ *has the following properties:*

(i) $\mathcal{L}_{18}(\mathbf{x},.;f)$ *is superadditive on real* n*-tuples, i.e.,*

$$\mathcal{L}_{18}(\mathbf{x},\mathbf{p}+\mathbf{q};f) \geq \mathcal{L}_{18}(\mathbf{x},\mathbf{p};f) + \mathcal{L}_{18}(\mathbf{x},\mathbf{q};f), \tag{4.1.24}$$

provided that $\sum_{i=1}^{n} (p_i+q_i)x_i \in I$ *and* $0 < \sum_{i=1}^{n} p_i x_i \leq \sum_{i=1}^{n} q_i x_i$.

(ii) If $0 < x_1 \leq \ldots \leq x_n$, $\mathbf{p} \geq \mathbf{q}$ *and there exist* m $(\leq n)$ *such that*

$$\begin{aligned} &\bar{P}_1 - \bar{Q}_1 \geq \bar{P}_2 - \bar{Q}_2 \geq \ldots \geq \bar{P}_m - \bar{Q}_m \geq 1, \\ &\bar{P}_{m+1} = \bar{Q}_{m+1} = \ldots = \bar{P}_n = \bar{Q}_n = 0, \end{aligned} \tag{4.1.25}$$

where $P_k = \sum_{i=1}^k p_i$, $Q_k = \sum_{i=1}^k q_i$, $\bar{P}_k - \bar{Q}_k = (P_n + Q_n) - (P_{k-1} + Q_{k-1})$, $k = 2, \ldots, n$, $\bar{P}_1 = P_n$ *and* $\bar{Q}_1 = Q_n$, *then*

$$\mathcal{L}_{18}(\mathbf{x}, \mathbf{p}; f) \geq \mathcal{L}_{18}(\mathbf{x}, \mathbf{q}; f) \geq 0, \tag{4.1.26}$$

i.e., $\mathcal{L}_{18}(\mathbf{x}, .; f)$ *is increasing on real* n*-tuples.*

(iii) If real n*-tuples* $\mathbf{p}$ *and* $\mathbf{q}$ *fulfill conditions as in* (4.1.7)*, then the signs of inequalities in* (4.1.24) *and* (4.1.26) *are reversed, that is,* $\mathcal{L}_{18}(\mathbf{x}, .; f)$ *is subadditive and decreasing on real* n*-tuples.*

Proof. (i) The proof follows the same lines as the proof of the previous theorem. Namely, the L. H. S. of (4.1.24) can be rewritten as

$$\begin{aligned} \mathcal{L}_{18}(\mathbf{x}, \mathbf{p}+\mathbf{q}; f) &= \sum_{i=1}^n (p_i + q_i) f(x_i) - f\left(\sum_{i=1}^n (p_i + q_i) x_i\right) \\ &= \sum_{i=1}^n p_i f(x_i) + \sum_{i=1}^n q_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i + \sum_{i=1}^n q_i x_i\right). \end{aligned} \tag{4.1.27}$$

Moreover, $f(x)/x$ is increasing, hence Theorem 4.1.2 for $n = 2$ yields inequality

$$f\left(\sum_{i=1}^n p_i x_i + \sum_{i=1}^n q_i x_i\right) \leq f(\sum_{i=1}^n p_i x_i) + f(\sum_{i=1}^n q_i x_i). \tag{4.1.28}$$

Finally, relations (4.1.27) and (4.1.28) imply inequality

$$\mathcal{L}_{18}(\mathbf{x}, \mathbf{p}+\mathbf{q}; f) \geq \sum_{i=1}^n p_i f(x_i) + \sum_{i=1}^n q_i f(x_i) - f(\sum_{i=1}^n p_i x_i) - f(\sum_{i=1}^n q_i x_i),$$

i.e., we obtain (4.1.24).
(ii) Considering $\mathbf{p} \geq \mathbf{q} \geq 0$, the real n-tuple $\mathbf{p}$ can be rewritten as $\mathbf{p} = (\mathbf{p} - \mathbf{q}) + \mathbf{q}$. Now, regarding relation (4.1.24) we have

$$\mathcal{L}_{18}(\mathbf{x}, \mathbf{p}; f) = \mathcal{L}_{18}(\mathbf{x}, \mathbf{p} - \mathbf{q} + \mathbf{q}; f) \geq \mathcal{L}_{18}(\mathbf{x}, \mathbf{p} - \mathbf{q}; f) + \mathcal{L}_{18}(\mathbf{x}, \mathbf{q}; f).$$

Finally, taking into account conditions as in (4.1.25), it follows by Theorem 4.1.2 that $\mathcal{L}_{18}(\mathbf{x}, \mathbf{p} - \mathbf{q}; f) \geq 0$, that is, $\mathcal{L}_{18}(\mathbf{x}, \mathbf{p}; f) \geq \mathcal{L}_{18}(\mathbf{x}, \mathbf{q}; f)$, which completes the proof.
(iii) This case is treated in the same way as in (i) and (ii), taking into account that the sign of the corresponding Petrović type inequality is reversed. □

Superadditivity and monotonicity properties stated in Theorem 4.1.5 play an important role in numerous applications of the Petrović type inequalities. In the sequel we utilize the monotonicity property of the Petrović type functional $\mathcal{L}_{18}$. More precisely, we derive some bounds for this functional, expressed in terms of the non-weighted functional of the same type.

Corollary 4.1.7. *Let $I = (0, a] \subseteq \mathbb{R}_+$, $\mathbf{x} \in I^n$, and let $f : I \to \mathbb{R}$ be such that $f(x)/x$ is decreasing on I. Further, suppose $\mathbf{p} \in \mathbb{R}_+^n$ is such that $\sum_{i=1}^n (p_i - m)x_i \geq x_j$ and $\sum_{i=1}^n (M - p_i)x_i \geq x_j$, $j = 1, 2, \ldots, n$, where $m = \min_{1\leq i\leq n}\{p_i\}$ and $M = \max_{1\leq i\leq n}\{p_i\}$.*

If $m > 1$ then the Petrović type functional $\mathcal{L}_{18}$ fulfills inequality

$$\mathcal{L}_{18}(\mathbf{x}, \mathbf{p}; f) \geq m\mathcal{L}_{18}^0(\mathbf{x}; f), \tag{4.1.29}$$

while for $M < 1$ we have

$$\mathcal{L}_{18}(\mathbf{x}, \mathbf{p}; f) \leq M\mathcal{L}_{18}^0(\mathbf{x}; f), \tag{4.1.30}$$

where

$$\mathcal{L}_{18}^0(\mathbf{x}; f) = \sum_{i=1}^n f(x_i) - f\left(\sum_{i=1}^n x_i\right). \tag{4.1.31}$$

Moreover, if $f(x)/x$ is increasing on I, then the signs of inequalities in (4.1.29) *and* (4.1.30) *are reversed.*

Proof. Since $\mathbf{p} = (p_1, \ldots, p_n) \geq \mathbf{m} = (m, \ldots, m)$, monotonicity of the Petrović type functional implies that $\mathcal{L}_{18}(\mathbf{x}, \mathbf{p}; f) \geq \mathcal{L}_{18}(\mathbf{x}, \mathbf{m}; f)$.

On the other hand, if $f(x)/x$ is decreasing function, we have

$$f(au) \leq af(u),\ a > 1 \quad \text{and} \quad f(au) \geq af(u),\ a < 1. \tag{4.1.32}$$

Now, regarding (4.1.32) we have

$$\mathcal{L}_{18}(\mathbf{x}, \mathbf{m}; f) = m\sum_{i=1}^n f(x_i) - f\left(m\sum_{i=1}^n x_i\right) \geq m\sum_{i=1}^n f(x_i) - mf\left(\sum_{i=1}^n x_i\right),$$

that is, we obtain (4.1.29). Inequality (4.1.30) is derived in a similar way, by using the second inequality in (4.1.32). □

Our next result provides superadditivity and monotonicity properties of the Petrović type functional defined by (4.1.10).

Theorem 4.1.8. *Let $I = [0, a] \subseteq \mathbb{R}_+$, $\mathbf{x} \in I^n$ and let $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^n$ fulfill conditions as in* (4.1.1). *If $f : I \to \mathbb{R}$ is a convex function, then the functional* (4.1.10) *has the following properties:*

(i) *$\mathcal{L}_{19}(\mathbf{x}, .; f)$ is superadditive on nonnegative n-tuples, i.e.,*

$$\mathcal{L}_{19}(\mathbf{x}, \mathbf{p} + \mathbf{q}; f) \geq \mathcal{L}_{19}(\mathbf{x}, \mathbf{p}; f) + \mathcal{L}_{19}(\mathbf{x}, \mathbf{q}; f), \tag{4.1.33}$$

provided that $\sum_{i=1}^{n} (p_i + q_i)x_i \in I$.

(ii) *If $\mathbf{p}, \mathbf{q}$ are such that $\mathbf{p} \geq \mathbf{q}$ and $\sum_{i=1}^{n} (p_i - q_i)x_i \geq x_j$, $j = 1, .., n$, then*

$$\mathcal{L}_{19}(\mathbf{x}, \mathbf{p}; f) \geq \mathcal{L}_{19}(\mathbf{x}, \mathbf{q}; f) \geq 0, \tag{4.1.34}$$

that is, $\mathcal{L}_{19}(\mathbf{x}, .; f)$ is increasing on nonnegative n-tuples.

(iii) *If $f : I \to \mathbb{R}$ is a concave function, then the signs of inequalities in* (4.1.33) *and* (4.1.34) *are reversed, i.e., $\mathcal{L}_{19}(\mathbf{x}, .; f)$ is subadditive and decreasing on nonnegative n-tuples.*

Proof. (i) The L. H. S. of inequality (4.1.33) can be rewritten as

$$\begin{aligned}
\mathcal{L}_{19}(\mathbf{x}, \mathbf{p} + \mathbf{q}; f) &= f\left(\sum_{i=1}^{n} (p_i + q_i)x_i\right) - \sum_{i=1}^{n} (p_i + q_i) f(x_i) \\
&\quad - \left(1 - \sum_{i=1}^{n}(p_i + q_i)\right) f(0) \\
&= f\left(\sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} q_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i) - \sum_{i=1}^{n} q_i f(x_i) \\
&\quad - \left(1 - (\sum_{i=1}^{n} p_i + \sum_{i=1}^{n} q_i)\right) f(0).
\end{aligned} \tag{4.1.35}$$

Further, Theorem 4.1.3 in the non-weighted case (for $n = 2$) yields inequality

$$f\left(\sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} q_i x_i\right) \geq f(\sum_{i=1}^{n} p_i x_i) + f(\sum_{i=1}^{n} q_i x_i) - f(0), \tag{4.1.36}$$

hence combining relations (4.1.35) and (4.1.38), we get

$$\mathcal{L}_{19}(\mathbf{x},\mathbf{p}+\mathbf{q};f) \geq f(\sum_{i=1}^{n} p_i x_i) - \sum_{i=1}^{n} p_i f(x_i) - \left(1-\sum_{i=1}^{n} p_i\right) f(0)$$
$$+ f(\sum_{i=1}^{n} p_i x_i) - \sum_{i=1}^{n} q_i f(x_i) - \left(1-\sum_{i=1}^{n} q_i\right) f(0). \quad (4.1.37)$$

Thus, considering definition (4.1.10) we obtain (4.1.33), as claimed.
(ii) Monotonicity property follows from the corresponding superadditivity property (4.1.33), as in Theorem 4.1.6.
(iii) The case of concave function f follows from the fact that the sign of the corresponding Petrović type inequality is reversed. □

To conclude this section we also derive the properties of the integral Petrović type functional, defined by (4.1.17).

Theorem 4.1.9. *Suppose $f : I = [0, a] \to \mathbb{R}$ is a convex function, $h : [a, b] \to I$ is continuous and monotone with $h(t_0) = 0$, where $t_0 \in [a, b]$ is fixed and let g_1, g_2 be functions of bounded variation with*

$$G_i(t) := \int_a^t dg_i(x), \quad \overline{G}_i(t) := \int_t^b dg_i(x) \;\; for \;\; i = 1, 2.$$

Then the functional $\mathcal{P}_3$, defined by (4.1.17), *has the following properties:*

(i) $\mathcal{L}_{20}(h, .; f)$ is subadditive with respect to functions of bounded variation, i.e.,

$$\mathcal{L}_{20}(h, g_1 + g_2; f) \leq \mathcal{L}_{20}(h, g_1; f) + \mathcal{L}_{20}(h, g_2; f), \quad (4.1.38)$$

where $\int_a^b h(t)dg_1(t) \geq 0$, $\int_a^b h(t)dg_2(t) \geq 0$ and $\int_a^b h(t)dg_1(t) + \int_a^b h(t)dg_2(t) \in I$.

(ii) If $\int_a^b h(t)d(g_1)(t) - \int_a^b h(t)d(g_2)(t) \in I$ and either there exists an $s \leq t_0$ such that $G_1(t) \leq G_2(t)$ for $t < s$, $G_1(t) - G_2(t) \geq 1$ for $s \leq t \leq t_0$ and $\overline{G}_1(t) \leq \overline{G}_2(t)$ for $t > t_0$, or there exists an $s \geq t_0$ such that $G_1(t) \leq G_2(t)$ for $t < t_0$, $\overline{G}_1(t) - \overline{G}_2(t) \geq 1$ for $t_0 < t < s$ and $\overline{G}_1(t) \leq \overline{G}_2(t)$ for $t \geq s$, then

$$\mathcal{L}_{20}(h, g_1; f) \leq \mathcal{L}_{20}(h, g_2; f). \quad (4.1.39)$$

Proof. (i) Regarding definition (4.1.17) of the Petrović type integral functional, we have

$$\mathcal{L}_{20}(h, g_1+g_2; f) = \int_a^b f(h(t))d(g_1+g_2)(t) - f\left(\int_a^b h(t)d(g_1+g_2)(t)\right) - \left(\int_a^b d(g_1+g_2)(t) - 1\right) f(0),$$

that is,

$$\begin{aligned}\mathcal{L}_{20}(h, g_1+g_2; f) &= \int_a^b f(h(t))dg_1(t) + \int_a^b f(h(t))dg_2(t) \\ &\quad -f\left(\int_a^b h(t)dg_1(t) + \int_a^b h(t)dg_2(t)\right) \\ &\quad -\left(\int_a^b dg_1(t) + \int_a^b dg_2(t) - 1\right) f(0), \qquad (4.1.40)\end{aligned}$$

by the linearity of the differential. Now, applying inequality (4.1.8) to the term $f\left(\int_a^b h(t)dg_1(t) + \int_a^b h(t)dg_2(t)\right)$, we obtain

$$\begin{aligned}&f\left(\int_a^b h(t)dg_1(t) + \int_a^b h(t)dg_2(t)\right) \\ &\quad \geq f\left(\int_a^b h(t)dg_1(t)\right) + f\left(\int_a^b h(t)dg_2(t)\right) - f(0). \qquad (4.1.41)\end{aligned}$$

Further, inserting (4.1.41) in (4.1.40), we have

$$\begin{aligned}\mathcal{L}_{20}(h, g_1+g_2; f) &\leq \int_a^b f(h(t))dg_1(t) + \int_a^b f(h(t))dg_2(t) \\ &\quad -f\left(\int_a^b h(t)dg_1(t)\right) - f\left(\int_a^b h(t)dg_2(t)\right) + f(0) \\ &\quad -\left(\int_a^b dg_1(t) + \int_a^b dg_2(t) - 1\right) f(0),\end{aligned}$$

i.e., by rearranging,

$$\mathcal{L}_{20}(h, g_1+g_2; f) \leq \mathcal{L}_{20}(h, g_1; f) + \mathcal{L}_{20}(h, g_2; f).$$

(ii) Monotonicity follows from the subadditivity property (4.1.38). Namely, representing g_1 as $g_1 = (g_1 - g_2) + g_2$, we have

$$\mathcal{L}_{20}(h, g_1; f) = \mathcal{L}_{20}(h, (g_1-g_2)+g_2; f) \leq \mathcal{L}_{20}(h, g_1-g_2; f) + \mathcal{L}_{20}(h, g_2; f).$$

Clearly, under assumptions as in the statement of theorem, we have $\mathcal{L}_{20}(h, g_1-g_2; f) \leq 0$ (see also Remark 4.1.7), hence it follows that $\mathcal{L}_{20}(h, g_1; f) \leq \mathcal{L}_{20}(h, g_2; f)$, which completes the proof. □

4.1.2 n-Exponential Convexity and Exponential Convexity of the Petrović Type Functionals

By virtue of the results from Section 4.1.1, in this section we define several new classes of Petrović type functionals and investigate their properties regarding exponential convexity.

Now, we are ready to study some new classes of Petrović-type functionals. For the sake of simplicity and to avoid many notions, we first introduce the following definitions:

(M_{21}) Under the assumptions of Theorem 4.1.1 equipped with conditions as in (4.1.5), we define linear functional as

$$\mathcal{L}_{21}(f) = -\mathcal{L}_{18}(\mathbf{x}, \mathbf{p}; f).$$

(M_{22}) Under the assumptions of Theorem 4.1.2 with conditions as in (4.1.6), we define linear functional as

$$\mathcal{L}_{22}(f) = -\mathcal{L}_{21}(f).$$

(M_{23}) Under the assumptions of Theorem 4.1.2 with conditions as in (4.1.6), we define linear functional as

$$\mathcal{L}_{23}(f) = \mathcal{L}_{21}(f).$$

(M_{24}) Under the assumptions of Theorem 4.1.5 with conditions as in (4.1.1) and provided that $\sum_{i=1}^{n} (p_i + q_i)x_i \in I$, we define linear functional as

$$\mathcal{L}_{24}(f)) = \mathcal{L}_{18}(\mathbf{x}, \mathbf{p}; f) + \mathcal{L}_{18}(\mathbf{x}, \mathbf{q}; f) - \mathcal{L}_{18}(\mathbf{x}, \mathbf{p} + \mathbf{q}; f).$$

(M_{25}) Under the assumptions of Theorem 4.1.6 with conditions as in (4.1.6) and provided that $\sum_{i=1}^{n} (p_i + q_i)x_i \in I$, $0 < \sum_{i=1}^{n} p_i x_i \leq \sum_{i=1}^{n} q_i x_i$, we define linear functional as

$$\mathcal{L}_{25}(f) = -\mathcal{L}_{24}(f).$$

(M_{26}) Under the assumptions of Theorem 4.1.6 with conditions as in (4.1.7) and provided that $\sum_{i=1}^{n} (p_i + q_i)x_i \in I$, $0 < \sum_{i=1}^{n} p_i x_i \leq \sum_{i=1}^{n} q_i x_i$, we define linear functional as

$$\mathcal{L}_{26}(f) = \mathcal{L}_{24}(f).$$

(M_{27}) Under the assumptions of Theorem 4.1.3 with conditions as in (4.1.1), we define linear functional as

$$\mathcal{L}_{27}(f) = \mathcal{L}_{19}(\mathbf{x}, \mathbf{p}; f).$$

(M_{28}) Under the assumptions of Theorem 4.1.4 with conditions as in (4.1.13), we define linear functional as

$$\mathcal{L}_{28}(f) = \mathcal{L}_{20}(h, g; f).$$

(M_{29}) Under the assumptions of Theorem 4.1.4 equipped with conditions (4.1.15) or (4.1.16), we define linear functional as

$$\mathcal{L}_{29}(f) = -\mathcal{L}_{20}(h, g; f).$$

(M_{30}) Under the assumptions of Theorem 4.1.8 with conditions as in (4.1.1) and provided that $\sum_{i=1}^{n}(p_i + q_i)x_i \in I$, we define linear functional as

$$\mathcal{L}_{30}(f) = \mathcal{L}_{19}(\mathbf{x}, \mathbf{p} + \mathbf{q}; f) - \mathcal{L}_{19}(\mathbf{x}, \mathbf{p}; f) - \mathcal{L}_{19}(\mathbf{x}, \mathbf{q}; f).$$

(M_{31}) Under the assumptions of Theorem 4.1.9, and provided that $\int_a^b h(t)dg_1(t) \geq 0$, $\int_a^b h(t)dg_2(t) \geq 0$, $\int_a^b h(t)dg_1(t) + \int_a^b h(t)dg_2(t) \in I$, we define linear functional as

$$\mathcal{L}_{31}(f) = \mathcal{L}_{20}(h, g_1; f) + \mathcal{L}_{20}(h, g_2; f) - \mathcal{L}_{20}(h, g_1 + g_2; f).$$

Remark 4.1.8. Considering the assumptions as in (M_k), $k = 21, \ldots, 26$ if $f(u)/u$ is increasing function on I, then

$$\mathcal{L}_k(f) \geq 0, \text{ for } k = 21, \ldots, 26.$$

Remark 4.1.9. Considering the assumptions as in (M_k), $k = 27, \ldots, 31$ if f is convex function on I then

$$\mathcal{L}_k(f) \geq 0 \text{ for } k = 27, \ldots, 31.$$

Now we are ready to investigate the properties of functionals as defined above, regarding n-exponential and exponential convexity. In order to prove exponential convexity of the functionals defined above, we define different families of functions. Let $I \subseteq \mathbb{R}_+$ not containing zero and $J \subseteq \mathbb{R}$ be intervals. For distinct points $u_0, u_1, u_2 \in I$, let $\mathbf{E}_i$, $i = 16, 17, 18$, denote a family of functions with the following property:

$\mathbf{E}_{16} = \{f_t : I \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1, F_t]$ is n-exponentially convex in the Jensen sense on J, where $F_t(u) = f_t(u)/u\}$.

$\mathbf{E}_{17} = \{f_t : I \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1, F_t]$ is exponentially convex in the Jensen sense on J, where $F_t(u) = f_t(u)/u\}$.

$\mathbf{E}_{18} = \{f_t : I \to \mathbb{R} \mid t \in J$ and $t \mapsto [u_0, u_1, F_t]$ is 2-exponentially convex in the Jensen sense on J, where $F_t(u) = f_t(u)/u\}$.

Theorem 4.1.10. *Let $\mathcal{L}_k$ be the linear functionals defined as in (M_k) for $k = 21, \ldots, 31$ associated with the families $\mathbf{E}_{16}$ and $\mathbf{E}_1$ in such a way that $f_t \in \mathbf{E}_{16}$, for $k = 21, \ldots, 26$ and $f_t \in \mathbf{E}_1$, for $k = 27, \ldots, 31$. Then $t \mapsto \mathcal{L}_k(f_t)$ is an n-exponentially convex function in the Jensen sense on J. If the function $t \mapsto \mathcal{L}_k(f_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. See proof of Theorem 3.2.5 and 2.3.5 respectively. □

The following corollary is an immediate consequence of the above theorem.

Corollary 4.1.11. *Let $\mathcal{L}_k$ be the linear functionals defined as in (M_k) for $k = 21, \ldots, 31$ associated with the families $\mathbf{E}_{17}$ and $\mathbf{E}_2$ in such a way that $f_t \in \mathbf{E}_{17}$, $k = 21, \ldots, 26$ and $f_t \in \mathbf{E}_2$, $k = 27, \ldots, 31$. Then $t \mapsto \mathcal{L}_k(f_t)$ is exponentially convex function in the Jensen sense on J. If $t \mapsto \mathcal{L}_k(f_t)$ is continuous on J, then it is exponentially convex on J.*

Proof. Follows from the previous theorem. □

Corollary 4.1.12. *Let $\mathcal{L}_k$ be the linear functionals defined as in (M_k) for $k = 21, \ldots, 31$ associated with the families $\mathbf{E}_{18}$ and $\mathbf{E}_3$ in such a way that $f_t \in \mathbf{E}_{18}$, $k = 21, \ldots, 26$ and $f_t \in \mathbf{E}_3$, $k = 27, \ldots, 31$. Then the following statements hold:*

(i) If the function $t \mapsto \mathcal{L}_k(f_t)$ is strictly positive and continuous on J, then it is 2-exponentially convex on J, and thus, log-convex. Also for $r, s, t \in J$ such that $r < s < t$, we have

$$(\mathcal{L}_k(f_s))^{t-r} \leq (\mathcal{L}_k(f_r))^{t-s}(\mathcal{L}_k(f_t))^{s-r}. \tag{4.1.42}$$

(ii) If the function $t \mapsto \mathcal{L}_k(f_t)$ is strictly positive and differentiable on J, then for all $t, r, u, v \in J$ such that $t \leq u$, $r \leq v$ and for fixed $i = 18$ or 3 we have

$$\mathfrak{B}(t, r; \mathcal{L}_k, \mathbf{E}_i) \leq \mathfrak{B}(u, v; \mathcal{L}_k, \mathbf{E}_i), \ k = 9, \ldots, 12$$

where

$$\mathfrak{B}(t, r; \mathcal{L}_k, \mathbf{E}_i) = \begin{cases} \left(\frac{\mathcal{L}_k(f_t)}{\mathcal{L}_k(f_r)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{\frac{d}{d_t}(\mathcal{L}_k(f_t))}{\mathcal{L}_k(f_t)}\right), & t = r. \end{cases} \tag{4.1.43}$$

Proof. Follow similar steps as in Corollary 3.2.7 and 2.3.7 respectively. □

Remark 4.1.10. Also, the comments given in Remark 2.3.4 are valid for the functionals $\mathcal{L}_k$ for $k = 21, \ldots, 31$.

4.1.3 Examples

We conclude this chapter with several examples related to the results from the previous section.

Example 4.1.1. *Consider a family of functions*

$$\Upsilon_{10} = \{\zeta_t : (0, \infty) \to \mathbb{R} : t > 0\}$$

defined by

$$\zeta_t(u) = \begin{cases} \frac{ut^{-u}}{-\log t}, & t \neq 1, \\ u^2, & t = 1. \end{cases}$$

Obviously, a family of functions $\zeta_t(u)/u$ is increasing for all $t > 0$, hence, by virtue of Theorem 4.1.5*, we obtain that the functional $\mathcal{L}_{18}(\mathbf{x}, .; \zeta_t)$ is subadditive and decreasing on nonnegative n-tuples.*

Moreover, since $(\zeta_t(u)/u)' = t^{-u}$, the mapping $t \mapsto (\zeta_t(u)/u)'$ is exponentially convex (see [22]). Now, regarding Corollary 4.1.11 *and Remark* 4.1.10*, we get exponential convexity of the functionals $\mathcal{L}_k(\zeta_t)$ for $k = 21, \ldots, 26$.*

In addition, Corollary 4.1.12 *provides the log-convexity of these functionals and we have*

$$\mathfrak{B}(t, r; \mathcal{L}_k, \Upsilon_{10}) = \begin{cases} \left(\frac{\mathcal{L}_k(\zeta_t)}{\mathcal{L}_k(\zeta_r)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{-1}{t\log t} - \frac{\mathcal{L}_k(u\zeta_t)}{t(\mathcal{L}_k(\zeta_t))}\right), & t = r \neq 1, \\ \exp\left(\frac{\mathcal{L}_k(u\zeta_1)}{-2(\mathcal{L}_k(\zeta_1))}\right), & t = r = 1, \end{cases}$$

for $k = 21, \ldots, 26$.

Example 4.1.2. *Consider a family of functions*

$$\Upsilon_{11} = \{\lambda_t : (0, \infty) \to \mathbb{R} : t > 0\}$$

defined by

$$\lambda_t(u) = \frac{u \exp(-u\sqrt{t})}{-\sqrt{t}}.$$

Since the function $\lambda_t(u)/u$ is increasing for every $t > 0$, utilizing Theorem 4.1.5, we obtain that the functional $\mathcal{L}_{18}(\mathbf{x}, .; \lambda_t)$ is subadditive and decreasing on nonnegative n-tuples.

Further, since $(\lambda_t(u)/u)' = \exp(-u\sqrt{t})$, the mapping $t \mapsto (\lambda_t(u)/u)'$ is exponentially convex (see [22]). Now, by using Corollary 4.1.11 and Remark 4.1.10, we get exponential convexity of the functionals $\mathcal{L}_k(\lambda_t)$ for $k = 21, \ldots, 26$.

In addition, Corollary 4.1.12 implies the log-convexity of these functionals and we have

$$\mathfrak{B}(t, r; \mathcal{L}_k, \Upsilon_{11}) = \begin{cases} \left(\frac{\mathcal{L}_k(\lambda_t)}{\mathcal{L}_k(\lambda_r)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{-1}{2t} - \frac{\mathcal{L}_k(u\lambda_t)}{2\sqrt{t}(\mathcal{L}_k(\lambda_t))}\right), & t = r, \end{cases}$$

for $k = 21, \ldots, 26$.

Example 4.1.3. *Consider a family of functions*

$$\Upsilon_{12} = \{\psi_t : \mathbb{R}_+ \to \mathbb{R} : t \in \mathbb{R}_+\}$$

defined by

$$\psi_t(u) = \begin{cases} \frac{u\exp(ut)}{t}, & t \neq 0, \\ u^2, & t = 0. \end{cases}$$

It is easy to see that the function $\psi_t(u)/u$ is increasing on $\mathbb{R}_+$ for all $t \in \mathbb{R}_+$. Hence, by virtue of Theorem 4.1.5, the functional $\mathcal{L}_{18}(\mathbf{x}, .; \psi_t)$ is subadditive and decreasing on nonnegative n-tuples.

In addition, $(\psi_t(u)/u)' = \exp(ut)$ and the mapping $t \mapsto (\psi_t(u)/u)'$ is exponentially convex (see [22]). Similarly as in the previous examples, Corollary 4.1.11 and Remark 4.1.10 provide exponential convexity of the functionals $\mathcal{L}_k(\psi_t)$ for $k = 1, \ldots, 6$.

Also, by Corollary 4.1.12, we get log-convexity of these functionals and we have

$$\mathfrak{B}(t, r; \mathcal{L}_k, \Upsilon_{12}) = \begin{cases} \left(\frac{\mathcal{L}_k(\psi_t)}{\mathcal{L}_k(\psi_r)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{-1}{t} + \frac{\mathcal{L}_k(u\psi_t)}{(\mathcal{L}_k(\psi_t))}\right), & t = r \neq 0, \\ \exp\left(\frac{\mathcal{L}_k(u\psi_0)}{2(\mathcal{L}_k(\psi_0))}\right), & t = r = 0, \end{cases}$$

for $k = 21, \ldots, 26$.

Example 4.1.4. *Consider a family of functions*

$$\Upsilon_{13} = \{\beta_t : (0, \infty) \to \mathbb{R} : t > 0\}$$

defined by

$$\beta_t(u) = \begin{cases} \frac{u^t}{t-1}, & t \neq 1, \\ u \log u, & t = 1. \end{cases}$$

Obviously, a family of functions $\beta_t(u)/u$ *is increasing for* $t > 0$*, hence, by virtue of Theorem* 4.1.5*, we obtain that the functional* $\mathcal{L}_{18}(\mathbf{x}, .; \beta_t)$ *is subadditive and decreasing on nonnegative* n*-tuples.*

Further, since $(\beta_t(u)/u)' = u^{t-2} = \exp((t-2)\log u)$*, the mapping* $t \mapsto (\beta_t(u)/u)'$ *is exponentially convex (see [22]). Similarly as in the previous examples, regarding Corollary* 4.1.11 *and Remark* 4.1.10*, we get exponential convexity of the functionals* $\mathcal{L}_k(\beta_t)$ *for* $k = 21, \ldots, 26$*.*

In addition, Corollary 4.1.12 *provides the log-convexity of these functionals and we have*

$$\mathfrak{B}(t, r; \mathcal{L}_k, \Upsilon_{13}) = \begin{cases} \left(\frac{\mathcal{L}_k(\beta_t)}{\mathcal{L}_k(\beta_r)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{1}{1-t} + \frac{\mathcal{L}_k(\log u \beta_t)}{(\mathcal{L}_k(\beta_t)}\right), & t = r \neq 1, \\ \exp\left(\frac{\mathcal{L}_k(\log u \beta_1)}{2(\mathcal{L}_k(\beta_1))}\right), & t = r = 1, \end{cases}$$

for $k = 21, \ldots, 26$*.*

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