Face Labelings of Graphs

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DECLARATION

I, **Mrs. Fozia Bashir** Registration No. **19–GCU–PHD–SMS–05** student at **Abdus Salam School of Mathematical Sciences GC University** in the subject of **Mathematics**, hereby declare that the matter printed in this thesis titled

"Face Labelings of Graphs"

is my own work and that

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(ii) No direct major work had already been done by me or anybody else on this topic; I worked on, for the Ph. D. degree.

(iii) The work, I am submitting for the Ph. D. degree has not already been submitted elsewhere and shall not in future be submitted by me for obtaining similar degree from any other institution.

Dated: ______________________ ______________________

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RESEARCH COMPLETION CERTIFICATE

Certified that the research work contained in this thesis titled

"Face Labelings of Graphs"

has been carried out and completed by Mrs. **Fozia Bashir**
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Sovereignty is due to almighty ALLAH and none else, Praise is to ALLAH, Lord of the world, the self sufficient, the responder, the omnipotent, the answer of prayers, the peace and source of peace, whose blessings and exaltation flourished my thoughts, thrived my ambitions and enabled me to contribute sincerely to the sacred wealth of knowledge. Special praise and regards for his last messenger, Hazrat Muhammad (May peace be upon him) who is for ever a tower of knowledge and guidance for humanity.

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Abstract

The thesis deals with the problem of labeling the vertices, edges and faces of a plane graph in such a way that the label of a face and the labels of vertices and edges surrounding that face add up to a weight of that face. A labeling of a plane graph is called *d-antimagic* if for every positive integer s, the s-sided face weights form an arithmetic progression with a difference d. Such a labeling is called *super* if the smallest possible labels appear on the vertices.

The thesis is devoted to study of super d-antimagic labelings of type $(1, 1, 1)$ for antiprisms and disjoint union of prisms.

We consider the antiprism and prism as three cycle parts: the outer cycle, the inner cycle and the middle cycle. To label the inner, the outer and the middle cycles we use the edge-antimagic total labelings and the vertex-antimagic total labelings. These labelings combine to a resulting super d-antimagic labeling of type $(1, 1, 1)$ for the required values of difference d.

List of publications arising from this thesis

[1] G. Ali, M. Bača, F. Bashir and A. Semaničová-Feňovčíková, *On face antimagic labelings of disjoint union of prisms*, **Utilitas Math.**, in press.

[2] M. Bača, F. Bashir and A. Semaničová, *Face antimagic labelings of antiprisms*, **Utilitas Math.**, in press.

[3] M. Bača and F. Bashir, *On super d-antimagic labelings of disjoint union of prisms*, **AKCE J. Graphs. Combin.** 6, No. 1 (2009), 31–39.

Further publications produced during my PhD candidature

[1] G. Ali, M. Bača and F. Bashir, *On super vertex-antimagic total labelings of disjoint union of paths*, **AKCE J. Graphs. Combin.** 6, No. 1 (2009), 11–20.

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Introduction

Motivated by the notion of magic squares in number theory, magic labeling was introduced by Sedláček [68] in 1963. He defined a graph to be *magic* if it had an edge labeling, with range the real numbers, such that the vertex-weights, obtained for each vertex by adding all the labels of its adjacent edges, are the same.

Later Ko-Wei Lih in [54] dealt with the problem of labeling the vertices, edges and faces of a plane graph in such a way that the label of a face and the labels of vertices and edges surrounding that face added up to a fixed value. He called such a labeling *magic* but this notion of being magic is entirely different from this defined by Sedláček. Therefore the Lih's labeling is called *face-magic*.

However, the subject of magic labeling can be traced back to the 13th century when similar notions were investigated by the Chinese mathematician Yang Hui and published in 1275. Surely, Yang Hui did not have the concept of a graph. He extended magic constructions (configurations similar to magic squares) to plane configurations (vertical and horizontal diagrams), see [54] and [55]. This theme was further developed by Chang Chhao, circa 1670. It finally reached the amazing achievements of an almost unknown amateur mathematician Pao Chhi-Shou, circa 1880, of constructing magic labelings for the platonic polyhedra and icosidodecahedron. Pao Chhi-Shou followed the traditional custom of not revealing the methods by which he obtained his results. Unfortunately, except labelings for the cube, his constructions were illustrated largely by plane net representations of polyhedra. Needham in [66] shows only one of his easier products.

Ko-Wei Lih [54] clarified the concepts behind Pao's labelings by using modern notions of the graph theory and extend these classical labelings of platonic polyhedra to certain families of plane graphs.

Assume that $G = G(V, E, F)$ is a plane graph with the vertex set $V = V(G)$, the edge set $E = E(G)$, the face set $F = F(G)$ and $\alpha, \beta, \gamma \in \{0, 1\}$. A labeling of type (α, β, γ) assigns the labels from the set $\{1, 2, 3, \dots, \alpha|V(G)| + \beta|E(G)| + \gamma|F(G)|\}$ to the vertices, edges and faces of G in such a way that each vertex receives α labels, each edge receives β labels and each face receives γ labels and each number is used exactly once as a label. Labelings of types $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are also called *vertex*, *edge* and *face* labelings, respectively.

The *weight* of a face under a labeling of type (α, β, γ) is the sum of labels (if present) carried by that face and the edges and vertices surrounding it. A labeling of type (α, β, γ) is said to be *face-magic*, if for every positive integer s, all s-sided faces have the same weight. We allow different weights for different s.

We mentioned that this notion of face-magic labeling of the plane graphs was introduced by Ko-Wei Lih [54]. He described face-magic labelings of type $(1, 1, 0)$ for the wheels, the friendship graphs and the prisms. The face-magic labelings of type $(1, 1, 1)$ for the grid graphs and the honeycomb are given in [8] and [9], respectively.

Another important labeling, that has also been the main subject of study of many papers, is the *edge-magic* labeling. This labeling was introduced by Kotzig and Rosa [51] as a one-to-one map taking the vertices and edges into the integers $1, 2, \ldots, |V(G)| + |E(G)|$ with the property that the edge-weights, obtained for each edge by adding the labels of an edge and its endpoints, are the same. It has been conjectured in [52] and also in [67] that all trees are edge-magic. However, proving or disproving this conjecture seems to be a difficult problem.

Hartsfield and Ringel [41] introduced the concept of an *antimagic labeling* as a natural extension of the notion of the magic labeling. In their terminology, a graph $G(V, E)$ is called *antimagic* if its edges are labeled with labels $1, 2, \ldots, |E(G)|$ in such a way that all vertex-weights are pairwise distinct. Hartsfield and Ringel point out that antimagic graphs include paths P_n, $n \geq 3$, cycles, wheels, and complete graphs K_n, $n \geq 3$. They conjecture that every connected graph, except K_2, is antimagic.

Often it is very easy to find many different antimagic labelings for a given graph. Therefore, it is reasonable to investigate antimagic labelings with some restrictions placed on the weights. Consequently, Bodendiek and Walther [28] introduced a restriction on the vertex-weights such that the vertex-weights form an arithmetic progression with a common difference d.

Another situation that is of interest is when all the edge-weights form an arithmetic progression with a common difference d. In such a case we call the labeled graph *edge-antimagic*. The definition of this kind of labeling was introduced by Simanjuntak, Miller and Bertault [71].

Bača and Miller [13] introduced the concept of a *d-antimagic labeling* as a natural extension of the notion of a face-magic labeling. A labeling of type (α, β, γ) of a plane graph is called *d-antimagic*, if for every positive integer s, the set of s-sided face-weights is $W_s = \{a_s, a_s + d, a_s + 2d, \ldots, a_s + (f_s - 1)d\}$ for some integers a_s and $d \geq 0$, where f_s is the number of the s-sided faces. We allow different sets W_s for different s.

While many researchers studied the properties of magic and antimagic labelings, other researchers examined their applications. Kalantari, Khosrovshahi and Mitchell in [47] and [64] tried to find applications of magic labeling in optimization theory, especially for the traveling salesmen problem. Baskoro, Simanjuntak and Adithia [23], [24] proposed a secret sharing scheme construction using an edge-magic labeling. Based on Bloom and Golomb's results [26], [27], Wallis [79] proposed the edge-magic total labeling for assigning addresses of communication networks and radar pulse codes. Recently, Hartnell and Rall [40] proposed a game based on vertex-magic labeling.

Graph labelings provide useful mathematical models for a wide range of applications, such as data security, cryptography, various coding theory problems, communication networks, mobile telecommunication systems, bioinformatics and x-ray crystallography. More detailed discussions about applications of graph labelings can be found in Bloom and Golomb's papers [26] and [27].

Outline of the thesis

In Chapter 1 we introduce some basic definitions and notation about graph theory that are used through the thesis. We give an overview of magic-type labelings, namely magic, supermagic, prime-magic and face-magic; and antimagic-type labelings, namely edge-antimagic, vertex-antimagic and d-antimagic. Additionally, we give certain known results for these considered labelings.

In Chapter 2 we present edge-antimagic and vertex-antimagic labelings of cycles and disjoint union of cycles. We use these labelings in next two chapters to obtain the desired super d-antimagic labelings of type $(1,1,1)$ for the antiprisms and the union of prisms.

In Chapter 3 we consider the antiprism as three cycle parts. To label the three cycle parts we use the (a,d)-edge-antimagic total labelings and the (a,d)-vertex-antimagic total labelings. The super d-antimagic labelings of type $(1,1,1)$ of antiprisms, for $d \in \{0,1,2,3,4,5,6\}$, we obtain as a combination of the labelings of these cycle parts.

In Chapter 4 we deal with the existence of super d-antimagic labelings of type $(1,1,1)$ of the disjoint union of prisms. We use a super (a,d)-edge-antimagic total labeling and an (a, d)-vertex-antimagic total labeling of a disjoint union of cycles for describing various d-antimagic labelings.

In Chapter 5 we summarize the open problems.

At the end of the thesis, in Appendix, we give the list of graph-theoretic symbols used in the thesis.

Chapter 1

Magic and antimagic labelings

As the first part of this chapter some basic definitions, notation and terminology in the graph theory are discussed. For other concepts which are not explicitly given here see [19], [31], [79] and [80]. The second part is dedicated to presenting some types of graph labelings.

1.1 Basic definitions

Throughout this thesis we consider only finite, undirected graphs without loops, multiple edges and isolated vertices. If $G = G(V, E)$ is a graph, then $V(G)$ is a finite non empty set of elements called *vertices* and $E(G)$ is a set (possibly empty) of unordered pairs $\{x, y\}$ of vertices $x, y \in V(G)$, called *edges*. The cardinality of the vertex set $V(G)$ is called the *order* of G, commonly denoted by $|V(G)| = v$. The cardinality of the edge set $E(G)$ is the *size* of G, often denoted by $|E(G)| = e$.

There are many ways to represent a graph. However, traditionally a graph is represented by a diagram. A dot represents a vertex and a curve, usually a line segment, represents an edge.

In a graph G, a vertex x is said to be *adjacent* to the vertex y if there is an edge xy between x and y. The vertices x and y are called the endpoints of an edge xy. The vertex y is then called a neighbor of x, or we say that x and y are incident with the edge xy. The set of all neighbors of the vertex x in a graph G is denoted by the symbol $N(x)$. The number of neighbors of x is called the *degree* of a vertex x, denoted by $d(x)$. A vertex of degree 1 is called a *pendant vertex* or a *leaf*.

The *minimum degree* of a graph G is denoted by $\delta = \delta(G)$ and the *maximum degree* of a graph G is denoted by $\Delta = \Delta(G)$. If every vertex in a graph G has the same degree r, that is $\delta = \Delta = r$, then G is called a *regular graph of degree* r, or an r*-regular graph*.

A graph H is a *subgraph* of G, denoted by $H \subseteq G$, if every vertex of H is a vertex of G and every edge of H is an edge of G. In other words, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We say that a subgraph H is a *factor* of G if H contains all vertices of G, i.e. $V(H) = V(G)$. A (k,t)*-factor* is a factor such that every vertex in the factor has a degree k or t.

By P_n we denote the path on n vertices and by C_n is denoted the cycle on n vertices. A graph is *bipartite* if it is possible to categorize its vertices into two partite sets, such that there are no edges between vertices in the same partite set. A bipartite graph is *complete* if each vertex in one partite set is adjacent to all vertices in the second partite set. The symbol $K_{m,n}$ denotes the complete bipartite graph with partite sets of cardinalities m and n.

A *complete graph* K_n of order n is a graph in which every two distinct vertices are adjacent. K_n is an $(n-1)$-regular graph.

A *generalized Petersen graph* $P(n,m)$, $n \geq 3$ and $1 \leq m \leq \lfloor (n-1)/2 \rfloor$, consists of an outer n-cycle $y_0 y_1 \dots y_{n-1}$, a set of n spokes $y_i x_i$, $0 \leq i \leq n-1$, and n edges $x_i x_{i+m}$, $0 \leq i \leq n-1$, with indices taken modulo n. Figure 1.1 shows the generalized Petersen graphs $P(9,2)$ and $P(9,4)$.

A *Cartesian product* of two graphs G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$, where two vertices (x,x') and (y,y') are adjacent if and only if $x = y$ and $x'y' \in E(H)$ or $x' = y'$ and $xy \in E(G)$. A *prism* D_n, $n \geq 3$, is a 3-regular graph which can be defined as the Cartesian product $C_n \times P_2$ of a cycle C_n with a path P_2. A *Generalized prism* or m*-prism* can be defined as the Cartesian product $C_n \times P_m$ of a cycle C_n with a path P_m. A *grid* G_n^m can be defined as the Cartesian product $P_n \times P_m$. Figure 1.2 shows the prism $D_9 \cong C_9 \times P_2$ that is also the generalized Petersen graph $P(9,1)$.

A *wheel* W_n, $n \geq 3$, with n spokes is a graph that has a center vertex connected to all n vertices in cycle C_n. A *fan graph* F_n can be constructed from a wheel by deleting one edge in C_n. A *friendship graph* $\mathbb{F}_m$ consists of m triangles with exactly one common vertex called the *center*. The parachute graph $P_{m,n}$ is obtained from the wheel W_{m+n} by deleting n consecutive spokes. The wheel W_8 and fan graph

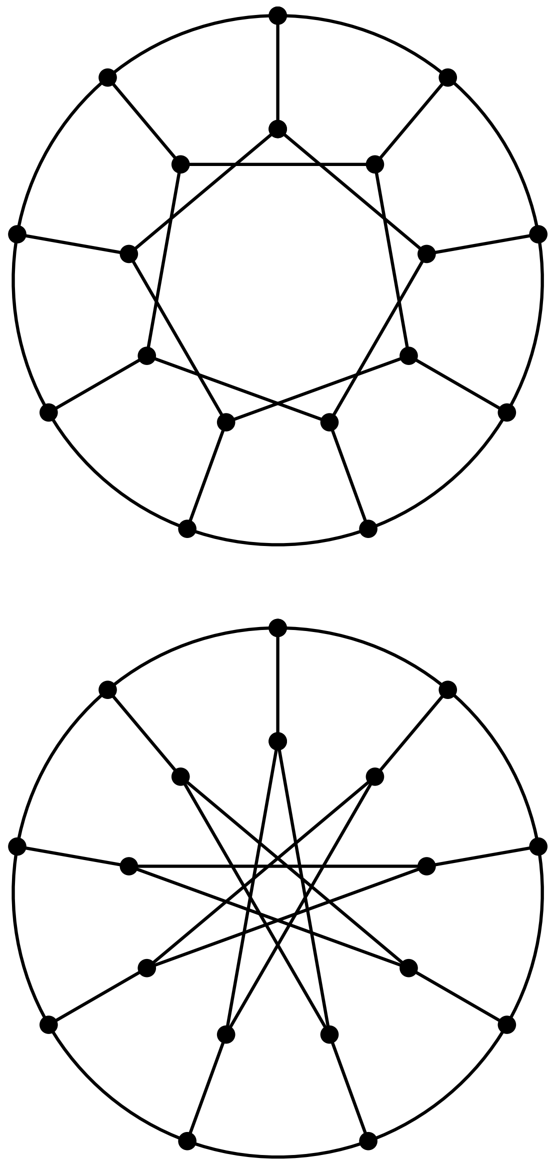

Figure 1.1: Generalized Petersen graphs $P(9, 2)$ and $P(9, 4)$.

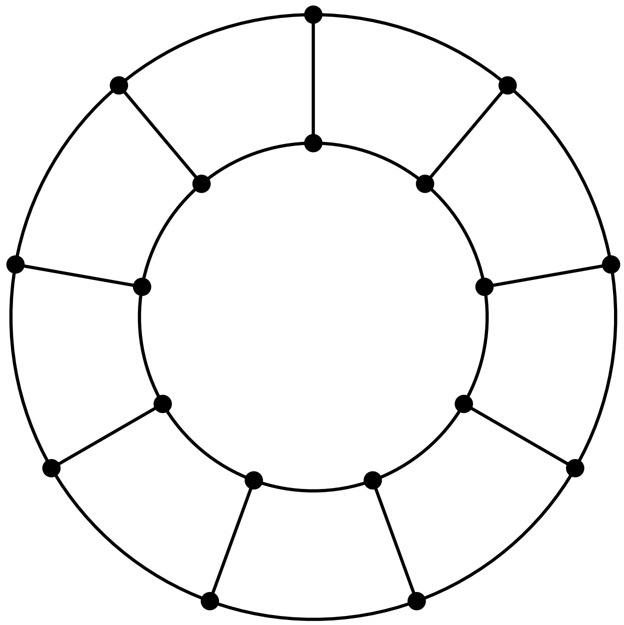

Figure 1.2: Prism D_9.

F_8 are displayed in Figure 1.3. Figure 1.4 depicts the friendship graph $\mathbb{F}_4$ and the pachachute graph $P_{6,5}$.

A *labeling* of a graph $G(V, E)$ is any mapping that sends some set of graph elements to a set of numbers, usually the non-negative integers. If the domain is the set of vertices or the set of edges, the labelings are called *vertex labelings* or *edge labelings*, respectively. Moreover, if the domain is $V(G) \cup E(G)$ then the labeling is called *total labeling*.

If we consider a *plane graph* which is drawn on the Euclidean plane in such a way that edges do not cross each other except at the vertices of the graph then for a plane graph $G(V, E, F)$, it makes sense to consider its faces, including the unique face of the infinite area. Let $F(G)$ be the face set and let $|F(G)| = f$.

The mapping usually produce partial sums of the labeled elements of the graph. The partial sums are either a set of *vertex-weights*, obtained for each vertex by adding all the labels of a vertex and its adjacent edges, or a set of *edge-weights*, obtained for each edge by adding the labels of an edge and its endpoints, or a set of *face-weights*, obtained for each face by adding the labels (if present) of that face and the edges and vertices on its boundary.

One of the situations that we are particularly interested in is when all the edge-

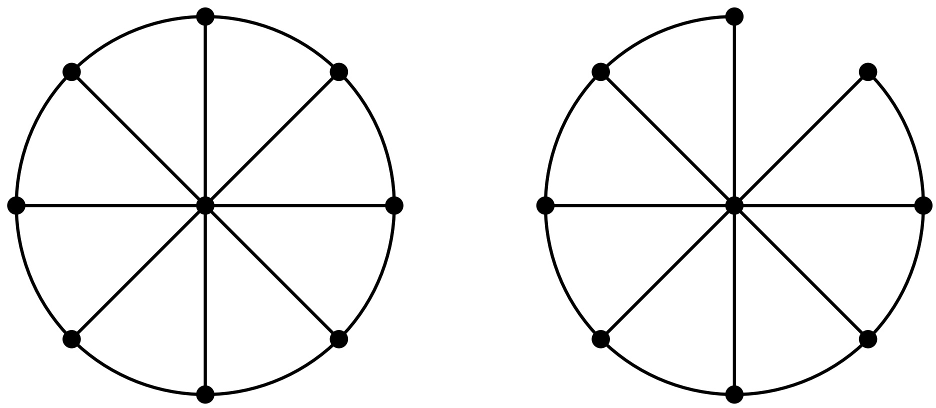

Figure 1.3: Wheel W_8 and fan graph F_8.

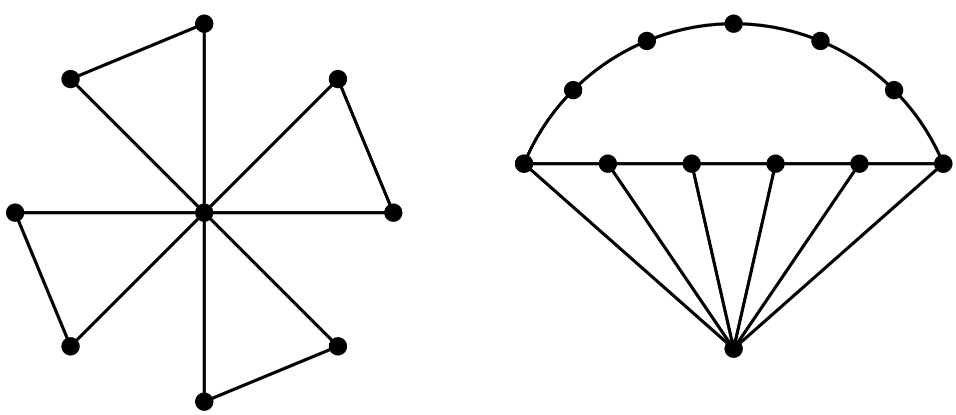

Figure 1.4: Friendship graph $\mathbb{F}_4$ and pachachute graph $P_{6,5}$.

weights or all the vertex-weights or all the face-weights are the same. In such a case we call the labeled graph *edge-magic* or *vertex-magic* or *face-magic*, respectively.

Another situation that is of interest is when all the edge-weights or all the vertex-weights or all the face-weights are different. In such a case we call the labeled graph *edge-antimagic* or *vertex-antimagic* or *face-antimagic*, respectively.

1.2 Magic-type labelings

1.2.1 Magic, supermagic and prime-magic labelings

Magic labelings were introduced more than forty years ago by Sedláček [68]. A graph is said to be *magic* if it has a real-valued edge labeling such that;

(i) distinct edges have distinct non-negative labels, and

(ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Some sufficient conditions for the existence of magic graphs are established in [65, 69, 72, 78]. A characterization of regular magic graphs in terms of circuits is given by Doob [33]. Műhlbacher [65] used matrix theory to prove two necessary conditions for the existence of a magic graph.

The problem of characterizing magic graphs was solved in 1980s when there were published two different characterizations of all magic graphs - those of Jeurissen's and Jezný-Trenkler's. Jeurissen [44, 45] used forbidden graphs and the cardinality of the neighborhood of independent set to characterize magic graphs. This characterization is divided into two parts - the characterization of non-bipartite and the characterization of bipartite graphs. Jezný and Trenkler [46] characterized magic graphs using the separation of edges by a $(1-2)$-factor.

Stewart [73] defined a graph G as *supermagic* if it is magic with the edge labeling consisting of consecutive positive integers. Note, that if f is a supermagic labeling of a regular graph $G(V, E)$, then $f+m$, for every integer $m > -min\{f(e) : e \in E(G)\}$, is a supermagic labeling of G, too. Thus, a regular graph G is supermagic if and only if it admits a supermagic labeling $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$, see [42]. Figure 1.5 illustrates a supermagic labeling of the complete bipartite graph $K_{3,3}$.

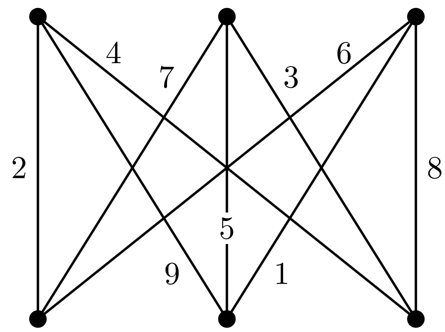

Figure 1.5: Supermagic labeling of the complete bipartite graph $K_{3,3}$.

There is no known characterization of all supermagic graphs. Only some special classes of supermagic graphs have been characterized. Stewart [73] proved that a complete graph K_n is supermagic if and only if either $n \geq 6$ and $n \not\equiv 0 \pmod 4$, or $n = 2$. The characterizations of supermagic regular complete multipartite graphs and cubes are given in [42] by Ivančo. Shiu, Lam and Cheng [70] proved that, for $n \geq 2$, $mK_{n,n}$ (disjoint union of m copies of complete bipartite graph) is supermagic if and only if n is even or both m and n are odd.

It is easy to see that the classical concept of a magic square of n^2 boxes corresponds to the fact that the complete bipartite graph $K_{n,n}$ is supermagic, for every positive integer $n \neq 2$, see [25].

In Stewart's terminology the graph is called *prime-magic* if it is magic and each value of an edge is a prime number. Figure 1.6 shows a prime-magic labeling of the complete bipartite graph $K_{3,3}$ with the smallest value of vertex sums.

There are infinitely many graphs that are magic, but not prime-magic. This answers, in the negative, a conjecture of Sedláček that every magic graph is prime-magic. Stewart also proposed the more plausible conjecture that every regular magic graph is prime-magic. The difficulty surrounding any general statement about prime-magic graphs is obviously due to the irregular distribution of primes.

1.2.2 Face-magic labelings

A graph is said to be *plane* if it is drawn on the Euclidean plane in such a way that edges do not cross each other except at the vertices of the graph. For a plane graph $G = (V, E, F)$, it makes sense to determine its faces, including the unique face of the infinite area.

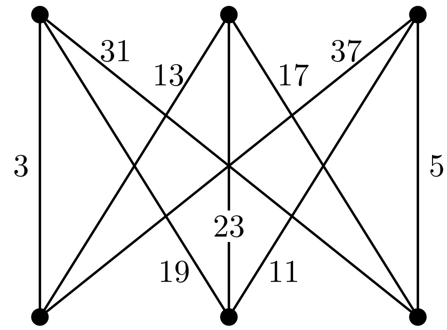

Figure 1.6: Prime-magic labeling of $K_{3,3}$.

Assume that $\alpha, \beta, \gamma \in \{0, 1\}$. A labeling of type (α, β, γ) assigns the labels from the set

$$\{1, 2, 3, \dots, \alpha|V(G)| + \beta|E(G)| + \gamma|F(G)|\}$$

to the vertices, edges and faces of G in such a way that each vertex receives α labels, each edge receives β labels and each face receives γ labels and each number is used exactly once as a label. Labelings of types $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are also called *vertex*, *edge* and *face* labelings, respectively.

The *weight* of a face (*face-weight*) under a labeling of type (α, β, γ) is the sum of labels (if present) carried by that face and the edges and the vertices surrounding that face.

A labeling of type (α, β, γ) is said to be *face-magic*, if for every positive integer s, all s-sided faces have the same weight. We allow different weights for different s.

The notion of face-magic labeling of the plane graphs was defined by Ko-Wei Lih in [54]. Ko-Wei Lih called such a labeling *magic* but this notion of being magic is entirely different from those defined in [51] and [68]. Ko-Wei Lih [54] described face-magic labelings of type $(1, 1, 0)$ for the wheels, the friendship graphs, the prisms and some of the platonic polyhedra. Figure 1.7 depicts a face-magic labeling of type $(1, 1, 0)$ for octahedron.

In [4] is shown that the fan graphs and planar bipyramids have the face-magic labelings of type $(1, 1, 1)$. The face-magic labelings of type $(1, 1, 1)$ for Mőbius ladder L_n^m, $n \geq 3$ odd, $m \geq 1$, for the grid graphs G_n^m, $n \geq 2$, $m \geq 1$, $n + m \neq 3$, and the hexagonal planar graphs H_n^m (honeycomb) are given in [5], [8] and [9], respectively. The face-magic labelings of type $(1, 1, 0)$ for the m-antiprisms Q_n^m, $n \geq 4$, and the m-prisms R_n^m, $m \geq 1$, $n \geq 3$, $n \neq 4$, are described in [6] and [7], respectively.

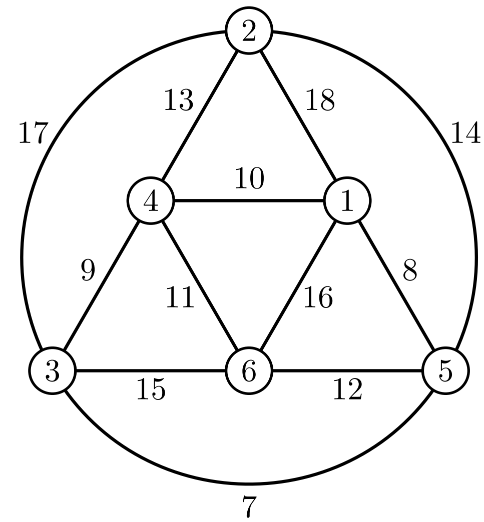

Figure 1.7: Face-magic labeling of type $(1, 1, 0)$ for octahedron.

Kathiresan and Gokulakrishnan [48] provided the face-magic labelings of type $(1, 1, 1)$ for the families of planar graphs with 3-sided faces, 5-sided faces, 6-sided faces and one external infinite face.

1.3 Antimagic-type labelings

A graph $G(V, E)$ is called *antimagic* if its edges are labeled with the labels $1, 2, 3, \dots,$ $|E(G)|$ in such a way that all the weights of vertices are pairwise distinct, where a weight of a vertex x is the sum of labels of all edges incident with x. The concept of an antimagic graph was introduced by Hartsfield and Ringel [41]. Figure 1.8 shows an antimagic labeling of the complete graph K_4.

Hartsfield and Ringel [41] point out that antimagic graphs include paths P_n, $n \geq 3$, cycles, wheels, and complete graphs K_n, $n \geq 3$. They conjecture that every connected graph, except K_2, is antimagic. Alon, Kaplan, Lev, Roditty and Yuster [3] used several probabilistic tools and some techniques from analytic number theory to show that this conjecture is true for all graphs having minimum degree $\Omega(\log |V(G)|)$.

It is not difficult to find many different antimagic labelings for a given graph. Therefore, it is reasonable to investigate antimagic labelings with some restrictions placed on the weights of vertices. One such restriction introduced Bodendiek and Walther [28]. They defined the concept of an (a, d)-antimagic graph as follows.

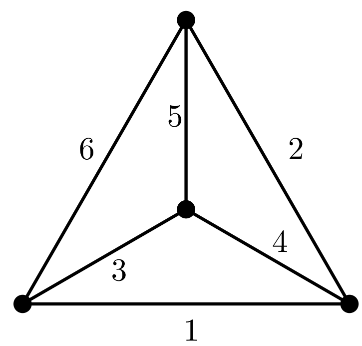

Figure 1.8: Antimagic labeling of the complete graph K_4.

A connected graph $G(V, E)$ is said to be (a, d)-*antimagic* if there exist positive integers a, d and a bijection

$$g : E(G) \to \{1, 2, \ldots, |E(G)|\}$$

such that the induced mapping $h_g : V(G) \to W$ is also a bijection, where

$$W = \{w(x) : x \in V(G)\} = \{a, a + d, a + 2d, \ldots, a + (|V(G)| - 1)d\}$$

is the set of the weights of vertices.

The problem of deciding whether a given graph is magic or antimagic is very difficult. A very good survey on magic and antimagic graphs can be found in [35]. Until now only few infinite families of graphs are known to be antimagic. The (a, d)-antimagic labelings for the special graphs called parachutes are described in [29] and [30], for prisms and antiprisms in [10], [11] and [63].

1.3.1 Edge-antimagic total labelings

A bijection

$$g : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$$

is called an (a, d)-*edge-antimagic total labeling* of G if the edge-weights

$$\{w_g(xy) = g(x) + g(xy) + g(y) : xy \in E(G)\}$$

form an arithmetic sequence starting at a and having a common difference d, where $a > 0$ and $d \geq 0$ are two fixed integers.

The definition of (a, d)-edge-antimagic total labeling was introduced by Simanjuntak, Miller and Bertault [71] as a natural extension of a notion of "magic valuation" defined by Kotzig and Rosa in [51]. Note that for $d = 0$ the (a, d)-edge-antimagic total labeling is an edge-magic labeling, see [67, 79].

An (a, d)-edge-antimagic total labeling is called *super* if the smallest possible labels appear on the vertices. A super (a, d)-edge-antimagic total labeling is a natural extension of a notion of super edge-magic labeling defined by Enomoto, Lladó, Nakamigawa and Ringel in [34]. A graph that admits a (super) (a, d)-edge-antimagic total labeling is called a (super) (a, d)-edge-antimagic total graph.

Figure 2.1 gives an example of a super $(16, 1)$-edge-antimagic total labeling of cycle C_7, where the integers in italic mean the edge-weights.

In [71] Simanjuntak, Miller and Bertault studied the properties of edge-antimagic total labeling and gave constructions of (a, d)-edge-antimagic total labelings for cycles and paths. Bača, Lin, Miller and Simanjuntak [12] presented some relationships between (a, d)-edge-antimagic total labelings and other labelings, namely edge-magic labeling. In the paper [18] there are studied the properties of super (a, d)-edge-antimagic total labelings of certain classes of graphs, including friendship graphs, wheels, fan graphs, complete graphs and complete bipartite graphs.

Many authors investigated the existence of super edge-antimagic labelings for disconnected graphs. Ivančo and Lučkaničová [43] described some constructions of super edge-magic labelings for disconnected graphs, namely, $nC_k \cup mP_k$ and $K_{1,m} \cup K_{1,n}$. Super (a, d)-edge-antimagic total labelings for $P_n \cup P_{n+1}$, $nP_2 \cup P_n$ and $nP_2 \cup P_{n+2}$ have been described by Sudarsana, Ismaimuza, Baskoro and Assiyatun in [74].

Dafik, Miller, Ryan and Bača investigated the super edge-antimagicness for the disjoint union of cycles mC_n and for disjoint union of paths mP_n in [32]. Super (a, d)-edge-antimagic total labelings of regular graphs, for d odd, are presented in [21]. Some other results on super (a, d)-edge-antimagic total graphs can be found in [19] and [35].

1.3.2 Vertex-antimagic total labelings

For a total labeling $g : V(G) \cup E(G) \to \{1, 2, \dots, |V(G)| + |E(G)|\}$ the associated vertex-weight of a vertex $x \in V(G)$ is

$$wt_g(x) = g(x) + \sum_{y \in N(x)} g(xy).$$

A total labeling g with the property that the set of the vertex-weights is

$$W = \{wt_g(x) : x \in V(G)\} = \{a, a+d, \dots, a+(|V(G)|-1)d\},$$

$a > 0$, $d \geq 0$, is called (a,d)-*vertex-antimagic total labeling*. If $d = 0$ then the (a,d)-vertex-antimagic total labeling is called *vertex-magic total*. As an illustration, Figure 2.5 provides an example of a $(19,2)$-vertex-antimagic total labeling of C_8, where the integers in italic mean vertex-weights.

The definition of an (a,d)-vertex-antimagic total labeling was introduced by Bača, Bertault, MacDougall, Miller, Simanjuntak and Slamin in [14] as a natural extension of the vertex-magic total labeling defined by MacDougall, Miller, Slamin and Wallis [60].

An (a,d)-vertex-antimagic total labeling g is called *super* if

$$g(V(G)) = \{g(x), x \in V(G)\} = \{1, 2, \dots, |V(G)|\}.$$

That is, in a super (a,d)-vertex-antimagic total labeling the smallest labels are assigned to the vertices. A graph which admits a (super) (a,d)-vertex-antimagic total labeling is said to be (super) (a,d)-vertex-antimagic total.

In [61], it is shown that wheels, fan graphs and friendship graphs have no vertex-magic total labelings except for the certain range of number of vertices n and for all n in this range the vertex-magic total labelings are found. A vertex-magic total labeling for K_n, for odd n, can be found in [56], [60] and [62], and for K_n, with n even, is given in [36] and [37]. A construction for a vertex-magic total labeling of complete bipartite graphs $K_{m,m}$ is presented in [60]. In [38], it is completely determined which complete bipartite graphs have vertex-magic total labelings. The constructions of vertex-magic total labelings of certain regular graphs are given in [39], [53] and [77].

The basic properties of (a,d)-vertex-antimagic total labelings are investigated in [14] and super (a,d)-vertex-antimagic total labelings are studied in [75]. In [75], it is shown how to construct the super (a,d)-vertex-antimagic total labelings for certain families of graphs, including complete graphs, complete bipartite graphs, cycles, paths and generalized Petersen graphs. In the paper [1] there are presented some new results on existence of super (a,d)-vertex-antimagic total labelings for disconnected graphs, namely a disjoint union of m copies of a regular graph.

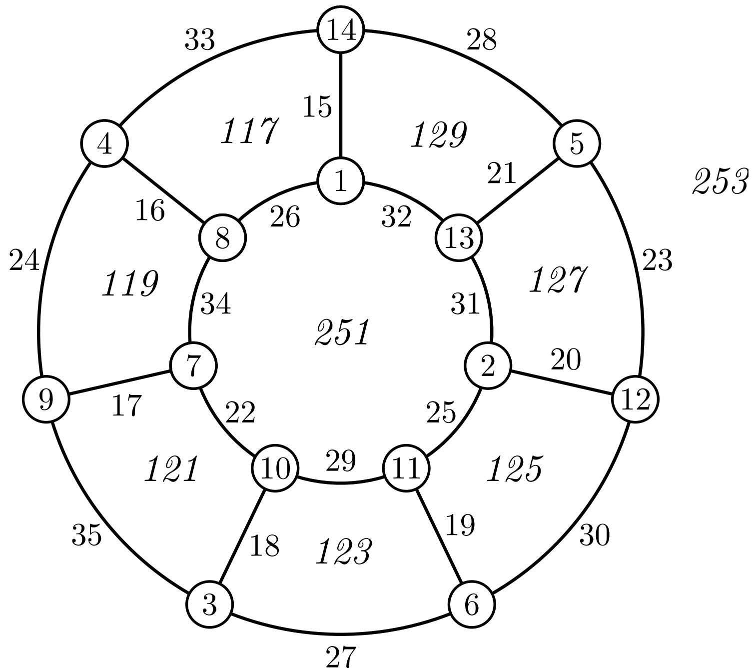

Figure 1.9: 2-antimagic labeling of type $(1, 1, 0)$ for prism D_7.

1.3.3 d-antimagic labelings

Consider a labeling of type (α, β, γ) with $\alpha, \beta, \gamma \in \{0, 1\}$. A labeling of type (α, β, γ) of a plane graph G is called *d-antimagic*, if for every positive integer s, the set of s-sided face-weights is

$$W_s = \{a_s, a_s + d, a_s + 2d, \ldots, a_s + (f_s - 1)d\}$$

for some integers a_s and $d \geq 0$, where f_s is the number of s-sided faces. We allow different sets W_s for different s.

If $d = 0$ then the d-antimagic labeling is face-magic labeling. Figure 1.9 gives an example of a 2-antimagic labeling type $(1, 1, 0)$ for prism D_7, where the integers in italic mean the face-weights.

The concept of the d-antimagic labeling of the plane graphs was introduced in [13], where are described d-antimagic labelings of type $(1, 1, 1)$ for prism D_n. The d-antimagic labelings of type $(1, 1, 1)$ for the generalized Petersen graph $P(n, 2)$, the hexagonal planar maps and the grids can be found in [15], [16] and [17]. Kathiresan and Ganesan [49] investigated the existence of d-antimagic labelings of type $(1, 1, 1)$ for a special plane graph $P_{a,b}$ with $2a$-sided faces. They have proved [50] that a plane graph P_a^b, with $b-1$ $(2i+2)$-sided faces, $1 \leq i \leq a-1$, admits a d-antimagic labeling of type $(1,1,1)$ for $d \in \{0,1,2,3,4,6\}$. The existence of d-antimagic labeling of type $(1,1,1)$ for P_a^b for many other values of parameter d can be found in [59].

Chapter 2

Labelings for 2-regular graphs

In this chapter we list the super (a,d)-edge-antimagic total labelings and the (super) (a,d)-vertex-antimagic total labelings for cycles and for disjoint union of cycles. These labelings are used in the next two chapters to obtain the super d-antimagic labelings of type $(1,1,1)$ for antiprisms and for the disjoint union of m copies of a prism.

2.1 Labelings for cycles

Let C_n be the cycle with the vertex set $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Note that the indices are taken modulo n.

a) The following super $(2n+2,1)$-edge-antimagic total labelings g_1 and g_2 for cycle C_n, $n \geq 3$, can be found in [71].

$$g_1(v_i) = n + 1 - i \quad \text{for} \quad i = 1, 2, \ldots, n,$$

$$g_1(v_i v_{i+1}) = \begin{cases} n + 1 + i & \text{for} \quad i = 1, 2, \ldots, n-1, \\ n + 1 & \text{for} \quad i = n. \end{cases}$$

$$g_2(v_i) = i \quad \text{for} \quad i = 1, 2, \ldots, n,$$

$$g_2(v_i v_{i+1}) = 2n + 1 - i \quad \text{for} \quad i = 1, 2, \ldots, n.$$

An example of the labeling g_1 for cycle C_7 is presented in Figure 2.1. Figure 2.2 shows the labeling g_2 for cycle C_8. The integers in italic mean the edge-weights.

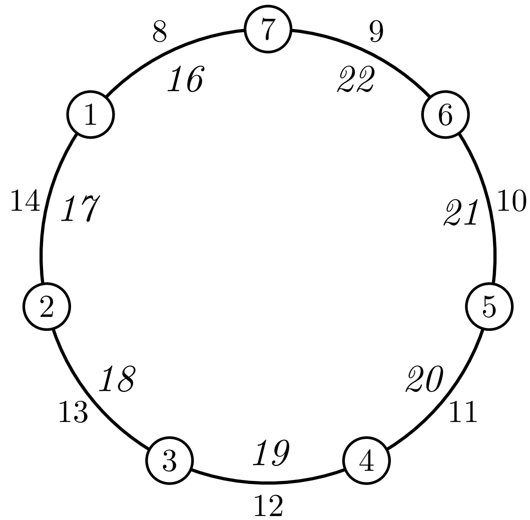

Figure 2.1: Super $(16, 1)$-edge-antimagic total labeling of C_7.

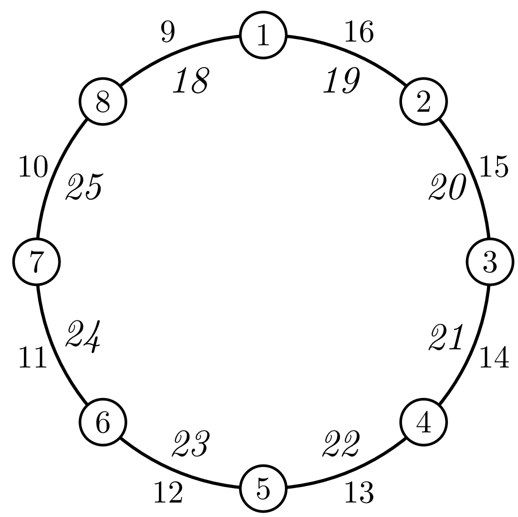

Figure 2.2: Super $(18, 1)$-edge-antimagic total labeling of C_8.

Now we define the (a, d)-vertex-antimagic total labelings of cycle C_m with the vertex set $V(C_m) = \{u_1, u_2, \dots, u_m\}$ in the following way.

b) The $(\frac{7m}{2} + 1, 0)$-vertex-antimagic total labelings g_3 and g_4:

for $m \equiv 0 \pmod 4$

$$g_3(u_i) = \begin{cases} 1 & \text{for } i = 1, \\ m - 2 + i & \text{for } i = 2, 3, \\ i - 1 & \text{for } i = 4, 5, \dots, \frac{m}{2} + 1, \\ 2 & \text{for } i = \frac{m}{2} + 2, \\ i - 1 & \text{for } i = \frac{m}{2} + 3, \frac{m}{2} + 4, \dots, m, \end{cases}$$

$$g_3(u_i u_{i+1}) = \begin{cases} 2m + \frac{1-i}{2} & \text{for } i = 1, 3, 5, \dots, \frac{m}{2} + 1, \\ \frac{m}{2} + 1 & \text{for } i = 2, \\ \frac{3m-i}{2} + 1 & \text{for } i = 4, 6, 8, \dots, \frac{m}{2}, \\ 2m - \frac{i}{2} & \text{for } i = \frac{m}{2} + 2, \frac{m}{2} + 4, \dots, m, \\ \frac{3m+3-i}{2} & \text{for } i = \frac{m}{2} + 3, \frac{m}{2} + 5, \dots, m - 1, \end{cases}$$

for $m \equiv 2 \pmod 4$

$$g_4(u_i) = \begin{cases} 2 & \text{for } i = 1, \\ m - 2 + i & \text{for } i = 2, 3, \\ i & \text{for } i = 4, 5, \dots, \frac{m}{2}, \\ 1 & \text{for } i = \frac{m}{2} + 1, \\ 3 & \text{for } i = \frac{m}{2} + 2, \\ i & \text{for } i = \frac{m}{2} + 3, \frac{m}{2} + 4, \dots, m - 1, \\ \frac{m}{2} + 2 & \text{for } i = m, \end{cases}$$

$$g_4(u_i u_{i+1}) = \begin{cases} 2m + \frac{1-i}{2} & \text{for } i = 1, 3, 5, \dots, \frac{m}{2}, \\ \frac{m}{2} + 1 & \text{for } i = 2, \\ \frac{3m-i}{2} & \text{for } i = 4, 6, 8, \dots, \frac{m}{2} - 1, \\ \frac{7m-2}{4} & \text{for } i = \frac{m}{2} + 1, \\ 2m - \frac{i+1}{2} & \text{for } i = \frac{m}{2} + 2, \frac{m}{2} + 4, \dots, m - 1, \\ \frac{3m-i}{2} + 1 & \text{for } i = \frac{m}{2} + 3, \frac{m}{2} + 5, \dots, m - 2, \\ \frac{3m}{2} - 1 & \text{for } i = m. \end{cases}$$

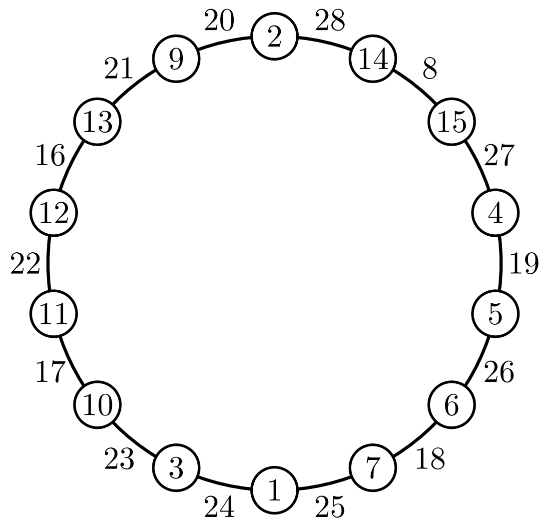

Figure 2.3: Vertex-magic total labeling of C_{14}.

Figure 2.3 illustrates the labeling g_3 for cycle C_{14} with the common vertex-weight 50.

c) The super $(3m+2,1)$-vertex-antimagic total labeling g_5, for every $m \geq 3$:

$$g_5(u_i) = i \quad \text{for} \quad i = 1, 2, \dots, m,$$

$$g_5(u_i u_{i+1}) = \begin{cases} 2m - i & \text{for} \quad i = 1, 2, \dots, m-1, \\ 2m & \text{for} \quad i = m. \end{cases}$$

d) The $(2m+3,2)$-vertex-antimagic total labeling g_6, for $m \geq 4$ even:

$$g_6(u_i) = \begin{cases} i & \text{for} \quad i = 1, 3, 5, \dots, m-1, \\ m - 1 + i & \text{for} \quad i = 2, 4, 6, \dots, m, \end{cases}$$

$$g_6(u_i u_{i+1}) = \begin{cases} 2m + 1 - i & \text{for} \quad i = 1, 3, 5, \dots, m-1, \\ m + 2 - i & \text{for} \quad i = 2, 4, 6, \dots, m. \end{cases}$$

The labeling g_5 for cycle C_7 is depicted in Figure 2.4. Figure 2.5 shows the labeling g_6 for cycle C_8. The integers in italic mean the vertex-weights.

e) The $(2m+2,3)$-vertex-antimagic total labeling g_7, for every $m \geq 3$:

$$g_7(u_i) = \begin{cases} i & \text{for} \quad i = 1, 2, \dots, m-1, \\ 2m & \text{for} \quad i = m, \end{cases}$$

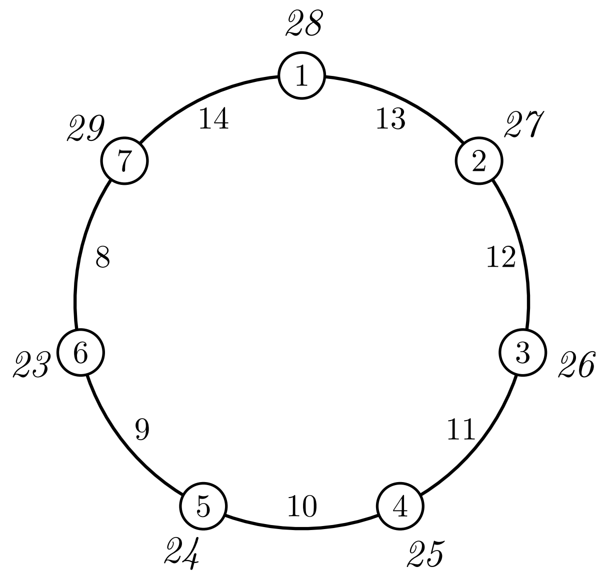

Figure 2.4: Super $(23, 1)$-vertex-antimagic total labeling of C_7.

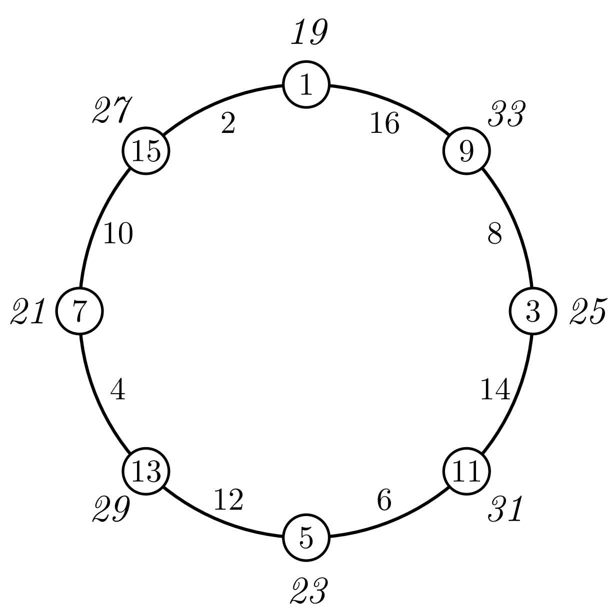

Figure 2.5: $(19, 2)$-vertex-antimagic total labeling of C_8.

$$g_7(u_iu_{i+1}) = \begin{cases} m+i & \text{for} \quad i = 1, 2, \ldots, m-1, \\ m & \text{for} \quad i = m. \end{cases}$$

f) The $\left(\frac{5m}{2} + 2, 2\right)$-vertex-antimagic total labeling g_8, for $m \geq 4$ even:

$$g_8(u_i) = \begin{cases} i & \text{for} \quad i = 1, 2, \ldots, m-1, \\ \frac{3m}{2} & \text{for} \quad i = m, \end{cases}$$

$$g_8(u_iu_{i+1}) = \begin{cases} \frac{3m+i+1}{2} & \text{for} \quad i = 1, 3, 5, \ldots, m-1, \\ m + \frac{i}{2} & \text{for} \quad i = 2, 4, 6, \ldots, m-2, \\ m & \text{for} \quad i = m. \end{cases}$$

g) The $(m + 4, 3)$-vertex-antimagic total labeling g_9, for every $m \geq 3$:

$$g_9(u_i) = \begin{cases} 1 & \text{for} \quad i = 1, \\ m+i & \text{for} \quad i = 2, 3, \ldots, m, \end{cases}$$

$$g_9(u_iu_{i+1}) = i + 1 \quad \text{for} \quad i = 1, 2, \ldots, m.$$

Figure 2.6 shows the labeling g_7 for cycle C_9 and Figure 2.7 gives an example of the labeling g_8 for cycle C_{10}. Figure 2.8 provides an example of the labeling g_9 for cycle C_8. The integers in italic mean the vertex-weights.

2.2 Labelings for union of cycles

Now, we consider a disjoint union of m copies of a cycle C_n, denoted by mC_n. For $m \geq 2$, it is a disconnected graph with the vertex set $V(mC_n) = \{u_i^j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and edge set $E(mC_n) = \{u_i^j u_{i+1}^j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ with the indices taken modulo n.

In [32], it is proved that the following bijective function $h_1 : V(mC_n) \cup E(mC_n) \to \{1, 2, \ldots, 2mn\}$ is a super $(2mn+2, 1)$-edge-antimagic total labeling of mC_n, for every $m \geq 2$ and $n \geq 3$,

$$h_1(u_i^j) = (i-1)m + j \qquad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad 1 \leq j \leq m,$$
$$h_1(u_i^j u_{i+1}^j) = (2n - i + 1)m + 1 - j \qquad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad 1 \leq j \leq m.$$

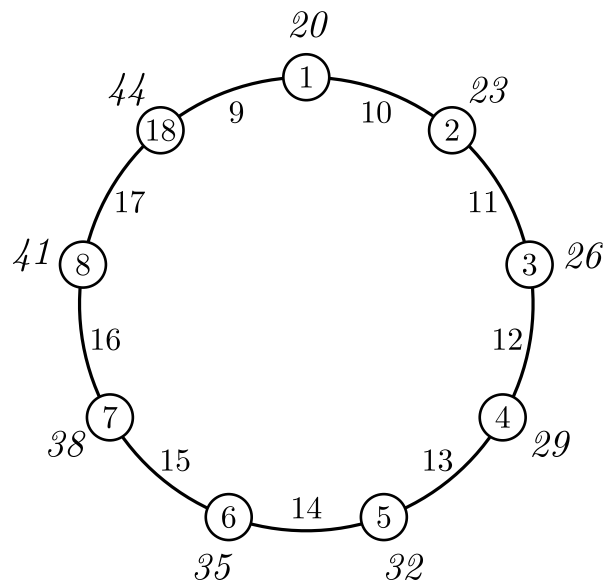

Figure 2.6: $(20, 3)$-vertex-antimagic total labeling of C_9.

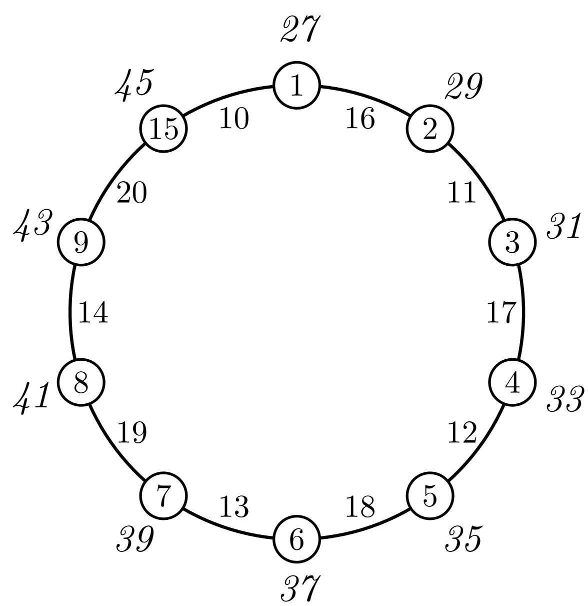

Figure 2.7: $(27, 2)$-vertex-antimagic total labeling of C_{10}.

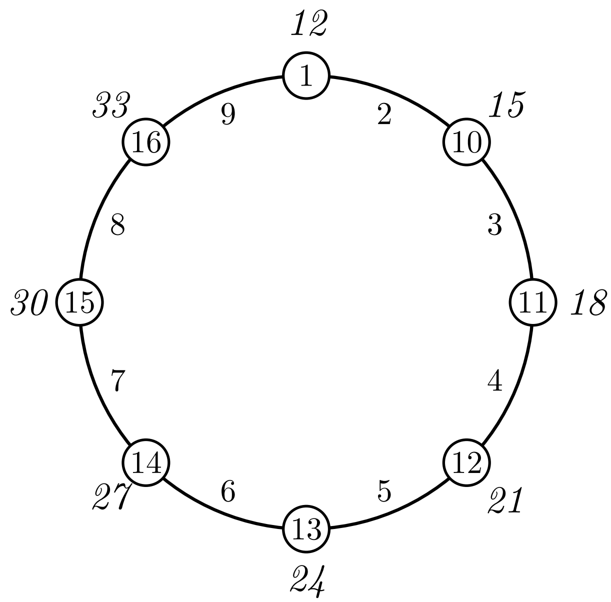

Figure 2.8: $(12, 3)$-vertex-antimagic total labeling of C_8.

Figure 2.9 depicts the super $(62, 1)$-edge-antimagic total labeling of $5C_6$ given by labeling h_1, where the integers in italic mean edge-weights.

It is not difficult to verify that for the 2-regular graphs (and only for the 2-regular graphs) an (a, d)-edge-antimagic total labeling is equivalent to an (a, d)-vertex-antimagic total labeling (see also [79] for $d = 0$).

Thus, the following bijection $h_2 : V(mC_n) \cup E(mC_n) \to \{1, 2, \ldots, 2mn\}$ defined by

$$
\begin{aligned}
h_2(u_1^j) &= (n+1)m + 1 - j && \text{for} \quad 1 \le j \le m, \\
h_2(u_i^j) &= (2n - i + 2)m + 1 - j && \text{for} \quad 2 \le i \le n \quad \text{and} \quad 1 \le j \le m, \\
h_2(u_i^j u_{i+1}^j) &= (i-1)m + j && \text{for} \quad 1 \le i \le n \quad \text{and} \quad 1 \le j \le m
\end{aligned}
$$

is the $(2mn + 2, 1)$-vertex-antimagic total labeling of mC_n, for every $m \ge 2$ and $n \ge 3$.

Theorem 1. [2] *The graph mC_n admits a $(2mn+2, 2)$-vertex-antimagic total labeling for every $m \ge 2$ and $n \ge 3$.*

Proof. Let us construct a total labeling h_3 of mC_n, $m \ge 2$ and $n \ge 3$, in the following

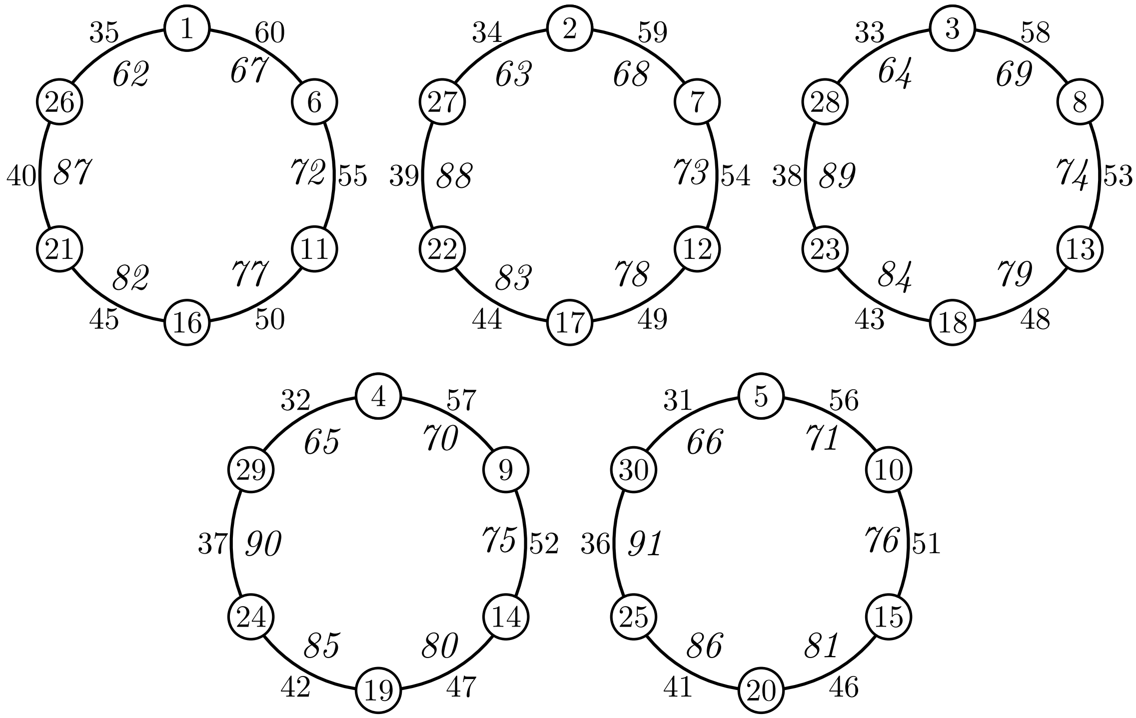

Figure 2.9: Super $(62, 1)$-edge-antimagic total labeling of $5C_6$.

way

$$h_3(u_i^j) = 2h_2(u_i^j) - 2mn,$$
$$h_3(u_i^j u_{i+1}^j) = 2h_2(u_i^j u_{i+1}^j) - 1$$

for $1 \le i \le n$ and $1 \le j \le m$.

We can see that the labeling h_3 is a bijective function from $V(mC_n) \cup E(mC_n)$ onto the set $\{1, 2, \dots, 2mn\}$ and moreover that the vertex-weights of mC_n constitute the sets

$$W_{h_3,1} = \{wt_{h_3}(u_1^j) = h_3(u_n^j u_1^j) + h_3(u_1^j) + h_3(u_1^j u_2^j) = 2mn + 2j : 1 \le j \le m\},$$
$$W_{h_3,2} = \{wt_{h_3}(u_i^j) = h_3(u_{i-1}^j u_i^j) + h_3(u_i^j) + h_3(u_i^j u_{i+1}^j) = 2m(n-1+i) + 2j : 2 \le i \le n,\ 1 \le j \le m\}.$$

Hence, the set $W_{h_3,1} \cup W_{h_3,2} = \{2mn+2, 2mn+4, \dots, 4mn\}$ contains an arithmetic sequence with the first term $2mn + 2$ and a common difference 2. Thus, h_3 is the $(2mn + 2, 2)$-vertex-antimagic total labeling. This concludes the proof. □

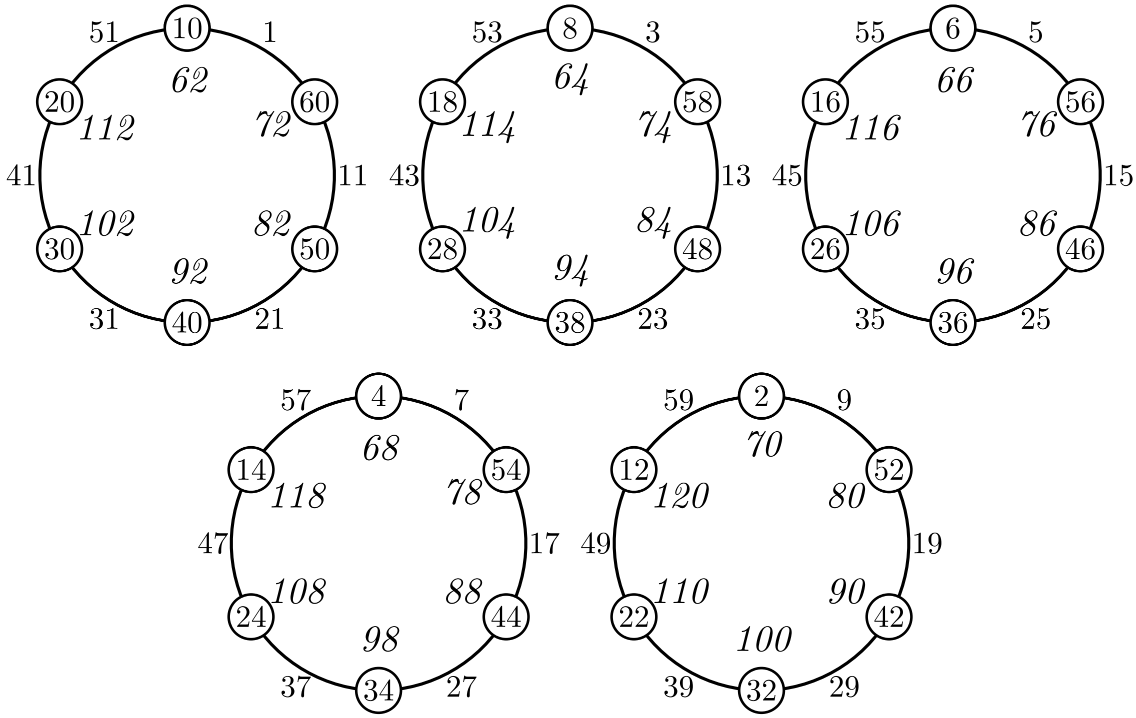

Figure 2.10: $(62, 2)$-vertex-antimagic total labeling of $5C_6$.

Figure 2.10 illustrates the $(62, 2)$-vertex-antimagic total labeling of $5C_6$ given by labeling h_3, where the integers in italic mean the vertex-weights.

Theorem 2. [2] *The graph mC_n admits a $(m(n+1)+3, 3)$-vertex-antimagic total labeling for every $m \geq 2$ and $n \geq 3$.*

Proof. For $m \geq 2$ and $n \geq 3$, we define the bijection $h_4 : V(mC_n) \cup E(mC_n) \to \{1, 2, \ldots, 2mn\}$ as follows:

$$h_4(u_1^j) = j \qquad \text{for} \quad 1 \leq j \leq m,$$
$$h_4(u_i^j) = m(n-1+i)+j \qquad \text{for} \quad 2 \leq i \leq n \quad \text{and} \quad 1 \leq j \leq m,$$
$$h_4(u_i^j u_{i+1}^j) = im+j \qquad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad 1 \leq j \leq m.$$

Then for the vertex-weights of mC_n we have

$$W_{h_4,1} = \{wt_{h_4}(u_1^j) = m(n+1)+3j : 1 \leq j \leq m\},$$
$$W_{h_4,2} = \{wt_{h_4}(u_i^j) = m(n+3i-2)+3j : 2 \leq i \leq n,\ 1 \leq j \leq m\}$$

and $W_{h_4,1} \cup W_{h_4,2} = \{m(n+1)+3, m(n+1)+6, \ldots, m(4n+1)\}$ contains an arithmetic sequence of the difference 3. This implies that h_4 is the $(m(n+1)+3, 3)$-vertex-antimagic total labeling. □

Figure 2.11 gives an example of $(51, 3)$-vertex-antimagic total labeling of $6C_7$, where the integers in italic mean the vertex-weights.

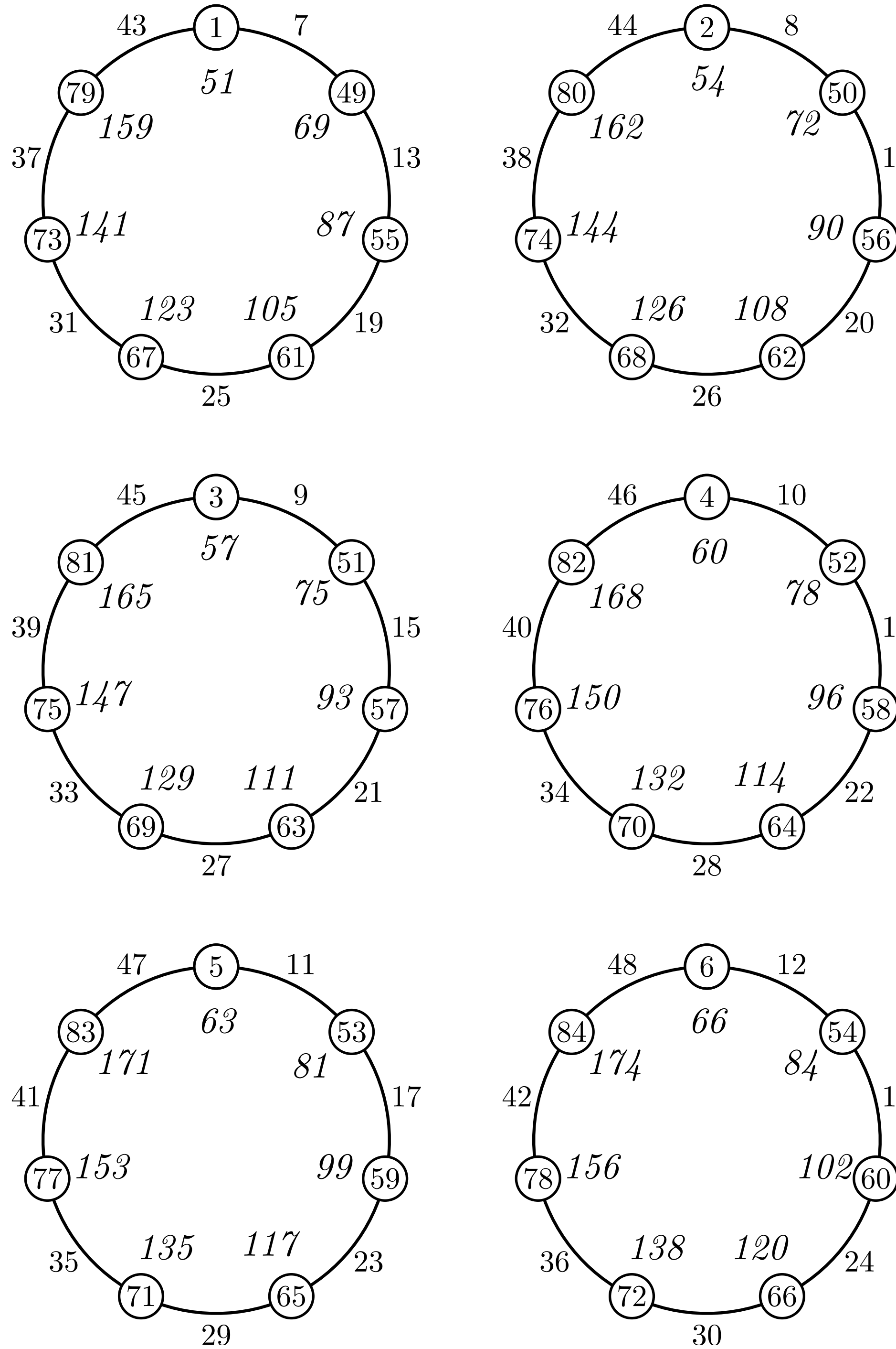

Figure 2.11: $(51, 3)$-vertex-antimagic total labeling of $6C_7$.

Chapter 3

Face antimagic labelings of antiprisms

The antiprism A_n, $n \geq 3$, is a 4-regular graph and for $n = 3$ it is the octahedron. The antiprism A_n, $n \geq 3$, consists of an outer n-cycle $y_1y_2 \ldots y_n$, an inner n-cycle $x_1x_2 \ldots x_n$, and a set of n spokes x_iy_i and $x_{i+1}y_i$, $i = 1, 2, \ldots, n$, with the indices taken modulo n. We define the 3-sided face $f_{1,i}$ as the face bounded by the edges $x_{i+1}y_{i+1}, x_{i+1}y_i, y_iy_{i+1}$ and the 3-sided face $f_{0,i}$ as the face bounded by the edges x_iy_i, x_ix_{i+1} and $x_{i+1}y_i$. We denote by $z_{n,1}$ and $z_{n,2}$ the inner n-sided face and the outer n-sided face, respectively. See Figure 3.1.

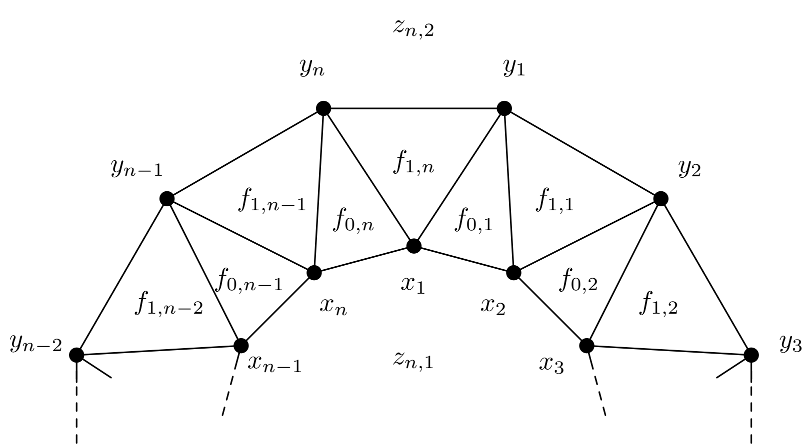

Figure 3.1: The antiprism A_n.

It was proved in [13] that for $n \geq 4$, the antiprism A_n is d-antimagic of type

$(1,1,1)$ for $d = 1,2,4$. Lin, Slamin and Miller [57] showed that for $n \geq 5$, the antiprism admits d-antimagic labeling of type $(1,1,1)$ for $d = 3,5$ and 6. The labelings for $d \in \{1,2,3,4,5\}$ are super d-antimagic of type $(1,1,1)$ and they have been obtained directly by describing the requested labelings. The 6-antimagic labeling of type $(1,1,1)$ described in [57] is not super.

In this chapter, we consider the antiprism as three cycle parts: the outer cycle, the inner cycle and the middle cycle. To label the inner, the outer and the middle cycles we used the (a,d)-edge-antimagic total and (a,d)-vertex-antimagic total labelings and combine these labelings to a resulting super d-antimagic labeling of type $(1,1,1)$.

3.1 Necessary conditions

Let us find the bounds for a feasible value of d for the super d-antimagic labeling of type $(1,1,1)$ of the antiprism A_n. Let g be such a labeling.

First, we will consider the weights of the n-sided faces. In this case the vertices x_i, y_i, $i = 1,2,\dots,n$, receive the smallest possible labels $1,2,\dots,2n$. The edges x_ix_{i+1}, y_iy_{i+1}, $i = 1,2,\dots,n$, could conceivably receive the next smallest $2n$ labels $2n+1, 2n+2, \dots, 4n$ and the n-sided faces the next two labels $4n+1, 4n+2$ or, at the other extreme, the edges x_ix_{i+1}, y_iy_{i+1}, $i = 1,2,\dots,n$, could receive the largest $2n$ labels $6n+1, 6n+2, \dots, 8n$ and the n-sided faces the labels $8n+1, 8n+2$, or anything in between. Consequently, we have

$$\sum_{i=1}^{2n}(2n+2i) + 8n + 3 \leq a_n + (a_n + d) \leq \sum_{i=1}^{2n}(6n+2i) + 16n + 3.$$

The minimum weight of n-sided face is

$$a_n \geq \sum_{i=1}^{n}(2n+2i) + 3n + 1 = 3n^2 + 4n + 1.$$

The maximum weight of n-sided face is

$$a_n + d \leq \sum_{i=1}^{n}(n+i) + \sum_{i=1}^{n}(7n+1+i) + 8n + 2 = 9n^2 + 10n + 2.$$

Thus

$$d \leq 9n^2 + 10n + 2 - a_n \leq 6n^2 + 6n + 1.$$

We can see that upper bound for the parameter d is very large in this case.

Now, we consider the weights of the 3-sided faces of the antiprism.

Theorem 3. [20] *For every antiprism A_n, $n \geq 4$, there is no super d-antimagic labeling of type $(1,1,1)$ with $d \geq 15$.*

Proof. Suppose that g is a super d-antimagic labeling of type $(1,1,1)$ of the antiprism A_n. To calculate the weights of 3-sided faces each vertex label is used three times, each edge label in outer (inner) cycle and face label is used once and each spoke label is used twice. In this case the vertices receive the smallest possible labels $1, 2, \dots, 2n$. The spokes could conceivably receive the smallest labels $2n+1, 2n+2, \dots, 4n$ and cycle edges and faces receive the larger labels $4n+1, 4n+2, \dots, 8n$ or, at the other extreme, the cycle edges and faces receive the smaller labels $2n+3, 2n+4, \dots, 6n+2$ and the spokes can receive the largest labels $6n+3, 6n+2, \dots, 8n+2$, or anything in between. So we get

$$42n^2 + 7n \leq 3\sum_{i=1}^{n}(g(x_i) + g(y_i)) + 2\sum_{i=1}^{n}(g(x_i y_i) + g(x_{i+1} y_i))$$

$$+ \sum_{i=1}^{n}(g(x_i x_{i+1}) + g(y_i y_{i+1})) + \sum_{i=1}^{n}(g(f_{0,i}) + g(f_{1,i})) \leq 50n^2 + 23n. \quad (3.1)$$

The sum of all the face-weights is

$$a_3 + (a_3 + d) + \cdots + (a_3 + (2n-1)d) = 2na_3 + n(2n-1)d. \quad (3.2)$$

The minimum weight of 3-sided face is $a_3 \geq 8n + 16$. Thus, from (3.1) and (3.2) we get the following inequality

$$(2n-1)d \leq 50n + 23 - 2a_3$$

and

$$d \leq 18.$$

On the other hand, the maximum weight of a 3-sided face is no more than $(2n - 2 + 2n - 1 + 2n) + (8n - 1 + 8n + 8n + 1 + 8n + 2) = 38n - 1$ and then

$$a_3 + (2n-1)d \leq 38n - 1.$$

We get that $d < 15$, which completes the proof. □

3.2 Antiprism as three cycle parts

In this section we are using a similar idea which was used for an investigation of d-antimagic labelings of prism in [76].

We denote by C_O, C_I, C_M the outer, inner and middle cycle of antiprism A_n, where

$$C_O = V(C_O) \cup E(C_O),$$
$$V(C_O) = \{y_1, y_2, \dots, y_n\} \subset V(A_n) \text{ and}$$
$$E(C_O) = \{y_1y_2, y_2y_3, \dots, y_ny_1\} \subset E(A_n),$$

$$C_I = V(C_I) \cup E(C_I),$$
$$V(C_I) = \{x_1, x_2, \dots, x_n\} \subset V(A_n) \text{ and}$$
$$E(C_I) = \{x_1x_2, x_2x_3, \dots, x_nx_1\} \subset E(A_n),$$

$$C_M = V(C_M) \cup E(C_M),$$
$$V(C_M) = \{f_{0,1}, f_{1,1}, f_{0,2}, f_{1,2}, \dots, f_{0,n-1}, f_{1,n-1}, f_{0,n}, f_{1,n}\} \subset F(A_n) \text{ and}$$
$$E(C_M) = \{f_{0,i}f_{1,i} = x_{i+1}y_i : i = 1, 2, \dots, n\} \cup \{f_{1,i}f_{0,i+1} = x_{i+1}y_{i+1} : i = 1, 2, \dots, n\}.$$

Let us assume that C_O and C_I is a cycle on n vertices and C_M is a cycle on $2n$ vertices.

Given a labeling schema α, by $\alpha(C_O)$ we mean that we apply α on C_O. The labeling denoted by $(\alpha(C_O), \beta(C_I))$ or $(\alpha(C_O), \beta(C_I), \gamma(C_M))$ means a combination of the labeling α for the outer cycle of antiprism, β for the inner cycle of antiprism and labeling γ for the middle cycle of antiprism.

The procedure for labeling the antiprism A_n by a super d-antimagic labeling of type $(1, 1, 1)$ is as follows;

1. Label C_O and C_I by the (a, d')-edge-antimagic total labelings, say α and β, such that the weights of the 3-sided faces under the labeling $(\alpha(C_O), \beta(C_I))$ form a requested arithmetic sequence.

2. Label C_M by an (a, d'')-vertex-antimagic total labeling, say γ, such that the weights of the 3-sided faces under the labeling $(\alpha(C_O), \beta(C_I), \gamma(C_M))$ form an arithmetic sequence with a common difference d.

3. Label the inner and outer n sided faces $z_{n,1}$ and $z_{n,2}$. Swap the label of the edge $x_i x_{i+1}$ in C_I with the label of the 3-sided face $f_{0,i}$ (as vertex of C_M), for $1 \le i \le n$, and/or swap the label of the edge $y_i y_{i+1}$ in C_O with the label of the 3-sided face $f_{1,i}$ (as vertex of C_M), for $1 \le i \le n$.

In every case, the applied swapping results in changes of face weight of $z_{n,1}$ and/or $z_{n,2}$ for obtaining the difference d between the weights of $z_{n,1}$ and $z_{n,2}$. Note that the swapping processes do not have any impact on the weights of the 3-sided faces.

3.3 d-antimagic labelings of antiprism

In this section we deal with the super d-antimagic labeling of type $(1,1,1)$ for the antiprisms and we describe those labelings for some values of d.

Theorem 4. [20] *The antiprism A_n, $n \ge 4$, has a super d-antimagic labeling of type $(1,1,1)$ for $d \in \{0,1,2,3,6\}$.*

Proof. First we define the labelings for C_I and C_O as follows.

$$\alpha_1(y_i) = 2n + 1 - g_1(y_i) \quad \text{for} \quad i = 1,2,\ldots,n,$$

$$\alpha_1(y_i y_{i+1}) = \begin{cases} 4n + 2 - g_1(y_i y_{i+1}) & \text{for} \quad i = 1,2,\ldots,n-1, \\ 2n+1 & \text{for} \quad i = n, \end{cases}$$

$$\beta_1(x_i) = g_1(x_i) \quad \text{for} \quad i = 1,2,\ldots,n,$$

$$\beta_1(x_i x_{i+1}) = 2n + g_1(x_i x_{i+1}) \quad \text{for} \quad i = 1,2,\ldots,n.$$

$$\alpha_2(y_i) = \begin{cases} 2n - g_2(y_i) & \text{for} \quad i = 1,2,\ldots,n-1, \\ 2n & \text{for} \quad i = n, \end{cases}$$

$$\alpha_2(y_i y_{i+1}) = \begin{cases} 4n + 3 - g_2(y_i y_{i+1}) & \text{for} \quad i = 1,2,\ldots,n-2, \\ 3n + 3 - g_2(y_i y_{i+1}) & \text{for} \quad i = n-1, n, \end{cases}$$

$$\beta_2(x_i) = g_2(x_i) \quad \text{for} \quad i = 1,2,\ldots,n,$$

$$\beta_2(x_i x_{i+1}) = 2n + g_2(x_i x_{i+1}) \quad \text{for} \quad i = 1, 2, \ldots, n.$$

We can see that all the weights of the 3-sided faces under the labeling $(\alpha_1(C_O), \beta_1(C_I))$ and $(\alpha_2(C_O), \beta_2(C_I))$ have the same value $6n+2$. Now, define the labelings γ_k, $1 \le k \le 6$, for C_M in the following way.

For $m = 2n \equiv 0 \pmod 4$

$$\gamma_1(f_{0,i}) = \begin{cases} 4n + g_3(u_{3+2i}) & \text{for} \quad 1 \le i \le n-2, \\ 4n + g_3(u_{3+2i-2n}) & \text{for} \quad n-1 \le i \le n, \end{cases}$$

$$\gamma_1(f_{1,i}) = \begin{cases} 4n + g_3(u_{4+2i}) & \text{for} \quad 1 \le i \le n-2, \\ 4n + g_3(u_{4+2i-2n}) & \text{for} \quad n-1 \le i \le n, \end{cases}$$

$$\gamma_1(x_{i+1} y_i) = \begin{cases} 4n + g_3(u_{3+2i} u_{4+2i}) & \text{for} \quad 1 \le i \le n-2, \\ 4n + g_3(u_{3+2i-2n} u_{4+2i-2n}) & \text{for} \quad n-1 \le i \le n, \end{cases}$$

$$\gamma_1(x_i y_i) = \begin{cases} 4n + g_3(u_{2+2i} u_{3+2i}) & \text{for} \quad 1 \le i \le n-1, \\ 4n + g_3(u_2 u_3) & \text{for} \quad i = n. \end{cases}$$

For $m = 2n \equiv 2 \pmod 4$

$$\gamma_2(f_{0,i}) = \begin{cases} 4n + g_4(u_{n-1-2i}) & \text{for} \quad 1 \le i \le \frac{n-3}{2}, \\ 4n + g_4(u_{3n-1-2i}) & \text{for} \quad \frac{n-1}{2} \le i \le n, \end{cases}$$

$$\gamma_2(f_{1,i}) = \begin{cases} 4n + g_4(u_{n-2-2i}) & \text{for} \quad 1 \le i \le \frac{n-3}{2}, \\ 4n + g_4(u_{3n-2-2i}) & \text{for} \quad \frac{n-1}{2} \le i \le n, \end{cases}$$

$$\gamma_2(x_{i+1} y_i) = \begin{cases} 4n + g_4(u_{n-2-2i} u_{n-1-2i}) & \text{for} \quad 1 \le i \le \frac{n-3}{2}, \\ 4n + g_4(u_{3n-2-2i} u_{3n-1-2i}) & \text{for} \quad \frac{n-1}{2} \le i \le n, \end{cases}$$

$$\gamma_2(x_i y_i) = \begin{cases} 4n + g_4(u_{n-1-2i} u_{n-2i}) & \text{for} \quad 1 \le i \le \frac{n-3}{2}, \\ 4n + g_4(u_{3n-1-2i} u_{3n-2i}) & \text{for} \quad \frac{n-1}{2} \le i \le n. \end{cases}$$

$$\gamma_3(f_{0,i}) = 4n + g_5(u_{2i-1}) \quad \text{for} \quad 1 \le i \le n,$$

$$\gamma_3(f_{1,i}) = 4n + g_5(u_{2i}) \quad \text{for} \quad 1 \le i \le n,$$

$$\gamma_3(x_{i+1}y_i) = 4n + g_5(u_{2i-1}u_{2i}) \ \text{ for } \ 1 \le i \le n,$$

$$\gamma_3(x_iy_i) = \begin{cases} 4n + g_5(u_{2i}u_{2i+1}) & \text{for } \ 1 \le i \le n-1, \\ 4n + g_5(u_{2n}u_1) & \text{for } \ i = n, \end{cases}$$

$$\gamma_4(f_{0,i}) = 4n + g_6(u_{2i-1}) \ \text{ for } \ 1 \le i \le n,$$

$$\gamma_4(f_{1,i}) = 4n + g_6(u_{2i}) \ \text{ for } \ 1 \le i \le n,$$

$$\gamma_4(x_{i+1}y_i) = 4n + g_6(u_{2i-1}u_{2i}) \ \text{ for } \ 1 \le i \le n,$$

$$\gamma_4(x_iy_i) = \begin{cases} 4n + g_6(u_{2i}u_{2i+1}) & \text{for } \ 1 \le i \le n-1, \\ 4n + g_6(u_{2n}u_1) & \text{for } \ i = n, \end{cases}$$

$$\gamma_4(z_{n,1}) = 8n + 2,$$
$$\gamma_4(z_{n,2}) = 8n + 1.$$

$$\gamma_5(f_{0,i}) = 4n + g_7(u_{2i-1}) \ \text{ for } \ 1 \le i \le n,$$

$$\gamma_5(f_{1,i}) = \begin{cases} 4n + g_7(u_{2i}) & \text{for } \ 1 \le i \le n-1, \\ 4n + g_7(u_{2n}) & \text{for } \ i = n, \end{cases}$$

$$\gamma_5(x_{i+1}y_i) = 4n + g_7(u_{2i-1}u_{2i}) \ \text{ for } \ 1 \le i \le n,$$

$$\gamma_5(x_iy_i) = \begin{cases} 4n + g_7(u_{2n}u_1) & \text{for } \ i = 1, \\ 4n + g_7(u_{2i-2}u_{2i-1}) & \text{for } \ 2 \le i \le n, \end{cases}$$

$$\gamma_1(z_{n,1}) = \gamma_2(z_{n,1}) = \gamma_3(z_{n,1}) = \gamma_5(z_{n,1}) = 8n + 1,$$

$$\gamma_1(z_{n,2}) = \gamma_2(z_{n,2}) = \gamma_3(z_{n,2}) = \gamma_5(z_{n,2}) = 8n + 2.$$

$$\gamma_6(f_{0,i}) = \begin{cases} 8n-2 & \text{for } \ i=1, \\ 4n-6+4i & \text{for } \ 2 \le i \le n, \end{cases}$$

$$\gamma_6(f_{1,i}) = \begin{cases} 8n & \text{for } \ i=1, \\ 4n-4+4i & \text{for } \ 2 \le i \le n, \end{cases}$$

$$\gamma_6(x_{i+1}y_i) = 4n-3+4i \ \text{ for } \ 1 \le i \le n,$$

$$\gamma_6(x_i y_i) = \begin{cases} 8n-1 & \text{for } \ i=1, \\ 4n-5+4i & \text{for } \ 2 \le i \le n, \end{cases}$$

$$\gamma_6(z_{n,1}) = \begin{cases} 8n+2 & \text{for } \ n=4, \\ 8n+1 & \text{for } \ n \ge 5, \end{cases}$$

$$\gamma_6(z_{n,2}) = \begin{cases} 8n+1 & \text{for } \ n=4, \\ 8n+2 & \text{for } \ n \ge 5. \end{cases}$$

Case $d = 0$.

Using the labeling $(\alpha_1(C_O), \beta_1(C_I), \gamma_1(C_M))$ for $m = 2n \equiv 0 \pmod 4$ and $(\alpha_1(C_O), \beta_1(C_I), \gamma_2(C_M))$ for $m = 2n \equiv 2 \pmod 4$, all the weights of the 3-sided faces have the same value $25n+3$ and the weight of $z_{n,2}$ is one more than the weight of $z_{n,1}$. If we swap the edge value of $\beta_1(x_{n-1}x_n) = 4n$ and the face value $\gamma_1(f_{0,n-1}) = \gamma_2(f_{0,n-1}) = 4n+1$, then the weights of $z_{n,2}$ and $z_{n,1}$ will be $4n^2+9n+2$. Thus we obtain a super 0-antimagic labeling of type $(1,1,1)$ for the antiprism A_n.

Case $d = 1$.

In the labeling $(\alpha_2(C_O), \beta_2(C_I), \gamma_3(C_M))$, the weights for the 3-sided faces constitute an arithmetic progression $24n+4, 24n+5, \ldots, 26n+3$ and the weight of the face $z_{n,2}$ $(z_{n,1})$ is $4n^2+9n+2$ $(4n^2+9n+1)$, respectively. It follows that we have a super 1-antimagic labeling of type $(1,1,1)$.

Case $d = 2$.

Under the labeling $(\alpha_2(C_O), \beta_2(C_I), \gamma_4(C_M))$, the weights of the 3-sided faces constitute an arithmetic progression with difference $d = 2$ with values $22n +$

$5, 22n+7, \ldots, 26n+3$ and the weight of $z_{n,1}$ is one more than the weight of $z_{n,2}$. If we swap the edge value of the inner cycle edge $\beta_2(x_1x_2) = 4n$ and the value of the 3-sided face $\gamma_4(f_{0,1}) = 4n+1$, then the weight of the face $z_{n,1}$ will be increased by 1. Thus the difference between the weights of $z_{n,1}$ and $z_{n,2}$ will be 2 and we have a super 2-antimagic labeling of type $(1,1,1)$ of A_n.

Case $d = 3$.

If we use the labeling $(\alpha_2(C_O), \beta_2(C_I), \gamma_5(C_M))$, then the weights of the 3-sided faces constitute an arithmetic progression with difference $d = 3$, namely $22n+4, 22n+7, \ldots, 28n+1$, and the weight of $z_{n,2}$ is $4n^2+9n+2$ and the weight of $z_{n,1}$ is $4n^2+9n+1$. If we swap the label of the inner cycle edge $\beta_2(x_2x_3) = 4n-1$ with $\gamma_5(f_{0,2}) = 4n+3$, then the weight of the face $z_{n,1}$ will increase by 4. Now, the difference between the weight of $z_{n,1}$ and the weight of $z_{n,2}$ is 3.

Case $d = 6$.

If we use the labeling $(\alpha_2(C_O), \beta_2(C_I), \gamma_6(C_M))$ and if we swap the edge value $\beta_2(x_1x_2) = 4n$ and the face value $\gamma_6(f_{1,1}) = 8n$, then the weights of the 3-sided faces form an arithmetic progression $18n+6, 18n+12, \ldots, 30n$.

If $n \geq 5$ then the weight of the face $z_{n,1}$ admits the value $4n^2+13n+1$ and the weight of the face $z_{n,2}$ admits the value $4n^2+9n+2$. Swapping the label of the edge $\beta_2(x_1x_2) = 8n$ with the face value $\gamma_6(f_{0,1}) = 8n-2$ will decrease the weight of the inner n-sided face $z_{n,1}$ by 2. On the other hand swapping the edge label $\alpha_2(y_2y_3) = 2n+4$ with the face value $\gamma_6(f_{1,2}) = 4n+4$, and the edge label $\alpha_2(y_3y_4) = 2n+5$ with the face value $\gamma_6(f_{1,3}) = 4n+8$ will increase the weight of the face $z_{n,2}$ by $4n+3$. It follows that the weight of $z_{n,2}$ is 6 more than the weight of $z_{n,1}$.

If $n = 4$ then we swap the edge value $\alpha_2(y_4y_1) = 10$ with the face value $\gamma_6(f_{1,4}) = 28$ and the edge value $\alpha_2(y_1y_2) = 11$ with the new value of the face $f_{1,1}$. We can see that the weight of the face $z_{4,1}$ is 118 and the weight of the face $z_{4,2}$ is 124.

Thus, the resulting labeling of antiprism A_n, $n \geq 4$, is a super 6-antimagic labeling of type $(1,1,1)$.

□

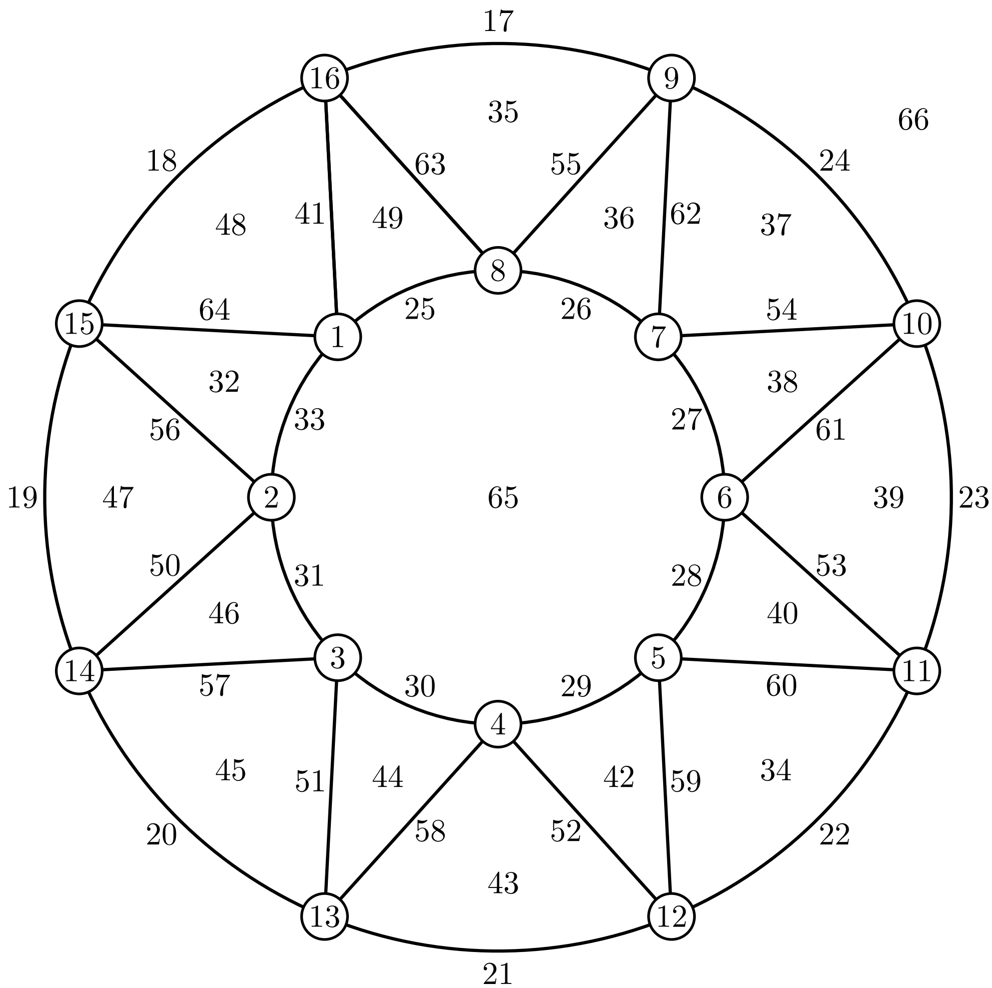

Figure 3.2: Super face-magic labeling of type $(1, 1, 1)$ for A_8.

Figure 3.2 gives an example of super face-magic labeling of type $(1, 1, 1)$, given by labeling $(\alpha_1(C_O), \beta_1(C_I), \gamma_1(C_M))$, for the antiprism A_8 with common face-weight 203 for all 3-sided faces and common face-weight 330 for both 8-sided faces.

Theorem 5. [20] *The antiprism A_n, $n \geq 4$, has a super 4-antimagic labeling of type $(1, 1, 1)$.*

Proof. First we construct the labelings α_3 and β_3 for the outer and the inner cycle of the antiprism such that

$$\alpha_3(y_i) = \begin{cases} 2 + 2g_2(y_i) & \text{for } \; i = 1, 2, \dots, n-1, \\ 2 & \text{for } \; i = n, \end{cases}$$

$$\alpha_3(y_iy_{i+1}) = \begin{cases} 2g_2(y_iy_{i+1}) - 4 & \text{for } \ i = 1, 2, \ldots, n-2, \\ 6n - 2 - 2i & \text{for } \ i = n-1, n, \end{cases}$$

$$\beta_3(x_i) = 2g_2(x_i) - 1 \ \text{ for } \ i = 1, 2, \ldots, n,$$

$$\beta_3(x_ix_{i+1}) = 2g_2(x_ix_{i+1}) - 1 \ \text{ for } \ i = 1, 2, \ldots, n.$$

It is not difficult to see that the weights of the 3-sided faces under the labeling $(\alpha_3(C_O), \beta_3(C_I))$ form an arithmetic progression with difference 2, more precisely $4n+3, 4n+5, \ldots, 8n+1$.

We construct the labeling γ_7 for the faces and spokes of the middle part of the antiprism A_n in the following way.

$$\gamma_7(f_{0,i}) = \begin{cases} 4n + 1 + g_8(u_{2i+1}) & \text{for } \ 1 \le i \le n-1, \\ 4n + 2 & \text{for } \ i = n, \end{cases}$$

$$\gamma_7(f_{1,i}) = \begin{cases} 4n + 1 + g_8(u_{2i+2}) & \text{for } \ 1 \le i \le n-2, \\ 7n + 1 & \text{for } \ i = n-1, \\ 4n + 3 & \text{for } \ i = n, \end{cases}$$

$$\gamma_7(x_{i+1}y_i) = \begin{cases} 4n + 1 + g_8(u_{2i+1}u_{2i+2}) & \text{for } \ 1 \le i \le n-1, \\ 7n + 2 & \text{for } \ i = n, \end{cases}$$

$$\gamma_7(x_iy_i) = \begin{cases} 4n + 1 + g_8(u_{2i}u_{2i+1}) & \text{for } \ 1 \le i \le n-1, \\ 6n + 1 & \text{for } \ i = n. \end{cases}$$

It is easy to verify that the labeling γ_7 uses each integer from the set $\{4n+2, 4n+3, \ldots, 8n+1\}$. Using the labeling $(\alpha_3(C_O), \beta_3(C_I), \gamma_7(C_M))$, we obtain a set of 3-sided face-weights $21n+8, 21n+12, \ldots, 29n+4$. To obtain an arithmetic progression with the difference 4 for the two n-sided faces we will distinguish the following two cases.

Case 1. n is even

If we label the outer face $z_{n,2}$ by $4n+1$ and the inner face $z_{n,1}$ by $8n+2$ then the weight of the outer face is $4n^2 + 6n + 1$ and the weight of the inner face is $4n^2 + 8n + 2$. For $n \geq 6$, we swap the edge value $\alpha_3(y_{\frac{n}{2}-2}y_{\frac{n}{2}-1}) = 3n + 2$ with the face value $\gamma_7(f_{1,\frac{n}{2}-2}) = 5n - 1$, and for $n = 4$, we swap the edge value $\alpha_3(y_4y_1) = 14$ with the face value $\gamma_7(f_{1,4}) = 19$. It is easy to see that the weight of the inner face is 4 more than the weight of the outer face.

Case 2. n is odd

If we label the face $z_{n,2}$ by $8n+2$ and the face $z_{n,1}$ by $4n+1$ then the weight of the face $z_{n,2}$ is $4n^2+10n+2$ and the weight of the face $z_{n,1}$ is $4n^2+4n+1$. Now, we swap the edge value $\beta_3(x_{\frac{n+1}{2}}x_{\frac{n+3}{2}}) = 3n$ with the face value $\gamma_7(f_{0,\frac{n+1}{2}}) = 5n+3$, the edge value $\beta_3(x_{\frac{n-1}{2}}x_{\frac{n+1}{2}}) = 3n+2$ with the face value $\gamma_7(f_{0,\frac{n-1}{2}}) = 5n+1$ and the edge value $\beta_3(x_{\frac{n-3}{2}}x_{\frac{n-1}{2}}) = 3n + 4$ with the face value $\gamma_7(f_{0,\frac{n-3}{2}}) = 5n - 1$. It is easy to check that the antiprism has been assigned a 4-antimagic labeling of type $(1, 1, 1)$.

□

Figure 3.3 illustrates a super 4-antimagic labeling of type $(1, 1, 1)$ for A_7, where the integers in italic mean face-weights.

Theorem 6. [20] *The antiprism A_n, $n \geq 4$, has a super 5-antimagic labeling of type $(1, 1, 1)$.*

Proof. Define a labeling γ_8 for C_M in the following way.

$$\gamma_8(f_{0,i}) = \begin{cases} 4n + g_9(u_{2i+1}) & \text{for } 1 \leq i \leq n-1, \\ 4n + g_9(u_1) & \text{for } i = n, \end{cases}$$

$$\gamma_8(f_{1,i}) = \begin{cases} 4n + g_9(u_{2i+2}) & \text{for } 1 \leq i \leq n-1, \\ 4n + g_9(u_2) & \text{for } i = n, \end{cases}$$

$$\gamma_8(x_{i+1}y_i) = \begin{cases} 4n + g_9(u_{2i+1}u_{2i+2}) & \text{for } 1 \leq i \leq n-1, \\ 4n + g_9(u_1u_2) & \text{for } i = n, \end{cases}$$

$$\gamma_8(x_iy_i) = 4n + g_9(u_{2i}u_{2i+1}) \quad \text{for } 1 \leq i \leq n,$$

$$\gamma_8(z_{n,1}) = 8n + 2,$$

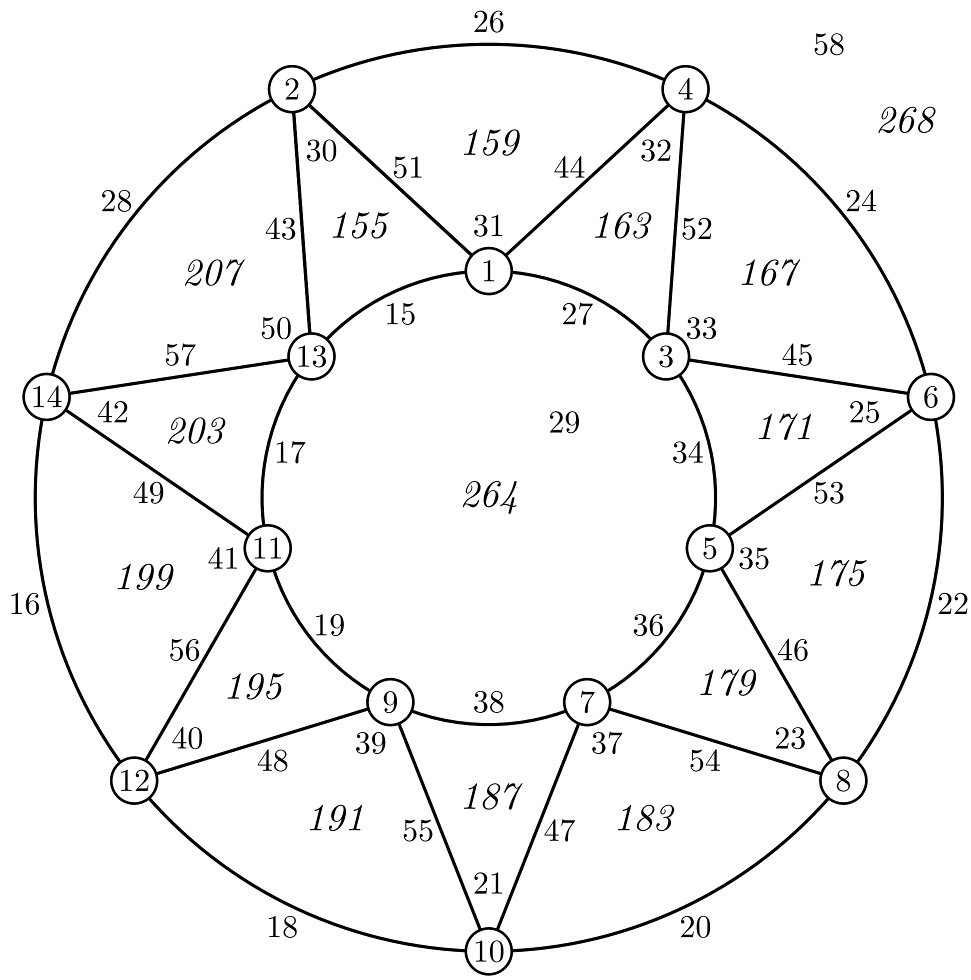

Figure 3.3: Super 4-antimagic labeling of type $(1, 1, 1)$ for A_7.

$$\gamma_8(z_{n,2}) = 8n + 1.$$

If we use the labeling $(\alpha_3(C_O), \beta_3(C_I), \gamma_8(C_M))$ then the weights of the 3-sided faces constitute an arithmetic progression $18n+7, 18n+12, \dots, 28n+2$. The weight of the face $z_{n,1}$ is $4n^2 + 8n + 2$ and the weight of the face $z_{n,2}$ is $4n^2 + 10n + 1$. If we swap the value of the inner cycle edge $\beta_3(x_1x_2) = 4n - 1$ with the value of the 3-sided face $\gamma_8(f_{0,1}) = 6n + 3$, then the weight of the face $z_{n,1}$ will be 5 more than the weight of the face $z_{n,2}$. Thus, the resulting labeling is super 5-antimagic labeling of type $(1, 1, 1)$. $\square$

Chapter 4

Face antimagic labelings of disjoint union of prisms

The prism $D_n \cong C_n \times P_2$, $n \geq 3$, is a cubic graph which consists of an outer n-cycle say $y_1, y_2, \dots, y_n$, an inner n-cycle say $x_1, x_2, \dots, x_n$, and a set of n spokes $x_i y_i$, $i = 1, 2, \dots, n$. Note that $x_{n+1} = x_1$ and $y_{n+1} = y_1$. We denote by $z_{4,i}$, $1 \leq i \leq n$, the 4-sided face bounded by the edges $x_i y_i$, $x_i x_{i+1}$, $x_{i+1} y_{i+1}$ and $y_i y_{i+1}$, and we denote by $z_{n,1}$ and $z_{n,2}$ the inner n-sided face and outer n-sided face, respectively. See Figure 4.1.

In [13], it was proved that the prism D_n has a d-antimagic labelings of type $(1,1,1)$ for $d \in \{2,3,4,6\}$ and $n \equiv 3 \pmod 4$. Lin, Slamin, Bača and Miller in [58] showed that D_n, $n \geq 3$, admits d-antimagic labelings of type $(1,1,1)$ for $d \in \{2,4,5,6\}$. The d-antimagic labelings of type $(1,1,1)$ for prism D_n and for $d \in \{7,8,9,10,12,14,15,16,17,18,20,21,24,26,27,30,36\}$ are described in [76].

In this chapter we investigate the existence of the super d-antimagic labeling of type $(1,1,1)$ for the disjoint union of m copies of prism D_n, denoted by mD_n.

The disconnected graph mD_n, for $n \geq 3$, $m \geq 2$, consists of the vertex set $V(mD_n) = \{x_i^j, y_i^j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and the edge set $E(mD_n) = \{x_i^j x_{i+1}^j, y_i^j y_{i+1}^j, x_i^j y_i^j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$, with indices taken modulo n. The face set $F(mD_n)$ contains mn 4-sided faces $\{z_{4,i}^j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$, m inner n-sided faces $\{z_{n,1}^j : 1 \leq j \leq m\}$ and one external face. So, $v = 2mn$, $e = 3mn$ and $f = m(n+1)+1$.

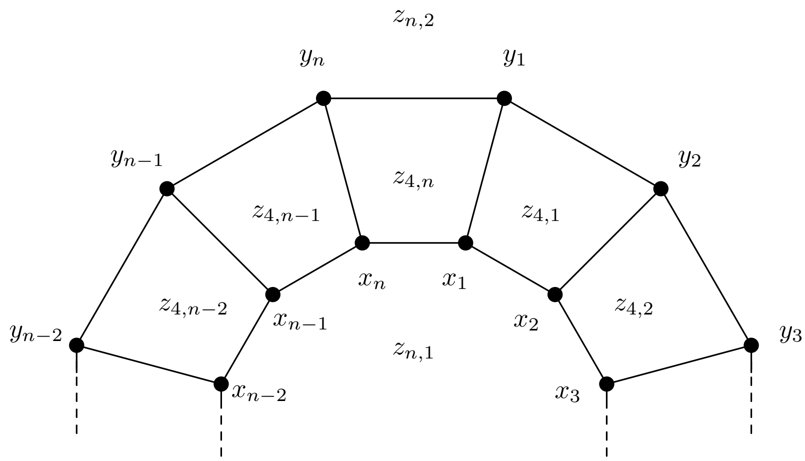

Figure 4.1: The prism D_n.

4.1 Necessary conditions

We start by finding the bounds for a feasible value of d for the super d-antimagic labeling of type $(1,1,1)$ of the disjoint union of m copies of prism.

Suppose that $g : V(mD_n) \cup E(mD_n) \cup F(mD_n) \to \{1, 2, \ldots, 6mn + m + 1\}$ is a super d-antimagic labeling of type $(1,1,1)$ of the mD_n. Let us consider the weights of the n-sided faces. Under the labeling g, the vertices x_i^j, y_i^j, $1 \le i \le n$, $1 \le j \le m$, receive the smallest possible labels $1, 2, \ldots, 2mn$. The edges $x_i^j x_{i+1}^j$, $1 \le i \le n$, $1 \le j \le m$ and the n-sided faces $z_{n,1}^j$, $1 \le j \le m$, could conceivably receive the next $m(n+1)$ smallest labels $2mn+1, 2mn+2, \ldots, 3mn+m$ or, at the other extreme, the edges $x_i^j x_{i+1}^j$, $1 \le i \le n$, $1 \le j \le m$, and the n-sided faces $z_{n,1}^j$, $1 \le j \le m$, could receive the largest possible labels $5mn+2, 5mn+3, \ldots, 6mn+m+1$.

The minimum weight of n-sided face is

$$a_n \ge \sum_{i=1}^{n} i + \sum_{i=1}^{n+1} (2mn + i) = (n+1)(2mn + n + 1).$$

The maximum weight of n-sided face is

$$\begin{aligned} a_n + (m-1)d &\le \sum_{i=1}^{n} (2mn - n + i) + \sum_{i=1}^{n+1} (6mn + m - n + i) \\ &= m(8n^2 + 7n + 1) - n^2 + n + 2. \end{aligned}$$

Thus,

$$d \le \frac{mn(6n+5) - n(2n+1) + m + 1}{m-1}.$$

In this case, the upper bound for the difference d is very large.

Thus, let us consider the weights of the 4-sided faces of mD_n.

Theorem 7. [2] *For every mD_n, $m \ge 2$, $n \ge 3$, there is no super d-antimagic labeling of type $(1,1,1)$ with $d \ge 30$.*

Proof. Let $G \cong mD_n$. Let $g : V(G) \cup E(G) \cup F(G) \to \{1, 2, \dots, 6mn + m + 1\}$ be a super d-antimagic labeling of type $(1,1,1)$ and the set $W_4 = \{a_4, a_4 + d, \dots, a_4 + (mn-1)d\}$ be the set of the 4-sided face-weights.

It is easy to see that the minimum possible weight a_4 of the 4-sided face, under the labeling g, is at least $(1+2+3+4) + \sum_{i=1}^{5}(2mn+i) = 10mn + 25$.

On the other hand, the maximum possible weight of a 4-sided face, under the labeling g, is at most $\sum_{i=1}^{4}(2mn+1-i) + \sum_{i=1}^{5}(6mn+m+2-i) = 38mn + 5m - 11$. So

$$a_4 + (mn-1)d \le 38mn + 5m - 11.$$

From the last inequality, for $n \ge 3$, we obtain $d < 30$, which completes the proof. □

4.2 mD_n decomposed into $3m$ disjoint cycles

The prism D_n can be considered as three cycle parts – the outer n-cycle, the inner n-cycle and the middle part which consists of the 4-sided faces and the spokes. The third part can be treated as an n-cycle for the labeling purposes. Thus, the face labels are considered as the vertex labels of a middle cycle and the spoke labels are considered as the edge labels of a middle cycle.

In this chapter we treat a graph mD_n also as three parts – the disjoint union of m outer cycles, the disjoint union of m inner cycles and the disjoint union of m middle cycles. We obtain the desired super d-antimagic labeling of mD_n by applying suitable labelings of these three parts. To simplify the proofs, we denote by

$\mathbb{C}_O$ – the disjoint union of m outer cycles of all D_n in mD_n, where

$V(\mathbb{C}_O) = \{y_i^j : 1 \le i \le n,\ 1 \le j \le m\}$ and

$E(\mathbb{C}_O) = \{y_i^j y_{i+1}^j : 1 \le i \le n,\ 1 \le j \le m\}$,

$\mathbb{C}_I$ – the disjoint union of m inner cycles of all D_n in mD_n, where
$V(\mathbb{C}_I) = \{x_i^j : 1 \le i \le n,\ 1 \le j \le m\}$ and
$E(\mathbb{C}_I) = \{x_i^j x_{i+1}^j : 1 \le i \le n,\ 1 \le j \le m\}$,

$\mathbb{C}_M$ – the disjoint union of m middle cycles of all D_n in mD_n, where the vertices of the cycles represent the 4-sided faces of mD_n, i.e.
$V(\mathbb{C}_M) = \{z_{4,i}^j : 1 \le i \le n,\ 1 \le j \le m\}$ and the edges of the cycles represent the spokes of mD_n, i.e.
$E(\mathbb{C}_M) = \{z_{4,i}^j z_{4,i+1}^j = x_{i+1}^j y_{i+1}^j : 1 \le i \le n,\ 1 \le j \le m\}$.

Note, that the indices are taken modulo n.

Given a labeling schema ρ, by $\rho(\mathbb{C}_O)$ we mean that we apply ρ on $\mathbb{C}_O$. The labeling denoted by $(\rho(\mathbb{C}_O), \phi(\mathbb{C}_I))$ or $(\rho(\mathbb{C}_O), \phi(\mathbb{C}_I), \psi(\mathbb{C}_M))$ means a combination of the labeling ρ for the disjoint union of m outer cycles in mD_n, ϕ for the disjoint union of m inner cycles in mD_n and labeling ψ for the disjoint union of m middle cycles in mD_n.

The process for labeling the graph mD_n by a super d-antimagic labeling of type $(1,1,1)$ is as follows;

1. Label the vertices and edges of $\mathbb{C}_O$ and $\mathbb{C}_I$ by the super (a', d')-edge-antimagic total labelings. The edge-weights of $\mathbb{C}_O$ and $\mathbb{C}_I$ form two different arithmetic sequences of the same difference d'.

2. Label the spokes and the 4-sided faces of $\mathbb{C}_M$ by an (a'', d'')-vertex-antimagic total labeling. The vertex-weights of $\mathbb{C}_M$ form an arithmetic sequence of difference d''.

   The labelings of $\mathbb{C}_O$, $\mathbb{C}_I$ and $\mathbb{C}_M$ combine to a labeling, where the weights of the 4-sided faces of mD_n form a requested arithmetic sequence with a common difference d which is depending on the differences d' and d''.

3. Label the n-sided faces $z_{n,1}^j$, $1 \le j \le m$. Swap the label of the edge $x_i^j x_{i+1}^j$ in $\mathbb{C}_I$ with the label of the 4-sided face $z_{4,i}^j$ (represented by the vertex in $\mathbb{C}_M$), for $1 \le j \le m$, and/or swap the label of the vertex x_i^j (respectively y_i^j) with the label of the vertex x_i^k (respectively y_i^k) in $\mathbb{C}_I$ (respectively $\mathbb{C}_O$), for $1 \le j, k \le m$, and/or swap the label of the edge $x_i^j y_i^j$ with the label of the edge $x_i^k y_i^k$ in $\mathbb{C}_M$, for $1 \le j, k \le m$.

In every case, the applied swapping results in changes of the face-weights of $z^j_{n,1}$ to obtain an arithmetic sequence of the difference d for the weights of the n-sided faces. Note that the swapping processes do not have any impact on the weights of the 4-sided faces $z^j_{4,i}$.

4.3 d-antimagic labelings of mD_n

In this section we present the super d-antimagic labelings of the disjoint union of m copies of prism for the certain values of d.

Theorem 8. [2] *For $d \in \{0,1,3,4,5\}$ and for every $m \geq 2$ and $n \geq 3$, $n \neq 4$, the graph mD_n has a super d-antimagic labeling of type $(1,1,1)$.*

Proof. Let us consider the super $(2mn+2,1)$-edge-antimagic total labeling h_1. Using the labeling h_1 we define the labeling ρ_1 for $\mathbb{C}_O$ and the labeling ϕ_1 for $\mathbb{C}_I$ in the following way

$$\begin{aligned}
\rho_1(y_i^j) &= mn + h_1(u_i^j),\\
\rho_1(y_i^j y_{i+1}^j) &= mn + h_1(u_i^j u_{i+1}^j),\\
\phi_1(x_i^j) &= h_1(u_i^j),\\
\phi_1(x_i^j x_{i+1}^j) &= 2mn + h_1(u_i^j u_{i+1}^j),
\end{aligned}$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$.

It is easy to verify that the labelings ρ_1 and ϕ_1 use each integer from the set $\{1,2,\dots,4mn\}$ exactly once and edge-weights of $\mathbb{C}_O$ and $\mathbb{C}_I$ constitute the sets

$$\begin{aligned}
&\{w_{\rho_1}(y_n^j y_1^j) = 5mn + j + 1 : 1 \leq j \leq m\}\\
&\qquad \cup \{w_{\rho_1}(y_i^j y_{i+1}^j) = 5mn + im + j + 1 : 1 \leq i \leq n-1,\ 1 \leq j \leq m\}
\end{aligned}$$

and

$$\begin{aligned}
&\{w_{\phi_1}(x_n^j x_1^j) = 4mn + j + 1 : 1 \leq j \leq m\}\\
&\qquad \cup \{w_{\phi_1}(x_i^j x_{i+1}^j) = 4mn + im + j + 1 : 1 \leq i \leq n-1,\ 1 \leq j \leq m\}.
\end{aligned}$$

We can see that all weights of the 4-sided faces, under the labeling $(\rho_1(\mathbb{C}_O), \phi_1(\mathbb{C}_I))$, form an arithmetic progression of the difference 2 with the values $9mn + 4, 9mn+6, \dots, 11mn+2$.

Case $d = 0$.

Using the $(2mn + 2, 2)$-vertex-antimagic total labeling h_3, we construct the labeling ψ_1 for $\mathbb{C}_M$ as follows

$$\psi_1(x_i^j y_i^j) = \begin{cases} 4mn + h_3\left(u_{n-i}^{m-j+1} u_{n-i+1}^{m-j+1}\right) & \text{for} \quad 1 \le i \le n-1,\ 1 \le j \le m, \\ 4mn + h_3\left(u_n^{m-j+1} u_1^{m-j+1}\right) & \text{for} \quad i = n,\ 1 \le j \le m, \end{cases}$$

$$\psi_1(z_{4,i}^j) = \begin{cases} 4mn + h_3\left(u_{n-i}^{m-j+1}\right) & \text{for} \quad 1 \le i \le n-2,\ 1 \le j \le m, \\ 4mn + h_3\left(u_1^{m-j+1}\right) & \text{for} \quad i = n-1,\ 1 \le j \le m, \\ 4mn + h_3\left(u_n^{m-j+1}\right) & \text{for} \quad i = n,\ 1 \le j \le m. \end{cases}$$

The labeling ψ_1 is constructed such that the all weights of the 4-sided faces under the labeling $(\rho_1(\mathbb{C}_O), \phi_1(\mathbb{C}_I), \psi_1(\mathbb{C}_M))$ have the same value $25mn + 4$. Also the all weights of the n-sided faces have the same value $n(4mn + 1)$.

Now, we swap the edge label $\phi_1(x_1^j x_2^j) = 4mn + 1 - j$ with the face label $\psi_1(z_{4,1}^j) = 4m(n + 1) + 2j$, for every $1 \le j \le m$, the vertex label $\phi_1(x_1^j) = j$ (respectively, $\rho_1(y_1^j) = mn + j$) with the vertex label $\phi_1(x_1^{m-j+1}) = m + 1 - j$ (respectively, $\rho_1(y_1^{m-j+1}) = m(n + 1) - j + 1$), for every $1 \le j \le \lfloor \frac{m}{2} \rfloor$, and the spoke label $\psi_1(x_1^j y_1^j) = 2m(3n - 1) - 2j + 1$ with the spoke label $\psi_1(x_1^{m-j+1} y_1^{m-j+1}) = 2m(3n - 2) + 2j - 1$, for every $1 \le j \le \lfloor \frac{m}{2} \rfloor$.

The swapping process does not have any impact on the weights of the 4-sided faces, however the weights of n-sided faces constitute the set $\{m(4n^2+5)+n+j : 1 \le j \le m\}$.

If we complete the face labels

$$\sigma_1(z_{n,1}^j) = m(6n + 1) + 1 - j \qquad \text{for} \quad 1 \le j \le m,$$

$$\sigma_1(external face) = m(6n + 1) + 1,$$

then the resulting labeling of type $(1, 1, 1)$ has the common weight for all n-sided faces equal to $m(4n^2 + 6n + 6) + n + 1$. This proves that mD_n has the 0-antimagic labeling of type $(1, 1, 1)$.

Case $d = 1$.

Using the $(2mn+2, 1)$-vertex-antimagic total labeling h_2, we define the labeling ψ_2 for $\mathbb{C}_M$ in the following way

$$\psi_2(x_i^j y_i^j) = \begin{cases} 4mn + h_2\left(u_{n-i}^{m-j+1} u_{n-i+1}^{m-j+1}\right) & \text{for} \quad 1 \le i \le n-1,\ 1 \le j \le m, \\ 4mn + h_2\left(u_n^{m-j+1} u_1^{m-j+1}\right) & \text{for} \quad i = n,\ 1 \le j \le m, \end{cases}$$

$$\psi_2(z_{4,i}^j) = \begin{cases} 4mn + h_2\left(u_{n-i}^{m-j+1}\right) & \text{for} \quad 1 \le i \le n-2,\ 1 \le j \le m, \\ 4mn + h_2\left(u_1^{m-j+1}\right) & \text{for} \quad i = n-1,\ 1 \le j \le m, \\ 4mn + h_2\left(u_n^{m-j+1}\right) & \text{for} \quad i = n,\ 1 \le j \le m. \end{cases}$$

Under the labeling $(\rho_1(\mathbb{C}_O), \phi_1(\mathbb{C}_I), \psi_2(\mathbb{C}_M))$, the weights of the 4-sided faces constitute an arithmetic progression $24mn+5, 24mn+6, \ldots, 25mn+4$ and the all weights of the n-sided faces have the same value $n(4mn+1)$. If we label the n-sided faces by the labeling σ_1 then the resulting labeling is required super 1-antimagic labeling of type $(1, 1, 1)$.

Case $d = 3$.

Again we use the $(2mn+2, 1)$-vertex-antimagic total labeling h_2 and define the labeling ψ_3 for $\mathbb{C}_M$ such that

$$\psi_3(x_i^j y_i^j) = 4mn + h_2(u_i^j u_{i+1}^j) \quad \text{for} \quad 1 \le i \le n,\ 1 \le j \le m,$$

$$\psi_3(z_{4,i}^j) = \begin{cases} 4mn + h_2(u_{i+1}^j) & \text{for} \quad 1 \le i \le n-1,\ 1 \le j \le m, \\ 4mn + h_2(u_1^j) & \text{for} \quad i = n,\ 1 \le j \le m. \end{cases}$$

In the labeling $(\rho_1(\mathbb{C}_O), \phi_1(\mathbb{C}_I), \psi_3(\mathbb{C}_M))$, the weights for the 4-sided faces constitute an arithmetic progression of the difference 3, namely $23mn+6, 23mn+9, \ldots, 26mn+3$, and the common weight for all n-sided faces is equal to $n(4mn+1)$.

If we swap the vertex label $\phi_1(x_1^j) = j$ with the vertex label $\phi_1(x_1^{m-j+1}) = m - j + 1$ and the face label $\psi_3(z_{4,n}^j) = m(5n+1) + 1 - j$ (respectively, $\psi_3(z_{4,1}^j) = 6mn+1-j$) with the face label $\psi_3(z_{4,n}^{m-j+1}) = 5mn+j$ (respectively, $\psi_3(z_{4,1}^{m-j+1}) = m(6n-1)+j$), for every $1 \le j \le \lfloor \frac{m}{2} \rfloor$, then the weights of 4-sided faces will not be changed, but the weights of n-sided faces will constitute the set $\{n(4mn+1)+m-2j+1 : 1 \le j \le m\}$.

After completing the face labels according the labeling σ_1, the resulting labeling of type $(1,1,1)$ is the super 3-antimagic labeling.

Case $d = 4$.

Define the labeling ψ_4 for the 4-sided faces and spokes of the middle part of the mD_n in the following way.

$$\psi_4(x_i^j y_i^j) = 4mn + h_3(u_i^j u_{i+1}^j) \quad \text{for} \quad 1 \le i \le n,\ 1 \le j \le m,$$

$$\psi_4(z_{4,i}^j) = \begin{cases} 4mn + h_3(u_{i+1}^j) & \text{for} \quad 1 \le i \le n-1,\ 1 \le j \le m, \\ 4mn + h_3(u_1^j) & \text{for} \quad i = n,\ 1 \le j \le m. \end{cases}$$

Using the labeling $(\rho_1(\mathbb{C}_O), \phi_1(\mathbb{C}_I), \psi_4(\mathbb{C}_M))$, the weights of the 4-sided faces form an arithmetic sequence of difference 4, namely $23mn+6, 23mn+10, \ldots, 27mn+2$.

If we swap the edge label $\phi_1(x_i^j x_{i+1}^j) = m(4n-i+1)+1-j$ with the face label $\psi_4(z_{4,i}^j) = 2m(3n-i+1)+2-2j$, for $1 \le i \le 3$ and $1 \le j \le m$, then the weights of the 4-sided faces will again not change and the weights of the n-sided faces will form the set $\{n(4mn+1)+3m(2n-1)-3j+3 : 1 \le j \le m\}$. It suffices to complete the face labeling σ_1 and the resulting labeling is super 4-antimagic labeling of type $(1,1,1)$.

Case $d = 5$.

Now, we use the $(m(n+1)+3, 3)$-vertex-antimagic total labeling h_4 and define the labeling ψ_5 for $\mathbb{C}_M$ as follows:

$$\psi_5(x_i^j y_i^j) = 4mn + h_4(u_i^j u_{i+1}^j) \quad \text{for} \quad 1 \le i \le n,\ 1 \le j \le m,$$

$$\psi_5(z_{4,i}^j) = \begin{cases} 4mn + h_4(u_{i+1}^j) & \text{for} \quad 1 \le i \le n-1,\ 1 \le j \le m, \\ 4mn + h_4(u_1^j) & \text{for} \quad i = n,\ 1 \le j \le m. \end{cases}$$

Under the labeling $(\rho_1(\mathbb{C}_O), \phi_1(\mathbb{C}_I), \psi_5(\mathbb{C}_M))$, the weights of the 4-sided faces constitute the set $\{m(22n+1)+7, m(22n+1)+12, \ldots, m(27n+1)+2\}$.

If we swap the edge label $\phi_1(x_i^j x_{i+1}^j) = m(4n - i + 1) + 1 - j$ with the face label $\psi_5(z_{4,i}^j) = m(5n + i) + j$, for $1 \le i \le 2$ and $1 \le j \le m$, then the weights of the 4-sided faces will not be changed but the weights of the n-sided faces constitute the set $\{n(4mn + 1) + 2m(n + 2) - 2 + 4j : 1 \le j \le m\}$. If we complete the face labels by

$$\begin{aligned} \sigma_2(z_{n,1}^j) &= 6mn + j && \text{for} \quad 1 \le j \le m, \\ \sigma_2(external face) &= m(6n + 1) + 1, \end{aligned}$$

then the weights of n-sided faces form an arithmetic progression of the difference 5. Hence, we have the super 5-antimagic labeling of type $(1, 1, 1)$.

□

Figure 4.2 depicts a super 4-antimagic labeling of type $(1, 1, 1)$ for $4D_6$, where the integers in italic mean the face-weights.

Theorem 9. [2] *For every $m \ge 2$ and $n \ge 3$, $n \ne 4$, the graph mD_n has a super 2-antimagic labeling of type* $(1, 1, 1)$.

Proof. Define the labeling ρ_2 for $\mathbb{C}_O$ in the following way

$$\rho_2(y_i^j) = \begin{cases} mn + h_1\left(u_{n-i}^{m-j+1}\right) & \text{for } 1 \le i \le n-1,\ 1 \le j \le m, \\ mn + h_1\left(u_n^{m-j+1}\right) & \text{for } i = n,\ 1 \le j \le m, \end{cases}$$

$$\rho_2(y_i^j y_{i+1}^j) = \begin{cases} mn + h_1\left(u_{n-i-1}^{m-j+1} u_{n-i}^{m-j+1}\right) & \text{for } 1 \le i \le n-2,\ 1 \le j \le m, \\ mn + h_1\left(u_n^{m-j+1} u_1^{m-j+1}\right) & \text{for } i = n-1,\ 1 \le j \le m, \\ mn + h_1\left(u_{n-1}^{m-j+1} u_n^{m-j+1}\right) & \text{for } i = n,\ 1 \le j \le m. \end{cases}$$

We label the vertices, edges and 4-sided faces of mD_n by the labeling $(\rho_2(\mathbb{C}_O), \phi_1(\mathbb{C}_I), \psi_4(\mathbb{C}_M))$. We obtain a labeling, where the weights of the 4-sided faces constitute an arithmetic sequence of difference 2, namely $24mn+5, 24mn+7, \ldots, 26mn+3$ and all n-sided faces attain the common weight $n(4mn + 1)$.

If we swap the edge label $\phi_1(x_1^j x_2^j) = 4mn + 1 - j$ with the face label $\psi_4(z_{4,1}^j) = 6mn+2-2j$, for $1 \le j \le m$, then the weights of the 4-sided faces will not be changed, but the weights of the n-sided faces will form the set $\{n(4mn+1)+2mn+1-j : 1 \le j \le m\}$. It only remains to complete the face labeling σ_1 and the resulting labeling of type $(1, 1, 1)$ is the required super 2-antimagic labeling. □

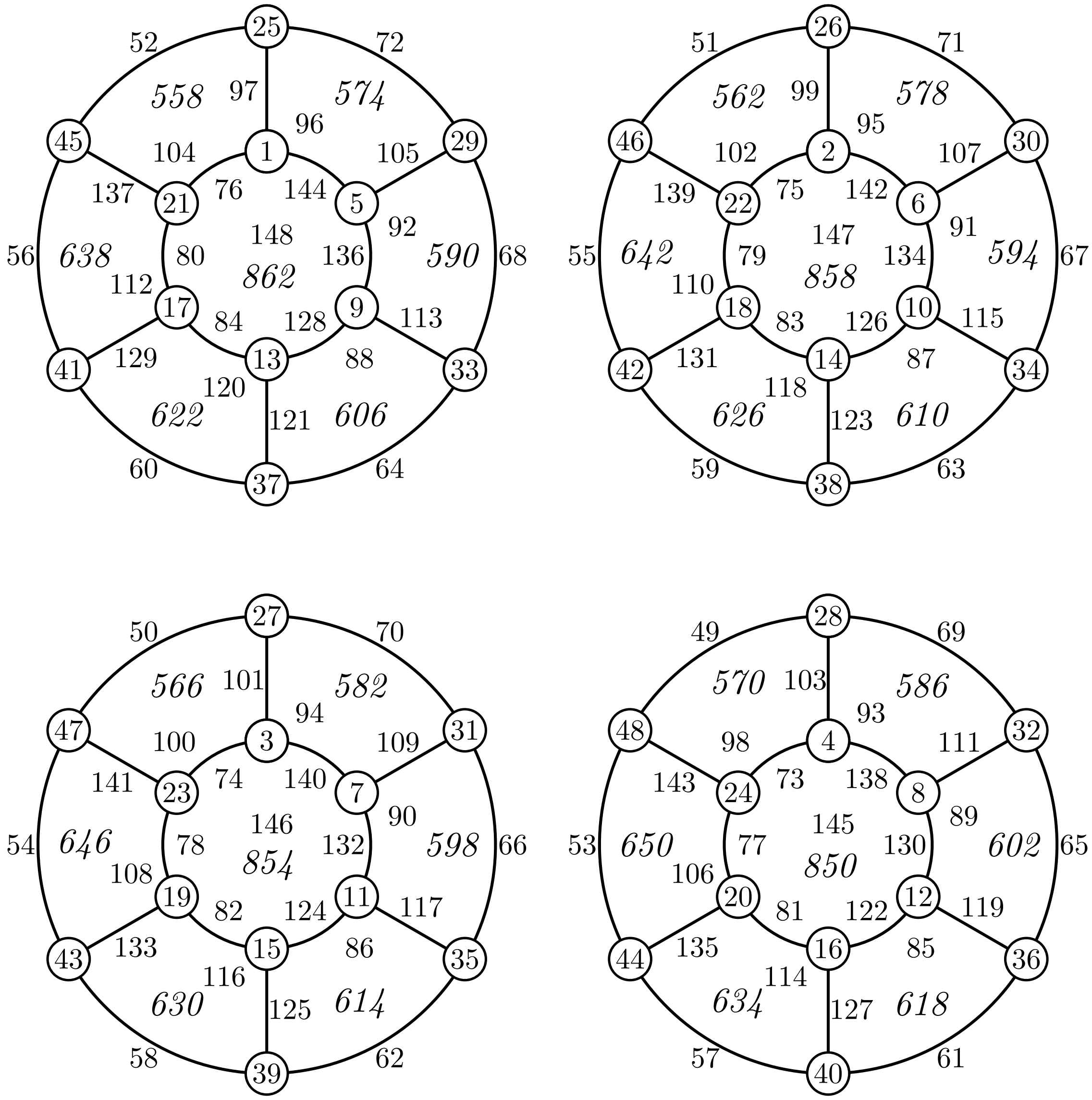

Figure 4.2: Super 4-antimagic labeling of type $(1, 1, 1)$ for $4D_6$.

Using the labeling h_1 we define the labeling ρ_3 for $\mathbb{C}_O$ and the labeling ϕ_2 for $\mathbb{C}_I$ in the following way

$$\begin{aligned}\rho_3(y_i^j) &= 2h_1(u_i^j),\\ \phi_2(x_i^j) &= 2h_1(u_i^j) - 1,\\ \rho_3(y_i^j y_{i+1}^j) &= 2h_1(u_i^j u_{i+1}^j) + 2mn,\\ \phi_2(x_i^j x_{i+1}^j) &= 2h_1(u_i^j u_{i+1}^j) + 2mn - 1,\end{aligned}$$

for $1 \le i \le n$ and $1 \le j \le m$.

Using the labeling h_4, we construct the labeling ψ_6 for $\mathbb{C}_M$ as follows

$$\psi_6(x_i^j y_i^j) = h_4(u_i^j u_{i+1}^j) + 2mn, \quad \text{for} \quad 1 \le i \le n,\ 1 \le j \le m$$

$$\psi_6(z_{4,i}^j) = \begin{cases} h_4(u_{i+1}^j) + 2mn, & \text{for} \quad 1 \le i \le n-1,\ 1 \le j \le m \\ h_4(u_1^j) + 2mn, & \text{for} \quad i = n,\ 1 \le j \le m. \end{cases}$$

It is easy to verify that the labeling ρ_3, ϕ_2 and ψ_6 use each integer from the set $\{1, 2, \ldots, 6mn\}$ exactly once. The following lemma gives a super 7-antimagic labeling of type $(1, 1, 1)$ of mD_n for a special case, when $n = 4$.

Lemma 1. [22] *For every $m \ge 2$, the graph mD_4 admits a super 7-antimagic labeling of type $(1, 1, 1)$.*

Proof. We consider the labeling $(\rho_3(\mathbb{C}_O), \phi_2(\mathbb{C}_I), \psi_6(\mathbb{C}_M))$ for all $m \ge 2$ and $n = 4$. All the weights of the 4-sided faces $z_{4,i}^j$, $1 \le i \le 4$, $1 \le j \le m$, form an arithmetic progression of the difference 7 with the values $77m + 8, 77m + 15, \ldots, 105m + 1$. On the other hand, all weights of the 4-sided faces z_4^j, $1 \le j \le m$, have the same value $96m$.

Now, we swap the vertex label $\phi_2(x_i^j) = 2(i-1)m + 2j - 1$ with the vertex label $\rho_3(y_i^j) = 2(i-1)m + 2j$, for every $1 \le i \le 3$ and $1 \le j \le m$, and the edge label $\phi_2(x_3^j x_4^j) = 20m + 1 - 2j$ (respectively, $\phi_2(x_4^j x_1^j) = 18m + 1 - 2j$) with the face label $\psi_6(z_{4,3}^j) = 15m + j$ (respectively, $\psi_6(z_{4,4}^j) = 8m + j$), for $1 \le j \le m$. The swapping process does not have any impact on the weights of the 4-sided faces $z_{4,i}^j$, however the weights of 4-sided faces z_4^j, $1 \le j \le m$, constitute the set $\{81m + 1 + 6j : 1 \le j \le m\}$.

If we complete the face labels

$$\sigma_3(z_4^j) = 24m + j, \qquad \text{for} \quad 1 \le j \le m$$
$$\sigma_3(external face) = 25m + 1,$$

then the weights of all 4-sided faces in mD_4 form the arithmetic progression $77m + 8, 77m + 15, \ldots, 105m + 1, 105m + 8, \ldots, 112m + 1$. Hence, the resulting labeling is super 7-antimagic of type $(1, 1, 1)$. □

Figure 4.3 illustrates a super 7-antimagic labeling of type $(1, 1, 1)$ for $6D_4$, where the integers in italic mean the face-weights.

Theorem 10. [22] *For every $m \ge 2$ and $n \ge 3$, the graph mD_n has a super 7-antimagic labeling of type $(1, 1, 1)$.*

Proof. For $m \ge 2$ and $n = 4$, the assertion follows from previous Lemma 1. Now, we consider the labeling $(\rho_3(\mathbb{C}_O), \phi_2(\mathbb{C}_I), \psi_6(\mathbb{C}_M))$ for all $m \ge 2$ and $n \ge 3$, $n \ne 4$. The weights of the 4-sided faces $z_{4,i}^j$, $1 \le i \le n$, $1 \le j \le m$, constitute an arithmetic progression of the difference 7, namely $m(19n+1)+8, m(19n+1)+15, \ldots, m(26n+1)+1$ and the common weight for all n-sided faces is equal to $6mn^2$.

If we swap the edge label $\phi_2(x_i^j x_{i+1}^j) = 2m(3n - i + 1) + 1 - 2j$ with the face label $\psi_6(z_{4,i}^j) = m(3n + i) + j$, for $1 \le i \le 2$ and $1 \le j \le m$, then the weights of the 4-sided faces will not be changed, but the weights of the n-sided faces will form the set $\{6mn(n-1) + 5m - 2 + 6j : 1 \le j \le m\}$. It suffices to complete the face labeling

$$\sigma_4(z_n^j) = \sigma_2(z_n^j), \qquad \text{for} \quad 1 \le j \le m$$
$$\sigma_4(external face) = \sigma_2(external face)$$

and the resulting labeling is the super 7-antimagic labeling of type $(1, 1, 1)$. □

Theorem 11. [22] *The graph mD_n admits a super 6-antimagic labeling of type $(1, 1, 1)$ for $2 \le m \le 2n$ and $n \ge 3$, $n \ne 4$.*

Proof. We construct the labeling ψ_7 for the 4-sided faces and spokes of the middle part of the mD_n as follows

$$\psi_7(x_i^j y_i^j) = 2mn + 2(i-1)m + 2j - 1, \quad \text{for} \quad 1 \le i \le n, \ 1 \le j \le m$$

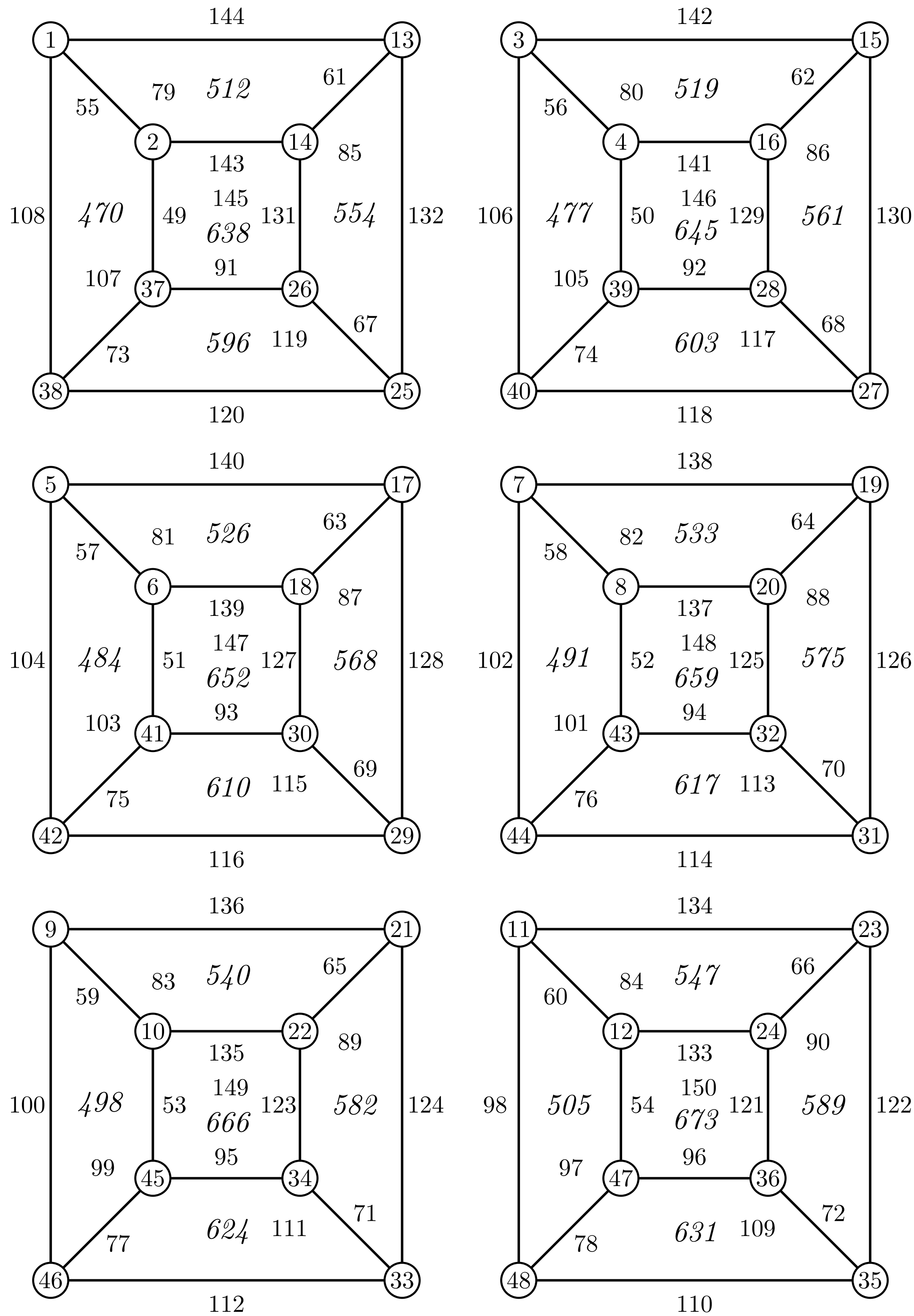

Figure 4.3: Super 7-antimagic labeling of type $(1, 1, 1)$ for $6D_4$.

$$\psi_7(z_{4,i}^j) = \begin{cases} 2(2n-i+1)m+2-2j, & \text{for} \quad 1 \le i \le n-1, \ 1 \le j \le m \\ 2(n+1)m+2-2j, & \text{for} \quad i=n, \ 1 \le j \le m. \end{cases}$$

Under the labeling $(\rho_3(\mathbb{C}_O), \phi_2(\mathbb{C}_I), \psi_7(\mathbb{C}_M))$, the weights of the 4-sided faces $z_{4,i}^j$, $1 \le i \le n$, $1 \le j \le m$, constitute the set $\{20mn+7, 20mn+13, \ldots, 26mn+1\}$ and all the weights of the n-sided faces have the same value $6mn^2$.

If we swap the vertex label $\phi_2(x_1^j) = 2j-1$ with the vertex label $\phi_2(x_1^{m-j+1}) = 2(m-j)+1$ and the face label $\psi_7(z_{4,n}^j) = 2(n+1)m+2-2j$ (respectively, $\psi_7(z_{4,1}^j) = 4mn+2-2j$) with the face label $\psi_7(z_{4,n}^{m-j+1}) = 2mn+2j$ (respectively, $\psi_7(z_{4,1}^{m-j+1}) = 2m(2n-1)+2j$), for every $1 \le j \le m$, then the weights of 4-sided faces will not be changed, but the weights of n-sided faces will constitute an arithmetic progression of the difference 4, namely $\{6mn^2+2m+2-4j : 1 \le j \le m\}$.

Only if $m \le 2n$ then for every k, $1 \le k \le m-1$, we swap the edge label $\rho_3(y_i^{m-k} y_{i+1}^{m-k}) = 6mn - 2(im-k)+2$ with the edge label $\phi_2(x_i^{m-k} x_{i+1}^{m-k}) = 6mn - 2(im-k)+1$, for $1 \le i \le \lfloor \frac{k+1}{2} \rfloor$, and the vertex label $\rho_3(y_i^{m-k}) = 2(im-k)$ with the vertex label $\phi_2(x_i^{m-k}) = 2(im-k)-1$, for $2 \le i \le \lceil \frac{k+1}{2} \rceil$. The applied swapping will result in changes of the face-weights of z_n^j to obtain an arithmetic sequence of the difference 5 for the weights of the n-sided faces.

If we complete the face labels

$$\sigma_5(z_n^j) = m(6n+1)+1-j, \qquad \text{for} \quad 1 \le j \le m$$

$$\sigma_5(external face) = m(6n+1)+1,$$

then the resulting labeling is required super 6-antimagic labeling of type $(1,1,1)$. $\square$

Chapter 5

Conclusion

In Chapter 3 we studied the super d-antimagic labelings for the antiprism A_n. We have shown that for $n \geq 4$ and $d \in \{0, 1, 2, 3, 4, 5, 6\}$ there exist super d-antimagic labelings of type $(1, 1, 1)$. According to Theorem 3, which gives the bound for the feasible values of the parameter d, we suggest the following open problem.

Open Problem 1. *Find other possible values of the parameter d and the corresponding super d-antimagic labelings of type $(1, 1, 1)$ for antiprism A_n.*

In Chapter 4 we have shown that for mD_n, $m \geq 2$, $n \geq 3$, $n \neq 4$ and $d \in \{0, 1, 2, 3, 4, 5\}$, there exists the super d-antimagic labeling of type $(1, 1, 1)$. We have tried to find a super d-antimagic labeling of type $(1, 1, 1)$ also for mD_4, $m \geq 2$ and $d \in \{0, 1, 2, 3, 4, 5\}$ but so far without success. Our method for describing a super d-antimagic labeling of type $(1, 1, 1)$ for mD_4 allows to obtain the required arithmetic sequence of the difference d only for the weights of 4-sided faces $z^j_{4,i}$, $1 \leq i \leq n$, $1 \leq j \leq m$, but not for the weights of the 4-sided faces $z^j_{n,1}$, $n = 4$, $1 \leq j \leq m$. So, we propose the following open problem for further investigation.

Open Problem 2. *For the graph mD_4 determine if there is a super d-antimagic labeling of type $(1, 1, 1)$, for every $m \geq 2$ and $d \in \{0, 1, 2, 3, 4, 5\}$.*

According to Theorem 7 we expect the existence of the other feasible values of the difference d. Therefore we propose the following open problem.

Open Problem 3. *Find other possible values of the difference d and the corresponding super d-antimagic labelings of type $(1, 1, 1)$ for mD_n, $m \geq 2$, $n \geq 3$ and $d > 7$.*

Appendix
Graph-theoretic symbols

$V(G)$	vertex set of G
$E(G)$	edge set of G
$F(G)$	face set of G
$\|V(G)\| = v$	order of G
$\|E(G)\| = e$	size of G
$\|F(G)\| = f$	number of faces of a plane graph G
$d(x)$	degree of vertex x (in G)
$N(x)$	set of all neighbors of the vertex x
$\delta(G)$	minimum degree of G
$\Delta(G)$	maximum degree of G
P_n	path on n vertices
C_n	cycle on n vertices
K_n	complete graph on n vertices
$K_{m,n}$	complete bipartite graph with partite sets of cardinalities m and n
$P(n,m)$	generalized Petersen graph
W_n	wheel on $n+1$ vertices
F_n	fan graph on $n+1$ vertices
$\mathbb{F}_n$	friendship graph on $2n+1$ vertices
mG	union of m disjoint copies of a graph G
$G \times H$	Cartesian product of graphs G and H
D_n	prism $C_n \times P_2$
A_n	antiprism
$H \subseteq G$	graph H is a subgraph of a graph G
$P_{m,n}$	parachute graph

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