# Homological and Combinatorial Properties of Monomial Ideals and Their Powers

Since 1864


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# Homological and Combinatorial Properties of Monomial Ideals and Their Powers

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in the partial fulfillment of the requirements for the award of degree of

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Certified that the research work contained in this thesis titled

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has been carried out and completed by Mr. **Adnan Aslam** Registration No. **12-GCU-PHD-SMS-07** under my supervision.

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## To Whom It May Concern

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Registration No. **12-GCU-PHD-SMS-07,** under my supervision.

The matter contained in this thesis is his original research work, and has already been published in prestigious referred journals of international repute. When material has been used from other sources it has been properly acknowledged. This thesis has been thoroughly checked by me, it is completed in all respects and is ready for submission.

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Date

Supervisor

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# HOMOLOGICAL AND COMBINATORIAL PROPERTIES OF MONOMIAL IDEALS AND THEIR POWERS

By

**Adnan Aslam**

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "**Homological and combinatorial properties of monomial ideals and their powers**" by **Adnan Aslam** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

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SCHOOL OF MATHEMATICAL SCIENCES

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Signature of Author

*To*

*my parents,*

*whom support, encouragement,*

*and fruitful prayers*

*made this*

*......*

*Possible*

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# Abstract

In this thesis we classify all unmixed monomial ideals $I$ of codimension 2 which are generically a complete intersection and which have the property that the symbolic power algebra $A(I)$ is standard graded. We give a lower bound for the highest degree of a generator of $A(I)$ in the case that $I$ is a modification of the vertex cover ideal of a bipartite graph, and show that this highest degree can be any given number. We furthermore give an upper bound for the highest degree of a generator of the integral closure of $A(I)$ in the case that $I$ is a monomial ideal which is generically a complete intersection.

Minh and Trung presented criteria for the Cohen-Macaulayness of a monomial ideal in terms of its primary decomposition. We extend their criteria to characterize the unmixed monomial ideals for which the equality $\operatorname{depth}(S/I) = \operatorname{depth}(S/\sqrt{I})$ holds. As an application we characterize all the pure simplicial complexes $\Delta$ which have rigid depth, that is, which satisfy the condition that for every unmixed monomial ideal $I \subset S$ with $\sqrt{I} = I_\Delta$ one has $\operatorname{depth}(I) = \operatorname{depth}(I_\Delta)$.

It is shown that a squarefree principal Borel ideal satisfies the persistence property for the associated prime ideals. For the graded maximal ideal we compute the index of stability with respect to squarefree principal Borel ideals and determine their stable set of associated prime ideals.

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# Introduction

Monomials ideals play an important role in studying the connections between commutative algebra and combinatorics. Problems in combinatorics are encoded into monomial ideals, which allow us to use techniques and methods in commutative algebra to solve the original combinatorial question. Stanley's proof of the Upper Bound Conjecture [36] for simplicial spheres is one of the early highlights of exploiting this connection between two fields. Square-free monomial ideals are used as a bridge between the two areas of mathematics. Over the last decade or so, commutative algebraists have become interested in studying the properties of finite simple graphs through monomial ideals. Fröberg [15], Villarreal [41], and Simis, Vasconcelos, and Villarreal [35] were among the early pioneers in this field. The starting point in these papers is to use the edges of the finite simple graph to construct a monomial ideal, usually called the edge ideal, and to study the properties of this monomial ideal using the properties of the graph, and vice versa. Monomial ideals and monomial algebras are studied in the books of Bruns-Herzog [6], Stanley [34], Hibi [26] and Villarreal [39] and Herzog-Hibi [18].

Since the monomial algebras defined by squarefree monomials of degree 2 have an underlying graph theoretical structure, it is natural that there is some interaction between monomial algebras, graph theory and polyhedra theory. Villareal [39] was the first to consider edge ideals associated to graphs, which are generated by monomials corresponding to the edges of the graph. Trees are the simplest graphs. Faridi [14], generalizes the notion of trees for simplicial complexes of any dimension. One can associate two squarefree monomial ideals with a simplicial complex

$\Delta$: the Stanley-Reisner ideal $I_\Delta$, generated by non-faces of $\Delta$, and the facet ideal $I(\Delta)$, generated the facets of $\Delta$. Stanley [34] work has shown that there is a deep relationship between the combinatorial properties of $\Delta$ and algebraic properties of $I_\Delta$.

In the first chapter we recall some basic notions concerning monomial ideals, Borel fixed ideals and simplicial complexes. This chapter is divided into two parts: in the first part, we define monomial ideals and recall some basic properties of monomial ideals, for instance, primary decomposition and symbolic powers of a monomial ideal are discussed and we describe some properties of symbolic powers of a monomial ideal which are studied in [39]. At the end of this section we define Borel fixed ideals and discuss some of the properties of strongly stable ideals, which can be found in [18].

The second part of the first chapter concerns with combinatorics. We define simplicial complexes and study the facet and Stanley-Reisner ideals associated to a simplicial complex. The Alexander dual of a simplicial complex plays an essential role in commutative algebra and combinatorics. One can compute the graded Betti numbers of the Stanley-Reisner ideal of a simplicial complex by using the well known Hochster's formula and it can also be used to prove a famous result by Eagon–Reiner [10].

In the second chapter, we first recall the definition of symbolic powers for monomial ideals. Classically one defines the symbolic power algebra of prime ideal $P$ in a Noetherian commutative ring $R$ as the graded algebra $\bigoplus_{k\geq 0} P^{(k)}t^k$ where $P^{(k)}$ is the $k$th symbolic power of $P$. In general, this algebra is not finitely generated, even though $R$ is Noetherian. Examples of this bad behavior have been given by Cowsik [7], Roberts [33] and others. On the other hand, it has been shown in [21] the symbolic power algebra of a monomial ideal is always finitely generated. In the case, that the monomial ideal $I$ is squarefree, it well understood when its symbolic power algebra $A(I)$ is standard graded, in other words, when the ordinary powers of $I$ coincide with the symbolic powers of $I$, see [21],[16], and [35].

The interest in symbolic powers of monomial ideals is partly due to the fact that symbolic powers of squarefree monomial ideals may be interpreted as vertex cover ideals of graphs and hypergraphs. This point of view has been stressed in the papers [21],[22] and [25]. On the other hand, not much is known about the generation of symbolic power algebras in the case that the monomial ideal is not squarefree. Chapter 2 is an attempt to better understand symbolic power algebras of non-squarefree monomial ideals. The situation in this more general case is much more complicated. While the symbolic power algebra of an unmixed squarefree monomial ideal of codimension 2 is generated in degree at most 2, the highest degree $d(A(I))$ of a generator of the algebra $A(I)$, where $I$ is an unmixed monomial ideal of codimension 2 can be any given number, as is shown in a family of examples in Section 2. In the first section we introduced some concepts and first recall a few facts about vertex cover algebras which will be needed in the following sections. The main result of Section 2 is Theorem 2.2.3 in which it is shown that the symbolic power algebra of monomial ideal $I$ of codimension 2 which is generically a complete intersection is standard graded if and only if $I$ is a trivial modification of a vertex cover ideal of a bipartite graph, which means that it is obtained from the vertex cover ideal simply by variable substitutions. In Section 3 we give a general lower bound for the number $d(A(I))$ in the case that $I$ is a modification of a vertex cover ideal of a bipartite graph. To give an upper bound for $d(A(I))$ is much harder. However, in Section 4, an explicit upper bound for the highest degree of a generator of the integral closure $\overline{A(I)}$ of $A(I)$ is given when $I$ is a monomial ideal which is an arbitrary generically complete intersection with no condition on the codimension. It remains an open question how the numbers $d(A(I))$ and $d(\overline{A(I)})$ are related to each other. The results described here are the content of a forthcoming paper [1].

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over a field $K$ and $I \subset S$ a monomial ideal. In [24], the authors compare the properties of $I$ with the properties of its radical by using the inequality $\beta_i(I) \geq \beta_i(\sqrt{I})$. In particular, from the inequality between the Betti numbers, one gets the inequality $\operatorname{depth}(S/I) \leq \operatorname{depth}(S/\sqrt{I})$, which implies, for instance, that $S/I$ is Cohen-Macaulay if $S/\sqrt{I}$ is so. In [30], the authors presented criteria for the Cohen-Macaulayness of a monomial ideal in terms of its primary decomposition. In Chapter 3 we extend their criteria to characterize the unmixed monomial ideals for which the equality $\operatorname{depth}(S/I) = \operatorname{depth}(S/\sqrt{I})$ holds. We recall that an ideal $I \subset S$ is *unmixed* if the associated prime ideals of $S/I$ are the minimal prime ideals of $I$.

Let $\Delta$ be a pure simplicial complex with the facet set denoted, as usual, by $\mathcal{F}(\Delta)$, and let $I_\Delta = \bigcap_{F\in\mathcal{F}(\Delta)} P_F$ be its Stanley-Reisner ideal. For any subset $F \subset [n]$, we denoted by $P_F$ the monomial prime ideal generated by the variables $x_i$ with $i \notin F$. Let $I \subset S$ be an unmixed monomial ideal such that $\sqrt{I} = I_\Delta$ and assume that $I = \bigcap_{F\in\mathcal{F}(\Delta)} I_F$ where $I_F$ is the $P_F$-primary component of $I$. Following [30], for every $\mathbf{a} \in \mathbb{N}^n$, $\mathbf{a} = (a_1, \ldots, a_n)$, we set $\mathbf{x}^\mathbf{a} = x_1^{a_1} \cdots x_n^{a_n}$ and denote by $\Delta_\mathbf{a}$ the simplicial complex on the set $[n]$ with the facet set $\mathcal{F}(\Delta_\mathbf{a}) = \{F \in \mathcal{F}(\Delta) \mid \mathbf{x}^\mathbf{a} \notin I_F\}$. Moreover, for every simplicial complex $\Gamma$ with $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)$, we set

$$L_\Gamma(I) = \{\mathbf{a} \in \mathbb{N}^n \mid \mathbf{x}^\mathbf{a} \in \bigcap_{F\in\mathcal{F}(\Delta)\setminus\mathcal{F}(\Gamma)} I_F \setminus \bigcup_{G\in\mathcal{F}(\Gamma)} I_G\}.$$

In Section 3.1 of Chapter 3, we prove the following theorem which will appear in a joint paper with Ene [2] and which is a natural extension of Theorem 1.6 in [30].

**Theorem 0.0.1.** *Let $\Delta$ be a pure simplicial complex with $\operatorname{depth} K[\Delta] = t$. Let $I \subset S$ be an unmixed monomial ideal with $\sqrt{I} = I_\Delta$. Then the following conditions are equivalent:*

*(a)* $\operatorname{depth}(S/I) = \operatorname{depth}(S/\sqrt{I})$,

*(b)* $\operatorname{depth} K[\Delta_\mathbf{a}] \geq t$ *for all* $\mathbf{a} \in \mathbb{N}^n$,

*(c)* $L_\Gamma(I) = \emptyset$ *for every simplicial complex* $\Gamma$ *with* $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)$ *and* $\operatorname{depth} K[\Gamma] < t$.

As a main application of the above theorem we study in Section 3.2 a special class of simplicial complexes. We say that a pure simplicial complex has *rigid depth* if for every unmixed monomial ideal $I \subset S$ with $\sqrt{I} = I_\Delta$ one has $\operatorname{depth}(S/I) = \operatorname{depth}(S/I_\Delta)$. In Theorem 3.2.3 which extends [24, Theorem 3.2], we give necessary and sufficient conditions for $\Delta$ to have rigid depth. In particular, from this characterization, it follows that if a pure simplicial complex has rigid depth over a field of characteristic 0, then it has rigid depth over any field. In the last part we discuss the behavior of rigid depth in connection to the skeletons of the simplicial complex.

In recent years the stable set of monomial ideals has been studied in various papers, see for example [23],[31], [32] and [40]. By Brodmann [5], for any graded ideal $I$ in the polynomial ring $S$ there exists an integer $k_0$ such that $\mathrm{Ass}(I^k) = \mathrm{Ass}(I^{k+1})$ for $k \geq k_0$. The smallest integer $k_0$ with this property is called the index of stability of $I$ and $\mathrm{Ass}(I^{k_0})$ is called the set of stable prime ideals of $I$. A prime ideal $P \in \bigcup_{k\geq 1} \mathrm{Ass}(I^k)$ is said to be persistent with respect to $I$ if whenever $P \in \mathrm{Ass}(I^k)$ then $P \in \mathrm{Ass}(I^{k+1})$, and the ideal $I$ is said to satisfy the persistence property if all prime ideals $P \in \bigcup_{k\geq 1} \mathrm{Ass}(I^k)$ are persistent. It is an open question whether all squarefree monomial ideal satisfy the persistence property.

Chapter 4 deals with the content of the paper [3]. We show that any squarefree principal Borel ideal satisfies the persistence property. The strategy of the proof is the same as that applied in [23]. Indeed, by using a result of De Negri [9] it follows that all powers of a squarefree principal Borel ideal have linear resolutions. This fact together with the result that all monomial localizations of a squarefree principal Borel ideal are again squarefree principal Borel ideals, as shown in Theorem 4.1.2, we conclude in Section 4.1 that this class of ideals satisfies the persistence property, see Corollary 4.1.3.

In Section 4.2 we answer the question for which squarefree principal Borel ideals the graded maximal ideal of $S$ is a stable associated prime ideal. The answer is given in Theorem 4.2.1. For the proof of this result we use a formula about linear quotients from [4, Section 2]. Then in the following Theorem 4.2.2 we give a precise formula for the index of stability of the graded maximal ideal with respect to the given squarefree principal Borel ideal. As a consequence of this formula we show in Corollary 4.2.3 that this index stability is bounded above by the degree of the generators of the

Borel ideal. Finally in the last Section 4.3 we describe in Theorem 4.3.2 the stable set of associated prime ideals for any squarefree principal Borel ideal and conclude the paper with an explicit example demonstrating this theorem.

# Chapter 1

# Preliminaries

## 1.1 Some monomial algebra

### 1.1.1 Basic concepts

In this section, we will recall some basic notions of monomial ideals. A comprehensive monograph on monomial ideals is the book [18]. For general results in the ideal theory we refer to the well known monographs [6], [8], [39].

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over a field $K$. Any product of the form $x_1^{a_1} \ldots x_n^{a_n}$ with $a_i \in \mathbb{Z}_+$ is called a ***monomial***. We often write the monomial $u = x_1^{a_1} \ldots x_n^{a_n}$ in the form $u = \mathbf{x}^{\mathbf{a}}$ with $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{Z}_+^n$.

The set of all monomials of $S$, denoted by $\mathrm{Mon}(S)$ forms a canonical $K$-basis of $S$. Any polynomial $f \in S$ can be written as unique $K$-linear combination of monomials.

$$f = \sum_{u \in \mathrm{Mon}(S)} a_u u \quad \text{with } a_u \in K.$$

We call the set

$$\mathrm{supp}(f) = \{u \in \mathrm{Mon}(S) \,:\, a_u \neq 0\}$$

the ***support*** of $f$.

**Definition 1.1.1.** An ideal $I \subset S$ is called a ***monomial ideal*** if it is generated by monomials.

An important property of monomial ideals is given in the following:

**Proposition 1.1.2.** [18] *Each monomial ideal has a unique minimal monomial set of generators.*

We usually denote the unique minimal set of monomial generators of the monomial ideal $I$ by $G(I)$.

**Definition 1.1.3.** Let $R$ be a commutative ring. The height of a prime ideal $P$, denoted by height$(P)$, is the supremum of all $n$ such that there is a chain $P_0 \subset P_1 \subset \ldots \subset P_n$, where all $P_i$ are distinct prime ideals. Then, ***Krull dimension*** of $R$, denoted by $\dim R$, is defined as supremum of all the heights of all its prime ideals.

Note that $\text{height}(P) = \dim(R_P)$, where $R_P$ is the localization of $R$ in $P$. If $I$ is an ideal of $R$, then the ***height*** of $I$, denoted by height$(I)$, is defined as

$$\text{height}(I) = \inf\{\text{height}(P) \mid I \subset P \text{ and } P \in Spec(R)\}.$$

In general, we have the following inequality,

$$\text{height}(I) + \dim(R/I) \leq \dim(R).$$

The difference of $\dim(R)$ and $\dim(R/I)$ is called the ***codimension*** of $I$.

Let $M$ be an $R$-module. The ***annihilator*** of $M$, denoted $\text{Ann}_R(M)$, is the set of all elements $r$ in $R$ such that $rm = 0$ for each $m$ in $M$. Recall that the ***dimension*** $M$ is defined as

$$\dim(M) = \dim(R/\operatorname{Ann}_R(M))$$

and the ***codimension*** of $M$ is

$$\text{codim}(M) = \dim(R) - \dim(M).$$

**Definition 1.1.4.** Let $I$ be an ideal of a ring $R$. The ***radical*** of $I$, denoted by $\sqrt{I}$ is

$$\sqrt{I} = (x \in R : x^n \in I \text{ for some } n > 0).$$

In particular, $\sqrt{(0)}$, is the set of ***nilpotent*** elements of $R$ and is called as the ***nilradical*** of $R$. It is often denoted by $\mathcal{N}_R$. A ring is ***reduced*** if its nilradical is zero.

**Proposition 1.1.5.** [39] *If $I$ is a proper ideal of a ring $R$, then the $\sqrt{I}$ is the intersection of all prime ideals containing $I$.*

### 1.1.2 Primary decomposition and associated primes

Let $R$ be a ring, $M$ a non-zero $R$- module, and $P$ a prime ideal of $R$. $P$ is called an ***associated prime*** of $M$ if there exists $m \in M$ such that $P = (0 :_R m) = \{r \in R \mid rm = 0\}$. The set of all associated primes of $M$ is denoted by $\mathrm{Ass}_R(M)$. We are mainly interested in the following situation. The ring is $S = K[x_1, \ldots, x_n]$, and the module is $S/I$, where $I \subset S$ is a homogenous ideal. In this case we denote $\mathrm{Ass}(S/I)$ simply by $\mathrm{Ass}(I)$. Since in the graded module $M$ over a graded ring $R$ the element $m \in M$ such that $P = (0 :_R m)$ may be taken to be homogenous, it follows that, in this case, all the associated primes of $M$ are graded. In particular, if $I \subset S$ is a monomial ideal, then $M = S/I$ is a $\mathbb{Z}^n$ graded $S$-module and all the associated primes are $\mathbb{Z}^n$ graded as well. Consequently, if $I \subset S$ is a monomial ideal, then $\mathrm{Ass}(I)$ consists of monomial prime ideals, that is, ideals generated by variables.

The main results on associated primes are collected below. For a comprehensive study of associated primes and primary decomposition we refer to the monograph of Eisenbud [8].

Let $R$ be a (graded) Neotherian ring and $M$ a (graded) finite $R$-module. Then

(1) $\mathrm{Ass}_R(M)$ is non-empty, finite, and contains all the minimal primes of $\mathrm{Ann}_R(M) = \{r \in R \mid rm = 0, \text{ for all } m \in M\}$.

Note that in the case $R = S = K[x_1, \ldots, x_n]$ and $M = S/I$, where $I \subset S$ is a homogenous ideal, we have $\mathrm{Ann}_S(M) = I$, hence $\mathrm{Ass}(I)$ contain $\mathrm{Min}(I)$, the set of all minimal primes of $I$.

(2) Let $Z(M) = \{r \in R \mid \text{ there exists } m \in M, m \neq 0, \text{ such that } rm = 0\}$ be the set of all zero divisors on $M$. Then

$$\operatorname{Ass}_R(M) = Z(M) \cup \{0\}.$$

(3) If $U \subset R$ is a multiplicative closed set, then

$$\operatorname{Ass}_{U^{-1}R}(U^{-1}M) = \{U^{-1}P \mid P \in \operatorname{Ass}_R(M), P \cap U = \emptyset\}.$$

(4) Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of (graded) $R$-modules. Then

$$\operatorname{Ass}(M') \subset \operatorname{Ass}(M) \subset \operatorname{Ass}(M') \cup \operatorname{Ass}(M'').$$

Let $M$ be an $R$- module and $N \subset M$ a submodule. $N$ is called ***$P$-primary*** if $\operatorname{Ass}(M/N) = \{P\}$. In particular, an ideal $Q \subset R$ is $P$-primary if $\operatorname{Ass}(R/Q) = \{P\}$. One may show that $Q \subset R$ is primary ideal if and only if for any $x, y \in R$, either $x \in Q$ or $y^n \in Q$ for some $n \geq 1$. In other words, if $x \notin Q$, then $y^n \in Q$ for some $n \geq 1$, which means that every zero divisor on $R/Q$ is a nilpotent element modulo $Q$.

**Theorem 1.1.6.** [8] *Let $R$ be a (graded) Noetherian ring and let $N \subset M$ be a submodule. Then $N$ has a decomposition of the form*

$$N = N_1 \cap \cdots \cap N_r \tag{1.1.1}$$

*with the following properties*

*(a) (irredundunce) $N \neq N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_r$, for every $1 \leq i \leq r$.*

*(b) for every $1 \leq i \leq r$, $\operatorname{Ass}(M/N_i) = \{P_i\}$ where $P_i$ is a prime ideal, that is, $N_i$ is a $P_i$-primary module for all $i$.*

*(c)* $P_i \neq P_j$ *if* $i \neq j$*.*

*In this case, we also have* $\mathrm{Ass}(M/N) = \{P_1, \ldots, P_r\}$*.*

*A decomposition (1.1.1) which satisfies* $(a), (b), (c)$ *is called an* **irredundant primary decomposition** *of* $N$*.*

In particular, if $I \subset S$ is a homogenous ideal, we may read the associated ideals of $I$ from an irredundant primary decomposition of $I$. We now take a closer look at monomial ideals (see [18]). A monomial ideal $I \subset S$ is called ***irreducible*** if for any monomial ideals $I_1, I_2 \subset S$ such that $I = I_1 \cap I_2$, we have $I = I_1$ or $I = I_2$. A monomial ideal is irreducible if and only if it is generated by powers of variables (see [18])

**Proposition 1.1.7.** [18] *The irreducible monomial ideal* $Q = (x_1^{a_1}, \ldots, x_r^{a_r})$ *is* $(x_1, \ldots, x_r)$*-primary*

**Proposition 1.1.8.** [18] *Let* $I \subset S$ *be a monomial ideal. Then* $I$ *may be written as an intersection of irreducible monomial ideals.*

Let $I = Q_1 \cap \cdots \cap Q_m$ be such a decomposition of monomial ideal $I$. We may assume that it is irredundant. Since the intersection of $P$-primary ideals is a $P$-primary ideal as well, we can collect all the $Q_i's$ which have the same radical in the decomposition of $I$ and get an irredundant primary decomposition of $I$ from where we may find all the associated primes of $I$. For instance, $I = (x_1^2 x_2, x_1 x_2^2) \subset K[x_1, x_2]$ has the irredundant primary decomposition

$$I = (x_1) \cap (x_2) \cap (x_1^2, x_2^2).$$

Note that $(x_1)$ and $(x_2)$ are minimal primes of $I$, while $(x_1, x_2)$ is a so called ***embedded prime ideal*** of $I$.

In particular, by the above discussion, it follows that all the associated prime ideals of $I$ are monomial prime ideals.

### 1.1.3 Cohen–Macaulay rings and modules

After dimension, depth is the most fundamental invariant of a Noetherian local ring $R$ or a finite $R$-module $M$. Depth is defined in terms of regular sequences.

#### Regular sequences

Let $R$ be a commutative ring and $M$ an $R$-module. An element $0 \neq r \in R$ is called ***$M$-regular*** if $r$ is not a zero divisor on $M$, and $M/rM \neq 0$. A ***regular sequence*** on $M$ is a sequence $r_1, \ldots, r_t$ in $R$ such that for each $i \leq t$, $r_i$ is $M/(r_1, \ldots, r_{i-1})M$-regular, where $M/(r_1, \ldots, r_{i-1})M$ is the quotient $R$-module. Such a sequence is also called an ***$M$-sequence***. An $R$-regular sequence is called simply a ***regular sequence***. Note that, after permutation, the elements of an $M$ sequence may no longer form an $M$ sequence. However, if $R$ is a (graded) local ring and all $r_i$ are (homogeneous) elements in the maximal ideal of $R$, then a sequence is an $R$-sequence only if every permutation of it is an $R$-sequence.

**Example 1.1.1.** Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over a field $K$. $x_1, \ldots, x_n$ is an $S$-sequence.

#### Depth of a module

Let $R$ be a Noetherian ring and $M$ be an $R$-module. An $M$-sequence $x = x_1, x_2, \ldots, x_n$ in an ideal $I$ is called a maximal $M$-sequence if any $y \in I$ is a zero divisor on $M/xM$. If $x = x_1, x_2, \ldots, x_n$ is an $M$-sequence in $I$, then we have a strictly ascending sequence of ideals:

$$(x_1) \subseteq (x_1, x_2) \subseteq \cdots \subseteq (x_1, x_2, \ldots, x_n) \subseteq \cdots \subset I.$$

Thus this sequence must terminate because $R$ is Noetherian. Hence any $M$-sequence can be extended to a maximal one.

**Theorem 1.1.9** (Rees). [6] *Let $R$ be a Noetherian ring, $M$ a finite $R$-module, and $I$ an ideal such that $IM \neq M$. Then all maximal $M$-sequences in $I$ have the same* *length n given by*

$$n = \min\{i \ : \ \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

**Definition 1.1.10.** Let $(R, \mathfrak{m})$ be a Noetherian local ring, and $M$ a finite $R$-module. The common length of all maximal $M$-sequences in $\mathfrak{m}$ is called the ***depth*** of $M$ and it will be denoted $\operatorname{depth}(M)$.

By theorem 1.1.9, $\operatorname{depth} M = \min\{i \mid \operatorname{Ext}^i_R(K, M) \neq 0\}$, where $K$ is the residue field of $R$. Let $S = K[x_1, x_2, \ldots, x_d]$, where $K$ is a field and $x_1, x_2, \ldots, x_n$ are indeterminates, then $x_1, x_2, \ldots, x_n$ form a regular sequence of length $n$ and there are no longer $S$-sequences, so $S$ has depth $n$.

**Proposition 1.1.11.** *(Depth's Lemma)*[6] *Let*$(R, \mathfrak{m})$ *be a local Noetherian ring and*

$$0 \to U \to M \to N \to 0$$

*an exact sequence of finitely generated R-modules. Then:*

*(a)* $\operatorname{depth}(M) \geq \min\{\operatorname{depth}(U), \operatorname{depth}(N)\}$*;*

*(b)* $\operatorname{depth}(U) \geq \min\{\operatorname{depth}(M), \operatorname{depth}(N) + 1\}$*;*

*(c)* $\operatorname{depth}(N) \geq \min\{\operatorname{depth}(U) - 1, \operatorname{depth}(M)\}$*;*

**Proposition 1.1.12.** [6] *Let* $(R, \mathfrak{m})$ *be a Noetherian (graded) local ring and M a (graded) finitely generated R-module. Then* $\operatorname{depth}(M) \leq \dim(R/p)$ *for all* $p \in \operatorname{Ass}(M)$*: In particular we get* $\operatorname{depth}(M) \leq \dim(M)$*.*

### Cohen–Macaulay rings and modules

Let $R$ be a Noetherian local ring and $M$ a finite $R$-module. $M$ is a Cohen–Macaulay module if its depth is equal to its Krull dimension. A ring $R$ is said to be Cohen–Macaulay if it is a Cohen–Macaulay module over itself. Next theorem characterize some basic properties of Cohen–Macaulay modules.

**Theorem 1.1.13.** [39] *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* $M \neq 0$ *a Cohen–Macaulay R-module. Then*

*(a)* $\dim(R/P) = \operatorname{depth}(M)$ *for all* $P \in \operatorname{Ass}(M)$.

*(b)* $\mathbf{x} = x_1, \ldots, x_r$ *is an* $M$*-sequence if and only if* $\dim(M/\mathbf{x}M) = \dim(M) - r$.

*(c)* $\mathbf{x}$ *is an* $M$*-sequence if and only if it is part of a system of parameters of* $M$.

**Definition 1.1.14.** An ideal $I$ of a ring $R$ is ***unmixed*** if the associated primes of S/I are the minimal primes and have all the same height.

**Theorem 1.1.15.** [39] *A Noetherian ring* $R$ *is Cohen–Macaulay if and only if every ideal* $I$ *generated by* $\operatorname{height}(I)$ *elements is unmixed.*

### 1.1.4 Symbolic powers of monomial ideal

In this section we define the symbolic powers of an ideal and give some basic results about symbolic powers of a monomial ideal that can be found in [39]. Throughout this section $R$ denotes a Noetherian ring.

**Definition 1.1.16.** Let $I \subset R$ be an ideal of a ring $R$ and $p_1, \ldots, p_r$ the minimal primes of $I$. Given an integer $m \geq 1$, the $m$-th ***symbolic power*** of $I$ is defined to be the ideal $I^{(m)} = q_1 \cap q_2 \cap \cdots \cap q_r$, where $q_j$ is the primary component of $I^m$ corresponding to $p_j$.

**Proposition 1.1.17.** [39] *Let* $I$ *be a proper ideal of a ring* $R$ *and* $S = R \setminus \cup_{i=1}^{k} p_i$*, where* $p_1, p_2, \ldots, p_k$ *are the minimal primes of* $I$*. Then*

$$I^{(m)} = S^{-1} I^m \cap R \quad \textit{for} \quad m \geq 1.$$

**Proposition 1.1.18.** [39] *Let* $I$ *be a radical ideal of a ring* $R$ *and* $p_1, \ldots, p_k$ *the minimal primes of* $I$*. Then*

$$I^{(m)} = p_1^{(m)} \cap p_2^{(m)} \cap \cdots \cap p_k^{(m)} \quad \textit{for} \quad m \geq 1.$$

A very interesting problem is to describe when the symbolic and ordinary powers of a given ideal $I$ coincide; see [27] and [29]. There are a few cases where equality of symbolic and ordinary powers can be described in terms of properties of the associated graded ring.

**Definition 1.1.19.** An ideal $I$ of a ring $R$ is called ***normally torsion free*** if $\mathrm{Ass}(R/I^k)$ is contained in $\mathrm{Ass}(R/I)$ for all $k \geq 1$ and $I \neq R$.

Let $R$ be a neotherian ring and $I \subset R$ be an ideal of $R$. M.Brodmann [5] showed that $\mathrm{Ass}(I^k) = \mathrm{Ass}(I^{k+1})$ for all $k \gg 0$. If $I$ is a radical ideal which is normally torsion free, then $\mathrm{Ass}(R/I^k) = \mathrm{Ass}(R/I)$ for all $k \geq 1$. Hence in this case the notion of normally torsion free is a strong form of stability.

The following results presents classes of ideals whose symbolic powers coincide with the ordinary powers.

**Proposition 1.1.20.** [39] *Let $I \subset R$ be an ideal of a ring $R$. If $I$ has no embedded primes, then $I$ is normally torsion free if and only if $I^m = I^{(m)}$ for all $m \geq 1$.*

**Proposition 1.1.21.** [39] *Let $R$ be Cohen-Macaulay ring and $I$ be an ideal of $R$. If $I$ is generated by a regular sequence, then $I^m = I^{(m)}$ for $m \geq 1$.*

**Definition 1.1.22.** A proper ideal $I \subset R$ of a ring $R$ is said to be ***Locally a complete intersection*** if $IR_p$, is a complete intersection for all $p \in V(I)$.

**Proposition 1.1.23.** [39] *Let $R$ be a Cohen-Macaulay ring and $I$ a prime ideal. If $I$ is locally a complete intersection, then $I^m = I^{(m)}$ for all $m \geq 1$.*

**Proposition 1.1.24.** [39] *Let $p$ be a prime ideal of a ring $R$ such that $pR_p$, is a complete intersection, then $p^{(m)} = p^m$ for all $m \geq 1$ if and only if $gr_p(R)$ is a domain.*

### 1.1.5 Borel-fixed monomial ideals

In this section, we will recall some basic facts about Borel-fixed (In particular strongly stable) monomial ideals which can be found in [18].

The subgroup $\mathcal{B} \subset GL_n(K)$ of all nonsingular upper triangular matrices is called the ***Borel subgroup*** of $GL_n(K)$. A matrix $A = (a_{ij}) \in \mathcal{B}$ is called an upper elementary matrix, if $a_{ii} = 1$ for all $i$ and if $a_{kl} \neq 0$ for some integers $1 \leq k < l \leq n$ and $a_{ij} = 0$ for all $i \neq j$ then $\{i, j\} \neq \{k, l\}$. The subgroup $\mathcal{D} \subset \mathcal{B}$ of the set of all upper elementary matrices and all nonsingular diagonal matrices generate $\mathcal{B}$.

**Definition 1.1.25.** A graded monomial ideal $I \subset S$ is called ***Borel-fixed*** if it is fixed under the action of $\mathcal{B}$.

If $\operatorname{char} K = 0$, then the Borel-fixed ideals can be easily characterized.

**Definition 1.1.26.** Let $I \subset S$ be a monomial ideal. Then $I$ is called ***strongly stable*** if for all monomials $u \in I$ and all $i < j$ such that $x_j$ divides $u$, one has $x_i(u/x_j) \in I$.

The property of a strongly stable ideal can be checked only for the set of monomial generators of a monomial ideal.

**Lemma 1.1.27.** [18] *Let $I$ be a monomial ideal. Then $I$ is strongly stable if for all $u \in G(I)$, and for all integers $1 \leq i < j \leq n$ such that $x_j$ divides $u$ one has $x_i(u/x_j) \in I$.*

Let $S = K[x_1, x_2, x_3]$ and $I = (x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3)$, then $I$ is a strongly stable ideal.

**Proposition 1.1.28.** [18]

*(a) Let $I \subset S$ be a graded ideal. Then $I$ is a monomial ideal, if $I$ is Borel-fixed.*

*(b) Let $I$ be a Borel-fixed ideal and $a$ the largest exponent appearing among the monomial generators of $I$. If $\operatorname{char} K = 0$ or $\operatorname{char} K > a$, then $I$ is strongly stable.*

*(c) If $I$ is strongly stable, then $I$ is Borel-fixed.*

**Definition 1.1.29.** a monomial ideal $I \subset S$ is called ***stable*** if for all monomials $u \in I$ and all $i < m(u)$ one has $x_i u/x_{m(u)} \in I$, where $m(u)$ denotes the largest index $j$ such that $x_j$ divides $u$.

Let $S = K[x_1, x_2, x_3, x_4]$ and $I = (x_1^2, x_1x_3, x_1x_2, x_2^2, x_2x_3, x_3^2, x_3x_4)$, then $I$ is a stable ideal.

**Projective Dimension, Betti numbers, Regularity**

Let $S = K[x_1, \dots, x_n]$ and $M$ a graded finitely generated $S$-module. Then $M$ has a finite minimal graded free resolution:

$$F_\bullet : 0 \to F_p \to \cdots \to F_1 \to F_0 \to M \to 0$$

where $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$ for $i \geq 0$.

The numbers $\beta_{ij}$ are uniquely determined by $M$. They are called the ***graded Betti numbers*** of $M$.

The ***projective dimension*** of $M$ is

$$\operatorname{proj\,dim} M = \max\{i : \beta_{ij} \neq 0 \text{ for some } j\}.$$

The ***regularity*** of $M$ is

$$\operatorname{reg} M = \max\{j : \beta_{i,i+j} \neq 0 \text{ for some } i\}.$$

By the theorem of Auslander-Buchsbaum, one has

$$\operatorname{depth} M + \operatorname{proj\,dim} M = n,$$

therefore

$$\operatorname{depth} M = \min\{i : \beta_{n-i,j} \neq 0 \text{ for some } j\}$$

**Theorem 1.1.30.** *(Eliahou.Kervaire)*[18] *Let $I \subset S$ be a stable ideal. Then*

*(a)* $\beta_{i,i+j}(I) = \sum_{u \in G(I)_j} \binom{m(u)-1}{i}$

*(b)* $\operatorname{proj\,dim} S/I = \{\max m(u) : u \in G(I)\}$*;*

*(c)* $\operatorname{reg}(I) = \{\max \deg(u) : u \in G(I)\}$.

*where $m(u)$ is the maximal number $j$ such that $x_j \mid u$.*

**Definition 1.1.31.** A monomial ideal $I$ is called ***squarefree strongly stable***, if all monomials $u \in G(I)$ are of the form $u = x_{i_1}x_{i_2}\cdots x_{i_r}$ with $1 \le i_1 < i_2 < \cdots < i_r \le n$, and for all $u \in G(I)$ and all integers $1 \le i < j \le n$ such that $x_j$ divides $u$ and $x_i \nmid u$ we have that $x_i(u/x_j) \in G(I)$.

Let $S = K[x_1, x_2, x_3, x_4]$ and $I = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4)$, then $I$ is a squarefree strongly stable ideal.

### 1.1.6 Cohen–Macaulay graphs

In this section we introduce the so-called ***edge ideal*** of a graph and recall some results of Cohen–Macaulay graphs. Let $I$ be an ideal of $S$. If the quotient ring $S/I$ is Cohen–Macaulay, then we say $I$ is a ***Cohen–Macaulay ideal***.

In this section we consider only simple graphs, that is, undirected, without loops and without multiple edges. Let $G$ be a graph on a vertex set $V(G) = \{1, \ldots, n\}$ with edge set $E(G)$. Let $S = K[x_1, \ldots, x_n]$ the polynomial ring in $n$ variables over the field $K$.

**Definition 1.1.32.** The ***edge ideal*** $I(G)$ of a graph $G$ is the ideal of $S$ generated by the monomials $x_ix_j$ such that $\{i, j\} \in E(G)$.

If all the vertices of $G$ are isolated, we set $I(G) = 0$. Since $G$ has no loop, the edge ideal $I(G)$ is a squarefree monomial ideal generated in degree 2.

**Definition 1.1.33.** A graph $G$ is said to be ***Cohen–Macaulay*** if the edge ideal $I(G)$ is a Cohen–Macaulay ideal.

**Remark 1.1.34.** In general, the Cohen–Macaulay property of a graph $G$ depends on the field $K$. For example, if we consider the ideal $I$ of a ring $R = K[x_1, \ldots, x_{11}]$,

$$I = (x_1x_3, x_1x_4, x_1x_7, x_1x_{10}, x_1x_{11}, x_2x_4, x_2x_5, x_2x_8, x_2x_{10}, x_2x_{11}, x_3x_5, x_3x_6,$$

$x_3x_8, x_3x_{11}, x_4x_6, x_4x_9, x_4x_{11}, x_5x_7, x_5x_9, x_5x_{11}, x_6x_8, x_6x_9, x_7x_9, x_7x_{10}, x_8x_{10})$

then $\dim(R/I) = 3$. If the field $K$ is of characteristic zero, then the resolution of $R/I$ over $R$ is

$$0 \to R^{11} \to R^{80} \to R^{245} \to R^{406} \to R^{396} \to R^{247} \to R^{105} \to R^{25} \to R \to R/I \to 0,$$

so $I$ is Cohen-Macaulay, and if the field $K$ has characteristic 2, the resolution of $R/I$ over $R$ is

$$0 \to R \to R^{12} \to R^{80} \to R^{245} \to R^{406} \to R^{396} \to R^{247} \to R^{105} \to R^{25} \to R \to R/I \to 0,$$

so $I$ is not Cohen-Macaulay. This example is due to N.Terai.

**Definition 1.1.35.** Let $G$ be a finite graph on the vertex set $[n]$. A subset $C \subset [n]$ is called a ***vertex cover*** of $G$ if $C \cap \{i, j\} \neq \emptyset$ for all edges $\{i, j\} \in E(G)$.

The next result establishes a one to one correspondence between the minimal vertex covers of a graph and the minimal primes of the corresponding edge ideal.

**Proposition 1.1.36.** [39] *Let $G$ be a graph over the vertex set $V(G) = [n]$ and $S = K[x_1, \dots, x_n]$ a polynomial ring over a field $K$. If $P$ is an ideal generated by $\{x_{i_1}, \dots, x_{i_s}\}$, then the following conditions are equivalent:*

1. *$P$ is a minimal prime of $I(G)$;*

2. *$\{i_1, \dots, i_s\}$ is a minimal vertex cover of $G$.*

**Definition 1.1.37.** A graph $G$ is ***unmixed*** if all minimal vertex covers of $G$ have same cardinality.

Since any Cohen–Macaulay ideal in a polynomial ring is unmixed, we have:

**Corollary 1.1.38.** [39] *If $G$ is a Cohen–Macaulay graph, then $G$ is unmixed.*

The class of Cohen–Macaulay graphs is huge. In [39], Villarreal gave several constructions of Cohen–Macaulay graphs. In particular, he gave the following effective description of Cohen–Macaulay trees and presented an interesting family of graphs containing all Cohen–Macaulay trees.

**Theorem 1.1.39** (Villarreal)**.** *Let $T$ be a tree with vertex set $V(G)$ and edge set $E(G)$. Then $T$ is Cohen–Macaulay if and only if $|V| \leq 2$ or $2 < |V| = 2r$ and there are vertices $a_1, \ldots, a_r, b_1, \ldots, b_r$ so that $\deg(b_i) \geq 2$, and $\{a_i, b_i\} \in E(G)$ for $i = 1, \ldots, r$.*

Recently, Herzog and Hibi classified all Cohen–Macaulay bipartite graphs in [18] by using the Alexander dual of some special simplicial complex. Their main result is

**Theorem 1.1.40** (Herzog-Hibi)**.** *Let $G$ be a bipartite graph with vertex set $V(G)$ and edge set $E(G)$. Then $G$ is Cohen–Macaulay if and only if after a suitable labeling of the vertices the following conditions hold:*

- $V = V_1 \cup V_2$ *where* $V_1 = \{x_1, \ldots, x_n\}$ *and* $V_2 = \{y_1, \ldots, y_n\}$;
- $\{x_i, y_i\} \in E(G)$ *for all* $i \in [n]$;
- *if* $\{x_i, y_i\} \in E(G)$, *then* $i \leq j$;
- *if* $\{x_i, y_j\}, \{x_j, y_k\} \in E(G)$, *with* $i < j < k$, *then* $\{x_i, y_k\} \in E(G)$.

## 1.2 Stanley-Reisner and facet ideal of a simplicial complex

In this section we introduce the concept of simplicial complex and some ideal such as Stanley-Reisner ideal and facet ideal. We also list some well known results of combinatorial commutative algebra, for example, Hochster's formula and Eagon-Reiner theroem.

### 1.2.1 Simplicial complexes and Stanley-Reisner rings

Here we recall the definition of simplicial complex and study two squarefree monomial ideals (Stanley-Reisner and facet ideals) associated to a simplicial complex.

**Definition 1.2.1.** Let $[n] = \{1, ..., n\}$ be the vertex set and $\Delta$ a ***simplicial complex*** on $[n]$. Then $\Delta$ is a collection of subsets of $[n]$ such that if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$. Often it is also required that $\{i\} \in \Delta$ for all $i \in [n]$, however we will not assume this condition. Each element $F \in \Delta$ is called a ***face*** of $\Delta$. The dimension of a face $F$ is $|F| - 1$. Let $d = \max\{|F| : F \in \Delta\}$ and define the ***dimension*** of $\Delta$ to be $\dim(\Delta) = d - 1$. An ***edge*** of $\Delta$ is a face of dimension 1. A ***vertex*** of $\Delta$ is a face of dimension 0. A ***facet*** is a maximal face of $\Delta$ (with respect to inclusion).

Let $\mathcal{F}(\Delta)$ denote the set of facets of $\Delta$. When $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$, we write $\Delta = \langle F_1, \ldots, F_m \rangle$. We say that a simplicial complex is ***pure*** if all facets have the same cardinality. Let $f_i = f_i(\Delta)$ denote the number of faces of $\Delta$ of dimension $i$. In particular, $f_0 = n$. The sequence $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ is called the ***f-vector*** of $\Delta$. Letting $f_{-1} = 1$, we define the ***h-vector*** $h(\Delta) = \{h_0, h_1, \ldots, h_d\}$ of $\Delta$ by the formula

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.$$

A ***nonface*** of $\Delta$ is a subset $F$ of $[n]$ with $F \notin \Delta$. let $\mathcal{N}(\Delta)$ denote the set of minimal nonfaces of $\Delta$.

**Definition 1.2.2.** Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over a field $K$ and $\Delta$ a simplicial complex on $[n]$. For each subset $F \subset [n]$ we set

$$x_F = \prod_{i \in F} x_i.$$

The ***Stanley-Reisner ideal*** of $\Delta$ is the ideal $I_\Delta$ of $S$ which is generated by those squarefree monomials $x_F$ with $F \notin \Delta$. In other words,

$$I_\Delta = (x_F : F \in \mathcal{N}(\Delta)).$$

The ***Stanley-Reisner ring*** of $\Delta$ (with respect to the field $K$) is the homogeneous $k$-algebra $K[\Delta] = K[x_1, \ldots, x_n]/I_\Delta$. The ***facet ideal*** of $\Delta$ is the ideal of $S$ which is generated by those squarefree monomials $x_F$ with $F \in \mathcal{F}(\Delta)$ and is denoted by

$I(\Delta)$. Thus if $\Delta = \langle F_1, \ldots, F_m \rangle$, then

$$I(\Delta) = (x_{F_1}, \ldots, x_{F_m}).$$

If we consider the simplicial complex $\Delta$ of dimension 1 on the vertex set [4] with

$$\mathcal{F}(\Delta) = \big\{\{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}\big\}$$

and with

$$\mathcal{N}(\Delta) = \big\{\{1,4\}, \{2,4\}, \{1,2,3\}\big\},$$

we have

$$I_\Delta = (x_3, x_4) \cap (x_2, x_4) \cap (x_1, x_4) \cap (x_1, x_2),$$

and

$$I(\Delta) = (x_1x_2, x_1x_3, x_2x_3, x_3x_4).$$

Let $G$ be a graph. If $G$ contains no isolated vertex, then the facet ideal of $G$ coincides with the so called edge ideal of $G$.

### 1.2.2 The Alexander dual

The Alexander dual of a simplicial complex plays an essential role in combinatorics and commutative algebra. Given a simplicial complex $\Delta$ on $[n]$, we define $\Delta^\vee$ by

$$\Delta^\vee = \{[n] \setminus F \,:\, F \notin \Delta\}.$$

**Lemma 1.2.3.** [18] *The collection of sets $\Delta^\vee$ is a simplicial complex and*

$$(\Delta^\vee)^\vee = \Delta.$$

This simplicial complex $\Delta^\vee$ is called the ***Alexander dual*** of $\Delta$. The facets of $\Delta^\vee$ are,

$$\mathcal{F}(\Delta^\vee) = \{[n] \setminus F \,:\, F \in \mathcal{N}(\Delta)\}.$$

For each subset $F \subset [n]$ we set $F^c = [n] \setminus F$. The simplicial complex

$$\Delta^c = \langle F^c \,:\, F \in \mathcal{F}(\Delta) \rangle.$$

is called the ***complement*** of $\Delta$.

**Lemma 1.2.4.** [18] *One has*

$$I_{\Delta^\vee} = I(\Delta^c).$$

For each subset $F \subset [n]$ we set,

$$P_F = (x_i \,:\, i \in F).$$

**Theorem 1.2.5.** [18] *The standard primary decomposition of $I_\Delta$ is*

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_{F^c}.$$

*In particular,*

$$\dim\, K[\Delta] = dim\Delta + 1.$$

**Corollary 1.2.6.** [18] *Let $I_\Delta = P_{F_1} \cap \ldots \cap P_{F_m}$ be the standard primary decomposition of $I_\Delta$, where each $F_j \subset [n]$. Then $G(I_{\Delta^\vee}) = \{x_{F_1}, \ldots, x_{F_m}\}$.*

Similar as in the graph case, we define

**Definition 1.2.7.** A ***vertex cover*** of a simplicial complex $\Delta$ is a set $G \subset [n]$ such that $G \cap F \neq \emptyset$ for all $F \in \mathcal{F}(\Delta)$. A vertex cover $G$ of $\Delta$ is ***minimal***, if any proper subset of $G$ is not a vertex cover of $\Delta$. We denote by $\mathcal{C}(\Delta)$ the set of minimal vertex covers of $\Delta$. If all the minimal vertex covers of $\Delta$ have the same cardinality, then we say $\Delta$ is ***unmixed***.

**Proposition 1.2.8.** *Let $\Delta$ be a simplicial complex on the vertex set $[n]$ and $I(\Delta)$ the facet ideal of $\Delta$. Then an ideal $P = (x_{i_1}, \ldots, x_{i_s})$ is a minimal prime of $I(\Delta)$ if and only if $\{i_1, \ldots, i_s\}$ is a minimal vertex cover of $\Delta$.*

### 1.2.3 The Eagon-Reiner theorem

A very useful result to compute the graded Betti numbers of the Stanley-Reisner ideal of a simplicial complex is the so-called Hochster formula. To state the formula we need some notation and terminologies.

**Definition 1.2.9.** Let $\Delta$ be a simplicial complex on $[n]$. For a face $F$ of $\Delta$, the ***link*** of $F$ in $\Delta$ is the subcomplex

$$\operatorname{link}_\Delta(F) = \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\}.$$

In particular, $\operatorname{link}_\Delta(\emptyset) = \Delta$. For a subset $W$ of $[n]$, the ***restriction*** of $\Delta$ on $W$ is the subcomplex

$$\Delta_W = \{F \in \Delta : F \subset W\}.$$

The notion $\tilde{H}_q(\Delta, K)$ stands for the qth ***reduced homology group of*** $\Delta$ ***with coefficients in K***, where $K$ is a field.

### Hochster's formula

The following fundamental theorem of Hochster gives a very useful description of the $\mathbb{Z}^n$-graded Betti numbers of a Stanley-Reisner ideal.

**Theorem 1.2.10** (Hochster)**.** *Let $\Delta$ be a simplicial complex and $\mathbf{a} \in \mathbb{Z}^n$. Then we have:*

1. $\operatorname{Tor}_i^S(K; I_\Delta)_{\mathbf{a}} = 0$ *if* $\mathbf{a}$ *is not squarefree;*

2. *if* $\mathbf{a}$ *is squarefree and* $W = \operatorname{supp}(\mathbf{a})$*, then*

$$\operatorname{Tor}_i^S(K; I_\Delta)_{\mathbf{a}} \cong \tilde{H}^{|W|-i-2}(\Delta_W; K) \quad \textit{for all } i.$$

We can compute graded Betti numbers of $I_\Delta$ using:

**Corollary 1.2.11.** *Let $\Delta$ be a simplicial complex, $\mathbf{a} \in \mathbb{Z}^n$ be squarefree and $F = [n] \setminus supp(\mathbf{a})$. Then*

$$Tor_i^S(K; I_\Delta)_{\mathbf{a}} \cong \tilde{H}_{i-1}(link_{\Delta^\vee}(F); K) \textit{ for all } i.$$

*In particular it follows that the graded Betti number $\beta_{ij}(I_\Delta)$ of $I_\Delta$ can be computed by the formula*

$$\beta_{ij}(I_\Delta) = \sum_{F \in \Delta^\vee, |F| = n-j} \dim_K \tilde{H}_{i-1}(link_{\Delta^\vee}; K).$$

**Reisner's criterion**

The $K$-algebra $K[\Delta] = S/I_\Delta$ is called the *Stanley–Reisner ring* of $\Delta$. We say that $\Delta$ is *Cohen–Macauably* over $K$ if $K[\Delta]$ is Cohen–Macaulay.

**Lemma 1.2.12.** [6] *Every Cohen–Macaulay simplicial complex is pure.*

The following result is the well known Reisner's criterion for the Cohen–Macaulay property of the Stanley-Reinser ring.

**Theorem 1.2.13** (Reisner)**.** *A simplicial complex $\Delta$ is Cohen–Macaulay over $K$ if and only if, for all faces $F$ of $\Delta$ including the empty face $\emptyset$ and for all $i < \dim(\mathrm{link}_\Delta(F))$, one has $\tilde{H}_i(\mathrm{link}_\Delta F; K) = 0$.*

Now the well-known Eagon-Reiner [10] theorem states:

**Theorem 1.2.14** (Eagon-Reiner)**.** *Let $\Delta$ be a simplicial complex on $[n]$ and let $K$ be a field. Then the Stanley–Reisner ideal $I_\Delta \subset K[x_1, \ldots, x_n]$ has a linear resolution if and only if $K[\Delta^\vee]$ is Cohen–Macaulay. More precisely, $I_\Delta$ has $q$-linear resolution if and only if $K[\Delta^\vee]$ is Cohen–Macaulay of dimension $n - q - 1$.*

# Chapter 2

# Symbolic powers of monomial ideals which are generically complete intersections

## 2.1 A review on symbolic powers of monomial ideals

Let $K$ be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring in $n$ indeterminates over $K$, and let $I \subset S$ be a monomial ideal. Then $I$ has a unique irredundant presentation

$$I = \bigcap_{i=1}^{m} C_i.$$

as an intersection of irreducible monomial ideals, see [18]. The irreducible monomial ideals $C_i$ are all of the form $(x_{i_1}^{a_1}, \ldots, x_{i_k}^{a_k})$. In particular, they are monomial complete intersections. One obtains from this presentation a canonical presentation of $I$ as an intersection of primary ideals

$$I = \bigcap_{i=1}^{r} Q_i. \tag{2.1.1}$$

where each $Q_i$ is $P_i$-primary and is defined to be the intersection of all $C_j$ with $\sqrt{C_j} = P_i$. For example, if $I = (x_1^3, x_2^3, x_1^2x_3^2, x_1x_2x_3^2, x_2^2x_3^2)$ then $I$ has the irredundant presentation as intersection of irreducible ideals

$$I = (x_1^3, x_2^3, x_3^2) \cap (x_1^2, x_2) \cap (x_1, x_2^2).$$

We have $\operatorname{Ass}(x_1^2, x_2) = (x_1, x_2) = \operatorname{Ass}(x_1, x_2^2)$. Intersecting $(x_1^2, x_2)$ and $(x_1, x_2^2)$ we obtain $(x_1, x_2)$-primary ideal $(x_1^2, x_1x_2, x_2)$ and hence the irredundant primary decomposition

$$I = (x_1^3, x_2^3, x_3^2) \cap (x_1^2, x_1x_2, x_2^2).$$

Given a primary decomposition as in (2.1.1) one defines the *kth symbolic power* of $I$ to be

$$I^{(k)} = \bigcap_{i=1}^{r} Q_i^k.$$

Obviously one has $I^k \subset I^{(k)}$ for all $k$. Note that there is no ambiguity in the definition of $I^{(k)}$ since the presentation of $I$ as an intersection of primary ideals as given in (2.1.1) is uniquely determined by $I$. Furthermore, if $I$ is generically complete intersection, as will be assumed in the following sections, this definition of $I^{(k)}$ coincide with one given in [21]

One of the basic questions related to symbolic powers is to find conditions when $I^k = I^{(k)}$ for all $k$. Since $I^{(k)}I^{(l)} \subset I^{(k+l)}$ for all $k$ and $l$, one may consider the so-called *symbolic power algebra*

$$A(I) = \bigoplus_{k \geq 0} I^{(k)}t^k \subset S[t].$$

This algebra contains the Rees algebra $R(I) = \bigoplus_{k\geq 0} I^k t^k$, and the following statements are equivalent:

(a) $I^k = I^{(k)}$ for all $k$.

(b) $R(I) = A(I)$.

(c) $A(I)$ is a standard graded $S$-algebra, i.e., $A(I)$ is generated over $S$ by elements of degree 1.

It is known that $A(I)$ is always finitely generated, see [21, Theorem 3.2]. For squarefree monomial ideals this has first been shown in [28, proposition1]. In case that $I$ is a squarefree monomial ideal it is well understood when $A(I)$ is standard graded, see [21, Theorem 5.1]. In particular, the following results will be of importance for our considerations. Assume that $I$ is a squarefree monomial ideal which is obtained as an intersection of ideals of the form $(x_i, x_j)$. To such an ideal we can associate a graph $G$ on the vertex set $[n]$ for which $\{i, j\}$ is an edge of $G$ if and only if $(x_i, x_j)$ appears in the intersection of $I$. Thus we can write

$$I = \bigcap_{\{i,j\} \in E(G)} (x_i, x_j).$$

The monomial generators of $I$ correspond to the vertex covers of $G$. Indeed, $u = x_{i_1} x_{i_2} \cdots x_{i_k}$ is a generator of $I$ if and only if the set $C = \{i_1, i_2, \ldots, i_k\}$ is a vertex cover of $G$, that is, $C \cap \{i, j\} \neq \emptyset$ for all edges $\{i, j\} \in E(G)$. One therefore calls $I$ the *vertex cover ideal* of $G$ and denotes it by $I_G$. The vertex cover ideal is in fact the Alexander dual of the edge ideal associated to the graph $G$. In the next section we will use the following result that can be found in [21, Theorem 5.1(b)],

**Theorem 2.1.1.** *The following conditions are equivalent:*

(a) $R(I_G) = A(I_G)$.

(b) $I_G^2 = I_G^{(2)}$.

(c) *$G$ is a bipartite graph.*

## 2.2 Unmixed monomial ideals of codimension 2 which are generically a complete intersection

A monomial ideal $I \subset S = K[x_1, \ldots, x_n]$ of codimension 2 is unmixed and generically a complete intersection if and only if

$$I = \bigcap_{1 \leq i < j \leq n} (x_i^{a_{ij}}, x_j^{a_{ji}}),$$

where $A = (a_{ij})$ is an $n \times n$-matrix of non-negative integers. Whenever $a_{ij} = 0$ or $a_{ji} = 0$, we have $(x_i^{a_{ij}}, x_j^{a_{ji}}) = S$, and we may omit this ideal in the intersection. Thus we may rewrite the above intersection as

$$I = \bigcap_{\{i,j\} \in E(G)} (x_i^{a_{ij}}, x_j^{a_{ji}}),$$

where $G$ is a simple graph on the vertex set $[n]$ and where for each edge $\{i, j\}$ of $G$ the numbers $a_{ij}$ and $a_{ji}$ are positive integers. The graph $G$ introduced is uniquely determined by $I$, and $\sqrt{I} = \bigcap_{\{i,j\} \in E(G)} (x_i, x_j)$ is the vertex cover ideal $I_G$ of $G$.

We say that $J = \bigcap_{1 \leq i < j \leq n} (x_i^{b_{ij}}, x_j^{b_{ji}})$ is a *trivial modification* of $I = \bigcap_{1 \leq i < j \leq n} (x_i^{a_{ij}}, x_j^{a_{ji}})$, if there exist integers $c_i > 0$ such that $b_{ij} = c_i a_{ij}$ for all $i$ and $j$. In particular, $I$ is a trivial modification of $\bigcap_{\{i,j\} \in E(G)} (x_i, x_j)$, if $a_{kl}$ does not depend on $l$ (and only on $k$).

We begin with an example which shows that for a non-squarefree monomial ideal $I$, the algebra $A(I)$ may have arbitrarily high degree generators.

**Example 2.2.1.** Given an integer $n$. Let $R$ be the polynomial ring $\mathbb{Q}[a, b, c]$ and $I = (a^n, ab, bc)$. Then the highest degree of an algebra generator of $A(I)$ is precisely $n$. In particular, the highest degree of a generator of the symbolic power of a monomial ideal of height 2 may be arbitrarily large. For the proof we note first that

$$I = (a^n, b) \cap (a, c).$$

Hence $I$ is a monomial ideal of codimension 2. We claim that the $k$th symbolic power $I^{(k)}$ of $I$ is minimally generated by the monomials

$$a^i b^{k-\lfloor i/n \rfloor} c^{k-i} \quad \text{with} \quad 0 \le i \le k, \qquad (2.2.1)$$
$$a^{nj} b^{k-j} \quad \text{with} \quad \lfloor k/n \rfloor < j \le k.$$

Indeed, we have that

$$I^{(k)} = (a^n, b)^k \cap (a, c)^k.$$

Therefore $I^{(k)}$ is generated by the monomials

$$a^{\max(nj,i)} b^{k-j} c^{k-i}, \quad 0 \le i, j \le k. \qquad (2.2.2)$$

Therefore, we have the generators

$$a^i b^{k-j} c^{k-i} \quad \text{with} \quad 0 \le i, j \le k \quad \text{and} \quad nj \le i, \qquad (2.2.3)$$

and

$$a^{nj} b^{k-j} c^{k-i} \quad \text{with} \quad 0 \le i, j \le k \quad \text{and} \quad nj > i. \qquad (2.2.4)$$

Among the generators (2.2.3) it is enough to take the generators

$$a^i b^{k-\lfloor i/n \rfloor} c^{k-i} \quad \text{with} \quad 0 \le i \le k.$$

And among the generators (2.2.4) it is enough to take the generators

$$a^{nj} b^{k-j} \quad \text{with} \quad \lfloor k/n \rfloor < j \le k.$$

This implies the assertion about the generators of $I^{(k)}$.

Next we claim that the element $a^n b^{n-1} \in I^{(n)}$ is a minimal generator of the algebra. Suppose this is not the case. Then, since $a^n b^{n-1}$ is a minimal generator of $I^{(n)}$, there exist integers $k_1, k_2 > 0$ with $k_1 + k_2 = n$, a minimal generators $u$ of $I^{(k_1)}$ and a minimal generator $v$ of $I^{(k_2)}$ such that $a^n b^{n-1} = uv$. Since $a^n b^{n-1}$ does not contain the factor $c$, it follows from (2.2.1) that $u = a^{k_1} b^{k_1 - \lfloor k_1/n \rfloor}$ and $v = a^{k_2} b^{k_2 - \lfloor k_2/n \rfloor}$ with $k_i < n$, $k_1 + k_2 = n$ and $k_1 + k_2 - \lfloor k_1/n \rfloor - \lfloor k_2/n \rfloor = n - 1$. Since $\lfloor k_i/n \rfloor = 0$ for $i = 1, 2$, this is impossible.

It remains to be shown that for $k > n$, the monomials in (2.2.1) can be presented as products of generators of the symbolic powers $I^{(l)}$ with $0 < l < k$. In fact,

- for $0 \leq i < k$ we have $a^i b^{k-\lfloor i/n \rfloor} c^{k-i} = (a^i b^{k-1-\lfloor i/n \rfloor} c^{k-1-i})(bc)$ with $a^i b^{k-1-\lfloor i/n \rfloor} c^{k-1-i} \in I^{(k-1)}$ and $bc \in I$
- $a^k b^{k-\lfloor k/n \rfloor} = (a^{k-1} b^{k-1-\lfloor k/n \rfloor})(ab)$ with $a^{k-1} b^{k-1-\lfloor k/n \rfloor} \in I^{(k-1)}$ and $ab \in I$.
- for $\lfloor k/n \rfloor < j < k$ we have $a^{nj} b^{k-j} = (a^{n(j-1)} b^{k-1-j})(ab)$ with $a^{n(j-1)} b^{k-1-j} \in I^{(k-1)}$ and $ab \in I$.
- $a^{nk} = a^{n(k-1)} a^n$ with $a^{n(k-1)} \in I^{k-1}$ and $a^n \in I$.

Next we classify all unmixed monomial ideals $I$ of codimension 2 which are generically a complete intersection and which have the property that $A(I)$ is standard graded.

Let $A$ be finitely generated graded $S$-algebra. We denote by $d(A)$ the highest degree of the minimal generators of $A$. By using this notation we have that $A(I)$ is standard graded if and only if $d(A(I)) = 1$.

Before proving the main result of this section we observe

**Proposition 2.2.2.** *Let $I$ be an arbitrary monomial ideal and $J$ a trivial modification of $I$. Then $d(A(J)) = d(A(I))$.*

*Proof.* Let $c_1, \ldots, c_n$ be positive integers such that $J$ arises from $I$ by the substitution $x_i \mapsto x_i^{c_i}$ for $i = 1, \ldots, n$, and let $\varphi\colon S \to S$ be the $K$-algebra homomorphism with $\varphi(x_i) = x_i^{c_i}$. For an ideal $J = (f_1, \ldots, f_m) \subset S$ we denote by $\varphi(J)$ the ideal in $S$ generated by the elements $\varphi(f_i)$, $i = 1, \ldots, m$. Then we have $\varphi(I^k) = \varphi(I)^k$ for all $k$. We claim that for any monomial ideal $I$ we also have $\varphi(I^{(k)}) = \varphi(I)^{(k)}$ for all $k$. To show this it suffices to prove that whenever $I$ and $J$ are monomial ideals, then $\varphi(I \cap J) = \varphi(I) \cap \varphi(J)$. Indeed, if $I = (u_1, \ldots, u_r)$ and $J = (v_1, \ldots, v_s)$, then $I \cap J = (\{\mathrm{lcm}(u_i, v_j)\}_{\substack{i=1,\ldots,r \\ j=1,\ldots,s}})$. Thus the claim follows from the fact that

$$\varphi(\mathrm{lcm}(u, v)) = \mathrm{lcm}(\varphi(u), \varphi(v)) \quad \text{for all monomials} \quad u, v.$$

Let $t$ be an indeterminate over $S$. Then $\varphi$ induces a graded $K$-algebra homomorphism $\hat{\varphi}\colon S[t] \to S[t]$ with $\hat{\varphi}(\sum_k f_k t^k) = \sum_k \varphi(f_k) t^k$. Restricting $\hat{\varphi}$ to $A(I) =$

$\bigoplus_k I^{(k)}t^k$ we obtain

$$\hat{\varphi}(A(I)) = \bigoplus_k \varphi(I^{(k)})t^k = \bigoplus_k \varphi(I)^{(k)}t^k = \bigoplus_k J^{(k)}t^k = A(J).$$

This shows that $\hat{\varphi}\colon A(I) \to A(J)$ is surjective. Since $\hat{\varphi}$ is injective, it follows that $A(I)$ and $A(J)$ are isomorphic, and in particular, $d(A(I)) = d(A(J))$, as desired. □

**Theorem 2.2.3.** *Let $I$ be an unmixed monomial ideal of codimension 2 which is generically a complete intersection. Then the following conditions are equivalent:*

(a) *$I^k = I^{(k)}$ for all $k$.*

(b) *$I^2 = I^{(2)}$.*

(c) *$I$ is a trivial modification of a vertex cover ideal of a bipartite graph.*

*Proof.* The implication (a) ⇒ (b) is trivial. Next we show that (c) ⇒ (a): The vertex cover ideal $J$ of a bipartite graph is standard graded, see Theorem 2.1.1. In other words, $d(A(J)) = 1$. Since $I$ is a trivial modification of $J$, Proposition 2.2.2 implies that $d(A(I))) = 1$

It remains to be shown that (b) ⇒ (c). Assume first that $\sqrt{I} = I_G$ for a bipartite graph $G$ and $I$ is a not trivial modification of vertex cover ideal of a bipartite graph $G$. It is convenient to identify the edges of $G$ with variables in the polynomial ring. Thus let the vertex set $V(G)$ of $G$ be $\{x_1, \ldots, x_r\} \cup \{y_1, \ldots, y_s\}$ with the edge set $E(G) \subset \{x_1, \ldots, x_r\} \times \{y_1, \ldots, y_s\}$. Then

$$I = \bigcap_{\{x_i, y_j\} \in E(G)} (x_i^{a_{ij}}, y_j^{b_{ij}}).$$

Since we assume that $I$ is not a trivial modification of $I_G$ there exits an integer $i$, say $i = 1$, such that

$$I_1 = \bigcap_{\{x_1, y_j\} \in E(G)} (x_1^{a_{1j}}, y_j^{b_{1j}})$$

is not a trivial modification of $\bigcap_{\{x_1,y_j\}\in E(G)}(x_1,y_j)$. Let $G_1$ be the subgraph of $G$ with $E(G_1)=\{\{x_1,y_j\}\colon\ \{x_1,y_j\}\in E(G)\}$. Then $I=I_1\cap I_2$, where

$$I_2=\bigcap_{\substack{\{x_i,y_j\}\in E(G)\\ i\neq 1}}(x_i^{a_{ij}},y_j^{b_{ij}}),$$

and $I_1$ is not a trivial modification of $I_{G_1}$.

Let $f=x_2\cdots x_r$. Then for the localization with respect to $f$ we have $(I_1^2)_f=(I^2)_f$ and $(I_1^{(2)})_f=(I^{(2)})_f$. Assume we know already $I_1^2\neq I_1^{(2)}$. If $(I_1^2)_f\neq(I_1^{(2)})_f$, then $(I^2)_f\neq(I^{(2)})_f$. But then also $I^2\neq I^{(2)}$, as we want to show. Thus we have reduced the problem to show that for an ideal of the type

$$I=(x_1^{a_1},y_1^{b_1})\cap\cdots\cap(x_1^{a_r},y_r^{b_r})$$

with $a_1\geq a_2\geq\cdots\geq a_r$ and one of the inequalities is strict. Since $I$ is a trivial modification of the ideal $(x_1^{a_1},y_1)\cap\cdots\cap(x_1^{a_r},y_r)$, we may as well assume that all $b_i=1$, see Proposition 2.2.2. Take the smallest $i$ such that $a_1>a_i$. Then for $a_1\geq 2a_i$,

$$x_1^{a_1}y_1y_2\cdots y_{i-1}\in I^{(2)}$$

and for $a_1<2a_i$,

$$x_1^{2a_i}y_1y_2\cdots y_{i-1}\in I^{(2)}$$

We have

$$I=(x_1^{a_1},x_1^{a_2}y_1,x_1^{a_3}y_1y_2,\ldots,x_1^{a_i}y_1y_2\cdots y_{i-1},\ldots,y_1\cdots y_r).$$

It is obvious that $x_1^{a_1}y_1y_2\cdots y_{i-1}\notin I^2$. On the other hand, the possible product of two generators of $I$ to get $x_1^{2a_i}y_1y_2\cdots y_{i-1}$ would be $x_1^{a_1}(x_1^{a_i}y_1y_2\cdots y_{i-1})$. But this is also not possible, because $a_1+a_i>2a_i$.

Assume now that $G$ is not bipartite. Then $G$ contains an odd cycle. If this cycle has a chord, then this chord decomposes this cycle into two smaller cycles, where one of them is again an odd cycle. This argument shows that $G$ contains an odd cycle with no chord. Suppose first that this cycle has length $>3$. By a localization argument as in the first part of the proof (c) $\Rightarrow$ (b) we may assume that $G$ coincides with this cycle. Say, the edges of $G$ are $\{1, n\}$ and $\{i, i+1\}$ for $i = 1, \ldots, n-1$ with $n > 3$. Localizing $I$ at $f = x_3 x_5 \cdots x_{n-1}$ we are reduced to show that for the ideal $J = (x_1^{a_1}, x_2^{b_1}) \cap (x_1^{a_2}, x_n^{b_2})$ we have that $J^2 \neq J^{(2)}$. Without loss of generality we may assume that $a_1 > a_2$ and $b_1 = 1 = b_2$ see Proposition 2.2.2. If $a_1 \geq 2a_2$ then $x_1^{a_1} x_2 \in J^{(2)} \setminus J^2$, if $a_1 < 2a_2$ then $x_1^{2a_2} x_2 \in J^{(2)} \setminus J^2$, as desired.

Finally we assume that $E(G) = \{\{1,2\}, \{2,3\}, \{1,3\}\}$. Then

$$I = (x_1^{a_1}, x_2^{b_1}) \cap (x_2^{b_2}, x_3^{c_1}) \cap (x_3^{c_2}, x_1^{a_2}).$$

We want so show that $I^{(2)} \neq I^2$. Notice that

$$\begin{aligned} I \;=\; & (x_1^{\max(a_1,a_2)} x_2^{b_2}, x_1^{a_1} x_2^{b_2} x_3^{c_2}, x_1^{\max(a_1,a_2)} x_3^{c_1}, x_1^{a_1} x_3^{\max(c_1,c_2)}, x_1^{a_2} x_2^{\max(b_1,b_2)}, x_2^{\max(b_1,b_2)} x_3^{c_2}, \\ & x_1^{a_2} x_2^{b_1} x_3^{c_1}, x_2^{b_1} x_3^{\max(c_1,c_2)}). \end{aligned}$$

Assume that $a_1 \geq a_2$ then

$$\begin{aligned} I \;=\; & (x_1^{a_1} x_2^{b_2}, x_1^{a_1} x_2^{b_2} x_3^{c_2}, x_1^{a_1} x_3^{c_1}, x_1^{a_1} x_3^{\max(c_1,c_2)}, x_1^{a_2} x_2^{\max(b_1,b_2)}, x_2^{\max(b_1,b_2)} x_3^{c_2}, x_1^{a_2} x_2^{b_1} x_3^{c_1}, \\ & x_2^{b_1} x_3^{\max(c_1,c_2)}). \end{aligned}$$

Then it follows that

$$x_1^{a_1} x_2^{\max(b_1,b_2)} x_3^{c_1} \in I^{(2)} \setminus I^2, \quad \text{if} \quad a_1 \geq 2a_2,$$

and

$$x_1^{a_1} x_2^{\max(b_1,b_2)} x_3^{\max(c_1,c_2)} \in I^{(2)} \setminus I^2, \quad \text{if} \quad a_2 \leq a_1 < 2a_2.$$

□

## 2.3 Bounds for the generators of the algebra $A(I)$

In this section we want to give a lower bound for $d(A(I))$ in the case that $I$ is a modification of the vertex cover ideal of a bipartite graph.

**Theorem 2.3.1.** *Let $I \subset S$ is an ideal, $S = K[x_1, \ldots, x_r, y_1, \ldots, y_s]$ and $a_{ij}$ and $b_{ij}$ are non-negative integers with $1 \leq i \leq r$ and $1 \leq j \leq s$. Let $G$ be a bipartite graph with vertex set $V(G) = \{x_1, \ldots, x_r, y_1, \ldots, y_s\}$ and edge set $E(G) \subset \{x_1, \ldots, x_r\} \times \{y_1, \ldots, y_s\}$, and let*

$$I = \bigcap_{\{x_i, y_j\} \in E(G)} (x_i^{a_{ij}}, y_j^{b_{ij}})$$

*Then*

$$d(A(I)) \geq \max\left\{ \max_{i,j,l} \left\{ \frac{a_{ij}}{\gcd(a_{ij}, a_{il})} \right\}, \max_{i,j,l} \left\{ \frac{b_{ij}}{\gcd(b_{ij}, b_{il})} \right\} \right\}.$$

*Proof.* Let $f \in S = A_0(I)$ be a squarefree monomial, then $A(I)_f$ is again naturally graded because $f$ is an element of degree 0 in $A(I)$, and the generators of $A(I)$ are also generators of $A(I)_f$. Hence it follows that $d(A(I)_f) \leq d(A(I))$. Thus any lower bound for $d(A(I)_f)$ is also lower bound for $d(A(I))$.

Notice that

$$(I^{(k)})_f = \bigcap_{\substack{\{x_i, y_j\} \in E(G) \\ \{x_i, y_j\} \cap \operatorname{supp}(f) = \emptyset}} (x_i^{a_{ij}}, y_j^{b_{ij}})^k S_f$$

Therefore

$$(I_f)^{(k)} = (I^{(k)})_f.$$

This implies that $A(I)_f = A(I_f)$. Let

$$J = \bigcap_{\substack{\{x_i, y_j\} \in E(G) \\ \{x_i, y_j\} \cap \operatorname{supp}(f) = \emptyset}} (x_i^{a_{ij}}, y_j^{b_{ij}}) \subset K[\{x_i, y_j : x_i, y_j \notin \operatorname{supp}(f)\}]$$

Then $d(A(I_f)) = d(A(J))$, because $A(I_f) = A(J) \otimes_k K[\{x_i, x_i^{-1}, y_j, y_j^{-1} \colon \; x_i, y_j \in \operatorname{supp}(f)\}]$.

We may assume that

$$m = \max_{i,j,l} \{a_{ij} / \gcd(a_{ij}, a_{il})\} \geq \max_{i,j,l} \{b_{ij} / \gcd(b_{ij}, b_{il})\}.$$

Let $i$ be an integer for which there exist $j$ and $l$ such that $a_{ij}/\gcd(a_{ij}, a_{il})$ takes the maximal value $m$. We may assume that $i = 1$ and $j = 1$. If $m = 1$, the assertion of the theorem is trivial. Thus we may assume that $m > 1$. This then implies that

$l \neq j$ and $a_{ij} > a_{il}$. We may assume that $l = 2$. Localizing at $f = x_2 \cdots x_r y_3 \cdots y_s$ and applying the above considerations, we may assume that

$$I = (x_1^{a_{11}}, y_1^{b_{11}}) \cap (x_1^{a_{12}}, y_2^{b_{12}}) \quad \text{with} \quad a_{11} > a_{12}.$$

Now $I$ is a trivial modification of the ideal $(x_1^a, y_1) \cap (x_1^b, y_2)$ with $a > b$, where $a = a_{11}/\gcd(a_{11}, a_{12})$ and $b = a_{12}/\gcd(a_{11}, a_{12})$.

Thus, according to the Proposition 2.2.2, it is enough to be shown that $d(A(I)) \geq a$ for $I = (x_1^a, y_1) \cap (x_1^b, y_2)$ with $a > b$ and $\gcd(a, b) = 1$.

In order to prove this, we first claim that the $k$th symbolic power $I^{(k)}$ of $I$ is minimally generated by the monomials

$$x_1^{ai} y_1^{k-i} y_2^{k-j} \quad \text{with} \quad 0 \leq i, j \leq k \quad \text{and} \quad j \leq \lfloor ai/b \rfloor. \tag{2.3.1}$$

and the monomials

$$x_1^{jb} y_1^{k-i} y_2^{k-j} \quad \text{with} \quad 0 \leq i, j \leq k \quad \text{and} \quad j > \lfloor ai/b \rfloor. \tag{2.3.2}$$

Indeed, we have that

$$I^{(k)} = (x_1^a, y_1)^k \cap (x_1^b, y_2)^k.$$

Therefore $I^{(k)}$ is generated by the monomials

$$x_1^{\max(ai, bj)} y_1^{k-i} y_2^{k-j}, \quad 0 \leq i, j \leq k.$$

Therefore, we have the generators

$$x_1^{ai} y_1^{k-i} y_2^{k-j}, \quad \text{with} \quad 0 \leq i, j \leq k \quad \text{and} \quad ai \geq bj, \tag{2.3.3}$$

and

$$x_1^{bj} y_1^{k-i} y_2^{k-j}, \quad \text{with} \quad 0 \leq i, j \leq k \quad \text{and} \quad ai < bj. \tag{2.3.4}$$

Thus we obtain exactly the monomials listed in (2.3.1) and (2.3.2).

Next we claim that the element $x_1^{ab} y_1^{a-b} \in I^{(a)}$ belongs to the minimal set of generators of the algebra. This will then show that $d(A(I)) \geq a$. Suppose $x_1^{ab} y_1^{a-b} t^a$

is not a minimal generator of $A(I)$. Then, since $x_1^{ab}y_1^{a-b}$ is a minimal generator of $I^{(a)}$, there exist integers $k_1, k_2 > 0$ with $k_1 + k_2 = a$, a minimal generator $u$ of $I^{(k_1)}$ and a minimal generator $v$ of $I^{(k_2)}$ such that $x_1^{ab}y_1^{a-b} = uv$. Since $x_1^{ab}y_1^{a-b}$ does not contain the factor $y_2$, it follows from (2.3.1) and (2.3.2) that the monomials $u$ and $v$ are of the form $x_1^{ai}y_1^{c-i}$ or $x_1^{bc}y_1^{c-i}$ where $c = k_1$ or $c = k_2$. We have to distinguish several cases:

(i) If $u = x_1^{ai_1}y_1^{k_1-i_1}$ with $ai_1 \geq bk_1$ and $v = x_1^{ai_2}y_1^{k_2-i_2}$ with $ai_2 \geq bk_2$, then we have

$$x_1^{ab}y_1^{a-b} = uv = x_1^{a(i_1+i_2)}y_1^{k_1+k_2-(i_1+i_2)} = x_1^{a(i_1+i_2)}y_1^{a-(i_1+i_2)}.$$

This implies that $i_1 + i_2 = b$. Adding the inequalities $ai_1 \geq bk_1$ and $ai_2 \geq bk_2$, and using $k_1 + k_2 = a$ and $i_1 + i_2 = b$, we get

$$ab = ai_1 + ai_2 \geq bk_1 + bk_2 = ba,$$

which implies that $ai_1 = bk_1$ and $ai_2 = bk_2$. Since $\gcd(a, b) = 1$, we see that $i_1 = bc$ and $k_1 = ac$ for some integer $c > 0$. This is not possible, because $k_1 < a$.

(ii) If $u = x_1^{ai_1}y_1^{k_1-i_1}$ with $ai_1 \geq bk_1$ and $v = x_1^{bk_2}y_1^{k_2-i_2}$ with $ai_2 < bk_2$, then we have

$$x_1^{ab}y_1^{a-b} = uv = x_1^{ai_1+bk_2}y_1^{k_1+k_2-(i_1+i_2)} = x_1^{ai_1+bk_2}y_1^{a-(i_1+i_2)}.$$

This implies that $ai_1 + bk_2 = ab$ and $i_1 + i_2 = b$. Using $ai_2 < bk_2$ in the equation $ai_1 + bk_2 = ab$, we get $a(i_1 + i_2) < ab$ which contradicts the fact that $i_1 + i_2 = b$.

(iii) If $u = x_1^{bk_1}y_1^{k_1-i_1}$ with $ai_1 < bk_1$ and $v = x_1^{bk_2}y_1^{k_2-i_2}$ with $ai_2 < bk_2$, then we have

$$x_1^{ab}y_1^{a-b} = uv = x_1^{b(k_1+k_2)}y_1^{k_1+k_2-(i_1+i_2)} = x_1^{2bk_2}y_1^{a-(i_1+i_2)}.$$

This implies that $b(k_1 + k_2) = ab$ and $i_1 + i_2 = b$. Using $ai_1 < bk_1$ and $ai_2 < bk_2$ in the equation $b(k_1 + k_2) = ab$, we get $a(i_1 + i_2) < ab$ which contradicts the fact that $i_1 + i_2 = b$. $\square$

One would like to obtain a similar lower bound for ideals which are modifications of the vertex cover ideals of an arbitrary graph. The first non-trivial case to be considered would be

$$I = (x^a, y^b)\cap(y^c, z^d)\cap(x^e, z^f) \quad \text{with} \quad \gcd(a, e) = 1,\ \gcd(b, c) = 1 \quad \text{and} \quad \gcd(d, f) = 1.$$

We conjecture that $d(A(I)) \geq \max\{a, b, c, d, e, f\}$. However, in general this lower bound, if correct, is not strict as the following examples shows: let $I = (a^2, b) \cap (b, c) \cap (a, c)$. Then $a^2b^2ct^3 \in I^{(3)}t^3$ is a minimal generator of $A(I)$.

## 2.4 The integral closure of $A(I)$ for a monomial ideal $I$ which is generically a complete intersection

It seems to be pretty hard to find a general upper bound for the number $d(A(I))$. However for the integral closure $\overline{A(I)}$ of $A(I)$ where $I$ is a monomial ideal which is generically a complete intersection such bound can be given, with respect to the number of indeterminates and degrees of the powers of the variables which appears in the irredundant irreducible decomposition of $I$. Since $I$ is generically a complete intersection it is of the form

$$I = \bigcap_{l=1}^{m} \mathfrak{m}^{\mathbf{a}_l},$$

where for a nonzero vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ we set $\mathfrak{m}^{\mathbf{a}} = (\{x_i^{a_i}\}_{i \in \mathrm{supp}(\mathbf{a})})$.

**Lemma 2.4.1.** *The integral closure of $A(I)$ is spanned as a $K$-vector space by all monomials $x_1^{c_1} \cdots x_n^{c_n} t^k$ for which the exponent vector $(c_1, \ldots, c_n, k)$ satisfies the following inequalities:*

$$c_1 \geq 0, \ldots, c_n \geq 0,\ k \geq 0 \quad \textit{and} \quad \sum_{i \in \mathrm{supp}(\mathbf{a}_l)} \frac{c_i}{a_{li}} - k \geq 0, \quad 1 \leq l \leq m.$$

*Proof.* . The integral closure $\overline{A(I)}$ of $A(I)$ is generated as a $K$-vector space by all monomials $u = x_1^{c_1}\cdots x_n^{c_n}t^k \in S[t]$ such that $u^r \in I^{(rk)}t^{rk}$. Now $u^r \in I^{(rk)}t^{rk}$ if and only if $x_1^{rc_1}\cdots x_n^{rc_n} \in (x_1^{a_{l1}},\ldots,x_n^{a_{ln}})^{rk}$ for all $l = 1,\ldots,m$. Note that $x_1^{rc_1}\cdots x_n^{rc_n} \in (x_1^{a_{l1}},\ldots,x_n^{a_{ln}})^{rk}$ if and only if there exists non-negative integers $j_1,\ldots,j_n$ with $j_1+\cdots+j_n = rk$ and such that $rc_1 \geq j_1a_{l1},\ldots,rc_n \geq j_na_{ln}$. These inequalities are satisfied for $c_1,\ldots,c_n$ if and only if $\sum_{i\in \mathrm{supp}(\mathbf{a}_l)} c_i/a_{li} \geq k$. This yields the desired conclusion. □

**Theorem 2.4.2.** *Let $I \subset S = K[x_1,\ldots,x_n]$ be a monomial ideal which is generically a complete intersection, and suppose that $I = \bigcap_{l=1}^m \mathfrak{m}^{\mathbf{a}_l}$. Then*

$$d(\overline{A(I)}) \leq \frac{(n+1)!}{2}d^{n(n-1)}, \quad \textit{where} \quad d = \max_{i,l}\{a_{il}\}.$$

*Proof.* The proof of the theorem follows the line of arguments in the proof of Theorem 5.6 in [21]. The inequalities given in Lemma 2.4.1 describe a positive cone $C \subset \mathbb{R}^{n+1}$. If $q = (c_1,\ldots,c_n,k)$ is an integral vector in $C$, we set $\deg q = k$ and call this number the degree of $q$. Let $E$ be a set of integral vectors spanning the extremal arrays. Then the maximal degree of a generator of $\overline{A(I)}$ is bounded by the maximum of the degree of the integral vectors of the form

$$d(\overline{A(I)}) \leq (n+1)\max\{\deg q\colon\ q \in F\}. \tag{2.4.1}$$

Note that each integral vector $q$ spanning an extremal array of $C$ is given as an integral solution of $n$ linear equations describing supporting hyperplanes of $C$. Thus there exist numbers $l_1 < l_2 < \cdots < l_r$ with $r \leq n$ and numbers $k_1 < k_2 < \cdots < k_{n-r}$ such that $q$ is an integral solution of the following system of $n$ homogeneous linear equations

$$\begin{aligned} \sum_{i\in\mathrm{supp}(\mathbf{a}_{l_t})}\ \prod_{\substack{j\in\mathrm{supp}(\mathbf{a}_{l_t})\\ j\neq i}} a_{l_t,j}z_i - \prod_{j\in\mathrm{supp}(\mathbf{a}_{l_t})} a_{l_t,j}y &= 0 \quad \text{for} \quad t = 1,\ldots,r \\ z_{k_s} &= 0 \quad \text{for} \quad s = 1,\ldots,n-r. \end{aligned}$$

which arise from the linear inequalities by clearing denominators.

The coefficient matrix of this linear equation is an $n \times n+1$ matrix $B$. Thus an integral solution of this system is the vector $q$ whose $i$th component is $(-1)^i \det B_i$ where $B_i$ is obtained from $B$ by skipping the $i$th column of $B$. In particular it follows that $\deg q = |\det B_{n+1}|$. Obviously, $\det B_{n+1}$ is equal to a suitable minor of the matrix $A$ whose rows are $\mathbf{a}_1, \ldots, \mathbf{a}_m$.

If follows that each such minor is a sum of terms $\pm g$ where each $g$ is a suitable product of the $a_{ij}$. If we add up in this sum only the positive terms, then this will give an upper bound for this minor. We denote this upper bound of $g$ by $g_+$. Thus it follows from (2.4.1) that the desired upper bound for $d(\overline{A(I)})$ is given by $(n+1)f$ where $f$ is bound for the maximal possible $g_+$. The maximal possible $g_+$ we obtain if we consider $n$ minors. Each of these $n$-minors has $n!/2$ positive terms where each of this terms is a product of $n(n-1)$ entries $a_{l_i}$. Thus $d^{n(n-1)}$ is a bound for each of these entries. Combining all this, we obtain the desired upper bound for $d(\overline{A(I)})$. $\square$

# Chapter 3

# Simplicial complexes with rigid depth

## 3.1 Criteria for $\operatorname{depth}(S/I) = \operatorname{depth}(S/\sqrt{I})$

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over a field $K$. Let $I \subset S$ be an unmixed monomial ideal such that $\sqrt{I} = I_\Delta$ where $\Delta$ is a pure simplicial complex with the facet set $\mathcal{F}(\Delta)$. Then $I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_F$, where $P_F = (x_i \mid i \notin F)$ for every $F \in \mathcal{F}(\Delta)$. Let $I = \bigcap_{F \in \mathcal{F}(\Delta)} I_F$ where $I_F$ is the $P_F$-primary component of $I$.

In order to prove the main result of this section we need to recall some facts from [30, Section 1]. For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, let $G_{\mathbf{a}} = \{i \mid a_i < 0\}$. We denote by $\Delta_{\mathbf{a}}$ the simplicial complex on $[n]$ of all the sets of the form $F \setminus G_{\mathbf{a}}$ where $G_{\mathbf{a}} \subset F \subset [n]$ and such that $F$ satisfies the condition $\mathbf{x}^{\mathbf{a}} \notin IS_F$ where $S_F = S[x_i^{-1} \mid i \in F]$. It is shown in [30, Section 1] that if $\Delta_{\mathbf{a}}$ is non-empty, then $\Delta_{\mathbf{a}}$ is a pure subcomplex of $\Delta$ of $\dim \Delta_{\mathbf{a}} = \dim \Delta - |G_{\mathbf{a}}|$.

For every simplicial subcomplex $\Gamma$ of $\Delta$ with $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$ we set

$$L_\Gamma(I) = \{\mathbf{a} \in \mathbb{N}^n \mid \mathbf{x}^{\mathbf{a}} \in \bigcap_{F \in \mathcal{F}(\Delta) \setminus \mathcal{F}(\Gamma)} I_F \setminus \bigcup_{G \in \mathcal{F}(\Gamma)} I_G\}.$$

By [30, Lemma 1.5], we have

$$\Delta_{\mathbf{a}} = \Gamma \text{ if and only if } \mathbf{a} \in L_\Gamma(I). \tag{3.1.1}$$

For the proof of the next theorem we also need to recall Takayama's formula

[38]. For every degree $\mathbf{a} \in \mathbb{Z}^n$ we denote by $H^i_{\mathfrak{m}}(S/I)_{\mathbf{a}}$ the $\mathbf{a}$-component of the $i$th local cohomology module of $S/I$ with respect to the homogeneous maximal ideal of $S$. For $1 \leq j \leq n$, let

$$\rho_j(I) = \max\{\nu_j(u) \mid u \text{ is a minimal generator of } I\},$$

where by $\nu_j(u)$ we mean the exponent of the variable $x_j$ in $u$. If $x_j$ does not divide $u$, then we use the usual convention, $\nu_j(u) = 0$.

**Theorem 3.1.1** (Takayama's formula)**.**

$$\dim_K H^i_{\mathfrak{m}}(S/I)_{\mathbf{a}} = \begin{cases} \dim_K \tilde{H}_{i-|G_{\mathbf{a}}|-1}(\Delta_{\mathbf{a}}, K), & \text{if } G_{\mathbf{a}} \in \Delta \text{ and} \\ & a_j < \rho_j(I) \text{ for } 1 \leq j \leq n, \\ 0, & \text{else.} \end{cases}$$

The next theorem is a natural extension of [30, Theorem 1.6].

**Theorem 3.1.2.** *Let $\Delta$ be a pure simplicial complex with* $\operatorname{depth} K[\Delta] = t$. *Let $I \subset S$ be an unmixed monomial ideal with $\sqrt{I} = I_{\Delta}$. The following conditions are equivalent:*

*(a)* $\operatorname{depth}(S/I) = t$,

*(b)* $\operatorname{depth} K[\Delta_{\mathbf{a}}] \geq t$ *for all* $\mathbf{a} \in \mathbb{N}^n$ *with* $\Delta_{\mathbf{a}} \neq \emptyset$,

*(c)* $L_{\Gamma}(I) = \emptyset$ *for every simplicial complex* $\Gamma$ *with* $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)$ *and* $\operatorname{depth} K[\Gamma] < t$.

*Proof.* The proof of this theorem follows closely the ideas of the proof of [30, Theorem 1.6]. For the equivalence $(a) \Leftrightarrow (b)$ we need to recall some known facts about local cohomology; see [18, Section A. 7]. For any finitely generated graded $S$-module $M$ we have $\operatorname{depth} M \geq t$ if and only if $H^i_{\mathfrak{m}}(M) = 0$ for all $i < t$. Therefore, in our hypothesis, and since $\operatorname{depth}(S/I) \leq \operatorname{depth}(S/\sqrt{I}) = t$, we get

$$\operatorname{depth}(S/I) = t \Leftrightarrow H^i_{\mathfrak{m}}(S/I) = 0 \text{ for } i < t. \tag{3.1.2}$$

In addition, for every $\mathbf{a} \in \mathbb{N}^n$, we get

$$\operatorname{depth}(K[\Delta_{\mathbf{a}}]) \geq t \Leftrightarrow H^i_{\mathfrak{m}}(K[\Delta_{\mathbf{a}}]) = 0 \text{ for } i < t. \tag{3.1.3}$$

For $\mathbf{b} \in \mathbb{Z}^n$, we set $G_{\mathbf{b}} = \{i \mid b_i < 0\}$ and $H_{\mathbf{b}} = \{i \mid b_i > 0\}$. By using [18, Theorem A.7.3], for every $\mathbf{b} \in \mathbb{Z}^n$, we obtain

$$\dim_K H^i_{\mathfrak{m}}(K[\Delta_{\mathbf{a}}])_{\mathbf{b}} = \dim_K \tilde{H}_{i-|G_{\mathbf{b}}|-1}(\operatorname{link}_{\operatorname{star} H_{\mathbf{b}}} G_{\mathbf{b}}; K).$$

Here we denoted by $\operatorname{star} H_{\mathbf{b}}$ the star of $H_{\mathbf{b}}$ in $\Delta_{\mathbf{a}}$, and by $\operatorname{link}_{\operatorname{star} H_{\mathbf{b}}} G_{\mathbf{b}}$ the link of $G_{\mathbf{b}}$ in the complex $\operatorname{star} H_{\mathbf{b}}$. We recall that if $\Gamma$ is a simplicial complex and $F$ is a face of $\Gamma$, then $\operatorname{star}_\Gamma F = \{G \mid F \cup G \in \Gamma\}$ and $\operatorname{link}_\Gamma F = \{G \mid F \cup G \in \Gamma \text{ and } F \cap G = \emptyset\}$. Therefore, the equivalence (3.1.3) my be written

$$\operatorname{depth} K[\Delta_{\mathbf{a}}] \geq t$$

$$\Leftrightarrow \tilde{H}_{i-|G_{\mathbf{b}}|-1}(\operatorname{link}_{\operatorname{star} H_{\mathbf{b}}} G_{\mathbf{b}}; K) = 0 \text{ for } i < t \text{ and for every } \mathbf{b} \in \mathbb{Z}^n. \tag{3.1.4}$$

Since $\operatorname{link}_{\operatorname{star} H_{\mathbf{b}}} G_{\mathbf{b}}$ is acyclic for $H_{\mathbf{b}} \neq \emptyset$ and $\operatorname{star} H_{\mathbf{b}} = \Delta_{\mathbf{a}}$ if $H_{\mathbf{b}} = \emptyset$, we get

$$\operatorname{depth} K[\Delta_{\mathbf{a}}] \geq t$$

$$\Leftrightarrow \tilde{H}_{i-|G_{\mathbf{b}}|-1}(\operatorname{link}_{\Delta_{\mathbf{a}}} G_{\mathbf{b}}; K) = 0 \text{ for } i < t \text{ and for every } \mathbf{b} \in \mathbb{Z}^n. \tag{3.1.5}$$

By Takayama's formula, the equivalence (3.1.2) may be rewritten

$$\operatorname{depth}(S/I) = t$$

$$\Leftrightarrow \dim_K \tilde{H}_{i-|G_{\mathbf{b}}|-1}(\Delta_{\mathbf{b}}; K) = 0 \text{ for } i < t \text{ and for every } \mathbf{b} \in \mathbb{Z}^n. \tag{3.1.6}$$

Now, the equivalence (a)$\Leftrightarrow$ (b) follows by relations (3.1.5) and (3.1.6) if we notice that, by the proof of (i) $\Rightarrow$ (ii) in [30, Theorem 1.6], we have $\operatorname{link}_{\Delta_{\mathbf{a}}} G_{\mathbf{b}} = \Delta_{\mathbf{b}}$ for any $G_{\mathbf{b}} \in \Delta_{\mathbf{a}}$.

For the rest of the proof we only need to use (3.1.1). Indeed, for (b) $\Rightarrow$ (c), let us assume that $L_\Gamma(I) \neq \emptyset$ for some subcomplex $\Gamma$ of $\Delta$ with $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$ and such that $\operatorname{depth}(K[\Gamma]) < t$. Then there exists $\mathbf{a} \in Ł_\Gamma(I)$, hence $\Gamma = \Delta_{\mathbf{a}}$. But this equality is impossible since $\operatorname{depth}(K[\Delta_{\mathbf{a}}]) \geq t$. For (c) $\Rightarrow$ (b), let us assume that there exists $\mathbf{a} \in \mathbb{N}^n$ such that $\operatorname{depth} K[\Delta_{\mathbf{a}}] < t$. Then, for $\Gamma = \Delta_{\mathbf{a}}$ we get $L_\Gamma(I) \neq \emptyset$, a contradiction. □

Obviously, for $t = \dim K[\Delta]$ in the above theorem we recover Theorem 1.6 in [30].

The above theorem is especially useful in the situation when $I$ is either an intersection of monomial prime ideal powers or an intersection of irreducible monomial ideals. The first class of ideals may be studied with completely similar arguments to those used in [30, Section 1]. In the sequel we discuss ideals which are intersections of irreducible monomial ideals.

Let $\mathcal{F}(\Delta) = \{F_1, \ldots, F_r\}$ and $I = \bigcap_{i=1}^r I_{F_i}$ be an intersection of irreducible monomial ideals, that is, for every $1 \leq i \leq r$, $I_{F_i} = (x_j^{a_{ij}} \mid j \notin F_i)$ for some positive exponents $a_{ij}$. As a consequence of the above theorem, one may express the condition $\operatorname{depth}(S/I) = \operatorname{depth}(S/\sqrt{I})$ in terms of linear inequalities on the exponents $a_{ij}$.

**Proposition 3.1.3.** *The set of exponents* $(a_{ij})$ *for which the equality* $\operatorname{depth}(S/I) = \operatorname{depth}(S/\sqrt{I})$ *holds consists of all points of positive integer coordinates in a finite union of rational cones in* $\mathbb{R}^{r(n-d)}$.

*Proof.* Let $\Gamma$ be a subcomplex of $\Delta$ with $\operatorname{depth}(K[\Gamma]) < t$ and $\mathcal{F}(\Delta) \setminus \mathcal{F}(\Gamma) = \{F_{i_1}, \ldots, F_{i_s}\}$ where $1 \leq i_1 < \cdots < i_s \leq r$. The condition $L_\Gamma(I) = \emptyset$ gives

$$\bigcap_{q=1}^{s} (x_j^{a_{i_q j}} : j \notin F_{i_q}) \subseteq \bigcup_{k \notin \{i_1, \ldots, i_s\}} I_{F_k}.$$

This implies that the following conditions must hold

$$\operatorname{lcm}(x_{j_1}^{a_{i_1 j_1}}, x_{j_2}^{a_{i_2 j_2}}, \ldots, x_{j_s}^{a_{i_s j_s}}) \in \bigcup_{k \notin \{i_1, \ldots, i_s\}} I_{F_k}$$

for all $s$-tuples $(j_1, j_2, \ldots, j_s)$, with $j_q \notin F_{i_q}$ for $1 \leq q \leq s$. This is equivalent to saying that for every $s$-tuple $(j_1, j_2, \ldots, j_s)$, with $j_q \notin F_{i_q}$ for $1 \leq q \leq s$, there exists

$1 \leq q \leq s$ such that

$$a_{i_q j_q} \geq \min\{a_{kj_q} : k \neq i_1, i_2, \ldots, i_s, \}.$$

□

In the following example we consider tetrahedral type ideals.

**Example 3.1.4.** Let $\Delta$ be the 4-cycle, that is, $I_\Delta = (x_1, x_2) \cap (x_1, x_4) \cap (x_2, x_3) \cap (x_3, x_4)$. Note that $S/I_\Delta$ is Cohen-Macaulay, hence $\operatorname{depth}(S/I_\Delta) = 2$.

Let $I = (x_1^{a_1}, x_2^{a_2}) \cap (x_1^{a_3}, x_4^{a_4}) \cap (x_2^{a_5}, x_3^{a_6}) \cap (x_3^{a_7}, x_4^{a_8})$. Then $\operatorname{depth}(S/I) = \operatorname{depth}(S/I_\Delta)$, that is, $I$ is a Cohen-Macaulay ideal, if and only if one of the following condition holds:

(1) $a_3 \leq a_1, \quad a_2 = a_5, \quad a_7 \leq a_6.$

(2) $a_2 \leq a_5, \quad a_6 = a_7, \quad a_4 \leq a_8.$

(3) $a_5 \leq a_2, \quad a_1 = a_3, \quad a_8 \leq a_4.$

(4) $a_1 \leq a_3, \quad a_4 = a_8, \quad a_6 \leq a_7.$

In order to prove the above claim, we first notice that any subcomplex $\Gamma$ of $\Delta$ which has $\operatorname{depth}(K[\Gamma]) < 2$ corresponds to a disconnected subgraph of $\Delta$. But $\Delta$ has two disconnected subgraphs which correspond to the pair of disjoint edges $\big\{\{1,2\},\{3,4\}\big\}$ and $\big\{\{1,4\},\{2,3\}\big\}$. Let $\Gamma$ be the subgraph $\big\{\{1,2\},\{3,4\}\big\}$. Then the inequalities of the proof of Proposition 3.1.3 give

$$(a_1 \leq a_3 \text{ or } a_2 \leq a_5) \text{ and } (a_1 \leq a_3 \text{ or } a_7 \leq a_6)$$

$$\text{and } (a_8 \leq a_4 \text{ or } a_2 \leq a_5) \text{ and } (a_8 \leq a_4 \text{ or } a_7 \leq a_6),$$

which is equivalent to

$$(a_1 \leq a_3 \text{ and } a_8 \leq a_4) \quad \text{or} \quad (a_2 \leq a_5 \text{ and } a_7 \leq a_6). \tag{3.1.7}$$

Now we consider the other disconnected subgraph which corresponds to the pair of disjoint edges $\{\{1,4\},\{2,3\}\}$ and get, similarly,

$$(a_3 \leq a_1 \text{ and } a_5 \leq a_2) \quad \text{or} \quad (a_6 \leq a_7 \text{ and } a_4 \leq a_8). \qquad (3.1.8)$$

By intersecting conditions (3.1.7) and (3.1.8), we get the desired relations.

Note that in this example the union of the four rational cones defined by the set of the linear inequalities $(1)-(4)$ is not a convex set. Indeed, if we take the exponent vectors $\mathbf{a} = (3,5,1,3,5,9,7,9)$ and $\mathbf{a}' = (1,3,1,1,7,11,11,1)$, then the corresponding ideals are both Cohen-Macaulay. However, for the vector $\mathbf{b} = (\mathbf{a} + \mathbf{a}')/2 = (2,4,1,2,6,10,9,5)$, the corresponding ideal is not Cohen-Macaulay.

## 3.2 Rigid depth

**Definition 3.2.1.** Let $\Delta$ be a pure simplicial complex. We say that $\Delta$ has *rigid depth* if for every unmixed monomial ideal $I \subset S$ with $\sqrt{I} = I_\Delta$ one has $\operatorname{depth}(S/I) = \operatorname{depth}(S/I_\Delta)$.

For example, any pure simplicial complex $\Delta$ with $\operatorname{depth}(K[\Delta]) = 1$ has rigid depth. In this section we characterize all the pure simplicial complexes which have rigid depth.

In the next theorem we will use the formula given in the following proposition for computing the depth of a Stanley-Reisner ring. We recall that the $i$th skeleton of a simplicial complex $\Delta$ is defined as $\Delta^{(i)} = \{F \in \Delta \mid \dim F \leq i\}$.

**Proposition 3.2.2.** *[37] Let $\Delta$ be a simplicial complex of dimension $d-1$. Then:*

$$\operatorname{depth}(K[\Delta]) = \max\{i \mid \Delta^{(i)} \textit{ is Cohen-Macaulay}\} + 1.$$

The following theorem generalizes [24, Theorem 3.2].

**Theorem 3.2.3.** *Let $\Delta$ be a pure simplicial complex with $\operatorname{depth}(K[\Delta]) = t$ and $I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_F$. The following statements are equivalent:*

(a) $\Delta$ *has rigid depth.*

(b) $\operatorname{depth}(S/I) = t$ *for every ideal* $I = \bigcap_{F\in\mathcal{F}(\Delta)} I_F$ *where* $I_F$ *are irreducible monomial ideals with* $\sqrt{I_F} = P_F$ *for all* $F \in \mathcal{F}(\Delta)$.

(c) $\operatorname{depth}(S/I) = t$ *for every ideal* $I = \bigcap_{F\in\mathcal{F}(\Delta)} P_F^{m_F}$ *where* $m_F$ *are positive integers.*

(d) $\operatorname{depth}(K[\Gamma]) \geq t$ *for every subcomplex* $\Gamma$ *of* $\Delta$ *with* $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$.

(e) *For every subcomplex* $\Gamma$ *of* $\Delta$ *with* $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$, *the skeleton* $\Gamma^{(t-1)}$ *is Cohen-Macaulay.*

(f) *Let* $\mathcal{F}(\Delta) = \{F_1, \dots, F_r\}$. *Then, for every* $1 \leq k \leq \min\{r, t\}$ *and for any indices* $1 \leq i_1 < \cdots < i_k \leq r$, *we have* $|F_{i_1} \cap \cdots \cap F_{i_k}| \geq t - k + 1$.

*Proof.* (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) are trivial.

(b) $\Rightarrow$ (d): Let $\Gamma$ be a subcompex of $\Delta$ with $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$. We have to show that $\operatorname{depth}(K[\Gamma]) \geq t$. For every $F \in \mathcal{F}(\Gamma)$, let $I_F = (x_i^2 \mid i \notin F)$, and for every $F \in \mathcal{F}(\Delta) \setminus \mathcal{F}(\Gamma)$ let $I_F = P_F = (x_i \mid i \notin F)$. Let $I = \bigcap_{F\in\mathcal{F}(\Delta)} I_F$. By assumption, $\operatorname{depth}(S/I) = t$. Let $S' \subset K[x_1, \dots, x_n, y_1, \dots, y_n]$ be the polynomial ring over $K$ in all the variables which are needed for the polarization of $I$, and let $I^p \subset S'$ be the polarization of $I$. We have $I^p = \bigcap_{F\in\mathcal{F}(\Delta)} I_F^p$, where

$$I_F^p = \begin{cases} (x_i y_i \mid i \notin F), & \text{if } F \in \mathcal{F}(\Gamma), \\ P_F, & \text{if } F \in \mathcal{F}(\Delta) \setminus \mathcal{F}(\Gamma). \end{cases}$$

Then $\operatorname{proj\,dim}(S'/I^p) = \operatorname{proj\,dim}(S/I)$. Let $N$ be the multiplicative set generated by all the variables $x_i$. Then $I_N^p = \bigcap_{F\in\mathcal{F}(\Gamma)} (y_i \mid i \notin F)$ and

$$\operatorname{proj\,dim}(S'/I^p)_N \leq \operatorname{proj\,dim}(S'/I^p) = \operatorname{proj\,dim}(S/I).$$

This inequality implies that $\operatorname{depth}(K[\Gamma]) \geq \operatorname{depth}(S/I) = t$.

(d) $\Leftrightarrow$ (e) follows immediately by applying the criterion given in Proposition 3.2.2.

(d) $\Rightarrow$ (f): We proceed by induction on $k$. The initial inductive step is trivial. Let $k > 1$ and assume that $|F_{i_1} \cap \cdots \cap F_{i_\ell}| \geq t - \ell + 1$ for $1 \leq \ell < k$ and for any $1 \leq i_1 < \cdots < i_\ell \leq r$. Obviously, it is enough to show that $|F_1 \cap \cdots \cap F_k| \geq t - k + 1$. By [**?**, Theorem 1.1], we have the following exact sequence of $S$-modules:

$$0 \to \frac{S}{\bigcap_{i=1}^k P_{F_i}} \to \bigoplus_{i=1}^k \frac{S}{P_{F_i}} \to \bigoplus_{1 \leq i < j \leq k} \frac{S}{P_{F_i} + P_{F_j}} \to \cdots \to \frac{S}{P_{F_1} + \cdots + P_{F_k}} \to 0. \quad (3.2.1)$$

By assumption, $\operatorname{depth}(S/\bigcap_{i=1}^k P_{F_i}) \geq t$. We decompose the above sequence in $k-1$ short exact sequences as follows:

$$0 \to \frac{S}{\bigcap_{i=1}^k P_{F_i}} \to \bigoplus_{i=1}^k \frac{S}{P_{F_i}} \to U_1 \to 0,$$

$$0 \to U_1 \to \bigoplus_{1 \leq i < j \leq k} \frac{S}{P_{F_i} + P_{F_j}} \to U_2 \to 0,$$

$$\vdots$$

$$0 \to U_{k-2} \to \bigoplus_{1 \leq j_1 < \cdots < j_{k-1} \leq k} \frac{S}{P_{F_{j_1}} + \cdots P_{F_{j_{k-1}}}} \to \frac{S}{P_{F_1} + \cdots + P_{F_k}} \to 0.$$

Note that, for all $\ell$ and any $1 \leq j_1 < \cdots < j_\ell \leq k$, we have

$$P_{F_{j_1}} + \cdots + P_{F_{j_\ell}} = P_{F_{j_1} \cap \cdots \cap F_{j_\ell}}.$$

In particular, $S/(P_{F_{j_1}} + \cdots + P_{F_{j_\ell}})$ is Cohen-Macaulay of depth equal to $|F_{j_1} \cap \cdots \cap F_{j_\ell}|$. Therefore,

$$\operatorname{depth}(\bigoplus_{1 \leq j_1 < \cdots < j_\ell \leq k} S/(P_{F_{j_1}} + \cdots P_{F_{j_\ell}})) \geq t - \ell + 1$$

for every $1 \leq \ell < k$ and any $1 \leq j_1 < \cdots < j_\ell \leq k$. Now, by using the inductive hypothesis and by applying Depth Lemma in the first $k-2$ above short exact sequences from top to bottom, step by step, we obtain $\operatorname{depth}(U_1) \geq t-1, \operatorname{depth}(U_2) \geq t-2, \ldots, \operatorname{depth}(U_{k-2}) \geq t-k+2$. Finally, by applying Depth Lemma in the last short exact sequence, since the depth of the middle term is $\geq t-k+2$, we get $\operatorname{depth}(S/(P_{F_1} + \cdots + P_{F_k})) = |F_1 \cap \cdots \cap F_k| \geq t - k + 1$.

(f)$\Rightarrow$(d): Let $\Gamma$ be a subcomplex of $\Delta$ with $\mathcal{F}(\Gamma) = \{F_{j_1}, \ldots, F_{j_k}\} \subset \mathcal{F}(\Delta)$. We have to show that $\operatorname{depth}(K[\Gamma]) \geq t$. We may obviously assume that $k < r$ and the facets of $\Gamma$ are $F_1, \ldots, F_k$. If $k \leq t$, then we use the short exact sequences derived from (3.2.1) in the proof of (d) $\Rightarrow$ (f) and, by applying successively Depth Lemma from bottom to the top, we get, step by step, $\operatorname{depth}(U_{k-2}) \geq t-k+2, \ldots, \operatorname{depth}(U_2) \geq t-2, \operatorname{depth}(U_1) \geq t-1$, and, finally, from the first exact sequence, $\operatorname{depth}(K[\Gamma]) \geq t$. If $t < k$, we use only the first $t$ short exact sequences, that is, we stop at

$$0 \to U_{t-1} \to \bigoplus_{1 \leq j_1 < \cdots < j_t \leq k} \frac{S}{P_{F_{j_1}} + \cdots + P_{F_{j_t}}} \to U_t \to 0.$$

Since the middle term in this short exact sequence has depth $\geq 1$, we get $\operatorname{depth}(U_{t-1}) \geq 1$. Next, by using the same arguments as before, we get $\operatorname{depth}(U_{t-2}) \geq 2, \ldots, \operatorname{depth}(U_1) \geq t-1$, and, finally, $\operatorname{depth}(K[\Delta]) \geq t$, as desired.

The implication (d) $\Rightarrow$ (a) follows by Theorem 3.1.2.

Finally, the implication (c) $\Rightarrow$ (e) follows similarly to the proof of Corollary 1.9 in [30]. □

In order to state the first consequence of the above theorem, we need to know the behavior of the depth of a Stanley-Reisner ring over a field when passing from characteristic 0 to characteristic $p > 0$. We show in the next lemma that the Betti numbers of the Stanley-Reisner ring can only go up when passing from characteristic 0 to a positive characteristic which, in particular, implies that the depth does not increase. This result is certainly known. However we include here its proof since we could not find any precise reference. The argument of the proof was communicated for the thesis by Ezra Miller.

**Lemma 3.2.4.** *Let $\Delta$ be a simplicial complex on the vertex set $[n]$ and let $K, L$ be two fields with* $\operatorname{char} K = 0$, $\operatorname{char} L = p > 0$. *Then $\beta_i(K[\Delta]) \leq \beta_i(L[\Delta])$ for all $i$.*

*Proof.* Any field is flat over its prime field. Therefore, since $\operatorname{char} K = 0$, we have $\beta_i(K[\Delta]) = \beta_i(\mathbb{Q}[\Delta])$ for all $i$, and since $\operatorname{char} L = p$, we have $\beta_i(K[\Delta]) = \beta_i(\mathbb{F}_p[\Delta])$ for all $i$, where $\mathbb{F}_p$ is the prime field of characteristic $p$. In other words, the Betti numbers depend only on the characteristic of the base field. Let $\mathbb{Z}_p$ be the local ring of the integers at the prime $p$. The ring $\mathbb{Z}_p[X]$ is *local ([6, Section 1.5]) and the Stanley-Reisner ideal $I_\Delta \subset \mathbb{Z}_p[X]$ is *homogeneous. Let $\mathcal{F}$ be a minimal free resolution of $\mathbb{Z}_p[\Delta]$ over $\mathbb{Z}_p[x_1, \ldots, x_n]$. Since $p$ is a nonzerodivisor on $\mathbb{Z}_p[\Delta]$, by [**?**, Lemma 8.27], the quotient $\mathcal{F}/p\mathcal{F}$ is a minimal free resolution of $\mathbb{F}_p[\Delta]$ over $\mathbb{F}_p[x_1, \ldots, x_n]$. On the other hand, the localization $\mathcal{F}[p^{-1}]$ by inverting $p$ is a free resolution, not necessarily minimal, of $\mathbb{Q}[\Delta]$ over $\mathbb{Q}[x_1, \ldots, x_n]$. Since the modules in $\mathcal{F}/p\mathcal{F}$ and $\mathcal{F}[p^{-1}]$ have the same ranks, it follows that $\beta_i(\mathbb{Q}[\Delta]) \le \beta_i(\mathbb{F}_p[\Delta])$ for all $i$ which leads to the desired inequalities. □

**Corollary 3.2.5.** *Let $\Delta$ be a pure simplicial complex with rigid depth over a field of characteristic* 0*. Then $\Delta$ has rigid depth over any field.*

*Proof.* Let $K$ be a field of characteristic 0 and $L$ a field of characteristic $p > 0$. The above lemma implies that $\operatorname{proj\,dim} K[\Delta] \le \operatorname{proj\,dim} L[\Delta]$. By Auslander-Buchsbaum formula, it follows that $\operatorname{depth} K[\Delta] \ge \operatorname{depth} L[\Delta]$. Therefore, the desired statement follows by applying the combinatorial condition (f) of Theorem 3.2.3.

□

**Example 3.2.6.** Let $\Delta$ be the six-vertex triangulation of the real projective plane; see [6, Section 5.3]. If $\operatorname{char} K \neq 2$, then $\Delta$ is Cohen-Macaulay over $K$, hence $\operatorname{depth}(K[\Delta]) = 2$, and, by condition (f) of Theorem 3.2.3, it follows that $\Delta$ does not have rigid depth over $K$. But if $\operatorname{char} K = 2$, then $\operatorname{depth}(K[\Delta]) = 1$, and, consequently, $\Delta$ has rigid depth over $K$.

The simplicial complexes with one or two facets have rigid depth.

**Lemma 3.2.7.** *Let $\Delta$ be a pure simplicial complex with at most two facets. Then $\Delta$ has rigid depth.*

*Proof.* We only need to consider the case of simplicial complexes with two facets since the other case is obvious. Let $\dim \Delta = d - 1$ and $\mathcal{F}(\Delta) = \{F, G\}$. We show that $\operatorname{depth}(K[\Delta]) = t$ if and only if $|F \cap G| = t - 1$. Then the claim follows by condition (f) in Theorem 3.2.3. We consider the exact sequence

$$0 \to K[\Delta] \to (S/P_F) \oplus (S/P_G) \to S/(P_F + P_G) \cong K[x_i \mid i \in F \cap G] \to 0.$$

As $(S/P_F) \oplus (S/P_G)$ and $S/(P_F + P_G)$ are Cohen-Macaulay of dimensions $d$ and, respectively, $|F \cap G|$, it follows that $\operatorname{depth}(K[\Delta]) = t$ if and only if $|F \cap G| = t - 1$.

□

**Example 3.2.8.** Let $\Delta$ and $\Gamma$ be the simplicial complexes with $\mathcal{F}(\Delta) = \{\{1,2,3\}, \{1,4,5\}\}$ and $\mathcal{F}(\Gamma) = \{\{1,2,3\},\{1,3,4\}\}$. Obviously, by Lemma 3.2.7, $\Delta$ is non-Cohen-Macaulay of rigid depth 2, while $\Gamma$ is Cohen-Macaulay of rigid depth.

In the sequel we investigate whether the rigid depth property is preserved by the skeletons of the simplicial complexes with rigid depth. The next example shows that this is not the case.

**Example 3.2.9.** Let $\Delta$ be the simplicial complex on the vertex set [8] with $\mathcal{F}(\Delta) = \{F, G\}$ where $F = \{1,2,3,4,5\}$ and $G = \{1,2,6,7,8\}$. Then, by Lemma 3.2.7 and its proof, it follows that $\operatorname{depth}(K[\Delta]) = 3$ and $\Delta$ has rigid depth. Let $\Delta^{(3)}$ be the 3-dimensional skeleton of $\Delta$ and $\Gamma$ the subcomplex of $\Delta^{(3)}$ with the facets $G_1 = \{1,2,3,5\}$ and $G_2 = \{2,6,7,8\}$. Then, again by the proof the above lemma, we get $\operatorname{depth}(K[\Gamma]) = 2$. But $\operatorname{depth} K[\Delta^{(3)}] = 3$, thus the skeleton $\Delta^{(3)}$ of $\Delta$ does not have rigid depth since it does not satisfy condition (d) in Theorem 3.2.3.

However, as an application of Theorem 3.2.3, we prove the following

**Proposition 3.2.10.** *Let $\Delta$ be a pure simplicial complex with rigid depth and let $t = \operatorname{depth}(K[\Delta])$. If $\Delta^{(i)}$ has rigid depth for some $i \geq t-1$, then $\Delta^{(j)}$ has rigid depth for every $j \geq i$.*

*Proof.* By [19], we know that $\operatorname{depth}(K[\Delta^{(i)}]) = t$ for $i \geq t - 1$. It is enough to show that if $\Delta^{(i)}$ has rigid depth for some $i \geq t-1$, then $\Delta^{(i+1)}$ has the same property.

Let $\Gamma \subset \Delta^{(i+1)}$ be a subcomplex with $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta^{(i+1)})$. Then $\Gamma^{(i)}$ is a subcomplex of $\Delta^{(i)}$ and $\mathcal{F}(\Gamma^{(i)}) \subset \mathcal{F}(\Delta^{(i)})$. By our assumption and by using condition (e) in Theorem 3.2.3, it follows that $\Gamma^{(t-1)}$ is Cohen-Macaulay. Therefore, $\Delta^{(i+1)}$ satisfies condition (e) in Theorem 3.2.3, which ends our proof. □

# Chapter 4

# The stable set of associated prime ideals of a squarefree principal Borel ideal

## 4.1 The persistence property for squarefree principal Borel ideals

Let $K$ be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring over $K$ in $n$ indeterminates. A monomial $u = x^{\mathbf{a}} = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ of degree $d$ we often write in the form $u = x_{i_1}x_{i_2}\cdots x_{i_r}$ with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n$. The monomial $u$ is squarefree, if $\deg_{x_i}(u) \leq 1$ for all $i$, or equivalently, if $i_1 < i_2 < \cdots < i_r$.

We set $\deg_{x_i}(u) = a_i$ for $i = 1, \ldots, n$, and set $\min(u) = \min\{i\colon\ a_i \neq 0\}$ and $\max(u) = \max\{i\colon\ a_i \neq 0\}$.

For a monomial ideal $I$ denote by $G(I)$ the unique minimal set of monomial generators of $I$.

We first recall some concepts related to stable ideals. Let $I$ be a monomial ideal in $S$ generated in single degree. Then $I$ is called a *$k$-strongly stable ideal*, if all monomials $u \in G(I)$ are of the form $x_1^{a_1}\cdots x_n^{a_n}$ with $a_i \leq k$, and for all $u \in G(I)$ and all integers $1 \leq i < j \leq n$ such that $x_j$ divides $u$ and $\deg_{x_i}(u) < k$ we have $x_i(u/x_j) \in G(I)$. A monomial ideal is called a squarefree strongly stable ideal if it is a 1-strongly stable ideal.

Let $I$ be a $k$-strongly stable ideal, then the elements $u_1,\dots,u_r\in G(I)$ are called *Borel generators* of $I$, if $I$ is the smallest $k$-strongly stable ideal containing $u_1,\dots,u_r$. If $u_1,\dots,u_r$ are Borel generators of $I$, we set $B_k(u_1,\dots,u_r)=G(I)$.

**Definition 4.1.1.** A monomial ideal is called $k$-strongly stable *principal* ideal, if there exist a monomial $u\in G(I)$ such that $I=(B_k(u))$.

Let $u=x_{i_1}x_{i_2}\cdots x_{i_d}$ with $1\le i_1<i_2<\cdots<i_d\le n$ be a squarefree monomial in $S$. We observe that

$$B_1(u)=\{x_{j_1}\cdots x_{j_d}\colon\ 1\le j_1<j_2<\cdots<j_d\le n \quad \text{and} \quad j_k\le i_k \quad \text{for} \quad 1\le k\le d\}$$

Let $P$ be a monomial prime ideal in $S$. Then $P$ is generated by a subset of $\{x_1,\dots,x_n\}$, thus there exist a set $A\subset[n]$ such that $P=P_A$, where $P_A=\{x_i\colon\ i\notin A\}$.

Now let $I$ be arbitrary monomial ideal in $S$. Then there exist a unique monomial ideal $I(P_A)\subset S_A$ where $S_A=K[x_i\colon\ i\in A]$ such that

$$I(P_A)S_{P_A}=IS_{P_A}.$$

Notice that $I(P_A)$ may also be viewed as saturation of $I$ with respect to $u=\prod_{i\in A}x_i$. In other words

$$I(P_A)=I:u^\infty=\bigcup_{k\ge 0}I:u^k.$$

The ideal $I(P_A)$ is called the *monomial localization* of $I$ with respect to the monomial prime ideal $P_A$.

Observe that for any two subsets $A\subset B$ we have

$$I(P_A)(P_{B\setminus A})=I(P_B). \tag{4.1.1}$$

**Theorem 4.1.2.** *Let $I\subset S$ be a squarefree strongly stable principal ideal, and $A$ a subset of $[n]$. Then $I(P_A)$ is squarefree strongly stable principal ideal in $S_A$.*

*Proof.* Let $u = x_{i_1} \cdots x_{i_d}$ be the Borel generator of $I$, and let $k \in [n]$. We first show that $I(P_{\{k\}})$ is again a squarefree strongly stable monomial ideal with Borel generator $v$ where

$$v = \begin{cases} u, & \text{if} \quad k > \max(u); \\ u/x_{i_j}, & \text{if} \quad i_{j-1} < k \leq i_j. \end{cases} \tag{4.1.2}$$

Indeed, we claim that

$$I(P_{\{k\}}) = (B_1(v)). \tag{4.1.3}$$

First we show that $I(P_{\{k\}})$ is a strongly stable ideal in $S_{\{k\}}$ generated either in degree $d-1$ or in degree $d$. Observe that $I(P_{\{k\}})$ is generated by the monomials $w \in G(I)$ such that $x_k \nmid w$ together with monomials $w \in S_{\{k\}}$ such that $wx_k \in G(I)$. It follows that $G(I)$ has generators at most in degree $d-1$ and in degree $d$, and generated only in degree $d$ if no element in $G(I)$ is divisible by $x_k$.

Suppose first that $k > i_d$. Then $x_k$ divides no element in $G(I)$, and hence $I(P_{\{k\}}) = (B_1(u)) = (B_1(v))$, and we are done.

Next suppose that $k \leq i_d$. We show that each $w \in G(I)$ which is not divisible by $x_k$ is a multiple of a generator of of degree $d-1$ in $I(P_{\{k\}})$. Let $w = x_{j_1} \cdots x_{j_d}$ be a monomial in $B_1(u)$, then $j_1 \leq i_1, \ldots, j_d \leq i_d$, and suppose that $x_k$ does not divide $w$. If $k > j_d$ we set $w' = x_k(w/x_{j_d}$. Otherwise there exists an integer $t$ such that $j_{t-1} < k < j_t$, and we set $w' = x_k(w/x_{j_t}$. In both cases, $w' = x_k(w/x_{i_j})$ belongs to $I$ and $x_k$ divides $w'$. Therefore $w'/x_k$ is an element of degree $d-1$ in $I(P_{\{k\}})$, and $w = x_{i_j}(w'/x_k)$. In conclusion we find that $I(P_{\{k\}})$ is generated in degree $d-1$ if $k \leq i_d$.

It remains to be shown that $(B_1(v))I(P_{\{k\}})$ if $k \leq i_d$. We first prove that $(B_1(v)) \subset I(P_{\{k\}})$. Since both ideals $I(P_{\{k\}})$ and $(B_1(v))$ are generated in degree $d-1$ in the case that $k \leq i_d$, and since $v \in I(P_{\{k\}})$, the inclusion $(B_1(v)) \subset I(P_{\{k\}})$ will follow, once we have shown that $I(P_{\{k\}})$ is squarefree strongly stable. To see this, let $w \in G(I(P_{\{k\}}))$, then $wx_k \in G(I)$. Let $1 \leq i < j \leq n$ with $i, j \neq k$ and such that $x_j|w$ and $x_i \nmid w$. It follows that $x_j|wx_k$ and $x_i \nmid wx_k$. Since $I$ is a squarefree strongly stable ideal, we have that $x_i(wx_k)/x_j = (x_iw/x_j)x_k \in I$, and hence $x_iw/x_j \in I(P_{\{k\}})$. This shows that $I(P_{\{k\}})$ is indeed squarefree strongly stable.

Next we show that $I(P_{\{k\}}) \subset (B_1(v))$, let $w = x_{l_1} \cdots x_{l_{d-1}} \in G(I(P_{\{k\}}))$. Then $wx_k = x_{l_1} \cdots x_{l_{t-1}} x_k x_{l_t} \cdots x_{l_{d-1}} \in G(I)$, where $l_{t-1} < k \leq l_t$ for some $t$. As $I$ is a squarefree strongly stable principal ideal with Borel generator $u = x_{i_1} \cdots x_{i_d}$, it follows that $l_1 \leq i_1, \ldots, l_{t-1} \leq i_{t-1}, k \leq i_t, l_t \leq i_{t+1} \ldots, l_{d-1} \leq i_d$. To show that $v$ is the unique Borel generator of $I(P_{\{k\}})$, it suffices to show that

$$l_1 \leq i_1, \ldots, l_{j-1} \leq i_{j-1}, l_j \leq i_{j+1} \ldots, l_{d-1} \leq i_d. \tag{4.1.4}$$

Suppose $t < j$, then $i_{j-1} < k \leq i_t \leq i_{j-1}$, a contradiction. Therefore $t \geq j$, and the inequalities (4.1.4) are satisfied and this proves our claim.

Finally we consider the general case. Let $A = \{j_1, \ldots, j_r\} \subset [n]$, and set $B = \{j_1, \ldots, j_{r-1}\}$. We proceed by induction on $r$. If $r = 1$, then the statement follows from what we have already shown. Let $B = \{j_1, \ldots, j_{r-1}\}$. Then $B \subset A$ and $I(P_B)(P_{\{j_r\}}) = I(P_A)$. By our induction hypothesis, $I(P_B)$ is a squarefree strongly stable principal ideal in $S_B$. Hence the desired conclusion follows from the case discussed in the first part of the proof. □

A prime ideal $P \in V(I)$ is said to be a *persistent prime ideal* of $I$, if whenever $P \in \operatorname{Ass}(I^k)$ for some exponent $k$, then $P \in \operatorname{Ass}(I^{k+1})$. If this happens to be so for $k$, then of course we have $P \in \operatorname{Ass}(I^\ell)$ for all $\ell \geq k$. The ideal $I$ is said to have the *persistence property* if all prime ideals $P \in \bigcup_k \operatorname{Ass}(I^k)$ are persistent prime ideals.

**Corollary 4.1.3.** *Let $I$ be squarefree strongly stable principal ideal. Then $I$ satisfies the persistence property.*

*Proof.* Let $P \in \operatorname{Ass}(I^k)$ for some $k > 0$. Since $I$ is a monomial ideal it follows that $P$ is a monomial prime ideal. Therefore there exist a subset $A \subset [n]$ such that $P = P_A$. One has $P_A \in \operatorname{Ass}(I^k)$ if and only if $\mathfrak{m} \in \operatorname{Ass}(I(P_A)^k)$, where $\mathfrak{m}$ is the graded maximal ideal of $S_A$. We know that $\mathfrak{m} \in \operatorname{Ass}(I(P_A)^k)$ if and only if $\operatorname{depth}(I(P_A)^k) = 0$. Since

$I$ is a squarefree strongly stable principal ideal, Theorem 4.1.2 implies that $I(P)$ is again a squarefree strongly stable principal ideal. In [9, Proposition 3.4], De Negri has shown that powers of squarefree strongly principal ideals are $k$-strongly stable principal ideals. Therefore, since for all $l$, $I(P_A)^l$ are $k$-strongly stable principal ideals, it follows from [17, Theorem 2.1] that $I(P_A)^l$ has a linear resolution for all $l$. Now applying [18, Proposition 2.1] we have that $\operatorname{depth}(I(P_A)^l) \geq \operatorname{depth}(I(P_A)^{l+1})$ for all $l$. In particular, if $\mathfrak{m} \in \operatorname{Ass}(I(P_A)^k)$ then $\mathfrak{m} \in \operatorname{Ass}(I(P_A)^{k+1})$. This implies that $P_A \in \operatorname{Ass}(I^{k+1})$, as desired. □

## 4.2 When is the maximal ideal a stable associated prime ideal?

The following theorem give an answer to the question raised in the title of this section. The trivial case that $u = x_1$ will not be considered in the following discussion.

**Theorem 4.2.1.** *Let $I \subset S = K[x_1, x_2, \ldots, x_n]$ be squarefree strongly stable principal ideal with Borel generator $u \neq x_1$. Then $\mathfrak{m} \in \operatorname{Ass}(I^k)$ for some $k \geq 1$ if and only if $\min(u) > 1$ and $\max(u) = n$.*

*Proof.* Let $u = x_{i_1} \cdots x_{i_d}$, with $1 \leq i_1 < \cdots < i_d \leq n$. First we show that, if $\max(u) < n$ then $\mathfrak{m} \notin \operatorname{Ass}(I^k)$ for all $k > 0$. By using [9, Proposition 3.4], $I^k$ is a $k$-strongly stable principal ideal with Borel generator $u^k$.

Let $B_k(u^k) = \{u_1, \ldots, u_r\}$. We may assume that $u_i >_{lex} u_j$ for $i < j$. Then by [4, Section 2], one has

$$\begin{aligned}(u_1, \ldots, u_{i-1}) : u_i &= (\{x_j : 1 \leq j \leq \max(u_i) - 1\} \\ &- \{x_j \in \operatorname{supp}(u_i) : \deg_{x_j}(u_i) = k\}). \qquad (4.2.1)\end{aligned}$$

Let $q(I^k)$ be the maximal of the number of generators of $(u_1, \ldots, u_{i-1}) : u_i$ for $i = i, \ldots, r$. By [20, Eq. 1] it is known that $\operatorname{depth}(S/I^k) = n - q(I^k) - 1$. Since

$\max(u) < n$, we also have $\max(u_i) < n$ for all $i$. It follows therefore from (4.2.1) that $q(I^k) < n-1$. Hence, $\operatorname{depth}(S/I^k) > 0$, and so $\mathfrak{m} \notin \operatorname{Ass}(I^k)$.

Now we show that if $\min(u) = 1$, then $\mathfrak{m} \notin \operatorname{Ass}(I^k)$. In this case, $u = x_1 x_{i_2} \cdots x_{i_d}$ with $1 < i_2 < \cdots < i_d \leq n$ and $d \geq 2$, because $u \neq x_1$. Since $\min(u) = 1$, we have $\deg_{x_1}(u_i) = k$ for all $i$. It follows therefore from (4.2.1) that $q(I^k) < n-1$. Therefore, $\operatorname{depth}(S/I^k) > 0$, and hence $\mathfrak{m} \notin \operatorname{Ass}(I^k)$.

Finally suppose that $\max(u) = n$ and $\min(u) > 1$. Then $u = x_{i_1} x_{i_2} \cdots x_{i_r} x_n$, where $1 < i_1 < i_2 < \cdots < i_r \leq n-1$. Let $k > r$. Then the element $v = x_1^r x_{i_1}^{k-1} x_{i_2}^{k-1} \cdots x_{i_r}^{k-1} x_n^k$ belongs to $G(I^k)$. Hence there exists an integer $i$ such that $v = u_i$. Therefore formula (4.2.1) implies that $q(I^k) = n-1$. As a consequence, we have $\operatorname{depth}(S/I^k) = 0$, and hence $\mathfrak{m} \in \operatorname{Ass}(I^k)$. $\square$

Let as before $I = (B_1(u))$ and assume that $\min(u) > 1$ and $\max(u) = n$. Then $u = x_{i_1} x_{i_2} \cdots x_{i_{d-1}} x_n$ with $1 < i_1 < i_2 < \cdots < i_{d-1} < n$. In the following we want to determine the smallest integer $k$ for which $\mathfrak{m} \in \operatorname{Ass}(I^k)$. To do this we have to introduce some notation. We write the set $\{i_1, \ldots, i_{d-1}, n\}$ in a unique way as a disjoint union of intervals. In other words,

$$\{i_1, \ldots, i_{d-1}, n\} = \bigcup_{j=1}^{m} [a_j, b_j] \quad \text{with} \quad a_i \leq b_j < a_{j+1} \leq b_{j+1} \quad \text{for} \quad j = 1, \ldots, m-1.$$

The number

$$l_j = \begin{cases} b_j - a_j + 1, & \text{if} \quad j < m; \\ n - a_m, & \text{if} \quad j = m. \end{cases} \tag{4.2.2}$$

is the length of interval $[a_j, b_j]$ for $j < m$, and $l_m$ is the length of the last interval minus one.

Similarly we define the length of the gap intervals

$$k_j = \begin{cases} a_1 - 1, & \text{if} \quad j = 1; \\ a_j - b_{j-1} - 1, & \text{if} \quad j > 1. \end{cases} \tag{4.2.3}$$

Let $I \subset S$ be monomial ideal and $P \subset S$ be a monomial prime ideal. Suppose that $P \in \operatorname{Ass}(I^k)$ for some $k$. We denote by $\lambda(P; I)$ the smallest index $k$ for which this is the case, and set $\lambda(P;I)=\infty$ if $P\notin \operatorname{Ass}(I^k)$ for all $k$. The number $\lambda(P;I)$ is called the *index of stability* of $P$ with respect to $I$.

**Theorem 4.2.2.** *Let $I$ be a squarefree strongly stable principal ideal with Borel generator*

$$x_{i_1}x_{i_2}\cdots x_{i_{d-1}}x_n \quad \textit{with} \quad 1<i_1<i_2<\cdots<i_{d-1}<n.$$

*With the notation introduced in (4.2.2) and (4.2.3) for the length of the intervals and gap intervals of the sequence $\{i_1,i_2,\dots,i_{d-1},n\}$, we have*

$$\lambda(\mathfrak{m};I)=\max_{j=1,\dots,m}\left\{\left\lceil\frac{l_1+l_2+\cdots+l_j}{k_1+k_2+\cdots+k_j}\right\rceil+1\right\} \tag{4.2.4}$$

*Proof.* By [20, Equation 1], $\mathfrak{m}\in\operatorname{Ass}(I^k)$ if and only if $q(I^k)=n-1$. So we want to find the smallest integer $k$ for which $q(I^k)=n-1$. In other words, we have to find the smallest integer $k$ for which there exist a monomial $v\in B_k(u^k)$ with the property that $\deg_{x_j}(v)<k$ for $j=1,\dots,n-1$.

If $v\in B_k(u^k)$ with $\deg_{x_i}(v)<k$ for all $i<n$. Then $v$ must be of the form

$$v=\prod_{s=1}^{m}(\prod_{i\in[b_{s-1}+1,a_{s-1}]}x_i^{c_i}\prod_{i\in[a_s,b_s]}x_i^{d_i}),$$

where all $c_i,d_i<k$, except possibly $d_m$ which may be equal to $k$, where $b_0$ is defined to be 0, and where

$$\sum_{s=1}^{j}\sum_{i\in[b_{s-1}+1,a_{s-1}]}c_i\geq\sum_{s=1}^{j}\sum_{i\in[a_s,b_s]}(k-d_i)$$

for all $j=1,\dots,m$. Since

$$(k_1+k_2+\cdots+k_j)(k-1)\geq\sum_{s=1}^{j}\sum_{i\in[b_{s-1}+1,a_{s-1}]}c_i,$$

and

$$l_1+l_2+\cdots+l_j\leq\sum_{j=s}^{j}\sum_{i\in[a_s,b_s]}(k-d_i)$$

it follows that

$$(k_1+k_2+\cdots+k_j)(k-1)\geq l_1+l_2+\cdots+l_j.$$

This shows that $\lambda(\mathfrak{m}; I) \geq \max_{j=1,\ldots,m}\left\{\left\lceil \frac{l_1+l_2+\cdots+l_j}{k_1+k_2+\cdots+k_j}\right\rceil + 1\right\}$.

In order to prove the opposite inequality, we let $k$ be the maximum on the right hand side of the above inequality. Then there exists $c_i < k$ such that

$$\sum_{s=1}^{j} \sum_{i\in[b_{s-1}+1, a_{s-1}]} c_i \geq l_1 + l_2 + \cdots + l_j$$

for all $j = 1, \ldots, m$. This implies that

$$v = \prod_{s=1}^{m}\Big(\prod_{i\in[b_{s-1}+1,a_{s-1}]} x_i^{c_i} \prod_{i\in[a_s,b_s]} x_i^{k-1}\Big)x_n$$

belongs to $B_k(u^k)$, and shows that $\lambda(\mathfrak{m}; I) \leq k$. □

**Corollary 4.2.3.** *Let $u \in K[x_1, x_2, \ldots, x_n]$ be a squarefree monomial of degree $d$, and let $I$ be the squarefree strongly stable principal ideal with Borel generator $u$. Assume that $\lambda(\mathfrak{m}; I) < \infty$. Then*

*(a)* $\lambda(\mathfrak{m}; I) \leq d$.

*(b)* $\lambda(\mathfrak{m}; I) = d$ *if and only if* $u = x_2x_3\cdots x_dx_n$.

*Proof.* (a) Since $l_1 + l_2 + \cdots + l_m = d - 1$, the maximum value of $\lceil \frac{l_1+l_2+\cdots+l_j}{k_1+k_2+\cdots+k_j}\rceil$ is at most $d - 1$. Hence (4.2.4) implies $\lambda(\mathfrak{m}; I) \leq d$.

(b) Let $u = x_2x_3\cdots x_dx_n$. Then by (4.2.4),

$$\lambda(\mathfrak{m}; I) = \lceil \frac{l_1}{k_1} \rceil + 1 = d - 1 + 1 = d.$$

Now let $u \neq x_2x_3\cdots x_dx_n$. Then either $k_1 \geq 2$, or there exists a $1 < j < m$ such that $k_j \geq 1$. So the maximum value of $\lceil \frac{l_1+l_2+\cdots+l_j}{k_1+k_2+\cdots+k_j}\rceil$ is less than $d - 1$, and hence by (4.2.4), $\lambda(\mathfrak{m}; I) < d$. □

**Corollary 4.2.4.** *Given an integer $d \geq 2$. Then for any integer $2 \leq i \leq d$, there exists a squarefree monomial $u$ of degree $d$ in $2d-i+1$ variables such that $\lambda(\mathfrak{m}; I) = i$ for $I = B_1(u)$.*

*Proof.* Choose the monomial $u = (\prod_{j=2}^{i} x_j \prod_{j=1}^{d-i} x_{i+2j})x_{2d-i+1}$. □

## 4.3 The stable set of prime ideals

Let $I$ be a squarefree strongly stable principal ideal with Borel generator $u = x_{i_1}\cdots x_{i_d}$. In this section we want to determine the subsets $A = \{k_1 < \cdots < k_s\}$ of $[n]$, for which the prime ideal $P_A$ belongs to $\mathrm{Ass}^\infty(I)$.

We have seen in Theorem 4.1.2 that $I(P_A)$ is a squarefree strongly stable principal ideal. We denote its Borel generator by $u_A$. We observe that

$$\max(u_A) \geq \max(u_B) \quad \text{for} \quad A \subset B \subset [n]. \tag{4.3.1}$$

For the main result of this section we need the following results.

**Lemma 4.3.1.** *Let $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ and $A = \{k_1 < k_2 < \cdots < k_s\}$ with $k_s \leq i_d$. Then the following conditions are equivalent:*

*(a)* $\max(u_A) < i_d$.

*(b)* $k_{s-j} > i_{d-j-1}$ *for some* $j \geq 0$.

*(c) there exist an integer $j$ with $0 \leq j \leq s-1$ such that $l(s-j) \geq d-j$, where* $l(t) = \min\{r\colon\ k_t \leq i_r\}$.

*Proof.* We prove the implication (a) $\Rightarrow$ (b) by induction on $d$. The assertion is trivial for $d = 1$. Now let $d > 1$, and suppose $k_{s-j} \leq i_{d-j-1}$ for all $j \geq 0$. Then we have,

$$k_1 \leq i_{d-s}, k_2 \leq i_{d-s+1}, \ldots, k_{s-1} \leq i_{d-2}, k_s \leq i_{d-1}. \tag{4.3.2}$$

Let $B = \{k_2, k_3, \ldots, k_s\}$, and set $v = u_{A\setminus B}$. Then it follows from (4.2.2) that $v = x_{i_1}x_{i_2}\cdots\widehat{x_{i_j}}\cdots x_{i_d}$, where $j = \min\{r\colon\ k_1 \leq i_r\}$. Since $k_1 \leq i_{d-s}$ it follows that $j \leq d-s$. Therefore the corresponding inequalities (4.3.2) for $B$ compared with the indices of $v$ hold. Since $\max(v) = i_d$ and $\deg(v) = d-1$, we may apply our induction hypothesis and obtain that $\max(v_B) = i_d$. Since $u_A = v_B$, we conclude that $\max(u_A) = i_d$, a contradiction.

(b) $\Rightarrow$ (a): By assumption, we have $k_{s-t} > i_{d-t-1}$ for some $t \geq 0$. Then $|\{r\colon\ k_{s-t} \leq i_r\}| \leq t+1$. Now we show by induction on $j$ that whenever

$$|\{r\colon\ k_{s-j} \leq i_r\}| \leq j+1 \tag{4.3.3}$$

for some $j$, then $\max(u_A) < i_d$. If $j = 0$, then (4.3.3) implies $i_r$ is the unique index of $u$ with the property that $k_s \leq i_r$. Therefore it follows from (4.2.2) that $\max(u_A) < i_d$. Suppose now that for $j > 0$, the inequality (4.3.3) holds. Let $v = u_{\{s-j\}} = x_{i_1}x_{i_2}\cdots x_{i_{d-j-1}}x_{i_{d-j+1}}\cdots x_{i_d}$. Therefore the set of indices in $v$ which are greater or equal to $k_{s-j+1}$ is contained in the set $\{i_{d-j+1}, \ldots, i_d\}$, whose cardinality is $j$. Our induction hypothesis implies that $\max(v) < i_d$. Since $\max(u_A) \leq \max(v)$, the assertion follows.

(b) $\Longleftrightarrow$ (c): It follows from the definition of the function $l(t)$ that $l(s-j) \geq d-j$ for some $j \geq 0$, if and only if $k_{s-j} > i_{d-j-1}$. □

Given $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ and $A = \{k_1 < k_2 < \cdots < k_s\}$. Set $k_0 = 0$ and $k_{s+1} = n+1$, we introduce the following numbers. Let $f = \max\{r \in [s]_0\colon\ k_r + 1 < k_{r+1}\}$ where $[s]_0 = 0, \ldots, s$. Then $\max(S_A) = k_{f+1} - 1$. Let $g = \max\{t\colon\ i_t \leq k_{f+1} - 1\}$, and set $B = \{k_{f+1}, \ldots, k_s\}$. Since $u_A = (u_B)_{A\setminus B}$, it follows by Lemma 4.3.1 that $\max(u_A) = \max(S_A)$ if and only if $l(f-j) < g-j$ for $j = 0, \ldots, r-1$. Let $h = \min\{j\colon,\ i_{j+1} > i_j + 1\}$ and set $i_0 = 0$. Now we are ready to state our main result.

**Theorem 4.3.2.** *Let $I \subset S = K[x_1, x_2, \ldots, x_n]$ be squarefree strongly stable principal ideal with Borel generator $u \neq x_1$. With the notation introduced above the following conditions are equivalent:*

*(a)* $P_A \in \mathrm{Ass}^\infty(I)$.

*(b)* (i) $\min(u_A) > \min(S_A)$ *and* (ii) $\max(u_A) = \max(S_A)$.

*Moreover,* (b)(i) *holds if and only if $k_t = t$ for $t = 0, \ldots, h$, and* (b)(ii) *holds if and only if $l(f-j) < g-j$ for $j = 0, \ldots, f-1$.*

*Proof.* One has $P_A \in \operatorname{Ass}^\infty(I)$ if and only if $\mathfrak{m}_A \in \operatorname{Ass}^\infty(I(P_A))$. Therefore, the equivalence of (a) and (b) follows from Theorem 4.2.1. The remaining statements follow for the definition of the the numbers $h$, $f$ and $g$ and from Lemma 4.3.1. □

Consider the following simple, concrete example of a squarefree strongly stable principal ideal $I$ with Borel generator $u = x_1x_3x_4x_5 \in K[x_1, \dots, x_5]$. We use Theorem 4.3.2 to determine the subsets $A \subset [5]$ for which $P_A \in \operatorname{Ass}^\infty(I)$. The last column of the table gives the smallest power of $I^k$ of $I$ with $P_A \in \operatorname{Ass}(I^k)$.

| $A$ | $u_A$ | $P_A$ | $\lambda(P_A; I)$ |
|---|---|---|---|
| $\{2,3,4,5\}$ | $x_1$ | $(x_1)$ | 1 |
| $\{1,2,5\}$ | $x_4$ | $(x_3, x_4)$ | 1 |
| $\{1,3,4\}$ | $x_5$ | $(x_2, x_5)$ | 1 |
| $\{1,3,5\}$ | $x_4$ | $(x_2, x_4)$ | 1 |
| $\{1,4,5\}$ | $x_3$ | $(x_2, x_3)$ | 1 |
| $\{1,2,3\}$ | $x_5$ | $(x_4, x_5)$ | 1 |
| $\{1,2,4\}$ | $x_5$ | $(x_3, x_5)$ | 1 |
| $\{1,2\}$ | $x_4x_5$ | $(x_3, x_4, x_5)$ | 2 |
| $\{1,3\}$ | $x_4x_5$ | $(x_2, x_4, x_5)$ | 2 |
| $\{1,4\}$ | $x_3x_5$ | $(x_2, x_3, x_5)$ | 2 |
| $\{1,5\}$ | $x_3x_4$ | $(x_2, x_3, x_4)$ | 2 |
| $\{1\}$ | $x_3x_4x_5$ | $(x_1, x_2, x_3, x_4)$ | 3 |

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