On an Inequality of G. H. Hardy

Since 1864

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DECLARATION

I, **Mr. Sajid Iqbal** Registration No. **108-GCU-PHD-SMS-08** student at **Abdus Salam School of Mathematical Sciences GC University** in the subject of **Mathematics**, **Year of Admission** (**2008**), hereby declare that the matter printed in this thesis titled

"On an Inequality of G. H. Hardy"

is my own work and that

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(ii) No direct major work had already been done by me or anybody else on this topic; I worked on, for the Ph. D. degree.

(iii) The work, I am submitting for the Ph. D. degree has not already been submitted elsewhere and shall not in future be submitted by me for obtaining similar degree from any other institution.

Dated: -------------------------   -----------------------------------

Signature

<u>RESEARCH COMPLETION CERTIFICATE</u>

Certified that the research work contained in this thesis titled

"On an Inequality of G. H. Hardy"

has been carried out and completed by Mr. **Sajid Iqbal** Registration No. **108-GCU-PHD-SMS-08** under my supervision.

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Submitted Through

Prof. Dr. A. D. Raza Choudary	**-------------------------------**
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Dedicated

to

my beloved parents

Naseem Akhtar and Muhammad Iqbal

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Abstract

Mathematical inequalities play very important role in development of all branches of mathematics. A huge effort has been made to discover the new types of inequalities and the basic work published in 1934 by Hardy, Littlewood and Pólya [36]. Later on Beckenbach and Bellman in 1961 in their book "Inequalities"[13], and the book "Analytic inequalities"by Mitronović [53] published in 1970 made considerable contribution in this field. The mathematical inequalities are useful because these are used as major tool in the development of modern analysis. A wide range of problems in various branches of mathematics are studied by well known Jensen, Hilbert, Hadamard, Hardy, Poinćare, Opial, Sobolev, Levin and Lyapunov inequalities. In 1992, J. Pečarić, F. Proschan and Y. L. Tong play their vital role in this field and they published famous book "Convex Functions, Partial Orderings and Statistical Application"which is considered as a brightening star in this field.

On the other hand, the applications of fractional calculus in mathematical inequalities have great importance. Hardy-type inequalities are very famous and play fundamental role in mathematical inequalities. Many mathematicians gave generalizations, improvements and application in the development of the Hardy's inequalities and they use fractional integrals and fractional derivatives to establish new integral inequalities. Further details concerning the rich history of the integral inequalities can be found in [58]–[64], [73]–[75] and the references given therein.

Čižmešija, Krulić, Pečarić and Persson establish some new refined Hardy-type inequalities with kernels in their recent papers [4], [25], [28], [29], [34], [52] (also see

[15]– [23]). Inequalities lies in the heart of the mathematical analysis and numerous mathematicians are attracted by these famous Hardy-type inequalities and discover new inequalities with kernels and applications of different fractional integrals and fractional derivatives, (see [25], [28], [38], [50], [52], [65]).

In this Ph.D thesis an integral operator with general non-negative kernel on measure spaces with positive σ-finite measure is considered. Our aim is to give the inequality of G. H. Hardy and its improvements for Riemann-Liouville fractional integrals, Canavati-type fractional derivative, Caputo fractional derivative, fractional integral of a function with respect to an increasing function, Hadamard-type fractional integrals and Erdélyi-Kober fractional integrals with respect to the convex and superquadratic functions. We will use different weights in this construction to obtain new inequalities of G. H. Hardy. Such type of results are widely discussed in [38](see also [28]). Also, we generalize and refine some inequalities of classical Hardy-Hilbert-type, classical Hardy-Littlewood-Pólya-type and Godunova-type inequalities [55] for monotone convex function.

The first chapter contains the basic concepts and notions from theory of convex functions and superquadratic functions. Some useful lemmas related to fractional integrals and fractional derivatives are given which we frequently use in next chapters to prove our results.

In the second chapter, we state, prove and discuss new general inequality for convex and increasing functions. Continuing the extension of our general result, we obtain new results involving different fractional integrals and fractional derivatives. We give improvements of an inequality of G. H. Hardy for convex and superquadratic functions as well.

In the third chapter, we give the new class of the G. H. Hardy-type integral inequalities with applications. We provide some generalized G. H. Hardy-type inequalities for fractional integrals and fractional derivatives.

In fourth chapter, we present generalized Hardy's and related inequalities involving monotone convex function. We generalize and refine some inequalities of classical

Pólya-Knopp's, Hardy-Hilbert, classical Hardy-Littlewood-Pólya, Hardy-Hilber-type and Godunova's. We also give some new fractional inequalities as refinements.

In the fifth chapter, we establish a generalization of the inequality introduced by D. S. Mitrinović and J. Pečarić in 1988. We prove mean value theorems of Cauchy type and discuss the exponential convexity, logarithmic convexity and monotonicity of the means. Also, we produce the n-exponential convexity of the linear functionals obtained by taking the non-negative difference of Hardy-type inequalities. At the end, some related examples are given.

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Lahore, Pakistan — Sajid Iqbal
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Chapter 1

Introduction

1.1 Convex functions

Convex functions has great importance in the theory of inequalities and they are widely discuss in the classical book of Hardy, Littlewood and Pólya [36] (see also [57]). In this section we give some of the results concerning convex functions.

Definition 1.1.1. Let I be an interval in $\mathbb{R}$. A function $\Phi\colon I \to \mathbb{R}$ is called *convex* if

$$\Phi(\lambda x + (1-\lambda)y) \leq \lambda\Phi(x) + (1-\lambda)\Phi(y) \tag{1.1.1}$$

for all points $x, y \in I$ and all $\lambda \in [0,1]$. It is called *strictly convex* if the inequality (1.1.1) holds strictly whenever $x \neq y$ and $\lambda \in (0,1)$.

In the following lemma the equivalent definition of convex functions is given.

Lemma 1.1.1. *If $\Phi : I \to \mathbb{R}$ is convex on an interval $I \subseteq \mathbb{R}$, then*

$$(x_3 - x_2)\Phi(x_1) + (x_1 - x_3)\Phi(x_2) + (x_2 - x_1)\Phi(x_3) \geq 0$$

holds for every $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

Definition 1.1.2. Let I be an interval in $\mathbb{R}$. A function $\Phi : I \to \mathbb{R}$ is called *convex in the Jensen sense*, or *J-convex on I (midconvex, midpoint convex)* if for all points $x, y \in I$ the inequality

$$\Phi\left(\frac{x+y}{2}\right) \leq \frac{\Phi(x) + \Phi(y)}{2} \tag{1.1.2}$$

holds. A J-convex function is said to be *strictly J-convex* if for all pairs of points $(x, y), x \neq y$, strict inequality holds in (1.1.2).

In the context of continuity the following criteria of equivalence of (1.1.1) and (1.1.2) is valid.

Theorem 1.1.2. *Let $\Phi : I \to \mathbb{R}$ be a continuous function. Then Φ is convex function if and only if Φ is J-convex function.*

Now, we introduce some necessary notations and recall some basic facts about convex functions, log-convex functions (see e.g. [50], [57], [65]) as well as exponentially convex functions (see e.g [12], [54], [56]).

In 1929, S. N. Bernstein introduced the notion of exponentially convex function in [12]. Later on D.V. Widder in [71] introduced these functions as a sub-class of convex function in a given interval (a, b) (for details see [71], [72]).

Definition 1.1.3. A positive function Φ is said to be *logarithmically convex* on an interval $I \subseteq \mathbb{R}$ if $\log \Phi$ is a convex function on I, or equivalently if for all $x, y \in I$ and all $\lambda \in [0, 1]$

$$\Phi(\lambda x + (1-\lambda)y) \leq \Phi^{\lambda}(x)\Phi^{1-\lambda}(y).$$

For such function Φ, we shortly say Φ is log-convex. A positive function Φ is *log-convex in the Jensen sense* if for each $x, y \in I$

$$\Phi^2\left(\frac{x+y}{2}\right) \leq \Phi(x)\Phi(y)$$

holds, i.e., if $\log \Phi$ is convex in the Jensen sense.

Remark 1.1.1. A function Φ is log-convex on an interval I, if and only if for all $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$, it holds

$$[\Phi(x_2)]^{x_3-x_1} \leq [\Phi(x_1)]^{x_3-x_2}[\Phi(x_3)]^{x_2-x_1}.$$

Furthermore, if $x_1, x_2, y_1, y_2 \in I$ are such that $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then

$$\left(\frac{\Phi(x_2)}{\Phi(x_1)}\right)^{\frac{1}{x_2-x_1}} \leq \left(\frac{\Phi(y_2)}{\Phi(y_1)}\right)^{\frac{1}{y_2-y_1}}. \tag{1.1.3}$$

Inequality (1.1.3) is known as Galvani's theorem for log-convex functions $\Phi \colon I \to \mathbb{R}$.

We continue with the definition of exponentially convex function as originally given in [12] by Berstein (see also [6], [54], [56]).

Definition 1.1.4. A function $\Phi\colon (a,b) \to \mathbb{R}$ is *exponentially convex* if it is continuous and

$$\sum_{i,j=1}^{n} t_i t_j \Phi(x_i + x_j) \geq 0 \tag{1.1.4}$$

holds for every $n \in \mathbb{N}$ and all sequences $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ of real numbers, such that $x_i + x_j \in (a,b)$, $1 \leq i,j \leq n$.

We continue this section by recalling some notions of our special interest about n-exponential convexity given in [66].

Definition 1.1.5. A function $\Phi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} t_i t_j \Phi\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all choices of $t_i \in \mathbb{R},\ x_i \in I,\ i = 1, \dots, n.$

A function $\Phi : I \to \mathbb{R}$ is n-exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 1.1.2. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact non-negative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N},\ k \leq n.$

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition.

Proposition 1.1.3. *Let I be an open interval in $\mathbb{R}$. If Φ is n-exponentially convex in the Jensen sense on J, then the matrix $\left[\Phi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ is positive semi-definite matrix for all $k \in \mathbb{N},\ k \leq n$. Particularly*

$$\det\left[\Phi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k} \geq 0, \ \textit{for all } k \in \mathbb{N},\ k \leq n.$$

Definition 1.1.6. Let I be an open interval in $\mathbb{R}$. A function $\Phi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense on I for $n \in \mathbb{N}$.

A function $\Phi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 1.1.3. It follows that a function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.
Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex.

Now, we continue with derivative of convex functions. The derivative of convex functions is best studied in terms of the left and right derivatives defined by

$$\Phi_{-}^{'}(x) = \lim_{y \nearrow x} \frac{\Phi(y) - \Phi(x)}{y - x}, \quad \Phi_{+}^{'}(x) = \lim_{y \searrow x} \frac{\Phi(y) - \Phi(x)}{y - x}.$$

Definition 1.1.7. Let $\Phi : I \longrightarrow \mathbb{R}$ be a convex function, then the sub-differential of Φ at x, denoted by $\partial\Phi(x)$, is defined as

$$\partial\Phi(x) = \{\alpha \in \mathbb{R} \,:\, \Phi(y) - \Phi(x) - \alpha(y - x) \geq 0,\, y \in I\}.$$

Many further information on convex and concave functions can be found e.g. in the monographs [57] and [65] and in references cited therein.

The concept of superquadratic and subquadratic functions is introduced by Abramovich, Jameson and Sinnamon in [2] (see also [1], [3]).

Definition 1.1.8. [2, Definition 2.1] A function $\varphi : [0, \infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi\left(|y - x|\right) \geq C_x \left(y - x\right), \tag{1.1.5}$$

for all $y \geq 0$. We say that φ is subquadratic if $-\varphi$ is superquadratic.

1.2 Fractional integrals and fractional derivatives

Here we give some definitions and useful lemmas, which we frequently use in next chapters to give the proof of our results. First, let us recall some facts about fractional derivatives needed in the sequel, for more details see e.g. [7], [33].

Let $0 < a < b \leq \infty$. By $C^m[a, b]$ we denote the space of all functions on $[a, b]$ which have continuous derivatives up to order m, and $AC[a, b]$ is the space of all absolutely continuous functions on $[a, b]$. By $AC^m[a, b]$ we denote the space of all functions $g \in C^{m-1}[a, b]$ with $g^{(m-1)} \in AC[a, b]$. For any $\alpha \in \mathbb{R}$ we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leq \alpha < k + 1$) and $\lceil\alpha\rceil$ is the ceiling of α ($\min\{n \in \mathbb{N}, n \geq \alpha\}$). By $L_1(a, b)$ we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

Now, we give well known definitions of Riemann-Liouville fractional integrals, see [51]. Let $[a,b]$ be a finite interval on real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I^\alpha_{a+}f$ and $I^\alpha_{b-}f$ of order $\alpha > 0$ are defined by

$$I^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x (x-y)^{\alpha-1} f(y)dy, \quad x > a$$

and

$$I^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_x^b (y-x)^{\alpha-1} f(y)dy, \quad x < b.$$

Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and right-sided fractional integrals respectively.

Let us recall the definition, for details see [7, p. 448].
The generalized Riemann–Liouville fractional derivative of f of order $\alpha > 0$ is given by

$$D^\alpha_a f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-y)^{n-\alpha-1} f(y)dy,$$

where $n = [\alpha]+1$, $x \in [a,b]$.

For $a, b \in \mathbb{R}$, we say that $f \in L_1(a,b)$ has an L_∞ fractional derivative $D^\alpha_a f$ $(\alpha > 0)$ in $[a,b]$, iff

1) $D^{\alpha-k}_a f \in C[a,b], \qquad k = 1, \ldots, n = [\alpha]+1$
2) $D^{\alpha-1}_a f \in AC[a,b]$,
3) $D^\alpha_a f \in L_\infty(a,b)$.

The following lemma is given in [7, p.449] (see also [33]).

Lemma 1.2.1. *Let $\beta > \alpha \geq 0$, let $f \in L_1(a,b)$ have an L_∞ fractional derivative $D^\beta_a f$ in $[a,b]$ and let $D^{\beta-k}_a f(a) = 0, k = 1, \ldots, [\beta]+1$. Then*

$$D^\alpha_a f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int\limits_a^x (x-y)^{\beta-\alpha-1} D^\beta_a f(y)dy$$

for all $a \leq x \leq b$.

Let $\alpha > 0$ and $n = [\alpha] + 1$, where $[\cdot]$ is the integral part and we define the generalized Riemann-Liouville fractional derivative of f of order α. In addition, we stipulate

$$D_a^0 f := f =: I_a^0 f, \quad I_a^{-\alpha} f := D_a^\alpha f \text{ if } \alpha > 0.$$

If $\alpha \in \mathbb{N}$, then $D_a^\alpha f = \frac{d^\alpha f}{dx^\alpha}$, the ordinary α-order derivative.

The space $I_a^\alpha(L(a,b))$ is defined as the set of all functions f on $[a,b]$ of the form $f = I_a^\alpha \varphi$ for some $\varphi \in L(a,b)$, [68, Chapter 1, Definition 2.3]. According to Theorem 2.3 in [68, p. 43], the latter characterization is equivalent to the condition

$$I_a^{n-\alpha} f \in AC^n[a,b]\,, \tag{1.2.1}$$

$$\frac{d^j}{dx^j} I_a^{n-\alpha} f(a) = 0\,, \quad j = 0, 1, \ldots, n-1\,.$$

A function $f \in L(a,b)$ satisfying (1.2.1) is said to have an integrable fractional derivative $D_a^\alpha f$, [68, Chapter1, Definition 2.4].

The following lemma summarizes conditions in identity for generalized Riemann-Liouville fractional derivative. For details see [9].

Lemma 1.2.2. *Let $\beta > \alpha \geq 0$, $n = [\beta] + 1$, $m = [\alpha] + 1$. Identity*

$$D_a^\alpha f(x) = \frac{1}{\Gamma(\beta - \alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D_a^\beta f(y)\, dy\,, \quad x \in [a,b],$$

is valid if one of the following conditions holds:

(i) $f \in I_a^\beta(L(a,b))$.

(ii) $I_a^{n-\beta} f \in AC^n[a,b]$ *and* $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots n$.

(iii) $D_a^{\beta-k} f \in C[a,b]$ *for* $k = 1, \ldots, n$, $D_a^{\beta-1} f \in AC[a,b]$ *and* $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots n$.

(iv) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta - \alpha \notin \mathbb{N}$, $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots, n$ *and* $D_a^{\alpha-k} f(a) = 0$ *for* $k = 1, \ldots, m$.

(v) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta - \alpha = l \in \mathbb{N}$, $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \ldots, l$.

(vi) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$ *and* $f(a) = f'(a) = \cdots = f^{(n-2)}(a) = 0$.

(*vii*) $f \in AC^n[a,b]$, $D_a^{\beta} f \in L(a,b)$, $D_a^{\alpha} f \in L(a,b)$, $\beta \notin \mathbb{N}$ *and* $D_a^{\beta-1} f$ *is bounded in a neighborhood of* $t = a$.

Next, we recall Canavati-type fractional derivative ($\nu-$fractional derivative of f), for details see [7, p. 446]. We consider

$$C^{\nu}[a,b] = \{f \in C^n[a,b] : I_{a+}^{n-\nu+1} f^{(n)} \in C^1[a,b]\},$$

$\nu > 0$, $n = [\nu]$. Let $f \in C^{\nu}[a,b]$. We define the generalized $\nu-$fractional derivative of f over $[a,b]$ as

$$D_a^{\nu} f = (I_{a+}^{n-\nu+1} f^{(n)})',$$

the derivative with respect to x.

Lemma 1.2.3. *Let* $\nu \geq \gamma + 1$, *where* $\gamma \geq 0$ *and* $f \in C^{\nu}[a,b]$. *Assume* $f^{(i)}(a) = 0$, $i = 0, 1, \ldots, [\nu] - 1$. *Then*

$$(D_a^{\gamma} f)(x) = \frac{1}{\Gamma(\nu - \gamma)} \int_a^x (x-t)^{\nu-\gamma-1} (D_a^{\nu} f)(t) dt,$$

for all $x \in [a,b]$.

The definition of Canavati-type fractional derivative is given in [7] but we will consider the Canavati-type fractional derivative given in [10] with some new conditions in our results. Now we recall the Canavati-type fractional derivative ($\nu-$fractional derivative of f). We consider

$$C^{\nu}[0,1] = \{f \in C^n[0,1] : I_{1-\bar{\nu}} f^{(n)} \in C^1[0,1]\},$$

$\nu > 0$, $n = [\nu], [.]$ is the integral part, and $\bar{\nu} = \nu - n, 0 \leq \bar{\nu} < 1$.
For $f \in C^{\nu}[0,1]$, the Canavati-ν fractional derivative of f is defined by

$$D^{\nu} f = D I_{1-\bar{\nu}} f^{(n)},$$

where $D = d/dx$.

Lemma 1.2.4. *Let* $\nu > \gamma \geq 0$, $n = [\nu]$, $m = [\gamma]$. *Let* $f \in C^{\nu}[0,1]$, *be such that* $f^{(i)}(0) = 0$, $i = m, m+1, \ldots, n-1$. *Then*

(*i*) $f \in C^{\gamma}[0,1]$,

(*ii*) $(D_a^{\gamma} f)(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-t)^{\nu-\gamma-1} (D_a^{\nu} f)(t) dt$, *for every* $x \in [a,b]$.

Now we recall Caputo fractional derivative, for details see [7, p. 449].

Let $\nu \geq 0$, $n = \lceil \nu \rceil$, $f \in AC^n[a,b]$. The Caputo fractional derivative is given by

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x \frac{f^{(n)}(y)}{(x-y)^{\nu-n+1}} dy,$$

for all $x \in [a,b]$. The above function exists almost everywhere for $x \in [a,b]$.

We continue with the following lemma that is given in [11].

Lemma 1.2.5. *Let* $\nu > \gamma \geq 0$, $n = [\nu] + 1$, $m = [\gamma] + 1$ *and* $f \in AC^n[a,b]$. *Suppose that one of the following conditions hold:*

(a) $\nu, \gamma \notin \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m, \ldots, n-1$.

(b) $\nu \in \mathbb{N}_0, \gamma \notin \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m, \ldots, n-2$.

(c) $\nu \notin \mathbb{N}_0, \gamma \in \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m-1, \ldots, n-1$.

(d) $\nu \in \mathbb{N}_0, \gamma \in \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m-1, \ldots, n-2$.

Then

$$D_{*a}^{\gamma} f(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-y)^{\nu-\gamma-1} D_{*a}^{\nu} f(y) dy$$

for all $a \leq x \leq b$.

The following result is given in [7, p. 450].

Lemma 1.2.6. *Let* $\alpha \geq \gamma + 1$, $\gamma > 0$ *and* $n = \lceil \alpha \rceil$. *Assume* $f \in AC^n[a,b]$ *such that* $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$, *and* $D_{*a}^{\alpha} f \in L_{\infty}(a,b)$. *Then* $D_{*a}^{\gamma} f \in C[a,b]$, *and*

$$D_{*a}^{\gamma} f(x) = \frac{1}{\Gamma(\alpha-\gamma)} \int_a^x (x-y)^{\alpha-\gamma-1} D_{*a}^{\alpha} f(y) dy$$

for all $a \leq x \leq b$.

Now, we recall the definition of Hadamard-type fractional integrals.
Let $[a, b]$ be finite or infinite interval of $\mathbb{R}_+$ and $\alpha > 0$. The left and right-sided Hadamard-type fractional integrals of order $\alpha > 0$ are given by

$$(J^{\alpha}_{a_+} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x \left(\log \frac{x}{y} \right)^{\alpha - 1} f(y) \frac{dy}{y}, \quad x > a$$

and

$$(J^{\alpha}_{b_-} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_x^b \left(\log \frac{y}{x} \right)^{\alpha - 1} f(y) \frac{dy}{y}, \quad x < b$$

respectively.

We continue with definitions and some properties of the fractional integrals of a function f with respect to given function g. For details see e.g. [51, p. 99]:

Let (a, b), $-\infty \leq a < b \leq \infty$ be a finitive or infinitive interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let g be an increasing function on $(a, b]$ and g' be a continuous function on (a, b). The left- and right-sided fractional integrals of a function f with respect to another function g in $[a, b]$ are given by

$$(I^{\alpha}_{a+;g} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x \frac{g'(t) f(t) dt}{[g(x) - g(t)]^{1-\alpha}}, \quad x > a$$

and

$$(I^{\alpha}_{b-;g} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_x^b \frac{g'(t) f(t) dt}{[g(t) - g(x)]^{1-\alpha}}, \quad x < b$$

respectively.

Remark 1.2.1. If $g(x) = x$, then $I^{\alpha}_{a+;x} f$ reduces to $I^{\alpha}_{a+} f$ and $I^{\alpha}_{b_-;x} f$ reduces to $I^{\alpha}_{b_-} f$, that is to Riemann–Liouville fractional integrals. Notice that Hadamard fractional integrals of order α are special case of the left- and right-sided fractional integrals of a function f with respect to another function $g(x) = \log(x)$, $x \in (a, b)$ where $0 \leq a < b \leq \infty$.

We also recall the definition of Erdélyi-Kober type fractional integrals. For details see [68] (also see [27, p, 154]).

Let $(a,b),(0\le a<b\le\infty)$ be finite or infinite interval of $\mathbb{R}_+$. Let $\alpha>0$, $\sigma>0$, and $\eta\in\mathbb{R}$. The left and right-sided Erdélyi-Kober type fractional integral of order $\alpha>0$ are defined by

$$(I^{\alpha}_{a+;\sigma;\eta}f)(x)=\frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int\limits_a^x\frac{t^{\sigma\eta+\sigma-1}f(t)dt}{(x^\sigma-t^\sigma)^{1-\alpha}},\quad x>a$$

and

$$(I^{\alpha}_{b-;\sigma;\eta}f)(x)=\frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)}\int\limits_x^b\frac{t^{\sigma(1-\eta-\alpha)-1}f(t)dt}{(t^\sigma-x^\sigma)^{1-\alpha}},\qquad x<b$$

respectively.

We recall multidimensional fractional integrals. Such type of fractional integrals are usually generalization of the corresponding one-dimensional fractional integral and fractional derivative.

For $\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n$ and $\alpha=(\alpha_1,\ldots,\alpha_n)$, we use the following notations:

$$\Gamma(\alpha)=(\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)),\ [\mathbf{a},\mathbf{b}]=[a_1,b_1]\times\ldots\times[a_n,b_n],$$

and by $\mathbf{x}>\mathbf{a}$ we mean $x_1>a_1,\ldots,x_n>a_n$.

We define the mixed Riemann-Liouville fractional integrals of order $\alpha>0$ as

$$(I^{\alpha}_{\mathbf{a}_+}f)(\mathbf{x})=\frac{1}{\Gamma(\alpha)}\int\limits_{a_1}^{x_1}\ldots\int\limits_{a_n}^{x_n}f(\mathbf{t})(\mathbf{x}-\mathbf{t})^{\alpha-1}d\mathbf{t},\quad \mathbf{x}>\mathbf{a}$$

and

$$(I^{\alpha}_{\mathbf{b}_-}f)(\mathbf{x})=\frac{1}{\Gamma(\alpha)}\int\limits_{x_1}^{b_1}\ldots\int\limits_{x_n}^{b_n}f(\mathbf{t})(\mathbf{t}-\mathbf{x})^{\alpha-1}d\mathbf{t},\quad \mathbf{x}<\mathbf{b}.$$

Conventions. All measures are assumed to be positive, all functions are assumed to be measurable, and expressions of the form $0\cdot\infty$, $\frac{0}{0}$, $\frac{a}{\infty}$ $(a\in\mathbb{R})$, and $\frac{\infty}{\infty}$ are taken to be equal to zero. For a real parameter $0\ne p\ne 1$, by p' we denote its conjugate exponent $p'=\frac{p}{p-1}$, that is, $\frac{1}{p}+\frac{1}{p'}=1$. Also, by a weight function (shortly: a weight) we mean a non-negative measurable function on the actual set. Further, we set $\mathbb{N}_k=\{1,\,2,\,\ldots,\,k\}$ for $k\in\mathbb{N}$. An interval I in $\mathbb{R}$ is any convex subset of $\mathbb{R}$, while by $Int\, I$ we denote its interior. By $\mathbb{R}_+$ we denote the set of all positive real numbers i.e. $\mathbb{R}_+ = (0, \infty)$. $B(\,\cdot\,;\,\cdot\,,\,\cdot\,)$ denotes the incomplete Beta function, defined by

$$B(x; a, b) = \int_0^x t^{a-1}(1-t)^{b-1}\, dt, \quad x \in [0,1],\ a, b > 0. \tag{1.2.2}$$

As usual, $B(a, b) = B(1; a, b)$ stands for the standard Beta function and ${}_2F_1(a, b; c; z)$ denotes the Euler type of hypergeometric function, defined by

$${}_2F_1(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt, \quad Re(c) > Re(b) > 0$$

provided $|z| < 1$ or $|z| = 1$.

Throughout this thesis, we denote

$${}_2F_1(x) = {}_2F_1\left(-\eta, \alpha; \alpha+1; 1-\left(\frac{a}{x}\right)^{\sigma}\right),$$

and

$${}_2F_1(y) = {}_2F_1\left(\eta, \alpha; \alpha+1; 1-\left(\frac{b}{y}\right)^{\sigma}\right).$$

Chapter 2

Inequality of G. H. Hardy and its improvements

In this chapter, we state, prove and discuss new general inequality for convex and increasing functions. As a special case of that general result, we obtain new fractional inequalities involving fractional integrals and derivatives of Riemann–Liouville type. Consequently, we get the inequality of G. H. Hardy from 1918. We also obtain new results involving fractional derivatives of Canavati and Caputo as well as fractional integrals of a function with respect to another function. We apply our result to multidimensional setting to obtain new results involving mixed Riemman-Liouville fractional integrals. We use the convex and superquadratic functions to get new improvements of inequalitity of G. H. Hardy. We also obtain means of Cauchy type and prove their monotonicity. The results given in this chapter can be seen in [38]–[43].

2.1 On an inequality of G. H. Hardy

First we denote some properties of the fractional integral operators $I^{\alpha}_{a^+}f$ and $I^{\alpha}_{b_-}f$ of order $\alpha > 0$, see also [68]. It is known that the fractional integral operators $I^{\alpha}_{a^+}f$ and $I^{\alpha}_{b_-}f$ are bounded in $L_p(a,b)$, $1 \leq p \leq \infty$, that is

$$\|I^{\alpha}_{a^+}f\|_p \leq K\|f\|_p, \quad \|I^{\alpha}_{b_-}f\|_p \leq K\|f\|_p, \tag{2.1.1}$$

where

$$K = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}.$$

Inequality (2.1.1), that is the result involving the left-sided fractional integral, was proved by G. H. Hardy in one of his initial papers, see [35]. He did not write down the constant, but the calculation of the constant was hidden inside his proof. Inequality (2.1.1) refer to an inequality of G. H. Hardy. Some recent results involving Riemann-Liouville fractional integrals can be seen in [38].

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a non-negative function, and

$$K(x) = \int\limits_{\Omega_2} k(x,y)\, d\mu_2(y),\ x \in \Omega_1. \tag{2.1.2}$$

Throughout this thesis we suppose $K(x) > 0$ a.e. on Ω_1.

Let $U(k)$ denote the class of measurable functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) = \int\limits_{\Omega_2} k(x,y) f(y) d\mu_2(y),$$

where $f : \Omega_2 \to \mathbb{R}$ is a measurable function.

2.1.1 Main results

Our first result is given in the following theorem.

Theorem 2.1.1. *Let u be a weight function on Ω_1, k a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (2.1.2). Assume that the function $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v on Ω_2 by*

$$v(y) := \int\limits_{\Omega_1} u(x) \frac{k(x,y)}{K(x)} d\mu_1(x) < \infty. \tag{2.1.3}$$

If $\Phi : (0, \infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int\limits_{\Omega_1} u(x) \Phi\left(\left| \frac{g(x)}{K(x)} \right| \right) d\mu_1(x) \le \int\limits_{\Omega_2} v(y) \Phi(|f(y)|) d\mu_2(y) \tag{2.1.4}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ and for all functions $g \in U(k)$.

Proof. By using Jensen's inequality and the Fubini theorem, since Φ is an increasing function, we find that

$$\begin{aligned}
\int_{\Omega_1} u(x)\Phi\left(\left|\frac{g(x)}{K(x)}\right|\right) d\mu_1(x) &= \int_{\Omega_1} u(x)\Phi\left(\left|\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)d\mu_2(y)\right|\right) d\mu_1(x) \\
&\leq \int_{\Omega_1} \frac{u(x)}{K(x)}\left(\int_{\Omega_2} k(x,y)\Phi(|f(y)|)d\mu_2(y)\right) d\mu_1(x) \\
&= \int_{\Omega_2} \Phi(|f(y)|)\left(\int_{\Omega_1} u(x)\frac{k(x,y)}{K(x)}d\mu_1(x)\right) d\mu_2(y) \\
&= \int_{\Omega_2} v(y)\Phi(|f(y)|)d\mu_2(y)
\end{aligned}$$

and the proof is complete. ∎

As a special case of Theorem 2.1.1 we get the following result.

Corollary 2.1.2. *Let u be a weight function on (a,b) and $\alpha > 0$. $I^{\alpha}_{a^+}f$ denotes the left-sided Riemann-Liouville fractional integral of f. Define v on (a,b) by*

$$v(y) := \alpha \int_y^b u(x)\frac{(x-y)^{\alpha-1}}{(x-a)^{\alpha}}dx < \infty.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\Phi\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}|I^{\alpha}_{a^+}f(x)|\right) dx \leq \int_a^b v(y)\Phi(|f(y)|)\, dy \tag{2.1.5}$$

holds.

Proof. Applying Theorem 2.1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)}, & a < y \leq x\,; \\ 0, & x < y \leq b, \end{cases} \tag{2.1.6}$$

we get that $K(x) = \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}$ and $g(x) = I^{\alpha}_{a^+}f(x)$, so (2.1.5) follows. ∎

Remark 2.1.1. In particular for the weight $u(x) = (x-a)^\alpha$, $x \in (a,b)$ in Corollary 2.1.2, we obtain the inequality

$$\int_a^b (x-a)^\alpha \Phi\left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}|I^\alpha_{a^+} f(x)|\right) dx \le \int_a^b (b-y)^\alpha \Phi(|f(y)|)\, dy. \tag{2.1.7}$$

Although (2.1.4) holds for all convex and increasing functions, some choices of Φ are of particular interest. Namely, we shall consider power function. Let $q > 1$ and the function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\Phi(x) = x^q$, then (2.1.7) reduces to

$$\int_a^b (x-a)^\alpha \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}|I^\alpha_{a^+} f(x)|\right)^q dx \le \int_a^b (b-y)^\alpha |f(y)|^q\, dy. \tag{2.1.8}$$

Since $x \in (a,b)$ and $\alpha(1-q) < 0$, then we obtain that the left-hand side of (2.1.8)

$$\begin{aligned}\int_a^b (x-a)^\alpha &\left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}|I^\alpha_{a^+} f(x)|\right)^q dx \\ &\ge (b-a)^{\alpha(1-q)}(\Gamma(\alpha+1))^q \int_a^b |I^\alpha_{a^+} f(x)|^q\, dx\end{aligned} \tag{2.1.9}$$

and the right-hand side of (2.1.8)

$$\int_a^b (b-y)^\alpha |f(y)|^q\, dy \le (b-a)^\alpha \int_a^b |f(y)|^q\, dy. \tag{2.1.10}$$

Combining (2.1.9) and (2.1.10) we get

$$\int_a^b |I^\alpha_{a^+} f(x)|^q\, dx \le \left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right)^q \int_a^b |f(y)|^q\, dy. \tag{2.1.11}$$

Taking power $\frac{1}{q}$ on both sides we obtain (2.1.1).

We can give the similar results like Corollary 2.1.2 and Remark 2.1.1 for the right-sided Riemann-Liouville fractional integral but here we omit the details.

Theorem 2.1.3. *Let $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $I^{\alpha}_{a^+} f$ and $I^{\alpha}_{b_-} f$ denote the Riemann-Liouville fractionals integrals of f. Then the following inequalities*

$$\int_a^b |I^{\alpha}_{a^+} f(x)|^q \, dx \leq C \int_a^b |f(y)|^q \, dy \tag{2.1.12}$$

and

$$\int_a^b |I^{\alpha}_{b_-} f(x)|^q \, dx \leq C \int_a^b |f(y)|^q \, dy \tag{2.1.13}$$

hold, where $C = \frac{(b-a)^{q\alpha}}{(\Gamma(\alpha))^q q\alpha (p(\alpha-1)+1)^{q-1}}$.

Proof. We will prove only inequality (2.1.12), since the proof of (2.1.13) is analogous. We have

$$|(I^{\alpha}_{a^+} f)(x)| \leq \frac{1}{\Gamma(\alpha)} \int_a^x |f(t)|(x-t)^{\alpha-1} dt.$$

Then by the Hölder inequality, the right-hand side of the above inequality is

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_a^x (x-t)^{p(\alpha-1)} dt \right)^{\frac{1}{p}} \left(\int_a^x |f(t)|^q dt \right)^{\frac{1}{q}} \\
&= \frac{1}{\Gamma(\alpha)} \frac{(x-a)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_a^x |f(t)|^q dt \right)^{\frac{1}{q}} \\
&\leq \frac{1}{\Gamma(\alpha)} \frac{(x-a)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_a^b |f(t)|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Consequently, we find

$$|(I^{\alpha}_{a^+} f)(x)|^q \leq \frac{1}{(\Gamma(\alpha))^q} \frac{(x-a)^{q(\alpha-1)+\frac{q}{p}}}{(p(\alpha-1)+1)^{\frac{q}{p}}} \left(\int_a^b |f(t)|^q dt \right)$$

and we obtain

$$\int_a^b |I^{\alpha}_{a^+} f(x)|^q \, dx \leq \frac{(b-a)^{q(\alpha-1)+\frac{q}{p}+1}}{(\Gamma(\alpha))^q (q(\alpha-1)+\frac{q}{p}+1)(p(\alpha-1)+1)^{\frac{q}{p}}} \int_a^b |f(t)|^q \, dt.$$

■

Remark 2.1.2. For $\alpha \geq 1$ inequalities (2.1.12) and (2.1.13) are refinements of (2.1.1) since

$$q\alpha(p(\alpha-1)+1)^{q-1} \geq q\alpha^q > \alpha^q, \text{ so } C < \left(\frac{(b-a)^\alpha}{\alpha\Gamma(\alpha)}\right)^q.$$

We proved that Theorem 2.1.3 is a refinement of (2.1.1) and Corollary 2.1.2 are generalizations of (2.1.1).

Corollary 2.1.4. *Let u be a weight function on (a,b) and let assumptions in Lemma* 1.2.1 *be satisfied. Define v on (a,b) by*

$$v(y) := (\beta-\alpha)\int_y^b u(x)\frac{(x-y)^{\beta-\alpha-1}}{(x-a)^{\beta-\alpha}}dx < \infty.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\Phi\left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}}|D_a^\alpha f(x)|\right)dx \leq \int_a^b v(y)\Phi(|D_a^\beta f(y)|)\,dy \tag{2.1.14}$$

holds.

Proof. Applying Theorem 2.1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}, & a < y \leq x\,; \\ 0, & x < y \leq b, \end{cases} \tag{2.1.15}$$

we get that $K(x) = \frac{(x-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}$. Replace f by $D_a^\beta f$. Then, by Lemma 1.2.1 $g(x) = D_a^\alpha f(x)$ and we get (2.1.14). ■

Remark 2.1.3. In particular for the weight function $u(x) = (x-a)^{\beta-\alpha}$, $x \in (a,b)$ and $\Phi(x) = x^q, q > 1, x \in \mathbb{R}_+$ in Corollary 2.1.4 after short calculations we obtain the following inequality

$$\int_a^b |D_a^\alpha f(x)|^q\,dx \leq \left(\frac{(b-a)^{(\beta-\alpha)}}{\Gamma(\beta-\alpha+1)}\right)^q \int_a^b |D_a^\beta f(y)|^q\,dy.$$

Corollary 2.1.5. *Let u be a weight function on (a,b) and let the assumptions in Lemma* 1.2.3 *be satisfied. Define v on (a,b) by*

$$v(y) := (\nu-\gamma)\int_y^b u(x)\frac{(x-y)^{\nu-\gamma-1}}{(x-a)^{\nu-\gamma}}dx < \infty.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\Phi\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}|D_a^{\gamma} f(x)|\right) dx \le \int_a^b v(y)\Phi(|D_a^{\nu} f(y)|)\, dy$$

holds.

Proof. Similar to Corollary 2.1.4. ■

Remark 2.1.4. In particular for the weight function $u(x) = (x-a)^{\nu-\gamma}$, $x \in (a,b)$ and $\Phi(x) = x^q, q > 1$, then after some calculations like Remark 2.1.1, we obtain the inequality

$$\|(D_a^{\gamma} f(x)\|_q \le \frac{(b-a)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)}\|D_a^{\nu} f(y)\|_q.$$

When $\gamma = 0$, we find

$$\|f\|_q \le \frac{(b-a)^{\nu}}{\Gamma(\nu+1)}\|D_a^{\nu} f(y)\|_q.$$

In the next corollary we give results for Caputo fractional derivative.

Corollary 2.1.6. *Let u be a weight function on (a,b) and $\alpha > 0$. $D_{*a}^{\alpha} f$ denotes the Caputo fractional derivative of f. Define v on (a,b) by*

$$v(y) := (n-\alpha)\int_y^b u(x)\frac{(x-y)^{n-\alpha-1}}{(x-a)^{n-\alpha}}dx < \infty.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\Phi\left(\frac{\Gamma(n-\alpha+1)}{(x-a)^{n-\alpha}}|D_{*a}^{\alpha} f(x)|\right) dx \le \int_a^b v(y)\Phi(|f^{(n)}(y)|)\, dy$$

holds.

Proof. Similar to Corollary 2.1.4. ■

Remark 2.1.5. For particular weight function $u(x) = (x-a)^{n-\alpha}$, $x \in (a,b)$, and $\Phi(x) = x^q, q > 1$ in Corollary 2.1.6, after some calculation like Remark 2.1.1, we obtain the inequality

$$\|D_{*a}^{\alpha} f(x)\|_q \le \frac{(b-a)^{(n-\alpha)}}{\Gamma(n-\alpha+1)}\|f^{(n)}(y)\|_q.$$

Theorem 2.1.7. *Let $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $n - \alpha > \frac{1}{q}$, $D^{\alpha}_{*a} f(x)$ denotes the Caputo fractional derivative of f. Then the following inequality*

$$\int_a^b |D^{\alpha}_{*a} f(x)|^q \, dx \leq \frac{(b-a)^{q(n-\alpha)}}{(\Gamma(n-\alpha))^q (p(n-\alpha-1)+1)^{\frac{q}{p}} q(n-\alpha)} \int_a^b |f^{(n)}(y)|^q \, dy$$

holds.

Proof. Similar to Theorem 2.1.3. ■

Corollary 2.1.8. *Let u be a weight function on (a, b) and $\alpha > 0$. $D^{\alpha}_{*a} f$ denotes the Caputo fractional derivative of f and let the assumptions in Lemma* 1.2.6 *be satisfied. Define v on (a, b) by*

$$v(y) := (\alpha - \gamma) \int_y^b u(x) \frac{(x-y)^{\alpha-\gamma-1}}{(x-a)^{\alpha-\gamma}} dx < \infty.$$

If $\Phi : (0, \infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x) \Phi \left(\frac{\Gamma(\alpha-\gamma+1)}{(x-a)^{\alpha-\gamma}} |D^{\gamma}_{*a} f(x)| \right) dx \leq \int_a^b v(y) \Phi(|D^{\alpha}_{*a} f(y)|) \, dy$$

holds.

Proof. Similar to Corollary 2.1.4. ■

Remark 2.1.6. In particular for the weight function $u(x) = (x-a)^{\alpha-\gamma}$, $x \in (a, b)$ and $\Phi(x) = x^q, q > 1$ in Corollary 2.1.8 after some calculations like Remark 2.1.1, we obtain the following inequality

$$\int_a^b |D^{\gamma}_{*a} f(x)|^q \; dx \leq \left(\frac{(b-a)^{(\alpha-\gamma)}}{\Gamma(\alpha-\gamma+1)} \right)^q \int_a^b |D^{\alpha}_{*a} f(y)|^q \, dy.$$

For $\gamma = 0$, we obtain

$$\int_a^b |f(x)|^q \, dx \leq \left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \right)^q \int_a^b |D^{\alpha}_{*a} f(y)|^q \, dy.$$

Corollary 2.1.9. *Let u be a weight function on (a,b), g be an increasing function on $(a,b]$ such that g' be a continuous function on (a,b) and $\alpha > 0$. $I^{\alpha}_{a+;g}f$ denotes the left-sided fractional integral of a function f with respect to another function g in $[a,b]$. Define v on (a,b) by*

$$v(y) := \alpha g'(y) \int_y^b u(x) \frac{(g(x)-g(y))^{\alpha-1}}{(g(x)-g(a))^{\alpha}} dx < \infty.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\Phi\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}}|I^{\alpha}_{a+;g}f(x)|\right) dx \le \int_a^b v(y)\Phi(|f(y)|)\, dy$$

holds.

Proof. Similar to Corollary 2.1.2. ■

Remark 2.1.7. In particular for the weight function $u(x) = g'(x)(g(x)-g(a))^{\alpha}$, $x \in (a,b)$ in Corollary 2.1.9, we obtain the inequality

$$\int_a^b g'(x)(g(x)-g(a))^{\alpha}\Phi\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}}|I^{\alpha}_{a+;g}f(x)|\right) dx$$

$$\le \int_a^b g'(y)(g(b)-g(y))^{\alpha}\Phi(|f(y)|)\, dy. \tag{2.1.16}$$

Let $q > 1$ and $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\Phi(x) = x^q$, then (2.1.16) reduces to

$$(\Gamma(\alpha+1))^q \int_a^b g'(x)(g(x)-g(a))^{\alpha(1-q)}|I^{\alpha}_{a+;g}f(x)|^q\, dx$$

$$\le \int_a^b g'(y)(g(b)-g(y))^{\alpha}|f(y)|^q\, dy.$$

Since $x \in (a,b)$ and $\alpha(1-q) < 0$, g is increasing, then $(g(x)-g(a))^{\alpha(1-q)} > (g(b)-g(a))^{\alpha(1-q)}$ and $(g(b)-g(y))^{\alpha} < (g(b)-g(a))^{\alpha}$, we obtain

$$\int_a^b g'(x)|I^{\alpha}_{a+;g}f(x)|^q\, dx \le \left(\frac{(g(b)-g(a))^{\alpha}}{\Gamma(\alpha+1)}\right)^q \int_a^b g'(y)|f(y)|^q\, dy. \tag{2.1.17}$$

Remark 2.1.8. If $g(x) = x$, then $I^{\alpha}_{a+;x}f(x)$ reduces to $I^{\alpha}_{a+}f(x)$ Riemann–Liouville fractional integral and (2.1.17) becomes (2.1.11).

Analogous to the Corollary 2.1.9, we obtain the similar result for right-sided fractional integral of a function f with respect to another function g.

The refinements of (2.1.17) and its analogous case for left-sided fractional integral of a function f with respect to another function g for $\alpha > \frac{1}{q}$ are given in the following theorem.

Theorem 2.1.10. *Let $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $I^{\alpha}_{a+;g}f$ denotes the left-sided fractional integral of a function f with respect to another function g in $[a, b]$. Then the following inequality*

$$\int_a^b |I^{\alpha}_{a+;g}f(x)|^q g'(x)\, dx \le \frac{(g(b) - g(a))^{\alpha q}}{\alpha q(\Gamma(\alpha))^q (p(\alpha - 1) + 1)^{\frac{q}{p}}} \int_a^b |f(y)|^q g'(y)\, dy$$

hold.

If $g(x) = \log(x)$, $x \in (a, b)$ where $0 \le a < b \le \infty$, so (2.1.17) reduces to

$$\int_a^b |(J^{\alpha}_{a+}f)(x)|^q \, \frac{dx}{x} \le \left(\frac{(\log \frac{b}{a})^{\alpha}}{\Gamma(\alpha + 1)} \right)^q \int_a^b |f(y)|^q \, \frac{dy}{y} \tag{2.1.18}$$

Also, from Theorem 2.1.10 we obtain refinements of (2.1.18) for $\alpha > \frac{1}{q}$

$$\int_a^b |(J^{\alpha}_{a+}f)(x)|^q \, \frac{dx}{x} \le \frac{(\log \frac{b}{a})^{q\alpha}}{q\alpha(\Gamma(\alpha))^q (p(\alpha - 1) + 1)^{\frac{q}{p}}} \int_a^b |f(y)|^q \, \frac{dy}{y}.$$

Some results involving Hadamard-type fractional integrals are given in [51, p. 110]. Here we mention the following result that can not be compared with our result.

Let $\alpha > 0$, $1 \le p \le \infty$ and $0 \le a < b \le \infty$. Then the operators $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ are bounded in $L_p(a, b)$ as follows:

$$\|J^{\alpha}_{a+}f\|_p \le K_1 \|f\|_p \text{ and } \|J^{\alpha}_{b-}f\|_p \le K_2 \|f\|_p,$$

where

$$K_1 = \frac{1}{\Gamma(\alpha)} \int_0^{\log(b/a)} t^{\alpha - 1} e^{\frac{t}{p}} dt \text{ and } K_2 = \frac{1}{\Gamma(\alpha)} \int_0^{\log(b/a)} t^{\alpha - 1} e^{-\frac{t}{p}} dt.$$

Corollary 2.1.11. *Let u be a weight function on (a,b), ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function and $I^{\alpha}_{a+;\sigma;\eta}f$ denotes the Erdélyi–Kober-type left-sided fractional integral. Define v by*

$$v(y) := \alpha\sigma y^{\sigma\eta+\sigma-1}\int_y^b u(x)\frac{x^{-\sigma\eta}(x^\sigma-y^\sigma)^{\alpha-1}}{(x^\sigma-a^\sigma)^\alpha\,{}_2F_1(-\eta,\alpha;\alpha+1;1-\left(\frac{a}{x}\right)^\sigma)}dx<\infty.$$

If $\Phi:(0,\infty)\to\mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\Phi\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha\,{}_2F_1(-\eta,\alpha;\alpha+1;1-\left(\frac{a}{x}\right)^\sigma)}|I^{\alpha}_{a+;\sigma;\eta}f(x)|\right)dx$$
$$\leq\int_a^b v(y)\Phi(|f(y)|)\,dy$$

holds.

Proof. Similar to Corollary 2.1.2. ∎

Remark 2.1.9. In particular for the weight function $u(x)=x^{\sigma-1}(x^\sigma-a^\sigma)^\alpha\,{}_2F_1(x)$ in Corollary 2.1.11 we obtain the inequality

$$\int_a^b x^{\sigma-1}(x^\sigma-a^\sigma)^\alpha\,{}_2F_1(x)\Phi\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha\,{}_2F_1(x)}|I^{\alpha}_{a+;\sigma;\eta}f(x)|\right)dx$$
$$\leq\int_a^b y^{\sigma-1}(b^\sigma-y^\sigma)^\alpha\,{}_2F_1(y)\Phi(|f(y)|)\,dy.$$

In the previous corollaries we derived only inequalities over some subsets of $\mathbb{R}$. However, Theorem 2.1.1 covers much more general situations. We give result for multidimensional fractional integrals. Such operations of fractional integration in the n-dimensional Euclidean space $\mathbb{R}^n$, $(n\in\mathbb{N})$ are natural generalizations of the corresponding one-dimensional fractional integrals and fractional derivatives, being taken with respect to one or several variables.

Corollary 2.1.12. *Let u be a weight function on $(\mathbf{a},\mathbf{b})$ and $\alpha>0$. $I^{\alpha}_{\mathbf{a}+}f$ denotes the mixed Riemann-Liouville fractional integral of f. Define v on $(\mathbf{a},\mathbf{b})$ by*

$$v(\mathbf{y}):=\alpha\int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n}u(\mathbf{x})\frac{(\mathbf{x}-\mathbf{y})^{\alpha-1}}{(\mathbf{x}-\mathbf{a})^{\alpha}}d\mathbf{x}<\infty.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(\mathbf{x}) \Phi\left(\frac{\Gamma(\alpha+1)}{(\mathbf{x}-\mathbf{a})^{\alpha}} |I_{\mathbf{a}^+}^{\alpha} f(\mathbf{x})|\right) d\mathbf{x} \leq \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} v(\mathbf{y}) \Phi(|f(\mathbf{y})|)\, d\mathbf{y}$$

holds for all measurable functions $f : (\mathbf{a}, \mathbf{b}) \to \mathbb{R}$.

Proof. Similar to Corollary 2.1.2. ■

Remark 2.1.10. Analogous to Remark 2.1.1 we obtain multidimensional version of inequality (2.1.1) for $q > 1$:

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} |I_{\mathbf{a}^+}^{\alpha} f(\mathbf{x})|^q\, d\mathbf{x} \leq \left(\frac{(\mathbf{b}-\mathbf{a})^{\alpha}}{\Gamma(\alpha+1)}\right)^q \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} |f(\mathbf{y})|^q\, d\mathbf{y}.$$

2.2 New inequalities involving fractional integrals and fractional derivatives

If we substitute $k(x,y)$ by $k(x,y)f_2(y)$ and f by $\dfrac{f_1}{f_2}$, where $f_i : \Omega_2 \to \mathbb{R}, (i = 1,2)$ are measurable functions in Theorem 2.1.1 we obtain the following result.

Theorem 2.2.1. *Let $f_i : \Omega_2 \to \mathbb{R}$ be measurable functions, $g_i \in U(k)$, where $g_2(x) > 0$ for every $x \in \Omega_1$. Let u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$. Assume that the function $x \mapsto u(x)\frac{f_2(y)k(x,y)}{g_2(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v on Ω_2 by*

$$v(y) := f_2(y) \int_{\Omega_1} \frac{u(x)k(x,y)}{g_2(x)} d\mu_1(x) < \infty. \tag{2.2.1}$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_{\Omega_1} u(x)\Phi\left(\left|\frac{g_1(x)}{g_2(x)}\right|\right) d\mu_1(x) \leq \int_{\Omega_2} v(y)\Phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right) d\mu_2(y)$$

holds.

Remark 2.2.1. If Φ is strictly convex and $\frac{f_1(x)}{f_2(x)}$ is non-constant, then in Theorem 2.2.1 the inequality is strict.

Remark 2.2.2. As a special case of Theorem 2.2.1 for $\Omega_1 = \Omega_2 = [a,b]$ and $d\mu_1(x) = dx,\ d\mu_2(y) = dy$ we obtain the result in [55] (see also [65, p. 236]).

As a special case of Theorem 2.2.1 we obtain the following result.

Corollary 2.2.2. *Let u be a weight function on (a,b) and let the assumptions in Lemma* 1.2.4 *be satisfied. Define v on (a,b) by*

$$v(y) = \frac{D_a^{\nu} f_2(y)}{\Gamma(\nu-\gamma)} \int_y^b \frac{u(x)(x-y)^{\nu-\gamma-1}}{D_a^{\gamma} f_2(x)} dx < \infty.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{D_a^{\gamma} f_1(x)}{D_a^{\gamma} f_2(x)}\right|\right) dx \le \int_a^b v(y)\Phi\left(\left|\frac{D_a^{\nu} f_1(y)}{D_a^{\nu} f_2(y)}\right|\right) dy \tag{2.2.2}$$

holds.

Proof. Applying Theorem 2.2.1 with $\Omega_1 = \Omega_2 = (a,b),\ d\mu_1(x) = dx,\ d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\nu-\gamma-1}}{\Gamma(\nu-\gamma)}, & a < y \le x; \\ 0, & x < y \le b, \end{cases} \tag{2.2.3}$$

and replacing f_i by $D_a^{\nu} f_i$, $i = 1,2$ we obtain (2.2.2).

Corollary 2.2.3. *Let u be a weight function on (a,b) and let the assumptions in Lemma* 1.2.2 *be satisfied. Define v on (a,b) by*

$$v(y) = \frac{D_a^{\beta} f_2(y)}{\Gamma(\beta-\alpha)} \int_y^b \frac{u(x)(x-y)^{\beta-\alpha-1}}{D_a^{\alpha} f_2(x)} dx < \infty.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{D_a^{\alpha} f_1(x)}{D_a^{\alpha} f_2(x)}\right|\right) dx \le \int_a^b v(y)\Phi\left(\left|\frac{D_a^{\beta} f_1(y)}{D_a^{\beta} f_2(y)}\right|\right) dy$$

holds.

Proof. Similar to the Corollary 2.2.2. ∎

Corollary 2.2.4. *Let u be a weight function on (a,b) and $\alpha \geq 0$. $D^{\alpha}_{*a} f$ denotes the Caputo fractional derivative of f. Define v on (a,b) by*

$$v(y) = \frac{f_2^{(n)}(y)}{\Gamma(n-\alpha)} \int_y^b \frac{u(x)(x-y)^{n-\alpha-1}}{D^{\alpha}_{*a} f_2(x)} dx < \infty.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{D^{\alpha}_{*a} f_1(x)}{D^{\alpha}_{*a} f_2(x)}\right|\right) dx \leq \int_a^b v(y)\Phi\left(\left|\frac{f_1^{(n)}(y)}{f_2^{(n)}(y)}\right|\right) dy$$

holds.

Proof. Similar to Corollary 2.2.2. ■

Corollary 2.2.5. *Let u be a weight function on (a,b) and let the assumptions in Lemma 1.2.5 be satisfied. Define v on (a,b) by*

$$v(y) = \frac{D^{\nu}_{*a} f_2(y)}{\Gamma(\nu-\gamma)} \int_y^b \frac{u(x)(x-y)^{\nu-\gamma-1}}{D^{\gamma}_{*a} f_2(x)} dx < \infty.$$

If $\Phi : (0,\infty) \to \mathbb{R}$ is a convex and increasing, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{D^{\gamma}_{*a} f_1(x)}{D^{\gamma}_{*a} f_2(x)}\right|\right) dx \leq \int_a^b v(y)\Phi\left(\left|\frac{D^{\nu}_{*a} f_1(y)}{D^{\nu}_{*a} f_2(y)}\right|\right) dy$$

holds.

Proof. Similar to Corollary 2.2.2. ■

Now we will show some new inequalities for fractional integrals.

Corollary 2.2.6. *Let u be a weight function on (a,b) and $\alpha > 0$. $I^{\alpha}_{a^+} f$ denotes the left-sided Riemann-Liouville fractional integral of f. Define v on (a,b) by*

$$v(y) = \frac{f_2(y)}{\Gamma(\alpha)} \int_y^b \frac{u(x)(x-y)^{\alpha-1}}{I^{\alpha}_{a^+} f_2(x)} dx < \infty.$$

If $\Phi : (0, \infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{I^{\alpha}_{a+}f_1(x)}{I^{\alpha}_{a+}f_2(x)}\right|\right)dx \leq \int_a^b v(y)\Phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right)dy$$

holds.

Proof. Similar to the Corollary 2.2.2. ■

Corollary 2.2.7. *Let u be a weight function, $\alpha > 0$ and $J^{\alpha}_{a+}f$ denotes the left-sided Hadamard-type fractional integral. Define v on (a, b) by*

$$v(y) = \frac{f_2(y)}{y\Gamma(\alpha)}\int_y^b u(x)\left(\log\frac{x}{y}\right)^{\alpha-1}\frac{1}{(J^{\alpha}_{a+}f_2)(x)}dx < \infty.$$

If $\Phi : (0, \infty) \to \mathbb{R}$ is a convex and increasing, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{J^{\alpha}_{a_+}f_1(x)}{J^{\alpha}_{a_+}f_2(x)}\right|\right)dx \leq \int_a^b v(y)\Phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right)dy$$

holds.

Proof. Similar to Corollary 2.2.2. ■

Corollary 2.2.8. *Let u be a weight function, $I^{\alpha}_{a+;\sigma;\eta}f$ denotes the left-sided Erdélyi-Kober-type fractional integral of function f of order $\alpha > 0$. Define v on (a, b) by*

$$v(y) = \frac{\sigma f_2(y)}{\Gamma(\alpha)}\int_y^b \frac{u(x)x^{-\sigma(\alpha+\eta)}y^{\sigma\eta+\sigma-1}}{(x^{\sigma} - y^{\sigma})^{1-\alpha}(I^{\alpha}_{a+;\sigma;\eta}f_2)(x)}dx < \infty.$$

If $\Phi : (0, \infty) \to \mathbb{R}$ is a convex and increasing, then the inequality

$$\int_a^b u(x)\Phi\left(\left|\frac{I^{\alpha}_{a+;\sigma;\eta}f_1(x)}{I^{\alpha}_{a+;\sigma;\eta}f_2(x)}\right|\right)dx \leq \int_a^b v(y)\Phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right)dy$$

holds.

Proof. Similar to Corollary 2.2.2. ■

Corollary 2.2.9. *Let u be a weight function on $(\mathbf{a}, \mathbf{b})$ and $\alpha > 0$. $I^{\alpha}_{\mathbf{a}^+} f$ denotes the mixed Riemann-Liouville fractional integral of f. Define v on $(\mathbf{a}, \mathbf{b})$ by*

$$v(\mathbf{y}) := \frac{f_2(\mathbf{y})}{\Gamma(\alpha)} \int\limits_{y_1}^{b_1} \cdots \int\limits_{y_n}^{b_n} u(\mathbf{x}) \frac{(\mathbf{x} - \mathbf{y})^{\alpha-1}}{(I^{\alpha}_{\mathbf{a}^+} f_2)(\mathbf{x})} d\mathbf{x} < \infty.$$

If $\Phi : (0, \infty) \to \mathbb{R}$ is a convex and increasing, then the inequality

$$\int\limits_{a_1}^{b_1} \ldots \int\limits_{a_1}^{b_1} u(\mathbf{x}) \Phi \left(\left| \frac{I^{\alpha}_{\mathbf{a}^+} f_1(\mathbf{x})}{I^{\alpha}_{\mathbf{a}^+} f_2(\mathbf{x})} \right| \right) d\mathbf{x} \leq \int\limits_{a_1}^{b_1} \ldots \int\limits_{a_1}^{b_1} v(\mathbf{y}) \Phi \left(\left| \frac{f_1(\mathbf{y})}{f_2(\mathbf{y})} \right| \right) d\mathbf{y}$$

holds.

Proof. Similar to Corollary 2.2.2. ∎

Note that Theorem 2.2.1 can be generalized for convex functions of several variables.

Theorem 2.2.10. *Let $g_i \in U(k)$, $(i = 1, 2, 3)$, where $g_2(x) > 0$ for every $x \in \Omega_1$. Let u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$. Let v be defined by (2.2.1). If $\Phi : (0, \infty) \times (0, \infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality*

$$\int\limits_{\Omega_1} u(x) \Phi \left(\left| \frac{g_1(x)}{g_2(x)} \right|, \left| \frac{g_3(x)}{g_2(x)} \right| \right) d\mu_1(x) \leq \int\limits_{\Omega_2} v(y) \Phi \left(\left| \frac{f_1(y)}{f_2(y)} \right|, \left| \frac{f_3(y)}{f_2(y)} \right| \right) d\mu_2(y) \quad (2.2.4)$$

holds.

Remark 2.2.3. Apply Theorem 2.2.10 with $\Omega_1 = \Omega_2 = [a, b]$ and $d\mu_1(x) = dx,\ d\mu_2(y) = dy$. Then

$$v(y) = f_2(y) \int\limits_{a}^{b} \frac{u(x) k(x, y)}{g_2(x)} dx$$

and (2.2.4) reduces to

$$\int\limits_{a}^{b} u(x) \Phi \left(\left| \frac{g_1(x)}{g_2(x)} \right|, \left| \frac{g_3(x)}{g_2(x)} \right| \right) dx \leq \int\limits_{a}^{b} v(y) \Phi \left(\left| \frac{f_1(y)}{f_2(y)} \right|, \left| \frac{f_3(y)}{f_2(y)} \right| \right) dy.$$

This result is given in [55] (see also [65, p. 236]).

2.3 Improvement of an an inequality of G. H. Hardy and Cauchy means

In this section, we give the improvement of an inequality of G. H. Hardy using fractional integrals and fractional derivatives. We obtain means of Cauchy type and prove their monotonicity. Also, using the concept of exponential convexity and log-convexity we establish some new inequalities.

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures and A_k be an integral operator defined by

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y) d\mu_2(y), \tag{2.3.1}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel, f is measurable function on Ω_2 and K be defined by (2.1.2).

The following theorem is give in [52](see also [28]).

Theorem 2.3.1. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$ and K be defined on Ω_1 by* (2.1.2). *Suppose that the function $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$ and that v is defined on Ω_2 by* (2.1.3). *If Φ is a convex function on the interval $I \subseteq \mathbb{R}$, then the inequality*

$$\int_{\Omega_1} u(x) \Phi\left(A_k f(x)\right) d\mu_1(x) \leq \int_{\Omega_2} v(y) \Phi\left(f(y)\right) d\mu_2(y) \tag{2.3.2}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $Imf \subseteq I$, where A_k is defined by (2.3.1).

2.3.1 Main results

Lemma 2.3.2. *Let $s \in \mathbb{R}$, $\varphi_s : \mathbb{R}_+ \to \mathbb{R}$ be a function defined as*

$$\varphi_s(x) := \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0, 1, \\ -\log x, & s = 0, \\ x \log x, & s = 1. \end{cases} \tag{2.3.3}$$

Then φ_s is strictly convex on $\mathbb{R}_+$ for each $s \in \mathbb{R}$.

The following theorem is given in [28].

Theorem 2.3.3. *Let the assumption in the Theorem 2.3.1 be satisfied and φ_s be defined by (2.3.3). Let f be a positive function. Then the function $\xi : \mathbb{R} \to [0, \infty)$ is defined by*

$$\xi(s) = \int_{\Omega_2} v(y)\varphi_s(f(y))d\mu_2(y) - \int_{\Omega_1} u(x)\varphi_s(A_k f(x))d\mu_1(x) \tag{2.3.4}$$

is exponentially convex.

The function ξ being exponentially convex is also log-convex function. Then by Remark 1.1.1(c) and Remark 1.1.1 the following inequality holds true:

$$[\xi(p)]^{q-r} \leq [\xi(q)]^{p-r}[\xi(r)]^{q-p} \tag{2.3.5}$$

for every choice $r, p, q \in \mathbb{R}$, such that $r < p < q$.

Our first result involving fractional integral of f with respect to another increasing function g is given in the following theorem as an application of Theorem 2.3.3.

Theorem 2.3.4. *Let $s > 1$, $\alpha > 0$, g be increasing function on $(a, b]$ such that g' be continuous on (a, b) and $\xi_1 : \mathbb{R} \to [0, \infty)$. Then the following inequality holds:*

$$\xi_1(s) \leq H_1(s),$$

where

$$\xi_1(s) = \frac{1}{s(s-1)} \left[\int_a^b g'(y)(g(b) - g(y))^\alpha f^s(y)dy \right.$$
$$\left. -(\Gamma(\alpha+1))^s \int_a^b g'(x)(g(x) - g(a))^{\alpha(1-s)} \left(I^\alpha_{a+;g} f(x)\right)^s dx \right],$$

and

$$H_1(s) = \frac{(g(b) - g(a))^{\alpha(1-s)}}{s(s-1)} \left[(g(b) - g(a))^{\alpha s} \int_a^b f^s(y)g'(y)dy \right.$$
$$\left. - (\Gamma(\alpha+1))^s \int_a^b (I^\alpha_{a+;g} f(x))^s g'(x)dx \right].$$

Proof. Applying Theorem 2.3.3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha)(g(x)-g(y))^{1-\alpha}}, & a < y \le x; \\ 0, & x < y \le b, \end{cases} \tag{2.3.6}$$

we get that $K(x) = \frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha}$, then (2.3.4) becomes

$$\xi_1(s) = \int_a^b v(y)\Phi_s(f(y))dy - \int_a^b u(x)\Phi_s\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g}f(x)\right)dx.$$

For the particular weight function $u(x) = g'(x)(g(x)-g(a))^{\alpha}$, we obtain $v(y) = g'(y)(g(b)-g(y))^{\alpha}$ and take $\Phi_s(x) = \frac{x^s}{s(s-1)},\ x \in \mathbb{R}_+$, we get

$$\begin{aligned} \xi_1(s) = \frac{1}{s(s-1)} & \left[\int_a^b g'(y)(g(b)-g(y))^{\alpha} f^s(y)dy \right. \\ & \left. - \int_a^b g'(x)(g(x)-g(a))^{\alpha}\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g}f(x)\right)^s dx \right]. \end{aligned}$$

Since $x \in (a,b)$ and $\alpha(1-s) < 0$, g is increasing, then $(g(x)-g(a))^{\alpha(1-s)} > (g(b)-g(a))^{\alpha(1-s)}$ and $(g(b)-g(y))^{\alpha} < (g(b)-g(a))^{\alpha}$, we obtain

$$\begin{aligned} \xi_1(s) &\le \frac{(g(b)-g(a))^{\alpha(1-s)}}{s(s-1)} \left[(g(b)-g(a))^{\alpha s} \int_a^b f^s(y)g'(y)dy \right. \\ & \quad \left. - (\Gamma(\alpha+1))^s \int_a^b (I^{\alpha}_{a+;g}f(x))^s g'(x)dx \right] \\ &= H_1(s). \end{aligned}$$

This complete the proof. ∎

If $g(x) = x$, then $I^{\alpha}_{a+;x}f(x)$ reduces to $I^{\alpha}_{a+}f(x)$ left-sided Riemann–Liouville fractional integral and next result follows.

Corollary 2.3.5. *Let $s > 1$, $\alpha > 0$ and $\xi_2 : \mathbb{R} \to [0,\infty)$. Then the following inequality holds true:*

$$\xi_2(s) \le H_2(s),$$

where

$$\xi_2(s) = \frac{1}{s(s-1)} \left[\int_a^b (b-y)^{\alpha} f^s(y) dy - (\Gamma(\alpha+1))^s \int_a^b (x-a)^{\alpha(1-s)} \left(I_{a+}^{\alpha} f(x)\right)^s dx \right],$$

and

$$H_2(s) = \frac{(b-a)^{\alpha(1-s)}}{s(s-1)} \left[(b-a)^{\alpha s} \int_a^b f^s(y) dy - (\Gamma(\alpha+1))^s \int_a^b (I_{a+}^{\alpha} f(x))^s dx \right].$$

Notice that Hadamard fractional integrals of order α are special case of the left- and right-sided fractional integrals of a function f with respect to another function $g(x) = \log(x)$, $x \in (a, b)$ where $0 \leq a < b \leq \infty$ and next result follows.

Corollary 2.3.6. *Let $s > 1$, $\alpha > 0$ and $\xi_3 : \mathbb{R} \to [0, \infty)$. Then the following inequality holds:*

$$\xi_3(s) \leq H_3(s),$$

where

$$\xi_3(s) = \frac{1}{s(s-1)} \left[\int_a^b (\log b - \log y)^{\alpha} f^s(y) \frac{dy}{y} \right.$$
$$\left. -(\Gamma(\alpha+1))^s \int_a^b (\log x - \log a)^{\alpha(1-s)} \left(J_{a+}^{\alpha} f(x)\right)^s \frac{dx}{x} \right],$$

and

$$H_3(s) = \frac{(\log b - \log a)^{\alpha(1-s)}}{s(s-1)} \left[(\log b - \log a)^{\alpha s} \int_a^b f^s(y) \frac{dy}{y} \right.$$
$$\left. - (\Gamma(\alpha+1))^s \int_a^b (J_{a+}^{\alpha} f(x))^s \frac{dx}{x} \right].$$

Next we give result with respect to the generalized Riemann-Liouville fractional derivative for details see [7, p. 448].

Theorem 2.3.7. *Let $s > 1$, $\beta > \alpha \geq 0$, let $f \in L_1(a,b)$ have an L_∞ fractional derivative $D_a^\beta f$ in $[a,b]$ and $\xi_4 : \mathbb{R} \to [0,\infty)$. Then the following inequality holds true:*

$$\xi_4(s) \leq H_4(s),$$

where

$$\xi_4(s) = \frac{1}{s(s-1)} \left[\int_a^b (b-y)^{\beta-\alpha} (D_a^\beta f(y))^s dy - (\Gamma(\beta-\alpha+1))^s \int_a^b (x-a)^{(\beta-\alpha)(1-s)} \left(D_a^\alpha f(x)\right)^s dx \right],$$

and

$$H_4(s) = \frac{(b-a)^{(\beta-\alpha)(1-s)}}{s(s-1)} \left[(b-a)^{(\beta-\alpha)s} \int_a^b (D_a^\beta f(y))^s dy - (\Gamma(\beta-\alpha+1))^s \int_a^b (D_a^\alpha f(x))^s dx \right].$$

Proof. Applying Theorem 2.3.3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$ and $k(x,y)$ is given in (2.3.6). Replace f by $D_a^\beta f$, then (2.3.4) becomes

$$\xi_4(s) = \int_a^b v(y) \Phi_s(D_a^\beta f(y)) dy - \int_a^b u(x) \Phi_s \left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x) \right) dx. \tag{2.3.7}$$

For the particular weight function $u(x) = (x-a)^{\beta-\alpha}$, we get $v(y) = (b-y)^{\beta-\alpha}$ and we choose $\Phi_s(x) = \frac{x^s}{s(s-1)}$, $x \in \mathbb{R}_+$, then (2.3.7) becomes

$$\begin{aligned} \xi_4(s) \;&=\; \frac{1}{s(s-1)} \left[\int_a^b (b-y)^{\beta-\alpha} (D_a^\beta f(y))^s dy \right. \\ &\quad \left. - \int_a^b (x-a)^{\beta-\alpha} \left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} (D_a^\alpha f(x)) \right)^s dx \right] \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{s(s-1)}\left[(b-a)^{\beta-\alpha}\int_a^b (D_a^{\beta}f(y))^s dy\right.\\
&\qquad \left. -(b-a)^{(\beta-\alpha)(1-s)}(\Gamma(\beta-\alpha+1))^s\int_a^b (D_a^{\alpha}f(x))^s dx\right]\\
&= H_4(s).
\end{aligned}
$$

■

In the following theorem, we will construct new inequality for the Canavati-type fractional derivative.

Theorem 2.3.8. *Let* $s > 1$, *let the assumptions in Lemma 1.2.4 be satisfied and* $\xi_5 : \mathbb{R} \to [0,\infty)$. *Then the following inequality holds:*

$$\xi_5(s) \leq H_5(s),$$

where

$$
\begin{aligned}
\xi_5(s) = \frac{1}{s(s-1)}&\left[\int_a^b (b-y)^{\nu-\gamma}(D_a^{\nu}f(y))^s dy\right.\\
&\left. -(\Gamma(\nu-\gamma+1))^s\int_a^b (x-a)^{(\nu-\gamma)(1-s)}\left(D_a^{\gamma}f(x)\right)^s dx\right],
\end{aligned}
$$

and

$$
\begin{aligned}
H_5(s) = \frac{(b-a)^{(\nu-\gamma)(1-s)}}{s(s-1)}&\left[(b-a)^{(\nu-\gamma)s}\int_a^b (D_a^{\nu}f(y))^s dy\right.\\
&\left. -(\Gamma(\nu-\gamma+1))^s\int_a^b (D_a^{\gamma}f(x))^s dx\right].
\end{aligned}
$$

Proof. Similar to Theorem 2.3.7. ■

As a special case of Theorem 2.3.3 to construct new inequality for the Caputo fractional derivative in the upcoming theorem.

Theorem 2.3.9. *Let $s > 1$, $\alpha \geq 0$, $n = \lceil \alpha \rceil$, $f \in AC^n[a,b]$ and $\xi_6 : \mathbb{R} \to [0,\infty)$. Then the following inequality holds true:*

$$\xi_6(s) \leq H_6(s),$$

where

$$\xi_6(s) = \frac{1}{s(s-1)} \left[\int_a^b (b-y)^{n-\alpha} (f^{(n)}(y))^s dy \right.$$
$$\left. -(\Gamma(n-\alpha+1))^s \int_a^b (x-a)^{(n-\alpha)(1-s)} \left(D_{*a}^{\alpha} f(x) \right)^s dx \right],$$

and

$$H_6(s) = \frac{(b-a)^{(n-\alpha)(1-s)}}{s(s-1)} \left[(b-a)^{(n-\alpha)s} \int_a^b (f^{(n)}(y))^s dy \right.$$
$$\left. - (\Gamma(n-\alpha+1))^s \int_a^b (D_{*a}^{\alpha} f(x))^s dx \right].$$

Proof. Similar to Theorem 2.3.7. ∎

Theorem 2.3.10. *Let $s > 1$, let the assumptions in Lemma 1.2.6 be satisfied and $\xi_7 : \mathbb{R} \to [0,\infty)$. Then the following inequality holds:*

$$\xi_7(s) \leq H_7(s),$$

where

$$\xi_7(s) = \frac{1}{s(s-1)} \left[\int_a^b (b-y)^{\alpha-\gamma} (D_{*a}^{\alpha} f(y))^s dy \right.$$
$$\left. -(\Gamma(\alpha-\gamma+1))^s \int_a^b (x-a)^{(\alpha-\gamma)(1-s)} \left(D_{*a}^{\gamma} f(x) \right)^s dx \right],$$

and

$$H_7(s) = \frac{(b-a)^{(\alpha-\gamma)(1-s)}}{s(s-1)} \left[(b-a)^{(\alpha-\gamma)s} \int_a^b (D_{*a}^{\alpha} f(y))^s dy \right.$$
$$\left. - (\Gamma(\alpha-\gamma+1))^s \int_a^b (D_{*a}^{\gamma} f(x))^s dx \right].$$

Proof. Similar to Theorem 2.3.7. ■

Now, we give the following result for Erdélyi–Kober type fractional integral of f.

Theorem 2.3.11. *Let* $s > 1$, ${}_2F_1(a,b;c;z)$ *denotes the hypergeometric function and* $\xi_8 : \mathbb{R} \to [0,\infty)$. *Then the following inequality holds:*

$$\xi_8(s) \leq H_8(s),$$

where

$$\xi_8(s) = \frac{1}{s(s-1)} \left[\int_a^b y^{\sigma-1} (b^\sigma - y^\sigma)^\alpha \, {}_2F_1(y) f^s(y) dy \right.$$
$$\left. - \int_a^b x^{\sigma-1} (x^\sigma - a^\sigma)^\alpha \, {}_2F_1(x) \left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha \, {}_2F_1(x)} I_{a+;\sigma;\eta}^{\alpha} f(x) \right)^s dx \right],$$

and

$$H_8(s) = \frac{(b^\sigma - a^\sigma)^{\alpha(1-s)}}{s(s-1)} \left[(b^\sigma - a^\sigma)^{\alpha s} b^{\sigma-1} \int_a^b {}_2F_1(y) f^s(y) dy \right.$$
$$\left. - a^{\sigma-1+\alpha\sigma s} (\Gamma(\alpha+1))^s \int_a^b \left(({}_2F_1(x))^{1-s} I_{a+;\sigma;\eta}^{\alpha} f(x) \right)^s dx \right].$$

Proof. Applying Theorem 2.3.3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{1}{\Gamma(\alpha)} \frac{\sigma x^{-\sigma(\alpha+\eta)}}{(x^\sigma - y^\sigma)^{1-\alpha}} y^{\sigma\eta+\sigma-1}, & a < y \leq x\,; \\ 0, & x < y \leq b, \end{cases} \tag{2.3.8}$$

we get that $K(x) = \frac{1}{\Gamma(\alpha+1)}\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha} {}_2F_1(-\eta,\alpha;\alpha+1;1-\left(\frac{a}{x}\right)^{\sigma})$ and (2.3.4) becomes

$$\xi_8(s) = \int_a^b v(y)\Phi_s(f(y))dy - \int_a^b u(x)\Phi_s\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha} {}_2F_1(x)} I^{\alpha}_{a+;\sigma;\eta} f(x)\right) dx. \quad (2.3.9)$$

For the particular weight function $u(x) = x^{\sigma-1}(x^{\sigma}-a^{\sigma})^{\alpha} {}_2F_1(x)$, we obtain $v(y) = y^{\sigma-1}(b^{\sigma}-y^{\sigma})^{\alpha} {}_2F_1(y)$, and if we choose $\Phi_s(x) = \frac{x^s}{s(s-1)}$, $x \in \mathbb{R}_+$, then (2.3.9) becomes

$$\begin{aligned}
\xi_8(s) &= \frac{1}{s(s-1)}\left[\int_a^b y^{\sigma-1}(b^{\sigma}-y^{\sigma})^{\alpha} {}_2F_1(y) f^s(y)dy\right. \\
&\quad \left. - \int_a^b x^{\sigma-1}(x^{\sigma}-a^{\sigma})^{\alpha} {}_2F_1(x)\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha} {}_2F_1(x)} I^{\alpha}_{a+;\sigma;\eta} f(x)\right)^s dx\right] \\
&\leq \frac{1}{s(s-1)}\left[b^{\sigma-1}(b^{\sigma}-a^{\sigma})^{\alpha}\int_a^b {}_2F_1(y) f^s(y)dy\right. \\
&\quad \left. - a^{\sigma-1+\alpha\sigma s}(b^{\sigma}-a^{\sigma})^{\alpha(1-s)}(\Gamma(\alpha+1))^s \int_a^b ({}_2F_1(x))^{1-s}(I^{\alpha}_{a+;\sigma;\eta} f(x))^s dx\right] \\
&= H_8(s).
\end{aligned}$$

This complete the proof. ■

In the following theorem we prove the three different cases for the above results.

Theorem 2.3.12. *For $i = 1, \ldots, 8$ the following inequalities hold true:*

$$(i). \qquad [\xi_i(p)]^{\frac{q-r}{q-p}}[\xi_i(q)]^{\frac{r-p}{q-p}} \leq H_i(r) \quad (2.3.10)$$

$$(ii). \qquad [\xi_i(r)]^{\frac{p-q}{p-r}}[\xi_i(p)]^{\frac{q-r}{p-r}} \leq H_i(q) \quad (2.3.11)$$

$$(iii). \qquad \xi_i(p) \leq [H_i(r)]^{\frac{q-p}{q-r}}[H_i(q)]^{\frac{p-r}{q-r}} \quad (2.3.12)$$

for every choice $p, q, r \in \mathbb{R}$, such that $1 < r < p < q$.

Proof. We will prove this theorem just in case $i = 1$, since all other case are proved analogous.

(i). Since the function ξ_1 is exponentially convex, it is also log-convex, then for $1 < r < p < q$, $r, p, q \in \mathbb{R}$, (2.3.5) can be written as:

$$[\xi_1(p)]^{q-r}[\xi_1(q)]^{r-p} \leq [\xi_1(r)]^{q-p},$$

this implies that

$$\begin{aligned}
[\xi_1(p)]^{\frac{q-r}{q-p}}[\xi_1(q)]^{\frac{r-p}{q-p}} &\leq \frac{(g(b)-g(a))^{\alpha(1-r)}}{r(r-1)}\left[(g(b)-g(a))^{\alpha r}\int_a^b f^r(y)g'(y)dy\right.\\
&\quad \left.-(\Gamma(\alpha+1))^r\int_a^b (I_{a^+}^{\alpha}f(x))^r g'(x)dx\right]\\
&= H_1(r).
\end{aligned}$$

It follows (2.3.10).

(ii). Now (2.3.5) can be written as:

$$[\xi_1(r)]^{p-q}[\xi_1(p)]^{q-r} \leq [\xi_1(q)]^{p-r},$$

this implies that

$$\begin{aligned}
[\xi_1(r)]^{\frac{p-q}{p-r}}[\xi_1(p)]^{\frac{q-r}{p-r}} &\leq \frac{(g(b)-g(a))^{\alpha(1-q)}}{q(q-1)}\left[(g(b)-g(a))^{\alpha q}\int_a^b f^q(y)g'(y)dy\right.\\
&\quad \left.-(\Gamma(\alpha+1))^q\int_a^b (I_{a^+}^{\alpha}f(x))^q g'(x)dx\right]\\
&= H_1(q).
\end{aligned}$$

It follows (2.3.11).

(iii). The (2.3.5) can be written as:

$$[\xi_1(p)]^{\frac{q-r}{p-r}} \leq [\xi_1(r)]^{\frac{q-p}{p-r}}\,\xi_1(q),$$

$$[\xi_1(p)]^{\frac{q-r}{p-r}} \leq [\xi_1(r)]^{\frac{q-p}{p-r}}H_1(q),$$

this implies that

$$\xi_1(p) \leq [H_1(r)]^{\frac{q-p}{q-r}}[H_1(q)]^{\frac{p-r}{q-r}}.$$

It follows (2.3.12). ∎

2.3.2 Mean value theorems and Cauchy means

Now we will give the mean value theorems and means of Cauchy type for different fractional integrals and fractional derivatives. For this purpose we define a notation

$$\xi_i(s) := \xi_i(v, \Phi_s(f(y));\, u, \Phi_s(A_k f(x)), \quad \text{for} \quad (i = 1, \dots, 8) \tag{2.3.13}$$

where $A_k f$ and v are defined by (2.3.1) and (2.1.3) respectively.

The following theorems are given in [28]. Such type of results are also given in [67] and [70].

Theorem 2.3.13. *Let* $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ *be measure spaces with* σ*-finite measures and* $u : \Omega_1 \to \mathbb{R}$ *be a weight function. Let* I *be compact interval of* $\mathbb{R}$, $h \in C^2(I)$, *and* $f : \Omega_2 \to \mathbb{R}$ *be a measurable function such that* $Imf \subseteq I$. *Then there exists* $\eta \in I$ *such that*

$$\begin{aligned}&\int_{\Omega_2} v(y)h(f(y))d\mu_2(y) - \int_{\Omega_1} u(x)h(A_k f(x))d\mu_1(x)\\ &\qquad = \frac{h''(\eta)}{2}\left[\int_{\Omega_2} v(y)f^2(y)d\mu_2(y) - \int_{\Omega_1} u(x)(A_k f(x))^2 d\mu_1(x)\right],\end{aligned}$$

where $A_k f$ *and* v *are defined by* (2.3.1) *and* (2.1.3) *respectively.*

Theorem 2.3.14. *Assume that all conditions of Theorem 2.3.13 are satisfied. Let* I *be a compact interval in* $\mathbb{R}$ *and* $g, h \in C^2(I)$ *such that* $h''(x) \neq 0$ *for every* $x \in I$. *Let* $f : \Omega_2 \to \mathbb{R}$ *be a measurable function such that* $Imf \subseteq I$ *and*

$$\int_{\Omega_2} v(y)h(f(y))d\mu_2(y) - \int_{\Omega_1} u(x)h(A_k f(x))d\mu_1(x) \neq 0.$$

Then there exists $\eta \in I$ *such that it holds*

$$\frac{g''(\eta)}{h''(\eta)} = \frac{\int_{\Omega_2} v(y)g(f(y))d\mu_2(y) - \int_{\Omega_1} u(x)g(A_k f(x))d\mu_1(x)}{\int_{\Omega_2} v(y)h(f(y))d\mu_2(y) - \int_{\Omega_1} u(x)h(A_k f(x))d\mu_1(x)}.$$

We will give some special cases of Theorems 2.3.13 and 2.3.14 for different fractional integrals and fractional derivatives in upcoming results.

Theorem 2.3.15. *Let u be weight functions on (a,b), $A_kf(x)$ be defined by* (2.3.1) *and v be defined by (2.1.3). Let I be a compact interval of $\mathbb{R}$, $h \in C^2(I)$ and $\xi_i : \mathbb{R} \to [0,\infty)$. Then there exists $\eta_i \in I$ such that*

$$\xi_i(v, h(f(y));\, u, h(A_kf(x))) = \frac{h''(\eta_i)}{2}\,\xi_i(v, (f(y))^2;\, u, (A_kf(x))^2), \qquad (i = 1, \ldots, 8).$$

Theorem 2.3.16. *Let u be weight functions on (a,b), $A_kf(x)$ be defined by* (2.3.1) *and v be defined by (2.1.3). Let I be a compact interval of $\mathbb{R}$, $g, h \in C^2(I)$ such that $h''(x) \neq 0$ for every $x \in I$, $\xi_i : \mathbb{R} \to [0,\infty)$ and*

$$\xi_i(v, h(f(y));\, u, h(A_kf(x))) \neq 0.$$

Then there exists $\eta_i \in I$ such that

$$\frac{g''(\eta_i)}{h''(\eta_i)} = \frac{\xi_i(v, g(f(y));\, u, g(A_kf(x)))}{\xi_i(v, h(f(y));\, u, h(A_kf(x)))}, \qquad (i = 1, \ldots, 8).$$

If we apply Theorem 2.3.16 with $g(x) = \frac{x^p}{p(p-1)},\ h(x) = \frac{x^s}{s(s-1)}, p \neq s,\ p, s \neq 0, 1$, we get the following result.

Corollary 2.3.17. *Let u be weight functions on (a,b), $A_kf(x)$ be defined by* (2.3.1) *and v be defined by (2.1.3). Let I be a compact interval of $\mathbb{R}_+$, $\xi_i : \mathbb{R} \to [0,\infty)$, $(i = 1, \ldots, 8)$, then for $p \neq s,\ p, s \neq 1$, there exist $\eta_i \in I$ such that*

$$\eta_i^{p-s} = \frac{\xi_i(p)}{\xi_i(s)} = \frac{s(s-1)}{p(p-1)}\frac{\xi_i(v, f^p(y);\, u, (A_kf(x))^p)}{\xi_i(v, f^s(y);\, u, (A_kf(x))^s)}. \tag{2.3.14}$$

Remark 2.3.1. Since $g''(x) = x^{p-2}$ and $h''(x) = x^{s-2}$, $\frac{g''}{h''}$ are invertible. Then from (2.3.14), we obtain

$$\inf_{t\in[a,b]} f(t) \leq \left(\frac{\xi_i(p)}{\xi_i(s)}\right)^{\frac{1}{p-s}} \leq \sup_{t\in[a,b]} f(t).$$

So,

$$M_i^{p,s}(v, \varphi_s(f(y));\, u, \varphi_s(A_kf(x))) = \left(\frac{\xi_i(p)}{\xi_i(s)}\right)^{\frac{1}{p-s}}$$

and for simplicity we denote

$$M_i^{p,s} := M_i^{p,s}(v, \varphi_s(f(y));\, u, \varphi_s(A_kf(x))$$

$p \neq s,\ p, s \neq 0, 1$ are means. Moreover we can extend these means to excluded cases. Taking a limit we can define

$$M_i^{p,s} = \begin{cases} \left(\frac{\xi_i(v,\varphi_p(f(y));u,\varphi_p(A_k f(x)))}{\xi_i(v,\varphi_s(f(y));u,\varphi_s(A_k f(x)))}\right)^{\frac{1}{p-s}}, & p \neq s, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\xi_i(v,\varphi_s(f(y))\varphi_0(f(y));u,\varphi_s(A_k f(x))\varphi_0(A_k f(x)))}{\xi_i(v,\varphi_s(f(y));u,\varphi_s(A_k f(x)))}\right), & p = s \neq 0, 1, \\ \exp\left(\frac{-\xi_i(v,\varphi_1(f(y))(\varphi_0(f(y)+2)));u,\varphi_1(A_k f(x))(\varphi_0(A_k f(x)+2)))}{2\xi_i(v,f(y)+\varphi_1(f(y));u,A_k f(x)+\varphi_1(A_k f(x)))}\right), & p = s = 1, \\ \exp\left(\frac{\xi_i(v,(2\varphi_0(f(y))-\varphi_0^2(f(y)));u,(2\varphi_0(A_k f(x))-\varphi_0^2(A_k f(x))))}{2\xi_i(v,1+\varphi_0 f(y);u,1+\varphi_0(A_k f(x)))}\right), & p = s = 0. \end{cases}$$

In the following theorem, we prove the monotonicity of means.

Theorem 2.3.18. *Let $r \leq s,\ l \leq p$, then the following inequality is valid,*

$$M_i^{l,r} \leq M_i^{p,s}, \quad (i = 1, \ldots, 8), \tag{2.3.15}$$

that is, the means $M_i^{p,s}$ are monotonic.

Proof. Since ξ_i are exponentially convex is also log-convex, we can apply Remark 1.1.1 (Galvani's theorem) and get (2.3.15). For $r = s,\ l = p$, we get the result by taking limit in (2.3.15). ∎

2.4 Further results on an inequality of G. H. Hardy

In [52], K. Krulić et. al. gave the new generalization of the Theorem 2.3.1 and they prove the following theorem.

Theorem 2.4.1. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$ and K be defined on Ω_1 by (2.1.2). Let $0 < p \leq q < \infty$ and that the function $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$ and that v is defined on Ω_2 by*

$$v(y) := \left(\int_{\Omega_1} u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(x)\right)^{\frac{p}{q}} < \infty. \tag{2.4.1}$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_{\Omega_1} u(x)[\Phi\left(A_k f(x)\right)]^{\frac{q}{p}} d\mu_1(x) \right)^{\frac{1}{q}} \leq \left(\int_{\Omega_2} v(y)\Phi\left(f(y)\right) d\mu_2(y) \right)^{\frac{1}{p}}, \tag{2.4.2}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $Imf \subseteq I$, where A_k is defined by (2.3.1).

Using Theorem 2.4.1, we will give some special cases for different fractional integrals and fractional derivatives to establish new Hardy-type inequalities as a generalization of the results given in [40].

Our next result involving fractional integral of f with respect to another increasing function g is given in the following theorem.

Theorem 2.4.2. *Let $0 < p \leq q < \infty$, u be a weight function on (a, b), g be increasing function on $(a, b]$ such that g' be continuous on (a, b). Let v be defined on (a, b) by*

$$v(y) := \alpha g'(y) \left(\int_y^b u(x) \left(\frac{(g(x) - g(y))^{\alpha-1}}{(g(x) - g(a))^{\alpha}} \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} < \infty. \tag{2.4.3}$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(\alpha+1)}{(g(x) - g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \leq \left(\int_a^b v(y)\Phi\left(f(y)\right) dy \right)^{\frac{1}{p}} \tag{2.4.4}$$

holds for all measurable functions $f : (a, b) \to \mathbb{R}$, such that $Imf \subseteq I$.

Proof. Applying Theorem 2.4.1 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$, and $k(x, y)$ is given in (2.3.6). Then $A_k f(x) = \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)$ and the inequality in (2.4.2) reduces to (2.4.4) with v defined by (2.4.3). ∎

Corollary 2.4.3. *Let $0 < p \leq q < \infty$, $s \geq 1$, $\alpha > 1 - \frac{p}{q}$, g be increasing function on $(a, b]$ such that g' be continuous on (a, b). Then the inequality*

$$\left(\int_a^b g'(x)(I^{\alpha}_{a+;g} f(x))^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \leq \frac{\alpha^{\frac{1}{p}} (g(b) - g(a))^{\frac{q(\alpha s-1)+p}{pq}}}{((\alpha-1)\frac{q}{p} + 1)^{\frac{1}{q}} (\Gamma(\alpha+1))^{\frac{s}{p}}} \left(\int_a^b g'(y) f^s(y) dy \right)^{\frac{1}{p}} \tag{2.4.5}$$

holds.

Proof. For particular convex function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\Phi(x) = x^s, s \geq 1$ and weight function $u(x) = g'(x)(g(x)-g(a))^{\frac{\alpha q}{p}}$, $x \in (a,b)$ in (2.4.4), we get $v(y) = (\alpha g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}})/(((\alpha-1)\frac{q}{p}+1)^{\frac{p}{q}})$ and (2.4.4) becomes

$$\left(\int_a^b g'(x)(g(x)-g(a))^{\frac{\alpha q}{p}(1-s)}(I^{\alpha}_{a+;g}f(x))^{\frac{sq}{p}}\,dx\right)^{\frac{1}{q}}$$
$$\leq \frac{\alpha^{\frac{1}{p}}}{((\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}}(\Gamma(\alpha+1))^{\frac{s}{p}}}\left(\int_a^b g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}}f^s(y)dy\right)^{\frac{1}{p}}.$$

Since $(g(x)-g(a))^{\frac{\alpha q}{p}(1-s)} \geq (g(b)-g(a))^{\frac{\alpha q}{p}(1-s)}$ and $(g(b)-g(y))^{\alpha-1+\frac{p}{q}} \leq (g(b)-g(a))^{\alpha-1+\frac{p}{q}}$, $\alpha > 1-\frac{p}{q}$ we obtain (2.4.5). ∎

If $g(x) = x$, then $I^{\alpha}_{a+;x}f(x)$ reduces to $I^{\alpha}_{a+}f(x)$ left-sided Riemann-Liouville fractional integral, so the following result follows.

Corollary 2.4.4. *Let $0 < p \leq q < \infty$, u be a weight function on (a,b) and let v be defined on (a,b) by*

$$v(y) := \alpha\left(\int_y^b u(x)\left(\frac{(x-y)^{\alpha-1}}{(x-a)^{\alpha}}\right)^{\frac{q}{p}}dx\right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}I^{\alpha}_{a+}f(x)\right)\right]^{\frac{q}{p}}dx\right)^{\frac{1}{q}} \leq \left(\int_a^b v(y)\Phi\left(f(y)\right)dy\right)^{\frac{1}{p}}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$, such that $Imf \subseteq I$.

Corollary 2.4.5. *Let $0 < p \leq q < \infty$, $s \geq 1$ and $\alpha > 1-\frac{p}{q}$. Then the inequality*

$$\left(\int_a^b (I^{\alpha}_{a+}f(x))^{\frac{sq}{p}}dx\right)^{\frac{1}{q}} \leq \frac{\alpha^{\frac{1}{p}}(b-a)^{\frac{q(\alpha s-1)+p}{pq}}}{((\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}}(\Gamma(\alpha+1))^{\frac{s}{p}}}\left(\int_a^b f^s(y)dy\right)^{\frac{1}{p}}$$

holds.

If $g(x) = \log(x)$, $x \in (a, b)$ where $0 \leq a < b \leq \infty$, the following result follows.

Corollary 2.4.6. *Let $0 < p \leq q < \infty$, $s \geq 1$ and $\alpha > 1 - \frac{p}{q}$. Then the following inequality holds*

$$\left(\int_a^b (J_{a+}^{\alpha} f(x))^{\frac{sq}{p}} \frac{dx}{x} \right)^{\frac{1}{q}} \leq \frac{\alpha^{\frac{1}{p}} (\log b - \log a)^{\frac{q(\alpha s - 1) + p}{pq}}}{((\alpha - 1)\frac{q}{p} + 1)^{\frac{1}{q}} (\Gamma(\alpha + 1))^{\frac{s}{p}}} \left(\int_a^b f^s(y) \frac{dy}{y} \right)^{\frac{1}{p}}.$$

Next we give result with respect to the generalized Riemann–Liouville fractional derivative.

Theorem 2.4.7. *Let $0 < p \leq q < \infty$, u be a weight function on (a, b), $\beta > \alpha \geq 0$ and let the assumptions in Lemma 1.2.2 be satisfied. Let v be defined on (a, b) by*

$$v(y) := (\beta - \alpha) \left(\int_y^b u(x) \left(\frac{(x - y)^{\beta - \alpha - 1}}{(x - a)^{\beta - \alpha}} \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x) \left[\Phi \left(\frac{\Gamma(\beta - \alpha + 1)}{(x - a)^{\beta - \alpha}} D_a^{\alpha} f(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \leq \left(\int_a^b v(y) \Phi \left(D_a^{\beta} f(y) \right) dy \right)^{\frac{1}{p}} \tag{2.4.6}$$

holds for all measurable functions $f : (a, b) \to \mathbb{R}$, such that $Imf \subseteq I$.

Proof. Applying Theorem 2.4.1 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$, and $k(x, y)$ is given in (2.1.15). Replace f by $D_a^{\beta} f$. Then $A_k f(x) = \frac{\Gamma(\beta - \alpha + 1)}{(x - a)^{\beta - \alpha}} D_a^{\alpha} f(x)$ and the inequality given in (2.4.2) reduces to (2.4.6). ∎

If we take $\Phi(x) = x^s, s \geq 1$ and $u(x) = (x - a)^{\frac{(\beta - \alpha)q}{p}}$, $x \in (a, b)$, similar to Corollary 2.4.3 we obtain the following result.

Corollary 2.4.8. *Let $0 < p \leq q < \infty$, $s \geq 1$, $\beta - \alpha > 1 - \frac{p}{q}$ and let assumption in Lemma 1.2.2 be satisfied.. Then the following inequality holds*

$$\left(\int_a^b (D_a^{\alpha} f(x))^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \leq \frac{(\beta - \alpha)^{\frac{1}{p}} (b - a)^{\frac{q((\beta - \alpha)s - 1) + p}{pq}}}{((\beta - \alpha - 1)\frac{q}{p} + 1)^{\frac{1}{q}} (\Gamma(\beta - \alpha + 1))^{\frac{s}{p}}} \left(\int_a^b (D_a^{\beta} f(y))^s dy \right)^{\frac{1}{p}}.$$

In the following theorem, we will construct new inequality for the Canavati-type fractional derivative.

Theorem 2.4.9. *Let $0 < p \leq q < \infty$, $\nu > \gamma > 0$, u be a weight function on (a,b) and assumptions in Lemma 1.2.4 be satisfied. Let v be defined on (a,b) by*

$$v(y) := (\nu-\gamma)\left(\int_y^b u(x)\left(\frac{(x-y)^{\nu-\gamma-1}}{(x-a)^{\nu-\gamma}}\right)^{\frac{q}{p}} dx\right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^{\gamma} f(x)\right)\right]^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \leq \left(\int_a^b v(y)\Phi\left(D_a^{\nu} f(y)\right) dy\right)^{\frac{1}{p}} \tag{2.4.7}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$, such that $Im f \subseteq I$.

Proof. Similar to Theorem 2.4.7. ∎

Example 2.4.1. *If we take $\Phi(x) = x^s, s \geq 1$, $\nu - \gamma > 1 - \frac{p}{q}$ and weight function $u(x) = (x-a)^{\frac{(\nu-\gamma)q}{p}}$, $x \in (a,b)$ in (2.4.7), after some calculations we obtain*

$$\left(\int_a^b (D_a^{\gamma} f(x))^{\frac{sq}{p}} dx\right)^{\frac{1}{q}} \leq \frac{(\nu-\gamma)^{\frac{1}{p}}(b-a)^{\frac{q((\nu-\gamma)s-1)+p}{pq}}}{((\nu-\gamma-1)\frac{q}{p}+1)^{\frac{1}{q}}(\Gamma(\nu-\gamma+1))^{\frac{s}{p}}}\left(\int_a^b (D_a^{\nu} f(y))^s dy\right)^{\frac{1}{p}}.$$

Using the definition of Caputo fractional derivative [7, p. 449], we will prove the following result.

Theorem 2.4.10. *Let $0 < p \leq q < \infty$, u be a weight function on (a,b) and $D_{*a}^{\alpha} f$ denotes the Caputo fractional derivative of f. Let v be defined on (a,b) by*

$$v(y) := (n-\alpha)\left(\int_y^b u(x)\left(\frac{(x-y)^{n-\alpha-1}}{(x-a)^{n-\alpha}}\right)^{\frac{q}{p}} dx\right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{\Gamma(n-\alpha+1)}{(x-a)^{n-\alpha}} D_{*a}^{\alpha} f(x)\right)\right]^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \leq \left(\int_a^b v(y)\Phi\left(f^{(n)}(y)\right) dy\right)^{\frac{1}{p}} \tag{2.4.8}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$, such that $Im f \subseteq I$.

Proof. Similar to Theorem 2.4.7. ■

Example 2.4.2. *If we take $\Phi(x) = x^s, s \geq 1$, $n - \alpha > 1 - \frac{p}{q}$ and weight function $u(x) = (x-a)^{\frac{(n-\alpha)q}{p}}$, $x \in (a,b)$, in (2.4.8), after some calculations we obtain*

$$\left(\int_a^b (D_{*a}^{\alpha} f(x))^{\frac{sq}{p}} dx\right)^{\frac{1}{q}} \leq \frac{(n-\alpha)^{\frac{1}{p}}(b-a)^{\frac{q((n-\alpha)s-1)+p}{pq}}}{((n-\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}}(\Gamma(n-\alpha+1))^{\frac{s}{p}}} \left(\int_a^b (f^{(n)}(y))^s dy\right)^{\frac{1}{p}}.$$

Now, we give the following result for Erdélyi–Kober type fractional integral.

Theorem 2.4.11. *Let $0 < p \leq q < \infty$, u be a weight function on (a,b) and ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function. Let v be defined on (a,b) by*

$$v(y) := \alpha\sigma \left(\int_y^b u(x) \left(\frac{x^{-\sigma\eta} y^{\sigma\eta+\sigma-1}}{(x^{\sigma} - y^{\sigma})^{1-\alpha}(x^{\sigma} - a^{\sigma})^{\alpha}}\right)^{\frac{q}{p}} dx\right)^{\frac{p}{q}} < \infty.$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha} {}_2F_1(x)} I_{a+;\sigma;\eta}^{\alpha} f(x)\right)\right]^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \leq \left(\int_a^b v(y)\Phi\left(f(y)\right) dy\right)^{\frac{1}{p}} \tag{2.4.9}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$, such that $Imf \subseteq I$.

Proof. Similar to Theorem 2.4.7. ■

Example 2.4.3. *If we take $\Phi(x) = x^s, s \geq 1$, $u(x) = x^{\sigma-1}\left((x^{\sigma} - a^{\sigma})^{\alpha} {}_2F_1(x)\right)^{\frac{q}{p}}$, $x \in (a,b)$ in (2.4.9), after some calculations we obtain*

$$\left(\int_a^b ({}_2F_1(x))^{\frac{q}{p}(1-s)} \left(I_{a+;\sigma;\eta}^{\alpha} f(x)\right)^{\frac{sq}{p}} dx\right)^{\frac{1}{q}} \leq C \left(\int_a^b ({}_2F_1(y)) f^s(y) dy\right)^{\frac{1}{p}},$$

where

$$C = \frac{\alpha^{\frac{1}{p}} \sigma^{\frac{q-p}{pq}} b^{\frac{\sigma-1}{p}} (b^{\sigma} - a^{\sigma})^{\frac{q(\alpha s-1)+p}{pq}}}{a^{\frac{p\sigma-p+qs\sigma\alpha}{pq}} ((\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}} (\Gamma(\alpha+1))^{\frac{s}{p}}}.$$

Remark 2.4.1. For $p = q = 1$ all above results reduce to results given in [38, 40].

2.5 Inequality of G. H. Hardy and superquadratic function

In [52] (also see [28], [30]) K. Krulić et. al. study some new weighted Hardy-type inequalities. But here we give the improvements of an inequality of G. H. Hardy via superquadratic function. We establish new inequalities of Hardy-type using different fractional integrals and fractional derivatives.

In [4] the following refined Hardy-type inequality is given:

Theorem 2.5.1. *Let u be a weight function, $k(x,y) \geq 0$. Assume that $\frac{k(x,y)}{K(x)}u(x)$ is locally integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v by (2.1.3). Suppose $I = [0,c)$, $c \leq \infty$, $\varphi : I \to \mathbb{R}$. If φ is a superquadratic function, then the inequality*

$$\int_{\Omega_1} \varphi(A_k f(x))u(x)d\mu_1(x) + \int_{\Omega_2}\int_{\Omega_1} \varphi\left(|f(y) - A_k f(x)|\right) \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)\, d\mu_2(y)$$
$$\leq \int_{\Omega_2} \varphi(f(y))v(y)d\mu_2(y) \quad (2.5.1)$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $Imf \subseteq I$, where A_k is defined by (2.3.1).

If φ is subquadratic, then the inequality sign in (2.5.1) *is reversed.*

Let us continue by defining a linear functional as a difference between the right-hand side and the left-hand side of the refined Hardy-type inequality (2.5.1) as:

$$A(\varphi) = \int_{\Omega_2} \varphi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \varphi(A_k f(x))u(x)d\mu_1(x)$$
$$- \int_{\Omega_2}\int_{\Omega_1} \varphi\left(|f(y) - A_k f(x)|\right) \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)d\mu_2(y) \quad (2.5.2)$$

It is clear, that if φ is superquadratic function, then $A(\varphi) \geq 0$.

Lemma 2.5.2. *Consider the function φ_s for $s > 0$ defined as*

$$\varphi_s(x) = \begin{cases} \frac{x^s}{s(s-2)}, & s \neq 2, \\ \frac{x^2}{2}\log x, & s = 2. \end{cases} \quad (2.5.3)$$

Then, with the convention $0\,log\,0 = 0$, it is superquadratic.

For linear functional A defined by (2.5.2) we have $A(\varphi_s) \geq 0$ for all $s > 0$.

2.5.1 Main results

Our first result is for Riemann-Liouville fractional integral.

Theorem 2.5.3. *Let $s > 2$, $\alpha > 0$. Then the function $A_1 : \mathbb{R} \to [0, \infty)$ defined by*

$$\begin{aligned} A_1(s) &= \frac{1}{s(s-2)} \left[\int_a^b f^s(y)(b-y)^\alpha dy - (\Gamma(\alpha+1))^s \int_a^b (x-a)^{\alpha(1-s)} \left(I_{a^+}^\alpha f(x)\right)^s dx \right. \\ &\quad \left. -\alpha \int_a^b \int_y^b \left(\left| f(y) - \frac{\Gamma(\alpha+1)}{(x-a)^\alpha} I_{a^+}^\alpha f(x) \right| \right)^s (x-y)^{\alpha-1} dx\, dy \right] \end{aligned} \tag{2.5.4}$$

is exponentially convex.

Proof. Applying Theorem 2.5.1 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$ and $k(x, y)$ is given in (2.1.6), then (2.5.2) reduces to

$$\begin{aligned} A_1(s) = \int_a^b \varphi_s(f(y))v(y)dy - \int_a^b \varphi_s \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} I_{a^+}^\alpha f(x) \right) u(x)dx \\ -\alpha \int_a^b \int_y^b \varphi_s \left(\left| f(y) - \frac{\Gamma(\alpha+1)}{(x-a)^\alpha} I_{a^+}^\alpha f(x) \right| \right) \frac{u(x)(x-y)^{\alpha-1}}{(x-a)^\alpha} dxdy, \end{aligned} \tag{2.5.5}$$

where φ_s is defined by (2.5.3). Function A_1 is exponentially convex. Applying (2.5.5) with particular weight function $u(x) = (x-a)^\alpha$, $x \in (a, b)$ we get (2.5.4). ■

Remark 2.5.1. Notice that for $s > 2$

$$\begin{aligned} A_1(s) &= \frac{1}{s(s-2)} \left[\int_a^b f^s(y)(b-y)^\alpha dy - \int_a^b \left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} I_{a^+}^\alpha f(x) \right)^s (x-a)^\alpha dx \right. \\ &\quad \left. -\alpha \int_a^b \int_y^b \left(\left| f(y) - \frac{\Gamma(\alpha+1)}{(x-a)^\alpha} I_{a^+}^\alpha f(x) \right| \right)^s (x-y)^{\alpha-1} dx\, dy \right] \\ &\le \frac{1}{s(s-2)} \left[(b-a)^\alpha \int_a^b f^s(y)dy - (b-a)^{\alpha(1-s)} (\Gamma(\alpha+1))^s \int_a^b \left(I_{a^+}^\alpha f(x)\right)^s dx \right] \\ &= \frac{(b-a)^{\alpha(1-s)}}{s(s-2)} \left[(b-a)^{\alpha s} \int_a^b f^s(y)dy - (\Gamma(\alpha+1))^s \int_a^b \left(I_{a^+}^\alpha f(x)\right)^s dx \right], \end{aligned}$$

so the following inequality

$$A_1(s) \le \widetilde{H}_1(s),$$

holds, where

$$\widetilde{H}_1(s) = \frac{(b-a)^{\alpha(1-s)}}{s(s-2)} \left[(b-a)^{\alpha s} \int_a^b f^s(y)dy - (\Gamma(\alpha+1))^s \int_a^b (I_{a+}^{\alpha} f(x))^s dx \right].$$

Next we give results with respect to the generalized Riemann–Liouville fractional derivative.

Theorem 2.5.4. *Let $s > 2, \alpha \ge 0$ and let the assumptions in Lemma* 1.2.2 *be satisfied. Then the function $A_2 : \mathbb{R} \to [0, \infty)$ defined by*

$$\begin{aligned} A_2(s) \quad = \quad & \frac{1}{s(s-2)} \left[\int_a^b (b-y)^{\beta-\alpha} (D_a^{\beta} f(y))^s dy \right. \\ & -(\Gamma(\beta-\alpha+1))^s \int_a^b (x-a)^{(\beta-\alpha)(1-s)} \left(D_a^{\alpha} f(x)\right)^s dx \\ & \left. -(\beta-\alpha) \int_a^b \int_y^b \left(\left| D^{\alpha} f(y) - \frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^{\alpha} f(x) \right| \right)^s (x-y)^{\beta-\alpha-1} dx\, dy \right] \end{aligned}$$

is exponentially convex and the following inequality holds true:

$$A_2(s) \le \widetilde{H}_2(s),$$

where

$$\begin{aligned} \widetilde{H}_2(s) = \frac{(b-a)^{(\beta-\alpha)(1-s)}}{s(s-2)} & \left[(b-a)^{(\beta-\alpha)s} \int_a^b (D_a^{\beta} f(y))^s dy \right. \\ & \left. - (\Gamma(\beta-\alpha+1))^s \int_a^b (D_a^{\alpha} f(x))^s dx \right]. \end{aligned}$$

Proof. Similar to Theorem 2.5.3 and Theorem 2.3.7. ■

In the following theorem, new inequality for the Canavati-type fractional derivative is given.

Theorem 2.5.5. *Let $s > 2$ and let the assumptions in Lemma 1.2.4 be satisfied. Then the function $A_3 : \mathbb{R} \to [0, \infty)$ defined by*

$$\begin{aligned} A_3(s) &= \frac{1}{s(s-2)} \Bigg[\int_a^b (b-y)^{\nu-\gamma} (D_a^\nu f(y))^s dy \\ &\quad - (\Gamma(\nu-\gamma+1)) \int_a^b (x-a)^{(\nu-\gamma)(1-s)} \left(D_a^\gamma f(x)\right)^s dx \\ &\quad - (\nu-\gamma) \int_a^b \int_y^b \left(\left| D_a^\nu f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x) \right| \right)^s (x-y)^{\nu-\gamma-1} dx\, dy \Bigg] \end{aligned}$$

is exponentially convex and the following inequality holds true:

$$A_3(s) \leq \widetilde{H}_3(s),$$

where

$$\widetilde{H}_3(s) = \frac{(b-a)^{(\nu-\gamma)(1-s)}}{s(s-2)} \Bigg[(b-a)^{(\nu-\gamma)s} \int_a^b (D_a^\nu f(y))^s dy - (\Gamma(\nu-\gamma+1))^s \int_a^b (D_a^\gamma f(x))^s dx \Bigg].$$

Proof. Similar to Theorem 2.5.3 and Theorem 2.3.7. ■

Next, new inequalities for the Caputo fractional derivative are given.

Theorem 2.5.6. *Let $s > 2$, $\nu \geq 0$. Then the function $A_4 : \mathbb{R} \to [0, \infty)$ defined by*

$$\begin{aligned} A_4(s) &= \frac{1}{s(s-2)} \Bigg[\int_a^b (b-y)^{n-\nu} (f^{(n)}(y))^s dy \\ &\quad - (\Gamma(n-\nu+1))^s \int_a^b (x-a)^{(n-\nu)(1-s)} \left(D_{*a}^\nu f(x)\right)^s dx \\ &\quad - (n-\nu) \int_a^b \int_y^b \left(\left| f^{(n)}(y) - \frac{\Gamma(n-\nu+1)}{(x-a)^{n-\nu}} D_{*a}^\nu f(x) \right| \right)^s (x-y)^{n-\nu-1} dx dy \Bigg] \end{aligned}$$

is exponentially convex and the following inequality holds:

$$A_4(s) \le \widetilde{H}_4(s),$$

where

$$\widetilde{H}_4(s) = \frac{(b-a)^{(n-\nu)(1-s)}}{s(s-2)}\left[(b-a)^{(n-\nu)s}\int_a^b (f^{(n)}(y))^s dy \right.$$
$$\left. - (\Gamma(n-\nu+1))^s \int_a^b (D^{\nu}_{*a}f(x))^s dx\right].$$

Proof. Similar to Theorem 2.5.3 and Theorem 2.3.7. ■

Theorem 2.5.7. *Let $s > 2$ and let the assumptions in Lemma 1.2.5 be satisfied. Then the function $A_5 : \mathbb{R} \to [0,\infty)$ defined by*

$$\begin{aligned} A_5(s) &= \frac{1}{s(s-2)}\left[\int_a^b (b-y)^{\nu-\gamma}(D^{\nu}_{*a}f(y))^s dy \right. \\ &\quad -(\Gamma(\nu-\gamma+1))^s \int_a^b (x-a)^{(\nu-\gamma)(1-s)}\left(D^{\gamma}_{*a}f(x)\right)^s dx \\ &\quad \left. -(\nu-\gamma)\int_a^b\int_y^b \left(\left|D^{\nu}_{*a}f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D^{\gamma}_{*a}f(x)\right|\right)^s (x-y)^{\nu-\gamma-1}dxdy\right] \end{aligned}$$

is exponentially convex and the following inequality holds true:

$$A_5(s) \le \widetilde{H}_5(s),$$

where

$$\widetilde{H}_5(s) = \frac{(b-a)^{(\nu-\gamma)(1-s)}}{s(s-2)}\left[(b-a)^{(\nu-\gamma)s}\int_a^b (D^{\nu}_{*a}f(y))^s dy \right.$$
$$\left. - (\Gamma(\nu-\gamma+1))^s \int_a^b (D^{\gamma}_{*a}f(x))^s dx\right].$$

Proof. Similar to Theorem 2.5.3 and Theorem 2.3.7. ∎

Now, we give the following result.

Theorem 2.5.8. *Let $s > 2$ and ${}_2F_1(a, b; c; z)$ denotes the hypergeometric function. Then the function $A_6 : \mathbb{R} \to [0, \infty)$ defined by*

$$\begin{aligned} A_6(s) \quad = \quad & \frac{1}{s(s-2)} \Bigg[\int_a^b y^{\sigma-1}(b^\sigma - y^\sigma)^\alpha \, {}_2F_1(y) f^s(y) dy \\ & - (\Gamma(\alpha+1))^s \int_a^b x^{\sigma+\sigma\alpha s-1}(x^\sigma - a^\sigma)^{\alpha(s-1)} \left({}_2F_1(x)\right)^{1-s} \left(I^\alpha_{a+;\sigma;\eta} f(x)\right)^s dx \\ & -\alpha\,\sigma \int_a^b \int_y^b \left(\left| f(y) - \frac{\Gamma(\alpha+1)}{\left(1 - \left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)} I^\alpha_{a+;\sigma;\eta} f(x) \right| \right)^s \\ & \times x^{-\sigma\eta+\sigma-1} y^{\sigma\eta+\sigma-1} (x^\sigma - y^\sigma)^{\alpha-1} dx\, dy \Bigg] \end{aligned}$$

is exponentially convex and the following inequality holds true:

$$A_6(s) \leq \widetilde{H}_6(s),$$

where

$$\begin{aligned} \widetilde{H}_6(s) = \frac{(b^\sigma - a^\sigma)^{\alpha(1-s)}}{s(s-2)} & \Bigg[(b^\sigma - a^\sigma)^{\alpha s} b^{\sigma-1} \int_a^b {}_2F_1(y) f^s(y) dy \\ & - a^{\sigma-1+\alpha\sigma s} (\Gamma(\alpha+1))^s \int_a^b \left({}_2F_1(x)\right)^{1-s} (I^\alpha_{a+;\sigma;\eta} f(x))^s dx \Bigg]. \end{aligned}$$

Proof. Similar to Theorem 2.5.3 and Theorem 2.3.7. ∎

The following result is about Hadamard-type fractional integrals.

Theorem 2.5.9. *Let $s > 2$, $\alpha > 0$. Then the function $A_7 : \mathbb{R} \to [0, \infty)$ defined by*

$$\begin{aligned}
A_7(s) \quad = \quad & \frac{1}{s(s-2)} \left[\int_a^b \frac{(\log b - \log y)^\alpha}{y} f^s(y) dy \right. \\
& -(\Gamma(\alpha+1))^s \int_a^b \frac{(\log x - \log a)^{\alpha(1-s)}}{x} (J_{a+}^\alpha f(x))^s dx \\
& \left. -\alpha \int_a^b \int_y^b \left(\left| f(y) - \frac{\Gamma(\alpha+1)}{(\log x - \log a)^\alpha} J_{a+}^\alpha f(x) \right| \right)^s \frac{(\log x - \log y)^{\alpha-1}}{xy} dx dy \right]
\end{aligned}$$

is exponentially convex and the following inequality holds:

$$A_7(s) \leq \widetilde{H}_7(s),$$

where

$$\begin{aligned}
H_7(s) = \frac{1}{s(s-2)} \frac{(\log b - \log a)^{\alpha(1-s)}}{ab} & \left[b(\log b - \log a)^{\alpha s} \int_a^b f^s(y) dy \right. \\
& \left. - \ a(\Gamma(\alpha+1))^s \int_a^b (J_{a+}^\alpha f(x))^s dx \right].
\end{aligned}$$

Proof. Similar to Theorem 2.5.3. ■

In the following theorem we will discuss three cases for results given in Theorems 2.5.3–2.5.9.

Theorem 2.5.10. *For $i = 1, \ldots, 7$ the following inequalities hold true:*

$$\begin{aligned}
&(i). \qquad [A_i(p)]^{\frac{q-r}{q-p}} [A_i(q)]^{\frac{r-p}{q-p}} \leq H_i(r) \\
&(ii). \qquad [A_i(r)]^{\frac{p-q}{p-r}} [A_i(p)]^{\frac{q-r}{p-r}} \leq H_i(q) \\
&(iii). \qquad A_i(p) \leq [H_i(r)]^{\frac{q-p}{q-r}} [H_i(q)]^{\frac{p-r}{q-r}}
\end{aligned}$$

for every choice $r, p, q \in \mathbb{R}_+$, such that $2 < r < p < q$.

Proof. Similar to Theorem 2.3.12. ■

2.6 Refinements of an inequality of G. H. Hardy

Here we are ready to give the applications of refined Hardy-type inequality for fractional integrals and fractional derivatives for arbitrary convex function. K. Krulić et. al. gave the refinement of Theorem 2.3.1 and they proved the following theorem in [25].

Theorem 2.6.1. *Let* $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$ *be measure spaces with* σ*-finite measures,* u *be a weight function on* Ω_1*,* k *be a non-negative measurable function on* $\Omega_1 \times \Omega_2$ *and* K *be defined on* Ω_1 *by* (2.1.2). *Suppose that the function* $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ *is integrable on* Ω_1 *for each fixed* $y \in \Omega_2$ *and that* v *is defined on* Ω_2 *by* (2.1.3). *If* Φ *is a convex function on the interval* $I \subseteq \mathbb{R}$ *and* $\varphi : I \to \mathbb{R}$ *is any function, such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in Int\, I$*, then the inequality*

$$\begin{aligned}
&\int_{\Omega_2} v(y)\Phi\left(f(y)\right) d\mu_2(y) - \int_{\Omega_1} u(x)\Phi\left(A_k f(x)\right) d\mu_1(x) \\
&\quad \geq \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,y) \,\big|\, |\Phi(f(y)) - \Phi(A_k f(x))| \\
&\quad - |\varphi(A_k f(x))|.|f(y) - A_k f(x)| \,\big|\, d\mu_2(y)\, d\mu_1(x) \qquad (2.6.1)
\end{aligned}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$*, such that* $f(y) \in I$*, for all fixed* $y \in \Omega_2$ *where* A_k *is defined by* (2.3.1).

2.6.1 Main results

Let us continue by taking a non-negative difference between the left-hand side and the right-hand side of refined Hardy-type inequality given in (2.6.1).

$$\begin{aligned}
\psi(\Phi) &= \int_{\Omega_2} v(y)\Phi\left(f(y)\right) d\mu_2(y) - \int_{\Omega_1} u(x)\Phi\left(A_k f(x)\right) d\mu_1(x) \\
&\quad - \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,y) \,\big|\, |\Phi(f(y)) - \Phi(A_k f(x))| \\
&\quad - |\varphi(A_k f(x))|.|f(y) - A_k f(x)| \,\big|\, d\mu_2(y)\, d\mu_1(x). \qquad (2.6.2)
\end{aligned}$$

We will show some new inequalities for different fractional integrals and fractional derivatives by using the non-negative difference given in (2.6.2).

Our first result is given in the following theorem.

Theorem 2.6.2. *Let $s \geq 1$, $\alpha > 0$, $f \geq 0$, g be increasing function on $(a,b]$ such that g' be continuous on (a,b), $\psi_1 : \mathbb{R} \to [0,\infty)$. Then the following inequality holds:*

$$\psi_1(s) \leq \widehat{H}_1(s),$$

where

$$\begin{aligned}
\psi_1(s) &= \int_a^b g'(y)(g(b)-g(y))^{\alpha} f^s(y)dy \\
&-(\Gamma(\alpha+1))^s \int_a^b g'(x)(g(x)-g(a))^{\alpha(1-s)} \left(I^{\alpha}_{a+;g} f(x)\right)^s dx \\
&-\alpha \int_a^b \int_a^x g'(x)g'(y)(g(x)-g(y))^{\alpha-1} \left| \left| f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)\right)^s \right| \right. \\
&\left. -s \left| \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x) \right|^{s-1} . \left| f(y) - \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x) \right| \right| dydx
\end{aligned} \tag{2.6.3}$$

and

$$\begin{aligned}
\widehat{H}_1(s) = (g(b)-g(a))^{\alpha(1-s)} \Bigg[& (g(b)-g(a))^{\alpha s} \int_a^b f^s(y)g'(y)dy \\
& - (\Gamma(\alpha+1))^s \int_a^b (I^{\alpha}_{a+;g} f(x))^s g'(x)dx \Bigg].
\end{aligned}$$

Proof. Rewrite (2.6.2) with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$ and $k(x,y)$ is given in (2.3.6). Then $A_k f(x) = \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)$. For the particular weight function $u(x) = g'(x)(g(x)-g(a))^{\alpha}$, we obtain $v(y) = g'(y)(g(b)-g(y))^{\alpha}$. If we take $\Phi(x) = x^s$, $x \in \mathbb{R}_+$, after some calculations we get (2.6.3). Since

$$\begin{aligned}
\alpha \int_a^b \int_a^x & g'(x)g'(y)(g(x)-g(y))^{\alpha-1} \left| \left| f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)\right)^s \right| \right. \\
&\left. -s \left| \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x) \right|^{s-1} . \left| f(y) - \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x) \right| \right| dy\, dx \geq 0.
\end{aligned}$$

Then

$$
\begin{aligned}
\psi_1(s) &\le \int_a^b g'(y)(g(b)-g(y))^{\alpha} f^s(y)dy \\
&\quad - \int_a^b g'(x)(g(x)-g(a))^{\alpha}\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)\right)^s dx \\
&\le (g(b)-g(a))^{\alpha(1-s)}\left[(g(b)-g(a))^{\alpha s}\int_a^b f^s(y)g'(y)dy\right. \\
&\quad \left. - (\Gamma(\alpha+1))^s \int_a^b (I^{\alpha}_{a+;g} f(x))^s g'(x)dx\right] \\
&= \widehat{H}_1(s).
\end{aligned}
$$

This complete the proof. ■

If $g(x)=x$, then $I^{\alpha}_{a+;x}f(x)$ reduces to $I^{\alpha}_{a^+}f(x)$ left-sided Riemann–Liouville fractional integral and the following result follows.

Corollary 2.6.3. *Let $s \ge 1$, $\alpha > 0$, $f \ge 0$ and $\psi_2 : \mathbb{R} \to [0,\infty)$. Then the following inequality holds true:*

$$\psi_2(s) \le \widehat{H}_2(s),$$

where

$$
\begin{aligned}
\psi_2(s) &= \int_a^b (b-y)^{\alpha} f^s(y)dy - (\Gamma(\alpha+1))^s \int_a^b (x-a)^{\alpha(1-s)} \left(I^{\alpha}_{a^+} f(x)\right)^s dx \\
&\quad -\alpha \int_a^b \int_a^x (x-y)^{\alpha-1} \left|\left|f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a^+} f(x)\right)^s\right|\right. \\
&\quad \left. -s\left|\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a^+} f(x)\right|^{s-1} . \left|f(y) - \frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a^+} f(x)\right|\right| dy\, dx,
\end{aligned}
$$

and

$$\widehat{H}_2(s) = (b-a)^{\alpha(1-s)}\left[(b-a)^{\alpha s}\int_a^b f^s(y)dy - (\Gamma(\alpha+1))^s \int_a^b (I^{\alpha}_{a^+} f(x))^s dx\right].$$

If we take $g(x) = \log x$, the following result is obtained.

Corollary 2.6.4. *Let $s \geq 1$, $\alpha > 0$, $f \geq 0$ and $\psi_3 : \mathbb{R} \to [0, \infty)$. Then the following inequality holds*

$$\psi_3(s) \leq \widehat{H}_3(s),$$

where

$$\begin{aligned}
\psi_3(s) &= \int_a^b (\log b - \log y)^{\alpha} f^s(y) \frac{dy}{y} \\
&\quad - (\Gamma(\alpha+1))^s \int_a^b (\log x - \log a)^{\alpha(1-s)} \left(J^{\alpha}_{a+} f(x)\right)^s \frac{dx}{x} \\
&\quad - \alpha \int_a^b \int_a^x (\log x - \log y)^{\alpha-1} \Bigg| \left| f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(\log x - \log a)^{\alpha}} J^{\alpha}_{a+} f(x)\right)^s \right| \\
&\quad - s \left| \frac{\Gamma(\alpha+1)}{(\log x - \log a)^{\alpha}} J^{\alpha}_{a+} f(x) \right|^{s-1} \cdot \left| f(y) - \frac{\Gamma(\alpha+1)}{(\log x - \log a)^{\alpha}} J^{\alpha}_{a+} f(x) \right| \Bigg| \frac{dy}{y} \frac{dx}{x},
\end{aligned}$$

and

$$\begin{aligned}
\widehat{H}_3(s) = (\log b - \log a)^{\alpha(1-s)} \Bigg[&(\log b - \log a)^{\alpha s} \int_a^b f^s(y) \frac{dy}{y} \\
&- (\Gamma(\alpha+1))^s \int_a^b (J^{\alpha}_{a+} f(x))^s \frac{dx}{x} \Bigg].
\end{aligned}$$

Theorem 2.6.5. *Let $s \geq 1$ and let the assumptions in Lemma 1.2.2 be satisfied. Let $\psi_4 : \mathbb{R} \to [0, \infty)$. Then for non-negative functions f, $D^{\beta}_a f$ and $D^{\alpha}_a f$ the following inequality holds true:*

$$\psi_4(s) \leq \widehat{H}_4(s),$$

where

$$\begin{aligned}
\psi_4(s) &= \int_a^b (b-y)^{\beta-\alpha} (D^{\beta}_a f(y))^s dy \\
&\quad - (\Gamma(\beta - \alpha + 1))^s \int_a^b (x-a)^{(\beta-\alpha)(1-s)} \left(D^{\alpha}_a f(x)\right)^s dx
\end{aligned}$$

$$-(\beta-\alpha)\int_a^b\int_a^x (x-y)^{\beta-\alpha-1}\left|\left|(D_a^\beta f(y))^s-\left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}}D_a^\alpha f(x)\right)^s\right|\right.$$

$$\left.-s\left|\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}}D_a^\alpha f(x)\right|^{s-1}\cdot\left|D_a^\beta f(y)-\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}}D_a^\alpha f(x)\right|\right|dy\,dx$$

and

$$\widehat{H}_4(s)=(b-a)^{(\beta-\alpha)(1-s)}\left[(b-a)^{(\beta-\alpha)s}\int_a^b (D_a^\beta f(y))^s dy\right.$$

$$\left.-\ (\Gamma(\beta-\alpha+1))^s\int_a^b (D_a^\alpha f(x))^s dx\right].$$

Proof. Similar to Theorem 2.6.2 and Theorem 2.3.7. ■

In the following theorem, we give result for the Canavati-type fractional derivative.

Theorem 2.6.6. *Let $s \geq 1$, let the assumptions in Lemma 1.2.4 be satisfied and $\psi_5 : \mathbb{R} \to [0,\infty)$. Then for non-negative functions f, $D_a^\nu f$ and $D_a^\gamma f$ the following inequality holds:*

$$\psi_5(s) \leq \widehat{H}_5(s), \tag{2.6.4}$$

where

$$\begin{aligned}\psi_5(s) \;=\; & \int_a^b (b-y)^{\nu-\gamma}(D_a^\nu f(y))^s dy \\ & -(\Gamma(\nu-\gamma+1))^s\int_a^b (x-a)^{(\nu-\gamma)(1-s)}\,(D_a^\gamma f(x))^s\,dx \\ & -(\nu-\gamma)\int_a^b\int_a^x (x-y)^{\nu-\gamma-1}\left|\left|(D_a^\nu f(y))^s-\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right)^s\right|\right. \\ & \left.-s\left|\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right|^{s-1}\cdot\left|D_a^\nu f(y)-\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right|\right|dy\,dx,\end{aligned}$$

and

$$\widehat{H}_5(s) = (b-a)^{(\nu-\gamma)(1-s)} \left[(b-a)^{(\nu-\gamma)s} \int_a^b (D_a^{\nu} f(y))^s dy - (\Gamma(\nu-\gamma+1))^s \int_a^b (D_a^{\gamma} f(x))^s dx \right].$$

Proof. Similar to Theorem 2.6.2 and Theorem 2.3.7. ■

As a special case of Theorem 2.6.1, we construct new inequality for the Caputo fractional derivative.

Theorem 2.6.7. *Let* $s \geq 1,\ \alpha \geq 0,\ f^{(n)} \geq 0$ *and* $\psi_6 : \mathbb{R} \to [0, \infty)$. *Then the following inequality holds true:*

$$\psi_6(s) \leq \widehat{H}_6(s),$$

where

$$\begin{aligned}
\psi_6(s) \;=\; & \int_a^b (b-y)^{n-\alpha} (f^{(n)}(y))^s dy \\
& -(\Gamma(n-\alpha+1))^s \int_a^b (x-a)^{(n-\alpha)(1-s)} \left(D_{*a}^{\alpha} f(x)\right)^s dx \\
& -(n-\alpha) \int_a^b \int_a^x (x-y)^{n-\alpha-1} \left| \left| (f^{(n)}(y))^s - \left(\frac{\Gamma(n-\alpha+1)}{(x-a)^{n-\alpha}} D_{*a}^{\alpha} f(x) \right)^s \right| \right. \\
& \left. -s \left| \frac{\Gamma(n-\alpha+1)}{(x-a)^{n-\alpha}} D_{*a}^{\alpha} f(x) \right|^{s-1} . \left| f^{(n)}(y) - \frac{\Gamma(n-\alpha+1)}{(x-a)^{n-\alpha}} D_{*a}^{\alpha} f(x) \right| \right| dy\, dx,
\end{aligned}$$

and

$$\widehat{H}_6(s) = (b-a)^{(n-\alpha)(1-s)} \left[(b-a)^{(n-\alpha)s} \int_a^b (f^{(n)}(y))^s dy - (\Gamma(n-\alpha+1))^s \int_a^b (D_{*a}^{\alpha} f(x))^s dx \right].$$

Proof. Similar to Theorem 2.6.2 and Theorem 2.3.7. ∎

Theorem 2.6.8. *Let $s \geq 1$, let the assumptions in Lemma 1.2.6 be satisfied and $\psi_7 : \mathbb{R} \to [0, \infty)$. Then for non-negative functions f, $D^{\nu}_{*a}f$ and $D^{\gamma}_{*a}f$ the following inequality holds:*

$$\psi_7(s) \leq \widehat{H}_7(s),$$

where

$$\begin{aligned}
\psi_7(s) \;=\; & \int_a^b (b-y)^{\nu-\gamma}(D^{\nu}_{*a}f(y))^s dy \\
& -(\Gamma(\nu-\gamma+1))^s \int_a^b (x-a)^{(\nu-\gamma)(1-s)} \left(D^{\gamma}_{*a}f(x)\right)^s dx \\
& -(\nu-\gamma)\int_a^b \int_a^x (x-y)^{\nu-\gamma-1} \left| \left| (D^{\nu}_{*a}f(y))^s - \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D^{\gamma}_{*a}f(x)\right)^s \right| \right. \\
& \left. -s\left|\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D^{\gamma}_{*a}f(x)\right|^{s-1} \cdot \left| D^{\nu}_{*a}f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D^{\gamma}_{*a}f(x)\right| \right| dy\, dx,
\end{aligned}$$

and

$$\begin{aligned}
\widehat{H}_7(s) = (b-a)^{(\nu-\gamma)(1-s)} & \left[(b-a)^{(\nu-\gamma)s} \int_a^b (D^{\nu}_{*a}f(y))^s dy \right. \\
& \left. - (\Gamma(\nu-\gamma+1))^s \int_a^b (D^{\gamma}_{*a}f(x))^s dx \right].
\end{aligned}$$

Proof. Similar to Theorem 2.6.2 and Theorem 2.3.7. ∎

Now, we give the following result.

Theorem 2.6.9. *Let $s \geq 1$, $\alpha > 0$, $f \geq 0$, ${}_2F_1(a, b; c; z)$ denotes the hypergeometric function and $\psi_8 : \mathbb{R} \to [0, \infty)$. Then the following inequality holds:*

$$\psi_8(s) \leq \widehat{H}_8(s),$$

where

$$\psi_8(s) = \int_a^b y^{\sigma-1}(b^\sigma - y^\sigma)^\alpha \,{}_2F_1(y) f^s(y)dy$$

$$- \int_a^b x^{\sigma-1}(x^\sigma - a^\sigma)^\alpha \,{}_2F_1(x) \left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha \,{}_2F_1(x)} I^\alpha_{a+;\sigma;\eta} f(x) \right)^s dx$$

$$-\alpha\sigma \int_a^b \int_a^x \left(\frac{y}{x}\right)^{\sigma\eta} \frac{(xy)^{\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha}} \left| \left| f^s(y) - \left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha \,{}_2F_1(x)} I^\alpha_{a+;\sigma;\eta} f(x) \right)^s \right| \right.$$

$$\left. -s \left| \frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha \,{}_2F_1(x)} I^\alpha_{a+;\sigma;\eta} f(x) \right|^{s-1} \left| f(y) - \frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha \,{}_2F_1(x)} I^\alpha_{a+;\sigma;\eta} f(x) \right| \right| dy\, dx$$

and

$$\widehat{H}_8(s) = (b^\sigma - a^\sigma)^{\alpha(1-s)} \left[(b^\sigma - a^\sigma)^{\alpha s} b^{\sigma-1} \int_a^b \,{}_2F_1(y) f^s(y)dy \right.$$
$$\left. - \; a^{\sigma-1+\alpha\sigma s}(\Gamma(\alpha+1))^s \int_a^b \left(({}_2F_1(x))^{1-s} I^\alpha_{a+;\sigma;\eta} f(x) \right)^s dx \right],$$

Proof. Similar to Theorem 2.6.2 and Theorem 2.3.11. ■

Chapter 3

G. H. Hardy-type integral inequalities

In this chapter, we show new class of general G. H. Hardy-type integral inequalities with fractional integrals and fractional derivatives. Also we give some improvements of generalize G. H. Hardy-type inequalities. The results given in this chapter can be found in [44], [45] (see also [48]).

3.1 New class of the G. H. Hardy-type inequalities with kernels

We establish some new integral inequalities of G. H. Hardy-type for fractional integrals and fractional derivatives using the following theorem given in [26].

Theorem 3.1.1. *Let $0 < p \leq q < \infty$. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, w be a μ_2- a.e. positive function on Ω_2, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$ and K be defined on Ω_1 by (2.1.2). Suppose that $K(x) > 0$ for all $x \in \Omega_1$ and that the function $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Let Φ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If*

$$A = \sup_{y \in \Omega_2} w^{\frac{-1}{p}}(y) \left(\int_{\Omega_1} u(x) \left(\frac{k(x,y)}{K(x)} \right)^{\frac{q}{p}} d\mu_1(x) \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, *such that the inequality*

$$\left(\int_{\Omega_1} u(x)[\Phi\left(A_k f(x)\right)]^{\frac{q}{p}} d\mu_1(x) \right)^{\frac{1}{q}} \leq C \left(\int_{\Omega_2} w(y)\Phi\left(f(y)\right) d\mu_2(y) \right)^{\frac{1}{p}} \tag{3.1.1}$$

holds for all measurable function $f : \Omega_2 \to \mathbb{R}$ *with values in* I *and* $A_k f$ *be defined by* (2.3.1). *Moreover, if* C *is smallest constant for* (3.1.1) *to hold, then* $C \leq A$.

The upcoming corollary is given in [26].

Corollary 3.1.2. *Let* $-\infty < q \leq p < 0$ *and let the assumption in Theorem* 3.1.1 *be satisfied with a positive convex function* Φ. *If*

$$B = \inf_{y \in \Omega_2} w^{\frac{-1}{p}}(y) \left(\int_{\Omega_1} u(x) \left(\frac{k(x,y)}{K(x)} \right)^{\frac{q}{p}} d\mu_1(x) \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive real constant C, *such that the inequality*

$$\left(\int_{\Omega_1} u(x)[\Phi\left(A_k f(x)\right)]^{\frac{q}{p}} d\mu_1(x) \right)^{\frac{1}{q}} \geq C \left(\int_{\Omega_2} w(y)\Phi\left(f(y)\right) d\mu_2(y) \right)^{\frac{1}{p}} \tag{3.1.2}$$

holds for all measurable function $f : \Omega_2 \to \mathbb{R}$ *with values in* Ω_2. *Moreover, if* C *is smallest constant for* (3.1.2) *to hold, then* $C \geq B$.

We will give some applications of Theorem 3.1.1 for different type of fractional integrals and fractional derivatives. Our first result deals with fractional integral of f with respect to another increasing function g.

Theorem 3.1.3. *Let* $0 < p \leq q < \infty$, $s \geq 1$, $\alpha > 0$, u *be a weight function on* (a, b), w *be a.e. positive function on* (a, b), g *be increasing function on* $(a, b]$ *such that* g' *be continuous on* (a, b). *If*

$$A = \sup_{y \in (a,b)} w^{\frac{-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{\alpha\, g'(y)(g(x) - g(y))^{\alpha - 1}}{(g(x) - g(a))^{\alpha}} \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, *such that the inequality*

$$\left(\int_a^b u(x) \left(\frac{\Gamma(\alpha+1)}{(g(x) - g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x) \right)^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b w(y) f^s(y) dy \right)^{\frac{1}{p}} \tag{3.1.3}$$

holds. Moreover, if C *is the smallest constant for* (3.1.3) *to hold, then* $C \leq A$.

Proof. Applying Theorem 3.1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$, $k(x,y)$ is given in (2.3.6) and $A_k f(x) = \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^\alpha} I^\alpha_{a+;g} f(x)$. Then the inequality given in (3.1.1) takes the form

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^\alpha} I^\alpha_{a+;g} f(x)\right)\right]^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \le C\left(\int_a^b w(y)\Phi(f(y))dy\right)^{\frac{1}{p}}. \tag{3.1.4}$$

If we choose the function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ defined by $\Phi(x) = x^s, s \ge 1$, then (3.1.4) becomes (3.1.3). ■

If $g(x) = x$, then $I^\alpha_{a+;x} f(x)$ reduces to $I^\alpha_{a+} f(x)$ left-sided Riemann–Liouville fractional integral, so the following result follows.

Corollary 3.1.4. *Let* $0 < p \le q < \infty,\ \alpha > 0,\ s \ge 1$, u *be a weight function on* (a,b), w *be a.e. positive function on* (a,b). *If*

$$A = \sup_{y\in(a,b)} w^{\frac{-1}{p}}(y)\left(\int_y^b u(x)\left(\frac{\alpha\,(x-y)^{\alpha-1}}{(x-a)^\alpha}\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, *such that the inequality*

$$\left(\int_a^b u(x)\left(\frac{\Gamma(\alpha+1)}{(x-a)^\alpha} I^\alpha_{a+} f(x)\right)^{\frac{sq}{p}} dx\right)^{\frac{1}{q}} \le C\left(\int_a^b w(y) f^s(y)dy\right)^{\frac{1}{p}} \tag{3.1.5}$$

holds. Moreover, if C *is the smallest constant for* (3.1.5) *to hold, then* $C \le A$.

If $g(x) = \log x$, $x \in (a,b)$ where $0 \le a < b \le \infty$, the following result follows.

Corollary 3.1.5. *Let* $0 < p \le q < \infty,\ s \ge 1,\ \alpha > 0$, u *be a weight function on* (a,b), w *be a.e. positive function on* (a,b). *If*

$$A = \sup_{y\in(a,b)} w^{\frac{-1}{p}}(y)\left(\int_y^b u(x)\left(\frac{\alpha\,(\log x - \log y)^{\alpha-1}}{y(\log x - \log a)^\alpha}\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, *such that the inequality*

$$\left(\int_a^b u(x)\left(\frac{\Gamma(\alpha+1)}{(\log x - \log a)^\alpha} J^\alpha_{a+} f(x)\right)^{\frac{sq}{p}} dx\right)^{\frac{1}{q}} \le C\left(\int_a^b w(y) f^s(y)dy\right)^{\frac{1}{p}} \tag{3.1.6}$$

holds. Moreover, if C *is the smallest constant for* (3.1.6) *to hold, then* $C \le A$.

Corollary 3.1.6. *Let $0 < p \leq q < \infty$, $s \geq 1$, $\beta > \alpha \geq 0$, u be a weight function on (a,b), w be a.e. positive function on (a,b) and let the assumptions in the Lemma 1.2.2 be satisfied. If*

$$A = \sup_{y\in(a,b)} w^{\frac{-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{(\beta-\alpha)(x-y)^{\beta-\alpha-1}}{(x-a)^{\beta-\alpha}} \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, such that the inequality

$$\left(\int_a^b u(x) \left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x) \right)^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b w(y) (D_a^\beta f(y))^s dy \right)^{\frac{1}{p}} \tag{3.1.7}$$

holds. Moreover, if C is the smallest constant for (3.1.7) to hold, then $C \leq A$.

Proof. Applying Theorem 3.1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$ and $k(x,y)$ is given in (2.1.15). Replace f by $D_a^\beta f$ and $A_k f(x) = \frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x)$. Then the inequality given in (3.1.1) takes the form

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b w(y) \Phi\left(D_a^\beta f(y) \right) dy \right)^{\frac{1}{p}}. \tag{3.1.8}$$

For $s \geq 1$, $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\Phi(x) = x^s$, then (3.1.8) becomes (3.1.7). ∎

Corollary 3.1.7. *Let $0 < p \leq q < \infty$, $s \geq 1$, u be a weight function on (a,b), w be a.e. positive function on (a,b) and assumptions in Lemma 1.2.4 be satisfied. If*

$$A = \sup_{y\in(a,b)} w^{\frac{-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{(\nu-\gamma)(x-y)^{\nu-\gamma-1}}{(x-a)^{\nu-\gamma}} \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, such that the inequality

$$\left(\int_a^b u(x) \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x) \right)^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b w(y) (D_a^\nu f(y))^s dy \right)^{\frac{1}{p}} \tag{3.1.9}$$

holds. Moreover, if C is the smallest constant for (3.1.9) to hold, then $C \leq A$.

Proof. Similar to Corollary 3.1.6. ∎

Now, we construct new inequality for the Caputo fractional derivative.

Corollary 3.1.8. *Let $0 < p \leq q < \infty$, $s \geq 1$, u be a weight function on (a,b), w be a.e. positive function on (a,b). If*

$$A = \sup_{y \in (a,b)} w^{\frac{-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{(n-\alpha)(x-y)^{n-\alpha-1}}{(x-a)^{n-\alpha}} \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, such that the inequality

$$\left(\int_a^b u(x) \left(\frac{\Gamma(n-\alpha+1)}{(x-a)^{n-\alpha}} D_{*a}^{\alpha} f(x) \right)^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b w(y)(f^{(n)}(y))^s dy \right)^{\frac{1}{p}} \quad (3.1.10)$$

holds. Moreover, if C is the smallest constant for (3.1.10) *to hold, then $C \leq A$.*

Proof. Similar to Corollary 3.1.6. ∎

Corollary 3.1.9. *Let $0 < p \leq q < \infty$, $s \geq 1$, u be a weight function on (a,b), w be a.e. positive function on (a,b) and let the assumptions in Lemma 1.2.6 be satisfied. If*

$$A = \sup_{y \in (a,b)} w^{\frac{-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{(\nu-\gamma)(x-y)^{\nu-\gamma-1}}{(x-a)^{\nu-\gamma}} \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, such that the inequality

$$\left(\int_a^b u(x) \left(\frac{\Gamma(n-\alpha+1)}{(x-a)^{\nu-\gamma}} D_{*a}^{\gamma} f(x) \right)^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b w(y)(D_{*a}^{\nu} f(y))^s dy \right)^{\frac{1}{p}} \quad (3.1.11)$$

holds. Moreover, if C is the smallest constant for (3.1.11) *to hold, then $C \leq A$.*

Proof. Similar to Corollary 3.1.6. ∎

Now, we give the following result.

Corollary 3.1.10. *Let $0 < p \leq q < \infty$, $s \geq 1$, u be a weight function on (a,b), w be a.e. positive function on (a,b), ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function. If*

$$A = \sup_{y\in(a,b)} w^{\frac{-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{\alpha\,\sigma\, x^{-\sigma\eta} y^{\sigma\eta+\sigma-1} (x^\sigma - y^\sigma)^{\alpha-1}}{(x^\sigma - a^\sigma)^\alpha \; {}_2F_1(x)} \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, such that the inequality

$$\left(\int_a^b u(x) \left(\frac{\Gamma(\alpha+1)}{\left(1 - \left(\frac{a}{x}\right)^\sigma\right)^\alpha \; {}_2F_1(x)} I^\alpha_{a+;\sigma;\eta} f(x) \right)^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b w(y) f^s(y) dy \right)^{\frac{1}{p}}. \tag{3.1.12}$$

holds. Moreover, if C is the smallest constant for (3.1.12) *to hold, then $C \leq A$.*

Proof. Similar to Corollary 3.1.6. ■

Remark 3.1.1. Analogous to the Corollary 3.1.2, we can obtain the results with infimum, but here we omit the details.

3.2 Generalized G. H. Hardy-type inequalities

This section is dedicated to the generalized G. H. Hardy-type inequalities for arbitrary convex functions. In [52], K. Krulić prove the Hardy-type inequality with general kernel in the following theorem.

Theorem 3.2.1. *Let $0 < p \leq q < \infty$, or $-\infty < q \leq p < 0$, $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$ and K be defined on Ω_1 by* (2.1.2)*, and that the function $x \mapsto u(x) \left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$ and that v is defined on Ω_2 by* (2.4.1)*. If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi \in \partial\Phi(x)$ for all $x \in IntI$, then the inequality*

$$\begin{aligned} &\left(\int_{\Omega_2} v(y) \Phi\left(f(y)\right) d\mu_2(y) \right)^{\frac{q}{p}} - \int_{\Omega_1} u(x) [\Phi\left(A_k f(x)\right)]^{\frac{q}{p}} d\mu_1(x) \\ &\quad \geq \frac{q}{p} \int_{\Omega_1} \frac{u(x)}{K(x)} \Phi^{\frac{q}{p}-1}(A_k f(x)) \int_{\Omega_2} k(x,y) r(x,y) d\mu_2(y) d\mu_1(x) \end{aligned} \tag{3.2.1}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $f(y) \in I$ for all $y \in \Omega_2$, where A_k defined by (2.3.1) *and $r : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is non-negative function defined by*

$$r(x,y) = \left|\, |\Phi(f(y)) - \Phi(A_k f(x))| - |\varphi(A_k f(x))|\, |f(y) - A_k f(x)|\, \right|. \tag{3.2.2}$$

Remark 3.2.1. For $p = q$, the Theorem 3.2.1 becomes [25, Theorem 2.1] and convex function Φ need not to be non-negative.

3.2.1 G. H. Hardy-type inequalities for fractional integrals

Let us continue by taking the non-negative difference of the left- and the right-hand side of the inequality given in (3.2.1) by taking $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\Phi(x) = x^s$, $s \geq 1$ as:

$$\begin{aligned} \rho(s) &= \left(\int_{\Omega_2} v(y) f^s(y) d\mu_2(y) \right)^{\frac{q}{p}} - \int_{\Omega_1} u(x) (A_k f(x))^{\frac{sq}{p}} d\mu_1(x) \\ &\quad - \frac{q}{p} \int_{\Omega_1} \frac{u(x)}{K(x)} (A_k f(x))^{s(\frac{q}{p}-1)} \int_{\Omega_2} k(x,y) r(x,y) d\mu_2(y) d\mu_1(x), \end{aligned} \tag{3.2.3}$$

where $r(x,y)$ is defined by (3.2.2).

Our first result involving fractional integral of f with respect to another increasing function g is given.

Theorem 3.2.2. *Let $0 < p \leq q < \infty$, $s \geq 1$, $\alpha > 1 - \frac{p}{q}$, $f \geq 0$, g be increasing function on $(a,b]$ such that g' be continuous on (a,b). Then the following inequality holds true:*

$$0 \leq \rho_1(s) \leq \overline{H}_1(s) - M_1(s) \leq \overline{H}_1(s),$$

where

$$\begin{aligned} \rho_1(s) = \frac{(\Gamma(\alpha+1))^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1} &\left((I_{a+;g}^{\alpha+\frac{p}{q}} f^s)(b) \right)^{\frac{q}{p}} \\ &- \alpha^{\frac{sq}{p}} (\Gamma(\alpha))^{\frac{sq}{p}+1} \left(I_{b-;g}^{\alpha\frac{q}{p}(1-s)+1} [(I_{a+;g}^{\alpha} f)(x)]^{\frac{sq}{p}} \right)(a) - M_1(s), \end{aligned}$$

$$\begin{aligned} M_1(s) = \frac{q(\Gamma(\alpha+1))^{s(\frac{q}{p}-1)+1}}{p} \int_a^b g'(x)(g(x)-g(a))^{\frac{\alpha(q-p)(1-s)}{p}} &\left(I_{a+;g}^{\alpha} f(x) \right)^{s(\frac{q}{p}-1)} \\ &\times (I_{a+;g}^{\alpha} r_1)(x,\cdot)\, dx, \end{aligned}$$

$$r_1(x,y) = \left| \left| f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^\alpha} I^\alpha_{a+;g} f(x) \right)^s \right| \right.$$
$$\left. - s \left| \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^\alpha} I^\alpha_{a+;g} f(x) \right|^{s-1} \cdot \left| f(y) - \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^\alpha} I^\alpha_{a+;g} f(x) \right| \right|,$$

and

$$\overline{H}_1(s) = (g(b)-g(a))^{\frac{\alpha q}{p}(1-s)} \left[\frac{\alpha^{\frac{q}{p}} (g(b)-g(a))^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1} \left(\int_a^b g'(y) f^s(y) dy \right)^{\frac{q}{p}} \right.$$
$$\left. - (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b g'(x) (I^\alpha_{a+;g} f(x))^{\frac{sq}{p}} dx \right]. \quad (3.2.4)$$

Proof. Applying Theorem 3.2.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, $k(x,y)$ is given in (2.3.6) and $A_k f(x) = \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^\alpha} I^\alpha_{a+;g} f(x)$. For the particular weight function $u(x) = g'(x)(g(x)-g(a))^{\frac{\alpha q}{p}}$, $x \in (a,b)$, we get $v(y) = (\alpha g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}})/(((\alpha-1)\frac{q}{p}+1)^{\frac{p}{q}})$, then (3.2.3) takes the form

$$\rho_1(s) = \frac{\alpha^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1} \left(\int_a^b g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}} f^s(y) dy \right)^{\frac{q}{p}}$$
$$- (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b g'(x)(g(x)-g(a))^{\frac{\alpha q(1-s)}{p}} \left(I^\alpha_{a+;g} f(x) \right)^{\frac{sq}{p}} dx - M_1(s).$$

Since $\frac{\alpha q}{p}(1-s) \leq 0$, g is increasing and $M_1(s) \geq 0$, we obtain that

$$\begin{aligned} 0 \leq \rho_1(s) &\leq \frac{\alpha^{\frac{q}{p}} (g(b)-g(a))^{(\alpha-1)\frac{q}{p}+1}}{(\alpha-1)\frac{q}{p}+1} \left(\int_a^b g'(y) f^s(y) dy \right)^{\frac{q}{p}} \\ &\quad - (g(b)-g(a))^{\frac{\alpha q}{p}(1-s)} (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b g'(x)(I^\alpha_{a+;g} f(x))^{\frac{sq}{p}} dx - M_1(s) \\ &= \overline{H}_1(s) - M_1(s) \\ &\leq \overline{H}_1(s). \end{aligned}$$

This completes the proof. ∎

If $g(x)=x$, then $I^{\alpha}_{a+;x}f(x)$ reduces to $I^{\alpha}_{a^+}f(x)$ left-sided Riemann-Liouville fractional integral and the following result follows.

Corollary 3.2.3. *Let $0<p\leq q<\infty,\ \alpha>1-\frac{p}{q},\ s\geq 1,\ f\geq 0$. Then the following inequality holds true:*

$$0\leq \rho_2(s)\leq \overline{H}_2(s)-M_2(s)\leq \overline{H}_2(s),$$

where

$$\begin{aligned}\rho_2(s)=&\frac{(\Gamma(\alpha+1))^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1}\left((I^{\alpha+\frac{p}{q}}_{a^+}f^s)(b)\right)^{\frac{q}{p}}\\&-\alpha^{\frac{sq}{p}}(\Gamma(\alpha))^{\frac{sq}{p}+1}\left(I^{\alpha\frac{q}{p}(1-s)+1}_{b^-}[(I^{\alpha}_{a^+}f)(x)]^{\frac{sq}{p}}\right)(a)-M_2(s),\end{aligned}$$

$$M_2(s)\ =\ \frac{q(\Gamma(\alpha+1))^{s(\frac{q}{p}-1)+1}}{p}\int\limits_a^b(x\ -\ a)^{\frac{\alpha(q-p)(1-s)}{p}}\ (I^{\alpha}_{a^+}f(x))^{s(\frac{q}{p}-1)}\ (I^{\alpha}_{a^+}r_2)(x,\cdot)\,dx,$$

$$\begin{aligned}r_2(x,y)=&\left|\left|f^s(y)-\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}I^{\alpha}_{a^+}f(x)\right)^s\right|\right.\\&\left.-s\left|\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}I^{\alpha}_{a+}f(x)\right|^{s-1}\cdot\left|f(y)-\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}I^{\alpha}_{a^+}f(x)\right|\right|,\end{aligned}$$

and

$$\begin{aligned}\overline{H}_2(s)=(b-a)^{\frac{\alpha q}{p}(1-s)}&\left[\frac{\alpha^{\frac{q}{p}}(b-a)^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1}\left(\int\limits_a^b f^s(y)dy\right)^{\frac{q}{p}}\right.\\&\left.-(\Gamma(\alpha+1))^{\frac{sq}{p}}\int\limits_a^b(I^{\alpha}_{a^+}f(x))^{\frac{sq}{p}}dx\right].\qquad(3.2.5)\end{aligned}$$

If $g(x)=\log x,\ x\in(a,b)$ where $0\leq a<b\leq\infty$, the following result follows.

Corollary 3.2.4. *Let $0<p\leq q<\infty,\ s\geq 1,\ \alpha>1-\frac{p}{q},\ f\geq 0$. Then the following inequality holds:*

$$0\leq \rho_3(s)\leq \overline{H}_3(s)-M_3(s)\leq \overline{H}_3(s),$$

where

$$\rho_3(s) = \frac{(\Gamma(\alpha+1))^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1}\left((J_{a_+}^{\alpha+\frac{p}{q}} f^s)(b)\right)^{\frac{q}{p}} - \alpha^{\frac{sq}{p}}(\Gamma(\alpha))^{\frac{sq}{p}+1}\left(J_{b_-}^{\alpha\frac{q}{p}(1-s)+1}[(J_{a_+}^{\alpha} f)(x)]^{\frac{sq}{p}}\right)(a) - M_3(s),$$

$$M_3(s) = \frac{q(\Gamma(\alpha+1))^{s(\frac{q}{p}-1)+1}}{p}\int_a^b (\log x - \log a)^{\frac{\alpha(q-p)(1-s)}{p}}\left(J_{a_+}^{\alpha} f(x)\right)^{s(\frac{q}{p}-1)} \times (J_{a_+}^{\alpha} r_3)(x,\cdot)\,\frac{dx}{x},$$

$$r_3(x,y) = \left|\left|f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(\log x-\log a)^{\alpha}} J_{a_+}^{\alpha} f(x)\right)^s\right| - s\left|\frac{\Gamma(\alpha+1)}{(\log x-\log a)^{\alpha}} J_{a_+}^{\alpha} f(x)\right|^{s-1}\cdot\left|f(y) - \frac{\Gamma(\alpha+1)}{(\log x-\log a)^{\alpha}} J_{a_+}^{\alpha} f(x)\right|\right|,$$

and

$$\overline{H}_3(s) = (\log b-\log a)^{\frac{\alpha q}{p}(1-s)}\left[\frac{\alpha^{\frac{q}{p}}(\log b-\log a)^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b f^s(y)\frac{dy}{y}\right)^{\frac{q}{p}} - (\Gamma(\alpha+1))^{\frac{sq}{p}}\int_a^b (J_{a_+}^{\alpha} f(x))^{\frac{sq}{p}}\frac{dx}{x}\right]. \quad (3.2.6)$$

Now, we give the following result for Erdélyi-Kober fractional integral.

Theorem 3.2.5. *Let $0 < p \le q < \infty$, $s \ge 1$, $\alpha > 1-\frac{p}{q}$, $f \ge 0$ and ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function. Then the following inequality holds true:*

$$0 \le \rho_4(s) \le \overline{H}_4(s) - M_4(s) \le \overline{H}_4(s),$$

where

$$\rho_4(s) = \frac{\alpha^{\frac{q}{p}}\sigma^{\frac{q}{p}-1}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b y^{\sigma-1}\,{}_2F_1(y)\,(b^{\sigma}-y^{\sigma})^{\alpha-1+\frac{p}{q}} f^s(y)dy\right)^{\frac{q}{p}} - (\Gamma(\alpha+1))^{\frac{sq}{p}}\int_a^b x^{\frac{\sigma\alpha sq}{p}+\sigma-1}\left((x^{\sigma}-a^{\sigma})^{\alpha}\,{}_2F_1(x)\right)^{\frac{q(1-s)}{p}}\left(I_{a_+;\sigma;\eta}^{\alpha} f(x)\right)^{\frac{sq}{p}} dx - M_4(s),$$

$$M_4(s) = \frac{\alpha q\sigma(\Gamma(\alpha+1))^{\frac{sq}{p}}}{p} \int_a^b x^{\sigma\alpha s(\frac{q}{p}-1)+\sigma-\sigma\eta-1} \left((x^\sigma - a^\sigma)^\alpha \, {}_2F_1(x)\right)^{\frac{(q-p)(1-s)}{p}}$$
$$\times \left(I^\alpha_{a+;\sigma;\eta} f(x)\right)^{s(\frac{q}{p}-1)} \int_a^x \frac{r_4(x,y) y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha}} dy\, dx,$$

$$r_4(x,y) = \Bigg| \left| f^s(y) - \left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha \, {}_2F_1(x)} I^\alpha_{a+;\sigma;\eta} f(x) \right)^s \right|$$
$$-s \left| \frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha \, {}_2F_1(x)} I^\alpha_{a+;\sigma;\eta} f(x) \right|^{s-1} \cdot \left| f(y) - \frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha \, {}_2F_1(x)} I^\alpha_{a+;\sigma;\eta} f(x) \right| \Bigg|,$$

and

$$\overline{H}_4(s) = (b^\sigma - a^\sigma)^{\alpha\frac{q}{p}(1-s)} \left[\frac{\alpha^{\frac{q}{p}} \sigma^{\frac{q}{p}-1} b^{(\sigma-1)\frac{q}{p}} (b^\sigma - a^\sigma)^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1} \left(\int_a^b {}_2F_1(y) f^s(y) dy \right)^{\frac{q}{p}} \right.$$
$$\left. -a^{\sigma\alpha s\frac{q}{p}+\sigma-1} (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b \left({}_2F_1(x)\right)^{\frac{q}{p}(1-s)} \left(I^\alpha_{a+;\sigma;\eta} f(x)\right)^{\frac{sq}{p}} dx \right], \quad (3.2.7)$$

Proof. Similar to Theorem 3.2.2. ∎

3.2.2 G. H. Hardy-type inequalities for fractional derivatives

In the following theorem, new inequality for the Canavati-type fractional derivative is given.

Theorem 3.2.6. *Let* $0 < p \leq q < \infty$, $s \geq 1$, $\nu - \gamma > 1 - \frac{p}{q}$ *and let the assumptions in Lemma 1.2.4 be satisfied. Then for non-negative functions* $D^\nu_a f$ *and* $D^\gamma_a f$ *the following inequality holds true:*

$$0 \leq \rho_5(s) \leq \overline{H}_5(s) - M_5(s) \leq \overline{H}_5(s),$$

where

$$\rho_5(s) = \frac{(\nu-\gamma)^{\frac{q}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b (b-y)^{\nu-\gamma-1+\frac{p}{q}}(D_a^\nu f(y))^s dy\right)^{\frac{q}{p}}$$
$$-\left(\Gamma(\nu-\gamma+1)\right)^{\frac{sq}{p}}\int_a^b (x-a)^{\frac{(\nu-\gamma)q(1-s)}{p}}\left(D_a^\gamma f(x)\right)^{\frac{sq}{p}}dx - M_5(s),$$

$$M_5(s) = \frac{q(\nu-\gamma)(\Gamma(\nu-\gamma+1))^{s(\frac{q}{p}-1)}}{p}\int_a^b (x-a)^{\frac{(\nu-\gamma)(q-p)(1-s)}{p}}\left(D_a^\gamma f(x)\right)^{s(\frac{q}{p}-1)}$$
$$\times\int_a^x r_5(x,y)(x-y)^{\nu-\gamma-1}dy\,dx,$$

$$r_5(x,y) = \left|\left|(D_a^\nu f(y))^s - \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^\gamma}D_a^\gamma f(x)\right)^s\right|\right.$$
$$\left.-s\left|\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right|^{s-1}\left|D_a^\nu f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right|\right|,$$

and

$$\overline{H}_5(s) = (b-a)^{(\nu-\gamma)\frac{q}{p}(1-s)}\left[\frac{(\nu-\gamma)^{\frac{q}{p}}(b-a)^{\frac{q((\nu-\gamma)s-1)+p}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b (D_a^\nu f(y))^s dy\right)^{\frac{q}{p}}\right.$$
$$\left.-\left(\Gamma(\nu-\gamma+1)\right)^{\frac{sq}{p}}\int_a^b \left(D_a^\gamma f(x)\right)^{\frac{sq}{p}}dx\right]. \quad (3.2.8)$$

Proof. Applying Theorem 3.2.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$, $k(x,y)$ is given in (2.2.3) and $A_k f(x) = \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)$. Replace f by $D_a^\nu f$. For particular weight function $u(x) = (x-a)^{\frac{(\nu-\gamma)q}{p}}$, $x \in (a,b)$ we get $v(y) = ((\nu-\gamma)(b-y)^{\nu-\gamma-1+\frac{p}{q}})/(((\nu-\gamma-1)\frac{q}{p}+1)^{\frac{p}{q}})$, then (3.2.3) takes the form

$$\rho_5(s) = \frac{(\nu-\gamma)^{\frac{q}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b (b-y)^{\nu-\gamma-1+\frac{p}{q}}(D_a^{\nu}f(y))^s dy\right)^{\frac{q}{p}}$$
$$-\left(\Gamma(\nu-\gamma+1)\right)^{\frac{sq}{p}}\int_a^b (x-a)^{\frac{(\nu-\gamma)q(1-s)}{p}}\left(D_a^{\gamma}f(x)\right)^{\frac{sq}{p}}dx - M_5(s).$$

Since $\frac{(\nu-\gamma)q}{p}(1-s) \leq 0$ and $M_5(s) \geq 0$, we obtain that

$$\begin{aligned}
\rho_5(s) &\leq \frac{(\nu-\gamma)^{\frac{q}{p}}(b-a)^{(\nu-\gamma-1)\frac{q}{p}+1}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b (D_a^{\nu}f(y))^s dy\right)^{\frac{q}{p}} \\
&\quad -(b-a)^{\frac{(\nu-\gamma)q}{p}(1-s)}(\Gamma(\nu-\gamma+1))^{\frac{sq}{p}}\int_a^b (D_a^{\gamma}f(x))^{\frac{sq}{p}}dx - M_5(s) \\
&= \overline{H}_5(s) - M_5(s) \\
&\leq \overline{H}_5(s).
\end{aligned}$$

This complete the proof. ■

Theorem 3.2.7. *Let $0 < p \leq q < \infty$, $s \geq 1$, $\nu - \gamma > 1 - \frac{p}{q}$ and assumptions in Lemma 1.2.6 be satisfied. Then for non-negative functions $D_{*a}^{\nu}f$ and $D_{*a}^{\gamma}f$ the following inequality holds true:*

$$0 \leq \rho_6(s) \leq \overline{H}_6(s) - M_6(s) \leq \overline{H}_6(s),$$

where

$$\rho_6(s) = \frac{(\nu-\gamma)^{\frac{q}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b (b-y)^{\nu-\gamma-1+\frac{p}{q}}(D_{*a}^{\nu}f(y))^s dy\right)^{\frac{q}{p}}$$
$$-\left(\Gamma(\nu-\gamma+1)\right)^{\frac{sq}{p}}\int_a^b (x-a)^{\frac{(\nu-\gamma)q(1-s)}{p}}\left(D_{*a}^{\gamma}f(x)\right)^{\frac{sq}{p}}dx - M_6(s),$$

$$M_6(s) = \frac{q(\nu-\gamma)(\Gamma(\nu-\gamma+1))^{s(\frac{q}{p}-1)}}{p} \int_a^b (x-a)^{\frac{(\nu-\gamma)(q-p)(1-s)}{p}} \left(D_{*a}^{\gamma} f(x)\right)^{s(\frac{q}{p}-1)}$$
$$\times \int_a^x r_6(x,y)(x-y)^{\nu-\gamma-1} dy\, dx,$$

$$r_6(x,y) = \Bigg| \left| (D_{*a}^{\nu} f(y))^s - \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\gamma}} D_{*a}^{\gamma} f(x) \right)^s \right|$$
$$-s \left| \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_{*a}^{\gamma} f(x) \right|^{s-1} \cdot \left| D_{*a}^{\nu} f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_{*a}^{\gamma} f(x) \right| \Bigg|,$$

and

$$\overline{H}_6(s) = (b-a)^{(\nu-\gamma)\frac{q}{p}(1-s)} \left[\frac{(\nu-\gamma)^{\frac{q}{p}} (b-a)^{\frac{q((\nu-\gamma)s-1)+p}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1} \left(\int_a^b (D_{*a}^{\nu} f(y))^s dy \right)^{\frac{q}{p}} \right.$$
$$\left. - \left(\Gamma(\nu-\gamma+1)\right)^{\frac{sq}{p}} \int_a^b (D_{*a}^{\gamma} f(x))^{\frac{sq}{p}} dx \right]. \quad (3.2.9)$$

Proof. Similar to Theorem 3.2.6. ∎

Chapter 4

Generalization of Hardy's and related inequalities involving monotone convex functions

In this chapter, we present generalized Hardy's and related inequalities involving monotone convex functions. Also, the generalize and refined inequalities of classical Pólya-Knopp's, Hardy-Hilbert, classical Hardy-Littlewood-Pólya, Hardy-Hilbert-type and Godunova are given. At the end, some refined Hardy-type inequalities for different kinds of fractional integrals and fractional derivatives are proved. The results given in this chapter can be seen in [46].

4.1 Introduction

We recall some well-known integral inequalities. First inequality is classical Hardy's inequality.

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{4.1.1}$$

where $1 < p < \infty, \mathbb{R}_+ = (0,\infty)$, and $f \in L^p(\mathbb{R}_+)$ is a non-negative function. By rewriting (4.1.1) with the function $f^{\frac{1}{p}}$ instead of f and then by letting limit $p \to \infty$, we get the limiting case of Hardy's inequality known as Pólya-Knopp's inequality, that is:

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right) dx \leq e\int_0^\infty f(x)dx, \tag{4.1.2}$$

which holds for all positive functions $f \in L^1(\mathbb{R}_+)$. Two important inequalities related to (4.1.1) are Hardy-Hilbert's inequality,

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{(x+y)} dx \right)^p dy \leq \left(\frac{\pi}{\sin \frac{\pi}{p}} \right)^p \int_0^\infty f^p(x) dx, \tag{4.1.3}$$

and the Hardy-Littlewood-Pólya inequality

$$\int_0^\infty \left(\int_0^\infty \frac{f(y)}{\max\{x,y\}} dx \right)^p dy \leq (pp')^p \int_0^\infty f^p(y) dy, \tag{4.1.4}$$

which hold for $1 < p < \infty$ and non-negative $f \in L^p(\mathbb{R}_+)$. The constants $\left(\frac{p}{p-1}\right)^p$, e, $\left(\frac{\pi}{\sin \frac{\pi}{p}}\right)^p$, $(pp')^p$ in the above inequalities are the best possible constants. For further details we refer [14]-[21] (also see [5]) and the references therein.

Godunova in [31] (see also [65]) proved the inequality

$$\int_{\mathbb{R}_+^n} \Phi \left(\frac{1}{x_1, \cdots x_n} \int_{\mathbb{R}_+^n} l\left(\frac{y_1}{x_1}, \cdots \frac{y_n}{x_n} \right) f(\mathbf{y}) d\mathbf{y} \right) \frac{d\mathbf{x}}{x_1, \cdots x_n} \leq \int_{\mathbb{R}_+^n} \frac{\Phi(f(y))}{x_1, \cdots x_n} d\mathbf{x}, \tag{4.1.5}$$

holds for all non-negative measurable functions $l : \mathbb{R}_+^n \to \mathbb{R}_+$, such that $\int_{\mathbb{R}_+^n} l(\mathbf{x}) d\mathbf{x} = 1$, convex function $\Phi : [0, \infty) \to [0, \infty)$, and a non-negative function f on $\mathbb{R}_+^n$, such that the function $\mathbf{x} \to \int_{\mathbb{R}_+^n} \frac{\Phi(f(\mathbf{x}))}{x_1, \cdots x_n}$ is integrable on $\mathbb{R}_+^n$.

4.2 Main results

In Theorem 3.2.1, the refinement of the inequality (2.4.2) is given (see [26]), but in the following theorem we provide another refinement of the inequality (2.4.2). Infact this is more general result related to Hardy-type inequalities involving monotone convex functions as an extension of the Theorem 3.2.1.

Theorem 4.2.1. *Let $0 < p \leq q < \infty$, or $-\infty < q \leq p < 0$, and let assumptions in Theorem 3.2.1 be satisfied. If Φ is a non-negative monotone convex on the interval $I \subseteq \mathbb{R}$, $f(y) > A_k f(x)$ for $y \in \Omega_2{}' \, (\Omega_2{}' \subset \Omega_2)$ and $\varphi : I \to \mathbb{R}$ is any function, such* *that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\, I$, then the inequality*

$$\left(\int\limits_{\Omega_2} v(y)\Phi\left(f(y)\right) d\mu_2(y)\right)^{\frac{q}{p}} - \int\limits_{\Omega_1} u(x)\Phi^{\frac{q}{p}}\left(A_k f(x)\right) d\mu_1(x)$$

$$\geq \frac{q}{p}\left|\int\limits_{\Omega_1} \frac{u(x)}{K(x)}\Phi^{\frac{q}{p}-1}\left(A_k f(x)\right)\int\limits_{\Omega_2} sgn(f(y)-A_k f(x))k(x,y)\Big[\Phi(f(y)) - \Phi(A_k f(x))\right.$$

$$\left. - |\varphi(A_k f(x))| \cdot (f(y)-A_k f(x))\Big] d\mu_2(y)\, d\mu_1(x)\right| \quad (4.2.1)$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $f(y) \in I$, for all fixed $y \in \Omega_2$ where $A_k f$ is defined by (2.3.1).

If Φ is a non-negative monotone concave, then the order of terms on the left-hand side of (4.2.1) *is reversed.*

Proof. Consider the case, when Φ is non-decreasing on the interval I. Then

$$\begin{aligned}
& \int\limits_{\Omega_2} k(x,y)|\Phi(f(y)) - \Phi(A_k f(x))| d\mu_2(y) \\
= & \int\limits_{\Omega_2'} k(x,y)[\Phi(f(y)) - \Phi(A_k f(x)))] d\mu_2(y) \\
& + \int\limits_{\Omega_2 \setminus \Omega_2'} k(x,y)[\Phi(A_k f(x)) - \Phi(f(y))] d\mu_2(y) \\
= & \int\limits_{\Omega_2'} k(x,y)\Phi(f(y)) d\mu_2(y) - \int\limits_{\Omega_2 \setminus \Omega_2'} k(x,y)\Phi(f(y)) d\mu_2(y) \\
& -\Phi(A_k f(x))\int\limits_{\Omega_2'} k(x,y) d\mu_2(y) + \Phi(A_k f(x))\int\limits_{\Omega_2 \setminus \Omega_2'} k(x,y) d\mu_2(y) \\
= & \int\limits_{\Omega_2} sgn(f(y)-A_k f(x))k(x,y)[\Phi(f(y)) - \Phi(A_k f(x)] d\mu_2(y). \qquad (4.2.2)
\end{aligned}$$

Similarly, we can write

$$\int_{\Omega_2} k(x,y)|f(y)-A_kf(x)|d\mu_2(y)$$
$$= \int_{\Omega_2} sgn(f(y)-A_kf(x))k(x,y)(f(y)-A_kf(x))d\mu_2(y). \quad (4.2.3)$$

From (3.2.1), (4.2.2) and (4.2.3), we get (4.2.1).

The case, when Φ is non-increasing can be discussed in the similar way. ■

For $p=q$, we get the following result which is in fact the new version of the [25, Theorem 2.1] involving monotone convex function and the function Φ not need to be non-negative.

Corollary 4.2.2. *Let $\Omega_1, \Omega_2, \mu_1, \mu_2, u, k, K,$ and v be as in Theorem 3.2.1. If Φ is a monotone convex on the interval $I \subseteq \mathbb{R}$, $f(y) > A_kf(x)$ for $y \in \Omega_2{}'\,(\Omega_2{}' \subset \Omega_2)$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\, I$, then the inequality*

$$\int_{\Omega_2} v(y)\Phi\left(f(y)\right)d\mu_2(y) - \int_{\Omega_1} u(x)\Phi\left(A_kf(x)\right)d\mu_1(x)$$
$$\geq \Bigg| \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} sgn(f(y)-A_kf(x))k(x,y)\Big[\Phi(f(y)) - \Phi(A_kf(x))$$
$$- |\varphi(A_kf(x))| \cdot (f(y)-A_kf(x))\Big]\, d\mu_2(y)\, d\mu_1(x)\Bigg| \quad (4.2.4)$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $f(y) \in I$, for all fixed $y \in \Omega_2$ where A_kf is defined by (2.3.1).

If Φ is a monotone concave, then the order of terms on the left-hand side of (4.2.4) *is reversed.*

Although the (4.2.1), holds for non-negative monotone convex functions some choices of Φ are of our particular interest. Here, we consider the power and exponential functions. Let the function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be defined by $\Phi(x) = x^p$. It is non-negative and monotone function. Obviously, $\varphi(x) = \Phi'(x) = px^{p-1}$, $x \in \mathbb{R}_+$, so Φ is convex for $p \in \mathbb{R} \setminus [0,1)$, concave for $p \in (0,1]$, and affine, that is, both convex and concave for $p = 1$.

Corollary 4.2.3. *Let $\Omega_1, \Omega_2, \mu_1, \mu_2, u, k, K,$ and v be as in Theorem 3.2.1. Let $p \in \mathbb{R}$ be such that $p \neq 0$, $f : \Omega_2 \to \mathbb{R}$ be a non-negative measurable function (positive for $p < 0$), $A_k f$ be defined by (2.3.1) and*

$$M_{p,k}f(x,y) = f^p(y) - A_k^p f(x) - |p| \cdot |A_k f(x)|^{p-1}(f(y) - A_k f(x)) \tag{4.2.5}$$

for $x \in \Omega_1$, $y \in \Omega_2$. If $p \geq 1$ or $p < 0$, then the inequality

$$\begin{aligned} &\left(\int_{\Omega_2} v(y) f^p(y) d\mu_2(y) \right)^{\frac{q}{p}} - \int_{\Omega_1} u(x) A_k^q f(x) d\mu_1(x) \\ &\geq \frac{q}{p} \left| \int_{\Omega_1} \frac{u(x)}{K(x)} \left(A_k f(x)\right)^{q-p} \int_{\Omega_2} sgn(f(y) - A_k f(x)) k(x,y) M_{p,k} f(x,y)\, d\mu_2(y)\, d\mu_1(x) \right| \end{aligned} \tag{4.2.6}$$

holds. If $p \in (0,1)$ relation (4.2.6) *holds with*

$$\int_{\Omega_1} u(x) A_k^q f(x) d\mu_1(x) - \left(\int_{\Omega_2} v(y) f^p(y) d\mu_2(y) \right)^{\frac{q}{p}}$$

on its left hand-side.

For the monotone convex function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ defined by $\Phi(x) = e^x, x \in \mathbb{R}_+$ the following result follows.

Corollary 4.2.4. *Let $\Omega_1, \Omega_2, \mu_1, \mu_2, u, k, K$ and v be defined as in Theorem 3.2.1 and let $p > 0$. Let $G_k f(x)$ be defined by*

$$G_k f(x) := \exp\left(\frac{1}{K(x)} \int_{\Omega_2} k(x,y) \ln f(y) d\mu_2(y) \right), \tag{4.2.7}$$

$$P_{p,k} f(x,y) = f^p(y) - G_k^p f(x) - p\, |G_k^p(x)| \ln \frac{f(y)}{G_k f(x)} \tag{4.2.8}$$

and $f : \Omega_2 \to \mathbb{R}$ be a positive measurable function, $f(y) > G_k f(x)$ for $y \in \Omega_2{}'$ $(\Omega_2{}' \subset$

Ω_2). *Then the following inequality holds*

$$\left(\int_{\Omega_2} v(y) f^p(y) d\mu_2(y)\right)^{\frac{q}{p}} - \int_{\Omega_1} u(x) G_k^q f(x) d\mu_1(x)$$
$$\geq \frac{q}{p}\left|\int_{\Omega_1} \frac{u(x)}{K(x)} G_k^{q-p} f(x) \int_{\Omega_2} sgn(f(y) - G_k f(x)) k(x,y) P_{p,k} f(x,y) d\mu_2(y)\, d\mu_1(x)\right|. \tag{4.2.9}$$

Proof. Apply (4.2.1) with $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(x) = e^x$, and replace the function f with $p \ln f$. Note that $G_k f = \exp(A_k(\ln f))$. ■

Here we give the results for one dimensional settings, with intervals in $\mathbb{R}$ and Lebesgue measures. Also we give the related dual results.

Theorem 4.2.5. *Let* $0 < b \leq \infty$ *and* $k : (0,b) \times (0,b) \to \mathbb{R}$ *be a non-negative measurable function, such that*

$$K(x) =: \int_0^x k(x,y) dy, \quad x \in (0,b). \tag{4.2.10}$$

Let u *be a weight function such that the function* $x \mapsto \frac{u(x)}{x} \cdot \left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ *is integrable on* (y,b) *for each fixed* $y \in (0,b)$, *and let the function* $w : (0,b) \to \mathbb{R}$ *be defined by*

$$w(y) = y\left(\int_y^b \left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}} u(x) \frac{dx}{x}\right)^{\frac{p}{q}}. \tag{4.2.11}$$

If Φ *is a non-negative monotone convex on the interval* $I \subseteq \mathbb{R}$ *and* $\varphi : I \to \mathbb{R}$ *is that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in IntI$, *then the following inequality*

$$\left(\int_0^b w(y)\Phi\left(f(y)\right)\frac{dy}{y}\right)^{\frac{q}{p}} - \int_0^b u(x)\Phi^{\frac{q}{p}}\left(A_k f(x)\right)\frac{dx}{x}$$
$$\geq \frac{q}{p}\left|\int_0^b \frac{u(x)}{K(x)}\Phi^{\frac{q}{p}-1}\left(A_k f(x)\right)\int_0^x sgn(f(y) - A_k f(x)) k(x,y)\Big[\Phi(f(y) - \Phi(A_k f(x))\right.$$
$$\left. - |\varphi(A_k f(x))|.(f(y) - A_k f(x))\Big] dy \frac{dx}{x}\right| \tag{4.2.12}$$

holds for all measurable functions $f:(0,b)\to\mathbb{R}$, $f(y)>A_kf(x)$ *for* $y\in I'(I'\subset(0,b))$, *such that* $f(y)\in I$, *for all fixed* $y\in(0,b)$ *where* A_kf *is defined by*

$$A_kf(x):=\frac{1}{K(x)}\int_0^x k(x,y)f(y)dy,\quad x\in(0,b). \tag{4.2.13}$$

Proof. Similar to [25, Theorem 3.1]. ■

By considering the power and exponential functions, we can give the following results.

Corollary 4.2.6. *Let* $0<b\le\infty$, u,k,K *and* w *be defined in Theorem 4.2.5. Let* $p\in\mathbb{R}$, $p\neq 0$, f *be a non-negative measurable function on* $(0,b)$, $f(y)>A_kf(x)$ *for* $y\in I'\,(I'\subset(0,b))$, *where* A_kf *and* $M_{p,k}$ *be defined by* (4.2.13) *and* (4.2.5) *respectively. If* $p>1$ *or* $p<0$, *then the following inequality holds*

$$\left(\int_0^b w(y)f^p(y)\frac{dy}{y}\right)^{\frac{q}{p}}-\int_0^b u(x)\left(A_kf(x)\right)^q\frac{dx}{x}$$
$$\ge\frac{q}{p}\left|\int_0^b\frac{u(x)}{K(x)}\left(A_kf(x)\right)^{q-p}\int_0^x sgn(f(y)-A_kf)k(x,y)M_{p,k}f(x,y)\,dy\,\frac{dx}{x}\right|. \tag{4.2.14}$$

If $p\in(0,1)$, *then the order of terms on the left-hand side of relation* (4.2.14) *is reversed.*

Corollary 4.2.7. *Let* $0<b\le\infty$ u,k,K *and* w *be defined in Theorem 4.2.5 and* $P_{p,k}$ *by* (4.2.8). *Let* $p>1$ *and* f *be a positive measurable function on* $(0,b)$, $f(y)>G_kf(x)$ *for* $y\in I'\,(I'\subset(0,b))$. *Then the following inequality holds:*

$$\left(\int_0^b w(y)f^p(y)\frac{dy}{y}\right)^{\frac{q}{p}}-\int_0^b u(x)G_k^qf(x)\frac{dx}{x}$$
$$\ge\frac{q}{p}\left|\int_0^b\frac{u(x)}{K(x)}G_k^{q-p}f(x)\int_0^x sgn(f(y)-G_kf(x))k(x,y)P_{p,k}f(x,y)dy\,\frac{dx}{x}\right|,$$

where $G_k f(x)$ is defined by

$$G_k f(x) := \exp\left(\frac{1}{K(x)}\int_0^b k(x,y)\ln f(y)d\mu_2(y)\right), \quad x \in (0,b). \tag{4.2.15}$$

Now we give the dual results to Theorem 4.2.5 with some related corollaries.

Theorem 4.2.8. *Let $0 < b \leq \infty$, let $k : (b,\infty)\times(b,\infty) \to \mathbb{R}$ be a non-negative measurable function and $\widetilde{K}(x)$ be defined by*

$$\widetilde{K}(x) := \int_x^\infty k(x,y)dy, \qquad x \in (b,\infty). \tag{4.2.16}$$

Let u be a weight function such that the function $x \mapsto \frac{u(x)}{x}\cdot\left(\frac{k(x,y)}{\widetilde{K}(x)}\right)^{\frac{q}{p}}$ is integrable on (b,y) for each fixed $y \in (b,\infty)$, and let the function $\widetilde{w} : (b,\infty) \to \mathbb{R}$ be defined by

$$\widetilde{w}(y) = y\left(\int_b^y \left(\frac{k(x,y)}{\widetilde{K}(x)}\right)^{\frac{q}{p}} u(x)\frac{dx}{x}\right)^{\frac{p}{q}}. \tag{4.2.17}$$

If Φ is a non-negative monotone convex on the interval $I \subseteq \mathbb{R}$, and $\varphi : I \to \mathbb{R}$ is that $\varphi(x) \in \partial\Phi(x)$ for all $x \in IntI$, then the following inequality

$$\begin{aligned}
&\left(\int_b^\infty \widetilde{w}(y)\Phi\left(f(y)\right)\frac{dy}{y}\right)^{\frac{q}{p}} - \int_b^\infty u(x)\Phi^{\frac{q}{p}}\left(\widetilde{A}_k f(x)\right)\frac{dx}{x} \\
&\geq \frac{q}{p}\left|\int_b^\infty \frac{u(x)}{\widetilde{K}(x)}\Phi^{\frac{q}{p}-1}\left(\widetilde{A}_k f(x)\right)\int_x^\infty sgn(f(y)-\widetilde{A}_k f)k(x,y)\left[\Phi(f(y)) - \Phi(\widetilde{A}_k f(x))\right.\right. \\
&\qquad\left.\left. - |\varphi(\widetilde{A}_k f(x))|.(f(y)-\widetilde{A}_k f(x))\right]dy\,\frac{dx}{x}\right|
\end{aligned} \tag{4.2.18}$$

holds for all measurable functions $f : (b,\infty) \to \mathbb{R}$, such that $f(y) \in I$, for all fixed $y \in (b,\infty)$, $f(y) > \widetilde{A}_k f$ for $y \in I''(I'' \subset (b,\infty))$, where $\widetilde{A}_k f$ is defined by

$$\widetilde{A}_k f(x) := \frac{1}{\widetilde{K}(x)}\int_x^\infty k(x,y)f(y)dy, \quad x \in (b,\infty). \tag{4.2.19}$$

Proof. Similar to [25, Theorem 3.2]. ■

Corollary 4.2.9. *Let $0 < b \leq \infty$ $u, k, \widetilde{K}$ and $\widetilde{w}$ be defined in Theorem 4.2.8. Let $p > 1$, f be a non-negative measurable function on (b, ∞), $f(y) > \widetilde{A}_k f$ for $y \in I''(I'' \subset (b, \infty))$, where $\widetilde{A}_k f$ be defined by (4.2.19). Then the following inequality holds:*

$$\left(\int_b^\infty \widetilde{w}(y) f^p(y) \frac{dy}{y}\right)^{\frac{q}{p}} - \int_b^\infty u(x) \left(\widetilde{A}_k f(x)\right)^q \frac{dx}{x}$$
$$\geq \frac{q}{p} \left| \int_b^\infty \frac{u(x)}{\widetilde{K}(x)} \left(\widetilde{A}_k f(x)\right)^{q-p} \int_x^\infty sgn(f(y) - \widetilde{A}_k f) k(x,y) \left[f^p(y) - (\widetilde{A}_k f(x))^p \right.\right.$$
$$\left.\left. - p|\widetilde{A}_k f(x)|^{p-1}.(f(y) - \widetilde{A}_k f(x))\right] dy \frac{dx}{x} \right|.$$

Corollary 4.2.10. *Let $0 < b \leq \infty$ $u, k, \widetilde{K}$ and $\widetilde{w}$ be defined in Theorem 4.2.8. Let f be a positive measurable function on (b, ∞), $f(y) > \widetilde{G}_k f$ for $y \in I''(I'' \subset (b, \infty))$. Then the following inequality holds:*

$$\left(\int_b^\infty \widetilde{w}(y) f^p(y) \frac{dy}{y}\right)^{\frac{q}{p}} - \int_b^\infty u(x) \widetilde{G}_k^q f(x) \frac{dx}{x}$$
$$\geq \frac{q}{p} \left| \int_b^\infty \frac{u(x)}{\widetilde{K}(x)} \widetilde{G}_k^{q-p} f(x) \int_x^\infty sgn(f(y) - \widetilde{G}_k f(x)) k(x,y) \left[f^p(y) - \widetilde{G}_k^p f(x) \right.\right.$$
$$\left.\left. - p|\widetilde{G}_k^p f(x)|.(\ln f(y) - \ln \widetilde{G}_k f(x))\right] dy \frac{dx}{x} \right|,$$

where $\widetilde{G}_k f(x)$ is defined by

$$\widetilde{G}_k f(x) := \exp\left(\frac{1}{\widetilde{K}(x)} \int_x^\infty k(x,y) \ln f(y) dy\right) \quad x \in (b, \infty). \tag{4.2.20}$$

4.2.1 Hardy-Hilbert and Godunova's inequalities

In this section, we give some examples for different kernels using Theorem 4.2.1. We take $\Omega_1 = \Omega_2 = \mathbb{R}_+, d\mu_1(x) = dx,\ d\mu_2(y) = dy$ and function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ defined by $\Phi(x) = x^p$, where $p \in \mathbb{R}$, $p \neq 0$.

In our first example, we generalize and refine the Hardy-Hilbert's inequality given in (4.1.3).

Example 4.2.1. *Let $p, q, s \in \mathbb{R}$ be such that $\frac{q}{p} > 0$ and $\frac{s-2}{p}, \frac{s-2}{\acute{p}} > -1$, and let $\alpha \in \left(-\frac{q}{p}\left(\frac{s-2}{\acute{p}}+1\right), \frac{q}{p}\left(\frac{s-2}{p}+1\right)\right)$. Denote*

$$B_1 = B\left(\frac{q}{p}\left(\frac{s-2}{p}+1\right)-\alpha, \frac{q}{p}\left(\frac{s-2}{\acute{p}}+1\right)+\alpha\right),$$

and

$$B_2 = B\left(\frac{s-2}{p}+1, \frac{s-2}{\acute{p}}+1\right),$$

where $B(\cdot,\cdot)$ is the usual beta function and define $k : \mathbb{R}_+^2 \to \mathbb{R}$ by $k(x,y) = \left(\frac{y}{x}\right)^{\frac{s-2}{p}}(x+y)^{-s}$, we obtain that $K(x) = x^{1-s}B_2$. For particular weight $u(x) = x^{\alpha-1}$, we get that $v(y) = y^{\alpha\frac{p}{q}-1}B_1^{\frac{p}{q}}B_2^{-1}$. Finally, let f be a non-negative function on $\mathbb{R}_+$ (positive for $p<0$) and Sf its generalized Steiltjes transform,

$$Sf(x) = \int\limits_0^\infty \frac{f(y)}{(x+y)^s}dy, \quad x \in \mathbb{R}_+,$$

(see [8] and [69] for further details). Rewrite (4.2.1) with the above parameters and with $f(y)y^{\frac{2-s}{p}}$ instead of $f(y)$, we get $A_k(f(x)x^{\frac{2-s}{p}}) = x^{\frac{s-2}{\acute{p}}+1}B_2^{-1}Sf(x)$ for $1 \le p \le q < \infty$ or $-\infty < q \le p \le 0$, the following inequality holds:

$$\begin{aligned}
&B_1 B_2^{\frac{q}{\acute{p}}}\left(\int\limits_0^\infty y^{\alpha\frac{p}{q}-s+1}f^p(y)dy\right)^{\frac{q}{p}} - \int\limits_0^\infty x^{\alpha-1+\frac{(s-1)q}{\acute{p}}+\frac{q}{p}}\left(Sf(x)\right)^q dx \\
&\ge \frac{q}{p}\left|B_2^{p-1}\int\limits_0^\infty x^{\alpha+q-p+\left(\frac{s-2}{\acute{p}}\right)(q-p+1)}\left(Sf(x)\right)^{q-p}\int\limits_0^\infty sgn\left(y^{\frac{s-2}{p}}f(y) - x^{\frac{s-2}{\acute{p}}+1}B_2^{-1}Sf(x)\right)\right. \\
&\qquad \times \frac{y^{\frac{s-2}{p}}}{(x+y)^s}\left[y^{2-s}f^p(y) - \left(x^{\frac{s-2}{\acute{p}}+1}B_2^{-1}Sf(x)\right)^p - p\left|x^{\frac{s-2}{\acute{p}}+1}B_2^{-1}Sf(x)\right|^{p-1}\right. \\
&\qquad\qquad \left.\left.\times\left(y^{\frac{2-s}{p}}f(y) - x^{\frac{s-2}{\acute{p}}+1}B_2^{-1}Sf(x)\right)\right]dy\,dx\right|. \qquad (4.2.21)
\end{aligned}$$

In next our example, similarly we generalize and refine Hardy-Littlewood-Pólya's inequality given in (4.1.4).

Example 4.2.2. *Let $p, q, s, \alpha \in \mathbb{R}$ and that the function u and f be same as in the Example 4.2.1. Define $k : \mathbb{R}^2_+ \to \mathbb{R}$ by $k(x,y) = \left(\frac{y}{x}\right)^{\frac{s-2}{p}} \max\{x,y\}^{-s}$ and we get that $K(x) = x^{1-s}L_2$. For particular weight function $x^{\alpha-1}$, we obtain $v(y) = y^{\alpha\frac{p}{q}-1} L_1^{\frac{p}{q}} L_2^{-1}$ and the transformation is*

$$Lf(x) = \int_0^\infty \frac{f(y)}{\max\{x,y\}^s} dy, \quad x \in \mathbb{R}_+.$$

Set

$$L_1 = \frac{p^2 \acute{p} q s}{(\alpha p \acute{p} + \acute{p} q + qs - 2q)(pq + qs - \alpha p^2 - 2q)},$$

and

$$L_2 = \frac{p \acute{p} s}{(p + s - 2)(\acute{p} + s - 2)}.$$

Consider $1 \le p \le q < \infty$ or $-\infty < q \le p \le 0$ and $f(y) y^{\frac{2-s}{p}}$ instead of $f(y)$, we obtain $A_k\left(f(x) x^{\frac{2-s}{p}}\right) = x^{\frac{s-2}{\acute{p}}+1} L_2^{-1} Lf(x)$, then the following inequality hold:

$$\begin{aligned}
& L_1 L_2^{\frac{q}{\acute{p}}} \left(\int_0^\infty y^{\alpha\frac{p}{q} - s + 1} f^p(y) dy \right)^{\frac{q}{p}} - \int_0^\infty x^{\alpha - 1 + \frac{(s-1)q}{\acute{p}} + \frac{q}{p}} \left(Lf(x)\right)^q dx \\
& \ge \frac{q}{p} \Bigg| L_2^{p-1} \int_0^\infty x^{\alpha + q - p + \left(\frac{s-2}{\acute{p}}\right)(q-p+1)} \left(Lf(x)\right)^{q-p} \int_0^\infty sgn\left(y^{\frac{2-s}{p}} f(y) - x^{\frac{s-2}{\acute{p}}+1} L_2^{-1} Lf(x) \right) \\
& \qquad \times \frac{y^{\frac{s-2}{p}}}{\max\{x,y\}^s} \Bigg[y^{2-s} f^p(y) - \left(x^{\frac{s-2}{\acute{p}}+1} L_2^{-1} Lf(x) \right)^p - p \left| x^{\frac{s-2}{\acute{p}}+1} L_2^{-1} Lf(x) \right|^{p-1} \\
& \qquad \qquad \times \left(y^{\frac{2-s}{p}} f(y) - x^{\frac{s-2}{\acute{p}}+1} L_2^{-1} Lf(x) \right) \Bigg] dy\, dx \Bigg|.
\end{aligned}$$

We continue with Hardy-Hilbert-type inequality, making the use of the well-known reflection formula for the Digamma function ψ,

$$\int_0^\infty \frac{\ln t}{t-1} t^{-\alpha} dt = \acute{\psi}(1-\alpha) + \acute{\psi}(\alpha) = \frac{\pi^2}{\sin^2 \pi\alpha}, \quad \alpha \in (0,1),$$

and of the fact that

$$Z(a,b)=\int_0^\infty t^b e^{-at}(1-e^{-t})^b dt<\infty, \quad a\in\mathbb{R}_+, b\geq 1.$$

More precisely, $Z(a,b)=\Gamma(b+1)\phi_b^*(1,b+1,a)$, where ϕ_μ^* is the so called unified Riemann-Zeta function,

$$\phi_\mu^*(z,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}(1-ze^{-t})^{-\mu}dt,$$

where $\mu\geq 1$, Re $a>0$ and either $|z|\leq 1, z\neq 1$, Re $s>0$ or $z=1$ and Re $s>\mu$ (for more details see [32])

In upcoming example, we generalize and refine another Hardy-Hilbert's inequality.

Example 4.2.3. *Suppose that $\alpha\in(0,1)$ and $p,q,\beta\in\mathbb{R}$ such that $\frac{q}{p}\geq 1$ and $\alpha\frac{q}{p}+\beta\in(-1,\frac{q}{p}-1)$. Define the kernel $k:\mathbb{R}_+^2\to\mathbb{R}$ by $k(x,y)=\frac{\ln y-\ln x}{y-x}\left(\frac{x}{y}\right)^\alpha$ we get that $K(x)=\frac{\pi^2}{sin^2\pi\alpha}$. We also define the weight function $u:\mathbb{R}_+\to\mathbb{R}$ by $u(x)=x^\beta$, we obtain $v(y)=y^{\beta\frac{p}{q}+\frac{p}{q}-1}M_1^{\frac{p}{q}}M_2^{-1}$ with the transformation*

$$Mf(x)=\int_0^\infty \frac{\ln y-\ln x}{y-x}f(y)dy, \quad x\in\mathbb{R}_+,$$

where f is a non-negative function on $\mathbb{R}_+$ (positive, if $p<0$),

$$M_1=\int_0^\infty\left(\frac{\ln t}{t-1}\right)^{\frac{q}{p}}t^{\alpha\frac{q}{p}+\beta}dt=Z\left(\alpha\frac{q}{p}+\beta+1,\frac{q}{p}\right)+Z\left(\frac{q}{p}-\alpha\frac{q}{p}-\beta-1,\frac{q}{p}\right),$$

and

$$M_2=\int_0^\infty\frac{\ln t}{t-1}t^{-\alpha}dt=\frac{\pi^2}{\sin^2\pi\alpha}.$$

Rewrite (4.2.1) *with the above details and $f(y)y^\alpha$ instead of $f(y)$, we get that $A_k\left(f(x)x^\alpha\right)=x^\alpha M_2^{-1}Mf(x)$, then the following inequality holds:*

$$M_1M_2^{\frac{q}{p}}\left(\int_0^\infty y^{\alpha p+(\beta+1)\frac{p}{q}-1}f^p(y)dy\right)^{\frac{q}{p}}-\int_0^\infty x^{\alpha q+\beta}\left(Mf(x)\right)^q dx$$

$$\geq \frac{q}{p}\left| M_2^{p-1} \int_0^\infty x^{\alpha(q-p+1)+\beta} \left(Mf(x)\right)^{q-p} \int_0^\infty sgn\left(y^\alpha f(y) - x^\alpha M_2^{-1} Mf(x)\right) \right.$$

$$\times\, y^{-\alpha}\frac{\ln y - \ln x}{y-x}\left[y^{\alpha p} f^p(y) - \left(x^\alpha M_2^{-1} Mf(x)\right)^p - p\left|x^\alpha M_2^{-1} Mf(x)\right|^{p-1}\right.$$

$$\left.\left. \times \left(y^\alpha f(y) - x^\alpha M_2^{-1} Mf(x)\right)\right] dy\, dx\right|.$$

We complete this section with the general Godunova's inequality for multidimensional result. Let us take $\Omega_1 = \Omega_2 = \mathbb{R}_+^n, d\mu_1(\mathbf{x}) = d\mathbf{x}, d\mu_2(\mathbf{y}) = d\mathbf{y}$, let $\frac{\mathbf{x}}{\mathbf{y}}$ and $\mathbf{x}^{\mathbf{y}}$ be defined for $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$:

$\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdot, \cdot, \cdot, y_n)$ and we denote

$$\frac{\mathbf{x}}{\mathbf{y}} = \left(\frac{x_1}{y_1}, \cdots, \frac{x_n}{y_n}\right) \text{ and } \mathbf{x}^{\mathbf{y}} = x_1^{y_1}, \cdots, x_n^{y_n}.$$

Especially, $\mathbf{x}^{\mathbf{1}} = \prod_{i=1}^n x_i$ and $\mathbf{x}^{-\mathbf{1}} = \left(\prod_{i=1}^n u_i\right)^{-\mathbf{1}}$.

Let the kernel $k : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}$ be of the form $k(\mathbf{x}, \mathbf{y}) = l(\frac{\mathbf{y}}{\mathbf{x}})$, where $l : \mathbb{R}_+^n \to \mathbb{R}$ is a non-negative measurable function.

In our next result we obtain the generalization and refinement of the Godunova's inequality given in (4.1.5) using Theorem 4.2.1.

Theorem 4.2.11. *Let $0 < p \leq q < \infty$, or $-\infty < q \leq p < 0$,. Let l and u be non-negative measurable functions on $\mathbb{R}_+^n$, such that $0 < L(\mathbf{x}) = \mathbf{x}^1 \int_{\mathbb{R}_+^n} l(\mathbf{y})d\mathbf{y} < \infty$ for all $\mathbf{x} \in \mathbb{R}_+^n$, and let the function $\mathbf{x} \to u(\mathbf{x}) \left(\frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}\right)^{\frac{q}{p}}$ is integrable on $\mathbb{R}_+^n$ for each fixed $\mathbf{y} \in \mathbb{R}_+^n$. Let the function v be defined on $\mathbb{R}_+^n$ by*

$$v(\mathbf{y}) = \left(\int_{\mathbb{R}_+^n} u(\mathbf{x}) \left(\frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}\right)^{\frac{q}{p}}\right)^{\frac{p}{q}} d\mathbf{x}.$$

If Φ is a non-negative monotone convex on the interval $I \subseteq \mathbb{R}$, $f(\mathbf{y}) > A_l f(\mathbf{x})$ for $\mathbf{y} \in \mathbb{R}_+^{n\prime} (\mathbb{R}_+^{n\prime} \subset \mathbb{R}_+^n)$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all

$x \in Int\, I$, *then the inequality*

$$\left(\int\limits_{\mathbb{R}^n_+} v(\mathbf{y})\Phi\left(f(\mathbf{y})\right)d\mathbf{y}\right)^{\frac{q}{p}} - \int\limits_{\mathbb{R}^n_+} u(\mathbf{x})\Phi^{\frac{q}{p}}\left(A_l f(\mathbf{x})\right)d\mathbf{x}$$

$$\geq \frac{q}{p}\Bigg|\int\limits_{\mathbb{R}^n_+} \frac{u(\mathbf{x})}{L(\mathbf{x})}\Phi^{\frac{q}{p}-1}\left(A_l f(\mathbf{x})\right)\int\limits_{\mathbb{R}^n_+} sgn(f(\mathbf{y})-A_l f(\mathbf{x}))l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)\Big[\Phi(f(\mathbf{y}))-\Phi(A_l f(\mathbf{x}))$$

$$-|\varphi(A_l f(\mathbf{x}))|\cdot(f(\mathbf{y})-A_l f(\mathbf{x}))\Big]d\mathbf{y}\,d\mathbf{x}\Bigg| \quad (4.2.22)$$

holds for all measurable functions $f:\mathbb{R}^n_+\to\mathbb{R}$, *with the values in* I, *for all fixed* $\mathbf{y}\in\mathbb{R}^n_+$ *where* $A_l f$ *is defined by*

$$A_l f(\mathbf{x}) = \frac{1}{L(\mathbf{x})}\int\limits_{\mathbb{R}^n_+}\frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}f(\mathbf{y})d\mathbf{y}.$$

If Φ *is a positive monotone concave function, then the order of terms on the left-hand side of* (4.2.22) *is reversed.*

4.2.2 G. H. Hardy-type inequalities for fractional integrals involving monotone convex function

We continue our analysis about improvements by taking the non-negative difference of left-hand side and right-hand side of inequality given in (4.2.1) by taking $\Phi:\mathbb{R}_+\to\mathbb{R}_+$, $\Phi(x)=x^s, s\geq 1$ as:

$$\pi(s) = \left(\int\limits_{\Omega_2} v(y)f^s(y)d\mu_2(y)\right)^{\frac{q}{p}} - \int\limits_{\Omega_1} u(x)\left(A_k f(x)\right)^{\frac{sq}{p}}d\mu_1(x)$$

$$-\frac{q}{p}\Bigg|\int\limits_{\Omega_1}\frac{u(x)}{K(x)}\Phi^{\frac{q}{p}-1}\left(A_k f(x)\right)\int\limits_{\Omega_2} sgn(f(y)-A_k f(x))k(x,y)\Big[f^s(y)-(A_k f(x))^s$$

$$-s|A_k f(x)|^{s-1}\cdot(f(y)-A_k f(x))\Big]d\mu_2(y)\,d\mu_1(x)\Bigg|. \quad (4.2.23)$$

Our first result involve fractional integral of f with respect to another increasing function g.

Theorem 4.2.12. *Let $0 < p \le q < \infty,\ s \ge 1, f \ge 0,\ \alpha > 1 - \frac{p}{q}$, g be increasing function on (a,b) such that g' be continuous on $(a,b]$, $f(y) > \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)$ for $y \in I(I \subset (a,b))$. Then the following inequality holds true:*

$$\pi_1(s) \le \overline{H}_1(s) - B_1(s) \le \overline{H}_1(s),$$

where

$$\pi_1(s) = \frac{\alpha^{\frac{q}{p}}}{((\alpha-1)\frac{q}{p}+1)} \left(\int_a^b g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}} f^s(y)dy \right)^{\frac{q}{p}}$$
$$- \left(\Gamma(\alpha+1)\right)^{\frac{sq}{p}} \int_a^b g'(x)(g(x)-g(a))^{\frac{\alpha q}{p}(1-s)} \left(I^{\alpha}_{a+;g} f(x)\right)^{\frac{sq}{p}} dx - B_1(s),$$

$$B_1(s) = \frac{\alpha q (\Gamma(\alpha+1))^{s(\frac{q}{p}-1)}}{p} \Bigg| \int_a^b \int_a^x sgn\left(f(y) - \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x) \right) g'(x)g'(y)$$
$$\times \frac{(g(x)-g(a))^{\frac{\alpha(q-p)(1-s)}{p}}}{(g(x)-g(y))^{1-\alpha}} \left(I^{\alpha}_{a+;g} f(x)\right)^{s(\frac{q}{p}-1)} \left[f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x) \right)^s \right.$$
$$\left. -s \left| \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x) \right|^{s-1} \left(f(y) - \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x) \right) \right] dy\, dx \Bigg|,$$

and $\overline{H}_1(s)$ is defined by (3.2.4).

Proof. Similar to Theorem 3.2.2. ∎

If $g(x) = x$, then $I^{\alpha}_{a+;x} f(x)$ reduces to $I^{\alpha}_{a+} f(x)$ left-sided Riemann–Liouville fractional integral and the following result follows.

Corollary 4.2.13. *Let $0 < p \le q < \infty,\ \alpha > 1 - \frac{p}{q},\ s \ge 1, f \ge 0,\ f(y) > \frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a^+} f(x)$, for $y \in I(I \subset (a,b))$. Then the following inequality holds true:*

$$\pi_2(s) \le \overline{H}_2(s) - B_2(s) \le \overline{H}_2(s),$$

where

$$\pi_2(s) = \frac{\alpha^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b (b-y)^{\alpha-1+\frac{p}{q}} f^s(y)dy\right)^{\frac{q}{p}}$$
$$-\left(\Gamma(\alpha+1)\right)^{\frac{sq}{p}}\int_a^b (x-a)^{\frac{\alpha q}{p}(1-s)}\left(I_{a^+}^{\alpha} f(x)\right)^{\frac{sq}{p}} dx - B_2(s),$$

$$B_2(s) = \frac{\alpha q(\Gamma(\alpha+1))^{s(\frac{q}{p}-1)}}{p}\Bigg|\int_a^b\int_a^x sgn\left(f(y)-\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}I_{a^+}^{\alpha}f(x)\right)(x-a)^{\frac{\alpha(q-p)(1-s)}{p}}$$
$$\times\left(I_{a^+}^{\alpha}f(x)\right)^{s(\frac{q}{p}-1)}(x-y)^{\alpha-1}\left[f^s(y)-\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}I_{a^+}^{\alpha}f(x)\right)^s\right.$$
$$\left.-s\left|\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}I_{a_+}^{\alpha}f(x)\right|^{s-1}\left(f(y)-\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}I_{a^+}^{\alpha}f(x)\right)\right]dy\,dx\Bigg|,$$

and $\overline{H}_2(s)$ *is defined by* (3.2.5).

Remark 4.2.1. If we take $g(x)=\log x$, then $I_{a_+;x}^{\alpha}f(x)$ reduces to $J_{a_+}^{\alpha}f(x)$ left-sided Hadamard-type fractional integral that is defined for $\alpha>0$ and we obtain the following inequality:

$$\pi_3(s)\le \overline{H}_3(s)-B_3(s)\le\overline{H}_3(s),$$

where

$$\pi_3(s) = \frac{\alpha^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b (\log b-\log y)^{\alpha-1+\frac{p}{q}} f^s(y)\frac{dy}{y}\right)^{\frac{q}{p}}$$
$$-\left(\Gamma(\alpha+1)\right)^{\frac{sq}{p}}\int_a^b (\log x-\log a)^{\frac{\alpha q}{p}(1-s)}\left(J_{a_+}^{\alpha} f(x)\right)^{\frac{sq}{p}}\frac{dx}{x} - B_3(s),$$

$$B_3(s) = \frac{\alpha q(\Gamma(\alpha+1))^{s(\frac{q}{p}-1)}}{p}\left|\int_a^b\int_a^x sgn\left(f(y) - \frac{\Gamma(\alpha+1)}{(\log x - \log a)^\alpha}J^\alpha_{a+}f(x)\right)\right.$$
$$\times\left(J^\alpha_{a+}f(x)\right)^{s(\frac{q}{p}-1)}\frac{(\log x-\log a)^{\frac{\alpha(q-p)(1-s)}{p}}}{(\log x - \log y)^{1-\alpha}}\left[f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(\log x - \log a)^\alpha}J^\alpha_{a+}f(x)\right)^s\right.$$
$$\left.\left.-s\left|\frac{\Gamma(\alpha+1)}{(\log x - \log a)^\alpha}J^\alpha_{a+}f(x)\right|^{s-1}\left(f(y) - \frac{\Gamma(\alpha+1)}{(\log x - \log a)^\alpha}J^\alpha_{a+}f(x)\right)\right]\frac{dy}{y}\frac{dx}{x}\right|,$$

and $\overline{H}_3(s)$ is defined by (3.2.6).

Now, we give the following result.

Theorem 4.2.14. *Let $0 < p \leq q < \infty$, $s \geq 1$, $\alpha > 1 - \frac{p}{q}$, $f \geq 0$, $f(y) > \frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)} I^\alpha_{a+;\sigma;\eta}f(x)$ for $y \in I(I \subset (a,b))$ and ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function. Then the following inequality holds true:*

$$0 \leq \pi_4(s) \leq \overline{H}_4(s) - B_4(s) \leq \overline{H}_4(s),$$

where

$$\pi_4(s) = \frac{\alpha^{\frac{q}{p}}\sigma^{\frac{p}{q}-1}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b y^{\sigma-1}{}_2F_1(y)(b^\sigma - y^\sigma)^{\alpha-1+\frac{p}{q}}f^s(y)dy\right)^{\frac{q}{p}}$$
$$-(\Gamma(\alpha+1))^{\frac{sq}{p}}\int_a^b x^{\frac{\sigma\alpha s q}{p}+\sigma-1}\left((x^\sigma - a^\sigma)^\alpha {}_2F_1(x)\right)^{\frac{q}{p}(1-s)}\left(I^\alpha_{a+;\sigma;\eta}f(x)\right)^{\frac{sq}{p}}dx - B_4(s),$$

$$B_4(s) = \frac{\alpha\sigma q(\Gamma(\alpha+1))^{s(\frac{q}{p}-1)}}{p}\left|\int_a^b\int_a^x sgn\left(f(y) - \frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)\right)\right.$$
$$\times x^{\sigma\alpha s(\frac{q}{p}-1)-\sigma\eta+\sigma-1}\left((x^\sigma - a^\sigma)^\alpha {}_2F_1(x)\right)^{(\frac{q}{p}-1)(1-s)}\left(I^\alpha_{a+;\sigma;\eta}f(x)\right)^{s(\frac{q}{p}-1)}$$
$$\times\frac{y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha}}\left[f^s(y) - \left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)\right)^s\right.$$
$$\left.\left.-s\left|\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)\right|^{s-1}\left(f(y) - \frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)}I^\alpha_{a+;\sigma;\eta}f(x)\right)\right]dy\,dx\right|,$$

and $\overline{H}_4(s)$ is defined by (3.2.7).

Proof. Similar to Theorem 3.2.2. ∎

4.2.3 G. H. Hardy-type inequalities for fractional derivatives involving monotone convex function

Here we give the improvements for different fractional derivatives.

Theorem 4.2.15. *Let $0 < p \leq q < \infty$, $s \geq 1$, $\nu - \gamma > 1 - \frac{p}{q}$, $D_a^\nu f(y) > \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x)$ for $y \in I (I \subset (a,b))$ and let the assumptions in Lemma 1.2.4 be satisfied. Then for non-negative functions $D_a^\nu f$ and $D_a^\gamma f$ the following inequality holds true:*

$$0 \leq \pi_5(s) \leq \overline{H}_5(s) - B_5(s) \leq \overline{H}_5(s),$$

where

$$\pi_5(s) = \frac{(\nu-\gamma)^{\frac{q}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1} \left(\int_a^b (b-y)^{\nu-\gamma-1+\frac{p}{q}} (D_a^\nu f(y))^s dy \right)^{\frac{q}{p}}$$
$$- (\Gamma(\nu-\gamma+1))^{\frac{sq}{p}} \int_a^b (x-a)^{\frac{(\nu-\gamma)q(1-s)}{p}} (D_a^\gamma f(x))^{\frac{sq}{p}} dx - B_5(s),$$

$$B_5(s) = \frac{q(\nu-\gamma)(\Gamma(\nu-\gamma+1))^{s(\frac{q}{p}-1)}}{p} \Bigg| \int_a^b \int_a^x sgn\left(D_a^\nu f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x) \right)$$
$$\times (x-a)^{\frac{(\nu-\gamma)(q-p)(1-s)}{p}} (D_a^\gamma f(x))^{s(\frac{q}{p}-1)} (x-y)^{\nu-\gamma-1}$$
$$\times \left[(D_a^\nu f(y))^s - \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^\gamma} D_a^\gamma f(x) \right)^s \right.$$
$$\left. -s \left| \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x) \right|^{s-1} \left(D_a^\nu f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x) \right) \right] dy\, dx \Bigg|,$$

and $\overline{H}_5(s)$ is defined by (3.2.8).

Proof. Similar to Theorem 3.2.6. ∎

Theorem 4.2.16. *Let $0 < p \leq q < \infty$, $s \geq 1$, $\nu - \gamma > 1 - \frac{p}{q}$, $D_{*a}^\nu f(y) > \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_{*a}^\gamma f(x)$ for $y \in I \subset (a,b)$ and let the assumptions in Lemma 1.2.6 be satisfied. Then for non-negative functions $D_{*a}^\nu f$ and $D_{*a}^\gamma f$ the following inequality holds true:*

$$\pi_6(s) \leq \overline{H}_6(s) - B_6(s) \leq \overline{H}_6(s),$$

where

$$\pi_6(s) = \frac{(\nu-\gamma)^{\frac{q}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b (b-y)^{\nu-\gamma-1+\frac{p}{q}}(D_{*a}^{\nu}f(y))^s dy\right)^{\frac{q}{p}}$$
$$-\left(\Gamma(\nu-\gamma+1)\right)^{\frac{sq}{p}}\int_a^b (x-a)^{\frac{(\nu-\gamma)q(1-s)}{p}}\left(D_{*a}^{\gamma}f(x)\right)^{\frac{sq}{p}}dx - B_6(s),$$

$$B_6(s) = \frac{q(\nu-\gamma)(\Gamma(\nu-\gamma+1))^{s(\frac{q}{p}-1)}}{p}\Bigg|\int_a^b\int_a^x sgn\left(D_{*a}^{\nu}f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_{*a}^{\gamma}f(x)\right)$$
$$\times (x-a)^{\frac{(\nu-\gamma)(q-p)(1-s)}{p}}\left(D_{*a}^{\gamma}f(x)\right)^{s(\frac{q}{p}-1)}(x-y)^{\nu-\gamma-1}$$
$$\times\Bigg[(D_{*a}^{\nu}f(y))^s - \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\gamma}}D_{*a}^{\gamma}f(x)\right)^s$$
$$-s\left|\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_{*a}^{\gamma}f(x)\right|^{s-1}\left(D_{*a}^{\nu}f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_{*a}^{\gamma}f(x)\right)\Bigg]dy\,dx\Bigg|,$$

and $\overline{H}_6(s)$ *is defined by* (3.2.9).

Proof. Similar to Theorem 3.2.6. ∎

Remark 4.2.2. For the case $p = q$ we can get the similar improvements of the inequality given in (4.2.4) for different fractional integrals and fractional derivative.

Remark 4.2.3. In Chapter 2, Chapter 3 and Chapter 4, we discussed only the results for left-sided fractional integrals. Similarly we can give the results for right-sided fractional integrals but here we omit the details.

Chapter 5

Generalization of an inequality for integral transforms with some results related to exponential convexity

This chapter is devoted to a generalization of an inequality introduced by D. S. Mitrinović and J. Pečarić in 1988. Mean value theorems of Cauchy type and the positive semi-definiteness of the matrices generated by the difference of the inequality which implies the exponential convexity and logarithmic convexity is proved also, new means of Cauchy type and their monotonicity is proved. Furthermore, n-exponential convexity of the linear functionals obtained by taking the positive difference of Hardy-type inequalities is discussed and some related examples are given. The results given in this chapter can be seen in [37] and [47].

5.1 Generalization of an inequality for integral transforms with kernel and related results

5.1.1 Introduction

Let $k(x, t)$ be a non-negative kernel. Consider a function $u : [a, b] \to \mathbb{R}$ where $u \in U(v, k)$ and the representation of u is

$$u(x) = \int_a^b k(x, t)v(t)dt$$

for any continuous function v on $[a,b]$. Throughout the chapter, it is assumed that all integrals under consideration exist and that they are finite.

The following theorem is given in [56] (see also [65, p. 235]).

Theorem 5.1.1. *Let* $u_i \in U(v,k)$ $(i=1,2)$ *and* $r(t) \geq 0$ *for all* $t \in [a,b]$. *Also let* $\Phi : \mathbb{R}_+ \to \mathbb{R}$ *be a function such that* Φ *be convex and increasing for* $x > 0$. *Then*

$$\int_a^b r(x)\Phi\left(\left|\frac{u_1(x)}{u_2(x)}\right|\right) dx \leq \int_a^b s(x)\Phi\left(\left|\frac{v_1(x)}{v_2(x)}\right|\right) dx,$$

where

$$s(x) = v_2(x)\int_a^b \frac{r(t)k(t,x)}{u_2(t)}dt, \quad u_2(t) \neq 0.$$

5.1.2 Main results

Our main result is given in the following theorem.

Theorem 5.1.2. *Let* $u_i \in U(v,k)$ $(i=1,2)$ *and* $r(x) \geq 0$ *for all* $x \in [a,b]$. *Also let* $I \subseteq \mathbb{R}$ *be an interval,* $\Phi : I \to \mathbb{R}$ *be convex and* $u_1(x)/u_2(x), v_1(x)/v_2(x) \in I$. *Then*

$$\int_a^b r(x)\Phi\left(\frac{u_1(x)}{u_2(x)}\right) dx \leq \int_a^b q(x)\Phi\left(\frac{v_1(x)}{v_2(x)}\right) dx, \tag{5.1.1}$$

where

$$q(x) = v_2(x)\int_a^b \frac{r(t)k(t,x)}{u_2(t)}dt, \qquad u_2(t) \neq 0. \tag{5.1.2}$$

Proof. Since $u_1 = \int_a^b k(x,t)v_1(t)dt$ and $v_2(t) > 0$, we have

$$\begin{aligned}\int_a^b r(x)\Phi\left(\frac{u_1(x)}{u_2(x)}\right) dx &= \int_a^b r(x)\Phi\left(\frac{1}{u_2(x)}\int_a^b k(x,t)v_1(t)dt\right) dx \\ &= \int_a^b r(x)\Phi\left(\int_a^b \frac{k(x,t)v_2(t)}{u_2(x)}\,\frac{v_1(t)}{v_2(t)}dt\right) dx.\end{aligned}$$

By Jensen's inequality and Fubini's theorem, we get

$$\begin{aligned}
\int_a^b r(x)\Phi\left(\frac{u_1(x)}{u_2(x)}\right)dx &\leq \int_a^b r(x)\left(\int_a^b \frac{k(x,t)v_2(t)}{u_2(x)}\ \Phi\left(\frac{v_1(t)}{v_2(t)}\right)dt\right)dx \\
&= \int_a^b \Phi\left(\frac{v_1(t)}{v_2(t)}\right)v_2(t)\left(\int_a^b \frac{r(x)k(x,t)}{u_2(x)}dx\right)dt \\
&= \int_a^b q(t)\Phi\left(\frac{v_1(t)}{v_2(t)}\right)dt.
\end{aligned}$$

This complete the proof. ■

Remark 5.1.1. If Φ is strictly convex on I and $\frac{v_1(x)}{v_2(x)}$ is non-constant, then the inequality in (5.1.1) is strict.

Remark 5.1.2. Let us note that the Theorem 5.1.1 follows from Theorem 5.1.2. Indeed, let the conditions of Theorem 5.1.1 be satisfied, and let $\tilde{u}_i \in U(|v|,k)$, i.e.

$$\tilde{u}_1(x) = \int_a^b k(x,t)|v_1(t)|\,dt.$$

So, by Theorem 5.1.2, we have

$$\begin{aligned}
\int_a^b q(x)\Phi\left(\left|\frac{v_1(x)}{v_2(x)}\right|\right)dx &= \int_a^b q(x)\Phi\left(\frac{|v_1(x)|}{v_2(x)}\right)dx, \\
&\geq \int_a^b r(x)\,\Phi\left(\frac{\tilde{u}_1(x)}{u_2(x)}\right)dx. \qquad (5.1.3)
\end{aligned}$$

On the other hand, Φ is increasing function, we have

$$\begin{aligned}
\Phi\left(\frac{\tilde{u}_1(x)}{u_2(x)}\right) &= \Phi\left(\frac{1}{u_2(x)}\int_a^b k(x,t)\,|v_1(t)|\,dt\right) \\
&\geq \Phi\left(\frac{1}{u_2(x)}\left|\int_a^b k(x,t)v_1(t)dt\right|\right)
\end{aligned}$$

$$= \Phi\left(\frac{|u_1(x)|}{u_2(x)}\right) = \Phi\left(\left|\frac{u_1(x)}{u_2(x)}\right|\right). \tag{5.1.4}$$

From (5.1.3) and (5.1.4), we get (5.1.1).

Here, we will give the corollaries by taking the kernels of Riemann-Liouville fractional integral, L_∞ fractional derivative and Caputo fractional derivative respectively.

Corollary 5.1.3. *Let $u_i \in C[a,b]$ $(i = 1,2)$ and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $I \subseteq \mathbb{R}$ be an interval, $\Phi : I \to \mathbb{R}$ be convex, $u_1(x)/u_2(x), I^\alpha_{a^+}u_1(x)/I^\alpha_{a^+}u_2(x) \in I$ and $u_1(x), u_2(x)$ have Riemann-Liouville fractional integral of order $\alpha > 0$. Then*

$$\int_a^b r(x)\Phi\left(\frac{I^\alpha_{a^+}u_1(x)}{I^\alpha_{a^+}u_2(x)}\right)dx \leq \int_a^b \Phi\left(\frac{u_1(t)}{u_2(t)}\right)Q_I(t)dt,$$

where

$$Q_I(t) = \frac{u_2(t)}{\Gamma(\alpha)}\int_t^b \frac{r(x)(x-t)^{\alpha-1}}{I^\alpha_{a^+}u_2(x)}dx, \quad I^\alpha_{a^+}u_2(x) \neq 0. \tag{5.1.5}$$

Corollary 5.1.4. *Let $u_i \in AC^n[a,b]$ $(i = 1,2)$ and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $I \subseteq \mathbb{R}$ be an interval, $\Phi : I \to \mathbb{R}$ be convex, $u_1^{(n)}(t)/u_2^{(n)}(t), D^\alpha_{*a}u_1(x)/D^\alpha_{*a}u_2(x) \in I$ and $u_1(x), u_2(x)$ have Caputo fractional derivative of order $\alpha > 0$. Then*

$$\int_a^b r(x)\Phi\left(\frac{D^\alpha_{*a}u_1(x)}{D^\alpha_{*a}u_2(x)}\right)dx \leq \int_a^b \Phi\left(\frac{u_1^{(n)}(t)}{u_2^{(n)}(t)}\right)Q_D(t)dt,$$

where

$$Q_D(t) = \frac{u_2^{(n)}(t)}{\Gamma(n-\alpha)}\int_t^b \frac{r(x)(x-t)^{n-\alpha-1}}{D^\alpha_{*a}u_2(x)}dx, \quad D^\alpha_{*a}u_2(x) \neq 0. \tag{5.1.6}$$

Corollary 5.1.5. *Let $\beta > \alpha \geq 0$, $u_i \in L_1(a,b)$ $(i = 1,2)$ has an L_∞ fractional derivative $D^\beta_a u_i$ in $[a,b]$ and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $D^{\beta-k}_a u_i(a) = 0$ for $k = 1,\ldots,[\beta]+1$ $(i = 1,2)$, $\Phi : I \to \mathbb{R}$ be convex and $D^\alpha_a u_1(x)/D^\alpha_a u_2(x)$, $D^\beta_a u_1(x)/D^\beta_a u_2(x) \in I$. Then*

$$\int_a^b r(x)\Phi\left(\frac{D^\alpha_a u_1(x)}{D^\alpha_a u_2(x)}\right)dx \leq \int_a^b \Phi\left(\frac{D^\beta_a u_1(t)}{D^\beta_a u_2(t)}\right)Q_L(t)dt,$$

where

$$Q_L(t) = \frac{D^\beta_a u_2(t)}{\Gamma(\beta-\alpha)}\int_t^b \frac{r(x)(x-t)^{\beta-\alpha-1}}{D^\alpha_a u_2(x)}dx, \quad D^\alpha_a u_2(x) \neq 0. \tag{5.1.7}$$

Lemma 5.1.6. *Let $f \in C^2(I)$, I be a compact interval, such that*

$$m \leq f''(x) \leq M, \quad \text{for all} \quad x \in I.$$

Consider two functions Φ_1 and Φ_2 defined as:

$$\Phi_1(x) = \frac{Mx^2}{2} - f(x),$$

and

$$\Phi_2(x) = f(x) - \frac{mx^2}{2}.$$

Then Φ_1 and Φ_2 are convex on I.

Theorem 5.1.7. *Let $f \in C^2(I)$, I be a compact interval, $u_i \in U(v,k)$ $(i = 1,2)$, and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $u_1(x)/u_2(x), v_1(x)/v_2(x) \in I$, $v_1(x)/v_2(x)$ be non-constant and $q(x)$ be given in* (5.1.2)*. Then there exists $\xi \in I$ such that*

$$\int_a^b \left(q(x) f\left(\frac{v_1(x)}{v_2(x)}\right) - r(x) f\left(\frac{u_1(x)}{u_2(x)}\right)\right) dx$$
$$= \frac{f''(\xi)}{2} \int_a^b \left(q(x) \left(\frac{v_1(x)}{v_2(x)}\right)^2 - r(x)\left(\frac{u_1(x)}{u_2(x)}\right)^2\right) dx.$$

Proof. Since $f \in C^2(I)$ and I is a compact interval therefore, suppose $m = \min f''$, $M = \max f''$. Using the Theorem 5.1.2 for the function Φ_1 defined in Lemma 5.1.6, we have

$$\int_a^b r(x) \left(\frac{M}{2}\left(\frac{u_1(x)}{u_2(x)}\right)^2 - f\left(\frac{u_1(x)}{u_2(x)}\right)\right) dx$$
$$\leq \int_a^b q(x) \left(\frac{M}{2}\left(\frac{v_1(x)}{v_2(x)}\right)^2 - f\left(\frac{v_1(x)}{v_2(x)}\right)\right) dx. \quad (5.1.8)$$

Using Remark 5.1.1, (5.1.8) can be written as

$$\frac{2\int_a^b \left(q(x) f\left(\frac{v_1(x)}{v_2(x)}\right) - r(x) f\left(\frac{u_1(x)}{u_2(x)}\right)\right) dx}{\int_a^b \left(q(x) \left(\frac{v_1(x)}{v_2(x)}\right)^2 - r(x)\left(\frac{u_1(x)}{u_2(x)}\right)^2\right) dx} \leq M. \quad (5.1.9)$$

We have the similar result for the function Φ_2 defined in Lemma 5.1.6 as follows.

$$\frac{2\int\limits_a^b \left(q(x)f\left(\frac{v_1(x)}{v_2(x)}\right) - r(x)f\left(\frac{u_1(x)}{u_2(x)}\right)\right)dx}{\int\limits_a^b \left(q(x)\left(\frac{v_1(x)}{v_2(x)}\right)^2 - r(x)\left(\frac{u_1(x)}{u_2(x)}\right)^2\right)dx} \geq m. \tag{5.1.10}$$

Combining (5.1.9) and (5.1.10), we have

$$m \leq \frac{2\int\limits_a^b \left(q(x)f\left(\frac{v_1(x)}{v_2(x)}\right) - r(x)f\left(\frac{u_1(x)}{u_2(x)}\right)\right)dx}{\int\limits_a^b \left(q(x)\left(\frac{v_1(x)}{v_2(x)}\right)^2 - r(x)\left(\frac{u_1(x)}{u_2(x)}\right)^2\right)dx} \leq M.$$

By Lemma 5.1.6, there exists $\xi \in I$ such that

$$\frac{\int\limits_a^b \left(q(x)f\left(\frac{v_1(x)}{v_2(x)}\right) - r(x)f\left(\frac{u_1(x)}{u_2(x)}\right)\right)dx}{\int\limits_a^b \left(q(x)\left(\frac{v_1(x)}{v_2(x)}\right)^2 - r(x)\left(\frac{u_1(x)}{u_2(x)}\right)^2\right)dx} = \frac{f''(\xi)}{2}.$$

This is the claim of theorem. ■

Let us note that a generalized mean value theorem for fractional derivative was given in [70]. Here we will give some related results as consequences of Theorem 5.1.7.

Corollary 5.1.8. *Let $f \in C^2(I)$, I be a compact interval, $u_i \in C[a,b]$ $(i = 1,2)$ and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $u_1(x)/u_2(x), I^{\alpha}_{a^+}u_1(x)/I^{\alpha}_{a^+}u_2(x) \in I$, $u_1(x)/u_2(x)$ be non-constant, $Q_I(t)$ be given in (5.1.5) and $u_1(x), u_2(x)$ has Riemman-Liouville fractional integral of order $\alpha > 0$. Then there exists $\xi \in I$ such that*

$$\int\limits_a^b \left(Q_I(x)f\left(\frac{u_1(x)}{u_2(x)}\right) - r(x)f\left(\frac{I^{\alpha}_{a^+}u_1(x)}{I^{\alpha}_{a^+}u_2(x)}\right)\right)dx$$
$$= \frac{f''(\xi)}{2}\int\limits_a^b \left(Q_I(x)\left(\frac{u_1(x)}{u_2(x)}\right)^2 - r(x)\left(\frac{I^{\alpha}_{a^+}u_1(x)}{I^{\alpha}_{a^+}u_2(x)}\right)^2\right)dx.$$

Corollary 5.1.9. *Let $f \in C^2(I)$, I be a compact interval, $u_i \in AC^n[a,b]$ $(i = 1,2)$ and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $u_1^{(n)}(t)/u_2^{(n)}(t)$ $D^{\alpha}_{*a}u_1(x)/D^{\alpha}_{*a}u_2(x) \in I$,*

$u_1^{(n)}(x)/u_2^{(n)}(x)$ *be non-constant,* $Q_D(t)$ *be given in* (5.1.6) *and* $u_1(x), u_2(x)$ *have Caputo derivative of order* $\alpha > 0$*. Then there exists* $\xi \in I$ *such that*

$$\int_a^b \left(Q_D(x) f\left(\frac{u_1^{(n)}(x)}{u_2^{(n)}(x)} \right) - r(x) f\left(\frac{D_{*a}^{\alpha} u_1(x)}{D_{*a}^{\alpha} u_2(x)} \right) \right) dx$$
$$= \frac{f''(\xi)}{2} \int_a^b \left(Q_D(x) \left(\frac{u_1^{(n)}(x)}{u_2^{(n)}(x)} \right)^2 - r(x) \left(\frac{D_{*a}^{\alpha} u_1(x)}{D_{*a}^{\alpha} u_2(x)} \right)^2 \right) dx.$$

Corollary 5.1.10. *Let* $\beta > \alpha \geq 0$, $f \in C^2(I)$, I *be a compact interval,* $u_i \in L_1(a,b)$ $(i = 1,2)$ *has an* L_∞ *fractional derivative and* $r(x) \geq 0$ *for all* $x \in [a,b]$*. Let* $D_a^{\beta-k} u_i(a) = 0$ *for* $k = 1, \ldots, [\beta]+1$ $(i = 1,2)$, $D_a^{\alpha} u_1(x)/D_a^{\alpha} u_2(x)$, $D_a^{\beta} u_1(x)/D_a^{\beta} u_2(x)$ $\in I$, $D_a^{\beta} u_1(x)/D_a^{\beta} u_2(x)$ *be non-constant and* $Q_L(t)$ *be given in* (5.1.7)*. Then there exists* $\xi \in I$ *such that*

$$\int_a^b \left(Q_L(x) f\left(\frac{D_a^{\beta} u_1(x)}{D_a^{\beta} u_2(x)} \right) - r(x) f\left(\frac{D_a^{\alpha} u_1(x)}{D_a^{\alpha} u_2(x)} \right) \right) dx$$
$$= \frac{f''(\xi)}{2} \int_a^b \left(Q_L(x) \left(\frac{D_a^{\beta} u_1(x)}{D_a^{\beta} u_2(x)} \right)^2 - r(x) \left(\frac{D_a^{\alpha} u_1(x)}{D_a^{\alpha} u_2(x)} \right)^2 \right) dx.$$

Theorem 5.1.11. *Let* $f, g \in C^2(I)$, I *be a compact interval,* $u_i \in U(v,k)$ $(i = 1,2)$ *and* $r(x) \geq 0$ *for all* $x \in [a,b]$*. Also let* $u_1(x)/u_2(x), v_1(x)/v_2(x) \in I$, $v_1(x)/v_2(x)$ *be non-constant and* $q(x)$ *be given in* (5.1.2)*. Then there exists* $\xi \in I$ *such that*

$$\frac{\int_a^b q(x) f\left(\frac{v_1(x)}{v_2(x)} \right) dx - \int_a^b r(x) f\left(\frac{u_1(x)}{u_2(x)} \right) dx}{\int_a^b q(x) g\left(\frac{v_1(x)}{v_2(x)} \right) dx - \int_a^b r(x) g\left(\frac{u_1(x)}{u_2(x)} \right) dx} = \frac{f''(\xi)}{g''(\xi)}.$$

It is provided that denominators are not equal to zero.

Proof. Let us take a function $h \in C^2(I)$ defined as

$$h(x) = c_1 f(x) - c_2 g(x),$$

where

$$c_1 = \int_a^b q(x) g\left(\frac{v_1(x)}{v_2(x)} \right) dx - \int_a^b r(x) g\left(\frac{u_1(x)}{u_2(x)} \right) dx,$$

and

$$c_2 = \int\limits_a^b q(x) f\left(\frac{v_1(x)}{v_2(x)}\right) dx - \int\limits_a^b r(x) f\left(\frac{u_1(x)}{u_2(x)}\right) dx.$$

By Theorem 5.1.7 with $f = h$, we have

$$0 = \left(\frac{c_1}{2} f''(\xi) - \frac{c_2}{2} g''(\xi)\right) \left(\int\limits_a^b q(x) \left(\frac{v_1(x)}{v_2(x)}\right)^2 dx - \int\limits_a^b r(x) \left(\frac{u_1(x)}{u_2(x)}\right)^2 dx\right).$$

Using Remark 5.1.1, we get

$$\frac{c_2}{c_1} = \frac{f''(\xi)}{g''(\xi)}.$$

This is the claim of the theorem. ■

Let us note that a generalized Cauchy mean valued theorem for fractional derivative was given in [67]. Here we will give some related results as consequences of Theorem 5.1.11.

Corollary 5.1.12. *Let $f, g \in C^2(I)$, I be a compact interval, $u_i \in C[a,b]$ $(i = 1,2)$ and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $u_1(x)/u_2(x), I^{\alpha}_{a^+}u_1(x)/I^{\alpha}_{a^+}u_2(x) \in I$, $u_1(x)/u_2(x)$ be non-constant, $Q_I(t)$ be given in* (5.1.5) *and $u_1(x), u_2(x)$ have Riemann-Liouville fractional derivative of order $\alpha > 0$. Then there exists $\xi \in I$ such that*

$$\frac{\int\limits_a^b Q_I(x) f\left(\frac{u_1(x)}{u_2(x)}\right) dx - \int\limits_a^b r(x) f\left(\frac{I^{\alpha}_{a^+}u_1(x)}{I^{\alpha}_{a^+}u_2(x)}\right) dx}{\int\limits_a^b Q_I(x) g\left(\frac{u_1(x)}{u_2(x)}\right) dx - \int\limits_a^b r(x) g\left(\frac{I^{\alpha}_{a^+}u_1(x)}{I^{\alpha}_{a^+}u_2(x)}\right) dx} = \frac{f''(\xi)}{g''(\xi)}.$$

It is provided that denominators are not equal to zero.

Corollary 5.1.13. *Let $f, g \in C^2(I)$, I be a compact interval, $u_i \in AC^n[a,b]$ $(i = 1,2)$ and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $u_1^{(n)}(t)/u_2^{(n)}(t), D^{\alpha}_{*a}u_1(x)/D^{\alpha}_{*a}u_2(x) \in I$, $u_1^{(n)}(x)/u_2^{(n)}(x)$ be non-constant, $Q_D(t)$ be given in* (5.1.6) *and $u_1(x), u_2(x)$ have Caputo fractional derivative of order $\alpha > 0$. Then there exists $\xi \in I$ such that*

$$\frac{\int\limits_a^b Q_D(x) f\left(\frac{u_1^{(n)}(x)}{u_2^{(n)}(x)}\right) dx - \int\limits_a^b r(x) f\left(\frac{D^{\alpha}_{*a}u_1(x)}{D^{\alpha}_{*a}u_2(x)}\right) dx}{\int\limits_a^b Q_D(x) g\left(\frac{u_1^{(n)}(x)}{u_2^{(n)}(x)}\right) dx - \int\limits_a^b r(x) g\left(\frac{D^{\alpha}_{*a}u_1(x)}{D^{\alpha}_{*a}u_2(x)}\right) dx} = \frac{f''(\xi)}{g''(\xi)}.$$

It is provided that denominators are not equal to zero.

Corollary 5.1.14. *Let $\beta > \alpha \geq 0$, $f, g \in C^2(I)$, I be a compact interval, $u_i \in L_1(a,b)$ $(i = 1,2)$ has an L_∞ fractional derivative $D_a^\beta u_i$ in $[a,b]$ and $r(x) \geq 0$ for all $x \in [a,b]$. Also let $D_a^{\beta-k} u_i(a) = 0$ for $k = 1, \ldots, [\beta]+1$ $(i = 1,2)$, $D_a^\alpha u_1(x)/D_a^\alpha u_2(x)$, $D_a^\beta u_1(x)/D_a^\beta u_2(x) \in I$, $D_a^\beta u_1(x)/D_a^\beta u_2(x)$ be non-constant and $Q_L(t)$ be given in (5.1.7). Then there exists $\xi \in I$ such that*

$$\frac{\int\limits_a^b Q_L(x) f\left(\frac{D_a^\beta u_1(x)}{D_a^\beta u_2(x)}\right) dx - \int\limits_a^b r(x) f\left(\frac{D_a^\alpha u_1(x)}{D_a^\alpha u_2(x)}\right) dx}{\int\limits_a^b Q_L(x) g\left(\frac{D_a^\beta u_1(x)}{D_a^\beta u_2(x)}\right) dx - \int\limits_a^b r(x) g\left(\frac{D_a^\alpha u_1(x)}{D_a^\alpha u_2(x)}\right) dx} = \frac{f''(\xi)}{g''(\xi)}.$$

It is provided that denominators are not equal to zero.

Corollary 5.1.15. *Let $I \subseteq \mathbb{R}_+$, I be a compact interval, $u_i \in U(v,k)$ $(i = 1,2)$ and $r(x) \geq 0$ for all $x \in [a,b]$. Let $u_1(x)/u_2(x), v_1(x)/v_2(x) \in I$, $v_1(x)/v_2(x)$ be non-constant and $q(x)$ be given in (5.1.2). Then for $s, t \in \mathbb{R}\backslash\{0,1\}$ and $s \neq t$, there exists $\xi \in I$ such that*

$$\xi = \left(\frac{s(s-1)}{t(t-1)} \frac{\int\limits_a^b q(x)\left(\frac{v_1(x)}{v_2(x)}\right)^t dx - \int\limits_a^b r(x)\left(\frac{u_1(x)}{u_2(x)}\right)^t dx}{\int\limits_a^b q(x)\left(\frac{v_1(x)}{v_2(x)}\right)^s dx - \int\limits_a^b r(x)\left(\frac{u_1(x)}{u_2(x)}\right)^s dx}\right)^{\frac{1}{t-s}}. \tag{5.1.11}$$

Proof. We set $f(x) = x^t$ and $g(x) = x^s$, $t \neq s$ and $s, t \neq 0, 1$. By Theorem 5.1.11, we have

$$\frac{\int\limits_a^b q(x)\left(\frac{v_1(x)}{v_2(x)}\right)^t dx - \int\limits_a^b r(x)\left(\frac{u_1(x)}{u_2(x)}\right)^t dx}{\int\limits_a^b q(x)\left(\frac{v_1(x)}{v_2(x)}\right)^s dx - \int\limits_a^b r(x)\left(\frac{u_1(x)}{u_2(x)}\right)^s dx} = \frac{t(t-1)\xi^{t-2}}{s(s-1)\xi^{s-2}}.$$

This implies

$$\xi^{t-s} = \frac{s(s-1)}{t(t-1)} \frac{\int\limits_a^b q(x)\left(\frac{v_1(x)}{v_2(x)}\right)^t dx - \int\limits_a^b r(x)\left(\frac{u_1(x)}{u_2(x)}\right)^t dx}{\int\limits_a^b q(x)\left(\frac{v_1(x)}{v_2(x)}\right)^s dx - \int\limits_a^b r(x)\left(\frac{u_1(x)}{u_2(x)}\right)^s dx}.$$

It follows (5.1.11). ■

Remark 5.1.3. Since the function $\xi \to \xi^{t-s}$ is invertible and from (5.1.11), we have

$$m \leq \left(\frac{s(s-1)}{t(t-1)} \frac{\int_a^b q(x) \left(\frac{v_1(x)}{v_2(x)}\right)^t dx - \int_a^b r(x) \left(\frac{u_1(x)}{u_2(x)}\right)^t dx}{\int_a^b q(x) \left(\frac{v_1(x)}{v_2(x)}\right)^s dx - \int_a^b r(x) \left(\frac{u_1(x)}{u_2(x)}\right)^s dx} \right)^{\frac{1}{t-s}} \leq M. \tag{5.1.12}$$

Now we can suppose that f''/g'' is invertible function, then from (5.1.11) we have

$$\xi = \left(\frac{f''}{g''}\right)^{-1} \left(\frac{\int_a^b q(x) f\left(\frac{v_1(x)}{v_2(x)}\right) dx - \int_a^b r(x) f\left(\frac{u_1(x)}{u_2(x)}\right) dx}{\int_a^b q(x) g\left(\frac{v_1(x)}{v_2(x)}\right) dx - \int_a^b r(x) g\left(\frac{u_1(x)}{u_2(x)}\right) dx} \right). \tag{5.1.13}$$

We see that on the right hand side of the (5.1.13) is mean, then for distinct $s, t \in \mathbb{R}$ it can be written as

$$M_{s,t} = \left(\frac{\wedge_t}{\wedge_s}\right)^{\frac{1}{t-s}} \tag{5.1.14}$$

as mean in broader sense. Moreover we can extend these means, so in limiting cases for $s, t \neq 0, 1$,

$$\begin{aligned} &\lim_{t \to s} M_{s,t} = M_{s,s} \\ &= \exp\left(\frac{\int_a^b q(x) \left(\frac{v_1(x)}{v_2(x)}\right)^s \log\left(\frac{v_1(x)}{v_2(x)}\right) dx - \int_a^b r(x) \left(\frac{u_1(x)}{u_2(x)}\right)^s \log\left(\frac{u_1(x)}{u_2(x)}\right) dx}{\int_a^b q(x) \left(\frac{v_1(x)}{v_2(x)}\right)^s dx - \int_a^b r(x) \left(\frac{u_1(x)}{u_2(x)}\right)^s dx} - \frac{2s-1}{s(s-1)} \right), \end{aligned}$$

$$\lim_{s \to 0} M_{s,s} = M_{0,0} = \exp\left(\frac{\int_a^b q(x) \log^2\left(\frac{v_1(x)}{v_2(x)}\right) dx - \int_a^b r(x) \log^2\left(\frac{u_1(x)}{u_2(x)}\right) dx}{2\left[\int_a^b q(x) \log\left(\frac{v_1(x)}{v_2(x)}\right) dx - \int_a^b r(x) \log\left(\frac{u_1(x)}{u_2(x)}\right) dx\right]} + 1 \right),$$

$$\begin{aligned} &\lim_{s \to 1} M_{s,s} = M_{1,1} \\ &= \exp\left(\frac{\int_a^b q(x) \left(\frac{v_1(x)}{v_2(x)}\right) \log^2\left(\frac{v_1(x)}{v_2(x)}\right) dx - \int_a^b r(x) \left(\frac{u_1(x)}{u_2(x)}\right) \log^2\left(\frac{u_1(x)}{u_2(x)}\right) dx}{2\left[\int_a^b q(x) \left(\frac{v_1(x)}{v_2(x)}\right) \log\left(\frac{v_1(x)}{v_2(x)}\right) dx - \int_a^b r(x) \left(\frac{u_1(x)}{u_2(x)}\right) \log\left(\frac{u_1(x)}{u_2(x)}\right) dx\right]} - 1 \right), \end{aligned}$$

$$\lim_{t\to 0} M_{s,t} = M_{s,0} = \left(\frac{\int_a^b q(x) \left(\frac{v_1(x)}{v_2(x)}\right)^s dx - \int_a^b r(x) \left(\frac{u_1(x)}{u_2(x)}\right)^s dx}{\left[\int_a^b q(x) \log\left(\frac{v_1(x)}{v_2(x)}\right) dx - \int_a^b r(x) \log\left(\frac{u_1(x)}{u_2(x)}\right) dx\right] s(s-1)} \right)^{\frac{1}{s}},$$

$$\lim_{t\to 1} M_{s,t} = M_{s,1}$$

$$= \left(\frac{\left[\int_a^b q(x) \left(\frac{v_1(x)}{v_2(x)}\right) \log\left(\frac{v_1(x)}{v_2(x)}\right) dx - \int_a^b r(x) \left(\frac{u_1(x)}{u_2(x)}\right) \log\left(\frac{u_1(x)}{u_2(x)}\right) dx\right] s(s-1)}{\int_a^b q(x) \left(\frac{v_1(x)}{v_2(x)}\right)^s dx - \int_a^b r(x) \left(\frac{u_1(x)}{u_2(x)}\right)^s dx} \right)^{\frac{1}{1-s}}.$$

Remark 5.1.4. In the case of Riemann-Liouville fractional integral of order $\alpha > 0$, we will use the notation $\bar{M}_{s,t}$ instead of $M_{s,t}$ and we replace $v_i(x)$ to $u_i(x)$, $u_i(x)$ to $I^{\alpha}_{a+}u_i(x)$ and $q(x)$ to $Q_I(x)$.

Remark 5.1.5. In the case of Caputo fractional derivative of order $\alpha > 0$, we will use the notation $\tilde{M}_{s,t}$ instead of $M_{s,t}$ and we replace $v_i(x)$ to $u_i^{(n)}(x)$, $u_i(x)$ to $D^{\alpha}_{*a}u_i(x)$ and $q(x)$ to $Q_D(x)$.

Remark 5.1.6. In the case of L_∞ fractional derivative, we will use the notation $\hat{M}_{s,t}$ instead of $M_{s,t}$ and we replace $v_i(x)$ to $D^{\beta}_a u_i(x)$, $u_i(x)$ to $D^{\alpha}_a u_i(x)$ and $q(x)$ to $Q_L(x)$.

5.1.3 Exponential Convexity

Theorem 5.1.16. *Let $u_i \in U(v,k)$ $(i=1,2)$, $u_i(x), v_i(x) > 0$, $(i=1,2)$, $r(x) \geq 0$ for all $x \in [a,b]$, $q(x)$ be given in* (5.1.2) *and*

$$\wedge_t = \int_a^b q(x)\, \varphi_t\left(\frac{v_1(x)}{v_2(x)}\right) dx - \int_a^b r(x)\, \varphi_t\left(\frac{u_1(x)}{u_2(x)}\right) dx. \tag{5.1.15}$$

Then the following statements are valid:
(a) For $n \in N$ and $s_i \in \mathbb{R}$, $i = 1,\ldots,n$, the matrix $\left[\wedge_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^n$ is positive semi-definite matrix. Particulary

$$det\left[\wedge_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^k \geq 0 \;\; for \;\; k = 1,...n.$$

(b) The function $s \mapsto \wedge_s$ is exponentially convex on $\mathbb{R}$.

(*c*) *The function* $s \mapsto \wedge_s$ *is* log-*convex on* $\mathbb{R}$ *and the following inequality holds for* $-\infty < r < s < t < \infty$:

$$\wedge_s^{t-r} \leq \wedge_r^{t-s} \wedge_t^{s-r} \,. \tag{5.1.16}$$

Proof. (a) Here we define a new function μ,

$$\mu(x) = \sum_{i,j=1}^{k} a_i a_j \varphi_{s_{ij}}(x)$$

for $k = 1, \ldots, n$, $a_i \in \mathbb{R}$, $s_{ij} \in \mathbb{R}$, where $s_{ij} = \frac{s_i+s_j}{2}$.

$$\mu^{''}(x) = \sum_{i,j=1}^{n} a_i a_j x^{s_{ij}-2} = \left(\sum_{i=1}^{n} a_i x^{\frac{s_i}{2}-1} \right)^2 \geq 0.$$

This shows that $\mu(x)$ is convex for $x \geq 0$. Using Theorem 5.1.2, we have

$$\sum_{i,j=1}^{k} a_i a_j \wedge_{s_{ij}} \geq 0.$$

From the above result, it shows that the matrix $\left[\wedge_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{n}$ is positive semi-definite matrix. Specially, we get

$$det \left[\wedge_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{k} \geq 0 \;\text{ for all }\; k = 1, \ldots, n.$$

(b) Since

$$\lim_{s \to 1} \wedge_s = \wedge_1 \text{ and } \lim_{s \to 0} \wedge_s = \wedge_0,$$

it follows that $\wedge_s$ is continuous for $s \in \mathbb{R}$. Then by using Definition 1.1.4, we get the exponential convexity of the function $s \mapsto \wedge_s$.

(c) Since $\wedge_s$ is continuous for $s \in \mathbb{R}$ and using Definition 1.1.3, we get $\wedge_s$ is log-convex. Now by Remark 1.1.1 with $f(t) = \log \wedge_t$ and $r, s, t \in \mathbb{R}$ such that $r < s < t$, we get

$$\log \wedge_s^{t-r} \leq \log \wedge_r^{t-s} + \log \wedge_t^{s-r},$$

which is equivalent to (5.1.16). ∎

Corollary 5.1.17. *Let $u_i \in C[a,b]$ $(i = 1, 2)$ and $r(x) \geq 0$ for all $x \in [a, b]$. Also let $u_1(x)/u_2(x)$, $I^{\alpha}_{a+}u_1(x)/I^{\alpha}_{a+}u_2(x) \in \mathbb{R}_+$, $u_1(x)$, $u_2(x)$ has Riemann-Liouville fractional integral of order $\alpha > 0$, let $Q_I(t)$ be given in* (5.1.5) *and*

$$\bar{\wedge}_t = \int_a^b Q_I(x)\,\varphi_t\left(\frac{u_1(x)}{u_2(x)}\right)dx - \int_a^b r(x)\,\varphi_t\left(\frac{I^{\alpha}_{a+}u_1(x)}{I^{\alpha}_{a+}u_2(x)}\right)dx.$$

Then the statement of Theorem 5.1.16 with $\bar{\wedge}_t$ instead of $\wedge_t$ is valid.

Corollary 5.1.18. *Let $u_i \in AC^n[a,b]$ $(i = 1, 2)$ and $r(x) \geq 0$ for all $x \in [a, b]$. Also let $u_1^{(n)}(t)/u_2^{(n)}(t)$, $D^{\alpha}_{*a}u_1(x)/D^{\alpha}_{*a}u_2(x) \in \mathbb{R}_+$, $u_1(x)$, $u_2(x)$ has Caputo fractional derivative of order $\alpha > 0$, let $Q_D(t)$ be given in* (5.1.6) *and*

$$\tilde{\wedge}_t = \int_a^b Q_D(x)\,\varphi_t\left(\frac{u_1^{(n)}(x)}{u_2^{(n)}(x)}\right)dx - \int_a^b r(x)\,\varphi_t\left(\frac{D^{\alpha}_{*a}u_1(x)}{D^{\alpha}_{*a}u_2(x)}\right)dx.$$

Then the statement of Theorem 5.1.16 with $\tilde{\wedge}_t$ instead of $\wedge_t$ is valid.

Corollary 5.1.19. *Let $\beta > \alpha \geq 0$, $u_i \in L_1\,(a, b)$ $(i = 1, 2)$ has L_∞ fractional derivative and $r(x) \geq 0$ for all $x \in [a, b]$. Also let $D^{\beta-k}_a u_i(a) = 0$ for $k = 1, \ldots, [\beta] + 1$ $(i = 1, 2)$, $D^{\alpha}_a u_1(x)/D^{\alpha}_a u_2(x)$, $D^{\beta}_a u_1(x)/D^{\beta}_a u_2(x) \in \mathbb{R}_+$, $Q_L(t)$ be given in* (5.1.7) *and*

$$\hat{\wedge}_t = \int_a^b Q_L(x)\,\varphi_t\left(\frac{D^{\beta}_a u_1(x)}{D^{\beta}_a u_2(x)}\right)dx - \int_a^b r(x)\,\varphi_t\left(\frac{D^{\alpha}_a u_1(x)}{D^{\alpha}_a u_2(x)}\right)dx.$$

Then the statement of Theorem 5.1.16 with $\hat{\wedge}_t$ instead of $\wedge_t$ is valid.

In the following theorem we prove the monotonicity of $M_{s,t}$ defined in (5.1.14).

Theorem 5.1.20. *Let the assumption of Theorem 5.1.16 be satisfied, also let $\wedge_t$ be defined in* (5.1.15) *and $t, s, m, n \in \mathbb{R}$ such that $s \leq m$, $t \leq n$. Then following inequality is true:*

$$M_{s,t} \leq M_{m,n}. \tag{5.1.17}$$

Proof. For a convex function φ, using the Remark 1.1.1, we get the following inequality

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(y_2) - \varphi(y_1)}{y_2 - y_1}$$

with $x_1 \le y_1,\ x_2 \le y_2,\ x_1 \ne x_2,\ y_1 \ne y_2$. Since by Theorem 5.1.16, we get that $\wedge_t$ is log-convex. We set $\varphi(t) = \log \wedge_t,\ x_1 = s,\ x_2 = t,\ y_1 = m,\ y_2 = n,\ s \ne t,\ m \ne n$. Therefore, we get

$$\frac{\log \wedge_t - \log \wedge_s}{t-s} \le \frac{\log \wedge_n - \log \wedge_m}{n-m},$$

$$\log \left(\frac{\wedge_t}{\wedge_s}\right)^{\frac{1}{t-s}} \le \log \left(\frac{\wedge_n}{\wedge_m}\right)^{\frac{1}{n-m}}, \tag{5.1.18}$$

which is equivalent to (5.1.17) for $s \ne t,\ m \ne n$.

For $s = t$, $m = n$, we get that the required result by taking limit in (5.1.18). ■

Corollary 5.1.21. *Let $u_i \in C[a,b]$ $(i = 1,2)$ and let the assumption of Corollary 5.1.17 be satisfied, also let $\bar{\wedge}_t$ be defined by* (5.1.17). *For $t, s, m, n \in \mathbb{R}$ such that $s \le m$, $t \le n$, then following inequality holds:*

$$\bar{M}_{s,t} \le \bar{M}_{m,n}.$$

Corollary 5.1.22. *Let $u_i \in AC^n[a,b]$ $(i = 1,2)$ and let the assumption of Corollary 5.1.18 be satisfied, also let $\tilde{\wedge}_t$ defined by* (5.1.18). *For $t, s, m, n \in \mathbb{R}$ such that $s \le m$, $t \le n$, then following inequality holds:*

$$\tilde{M}_{s,t} \le \tilde{M}_{m,n}.$$

Corollary 5.1.23. *Let $\beta > \alpha \ge 0$, $u_i \in L_1(a,b)$ $(i = 1,2)$ and let the assumption of Corollary 5.1.19 be satisfied, also let $\hat{\wedge}_t$ be defined by* (5.1.19). *For $t, s, m, n \in \mathbb{R}$ such that $s \le m$, $t \le n$, then following inequality holds:*

$$\hat{M}_{s,t} \le \hat{M}_{m,n}.$$

5.2 n-Exponential convexity of Hardy-type functionals

In this part, we discuss and produce the exponential convexity and n-exponential convexity of the linear functionals obtained by taking the positive difference of Hardy-type inequalities.

Under assumptions of the Theorem 2.3.1 and Theorem 5.1.2, we define linear functionals by taking the positive differences of left hand-side and right hand-side of the inequalities stated in (2.3.2) and (5.1.1) respectively:

$$\Delta_1(\Phi) = \int\limits_{\Omega_2} v(y)\Phi\left(f(y)\right) d\mu_2(y) - \int\limits_{\Omega_1} u(x)\Phi\left(A_k f(x)\right) d\mu_1(x). \tag{5.2.1}$$

$$\Delta_2(\Phi) = \int_a^b q(x)\Phi\left(\frac{v_1(x)}{v_2(x)}\right)dx - \int_a^b r(x)\Phi\left(\frac{u_1(x)}{u_2(x)}\right)dx. \tag{5.2.2}$$

The discrete results about Hardy-type inequalities are given in [24, Theorem 2.1]. Here, we consider a special case of [24, Theorem 2.1], that is for convex functions this result holds.

Theorem 5.2.1. *Let $M, N \in \mathbb{N}$ and let non-negative real numbers u_m, v_n, k_{mn}, where $m \in \mathbb{N}_M$, $n \in \mathbb{N}_N$, be such that*

$$K_m = \sum_{n=1}^{N} k_{mn} > 0, \; m \in \mathbb{N}_M,$$

and

$$v_n = \sum_{m=1}^{M} u_m \frac{k_{mn}}{K_m}, \; n \in \mathbb{N}_N.$$

If Φ is a convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\sum_{n=1}^{N} v_n \Phi(a_n) - \sum_{m=1}^{M} u_m \Phi(A_m) \geq 0 \tag{5.2.3}$$

holds for all real numbers $a_n \in I$, for $n \in \mathbb{N}_N$, where

$$A_m = \frac{1}{K_m} \sum_{n=1}^{N} k_{mn} a_n.$$

We define linear functional from (5.2.3) as:

$$\Delta_3(\Phi) = \sum_{n=1}^{N} v_n \Phi(a_n) - \sum_{m=1}^{M} u_m \Phi(A_m). \tag{5.2.4}$$

The main purpose of this part is to discuss the n-exponential convexity of the three non-negative Hardy-type linear functionals obtained by taking the positive difference of Hardy-type inequalities stated in [52] (see also [28]), [37], [24] and defined by (5.2.1), (5.2.2) and (5.2.4).

5.2.1 Main results

First we give some necessary details about the divided differences.

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. Then for distinct points $z_i \in I$, $(i = 0, 1, 2)$, the divided differences of first and second order are defined by:

$$[z_i, z_{i+1}; f] = \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i}, \quad (i = 0, 1),$$

$$[z_0, z_1, z_2; f] = \frac{[z_1, z_2; f] - [z_0, z_1; f]}{z_2 - z_0}. \tag{5.2.5}$$

One can observe that if for all $z_0, z_1 \in I$, $[z_0, z_1, f] \geq 0$, then f is increasing on I and if for all $z_0, z_1, z_2 \in I$, $[z_0, z_1, z_2; f] \geq 0$, then f is convex on I.

Now we will produce n-exponentially convex and exponentially convex functions applying functionals Δ_i, $(i = 1, 2, 3)$ on a given family with the same property. In the sequel J and I will be intervals in $\mathbb{R}$. The proofs of our results are similar to the proofs in [66] but for completeness of results and for the reader's convenience we will also give them.

Theorem 5.2.2. *Let $\Gamma = \{\Phi_p : p \in J\}$ be a family of functions defined on I such that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n-exponentially convex in the Jensen sense on J for every three distinct points z_0, z_1, $z_2 \in I$. Let Δ_i $(i = 1, 2, 3)$ be linear functionals defined by* (5.2.1), (5.2.2) *and* (5.2.4). *Then the functions $p \mapsto \Delta_i(\Phi_p)$ $(i = 1, 2, 3)$ are n-exponentially convex in the Jensen sense on J. If the functions $p \mapsto \Delta_i(\Phi_p)$ are continuous on J, then it is n-exponentially convex on J.*

Proof. For $a_i \in \mathbb{R}$, $i = 1, \ldots, n$ and $p_i \in J$, $i = 1, \ldots, n$, we define the function

$$\Upsilon(z) = \sum_{i,j=1}^{n} a_i a_j \Phi_{\frac{p_i + p_j}{2}}(z).$$

Using the assumption that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n-exponentially convex in the Jensen sense, we have

$$[z_0, z_1, z_2; \Upsilon] = \sum_{i,j=1}^{n} a_i a_j [z_0, z_1, z_2; \Phi_{\frac{p_i + p_j}{2}}] \geq 0,$$

which shows that Υ is convex on I and therefore we have $\Delta_i(\Upsilon) \geq 0$ for $(i = 1, 2, 3)$. Hence

$$\sum_{i,j=1}^{n} a_i a_j \Delta_i(\Phi_{\frac{p_i + p_j}{2}}) \geq 0.$$

We conclude that the function $p \mapsto \Delta_i(\Phi_p)$ for $(i = 1, 2, 3)$ are n-exponentially convex in Jensen sense on J.

If the functions $p \mapsto \Delta_i(\Phi_p)$ for $(i = 1, 2, 3)$ are also continuous on J, then $p \mapsto \Delta_i(\Phi_p)$ is n-exponentially convex by definition. ■

As a direct consequence of the above theorem, we can give the following corollary.

Corollary 5.2.3. *Let $\Gamma = \{\Phi_p : I \to \mathbb{R}, p \in J \subseteq \mathbb{R}\}$ be a family of functions such that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is exponentially convex in the Jensen sense on J for every three distinct points z_0, z_1, $z_2 \in I$. Let Δ_i $(i = 1, 2, 3)$ be linear functionals defined by* (5.2.1)*,* (5.2.2) *and* (5.2.4)*. Then $p \mapsto \Delta_i(\Phi_p)$ are exponentially convex in the Jensen sense on J. If the functions $p \mapsto \Delta_i(\Phi_p)$ are continuous on J, then these are exponentially convex on J.*

Using analogous arguing as in the proof of [66, Corollary 3.11], we have the following corollary.

Corollary 5.2.4. *Let $\Gamma = \{\Phi_p : I \to \mathbb{R}, p \in J \subseteq \mathbb{R}\}$ be a family, such that the function $p \to [z_0, z_1, z_2; \Phi_p]$ is 2-exponentially convex in the Jensen sense on J for every three distinct points z_0, z_1, $z_2 \in I$. Let Δ_i $(i = 1, 2, 3)$ be a linear functionals defined in* (5.2.1)*,* (5.2.2) *and* (5.2.4)*. Then the following statements hold:*

(i) If the function $p \mapsto \Delta_i(\Phi_p)$ is continuous on J, then it is 2-exponentially convex function on J, thus log*-convex on J and for $p, q, r \in I$ such that $p < q < r$, we have*

$$\Delta_i(\Phi_q)^{r-p} \leq \Delta_i(\Phi_p)^{r-q} \Delta_i(\Phi_r)^{q-p}, \quad (i = 1, 2, 3).$$

(ii) If the function $p \mapsto \Delta_i(\Phi_p)$ is strictly positive and differentiable on J, then for every $p, q, m, n \in J$ such that $p \leq m$, $q \leq n$, we have

$$\mathcal{B}_{p,q}(f, \Delta_i; \Gamma) \leq \mathcal{B}_{m,n}(f, \Delta_i; \Gamma), \quad (i = 1, 2, 3), \tag{5.2.6}$$

where

$$\mathcal{B}_{p,q}(f, \Delta_i; \Gamma) = \begin{cases} \left(\frac{\Delta_i(\Phi_p)}{\Delta_i(\Phi_q)} \right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}(\Delta_i(\Phi_p))}{\Delta_i(\Phi_p)} \right), & p = q, \end{cases} \tag{5.2.7}$$

for $\Phi_p, \Phi_q \in \Gamma$.

Proof. (i) This can be obtained as a direct consequence of Theorem 5.2.2 and Remark 1.1.3.
(ii) Since by (i) the function $p \mapsto \Delta_i(\Phi_p)$ for $(i = 1, 2, 3)$ is log-convex on J, that is the function $p \mapsto \log \Delta_i(\Phi_p)$ for $(i = 1, 2, 3)$ is convex on J. Applying Remark 1.1.1, we obtain

$$\frac{\log \Delta_i(\Phi_p) - \log \Delta_i(\Phi_q)}{p-q} \leq \frac{\log \Delta_i(\Phi_m) - \log \Delta_i(\Phi_n)}{m-n} \tag{5.2.8}$$

for $p \leq m,\ q \leq n,\ p \neq q,\ m \neq n$ and we conclude that

$$\mathcal{B}_{p,q}(f, \Delta_i; \Gamma) \leq \mathcal{B}_{m,n}(f, \Delta_i; \Gamma), \quad (i = 1, 2, 3).$$

Cases for $p = q,\ m = n$ follows from (5.2.8) as limiting case. ■

Remark 5.2.1. Note that the results of Theorem 5.2.2, Corollary 5.2.3 and Corollary 5.2.4 still hold when two of the points $z_0,\ z_1,\ z_2 \in I$ coincides for a family of differentiable functions Φ_p such that $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense), further, they still hold when all three point coincide for a family of twice differentiable functions with the same property. The proofs are obtained using (5.2.1) and some facts about the exponential convexity.

5.2.2 Examples

Example 5.2.1. *Consider a family of functions*

$$\Gamma_1 = \{g_p : (0, \infty) \to (0, \infty) : p \in (0, \infty)\},$$

defined by

$$g_p(t) = \frac{e^{-t\sqrt{p}}}{p}.$$

Since $p \mapsto \frac{d^2 g_p(t)}{dt^2} = e^{-t\sqrt{p}}$ *is the Laplace transform of a non-negative function, it is exponentially convex (see [49], also see [71]). Clearly* g_p *are convex functions for each* $p > 0$*. It is obvious that* $\Delta_i(g_p)$ *for* $(i = 1, 2, 3)$ *are continuous. It is easy to prove that the function* $p \mapsto [z_0, z_1, z_2; g_p]$ *is also exponentially convex for arbitrary points* $z_0, z_1, z_2 \in I$*. For this family of functions,* $\mathcal{B}_{p,q}(f, \Delta_i; \Gamma_1)$ *becomes*

$$\mathcal{B}_{p,q}(f, \Delta_i(g_p); \Gamma_1) = \begin{cases} \left(\frac{\Delta_i(g_p)}{\Delta_i(g_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(-\frac{\Delta_i(id \cdot g_p)}{2\sqrt{p}\Delta_i(g_p)} - \frac{1}{p}\right), & p = q, \end{cases}$$

and from (5.2.6) *it follows that the function* $\mathcal{B}_{p,q}(f, \Delta_i; \Gamma_1)$ *are monotonous in parameters* p *and* q*.*

Example 5.2.2. *Let*

$$\Gamma_2 = \{h_p : (0,\infty) \to (0,\infty) : p \in (0,\infty)\},$$

be a family of functions defined by

$$h_p(t) = \begin{cases} \frac{p^{-t}}{(\ln p)^2}, & p \in \mathbb{R}_+ \setminus \{1\}, \\ \frac{t^2}{2}, & p = 1. \end{cases}$$

Since $p \mapsto \frac{d^2}{dt^2} h_p(t) = p^{-t}$ *is the Laplace transform of a non-negative function (see [71]), it is exponentially convex. Obviously* h_p *are convex functions for every* $p > 0$*. It is easy to prove that the function* $p \mapsto [z_0, z_1, z_2; h_p]$ *is also exponentially convex for arbitrary points* $z_0, z_1, z_2 \in I$*. Using Corollary 5.2.3, it follows that* $p \mapsto \Delta_i(h_p)$ *for* $(i = 1, 2, 3)$ *are exponentially convex (it is easy to verify that these are continuous) and thus* log*-convex. From* (5.2.7)*, we can write*

$$\mathcal{B}_{p,q}(f, \Delta_i(h_p); \Gamma_2) = \begin{cases} \left(\frac{\Delta_i(h_p)}{\Delta_i(h_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(-\frac{\Delta_i(id \cdot h_p)}{p\Delta_i(h_p)} - \frac{2}{p \ln p}\right), & p = q \neq 1, \\ \exp\left(-\frac{\Delta_i(id \cdot h_1)}{3\Delta_i(h_1)}\right), & p = q = 1, \end{cases}$$

and from (5.2.6) *it follows monotonicity of the function* $\mathcal{B}_{p,q}(f, \Delta_i(h_p); \Gamma_2)$ *in parameters* p *and* q *for* $h_p, h_q \in \Gamma_2$*.*

Example 5.2.3. *Consider a family of functions*

$$\Gamma_3 = \{\psi_p : \mathbb{R} \to [0,\infty) : p \in (0,\infty)\},$$

defined with

$$\psi_p(t) = \begin{cases} \frac{1}{p^2} e^{tp}, & p \in \mathbb{R} \setminus \{0\}, \\ \frac{1}{2} t^2, & p = 0, \end{cases}$$

Since $\frac{d^2}{dt^2}(\psi_p(t)) = e^{tp} > 0$ *which shows that* ψ_p *is convex on* $\mathbb{R}$ *for every* $p \in \mathbb{R}$ *and* $p \mapsto \frac{d^2}{dt^2}(\psi_p(t))$ *is exponentially convex by definition. Using the analogous arguments as in Theorem 5.2.2, we also have that* $p \mapsto [z_0, z_1, z_2; \psi_p]$ *is exponentially convex (also exponentially convex in J-sense). For the family of the function* $\mathcal{B}_{p,q}(f, \Delta_i; \Gamma_3)$ *for* $(i = 1, 2, 3)$*, then* (5.2.7) *becomes*

$$\mathcal{B}_{p,q}(f, \Delta_i(\psi_p); \Gamma_3) = \begin{cases} \left(\frac{\Delta_i(\psi_p)}{\Delta_i(\psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\Delta_i(id \cdot \psi_p)}{\Delta_i(\psi_p)} - \frac{2}{p}\right), & p = q \neq 0, \\ \exp\left(\frac{\Delta_i(id \cdot \psi_0)}{3\Delta_i(\psi_0)}\right), & p = q = 0, \end{cases}$$

and using (5.2.6) *we can see that these are monotonous function in parameters* p *and* q *for* $\psi_p, \psi_q \in \Gamma_3$.

Example 5.2.4. *Consider a family of functions*

$$\Gamma_4 = \{\phi_p : (0,\infty) \to \mathbb{R} : p \in \mathbb{R}\},$$

defined by

$$\phi_p(t) = \begin{cases} \frac{t^p}{p(p-1)} & p \neq 1, 0, \\ -\ln t & p = 0, \\ t \ln t & p = 1. \end{cases}$$

Since $p \mapsto \frac{d^2}{dt^2}(\phi_p(t)) = t^{p-2} = e^{(p-2)\ln t} > 0$, *is the Laplace transform of a non-negative function (see [71]), it is exponentially convex. Obviously* ϕ_p *are convex functions for every* $t > 0$. *It is easy to prove that the function* $p \mapsto [z_0, z_1, z_2; \phi_p]$ *is also exponentially convex for arbitrary points* $z_0, z_1, z_2 \in I$. *Using Corollary 5.2.3 it follows that* $p \mapsto \Delta_i(\phi_p)$ *for* $(i = 1, 2, 3)$ *are exponentially convex (it is easy to verify that these are continuous), and thus* log-*convex. From* (5.2.7), *we see that*

$$\mathcal{B}_{p,q}(f, \Delta_i(\phi_p); \Gamma_4) = \begin{cases} \left(\frac{\Delta_i(\phi_p)}{\Delta_i(\phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{1-2p}{p(p-1)} - \frac{\Delta_i(\phi_p\phi_0)}{\Delta_i(\phi_p)}\right), & p = q \neq 0, 1, \\ \exp\left(1 - \frac{\Delta_i(\phi_0^2)}{2\Delta_i(\phi_0)}\right), & p = q = 0, \\ \exp\left(-1 - \frac{\Delta_i(\phi_0\phi_1)}{2\Delta_i(\phi_1)}\right), & p = q = 1, \end{cases} \tag{5.2.9}$$

for $\phi_p, \phi_q \in \Gamma_4$.

Remark 5.2.2. For the case $i = 1$, the means given in (5.2.9) were already presented in [28] in explicit form.

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