Müntz Space Embeddings of Schatten-Von Neumann Class

Since 1864

Name : **Sahibzada Waleed Noor**

Year of Admission : **2007**

Registration No. : **28-GCU-PHD-SMS-07**

Abdus Salam School of Mathematical Sciences

GC University Lahore, Pakistan

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Name : Sahibzada Waleed Noor_

Year of Admission : 2007

Registration No. : 28-GCU-PHD-SMS-07

Abdus Salam School of Mathematical Sciences

GC University Lahore, Pakistan

DECLARATION

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Certified that the research work contained in this thesis titled

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has been carried out and completed by Mr. **Sahibzada Waleed Noor,** Registration No. **28-GCU-PHD-SMS-07,** under my supervision.

----------------------------- -------------------------------

Date **Prof. Dan Timotin**

Supervisor

Submitted Through

Prof. Dr. A. D. Raza Choudary
Director General
Abdus Salam School of Mathematical Sciences
GC University, Lahore, Pakistan

Controller of Examination
GC University, Lahore
Pakistan

To Ami, Baba and Aisha...

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Abstract

This thesis contains results about the embeddings of Müntz spaces in the Hilbert space scenario and its applications to composition operators on Müntz spaces. In the main, we shall be concerned with the embedding $M_\Lambda^2 \subset L^2(\mu)$, where the Hilbert-Müntz space M_Λ^2 is the closed linear span of the monomials x^{λ_n} in $L^2([0,1])$ and μ is a finite Borel measure on $[0,1]$.

After gathering together the mathematical preliminaries required for this work in Chapter 1, we shall use the notion of a sublinear measure introduced by I. Chalendar, E. Fricain and D. Timotin [8] to investigate the properties of boundedness, compactness and belonging to Schatten-von Neumann ideals of these Hilbert space embeddings. This will be the content of Chapters 2 and 3. In Chapter 4, we give examples of sublinear measures for bounded and compact embeddings with interesting properties. Finally, in Chapter 5 the general embedding theory is applied to initiate the study of composition operators on M_Λ^2.

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Introduction

The classical Müntz-Szász Theorem proved by Herman Müntz [17] in 1914 and Otto Szäsz [21] in 1916 says that for a sequence $\Lambda = (\lambda_n)_{n\geq 0}$ of positive real numbers with $\lambda_0 = 0$, the linear span of the collection $\{x^\lambda : \lambda \in \Lambda\}$ is dense in the space of complex continuous functions $\mathcal{C}([0,1])$ if and only if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$. Later results extended this to the Lebesgue spaces $L^p([0,1])$, that is, if $1 \leq p < \infty$ then M_Λ^p is a proper Banach subspace of $L^p([0,1])$ if and only if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$, where m is the Lebesgue measure on $[0,1]$ and M_Λ^p denotes the closed linear span of the Müntz monomials $\{x^{\lambda_n} : \lambda_n \in \Lambda\}$ in the topology of $L^p([0,1])$. These Banach spaces M_Λ^p are called Müntz spaces. The functions in M_Λ^p are continuous in $[0,1)$. Interest in these spaces grew when J. A. Clarkson and P. Erdös [10] established that M_Λ^p, when Λ consists of positive integers, is in fact a space of real analytic functions on $(0,1)$ that can be continued holomorphically to the unit disk in the complex plane minus the negative real axis.

The mathematical literature on Müntz spaces is not very vast, and recent interest has been rekindled in two directions. On the one hand due to many unresolved questions related to the Banach space geometry of Müntz spaces such as were investigated by I. Al Alam in the article [3] and his thesis [1], and the text by V. I. Gurairy and W. Lusky [13]. On the other hand interest has been to do with operator theory, particularly the study of composition operators on Müntz spaces. The study of weighted composition operators and essential norm formulas on M_Λ^∞ was initiated by Alam [2] and further extended to M_Λ^1 by Chalendar et al. [8]. Further expansions and generalizations relevant to the Hilbert space case M_Λ^2 were conducted by the author [19]. The general embedding theory for the Müntz space M_Λ^1 and the notion of embedding measures was introduced by Chalendar et al. [8]. The corresponding embedding theory for M_Λ^2 was dealt with by the author and his doctoral supervisor D. Timotin [18].

In this thesis we shall deal primarily with these recent developments in the embedding theory for the Hilbert Müntz space M_Λ^2 and its applications to composition operators. The presentation can be viewed as an elaboration of the articles [8][18][19]. Along the way we shall also draw upon ideas that have been inspired by the now well established theory of embeddings of Hardy spaces of holomorphic functions on the unit disk in $\mathbb{C}$.

The plan of the thesis is the following. In Chapter 1 we gather all the necessary preliminaries for our work. Section 1.1 contains a self-contained exposition of compact operators of Schatten-von Neumann class. The basic properties of these operators are derived with proofs. In Section 1.2 we introduce the main objects of our study, the Müntz spaces. Some important inequalities and the notion of minimality with respect to the Müntz monomials are duscussed. In Section 1.3, we define the lacunary and quasilacunary classes of sequences. When Λ belongs to one of these classes, the corresponding Müntz spaces M_Λ^p, now equiped with nice bases, become much more tractable. In Section 1.4 we show that the Hilbert Müntz space M_Λ^2, when Λ is lacunary, has a special Riesz basis which is equivalent to an orthonormal basis. A result on Grammian matrices is also included which shall prove useful.

With Chapter 2, we begin the study of embedding theorems for Müntz spaces in earnest. In the first three sections, we treat ideas and prove embedding results for the M_Λ^p case that are essentially contained in the article of Chalendar et al. [8] for the M_Λ^1 case. That is, in Section 2.1 we shall define the embedding operator $i_\mu : M_\Lambda^p \hookrightarrow L^p(\mu)$ and state some basic boundedness results for it. In Section 2.2, we derive some essential norm formulas for i_μ and give some compactness criterions. In Section 2.3, we recall the notions of sublinear and vanishing sublinear measures introduced in [8] for characterizing boundedness and compactness of the embedding $M_\Lambda^1 \subset L^1(\mu)$. We also introduce α-sublinear measures for studying the embeddings $M_\Lambda^2 \subset L^2(\mu)$ of Schatten-von Neumann class in Chapter 3. In Section 2.4 we state and prove the main embedding results for M_Λ^2 that were proved in [8] for M_Λ^1 by a different method, and in Section 2.5 use interpolation to extend these boundedness and compactness theorems to all $1 \leq p \leq 2$.

In Chapter 3 we study embeddings $M_\Lambda^2 \subset L^2(\mu)$ of Schatten-von Neumann class $\mathcal{S}_q$ for $q \geq 0$. An interesting kernel function ψ depending only on the sequence Λ is introduced in Section 3.1 and is used to obtain some embedding results. In Section 3.2 we prove the sufficiency of α-sublinearity for embeddings $M_\Lambda^2 \subset L^2(\mu)$, with quasilacunary Λ, to belong to $\mathcal{S}_q$ for all $q \geq 0$.

Chapter 4 contains two illustrative examples of embeddings. In the first we construct a lacunary sequence Λ and a discrete measure μ on $[0, 1]$ such that the embedding $M_\Lambda^2 \subset L^2(\mu)$ is continuous but $M_\Lambda^1 \subset L^1(\mu)$ fails to be continuous. In the second example we construct, for any given pair of indices $0 < p < q$, a lacunary sequence Λ and a discrete measure μ such that the embedding $M_\Lambda^2 \subset L^2(\mu)$ is compact and belongs to the q-th Schatten class but not to the p-th Schatten class.

In Chapter 5, we shall apply the preceding embedding theory to study composition operators on M_Λ^2. Alam [1][2] first studied composition operators on the Müntz spaces M_Λ^∞ of essentially bounded functions. Later Chalendar et al. [8] were able to recapture many of Alam's results using the embedding theory for M_Λ^1. Whereas previously the study of composition operators on Müntz spaces was restricted to symbols with strict regularity assumptions, we shall generalize the study to abstract Borel functions. In Section 5.1, we demonstrate that Müntz spaces are generally not invariant with respect to the operation of composition. In Section 5.2 we give sufficient conditions for composition operators on M_Λ^2 to be bounded, compact and belong to Schatten classes and Section 5.3 contains the inverse results.

Chapter 1

Preliminaries

We denote by m the Lebesgue measure on $[0,1]$. $L^p(\mu)$ shall be used to denote the space of Lebesgue integrable functions of order $p \in [1,\infty]$ with respect to the measure μ on $[0,1]$. We will frequently use L^p to mean $L^p(m)$, and denote by $\|\cdot\|_p$ and $||\cdot||_{L^p(\mu)}$ the norms in $L^p(m)$ and $L^p(\mu)$ respectively. We will use the notation $\mathcal{C}([a,b])$ for the space of continuous complex functions on the interval $[a,b]$.

1.1 Compact Operators of Schatten-Von Neumann Class

In this section we shall introduce the *Schatten-Von Neumann* ideals $\mathcal{S}_q$ of compact operators on Hilbert spaces. For this we shall review operator theoretic preliminaries as we go along.

An operator $T : \mathcal{E} \to \mathcal{F}$ on Banach spaces is said to be bounded if the image of the unit ball of $\mathcal{E}$ under T is bounded in $\mathcal{F}$. The usual operator norm of T shall be denoted by $||T||$. Furthermore, T is a compact operator if the image of the unit ball of $\mathcal{E}$ under T is relatively compact in $\mathcal{F}$. If $T : \mathcal{E} \to \mathcal{F}$ is a bounded operator, we define by $\|T\|_e = \inf_{\mathcal{K}} \|T + \mathcal{K}\|$ the *essential norm* of an operator, where the infimum is taken over all compact operators $\mathcal{K} : \mathcal{E} \to \mathcal{F}$. This norm measures how far an operator is from being compact. In particular, T is compact if and only if $\|T\|_e = 0$. An operator T on a Hilbert space $\mathcal{H}$ is called *positive* if $(Tf, f) \geq 0$ for all $f \in \mathcal{H}$.

If T^* is the Hilbert space adjoint of T, then T^*T is a positive operator since $(T^*Tf, f) = ||Tf||^2 \geq 0$ for all $f \in \mathcal{H}$. The spectrum of a positive operator consists of non-negative real numbers. It is a basic fact that for a positive operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ there exists a unique positive operator $\sqrt{T} : \mathcal{H} \longrightarrow \mathcal{H}$ such that $(\sqrt{T})^2 = T$. Generally, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a compact operator, then $|T| := \sqrt{T^*T}$ is a positive compact operator on $\mathcal{H}_1$. Hence by the *polar decomposition* for an operator, there exists a partial isometry $U : \overline{|T|(\mathcal{H}_1)} \longrightarrow \mathcal{H}_2$ such that $T = U|T|$. Now, the *spectral theorem* for positive compact operators says that there exists an orthonormal system $(u_n)_{n=1}^{\infty}$ in $\mathcal{H}_1$ of eigenvectors of $|T|$ such that the corresponding eigenvalues of $|T|$ called the *singular numbers* of T and denoted by $s_n(T)$ are non-negative. We may arrange so that $0 \leq s_{n+1}(T) \leq s_n(T)$. From this follows the orthonormal representation of $|T|$

$$|T|(f) = \sum_{n=1}^{\infty} s_n(T)(f, u_n)u_n$$

for all $f \in \mathcal{H}_1$. Since $T = U|T|$, we have that for an arbitrary compact operator $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ there exist orthonormal systems $(u_n)_{n=1}^{\infty}$ in $\mathcal{H}_1$ and $(v_n)_{n=1}^{\infty}$ in $\mathcal{H}_2$ such that

$$T(f) = \sum_{n=1}^{\infty} s_n(T)(f, u_n)v_n \tag{1.1.1}$$

for all $f \in \mathcal{H}_1$, where $v_n = Uu_n$. This is called the *Schmidt decomposition* of T. We want to show that the singular numbers $s_n(T)$ are uniquely determined by T and are independent of the orthonormal system chosen. First we need a definition.

Definition 1.1.1. Given an operator $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$, its *approximation numbers* are defined as

$$\alpha_n(T) = \inf\{||T - L_n|| : L_n : \mathcal{H}_1 \longrightarrow \mathcal{H}_2,\ rank\ L_n < n\}$$

where $rank\ L_n$ is the dimension of $L_n(\mathcal{H}_1)$.

Clearly $\alpha_1(T) = ||T||$ and the sequence $(\alpha_n(T))_{n=1}^{\infty}$ is decreasing. Infact, T is compact if and only if $\alpha_n(T) \to 0$ as $n \to \infty$, being the norm limit of finite-rank operators. The next proposition establishes our claim by showing that the singular numbers of a compact operator T are precisley its approximation numbers.

Proposition 1.1.2. *Let $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a compact operator. Then $s_n(T) = \alpha_n(T)$ for $n = 1, 2, \ldots \infty$.*

Proof. Fix $n \in \mathbb{N}$. Define finite rank operators $T_n : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ as the partial sums of (1.1.1), that is for $f \in \mathcal{H}_1$

$$T_n(f) = \sum_{k=1}^{n-1} s_k(T)(f, u_k)v_k.$$

Then rank $T_n < n$ and hence

$$\alpha_n(T) \leq ||T - T_n|| = \sup\{||\sum_{k \geq n} s_k(T)(f, u_k)v_k||_{\mathcal{H}_2} : ||f||_{\mathcal{H}_1} \leq 1\}$$
$$= \sup\{(\sum_{k \geq n} s_k^2(T)|(f, u_k)|^2)^{1/2} : ||f||_{\mathcal{H}_1} \leq 1\} \leq s_n(T)$$

because $s_k(T) \leq s_n(T)$ for $k \geq n$. To prove the opposite inequality, take any operator $L_n : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ with rank $L_n < n$. Choose $g = \sum_{k=1}^{n} \beta_k u_k$ such that $||g||_{\mathcal{H}_1} = 1$ and $g \in kerL_n$, where the u_k are as in (1.1.1). By the Schmidt decomposition of T we get

$$||Tg||_{\mathcal{H}_2} = ||\sum_{k=1}^{n} s_k(T)\beta_k v_k||_{\mathcal{H}_2} = (\sum_{k=1}^{n} s_k^2(T)|\beta_k|^2)^{1/2} \geq s_n(T)||g||_{\mathcal{H}_1}$$

hence

$$||T - L_n|| \geq ||Tg - L_n g||_{\mathcal{H}_2} = ||Tg||_{\mathcal{H}_2} \geq s_n(T).$$

So $\alpha_n(T) \geq s_n(T)$ and we are done. □

We are now ready to define the Schatten-Von Neumann classes.

Definition 1.1.3. For $0 < q < \infty$, the Schatten-Von Neumann class $\mathcal{S}_q(\mathcal{H}_1, \mathcal{H}_2)$ consists of all compact operators $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ with singular numbers $(s_n(T))_{n=1}^{\infty} \in \ell^q(\mathbb{N})$.

It is true that $\mathcal{S}_q(\mathcal{H}_1, \mathcal{H}_2)$ equipped with the norm

$$||T||_{\mathcal{S}_q} = (\sum_{n=1}^{\infty} s_n^q(T))^{1/q} = (\sum_{n=1}^{\infty} \alpha_n^q(T))^{1/q} \tag{1.1.2}$$

is a Banach space, though it is non-trivial and [11] may be referred to for a proof.

The following proposition enlists some immediate consequences of the definition.

Proposition 1.1.4. *(a) For each $0 < q < \infty$, the finite rank operators form a dense subset of $\mathcal{S}_q(\mathcal{H}_1, \mathcal{H}_2)$.*
(b) When $0 < p \leq q < \infty$, $||T||_{\mathcal{S}_q} \leq ||T||_{\mathcal{S}_p}$ $\forall$ $T \in \mathcal{S}_p(\mathcal{H}_1, \mathcal{H}_2)$, hence $\mathcal{S}_p(\mathcal{H}_1, \mathcal{H}_2) \subset \mathcal{S}_q(\mathcal{H}_1, \mathcal{H}_2)$.
(c) $T \in \mathcal{S}_q(\mathcal{H}_1, \mathcal{H}_2)$ if and only if $T^* \in \mathcal{S}_q(\mathcal{H}_2, \mathcal{H}_1)$ and $||T||_{\mathcal{S}_q} = ||T^*||_{\mathcal{S}_q}$.
(d) Let $0 < q < \infty$ and $S : \mathcal{H}_0 \longrightarrow \mathcal{H}_1$, $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ and $V : \mathcal{H}_2 \longrightarrow \mathcal{H}_3$ be bounded Hilbert space operators. If $T \in \mathcal{S}_q(\mathcal{H}_1, \mathcal{H}_2)$, then $STV \in \mathcal{S}_q(\mathcal{H}_0, \mathcal{H}_3)$ and

$$||STV||_{\mathcal{S}_q} \leq ||S||||T||_{\mathcal{S}_q}||V||.$$

If $\mathcal{H}_0 = \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3$ then part (d) says that $\mathcal{S}_q(\mathcal{H}_0)$ is a *two-sided ideal* in $\mathfrak{B}(\mathcal{H}_0)$, the algebra of all bounded operators on $\mathcal{H}_0$.

Proof. (a) If $T_n : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is any operator of rank n, then by definition $\alpha_k(T_n) = 0$ for all $k > n$. Hence all finite rank operators belong to $\mathcal{S}_q(\mathcal{H}_1, \mathcal{H}_2)$ and denseness follows from the definition of approximation numbers and (1.1.2).

(b) Follows from the familiar relation between the ℓ^p spaces, that is $||(s_n(T))_n||_{\ell^q} \leq ||(s_n(T))_n||_{\ell^p}$.

(c) If $T \in \mathcal{S}_q(\mathcal{H}_1, \mathcal{H}_2)$ then

$$T(f) = \sum_{n=1}^{\infty} s_n(T)(f, u_n)v_n$$

for $f \in \mathcal{H}_1$. Hence $T^* : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ is given by

$$T^*(g) = \sum_{n=1}^{\infty} s_n(T)(g, v_n)u_n$$

for $g \in \mathcal{H}_2$. So both T and T^* have the same singular numbers $s_n(T)$.

(d) If $L_n : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ has rank $< n$ then so does SL_nT, hence

$$\alpha_n(STV) \leq ||STV - SL_NV|| \leq ||S||||T - L_n||||V||.$$

Taking the infimum on the right side over all finite rank operators L_n with rank $< n$, we get $\alpha_n(STV) \leq ||S||\alpha_n(T)||V||$. Therefore by (1.1.2),

$$||STV||_{\mathcal{S}_q} \leq ||S||||T||_{\mathcal{S}_q}||V||. \quad \square$$

Charles A. McCarthy in [15] found useful expressions for the $\mathcal{S}_q$ norm of an operator:

Lemma 1.1.5. *Suppose* $T' : \mathcal{H}_0 \longrightarrow \mathcal{H}_1$, $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ *are bounded operators,* $0 < q, q' \leq \infty$, $T \in \mathcal{S}_q(\mathcal{H}_1, \mathcal{H}_2)$, $T' \in \mathcal{S}_{q'}(\mathcal{H}_0, \mathcal{H}_1)$.

(i) If $1/r = 1/q + 1/q'$, *then* $T'T \in \mathcal{S}_r$, *and*

$$\|TT'\|_r \leq \|T\|_s\|T'\|_{s'}.$$

(ii) If $0 < q \leq 2$*, then*

$$\|T\|_{\mathcal{S}_q}^q = \inf \sum_n \|T\phi_n\|_{\mathcal{H}_2}^q,$$

where the infimum is taken over all orthonormal bases $(\phi_n)_n$ *of* $\mathcal{H}_1$*.*

(iii) If $2 \leq q < \infty$*, then*

$$\|T\|_{\mathcal{S}_q}^q = \sup \sum_n \|T\phi_n\|_{\mathcal{H}_2}^q$$

where the supremum is taken over all orthonormal bases $(\phi_n)_n$ *of* $\mathcal{H}_1$*.*

The case $q = 2$ is of fundamental importance in the theory of Schatten class operators. The elements of $\mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$ are called *Hilbert-Schmidt operators* and by Lemma 1.1.5 the *Hilbert-Schmidt norm* of $T \in \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$ is given by

$$||T||_{\mathcal{S}_2} = (\sum_n ||T\phi_n||_{\mathcal{H}_2}^2)^{1/2}$$

where $(\phi_n)_n$ is any orthonormal basis of $\mathcal{H}_1$. Furthermore, the norm $||\cdot||_{\mathcal{S}_2}$ is induced by the inner product

$$(S, T) = \sum_n (S\phi_n, T\phi_n)_{\mathcal{H}_2}$$

for $S, T \in \mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$, where $(\phi_n)_n$ is an orthonormal basis for $\mathcal{H}_1$. By polarization, this inner product is independent of the choice of orthonormal basis for $\mathcal{H}_1$ because the norm $||\cdot||_{\mathcal{S}_2}$ is independent. Hence $\mathcal{S}_2(\mathcal{H}_1, \mathcal{H}_2)$ is a Hilbert space.

Operators in $\mathcal{S}_1(\mathcal{H}_1, \mathcal{H}_2)$ are called *Trace-class operators.* For $T \in \mathcal{S}_1(\mathcal{H}_1, \mathcal{H}_2)$ the trace-class norm of T is given by

$$\begin{aligned}||T||_{\mathcal{S}_1} &= \sum_{n=1}^{\infty} s_n(T) = \text{trace}(|T|) = \sum_{n=1}^{\infty} (|T|\phi_n, \phi_n)_{\mathcal{H}_1} \\ &= \sum_{n=1}^{\infty} (\sqrt{|T|}\phi_n, \sqrt{|T|}\phi_n)_{\mathcal{H}_1} = ||\sqrt{|T|}||_{\mathcal{S}_2}^2\end{aligned}$$

where $(\phi_n)_n$ is an orthonormal basis for $\mathcal{H}_1$. Hence T is a trace-class operator if and only if $\sqrt{|T|}$ is a Hilbert-Schmidt operator.

Lemma 1.1.5 will be summoned frequently in the following sections, where under various conditions we seek to show that our Müntz embedding operator $i_\mu \in \mathcal{S}_q(M^2_\Lambda, L^2(\mu))$.

1.2 Müntz Spaces

Let us denote, for a set S of nonnegative real numbers, the subspace

$$L^p_S = \text{closed span}\{x^t : t \in S\} \subset L^p.$$

When clear from the context, we shall denote by L_S the space L^p_S. Next we define the spaces that we shall work on for the remainder of this work.

Definition 1.2.1. Let Λ be an increasing sequence of nonnegative real numbers with $\sum_{\lambda\in\Lambda} \frac{1}{\lambda} < \infty$. The Müntz space M^p_Λ is defined to be the space L^p_Λ.

In this work, Λ shall always denote an increasing sequence of nonnegative real numbers with $\sum_{\lambda\in\Lambda} \frac{1}{\lambda} < \infty$. The functions in M^p_Λ are continuous on $[0, 1)$ and real analytic in (0,1). A feature of the Müntz monomials $(x^\lambda)_{\lambda\in\Lambda}$ is that they form a *minimal system* in M^p_Λ, which means that for any $\lambda' \in \Lambda$

$$\text{dist}\,(x^{\lambda'}, L_{\Lambda\setminus\{\lambda'\}}) = \inf_{g\in L_{\Lambda\setminus\{\lambda'\}}} ||x^{\lambda'} - g||_{L^p} > 0. \tag{1.2.1}$$

This can easily be extended to show that if $\Lambda' \subset \Lambda$ is a finite subset, then

$$L_{\Lambda'} \cap L_{\Lambda\setminus\Lambda'} = \{0\}. \tag{1.2.2}$$

We begin by highlighting a consequence of minimality: Let $\Lambda = (\lambda_k)_{k=1}^{\infty}$ with $\sum_k 1/\lambda_k < \infty$. If we define the functionals $\phi_n : \mathrm{span}(x^{\lambda_k})_{k=1}^{\infty} \longrightarrow \mathbb{C}$ by

$$\phi_n(\sum_k a_k x^{\lambda_k}) = a_n,$$

then

$$\begin{aligned} ||\phi_n|| &= \sup \frac{|\phi_n(\sum_k a_k x^{\lambda_k})|}{||\sum_k a_k x^{\lambda_k}||_{L^p}} = (\inf \frac{||\sum_k a_k x^{\lambda_k}||_{L^p}}{|a_n|})^{-1} \\ &= (\inf_{g \in L_{\Lambda \setminus \{\lambda_n\}}} ||x^{\lambda_n} - g||_{L^p})^{-1} \\ &= (\mathrm{dist}\,(x^{\lambda_n}, L_{\Lambda\setminus\{\lambda_n\}}))^{-1} \end{aligned} \tag{1.2.3}$$

where the supremum and infimum were taken over all elements $\sum_k a_k x^{\lambda_k} \in \mathrm{span}(x^{\lambda_k})_{k=1}^{\infty}$. Hence by (1.2.1) and (1.2.3), $(x^{\lambda_k})_{k=1}^{\infty}$ is a minimal system if and only if each ϕ_n is a bounded linear functional on $\mathrm{span}(x^{\lambda_k})_{k=1}^{\infty}$ with the L^p topology on $[0,1]$. Hence the ϕ_n may be extended uniquely to a bounded linear functional on M_Λ^p. For the special case $p = \infty$, we note that for any $0 < \varepsilon < 1$, we can scale $x \mapsto (1-\varepsilon)x$ to show that $(x^{\lambda_k})_{k=1}^{\infty}$ is a minimal system in the supremum norm topology of $\mathcal{C}([0, 1-\varepsilon])$ and everything discussed above will hold on $[0, 1-\varepsilon]$.

We state a Bernstein-type inequality from ([5], p.182, E.5.b) to prove a result about the uniform convergence of a uniformly bounded family in M_Λ^1:

Proposition 1.2.2. *Suppose $\lambda_1 \geq 1$. For every $\varepsilon \in (0,1)$ there is a constant c_ε depending only on ε and $(\lambda_i)_{i=0}^{\infty}$ such that*

$$||p'||_{[0,1-\varepsilon]} \leq c_\varepsilon ||p||_{[0,1]}$$

for every $p \in \mathrm{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$.

The next lemma has been taken from Lemma 4.2.5 in [1]. Minimality of the sequence $(x^{\lambda_k})_{k=1}^{\infty}$ will play a prominent role in the proof.

Lemma 1.2.3. *If* $(f_m)_m \subset M_\Lambda^1$, $||f_m||_1 \leq 1$ *for all* m, *then there exists a subsequence* $(f_{m_k})_k$ *which converges uniformly on every compact subset of* $[0,1)$.

Proof. For any $\varepsilon \in (0,1]$ the restrictions of f_m to $[0,1-\varepsilon]$ are uniformly bounded by Lemma 2.1.2. It is enough to prove the result for a sequence $(f_m)_m$ of polynomials in x^{λ_k} and then pass onto limits to prove it for arbitrary elements in M_Λ^1. Notice first that there are only finitely many $\lambda_m < 1$, say for all $m \leq N$, and the rest are greater than or equal to 1. So if $f_m \in \text{span}(x^{\lambda_k})_{k=1}^\infty$ then $f_m = g_m + h_m$ where $g_m \in \text{span}\{x^{\lambda_1}, \dots, x^{\lambda_N}\}$ and $h_m \in \text{span}\{x^{\lambda_{N+1}}, x^{\lambda_{N+2}}, \dots\}$.

Suppose $f = \sum_k a_k x^{\lambda_k} \in \text{span}(x^{\lambda_k})_{k=1}^\infty$. Then $g = \sum_{k=1}^N a_k x^{\lambda_k}$ and $h = \sum_{k \geq N+1} a_k x^{\lambda_k}$ so $f = g + h$. We know that $(x^{\lambda_k})_{k=1}^\infty$ is a minimal system in the topology of $\mathcal{C}([0,1-\varepsilon])$ and we shall denote by $M_\Lambda^{[0,1-\varepsilon]}$ the closed span of these monomials in this topology. Hence the discussion preceding Proposition 1.2.2 shows that the coefficient functionals ϕ_n are bounded linear functionals on $M_\Lambda^{[0,1-\varepsilon]}$ for all $0 < \varepsilon < 1$. So we get

$$\begin{aligned} ||g||_{[0,1-\varepsilon]} &= ||\sum_{k=1}^N a_k x^{\lambda_k}||_{[0,1-\varepsilon]} \leq \sum_{k=1}^N |a_k| = \sum_{k=1}^N |\phi_k(f)| \\ &\leq \sum_{k=1}^N ||\phi_k|| ||f||_{[0,1-\varepsilon]} = C_\varepsilon ||f||_{[0,1-\varepsilon]}. \end{aligned} \tag{1.2.4}$$

Hence

$$||g_m||_{[0,1-\varepsilon]} \leq C_\varepsilon ||f_m||_{[0,1-\varepsilon]} \tag{1.2.5}$$

for all m. From this it follows that

$$||h_m||_{[0,1-\varepsilon]} \leq ||f_m||_{[0,1-\varepsilon]} + ||g_m||_{[0,1-\varepsilon]} \leq (1 + C_\varepsilon,)||f_m||_{[0,1-\varepsilon]} \tag{1.2.6}$$

for all m. It follows from Proposition 1.2.2 and (1.2.6) that

$$||h'_m||_{[0,1-2\varepsilon]} \leq c_\varepsilon ||h_m||_{[0,1-\varepsilon]} \leq K_\varepsilon ||f_m||_{[0,1-\varepsilon]}$$

for all m. Hence h'_m restricted to $[0, 1-2\varepsilon]$ are uniformly bounded. By the Arzela-Ascoli theorem, the sequence $h_m|_{[0,1-2\varepsilon]}$ contains a subsequence which is uniformly convergent on $[0, 1-2\varepsilon]$. Now applying this to $\varepsilon = 1/N$ for all positive integers N and using a diagonal process, we obtain a subsequence $(h_{m_j})_j$ that converges uniformly on all compact subsets of $[0, 1)$.

The case for $(g_m)_m$ is simpler, since by (1.2.5) it is a uniformly bounded sequence in an N-dimensional space. Hence g_m has a subsequence that converges uniformly on $[0, 1-\varepsilon]$. Again using a diagonal argument we obtain a subsequence $(g_{m_k})_k$, where we may take $(m_k)_k$ a subsequence of $(m_j)_j$, that converges uniformly on compact subsets of $[0, 1)$.

Therefore $(f_{m_k})_k = (g_{m_k} + h_{m_k})_k$ is the desired subsequence of $(f_m)_m$ that converges uniformly on compact subsets of $[0, 1)$. □

The next result is Lemma 2.2 in [8] and will frequently prove useful to us in the remainder of this work, hence we prove it for completeness.

Lemma 1.2.4. *Suppose $\rho : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is an increasing, $\mathcal{C}^1$ function with $\rho(0) = 0$ such that $\mu(J_\varepsilon) \leq \rho(\varepsilon)$ for all $\varepsilon \in (0, 1]$. Then for any continuous, positive, increasing function g we have*

$$\int_{[0,1]} g d\mu \leq \int_0^1 g(x)\rho'(1-x)dx.$$

Proof. If $F(x) = \mu([0, x))$, then integration by parts yields

$$\int_{[0,1]} g d\mu = \int_0^1 g(x)dF(x) = g(1)F(1) - \int_0^1 F(x)dg(x)$$

since $F(0) = 0$. Since $\mu(J_\varepsilon) \leq \rho(\varepsilon)$ for all $\varepsilon \in (0, 1]$ and $\mu(\{0\}) = 1$, it follows that $F(1) - F(x) \leq \rho(1-x)$, or $-F(x) \leq \rho(1-x) - F(1)$. Plugging this into the above equation, using the fact that g is increasing hence dg is a positive measure, integration by parts and $\rho(0) = 0$, we obtain

$$\begin{aligned}\int_{[0,1]} g d\mu &\leq g(1)F(1) - F(1)(g(1) - g(0)) + \int_0^1 \rho(1-x)dg(x) \\ &= F(1)g(0) + \int_0^1 \rho(1-x)dg(x) \\ &= F(1)g(0) - \rho(1)g(0) - \int_0^1 g(x)d(\rho(1-x)) \\ &= g(0)(F(1) - \rho(1)) + \int_0^1 g(x)\rho'(1-x)dx.\end{aligned}$$

The proof is complete by noting that $F(1) = \mu([0,1)) \leq \rho(1)$. □

1.3 Lacunary and Quasilacunary Sequences

We introduce two important classes of sequences, the lacunary and quasilacunary sequences. Their importance stems from the result that quasilacunary Müntz spaces M_Λ^p are isomorphic to ℓ^p (see Lemma 1.3.5). Furthermore lacunary Müntz monomials x^Λ, where Λ is a lacunary sequence, form a basis for M_Λ^p.

Definition 1.3.1. A sequence Λ is called *lacunary*, if for some $q > 1$ we have $\lambda_{n+1}/\lambda_n \geq q,\ n \geq 1$. Generally, a sequence Λ is called *quasilacunary* if for some increasing sequence $\{n_k\}$ of integers and some $q > 1$ we have $\lambda_{n_{k+1}}/\lambda_{n_k} \geq q$ and $N := \sup_k(n_{k+1} - n_k) < \infty$.

For example, $\Lambda = \{2^k\}_{k=1}^\infty$ is lacunary while $\{2^k\}_{k=1}^\infty \cup \{2^k + 1\}_{k=1}^\infty$ is quasilacunary but not lacunary.

Remark 1.3.2. Any quasilacunary sequence Λ can be embedded in a larger quasilacunary sequence such that, for $q > 1$ as in the definition, we get $\lambda_{n+1}/\lambda_n \leq q^2$ for all n. This can be done as follows: if for some $j \geq 2$, we have $q^j \leq \lambda_{n+1}/\lambda_n \leq q^{j+1}$, then replace Λ with $\Lambda \cup \{q\lambda_n, q^2\lambda_n, \ldots, q^{j-1}\lambda_n\}$ where the new terms are placed between λ_n and λ_{n+1}. Continuing inductively in this way, we can arrange Λ to always satisfy $\lambda_{n+1}/\lambda_n \leq q^2$ for all n and also retain quasilacunarity with the same N. So in particular we may assume that

$$q \leq \frac{\lambda_{n_{m+1}}}{\lambda_{n_m}} \leq q^{2N} \tag{1.3.1}$$

for all n.

We will be needing some results from the book of Gurairy and Lusky [13]:

Lemma 1.3.3. *([13], Corollary 8.1.2) Any Müntz polynomial* $f(x) = \sum_{k=1}^m \alpha_k x^{\lambda_k}$ *satisfies*

$$|f(x)| \leq 2(\sum_{k=1}^m x^{\lambda_k \beta_k}) \|f\|_\infty$$

for any $x \in [0,1]$ *and any* $\beta_k \geq 0$ *with* $\sum_{k=1}^m \beta_k = 1$.

Lemma 1.3.4. *[13], Proposition 8.2.2) There is a constant* $K > 0$ *(depending only on* Λ*) such that, if* $f(x) = \sum_{n=1}^m \alpha_n x^{\lambda_n}$*, then*

$$\|f'\|_\infty \leq K(\sum_{n=1}^m \lambda_n) \|f\|_\infty .$$

Lemma 1.3.5. *([3], Theorem 9.3.3) If* Λ *is quasilacunary,* $F_k =$*span* $\{x^{\lambda_{n_k+1}}, \ldots, x^{\lambda_{n_{k+1}}}\}$, *then* $\exists\, d_1, d_2 > 0$ *such that for any sequence of functions* $f_k \in F_k$ *we have*

$$d_1(\sum_k \|f_k\|_p^p)^{1/p} \leq \left\| \sum_k f_k \right\|_p \leq d_2(\sum_k \|f_k\|_p^p)^{1/p}. \tag{1.3.2}$$

In particular, M_Λ^p *is isomorphic to* ℓ^p *and* M_Λ^p *has a basis.*

Lemma 1.3.5 has an important implication when Λ is lacunary. To see this, note that when Λ is lacunary each F_k is just the 1-dimensional span of x^{λ_k}. Hence if $\sum_k a_k \lambda_k^{1/p} x^{\lambda_k} \in \text{span}\{\lambda_1^{1/p} x^{\lambda_1}, \lambda_2^{1/p} x^{\lambda_2}, \ldots\}$ then (1.3.2) becomes

$$C_1(\sum_k |a_k|^p)^{1/p} \leq \left\| \sum_k a_k \lambda_k^{1/p} x^{\lambda_k} \right\|_p \leq C_2(\sum_k |a_k|^p)^{1/p} \tag{1.3.3}$$

for some constants $C_1, C_2 > 0$. This is due to the fact that the norm of f_k

$$||f_k||_p^p = ||a_k \lambda_k^{1/p} x^{\lambda_k}||_p^p = |a_k|^p \frac{\lambda_k}{p\lambda_k + 1}$$

for all k and $0 < \inf_k \frac{\lambda_k}{p\lambda_k+1} \leq \sup_k \frac{\lambda_k}{p\lambda_k+1} = \frac{1}{p}$. From relation (1.3.3), we conclude that when Λ is lacunary, the isomorphism $M_\Lambda^p \simeq \ell^p$ specified in Lemma 1.3.5 maps the sequence $\{\lambda_k^{1/p} x^{\lambda_k}\}_k$ onto the standard basis of ℓ^p. Hence $\{\lambda_k^{1/p} x^{\lambda_k}\}$ is a basis for M_Λ^p. We shall also need the Clarkson-Erdös Theorem from [13]:

Theorem 1.3.6. *Assume that* $\sum_k \frac{1}{\lambda_k} < \infty$ *and* $\inf_k(\lambda_{k+1} - \lambda_k) > 0$. *Then* $f \in M_\Lambda^p$ *if and only* $f \in L^p$ *and there exist* $b_k \in \mathbb{R}$ *such that*

$$f(x) = \sum_{k=1}^{\infty} b_k x^{\lambda_k} \quad \text{for } x \in [0,1),$$

where the series converges uniformly on compact subsets of $[0,1)$. *Also, for any* $\varepsilon > 0$, *there is a constant* $M > 0$ *such that*

$$|b_k|.||x^{\lambda_k}||_{L^p} \leq (1+\varepsilon)^{\lambda_k} ||f||_{L^p} \quad \text{if } k \geq M. \tag{1.3.4}$$

The following Lemma was stated and proved in [8], and we will need it in proving some theorems in the next chapter. We include that proof here for completeness.

Lemma 1.3.7. *If* $f : [0,1] \to \mathbb{R}$ *is a nonconstant differentiable function, then*

$$\|f\|_1 \geq \min\{\frac{\|f\|_\infty^2}{2\,\|f'\|_\infty}, \frac{\|f\|_\infty}{4}\}.$$

Proof. Let $x_0 \in [0,1]$ such that $|f(x_0)| = M$, where $M := ||f||_\infty > 0$. Replacing f by $-f$ we may assume that $f(x_0) = M$. Obviously one of the intervals $[x_0 - 1/2, x_0]$ or $[x_0, x_0 + 1/2]$ is in $[0,1]$. Suppose that the first interval lies in $[0,1]$.

If $\frac{||f||_\infty}{4} \leq \frac{||f||_\infty^2}{2||f'||_\infty}$, that is if $||f'||_\infty < 2M$, then for all $x \in [x_0 - 1/2, x_0]$,

$$f(x) \geq 2M(x - x_0) + M.$$

It follows that

$$||f||_1 \geq \int_{x_0-1/2}^{x_0} f(x)dx \geq \frac{M}{4}.$$

If $||f'||_\infty > 2M$ then $x_0 - \frac{M}{||f'||_\infty} \in [x_0 - 1/2, x_0]$ and for all $x \in [x_0 - M/||f'||_\infty, x_0]$,

$$f(x) \geq ||f'||_\infty(x - x_0) + M.$$

It follows that

$$||f'||_1 \geq \int_{x_0-M/||f'||_\infty}^{x_0} f(x)dx \geq \frac{M^2}{2||f'||_\infty}.$$

The proof in the case where $[x_0, x_0 + 1/2]$ lies in $[0,1]$ follows the same lines. $\square$

1.4 Riesz Bases and Gram Matrices

In this section we show that the Hilbert-Müntz space M_Λ^2 has a special basis when Λ is lacunary that is equivalent to an orthonormal basis and which will be used freely in the rest of this work.

Definition 1.4.1. A sequence $\{f_k\}_{k=1}^\infty$ in a Hilbert space H is said to be *complete* if $\text{span}\{f_k\}_k$ is dense in H. A *Riesz basis* for a seperable Hilbert space H is a complete sequence $\{f_k\}_{k=1}^{\infty}$ such that there exist constants $A, B > 0$ for which

$$A\sum_{k=1}^{m}|c_k|^2 \le \left\|\sum_{k=1}^{m} c_k f_k\right\|^2 \le B\sum_{k=1}^{m}|c_k|^2 \tag{1.4.1}$$

for every finite scalar sequence $\{c_k\}_{k=1}^{\infty}$.

Remark 1.4.2. If a sequence $\{g_k\}_{k=1}^{\infty}$ is a Riesz basis for a Hilbert space H then $\exists$ a bounded invertible operator $T : H \to H$ and an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of H such that $Te_k = g_k$ for each $k = 1, \dots, \infty$. In other words a Riesz basis is equivalent to an orthonormal basis from a vector space point of view.

Indeed, consider the operator $\sigma : \ell^2(\mathbb{N}) \to H$ defined by $\sigma(\{c_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} c_k g_k$. Then σ is well-defined and bounded by the right side of (1.4.1) after taking limits and noting that $\sum_{k=1}^{\infty} c_k g_k := \lim_{n\to\infty} \sum_{k=1}^{n} c_k g_k$. We next show that σ is invertible. Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be the standard orthonormal basis in $\ell^2(\mathbb{N})$. Then $U(\sum_k c_k g_k) := \sum_k c_k \varepsilon_k$ defines an operator from span$\{g_k\}_{k=1}^{\infty}$ into $\ell^2(\mathbb{N})$. The operator U is well-defined since for $g = \sum_k c_k g_k \in \text{span}\{g_k\}_{k=1}^{\infty}$ we get $g = 0$ implies $c_k = 0$ for all k by the left side of (1.4.1). The operator U is clearly linear and to show boundedness take a finite sequence $\{c_k\}_k$, then

$$||U(\sum_k c_k g_k)||^2 = ||\sum_k c_k \varepsilon_k||^2 = \sum_k |c_k|^2 \le \frac{1}{A}||\sum_k c_k g_k||^2$$

the last inequality following from the left side of (1.4.1). Now since $\{g_k\}_{k=1}^{\infty}$ is complete, U has a unique extension to a bounded operator from H onto $\ell^2(\mathbb{N})$ (also denoted by U). It follows from the definitions that $U = \sigma^{-1}$ and hence σ is invertible. Since H is seperable, $\exists$ an isometric isomorphism $\tau : H \to \ell^2(\mathbb{N})$ and an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ in H such that $\tau(e_k) = \varepsilon_k$ for all k. Define $T := \tau \cdot \sigma : H \to H$. Then T is the required invertible operator on H with $Te_k = g_k$ for all k.

The previous remark justifies the term Riesz *basis* by showing that they are infact bases. This is a consequence of the bijectivity of the operator $\sigma : \ell^2(\mathbb{N}) \to H$ defined above. It also shows that the representation of any element of H with respect to this basis has square-summable coefficients.

Proposition 1.4.3. *The sequence of functions* $(\lambda_k^{1/2} x^{\lambda_k})_k$ *is a Riesz basis for* M_Λ^2 *when* Λ *is lacunary.*

Proof. It follows from Lemma 1.3.5 with $p = 2$ and Λ lacunary that the sequence $(g_k)_k = (\lambda_k^{1/2} x^{\lambda_k})_k$ is a Riesz basis for M_Λ^2. Since for any scalar sequence $(c_k)_k$ we have

$$d_1 \sum_k |c_k|^2 \|g_k\|_2^2 \leq \left\| \sum_k c_k g_k \right\|_2^2 \leq d_2 \sum_k |c_k|^2 \|g_k\|_2^2 .$$

Now

$$\|g_k\|_2^2 = \int_0^1 \lambda_k x^{2\lambda_k} dx = \frac{\lambda_k}{2\lambda_k + 1} \longrightarrow 1/2$$

as $k \longrightarrow \infty$ and $\inf_k \|g_k\|_2^2 > 0$. Therefore $\exists$ constants $A, B > 0$ such that

$$A \sum_k |c_k|^2 \leq \left\| \sum_k c_k g_k \right\|_2^2 \leq B \sum_k |c_k|^2 .$$

Hence $(\lambda_k^{1/2} x^{\lambda_k})_{k=1}^{\infty}$ is a Riesz basis for M_Λ^2. □

Remark 1.4.4. By Remark 1.4.2, there exists an invertible operator $T : M_\Lambda^2 \longrightarrow M_\Lambda^2$ and an orthonormal basis $(e_k)_{k=1}^{\infty}$ for M_Λ^2 such that $\lambda_k^{1/2} x^{\lambda_k} = Te_k$ for all k. This operator T will be referred to as the *orthogonalizer* of the Riesz basis $(\lambda_k^{1/2} x^{\lambda_k})_k$.

We state and prove a basic lemma about Riesz bases and Gramian matrices:

Lemma 1.4.5. *Suppose that* $(g_n)_{n\in\mathbb{N}}$ *is a Riesz basis in the Hilbert* $\mathcal{E}$*, and* $(x_n)_{n\in\mathbb{N}}$ *is an arbitrary sequence in the Hilbert space* $\mathcal{F}$*. Define the Gramian of the sequence*

$(x_n)_{n\in\mathbb{N}}$ to be the infinite matrix $\Gamma = (\langle x_n, x_m\rangle)_{n,m\in\mathbb{N}}$. The following are equivalent:

(1) The map $g_n \mapsto x_n$ can be extended to a bounded linear operator $J : \mathcal{E} \to \mathcal{F}$.

(2) Γ defines a bounded operator on $\ell^2(\mathbb{N})$.

Proof. First, if (e_n) is an orthonormal basis, the Riesz condition on (g_n) implies that there is an invertible map that extends $e_n \mapsto g_n$. This means that we may suppose from the beginning that (g_n) is an orthonormal basis, and we shall do so in what follows.

If (1) is true, then

$$\langle x_n, x_m\rangle = \langle Jg_n, Jg_m\rangle = \langle J^*Jg_n, g_m\rangle.$$

Thus Γ is the matrix of the bounded operator J^*J.

Conversley, if Γ defines a bounded linear operator, then all truncations Γ_n to $0 \le n, m \le N$ are bounded matrices, that represent $J_N^*J_N$, where J_N is the operator defined by $g_n \mapsto x_n$ on the span of $\{g_n\}_{n=0}^N$. Thus all J_N are uniformly bounded, whence it follows that J is bounded. □

Lemma 1.4.6. *Suppose A, B are infinite matrices with positive entries, and such that $a_{ij} \le b_{ij}$ for all i, j. If B defines a bounded operator, then A defines a bounded operator.*

Proof. For a general infinite matrix $M = (m_{ij})$ we have

$$||M|| = \sup |\sum_{i,j} m_{ij}x_iy_j|,$$

where the supremum is taken with respect to all sequences $(x_i), (y_i)$ of finite support and such that $\sum_i |x_i|^2 = \sum_i |y_i|^2 = 1$. So M defines a bounded operator if the above supremum is finite. Since A and B have positive entries, it follows immediately that

$$||A|| = \sup \sum_{i,j} a_{ij} x_i y_j,$$

$$||B|| = \sup \sum_{i,j} b_{ij} x_i y_j,$$

where the two suprema are taken only with respect to non-negative sequences with finite support $(x_i), (y_i)$ such that $\sum_i x_i^2 = \sum_i y_i^2 = 1$. It follows then that, if $a_{ij} \leq b_{ij}$ for all i, j, then $||A|| \leq ||B||$. $\square$

Chapter 2

Embeddings of Müntz Spaces

2.1 Boundedness of Embeddings

Definition 2.1.1. For a fixed $p \geq 1$, a positive measure μ on $[0,1]$ is called Λ_p *-embedding* if there is a constant $C > 0$ such that

$$\|g\|_{L^p(\mu)} \leq C \|g\|_p \tag{2.1.1}$$

for all polynomials $g \in M_\Lambda^p$. Whenever p is clear from the context, we will remove subscript p and use the notation Λ-embedding.

If μ is Λ -embedding, then condition (2.1.1) extends to all $f \in M_\Lambda^p$. Indeed, since polynomials are dense in M_Λ^p, it will be shown in remark 2.1.4 that for any $f \in M_\Lambda^p$ there exists a sequence of polynomials $(p_n)_{n\geq 1}$ in M_Λ^p such that $p_n \to f$ in M_Λ^p and $p_n \to f$ in $L^p(\mu)$. Then after applying limits, (2.1.1) holds for all $f \in M_\Lambda^p$.

For a Λ-embedding μ we denote by i_μ the embedding operator $i_\mu : M_\Lambda^p \hookrightarrow L^p(\mu)$. Infact (2.1.1) is precisely the boundedness condition for i_μ. If $0 < \varepsilon < 1$, then the interval $[1-\varepsilon, 1]$ will be denoted by J_ε.

The first thing we notice is that if μ is Λ-embedding, then $\mu(\{1\}) = 0$. This can be seen as follows: since $x^{\lambda_n} \longrightarrow \chi_{\{1\}}$ pointwise monotonically on $[0,1]$ as $n \to \infty$, where $\chi_{\{1\}}$ is the characteristic function of $\{1\}$, then the convergence is also in $L^p(m)$ and $L^p(\mu)$ by Lebesgue dominated convergence. Hence taking the limit on both sides of (2.1.1) with $g(x) = x^{\lambda_n}$ we get $\mu(\{1\}) = ||\chi_{\{1\}}||^p_{L^p(\mu)} \leq C^p||\chi_{\{1\}}||^p_p = 0$.

In order to investigate conditions for a measure μ to be Λ-embedding, the next important result ([5], p.185, E.8.a) has been used in [8] and we state it here:

Lemma 2.1.2. *For any $\varepsilon \in (0,1]$ there exists a constant $c_\varepsilon > 0$ such that for any function $f \in M^p_\Lambda$ we have*

$$\sup_{0\leq t\leq 1-\varepsilon} |f(t)| \leq c_\varepsilon\Big(\int_{J_\varepsilon} |f(x)|^p dx\Big)^{1/p} \tag{2.1.2}$$

In particular, the supremum is majorized by $c_\varepsilon||f||_p$

The most important implication of this Lemma is that convergence in Müntz space M^p_Λ implies uniform convergence on compact subsets of $[0,1)$. Some immediate consequences of Lemma 2.1.2 are:

Corollary 2.1.3. *([8], Corollary 2.4)*

(i) If, for some $\varepsilon > 0$, supp $\mu \subset [0, 1-\varepsilon)$, then μ is Λ-embedding for any Λ, and

$$\|f\|_{L^p(\mu)} \leq c_\varepsilon \|\mu\|^{1/p} \|f\|_p \,.$$

(ii) More generally, if for some $\varepsilon > 0$ the restriction $\mu|_{J_\varepsilon}$ is absolutely continuous with respect to $m\,|_{J_\varepsilon}$, with essentially bounded density, then μ is Λ-embedding for any Λ.

Proof. (i) Using (2.1.2), for $f \in M^p_\Lambda$

$$\begin{aligned} ||f||^p_{L^p(\mu)} &= \int_{[0,1]} |f|^p d\mu = \int_{[0,1-\varepsilon)} |f|^p d\mu \leq \sup_{[0,1-\varepsilon)} |f|^p \mu([0,1-\varepsilon)) \\ &\leq c^p_\varepsilon ||\mu|| ||f||^p_p. \end{aligned}$$

Hence in particular, μ is Λ-embedding.

(ii) Again using (2.1.2) and the absolute continuity of the restriction of μ to J_ε with respect to Lebesgue measure m,

$$||f||^p_{L^p(\mu)} = \int_{[0,1-\varepsilon)} |f|^p d\mu + \int_{[1-\varepsilon,1]} |f|^p d\mu \leq \sup_{[0,1-\varepsilon)} |f|^p ||\mu|| + ||h||_\infty ||f||^p_p$$

$$\leq c^p_\varepsilon ||\mu|| ||f||^p_p + ||h||_\infty ||f||^p_p = (c^p_\varepsilon ||\mu|| + ||h||_\infty) ||f||^p_p.$$

So

$$||f||_{L^p(\mu)} \leq (c^p_\varepsilon ||\mu|| + ||h||_\infty)^{1/p} ||f||_p$$

hence μ is Λ-embedding for any Λ. □

Remark 2.1.4. As in remark 2.5 of [8], it is easy to show that μ is Λ-embedding if and only if $M^p_\Lambda \subset L^p(\mu)$, understood as a set inclusion of continuous functions on $[0, 1)$ representing elements of M^p_Λ. For necessity, let $f \in M^p_\Lambda$, then there exists a sequence of polymonials $(p_n)_{n\geq 1}$ in M^p_Λ such that $||f - p_n||_p \longrightarrow 0$ as $n \to +\infty$. By (2.1.1), $(p_n)_n$ is a Cauchy sequence in $L^p(\mu)$ and hence converges to a function $g \in L^p(\mu)$. So there exists a subsequence $(p_{n_k})_k$ which converges almost everywhere (w.r.t μ) to g. But by the comment after Lemma 2.1.2, $(p_n)_n$ converges to f uniformly on every compact subset of $[0, 1)$, so $g(t) = f(t)$ for almost every $t \in [0, 1)$ with respect to μ. And further since $\mu(\{1\}) = 0$, we have $f \in L^p(\mu)$. Therefore $M^p_\Lambda \subset L^p(\mu)$. Conversely, suppose $f_n \longrightarrow f$ in M^p_Λ and $i_\mu(f_n) \longrightarrow g$ in $L^p(\mu)$. Then again $f_n \longrightarrow f$ uniformly on compact subsets of $[0, 1)$ where $f \in M^p_\Lambda \subset L^p(\mu)$ and some subsequence $f_{n_k} \longrightarrow g$ almost everywhere w.r.t μ. Hence $i_\mu(f) = g$ almost everywhere w.r.t μ since $\mu(\{1\}) = 0$. So closed graph theorem implies that i_μ is continuous.

2.2 Essential Norm Formulae for Embeddings

In this section we set out to state and prove results from [1] and [8] about compactness and essential norms of i_μ, but for all M^p_Λ where $p \geq 1$, that will be used in the remainder of this thesis. What will become apparent in the course of our investigations is the special role played by the point $\{1\}$ in $[0,1]$ and how the *regularity* of μ near $\{1\}$ determines various compactness properties for i_μ. For the case of weighted composition operators, this will become clearer in chapter 4.

With this we can now improve Corollary 2.1.3 (i).

Proposition 2.2.1. *([8], Proposition 3.2) If supp $\mu \subset [0, 1-\varepsilon]$, then $i_\mu : M^p_\Lambda \to L^p(\mu)$ is compact for $p \geq 1$.*

Proof. Let $(f_n)_n$ be a sequence in the unit ball of M^p_Λ. Since $||f_n||_1 \leq ||f_n||_p \leq 1$, $(f_n)_n$ is a sequence in the unit ball of M^1_Λ. By Lemma 1.2.3 there exists a subsequence $(f_{n_k})_k$ that converges uniformly on $[0, 1-\varepsilon]$, hence converges in $L^p(\mu)$. So i_μ is compact. □

If μ is a positive measure on $[0,1]$ and $m \in \mathbb{N}^+$, we will denote by μ_m the measure equal to μ on $[0, 1-\frac{1}{m}]$ and 0 elsewhere, and $\mu'_m = \mu - \mu_m$.

The next general lemma helps us to prove Theorem 2.2.3 which is suggestive of the point we made earlier about the behaviour of our measure μ near the *boundary* $\{1\}$ and compactness phenomena.

Lemma 2.2.2. *([8], Lemma 3.4) Let X be a Banach space, (E, ν) a finite measure space, $T : X \to L^p(\nu)$ a bounded operator, $(E_m)_m$ a decreasing sequence of measurable subsets of E such that $\nu(\bigcap_m E_m) = 0$. Suppose that $(T - P_m T)$ is compact for all m, where P_m denotes the natural projection of $L^p(\nu)$ onto $L^p(E_m, \nu)$. Then the essential* *norm of T is given by*

$$||T||_e = \lim_{m\to\infty} ||P_m T||.$$

Proof. Since the sequence $(||P_m T||)_m$ is decreasing, its limit must exist as $m \to \infty$. Since $T - P_m T$ is compact for each m, it is clear then that $||T||_e \leq \lim_{m\to\infty} ||T - (T - P_m T)|| \leq \lim_{m\to\infty} ||P_m T||$. The prove the reverse inequality, let $\varepsilon > 0$ and $K : X \to L^p(\nu)$ be a compact operator. Take a sequence $(x_m)_m \subset X$ such that $||x_m|| = 1$ and $||P_m T x_m||_{L^p(E_m,\nu)} > ||P_m T|| - \varepsilon$. Then $(Kx_m)_m$ contains a convergent subsequence, say $Kx_{m_j} \to g \in L^p(\nu)$. We have

$$||(T-K)x_{m_j}||_{L^p(\nu)} \geq ||Tx_{m_j} - g||_{L^p(\nu)} - ||Kx_{m_j} - g||_{L^p(\nu)},$$

whence

$$\limsup_j ||(T-K)x_{m_j}||_{L^p(\nu)} \geq \limsup_j ||Tx_{m_j} - g||_{L^p(\nu)}. \tag{2.2.1}$$

Since $g \in L^p(\nu)$ and $\nu(\bigcap_m E_m) = 0$, there exists a positive integer N such that $\int_{E_m} |g|^p d\nu \leq \varepsilon^p$ for all $m \geq N$. Then if $m_j \geq N$, we have

$$\begin{aligned} ||Tx_{m_j} - g||_{L^p(\nu)} &\geq ||Tx_{m_j} - g||_{L^p(E_m,\nu)} \geq ||Tx_{m_j}||_{L^p(E_m,\nu)} - \varepsilon \\ &= ||P_{m_j} T x_{m_j}||_{L^p(E_{m_j},\nu)} - \varepsilon > ||P_{m_j} T|| - 2\varepsilon. \end{aligned}$$

From (2.2.1) it follows then that

$$||T - K|| \geq \lim_m ||P_m T|| - 2\varepsilon.$$

Since this is true for any compact K, we have

$$||T||_e \geq \lim_m ||P_m T|| - 2\varepsilon.$$

Letting $\varepsilon \to 0$ yields the desired inequality. □

Theorem 2.2.3. *([8], Theorem 3.5) Let M^p_Λ be a Müntz space, and suppose that μ is an embedding measure. Then*

$$\|i_\mu\|_e = \lim_{m\to\infty} \left\| i_{\mu'_m} \right\| .$$

Proof. This follows immediately from Lemma 2.2.2 by inserting $X = M^p_\Lambda$, $(E,\nu) = ([0,1],\mu)$, $T = i_\mu$ and $E_m = J_{1/m}$. Then $P_m T$ becomes $i_{\mu'_m}$. The compactness of $i_\mu - i_{\mu'_m} = i_{\mu_m}$ follows from Proposition 2.2.1. □

Corollary 2.2.4. *([8], Corollary 3.6) Let M^p_Λ be a Müntz space, and suppose there exists $\delta > 0$ such that $d\mu|_{J_\delta} = hdm|_{J_\delta}$ for some bounded measurable function h with $\lim_{t\to 1} h(t) = a$. Then i_μ is bounded and $\|i_\mu\|_e = a^{1/p}$.*

Proof. Boundedness of i_μ is a consequence of Corollary 2.1.3 (ii). By Theorem 2.2.3, we have to show that

$$\lim_{m\to\infty} ||i_{\mu'_m}|| = a^{1/p}.$$

For any $\varepsilon > 0$, if m is sufficiently large, then

$$d\mu'_m \leq (a+\varepsilon)dm$$

on $J_{1/m}$, which implies

$$||i_{\mu'_m}|| \leq (a+\varepsilon)^{1/p}.$$

Therefore

$$\lim_{m\to\infty} ||i_{\mu'_m}|| \leq a^{1/p}.$$

Conversely fix again an $\varepsilon > 0$. If m is sufficently large then $d\mu'_m \geq (a-\varepsilon)dm$ on $J_{1/m}$. Take the function

$$g_n(x) = (p\lambda_n + 1)^{1/p} x^{\lambda_n}$$

then $g_n \in M_\Lambda^p$ and $||g_n||_p = 1$, while

$$||i_{\mu'_m} g_n||^p_{L^p(\mu'_m)} = \int_{J_{1/m}} g_n^p d\mu \geq (a-\varepsilon)\int_{1-1/m}^1 g_n^p(x)dx$$

$$= (a-\varepsilon)[1-(1-1/m)^{p\lambda_n+1}] \longrightarrow a-\varepsilon$$

as $n \to \infty$. Therefore $||i_{\mu'_m}|| \geq (a-\varepsilon)^{1/p}$ for arbitrary $\varepsilon > 0$, hence

$$\lim_{m\to\infty} ||i_{\mu'_m}|| \geq a^{1/p}.$$

So $||i_\mu||_e = a^{1/p}$ by Theorem 2.2.3. $\square$

2.3 Sublinear Measures

Sublinear measures play for Müntz spaces, the role that *Carleson measures* play for the Hardy spaces H^p. As motivation for studying sublinear measures, we introduce Carleson measures first. Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk and $\mathbb{T}$ the unit circle. For $0 < p < \infty$, the Hardy space H^p consists of holomorphic functions f on $\mathbb{D}$ such that

$$||f||_p^p = \sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|^p d\theta < \infty.$$

It is well known that H^p is a Banach space with norm $||.||_p$ when $1 \leq p < \infty$. For $\xi \in \mathbb{T}$ and $0 < h < 1$, the *Carleson window* $W(\xi, h)$ centered at ξ and of size h is the set

$$W(\xi,h) = \{z \in \overline{\mathbb{D}} : |z| > 1-h, \quad |\arg(z\overline{\xi})| \leq h\}.$$

Then a finite positive measure μ on $\overline{\mathbb{D}}$ is called a *Carleson measure* if there exists a constant $C > 0$ such that

$$\sup_{|\xi|=1} \mu(W(\xi,h)) \leq Ch \tag{2.3.1}$$

for all $0 < h < 1$.

Carleson measures were introduced by L. Carleson in [6],[7] where he showed that μ is a Carleson measure if and only if there exists a constant $K > 0$ such that

$$||f||_{L^p(\mu)} \leq K||f||_{H^p} \tag{2.3.2}$$

for all $f \in H^p$, $0 < p < \infty$. A glance at (2.3.2) reveals a strong analogy with Λ-embedding measures (2.1.1). It is this analogy that motivates our definition of sublinear measures, where the role of the sets $W(\xi, h)$ is played by J_ε, in the hope of deducing necessary and sufficient conditions for μ to be Λ-embedding.

The class of sublinear measures was introduced in [8]. There they were used to characterize embedding operators $i_\mu : M_\Lambda^1 \hookrightarrow L^1(\mu)$ for the class of quasilacunary sequences Λ.

Definition 2.3.1. A measure μ is called *sublinear* if there is a constant $C > 0$ such that for any $0 < \varepsilon < 1$ we have $\mu(J_\varepsilon) \leq C\varepsilon$. The smallest such C will be denoted by $\|\mu\|_S$. The measure μ is called *vanishing sublinear* if $\lim_{\varepsilon\to 0} \frac{\mu(J_\varepsilon)}{\varepsilon} = 0$. Furthermore, a measure μ is called *α-sublinear* if $\mu(J_\varepsilon) \leq C\varepsilon^\alpha$ for some $\alpha > 1$.

There is little that Müntz spaces have in common with Hardy spaces. Hence it is striking that the cousin of Carleson measures on $[0, 1]$, namely, sublinear measures allow us to study the regularity of composition operators on M_Λ^p (their compactness or membership in Schatten classes) just as Carleson measures have done for composition operators on Hardy spaces H^p (see [14] and [16]). Composition operators on Müntz spaces will be the topic of Chapter 4.

We first prove a simple lemma.

Lemma 2.3.2. *If μ is Λ_p-embedding for $p \geq 1$, then for any $n \geq 1$ we have $\mu(J_{\frac{1}{\lambda_n}}) \leq C\frac{1}{\lambda_n}$, where C depends on the norm of the embedding $||i_\mu||$.*

Proof. Since $\lim_{n\to\infty}(1-\frac{1}{\lambda_n})^{\lambda_n} = \frac{1}{e}$, there exists a positive integer N such that, for all $n \geq N$ and for all $x \in [1-\frac{1}{\lambda_n}, 1]$, we have $x^{p\lambda_n} \geq \frac{1}{3^p}$. It follows that for all $n \geq N$

$$\frac{1}{3^p}\mu(J_{1/\lambda_n}) \leq \int_{J_{1/\lambda_n}} x^{p\lambda_n} d\mu \leq \int_{[0,1]} x^{p\lambda_n} d\mu \leq ||i_\mu||^p \int_0^1 x^{p\lambda_n} dx = \frac{||i_\mu||^p}{p\lambda_n + 1}.$$

Therefore, for all $n \geq N$, we have

$$\mu(J_{1/\lambda_n}) \leq \frac{3^p||i_\mu||^p}{p\lambda_n + 1} \leq \frac{3^p||i_\mu||^p}{\lambda_n} = \frac{C}{\lambda_n}.$$

□

Using this lemma, we obtain sublinearity as a necessary condition for Λ-embedding measures.

Proposition 2.3.3. *([8], Proposition 4.3) Suppose that there exists $M > 0$ such that $\frac{\lambda_{n+1}}{\lambda_n} \leq M$. If μ is Λ_p-embedding for any $p \geq 1$, then it is sublinear.*

Proof. It suffices to check sublinearity for ε sufficiently small. Since $1/\lambda_n$ is decreasing to 0, we may assume that for some n we have $\frac{1}{\lambda_{n+1}} \leq \varepsilon \leq \frac{1}{\lambda_n}$. Then by Lemma 2.3.2,

$$\mu(J_\varepsilon) \leq \mu(J_{\frac{1}{\lambda_n}}) \leq \frac{C}{\lambda_n} \leq \frac{CM}{\lambda_{n+1}} \leq CM\varepsilon.$$

Hence μ is a sublinear measure. □

We shall have occasion to utilize the next lemma in the final chapter to prove necessary conditions for boundedness and compactness of composition operators on M^2_Λ.

Lemma 2.3.4. *Let μ be a positive measure on $[0,1]$. Then the following hold:*

(i) If i^2_μ is bounded, then $\liminf_{\delta\to 0} \frac{\mu(J_\delta)}{\delta} < \infty$

(ii) If i^2_μ is compact, then $\liminf_{\delta\to 0} \frac{\mu(J_\delta)}{\delta} = 0$.

Proof. (i) Follows immediately from Lemma 2.3.2.

(ii) Choosing $f_n(x) = \lambda_n^{1/2} x^{\lambda_n}$, we see that

$$\langle f_n\,, x^{\lambda_k}\rangle = \int_{[0,1]} \lambda_n^{1/2} x^{\lambda_n+\lambda_k} dx = \frac{\lambda_n^{1/2}}{\lambda_n + \lambda_k + 1} \longrightarrow 0$$

as $n \to \infty$ for all $k \in \mathbb{N}$. Noting that $||f_n||_{L^2}$ is bounded and the linear span of the sequence $(x^{\lambda_k})_k$ is dense in M^2_Λ, it follows that $f_n \to 0$ weakly in M^2_Λ, as $n \to \infty$. If i^2_μ is compact, this implies that $(i^2_\mu\, f_n)_n$ converges strongly to 0 in $L^2(\mu)$ and hence $||f_n||_{L^2(\mu)} \to 0$ as $n \to 0$. Therefore

$$||f_n||^2_{L^2(\mu)} = \int_{[0,1]} \lambda_n x^{2\lambda_n} d\mu \geq \int_{J_{1/\lambda_n}} \lambda_n x^{2\lambda_n} d\mu \geq (1 - \frac{1}{\lambda_n})^{2\lambda_n} \frac{\mu(J_{1/\lambda_n})}{1/\lambda_n}.$$

Since $(1 - \frac{1}{\lambda_n})^{2\lambda_n} \to e^{-2}$ as $n \to \infty$, we get

$$\frac{\mu(J_{1/\lambda_n})}{1/\lambda_n} \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

and the result follows. □

As an immediate consequence of Lemma 1.2.4 with $\rho(x) = x$, we get

Lemma 2.3.5. *([8], Lemma 4.4) Suppose that $\|\mu\|_S < \infty$. If g is continuous, positive, and increasing, we have*

$$\int_{[0,1]} g d\mu \leq \|\mu\|_S \int_{[0,1]} g dm.$$

Corollary 4.5 in [8] was stated for $L^1(\mu)$ and can now be extended to other $L^p(\mu)$ for $p > 1$.

Corollary 2.3.6. *(i) If μ is sublinear, then*

$$\sup_{\lambda} \left\|(p\lambda+1)^{1/p}x^{\lambda}\right\|_{L^p(\mu)} \leq \|\mu\|_S^{1/p}.$$

(ii) If μ is vanishing sublinear, then

$$\lim_{\lambda\to\infty} ||\lambda^{1/p}x^{\lambda}||_{L^p(\mu)} = 0.$$

Proof. (i) Put $g = x^{p\lambda}$ in Lemma 2.3.5, then

$$\int_{[0,1]} x^{p\lambda}d\mu \leq ||\mu||_S \int_{[0,1]} x^{p\lambda}dm = ||\mu||_S \frac{1}{p\lambda+1}$$

$$\Rightarrow \quad \int_{[0,1]} (p\lambda+1)x^{p\lambda}d\mu \leq ||\mu||_S$$

$$\therefore \quad \sup_{\lambda} ||(p\lambda+1)^{1/p}x^{\lambda}||_{L^p(\mu)} \leq ||\mu||_S^{1/p}.$$

(ii) Fix $\varepsilon > 0$. By the definition of vanishing sublinear, $\exists\ \delta > 0$ such that $\mu(J_{\delta'}) \leq \varepsilon\delta'$ for all $\delta' \leq \delta$. If λ is large enough, then $\lambda x^{p\lambda} \leq \varepsilon$ for all $x \leq 1-\delta$, and we have

$$\int_{[0,1]} \lambda x^{p\lambda}d\mu = \int_{[0,1-\delta)} \lambda x^{p\lambda}d\mu + \int_{[1-\delta,1)} \lambda x^{p\lambda}d\mu$$

$$\leq \varepsilon||\mu|| + \int_{J_\delta} \lambda x^{p\lambda}d\mu.$$

For the second term above, let $\mu_\delta = \mu|_{J_\delta}$ and 0 elsewhere. Applying Lemma 2.3.5 to μ_δ and $g(x) = \lambda x^{p\lambda}$, we get

$$\int_{[0,1]} \lambda x^{p\lambda}d\mu_\delta \leq \varepsilon \int_{[0,1]} \lambda x^{p\lambda}dm = \frac{\varepsilon\lambda}{p\lambda+1} \leq \varepsilon.$$

Therefore

$$\lim_{\lambda\to 0} ||\lambda^{1/p}x^{\lambda}||_{L^p(\mu)} = 0$$

proves part (ii). □

2.4 Main Embedding Theorem

In this section we prove the main theorem of the chapter which is a culmination of the results that came before it and says that sublinear measures are Λ-embedding when Λ is lacunary in the Hilbert space case $p = 2$. This section will deal exclusively with the Hilbert-Müntz space M_Λ^2 and $i_\mu : M_\Lambda^2 \to L^2(\mu)$, and hence the results are understood to hold for $p = 2$.

Chalender, Fricain and Timotin in their paper [8] Theorem 5.5 proved the following: *If Λ is quasilacunary, then any sublinear measure μ is Λ-embedding for $p = 1$.* In Theorem 4.3 of [18] we were able to prove this for $p = 2$, but with Λ lacunary. Our proof does not extend easily to quasilacunary sequences and it remains an open problem.

Theorem 2.4.1. *([18], Theorem 4.3) Suppose Λ is lacunary. If μ is a sublinear measure, then μ is Λ_2-embedding.*

Proof. Since Λ is lacunary, the sequence of functions $g_n = \lambda_n^{1/2} x^{\lambda_n}$ form a Riesz basis in M_Λ^2. The embedding $i_\mu^2 : M_\Lambda^2 \to L^2(\mu)$ is defined by $i_\mu^2(g_n) = g_n$. In order to show that it is continuous, it is enough, by Lemma 1.4.5, to show that the matrix $A = (\langle g_n, g_m \rangle\rangle_{L^2(\mu)})$ defines a bounded operator in $\ell^2(\mathbb{N})$.

We have

$$\langle g_n, g_m \rangle_{L^2(\mu)} = \lambda_n^{1/2} \lambda_m^{1/2} \int_{[0,1]} x^{\lambda_n + \lambda_m} d\mu.$$

Since μ is sublinear and the function $x^{\lambda_n+\lambda_m}$ is continuous, positive, and increasing, it follows from Lemma 2.3.5 that

$$\int_{[0,1]} x^{\lambda_n+\lambda_m} d\mu \leq ||\mu||_S \int_{[0,1]} x^{\lambda_n+\lambda_m} dm,$$

and thus

$$\langle g_n, g_m\rangle_{L^2(\mu)} \leq ||\mu||_S \langle g_n, g_m\rangle_{M^2_\Lambda}.$$

We may then apply Lemma 1.4.6 to $a_{nm} = \langle g_n, g_m\rangle_{L^2(\mu)}$ and $b_{nm} = ||\mu||_S\langle g_n, g_m\rangle_{M^2_\Lambda}$. The matrix B is bounded since (g_n) is a Riesz basis in M^2_Λ; it follows that A is also bounded, which proves the theorem. □

Combining Proposition 2.3.3 and Theorem 2.4.1, we obtain

Corollary 2.4.2. *([18], Corollary 4.4) If Λ is lacunary with $\frac{\lambda_{n+1}}{\lambda_n} \leq M$ for some $M > 0$, then a measure μ is Λ-embedding if and only if it is sublinear.*

Using Theorem 2.4.1, we get vanishing sublinearity (definition 2.3.1) as a sufficient condition for compactness of the embedding.

Corollary 2.4.3. *([18], Corollary 4.5) If Λ is lacunary, then for any vanishing sublinear measure μ the embedding $i_\mu : M^2_\Lambda \to L^2(\mu)$ is compact.*

Proof. Recalling that μ_m is the measure equal to μ on $[0, \frac{1}{m}]$ and equal to 0 elsewhere on $[0, 1]$, we can view i_{μ_m} as the embeddings $M^2_\Lambda \hookrightarrow L^2(\mu_m)$ and regard $L^2(\mu_m)$ as a subspace of $L^2(\mu)$. Then $\mu'_m = \mu - \mu_m$ is the measure μ restricted to $J_{\frac{1}{m}}$ and by vanishing sublinearity and Theorem 2.4.1, μ'_m is Λ-embedding with embedding constants $||\mu'_m||_S \to 0$ as $m \to \infty$, that is

$$\lim_{m\to\infty} ||\mu'_m||_S \leq \lim_{m\to\infty} \frac{\mu'_m(J_{\frac{1}{m}})}{1/m} = \lim_{m\to\infty} \frac{\mu(J_{\frac{1}{m}})}{1/m} \longrightarrow 0.$$

Therefore by Lemma 2.3.5 we get

$$||i_\mu - i_{\mu_m}|| = ||i_{\mu'_m}|| \leq ||\mu'_m||_S \longrightarrow 0$$

as $m \to \infty$. Since i_{μ_m} is compact by Proposition 2.2.1, we see that i_μ is compact. □

2.5 Interpolation of Müntz Spaces

We have established that when Λ is lacunary, μ is Λ_p-embedding for the cases $p = 1$ and $p = 2$. It is interesting that, although the Müntz spaces do not form an interpolation scale of spaces, we may still apply the proof of the Riesz–Thorin theorem in order to extend the result to values $1 < p < 2$. We will actually obtain below a more general result concerning interpolation of embeddings.

As is often a necessary ingredient for applying complex interpolation, the *Three Lines Lemma* from complex function theory will be stated and proved for completeness sake.

Lemma 2.5.1. *Let ϕ be a bounded continuous function on the strip $0 \leq Re(z) \leq 1$ that is holomorphic on the interior of the strip. If $|\phi(z)| \leq M_0$ for $Re(z) = 0$ and $|\phi(z)| \leq M_1$ for $Re(z) = 1$, then $|\phi(z)| \leq M_0^{1-t} M_1^t$ for $Re(z) = t$, $0 < t < 1$.*

Proof. For $\varepsilon > 0$ let $\phi_\varepsilon(z) = \phi(z) M_0^{z-1} M_1^{-z} \exp(\varepsilon z(z-1))$. Then for $Re(z) = 0$, ϕ_ε satisfies

$$|\phi_\varepsilon(z)| = |\phi(z)| M_0^{-1} \exp(-\varepsilon y^2) \leq 1$$

where $y = Im(z)$, and for $Re(z) = 1$ we get

$$|\phi_\varepsilon(z)| = |\phi(z)| M_1^{-1} \exp(-\varepsilon y^2) \leq 1.$$

And also $|\phi_\varepsilon(z)| \to 0$ as $|Im(z)| \to \infty$. Thus $|\phi_\varepsilon(z)| \leq 1$ on the boundary of the rectangle $0 \leq Re(z) \leq 1$, $-A \leq Im(z) \leq A$ provided that A is large, and the maximum modulus principle therefore implies that $|\phi_\varepsilon(z)| \leq 1$ on the strip $0 \leq Re(z) \leq 1$. Letting $\varepsilon \to 0$, we obtain the desired result

$$|\phi(z)| M_0^{t-1} M_1^{-t} = \lim_{\varepsilon \to 0} |\phi_\varepsilon(z)| \leq 1$$

for $Re(z) = t$. $\square$

Theorem 2.5.2. *([18], Theorem 5.1) Suppose* $1 \leq p_0 < p_1 < \infty$. *For* $0 < t < 1$, *define* p_t *by*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}.$$

If a positive measure μ *on* $[0,1]$ *is* Λ_{p_0}*-embedding and* Λ_{p_1}*-embedding, then it is also* Λ_{p_t}*-embedding with*

$$||i_\mu||_{p_t} \leq ||i_\mu||_{p_0}^{1-t} ||i_\mu||_{p_1}^{t}$$

where $||i_\mu||_{p_s}$ *is the operator norm of* $i_\mu : M_\Lambda^{p_s} \to L^{p_s}(\mu)$ *for* $0 \leq s \leq 1$.

Proof. Let P_Λ be the space of all polynomials in $\text{span}\{x^\lambda : \lambda \in \Lambda\}$ and Σ the space of all simple functions on $[0,1]$ that vanish outside sets of finite measure with respect to μ. Then P_Λ is dense in M_Λ^p and Σ is dense in $L^p(\mu)$ for all $1 \leq p < \infty$. The theorem will be proved once we show that

$$||f||_{L^{p_t}(\mu)} \leq ||i_\mu||_{p_0}^{1-t} ||i_\mu||_{p_1}^{t} ||f||_{p_t}$$

for all $f \in P_\Lambda$. It is a consequence of the Riesz representation theorem that the dual space of $L^{p_t}(\mu)$ is $L^{p_t'}(\mu)$ where p_t' is the exponent conjugate to p_t and

$$||f||_{L^{p_t}(\mu)} = \sup_{g \in \Sigma} \{ | \int_{[0,1]} fg d\mu | : \ ||g||_{L^{p_t'}(\mu)} = 1 \}$$

for all $f \in P_\Lambda$, since $i_\mu(f) \in L^{p_0}(\mu) \cap L^{p_1}(\mu) \subset L^{p_t}(\mu)$. We may assume that $f \neq 0$ and rescale f so that $||f||_{p_t} = 1$. We therefore wish to establish the claim that if $f \in P_\Lambda$ and $||f||_{p_t} = 1$, then $| \int_{[0,1]} fg d\mu | \leq ||i_\mu||_{p_0}^{1-t} ||i_\mu||_{p_1}^{t}$ for all $g \in \Sigma$ such that $||g||_{L^{p_t'}(\mu)} = 1$.

Let $g = \sum_1^n d_k \chi_{E_k}$ where the E_k are disjoint measurable subsets of $[0,1]$ and the d_k are nonzero. Written in polar form, $d_k = |d_k| e^{i\psi_k}$. Now define

$$\alpha(z) = (1-z)p_0^{-1} + z p_1^{-1}$$

hence $\alpha(t) = p_t^{-1}$ for $0 < t < 1$. Fix $t \in (0,1)$ and since $\alpha(t) < 1$ define

$$g_z = \sum_{k=1}^{n} |d_k|^{(1-\alpha(z))/(1-\alpha(t))} e^{i\psi_k} \chi_{E_k}.$$

Notice that $g_t = g$. Also define

$$f_z(x) = |f(x)|^{p_t \alpha(z)} \frac{f(x)}{|f(x)|}$$

so that $f_t = f$ and finally define

$$\phi(z) = \int_{[0,1]} f_z g_z d\mu.$$

Thus

$$\phi(z) = \sum_k |d_k|^{(1-\alpha(z))/(1-\alpha(t))} e^{i\psi_k} \int_{[0,1]} f_z \chi_{E_k} d\mu.$$

So ϕ is an entire holomorphic function of z that is bounded in the strip $0 \leq Re(z) \leq 1$. Since $\phi(t) = \int_{[0,1]} fg d\mu$, by the three lines lemma it will suffice to show that $|\phi(z)| \leq ||i_\mu||_{p_0}$ for $Re(z) = 0$ and $|\phi(z)| \leq ||i_\mu||_{p_1}$ for $Re(z) = 1$.

Since $\alpha(is) = p_0^{-1} + is(p_1^{-1} - p_0^{-1})$ and $\alpha(1+is) = p_1^{-1} + is(p_1^{-1} - p_0^{-1})$ for $s \in \mathbb{R}$, we have

$$|f_{is}| = |f|^{Re[p_t \alpha(is)]} = |f|^{p_t/p_0}$$

$$|f_{1+is}| = |f|^{Re[p_t \alpha(1+is)]} = |f|^{p_t/p_1}$$

$$|g_{is}| = |g|^{Re[(1-\alpha(is))/(1-\alpha(t))]} = |g|^{p'_t/p'_0}$$

$$|g_{1+is}| = |g|^{Re[(1-\alpha(1+is))/(1-\alpha(t))]} = |g|^{p'_t/p'_1}$$

Therefore, by Hölder's inequality,

$$\begin{aligned} |\phi(is)| &\leq ||f_{is}||_{L^{p_0}(\mu)} ||g_{is}||_{L^{p'_0}(\mu)} \leq ||i_\mu||_{p_0} ||f_{is}||_{p_0} ||g_{is}||_{L^{p'_0}(\mu)} \\ &= ||i_\mu||_{p_0} ||f||_{p_t}^{p_t/p_0} ||g||_{L^{p'_t}(\mu)}^{p'_t/p'_0} = ||i_\mu||_{p_0} \end{aligned}$$

and

$$|\phi(1+is)| \leq ||f_{1+is}||_{L^{p_1}(\mu)}||g_{1+is}||_{L^{p_1'}(\mu)} \leq ||i_\mu||_{p_1}||f_{1+is}||_{p_1}||g_{1+is}||_{L^{p_1'}(\mu)}$$
$$= ||i_\mu||_{p_1}||f||_{p_t}^{p_t/p_1}||g||_{L^{p_t'}(\mu)}^{p_t'/p_1'} = ||i_\mu||_{p_1}$$

Hence we get $|\phi(is)| \leq ||i_\mu||_{p_0}$ and $|\phi(1+is)| \leq ||i_\mu||_{p_1}$ as required, so the three lines lemma completes the proof. □

Corollary 2.5.3. *([18], Corollary 5.2(i)) If Λ is lacunary, then every sublinear measure μ is Λ_p-embedding for $1 \leq p \leq 2$.*

Proof. Follows from the theorem since μ is Λ_1-embedding and Λ_2-embedding. □

Corollary 2.4.3 can now also be extended and the proof remains unchanged:

Corollary 2.5.4. *([18], Corollary 5.2(ii)) If Λ is lacunary, then for any vanishing sublinear measure μ the embedding $i_\mu : M_\Lambda^p \to L^p(\mu)$ is compact for $1 \leq p \leq 2$.*

Proof. Recall that μ_m is the measure equal to μ on $[0, \frac{1}{m}]$ and equal to 0 elsewhere on $[0,1]$. Then $\mu_m' = \mu - \mu_m$ is the measure μ restricted to $J_{\frac{1}{m}}$. Since vanishing sublinearity implies sublinearity, μ is Λ_p-embedding for all $1 \leq p \leq 2$, and hence so is μ_m'. Under these conditions, we showed in the proof of Corollary 2.4.3 that $||\mu_m'||_S \to 0$ as $m \to \infty$. Fixing some $p \in [1,2]$, by Lemma 2.3.5 and Theorem 2.5.2 we get

$$||i_\mu - i_{\mu_m}||_p = ||i_{\mu_m'}||_p \leq ||i_{\mu_m'}||_1^{1-t}||i_{\mu_m'}||_2^t \leq ||\mu_m'||_S^{1-t/2} \to 0$$

as $m \to \infty$, where $\frac{1}{p} = (1-t) + \frac{t}{2}$ for some $0 < t < 1$. Now $i_{\mu_m} : M_\Lambda^p \to L^p(\mu)$ is compact for each $m > 1$ by Proposition 2.2.1, hence $i_\mu : M_\Lambda^p \to L^p(\mu)$ is compact. □

Chapter 3

Müntz Space Embeddings of $\mathcal{S}_q$ Class

3.1 The Kernel Function ψ and Embeddings in M^2_Λ

Given that the Schatten-Von Neumann classes deal with Hilbert spaces, in this chapter we will consider only M^2_Λ and therefore we will drop the index p and write "Λ-embedding" and "i_μ". On the other hand, we will complicate things slightly by introducing also $M^2_{\Lambda,a}$ to be the closure of the same monomials x^{λ_n} in $L^2([0,a])$; so $M^2_\Lambda = M^2_{\Lambda,1}$.

In this section we introduce a special kernel function ψ that will help us to deduce interesting embedding and compactness criteria for M^2_Λ.

In section 1.2 we introduced the notion of minimal systems. It is known (see [5], pp. 177–178) that the condition $\sum_n 1/\lambda_n < \infty$ ensures that the system x^{λ_n} is minimal in M^2_Λ, and that, if d_n is the distance from x^{λ_n} to the linear space $\hat{\mathcal{P}}_n$ spanned by x^{λ_m} with $m \neq n$, then $d_n \geq e^{-\gamma_n \lambda_n}$, with $\gamma_n \to 0$. Let us denote

$$\psi(x) = \sum_{n \geq 1} \frac{x^{\lambda_n}}{d_n}. \tag{3.1.1}$$

The function ψ is clearly a non-negative increasing function of x and

$$|\psi(x)| \leq \sum_{n\geq 1} e^{\gamma_n \lambda_n} x^{\lambda_n} = \sum_{n\geq 1} (e^{\gamma_n} x)^{\lambda_n} < \infty$$

because for n sufficiently large we have $e^{\gamma_n} x < 1$ for any fixed $x < 1$. Hence the sum is convergent for any $x < 1$ and ψ is well-defined.

A simple argument of Hilbert space (also reproduced in [5], pp. 177–178) says that, if $p = \sum_i \alpha_i x^{\lambda_i}$, then

$$|\alpha_n| \leq d_n^{-1} \|p\|_2.$$

It follows then that, for any such polynomial $p \in M_\Lambda^2$, we have the estimate

$$\begin{aligned} |p^{(k)}(x)| &\leq \sum_i \lambda_i(\lambda_i - 1)\dots(\lambda_i - k + 1)|\alpha_i| x^{\lambda_i - k} \\ &\leq \sum_i \lambda_i(\lambda_i - 1)\dots(\lambda_i - k + 1) d_n^{-1} x^{\lambda_i - k} ||p||_2 \\ &= \psi^{(k)}(x)||p||_2, \qquad k = 0, 1, \dots \end{aligned}$$

We know that convergence in M_Λ^2 implies uniform convergence on compact subsets of $[0, 1)$ by Lemma 2.1.2, and since polynomials are dense in M_Λ^2 we may take limits on both sides of the above estimate to get

$$|f^{(k)}(x)| \leq \psi^{(k)}(x)\|f\|_2, \qquad k = 0, 1, \dots \tag{3.1.2}$$

($f^{(k)}$ denoting, as usual, the kth derivative of f).

Consider now $0 < a < 1$. If $d_n(a)$ is the distance in $M_{\Lambda,a}^2$ from x^{λ_n} to the space spanned by x^{λ_m} with $m \neq n$, then

$$d_n(a)^2 = \inf_{p\in\hat{\mathcal{P}}_n} \int_0^a |x^{\lambda_n} - p(x)|^2\, dx = \inf_{p\in\hat{\mathcal{P}}_n} \int_0^1 |a^{\lambda_n} t^{\lambda_n} - p(t)|^2 a\, dt = a^{2\lambda_n+1} d_n,$$

whence

$$\psi_a(x) := \sum_{n\geq 1} d_n(a)^{-1} x^{\lambda_n} = a^{-1/2}\psi(a^{-1}x).$$

We have thus the estimate, for functions in $M^2_{\Lambda,a}$,

$$|f(x)| \leq a^{-1/2}\psi(a^{-1}x)\|f\|_2. \tag{3.1.3}$$

In particular, if $a = 1$, we recapture (3.1.2) for $k = 1$.

Although the function ψ is a rather rough indicator of the properties of the sequences Λ, it is useful in obtaining sufficient conditions for embedding results. A first example is an analogue for M^2_Λ of [8, Theorem 2.6].

Theorem 3.1.1. *([18], Theorem 3.1) If $\psi \in L^2(\mu)$, then μ is Λ-embedding and $\|i_\mu\| \leq \|\psi\|_{L^2(\mu)}$.*

Proof. The proof follows immediately from the relation $|f(x)| \leq \psi(x)\|f\|_2$, which, as noted above, is the case $k = 1$ of (3.1.2). □

We obtain then the analogue for M^2_Λ of [8, Corollary 2.7].

Corollary 3.1.2. *([18], Corollary 3.2) Suppose $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing C^1 function with $\rho(0) = 0$, such that $\int_0^1 (\psi(x))^2 \rho'(1-x)\, dx < \infty$. If $\mu(J_\epsilon) \leq \rho(\epsilon)$ for all $\epsilon \in (0, 1]$, then μ is Λ-embedding.*

Proof. Similar to the L^1 case, the proof follows from Theorem 3.1.1 by applying Lemma 1.2.4 to $g = \psi^2$. □

More interestingly, we may improve Proposition 2.2.1: if the support of μ is compact in $[0, 1)$, then the embedding is not only compact, but inside any Schatten–von Neumann class.

Theorem 3.1.3. *([18], Theorem 3.3) If* $\operatorname{supp}\mu \subset [0, 1-\varepsilon]$*, then* $i_\mu \in \mathcal{S}_q$ *for any* $q > 0$*.*

Proof. Denote $b = 1 - \epsilon$. We have $\int \psi(x)^2 \, d\mu(x) \le \psi(b)^2 \|\mu\|$, and thus, by Theorem 3.1.1,

$$\|i_\mu\| \le \psi(b)\sqrt{\|\mu\|}. \tag{3.1.4}$$

Let us fix a positive integer k and a number $b < b' < 1$. If $f \in M^2_\Lambda$, then, by (3.1.2), we have

$$\int_0^{b'} |f^{(k)}(x)|^2 \, dx \le (\psi^{(k)}(b'))^2 \|f\|_2^2$$

and thus the k times differentiation operator D_k is bounded from M^2_Λ to $L^2([0, b'])$, of norm at most $\psi^{(k)}(b')$.

On the other hand, integration is a Hilbert–Schmidt operator on $L^2([0, b'])$, of $\mathcal{S}_2$ norm less than $2^{-1/2}$. It follows by Lemma 1.1.5 (i) that k times integration is an operator J_k in $\mathcal{S}_{2/k}$ on $L^2([0, b'])$, of norm at most $2^{-k/2}$. Denote by N_k the finite set of integers $\{0, 1, 2, \ldots, k-1\}$. Then the composition $J_k \circ D_k$ is the projection of functions in M^2_Λ onto functions in $M^2_{\Lambda \setminus N_k}$, but restricted to $[0, b']$; it is thus (again by Lemma 1.1.5 (i)) an operator of class $\mathcal{S}_{2/k}$ from M^2_Λ to $L^2([0, b'])$, whose image is in $M^2_{\Lambda \setminus N_k, b'}$. If we let $I_k : M^2_\Lambda \to L^2([0, b'])$ denote the finite-rank operator that projects functions in M^2_Λ onto functions in $M^2_{N_k}$ restricted to $[0, b']$, then $R_k = J_k \circ D_k + I_k$ is just the restriction of functions in M^2_Λ to $[0, b']$. Since finite-rank operators belong to all Schatten classes, we get $R_k \in \mathcal{S}_{2/k}$ and

$$\|R_k\|_{2/k} \le 2^{-k/2} \psi^{(k)}(b') + ||I_k||_{2/k}.$$

Consider then the embedding i'_μ from $M^2_{\Lambda, b'}$ into $L^2(\mu)$. According to (3.1.3) and (3.1.4), i'_μ is bounded and

$$\|i'_\mu\| \le \psi_{b'}(b)\sqrt{\|\mu\|} = b'^{-1/2}\psi(b/b')\sqrt{\|\mu\|}.$$

Finally, $i_\mu = i'_\mu R_k$, and thus

$$\|i_\mu\|_{2/k} \le (2^{-k/2}\psi^{(k)}(b') + ||I_k||_{2/k})b'^{-1/2}\psi(b/b')\sqrt{\|\mu\|}. \tag{3.1.5}$$

Choosing k such that $2/k \le q$, inequality (3.1.5) proves the theorem for any $q > 0$. □

If μ is a general measure, Theorem 3.1.3 can still be used in order to obtain sufficient conditions for the embedding to be in $\mathcal{S}_2$. Namely, we take a sequence $b_n \nearrow 1$ and define $\mu_j = \mu|[b_j, b_{j+1})$; then $i^2_\mu = \sum_j i^2_{\mu_j}$, and thus we have (for $q \ge 1$) $\|i^2_\mu\|_q \le \sum_j \|i^2_{\mu_j}\|_q$. We may then apply Theorem 3.1.3 to each of the measures μ_j. The statements obtained are, however, not very natural, depending on the arbitrary sequence (b_j). We prefer to give a more elegant result, valid for embedding in $\mathcal{S}_2$.

Theorem 3.1.4. *([18], Theorem 3.4) Define $\Psi(x) = \psi'(x^{1/4})\psi(x^{1/4})$. If $\Psi \in L^2(\mu)$, then $i_\mu \in \mathcal{S}_2$.*

Proof. Consider the sequence b_n defined by $b_0 = 0$, $b_1 = 1/2$, and $b_{j+1} = \sqrt{b_j}$; obviously it is an increasing sequence tending to 1. Define also, for $j \ge 0$, $\mu_j = \mu|[b_j, b_{j+1})$; we have

$$L^2(\mu) = \bigoplus_{j=0}^{\infty} L^2(\mu_j). \tag{3.1.6}$$

We have $\|i_\mu\|_2^2 = \sum_{j=0}^\infty \|i_{\mu_j}\|_2^2$. By Theorem 3.1.3 applied with $b = b_{j+1}$ and $b' = b_{j+2}$, we have, for some constant $C > 0$,

$$\|i_{\mu_j}\|_2^2 \le C(\psi'(b_{j+2}))^2 \left(\psi\left(\frac{b_{j+1}}{b_{j+2}}\right)\right)^2 \|\mu_j\| = C(\psi'(b_{j+2}))^2(\psi(b_{j+2}))^2\|\mu_j\| = C(\Psi(b_j))^2\|\mu_j\|.$$

Since Ψ is increasing the last term is less or equal $C\int \Psi(x)^2\, d\mu_j(x)$. Therefore

$$\|i_\mu\|_2^2 = \sum_{j=0}^{\infty} \|i_{\mu_j}\|_2^2 \le C\sum_{j=0}^{\infty} \int \Psi(x)^2\, d\mu_j(x) = \int \Psi(x)^2\, d\mu(x),$$

which proves the theorem. □

3.2 Embeddings of Schatten-Von Neumann Class

In this section we investigate conditions when $i_\mu \in \mathcal{S}_q(M_\Lambda^2, L^2(\mu))$ for $q > 0$.

If Λ is lacunary, then with the help of the asymptotic formula for the Euler beta function, we deduce the following sufficient condition for $\mathcal{S}_q$:

Theorem 3.2.1. *For Λ lacunary, if a positive measure μ is α-sublinear then $i_\mu \in \mathcal{S}_q(M_\Lambda^2, L^2(\mu))$ for all $q > 0$.*

Proof. The condition $\mu(J_\varepsilon) \le C\varepsilon^\alpha$ for $\alpha > 1$ implies that μ is vanishing sublinear. Hence i_μ is compact by Corollary 2.4.3. To show that $i_\mu \in \mathcal{S}_q(M_\Lambda^2, L^2(\mu))$ for all $q > 0$, we will make use of the following asymptotic formula for the Euler beta function: if s is large and t is fixed, then

$$B(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}dx \sim \Gamma(t)s^{-t}.$$

Using Lemma 1.2.4 with $\rho(x) = Cx^\alpha$ and this formula, we get

$$\begin{aligned}
\left\|i_\mu(\lambda_n^{1/2}x^{\lambda_n})\right\|_{L^2(\mu)}^2 &= \int_{[0,1]} \lambda_n x^{2\lambda_n} d\mu \le \int_0^1 \lambda_n x^{2\lambda_n}(\alpha C(1-x)^{\alpha-1})dx \\
&= \alpha C\lambda_n \int_0^1 x^{2\lambda_n}(1-x)^{\alpha-1}dx \\
&= \alpha C\lambda_n B(2\lambda_n+1, \alpha) \\
&\sim \alpha C\lambda_n \Gamma(\alpha)(2\lambda_n+1)^{-\alpha} \\
&\le \alpha C\Gamma(\alpha)\frac{1}{\lambda_n^{\alpha-1}}.
\end{aligned}$$

So

$$\sum_{n=1}^{\infty} \left\| i_\mu(\lambda_n^{1/2} x^{\lambda_n}) \right\|_{L^2(\mu)}^q \lesssim \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{(\alpha-1)q/2}}.$$

Now just like before, $\exists\, r > 1$ such that $\lambda_{n+1}/\lambda_n \geq r$ for all $n \geq 1$, so

$$(\frac{1}{\lambda_n})^{(\alpha-1)q/2} \leq \cdots \leq (\frac{1}{r^{n-1}\lambda_1})^{(\alpha-1)q/2} = \lambda_1^{(1-\alpha)q/2}(\frac{1}{r^{(\alpha-1)q/2}})^{n-1}$$

and clearly $1/r^{(\alpha-1)q/2} < 1$ for $\alpha > 1$. Therefore

$$\sum_{n=1}^{\infty} \left\| i_\mu(\lambda_n^{1/2} x^{\lambda_n}) \right\|_{L^2(\mu)}^q \lesssim \sum_{n=1}^{\infty} (\frac{1}{r^{(\alpha-1)q/2}})^{n-1} < \infty.$$

Now using the orthogonalizer T introduced at the end of section 1.3, we get

$$\|i_\mu T\|_q^q = \inf \sum_{\alpha} \|i_\mu T \omega_\alpha\|_{L^2(\mu)}^q \leq \sum_{n=1}^{\infty} \|i_\mu T e_n\|_{L^2(\mu)}^q = \sum_{n=1}^{\infty} \|i_\mu(\lambda_n^{1/2} x^{\lambda_n})\|_{L^2(\mu)}^q < \infty$$

the infimum being taken over all orthonormal bases $\{\omega_\alpha\}$ of M_Λ^2. Therefore $i_\mu T \in \mathcal{S}_q(M_\Lambda^2, L^2(\mu))$ for all $q \leq 2$. So by Proposition 1.1.4(b), $i_\mu T \in \mathcal{S}_q(M_\Lambda^2, L^2(\mu))$ for all $q > 0$. Hence $i_\mu = (i_\mu T)T^{-1} \in \mathcal{S}_q(M_\Lambda^2, L^2(\mu))$ for all $q > 0$ by Proposition 1.1.4(d). □

For the quasilacunary case, the standard arguments involve finding uniform estimates for each block F_k, and then trying to *glue* them together:

Theorem 3.2.2. *([18], Theorem 6.1) For Λ quasilacunary, if a positive measure μ is α-sublinear then $i_\mu \in \mathcal{S}_q(M_\Lambda^2, L^2(\mu))$ for all $q > 0$.*

Proof. Firstly for each $F_k = \operatorname{span}\{x^{\lambda_{n_k+1}}, \ldots, x^{\lambda_{n_{k+1}}}\}$, $k = 0, 1, 2, \ldots, \infty$, we choose an arbitrary orthonormal basis $\{\phi_i\}_{i=n_k+1}^{n_{k+1}}$. We set $n_0 = 0$. We then show that the union of these bases $\cup_k \{\phi_i\}_{i=n_k+1}^{n_{k+1}} = \{\phi_i\}_{i=1}^{\infty}$ is a Riesz basis for M_Λ^2. For any sequence

$\{c_i\}_{i=1}^{\infty}$ of scalars, define $f_k = \sum_{i=n_k+1}^{n_{k+1}} c_i\phi_i \in F_k$. Then by Lemma 1.3.5 with $p = 2$ there exist constants $A, B > 0$ such that

$$A \sum_k \|f_k\|_2^2 \le \left\| \sum_k f_k \right\|_2^2 \le B \sum_k \|f_k\|_2^2$$

but since the ϕ_i are orthonormal

$$A \sum_k (\sum_{i=n_k+1}^{n_{k+1}} |c_i|)^2 \le \left\| \sum_k f_k \right\|_2^2 \le B \sum_k (\sum_{i=n_k+1}^{n_{k+1}} |c_i|)^2.$$

Since $(a+b)^2 \le 2^2(a^2+b^2)$ for $a, b > 0$ and $N = \sup_k(n_{k+1} - n_k) < \infty$, we get

$$A \sum_k \sum_{i=n_k+1}^{n_{k+1}} |c_i|^2 \le \left\| \sum_{i=1}^{\infty} c_i\phi_i \right\|_2^2 \le 2^{2N} B \sum_k \sum_{i=n_k+1}^{n_{k+1}} |c_i|^2 .$$

Therefore

$$A \sum_{i=1}^{\infty} |c_i|^2 \le \left\| \sum_{i=1}^{\infty} c_i\phi_i \right\|_2^2 \le 2^{2N} B \sum_{i=1}^{\infty} |c_i|^2$$

and hence $\{\phi_i\}_{i=1}^{\infty}$ is a Riesz basis by definition 1.4.1. So $\exists$ an invertible operator $S : M_\Lambda^2 \longrightarrow M_\Lambda^2$ and an orthonormal basis $\{\psi_i\}_{i=1}^{\infty}$ of M_Λ^2 such that $S\psi_i = \phi_i$. So by Lemma 1.1.5 for $q \le 2$

$$\|i_\mu S\|_q^q = \inf \sum_\alpha \|i_\mu S\omega_\alpha\|_{L^2(\mu)}^q \le \sum_{i=1}^{\infty} \|i_\mu S\psi_i\|_{L^2(\mu)}^q = \sum_{i=1}^{\infty} \|i_\mu \phi_i\|_{L^2(\mu)}^q \qquad (3.2.1)$$

the infimum being taken over all orthonormal bases $\{\omega_\alpha\}$ of M_Λ^2. We want to show that the last series in (3.2.1) converges. To do so we estimate each term of the sum.

We fix a k and select a $\phi_i \in F_k$ where i is one of $n_k + 1, \dots, n_{k+1}$. First applying Lemma 1.3.3 to ϕ_i with $\beta_j = \beta = 1/(n_{k+1} - n_k)$, then using Lemma 1.2.4 with

$g(x) = x^{2\beta\lambda_{n_k+1}}$ and $\rho(x) = Cx^\alpha$, and finally the Euler beta function, we have

$$\begin{aligned}
\|i_\mu \phi_i\|^2_{L^2(\mu)} = \int_0^1 |\phi_i(x)|^2\, d\mu(x) &\leq 4\,\|\phi_i\|^2_\infty \int_0^1 (\sum_{j=n_k+1}^{n_{k+1}} x^{\lambda_j\beta_j})^2 d\mu(x) \\
&\leq 4^{(N+1)}\,\|\phi_i\|^2_\infty \sum_{j=n_k+1}^{n_{k+1}} \int_0^1 x^{2\lambda_j\beta_j} d\mu(x) \\
&\leq 4^{(N+1)}\,\|\phi_i\|^2_\infty N \int_0^1 x^{2\beta\lambda_{n_k+1}} d\mu(x) \\
&\leq 4^{(N+1)}\,\|\phi_i\|^2_\infty N \int_0^1 x^{2\beta\lambda_{n_k+1}} (\alpha C(1-x)^{\alpha-1}) dx \\
&= 4^{(N+1)}\alpha C\,\|\phi_i\|^2_\infty NB(2\beta\lambda_{n_k+1}+1, \alpha) \\
&\sim 4^{(N+1)}\alpha C\,\|\phi_i\|^2_\infty N\Gamma(\alpha)(2\beta\lambda_{n_k+1}+1)^{-\alpha} \\
&\leq (4^{(N+1)}\alpha CN\Gamma(\alpha)\beta^{-\alpha}) \frac{\|\phi_i\|^2_\infty}{\lambda^\alpha_{n_k+1}}
\end{aligned}$$

hence

$$\|i_\mu \phi_i\|_{L^2(\mu)} \lesssim \frac{\|\phi_i\|_\infty}{\lambda^{\alpha/2}_{n_k+1}}. \tag{3.2.2}$$

Based on Lemma 1.3.7 we consider two cases:

Case 1: If $\|\phi_i\|_\infty \leq 4\,\|\phi_i\|_1 \leq 4\,\|\phi_i\|_2 = 4$, then

$$\|i_\mu \phi_i\|_{L^2(\mu)} \lesssim \frac{4}{\lambda^{\alpha/2}_{n_k}} \leq \frac{1}{q^{\alpha/2}\lambda^{\alpha/2}_{n_{k-1}}} \cdots \leq \frac{1}{q^{\alpha(k-1)/2}\lambda_{n_1}} \lesssim \frac{1}{q^{\alpha(k-1)/2}} \tag{3.2.3}$$

$\forall\ i = n_k+1, \ldots, n_{k+1}$, for some $q > 1$ since Λ is quasilacunary.

Case 2: Now if the minimum in Lemma 1.3.7 is given by the first term, then apply the lemma to $g = \phi_i^2$. We get

$$\|g\|_1 \geq \frac{\|g\|^2_\infty}{2\,\|g'\|_\infty} \quad \Rightarrow \quad \|g\|^2_\infty \leq 2\,\|g\|_1\,\|g'\|_\infty\,. \tag{3.2.4}$$

Now $\|g\|_\infty = \|\phi_i\|_\infty^2$, $\|g\|_1 = \|\phi_i\|_2^2$. so one gets

$$\|g'\|_\infty = 2\|\phi_i'\phi_i\|_\infty \le 2\|\phi_i'\|_\infty \|\phi_i\|_\infty .$$

Then inequality (3.2.4) becomes

$$\begin{aligned}\|\phi_i\|_\infty^4 &\le 4\|\phi_i\|_2^2 \|\phi_i'\|_\infty \|\phi_i\|_\infty \le 4KN\lambda_{n_{k+1}} \|\phi_i\|_\infty \|\phi_i\|_\infty \\ &= 4KN\lambda_{n_{k+1}} \|\phi_i\|_\infty^2\end{aligned}$$

using Lemma 1.3.4 for the second inequality and the fact that $\|\phi_i\|_2 = 1$. After cancelling $\|\phi_i\|_\infty^2$ from both sides and taking the power 1/2, one obtains

$$\|\phi_i\|_\infty \le 2(KN\lambda_{n_{k+1}})^{1/2}.$$

Plugging this into (3.2.2), one obtains

$$\|i_\mu\phi_i\|_{L^2(\mu)} \lesssim \frac{\|\phi_i\|_\infty}{\lambda_{n_k+1}^{\alpha/2}} \le \frac{2(KN\lambda_{n_{k+1}})^{1/2}}{\lambda_{n_k+1}^{\alpha/2}} \le 2(KN)^{1/2}(\frac{\lambda_{n_{k+1}}}{\lambda_{n_k+1}})^{1/2}\frac{1}{\lambda_{n_k+1}{}^{(\alpha-1)/2}}.$$

The factor $(\lambda_{n_{k+1}}/\lambda_{n_k+1})^{1/2}$ is bounded by (1.3.1). Therefore just as in the previous theorem

$$\|i_\mu\phi_i\|_{L^2(\mu)} \lesssim \frac{1}{\lambda_{n_k}^{\frac{\alpha-1}{2}}} \le \frac{1}{q^{\frac{\alpha-1}{2}}\lambda_{n_{k-1}}^{\frac{\alpha-1}{2}}} \le \ldots \le \frac{1}{r^{\frac{(\alpha-1)(k-1)}{2}}\lambda_{n_1}^{\frac{\alpha-1}{2}}} \lesssim \frac{1}{r^{\frac{(\alpha-1)(k-1)}{2}}}. \qquad (3.2.5)$$

$\forall\ i = n_k + 1, \ldots, n_{k+1}$.

Since each ϕ_i for $i = 1, 2, \ldots, \infty$ can belong to either case 1 or 2, by (3.2.1), (3.2.3) and (3.2.5) we get

$$\begin{aligned}\|i_\mu S\|_q^q &\le \sum_{i=1}^{\infty} \|i_\mu\phi_i\|_{L^2(\mu)}^q = \sum_{k=0}^{\infty}\sum_{i=n_k+1}^{n_{k+1}} \|i_\mu\phi_i\|_{L^2(\mu)}^q \\ &\lesssim N\sum_{k=0}^{\infty}\frac{1}{(r^{\alpha q/2})^{k-1}} + N\sum_{k=0}^{\infty}\frac{1}{(r^{(\alpha-1)q/2})^{k-1}} < \infty\end{aligned}$$

hence $i_\mu S \in \mathcal{S}_q(M^2_\Lambda, L^2(\mu))$ for all $q \leq 2$. But since $\mathcal{S}_q \subset \mathcal{S}_{q'}$ for $q < q'$, $i_\mu S \in \mathcal{S}_q(M^2_\Lambda, L^2(\mu))$ for all $q > 0$. Since S is invertible and $\mathcal{S}_q(M^2_\Lambda, L^2(\mu))$ is a right ideal, we get $i_\mu \in \mathcal{S}_q(M^2_\Lambda, L^2(\mu))$ for all $q > 0$. □

Chapter 4

Examples of Discrete Embedding Measures

In this chapter we collect together two examples from our article [18].

In the first example we construct a discrete measure μ and a lacunary sequence Λ such that μ is Λ-embedding for $p = 2$ but not for $p = 1$.

We will take $\mu = \sum_k c_k \delta_{a_k}$, with $0 < a_k < 1$. We construct recurrently λ_n, a_n, c_n such as to have:

(A) $\sup_n \lambda_n c_n a_n^{\lambda_n} = \infty$

(B) $\sum_k \lambda_n c_k a_k^{2\lambda_n} \lesssim \frac{\ln n}{n^2}$.

Suppose λ_k, a_k, c_k have been obtained for $k \leq n-1$. We choose λ_n sufficiently large such that

(i) $\lambda_n \sum_{k \leq n-1} a_k^{\lambda_n} \leq \frac{1}{n^2}$.

(ii) $\lambda_{n+1} \geq n^4 \lambda_n$.

We put $a_n = 1 - \frac{2 \ln n}{\lambda_n}$ and $c_n = \frac{2n^2 \ln n}{\lambda_n}$. Then

$$a_n^{\lambda_n} = (1 - \frac{2 \ln n}{\lambda_n})^{\lambda_n} \sim e^{-2 \ln n} = \frac{1}{n^2}, \tag{4.0.1}$$

whence

$$\lambda_n c_n a_n^{\lambda_n} \sim \ln n \tag{4.0.2}$$

and thus (A) is satisfied. To, achieve (B), we write

$$\sum_k \lambda_n c_k a_k^{2\lambda_n} = \sum_{k\le n-1} \lambda_n c_k a_k^{2\lambda_n} + \lambda_n c_n a_n^{2\lambda_n} + \sum_{k\ge n+1} \lambda_n c_k a_k^{2\lambda_n}. \tag{4.0.3}$$

The first sum is smaller than $\frac{1}{n^2}$ by (i). The second term is of order $\frac{\ln n}{n^2}$ by (4.0.1) and (4.0.2). For the third term, we have

$$\sum_{k\ge n+1} \lambda_n c_k a_k^{2\lambda_n} \le \lambda_n \sum_{k\ge n+1} c_k.$$

From (ii) it follows in particular that c_n decreases faster than a geometric progression (which also proves the convergence of the sum defining μ), and thus, for some constant C we have

$$\sum_{k\ge n+1} c_k \le C c_{n+1} = \frac{C2(n+1)^2 \ln(n+1)}{\lambda_{n+1}}$$

and thus again by (ii),

$$\lambda_n \sum_{k\ge n+1} c_k \le C' \frac{\ln n}{n^2}.$$

So we have estimated all three terms of (4.0.3) by $\frac{\ln n}{n^2}$, thus (B) is satisfied. Now (ii) implies that Λ is lacunary and since M_Λ^2 is a Hilbert space, the functions $g_k(x) = \lambda_k^{1/2} x^{\lambda_k}$ form a Riesz basis in M_Λ^2 by section 1.4, thus by definition

$$\| \sum_k b_k g_k \|_2^2 \sim \sum_k |b_k|^2 . \tag{4.0.4}$$

for any finite sequence $\{b_k\}_k$. We have then

$$\| \sum_k b_k g_k \|_{L^2(\mu)}^2 \le (\sum_k |b_k| \, \|g_k\|_{L^2(\mu)})^2 \le (\sum_k |b_k|^2)(\sum_k \|g_k\|_{L^2(\mu)}^2). \tag{4.0.5}$$

According to (B), we have

$$\|g_n\|_{L^2(\mu)}^2 = \sum_k c_k \lambda_n a_k^{2\lambda_n} \le C'' \frac{\ln n}{n^2},$$

and thus

$$\sum_k \|g_k\|^2_{L^2(\mu)} < \infty.$$

So it follows from (4.0.4) and (4.0.5) that

$$\|\sum_k b_k g_k\|^2_{L^2(\mu)} \lesssim (\sum_k \|g_k\|^2_{L^2(\mu)})(\|\sum_k b_k g_k\|^2_2).$$

So for $p = 2$, μ is Λ-embedding. On the contrary, for $p = 1$ we have

$$\left\|\lambda_n x^{\lambda_n}\right\|_{L^1(\mu)} = \int_{[0,1]} \lambda_n x^{\lambda_n} d\mu = \sum_k c_k \lambda_n a_k^{\lambda_n} \geq c_n \lambda_n a_n^{\lambda_n}.$$

So by (A)

$$\sup_n \left\|\lambda_n x^{\lambda_n}\right\|_{L^1(\mu)} \geq \sup_n c_n \lambda_n a_n^{\lambda_n} = \infty$$

whereas

$$\left\|\lambda_n x^{\lambda_n}\right\|_{L^1(m)} \leq 1 \quad \forall\ n = 1, 2, \ldots.$$

Hence μ is not Λ-embedding for $p = 1$. $\square$

In the second example we will show that for *any* $0 < p < q$ we can construct a lacunary sequence Λ and a measure μ (both depending on p and q), such that $i_\mu : M^2_\Lambda \to L^2(\mu)$ is bounded and $i_\mu \notin \mathcal{S}_p(M^2_\Lambda, L^2(\mu))$ but $i_\mu \in \mathcal{S}_q(M^2_\Lambda, L^2(\mu))$.

Fix $q > p > 0$. We take $\mu = \sum_j c_j \delta_{a_j}$, with $0 < a_j < 1$. We choose a sequence $\{\alpha_n\} \in \ell^q$ with $|\alpha_n| < 1$ but $\{\alpha_n\} \notin \ell^p$. We now construct a_n, c_n, λ_n recurrently. Choose a double sequence $\{\beta_{nm}\}_{n,m=1}^\infty$ in such a way that

$$\sum_n \sum_m \beta_{nm} < \frac{1}{4}$$

for example $\beta_{nm} = c \cdot 2^{-(n+m)}$ for a suitable $c > 0$. First we note that for $0 < a < 1$ the function $\lambda \mapsto \lambda^{1/2} a^\lambda$ attains its maximum at the point $\lambda_M = -\frac{1}{2\ln a}$. Since

$a^{\lambda_M} = e^{\lambda_M \ln a} = e^{-1/2}$, it follows that

$$\frac{1}{e}\lambda_M^{1/2} < \lambda_M^{1/2} a^{\lambda_M} < \lambda_M^{1/2}. \tag{4.0.6}$$

Suppose a_j, c_j, λ_j have been obtained for $j \leq n-1$. We choose a_n, c_n, λ_n as follows:

(A) Choose λ_n sufficiently large such that Λ is lacunary, and

$$\sum_{i=1}^{n-1} c_i \lambda_n a_i^{2\lambda_n} \leq \frac{1}{8}\alpha_n^2 \tag{4.0.7}$$

and for $j = 1, \ldots, n-1$

$$\sum_{i=1}^{n-1} c_i \lambda_j^{1/2} \lambda_n^{1/2} a_i^{\lambda_j + \lambda_n} \leq \frac{1}{4}\alpha_j \alpha_n \beta_{jn}^{1/2} \tag{4.0.8}$$

and for $i, j < n$

$$\frac{\alpha_n^2 \lambda_i^{1/2} \lambda_j^{1/2}}{\lambda_n} \leq \frac{1}{2^{n+2-\max\{i,j\}}} \alpha_i \alpha_j \beta_{ij}^{1/2} \tag{4.0.9}$$

and finally for $i = 1, \ldots, n-1$

$$\frac{\alpha_n^2 \lambda_i^{1/2}}{\lambda_n^{1/2}} \leq \frac{1}{2}\alpha_i \alpha_n \beta_{in}^{1/2}. \tag{4.0.10}$$

(B) Take $a_n = e^{-1/2\lambda_n}$, then $\lambda_n^{1/2} a_n^{\lambda_n} \sim \lambda_n^{1/2}$ by (4.0.6). So $\lambda_n a_n^{2\lambda_n} \sim \lambda_n$. Now choosing $c_n = \frac{\alpha_n^2}{\lambda_n}$, we get

$$c_n \lambda_n a_n^{2\lambda_n} \sim \alpha_n^2 \tag{4.0.11}$$

Using our Riesz basis $\{g_n\} = \{\lambda_n^{1/2} x^{\lambda_n}\}$ for M_Λ^2, (4.0.7), (4.0.9) and (4.0.11) give us

$$\|i_\mu g_n\|_{L^2(\mu)} \sim \alpha_n \tag{4.0.12}$$

since

$$
\begin{aligned}
\|i_\mu g_n\|^2_{L^2(\mu)} &= \int_{[0,1]} \lambda_n x^{2\lambda_n} d\mu = \sum_j c_j \lambda_n a_j^{2\lambda_n} \\
&= \sum_{j\le n-1} c_j \lambda_n a_j^{2\lambda_n} + c_n \lambda_n a_n^{2\lambda_n} + \sum_{j\ge n+1} c_j \lambda_n a_j^{2\lambda_n} \\
&\lesssim \frac{1}{8}\alpha_n^2 + \alpha_n^2 + \sum_{j\ge n+1} \frac{\alpha_j^2 \lambda_n a_j^{2\lambda_n}}{\lambda_j} \\
&\le \frac{1}{8}\alpha_n^2 + \alpha_n^2 + \sum_{j\ge n+1} \frac{1}{2^{j+2-n}} \alpha_n^2 \beta_{nn} \\
&\le \frac{1}{8}\alpha_n^2 + \alpha_n^2 + \frac{1}{4}\alpha_n^2 < \frac{3}{2}\alpha_n^2
\end{aligned}
$$

and clearly $\|i_\mu g_n\|_{L^2(\mu)} \gtrsim \alpha_n$ by (4.0.11).

Now if we define $f_n = i_\mu g_n / \|i_\mu g_n\|_{L^2(\mu)}$, then using (4.0.8), (4.0.9), (4.0.10) and (4.0.12) we get for $n > m$

$$
\begin{aligned}
&\langle f_n, f_m\rangle_{L^2(\mu)} \lesssim \alpha_n^{-1}\alpha_m^{-1}\lambda_n^{1/2}\lambda_m^{1/2} \int_{[0,1]} x^{\lambda_n+\lambda_m} d\mu(x) = \alpha_n^{-1}\alpha_m^{-1}\lambda_n^{1/2}\lambda_m^{1/2} \sum_j c_j a_j^{\lambda_n+\lambda_m} \\
&= \alpha_n^{-1}\alpha_m^{-1}\Big(\sum_{j\le n-1} c_j \lambda_n^{1/2}\lambda_m^{1/2} a_j^{\lambda_n+\lambda_m} + c_n \lambda_n^{1/2}\lambda_m^{1/2} a_n^{\lambda_n+\lambda_m} + \sum_{j\ge n+1} c_j \lambda_n^{1/2}\lambda_m^{1/2} a_j^{\lambda_n+\lambda_m}\Big) \\
&\le \alpha_n^{-1}\alpha_m^{-1}\Big(\frac{1}{4}\alpha_n\alpha_m\beta_{nm}^{1/2} + \frac{\alpha_n^2}{\lambda_n}\lambda_n^{1/2}\lambda_m^{1/2} a_n^{\lambda_n+\lambda_m} + \sum_{j\ge n+1} \frac{\alpha_j^2}{\lambda_j}\lambda_n^{1/2}\lambda_m^{1/2} a_j^{\lambda_n+\lambda_m}\Big) \\
&\le \alpha_n^{-1}\alpha_m^{-1}\Big(\frac{1}{4}\alpha_n\alpha_m\beta_{nm}^{1/2} + \frac{\alpha_n^2\lambda_m^{1/2}}{\lambda_n^{1/2}} + \sum_{j\ge n+1} \frac{\alpha_j^2\lambda_n^{1/2}\lambda_m^{1/2}}{\lambda_j}\Big) \\
&\le \alpha_n^{-1}\alpha_m^{-1}\Big(\frac{1}{4}\alpha_n\alpha_m\beta_{nm}^{1/2} + \frac{1}{2}\alpha_m\alpha_n\beta_{mn}^{1/2} + \sum_{j\ge n+1} \frac{1}{2^{j+2-n}}\alpha_m\alpha_n\beta_{mn}^{1/2}\Big) \\
&\le \frac{1}{4}\beta_{nm}^{1/2} + \frac{1}{2}\beta_{nm}^{1/2} + \frac{1}{4}\beta_{nm}^{1/2} = \beta_{nm}^{1/2}.
\end{aligned}
$$

Therefore

$$
\sum_{n\ne m} \left|\langle f_n, f_m\rangle_{L^2(\mu)}\right|^2 = 2\sum_{n>m} \langle f_n, f_m\rangle^2_{L^2(\mu)} \le \sum_{n,m} \beta_{nm} < \frac{1}{4}. \tag{4.0.13}
$$

We claim that with the measure μ and sequence Λ constructed above, the embedding $i_\mu : M^2_\Lambda \to L^2(\mu)$ has the desired properties, i.e. $i_\mu \in \mathcal{S}_q(M^2_\Lambda, L^2(\mu))$ but $i_\mu \notin \mathcal{S}_p(M^2_\Lambda, L^2(\mu))$. To justify our claims, we first show that the sequence $f_n = i_\mu g_n / \|i_\mu g_n\|_{L^2(\mu)}$ is a Riesz sequence in $L^2(\mu)$ (not necessarily a basis). This amounts to showing that the Gram matrix $\Gamma = \{\langle f_n, f_m\rangle\}_{n,m=1}^\infty$ defines an invertible operator on $\ell^2(\mathbb{N})$. This result can be found in Theorem 3.6.6 of [9]. Define $\Gamma_0 = \Gamma - I$ so that $\Gamma = I + \Gamma_0$. To show Γ is invertible it is sufficient to show that $\|\Gamma_0\| < \frac{1}{2}$. Further since $\|\Gamma_0\| \leq \|\Gamma_0\|_2$, the Hilbert-Schmidt norm, it suffices to show $\|\Gamma_0\|_2 \leq \frac{1}{2}$. If $\{e_n\}$ is the standard ℓ^2 orthonormal basis, then

$$\begin{aligned}\|\Gamma_0\|_2^2 &= \sum_n \|\Gamma_0 e_n\|_{\ell^2}^2 = \sum_{n,m} |\langle \Gamma_0 e_n, e_m\rangle_{\ell^2}|^2 = \sum_{n\neq m} |\langle \Gamma e_n, e_m\rangle_{\ell^2}|^2 \\ &= \sum_{n\neq m} |\langle f_n, f_m\rangle_{L^2(\mu)}|^2 < \frac{1}{4}\end{aligned}$$

by (4.0.13). Hence $\{f_n\}$ is a Riesz sequence in $L^2(\mu)$. This implies that we can choose an orthonormal sequence $\{\varepsilon_n\}$ in $\ell^2(\mathbb{N})$ and an invertible operator $\tilde{\sigma} : E \to F$, where $E \subset L^2(\mu)$ is the closed subspace spanned by the f_n and $F \subset \ell^2$ is the closed subspace spanned by the ε_n, such that $\tilde{\sigma}(f_n) = \varepsilon_n$ and $\dim E^\perp = \dim F^\perp$. Let $\sigma : L^2(\mu) \to \ell^2$ be an invertible operator such that $\sigma_{|E} = \tilde{\sigma}$, for instance $\sigma = \tilde{\sigma} + \varsigma$ where $\varsigma : E^\perp \to F^\perp$ is a unitary operator. Let $\tau : \ell^2 \to M^2_\Lambda$ be the invertible operator corresponding to the Riesz basis g_n in M^2_Λ since Λ is lacunary. So $\tau(e_n) = g_n$ for an orthonormal basis $\{e_n\}$ in ℓ^2.

Now define $\pi = \sigma \circ i_\mu \circ \tau : \ell^2 \to \ell^2$. Then

$$\pi(e_n) = (\sigma \circ i_\mu)(g_n) = \|i_\mu g_n\|_{L^2(\mu)} \sigma(f_n) = \|i_\mu g_n\|_{L^2(\mu)} \varepsilon_n.$$

So $\pi(\ell^2) = F$ and it is easily seen that for the adjoint $\pi^* : \ell^2 \to \ell^2$, we have

$\pi^*(\varepsilon_n) = \|i_\mu g_n\|_{L^2(\mu)}\, e_n$ and $\pi^*_{|F^\perp} = 0$. Hence for $n = 1, 2, \ldots$

$$\pi^*\pi(e_n) = \|i_\mu g_n\|^2_{L^2(\mu)}\, e_n$$

and so $\{\|i_\mu g_n\|_{L^2(\mu)}\}_n$ are precisely the eigenvalues of $\sqrt{\pi^*\pi}$. So we can conclude that for any $r > 0$, $i_\mu \in \mathcal{S}_r(M^2_\Lambda, L^2(\mu))$ if and only if $\pi = \sigma \circ i_\mu \circ \tau \in \mathcal{S}_r(\ell^2)$ if and only if $\{\|i_\mu g_n\|_{L^2(\mu)}\}_n \in \ell^r$ if and only if $\{\alpha_n\} \in \ell^r$ by (4.0.12). But since we chose α_n so that $\{\alpha_n\} \in \ell^q$ but not in ℓ^p, we get $i_\mu \in \mathcal{S}_q(M^2_\Lambda, L^2(\mu))$ but $i_\mu \notin \mathcal{S}_p(M^2_\Lambda, L^2(\mu))$. $\square$

Chapter 5

Composition Operators on Müntz Spaces

In the thesis of Al Alam [1], certain weighted composition operators on M_Λ^1 were studied. In [8], the general framework introduced for embedding operators on M_Λ^1 was applied to recapture some of these results. In this section we make preparation for the remainder of this work where we shall be interested in the boundedness, compactness and $\mathcal{S}_q$ properties of composition operators on the Müntz space M_Λ^2.

Our goal is to apply these embedding results to composition operators. Recall that the pullback of a measure ν by ϕ is the measure $\phi^*\nu$ on $[0,1]$ defined by

$$\phi^*\nu(E) = \nu(\phi^{-1}(E))$$

for any Borel set E. If g is a positive measurable function, then the formula

$$\int_0^1 g(\phi(x))dx = \int_{[0,1]} g\, d(\phi^* m)$$

is easily checked on characteristic functions, hence the usual argument extends it to all positive Borel functions on $[0,1]$. In particular, if we define $\mu = \phi^* m$ and choose $g = |f|^p$ for some $f \in L^p(\mu)$, then the map $J : L^p(\mu) \longrightarrow L^p$ defined by $J(f) = f \circ \phi$ is an isometry.

Let ϕ be a Borel function on $[0,1]$ such that $\phi([0,1]) \subset [0,1]$. The *composition operator* C_ϕ is defined as

$$C_\phi(g) = g \circ \phi$$

for all polynomials $g \in M^p_\Lambda$. Just as we did for i^p_μ, we can extend $C_\phi = J \circ i^p_\mu$ to all $f \in M^p_\Lambda$. Since J is an isometry, we obtain the following results for composition operators.

Lemma 5.0.3. *Define the measure $\mu = \phi^* m$. Then*

(i) C_ϕ is bounded from M^p_Λ to L^p if and only if μ is a Λ_p-embedding measure.

(ii) C_ϕ is compact from M^p_Λ to L^p if and only if i^p_μ is compact.

(iii) $C_\phi \in \mathcal{S}_q(M^2_\Lambda, L^2)$ if and only if $i^2_\mu \in \mathcal{S}_q(M^2_\Lambda, L^2(\mu))$.

5.1 Müntz Spaces are not Invariant to Most Composition Operators

It has already appeared above that we study composition operators defined on M^p_Λ, but whose range space is L^p. The reason is that Müntz spaces are usually not invariant with respect to composition. This has already been noticed by Al Alam [2], in the case of Müntz space M^∞_Λ, i.e. the closure of the span of monomials x^{λ_n} in L^∞, and operators C_ϕ with continuous ϕ. The following result was proved therein.

Proposition 5.1.1. *Let $\Lambda = (\lambda_k)_k \subset \mathbb{N}$ and $\sum_k \frac{1}{\lambda_k} < \infty$. Then*

(i) $C_\phi M^\infty_\Lambda \nsubseteq M^\infty_\Lambda$ if $\phi = \alpha x^m + \beta x^n$ with $\alpha, \beta \neq 0$ and $m, n \in \mathbb{N}$.

(ii) $C_\phi M^\infty_\Lambda \nsubseteq M^\infty_\Lambda$ if ϕ is a polynomial with positive coefficients and more than one term.

In this section we will significantly extend these results to other values of $p \geq 1$ and functions ϕ. We prove in Theorem 5.1.5 that $C_\phi M_\Lambda^p \not\subseteq M_\Lambda^p$ whenever ϕ is a function of the form $c_1 x^{s_1} + \ldots + c_l x^{s_l}$ with $c_i \in \mathbb{R}$ and $s_i \in \mathbb{R}^+$. These functions will be called *real-exponent polynomials*. This generalizes Proposition 5.1.1 and Λ may not even satisfy the *gap condition* $\inf_k(\lambda_{k+1} - \lambda_k) > 0$. If we assume the gap condition, then Theorem 5.1.8 generalizes Proposition 5.1.1(i) for arbitrary $\Lambda \subset \mathbb{R}^+$.

We start with a result of A. Schinzel [20]:

Lemma 5.1.2. *If ϕ is a polynomial with at least two terms and $\lambda \in \mathbb{N}$, then ϕ^λ has at least $\lambda + 1$ terms.*

The next result is an analog of Lemma 5.1.2 for real-exponent polynomials.

Lemma 5.1.3. *([19], Lemma 2.3) If ϕ is a real-exponent polynomial with at least two terms and $\lambda \in \mathbb{N}$, then ϕ^λ has at least $\lambda + 1$ terms.*

Proof. Let $\phi(x) = c_1 x^{s_1} + \ldots + c_l x^{s_l}$ with $c_i \in \mathbb{R}\backslash\{0\}$ and $s_i \in \mathbb{R}^+$. Considering $\mathbb{R}$ as a vector space over the rationals $\mathbb{Q}$, choose a basis $r_1, \ldots, r_\tau > 0$ for the space spanned by $s_1, \ldots, s_l$ where $\tau \leq l$. Therefore

$$s_i = \sum_{j=1}^{\tau} a_{ij} r_j \quad \text{for} \quad i = 1, \ldots, l$$

where $a_{ij} \in \mathbb{Q}$. We may assume that $a_{ij} \in \mathbb{Z}$ by adjusting the r_j suitably. We note that for any positive real number N, ϕ^λ has the same number of terms as $(x^N \phi)^\lambda$. So by choosing $N = b_1 r_1 + \ldots + b_\tau r_\tau$ with integers $b_j > |a_{ij}|$ for $i = 1, \ldots, l$ and $j = 1, \ldots, \tau$, we may also assume that each $a_{ij} r_j > 0$ hence $a_{ij} \in \mathbb{N}$. We then obtain

$$\phi(x) = \sum_{i=1}^{l} c_i x^{s_i} = \sum_{i=1}^{l} c_i (x^{r_1})^{a_{i1}} \ldots (x^{r_\tau})^{a_{i\tau}}.$$

We define a polynomial ψ in τ variables by

$$\psi(Y_1, \ldots, Y_\tau) = \sum_{i=1}^{l} c_i Y_1^{a_{i1}} \ldots Y_\tau^{a_{i\tau}}.$$

Define Φ to be the collection of monomial terms in ϕ^λ after reduction and cancelation, and Ψ similarly for ψ^λ. Hence our goal is to prove that card$\Phi \geq \lambda + 1$. Since both ϕ and ψ each have l distinct monomial terms, the total number of possible products while computing ϕ^λ or ψ^λ is l^λ.

We claim that whenever two such products $p(x) = k.(x^{r_1})^{m_1} \ldots (x^{r_\tau})^{m_\tau}$ and $q(x) = k'.(x^{r_1})^{m'_1} \ldots (x^{r_\tau})^{m'_\tau}$ reduce (respectively cancel) in ϕ^λ, the corresponding products $p_\psi(Y_1, \ldots, Y_\tau) = k.Y_1^{m_1} \ldots Y_\tau^{m_\tau}$ and $q_\psi(Y_1, \ldots, Y_\tau) = k'.Y_1^{m'_1} \ldots Y_\tau^{m'_\tau}$ also reduce (resp. cancel) in ψ^λ, where $m_j, m'_j \in \mathbb{N}$. Indeed, it is obvious that p and q combine (resp. cancel) if and only if $m_1 r_1 + \ldots + m_\tau r_\tau = m'_1 r_1 + \ldots + m'_\tau r_\tau$. Since $r_1, \ldots, r_\tau$ are linearly independent over $\mathbb{Q}$, this is possible if and only if $m_j = m'_j$ for $j = 1, \ldots, \tau$. And this is equivalent to the reducing (resp. cancelling) of p_ψ and q_ψ. This proves that card$\Phi =$ cardΨ.

Note that ψ has at least two terms because ϕ has at least two terms. This implies that for some $1 \leq j' \leq \tau$, ψ as a polynomial in $Y_{j'}$ has at least two terms. Applying Lemma 5.1.2 to $\psi'(Y_{j'}) := \psi(1, \ldots, Y_{j'}, \ldots, 1) = \sum_{i=1}^{l} c_i Y_{j'}^{a_{ij'}}$, we see that $(\psi')^\lambda$ has at least $\lambda+1$ terms. Therefore ψ^λ has at least $\lambda+1$ terms and card$\Psi \geq \lambda+1$. Therefore card$\Phi \geq \lambda + 1$. □

The next lemma is a consequence of formula (1.2.2).

Lemma 5.1.4. *Let* $\Lambda = (\lambda_k)_k$ *and* $\sum_k \frac{1}{\lambda_k} < \infty$. *If a real-exponent polynomial* $c_1 x^{s_1} + \ldots + c_l x^{s_l}$ *belongs to* M_Λ^p , *then* $s_1, \ldots, s_l \in \Lambda$.

Proof. Given $c_1x^{s_1}+\ldots+c_lx^{s_l} \in M_\Lambda^p = L_\Lambda$, suppose on the contrary that some subset $\Lambda' = \{s_{k_1}, \ldots, s_{k_m}\} \subset \{s_1, \ldots, s_l\}$ does not belong to Λ and $\{s_1, \ldots, s_l\} \backslash \Lambda' \subset \Lambda$. Then

$$p(x) = (c_1x^{s_1} + \ldots + c_lx^{s_l}) - (c_{k_1}x^{s_{k_1}} + \ldots + c_{k_m}x^{s_{k_m}}) \in L_\Lambda.$$

This implies that $c_{k_1}x^{s_{k_1}} + \ldots + c_{k_m}x^{s_{k_m}} = c_1x^{s_1} + \ldots + c_lx^{s_l} - p(x) \in L_{\Lambda'} \cap L_\Lambda$. But $L_{\Lambda'} \cap L_\Lambda = \{0\}$ by (1.2.2), a contradiction. $\square$

Theorem 5.1.5. *([19], Theorem 2.5) Suppose $\Lambda = (\lambda_k)_k \subset \mathbb{N}$ with $\sum_k \frac{1}{\lambda_k} < \infty$. If ϕ is a real-exponent polynomial with more than one term, then $C_\phi M_\Lambda^p \not\subseteq M_\Lambda^p$.*

Proof. Let $\phi(x) = c_1x^{s_1} + \ldots + c_lx^{s_l}$ with $c_i \in \mathbb{R}\backslash\{0\}$ and $s_i \in \mathbb{R}^+$. Then for any $\lambda \in \Lambda$, we get $C_\phi(x^\lambda) = \phi^\lambda$ which has at least $\lambda + 1$ terms by Lemma 5.1.3. We may assume that these $\lambda + 1$ terms are nonzero multiples of

$$x^{s_1\lambda}, x^{t_1}, \ldots, x^{t_{\lambda-1}}, x^{s_l\lambda} \ \text{ where } \ s_1\lambda < t_1 < \ldots < t_{\lambda-1} < s_l\lambda.$$

Suppose that $C_\phi M_\Lambda^p \subset M_\Lambda^p$, then Theorem 5.1.4 gives us $s_1\lambda, t_1, \ldots, t_{\lambda-1}, s_l\lambda \in \Lambda$. We construct a subsequence $(\lambda_{k_j})_j$ of Λ as follows: Let $\lambda_{k_1} = \lambda_1$ and inductively choose λ_{k_j} such that $s_1\lambda_{k_j} > s_l\lambda_{k_{j-1}}$ for $j \geq 2$. Then the sequence

$$\Lambda^* := \bigcup_{j=1}^{\infty} \{s_1\lambda_{k_j}, t_1, \ldots, t_{\lambda_{k_j}-1}, s_l\lambda_{k_j}\}$$

is increasing and has distinct elements; moreover, $\Lambda^* \subset \Lambda$. So

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} \geq \sum_{s \in \Lambda^*} \frac{1}{s} \geq \sum_{j=1}^{\infty} \sum_{i=1}^{\lambda_{k_j}+1} \frac{1}{s_l\lambda_{k_j}} \geq \sum_{j=1}^{\infty} \frac{1}{s_l} = \infty$$

and hence the contradiction implies $C_\phi M_\Lambda^p \not\subseteq M_\Lambda^p$. $\square$

Corollary 5.1.6. *([19], Corollary 2.6) Let $\Lambda \subset \mathbb{N}$ and ϕ be a real-exponent polynomial. Then the following are equivalent:*

(i) $C_\phi M^p_\Lambda \subset M^p_\Lambda$

(ii) $\phi(x) = \alpha x^\eta$ *and* $\Lambda = \Lambda.\{1, \eta, \eta^2, \ldots\}$ *for some* $0 \le \alpha \le 1$ *and* $\eta \in \mathbb{R}^+$

(iii) $C_\phi : M^p_\Lambda \to M^p_\Lambda$ *is a bounded operator.*

Proof. $(i) \Rightarrow (ii)$. Theorem 5.1.5 implies that $\phi(x) = \alpha x^\eta$ for $\eta \in \mathbb{R}^+$ and $0 \le \alpha \le 1$ because $\phi([0,1]) \subset [0,1]$. Then $C^m_\phi(x^\lambda) = C^{m-1}_\phi(\alpha^\lambda x^{\lambda\eta}) = \ldots = Kx^{\lambda\eta^m} \in M^p_\Lambda$ for any $\lambda \in \Lambda$, $m \in \mathbb{N}$ and some constant K. Hence $\lambda\eta^m \in \Lambda$ for all $\lambda \in \Lambda$ and $m \in \mathbb{N}$ by Lemma 5.1.4. Therefore $\Lambda = \cup_{\lambda\in\Lambda}\lambda.\{1, \eta, \eta^2, \ldots\} = \Lambda.\{1, \eta, \eta^2, \ldots\}$.

$(ii) \Rightarrow (iii)$. Suppose $\phi(x) = \alpha x^\eta$ with $0 \le \alpha \le 1$ and $\eta \in \mathbb{R}^+$. If $\alpha < 1$, then $\mu = \phi^* m$ is supported on $[0, \alpha]$ and $d\mu|_{J_{1-\alpha}} = 0$. Hence $||i^p_\mu||_e = 0$ by Proposition 2.2.4 and $C_\phi = J \circ i^p_\mu$ is compact. For $\alpha = 1$, the measure $\mu = \phi^* m$ satisfies

$$\int_{J_\delta} f d\mu = \int_{\phi^{-1}(J_\delta)} f \circ \phi \, dm = \int_{J_\delta} f \cdot (\phi^{-1})' dm = \int_{1-\delta}^{1} f(x)\, \eta^{-1} x^{\frac{1}{\eta}-1} dx$$

for any continuous f and $0 < \delta < 1$. Therefore $d\mu|_{J_\delta} = h\, dm|_{J_\delta}$ where $h(x) = \eta^{-1}x^{\frac{1}{\eta}-1}$ is bounded on J_δ, and hence C_ϕ is bounded by Proposition 2.2.4. Moreover, for any $\lambda \in \Lambda$ we see that $C_\phi x^\lambda = \alpha^\lambda x^{\lambda\eta} \in M^p_\Lambda$. Hence by the density of linear span of monomials x^λ in M^p_Λ and continuity of C_ϕ, we get $C_\phi M^p_\Lambda \subset M^p_\Lambda$. The last part $(iii) \Rightarrow (i)$ is trivial. □

It is easy to see that Theorem 5.1.5 and Corollary 5.1.6 can be extended to the case when $\Lambda \not\subseteq \mathbb{N}$, but contains a subsequence of integers. To go beyond this case, we need some preparation about *real-exponent power series.*

Lemma 5.1.7. *([19], Lemma 2.7) Suppose* $f(x) = \sum_k a_k x^{s_k}$ *is a series such that* $(s_k)_k \subset \mathbb{R}^+$ *is the finite union of sequences that satisfy the gap condition. Then* f *is uniformly convergent on some interval* $[0, \rho]$ *if* $L := \limsup_k |a_k|^{1/s_k} < \infty$*. Furthermore, if* $f \equiv 0$ *on* $[0, \rho_0]$ *for* $\rho_0 \le \rho$ *then* $a_k = 0$ *for all* k*.*

Proof. It is sufficient to prove the first part for the case when $(s_k)_k$ itself satisfies the gap condition; in the general case, we can write f as a finite sum of uniformly convergent series.

Since $|a_k x^{s_k}|^{1/s_k} = |a_k|^{1/s_k}|x|$, we get $\limsup_k |a_k x^{s_k}|^{1/s_k} < 1$ if and only if $|x| < L^{-1}$ (taking $L^{-1} = \infty$ if $L = 0$). So, for $L|x| < 1$, we get $\limsup_k |a_k x^{s_k}|^{1/s_k} < r < 1$ for some r and hence there exists a positive integer N such that $|a_k x^{s_k}|^{1/s_k} < r$ for $k \geq N$. Therefore

$$\sum_{k \geq N}^{\infty} |a_k x^{s_k}| \leq \sum_{k \geq N}^{\infty} r^{s_k} < \infty$$

where the convergence follows from the ratio test and the gap condition because

$$\lim_{k \to \infty} \frac{r^{s_{k+1}}}{r^{s_k}} = \lim_{k \to \infty} r^{s_{k+1} - s_k} \leq r^{\inf_k (s_{k+1} - s_k)} < 1.$$

So $f(x)$ converges absolutely for $L|x| < 1$, and in particular converges uniformly on $[0, \rho]$ for some $\rho > 0$.

For the second part, suppose on the contrary that a_1 is the first non-zero coefficient. We see that

$$f(x) = \sum_{k \geq 1} a_k x^{s_k} = a_1 x^{s_1} (1 + \sum_{k > 1} \frac{a_k}{a_1} x^{s_k - s_1})$$

where $(s_k - s_1)_k$ is again a union of finitely many series satisfying the gap condition and

$$\limsup_k \left| \frac{a_k}{a_1} \right|^{\frac{1}{s_k - s_1}} \leq \limsup_k (|a_k|^{\frac{1}{s_k}})^{\frac{s_k}{s_k - s_1}} . \limsup_k \left(\frac{1}{|a_1|} \right)^{\frac{1}{s_k - s_1}} = L < \infty$$

hence $g(x) = 1 + \sum_{k>1} \frac{a_k}{a_1} x^{s_k - s_1}$ converges uniformly on some interval $[0, \rho_1]$. So $f(x) = a_1 x^{s_1} g(x) = 0$ on $[0, r]$, where $r = \min\{\rho_0, \rho_1\}$. Therefore $g = 0$ on $(0, r]$ and hence on $[0, r]$ by continuity. A contradiction, since $g(0) = 1$. □

Theorem 5.1.8. *([19], Theorem 2.8) Suppose* $\Lambda \subset \mathbb{R}^+$ *with* $\sum_k \frac{1}{\lambda_k} < \infty$ *satisfies the gap condition* $\inf_k(\lambda_{k+1} - \lambda_k) > 0$*. If* $\phi = \alpha x^{\zeta_1} + \beta x^{\zeta_2}$ *with* $\alpha, \beta \neq 0$ *and* $\zeta_1 < \zeta_2 \in \mathbb{R}^+$*, then* $C_\phi M_\Lambda^p \not\subseteq M_\Lambda^p$*.*

Proof. If $\Lambda \subset \mathbb{N}$, then Theorem 5.1.5 proves the result. So we assume $\Lambda \not\subseteq \mathbb{N}$, hence there exists $\lambda \in \Lambda$ that is not an integer. Suppose that $C_\phi M_\Lambda^p \subset M_\Lambda^p$; then

$$C_\phi(x^\lambda) = (\alpha x^{\zeta_1} + \beta x^{\zeta_2})^\lambda = \alpha^\lambda x^{\lambda\zeta_1}(1 + \frac{\beta}{\alpha} x^{\zeta_2 - \zeta_1})^\lambda \in M_\Lambda^p.$$

Hence by the binomial series we can represent $C_\phi(x^\lambda)$ as

$$C_\phi(x^\lambda)(t) = \alpha^\lambda t^{\lambda\zeta_1} \sum_{k=0}^{\infty} a_k t^{k(\zeta_2 - \zeta_1)} = \alpha^\lambda \sum_{k=0}^{\infty} a_k t^{\lambda\zeta_1 + k(\zeta_2 - \zeta_1)}$$

where the series converges for $|t| < |\frac{\alpha}{\beta}|^{\frac{1}{\zeta_2 - \zeta_1}}$, in particular on $[0, \eta]$ for some $\eta < 1$. The sequence of exponents $(\lambda\zeta_1 + k(\zeta_2 - \zeta_1))_k$ clearly satisfies the gap condition, while the coefficients

$$a_k = \left(\frac{\beta}{\alpha}\right)^k \frac{\lambda(\lambda - 1)(\lambda - 2)\ldots(\lambda - k + 1)}{k!}$$

satisfy

$$L_1 := \limsup_{k\to\infty} |a_k|^{1/\lambda\zeta_1 + k(\zeta_2 - \zeta_1)} < \infty.$$

Similarly, by Theorem 1.3.6 there exists a sequence of scalars $b_k \in \mathbb{R}$ such that

$$C_\phi(x^\lambda)(t) = \sum_{k=1}^{\infty} b_k t^{\lambda_k}$$

and the series converges uniformly on compact subsets of $[0, 1)$. By (1.3.4), the coefficients $(b_k)_k$ satisfy

$$L_2 := \limsup_{k\to\infty} |b_k|^{1/\lambda_k} \leq \limsup_{k\to\infty} [(1 + \varepsilon)(2\lambda_k + 1)^{1/2\lambda_k} ||f||_{L^2}^{1/\lambda_k}] < \infty.$$

Since both series representations coincide on $[0,\eta]$, the series defined by

$$f(t) = \sum_{k=1}^{\infty} b_k t^{\lambda_k} - \alpha^{\lambda} \sum_{k=0}^{\infty} a_k t^{\lambda\zeta_1 + k(\zeta_2-\zeta_1)} = \sum_k \gamma_k t^{s_k}$$

vanishes on $[0,\eta]$. Since $(s_k)_k$ is the union of two series satisfying the gap condition and $\limsup_k |\gamma_k|^{1/s_k} \le L_1 + L_2 < \infty$, by Lemma 5.1.7 we get $\gamma_k = 0$ for all k. Since λ is not an integer, all the a_k are non-zero; this implies that $\lambda\zeta_1 + k(\zeta_2 - \zeta_1) \in \Lambda$ for all k. This contradicts the fact that $\sum_k \frac{1}{\lambda_k} < \infty$ and hence $C_\phi M_\Lambda^p \not\subseteq M_\Lambda^p$. $\square$

5.2 Composition Operators on M_Λ^2: direct results

The next result is essentially contained in the work of Chalendar, Fricain, Timotin [8]:

Proposition 5.2.1. *Suppose the Borel function $\phi : [0,1] \longrightarrow [0,1]$ satisfies:*

(a) $\phi^{-1}(1) = \{x_1, \ldots, x_k\}$ is finite.

(b) There exists $\epsilon > 0$ such that, for each $i = 1, ..., k$, ϕ is continuous on $(x_i - \epsilon, x_i + \epsilon)$, $\phi \in C^1((x_i - \epsilon, x_i))$ and $\phi \in C^1((x_i, x_i + \epsilon))$.

(c) $\phi'_-(x_i) > 0$ and $\phi'_+(x_i) < 0$ for all $i = 1, \ldots, k$.

($\phi'_-(x)$ and $\phi'_+(x)$ denote the left and right derivatives at x respectively, which may be infinite).

(d) There exists $\alpha < 1$ such that, if $x \notin \cup_{i=1}^k (x_i - \epsilon, x_i + \epsilon)$, then $\phi(x) < \alpha$.

Then $C_\phi : M_\Lambda^2 \longrightarrow L^2$ is bounded and $||C_\phi||_e = \sum_{i=1}^k L(x_i)$, where

$$L(x_i) = \begin{cases} \frac{1}{\phi'_-(x_i)} + \frac{1}{|\phi'_+(x_i)|} & if\ x_i \in (0,1), \\ \frac{1}{\phi'_-(x_i)} & if\ x_i = 1, \\ \frac{1}{|\phi'_+(x_i)|} & if\ x_i = 0. \end{cases}$$

In particular, if $\phi'_-(x_i) = \infty$ and $\phi'_+(x_i) = -\infty$ for all $i = 1, \ldots, k$, then C_ϕ is compact.

We intend to go beyond the regularity assumptions in Proposition 5.2.1.

Definition 5.2.2. If $\phi : [0,1] \to [0,1]$ is a Borel function and $\alpha = \operatorname{ess\,sup}_{[0,1]} \phi$, then a point $x \in [0,1]$ is an *essential point of maximum* for ϕ if ess $\sup_E \phi = \alpha$ for every neighborhood E of x. Denote by $\mathfrak{M}_\phi$ the set of all essential points of maxima of ϕ, and by V_ε the neighborhood of $\mathfrak{M}_\phi$ defined for each $\varepsilon > 0$ by

$$V_\varepsilon = \{x \in [0,1] : \operatorname{dist}(x, \mathfrak{M}_\phi) < \varepsilon\}.$$

Lemma 5.2.3. *The following statements are true:*

(i) $\mathfrak{M}_\phi$ *is non-empty and closed,*

(ii) $\operatorname{ess\,sup} \phi|_{[0,1]\backslash V_\varepsilon} < \alpha$ *for all* $\varepsilon > 0$*,*

(iii) for every $\varepsilon > 0$ *there exists a* $\delta_0 > 0$ *such that* $\phi^{-1}([\alpha - \delta, \alpha]) \subset V_\varepsilon$ *almost everywhere whenever* $0 < \delta < \delta_0$*.*

Proof. (i). If $\mathfrak{M}_\phi$ were empty, then every point $x \in [0,1]$ would have a neighborhood $\mathcal{N}_x$ such that ess $\sup_{\mathcal{N}_x} \phi < \alpha$ and all such $\mathcal{N}_x$ would cover $[0,1]$. Choosing a finite subcover so that $\cup_{k=1}^m \mathcal{N}_{x_k} = [0,1]$, we see that

$$\operatorname*{ess\,sup}_{[0,1]} \phi = \max_k \{\operatorname*{ess\,sup}_{\mathcal{N}_{x_k}} \phi\} < \alpha.$$

The contradiction yields $\mathfrak{M}_\phi \neq \varnothing$. To prove that $\mathfrak{M}_\phi$ is closed, consider the set $\mathcal{S} := \cup_{x \in [0,1]\backslash \mathfrak{M}_\phi} \mathcal{N}_x$, where $\mathcal{N}_x$ again represents a neighborhood of x on which ess $\sup_{\mathcal{N}_x} \phi < \alpha$. So clearly $\mathcal{S}$ is open, and $\mathcal{S} \cap \mathfrak{M}_\phi = \varnothing$ since otherwise some $\mathcal{N}_{x'}$ for $x' \in [0,1]\backslash \mathfrak{M}_\phi$ would contain an essential point of maximum. Hence $\mathcal{S} = [0,1]\backslash \mathfrak{M}_\phi$ and $\mathfrak{M}_\phi$ is closed.

For (ii), suppose that $\operatorname{ess\,sup} \phi|_{[0,1]\backslash V_{\varepsilon'}} = \alpha$ for some $\varepsilon' > 0$. Then the argument in the proof of (i) applied to the compact set $[0,1]\backslash V_{\varepsilon'}$, shows that it contains an essential point of maximum.

Finally for (iii), it follows from (ii) that for every $\varepsilon > 0$ there exists a $\delta_0 > 0$ such that $\operatorname{ess\,sup} \phi|_{[0,1]\setminus V_\varepsilon} < \alpha - \delta_0 < \alpha$ and hence

$$\phi^{-1}([\alpha-\delta,\alpha]) = \{x \in [0,1] : \alpha - \delta \le \phi(x) \le \alpha\} \subset V_\varepsilon$$

except possibly for a subset of measure 0, whenever $0 < \delta < \delta_0$. □

We recall that the left and right derivatives of ϕ at the point y are defined as

$$D^i_-(y) = \liminf_{t\to y-} \frac{\phi(y)-\phi(t)}{y-t}$$

$$D^i_+(y) = \liminf_{t\to y+} \frac{\phi(y)-\phi(t)}{y-t}$$

$$D^s_-(y) = \limsup_{t\to y-} \frac{\phi(y)-\phi(t)}{y-t}$$

$$D^s_+(y) = \limsup_{t\to y+} \frac{\phi(y)-\phi(t)}{y-t}$$

respectively.

Suppose $\phi : [0,1] \to [0,1]$ is a Borel function such that $\alpha = \operatorname{ess\,sup}_{[0,1]} \phi < 1$. Then it is easy to show that the measure defined by $\mu = \phi^* m$ has support in $[0,\alpha]$. In fact

$$\mu((\alpha,1]) = \int_{(\alpha,1]} d(\phi^* m) = \int_{\phi^{-1}((\alpha,1])} dm = m(\phi^{-1}(\alpha,1]) = 0.$$

Hence in this case $i^2_\mu \in \mathcal{S}_q$ by Theorem 3.1.3(i). Therefore $C_\phi \in \mathcal{S}_q$ by Lemma 5.0.3, so from here onwards we assume that $\alpha = \operatorname{ess\,sup}_{[0,1]} \phi = 1$.

Since changing the values of ϕ on a set of measure zero does not effect $\mu = \phi^* m$, whenever $m(\mathfrak{M}_\phi) = 0$, one may take $\phi \equiv 1$ on $\mathfrak{M}_\phi$. This will be assumed in the rest of the paper.

Lemma 5.2.4. *([19], Lemma 3.4) Suppose ϕ is a Borel function with $\mathfrak{M}_\phi = \{x_1, \ldots, x_k\}$ and $\mu = \phi^* m$. If for some $s \geq 1$ there exists an $\varepsilon > 0$ and a constant $c > 0$ such that*

$$|x - x_i| \leq c|\phi(x) - 1|^s \quad \text{whenever} \quad |x - x_i| < \varepsilon$$

for all $i = 1, \ldots, k$, then there exists a $\delta_0 > 0$ such that $\mu(J_\delta) \leq 2kc\delta^s$ whenever $0 < \delta < \delta_0$.

Proof. By Lemma 5.2.3(iii), there exists a $\delta_0 > 0$ such that $\phi^{-1}(J_\delta) \subset V_\varepsilon$ almost everywhere whenever $0 < \delta < \delta_0$. Since $\sup_{\phi^{-1}(J_\delta)} |\phi(x) - 1| \leq \delta$, we get

$$\begin{aligned} m(\phi^{-1}(J_\delta)) &\leq \sum_{i=1}^{k} m(\phi^{-1}(J_\delta) \cap \{\operatorname{dist}(x, x_i) < \varepsilon\}) \leq 2 \sum_{i=1}^{k} \sup_{\phi^{-1}(J_\delta) \cap \{|x - x_i| < \varepsilon\}} |x - x_i| \\ &\leq 2 \sum_{i=1}^{k} \sup_{\phi^{-1}(J_\delta) \cap \{|x - x_i| < \varepsilon\}} c|\phi(x) - 1|^s \leq 2kc\delta^s. \end{aligned}$$

Therefore we get

$$\mu(J_\delta) = \int_{J_\delta} d\mu = \int_{J_\delta} d(\phi^* m) = \int_{\phi^{-1}(J_\delta)} dm = m(\phi^{-1}(J_\delta)) \leq 2kc\delta^s$$

whenever $0 < \delta < \delta_0$. □

We arrive at the main theorem that gives sufficient conditions for composition operators on M_Λ^2 to be bounded, compact or in $\mathcal{S}_q$.

Theorem 5.2.5. *([19], Theorem 3.5) Let Λ be lacunary and $\mathfrak{M}_\phi = \{x_1, \ldots, x_k\}$.*
(i) If $D_-^i > 0$ and $D_+^s < 0$ on $\mathfrak{M}_\phi$, then $C_\phi : M_\Lambda^2 \to L^2$ is bounded.
(ii) If $D_-^i = +\infty$ and $D_+^s = -\infty$ on $\mathfrak{M}_\phi$, then $C_\phi : M_\Lambda^2 \to L^2$ is compact.
(iii) If for some $\varepsilon > 0$, $\beta > 1$ and constant c we have

$$|x - x_i| \leq c\,|\phi(x) - 1|^\beta \quad \forall \quad |x - x_i| < \varepsilon \tag{5.2.1}$$

for $i = 1, \ldots, k$, then $C_\phi \in \mathcal{S}_q(M_\Lambda^2, L^2)$ $\forall$ $q > 0$.

Proof. (i) The hypothesis about the derivatives implies that for some constant $M > 0$ there exists an $\varepsilon > 0$ such that

$$\frac{|\phi(x) - 1|}{|x - x_i|} \geq M > 0 \iff |x - x_i| \leq M^{-1}|\phi(x) - 1|$$

whenever $|x - x_i| < \varepsilon$ for all $i = 1, \ldots, k$. Hence by Lemma 5.2.4, we get $\mu(J_\delta) \leq 2kM^{-1}\delta$ for $0 < \delta < \delta_0$. Therefore μ is sublinear and i_μ^2 is bounded by Theorem 2.4.1. So Lemma 5.0.3 implies that $C_\phi : M_\Lambda^2 \to L^2$ is bounded.

(ii) By our hypothesis, for any $M > 0$ there exists an $\varepsilon > 0$ such that

$$\frac{|\phi(x) - 1|}{|x - x_i|} \geq M \iff |x - x_i| \leq M^{-1}|\phi(x) - 1|$$

whenever $|x - x_i| < \varepsilon$. And for every such $\varepsilon > 0$ there exists a $\delta_0 > 0$ such that $\mu(J_\delta) \leq 2kM^{-1}\delta$ whenever $0 < \delta < \delta_0$ by Lemma 5.2.4. Therefore $\frac{\mu(J_\delta)}{\delta} \to 0$ as $\delta \to 0$. So the measure μ defined above is a vanishing sublinear measure hence $i_\mu^2 : M_\Lambda^2 \to L^2(\mu)$ is compact by Corollary 2.4.3, and so is $C_\phi = J \circ i_\mu^2$.

(iii) Applying Lemma 5.2.4 directly to condition (5.2.1), we get $\mu(J_\delta) \leq 2kc\delta^\beta$ whenever $0 < \delta < \delta_0$. Hence by Theorem 3.2.2, $i_\mu \in \mathcal{S}_q(M_\Lambda^2, L^2(\mu))$ for all $q > 0$. So $C_\phi \in \mathcal{S}_q(M_\Lambda^2, L^2)$ for all $q > 0$. $\square$

Remark 5.2.6. If $\psi \in L^\infty$ then these results still hold true for the weighted composition operator $M_\psi \circ C_\phi$ where M_ψ is the multiplication operator with symbol ψ, which is a bounded operator on L^2.

5.3 Composition Operators on M_Λ^2: Inverse results

We conclude by presenting some results that serve as converses to the boundedness and compactness theorems given above for composition operators on M_Λ^2.

The next lemma might be compared to Lemma 5.2.4.

Lemma 5.3.1. *([19], Lemma 4.2) Suppose* $\phi : [0,1] \to [0,1]$ *is a Borel function and* $\mu = \phi^* m$*. If for some* $x_0 \in [0,1]$ *with* $\phi(x_0) = 1$ *and* $\eta > 0$*, there exists an* $\varepsilon > 0$ *such that*

$$x_0 - x > \frac{1}{\eta}(1 - \phi(x)) \quad \text{whenever} \quad 0 < x_0 - x < \varepsilon,$$

then $\mu(J_\delta) \geq \frac{\delta}{\eta}$ *for* $0 < \delta < \eta\,\varepsilon$*.*

Proof. Since there exists an $\varepsilon > 0$ such that for $0 < x_0 - x < \varepsilon$, we have

$$\frac{1 - \phi(x)}{x_0 - x} < \eta \iff 1 - \phi(x) < \eta(x_0 - x).$$

Then suppose $0 < \delta < \delta_0 = \eta\,\varepsilon$. If $0 < x_0 - x < \frac{\delta}{\delta_0}\varepsilon = \frac{\delta}{\eta}$ then $1 - \phi(x) < \eta\frac{\delta}{\eta} = \delta$ which implies $\phi(x) > 1 - \delta$. So $\phi^{-1}(J_\delta)$ contains the interval $(x_0 - \frac{\delta}{\eta}, x_0)$ of Lebesgue measure $\frac{\delta}{\eta}$. Therefore

$$m(\phi^{-1}(J_\delta)) \geq \frac{\delta}{\eta} \;\Rightarrow\; \mu(J_\delta) \geq \frac{\delta}{\eta} \;\Rightarrow\; \frac{\mu(J_\delta)}{\delta} \geq \frac{1}{\eta}$$

□

For the partial converses to parts (i) and (ii) of Theorem 5.2.5, we need neither lacunarity nor any assumption on $\mathfrak{M}_\phi$:

Theorem 5.3.2. *([19], Theorem 4.3) Suppose* $\phi : [0,1] \to [0,1]$ *is a Borel function, and* $\phi(x_0) = 1$ *for some* $x_0 \in [0,1]$*.*
(i) If C_ϕ *is bounded, then* $D^s_-(x_0) > 0$ *and* $D^i_+(x_0) < 0$*.*
(ii) If C_ϕ *is compact, then* $D^s_-(x_0) = +\infty$ *and* $D^i_+(x_0) = -\infty$*.*

Proof. (i) Suppose on the contrary that either $D^s_-(x_0) = 0$ or $D^i_+(x_0) = 0$. We shall deduce a contradiction for one of these cases since both are analogous. So suppose

$D^s_-(x_0) = 0$, that is

$$\lim_{x \to x_0 -} \frac{1 - \phi(x)}{x_0 - x} = 0.$$

For each $\eta > 0$ there exists an $\varepsilon > 0$ such that for $0 < x_0 - x < \varepsilon$, we have

$$\frac{1 - \phi(x)}{x_0 - x} < \eta.$$

Therefore by Lemma 5.3.1 we get $\frac{\mu(J_\delta)}{\delta} \geq \frac{1}{\eta}$ whenever $\delta < \eta\,\varepsilon$. So $\frac{\mu(J_\delta)}{\delta} \to +\infty$ as $\delta \to 0$. Since C_ϕ is bounded we get that μ is Λ_2-embedding by Lemma 5.0.3. This leads to a contradiction since Lemma 2.3.4 gives $\liminf_{\delta \to 0} \frac{\mu(J_\delta)}{\delta} < \infty$.

For (ii), suppose to the contrary that either $D^s_-(x_0) < +\infty$ or $D^i_+(x_0) > -\infty$ for some $x_0 \in \mathfrak{M}_\phi$. Again due to similarities we shall deal with one case. So suppose $D^s_-(x_0) < \infty$, that is

$$\lim_{x \to x_0 -} \frac{1 - \phi(x)}{x_0 - x} < \infty.$$

So there exists a $\zeta > 0$ and an $\varepsilon > 0$ such that for $0 < x_0 - x < \varepsilon$, we have

$$\frac{1 - \phi(x)}{x_0 - x} < \zeta.$$

Therefore by Lemma 5.3.1 we get $\frac{\mu(J_\delta)}{\delta} \geq \frac{1}{\zeta}$ for $\delta < \zeta\,\varepsilon$. This contradicts Lemma 2.3.4 because i^2_μ is compact by Lemma 5.0.3. $\square$

Corollary 5.3.3. *([19], Corollary 4.4) Suppose ϕ is a polynomial with $\phi^{-1}(1)$ non-empty. Then C_ϕ is not compact, and if it is bounded then $\phi^{-1}(1) \subset \{0, 1\}$.*

Proof. If C_ϕ is bounded and some $x_0 \in \phi^{-1}(1)$ is an interior point of $[0, 1]$, then clearly x_0 must be a local maximum and hence $\phi'(x_0) = 0$. This contradicts Theorem 5.3.2 (i) and hence $\phi^{-1}(1) \subset \{0, 1\}$. Similarly, by part (ii) of the theorem we get the conclusion that C_ϕ can never be compact because ϕ is differentiable everywhere. $\square$

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