Metric Dimension and Distances in Graphs

Since 1864

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DECLARATION

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Certified that the research work contained in this thesis titled

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has been carried out and completed by Miss Ayesha Riasat Registration No. **14-GCU-PHD-SMS-2007** under my supervision.

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To my dear parents

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Preface

The metric dimension of a connected graph has been defined in [18], [34], [35]; an equivalent definition was given in [9]: Let $W = (w_1, w_2, ..., w_k)$ be an ordered set of vertices of G and let $v \in V(G)$. The representation $r(v|W)$ of v with respect to W is the k-tuple $(d(v, w_1), d(v, w_2), ..., d(v, w_k))$, defied in terms of distaces between v and vertices of W. If distinct vertices of G have distinct representations with respect to W, then W is called a resolving set for G. A resolving set of minimum cardinality is called a basis for G and the number of elements in a basis is the metric dimension $dim(G)$ of G. The problem of determining whether $dim(G) < k$ is an NP-complete problem [16], [26]. A survey of results on the metric dimension and its applications is [8]; also see [19], [26].

The wheel W_n consists of a cycle C_n and a new vertex v, called central vertex, adjacent with all vertices of C_n. The gear graph GR_n [15] is obtained from the wheel W_n by inserting a new vertex on each edge of C_n. Buczkowski, Chartrand, Poisson and Zhang [7] have shown that $dim(W_n) = \lfloor (2n + 2)/5 \rfloor$ for every $n \geq 7$ and Tomescu and Javaid have proved that $dim(GR_n) = \lfloor 2n/3 \rfloor$ for every $n \geq 4$ [38]. Tomescu and Imran deduced metric dimensions of some infinite regular graphs, which are subgraphs of the square and hexagonal grids [37] and Kousar, Tomescu and Husnine characterized all graphs having both metric dimension and diameter equal to 2 [28]. Very recently Tomescu and Imran [36] deduced new inequalities concerning metric dimension, introduced the notion of resolving set (or R-set) relative to a pair of vertices and made the conjecture called RP which asserts that for a graph G of order n the number of pairs of vertices such that the resolving set relative to this pair equals $V(G)$ (called resolving pairs) is bounded above by $\lfloor n^2/4 \rfloor$. This conjecture holds for graphs of diameter equal to 2.

Imran, Baig, Shafiq and Tomescu determined the metric dimension of generalized Petersen graphs $P(n, 3)$ [22] when $n \equiv 0$ or $n \equiv 1 \pmod 6$ and deduced inequalities for the remaining cases.

Let $W_{n,k}(k \geq 1)$ be the graph obtained from W_n by inserting $k-1$ new vertices on each edge of C_n, or equivalently, by subdividing each edge of C_n by $k-1$ vertices. Thus, $W_{n,k}$ consists of a kn-cycle C_{kn} with a hub adjacent to every $k-th$ vertex on the cycle. $W_{n,k}$ has order $nk+1$ and size $n(k+1)$. In particular, $W_{n,1} = W_n$ and $W_{n,2} = GR_n$.

A graph, denoted $f_{n\times m}$, is called an $(n \times m)$-flower graph if it has n vertices which form an n-cycle and n sets of $m-2$ vertices each which form m-cycles around the n-cycle so that each m-cycle uniquely intersects the n-cycle on a single edge.

For a pair $\{x, y\}$ of distinct vertices of G we shall denote by $RS(x, y)$ the set of vertices $z \in V(G)$ such that $d(z, x) \neq d(z, y)$ [36]. Such a set will be called the resolving set (or the R-set) relative to the pair $\{x, y\}$. It is clear that $\{x, y\} \subseteq RS(x, y) \subseteq V(G)$ for any pair $\{x, y\}$.

Some properties of R-sets were described in [36].

This thesis is divided into five chapters. The first two chapters consist of basic concepts and terminology of graphs and distances in graphs. In the third chapter, the metric dimension of plane graphs induced by some classes of convex polytopes and of flower graphs $f_{n\times m}$ has been determined and it was proved that most of these families of graphs have constant metric dimension. Fourth chapter deals with the metric dimension of $W_{n,k}$, a subdivision of the wheel W_n. In fifth chapter properties of resolving pairs of connected graphs have been discussed. In [36] the following conjecture has been proposed :

Conjecture RP: For every connected graph G of order $n \geq 2$ the number of resolving pairs is bounded above by $\lfloor n^2/4 \rfloor$. This conjecture is valid for graphs of diameter two, paths and cycles. Other classes of graphs verifying conjecture RP have been studied in this chapter. Also, in sixth chapter some open problems are suggested.

Chapter 1

Basic Notions

In this chapter we recall basic concepts and some results that will be used later.

1.1 Preliminaries

This section includes some definitions, notations and terminologies in graph theory.

A graph G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$, of edges, which are unordered pairs of distinct vertices. If e is an edge, $e = \{u, v\}$, then e is said to *join* u and v, and the vertices u and v are called the *ends* of e. We shall denote e by uv. Numbers of vertices and edges in G are called the *order* and *size* of G, repectively.

An edge whose end points are equal is called *loop*. Edges having same pair of endpoints are called *multiple edges*. A graph having no loops or multiple edges is called a *simple graph*. The graphs we consider in this thesis are all simple graphs.

Two distinct vertices are called *adjacent* if they are joined by an edge. In this case, these vertices are called the *neighboring* vertices. The set of neighbors of a vertex v in a graph G is denoted by $N(v)$. The cardinality of the set of neighbors of a vertex is called its *degree*. Equivalently, the degree of a vertex v in a graph G is the number of edges of G incident with v. The degree of a vertex v is denoted by $d_G(v)$ or $d(v)$. The maximum and minimum of the degrees of the vertices of a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A vertex with degree zero is an *isolated vertex*. If degree of each vertex v of a graph G is equal to k we say that G is k-regular.

1.2 Connectivity

A graph H is called a *subgraph* of a graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $H \subseteq G$ and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is called a *proper subgraph* of G. If a subgraph of a graph G has the same vertex set as G, then it is a *spanning subgraph* of G.

A subgraph F of a graph G is called an induced subgraph of G if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F as well. Let v and w be two vertices in a graph. A walk in a graph G, is a sequence of vertices of G of the form:

$$< v_1, v_2, v_3, ..., v_{n-1}, v_n >$$

such that $v_1v_2,\ v_2v_3, \ldots, v_{n-1}v_n$ are edges of G [14]. The vertices and edges of the walk need not be distinct. Two walks $< v_1, v_2, v_3, ..., v_{n-1}, v_n >$ and $< u_1, u_2, u_3, ..., u_{m-1}, u_m >$ are said to be equal if $n = m$, $v_i = u_i$ for $0 \leq i \leq n$. The number of edges in a walk is the length of the walk. A walk in which all vertices (and necessarily all of its edges) are distinct is called a *path*. A closed walk with no edge repeated is a *circuit*. A cycle is a circuit in which no vertex is repeated. In a simple graph G, any cycle consisting of k vertices is a k-cycle in G. It is an odd cycle if k is odd and it is an even cycle if k is even. A graph G is *connected* if for any two distinct vertices of G there is a path joining them (otherwise, G is called disconnected). If G has a u, v-path, then u is *connected* to v in G. The *connection relation* on $V(G)$ consists of the ordered pairs (u, v) such that u is connected to v.
A connected subgraph of G which is maximal relatively to set-inclusion is called a component of G. A component (or graph) is *trivial* if it has no edges; otherwise it is *nontrivial*. An *isolated vertex* is a vertex of degree 0.

In other words a graph is connected if and only if the number of its components is one. If F is a set of edges in a graph $G = (V, E)$, the graph obtained from G by deleting all the edges belonging to F is denoted by $G - F$. If F consists of the single edge f then $G - F$ is written as $G - f$. A set F of edges in a connected graph G is called a disconnecting set in G if $G - F$ is not connected. An edge which constitutes a disconnecting set is called a bridge.

The following theorem [11] provides a necessary and sufficient condition for an edge of a graph G to be bridge.

Theorem 1.2.1. *[11] An edge e of a graph G is a bridge if and only if e does not belong to any cycle of G.*

In a graph G a subset of vertices W such that $G - W$ has more components than G is called a *vertex-cut.* A *cut-vertex* (or *cutpoint*) is a vertex-cut consisting of a single vertex. Similarly, an *edge-cut* in a graph G is a set of edges D such that $G - D$ has more components than G. In this way a *bridge* is an edge-cut consisting of a single edge.

The *vertex-connectivity* of a connected graph G, denoted $\varkappa_v(G)$, is the minimum number of vertices whose removal can either disconnect G or reduce it to a $1-$vertex graph. Thus, if G is not a complete graph, then $\varkappa_v(G)$ is the size of a smallest vertex-cut. A graph G is said to be *k-connected* if G is connected and $\varkappa_v(G) \geq k$. If G is not a complete graph, then G is $k-$connected if every vertex-cut has at least k vertices. The *edge-connectivity* of a connected graph G, denoted $\varkappa_e(G)$, is the minimum number of edges whose removal can disconnect G. Thus, if G is a connected graph, the edge-connectivity $\varkappa_e(G)$ is the size of a smallest edge-cut. A graph is *k-edge-connected* if G is connected and every edge-cut has at least k edges (i.e., $\varkappa_e(G) \geq k$). The following properties have been quoted from [17].

Proposition 1.2.2. *Let G be a graph. Since by deleting all edges incident with a vertex having minimum degree disconnects G it follows that the edge-connectivity $\varkappa_e(G)$ is less than or equal to the minimum degree $\delta_(G)$.*

A *partition-cut* (X_1, X_2) is an edge-cut each of whose edges has one endpoint in each of the vertex bipartition sets X_1 and X_2. The following proposition characterizes the edge-connectivity of a graph in terms of the size of its partition-cuts.

Proposition 1.2.3. *A graph G is k-edge-connected if and only if every partition-cut contains at least k edges.*

Next result concerns the relationship between vertex-connectivity and edge-connectivity.

Corollary 1.2.4. *Let G be a connected graph. Then*

$$\varkappa_v(G) \leq \varkappa_e(G) \leq \delta_{min}(G).$$

A vertex of a path P is called an *internal vertex* of P if it is different from the initial and the final vertex of that path. Let u and v be two vertices in a graph G. A collection of $u - v$ paths in G is said to be *internally disjoint* if no two paths in the collection have an internal vertex in common.

Theorem 1.2.5 (Whitney, 1932)**.** *Let G be a connected graph with three or more vertices. Then G is $2-$connected if and only if for each pair of vertices in G, there are two internally disjoint paths connecting them.*

1.3 Bipartite Graphs

Definition 1.3.1. *A graph G is called a bipartite graph if $V(G)$ can be partitioned into two subsets U and W, such that every edge has one end in U and another one in W. These subsets are called partite sets.*

It is not always easy to tell at a glance whether a graph is bipartite.

Example 1.3.2. *The connected graphs G_1 and G_2 of Fig.* 1.1 *are bipartite, as every edge of G_1 joins a vertex of $U_1 = \{u_1, x_1, y_1\}$ and a vertex of $W_1 = \{v_1, w_1\}$, while every edge of G_2 joins a vertex of $U_2 = \{u_2, w_2, y_2\}$ and a vertex of $W_2 = \{v_2, x_2, z_2\}$.*

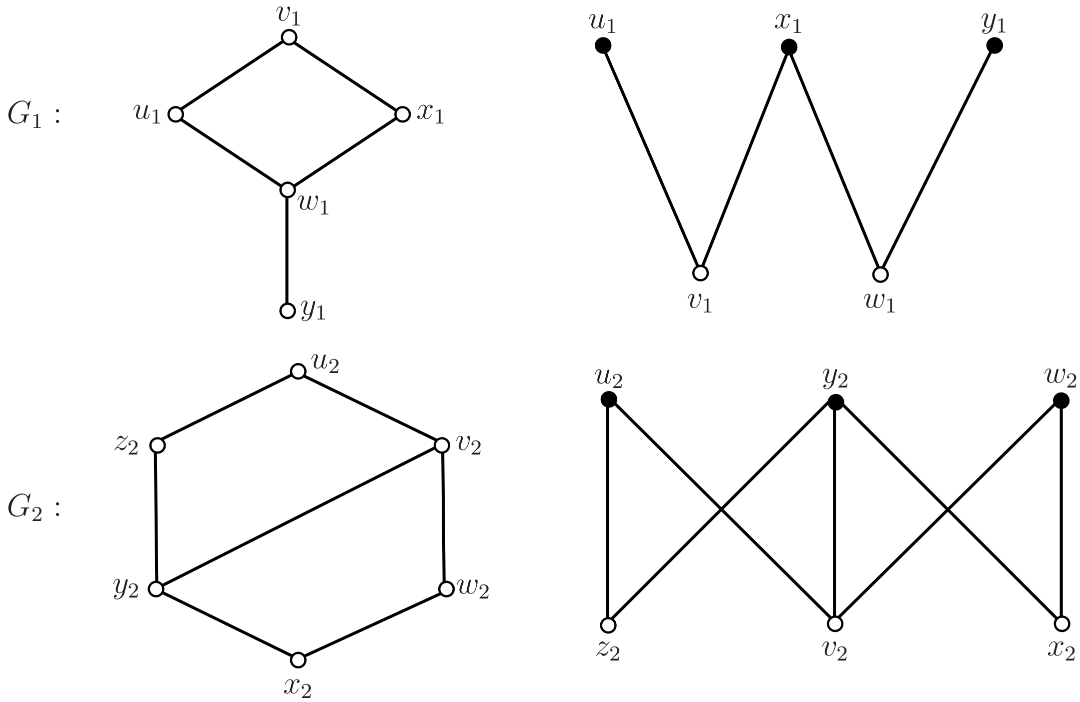

Figure 1.1: *Bipartite graphs*

The bipartite nature of these graphs is illustrated in figure below.
By letting $U = U_1 \cup U_2$ and $W = W_1 \cup W_2$, we see that every edge of $G = G_1 \cup G_2$ joins a vertex of U and a vertex of W. This illustrates the observation that a graph is bipartite if and only if each of its components is bipartite.

If $|U| = |W|$ then the graph is called *balanced bipartite graph.*
Certainly not every graph is bipartite. For example, consider the 5-cycle C_5 in Fig. 1.2. If C_5 is bipartite, then its vertex set can be partitioned into two sets U and W such that every edge of C_5 joins a vertex of U and a vertex of W. The vertex v_1 must belong to either U or W, say $v_1 \in U$. Since v_1v_2 is an edge of C_5, it follows that $v_2 \in W$. Since v_2v_3 is an edge of C_5, it follows that $v_3 \in U$. Similarly $v_4 \in W$ and $v_5 \in U$. However, $v_1, v_5 \in U$ and v_1v_5 is an edge of C_5. This is a contradiction. Therefore, C_5 is not bipartite. In fact, no odd cycle is bipartite.

Following properties have been discussed in [1].

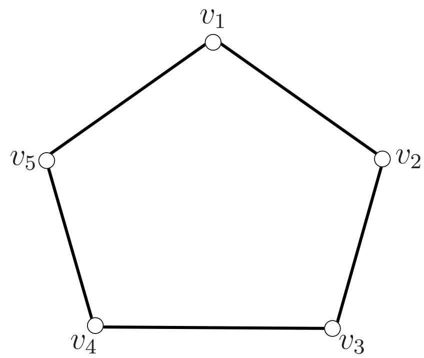

Figure 1.2: A 5-cycle: A graph that is not bipartite

Property 1.3.3. *A connected bipartite graph has a unique bipartition.*

Property 1.3.4. *A bipartite graph, without isolated vertices, which has t connected components has 2^{t-1} bipartitions.*

Theorem 1.3.5. $[König's\ Theorem]$ *A graph G is bipartite if and only if G has no cycle of odd length.*

Corollary 1.3.6. *[1] A connected graph G is bipartite if and only if for every vertex v there is no edge xy with $d(v,x) = d(v,y)$ ($d(v,x)$ denotes the distance between vertices v and x, see section* 1.5).

1.4 Subdivision of Graphs

Subdividing an edge is an operation that, from a geometric perspective, inserts a new vertex into the interior, thereby splitting that edge into two edges. Smoothing away a vertex is the inverse operation, which replaces two edges that meet at a vertex of degree 2 by a single edge that joins their other endpoints.

The operation of subdivision can be used to convert a general graph into a simple graph. This topic is comprehensively discussed in [17].

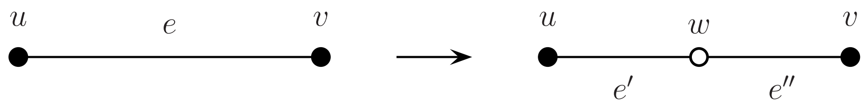

Figure 1.3: Subdividing an edge

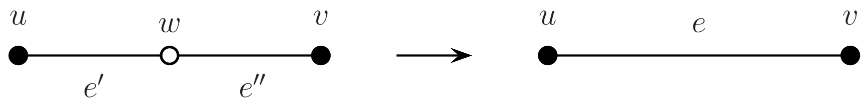

Figure 1.4: Smoothing away a vertex

Definition 1.4.1. *Let e be an edge with endpoints u, v in a graph G.* Subdividing the edge e *means that a new vertex w is added to $V(G)$, and that the edge e is replaced in $E(G)$ by an edge e' with endpoints u, w and an edge e'' with endpoints w, v. This concept is illustrated in Fig.* 1.3.

Definition 1.4.2. *Let w be a vertex of degree 2 in a graph G, such that two proper edges e' and e'' meet at w.* Smoothing away or (smoothing out) *the vertex w means replacing edges e' and e'' by a new edge e that joins the other endpoints of e' and e'', as shown in Fig.* 1.4.

Example 1.4.3. *The $(n+1)$ - cycle graph can be obtained from the n-cycle graph by subdividing any edge, as illustrated in Fig.* 1.5*. Inversely, smoothing away any vertex on the $(n+1)$-cycle, for $n \geq 1$, yields the n-cycle.*

Definition 1.4.4. *Subdividing a graph G means performing a sequence of edge-subdivision operations. The resulting graph is called a* subdivision of the graph G.

Example 1.4.5. *Performing any k subdivisions on the n-cycle graph C_n yields the $(n+k)$-cycle graph C_{n+k}, and C_n can be obtained by any k smoothing operations on C_{n+k}. Thus, C_{n+k} is a subdivision of C_k, for all $n \geq 1, k \geq 1$.*

Definition 1.4.6. *Suppose G is a connected plane graph. To* subdivide a face f *of G is to add a new edge e joining two vertices on its boundary in such a way that,*

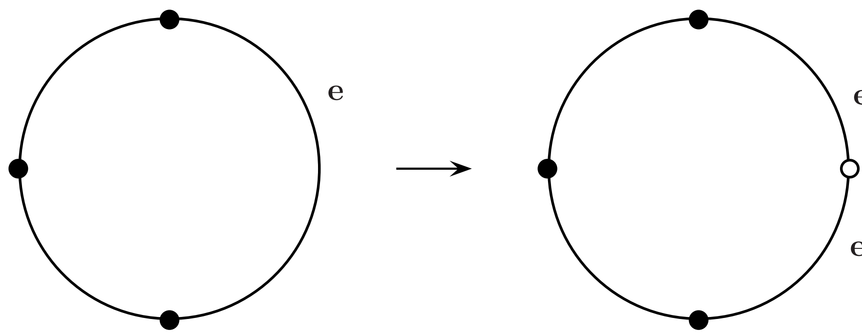

Figure 1.5: Subdividing an edge of the 3-cycle yields the 4-cycle

apart from its endpoints, e lies entirely in the interior of f [5].
This operation results in a plane graph $G + e$ with exactly one more face than; all faces of G except f are also faces of $G + e$, and the face f is replaced by two new faces, f_1 and f_2, which meet in the edge e, as illustrated in Fig. 1.6.

A graph G is *planar* if it can be drawn in the plane so that its edges intersect only at their ends. Subdivision of graphs plays a very important role in characterization of planar graphs. A graph G is planar if and only if every subdivision of G is planar. Two graphs are said to be homeomorphic if they are subdivisions of same graph G.

Theorem 1.4.7. [Kuratowski (1930)] *A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.*

1.5 Distances in graphs

Let G be a connected graph. The distance between two vertices u and v, denoted by $d(u, v)$ is the length of a shortest path between u and v. Let v be a node of G. The *eccentricity* of v, denoted by $e(v)$ is

$$e(v) = max\{d(u, v) : u \in V(G)\}.$$

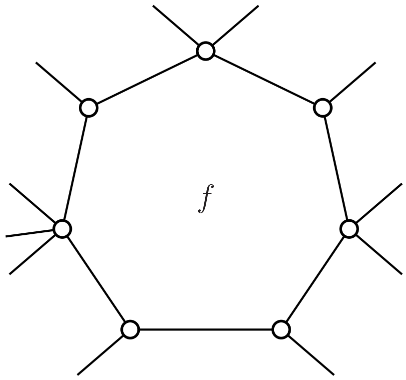

Figure 1.6: Subdivision of a face f by an edge e

The *radius* $r(G) = min\{e(v) : v \in V(G)\}$ is the minimum eccentricity of the nodes, whereas the *diameter* $diam(G)$ is the maximum eccentricity, or $diam(G) = max\{d(u,v) : u, v \in V(G\}$.

Theorem 1.5.1. *[11] For every nontrivial connected graph G,*

$$r(G) \leq diam(G) \leq 2r(G).$$

Now v is called a *central node* if $e(v) = r(G)$, and the center $C(G)$ is the set of all central nodes. Thus the center consists of all nodes having minimum eccentricity. Node v is *peripheral node* if $e(v) = diam(G)$, and the *periphery* is the set of all such nodes. For a node v, each node at distance $e(v)$ from v is an *eccentric node* for v. These concepts are illustrated in Fig. 1.7, where the eccentricity of each node is shown in parentheses. Graph G has radius 2, diameter 4 and central nodes d and g; nodes f and i are eccentric nodes for e.

Theorem 1.5.2. *[11] For every two adjacent vertices u and v in a connected graph,*

$$|e(u) - e(v)| \leq 1.$$

Theorem 1.5.3. *[11] The center of a tree consists of either a single node or a pair of adjacent nodes.*

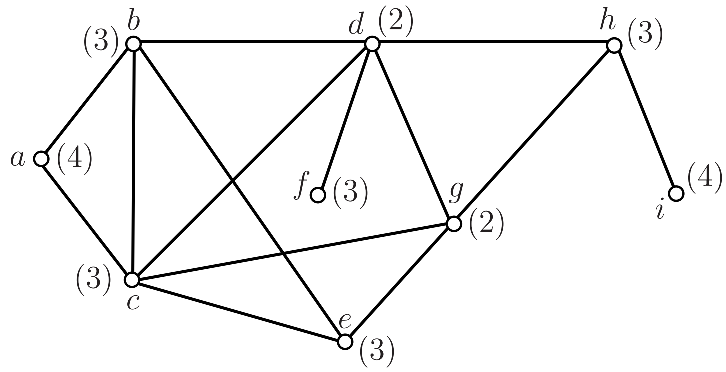

Figure 1.7: A graph and its eccentricities

Theorem 1.5.4. *[9] Every graph is the center of some graph.*

The notion of center of a graph is important in some applications involving location of the vertices of a graph representing service facilities such as police office, shopping center,power station or bank. When deciding where locating such facilities it is necessary to minimize the average distance that a person must travel. This is equivalent to minimizing the total distance traveled by all people within the sectors. For such situations, the concept of *median* is introduced.

For a connected graph G, the *status* $s(v)$ of a vertex v in G is defined as the sum of the distances from v to each other vertex in G. This concept was introduced by Harary [6]. The median $M(G)$ of a graph G is the set of vertices with minimum status.

1.6 Trees and spanning trees

A *forest* is a graph containing no cycle, i.e., one graph such that each component is a tree, a tree is a connected graph without cycles. A connected forest is a *tree*.

The vertices of degree 1 in a tree are called *leaves*. Every non-trivial tree has at least two leaves since the ends of a longest path are leaves.

Theorem 1.6.1. *[13] The following assertions are equivalent for a graph* T*:*

(i) *T is a connected graph without cycles;*

(ii) *There is a unique path of T joining any two distinct vertices of T;*

(iii) *T is minimally connected, i.e. T is connected but $T - e$ is disconnected for every edge $e \in T$;*

(iv) *T is maximally acyclic, i.e. T contains no cycle but for any two non-adjacent vertices $x, y \in T$, $T + xy$ does.*

Corollary 1.6.2. *[13] A connected graph with n vertices is a tree if and only if it has $n - 1$ edges.*

Corollary 1.6.3. *[11] Every forest of order n with k components has size $n - k$.*

Chapter 2

Metric dimension and resolving pairs of connected graphs

In this chapter, we give the definitions of resolving sets, resolving pairs and metric dimension which are the basic notions of this thesis. We describe some known results for these invariants.

2.1 Resolving Sets and Metric Dimension

Let $d(x, y)$ denote the distance between vertices x and y of a connected graph G. If $d(z, x) \neq d(z, y)$, then z is said to *resolve* x and y. A *resolving set* of G is a subset W of $V(G)$ having the property that for any two distinct vertices $x, y \in V(G)$ there exists a vertex $z \in W$ that resolves x and y, i. e., $d(z, x) \neq d(z, y)$.
To find whether a given set $W \subset V(G)$ is a resolving set of G we must only verify this property for the vertices in $V(G)\backslash W$, since every vertex $z \in W$ is the only vertex of G whose distance from W is 0. A resolving set of minimum cardinality is called a *metric basis* for G [8], [18] or just a basis [7], [9] for the graph. The *metric dimension* $dim(G)$ of G is by definition the number of elements in a basis [9], [18], [34], [35]. The metric dimension of a connected graph has been defined in [18], [34], [35]; an equivalent definition was given in [9]: Let $v \in V(G)$. The *representation*

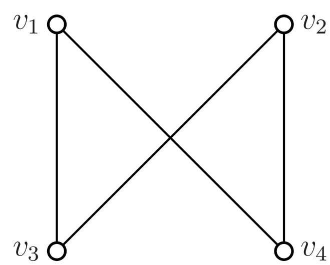

Figure 2.1: A graph with $dim(G) = 2$

$r(v|W)$ of v with respect to an ordered set of vertices of G, $W = \{w_1, w_2, \ldots, w_k\}$ is the k-tuple $(d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. W is called a resolving set for G if distinct vertices of G have distinct representations with respect to W. The problem of determining whether $dim(G) < k$ is an NP-complete problem [16], [26].

A survey of results on the metric dimension and its applications is [8]; also see [19], [30].

For example, consider the graph G of Fig. 2.1. Since $r(v_3|W_1) = (1, 1) = r(v_4|W_1)$ the set $W_1 = \{v_1, v_2\}$ is not a resolving set of G. On the other hand, $W_2 = \{v_1, v_2, v_4\}$ is a resolving set for G since the representation for the vertices of G with respect to W_2 are $r(v_1|W_2) = (0, 2, 1)$, $r(v_2|W_2) = (2, 0, 1)$, $r(v_3|W_2) = (1, 1, 2)$, $r(v_4|W_2) = (1, 1, 0)$. However, W_2 is not a minimum resolving set since $W_3 = \{v_1, v_4\}$ is also a resolving set. It follows that W_3 is a minimum resolving set implying that $dim(G) = 2$, since no single vertex constitutes a resolving set for G.

The following property is useful in finding $dim(G)$:

Lemma 2.1.1. *[37] Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all vertices $w \in V(G) \setminus \{u, v\}$, then $\{u, v\} \cap W \neq \emptyset$.*

Slater called the metric dimension of a graph its location number and motivated the study of this invariant by its application to the placement of a minimum number of sonar/loran detecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set ([34],[35]).

2.2 Resolving Pairs of Graphs

For a pair $\{x, y\}$ of distinct vertices of G we shall denote by $RS(x, y)$ the set of vertices $z \in V(G)$ such that $d(z, x) \neq d(z, y)$ [36]. Such a set will be called the resolving set (or the R-set) relative to the pair $\{x, y\}$. It is clear that $\{x, y\} \subseteq RS(x, y) \subseteq V(G)$ for any pair $\{x, y\}$.

This notion implicitly appeared in [12], whose authors constructed for a given connected graph G of order n an associated bipartite graph as follows: Let V_p be the collection of all $\binom{n}{2}$ pairs of vertices in G. The associated bipartite graph has partite sets $V(G)$ and V_p. A vertex $v \in V(G)$ is joined to a vertex $s \in V_p$ if v has distinct distances to the vertices in s.

So the neighborhood of s in this associated bipartite graph is precisely the R-set for s referred to above. Some properties of R-sets were described in [36].

A pair $\{x, y\}$ such that $RS(x, y) = V(G)$ will be called a resolving pair of G.

Now we will list some results about R-sets of graphs which has been proved already. These results have been quoted from [36]. The following lemma is based on the observation that if for a pair $\{x, y\}$ of vertices, the distance $d(x, y)$ is even, then the middle vertex v of a shortest path between x and y has equal distances to x and to y and so $v \notin RS(x, y)$.

Lemma 2.2.1. *[36] If $RS(x, y) = V(G)$ for a pair $\{x, y\}$ of distinct vertices of a connected graph G, then $d(x, y)$ is odd.*

Lemma 2.2.2. *[36] If $RS(x, y) = \{x, y\}$ for each pair $\{x, y\}$ of distinct vertices of a connected graph G, then G is a complete graph.*

Theorem 2.2.3. *[36] If G has n vertices and $diam(G) = 2$ then the number of pairs $\{x, y\}$ such that $RS(x, y) = V(G)$ is bounded above by $\lfloor n^2/4 \rfloor$. This bound is attained only for $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.*

If a graph G of order n has $diam(G) = n - 1$ then G is a path P_n and every pair of vertices $\{x, y\}$ of P_n such that $d(x, y)$ is odd satisfies $RS(x, y) = V(P_n)$; the number of such pairs equals $\lfloor n^2/4 \rfloor$. By joining by an edge the centers of the stars $K_{1,\lfloor n/2 \rfloor - 1}$ and $K_{1,\lceil n/2 \rceil - 1}$ for any $n \geq 4$ the resulting graph G has diameter 3 and the number of pairs such that $RS(x, y) = V(G)$ also equals $\lfloor n^2/4 \rfloor$.

Conjecture 2.2.4. *[36] For every connected graph G of order $n \geq 2$ the number of pairs $\{x, y\}$ such that $RS(x, y) = V(G)$ is bounded above by $\lfloor n^2/4 \rfloor$.*

R-sets can be used for obtaining bounds for the metric dimension of a graph G.

Chapter 3

On metric dimension of flower graphs $f_{n\times m}$ and convex polytopes

The motivation of this chapter is the study of metric dimension of flower graphs $f_{n\times m}$ and two classes of graphs associated to convex polytopes.

3.1 Notation and preliminary results

Let $\mathcal{F}$ be a family of connected graphs $G_n : \mathcal{F} = (G_n)_{n\geq 1}$ depending on n as follows: the order $|V(G)| = \varphi(n)$ and $\lim\limits_{n\to\infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $dim(G) \leq C$ for every $n \geq 1$ then we shall say that $\mathcal{F}$ has bounded metric dimension; otherwise $\mathcal{F}$ has unbounded metric dimension.

If all graphs in $\mathcal{F}$ have the same metric dimension (which does not depend on n), then $\mathcal{F}$ is called a family with constant metric dimension. A connected graph G has $dim(G) = 1$ if and only if G is a *path* [9]; *cycles* C_n have metric dimension 2 for every $n \geq 3$. Also *generalized Petersen graphs* $P(n, 2)$, *antiprisms* A_n and *Harary graphs* $H_{4,n}$ are families of graphs with constant metric dimension [23].

A *Cartesian product* of two graphs G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$, where two vertices (x, x') and (y, y') are adjacent if and only if $x = y$ and $x'y' \in E(H)$ or $x' = y'$ and $xy \in E(G)$. A *grid* G_n^m is obtained by the Cartesian product of two paths P_n by P_m. In [26] it was shown that $dim(P_n \times P_m) = 2$, so grids constitute a family of plane graphs with constant metric dimension as their metric dimension is 2 and does not depend upon the number of vertices in the graph. The metric dimension of the Cartesian product of graphs has been studied in [8] and [31].

Other families of graphs have unbounded metric dimension: if W_n denotes a *wheel* with n spokes and J_{2n} the graph deduced from the wheel W_{2n} by alternately deleting n spokes, then $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for every $n \geq 7$ [7] and $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$ [38] for every $n \geq 4$.

An example of a family which has bounded metric dimension is the *prism* denoted by D_n [8]. The prism and the antiprism are *Archimedean* convex polytopes defined e.g. in [25]. Also, it is shown in [22] that the generalized Petersen graphs $P(n, 3)$ have bounded metric dimension. The metric dimension of some classes of convex polytopes has been determined in [20] and it was shown that these classes of convex polytopes have constant metric dimension. Some bounds for this invariant, in terms of the diameter of the graph, are given in [27] and it was shown in [9], [18], [27] that the metric dimension of the trees can be determined efficiently.

In this chapter the metric dimension of some families of plane graphs is studied. In the second section the metric dimension of $n \times m$ flower graph $f_{n\times m}$ is deduced. In the third section, the metric dimension of the graph S_n of a convex polytope consisting of 3-sided, 5-sided, 6-sided and n-sided faces is determined by considering an open problem raised in [20]. In the fourth section is deduced the metric dimension of the graph T_n of a convex polytope, which is obtained from the graph R_n of a convex polytope described in [3] by deleting some edges and having the same vertex set.

3.2 The flower graphs $f_{n\times m}$

A graph G is called an $(n \times m)$-flower graph [4] if it has n vertices which form an n-cycle and n sets of $m - 2$ vertices each which form m-cycles around the n-cycle

so that each m-cycle uniquely intersects the n-cycle on a single edge. This graph will be denoted by $f_{n\times m}$. It is clear that $f_{n\times m}$ has $n(m-1)$ vertices and nm edges. The m-cycles are called the *petals* and the n-cycle is called the *center* of $f_{n\times m}$. The n vertices which form the center are all of degree 4 and all the other vertices have degree 2. The flower graphs $f_{8\times 4}$ and $f_{4\times 6}$ have been shown in Fig. 3.1.
In the next theorem it is shown that every $(n \times 3)$-flower graph $f_{n\times 3}$ has bounded metric dimension.

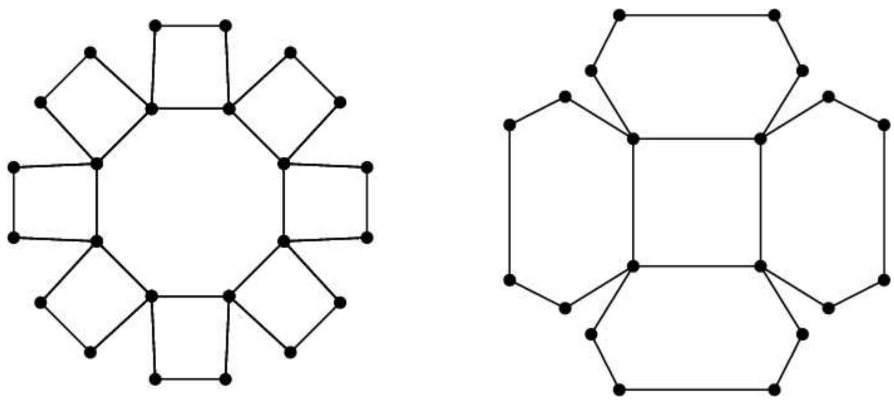

Figure 3.1: Flower graphs $f_{8\times 4}$ and $f_{4\times 6}$

For our purpose, we denote the vertices of the center of $f_{n\times 3}$ by $x_1, x_2, \ldots, x_n$ and the vertices of the petals of $f_{n\times 3}$ by $y_1, y_2, \ldots, y_n$ and we call the cycle induced by $\{x_i : 1 \leq i \leq n\}$, the inner cycle and vertices $y_i : 1 \leq i \leq n$, the outer vertices. Note that the choice of appropriate basis vertices (also referred to as landmarks in [26]) is the core of the problem.

Theorem 3.2.1. *[21] Let $f_{n\times 3}$ be a flower graph; then*

$$dim(f_{n\times 3}) = \begin{cases} 2, & \text{for } n \text{ even} \\ 3, & \text{otherwise} \end{cases}$$

for every $n \geq 6$.

Proof. We will prove the above equality by double inequality. We consider two cases.
Case(i) n is even.

In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{y_1, y_k\} \subset V(f_{n\times 3})$. We show that W is a resolving set of $f_{n\times 3}$ in this case. For this purpose, we give representation of any vertex of $V(f_{n\times 3})$.

Representations of the vertices on the inner cycle are

$$r(x_i|W) = \begin{cases} (1, k), & i = 1; \\ (i-1, k-i+1), & 2 \leq i \leq k; \\ (k, 1), & i = k+1; \\ (2k-i+2, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of the set of outer vertices are

$$r(y_i|W) = \begin{cases} (i, k-i+1), & 2 \leq i \leq k-1; \\ (2k-i+2, i-k+1), & k+1 \leq i \leq 2k. \end{cases}$$

One can easy to see that there are no two vertices having the same representations, which implies that $dim(f_{n\times 3}) \leq 2$. On the other hand, since $f_{n\times 3}$ is not a path, from [9] it follows that $dim(f_{n\times 3}) = 2$ in this case.

Case(ii) n is odd.

In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. We show that $W = \{x_1, x_2, x_{k+1}\} \subset V(f_{n\times 3})$ is a resolving set for $f_{n\times 3}$ in this case. For this we give representation of any vertex of $V(f_{n\times 3})$.

Representations of the vertices on the inner cycle are

$$r(x_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k+1; \\ (k, k, 1), & i = k+2; \\ (2k-i+2, 2k-i+3, i-k-1), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations of the set of outer vertices are

$$r(y_i|W) = \begin{cases} (1,1,k), & i=1; \\ (i,i-1,k-i+1), & 2 \le i \le k; \\ (k+1,k,1), & i=k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \le i \le 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations, which implies that $dim(f_{n\times 3}) \le 3$.

On the other hand, suppose that $dim(f_{n\times 3}) = 2$, then there are the following possibilities to be discussed.

(1) Both vertices are in the inner cycle. Without loss of generality, we can suppose that one resolving vertex is x_1. Suppose that second resolving vertex is x_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(x_n|\{x_1, x_t\}) = r(y_n|\{x_1, x_t\}) = (1, t)$ and for $t = k+1$, $r(x_n|\{x_1, x_t\}) = r(y_1|\{x_1, x_t\}) = (1, t-1)$, a contradiction.

(2) Both vertices belong to the set of outer vertices. Without loss of generality, we can suppose that one resolving vertex is y_1. Suppose that second resolving vertex is y_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(x_n|\{y_1, y_t\}) = r(y_n|\{y_1, y_t\}) = (2, t+1)$ and for $t = k+1$, $r(x_n|\{y_1, y_t\}) = r(y_2|\{y_1, y_t\}) = (2, t-1)$, a contradiction.

(3) One vertex is in the inner cycle and other belongs to the set of outer vertices. Without loss of generality, we can suppose that one resolving vertex is x_1. Suppose that second resolving vertex is y_t $(1 \le t \le k+1)$. Then for $1 \le t \le k$, we have $r(x_n|\{x_1, y_t\}) = r(y_n|\{x_1, y_t\}) = (1, t+1)$ and for $t = k+1$, $r(y_1|\{x_1, y_t\}) = r(y_n|\{x_1, y_t\}) = (1, t)$, a contradiction.

Hence in all possible cases, there is no resolving set with two vertices for $V(f_{n\times 3})$ thus implying that $dim(f_{n\times 3}) \ge 3$, which completes the proof.

□

In the next theorem we show that for every $m \ge 4$, the metric dimension of the flower graph $f_{n\times m}$ is unbounded as n tends to infinity. For our purpose, any two m-cycles which share one vertex with each other at the center of $f_{n\times m}$ will be called neighboring m-cycles.

Theorem 3.2.2. *[21] For every $n \geq 5$ and $m \geq 4$ $dim(f_{n\times m}) = \lceil n/2 \rceil$.*

Proof. Denote the m-cycles of $f_{n\times m}$ by $C_m^i (1 \leq i \leq n)$, C_m^i having vertices $a_i^1, a_i^2, \ldots, a_i^m$ as in Fig. 3.2. In order to simplify the notation we also make the convention that common vertices of these m-cycles are denoted as $a_n^m = a_1^1, a_1^m = a_2^1, \ldots, a_{n-1}^m = a_n^1$.

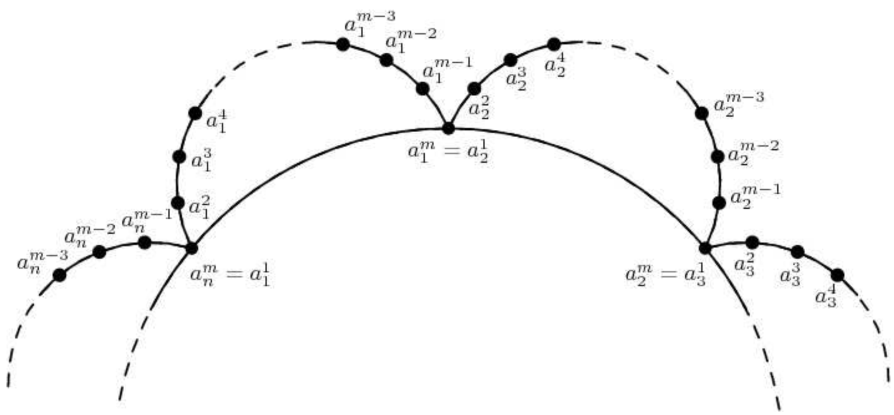

Figure 3.2: Graph $f_{n\times m}$

Let C_m^1 and C_m^2 be two neighboring m-cycles. Vertices a_1^{m-1} and a_2^2 have equal distances to all other vertices of $f_{n\times m}$ except the vertices a_1^{m-k}, where $2 \leq k \leq (m+1)/2$ for odd m and $2 \leq k \leq m/2$ for even m and a_2^{k+2}, where $1 \leq k \leq (m-1)/2$ for odd m and $1 \leq k \leq m/2 - 1$ for even m.

It follows that any two neighboring m-cycles must have at least one vertex which does not belong to the central n-cycle into any resolving set of $f_{n\times m}$. Let R be a resolving set of $f_{n\times m}$ and denote by m_i the number of non-central vertices of C_m^i which belong to R. We can write $m_i + m_{i+1} \geq 1$ for every $1 \leq i \leq n-1$ and also $m_n + m_1 \geq 1$. By adding up these inequalities we get $2\sum_{i=1}^n m_i \geq n$. We deduce $|R| \geq \sum_{i=1}^n m_i \geq n/2$, which implies $dim(f_{n\times m}) \geq \lceil n/2 \rceil$.

We will prove that the opposite inequality is also true for $n \geq 5$. For this it is necessary to find a resolving set $A_{n,m}$ of cardinality equal to $\lceil n/2 \rceil$. We shall define $A_{n,m}$ as follows:

(i) For n even: $A_{n,m} = \{a_1^{m/2}, a_3^{m/2}, \ldots, a_{n-1}^{m/2}\}$ for even m and $A_{n,m} = \{a_1^{(m-1)/2}, a_3^{(m-1)/2}, \ldots, a_{n-1}^{(m-1)/2}\}$ for odd m;
(ii) for n odd: $A_{n,m} = \{a_1^{m/2}, a_3^{m/2}, \ldots, a_n^{m/2}\}$ for even m and $A_{n,m} = \{a_1^{(m-1)/2}, a_3^{(m-1)/2}, \ldots, a_n^{(m-1)/2}\}$ for odd m.
Such a set $A_{5,7}$ is illustrated in Fig. 3.3.
For a subset of vertices $S \subset V(G)$ and a vertex $x \in V(G)$ we define $d(x, S) = \min_{y \in V(S)} d(x, y)$; it is easy to see that all central vertices lying on the n-cycle have a distance to $A_{n,m}$ equal to $(m-1)/2$ or $(m-3)/2$ for odd m and to $m/2$ or $(m-2)/2$ for even m.
For any two distinct cycles C_m^i and C_m^j $(i \neq j)$ we define the following parameter, called ρ-distance as follows:

$$\rho(C_m^i, C_m^j) = \min_{x \in V(C_m^i), y \in V(C_m^j)} d(x, y).$$

It is clear that if C_m^i and C_m^j are neighboring, then $\rho(C_m^i, C_m^j) = 0$. Since $n \geq 5$

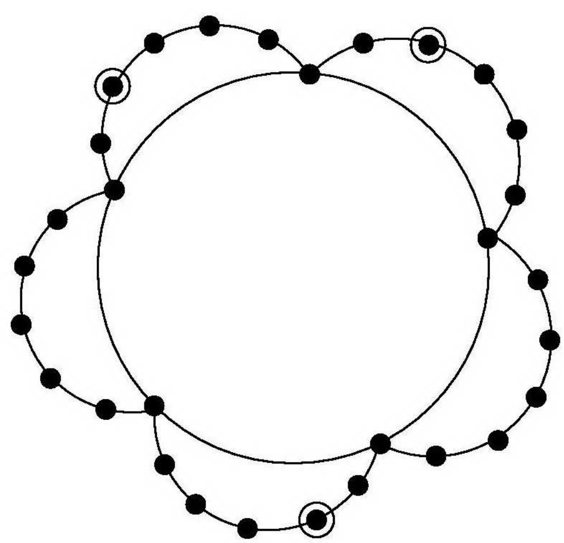

Figure 3.3: A basis of $f_{5\times 7}$

for any cycle C_m^i there exist exactly two distinct cycles C_m^p and C_m^q such that $\rho(C_m^i, C_m^p) = \rho(C_m^i, C_m^q) = 1$. It is necessary to show that for any two distinct vertices $x, y \in V(f_{n\times m})\backslash A_{n,m}$ there exists a vertex $w \in A_{n,m}$ that resolves x and y, i.e., $d(w,x) \neq d(w,y)$. There are three subcases to be considered.

A. First let us consider the case when both x and y are central vertices. If $d(x,y) = 1$ then x and y belong to an m-cycle C_m^i. If C_m^i contains a vertex $w \in A_{n,m}$, then w resolves x and y. Otherwise, both neighboring cycles of C_m^i contain each a vertex from $A_{n,m}$ and these vertices resolve x and y.

If $d(x,y) = 2$ then x and y belong to two neighboring cycles; if both these cycles contain a vertex in $A_{n,m}$ (for odd n), then one of them resolves x and y. If only one cycle C_m^j has a vertex in $A_{n,m}$ and this vertex has equal distances to x and y (see Fig. 3.3), then there exists an m-cycle C_m^k that contains a vertex $w \in A_{n,m}$ such that $\rho(C_m^j, C_m^k) = 1$ and w resolves x and y. If $n \geq 6$ and $d(x,y) \geq 3$ then one of x and y belongs to an m-cycle containing a vertex w in $A_{n,m}$ and this vertex resolves x and y.

B. Consider now the case when x is a central vertex but y is not a central vertex. If x and y belong to the same m-cycle C_m^i and C_m^i has no vertex in $A_{n,m}$, then x also belongs to a neighboring m-cycle containing a vertex w in $A_{n,m}$ and w resolves x and y.

If C_m^i has a vertex z in $A_{n,m}$ and z does not resolve x and y then there exists an m-cycle C_m^j such that $\rho(C_m^i, C_m^j) = 1$ and C_m^j has a vertex w in $A_{n,m}$ which resolves x and y. If x and y do not belong to the same m-cycle, but they belong to neighboring cycles C_m^i and C_m^j, respectively ($j = i + 1$ or $i = n$ and $j = 1$), then: if C_m^i has no vertex in $A_{n,m}$ then its neighboring cycle $C_m^k \neq C_m^j$ has a vertex w in $A_{n,m}$ which resolves x and y; if C_m^i has a vertex w in $A_{n,m}$, then $d(x,w) < d(y,w)$, hence w resolves x and y.

Otherwise x and y belong to m-cycles C_m^i and C_m^j, respectively such that $\rho(C_m^i, C_m^j) \geq 1$. In this case x and y are resolved by a vertex $w \in A_{n,m}$ belonging either to C_m^i or to an m-cycle which is one of the two neighbors of C_m^i.

C. The remaining case is that when x and y are not central vertices. If they belong to the same m-cycle C_m^i and C_m^i has no vertex in $A_{n,m}$, then neighboring m-cycles of C_m^i have each a vertex w_1 and w_2, respectively in $A_{n,m}$. If $d(x, w_1) = d(y, w_1)$ then $d(x, w_2) \neq d(y, w_2)$ and w_1 or w_2 resolves x and y.

If C_m^i has a vertex w in $A_{n,m}$ and w does not resolve x and y, then $d(x, w) = d(y, w)$, which implies that there exists an m-cycle C_m^j such that $\rho(C_m^i, C_m^j) = 1$ and C_m^j includes a vertex w in $A_{n,m}$ which resolves x and y.

If x and y do not belong to the same m-cycle, but they belong to two neighboring cycles: $x \in V(C_m^i)$ and $y \in V(C_m^j)$, then at least one of these two cycles, say C_m^i has a vertex w in $A_{n,m}$. In both cases m even and m odd we have $d(x, w) \neq d(y, w)$, hence w resolves x and y.

If x and y do not belong to the same m-cycle, nor to two neighboring cycles, then $x \in V(C_m^i), y \in V(C_m^j)$ and $\rho(C_m^i, C_m^j) \geq 1$. If C_m^i or C_m^j contains a vertex $w \in A_{n,m}$ then w resolves x and y. Otherwise, C_m^i and C_m^j have no vertex in $A_{n,m}$.

Suppose that $\rho(C_m^i, C_m^j) = 1$ and denote by: C_m^p the common neighbor of C_m^i and C_m^j; C_m^q the other neighbor of C_m^i and C_m^r the neighbor of C_m^j different from C_m^p (these neighbors exist for $n \geq 5$). They contain some vertices in $A_{n,m}$: w_p in C_m^p, w_q in C_m^q and w_r in C_m^r. If $d(x, w_p) = d(y, w_p)$ and $d(x, w_q) = d(y, w_q)$ then $d(x, w_r) \neq d(y, w_r)$ and w_r resolves vertices x and y.

If $\rho(C_m^i, C_m^j) > 1$ it follows that $\rho(C_m^i, C_m^j) \geq 3$ (consequently $n \geq 8$). In this case let C_m^a and C_m^b be the two neighbors of C_m^i, containing vertices w_a and w_b, respectively in $A_{n,m}$. In this case at least one vertex from the set $\{w_a, w_b\}$ resolves x and y. □

3.3 The graph S_n

The metric dimension of some classes of graphs of convex polytopes has been determined in [20]. In this section the metric dimension of the graph S_n of a convex polytope consisting of 3-sided, 5-sided, 6-sided and n-sided faces (Fig. 3.4) is studied. The graph S_n has the following vertex and edge sets:

$$V(S_n) = \{a_i; b_i; c_i; d_i; e_i : 1 \leq i \leq n\}$$

and

$$E(S_n) = \{a_i a_{i+1} : 1 \leq i \leq n\} \cup \{a_i b_i : 1 \leq i \leq n\} \cup \{a_{i+1} b_i : 1 \leq i \leq n\} \cup \{b_i c_i : 1 \leq i \leq n\} \cup \{c_i d_i : 1 \leq i \leq n\} \cup \{c_{i+1} d_i : 1 \leq i \leq n\} \cup \{d_i e_i : 1 \leq i \leq n\} \cup \{e_i e_{i+1} : 1 \leq i \leq n\},$$

where the subscript $n+1$ must be replaced by 1.

The graph S_n has $5n$ vertices, $8n$ edges and $3n+2$ faces.

For our purpose, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$ the inner cycle,

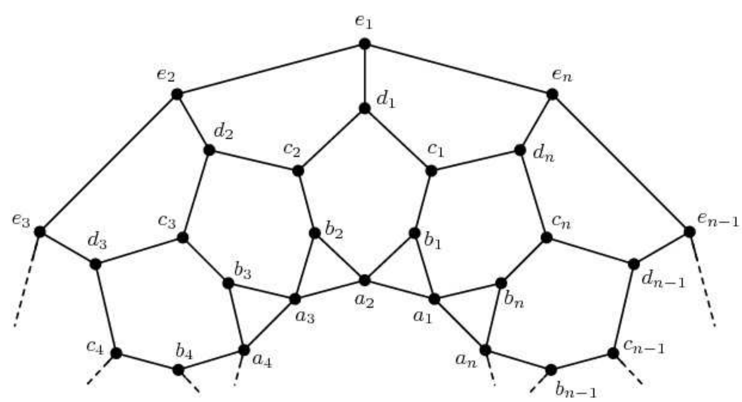

Figure 3.4: Graph S_n

the set of vertices $\{b_1, b_2, ..., b_n\}$ the set of central vertices, the cycle induced by $\{c_i : 1 \leq i \leq n\} \cup \{d_i : 1 \leq i \leq n\}$ the middle cycle and the cycle induced by $\{e_i : 1 \leq i \leq n\}$ the outer cycle. Again a choice of an appropriate vertex basis is crucial.

Theorem 3.3.1. *[21] We have $dim(S_n) = 3$ for every $n \geq 6$.*

Proof. **Case (i)** n is even.

In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(S_n)$, we show that W is a resolving set for S_n in this case. For this we give representations of any vertex of $V(S_n)$.

Representations of the vertices on the inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (k-1, k, 1), & i = k+2; \\ (2k-i+1, 2k-i+2, i-k-1), & k+3 \leq i \leq 2k. \end{cases}$$

Representations of the set of central vertices are

$$r(b_i|W) = \begin{cases} (1, 1, k), & i = 1; \\ (i, i-1, k-i+1), & 2 \leq i \leq k; \\ (k, k, 1), & i = k+1; \\ (2k-i+1, 2k-i+2, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of the vertices on the middle cycle are

$$r(c_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \leq i \leq k; \\ (k+1, k+1, 2), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k+1), & k+2 \leq i \leq 2k \end{cases}$$

and

$$r(d_i|W) = \begin{cases} (3, 3, k+1), & i = 1; \\ (i+2, i+1, k-i+2), & 2 \leq i \leq k-1; \\ (k+2, k+1, 3), & i = k; \\ (2k-i+2, 2k-i+3, i-k+2), & k+1 \leq i \leq 2k-1 \\ (3, 3, k+2), & i = 2k. \end{cases}$$

Representations of the vertices on the outer cycle are

$$r(e_i|W) = \begin{cases} (4,4,k+2), & i=1; \\ (i+3,i+2,k-i+3), & 2 \le i \le k-1; \\ (k+3,k+2,4), & i=k; \\ (2k-i+3,2k-i+4,i-k+3), & k+1 \le i \le 2k-1 \\ (4,4,k+3), & i=2k. \end{cases}$$

One can easily verify that there are no two vertices having the same representations, thus implying that $dim(S_n) \le 3$ in this case.

On the other hand, we show that $dim(S_n) \ge 3$. Suppose on the contrary that $dim(S_n) = 2$, then there are the following possibilities to be discussed.

(**1**) Both vertices are in the inner cycle. Without loss of generality we can suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is a_t $(2 \le t \le k+1)$. Thus $W = \{a_1, a_t\}$.

If $2 \le t \le k$, $r(a_n|\{a_1,a_t\}) = r(b_n|\{a_1,a_t\}) = (1,t)$ and

if $t = k+1$, $r(a_2|\{a_1,a_{k+1}\}) = r(a_n|\{a_1,a_{k+1}\}) = (1,t-2)$, a contradiction.

(**2**) Both vertices belong to the set of central vertices. Without loss of generality we can suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is b_t $(2 \le t \le k+1)$. Thus $W = \{b_1, b_t\}$.

If $2 \le t \le k-1$, $r(a_n|\{b_1,b_t\}) = r(b_n|\{b_1,b_t\}) = (2,t+1)$;

if $t = k$, $r(b_n|\{b_1,b_t\}) = r(d_1|\{b_1,b_t\}) = (2,t+1)$ and

if $t = k+1$, $r(b_2|\{b_1,b_{k+1}\}) = r(b_n|\{b_1,b_{k+1}\}) = (2,t-1)$, a contradiction.

(**3**) Both vertices are in the middle cycle. Here there are the following subcases.

• Both vertices are in the set $\{c_i : 1 \le i \le n\}$. Without loss of generality we can suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is c_t, $2 \le t \le k+1$. Thus $W = \{c_1, c_t\}$.

We have $r(a_1|\{c_1,c_t\}) = r(e_n|\{c_1,c_t\}) = (2,t+1)$ for $2 \le t \le k+1$, a contradiction.

• Both vertices are in the set $\{d_i : 1 \le i \le n\}$. Without loss of generality we can suppose that one resolving vertex is d_1. Suppose that the second resolving vertex is d_t, $2 \le t \le k+1$. Thus $W = \{d_1, d_t\}$.

If $2 \leq t \leq k$, $r(b_1|\{d_1, d_t\}) = r(d_n|\{d_1, d_t\}) = (2, t+2)$ and

if $t = k+1$, $r(b_1|\{d_1, d_{k+1}\}) = r(b_2|\{d_1, d_{k+1}\}) = (2, t+1)$ a contradiction.

• One vertex is in the set $\{c_i : 1 \leq i \leq n\}$ and other in the set $\{d_i : 1 \leq i \leq n\}$. Without loss of generality we can suppose that the resolving set is $W = \{c_1, d_t\}$, $1 \leq t \leq k+1$.

In this case one can easily verify that we also have $r(b_1|\{c_1, d_t\}) = r(d_n|\{c_1, d_t\})$, a contradiction.

(**4**) Both vertices are in the outer cycle.Without loss of generality we can suppose that one resolving vertex is e_1. Suppose that the second resolving vertex is e_t $(2 \leq t \leq k+1)$. Thus $W = \{e_1, e_t\}$.

If $2 \leq t \leq k$, $r(d_1|\{e_1, e_t\}) = r(e_n|\{e_1, e_t\}) = (1, t)$ and

if $t = k+1$, $r(e_2|\{e_1, e_{k+1}\}) = r(e_n|\{e_1, e_{k+1}\}) = (1, t-2)$, a contradiction.

(**5**) One vertex is in the inner cycle and other belongs to the set of central vertices. Without loss of generality we can suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is b_t $(1 \leq t \leq k+1)$. Thus $W = \{a_1, b_t\}$.

If $1 \leq t \leq k-1$, $r(a_n|\{a_1, b_t\}) = r(b_n|\{a_1, b_t\}) = (1, t+1)$

if $t = k$, $r(a_n|\{a_1, b_t\}) = r(b_1|\{a_1, b_t\}) = (1, t)$ and

if $t = k+1$, $r(a_2|\{a_1, b_{k+1}\}) = r(b_n|\{a_1, b_{k+1}\}) = (1, t-1)$, a contradiction.

(**6**) One vertex is in the inner cycle and other in the middle cycle. Here there are two subcases.

• One vertex is in the inner cycle and other belongs to the set $\{c_t : 1 \leq t \leq n\}$. Thus $W = \{a_1, c_t\}$, $1 \leq t \leq k+1$.

If $1 \leq t \leq k-1$, $r(a_n|\{a_1, c_t\}) = r(b_n|\{a_1, c_t\}) = (1, t+2)$;

if $t = k$, $r(a_n|\{a_1, c_k\}) = r(b_1|\{a_1, c_k\}) = (1, t+1)$;

if $t = k+1$, $r(a_2|\{a_1, c_{k+1}\}) = r(b_n|\{a_1, c_{k+1}\}) = (1, t)$, a contradiction.

• One vertex is in the inner cycle and other belongs to the set $\{d_t : 1 \leq t \leq n\}$. Thus $W = \{a_1, d_t\}$, $1 \leq t \leq k+1$.

If $1 \leq t \leq k-1$, $r(a_n|\{a_1, d_t\}) = r(b_n|\{a_1, d_t\}) = (1, t+3)$;

if $t = k$, $r(a_2|\{a_1, d_k\}) = r(a_n|\{a_1, d_k\}) = (1, t)$;

if $t = k+1$, $r(a_2|\{a_1, d_{k+1}\}) = r(b_1|\{a_1, d_{k+1}\}) = (1, t+1)$, a contradiction.

(**7**) One vertex is in the inner cycle and other in the outer cycle. Without loss of generality we can suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is e_t $(1 \leq t \leq k+1)$. Thus $W = \{a_1, e_t\}$.

If $t = 1$, $r(b_2|\{a_1, e_t\}) = r(c_n|\{a_1, e_t\}) = (2, t+2)$;

if $2 \leq t \leq k$, $r(b_2|\{a_1, e_t\}) = r(c_1|\{a_1, e_t\}) = (2, t+1)$ and

if $t = k+1$, $r(a_n|\{a_1, e_{k+1}\}) = r(b_n|\{a_1, e_{k+1}\}) = (1, t)$, a contradiction.

(**8**) One vertex is in the set of central vertices and other in the middle cycle. Here there are two subcases.

• One vertex is in the set of central vertices and other belongs to the set $\{c_i : 1 \leq i \leq n\}$. Thus $W = \{b_1, c_t\}$, $1 \leq t \leq k+1$.

If $1 \leq t \leq k-1$, $r(a_n|\{b_1, c_t\}) = r(b_n|\{b_1, c_t\}) = (2, t+2)$;

if $t = k$, $r(b_n|\{b_1, c_k\}) = r(d_n|\{b_1, c_k\}) = (2, t+2)$;

if $t = k+1$, $r(a_1|\{b_1, c_{k+1}\}) = r(a_2|\{b_1, c_{k+1}\}) = (1, t)$, a contradiction.

• One vertex is in the set of central vertices and other belongs to the set $\{d_i : 1 \leq i \leq n\}$. Thus $W = \{b_1, d_t\}$, $1 \leq t \leq k+1$.

If $1 \leq t \leq k-1$, $r(a_n|\{b_1, d_t\}) = r(b_n|\{b_1, d_t\}) = (2, t+3)$;

if $k \leq t \leq k+1$, $r(d_n|\{b_1, d_t\}) = r(b_n|\{b_1, d_t\}) = (2, t+2)$, a contradiction.

(**9**) One vertex is in the set of central vertices and other in the outer cycle. Without loss of generality we can suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is e_t $(1 \leq t \leq k+1)$. Thus $W = \{b_1, e_t\}$.

If $t = 1$, $r(a_1|\{b_1, e_1\}) = r(a_2|\{b_1, e_1\}) = (1, 4)$.

If $2 \leq t \leq k$, $r(b_2|\{b_1, e_t\}) = r(d_n|\{b_1, e_t\}) = (2, t+1)$ and

if $t = k+1$, $r(d_1|\{b_1, e_{k+1}\}) = r(b_n|\{b_1, e_{k+1}\}) = (2, t)$, a contradiction.

(**10**) One vertex is in the middle cycle and other in the outer cycle. We have two subcases.

• One vertex is in the set $\{c_i : 1 \leq i \leq n\}$ and other belongs to the outer cycle. Thus $W = \{c_1, e_t\}$, $1 \leq t \leq k+1$.

If $t = 1$, $r(a_1|\{c_1, e_1\}) = r(a_2|\{c_1, e_1\}) = (2, 4)$.

If $2 \leq t \leq k-1$, $r(a_2|\{c_1, e_t\}) = r(c_n|\{c_1, e_t\}) = (2, t+2)$;

if $t = k$, $r(c_2|\{c_1, e_k\}) = r(e_n|\{c_1, e_k\}) = (2, t)$ and

if $t = k+1$, $r(e_1|\{c_1, e_{k+1}\}) = r(c_n|\{c_1, e_{k+1}\}) = (2, t-1)$, a contradiction.

• One vertex is in the set $\{d_i : 1 \leq i \leq n\}$ and other belongs to the outer cycle. Thus $W = \{d_1, e_t\}$, $1 \leq t \leq k+1$.

If $t = 1$, $r(c_1|\{d_1, e_1\}) = r(c_2|\{d_1, e_1\}) = (1, 2)$.

If $2 \leq t \leq k$, $r(b_2|\{d_1, e_t\}) = r(d_n|\{d_1, e_t\}) = (2, t+1)$ and

if $t = k+1$, $r(e_2|\{d_1, e_{k+1}\}) = r(e_n|\{d_1, e_{k+1}\}) = (2, t-2)$, a contradiction.

From the above discussion it follows that there is no resolving set with two vertices for $V(S_n)$, thus implying that $dim(S_n) = 3$ in this case.

Case (ii) n is odd.

In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(S_n)$, again we show that W is a resolving set for S_n in this case. For this we give representations of any vertex of $V(S_n)$.

Representations of the vertices on the inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (k, k, 1), & i = k+2; \\ (2k-i+2, 2k-i+3, i-k-1), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations of the set of central vertices are

$$r(b_i|W) = \begin{cases} (1, 1, k), & i = 1; \\ (i, i-1, k-i+1), & 2 \leq i \leq k; \\ (k+1, k, 1), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on the middle cycle are

$$r(c_i|W) = \begin{cases} (2,2,k+1), & i=1; \\ (i+1,i,k-i+2), & 2 \le i \le k; \\ (k+2,k+1,2), & i=k+1; \\ (2k-i+3,2k-i+4,i-k+1), & k+2 \le i \le 2k+1 \end{cases}$$

and

$$r(d_i|W) = \begin{cases} (3,3,k+1), & i=1; \\ (i+2,i+1,k-i+2), & 2 \le i \le k-1; \\ (k+2,k+1,3), & i=k; \\ (k+2,k+2,3), & i=k+1; \\ (2k-i+3,2k-i+4,i-k+2), & k+2 \le i \le 2k \\ (3,3,k+2), & i=2k+1. \end{cases}$$

Representations of the vertices on the outer cycle are

$$r(e_i|W) = \begin{cases} (4,4,k+2), & i=1; \\ (i+3,i+2,k-i+3), & 2 \le i \le k-1; \\ (k+2,k+3,4), & i=k; \\ (k+3,k+3,4), & i=k+1; \\ (2k-i+4,2k-i+5,i-k+3), & k+2 \le i \le 2k \\ (4,4,k+3), & i=2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(S_n) \le 3$ in this case.

On the other hand, suppose that $dim(S_n) = 2$, then there are the same subcases as in case (**i**) and a contradiction can be deduced analogously. This implies that $dim(S_n) = 3$ in this case, which completes the proof.

□

3.4 The graph T_n

The graph of the convex polytope $\mathbb{D}_n$ [2] can be obtained from the graph of the convex polytope Q_n [3] by deleting the edges b_ib_{i+1}, i.e., $V(\mathbb{D}_n) = V(Q_n)$ and $E(\mathbb{D}_n) = E(Q_n)\backslash\{b_ib_{i+1} : 1 \leq i \leq n\}$. It was proved in [20] that both graphs of the convex polytopes $\mathbb{D}_n$ and Q_n have same metric dimension.

In this section this study is extended by considering the graph T_n of a convex polytope, consisting of 3-sided, 5-sided and n-sided faces (Fig. 3.5). Note that the graph T_n can be obtained from the graph R_n [3] by deleting the edges b_ib_{i+1}, i.e. $V(T_n) = V(R_n)$ and $E(T_n) = E(R_n)\backslash\{b_ib_{i+1} : 1 \leq i \leq n\}$. It was shown in [20] that the graph R_n has constant metric dimension 3.

For our purpose the cycle induced by $\{a_i : 1 \leq i \leq n\}$ is called the inner cycle, the set of vertices $\{b_1, b_2, \cdots, b_n\}$ the set of central vertices and the cycle induced by $\{c_i : 1 \leq i \leq n\}$ the outer cycle. In the next theorem it is shown that the graph T_n has the same metric dimension as the graph R_n. Once again, the choice of an appropriate vertex basis is very important.

Theorem 3.4.1. *[21] We have* $dim(T_n) = 3$ *for every* $n \geq 6$.

Proof. This theorem will be also proved by double inequality. Two cases will be considered.

Case (i) n is even.

In this case we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_3, a_{k+1}\} \subset V(T_n)$, we show that W is a resolving set for T_n in this case.

For this we give representations of any vertex of T_n.

Representations of the vertices on the inner cycle are

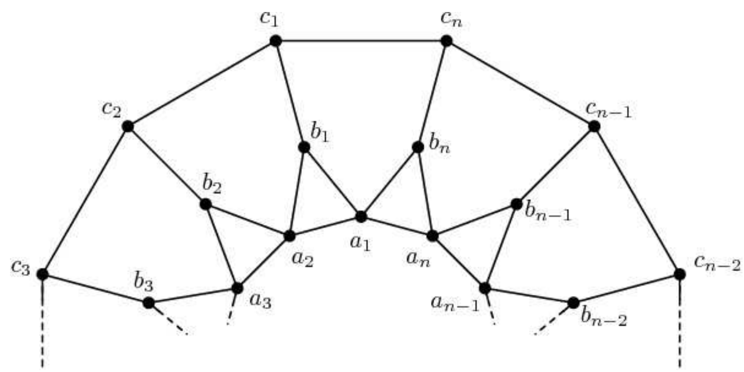

Figure 3.5: Graph T_n

$$r(a_i|W) = \begin{cases} (1,1,k-1), & i=2; \\ (i-1,i-3,k-i+1), & 4 \le i \le k; \\ (k-1,k-1,1), & i=k+2; \\ (k-2,k,2), & i=k+3; \\ (2k-i+1,2k-i+3,i-k-1), & k+4 \le i \le 2k. \end{cases}$$

Representations of vertices on the middle cycle are

$$r(b_i|W) = \begin{cases} (1,2,k), & i=1; \\ (2,1,k-1), & i=2; \\ (i,i-2,k-i+1), & 3 \le i \le k; \\ (k,k-1,1), & i=k+1; \\ (k-1,k,2), & i=k+2; \\ (2k-i+1,2k-i+3,i-k), & k+3 \le i \le 2k. \end{cases}$$

Representations of the vertices on the outer cycle are

$$r(c_i|W) = \begin{cases} (2,3,k+1), & i=1; \\ (3,2,k), & i=2; \\ (i+1,i-1,k-i+2), & 3 \leq i \leq k; \\ (k+1,k,2), & i=k+1; \\ (k,k+1,3), & i=k+2; \\ (2k-i+2,2k-i+4,i-k+1), & k+3 \leq i \leq 2k. \end{cases}$$

One can easily verify that there are no two vertices having the same representations implying that $dim(T_n) \leq 3$ in this case.

On the other hand, we show that $dim(T_n) \geq 3$. Suppose on the contrary that $dim(T_n) = 2$, then there are the following possibilities to be discussed.

(**1**) Both vertices are in the inner cycle. Without loss of generality we can suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is $a_t(2 \leq i \leq k+1)$. Thus $W = \{a_1, a_t\}$.

If $2 \leq t \leq k$, we have $r(a_n|\{a_1,a_t\}) = r(b_n|\{a_1,a_t\}) = (1,t)$ and if $t = k+1$ we have $r(a_2|\{a_1,a_{k+1}\}) = r(a_n|\{a_1,a_{k+1}\}) = (1,t-1)$, a contradiction.

(**2**) Both vertices belong to the set of central vertices. Without loss of generality we can suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is $b_t(2 \leq t \leq k+1)$. Thus $W = \{b_1, b_t\}$.

If $2 \leq t \leq k-1$ we have $r(a_n|\{b_1,b_t\}) = r(c_n|\{b_1,b_t\}) = (1,t+1)$, if $t = k$ we have $r(b_n|\{b_1,b_t\}) = r(c_n|\{b_1,b_t\}) = (2,t+1)$ and if $t = k+1$ we have $r(b_2|\{b_1,b_{k+1}\}) = r(b_n|\{b_1,b_{k+1}\}) = (1,t-1)$, a contradiction.

(**3**) Both vertices are in the outer cycle. Without loss of generality we can suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is $c_t(2 \leq t \leq k+1)$. Thus $W = \{c_1, c_t\}$.

If $2 \leq t \leq k$, we have $r(b_1|\{c_1,c_t\}) = r(c_n|\{c_1,c_t\}) = (1,t)$ and if $t = k+1$ we have $r(c_2|\{c_1,c_{k+1}\}) = r(c_n|\{c_1,c_{k+1}\}) = (1,t-2)$, a contradiction.

(**4**) One vertex is in the inner cycle and other belongs to the set of central vertices. Without loss of generality we can suppose that one resolving vertex is a_1. Suppose

that the second resolving vertex is $b_t (1 \leq t \leq k+1)$. Thus $W = \{a_1, b_t\}$.
If $1 \leq t \leq k-1$, $r(a_n|\{a_1, b_t\}) = r(b_n|\{a_1, b_t\}) = (1, t)$, if $t = k$ we have $r(b_1|\{a_1, b_k\}) = r(a_n|\{a_1, b_k\}) = (1, t)$ and for $t = k+1$, $r(a_2|\{a_1, b_{k+1}\}) = r(b_n|\{a_1, b_{k+1}\}) = (1, t-1)$, a contradiction.

(**5**) One vertex is in the inner cycle and other in the outer cycle. Without loss of generality we can suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is $c_t (1 \leq t \leq k+1)$. Thus $W = \{a_1, c_t\}$.
If $t = 1$, $r(a_2|\{a_1, c_1\}) = r(b_n|\{a_1, c_1\}) = (1, 2)$; if $2 \leq t \leq k+1$, $r(a_2|\{a_1, c_t\}) = r(b_1|\{a_1, c_t\}) = (1, t)$, a contradiction.

(**6**) One vertex is in the set of central vertices and other in the outer cycle. Without loss of generality we can suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is $c_t (1 \leq t \leq k+1)$. Thus $W = \{b_1, c_t\}$.
If $t = 1$, $r(a_1|\{b_1, c_1\}) = r(a_2|\{b_1, c_1\}) = (1, 2)$;
if $t = 2$, $r(a_3|\{b_1, c_2\}) = r(c_n|\{b_1, c_2\}) = (2, 3)$;
if $3 \leq t \leq k+1$, $r(a_3|\{b_1, c_t\}) = r(b_2|\{b_1, c_t\}) = (2, t-1)$, a contradiction.
Hence, from above it follows that there is no resolving set with two vertices for $V(T_n)$, thus implying that $dim(T_n) = 3$ in this case.

Case (ii) n is odd.
In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_3, a_{k+1}\} \subset V(T_n)$, we show that W is a resolving set for T_n in this case. For this we give representations of any vertex of T_n.
Representations of the vertices on the inner cycle are

$$r(a_i|W) = \begin{cases} (1, 1, k-1), & i = 2; \\ (i-1, i-3, k-i+1), & 4 \leq i \leq k; \\ (k, k-1, 1), & i = k+2; \\ (k-1, k, 2), & i = k+3; \\ (2k-i+2, 2k-i+4, i-k-1), & k+4 \leq i \leq 2k+1. \end{cases}$$

Representations of vertices on the middle cycle are

$$r(b_i|W) = \begin{cases} (1,2,k), & i = 1; \\ (2,1,k-1), & i = 2; \\ (i,i-2,k-i+1), & 3 \leq i \leq k; \\ (k+1,k-1,1), & i = k+1; \\ (k,k,2), & i = k+2; \\ (2k-i+2, 2k-i+4, i-k), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on the outer cycle are

$$r(c_i|W) = \begin{cases} (2,3,k+1), & i = 1; \\ (3,2,k), & i = 2; \\ (i+1,i-1,k-i+2), & 3 \leq i \leq k; \\ (k+2,k,2), & i = k+1; \\ (k+1,k+1,3), & i = k+2; \\ (2k-i+3, 2k-i+5, i-k+1), & k+3 \leq i \leq 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations, thus implying that $dim(T_n) \leq 3$ in this case.

On the other hand, suppose that $dim(T_n) = 2$, then there are the same subcases as in case (**i**) and a contradiction can be deduced analogously. This implies that $dim(T_n) = 3$ in this case, which completes the proof.

□

Chapter 4

On metric dimension of uniform subdivisions of the wheel

In this chapter metric dimension of $W_{n,k}$, a subdivision of the wheel W_n is discussed.

4.1 Introduction and a preliminary result

The *wheel* W_n consists of a cycle C_n and a new vertex v, called *central vertex* or the *hub*, adjacent with all vertices of C_n. The *gear graph* GR_n [15] is obtained from the wheel W_n by inserting a new vertex on each edge of C_n.

It was shown that $dim(W_n) = \lfloor (2n+2)/5 \rfloor$ for every $n \geq 7$ [7] and $dim(GR_n) = \lfloor 2n/3 \rfloor$ for every $n \geq 4$ [38].

Let $W_{n,k}$ $(k \geq 1)$ be the graph obtained from W_n by inserting $k-1$ new vertices on each edge of C_n, or equivalently, by subdividing each edge of C_n with $k-1$ vertices. Thus $W_{n,k}$ consists of a kn-cycle C_{kn} with a hub adjacent to every kth vertex on the cycle. $W_{n,k}$ has the order $nk+1$ and the size $n(k+1)$. In particular, $W_{n,1} = W_n$ and $W_{n,2} = GR_n$.

In the next sections we will determine $dim(W_{n,k})$ for every $k \geq 3$ and $n \geq 11$ for $k = 3$ and $n \geq 9$ for $k \geq 4$.

Suppose that the vertices of C_{kn} are denoted by $v_1, v_2, \ldots, v_{kn}$ and the central vertex

v is adjacent to v_{ki+1} for $0 \leq i \leq n-1$. These vertices will be partitioned into n *sectors* $S_0, \ldots, S_{n-1}$, where $S_i = \{v_{ki+1}, v_{ki+2}, \ldots, v_{k(i+1)}\}$ for $0 \leq i \leq n-1$. Sectors S_i and S_{i+1} for $0 \leq i \leq n-1$ are called *neighboring* (indices are taken modulo n). There are two kinds of vertices on C_{kn}: vertices of degree 2 and vertices of degree 3, referred to as *minor* and *major* vertices, respectively. If A is a subset of $V(C_{kn})$, $A = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ such that $r \geq 2$ and $i_1 < i_2 < \ldots < i_r$, we shall say that the pairs $\{v_{i_a}, v_{i_{a+1}}\}$ for $1 \leq a \leq r-1$ and $\{v_{i_r}, v_{i_1}\}$ are *pairs of neighboring vertices*. Given such an ordering, we will define as in [7], [38] the *gap* G_a for $1 \leq a \leq r-1$ as the path induced by the set $\{v_j | i_a < j < i_{a+1}\}$ and G_r the path induced by $\{v_j | 1 \leq j < i_1 \text{ or } i_r < j \leq kn\}$. Thus we have r gaps relatively to A, some of which may be empty. We will say that gaps G_a and G_b are *neighboring gaps* when $|a-b| = 1$ or $r-1$. A gap determined by neighboring vertices v_i and v_j will be called an $\alpha - \beta$ gap with $\alpha \leq \beta$ when $d(v_i) = \alpha$ and $d(v_j) = \beta$ or when $d(v_i) = \beta$ and $d(v_j) = \alpha$, hence we have 3 kinds of gaps: $2-2, 2-3$ and $3-3$.
The following lemma extends the corresponding property of the bases of $W_{n,2}$ for $n \geq 6$ [38].

Lemma 4.1.1. *[39] Let $n \geq 11$ for $k = 3$ and $n \geq 9$ for $k \geq 4$. Then the central vertex v does not belong to any basis of $W_{n,k}$.*

Proof. First we show that for $k = 3$, for any 3 consecutive sectors S_i, S_{i+1}, S_{i+2} $(0 \leq i \leq n-1)$ any resolving set R of $W_{n,3}$ contains at least one vertex in the set $S_i \cup S_{i+1} \cup S_{i+2} \backslash \{v_{ki+1}\}$. Indeed, the major vertices $v_{k(i+1)+1}$ and $v_{k(i+2)+1}$ have equal distances to all vertices not belonging to $S_i \cup S_{i+1} \cup S_{i+2} \backslash \{v_{ki+1}\}$. It follows that if $v_{k(i+1)+1}$ and $v_{k(i+2)+1}$ do not belong to R, then $R \cap ((S_i \cup S_{i+1} \cup S_{i+2}) \backslash \{v_{ki+1}\}) \neq \emptyset$. Similarly, for $k \geq 4$ and for any 2 consecutive sectors S_i and S_{i+1} $(0 \leq i \leq n-1)$, any resolving set R of $W_{n,k}$ contains at least one vertex in the set $S_i \cup S_{i+1} \backslash \{v_{ki+1}\}$ since the minor vertices $v_{k(i+1)}$ and $v_{k(i+1)+2}$, adjacent to the major vertex $v_{k(i+1)+1}$ have equal distances to all the vertices not in $S_i \cup S_{i+1} \backslash \{v_{ki+1}\}$.
Now, for $k = 3$ with $n \geq 11$ and for $k \geq 4$ with $n \geq 9$, assume that there exists a

basis B of $W_{n,k}$ that contains v. Since $\{v\}$ is not a basis, $B\backslash\{v\} \neq \emptyset$. Also $B\backslash\{v\}$ is not a basis, hence there exist distinct vertices u and u' such that $d(u,x) = d(u',x)$ for every $x \in B\backslash\{v\}$. Suppose that $u, u' \neq v$. Since B is a basis it follows that $d(u,v) \neq d(u',v)$. We shall consider 2 subcases: (1) u, u' belong to different sectors S_i and S_j; (2) u, u' belong to the same sector.

(1) Let $k \geq 4$; if $n \geq 9$ then there exist 2 consecutive sectors S_m and S_{m+1} such that S_m, S_{m+1} are different from S_{i-1}, S_i, S_{i+1} and S_{j-1}, S_j, S_{j+1} (some of them may coincide). There exists a vertex $w \in B$ in $S_m \cup S_{m+1}$. If $k = 3$ and $n \geq 11$ there exist 3 consecutive sectors S_m, S_{m+1}, S_{m+2} which are different from S_{i-1}, S_i, S_{i+1} and S_{j-1}, S_j, S_{j+1}. We have seen that we can find a vertex $w \in B$ in $S_m \cup S_{m+1} \cup S_{m+2}$. We get $d(u,w) = d(u,v) + d(v,w)$; $d(u',w) = d(u',v) + d(v,w)$. Since $d(u,v) \neq d(u',v)$ it follows that $d(u,w) \neq d(u',w)$, which contradicts the property that $d(u,x) = d(u',x)$ for every $x \in B\backslash\{v\}$.

(2) The proof is the same as in the previous case; for $k \geq 4$ and for $k = 3$ it is necessary that $n \geq 5$ and $n \geq 6$, respectively.

Consequently, we have $u = v$ or $u' = v$. So we can assume that $u' = v$. If $u \in S_i$, by the same argument we can find $w \in B$ such that $w \notin S_{i-1} \cup S_i \cup S_{i+1}$, therefore $d(u,w) = d(u,v) + d(v,w)$. Since $d(u,w) = d(v,w)$ it follows that $d(u,v) = 0$ or $u = v$, a contradiction.

□

4.2 Metric dimension of $W_{n,k}$

Case $k = 3$

Let $n \geq 11$ and B be a basis of $W_{n,3}$ which contains $r \geq 2$ vertices on C_{3n}, $B = \{v_{i_1}, \ldots, v_{i_r}\}$ such that $i_1 < \ldots < i_r$. Since the ends of a gap G_a relatively to B may be major or minor vertices, we shall denote by $\alpha - \beta(\gamma - \delta)$ a gap of kind $\alpha - \beta$ determined by v_i and v_j such that $d(v_i) = \alpha$ and $d(v_j) = \beta$ having ends of degrees γ and δ, respectively. The following 6 types of gaps are possible: $3 - 3(2 - 2), 2 -$

$2(3-3), 2-2(2-3), 2-2(2-2), 2-3(3-2), 2-3(2-2)$. In what follows we shall label major vertices by a star.

Lemma 4.2.1. *[39] If B is a basis of $W_{n,3}$ then every $3-3(2-2), 2-2(3-3), 2-2(2-3), 2-2(2-2), 2-3(3-2), 2-3(2-2)$ gap of B contains at most $5, 7, 8, 6, 6$ and 7 vertices, when they have the following patterns:*
$x_1x_2x_3^*x_4x_5, x_1^*x_2x_3x_4^*x_5x_6x_7^*, x_1x_2^*x_3x_4x_5^*x_6x_7x_8^*, x_1x_2^*x_3x_4x_5^*x_6, x_1^*x_2x_3x_4^*x_5x_6,$
$x_1x_2^*x_3x_4x_5^*x_6x_7$, respectively.

Proof. If a $3-3(2-2)$ gap has 8 vertices it has the pattern $x_1x_2x_3^*x_4x_5x_6^*x_7x_8$, which implies $r(x_3^*|B) = r(x_6^*|B)$, a contradiction. Similarly, if a $2-2(3-3)$ gap includes 10 vertices it has the pattern $x_1^*x_2x_3x_4^*x_5x_6x_7^*x_8x_9x_{10}^*$; we get $r(x_4^*|B) = r(x_7^*|B)$; if a $2-2(2-3)$ gap has 11 vertices $x_1x_2^*x_3x_4x_5^*x_6x_7x_8^*x_9x_{10}x_{11}^*$ then $r(x_5^*|B) = r(x_8^*|B)$; if a $2-2(2-2)$ gap has 9 vertices $x_1x_2^*x_3x_4x_5^*x_6x_7x_8^*x_9$ we deduce $r(x_4|B) = r(x_6|B)$; for a $2-3(3-2)$ gap with 9 vertices $x_1^*x_2x_3x_4^*x_5x_6x_7^*x_8x_9$ we have $r(x_4^*|B) = r(x_7^*|B)$; for a $2-3(2-2)$ gap with 10 vertices $x_1x_2^*x_3x_4x_5^*x_6x_7x_8^*x_9x_{10}$ we obtain $r(x_5^*|B) = r(x_8^*|B)$.

□

The gaps described in Lemma 4.2.1 having maximum length will be referred to as *major gaps*; the remaining ones are called *minor gaps*.

Lemma 4.2.2. *[39] If B is a basis of $W_{n,3}$, then it contains at most one major gap different from $2-2(2-2)$.*

Proof. For 2 gaps $3-3(2-2)$, $x_1x_2x_3^*x_4x_5$ and $y_1y_2y_3^*y_4y_5$ we get $r(x_3^*|B) = r(y_3^*|B)$. This is true for every pair of major gaps different from $2-2(2-2)$, since every major gap G_a different from $2-2(2-2)$ contains a major vertex x_i^* such that the minimum distance between x_i^* and the ends of G_a is at least equal to 2. This implies that every shortest path between x_i^* and the vertices not belonging to G_a passes through the center v. By considering 2 such vertices x_i^* and y_i^* in 2 major gaps we deduce that $d(x_i^*, z) = 1 + d(v, z) = d(y_i^*, z)$ for every $z \in B$, hence $r(x_i^*|B) = r(y_i^*|B)$, a contradiction.

Theorem 4.2.3. *[39] For every $n \geq 11$ we have $dim(W_{n,3}) = \lfloor n/2 \rfloor$.*

Proof. We will prove this equality by double inequality. We deduce that the patterns of the minor gaps of maximum length are obtained from those enumerated by Lemma 4.2.1 by deleting the last 3 letters, i. e., they are $x_1x_2, x_1^*x_2x_3x_4^*, x_1x_2^*x_3x_4x_5^*, x_1x_2^*x_3,$ $x_1^*x_2x_3, x_1x_2^*x_3x_4$ having length at most 5. Two adjacent sectors S_i and S_{i+1} have pattern $y_1^*y_2y_3y_4^*y_5y_6$ of length 6. It follows that the vertices of $S_i \cup S_{i+1}$ cannot be included in a major gap of type $2-2(2-2)$ having pattern $x_1x_2^*x_3x_4x_5^*x_6$ nor in any minor gap of any type. Since by Lemma 4.2.2 any basis B of $W_{n,3}$ induces at most one major gap different from $2-2(2-2)$, it follows that with at most one exception, say S_0 and S_1, any other pair of adjacent sectors contains at least a vertex from B. If $S_0 \cup S_1$ are covered by a major gap of length 6, 7 or 8, then S_{n-1} and S_2 contain each at least a vertex from B. It follows that if we denote by m_i $(0 \leq i \leq n-1)$ the number of vertices of B contained by $S_i \cup S_{i+1}$, we can write $m_i \geq 1$ for every $i = 1, \ldots, n-1$, which implies

$$2|B| = m_0 + m_1 + \ldots + m_{n-1} \geq m_1 + \ldots + m_{n-1} \geq n - 1.$$

We have obtained that $|B| \geq \lfloor n/2 \rfloor$, therefore $dim(W_{n,3}) \geq \lfloor n/2 \rfloor$.

We shall define a resolving set R_n containing exactly $\lfloor n/2 \rfloor$ vertices in both cases n even and n odd.

For n even let $R_n = \{v_{6k+2} : k = 0, \ldots, n/2 - 1\}$. In this case all gaps are minor gaps of type $2-2(2-3)$ and they have 5 vertices and all vertices of R_n are minor.

For n odd let $R_n = \{v_{6k+2} : k = 0, \ldots, (n-1)/2 - 1\}$. In this case one gap is major and has 8 vertices and other $\lfloor n/2 \rfloor - 1$ gaps are minor with 5 vertices, all having type $2-2(2-3)$.

The set R_{12} is illustrated in Fig. 4.1 for $W_{12,3}$. The set R_n is a resolving set for $W_{n,3}$ because any 2 vertices lying in different gaps are resolved by at least one

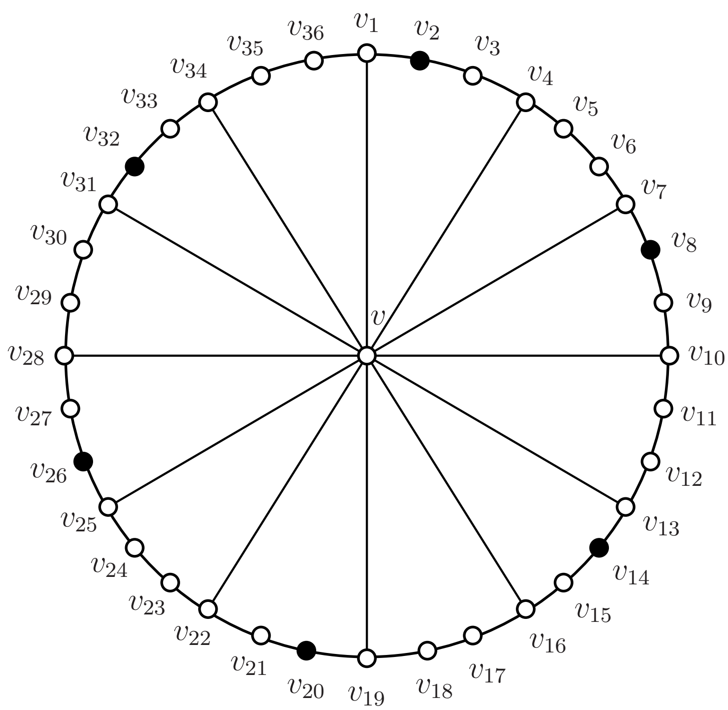

Figure 4.1: $W_{12,3}$

vertex in the set of 3 or 4 vertices of R_n determining these 2 gaps (neighboring or not). This property also holds for vertices lying in the same gap. Note that $r(v|R_n) = (2, 2, \ldots, 2)$ and $r(x|R_n) \neq r(v|R_n)$ for every vertex x of $W_{n,3}$, $x \neq v$. Consequently, $dim(W_{n,3}) \leq \lfloor n/2 \rfloor$, which concludes the proof.

□

Case k odd, $k \geq 5$

In this case $dim(W_{n,k})$ depends only on n.

Theorem 4.2.4. *[39] If $n \geq 9$, $k \geq 5$ and k is odd then $dim(W_{n,k}) = \lceil n/2 \rceil$.*

Proof. Let R be a resolving set of $W_{n,k}$. In the proof of Lemma 4.1.1 we have seen that for $k \geq 4$, for any 2 consecutive sectors S_i and S_{i+1} we have $R \cap (S_i \cup S_{i+1} \backslash \{v_{ki+1}\}) \neq \emptyset$. By denoting $m_i = |R \cap (S_i \cup S_{i+1})|$, $m_i \geq 1$ holds for every $0 \leq i \leq n-1$, which implies $2|R| = \sum_{i=0}^{n-1} m_i \geq n$, therefore $|R| \geq \lceil n/2 \rceil$. Consequently, we have $dim(W_{n,k}) \geq \lceil n/2 \rceil$.

We shall prove the reverse inequality by defining a resolving set R_n in both even and odd cases of n, containing exactly $\lceil n/2 \rceil$ vertices on C_{kn}.

Let n be even. In this case we shall choose vertices belonging to R_n in the sectors $S_0, S_2, \ldots, S_{n-2}$: $R_n = \{v_{2ki+(k+1)/2} : i = 0, 1, \ldots, n/2-1\}$. If n is odd, vertices in R_n belong to the sectors $S_0, S_2, \ldots, S_{n-1}$: $R_n = \{v_{2ki+(k+1)/2} : i = 0, 1, \ldots, (n-1)/2\}$. The set R_9 for $W_{9,5}$ is represented in Fig. 4.2.

If n is even then all the gaps generated by R_n are major gaps including each $2k-1$ vertices; if n is odd, there exists only one minor gap with $k-1$ vertices, other gaps being major with $2k-1$ vertices. Note that when n is even every 2 adjacent sectors include exactly one vertex in R_n; for odd n only S_{n-1} and S_0 have each a vertex in R_n, remaining pairs of sectors verify the same property as for even n.

It is necessary to show that for any 2 distinct vertices v_j and v_h from $V(W_{n,k}) \backslash R_n$ there exists $z \in R_n$ such that $d(z, v_j) \neq d(z, v_h)$.

If $v_j = v$, the central vertex, since $n \geq 9$, as in the proof of Lemma 4.1.1 we can find a vertex $z \in R_n$ such that the shortest path from v_h to z passes through v. We can write $d(v_h, z) = d(v_h, v) + d(v, z) > d(v, z)$ and we are done. If v_j and v_h belong to the same gap G_a (which is major or minor when n is odd), then z can be chosen one of the neighboring vertices w_1 and w_2 of R_n determining G_a. Indeed, if $d(v_j, w_1) = d(v_h, w_1)$ then $d(v_j, w_2) \neq d(v_h, w_2)$.

Suppose now that v_h and v_j belong to different gaps G_a and G_b, respectively. If G_a is a minor gap then n is odd, G_a is determined by $w_1 = v_{kn-(k-1)/2}$ and $w_2 = v_{(k+1)/2}$, $w_1, w_2 \in R_n$ and G_b is a major gap. Without loss of generality we can suppose that $d(v_h, w_1) < d(v_h, w_2)$, which implies that $d(v_h, w_1) \leq (k-1)/2$. If $d(v_j, w_1) \neq d(v_h, w_1)$ we can take $z = w_1$. Otherwise we have $d(v_j, w_1) = d(v_h, w_1) \leq (k-1)/2$. It follows that $v_j \in S_{n-1}$, like v_h and $d(v_j, v) = d(v_h, v) - 1$. We can find as above a vertex $z \in R_n$ such that $d(v_h, z) = d(v_h, v) + d(v, z) \neq d(v_h, v) - 1 + d(v, z) = d(v_j, z)$ and we are done.

The last case is when both G_a and G_b are distinct major gaps containing $2k-1$ vertices each.

Suppose that G_a is determined by the vertices $w_1 = v_{2ki+(k+1)/2}$ and $w_2 = v_{2k(i+1)+(k+1)/2}$

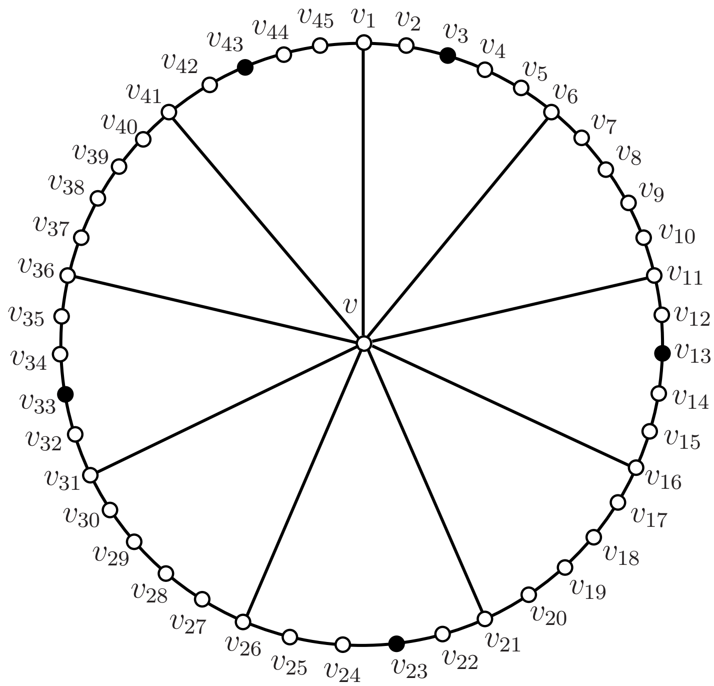

Figure 4.2: $W_{9,5}$

and denote by $\alpha(x)$ the minimum distance between a vertex $x \in C_{kn}$ and a major vertex of C_{kn}. It follows that $0 \leq \alpha(x) \leq (k-1)/2$. Suppose that $d(v_h, w_1) \leq d(v_h, w_2)$, which implies that $d(v_h, w_1) \leq k$ and $v_h \in \{v_{2ki+1+(k+1)/2}, v_{2ki+2+(k+1)/2}, \ldots, v_{2ki+k+(k+1)/2}\}$. Let G_c be the major gap determined by neighboring vertices $w_3 = v_{2k(i-1)+(k+1)/2}$ and w_1. If $v_j \in G_b$ and $d(v_h, w_1) \neq d(v_j, w_1)$ then we are done. Otherwise, it is not difficult to see that if $G_b \neq G_c$ and $d(v_h, w_1) = d(v_j, w_1)$ then $d(v_j, w_1) \geq (k+3)/2$ and $\alpha(v_j) = \alpha(v_h) - 1$. This implies that $d(v_j, v) = d(v_h, v) - 1$ and we have seen that there exists $z \in R_n$ such that $d(v_j, z) \neq d(v_h, z)$.

If $G_b = G_c$ and $d(v_h, w_1) = d(v_j, w_1)$ then $\alpha(v_j) = \alpha(v_h) + 1$ and the same conclusion holds, unless v_j and v_h verify $d(v_j, w_3) = d(v_j, w_1)$ and $d(v_h, w_1) = d(v_h, w_2)$. But in this case we can choose $z = w_2$ since $d(v_j, w_2) = k+1 \neq d(v_h, w_2) = k$. Therefore the reverse inequality $dim(W_{n,k}) \leq \lceil n/2 \rceil$ has been proved. $\square$

Case k even, $k \geq 4$

A similar situation occurs in this case: $dim(W_{n,k})$ depends only on n.

Theorem 4.2.5. *[39] Let $n \geq 9$ and k is even with $k \geq 4$. Then $dim(W_{n,k}) = \lceil 2n/3 \rceil$.*

Proof. Let R be a resolving set of $W_{n,k}$. As for the case k odd, any 2 consecutive sectors S_i and S_{i+1} contain at least one vertex in R. When k is even we shall prove that any 3 consecutive sectors contain at least 2 vertices in R. Suppose that this property is not true, hence there exist 3 sectors S_i, S_{i+1} and S_{i+2} $(0 \leq i \leq n-1)$ containing only one vertex $w \in R$. Since any 2 consecutive sectors contain at least one vertex in R, it follows that $w \in S_{i+1} = \{v_{k(i+1)+1}, v_{k(i+1)+2}, \ldots, v_{k(i+2)}\}$. If $w = v_{k(i+1)+1}$ then $r(v_{k(i+1)}|R) = r(v_{k(i+1)+2}|R)$, which contradicts the definition of a resolving set. Also, if $w = v_{k(i+1)+k/2+j}$ $(1 \leq j \leq k/2)$ then $r(v_{k(i+1)+j}|R) = r(v_{k(i+2)+j}|R)$; a similar situation holds if $w = v_{k(i+1)+k/2-j}$ for $0 \leq j \leq k/2-2$. It follows that $|R \cap (S_i \cup S_{i+1} \cup S_{i+2})| \geq 2$ for every $0 \leq i \leq n-1$. By letting $m_i = |R \cap (S_i \cup S_{i+1} \cup S_{i+2})|$, we deduce $m_i \geq 2$ for every $0 \leq i \leq n-1$, which implies as above, $3|R| = \sum_{i=0}^{n-1} m_i \geq 2n$. Thus $|R| \geq \lceil 2n/3 \rceil$, so $dim(W_{n,k}) \geq \lceil 2n/3 \rceil$.
In order to prove that $dim(W_{n,k}) \leq \lceil 2n/3 \rceil$ we shall define a resolving set R_n containing exactly $\lceil 2n/3 \rceil$ vertices of C_{kn} : $R_n = \{v_{ki+k/2+1} : 0 \leq i \leq n-1, i \equiv 0 \pmod 3$ or $i \equiv 1 \pmod 3\}$. Consequently, R_n consists of the median vertices of the sectors: $S_0, S_1, S_3, S_4, \ldots, S_{n-3}, S_{n-2}$ for $n \equiv 0 \pmod 3$; $S_0, S_1, S_3, S_4, \ldots, S_{n-4}, S_{n-3}, S_{n-1}$ for $n \equiv 1 \pmod 3$; $S_0, S_1, S_3, S_4, \ldots, S_{n-2}, S_{n-1}$ for $n \equiv 2 \pmod 3$.
The set R_9 for $W_{9,4}$ is shown in Fig. 4.3.

Note that minor gaps contain $k-1$ vertices each and major gaps $2k-1$ vertices and there are no two major neighboring gaps . Also, every 3 consecutive sectors contain at least 2 vertices in R_n; when $n \equiv 1$ or $2 \pmod 3$ there exists one triplet of consecutive sectors having 3 vertices in R_n.
We will prove that given 2 distinct vertices $v_j, v_h \in V(W_{n,k}) \backslash R_n$ there is $z \in R_n$ such that $d(z, v_j) \neq d(z, v_h)$, i. e. , R_n is a resolving set.

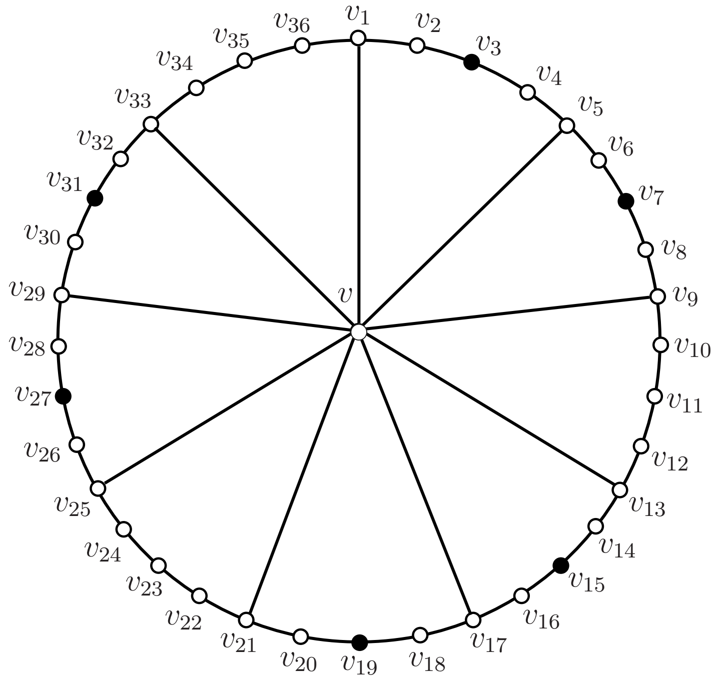

Figure 4.3: $W_{9,4}$

If $v_j = v$, by the same argument as for the case when k was odd there exists such a resolving vertex z. When v_j and v_h belong to the same gap G_a (major or minor), as for odd k, z can be chosen one of the neighboring vertices w_1 and w_2 of R_n determining G_a.

Let the case when v_h and v_j belong to different gaps G_a and G_b, respectively. We shall distinguish the following 5 subcases: G_a and G_b are: 1a) neighboring and minor; 1b) not neighboring and minor; 2a) neighboring, G_a is major, G_b is minor; 2b) not neighboring, G_a is major, G_b is minor; 3) not neighboring and major.

1a) Let G_a and G_b be determined by w_1, w_2 and w_2, w_3, respectively, where distinct vertices $w_1, w_2, w_3 \in R_n$. If $d(v_h, w_1) \neq d(v_j, w_1)$ then we take $z = w_1$. Otherwise $d(v_h, w_1) = d(v_j, w_1)$, but in this case $d(v_h, w_3) \neq d(v_j, w_3)$ and we can choose $z = w_3$.

1b) If G_a and G_b are determined by w_1, w_2 and w_3, w_4, where distinct vertices $w_1, w_2, w_3, w_4 \in R_n$, suppose that $d(v_h, w_1) \leq d(v_h, w_2)$. It follows that $d(v_h, w_1) \leq k/2$ and $d(v_j, w_1) \geq k/2 + 2$, therefore $d(v_h, w_1) \neq d(v_j, w_1)$ and we can choose

$z = w_1$.

2a) Suppose that G_a and G_b are determined by w_1, w_2 and w_2, w_3, respectively. If $d(v_j, w_3) \leq d(v_j, w_2)$ then $d(v_j, w_3) \leq k/2$, but $d(v_h, w_3) \geq k/2 + 2$ and we can choose $z = w_3$. If $d(v_j, w_2) \leq d(v_j, w_3)$ and $d(v_h, w_2) \neq d(v_j, w_2)$ then we choose $z = w_2$. If $d(v_h, w_2) = d(v_j, w_2) \leq k/2$ then we have $d(v_j, w_3) \neq d(v_h, w_3)$ and we put $z = w_3$.

2b) This case is similar to 1b).

3) If G_a and G_b are determined by w_1, w_2 and w_3, w_4 respectively, where distinct vertices $w_1, w_2, w_3, w_4 \in R_n$, then it is possible to show that at least one vertex from $w_1, \ldots, w_4$ resolves v_h and v_j. An alternative proof is the following: Without loss of generality we can suppose that $d(v_h, w_1) \leq d(v_h, w_2)$, thus implying that $d(v_h, w_1) \leq k$. If $d(v_h, w_1) \neq d(v_j, w_1)$ we take $z = w_1$. Otherwise $d(v_h, w_1) = d(v_j, w_1)$. Since $\min_{x \in G_b} d(x, w_1) = k/2 + 2$, it also follows that $k/2 + 2 \leq d(v_h, w_1) \leq k$ and the vertices v_h verifying these inequalities can belong to a single sector. But in this case we have $\alpha(v_h) = \alpha(v_j) + 2$, thus implying that $d(v_h, v) = d(v_j, v) + 2$. By choosing again a vertex $z \in R_n$ such that $d(v_h, z) = d(v_h, v) + d(v, z)$ and $d(v_j, z) = d(v_j, v) + d(v, z)$ we have $d(v_h, z) \neq d(v_j, z)$, which concludes the proof. $\square$

Chapter 5

On a conjecture concerning resolving pairs

In [36] the following conjecture has been proposed :

Conjecture RP. For every connected graph G of order $n \geq 2$ the number of resolving pairs is bounded above by $\lfloor n^2/4 \rfloor$. This conjecture is valid for graphs of diameter two, paths and cycles. Other classes of graphs verifying conjecture RP will be studied in the next sections.

5.1 Bipartite graphs and graphs of diameter $n-2$

For bipartite graphs resolving pairs are easy to characterize.

Theorem 5.1.1. *[40] Let G be a connected bipartite graph of order n having partite sets A and B. Resolving pairs of G are precisely the pairs $\{x, y\}$ where $x \in A$ and $y \in B$. The minimum number of resolving pairs is $n-1$ and this holds if and only if G is $K_{1,n-1}$ and the maximum number equals $\lfloor n^2/4 \rfloor$ and this bound is reached if and only if*

$$-1 \leq |A| - |B| \leq 1$$

Proof. If x and y belong to the same partite set then $d(x, y) \equiv 0\,(mod\,2)$ and by Lemma 2.2.1 $\{x, y\}$ is not a resolving pair.

Otherwise, let $x \in A$ and $y \in B$. If $z \in A$ then $d(x, z) \equiv 0\,(mod\,2)$ and $d(y, z) \equiv 1\,(mod\,2)$; if $z \in B$ then $d(x, z) \equiv 1\,(mod\,2)$ and $d(y, z) \equiv 0\,(mod\,2)$.

It follows that $d(x, z) \neq d(y, z)$ for any $z \in V(G)$.

In a bipartite graph $\{x, y\}$ is a resolving pair if and only if $d(x, y)$ is odd. Consequently, the number of resolving pairs of G is equal to $|A||B|$. Since $|A| + |B| = n$ the minimum value of this product is equal to $n-1$, when $\{|A|, |B|\} = \{1, n-1\}$ and the extremal graph is $K_{1,n-1}$, and the maximum value is $\lfloor n^2/4 \rfloor$, which is attained if and only if $-1 \leq |A| - |B| \leq 1$.

□

Let $x \in V(G)$. Denote the number of vertices $y \in V(G)$ such that $dist(y, x) = i$ by $v_i(x)$. Assume that $x \in A$. Then $|A| = \sum_{i \equiv 0 (mod\,2)} v_i(x)$ and $|B| = \sum_{j \equiv 1 (mod\,2)} v_j(x)$.

It follows that the condition $-1 \leq |A| - |B| \leq 1$ is equivalent to

$$-1 \leq \sum_{i \equiv 0 (mod\,2)} v_i(x) - \sum_{j \equiv 1 (mod\,2)} v_j(x) \leq 1 \tag{5.1}$$

Note that the number of resolving pairs of a bipartite graph G is also equal to $\sum_{i \equiv 0 (mod\,2)} v_i(x) \sum_{j \equiv 1 (mod\,2)} v_j(x) = |A||B|$ and this product does not depend on the choice of the vertex x in $V(G)$.

If $G = P_n$ or G is an even cycle C_n with $n \equiv 0(mod\,2)$ then condition (5.1) is obviously satisfied and these graphs have a maximum number of resolving pairs, equal to $\lfloor n^2/4 \rfloor$.

Another example of an extremal bipartite graph is the n-dimensional hypercube Q_n which has 2^n vertices representing binary n-tuples $(x_1, \ldots, x_n)$ and where two vertices are adjacent if they differ in exactly one coordinate.

Q_n has partite sets $A = \{(x_1, \ldots, x_n) : \sum_{i=1}^{n} x_i \equiv 0(mod\,2)\}$ and $B = \{(y_1, \ldots, y_n) : \sum_{i=1}^{n} y_i \equiv 1(mod\,2)\}$; one has $|A| = |B| = 2^{n-1}$.

Theorem 5.1.2. *[40] All graphs G of order $n \geq 5$ and diameter $n-2$ have a number of resolving pairs less than or equal to $\lfloor n^2/4 \rfloor$. Equality holds if and only if: n is odd and G consists of P_{n-1} and a new vertex x adjacent to one interior vertex a of P_{n-1} or to two vertices a, b of P_{n-1} such that $d(a, b) = 2$ or n is even and in both cases the distance between a and an endvertex of P_{n-1} is also even.*

Proof. Let G be a graph of order n and diameter $n-2$. G consists of a path P_{n-1} with endvertices u and v and another vertex x such that:

$a)$ $d(x) = 1$, when x is adjacent to a vertex a of P_{n-1}, different from u and v;

$b)$ $d(x) = 2$ and x is adjacent to two vertices a, b of P_{n-1} such that $d(a, b) \in \{1, 2\}$;

$c)$ $d(x) = 3$ and x is adjacent to three consecutive vertices a, b, c of P_{n-1}.

We shall consider these cases separately.

$a)$ In this case G is a tree, hence a bipartite graph.

If $v_i = v_i(u)$ denotes the number of vertices y of G with $d(u, y) = i$, then there exists a unique index k, $2 \leq k \leq n-2$ such that $v_k = 2$ and $v_i = 1$ for every $i \neq k$.

If n is odd, then (5.1) becomes

$$-1 \leq v_1 + v_3 + \ldots + v_{n-2} - (1 + v_2 + v_4 + \ldots + v_{n-3}) \leq 1 \qquad (5.2)$$

Both sums have $(n-1)/2$ terms and (5.2) is satisfied for any k.

It follows that all graphs consisting of P_{n-1} and another pendant vertex adjacent to any vertex $a \neq u, v$ of P_{n-1} are extremal graphs.

If n is even, (5.1) can be written as

$$-1 \leq v_1 + v_3 + \ldots + v_{n-3} - (1 + v_2 + v_4 + \ldots + v_{n-2}) \leq 1 \qquad (5.3)$$

The sum with odd indices contains $n/2 - 1$ terms and another sum $n/2$ terms. It follows that (5.3) is satisfied if and only if the index k such that $v_k = 2$ is odd. This means that $d(u, a)$ is even.

$b)$ In this case if $d(a, b) = 1$ consider $G_1 = G - xb$.

The resolving pairs in G consisting of vertices belonging to P_{n-1} remain resolving for

G_1, but in G_1 may appear new resolving pairs of vertices on P_{n-1}. Also the number of resolving pairs $\{x, t\}$ in G where $t \in P_{n-1}$ remains unchanged or increases by one and $\{x, a\}$ and $\{a, b\}$ become resolving in G_1, which is in case a).

This implies that the number of resolving pairs of G is strictly less than $\lfloor n^2/4 \rfloor$.

If $d(a, b) = 2$ then G is bipartite and numbers v_i are the same as in case a). It follows that G is extremal if n is odd or n is even and $d(a, u)$ is even (which implies also that $d(b, u)$ is even).

c) In this case $G_1 = G - xb$ has four new resolving pairs relatively to G, namely $\{x, a\}, \{x, c\}, \{a, b\}, \{b, c\}$ thus implying that G is not extremal since G_1 is in case b).

□

Proposition 5.1.3. *[40] For every integers n, k such that $n \geq 3$ and $2 \leq k \leq n-1$ there exists a connected graph G of order n and diam$(G) = k$ containing $\lfloor n^2/4 \rfloor$ resolving pairs.*

Proof. For $k = 2, n-2, n-1$ we have seen that the statement is true. Let $n \geq 6$ and k be such that $3 \leq k \leq n-3$ and consider a path $P_{k+1} : u, a, b, \ldots, v$ of diameter k. We shall add $v_2 - 1$ pendant vertices adjacent to a and $v_3 - 1$ pendant vertices adjacent to b, by obtaining a caterpillar G of diameter k. Since G must have n vertices we get $v_2 + v_3 = n - k + 1$.

If k is odd (5.1) is equivalent to $-1 \leq v_3 - v_2 \leq 1$, which is satisfied for example by choosing $v_2 = \lceil (n-k+1)/2 \rceil$ and $v_3 = \lfloor (n-k+1)/2 \rfloor$. If k is even (5.1) yields $-1 \leq v_3 - v_2 - 1 \leq 1$ and we can choose $v_2 = \lceil (n-k)/2 \rceil$ and $v_3 = \lfloor (n-k/2) \rfloor + 1$.

□

5.2 Graphs of diameter three

If G is a graph of diameter equal to three, every resolving pair $\{x, y\}$ of G must have $d(x, y) \in \{1, 3\}$.
There exist graphs of diameter three without resolving pairs (e.g. the odd cycle C_7) or without resolving pairs at distance one or at distance three, respectively (see graphs G_1 and G_2 from $Fig.\, 5.1$).

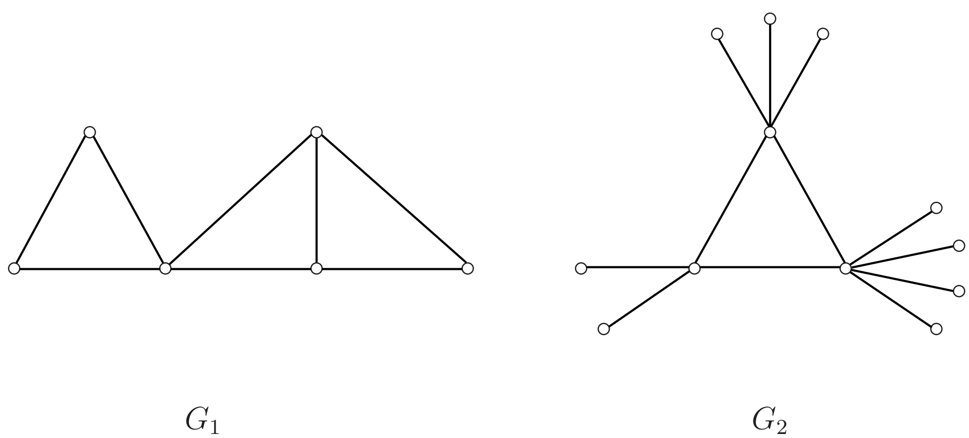

Figure 5.1: Two graphs of diameter 3

Theorem 5.2.1. *[40] Let G be a connected graph of order n and diameter three containing a resolving pair $\{x, y\}$ such that:*

i) $d(x, y) = 1$ and $N(x) \cup N(y) = V(G)$ or
ii) $d(x, y) = 3$ and there exists a shortest path x, u, v, y such that $N(x) \cup N(y) \cup N(u) \cup N(v) = V(G)$.
Then the number of resolving pairs of G is bounded above by $\lfloor n^2/4 \rfloor$ and this bound is tight.

Proof. In case $i)$ by denoting $A = N(x)$ and $B = N(y)$ we deduce that $A \cap B = \emptyset$ since $\{x, y\}$ is a resolving pair and $A \cup B = V(G)$.
Any pair $\{w, z\}$ of vertices from A or from B is not resolving since w and z have

a common neighbor. It follows that the number of resolving pairs of G is bounded above by $|A||B| \leq \lfloor n^2/4 \rfloor$. It can be easily seen that this bound can be reached if and only if A and B are independent sets of vertices, i.e., G is bipartite, and $-1 \leq |A| - |B| \leq 1$, or $-1 \leq |N(x)| - |N(y)| \leq 1$. Since G has diameter three there exists at least a pair $\{a, b\}, a \in A$ and $b \in B$ such that $ab \notin E(G)$.

Note that this class of extremal graphs contains 4-cycles book graph $B_{4,n}$ [29] consisting of $n \geq 2$ copies of the cycle C_4 with a common edge; the copies of the cycle C_4 are called the pages of $B_{4,n}$.

ii) In this case we also have $N(x) \cap N(y) = \emptyset$ and $N(u) \cap N(v) = \emptyset$; denote $A = N(x) - \{u\}$, $B = N(y) - \{v\}$, $C = N(u) - (N(x) \cup N(y) \cup \{x, v\})$, $D = N(v) - (N(x) \cup N(y) \cup \{u, y\})$ (see Fig. 5.2).

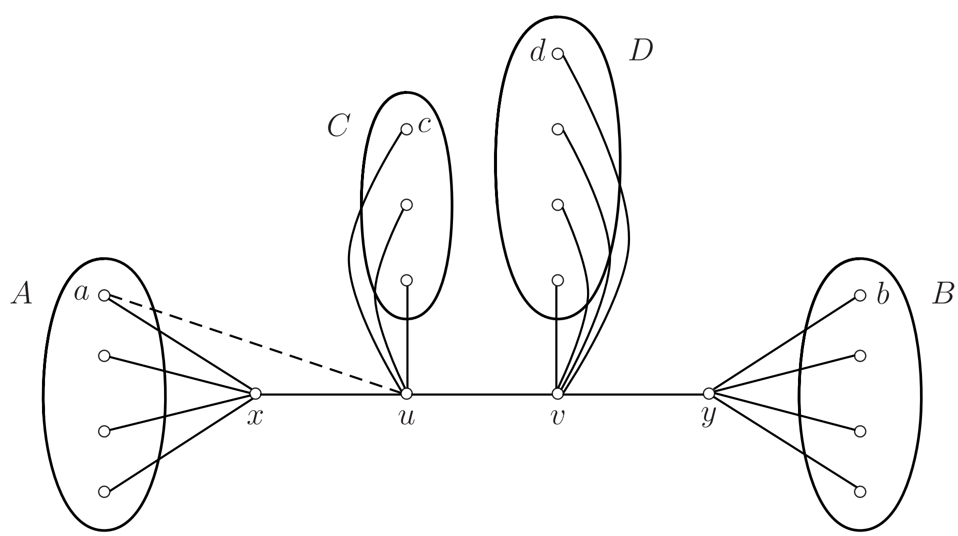

Figure 5.2: A graph of diameter 3 with a shortest path of length 3.

The pairs of distinct vertices from A cannot be resolving since x has equal distances to them; a similar situation occurs for pairs in B, C, D.

Suppose first that G has the following property: for any $a \in A$ we have $au \notin E(G)$ or $av \in E(G)$ and for any $b \in B$, $bv \notin E(G)$ or $bu \in E(G)$ holds.

In this case any pair $\{a, d\}$ with $a \in A$ and $d \in D$ is not resolving since $d(u,a) = d(u,d) = 2$ or $d(v,a) = d(v,d) = 1$; similarly any pair $\{b, c\}$ with $b \in B$ and $c \in C$ is not a resolving pair.

Also any pair $\{a, y\}$ with $a \in A$ is not resolving since $d(a,u) = d(y,u) = 2$ or $d(a,v) = d(y,v) = 1$ and in a similar way any pair $\{x, b\}$ is not resolving.

Another pairs which are not resolving are $\{x, c\}$ and $\{y, d\}$, where $c \in C$ and $d \in D$.

It follows that at most the following pairs of vertices can be resolving: $\{a, b\}$, $a \in A$; $b \in B$; $\{c, d\}$, $c \in C$; $d \in D$; $\{u, c\}$, $c \in C$; $\{v, d\}$, $d \in D$; $\{a, c\}$, $a \in A$; $c \in C$; $\{b, d\}$, $b \in B$; $d \in D$; $\{u, b\}$, $b \in B$; $\{v, a\}$, $a \in A$; $\{y, b\}$, $b \in B$; $\{x, a\}$, $a \in A$; $\{x, y\}$, $\{x, u\}$, $\{u, v\}$, $\{v, y\}$. By denoting $|A| = \alpha$, $|B| = \beta$, $|C| = p$ and $|D| = q$, the number of these possible resolving pairs is equal to

$$E = \alpha\beta + p\,q + \alpha p + \beta q + 2(\alpha + \beta + p + q) + 4 = \alpha\beta + p\,q + \alpha p + \beta q + 2n - 4$$

since $\alpha + \beta + p + q = n - 4$.

Substitution $\alpha = n - 4 - \beta - p - q$ yields

$$E = (p + \beta)(n - 4 - p - \beta) + 2n - 4 \le \lfloor (n-4)^2/4 \rfloor + 2n - 4 = \lfloor n^2/4 \rfloor.$$

Suppose now that there exist subsets of vertices $A_1 \subseteq A$ and $B_1 \subseteq B$, $|A_1| = s$, $|B_1| = t$, $0 \le s \le \alpha$, $0 \le t \le \beta$, $s + t \ge 1$ such that every vertex $a \in A_1$ and every vertex $b \in B_1$ verifies $au \in E(G)$ and $av \notin E(G)$ and $bv \in E(G)$ and $bu \notin E(G)$, respectively. We will prove that the number of resolving pairs in this case is strictly less than $\lfloor n^2/4 \rfloor$.

It follows that the following modifications have been produced relatively to the case when $s = t = 0$:

All pairs $\{a, c\}$ with $a \in A_1$ and $c \in C$, $\{a, b\}$ with $a \in A_1$ and $b \in B - B_1$, $\{a, v\}$ with $a \in A_1$, $\{a, x\}$ with $a \in A_1$ and the pair $\{x, u\}$ if $s > 0$ are not resolving (they are counted as resolving ones in expression E).

All pairs $\{a, d\}$ with $a \in A_1$ and $d \in D$ and $\{a, y\}$ with $a \in A_1$ may become resolving.

A similar situation holds for the pairs containing vertices $b \in B_1$. We get that the number of resolving pairs is at most equal to

$$E_1 = E + s\,q + t\,p - s(p+\beta-t) - t(q+\alpha-s) - s - t - 1 =$$
$$(p+\beta)(n-4-p-\beta) + 2\,n - 5 + s\,q - s(p+\beta) + t\,p - t(q+\alpha) + 2\,s\,t - s - t.$$

By denoting $p+\beta = k$ we deduce $q = n-4-k-\alpha \leq n-4-k-s$; $p = k-\beta \leq k-t$, which implies

$$E_1 \leq k(n-4-k)+2\,n-5+s(n-4-k-s)-k\,s+k\,t-t^2-t(n-4-k)+2\,s\,t-s-t =$$
$$(k+s)(n-4-k-s) + 2\,n - 5 + \varphi(t) - s,$$

where $\varphi(t) = -t^2 - t(n-4-2\,k-2\,s+1)$.
Suppose that n is even and denote $\gamma = (n-4)/2-(k+s)$. We get $(k+s)(n-4-k-s) = (n-4)^2/4 - \gamma^2$ and $\varphi(t) = -t^2 - t(2\gamma+1)$. If $2\gamma+1 \geq 0$ then $\varphi(t) \leq 0$ and $E_1 \leq \lfloor (n-4)^2/4 \rfloor + 2n - 5 < \lfloor n^2/4 \rfloor$. Otherwise $\gamma < -\frac{1}{2}$. The maximum value of $\varphi(t)$ is $\varphi(-\gamma-\frac{1}{2}) = \gamma^2 + \gamma - \frac{1}{4}$, which implies

$$E \leq (n-4)^2/4 + 2n - 5 + \gamma - \tfrac{1}{4} - s < \lfloor n^2/4 \rfloor.$$

A similar situation occurs for n odd by denoting $\gamma = (n-5)/2 - (k+s)$, when $(k+s)(n-4-k-s) = \lfloor (n-4)^2/4 \rfloor - \gamma^2 - \gamma$.

To see that this bound can also be reached in this case it is sufficient to consider $C = D = \emptyset$ $(p = q = 0), -1 \leq |A| - |B| \leq 1$ and any vertex $a \in A$ is adjacent to v or to a vertex $b \in B$ and any vertex $b \in B$ is adjacent to u or to a vertex $a \in A$. All these graphs are bipartite and by Theorem 5.1.1 the number of resolving pairs equals $\lfloor n^2/4 \rfloor$ since partite sets have $|A|+2$ and $|B|+2$ vertices, respectively. □

Concluding remarks and open problems

In chapter 3 the metric dimension of the flower graphs $f_{n\times m}$ and of two classes of graphs generated by convex polytopes has been studied. It was proved that the flower graphs $f_{n\times m}$ have bounded metric dimension for $m = 3$ but for every $m \geq 4$ the metric dimension of $f_{n\times m}$ is unbounded as n tends to infinity. The metric dimension of two classes of graphs of some convex polytopes has also been studied by considering the open problem raised in [20] and it was proved that these classes of graphs have constant metric dimension and only three vertices appropriately chosen suffice to resolve all the vertices.

Note that in [30] Melter and Tomescu gave an example of infinite regular graphs (namely the digital plane endowed with city-block and chessboard distances, respectively) having no finite metric basis. We conclude the topic by raising an open problem and a conjecture.

Open Problem. Let G' be the graph of a convex polytope obtained from the graph G of a convex polytope by deleting some edges. When G' and G have the same metric dimension.

All extremal graphs found in this chapter 5 are bipartite. Thus, the following conjecture seems to be plausible:

Conjecture . All non-bipartite graphs of order n have a number of resolving pairs less than $\lfloor n^2/4 \rfloor$.

The most striking example is the odd cycle which has no resolving pair.

SUBJECT INDEX

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