LIMITING REITERATION FOR REAL INTERPOLATION AND OPTIMAL SOBOLEV EMBEDDINGS

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By

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DECLARATION

I, Mr. Irshaad Ahmed Registration No. 66-GCU-PHD-SMS-06 student at Abdus Salam School of Mathematical Sciences GC University in the subject of Mathematics, year of admission 2006, hereby declare that the matter printed in this thesis titled

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Controller of Examination

GC University Lahore

Pakistan.
Dedicated

to

my family and dear friends.
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Abstract

Firstly, sharp reiteration theorems for the $K$–interpolation method in limiting cases are proved using two-sided estimates of the $K$–functional. As an application, sharp mapping properties of the Riesz potential are derived in a limiting case. Secondly, we prove optimal embeddings of the homogeneous Sobolev spaces built-up over function spaces in $\mathbb{R}^n$ with $K$–monotone and rearrangement invariant norm into another rearrangement invariant function spaces. The investigation is based on pointwise and integral estimates of the rearrangement or the oscillation of the rearrangement of $f$ in terms of the rearrangement of the derivatives of $f$. 
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Lahore, Pakistan

Irshaad Ahmed
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The objective of this thesis is twofold. Firstly, we prove reiteration theorems in limiting cases for the $K$–interpolation method for general weights. Secondly, we study the optimal embeddings of homogeneous Sobolev spaces in the framework of rearrangement invariant spaces on $\mathbb{R}^n$.

The classical reiteration theorem for the $K$–interpolation method characterizes the spaces $\{A_{t^{\theta_0}, p_0}, A_{t^{\theta_1}, p_1}\}$ where $0 < \theta_0 < \theta_1 < 1$. This reiteration has been generalized in [24] to the spaces $\{A_{w_0, p_0}, A_{w_1, p_1}\}_{w,q}$ with $w_j(t) = t^{-\theta_j}b_j(t)$ $(j = 0, 1)$, $w(t) = t^{-\theta}b(t)$ where $0 < \theta_0 < \theta_1 < 1$, $0 < \theta < 1$ and $b_0$, $b_1$, $b$ are slowly varying. The limiting cases $\theta = 0$ or $1$ have been considered in [20], [21] and [23], leading to the characterization of spaces $\{A_{w_0, p_0}, A_{w_1, p_1}\}_{b,q}$ and $\{A_{w_0, p_0}, A_{w_1, p_1}\}_{t^{-1}b,q}$. The characterization of former space is based on the next formula for the $K$–functional:

\[ K(\rho(t), f; A_{w_0, p_0}, A_{w_1, p_1}) \approx I(t, f) + \rho(t)J(t, f), \]

where $\rho = w_0/w_1$ and

\[ I(t, f) = \left( \int_0^t w_0^{p_0}(u)K^{p_0}(u, f)\frac{du}{u} \right)^{1/p_0}, \]

\[ J(t, f) = \left( \int_t^{\infty} w_1^{p_1}(u)K^{p_1}(u, f)\frac{du}{u} \right)^{1/p_1}. \]

In the first Chapter of the thesis we extend these results to general weights $w_1, w_2$ and $w$ including the case $w_j(t) = t^{-\theta_j}b_j(t)$ $(j = 0, 1)$, $w(t) = t^{-\theta}b(t)$ where $0 \leq \theta_0 \leq \theta_1 \leq 1$, $0 \leq \theta \leq 1$ and $b_0$, $b_1$, $b$ are slowly varying. We show that a variant of Holmstedt’s argument works for general weights. For example, we have

\[ K(W_0(t), f; A_{w_0, p_0}, A_1) \approx I(t, f) + \rho_0(t)K(t, f)/t, \]

and

\[ K(W_1(t), f; A_0, A_{w_1, p_1}) \approx W_1(t)J(t, f) + K(t, f), \]
where
\[ W_0(t) = \max(\rho_0(t), h_0(t)), \quad W_1(t) = \min(g_1(t), \sigma_1(t)), \]
\[ \rho_0(t) = t \left( \int_t^\infty w_0^{p_0}(u) \frac{du}{u} \right)^{1/p_0}, \quad h_0(t) = \left( \int_0^t u^{p_0} w_0^{p_0}(u) \frac{du}{u} \right)^{1/p_0}, \]
\[ \sigma_1(t) = t \left( \int_0^t u^{p_1} w_1^{p_1}(u) \frac{du}{u} \right)^{-1/p_1}, \quad g_1(t) = \left( \int_t^\infty u^{p_1} w_1^{p_1}(u) \frac{du}{u} \right)^{-1/p_1}. \]

We have only estimates from below and from above for the $K$–functional corresponding to the case \( \{ A_{w_0, p_0}, A_{w_1, p_1} \} \).

We investigate the problem of optimal embeddings for homogeneous Sobolev spaces in the second Chapter of the thesis. The problem of optimal embeddings for inhomogeneous Sobolev spaces $W^k E$, built-up over rearrangement invariant spaces $E$ on a bounded domain in $\mathbb{R}^n$, is treated in [19], [18], [29], [36], [33], [22]. A different method, based on the theory of capacities, is applied in [25], [35]. The case of homogeneous Sobolev spaces, $w^k E$ is treated in [34] in the class of rearrangement invariant Banach function spaces $E$ as in [7]. Our domain spaces are more general. In particular, we do not use the Fatou property and duality arguments. The construction of the optimal target space in the subcritical case is rather simple and gives an optimal couple (see Theorem 2.5.8 below). In the critical case we construct a large class of domain spaces for which the corresponding optimal target space is found. In [34] the optimal target set is not linear (see also Theorem 2.5.1 below).

The main results of the thesis are contained in the papers [1, 2, 3] and they are presented on the seminar of Analysis at Abdus Salam School of Mathematical Sciences, GC University Lahore and on the conferences: Mathematical Inequalities and Applications Conference, Trogir-Split, Crotia 2008, 4th World Conference on 21st Century Mathematics, Lahore, Pakistan 2009 and International Conference on Mathematical Inequalities and Applications, Lahore, Pakistan 2010.
Chapter 1

Reiteration for the $K$–interpolation method in limiting cases

Our main goal in this chapter is to extend the results in [20], [21] and [23]. To explain it we shall use notions given in section 1.1. Let $b$, $b_0$ and $b_1$ be slowly varying, suppose that $0 < \theta_0 < \theta_1 < 1$, put $w_j(t) = t^{-\theta_j} b_j(t)$ $(j = 0, 1)$, $\rho(t) = w_0(t)/w_1(t)$ and set

$$w(t) = t^{-\theta} b(t) \quad (0 < \theta < 1), \quad c(t) = w_0(t)^{1-\theta} w_1(t)^\theta \rho(t).$$

The following reiteration theorem is proved in [24]:

$$\{A_{w_0,p_0}, A_{w_1,p_1}\}_{w,q} = A_{c,q} \quad (p_0, p_1, q \in (0, \infty]).$$

More recently, the limiting cases $\theta = 0$ or $1$ have been considered in [20], [21] and [23], leading to the result (corresponding to $\theta = 0$) that

$$\{A_{w_0,p_0}, A_{w_1,p_1}\}_{b,q} = A_{w_0,p_0; b(\rho),q} \quad (0 < \theta_0 < \theta_1 < 1).$$

The result corresponding to $\theta = 1$ is that

$$\{A_{w_0,p_0}, A_{w_1,p_1}\}_{t^{-1}b,q} = A_{w_1,p_1; b(\rho),q} \quad (0 < \theta_0 < \theta_1 < 1).$$

We shall extend these results to general weights $w_j$, including the case

$$w_j(t) = t^{-\theta_j} b_j(t), \quad 0 \leq \theta_0 \leq \theta_1 \leq 1 \quad \text{and} \quad b_j \text{ is slowly varying} \quad (j = 0, 1).$$

Note that $\theta_0$, $\theta_1$ are allowed to take the values 0 and 1, respectively. As a special case of the results obtained we have, corresponding to the weight $w_0$ of the above form with $\theta_0 = 0$,

$$\{A_{w_0,p_0}, A_1\}_{t^{-1}b,q} = A_{t^{-1}b(\rho_0),q}.$$
When \( \theta_1 = 1 \), a sample result is that
\[
\{ A_0, A_{w_1,p_1} \}_{b,q} = A_{b(\sigma_1),q}.
\]

We also allow weights of the form
\[
w(t) = \begin{cases} 
  w_0(t), & 0 < t < 1, \\
  w_1(t), & 1 < t < \infty,
\end{cases}
\]
which arise in extrapolation theory (see [27]). To this end we show that a variant of Holmstedt’s argument works for general weights. For example, we have (see Theorem 1.2.1 below)
\[
\rho_0(t)K(t,f)/t \lesssim K(\rho_0(t),f;A_{w_0,p_0},A_1) \lesssim I(t,f) + \rho_0(t)K(t,f)/t
\]
and
\[
I(t,f) \lesssim K(h_0(t),f;A_{w_0,p_0},A_1) \lesssim I(t,f) + \rho_0(t)K(t,f)/t.
\]
These inequalities imply that
\[
K(\rho_0(t) + h_0(t),f;A_{w_0,p_0},A_1) \approx I(t,f) + \rho_0(t)K(t,f)/t.
\]

The chapter is organized as follows. The first section contains basic definitions, notations and known results which we shall need in the forthcoming sections. In addition, some new technical results are obtained. In the second section the estimates of the \( K \)-functional are obtained. The embeddings of \( K \)-interpolation spaces are exhibited in the third section. The reiteration theorems along with examples are presented in the fourth section. The chapter concludes by applying some of the general results to obtain sharp mapping properties of the Riesz potential.

1.1 Introduction

Let \( A \) denote a linear space over \( \mathbb{R} \). We shall use the notations \( a_1 \lesssim a_2 \) or \( a_2 \gtrsim a_1 \) for nonnegative functions or functionals to mean that the quotient \( a_1/a_2 \) is bounded; also, \( a_1 \approx a_2 \) means that \( a_1 \lesssim a_2 \) and \( a_1 \gtrsim a_2 \). We say that \( a_1 \) is equivalent to \( a_2 \) if \( a_1 \approx a_2 \).

Definition 1.1.1. A non-negative real-valued function \( \| \cdot \| \) defined on \( A \) is said to be a quasi-norm if it has the following properties:
(1.) \( \|x\| = 0 \) if and only \( x = 0 \);  
(2.) \( \|\alpha x\| = |\alpha|\|x\| \);  
(3.) \( \|x + y\| \lesssim \|x\| + \|y\| \); 
\[ \forall x, y \in A \text{ and } \alpha \in \mathbb{R}. \]

A linear space equipped with a quasi-norm is called a quasi-normed space. Henceforth, we shall denote a quasi-norm on a space \( A \) by \( \|\cdot\|_A \) or \( \|\cdot| \_A \), depending on the complexity of the expression \( A \).

**Definition 1.1.2.** A quasi-normed space \( A \) is said to be continuously embedded in another quasi-normed space \( B \) if the following estimate 
\[ \|x\|_B \lesssim \|x\|_A, \quad x \in A \]
holds. We use the symbol \( A \hookrightarrow B \) to indicate that \( A \) is continuously embedded in \( B \).

**Definition 1.1.3.** A pair \((A_0, A_1)\) of quasi-normed spaces is called a compatible couple if both \( A_0 \) and \( A_1 \) are continuously embedded in some common quasi-normed space \( A \). We shall also use the notation \( \{A_0, A_1\} \) to denote a compatible couple.

For any compatible couple \((A_0, A_1)\), there are always associated two quasi-normed spaces namely the sum and intersection of \( A_0 \) and \( A_1 \).

**Definition 1.1.4.** The sum space \( A_0 + A_1 \) is defined by 
\[ A_0 + A_1 := \{f \in A : f = f_0 + f_1, \ f_0 \in A_0, \ f_1 \in A_1 \}. \]

The space \( A_0 + A_1 \) turns out to be quasi-normed space under the quasi-norm 
\[ \|f\|_{A_0 + A_1} := \inf\{\|f_0\|_{A_0} + \|f_1\|_{A_1} : f = f_0 + f_1 \}; \]
where the infimum extends over all representations \( f = f_0 + f_1 \) of \( f \) with \( f_0 \in A_0 \) and \( f_1 \in A_1 \).

On the intersection space \( A_0 \cap A_1 \), we specify a quasi-norm structure as 
\[ \|f\|_{A_0 \cap A_1} := \max\{\|f\|_{A_0}, \|f\|_{A_1}\}. \]
Definition 1.1.5. Let \((A_0, A_1)\) be a compatible couple of quasi-normed spaces. The \(K\)-functional, denoted by \(K(t, f; A_0, A_1)\), is defined for each \(f \in A_0 + A_1\) and \(t > 0\) by
\[
K(t, f; A_0, A_1) := \inf \{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : f = f_0 + f_1\},
\]
where the infimum extends over all representations \(f = f_0 + f_1\) of \(f\) with \(f_0 \in A_0\) and \(f_1 \in A_1\).

In the sequel we let \(K(t, f) = K(t, f; A_0, A_1)\) whenever it is understood that the underlying compatible couple is \((A_0, A_1)\).

The \(K\)-functional will be the main object of study in what follows, and its properties, collected in the following lemma, will be used time and again.

Lemma 1.1.1. (\cite{7}, \cite{26}) Let \(f\) and \(g\) be in \(A_0 + A_1\). The \(K\)-functional \(K(t, f)\), as a function of \(t\), is increasing on \((0, \infty)\) and \(t^{-1}K(t, f)\), as a function of \(t\), is decreasing on \((0, \infty)\). Moreover,
\[
K(t, f) \leq \|f\|_{A_0}, \tag{1.1.1}
\]
\[
K(t, f) \leq t\|f\|_{A_1}, \tag{1.1.2}
\]
\[
K(t, f; A_0, A_1) = tK(1/t, f; A_1, A_0), \tag{1.1.3}
\]
and
\[
K(t, f + g) \lesssim K(t, f) + K(t, g). \tag{1.1.4}
\]

Definition 1.1.6. Let \(\Phi\) be a space consisting of functions defined on \((0, \infty)\). Let \(\Phi\) be endowed with a monotone quasi-norm such that
\[
t \mapsto \min\{1, t\} \in \Phi. \tag{1.1.5}
\]
By definition, the \(K\)-interpolation space \(A_\Phi = (A_0, A_1)_\Phi\) has the quasi-norm
\[
\|f\|_{A_\Phi} = \|K(t, f)\|_{\Phi}.
\]

We note that the embedding
\[
A_0 \cap A_1 \hookrightarrow A_\Phi \hookrightarrow A_0 + A_1
\]
follows from (1.1.5). Before stating the interpolation property of \(K\)-interpolation spaces, we let the symbol \(T : A \to B\) denote that the linear operator \(T\) is continuous. We shall also make use of this notation in section 1.5 and subsection 2.5.2.
**Theorem 1.1.2.** *(Real Interpolation Theorem)*[9] Let \((A_0, A_1)\) and \((B_0, B_1)\) be two compatible couples, and suppose \(A_{\Phi}^\ast\) and \(B_{\Phi}^\ast\) be corresponding \(K\)-interpolation spaces. Let \(T\) be a linear operator such that

\[ T : A_j \to B_j, \quad (j = 0, 1) \]

then we also have

\[ T : A_{\Phi} \to B_{\Phi}. \]

**Definition 1.1.7.** Let \(w\) be a positive real-valued function defined on \((0, \infty)\). We say that \(w\) is a weight if it is locally integrable on \((0, \infty)\).

We shall assume throughout this chapter that \(p_0, p_1, q \in (0, \infty]\) and that \(w, w_0, w_1, b\) are weights defined on \((0, \infty)\). We shall be mainly concerned with three \(K\)-interpolation spaces namely \(A_{w,q}^{-}, A_{w,p;b,q}^{-}\) and \(A_{w,p;b,q}^{+}\) Let us define them.

**Definition 1.1.8.** \(A_{w,q}^{-}\) is the \(K\)-interpolation space \(A_{\Phi}^{-}\) where \(\Phi^{-}\) is the weighted Lebesgue space \(L^q(w)\) defined by the quasi-norm

\[ \|g\|_{L^q(w)} := \begin{cases} \left( \int_0^\infty w^q(t) |g(t)|^q \frac{dt}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup_{0 < t < \infty} \{w(t) |g(t)|\}, & q = \infty. \end{cases} \quad (1.1.6) \]

We assume that (1.1.5) is satisfied, i.e.

\[ \left( \int_0^\infty (\min(1, t)w(t))^{q/p} \frac{dt}{t} \right)^{1/q} < \infty. \]

We shall write

\[ A_{w,q}^{-} = \{A_0, A_1\}_{w,q}^{-} = A_{L^q(w)}, \]

and shall use whichever of these representations is the most appropriate in particular situations.

**Definition 1.1.9.** By the space \(A_{w,p;b,q}^{-} = \{A_0, A_1\}_{w,p;b,q}^{-}\) we mean the \(K\)-interpolation space \(A_{\Phi}^{-}\), where \(\Phi^{-}\) has the quasi-norm

\[ \|g\|_{\Phi^{-}} := \left( \int_0^\infty b^2(t) \left( \int_0^t w^p(u) |g(u)|^p \frac{du}{u} \right)^{q/p} \frac{dt}{t} \right)^{1/q}, \quad (1.1.7) \]

when \(q < \infty\), with an obvious modification when \(q = \infty\).
Definition 1.1.10. $A_{w,p;b,q}^+ = \{A_0, A_1\}_{w,p;b,q}^+$ is the $K$-interpolation space $A_{\Phi^+}$, where $\Phi^+$ has the quasi-norm

$$
\|g\|_{\Phi^+} := \left( \int_0^\infty b^q(t) \left( \int_t^\infty w^p(u) |g(u)|^p \frac{du}{u} \right)^{q/p} \frac{dt}{t} \right)^{1/q},
$$

(1.1.8)

when $q < \infty$; when $q = \infty$ this is modified appropriately.

Before we proceed further we give the notion of a slowly varying function.

Definition 1.1.11. We say that the weight $b$ is slowly varying on $(0, \infty)$ (in the sense of Karamata) if, for all $\varepsilon > 0$, the function $t \mapsto t^\varepsilon b(t)$ is equivalent to an increasing function and $t \mapsto t^{-\varepsilon} b(t)$ is equivalent to a decreasing function. Here we say that a function $g$ is an increasing function if $t_1 < t_2$ implies that $g(t_1) \leq g(t_2)$.

Some of the elementary properties of slowly varying functions are singled out in the next lemma.

Lemma 1.1.3. Let $b$ be a slowly varying function. Then

(i) Given any other slowly varying function $d$, then $\frac{b}{d}$ or $\frac{d}{b}$ is also a slowly varying function.

(ii) If $f \approx g$, then $b(f) \approx b(g)$.

(iii) If $a > 0$, then for all $t > 0$,

$$
\int_0^t s^a b(s) \frac{ds}{s} \approx t^a b(t)
$$

(1.1.9)

and

$$
\int_t^\infty s^{-a} b(s) \frac{ds}{s} \approx t^{-a} b(t).
$$

(1.1.10)

Proof. (i) This property is a simple consequence of the definition.

(ii) Suppose $f \approx g$, then there exist positive constants $c_1$ and $c_2$ such that

$$
c_1 g \leq f \leq c_2 g.
$$

Now by the definition of a slowly varying function, there is an increasing function $d$ so that

$$
fb(f) \approx d(f)
$$

$$
\lesssim d(c_2 g) \quad \text{(as } f \leq c_2 g)\]

$$
\approx c_1 (c_2 g)
$$

$$
\lesssim (c_2 c_1^{-1} f) b(c_2 g).
$$

(1.1.9)
which gives

\[ b(f) \lesssim c_2 b(c_2 g). \quad (1.1.11) \]

Similarly, we have

\[
\begin{align*}
fb(f) & \approx d(f) \\
& \gtrsim d(c_1 g) \quad (\text{as } f \geq c_1 g) \\
& \approx (c_1 g)b(c_1 g) \\
& \gtrsim (c_1 c_2^{-1} f)b(c_1 g),
\end{align*}
\]

giving

\[ b(f) \gtrsim c_1 b(c_1 g). \quad (1.1.12) \]

Next we obtain the estimate \( b(f) \lesssim b(g) \) from (1.1.11). To this end, we first let \( c_2 \geq 1 \). Then we have a decreasing function \( d_1 \) such that

\[
\begin{align*}
 g^{-1}b(g) & \approx d_1(g) \\
& \gtrsim d_1(c_2 g) \quad (\text{as } c_2 g \geq g) \\
& \approx (c_2 g)^{-1}b(c_2 g),
\end{align*}
\]

thus,

\[ c_2 b(g) \gtrsim b(c_2 g), \]

inserting this estimate in (1.1.11), we get \( b(f) \lesssim b(g) \). If \( c_2 < 1 \), then

\[ gb(g) \approx d_2(g), \]

where \( d_2 \) is an increasing function. Thus \( d_2(g) \geq d_2(c_2 g) \) as \( g > c_2 g \). Therefore,

\[
\begin{align*}
 gb(g) & \gtrsim d(c_2 g) \\
& \approx (c_2 g)b(c_2 g),
\end{align*}
\]

whence we have \( b(g) \gtrsim c_2 b(c_2 g) \). With this estimate, (1.1.11) turns into \( b(f) \lesssim b(g) \). In the same way we can derive the remaining estimate \( b(f) \gtrsim b(g) \) from (1.1.12).

(iii) Let \( a > 0 \), and \( b \) be slowly varying function. Consider

\[
\int_0^t s^a b(s) \frac{ds}{s} = \int_0^t s^{a+1} s^{-1} b(s) \frac{ds}{s},
\]
by definition $s^{-1}b(s)$ is equivalent to a decreasing function, therefore

$$
\int_0^t s^a b(s) \frac{ds}{s} \geq t^{-1}b(t) \int_0^t s^{a+1} \frac{ds}{s} \\
\approx t^{-1}b(t)t^{a+1} \\
\approx t^a b(t),
$$

showing one of the estimates in (1.1.9). To derive the other estimate, we write as

$$
\int_0^t s^a b(s) \frac{ds}{s} = \int_0^t s^\frac{a}{2} s^\frac{a}{2} b(s) \frac{ds}{s},
$$

by definition $s^\frac{a}{2} b(s)$ is equivalent to an increasing function, thus

$$
\int_0^t s^a b(s) \frac{ds}{s} \lesssim t^\frac{a}{2} b(t) \int_0^t s^{\frac{a}{2}} \frac{ds}{s} \\
\approx t^\frac{a}{2} b(t) t^{\frac{a}{2}} \\
\approx t^a b(t),
$$

hence (1.1.9) is true. Similarly,

$$
\int_t^\infty s^{-a} b(s) \frac{ds}{s} = \int_t^\infty s^{-a-1} s b(s) \frac{ds}{s} \\
\approx t b(t) t^{-a-1} \\
\approx t^{-a} b(t),
$$

and

$$
\int_t^\infty s^{-a} b(s) \frac{ds}{s} = \int_t^\infty s^{-\frac{a}{2}} s^{-\frac{a}{2}} b(s) \frac{ds}{s} \\
\lesssim t^{-\frac{a}{2}} b(t) t^{-\frac{a}{2}} \\
\approx t^{-a} b(t).
$$

Therefore (1.1.10) is also valid.
Let us introduce four weights which will be crucial in the formation of our main results. For each $t > 0$, we define

$$
\rho_0(t) = t \left( \int_t^\infty w_0^p(u) \frac{du}{u} \right)^{1/p_0},
$$

$$
\rho_0(t) = \left( \int_0^t u^{p_0} w_0^{p_0}(u) \frac{du}{u} \right)^{1/p_0},
$$

$$
\sigma_1(t) = t \left( \int_0^t u^{p_1} w_1^{p_1}(u) \frac{du}{u} \right)^{-1/p_1},
$$

and

$$
\sigma_1(t) = t \left( \int_0^t u^{p_1} w_1^{p_1}(u) \frac{du}{u} \right)^{-1/p_1}.
$$

It would be useful in the subsequent sections to have the next results.

**Lemma 1.1.4.** Let $w_j(t) = t^{-\theta_j} b_j(t)$, where $0 \leq \theta_0 < 1$, $0 < \theta_1 \leq 1$ and $b_j$ is slowly varying $(j = 0, 1)$. Then

$$
h_0(t) \approx tw_0(t),
$$

$$
\rho_0(t) \gtrsim tw_0(t),
$$

$$
\sigma_1(t) \lesssim \frac{1}{w_1(t)},
$$

and

$$
g_1(t) \approx \frac{1}{w_1(t)}.
$$

**Proof.** Now

$$
h_0(t) = \left( \int_0^t u^{(1-\theta_0)p_0} b_0^{p_0}(u) \frac{du}{u} \right)^{1/p_0},
$$

we note that $b_0^{p_0}$ is slowly varying, therefore, by applying (1.1.9) we obtain

$$
h_0(t) \approx t^{(1-\theta_0)} b_0(t),
$$

or,

$$
h_0(t) \approx tw_0(t),
$$
as required.
To derive (1.1.18), we first let that $0 < \theta < 1$. Then from
\[
\rho_0(t) = t \left( \int_t^\infty u^{-\theta_0 p_0} b_0^{p_0}(u) \frac{du}{u} \right)^{1/p_0}
\]
we get
\[
\rho_0(t) \approx tw_0,
\]
by (1.1.10). If $\theta_0 = 0$ then we have
\[
\rho_0(t) = t \left( \int_t^\infty u^{-p_0} b_0^{p_0}(u) \frac{du}{u} \right)^{1/p_0}
\]
\[
\approx t^2 b_0(t) t^{-1}
\]
hence, it follows that the estimate (1.1.18) is valid. By the same way we can derive (1.1.19) and (1.1.20).

Remark 1.1.1. If $0 < \theta_0 < 1$, then we would have $\approx$ instead of $\gtrsim$ in (1.1.18). Similarly we can replace $\lesssim$ by $\approx$ in (1.1.19) if $0 < \theta_1 < 1$.

It is convenient here to introduce two integrals which will appear in what follows. They are
\[
I(t, f) = \left( \int_0^t w_0^{p_0}(u) K^{p_0}(u, f) \frac{du}{u} \right)^{1/p_0}, \tag{1.1.21}
\]
and
\[
J(t, f) = \left( \int_t^\infty w_1^{p_1}(u) K^{p_1}(u, f) \frac{du}{u} \right)^{1/p_1}. \tag{1.1.22}
\]
We note, in view of (1.1.4) and the equivalence $(a + b)^p \approx a^p + b^p, a, b, p > 0$, that for any decomposition $f = f_0 + f_1$, we have
\[
I(t, f) \lesssim I(t, f_0) + I(t, f_1), \tag{1.1.23}
\]
and

\[ J(t, f) \lesssim J(t, f_0) + J(t, f_1). \tag{1.1.24} \]

The next result is proved in [20], Lemma 3.4, for \(0 < s < 1\), and the same technique can be used to establish it for \(1 \leq s < \infty\).

**Lemma 1.1.5.** Let \(s \in (0, \infty)\) and suppose that \(v, w, \phi\) and \(h\) are positive functions on \((0, \infty)\) such that

\[ v(t) = (w(t))^{1-s} \left( \phi(t) \int_t^\infty w(u)du \right)^s. \tag{1.1.25} \]

Then

\[ \int_0^\infty \left( \int_0^t \phi(u)h(u)du \right)^s w(t)dt \geq s^s \int_0^\infty h^s(t)v(t)dt \text{ if } 0 < s < 1, \tag{1.1.26} \]

and

\[ \int_0^\infty \left( \int_0^t \phi(u)h(u)du \right)^s w(t)dt \leq s^s \int_0^\infty h^s(t)v(t)dt \text{ if } 1 \leq s < \infty. \tag{1.1.27} \]

We also have the following variant of this result.

**Lemma 1.1.6.** Let \(s \in (0, \infty)\), suppose that \(v, w, \phi\) and \(h\) are positive functions on \((0, \infty)\) such that

\[ \int_0^t \phi(u)h(u)du \geq h(t) \int_0^t \phi(u)du, \]

and let

\[ v_0(t) = \phi(t) \left( \int_0^t \phi(u)du \right)^{s-1} \int_t^\infty w(u)du. \tag{1.1.28} \]

Then

\[ \int_0^\infty \left( \int_0^t \phi(u)h(u)du \right)^s w(t)dt \geq s \int_0^\infty h^s(t)v_0(t)dt \text{ if } 1 < s < \infty, \tag{1.1.29} \]

and

\[ \int_0^\infty \left( \int_0^t \phi(u)h(u)du \right)^s w(t)dt \leq s \int_0^\infty h^s(t)v(t)dt \text{ if } 0 < s \leq 1. \tag{1.1.30} \]
Proof. Let 

\[ G(t) = \int_0^t \phi(u)h(u)du, \]

since \( G(0) = 0 \), we can write

\[ G^s(t) = \int_0^t sG^{s-1}(u)du \]

use of Fubini’s theorem shows that

\[ \int_0^\infty G^s(t)w(t)dt = s\int_0^\infty G^{s-1}(u)\phi(u)h(u)\int_u^\infty w(t)dtdu. \quad (1.1.31) \]

It remains to use the fact that \( G(t) \geq h(t)\int_0^t \phi(u)du \).

We shall apply the next lemma in the last section of the present chapter.

**Lemma 1.1.7.** ([37]) Let \( f \) be a non-negative function on \((0, \infty)\), and suppose that \( g \) and \( h \) are weights on \((0, \infty)\). Let \( 1 \leq q \leq \infty \), then

\[ \int_0^\infty g^q(t) \left( \int_t^\infty f(u)\frac{du}{u} \right)^q \frac{dt}{t} \lesssim \left( \int_0^\infty (f(t)h(t))^q \frac{dt}{t} \right)^{1/q}, \]

if and only if

\[ \left( \int_0^t g^q(u)\frac{du}{u} \right)^{1/q} \left( \int_t^\infty h^{q/(1-q)}(u)\frac{du}{u} \right)^{(q-1)/q} \lesssim 1. \]

**1.2 Estimates of the \( K \)-functional**

**1.2.1 The case \( \{A_{w_0,p_0}, A_1\} \)**

We assume that the weight \( w_0 \) is such that the functions \( \rho_0 \) and \( h_0 \) are finite on \((0, \infty)\).

**Theorem 1.2.1.** For any weight \( w \), any \( f \in A_0 + A_1 \) and all \( t > 0 \),

\[ K(w(t), f; A_{w_0,p_0}, A_1) \lesssim I(t, f) + w(t)K(t, f)/t + \rho_0(t)K(t, f)/t, \quad (1.2.1) \]
\[
\rho_0(t)K(t, f)/t \lesssim K(\rho_0(t); f; A_{w_0,p_0}, A_1), \quad \text{(1.2.2)}
\]
\[
I(t, f) \lesssim K(h_0(t), f; A_{w_0,p_0}, A_1), \quad \text{(1.2.3)}
\]
\[
K(h_0(t), f; A_{W_0,p_0}, A_1) \approx I(t, f) \text{ if } \rho_0 \lesssim h_0, \quad \text{(1.2.4)}
\]

and, with \(W_0(t) := \max(\rho_0(t), h_0(t))\),
\[
K(W_0(t), f; A_{w_0,p_0}, A_1) \approx I(t, f) + \rho_0(t)K(t, f)/t. \quad \text{(1.2.5)}
\]

**Proof.** To establish the estimates of the \(K\)-functional from below we take an arbitrary decomposition \(f = f_0 + f_1\), with \(f_j \in A_j \ (j = 0, 1)\), of \(f \in A_0 + A_1\). Then by (1.1.23),
\[
I(t, f) \lesssim \left( \int_0^t w_0^{p_0}(u)K^{p_0}(u, f_0) \frac{du}{u} \right)^{1/p_0} + \left( \int_0^t w_0^{p_0}(u)K^{p_0}(u, f_1) \frac{du}{u} \right)^{1/p_0},
\]
using (1.1.2),
\[
I(t, f) \lesssim \left( \int_0^t w_0^{p_0}(u)K^{p_0}(u, f_0) \frac{du}{u} \right)^{1/p_0} \| f_1 \| A_1 \| \left( \int_0^t w_0^{p_0}(u)u^{p_0} \frac{du}{u} \right)^{1/p_0},
\]
taking infimum over all decompositions \(f = f_0 + f_1\), of \(f\), we get (1.2.3). Altogether from (1.1.4) and (1.1.2), we can write
\[
K(t, f) \lesssim K(t, f_0) + t \| f_1 \| A_1 \|,
\]
from which we have
\[
\rho_0(t)K(t, f)/t \lesssim (\rho_0(t)/t) K(t, f_0) + \rho_0(t) \| f_1 \| A_1 \|
\]
\[
\approx \left( \int_t^\infty w_0^{p_0}(u) \frac{du}{u} \right)^{1/p_0} K(t, f_0) + \rho_0(t) \| f_1 \| A_1 \|
\]
\[
\lesssim \left( \int_t^\infty w_0^{p_0}(u)K^{p_0}(u, f_0) \frac{du}{u} \right)^{1/p_0} + \rho_0(t) \| f_1 \| A_1 \|
\]
\[
\lesssim \left( \int_0^\infty w_0^{p_0}(u)K^{p_0}(u, f_0) \frac{du}{u} \right)^{1/p_0} + \rho_0(t) \| f_1 \| A_1 \|
\]
\[
\approx \| f_0 \| A_{w_0,p_0} \| + \rho_0(t) \| f_1 \| A_1 \|,
\]
from which it follows (1.2.2).

To obtain the estimates of the \(K\)-functional from above we always take a particular representation \(f = f_0 + f_1\), of \(f \in A_0 + A_1\) such that
\[
K(t, f; A_0, A_1) \approx \| f_0 \| A_0 \| + t \| f_1 \| A_1 \|. \quad \text{(1.2.6)}
\]
We observe that \( \| f_0 \|_{\mathcal{A}_0} \lesssim K(t, f) \) and \( \| f_1 \|_{\mathcal{A}_1} \lesssim K(t, f)/t \). Inserting these in (1.1.1) and (1.1.2), we obtain

\[
K(u, f_0) \lesssim K(t, f),
\]

and

\[
K(u, f_1) \lesssim uK(t, f)/t.
\]

Therefore,

\[
\| f_0 \|_{\mathcal{A}_{w_0,p_0}} \approx \left( \int_0^t w_0^{p_0}(u)K^{p_0}(u, f_0)\frac{du}{u} \right)^{1/p_0} + \left( \int_t^\infty w_0^{p_0}(u)K^{p_0}(u, f_0)\frac{du}{u} \right)^{1/p_0} \lesssim I(t, f) + \left( \int_0^t w_0^{p_0}(u)K^{p_0}(u, f_1)\frac{du}{u} \right)^{1/p_0} K(t, f)/t + \left( \int_t^\infty w_0^{p_0}(u)\frac{du}{u} \right)^{1/p_0} K(t, f),
\]

consequently,

\[
\| f_0 \|_{\mathcal{A}_{w_0,p_0}} \lesssim I(t, f) + h_0(t)K(t, f)/t + \rho_0(t)K(t, f)/t.
\]

We note that

\[
I(t, f) \gtrsim h_0(t)K(t, f)/t,
\]

since \( K(t, f)/t \) is decreasing in \( t \). Thus, (1.2.9) turns into

\[
\| f_0 \|_{\mathcal{A}_{w_0,p_0}} \lesssim I(t, f) + \rho_0(t)K(t, f)/t.
\]

Combining above estimate and \( \| f_1 \|_{\mathcal{A}_1} \lesssim K(t, f)/t \) gives

\[
\| f_0 \|_{\mathcal{A}_{w_0,p_0}} + w(t)\| f_1 \|_{\mathcal{A}_1} \lesssim I(t, f) + w(t)K(t, f)/t + \rho_0(t)K(t, f)/t,
\]

which yields (1.2.1). Let \( \rho_0 \lesssim h_0 \), and take \( w(t) = h_0(t) \) in (1.2.1) to get

\[
K(h_0, f; \mathcal{A}_{w_0,p_0}, A_1) \lesssim I(t, f) + h_0(t)K(t, f)/t + \rho_0(t)K(t, f)/t,
\]
using $\rho_0 \lesssim h_0$,
\[ K(h_0, f; A_{w_0,p_0}, A_1) \lesssim I(t, f) + h_0(t)K(t, f)/t, \]
in view of (1.2.10),
\[ K(h_0, f; A_{w_0,p_0}, A_1) \lesssim I(t, f), \]
using this estimate along with (1.2.3), we deduce (1.2.4).
Let $\rho_0 \lesssim h_0$, then $W_0 \approx h_0$. Hence, it follows from (1.2.4) that
\[ K(W_0, f; A_{w_0,p_0}, A_1) \approx I(t, f). \]  
(1.2.12)
But inserting $\rho_0 \lesssim h_0$ in (1.2.10), we obtain
\[ I(t, f) \gtrsim \rho_0(t)K(t, f)/t \]
which leads us to
\[ K(W_0, f; A_{w_0,p_0}, A_1) \approx I(t, f) + \rho_0(t)K(t, f)/t. \]
Now let $\rho_0 \gtrsim h_0$, so that $W_0 \approx \rho_0$. Take $w = \rho_0$ in (1.2.1) to get
\[ K(\rho_0, f; A_{f_0,p_0}, A_1) \lesssim I(t, f) + \rho_0(t)K(t, f)/t, \]
or,
\[ K(W_0, f; A_{w_0,p_0}, A_1) \lesssim I(t, f) + \rho_0(t)K(t, f)/t. \]  
(1.2.13)
With the estimate $\rho_0 \gtrsim h_0$, (1.2.3) yields
\[ I(t, f) \lesssim K(W_0, f; A_{w_0,p_0}, A_1), \]
adding above estimate to (1.2.2),
\[ I(t, f) + \rho_0(t)K(t, f)/t \lesssim K(W_0, f; A_{w_0,p_0}, A_1). \]  
(1.2.14)
Combining (1.2.13) and (1.2.14), we get (1.2.5). This completes the proof.

1.2.2 The case $\{A_0, A_{w_1,p_1}\}$

In the following theorem, it is assumed that the weight $w_1$ is such that the functions $\sigma_1$ and $g_1$ are finite on $(0, \infty)$. 

**Theorem 1.2.2.** For any weight $w$, any $f \in A_0 + A_1$ and all $t > 0$,

$$K(w(t), f; A_0, A_{w_1,p_1}) \lesssim w(t) J(t, f) + (w(t)/\sigma_1(t)) K(t, f) + K(t, f),$$  \hspace{1cm} (1.2.15)

$$K(t, f) \lesssim K(\sigma_1(t), f; A_0, A_{w_1,p_1}),$$  \hspace{1cm} (1.2.16)

$$g_1(t) J(t, f) \lesssim K(g_1(t), f; A_0, A_{w_1,p_1}),$$  \hspace{1cm} (1.2.17)

$$K(g_1(t), f; A_0, A_{w_1,p_1}) \approx g_1(t) J(t, f) \text{ if } g_1 \lesssim \sigma_1,$$  \hspace{1cm} (1.2.18)

and, with $W_1(t) := \min(g_1(t), \sigma_1(t))$,

$$K(W_1(t), f; A_0, A_{w_1,p_1}) \approx W_1(t) J(t, f) + K(t, f).$$  \hspace{1cm} (1.2.19)

**Proof.** We take an arbitrary decomposition $f = f_0 + f_1$, with $f_j \in A_j \ (j = 0, 1)$, of $f \in A_0 + A_1$.

We use (1.1.24) together with (1.1.1) to have

$$J(t, f) \lesssim \|f_0|_{A_0}\left(\int_{t}^{\infty} w_1^{p_1}(u) \frac{du}{u} \right)^{1/p_1} + \|f_1|_{A_{w_1,p_1}};$$

which gives

$$g_1(t) J(t, f) \lesssim \|f_0|_{A_0}\| + g_1(t) \|f_1|_{A_{w_1,p_1}},$$  \hspace{1cm} (1.2.20)

from which follows (1.2.17).

We note that

$$\|f_1|_{A_{w_1,p_1}} = \left(\int_{0}^{\infty} w_1^{p_1}(u) K^{p_1}(u, f_1) \frac{du}{u} \right)^{1/p_1} \geq \left(\int_{0}^{t} w_1^{p_1}(u) K^{p_1}(u, f_1) \frac{du}{u} \right)^{1/p_1} \geq K(t, f_1)/t \left(\int_{t}^{\infty} w_1^{p_1} w_1^{p_1}(u) \frac{du}{u} \right)^{1/p_1},$$

in other words,

$$\|f_1|_{A_{w_1,p_1}} \gtrsim K(t, f_1)/\sigma_1(t).$$  \hspace{1cm} (1.2.21)

Since $f = f_0 + f_1$, (1.1.4) gives

$$K(t, f) \lesssim K(t, f_0) + K(t, f_1),$$
inserting the estimates of $K(t, f_0)$ and $K(t, f_1)$ from (1.1.1) and (1.2.21),

$$K(t, f) \lesssim \|f_0 | A_0\| + \sigma_1(t) \|f_1 | A_{w_1,p_1}\|,$$

which yields (1.2.16).

For the estimates of the $K-$functional from above, we take a representation $f = f_0 + f_1$ of $f$ satisfying (1.2.6).

Now,

$$\|f_1 | A_{w_1,p_1}\| = \left(\int_0^\infty w_1^{p_1}(u)K^{p_1}(u, f_1) \frac{du}{u}\right)^{1/p_1},$$

$$\lesssim \left(\int_t^\infty w_1^{p_1}(u)K^{p_1}(u, f_1) \frac{du}{u}\right)^{1/p_1} + \left(\int_0^t w_1^{p_1}(u)K^{p_1}(u, f_1) \frac{du}{u}\right)^{1/p_1},$$

since $f_1 = f - f_0$, therefore by (1.1.4)

$$\|f_1 | A_{w_1,p_1}\| \lesssim J(t, f) + K(t, f)/g_1(t) + K(t, f)/\sigma_1(t). \quad (1.2.22)$$

In view of the fact that $K(t, f)$ is increasing, we obtain

$$J(t, f) \gtrsim K(t, f) \left(\int_t^\infty w^{p_1}(u) \frac{du}{u}\right)^{1/p_1},$$

or,

$$J(t, f) \gtrsim K(t, f)/g_1(t), \quad (1.2.23)$$

with this estimate (1.2.22) becomes

$$\|f_1 | A_{w_1,p_1}\| \lesssim J(t, f) + K(t, f)/\sigma_1(t), \quad (1.2.24)$$

hence, in view of above estimate and $\|f_0 | A_0\| \leq K(t, f)$, we arrive at

$$\|f_0 | A_0\| + w(t) \|f_1 | A_{w_1,p_1}\| \leq w(t)J(t, f) + w(t)/\sigma_1(t)K(t, f) + K(t, f),$$
which produces (1.2.15).
To prove (1.2.18) we use (1.2.15) with \( w = g_1 \) to have
\[
K(g_1(t), f; A_0, A_{w_1,p_1}) \lesssim g_1(t)J(t, f) + g_1(t)/\sigma_1(t)K(t, f) + K(t, f),
\]
by the given relation \( g_1 \lesssim \sigma_1 \),
\[
K(g_1(t), f; A_0, A_{w_1,p_1}) \lesssim g_1(t)J(t, f) + K(t, f),
\]
making use of (1.2.23),
\[
K(g_1(t), f; A_0, A_{w_1,p_1}) \lesssim g_1(t)J(t, f),
\]
combining this with (1.2.17), we get (1.2.18)
To drive (1.2.19), we first suppose that \( g_1 \lesssim \sigma_1 \) so that \( W_1 \approx g_1 \). Thus, (1.2.18) turns into
\[
K(W_1(t), f; A_0, A_{w_1,p_1}) \approx W_1(t)J(t, f),
\]
appealing to (1.2.23), we arrive at
\[
K(W_1(t), f; A_0, A_{w_1,p_1}) \approx W_1(t)J(t, f) + K(t, f),
\]
as required.
Now we take up the case when \( g_1 \gtrsim \sigma_1 \), so that \( W_1 \approx \sigma_1 \). With \( w = \sigma \), (1.2.16) turns into
\[
K(\sigma(t), f; A_0, A_{w_1,p_1}) \approx \sigma(t)J(t, f) + K(t, f),
\]
or,
\[
K(W_1(t), f; A_0, A_{w_1,p_1}) \approx W_1(t)J(t, f) + K(t, f).
\]
On the other hand, from the estimate (1.2.20) we can write
\[
\sigma(t)J(t, f) \lesssim \sigma(t)/g_1(t) \|f_0 \| A_0 + \sigma(t) \|f_1 \| A_{w_1,p_1} ,
\]
which, in view of the relation \( g_1 \gtrsim \sigma_1 \), becomes
\[
\sigma(t)J(t, f) \lesssim \|f_0 \| A_0 + \sigma(t) \|f_1 \| A_{w_1,p_1} ,
\]
implying
\[
\sigma(t)J(t, f) \lesssim K(\sigma(t), f; A_0, A_{w_1,p_1}),
\]
the above estimate and (1.2.16) together give
\[
K(t, f) + \sigma(t)J(t, f) \lesssim K(\sigma(t), f; A_0, A_{w_1,p_1}),
\]
where \( g_1 \gtrsim \sigma_1 \). In terms of \( W_1 \), it reads as
\[
K(t, f) + W_1(t)J(t, f) \lesssim K(W_1(t), f; A_0, A_{w_1,p_1}),
\]
The proof of (1.2.19) is complete, so is that of the theorem.
1.2.3 The case \( \{A_{w_0,p_0}, A_{w_1,p_1}\} \)

This time the results are best expressed in terms of the following functions, defined for all \( t > 0 \) by

\[
\rho_1(t) = \frac{\rho_0(t)}{t} \min\{g_1(t), \sigma_1(t)\}, \tag{1.2.26}
\]

\[
\rho_2(t) = \frac{h_0(t)}{t} \min\{g_1(t), \sigma_1(t)\}, \tag{1.2.27}
\]

\[
g(t) = \left( \int_t^\infty w_1^{p_1}(u) \min\left\{ \frac{u}{h_0(u)}, \frac{u}{\rho_0(u)} \right\} \frac{p_1}{u} \, du \right)^{-1/p_1}, \tag{1.2.28}
\]

and

\[
h(t) = \left( \int_0^t w_0^{p_0}(u) \min\{g_1(u), \sigma_1(u)\} \frac{p_0}{u} \, du \right)^{1/p_0}. \tag{1.2.29}
\]

When the weights \( w_0 \) and \( w_1 \) are of the form \( w_j(t) = t^{-\theta_j}b_j(t) \), where \( 0 \leq \theta_0 < \theta_1 \leq 1 \) and \( b_j \) is slowly varying \( (j = 0, 1) \), we define \( g \) and \( h \) in a simpler way, namely

\[
g(t) = \left( \int_t^\infty \rho^{-p_1}(u) \frac{du}{u} \right)^{-1/p_1}, \quad h(t) = \left( \int_0^t \rho^{p_0}(u) \frac{du}{u} \right)^{1/p_0}; \tag{1.2.30}
\]

where \( \rho = w_0/w_1 \). With this understanding, we have

\[
g \approx \rho \approx h \text{ if } w_j(t) = t^{-\theta_j}b_j(t) \text{ and } 0 \leq \theta_0 < \theta_1 \leq 1. \tag{1.2.31}
\]

Indeed, as

\[
g(t) = \left( \int_t^\infty u^{-p_1(\theta_1-\theta_0)} \left( \frac{b_1(u)}{b_0(u)} \right) \frac{p_1}{u} \, du \right)^{-1/p_1}
\]

\[
\approx t^{(\theta_1-\theta_0)} \left( \frac{b_1(t)}{b_0(t)} \right)^{-1} \quad \text{(by (1.1.10))}
\]

\[
\approx \rho(t),
\]

and

\[
h(t) = \left( \int_0^t u^{(\theta_1-\theta_0)p_1} \left( \frac{b_0(u)}{b_1(u)} \right) \frac{p_1}{u} \, du \right)^{1/p_1}
\]

\[
\approx t^{(\theta_1-\theta_0)} \left( \frac{b_0(t)}{b_1(t)} \right) \quad \text{(by (1.1.9))}
\]

\[
\approx \rho(t).
\]

In the next theorem, the weights \( w_0 \) and \( w_1 \) are such that the functions \( \rho_0, h_0, \sigma_1, g_1, g \) and \( h \) are finite on \( (0, \infty) \).
Theorem 1.2.3. For any weight $w$, all $f \in A_0 + A_1$ and all $t > 0$,

\[
K(w(t), f; A_{w_0,p_0}, A_{w_1,p_1}) \lesssim I(t, f) + w(t)J(t, f) + \frac{w(t)}{\sigma_1(t)}K(t, f) + \frac{\rho_0(t)}{t}K(t, f),
\]

(1.2.32)

\[
\frac{\rho_0(t)}{t}K(t, f) \lesssim K(\rho_1(t), f; A_{w_0,p_0}, A_{w_1,p_1}),
\]

(1.2.33)

\[
\frac{h_0(t)}{t}K(t, f) \lesssim K(\rho_2(t), f; A_{w_0,p_0}, A_{w_1,p_1}),
\]

(1.2.34)

\[
I(t, f) \lesssim K(h(t), f; A_{w_0,p_0}, A_{w_1,p_1}),
\]

(1.2.35)

and

\[
g(t)J(t, f) \lesssim K(g(t), f; A_{w_0,p_0}, A_{w_1,p_1}).
\]

(1.2.36)

If $p_0 = p_1 = p$ and $\rho = w_0/w_1$ is increasing, then

\[
I(t, f) + \rho(t)J(t, f) \lesssim K(\rho(t), f; A_{w_0,p}, A_{w_1,p}).
\]

(1.2.37)

Proof. Starting with an arbitrary decomposition $f = f_0 + f_1$ and using (1.1.4), we obtain

\[
\frac{\rho_0(t)}{t}K(t, f) \lesssim \left( \int_t^\infty w_0^{p_0}(u) \frac{du}{u} \right)^{1/p_0} K(t, f_0) + \frac{\rho_0(t)}{t}K(t, f_1),
\]

since $K(t, f)$ is increasing, we get

\[
\frac{\rho_0(t)}{t}K(t, f) \lesssim \left( \int_t^\infty w_0^{p_0}(u)K^{p_0}(u, f_0) \frac{du}{u} \right)^{1/p_0} + \frac{\rho_0(t)}{t}K(t, f_1),
\]

thus,

\[
\frac{\rho_0(t)}{t}K(t, f) \lesssim \|f_0 \mid A_{w_0,p_0}\| + \frac{\rho_0(t)}{t}K(t, f_1).
\]

(1.2.38)

We need to estimate $K(t, f_1)$ from above by $\|f_1 \mid A_{w_1,p_1}\|$. For this consider

\[
\|f_1 \mid A_{w_1,p_1}\| = \left( \int_0^\infty w_1^{p_1}(u)K^{p_1}(u, f_1) \frac{du}{u} \right)^{1/p_1}
\]

\[
\approx \left( \int_t^\infty w_1^{p_1}(u)K^{p_1}(u, f_1) \frac{du}{u} \right)^{1/p_1} + \left( \int_t^\infty w_1^{p_1}(u)K^{p_1}(u, f_1) \frac{du}{u} \right)^{1/p_1},
\]

as $K(t, f)$ is increasing and $t^{-1}K(t, f)$ is decreasing, we have
\[
\|f_1 \mid A_{w_1, p_1}\| \gtrsim t^{-1}K(t, f_1) \left( \int_0^t w_1^{p_1}(u) K_1(u, f) \frac{du}{u} \right)^{1/p_1} + K(t, f_1) \left( \int_t^\infty w_1^{p_1}(u) K_1(u, f) \frac{du}{u} \right)^{1/p_1}
\approx K(t, f_1) \left( \frac{1}{\sigma_1(t)} + \frac{1}{g_1(t)} \right),
\]
which implies that
\[
\min\{g_1(t), \sigma_1(t)\} \|f_1 \mid A_{w_1, p_1}\| \gtrsim K(t, f_1).
\tag{1.2.39}
\]
Inserting this estimate in (1.2.38), we get
\[
\frac{\rho_0(t)}{t} K(t, f) \leq \|f_0 \mid A_{w_0, p_0}\| + \frac{\rho_0(t)}{t} \min\{g_1(t), \sigma_1(t)\} \|f_1 \mid A_{w_1, p_1}\|,
\]
from which (1.2.33) follows.
Again using (1.1.4), we have
\[
\frac{h_0(t)}{t} K(t, f) \lesssim \frac{h_0(t)}{t} K(t, f_0) + \frac{h_0(t)}{t} K(t, f_1)
\lesssim \frac{1}{t} \left( \int_0^t w_0^{p_0}(u) w_0^{p_0} \frac{du}{u} \right)^{1/p_0} K(t, f_0) + \frac{h_0(t)}{t} K(t, f_1),
\]
as $\frac{K(t, f)}{t}$ is decreasing, we deduce that
\[
\frac{h_0(t)}{t} K(t, f) \lesssim \left( \int_0^t w_0^{p_0}(u) K_0(u, f) \frac{du}{u} \right)^{1/p_0} + \frac{h_0(t)}{t} K(t, f_1),
\]
so that
\[
\frac{h_0(t)}{t} K(t, f) \lesssim \|f_0 \mid A_{w_0, p_0}\| + \frac{h_0(t)}{t} K(t, f_1),
\]
by (1.2.39),
\[
\frac{h_0(t)}{t} K(t, f) \lesssim \|f_0 \mid A_{w_0, p_0}\| + \frac{h_0(t)}{t} \min\{g_1(t), \sigma_1(t)\} \|f_1 \mid A_{w_1, p_1}\|
\lesssim \|f_0 \mid A_{w_0, p_0}\| + \rho_2(t) \|f_1 \mid A_{w_1, p_1}\|,
\]
whence we get (1.2.34).
To drive (1.2.35), we start with (1.1.23) and immediately get
\[
I(t, f) \lesssim \|f_0 \mid A_{w_0, p_0}\| + \left( \int_0^t w_0^{p_0}(u) K_0(u, f_1) \frac{du}{u} \right)^{1/p_0},
\]
with the aid of the estimate (1.2.39), we write

\[ I(t, f) \lesssim \| f_0 \|_{A_{w_0,p_0}} + \left( \int_0^t w_0^{p_0}(u) \min \{g_1(u), \sigma_1(u)\} \frac{du}{u} \right)^{1/p_0} \| f_1 \|_{A_{w_1,p_1}} \]

\[ \lesssim \| f_0 \|_{A_{w_0,p_0}} + h(t) \| f_1 \|_{A_{w_1,p_1}}, \]

which implies that (1.2.35) is valid. If \( h \) is defined by (1.2.30) in the case \( w_j(t) = t^{-\theta_j} b_j(t) \), then we can derive (1.2.35) in the same way since \( g_1 \lesssim 1/w_1 \) and \( \sigma_1 \lesssim 1/w_1 \) by Lemma 1.1.4.

Analogously, starting with (1.1.24) we have

\[ J(t, f) \lesssim \left( \int_t^{\infty} w_1^{p_1}(u) K^{p_1}(u, f_0) \frac{du}{u} \right)^{1/p_1} + \| f_1 \|_{A_{w_1,p_1}}. \]  

Next we want to estimate \( K(u, f_0) \) from above by \( \| f_0 \|_{A_{w_0,p_0}} \). Now

\[ \| f_0 \|_{A_{w_0,p_0}} \approx \left( \int_0^t w_0^{p_0}(u) K^{p_0}(u, f_0) \frac{du}{u} \right)^{1/p_0} + \left( \int_0^{\infty} w_0^{p_0}(u) K^{p_0}(u, f_0) \frac{du}{u} \right)^{1/p_0}, \]

using once again that \( K(t, f) \) is increasing and \( t^{-1} K(t, f) \) is decreasing, we obtain

\[ \| f_0 \|_{A_{w_0,p_0}} \gtrsim \left( \frac{h_0(t)}{t} + \frac{\rho_0(t)}{t} \right) K(t, f_0) \approx \max \left\{ \frac{h_0(t)}{t}, \frac{\rho_0(t)}{t} \right\} K(t, f_0), \]

from which we have,

\[ \min \left\{ \frac{t}{h_0(t)}, \frac{t}{\rho_0(t)} \right\} \| f_0 \|_{A_{w_0,p_0}} \gtrsim K(t, f_0), \]

with this estimate at hand, (1.2.40) turns into

\[ J(t, f) \lesssim \left( \int_t^{\infty} w_1^{p_1}(u) \min \left\{ \frac{u}{h_0(u)}, \frac{u}{\rho_0(u)} \right\} \frac{du}{u} \right)^{1/p_1} \| f_0 \|_{A_{w_0,p_0}} + \| f_1 \|_{A_{w_1,p_1}} \]

\[ \lesssim \frac{1}{g(t)} \| f_0 \|_{A_{w_0,p_0}} + \| f_1 \|_{A_{w_1,p_1}}, \]

or,

\[ g(t) J(t, f) \lesssim \| f_0 \|_{A_{w_0,p_0}} + g(t) \| f_1 \|_{A_{w_1,p_1}}. \]
from which we get (1.2.36). The same proof is valid if \( g(t) \) is defined by (1.2.30) in the case \( w_j(t) = t^{-b_j} b_j(t) \) since then \( h_0(t) \gtrsim tw_0(t) \) and \( \rho_0 \gtrsim tw_0(t) \) by Lemma 1.1.4. With \( p_0 = p_1 = p \), (1.1.23) becomes
\[
I(t, f) \lesssim \left( \int_0^t w_0^p(u) K^p(u, f_0) \frac{du}{u} \right)^{1/p} + \left( \int_0^t w_0^p(u) K^p(u, f_1) \frac{du}{u} \right)^{1/p},
\]
as \( \rho = w_0/w_1 \) is increasing, thus
\[
I(t, f) \lesssim \left( \int_0^t w_0^p(u) K^p(u, f_0) \frac{du}{u} \right)^{1/p} + \rho(t) \left( \int_0^t w_1^p(u) K^p(u, f_1) \frac{du}{u} \right)^{1/p},
\]
which gives
\[
I(t, f) \lesssim \| f_0 \|_{A_{w_0,p_0}} + \rho(t) \| f_1 \|_{A_{w_1,p_1}}. \tag{1.2.41}
\]
By the same token, we derive
\[
\rho(t) J(t, f) \lesssim \| f_0 \|_{A_{w_0,p_0}} + \rho(t) \| f_1 \|_{A_{w_1,p_1}} \tag{1.2.42}
\]
from (1.1.24). Adding (1.2.41) and (1.2.42),
\[
I(t, f) + \rho(t) J(t, f) \lesssim \| f_0 \|_{A_{w_0,p_0}} + \rho(t) \| f_1 \|_{A_{w_1,p_1}},
\]
from which we infer (1.2.37).

Finally, to prove (1.2.32) we take a particular representation \( f = f_0 + f_1 \) satisfying (1.2.6). Altogether from (1.2.11) and (1.2.24), we write
\[
\| f_0 \|_{A_{w_0,p_0}} + w(t) \| f_1 \|_{A_{w_1,p_1}} \lesssim I(t, f) + w(t) J(t, f) + \frac{w(t)}{\sigma_1(t)} K(t, f) + \frac{\rho_0(t)}{t} K(t, f),
\]
which immediately gives (1.2.32).

\[\square\]

### 1.3 Auxiliary results and embeddings

We prove some useful technical results. To this end we introduce two conditions on a positive weight \( v \) defined on \((0, \infty)\). We say that \( v \) satisfies condition \( H \) if it has the following properties:

(i) \( v \) is strictly increasing, locally absolutely continuous;

(ii) \( v(0) := \lim_{t \to 0^+} v(t) = 0; \)
(iii) \( v(\infty) := \lim_{t \to \infty} v(t) = \infty. \)

The condition \( H_0 \) is said to be satisfied by \( v \) if for some \( \alpha > 0 \) and some slowly varying function \( v_0 \), it can be written as \( v(t) = t^\alpha v_0(t) \).

**Lemma 1.3.1.** Suppose that \( v \) satisfies condition \( H_0 \). Then there is a weight \( v_1 \) that satisfies \( H \) and is such that

\[
v_1 \approx v \quad \text{and} \quad \frac{dv_1}{dt} \approx \frac{v_1}{t}.
\]

**Proof.** By the definition of condition \( H_0 \), we can find a \( \alpha > 0 \) and a slowly varying function \( v_0 \) such that

\[
v(t) = t^\alpha v_0(t),
\]

choose \( \alpha_1 \) as \( 0 < \alpha_1 < \alpha \) so that

\[
v(t) = t^{\alpha-\alpha_1+\alpha_1} v_0(t) \\
\approx t^{\alpha-\alpha_1} d_0(t) \quad \text{(\( d_0 \) is increasing)} \\
\approx t^\beta t^\beta d_0(t) \quad \text{\((\beta = \frac{\alpha-\alpha_1}{2} > 0)\)} \\
\approx t^\beta c(t),
\]

where \( c \) is a strictly increasing function. Therefore, in order to prove that \( v_1 \approx v \), it will suffice to show that \( v_1 \approx t^\beta c(t) \). For this reason we can take \( v(t) = t^\beta c(t) \).

Setting

\[
v_1(t) = \frac{1}{t} \int_0^t v(u)du,
\]

we obtain

\[
v_1(t) = t^{-1} \int_0^t \frac{t^\beta c(u)}{u}du \\
= t^\beta \int_0^1 \frac{1}{u} c(ut)du, \quad \text{(by a change of variable)}
\]

denote \( \int_0^1 \frac{1}{u} c(ut)du \) by \( c_1(t) \) to get

\[
v_1(t) = t^\beta c_1(t).
\]
Next we verify that $v_1$ satisfies the condition $H$. Assume that $0 < t_1 < t_2$ and consider

$$v_1(t_1) = \frac{1}{t_1} \int_0^{t_1} v(u)du$$

$$= \int_0^1 v(t_1u)du \quad \text{(by a change of variable)}$$

$$< \int_0^1 v(t_2u)du \quad \text{($v$ is strictly increasing)}$$

$$= \frac{1}{t_2} \int_0^{t_2} v(u)du$$

$$= v_1(t_2),$$

showing that $v_1$ is strictly increasing.

Moreover,

$$\lim_{t \to 0^+} v_1(t) = \lim_{t \to 0^+} t^{-1} \int_0^t v(u)du$$

$$\leq \lim_{t \to 0^+} t^{-1} v(t) \int_0^t du$$

$$= \lim_{t \to 0^+} v(t)$$

$$= 0,$$

which implies that $\lim_{t \to 0^+} v_1(t) = 0$.

And

$$\lim_{t \to \infty} v_1(t) \geq \lim_{t \to \infty} t^{-1} \int_{t/2}^t v(u)du$$

$$\geq \lim_{t \to \infty} t^{-1} v(t/2) \int_{t/2}^t du$$

$$\geq \lim_{t \to \infty} v(t/2)$$

$$= \infty,$$
showing that \( \lim_{t \to \infty} v_1(t) = \infty \). Therefore, we conclude that \( v_1 \) satisfies \( H \).

Now we want to show that the estimate \( v_1 \approx v \) is valid. In this connection, the fact that \( v \) is strictly increasing gives us \( v_1 \lesssim v \). In order to obtain the reverse estimate we proceed as follows:

\[
v_1(t) = \frac{1}{t} \int_0^t v(u) \, du
\]

\[
= \frac{1}{t} \int_0^t u^\alpha v_0(u) \, du
\]

\[
= \frac{1}{t} \int_0^t u^{\alpha+\epsilon} u^{-\epsilon} v_0(u) \, du \quad (\epsilon > 0)
\]

\[
\gtrsim \frac{1}{t} t^{-\epsilon} v_0(t) \int_0^t u^{\alpha+\epsilon} \, du \quad (v_0 \text{ is slowly varying})
\]

\[
\approx \frac{1}{t} t^{-\epsilon} v_0(t) t^{\alpha+\epsilon+1
\]

\[
\approx v(t),
\]

as we wished to do. Hence we get \( v_1 \approx v \). It remains only to establish that \( \frac{dv_1}{dt} \approx \frac{v_1}{t} \).

First we start with \( (1.3.2) \) to get

\[
t v_1(t) = \int_0^t v(u) \, du,
\]

differentiating with respect to \( t \), we obtain

\[
v_1(t) + tv'_1(t) = v(t),
\]

thus,

\[
\frac{v'_1(t)}{v_1(t)} = t^{-1} \left( \frac{v(t)}{v_1(t)} - 1 \right). \quad (1.3.4)
\]

Altogether the estimates \( v_1(t) \leq v(t) \) and \( v_1(t) \gtrsim v(t) \) give

\[
\left( \frac{v(t)}{v_1(t)} - 1 \right) \lesssim 1,
\]
with this estimate, \[1.3.4\] turns into
\[
\frac{v'_1(t)}{v_1(t)} \lesssim \frac{1}{t},
\]
from which it follows
\[
\frac{dv_1(t)}{v_1(t)} \lesssim \frac{dt}{t}.
\] (1.3.5)

On the other hand, we differentiate \[1.3.3\] with respect to \( t \) to have
\[
v'_1(t) = \beta t^{\beta - 1} c_1(t) + t^\beta c'_1(t),
\]
from which we can write
\[
\frac{tv'_1(t)}{v_1(t)} = \beta + \frac{tc'_1(t)}{c_1(t)} > \beta \quad \left( \frac{tc'_1(t)}{c_1(t)} > 0 \right)
\]
implying that
\[
\frac{dv_1(t)}{v_1(t)} \gtrsim \frac{dt}{t}.
\]
The proof is complete.

\textbf{Corollary 1.3.2.} Let \( b \) be slowly varying on \((0, \infty)\). If \( 0 < p \leq q < \infty \), then
\[
A_{b_0,q} \hookrightarrow A_{g,p;b,q}^{-} \hookrightarrow A_{b_1,q},
\] (1.3.6)
and if \( 0 < q < p < \infty \), then
\[
A_{b_1,q} \hookrightarrow A_{g,p;b,q}^{-} \hookrightarrow A_{b_0,q},
\] (1.3.7)
where
\[
b_0(t) = g(t)(b(t))^{1-q/p} \left( \int_t^\infty b^q(u) \frac{du}{u} \right)^{1/p}
\] (1.3.8)
and
\[
b_1(t) = (g(t))^{p/q} q^{p/q-1} \left( \int_0^t u^p g^p(u) \frac{du}{u} \right)^{1/p-1/q} \left( \int_t^\infty b^q(u) \frac{du}{u} \right)^{1/q}.
\] (1.3.9)
Proof. Take \(s = q/p\), \(h(u) = K^p(u, f)\), \(\phi(u) = g^p(u)u^{-1}\) and \(w(t) = b^q(t)t^{-1}\) in Lemma 1.1.5. Thus, if \(0 < p \leq q < \infty\), then by (1.1.27)

\[
\int_0^\infty \left( \int_0^t g^p(u)K^p(u, f) \frac{du}{u} \right)^{q/p} b^q(t) \frac{dt}{t} \lesssim \int_0^\infty K^q(t, f)v(t)dt, \tag{1.3.10}
\]

where

\[
v(t) = \left( \frac{b^q(t)}{t} \right)^{1-\frac{q}{p}} \left( \frac{g^p(t)}{t} \int_t^\infty \frac{b^q(u)}{u} du \right)^{\frac{q}{p}} \]

\[
= \frac{1}{t} \left( b^{1-\frac{q}{p}}(t)g(t) \left( \int_t^\infty \frac{b^q(u)}{u} du \right)^{1/p} \right)^q \]

\[
= \frac{b_0^q(t)}{t},
\]

therefore (1.3.10) becomes

\[
\int_0^\infty \left( \int_0^t g^p(u)K^p(u, f) \frac{du}{u} \right)^{q/p} b^q(t) \frac{dt}{t} \lesssim \int_0^\infty K^q(t, f)b_0^q(t) \frac{dt}{t},
\]

which gives the left embedding of (1.3.6). Similarly, the right embedding of (1.3.7) follows from (1.1.26) if \(0 < q < p < \infty\).

In order to obtain the right embedding of (1.3.6) and the left embedding of (1.3.7), we apply Lemma 1.1.6 with \(s = q/p\), \(h(u) = K^p(u, f)u^{-p}\), \(\phi(u) = g^p(u, f)w^{p-1}\) and \(w(t) = \frac{b^q(t)}{t}\).

First of all we observe that the inequality

\[
\int_0^t \phi(u)h(u)du \geq h(t)\int_0^t \phi(u)du
\]

is satisfied as \(h(u) = K^p(u, f)u^{-p}\) is decreasing. So if \(0 < p \leq q < \infty\), then it follows from (1.1.29) that

\[
\int_0^\infty \left( \int_0^t g^p(u)w^p \frac{K^p(u, f)}{u^p} \frac{du}{u} \right)^{q/p} b^q(t) \frac{dt}{t} \gtrsim \int_0^\infty t^{-q}K^q(t, f)v_0(t)dt, \tag{1.3.11}
\]
where

\[ v_0(t) = t^{p-1} g^p(t) \int_0^\infty \left( \int_0^t g^p(u) u^{\varphi} \frac{du}{u} \right)^{q/p-q} \left( \int_0^{\infty} b^q(u) \frac{du}{u} \right)^{1/q} dt \]

\[ = t^{q-1} \left( g^{p/q}(t) t^{p/q-1} \left( \int_0^t u^p g^p(u) \frac{du}{u} \right)^{1/p-1/q} \left( \int_0^{\infty} b^q(u) \frac{du}{u} \right)^{1/q} \right)^{1/q} \]

\[ = t^{q-1} b_1^q(t), \]

inserting this in (1.3.11), we achieve

\[ \int_0^\infty \left( \int_0^t g^p(u) K^p(u, f) \frac{du}{u} \right)^{q/p} b^q(t) \frac{dt}{t} \gtrsim \int_0^\infty K^q(t, f) b_1^q(t) \frac{dt}{t}, \]

which yields the right embedding of (1.3.6). Similarly if \(0 < q < p < \infty\), then we can derive the left embedding of (1.3.7) from (1.1.30). Note that \(b_0(t) = b_1(t)\) if \(q = p\), therefore we have

\[ A_{g,q,b,q}^- = A_{B,q} \]

where

\[ B(t) = g(t) \left( \int_t^\infty b^q(u) \frac{du}{u} \right)^{1/q}. \]

\[ \square \]

**Corollary 1.3.3.** Let \(v\) satisfy \(H_0\), let \(b\) be slowly varying and suppose that \(h_0(t) \gtrsim tw_0(t)\). Then

\[
\left( \int_0^\infty b^q(v(t)) v^{-q}(t) \left( \int_0^t w_0^{p_0}(u) K^{p_0}(u, f) \frac{du}{u} \right)^{q/p_0} \frac{dt}{t} \right)^{1/q} \\
\approx \left( \int_0^\infty b^q(v(t)) v^{-q}(t) w_0^q(t) K^q(t, f) \frac{dt}{t} \right)^{1/q}.
\]  

(1.3.12)

Equivalently,

\[
A_{w_0,p_0,b(v)/v,q}^- = A_{w_0 b(v)/v,q}.
\]  

(1.3.13)

**Proof.** Inserting \(h_0(t) \gtrsim tw_0(t)\) in (1.2.10), we get

\[ w_0(t) K(t, f) \lesssim I(t, f), \]
from which we obtain
\[
\left( \int_0^\infty \left( \frac{b(v)}{v} w_0(t) K(t, f) \right)^q \frac{dt}{t} \right)^{1/q} \lesssim \left( \int_0^\infty \left( \frac{b(v(t))}{v(t)} \right)^q I_q(t, f) \frac{dt}{t} \right)^{1/q},
\]
which gives the estimate of the left-hand side of (1.3.12) from below. To derive the estimate from above, we distinguish several cases. If \(0 < p_0 \leq q < \infty\), then by (1.3.6)
\[
A_{B,q} \hookrightarrow A_{w_0,p_0; b(v)/v,q},
\]
where
\[
B = w_0(t) \left( \frac{b(v(t))}{v(t)} \right)^{1-q/p_0} \left( \int_t^\infty \left( \frac{b(v(u))}{v(u)} \right)^q \frac{du}{u} \right)^{1/p_0},
\]
We claim that
\[
\int_t^\infty \left( \frac{b(v(u))}{v(u)} \right)^q \frac{du}{u} \approx \left( \frac{b(v(t))}{v(t)} \right)^q.
\]
As \(v\) satisfies condition \(H_0\) so by Lemma 1.3.1 we can find a weight \(v_1\) satisfying \(H\) with \(v \approx v_1\) and \(\frac{dv_1(t)}{v_1(t)} \approx \frac{dt}{t}\). By Lemma 1.1.3 then it also follows that \(b(v) \approx b(v_1)\) since \(b\) is slowly varying. Taking into account all this, we can write
\[
\int_t^\infty \left( \frac{b(v(u))}{v(u)} \right)^q \frac{du}{u} \approx \int_t^\infty \left( \frac{b(v_1(u))}{v_1(u)} \right)^q \frac{dv_1(u)}{v_1(u)},
\]
substituting \(s = v_1(u)\) we get
\[
\int_t^\infty \left( \frac{b(v(u))}{v(u)} \right)^q \frac{du}{u} \approx \int_{v_1(t)}^\infty \left( \frac{b(s)}{s} \right)^q \frac{ds}{s},
\]
from which it follows (1.3.16) on the same lines of the proof of (1.1.10). Now inserting (1.3.16) in (1.3.15) we obtain
\[
B \approx w_0(t) \left( \frac{b(v(t))}{v(t)} \right)
\]
which turns the embedding (1.3.14) into
\[
A_{w_0, p_0; b(v)/v,q} \hookrightarrow A_{w_0, p_0; b(v)/v,q},
\]
as required.
If $0 < q < q_0 < \infty$, then by \[1.3.7\]
\[
A_{b_1,q} \hookrightarrow A_{w_0,p_0; b(v)/v,q}^{-},
\]
(1.3.18)
where
\[
b_1(t) = (w_0(t))^{p_0/q} t^{p_0/q-1} \left( \int_0^t w_0^{p_0}(u) \frac{du}{u} \right)^{1/p_0-1/q_0} \left( \int_t^\infty \left( \frac{b(v(u))}{v(u)} \right)^q \frac{du}{u} \right)^{1/q}
\]
\[
\approx (w_0(t))^{p_0/q} t^{p_0/q-1} (h_0(t))^{1-p_0/q} \frac{b(v(t))}{v(t)}
\]
\[
\lesssim w_0(t) \frac{b(v(t))}{v(t)},
\]
in view of the above estimate it follows from \[1.3.18\] that
\[
A_{w_0b(v)/v,q} \hookrightarrow A_{w_0,p_0; b(v)/v,q}^{-},
\]
as we wished to do.
Now we take up the case when $q = \infty$. We have
\[
\sup_t \frac{b(v(t))}{v(t)} \left( \int_0^t w_0^{p_0}(u) K^{p_0}(u,f) \frac{du}{u} \right)^{1/p_0}
\]
\[
\lesssim \sup_t \frac{b(v(t))}{v(t)} \left( \int_0^t \left( \frac{v(u)}{b(v(u))} \right)^{p_0} \frac{du}{u} \right)^{1/p_0} \sup_t \frac{b(v(t))}{v(t)} w_0(t) K(t,f)
\]
\[
\lesssim \sup_t \frac{b(v(t))}{v(t)} w_0(t) K(t,f),
\]
where the last estimate is due to the following equivalence
\[
\left( \int_0^t \left( \frac{v(u)}{b(v(u))} \right) \frac{du}{u} \right)^{1/p_0} \approx \frac{v(t)}{b(v(t))},
\]
which we can derive following the proof of \[1.3.16\]. Hence we have shown that
\[
A_{w_0b(v)/v,\infty} \hookrightarrow A_{w_0,p_0; b(v)/v,\infty}^{-},
\]
The remaining cases can be dealt with similar arguments.
Corollary 1.3.4. Let \( v \) satisfy the condition \( H_0 \), let \( b \) be slowly varying and suppose that \( g_1 \lesssim 1/w_1 \). Then

\[
\left( \int_0^\infty b^q(v(t))v^q(t) \left( \int_t^\infty w_1^{p_1}(u)K^{p_1}(u,f)\frac{du}{u} \right)^{q/p_1} \frac{dt}{t} \right)^{1/q} \\
\approx \left( \int_0^\infty b^q(v(t))v^q(t)w_1^q(t)K^q(t,f)\frac{dt}{t} \right)^{1/q}.
\]

(1.3.19)

Equivalently,

\[
A_{w_1,p_1;vb(v),q}^+ = A_{w_1vb(v),q}. \tag{1.3.20}
\]

Proof. Inserting the given relation \( g \lesssim 1/w_1 \) in (1.2.23), we get

\[
J(t,f) \gtrsim K(t,f)w_1(t),
\]

which yields

\[
\left( \int_0^\infty b^q(v(t))v^q(t)J^q(t,f)\frac{dt}{t} \right)^{1/q} \gtrsim \left( \int_0^\infty b^q(v(t))v^q(t)w_1^q(t)K^q(t,f)\frac{dt}{t} \right)^{1/q},
\]

which gives the estimate from below of the left-hand side of (1.3.19).

For the estimate from above, we give the argument when \( 0 < p < q < \infty \), the other cases being handled in a similar manner. We start off by using the embedding (1.3.6) for the couple \((A_1, A_0)\) with \( g \) replaced by \( t^{-1}w_1(1/t) \) and \( b \) by \( v(1/t)b(v(1/t)) \) so that we have

\[
\{A_1, A_0\}_{C,q} \hookrightarrow \{A_{t^{-1}w_1(1/t), p_1; v(1/t)b(v(1/t))}, q\}, \tag{1.3.21}
\]

where

\[
C(t) = t^{-1}w_1(1/t)(v(1/t)b(v(1/t)))^{1-q/p_1} \left( \int_t^\infty (v(1/t)b(v(1/t)))^{q/p_1} \frac{dt}{t} \right)^{1/p_1},
\]

now we can derive the estimate

\[
\left( \int_t^\infty (v(1/t)b(v(1/t)))^{q} \right)^{1/p_1} \approx (v(t^{-1})b(v(t^{-1})))^{q/p_1}
\]
in the same way as we obtained (1.3.16). Hence,
\[ C(t) \approx t^{-1}w_1(t^{-1}) \left( v(t^{-1})b(v(t^{-1})) \right)^{1-g/p_1} \left( v(t^{-1})b(v(t^{-1})) \right)^{q/p_1} \]
\[ \approx t^{-1}w_1(t^{-1})v(t^{-1})b(v(t^{-1))). \]
Thus,
\[ \| f | \{ A_1, A_0 \}_C,q \| = \left( \int_0^\infty C^q(t)K^q(t,f,A_1,A_0) \frac{dt}{t} \right)^{1/q}, \]
(by (1.1.3))
\[ = \left( \int_0^\infty C^q(t)t^qK^q(t^{-1},f,A_0,A_1) \frac{dt}{t} \right)^{1/q}, \]
\[ = \left( \int_0^\infty (w_1(t)v(t)b(v(t)))^q K^q(t,f,A_0,A_1) \frac{dt}{t} \right)^{1/q}, \]
\[ = \| f | \{ A_0, A_1 \}_{w,v(b(v)),q}, \]
showing that (1.3.21) becomes
\[ \{ A_0, A_1 \}_{w,v(b(v)),q} \leftrightarrow \{ A_1, A_0 \}_{t^{-1}w_1(1/t),p_1v(1/t)b(v(1/t)),q}. \]
Finally, the use of (1.1.3) immediately yields that
\[ \{ A_1, A_0 \}_{t^{-1}w_1(1/t),p_1v(1/t)b(v(1/t)),q} = \{ A_0, A_1 \}^{+}_{w_1,p_1v(b(v)),q}, \]
hence,
\[ A_{w_1v(b(v)),q} \leftrightarrow A^{+}_{w_1,p_1v(b(v)),q}, \]
as required.

1.4 Reiteration theorems

1.4.1 The case \( \{ A_{w_0,p_0}, A_1 \} \)

Throughout this subsection we shall need the functions \( c \) and \( d \) defined by
\[ c(t) = b(h_0(t)) \left( \frac{th_0'(t)}{h_0(t)} \right)^{1/q}, \]
\[ d(t) = b(W_0(t)) \left( \frac{tW_0'(t)}{W_0(t)} \right)^{1/q}; \]
we recall that \( W_0 := \max(\rho_0, h_0). \)
Theorem 1.4.1. Let $W_0$ satisfy the condition $H$ and suppose that $b$ is slowly varying. Then

$$\{A_{w_0,p_0}, A_1\}_{b,q} = A_{w_0,p_0:d,q} \cap A_{t^{-1}p_0:d,q}.$$  \hfill (1.4.2)

If $\rho_0 \lesssim h_0$ and $h_0$ satisfies $H$, then

$$\{A_{w_0,p_0}, A_1\}_{b,q} = A_{w_0,p_0:c,q}.$$  \hfill (1.4.3)

Proof. Let

$$C_1 = \int_0^\infty b^q(W_0(t)) I^q(t,f) \frac{W'_0(t)}{W_0(t)} dt,$$

and

$$C_2 = \int_0^\infty b^q(W_0(t)) \left( \frac{\rho_0(t)}{t} \right)^q K^q(t,f) \frac{W'_0(t)}{W_0(t)} dt.$$

Then by (1.2.5),

$$C_1 + C_2 \approx \int_0^\infty b^q(W_0(t)) K^q(W_0(t), f; A_{w_0,p_0}, A_1) \frac{W'_0(t)}{W_0(t)} dt,$$

since $W_0$ satisfies the condition $H$, we can make a change of variable $s = W_0(t)$ to obtain

$$C_1 + C_2 \approx \int_0^\infty b^q(s) K^q(s, f; A_{w_0,p_0}, A_1) \frac{ds}{s},$$

in other words,

$$C_1 + C_2 \approx \| f | \{A_{w_0,p_0}, A_1\}_{b,q} \|^q.$$  \hfill (1.4.4)

From the definitions we note that

$$\| f | A_{w_0,p_0:c,q} \|^q = \left( \int_0^\infty b^q(W_0(t)) I^q(t,f) \frac{W'_0(t)}{W_0(t)} dt \right)^{\frac{1}{q}} = C_1^{1/q},$$

and

$$\| f | A_{t^{-1}p_0:d,q} \|^q = \left( \int_0^\infty K^q(t,f) (t^{-1} \rho_0(t)d(t))^q dt \right)^{\frac{1}{q}} = C_2^{1/q}.$$
Therefore, it follows from (1.4.4) that

$$\| f | A_{W_0,p_0;c,q}^- \|_q^q + \| f | A_{t^{-1}p_0d,q}^- \|_q^q \approx \| f | \{ A_{W_0,p_0}, A_1 \}_b \|_q^q,$$

which yields (1.4.2) immediately. In the same way, we derive (1.4.3) from (1.2.4). □

**Theorem 1.4.2.** Let $W_0$ satisfy $H$ and suppose that $b$ is slowly varying. Then

$$\{ A_{W_0,p_0}, A_1 \}_{t^{-1}b,q} = A_{W_0,p_0;c/W_0,q}^- \cap A_{t^{-1}p_0d/W_0,q}^-.$$  \hspace{1cm} (1.4.5)

If $\rho_0 \lesssim h_0$ and $h_0$ satisfies $H$, then

$$\{ A_{W_0,p_0}, A_1 \}_{t^{-1}b,q} = A_{W_0,p_0;c/h_0,q}^-.$$  \hspace{1cm} (1.4.6)

Finally, if $\rho_0$ satisfies $H_0$ and $\rho_0 \gtrsim tw_0$, $h_0 \gtrsim tw_0$, then

$$\{ A_{W_0,p_0}, A_1 \}_{t^{-1}b,q} = A_{t^{-1}b(\rho_0),q}.$$  \hspace{1cm} (1.4.7)

**Proof.** Let

$$C_1 = \int_0^\infty \left( \frac{b(W_0(t))}{W_0(t)} I(t,f) \right)^q \frac{W_0'(t)}{W_0(t)} dt,$$

$$C_2 = \int_0^\infty \left( \frac{b(W_0(t))}{W_0(t)} \right)^q \left( \frac{\rho_0(t)}{t} K(t,f) \right)^q \frac{W_0'(t)}{W_0(t)} dt,$$

using (1.2.5) and then making a change of variable $s = W_0(t)$. We have

$$C_1 + C_2 \approx \int_0^\infty t^{-1}b(s)^q K^q(s,f; A_{W_0,p_0}, A_1) \frac{ds}{s},$$

or,

$$C_1 + C_2 \approx \| f | \{ A_{W_0,p_0}, A_1 \} \|_q^q.$$

We observe that

$$C_1 = \| f | A_{W_0,p_0;d/W_0,q}^- \|_q^q,$$

and

$$C_2 = \| f | A_{t^{-1}p_0d/W_0,q}^- \|_q^q.$$

Therefore,

$$\| f | A_{W_0,p_0;d/W_0,q}^- \|_q^q + \| f | A_{t^{-1}p_0d/W_0,q}^- \|_q^q \approx \| f | \{ A_{W_0,p_0}, A_1 \} \|_q^q,$$
proving (1.4.5). On the same lines we can prove (1.4.5) using (1.2.4). To drive (1.4.6) we start by taking \( w = \rho_0 \) in (1.2.1) to arrive at
\[
K(\rho_0(t); f; A_{w_0,p_0}, A_1) \lesssim I(t, f) + \rho_0(t)K(t, f)/t,
\]
from which we drive
\[
\| f \| \{ A_{w_0,p_0}, A_1 \}_{t^{-1}b,q} \lesssim \| f \| A_{w_0,p_0,b(\rho_0)/\rho_0,q} \| + \| f \| A_{t^{-1}b(\rho_0)q} \|. \]

as \( \rho_0 \) satisfies \( H \) and \( h_0(t) \gtrsim tw_0(t) \), we appeal to Corollary 1.3.3 to obtain
\[
\| f \| \{ A_{w_0,p_0}, A_1 \}_{t^{-1}b,q} \lesssim \| f \| A_{w_0,b(\rho_0)/\rho_0,q} \| + \| f \| A_{t^{-1}b(\rho_0)q} \|,
\]
then applying the relation \( \rho_0(t) \gtrsim tw_0 \) gives
\[
\| f \| \{ A_{w_0,p_0}, A_1 \}_{t^{-1}b,q} \lesssim \| f \| A_{t^{-1}b(\rho_0)q} \|.
\]
On the other hand, (1.2.2) yields
\[
\| f \| A_{t^{-1}b(\rho_0)q} \| \lesssim \| f \| \{ A_{w_0,p_0}, A_1 \}_{t^{-1}b,q} \|.
\]
Combining this with the previous estimate we obtain (1.4.6).

\[\square\]

1.4.2 The case \( \{ A_0, A_{w_1,p_1} \} \)

We shall need the functions \( c_1 \) and \( d_1 \) defined by
\[
c_1(t) = b(g_1(t)) \left( \frac{tg_1'(t)}{g_1(t)} \right)^{1/q} , \quad d_1(t) = b(W_1(t)) \left( \frac{tW_1'(t)}{W_1(t)} \right)^{1/q};
\]
recall that \( W_1 := \min(\sigma_1, g_1) \).

Theorem 1.4.3. Let \( W_1 \) satisfy the condition \( H \) and suppose that \( b \) is slowly varying. Then
\[
\{ A_0, A_{w_1,p_1} \}_{t^{-1}b,q} = A_{w_1,p_1,d_1,q}^+ \cap A_{d_1/W_1,q}. \tag{1.4.9}
\]

If \( g_1 \lesssim \sigma_1 \) and \( g_1 \) satisfies \( H \), then
\[
\{ A_0, A_{w_1,p_1} \}_{t^{-1}b,q} = A_{w_1,p_1,c_1,q}^+. \tag{1.4.10}
\]
Proof. This time the proof is based on Theorem 1.2.2 rather than Theorem 1.2.1 but otherwise it is quite similar to the previous proofs. As before we introduce various constants; let

$$C_1 = \int_0^\infty b^q(W_1(t))J^q(t,f)\frac{W'_1(t)}{W_1(t)}dt, \quad C_2 = \int_0^\infty \left(b^q(W_1(t))\right)^q K^q(t,f)\frac{W'_1(t)}{W_1(t)}dt.$$ 

We note that

$$C_1 = \|f|A_{w_1,p_1;d_1,q}^+\|^q,$$

and

$$C_2 = \|f|A_{d_1/W_1,q}\|^q.$$ 

Therefore, it follows from (1.2.19) that

$$\|f|A_{w_1,p_1;d_1,q}^+\|^q + \|f|A_{d_1/W_1,q}\|^q \approx \|f|\{A_0, A_{w_1,p_1}\}_{t^{-1}b,q}\|^q$$

implying (1.4.9). The remaining claim (1.4.10) follows, on the same lines, from (1.2.18).

**Theorem 1.4.4.** Let $W_1 := \min(\sigma_1, g_1)$ satisfy $H$ and suppose that $b$ is slowly varying. Then

$$\{A_0, A_{w_1,p_1}\}_{b,q} = A_{w_1,p_1;d_1,W_1,q} \cap A_{d_1,q}. \quad (1.4.11)$$

If $g_1 \preceq \sigma_1$ and $g_1$ satisfies $H$, then

$$\{A_0, A_{w_1,p_1}\}_{b,q} = A_{w_1,p_1;e_1,g_1,q}. \quad (1.4.12)$$

Finally, if $\sigma_1$ satisfies $H_0$ and $\sigma_1 \preceq 1/w_1$, $g_1 \preceq 1/w_1$, then

$$\{A_0, A_{w_1,p_1}\}_{b,q} = A_{b(\sigma_1),q}. \quad (1.4.13)$$

Proof. We introduce the quantities

$$C_1 = \int_0^\infty b^q(W_1(t))W_1^q(t)J^q(t,f)\frac{W'_1(t)}{W_1(t)}dt, \quad C_2 = \int_0^\infty b^q(W_1(t))K^q(t,f)\frac{W'_1(t)}{W_1(t)}dt.$$ 

By the equivalence (1.2.19), we have

$$C_1 + C_2 \approx \|f|\{A_0, A_{w_1,p_1}\}_{b,q}\|^q,$$
which proves (1.4.11) since
\[ C_1 = \| f \mid A_{w_1, p_1; d_1/q}^+ \|^q, \]
and
\[ C_2 = \| f \mid A_{d_1/W_1, q} \|^q. \]
Similarly, we derive (1.4.12) using (1.2.18). To prove (1.4.13), we notice that (1.2.15) with \( w = \sigma_1 \) and (1.2.16) imply that
\[ K(t, f) \lesssim K(\sigma_1, f; A_0, A_{w_1, p_1}) \lesssim \sigma_1 J(t, f) + K(t, f), \]
from which we have
\[ \| f \mid A_{b(\sigma_1), q} \| \lesssim \| f \mid \{ A_0, A_{w_1, p_1} \}_{b, q} \| \lesssim \| f \mid A_{b(\sigma_1), q} \| + \| f \mid A_{w_1, p_1; \sigma_1 b(\sigma_1), q} \|. \]
As \( \sigma_1 \) satisfies \( H_0 \) and \( g_1 \lesssim \frac{1}{w_1} \), we can invoke Corollary 1.3.4 to obtain
\[ \| f \mid A_{w_1, p_1; \sigma_1 b(\sigma_1), q}^+ \| \approx \| f \mid A_{w_1, \sigma_1 b(\sigma_1), q} \|, \]
the relation \( w_1 \sigma_1 \lesssim 1 \) leads us to
\[ \| f \mid A_{w_1, p_1; \sigma_1 b(\sigma_1), q}^+ \| \lesssim \| f \mid A_{b(\sigma_1), q} \|, \]
inserting this estimate in (1.4.14) yields
\[ \| f \mid A_{b(\sigma_1), q} \| \approx \| f \mid \{ A_0, A_{w_1, p_1} \}_{b, q} \|, \]
which shows that (1.4.13) is true.

1.4.3 The case \( \{ A_{w_0, p_0}, A_{w_1, p_1} \} \)

Theorem 1.4.5. Let \( \rho_2, g, h \) and \( \rho_1 \) satisfy \( H \) suppose that \( w \) is a weight that satisfies \( H \),
\[ b(w)(w'/w)^{1/q} \lesssim \min \{ b(\rho_1)(\rho'_1/\rho_1)^{1/q}, b(h)(h'/h)^{1/q} \}, \]
where \( b \) is slowly varying, and
\[ wb(w)(w'/w)^{1/q} \lesssim \min \{ \rho_2 b(\rho_2)(\rho'_2/\rho_2)^{1/q}, gb(g')(g'/g)^{1/q} \}. \]
Then

\[
\{A_{w_0,p_0}, A_{w_1,p_1}\}_{b,q} = A_{w_0,p_0; c_2,q}^{-} \cap A_{w_1,p_1; d_2,q}^{+} \cap A_{t^{-1}, c_3, p_0, q}^{+} \cap A_{t^{-1}, d_3, h_0, q}, \quad (1.4.17)
\]

where

\[
c_2(t) = b(h(t))(th'(t)/h(t))^{1/q}, \quad d_2(t) = g(t)b(g(t))(tg'(t)/g(t))^{1/q} \quad (1.4.18)
\]

and

\[
c_3(t) = b(\rho_1(t))(t\rho_1'(t)/\rho_1(t))^{1/q}, \quad d_3(t) = b(\rho_2(t))(t\rho_2'/\rho_2(t))^{1/q}. \quad (1.4.19)
\]

If, in addition, \(g\) satisfies \(H_0\) and \(g_1 \lesssim 1/w_1\), then

\[
\{A_{w_0,p_0}, A_{w_1,p_1}\}_{b,q} = A_{w_0,p_0; c_2,q}^{-} \cap A_{b(g), p_1, q}^{+} \cap A_{t^{-1}, c_3, p_0, q}^{+} \cap A_{t^{-1}, d_3, h_0, q}. \quad (1.4.20)
\]

Moreover, if \(\rho_0 \lesssim h_0, g_1 \lesssim \sigma_1, g_1 \lesssim 1/w_1, h_0 \geq tw_0\) and \(g, h\) satisfy \(H_0\) with \(g \lesssim \rho, b(g) \lesssim b(h)\), then

\[
\{A_{w_0,p_0}, A_{w_1,p_1}\}_{b,q} = A_{w_0,p_0; b(h), q}. \quad (1.4.21)
\]

In particular,

\[
\{A_{w_0,p_0}, A_{w_1,p_1}\}_{b,q} = \{A_{w_0,p_0}, A_{1}\}_{a,q}, \quad (1.4.22)
\]

where \(a\) is defined by

\[
a(h_0(t))(th_0'(t)/h_0(t))^{1/q} = b(g(t)), \quad (1.4.23)
\]

provided that \(h_0\) satisfies \(H\).

Proof. This time we let

\[
C_1 = \int_0^\infty (b(h(t))I(t, f))^q \frac{h'(t)}{h(t)} dt,
\]

\[
C_2 = \int_0^\infty (b(g(t))J(t, f))^q \frac{g'(t)}{g(t)} dt,
\]

\[
C_3 = \int_0^\infty \left( \frac{b(\rho_1(t))\rho_0(t)}{t} K(t, f) \right)^q \frac{\rho_1'(t)}{\rho_1(t)} dt,
\]

\[
C_4 = \int_0^\infty \left( \frac{b(\rho_2(t))h_0(t)}{t} K(t, f) \right)^q \frac{\rho_2'(t)}{\rho_2(t)} dt,
\]

\[
D_1 = \int_0^\infty b^q(w(t)) L^q(t, f) \frac{w'(t)}{w(t)} dt,
\]
$$D_2 = \int_0^\infty (b(w(t))w(t)J(t,f))^q \frac{w'(t)}{w(t)} dt,$$

$$D_3 = \int_0^\infty \left( \frac{b(w(t))\rho_0(t)}{t} K(t,f) \right)^q \frac{w'(t)}{w(t)} dt$$

and

$$D_4 = \int_0^\infty \left( \frac{b(w(t))w(t)}{\sigma_1(t)} K(t,f) \right)^q \frac{w'(t)}{w(t)} dt.$$ 

By (1.2.32),

$$\int_0^\infty b^q(w(t)) K^q(w(t),f;A_{w_0,p_0}, A_{w_1,p_1}) \frac{w'(t)}{w(t)} dt \lesssim \sum_{j=1}^4 D_j,$$

since $w$ satisfies $H$, we can make a change of variable $s = w(t)$ to get

$$\left\| f \mid \{A_{w_0,p_0}, A_{w_1,p_1}\}_{b,q} \right\|^q \lesssim \sum_{j=1}^4 D_j.$$ 

(1.4.24)

Using (1.2.33), (1.2.35), (1.2.36) and (1.2.37), we have

$$C_j \lesssim \left\| f \mid \{A_{w_0,p_0}, A_{w_1,p_1}\}_{b,q} \right\|^q, \quad j = 1, 2, 3, 4$$

which gives

$$\sum_{j=1}^4 C_j \lesssim \left\| f \mid \{A_{w_0,p_0}, A_{w_1,p_1}\}_{b,q} \right\|^q,$$

so that, in view of (1.4.24), we have

$$\sum_{j=1}^4 C_j \lesssim \left\| f \mid \{A_{w_0,p_0}, A_{w_1,p_1}\}_{b,q} \right\|^q \lesssim \sum_{j=1}^4 D_j.$$ 

(1.4.25)

Since $\rho_2(t) \lesssim t^{-1} h_0(t) \sigma_1(t)$ and (1.4.15) and (1.4.16) hold, it follows that $D_j \lesssim C_j$ ($j = 1, 2, 3, 4$), which implies that

$$\sum_{j=1}^4 C_j \approx \left\| f \mid \{A_{w_0,p_0}, A_{w_1,p_1}\}_{b,q} \right\|^q,$$

(1.4.26)

establishing (1.4.17). The relation (1.4.20) simply follows from Corollary 1.3.4 applied with $v = g$ as $g_1 \lesssim 1/w_1$ and $g$ satisfies $H_0$, and (1.3.20).
To prove (1.4.21), we first notice that (1.2.32), with \( w = h \) and the condition \( \rho_0 \lesssim h_0 \), imply that
\[
K(h(t), f; A_{w_0, p_0}, A_{w_1, p_1}) \lesssim I(t, f) + h(t)J(t, f) + \frac{h_0(t)}{t} K(t, f),
\]
altogether with (1.2.10) and (1.2.23), we have
\[
K(h(t), f; A_{w_0, p_0}, A_{w_1, p_1}) \lesssim I(t, f) + h(t)J(t, f),
\]
combining this with (1.2.35) gives
\[
I(t, f) \lesssim K(h, f; A_{w_0, p_0}, A_{w_1, p_1}) \lesssim I(t, f) + h(t)J(t, f).
\]
(1.4.27)

Similarly, (1.2.32) with \( w = g \), the condition \( g_1 \lesssim \sigma_1 \) together with (1.2.10), (1.2.23) and (1.2.36) gives
\[
g(t)J(t, f) \lesssim K(g, f; A_{w_0, p_0}, A_{w_1, p_1}) \lesssim I(t, f) + g(t)J(t, f).
\]
(1.4.28)

Altogether from (1.4.27), (1.4.28) and the fact \( b(g) \lesssim b(h) \), we have
\[
\{A_{w_0, p_0}, A_{w_1, p_1}\}_{b, q} = A_{w_0, p_0; b(h), q} \cap A_{w_1, p_1; b(g), q}.
\]
(1.4.29)

From (1.3.20) it follows that \( A_{gb(g)w_1, q} = A_{w_1, p_1; b(g), q} \), and since \( g \lesssim \rho = w_0/w_1 \),
\[
A_{w_0, p_0; b(h), q} \hookrightarrow A_{w_1, p_1; b(g), q}.
\]

On the other hand, since \( h_0(t) \gtrsim tw_0(t) \) and \( b(g) \lesssim b(h) \),
\[
A_{w_0, p_0; b(h), q} \hookrightarrow A_{h_0(t)b(h(t)), t, q} \hookrightarrow A_{w_0, p_0; b(h), q} \hookrightarrow A_{w_0, b(g), q}.
\]

These embeddings and (1.4.29) imply (1.4.21). Finally, (1.4.22) follows from (1.4.21) and (1.4.3).

**Theorem 1.4.6.** Let \( \rho_2, g, h \) and \( \rho_1 \) satisfy \( H \) and suppose that \( w \) is a weight that satisfies \( H \),
\[
b(w)(w'/w)^{1/q} \lesssim \min \{b(\rho_2)(\rho'_2/\rho_2)^{1/q}, b(g)(g'/g)^{1/q}\},
\]
(1.4.30)

where \( b \) is slowly varying, and
\[
b(w)(w'/w)^{1/q}/w \lesssim \min \{b(\rho_1)(\rho'_1/\rho_1)^{1/q}/\rho_1, b(h)(h'/h)^{1/q}/h\}.
\]
(1.4.31)
Then

\[ \{A_{w_0,p_0}, A_{w_1,p_1}\}_{t^{-1}b,q} = A_{w_0,p_0,c_4,q} \cap A_{w_1,p_1,c_4,q} \cap A_{t^{-1}c_5,p_0,q} \cap A_{t^{-1}d_5,h_0,q}, \]  

where

\[ c_4(t) = \frac{b(h(t))}{h(t)} \left( \frac{h'(t)}{h(t)} \right)^{1/q}, \quad d_4(t) = b(g(t)) \left( \frac{g'(t)}{g(t)} \right)^{1/q}, \]

\[ c_5(t) = \frac{b(h_1(t))}{h_1(t)} \left( \frac{h_1'(t)}{h_1(t)} \right)^{1/q}, \quad d_5(t) = b(g_1(t)) \left( \frac{g_1'(t)}{g_1(t)} \right)^{1/q}. \]

If, in addition, \( w \) satisfies \( H_0 \) and \( h_0 \geq tw_0 \), then

\[ \{A_{w_0,p_0}, A_{w_1,p_1}\}_{t^{-1}b,q} = A_{w_0,p_0,c_4,q} \cap A_{w_1,p_1,c_4,q} \cap A_{t^{-1}c_5,p_0,q} \cap A_{t^{-1}d_5,h_0,q}. \]

Moreover, if \( \rho_0 \lesssim h_0, g_1 \lesssim \sigma_1, h_0 \gtrsim 1/w_1, h_0 \gtrsim tw_0 \) and \( g, h \) satisfy \( H_0 \) with \( \rho \lesssim h, b(h) \lesssim b(g) \), then

\[ \{A_{w_0,p_0}, A_{w_1,p_1}\}_{t^{-1}b,q} = A_{w_1,p_1,b(h),q}. \]

In particular,

\[ \{A_{w_0,p_0}, A_{w_1,p_1}\}_{t^{-1}b,q} = \{A_0, A_{w_1,p_1}\}_{a,q}, \]

where \( a(t) \) is defined by

\[ a(g_1(t))(tg_1'(t)/g_1(t))^{1/q} = b(h(t)), \]

provided that \( g_1 \) satisfies \( H_0 \).

**Proof.** Put

\[ C_1 = \int_0^\infty \left( \frac{b(h(t))}{h(t)} I(t,f) \right)^q \frac{h'(t)}{h(t)} dt, \]

\[ C_2 = \int_0^\infty b^q(g(t)) J^q(t,f) \frac{g'(t)}{g(t)} dt, \]

\[ C_3 = \int_0^\infty \left( \frac{b(t)}{\rho(t)} \rho(t) K(t,f) \right)^q \frac{\rho'(t)}{\rho(t)} dt, \]

\[ C_4 = \int_0^\infty \left( \frac{b(t)}{\rho(t)} \rho(t) K(t,f) \right)^q \frac{\rho'(t)}{\rho(t)} dt, \]

\[ D_1 = \int_0^\infty \left( \frac{b(t)}{w(t)} I(t,f) \right)^q \frac{w'(t)}{w(t)} dt, \]

\[ D_2 = \int_0^\infty b^q(w(t)) J^q(t,f) \frac{w'(t)}{w(t)} dt, \]

\[ D_3 = \int_0^\infty \left( \frac{b(t)}{w(t)} \rho(t) K(t,f) \right)^q \frac{w(t)}{w(t)} dt. \]
and
\[ D_4 = \int_0^\infty \left( \frac{b(w(t)) w(t)}{w(t) \sigma_1(t)} K(t, f) \right)^q w'(t) dt. \]

By Theorem 1.2.3,
\[ \sum_{j=1}^{4} C_j \lesssim \left\| f \mid \{A_0, A_{w_1,p_1}\}_{t^{-1}b,q} \right\|^q \lesssim \sum_{j=1}^{4} D_j. \]

As in the proof of the last theorem, we see that \( D_j \lesssim C_j \) \((j = 1, 2, 3, 4)\) since the conditions (1.4.30) and (1.4.31) hold; thus (1.4.32) follows. The relation (1.4.34) follows from (1.4.32), (1.3.1), (1.3.13) since \( h_0 \gtrsim tw_0 \) and \( h \) satisfies \( H_0 \).

Now we prove (1.4.35). Using the inequalities (1.4.27) and (1.4.28) established in the proof of the last theorem we obtain, since \( b(h) \lesssim b(g) \),
\[ \{A_{w_0,p_0}, A_{w_1,p_1}\}_{t^{-1}b,q} = A_{w_0,p_0,b(h)/h,q}^- \cap A_{w_1,p_1,b(h),q}^+. \tag{1.4.37} \]

On the other hand, \( h_0 \gtrsim tw_0 \), (1.3.13) and \( w_0 \lesssim hw_1 \) give
\[ A_{w_0,p_0,b(h)/h,q}^- = A_{w_0b(h)/h,q} \hookrightarrow A_{w_1b(h),q}, \]
while \( g_1 \lesssim 1/w_1 \) and (1.2.23) produce
\[ A_{w_1,p_1,b(h),q}^+ \hookrightarrow A_{b(h)/g_1,q} \hookrightarrow A_{w_1b(h),q}. \]

These embeddings and (1.4.37) imply (1.4.35). Finally, (1.4.36) follows from (1.4.35) and (1.4.11).

Remark 1.4.1. When \( p_0 = p_1 = p \), the results of the last two theorems are valid with some simplifications due to (1.2.37): we can replace \( g \) and \( h \) by \( \rho \) in conditions (1.4.15), (1.4.16), (1.4.30) and (1.4.31). For example, if \( \rho_0 \lesssim h_0 \), \( g_1 \lesssim \sigma_1 \), \( g_1 \lesssim 1/w_1 \), \( h_0 \gtrsim tw_0 \) and \( \rho \) satisfies \( H_0 \), then (cf. (1.4.21))
\[ \{A_{w_0,p}, A_{w_1,p}\}_{b,q} = A_{w_0,p,b(\rho),q}^- \tag{1.4.38} \]
and
\[ \{A_{w_0,p}, A_{w_1,p}\}_{t^{-1}b,q} = A_{w_1,p,b(\rho),q}^+. \tag{1.4.39} \]
1.4.4 Examples

Here we test our main results for the weights \( w_j(t) = t^{-\theta_j}b_j(t) \) (\( 0 \leq \theta_0 \leq \theta_1 \leq 1 \)) in some limiting cases with \( \theta_0 = 0 \) or \( \theta_1 = 1 \), not considered in [23].

Example 1.4.1. (The case \( 0 = \theta_0 < \theta_1 < 1 \))

First we make several observations. By Lemma 1.1.4 we have \( h_0 \approx tw_0 \lesssim \rho_0 \).

In view of the same lemma and Remark 1.1.1 we get \( \sigma_1 \approx 1/w_1 \approx g_1 \). Therefore, \( \rho_1 \approx \rho_0/(tw_1) \gtrsim \rho \), and \( \rho_2 \approx h_0/(tw_1) \approx \rho \). Then it follows from (1.2.31) that \( \rho_2 \approx \rho \approx h \approx g \). Moreover we note that \( h_0, \rho_0, \sigma_1, g_1, \rho_1, \rho \) all satisfy \( H_0 \). Since \( h_0 \lesssim \rho_0 \), we have from (1.4.2),

\[
\{A_{w_0,p_0}, A_1\}_{b,q} = A^-_{w_0,p_0;b(\rho_0),q} \cap A_{t^{-1}b(\rho_0)p_0,q}. \tag{1.4.40}
\]

Indeed, as \( W_0 \approx \rho_0 \) and \( d \approx b(\rho_0) \) by Lemma 1.3.1

As \( \rho_0 \) satisfies \( H_0 \) and the relations \( \rho_0 \gtrsim tw_0 \), \( h_0(t) \gtrsim tw_0 \) hold, so from (1.4.7) it follows that

\[
\{A_{w_0,p_0}, A_1\}_{t^{-1}b,q} = A_{t^{-1}b(\rho_0),q}. \tag{1.4.41}
\]

If we choose \( w = \rho \) in Theorem 1.4.5 we obtain

\[
\{A_{w_0,p_0}, A_{w_1,p_1}\}_{b,q} = A^-_{w_0,p_0;b(\rho),q} \cap A_{b(\rho_1)p_0/t,q} \text{ if } b(\rho) \lesssim b(\rho_1), \tag{1.4.42}
\]

while the choice \( w = \rho_1 \) in Theorem 1.4.6 gives

\[
\{A_{w_0,p_0}, A_{w_1,p_1}\}_{t^{-1}b,q} = A^+_{w_1,p_1;b(\rho),q} \text{ if } b(\rho_1) \lesssim b(\rho). \tag{1.4.43}
\]

To verify (1.4.42) we apply Theorem 1.4.5 (1.4.20) with \( w = \rho \) to obtain

\[
\{A_{w_0,p_0}, A_{w_1,p_1}\}_{b,q} = A^-_{w_0,p_0;c_2,q} \cap A_{b(g)gw_1,q} \cap A_{t^{-1}c_3p_0,q} \cap A_{t^{-1}d_3h_0,q}. \tag{1.4.44}
\]

Using Lemma 1.3.1 and the relation \( g \approx \rho \), we reach at

\[
\{A_{w_0,p_0}, A_{w_1,p_1}\}_{b,q} = A^-_{w_0,p_0;b(\rho),q} \cap A_{b(\rho_1)p_0/t,q} \cap A_{b(\rho_2)w_0,q}, \tag{1.4.44}
\]

since \( A_{b(\rho_2)w_0,q} = A_{b(\rho)w_0,q} \) as \( \rho_2 \approx \rho \), therefore

\[
\{A_{w_0,p_0}, A_{w_1,p_1}\}_{b,q} = A^-_{w_0,p_0;b(\rho),q} \cap A_{b(\rho)w_0,q} \cap A_{b(\rho_1)p_0/t,q}. \tag{1.4.45}
\]

The relation \( b(\rho_1) \gtrsim b(\rho) \) yields

\[
A_{b(\rho_1)p_0/t,q} \leftrightarrow A_{w_0,b(\rho),q}.
\]
use of this embedding in (1.4.45) gives (1.4.42). Similarly, (1.4.41) follows from Theorem 1.4.6 (1.4.32) with \( w = \rho_1 \),

\[
\{A_{w_0,p_0}, A_{w_1,p_1}\}_{t^{-1}b,q} = A^+_{w_1,p_1;b(\rho),q} \cap A_b(\rho_1)_{1,t} \cap A_b(\rho_2)_{w_0,q}, \tag{1.4.46}
\]

and the relations

\[
A^+_{w_1,p_1;b(\rho),q} \hookrightarrow A_b(\rho_1)_{1,q} \quad A_b(\rho_1)_{1,t^{-1}q} \leftrightarrow A_b(\rho_1)_{1,q},
\]

since \( \rho_0(t)/(t\rho_1(t)) \approx w_1(t), b(\rho_1) \lesssim b(\rho) \) and

\[
A_b(\rho_1)_{1,q} = A_b(\rho_1)_{1,q},
\]

since \( \rho_2 \approx \rho \).

**Example 1.4.2.** (The case \( 0 < \theta_0 < \theta_1 = 1 \))

By Lemma 1.1.4 and Remark 1.1.1 we note that \( \rho_0 \approx tw_0 \approx h_0 \) and \( \sigma_1 \lesssim 1/w_1 \approx g_1 \). It follows that \( \rho_1 \approx \rho_2 \) and \( \rho_2 \lesssim \rho \). From (1.2.31) we have \( h \approx g \approx \rho \). In addition, \( h_0, \rho_0, \sigma_1, g_1, \rho_1, \rho \) all satisfy \( H_0 \). Since \( g_1 \gtrsim \sigma_1 \) we have from (1.4.11),

\[
\{A_0, A_{w_1,p_1}\}_{t^{-1}b,q} = A^+_{w_1,p_1;b(\sigma_1),q} \cap A_b(\sigma_1)_{1,q}, \tag{1.4.47}
\]

and from (1.4.16) there follows

\[
\{A_0, A_{w_1,p_1}\}_{b,q} = A_b(\sigma_1)_{1,q}. \tag{1.4.48}
\]

The choice \( w = \rho_2 \) in Theorem 1.4.5 gives

\[
\{A_{w_0,p_0}, A_{w_1,p_1}\}_{b,q} = A^-_{w_0,p_0;b(\rho),q} \quad \text{if} \quad b(\rho_2) \lesssim b(\rho), \tag{1.4.49}
\]

while taking \( w = \rho \) in Theorem 1.4.6 we obtain

\[
\{A_{w_0,p_0}, A_{w_1,p_1}\}_{t^{-1}b,q} = A^+_{w_1,p_1;b(\rho),q} \cap A_{w_0,b(\rho_2)/2,q} \quad \text{if} \quad b(\rho) \lesssim b(\rho_2). \tag{1.4.50}
\]

Verification of (1.4.49) follows from the application of Theorem 1.4.5 with \( w = \rho_2 \) and recovery of (1.4.41); it remains to use the relations

\[
A^-_{w_0,p_0;b(\rho),q} \hookrightarrow A_{w_0,b(\rho),q}
\]

and

\[
A_b(\rho_2)_{w,q} = A_b(\rho_2)_{w,0} \leftrightarrow A_{w_0,b(\rho_0),q}.
\]

In the same way, (1.4.50) follows from Theorem 1.4.6 with \( w = \rho \), so that we have (1.4.43), and the relations

\[
A^+_{w_1,p_1;b(\rho),q} \hookrightarrow A_b(\rho)_{w_1,q}
\]

and

\[
A_b(\rho_1)_{1,q} \leftrightarrow A_{w_0,b(\rho_2)/2,q} \leftrightarrow A_{w_0,b(\rho)/\rho,q} = A_{w_1,b(\rho),q}.
\]
Example 1.4.3. (The case $\theta_0 = 0, \theta_1 = 1$)

This time we have $h_0 \approx tw_0 \lesssim \rho_0, \sigma_1 \lesssim 1/w_1 \approx g_1, \rho_1(t) \approx \rho_0(t)\sigma_1(t)/t, \rho_2 \approx w_0\sigma_1 \lesssim \rho, h \approx g \approx \rho$ and $h_0, \rho_0, \sigma_1, g_1, \rho_1, \rho_2, \rho$ all satisfy $H_0$. The choice $w = \rho_2$ in Theorem 1.4.5 gives

$$\{A_{w_0,p_0}, A_{w_1,p_1}\}_b\{A_{w_0,p_0; b(p_0), q} \cap A_{t^{-1}b(p_1)p_0, q}\} (1.4.51)$$

if $b(\rho_2) \lesssim \min\{b(\rho), b(\rho_1)\}$, while if we take $w = \rho$ in Theorem 1.4.6 we obtain

$$\{A_{w_0,p_0}, A_{w_1,p_1}\}_t^{-1}\{A_{b(p_0)/p_0, t}\} \cap A_{(b(p_1)/p_1)p_0/t, q} \cap A_{(b(p_2)/p_2)w_0, q} \} (1.4.52)$$

if $b(\rho) \lesssim b(\rho_2)$ and $b(\rho)/\rho \lesssim b(\rho_1)/\rho_1$.

To verify (1.4.51), we apply Theorem 1.4.5 with $w = \rho_2$, from which (1.4.42) follows, and use the relations

$$A_{w_0,p_0; b(p_0), q} \hookrightarrow A_{w_0b(p_0), q} \hookrightarrow A_{w_0b(\rho_2), q}.$$

Similarly, (1.4.52) follows from Theorem 1.4.6 with $w = \rho$ from which (1.4.46) follows, together with the relations

$$A_{w_1,p_1; b(p_1), q} \hookrightarrow A_{b(p_1)w_1, q}.$$

Finally, we consider the example of a weight of mixed type as follows:

$$w_0(t) = \begin{cases} b_0(t), & 0 < t < 1, \\ t^{-1}b_1(t), & 1 < t < \infty, \end{cases}$$

where the $b_j$ are slowly varying. Then $h_0 \approx tw_0 \lesssim \rho_0$ and $\rho_0$ satisfies $H_0$. Hence (1.4.2) implies (1.4.40) and (1.4.7) gives (1.4.41).

1.5 An application

Here we use the previous results to obtain sharp mapping properties of Riesz potentials in a limiting case.

For $0 < \alpha < n$ let $R^\alpha$ be the Riesz potential in $\mathbb{R}^n$:

$$(R^\alpha h)(x) = c_{\alpha,n} \int_{\mathbb{R}^n} |x - y|^{\alpha-n} h(y)dy,$$

where $c_{\alpha,n}$ is a certain positive constant. For a slowly varying $b$, we define $L^r_b$, $1 \leq r, q \leq \infty$, to be the Lorentz-Karamata space with norm

$$\|f\|_{L^r_b} := \left(\int_0^\infty |t^{1/r} b(t) f^{**}(t)|^q \frac{dt}{t}\right)^{1/q}.$$
Here $f^{**}(t) = t^{-1} \int_0^t f^*(s) ds$, where $f^*$ is the nonincreasing rearrangement of $f$ (see \[8\], \[40\]).

**Theorem 1.5.1.** Let $l$ and $m$ be slowly varying and suppose that $1 < p < \infty$. Then

\[
R^{n/p} : L_l^{p,q} \to L_m^{\infty,q}
\]

if and only if

\[
\left( \int_0^t m^q(u) \frac{du}{u} \right)^{1/q} \left( \int_t^\infty \frac{u^{q/(1-q)}}{u} \frac{du}{u} \right)^{(q-1)/q} \lesssim 1
\]

for all $t > 0$.

**Proof.** We start with the well-known properties

\[
R^{n/p} : L_1 \to L_{p',\infty} \quad (1/p + 1/p' = 1)
\]

and

\[
R^{n/p} : L_{p,1} \to L_\infty.
\]

Here $L_{p',\infty}$ and $L_{p,1}$ are the familiar Lorentz spaces. Since

\[L_{p',\infty} = \{L_1, L_\infty\}_{t^{-1/p}, \infty}, L_{p,1} = \{L_1, L_\infty\}_{t^{-1/p'}, 1},\]

by interpolation we have

\[
R^{n/p} : \{L_1, \{L_1, L_\infty\}_{t^{-1/p'}, 1}\}_{t^{-1/m(p')}, q} \to \{\{L_1, L_\infty\}_{t^{-1/p}, \infty}, L_\infty\}_{t^{-1/m(p')}, q}.
\]

The interpolation space $\{L_1, \{L_1, L_\infty\}_{t^{-1/p'}, 1}\}_{t^{-1/m(p')}, q}$ is characterized by Theorem \[1.4.3\] as follows:

Take $A_0 = L_1$, $A_1 = L_\infty$, $w_1 = t^{-1/p'}$ and $p_1 = 1$. Then $g_1 = \frac{1}{p'} t^{1/p'}$ and $\sigma_1 \approx g_1$.

Clearly $g_1$ satisfies $H$, therefore by (1.4.10) we obtain

\[
\{L_1, \{L_1, L_\infty\}_{t^{-1/p'}, 1}\}_{t^{-1/m(p')}, q} = \{L_1, L_\infty\}_{t^{-1/p'}, 1}^{+} m(m(t), q).
\]

Similarly, by Theorem \[1.4.2\] we have

\[
\{\{L_1, L_\infty\}_{t^{-1/p}, \infty}, L_\infty\}_{t^{-1/m(p')}, q} = \{L_1, L_\infty\}_{t^{-1/m(t)}, q} = L_\infty^{q}.q
\]

Hence, \[
R^{n/p} : \{L_1, L_\infty\}_{t^{-1/p'}, 1}^{+} m(t), q} \to L_\infty^{q}.
\]
giving

\[ \int_0^\infty \left\{ m(t) (R^{n/p}h)^{(t)} \right\}^q \frac{dt}{t} \lesssim \int_0^\infty m^q(t) \left( \int_t^\infty u^{1/p}h^{(u)}(u) \frac{du}{u} \right)^q \frac{dt}{t}. \]

using Lemma 1.1.7

\[ \int_0^\infty \left\{ m(t) (R^{n/p}h)^{(t)} \right\}^q \frac{dt}{t} \lesssim \int_0^\infty \left\{ t^{1/p}l(t)h^{(t)} \right\}^q \frac{dt}{t}, \]

which yields (1.5.1). To prove that condition (1.5.2) is necessary, we choose a test function \( \mu \) in the form

\[ \mu(x) = \int_0^\infty u^{-n/p}g(u)\phi(x/u) \frac{du}{u}, \]

where \( g > 0 \) and \( \phi \) is a smooth function with compact support such that \( \psi(x) := R^{n/p}\phi(x) > 1 \) if \( |x| < c \). Then

\[ f(x) := R^{n/p}\mu(x) = \int_0^\infty g(u)\psi(x/u) \frac{du}{u}. \]

Let \( B_t \) be the ball in \( \mathbb{R}^n \) centred at the origin and with volume \( t \). Choosing the constant \( c \) appropriately, we obtain

\[ f(x) \geq \int_{t^{1/n}}^\infty g(u) \frac{du}{u}, \quad x \in B_t. \]

In particular,

\[ f^*(t) \geq \int_{t^{1/n}}^\infty g(u) \frac{du}{u}. \quad (1.5.3) \]

On the other hand,

\[ \mu^{**}(t) \lesssim \int_0^\infty u^{-n/p}g(u)\phi^{**}(tu^{-n}) \frac{du}{u}, \]

or,

\[ \mu^{**}(t) \lesssim \int_0^\infty (t/u)^{-1/p}g((t/u)^{1/n})\phi^{**}(u) \frac{du}{u}. \]

Since \( \phi \) has a compact support, we can suppose that \( \phi^* \lesssim \chi_{(0,1)} \), the characteristic function of \( (0,1) \), so that \( \phi^{**}(t) \lesssim \min(1,1/t) \). Hence,

\[ \mu^{**}(t) \lesssim \int_0^\infty (t/u)^{-1/p}g((t/u)^{1/n}) \min(1,1/u) \frac{du}{u}, \]
setting \( h(t) = t^{-1/p} g(t^{1/n}) \), we can write

\[
\mu^{**}(t) \lesssim \int_0^\infty h(t/u) \min(1, 1/u) \frac{du}{u},
\]

which gives after using Minkowski inequality for integrals

\[
\left( \int_0^\infty \left| t^{1/p} l(t) \mu^{**}(t) \right|^q \, dt \right)^{1/q} \lesssim \left( \int_0^\infty \left| t^{1/p} l(t) h(t/u) \right|^q \, dt \right)^{1/q},
\]

in other words,

\[
\| \mu \|_{L_{t}^{p,q}} \lesssim \left( \int_0^\infty \left| u^{1/p} l(t) g((t/u)^{1/n}) \right|^q \, dt \right)^{1/q},
\]

by changing variables from \((t/u)^{1/n}\) to \( t \) and then using the fact that \( l \) is slowly varying, we obtain

\[
\| \mu \|_{L_{t}^{p,q}} \lesssim \left( \int_0^\infty \frac{t^n}{l^n(t)} g^q(t) \frac{dt}{t} \right)^{1/q}.
\]

From this and (1.5.3), provided that the embedding (1.5.1) is valid, we have

\[
\int_0^\infty m^q(t) \left( \int_{t^{1/n}}^\infty g(u) \frac{du}{u} \right)^q \frac{dt}{t} \lesssim \int_0^\infty \frac{t^n}{l^n(t)} g^q(t) \frac{dt}{t}.
\]

Now (1.5.2) follows from Lemma 1.1.7.

Remark 1.5.1. As a consequence, using the relation between Riesz potentials and homogeneous Sobolev spaces (see [18]), we have the same results for the homogeneous Sobolev spaces. Namely,

\[
\frac{w^{n/p}}{l} \rightarrow L_{\infty}^{\infty,q} \quad \text{if and only if (1.5.2) is satisfied.}
\]

More general results will be proved in the next chapter.
Chapter 2

Optimal embeddings of generalized homogeneous Sobolev spaces

One of the classical Sobolev embedding states that

\[ W^k_p \hookrightarrow L^{r,p}, \quad \frac{1}{r} = \frac{1}{p} - \frac{k}{n}, \quad \frac{k}{n} < 1 < \frac{1}{p}. \]

The problem of optimal embeddings for inhomogeneous Sobolev spaces \( W^k E \), built-up over rearrangement invariant spaces \( E \) on a bounded domain in \( \mathbb{R}^n \), is treated in [19], [18], [29], [36], [33], [22]. A different method, based on the theory of capacities, is applied in [25], [35]. The case of homogeneous Sobolev spaces, \( w^k E \) is treated in [34] in the class of rearrangement invariant Banach function spaces \( E \) as in [7]. Our domain spaces are more general. In particular, we do not use the Fatou property and duality arguments. The construction of the optimal target space in the subcritical case is rather simple and gives an optimal couple (see Theorem 2.5.8 below). In the critical case we construct a large class of domain spaces for which the corresponding optimal target space is found. In [34] the optimal target set is not linear (see also Theorem 2.5.1 below).

The present chapter consists of five sections. In the first section we collect some fundamental definitions, notations and known results. Specifically, the notion of rearrangement invariant norm is introduced, and its properties in terms of Boyd indices are described. It is worthy of mention that the underlying axioms are different from those in [7]. In particular, we do not include Fatou property of Definition 1.1 of [7] (chapter 1), which makes our definition more general. The second section presents certain known and newly derived inequalities involving decreasing rearrangements. In the third section we define the generalized homogeneous Sobolev spaces and the classes
of norms where optimality is investigated. We introduce the notion of admissible couples and optimal target and domain norms. The characterization of admissible couples is given in the fourth section, and the constructions of optimal norms are carried out in the fifth section. We illustrate these constructions with examples, given at the end of fifth section.

2.1 Introduction

Let $\mathbb{R}$ be a Lebesgue Measure space, and let $\mathcal{M}$ be the collection of all extended real-valued measurable functions on $\mathbb{R}$ which are finite a.e.

Definition 2.1.1. The distribution function $\mu_f$ of $f$ in $\mathcal{M}$ is defined by

$$\mu_f(\lambda) = |\{x \in \mathbb{R} : |f(x)| > \lambda\}|, \quad \lambda \geq 0$$

where $|\cdot|$ denotes Lebesgue’s measure.

Definition 2.1.2. The decreasing rearrangement of $f$ in $\mathcal{M}$, denoted by $f^*$, is given by

$$f^*(t) = \inf \{\lambda > 0 : \mu_f(\lambda) \leq t\}, \quad t \geq 0.$$ 

In the next two lemmas we recall important facts about $f^*$, which we shall need in what follows.

Lemma 2.1.1. ([7]) Suppose $f$ and $g$ belong to $\mathcal{M}$, and let $a$ be any real number. The decreasing rearrangement is nonnegative, decreasing and right-continuous on $[0, \infty)$. Moreover,

\begin{align*}
|g| \leq |f| & \Rightarrow g^* \leq f^*, \quad (2.1.1) \\
(af)^* &= |a|f^*, \quad (2.1.2) \\
(f + g)^*(t_1 + t_2) &\leq f^*(t_1) + g^*(t_2), \quad (2.1.3) \\
(f^*)^* &= f^*, \quad (2.1.4)
\end{align*}

and

\begin{equation}
(|f|^p)^* = (f^*)^p, \quad 0 < p < \infty. \quad (2.1.5)
\end{equation}

Lemma 2.1.2. ([7]) Let $f$ be in $\mathcal{M}$, and $E$ be any subset of $\mathbb{R}$ of positive measure $t$, then

$$\int_E |f(x)|dx \leq \int_0^t f^*(s)ds.$$
**Definition 2.1.3.** Let f belong to \( \mathcal{M} \). Then \( f^{**} \) denotes the maximal function of \( f^* \), and is defined by

\[
f^{**}(t) := \frac{1}{t} \int_0^t f^*(s)ds.
\] (2.1.6)

We give some important properties of \( f^{**} \).

**Lemma 2.1.3.** ([7]) Suppose \( f \) and \( g \) belong to \( \mathcal{M} \) and \( a \) be any real number. Then \( f^{**} \) is nonnegative, decreasing and continuous on \((0, \infty)\). In addition it enjoys the following properties:

\[
\begin{align*}
\delta f^{**} = 0 & \iff f = 0 \text{ a.e.,} \quad (2.1.7) \\
f^* & \leq f^{**}, \\
(f^*)^{**} &= f^{**}, \\
|g| & \leq |f| \Rightarrow g^{**} \leq f^{**}, \\
(af)^{**} &= |a|f^{**}, \quad (2.1.10)
\end{align*}
\]

and

\[
(f + g)^{**}(t) \leq f^{**}(t) + g^{**}(t). \quad (2.1.12)
\]

**Definition 2.1.4.** The quantity \( f^{**} - f^* \) is usually termed as the measure of the oscillation of \( f^* \) and is denoted by \( \delta f^{**} \). In 1981 Bennet et al. [12] defined certain function spaces in terms of \( \delta f^{**} \), and since then it has been used as a useful replacement for \( f^* \) in certain contexts.

The results in the following lemma are elementary and well-known; their proofs can be seen, for example, in [28], [11] and [17].

**Lemma 2.1.4.** Let \( f \) be in \( \mathcal{M} \), then \( t\delta f^{**}(t) \) is an increasing function. Moreover,

\[
\delta f^{**}(t) \lesssim f^{**}(t) - f^{**}(2t), \quad (2.1.13)
\]

and

\[
f^{**}(t) = \int_t^\infty \delta f^{**}(u) \frac{du}{u}, \quad f^*(\infty) = 0. \quad (2.1.14)
\]

Let \( \mathcal{M}^+ \) be the space of all non-negative measurable functions \( g \) on \((0, \infty)\) with the Lebesgue measure that are in \( L^1 + L^\infty \) and \( g^*(\infty) = 0 \).

**Definition 2.1.5.** A functional \( \rho : \mathcal{M}^+ \rightarrow [0, \infty] \), is said to be a norm if the following axioms hold:
(A_1) \( \rho(f) \geq 0 \), with \( \rho(f) = 0 \Leftrightarrow f = 0 \text{ a.e.} \);

(A_2) \( \rho(cf) = c\rho(f) \), \( c \geq 0 \);

(A_3) \( \rho(f + g) \leq \rho(f) + \rho(g) \).

This norm is called rearrangement invariant norm if it satisfies

(A_4) \( \rho(f) = \rho(f^*) \).

We shall impose certain conditions on these norms.

**Definition 2.1.6.** A norm \( \rho \) is said to be monotone if for \( f, g \in M^+ \) with \( f \leq g \), we have \( \rho(f) \leq \rho(g) \).

**Definition 2.1.7.** We say that a norm \( \rho \) is \( K \)-monotone if \( f^{**} \leq g^{**} \) implies that \( \rho(f^*) \leq \rho(g^*) \).

**Remark 2.1.1.** It follows from (2.1.10) that every \( K \)-monotone norm is monotone as well.

**Definition 2.1.8.** We say that a norm \( \rho \) satisfies Minkovski inequality if

\[
\rho\left( \sum g_j \right) \leq \sum \rho(g_j), \quad g_j \in M^+.
\]

(2.1.15)

With every norm \( \rho \), there are associated two indices; they are called as lower and upper Boyd indices. To define them we need the following definition.

**Definition 2.1.9.** Let

\[
h(s) = \sup \left\{ \frac{\rho(g^*_s)}{\rho(g^*)} : g \in M^+ \right\}, \quad g_s(t) := g(t/s)
\]

then \( h \) is called the dilation function generated by \( \rho \).

**Definition 2.1.10.** Let \( \alpha \) and \( \beta \) denote the lower Boyd index and the upper Boyd index respectively, then

\[
\alpha := \sup_{0 < t < 1} \frac{\log h(t)}{\log t} \quad \text{and} \quad \beta := \inf_{1 < t < \infty} \frac{\log h(t)}{\log t}.
\]

We present some results concerning these indices.
Lemma 2.1.5. ([30]) Let $\rho$ be a $K$–monotone norm, then

$$0 \leq \alpha \leq \beta \leq 1.$$  \hspace{1cm} (2.1.16)

Lemma 2.1.6. ([7]) The condition $\beta < 1$ is equivalent to

$$\int_0^1 h(1/v) dv < \infty.$$ 

Lemma 2.1.7. ([7]) The condition $\alpha > a$ is equivalent to

$$\int_0^1 s^{-a}h(s)ds/s < \infty.$$ 

Lemma 2.1.8. Let $\rho$ be a monotone norm satisfying Minkovski inequality. If $\beta < 1$, then

$$\rho(f^*) \approx \rho(f^{**}).$$

Proof. The estimate $\rho(f^*) \lesssim \rho(f^{**})$ follows at once from (2.1.8) since $\rho$ is monotone. To prove the reverse estimate we make a start with a change of variable $s = vt$ in (2.1.6) so that we have

$$f^{**} = \int_0^1 f^*(vt) dv,$$

or,

$$f^{**}(t) = \sum_{k=-\infty}^{-1} \int_{2^k}^{2^{k+1}} f^*(vt) dv,$$

since $f^*$ is decreasing, so

$$f^{**}(t) \leq \sum_{k=-\infty}^{-1} f^*(2^k t) \int_{2^k}^{2^{k+1}} dv = \sum_{k=-\infty}^{-1} f^*(2^k t) 2^k,$$

at this point Minkovski inequality gives

$$\rho(f^{**}) \leq \sum_{k=-\infty}^{-1} \rho(f^*(2^k t)) 2^k,$$

we then observe that $\rho(f^*(\frac{vt}{2}))$, as a function of $t$, is decreasing, hence

$$\rho(f^{**}) \leq \sum_{k=-\infty}^{-1} \int_{2^k}^{2^{k+1}} \rho(f^*(\frac{vt}{2})) dv = \int_0^1 \rho(f^*(\frac{vt}{2})) dv.$$
now we note, by the definition of dilation function $h$, that $\rho(f^{*(vt)}) \lesssim h(2/v)\rho(f^*)$, therefore

$$\rho(f^{**}) \leq \rho(f^*) \int_0^1 h(2/v)dv \approx \rho(f^*) \int_0^{1/2} h(1/v)dv,$$

finally, the condition $\beta < 1$, by Lemma 2.1.6 yields the estimate

$$\rho(f^{**}) \lesssim \rho(f^*),$$

and the result follows.

To present more results, we need to give further definitions.

**Definition 2.1.11.** The usual Hardy operators defined on $\mathcal{M}^+$ are

$$P_g(t) := \frac{1}{t} \int_0^t g(s)ds,$$  \hspace{1cm} (2.1.17)

and

$$Q_g(t) := \int_t^\infty g(s)ds/s.$$  \hspace{1cm} (2.1.18)

The following is well-known.

**Lemma 2.1.9.** The operators $P$ and $Q$ commute, i.e.

$$PQ = QP.$$  \hspace{1cm} (2.1.19)

**Definition 2.1.12.** A special operator of weighted Hardy type is defined on $\mathcal{M}^+$ as

$$T_g(t) := \int_t^\infty s^{k/n}g(s)ds/s, \hspace{1cm} t > 0.$$  \hspace{1cm} (2.1.20)

We shall need the following two subclasses of $\mathcal{M}^+$ in the formulation of our main results.

**Definition 2.1.13.** The subclass $M$ consists of those elements $g$ of $\mathcal{M}^+$ for which there is a $m > 0$ such that $t^mg(t)$ is equivalent to an increasing function, and for which we have $T_g(t) < \infty$.

The subclass $M_0$ consists of those elements $g$ of $\mathcal{M}^+$ which are decreasing and satisfy $T_g(t) < \infty$.

An important property of the elements of the class $M$ is given in the next lemma.

**Lemma 2.1.10.** Let $g \in M$, then $tg(t) \to 0$ as $t \to 0$. 

Proof. If \( g \in M \), then \( g \in L^1 + L^\infty \) so that
\[
\int_0^1 g^*(s) ds < \infty,
\]
thus, by Lemma 2.1.2
\[
\int_0^1 g(s) ds < \infty,
\]
which implies that for every \( t \) between 0 and 1, we also have
\[
\int_{t/2}^t g(s) ds < \infty,
\]
from which it follows that \( \int_{t/2}^t g(s) ds \) goes to zero as \( t \to 0 \). As \( g \in M \), we can choose a \( m > 1 \) such that \( t^mg(t) \) is equivalent to an increasing function. Therefore,
\[
\int_{t/2}^t g(s) ds = \int_{t/2}^t s^m g(s) s^{-m} ds \geq t^m g(t) \int_{t/2}^t s^{-m} ds \approx t^m g(t) t^{-m+1} = tg(t),
\]
showing that \( tg(t) \to 0 \) as \( t \to 0 \).

Remark 2.1.2. Similar arguments indicate that the elements of \( M_0 \) enjoy the same property.

Lemma 2.1.11. Let \( \rho \) be a monotone norm satisfying Minkovski inequality. If \( a > 0, \ 0 \leq a < 1, \ g \in M, \) then
\[
\rho(t^{-a}Qg(t)) \lesssim \rho(t^{-a}g(t)).
\]
Proof.
\[
Qg(t) = \int_t^{\infty} g(s) \frac{ds}{s},
\]
by a simple change of variables,
\[
Qg(t) = \int_0^1 g \left( \frac{t}{v} \right) \frac{dv}{v},
\]
or,
\[
Qg(t) = \sum_{k=-\infty}^{-1} \int_{2^k}^{2^{k+1}} g\left(\frac{t}{v}\right) \frac{dv}{v},
\]
by definition, we can find a \(m > 0\) such that \(t^mg(t)\) is equivalent to an increasing, then it follows that \(t^{-m}g(1/t)\) will be equivalent to a decreasing. Hence
\[
Qg(t) \lesssim \sum_{k=-\infty}^{-1} 2^{-km} g\left(\frac{t}{2^k}\right) \int_{2^k}^{2^{k+1}} v^m \frac{dv}{v}
\]
\[
\approx \sum_{k=-\infty}^{-1} g\left(\frac{t}{2^k}\right),
\]
since \(\rho\) is monotone and satisfies Minkovski inequality, therefore
\[
\rho(t^{-a}Qg(t)) \lesssim \sum_{k=-\infty}^{-1} \rho\left(t^{-a}g\left(\frac{t}{2^k}\right)\right),
\]
now we observe that \(\rho\left(t^{-a}s^{-m}g(\frac{2t}{s})\right)\) is a decreasing function of \(s\), consequently it follows that
\[
\rho(t^{-a}Qg(t)) \lesssim \sum_{k=-\infty}^{-1} \int_{2^k}^{2^{k+1}} \rho\left(t^{-a}g\left(\frac{2t}{s}\right)\right) \frac{ds}{s}
\]
\[
\approx \int_0^1 \rho\left(t^{-a}g\left(\frac{2t}{s}\right)\right) \frac{ds}{s},
\]
we note, by definition, that \(\rho(t^{-a}g(\frac{2t}{s})) \lesssim \rho(t^{-a}g(t))s^{-a}h(s/2)\), thus we have
\[
\rho(t^{-a}Qg(t)) \lesssim \rho(t^{-a}g(t)) \int_0^1 s^{-a}h(s/2) \frac{ds}{s}
\]
\[
\approx \rho(t^{-a}g(t)) \int_0^{1/2} s^{-a}h(s) \frac{ds}{s},
\]
finally, Lemma 2.1.7 yields
\[
\rho(t^{-a}Qg(t)) \lesssim \rho(t^{-a}g(t)),
\]
as required.

Remark 2.1.3. Similar proof shows that Lemma 2.1.11 is valid if \(g\) is decreasing.
We conclude this section by giving some examples of rearrangement invariant norms on $\mathcal{M}^+$. In what follows $w$ will be positive function from $\mathcal{M}^+$.

**Example 2.1.1.** The Lebesgue norm $\rho_p$, $1 \leq p \leq \infty$, is given by

$$\rho_p(g) = \left( \int_0^\infty [g^*(t)]^p dt \right)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\rho_p(g) = \sup_{t>0} f^*(t) = f^*(0), \quad p = \infty.$$  

It is well-known that it is a $K$–monotone and satisfies Minkovski inequality.

**Example 2.1.2.** The classical Lorentz functionals $\rho_{w,q}$, $1 \leq q \leq \infty$, are defined as

$$\rho_{w,q}(g) := \left( \int_0^\infty [g^*(t)w(t)]^q dt/t \right)^{1/q}.$$  

In general, $\rho_{w,q}$ fails to satisfy axiom $(A_3)$, but if $1 \leq q < \infty$ and $w^q(t)/t$ is not increasing, then $\rho_{w,q}$ becomes a rearrangement norm satisfying Minkovski inequality and $K$–monotonicity condition; see [7]. However, the class of weights $w$ for which it becomes equivalent to rearrangement norms satisfying Minkovski inequality and $K$–monotonicity condition, is much wider. Consider the functionals $\sigma_{w,q}$, $1 \leq q \leq \infty$, defined by

$$\sigma_{w,q}(g) := \left( \int_0^\infty [g^{**}(t)w(t)]^q dt/t \right)^{1/q}.$$  

By (2.1.12), $\sigma_{w,q}$ verifies $(A_3)$. Moreover it satisfies Minkovski inequality and $K$–monotonicity condition. Now we have the the equivalence

$$\rho_{w,q} \approx \sigma_{w,q} \quad (2.1.21)$$

in the following cases.

If $1 < q < \infty$ then (2.1.21) is satisfied if and only if $w$ is such that (see [4]),

$$t^q \int_t^\infty s^{-q}[w(s)]^q ds/s \lesssim \int_0^t [w(s)]^q ds/s. \quad (2.1.22)$$

If $q = 1$ then (2.1.21) holds if and only if $w$ is such that (see [10]),

$$t^{-1} \int_0^t w(s) ds \lesssim v^{-1} \int_0^v w(s) ds, \quad 0 < v \leq t < \infty. \quad (2.1.23)$$

If $q = \infty$ then (2.1.21) is valid if and only if [15],
\[ \frac{1}{t} \int_0^t \frac{1}{w(s)} ds \lesssim \frac{1}{w(t)}, \text{ where } w(t) := \int_0^t v(s) ds \text{ for some } v. \quad (2.1.24) \]

Note also that if \( \sigma = \sigma_{w,q} \), then (2.1.21) is equivalent to \( \beta < 1 \) (see [7], p. 150).

### 2.2 Pointwise estimates involving the decreasing rearrangements

We need certain rearrangement inequalities while proving the main results. In this connection the following is crucial.

**Lemma 2.2.1.** ([17]) For \( k = 1 \) and \( k = 2 \)

\[ f^{**}(t) - f^{**}(2t) \lesssim t^{k/n} |D^k f|^{**}(t), \quad f \in C_0^\infty, \quad (2.2.1) \]

where \( C_0^\infty \) is the class of \( C^\infty \) functions in \( \mathbb{R}^n \) with compact support.

When \( n = 1, k = 1 \) the estimate (2.2.1) is equivalent to one given in [28], Lemma 5. For \( k = 1 \) it was proved in [5] using another method.

In view of (2.1.13), we have

**Corollary 2.2.2.** For \( k = 1 \) and \( k = 2 \)

\[ \delta f^{**}(t) \lesssim t^{k/n} |D^k f|^{**}(t), \quad f \in C_0^\infty. \quad (2.2.2) \]

**Lemma 2.2.3.** If \( f \in C_0^\infty \), then

\[ f^{**}(t) \lesssim \int_t^\infty u^{k/n} |D^k f|^{**}(u) \frac{du}{u}, \quad (2.2.3) \]

and

\[ \delta f^{**}(t) \lesssim t^{2/n} \int_t^\infty u^{(k-2)/n} |D^k f|^{**}(u) \frac{du}{u}. \quad (2.2.4) \]

**Proof.** Since \( u|D^k f|^{**}(u) \) is increasing, we readily obtain the estimate

\[ t^{2/n} \int_t^\infty u^{(k-2)/n} |D^k f|^{**}(u) \frac{du}{u} \lesssim t^{k/n} |D^k f|^{**}(t), \]
altogether from this estimate, (2.2.1) and (2.1.13), we get (2.2.4) for \( k = 1 \) and \( k = 2 \). We prove it by induction for \( k > 2 \). Suppose

\[
\delta f^{**}(t) \lesssim t^{2/n} \int_t^\infty u^{(m-2)/n} |D^m f|^{**}(u) \frac{du}{u},
\]

using (2.1.14) for \( |D^m f|^{**} \), we have

\[
\delta f^{**}(t) \lesssim t^{2/n} \int_t^\infty u^{(m-2)/n} \int_u^\infty \delta |D^m f|^{**}(s) \frac{ds}{s} \frac{du}{u},
\]

now making use of (2.2.1) for \( k = 1 \), we obtain

\[
\delta f^{**}(t) \lesssim t^{2/n} \int_t^\infty u^{(m-2)/n} \int_u^\infty s^{1/n} |D^{m+1} f|^{**}(s) \frac{ds}{s} \frac{du}{u},
\]

a simple application of Fubini theorem yields

\[
\delta f^{**}(t) \lesssim t^{2/n} \int_t^\infty u^{(m-1)/n} |D^{m+1} f|^{**}(u) \frac{du}{u},
\]

which is precisely (2.2.4) when \( k = m + 1 \), and the result follows. To derive (2.2.3), we insert the estimate (2.2.4) in (2.1.14) to obtain

\[
f^{**}(t) \lesssim \int_t^\infty s^{2/n} \int_s^\infty u^{(k-2)/n} |D^k f|^{**}(u) \frac{du}{u} \frac{ds}{s},
\]

now applying Fubini theorem, we get (2.2.3) immediately.

We shall need the following estimate which is proved in \([12]\).

**Lemma 2.2.4.** The following estimate holds:

\[
\int_0^t (s^{1-2/n} (-f^*)(s))' \frac{du}{u} \lesssim \int_0^t |D^2 f|^{**}(s) ds, \ n > 2.
\]  

(2.2.5)

We have an important integrated oscillation inequality.

**Lemma 2.2.5.** The estimate

\[
\int_0^t s^{-k/n} \delta f^{**}(s) ds \lesssim \int_0^t |D^k f|^{**}(s) ds, \ f \in C_0^\infty
\]  

(2.2.6)

is valid when \( k = 1 \) and when \( k = 2 \) (\( n > 2 \)).
Proof. It is proved in [34] for $k = 1$. To prove it for $k = 2$ with $n > 2$, consider the function
\[ g(t) = t^{1-2/n} \delta f(t). \]
First thing we note about it is that it can be rewritten as
\[ g(t) := t^{-2/n} \int_0^t u(-f^*(u))' du \]
since $\delta f(t) = \frac{1}{t} \int_0^t u(-f^*(u))' du$.
Secondly, we have $g(0) = 0$, which is a consequence of the estimate
\[ g(t) \lesssim t |D^2 f|^*(t), \]
which follows from (2.2.1). Therefore, we can write
\[ \int_0^t s^{-2/n} \delta f^*(s) ds = \int_0^t g(s) \frac{ds}{s}, \]
or,
\[ \int_0^t s^{-2/n} \delta f^*(s) ds = \int_0^t s^{-2/n} \left( \int_0^s u(-f^*(u))' du \right) \frac{ds}{s}, \]
on integrating by parts on right hand side,
\[ \int_0^t s^{-2/n} \delta f^*(s) ds = -\frac{n}{2} t^{-2/n} \int_0^t u(-f^*(u))' du + \frac{n}{2} \int_0^t s^{1-2/n} (-f^*(s))' ds, \]
which gives
\[ \int_0^t s^{-2/n} \delta f^*(s) ds \lesssim \frac{n}{2} \int_0^t (s^{1-2/n} (-f^*(s))')^*(u) du, \]
by (2.2.5),
\[ \int_0^t s^{-2/n} \delta f^*(s) ds \lesssim \int_0^t |D^2 f|^*(s) ds, \]
which is (2.2.6) when $k = 2$. The proof is complete.

2.3 The generalized homogeneous Sobolev spaces

Let $L_{loc}$ be the linear space of all locally integrable functions $f$ on $\mathbb{R}^n$, $n \geq 2$ with the Lebesgue measure, finite almost everywhere.
Definition 2.3.1. We define $E$ as the class of all functions $f$ in $L_{loc}^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ with $f^*(\infty) = 0$, and for which

$$\|f\|_E = \rho_E(f^*) < \infty$$

where $\rho_E$ is $K-$monotone rearrangement invariant norm on $\mathcal{M}^+$. 

Proposition 2.3.1. $E$ is a normed space under the norm $\|\cdot\|_E$.

Proof. We first show that $E$ is a linear space. Let $f,g \in E$ and $a$ any real number. In view of (2.1.8) and (2.1.12), we have

$$(f + g)^{**} \leq (f^* + g^*)^{**},$$

from which we obtain by $K-$monotonicity of $\rho_E$,

$$\rho_E((f + g)^*) \leq \rho_E(f^*) + \rho_E(g^*),$$

hence,

$$\|f + g\|_E \leq \|f\|_E + \|g\|_E, \tag{2.3.1}$$

it follows that $f + g \in E$.

Now by (2.1.2),

$$\rho_E((af)^*) = |a|\rho_E(f^*),$$

or,

$$\|af\|_E = |a|\|f\|_E, \tag{2.3.2}$$

thus $af \in E$ as well. Therefore, $E$ is linear subspace of $L_{loc}$, hence, itself a linear space. It remains to show that $\|\cdot\|_E$ is a norm on $E$. Let $f = 0$, then $f^* = 0$ which implies that $\rho_E(f^*) = 0$ as $\rho_E$ is a rearrangement invariant norm. Then, by definition, $\|f\|_E = 0$. Conversely, suppose that $\|f\|_E = 0$ for some $f \in E$. Then we have $\rho_E(f^*) = 0$, from which we get $f^* = 0$ as $\rho_E$ is a rearrangement invariant norm. But $f^* = 0$ implies that $f = 0$. Moreover, the functional $\|\cdot\|_E$ satisfies the triangle inequality (2.3.1) and (2.3.2), thus it is a well-defined norm on $E$. \qed

For example, if $E$ is a rearrangement invariant Banach function space as in [7], then by the Luxemburg representation theorem $\|f\|_E = \rho_E(f^*)$ for some $K-$monotone rearrangement invariant norm $\rho_E$ satisfying Minkovski inequality. More general example is given by the Riesz-Fischer monotone spaces as in [7], p. 305.

Definition 2.3.2. Let $1 \leq q \leq \infty$ and $w$ be a positive function from $\mathcal{M}^+$. The classical Lorentz space $\Lambda^q(w)$ consists of those elements from $L_{loc}$ for which $\|f\|_{\Lambda^q(w)} := \rho_{w,q}(f^*) < \infty$; where $\rho_{w,q}$ is defined in Example 2.1.2. By $\Lambda^q_0(w)$, we mean the the closure of $C^\infty_0$ in $\Lambda^q(w)$. 
**Definition 2.3.3.** Let \(1 \leq q \leq \infty\) and \(w\) be a positive function from \(\mathcal{M}^+\). The \(\Gamma\) space \(\Gamma^q(w)\) is the family of the elements of \(L_{\text{loc}}\) which satisfy

\[
\|f\|_{\Gamma^q(w)} := \sigma_{w,q}(f^*) < \infty,
\]

where \(\sigma_{w,q}\) is defined in Example 2.1.2.

The following estimates of the \(K\)–functionals for the couples \((L^1, \Lambda^1(t^{1/n}))\) and \((\Lambda^1(t^{1-k/n}), L^{\infty})\) are well-known; see for example [8].

**Lemma 2.3.2.** Let \(k < n\), then

\[
K(u^{1-k/n}, f; L^1, \Lambda^1(t^{1/n})) \approx \int_0^u f^*(s)ds + u^{1-k/n} \int_u^{\infty} s^{k/n} f^*(s) \frac{ds}{s}; \quad (2.3.3)
\]

\[
K(u^{1-k/n}, f; \Lambda^1(t^{1-k/n}), L^{\infty}) \approx \int_0^u s^{-k/n} f^*(s)ds. \quad (2.3.4)
\]

Before we proceed further we introduce two notations. Let \(|D^k f| := \sum_{|\alpha|=k} |D^\alpha f|\), where \(D^\alpha f\) denotes the generalized partial derivatives of \(f\).

Let \(M_k = \{f \in L^1 + L^{\infty} : |D^j f|^*(\infty) = 0, 0 \leq j \leq k, \rho_E(|D^k f^*|) < \infty\}\).

**Remark 2.3.1.** We can extend the validity of pointwise inequalities, presented in the previous section, from the functions of \(C_0^\infty\) to the functions of \(M_k\) using standard approximation arguments as in [31].

**Definition 2.3.4.** The generalized homogeneous Sobolev norm is the functional

\[
\|f\|_{w^k E} := \rho_E(|D^k f|^*),
\]

defined on \(M_k, k < n\).

The main goal of this chapter is to prove optimal embeddings of \(w^k E\) into rearrangement invariant function spaces \(G\) with a norm \(\|f\|_G \approx \rho_G(f^*)\), where \(\rho_G\) is a monotone norm. Observe that we have two well known limiting embeddings: \(w^k L^1 \hookrightarrow \Lambda^1(t^{1-k/n})\) and \(w^k \Lambda^1(t^{k/n}) \hookrightarrow L^{\infty}\). For this reason we shall suppose that the domain space \(E\) and the target space \(G\) satisfy \(E \hookrightarrow L^1 + \Lambda^1(t^{k/n})\) and \(G \hookrightarrow \Lambda^1(t^{1-k/n}) + L^{\infty}\). In particular, \(\alpha_E > 0\) and \(f \in w^k E\) implies \(\int_1^{\infty} u^{k/n} |D^k f|^*(u)du/u < \infty\). It is convenient to introduce the following classes of norms:

- \(N_{d,0}\) consists of all norms \(\rho_E\) that are \(K\)–monotone and rearrangement invariant and \(\beta_E < 1\);

- \(N_{d,1}\) consists of all norms \(\rho_E\) that are \(K\)–monotone, rearrangement invariant and satisfy Minkovski inequality and \(\alpha_E > (k-2)/n, \beta_E < 1\);
$N_{d,2}$ consists of all norms $\rho_E$ that are monotone, rearrangement invariant, satisfy Minkovski inequality and $\alpha_E > (k - 2)/n$, $\beta_E = 1$;

$N_d$ is used for a shorter notation for any of the above classes;

$N_t$ consists of all norms $\rho_G$ that are monotone;

$N_{t,1}$ consists of all norms $\rho_G$ that are monotone, satisfy Minkovski inequality and $\beta_G < 1 - k/n$;

**Definition 2.3.5.** (admissible couple) We say that the couple $\rho_E \in N_d$, $\rho_G \in N_t$ is admissible if the a priori estimate is valid:

$$ \rho_G(f^*) \lesssim \rho_E(|D^k f|^*), \ f \in M_k. \quad (2.3.5) $$

If $E$ and $G$ are Banach spaces, then (2.3.5) allows us to define the Sobolev space $w^k E$ as the closure of $C_0^\infty$; then we have the continuous embedding $w^k E \hookrightarrow G$. The couples $\rho_E$, $\rho_G$ or $E, G$ are called admissible if (2.3.5) is true. Moreover, $\rho_E$ (E) is called domain norm (domain space), and $\rho_G$ (G) is called target norm (target space). In this way we shall reserve the letter $E$ for the domain space and $\rho_E$ for the domain norm, while the letter $G$ is reserved for the target space and $\rho_G$ for the target norm.

Now we recall the definition of optimal norms (see for example [18]).

**Definition 2.3.6.** (optimal target norm) Given the domain norm $\rho_E \in N_d$, the optimal target norm, denoted by $\rho_E(G)$, is the strongest target norm in $N_t$, i.e.

$$ \rho_G(g^*) \lesssim \rho_E(|g|^*), \ g \in \mathcal{M}^+, \quad (2.3.6) $$

for any target norm $\rho_G \in N_t$ such that the couple $\rho_E, \rho_G$ is admissible.

**Definition 2.3.7.** (optimal domain norm) Given the target norm $\rho_G \in N_t$, the optimal domain norm, denoted by $\rho_G(E)$, is the weakest domain norm in $N_d$, i.e.

$$ \rho_E(g^*) \lesssim \rho_G(g^*), \ g \in \mathcal{M}^+, \quad (2.3.7) $$

for any domain norm $\rho_E \in N_d$ such that the couple $\rho_E, \rho_G$ is admissible.

**Definition 2.3.8.** (optimal couple) The admissible couple $\rho_E, \rho_G$ in $N$ is said to be optimal if $\rho_E = \rho_{E(G)}$ and $\rho_G = \rho_{G(E)}$.

The optimal norms are uniquely determined up to equivalence, while the corresponding optimal Banach spaces are unique.
2.4 Admissible couples

Here we give a characterization of all admissible couples $\rho_E \in N_d, \rho_G \in N_t$.

**Theorem 2.4.1.** (Case $\rho_E \in N_{d,0}, \rho_G \in N_t$)

The couple $\rho_E \in N_{d,0}, \rho_G \in N_t$ is admissible if and only if

$$\rho_G(Tg) \lesssim \rho_E(g), \ g \in M,$$

(2.4.1)

where $T$ is the integral operator defined in (2.1.20).

**Proof.** We suppose that (2.4.1) is valid, and prove (2.3.5). Our starting point is the estimate (2.2.3) which reads, in terms of the operator $T$, as

$$f^{**}(t) \lesssim T(|D^k f|^{**})(t), \ f \in C_0^\infty$$

in view of Remark 2.3.1

$$f^{**}(t) \lesssim T(|D^k f|^{**})(t), \ f \in M_k$$

by the inequality (2.1.8),

$$f^{*}(t) \lesssim T(|D^k f|^{**})(t),$$

now monotonicity of $\rho_G$ gives

$$\rho_G(f^{*}) \lesssim \rho_G(T(|D^k f|^{**})),$$

since $|D^k f|^{**} \in M$, by (2.4.1) we have

$$\rho_G(f^{*}) \lesssim \rho_E(|D^k f|^{**})$$

at this point we can apply Lemma 2.1.8 to get

$$\rho_G(f^{*}) \lesssim \rho_E(|D^k f|^{*}).$$

The proof of the sufficiency part is complete.

To give the proof of the necessity part, we define a test function in the form

$$f(x) = \int_0^\infty u^k g(u^n)\psi(|x/u|)\frac{du}{u},$$

(2.4.2)

where $g \in M$ and $\psi \in C_0^\infty$ is non-negative such that $\psi(|x|) = 1$ if $|x| \leq c$ where the constant is chosen so as the ball $B_t := \{x: |x| \leq ct^{1/n}\}$ has volume $t$.

We shall need the several properties of this function which we obtain in the following
steps.

**STEP1.**

First of all, we prove that the decreasing rearrangement of \( f \) satisfies the estimate

\[
f^*(t) \gtrsim Tg(t), \quad t > 0.
\]

(2.4.3)

To this end, let \( x \in B_t \) and \( u > t^{1/n} \), so that \( \psi(|x/u|) = 1 \) as \( |x/u| < c \). Therefore,

\[
f(x) \geq \int_{t^{1/n}}^\infty u^k g(u^n) \psi(|x/u|) \frac{du}{u},
\]

it follows that

\[
\mu_f \left( \int_{t^{1/n}}^\infty u^k g(u^n) \frac{du}{u} \right) \geq t,
\]

which yields

\[
f^*(t) \geq \int_{t^{1/n}}^\infty u^k g(u^n) \frac{du}{u},
\]

by a change of variable \( v = u^n \), we get at

\[
f^*(t) \gtrsim \int_t^\infty v^{k/n} g(v) \frac{dv}{v}.
\]

Expressing the above estimate in terms of the operator \( T \), we obtain (2.4.3).

**STEP2.**

Next we show that \( f \) is smooth away from zero, and its partial derivatives obey the estimate

\[
|d^\alpha f(x)| \lesssim |x|^{k-|\alpha|} g(c^{-n}|x|^n), \quad x \neq 0, 1 \leq |\alpha| \leq k.
\]

(2.4.4)

where \( d^\alpha f \) denotes the classical partial derivative of order \( |\alpha| \).

If \( x \neq 0 \), then there is a \( d > 0 \) such that \( \psi(|x/u|) = 0 \) whenever \( |x|/u \geq d \). Thus, for every \( u \) such that \( u \leq |x|/d \), we would have \( \psi(|x/u|) = 0 \). It follows that

\[
f(x) = \int_{|x|/d}^\infty u^k g(u^n) \psi(|x/u|) \frac{du}{u},
\]

since \( \psi \) is bounded on \((0, \infty)\), so

\[
f(x) \lesssim \int_{|x|/d}^\infty u^k g(u^n) \frac{du}{u}.
\]
again by changing the variable,

\[ f(x) \lesssim \int_{\frac{|x|}{d_n}}^{\infty} u^{k/n} g(u) \frac{du}{u}, \tag{2.4.5} \]

or,

\[ f(x) \lesssim T g((\frac{|x|}{d})^{n}), \]

now \( T g(t) < \infty \) for every \( t > 0 \), in particular, \( T g(\frac{|x|}{d_n}) \) is finite. Hence

\[ f(x) < \infty, \text{ for all } x \neq 0. \]

Now as \( \psi \) is smooth, so is \( f(x) \) whenever \( x \neq 0 \). That is \( \partial_{\alpha} f(x) \) of any order exits whenever \( x \neq 0 \). To prove \((2.4.4)\), first we prove, by induction, that \( \partial_{\alpha} \psi(|x/u|) \) can be written as a sum of the terms of the form

\[ u^{-|\alpha|} \left( \psi_j(|x/u|) \frac{x_{i_1} x_{i_2} \ldots x_{i_p}}{u^p} \right) \tag{2.4.6} \]

where \( \psi_j(s) = \frac{\psi^{(j)}(s)}{s}, 1 \leq j \leq |\alpha| \) and \( 0 \leq p \leq |\alpha| \).

Let \( |\alpha| = 1 \), then \( \partial_{\alpha} f(x) = \frac{\partial f}{\partial x_i}, 1 \leq i \leq n \). Thus,

\[ \partial_{\alpha} \psi(|x/u|) = \frac{\partial \psi(|x/u|)}{\partial x_i}, \]

\[ = \psi'(|x/u|) \frac{\partial(|x/u|)}{\partial x_i}, \]

\[ = \psi'(|x/u|) \frac{1}{u} \frac{x_i}{|x|}, \]

\[ = u^{-1} \psi_1(|x/u|) \frac{x_i}{u}, \]

Thus, our statement is true for \( |\alpha| = 1 \). Suppose it is true for all \( \beta \) such that \( |\beta| = k \), and let \( |\alpha| = k + 1 \). Then, by our supposition, \( \partial_{\alpha} \psi(|x/u|) \) can be written as a sum of the terms of the form

\[ \frac{\partial}{\partial x_{i_m}} \left( u^{-|\alpha|+1} \psi_j(|x/u|) \frac{x_{i_1} x_{i_2} \ldots x_{i_p}}{u^p} \right), \tag{2.4.7} \]

where \( 1 \leq j \leq |\alpha| - 1, 0 \leq p \leq |\alpha| - 1, \) and \( 1 \leq m \leq |\alpha| \). The expression \((2.4.7)\) turns out to be

\[ u^{-|\alpha|} \psi_{j+1}(|x/u|) \frac{x_{i_1} x_{i_2} \ldots x_{i_p} x_{i_m}}{u^{p+1}} + u^{-|\alpha|} \psi_j(|x/u|) \left( \sum_{k=1}^{p} \frac{x_{i_1} x_{i_2} \ldots x_{i_k} \delta_{km} x_{i_{k+1}} \ldots x_{i_p}}{u^{p-1}} \right), \]
changing \( j \) by \( j + 1 \) and \( p \) by \( p + 1 \), we see that the above expression is the sum of terms of the form in (2.4.6). Therefore it follows that \( d^\alpha \psi(|x/u|) \) can be written as a sum of the terms of the form in (2.4.6). Thus, our statement that \( d^\alpha \psi(|x/u|) \) for every \( \alpha \) can be expressed as a sum of the terms of the form in (2.4.6) is true.

Let \( 1 \leq |\alpha| \leq k \), and consider

\[
|d^\alpha f(x)| \leq \int_0^\infty u^k g(u^n) |d^\alpha \psi(|x/u|)| \frac{du}{u},
\]

\[
\leq \int_{|x|/d}^{|x|/c} u^{k-|\alpha|} g(u^n) \frac{|x_{i_1} x_{i_2} \ldots x_{i_p}|}{u^p} \frac{du}{u},
\]

\[
\approx \int_{|x|/d}^{|x|/c} u^{k-|\alpha|+m-m} g(u^n) \frac{|x_i x_{i_2} \ldots x_{i_p}|}{|x|^p} \frac{du}{u},
\]

\[
\approx |x|^{k-|\alpha|} g(c^{-m} |x|^n) \int_{|x|/d}^{|x|/c} u^{-m} \frac{du}{u}, \quad g \in M,
\]

\[
\approx |x|^{k-|\alpha|} g(c^{-m} |x|^n).
\]

Hence, (2.4.4) is valid.

**STEP3.**

In this step we prove that \( f \) is locally integrable. To show that \( f \) is locally integrable, it is sufficient, in view of the fact that \( f \) is smooth away from zero, to prove that the integral \( \int_{|x|\leq 1} f(x) dx \) is finite. Now

\[
\int_{|x|\leq 1} f(x) dx = \int_{|x|\leq 1} \left( \int_0^\infty u^k g(u^n) \psi(|x/u|) \frac{du}{u} \right) dx,
\]

using Fubini theorem,

\[
\int_{|x|\leq 1} f(x) dx = \int_0^\infty u^k g(u^n) \left( \int_{|x|\leq 1} \psi(|x/u|) dx \right) \frac{du}{u},
\]

or,

\[
\int_{|x|\leq 1} f(x) dx = \int_0^\infty u^k g(u^n) \left( \int_0^1 \psi(r/u) r^{n-1} dr \right) \frac{du}{u},
\]

by a change of variable \( r = us \), we get

\[
\int_{|x|\leq 1} f(x) dx = \int_0^\infty u^k g(u^n) u^n \left( \int_0^{1/u} \psi(s) s^{n-1} ds \right) \frac{du}{u},
\]
splitting the outer integral on right hand side,

\[
\int_{|x| \leq 1} f(x) dx = \int_0^1 u^k g(u^n) u^n \left( \int_{0}^{1/u} \psi(s) s^{n-1} ds \right) \frac{du}{u} \\
+ \int_1^\infty u^k g(u^n) u^n \left( \int_{0}^{1/u} \psi(s) s^{n-1} ds \right) \frac{du}{u},
\]

\[= I_1 + I_2.\]

So we have to establish the convergence of both the integrals \(I_1\) and \(I_2\). As \(\psi(s) = 0 \forall s \geq d\), therefore, we have,

\[I_1 \leq \int_0^1 u^k g(u^n) u^n \left( \int_0^d \psi(s) s^{n-1} ds \right) \frac{du}{u},\]

since \(\int_0^d \psi(s) s^{n-1} ds < \infty\), thus,

\[I_1 \lesssim \int_0^1 u^k g(u^n) u^n \frac{du}{u},\]

by changing variables,

\[I_1 \lesssim \int_0^1 u^{k/n} g(u) du,\]

the fact that \(u^{k/n}\) is increasing leads us to

\[I_1 \lesssim \int_0^1 g(u) du,\]

since \(g \in L^1 + L^\infty\) implies that \(\int_0^1 g^*(u) du < \infty\) whence, using (2.1.2), we have \(\int_0^1 g(u) du < \infty\). Thus \(I_1\) is convergent. Next we show the convergence of \(I_2\) as:

\[I_2 = \int_1^\infty u^k g(u^n) u^n \left( \int_{0}^{1/u} \psi(s) s^{n-1} ds \right) \frac{du}{u} \]

\[\lesssim \int_1^\infty u^k g(u^n) \frac{du}{u} \quad (\psi \text{ is bounded on } (0, \infty))\]

\[\approx \int_1^\infty u^{k/n} g(u) \frac{du}{u} \quad \text{(by change of variables)}\]

\[\approx Tg(1)\]

\[\approx 1. \quad (g \in M)\]
Hence, \( f \) is locally integrable.

**STEP 4.**

Now we exhibit that the classical derivatives \( d^\alpha f, 1 \leq |\alpha| \leq k \), can be identified as the generalized derivatives \( D^\alpha f \) of \( f \). In the first place, every \( d^\alpha f \) is locally integrable. Indeed, as

\[
\int_{|x| \leq 1} d^\alpha f(x) dx \lesssim \int_{|x| \leq 1} |x|^{k-|\alpha|} g(c^{-n}|x|^n) dx \quad \text{by (2.4.4)}
\]

\[
\approx \int_0^1 r^{k-|\alpha|} g(c^{-n}r^n)r^{n-1} dr
\]

\[
\approx \int_0^c s^{k-|\alpha|} g(s) ds \quad \text{(by change of variable)}
\]

\[
\approx \int_0^1 s^{k-|\alpha|} g(s) ds + \int_1^c s^{k/n} g(s) ds
\]

\[
\approx \int_0^1 g(s) ds + \int_1^c s^{k/n} g(s) ds
\]

\[
\approx \|g\|_{L^1} + Tg(1) \quad (g \in M)
\]

Let \( U \) denotes the unit ball with center at zero, it is sufficient to show that

\[
\int_U d^\alpha f(x) \phi(x) dx = (-1)^{|\alpha|} \int_U d^\alpha f(x) \phi(x) dx,
\]

for every \( \phi \in C_0^\infty \). For this we fix \( \epsilon > 0 \), and apply integration by parts formula for classical derivatives to obtain

\[
\int_{U-B(0,\epsilon)} f(x) d^\alpha \phi(x) dx = (-1)^{|\alpha|} \int_{U-B(0,\epsilon)} d^\alpha f(x) \phi(x) dx
\]

\[
+ (-1)^{|\alpha|+1} \int_{\partial B(0,\epsilon)} \sum_{0 \leq |\beta| \leq |\alpha|-1} d^\beta f . d^{\alpha-\beta} \phi dx.
\]

We will obtain (2.4.8) if we show that the second term on the right hand side of the above equation goes to zero as \( \epsilon \) goes to zero. Now we have \( |d^\alpha \phi| \lesssim 1 \) for every \( \alpha \), therefore

\[
\left| \int_{\partial B(0,\epsilon)} \sum_{0 \leq |\beta| \leq |\alpha|-1} d^\beta f . d^{\alpha-\beta} \phi dx \right| \lesssim \sum_{0 \leq |\beta| \leq |\alpha|-1} \int_{\partial B(0,\epsilon)} |d^\beta f| dx,
\]
by (2.4.4),

\[
\left| \int_{\partial B(0, \epsilon)} \sum_{0 \leq |\beta| \leq |\alpha|-1} d^\beta f. d^{\alpha-\beta} \phi dx \right| \lesssim \int_{\partial B(0, \epsilon)} |f(x)| dx + \sum_{1 \leq |\beta| \leq |\alpha|-1} \int_{\partial B(0, \epsilon)} |x|^{k-|\beta|} g(|x|^n) dx,
\]

thus,

\[
\left| \int_{\partial B(0, \epsilon)} \sum_{0 \leq |\beta| \leq |\alpha|-1} d^\beta f. d^{\alpha-\beta} \phi dx \right| \lesssim \int_{\partial B(0, \epsilon)} |f(x)| dx + \sum_{1 \leq |\beta| \leq |\alpha|-1} \epsilon^{n-1} \epsilon^{k-|\beta|} g(\gamma^n) dx.
\]

Denote \( \epsilon^n g(\gamma^n) \) by \( h(\epsilon) \), then, in view of lemma (2.1.10), \( h(\epsilon) \to 0 \) as \( \epsilon \to 0 \). Now each term in the above sum is of the form \( \epsilon^l h(\epsilon) \) where \( l \geq 0 \). Therefore, each term in the sum goes to zero as \( \epsilon \) approaches zero. Thus the second term of the above estimate goes to zero as \( \epsilon \) goes to zero. It remains to show that \( \int_{\partial B(0, \epsilon)} |f(x)| dx \to 0 \) as \( \epsilon \to 0 \). Now

\[
\int_{\partial B(0, \epsilon)} |f(x)| dx = \int_{|x|=\epsilon} \left( \int_0^\infty u^k g(u^n) \psi(|x/u|) \frac{du}{u} \right) dx,
\]

\[
= \int_{|x|=\epsilon} \left( \int_0^1 u^k g(u^n) \psi(|x/u|) \frac{du}{u} + \int_1^\infty u^k g(u^n) \psi(|x/u|) \frac{du}{u} \right) dx,
\]

\[
= I_1 + I_2,
\]

where

\[
I_1 = \int_{|x|=\epsilon} \left( \int_0^1 u^k g(u^n) \psi(|x/u|) \frac{du}{u} \right) dx,
\]

and

\[
I_2 = \int_{|x|=\epsilon} \left( \int_1^\infty u^k g(u^n) \psi(|x/u|) \frac{du}{u} \right) dx.
\]

As \( \psi \) is bounded on \((0, \infty)\),

\[
I_2 \lesssim \int_{|x|=\epsilon} \left( \int_1^\infty u^k g(u^n) \frac{du}{u} \right) dx,
\]

the fact \( Tg(1) \approx 1 \) gives

\[
I_2 \lesssim \int_{|x|=\epsilon} dx,
\]
hence,
\[ I_2 \lesssim \epsilon^{n-1}. \]

Therefore, \( I_2 \to 0 \) as \( \epsilon \to 0 \). Next we show that \( I_1 \to 0 \) as \( \epsilon \to 0 \). Now
\[
I_1 = \int_{|x|=\epsilon} \left( \int_{|x|=\epsilon} u^k g(u^n) \psi(\frac{|x/u|}{u}) \frac{du}{u} \right) dx,
\]
\[
\lesssim \int_{|x|=\epsilon} \left( \int_{|x|=\epsilon} u^k g(u^n) \frac{du}{u} \right) dx,
\]
\[
\lesssim \int_{|x|=\epsilon} \left( \int_{|x|=\epsilon} u^k g(u^n) \frac{du}{u} \right) dx,
\]
\[
\lesssim \int_{|x|=\epsilon} \left( \int_{|x|=\epsilon} u^- g(u^n) \frac{du}{u} \right) dx,
\]
\[
\lesssim \epsilon^{-1} \int_{|x|=\epsilon} u^{1/n} g(u) \frac{du}{u},
\]
To show that \( \epsilon^{n-1} \int_{\frac{1}{n}}^{1} u^{1/n} g(u) \frac{du}{u} \to 0 \) as \( \epsilon \to 0 \), it is sufficient to exhibit that \( \epsilon^{1-1/n} \int_{\epsilon}^{1} u^{1/n} g(u) \frac{du}{u} \to 0 \) as \( \epsilon \to 0 \). For this consider
\[
\int_{0}^{1} \epsilon^{1-1/n} \left( \int_{\epsilon}^{1} u^{1/n} g(u) \frac{du}{u} \right) \frac{de}{\epsilon} = \int_{0}^{\epsilon} u^{1/n} g(u) \left( \int_{0}^{u} \epsilon^{1-1/n} \frac{de}{\epsilon} \right) \frac{du}{u},
\]
\[
\approx \int_{0}^{\epsilon} g(u) du,
\]
\[
\approx 1.
\]
Thus, \( \lim_{s \to 0} \int_{0}^{s} \epsilon^{1-1/n} \left( \int_{\epsilon}^{1} u^{1/n} g(u) \frac{du}{u} \right) \frac{de}{\epsilon} = 0 \). But
\[
\int_{0}^{s} \epsilon^{1-1/n} \left( \int_{\epsilon}^{1} u^{1/n} g(u) \frac{du}{u} \right) \frac{de}{\epsilon} \geq \left( \int_{\epsilon}^{1} u^{1/n} g(u) \frac{du}{u} \right) \int_{0}^{s} \epsilon^{1/n} \frac{de}{\epsilon},
\]
\[
\approx s^{1-1/n} \int_{s}^{1} u^{1/n} g(u) \frac{du}{u},
\]
therefore, \( s^{1-1/n} \int_{s}^{1} u^{1/n} g(u) \frac{du}{u} \to 0 \) as \( s \to 0 \). Hence, (2.4.8) is proved.

**STEP5.**

By virtue of previous step, we can write (2.4.4) as
\[
|D^\alpha f(x)| \lesssim |x|^{k-|\alpha|} g(e^{-n}|x|^n), \quad x \neq 0, 1 \leq |\alpha| \leq k,
\]
in particular, 
\[|D^k f| \lesssim g(c^{-n}|x|^n), \]  
(2.4.9)
from which we obtain  
\[\rho_E(|D^k f|^* ) \lesssim \rho_E(g^*). \]  
(2.4.10)
In addition, in view of (2.4.9) and (2.4.5), we observe that \(|D^j f|^*(\infty) = 0, 0 \leq j \leq k\). Therefore, \(f \in M_k\). Hence by (2.3.5),  
\[\rho_G(f^*) \lesssim \rho_E(|D^k f|^*). \]  
(2.4.11)
From (2.4.3) we have  
\[\rho_G(Tg) \lesssim \rho_G(f^*). \]  
(2.4.12)
Altogether from (2.4.12), (2.4.11) and (2.4.10), we get  
\[\rho_G(Tg) \lesssim \rho_E(g), \]  
as required.

**Remark 2.4.1.** We do not use the conditions on \(\beta_E\) and \(\alpha_E\) while proving the necessity of (2.4.1). Also, we can replace \(M\) by \(M_0\).

**Theorem 2.4.2.** (Case \(\rho_E \in \mathbb{N}_{d,2}, \rho_G \in \mathbb{N}_t\))

The couple \(\rho_E \in \mathbb{N}_{d,2}, \rho_G \in \mathbb{N}_t\) is admissible if and only if the condition (2.4.1) is satisfied.

A similar result is proved in [34] under the condition \(k < 1 + n\alpha_E, n > 1\), if \(E\) is a rearrangement invariant Banach space as in [7].

**Proof.** In view of Remark 2.4.1 the proof of the necessity part in the present case is the same as in the previous one. To establish the proof of the sufficiency part, we first prove that the next estimate holds:  
\[\rho_E(t^{-k/n}\delta f^{**}(t)) \lesssim \rho_E(|D^k f|^*), \quad k \geq 1, \quad f \in M_k. \]  
(2.4.13)
To verify (2.4.13) for \(k = 1, 2\), we start by expressing the estimate in (2.2.6), in terms of operator \(P\), as  
\[P(s^{-k/n}\delta f^{**}(s)) \lesssim P(|D^k f|^*(s)), \]
where \(k = 1, 2\) and \(f \in M_K\) by Remark 2.3.1. Applying operator \(Q\) on both sides of the above estimate,  
\[QP(s^{-k/n}\delta f^{**}(s)) \lesssim QP(|D^k f|^*(s)), \]
the commutativity of the operators $P$ and $Q$ (see Lemma 2.1.9) gives

$$PQ(s^{-k/n}\delta f** (s)) \lesssim PQ(|D^k f|^*(s)),$$

or,

$$\int_0^t Q(s^{-k/n}\delta f** (s)) ds \lesssim \int_0^t Q(|D^k f|^*(s)) ds,$$

whence by $K$–monotonicity of $\rho_E$,

$$\rho_E(Q(t^{-k/n}\delta f** (t))) \lesssim \rho_E(|D^k f|^*).$$

Next we want to have an estimate of $\rho_E(Q(t^{-k/n}\delta f** (t)))$ from below. In this connection, the fact that $t\delta f** (t)$ is increasing gives us

$$Q(t^{-k/n}\delta f** (t)) \geq t\delta f** (t) \int_t^{\infty} s^{-k/n-1} ds/s,$$

which implies

$$Q(t^{-k/n}\delta f** (t)) \gtrsim s^{-k/n} \delta f** (t),$$

monotonicity of $\rho_E$ yields

$$\rho_E(Q(t^{-k/n}\delta f** (t))) \gtrsim \rho_E(t^{-k/n}\delta f** (t)),$$

the above estimate and the estimate (2.4.14) together leads us to

$$\rho_E(t^{-k/n}\delta f** (t)) \lesssim \rho_E(|D^k f|^*),$$

where $k = 1, 2$.

Suppose (2.4.13) is true for $k = j - 1$ where $j \geq 2$. Introduce the following notation

$$h_k(t) = t^{-k/n}\delta f** (t),$$

(2.4.15)

thus,

$$\rho_E(h_{j+1}) = \rho_E(t^{-(j+1)/n}\delta f** (t)),$$

or,

$$\rho_E(h_{j+1}) = \rho_E(t^{-(j-1)/n}\delta f** (t)),$$

using Corollary 2.2.2 for $k = 2$, we get at

$$\rho_E(h_{j+1}) \lesssim \rho_E(t^{-(j-1)/n}|D^2 f||** (t)),$$
in view of (2.1.14) and (2.1.18),
\[ \rho_E(h_{j+1}) \lesssim \rho_E(t^{-(j-1)/n}Q(\delta|D^2f|^*)(t)), \]
if \( k = j + 1 \), then \( \alpha_E > (j - 1)/n \), and \( \delta|D^2f|^* \in M \). Therefore, we can invoke Lemma 2.1.11 with \( \alpha = (j - 1)/n \) and \( g = \delta|D^2f|^* \) to get
\[ \rho_E(h_{j+1}) \lesssim \rho_E(t^{-(j-1)/n}\delta|D^2f|^*(t)), \]
since (2.4.13) is valid when \( k = j - 1 \), we have
\[ \rho_E(h_{j+1}) \lesssim \rho_E(|D^{j+1}f|), \quad \alpha_E > (j - 1)/n, \]
Thus, (2.4.13) is satisfied for \( k = j + 1 \) as well. Hence, by induction, (2.4.13) is valid for every \( k \geq 1 \). Now by (2.1.14) and (2.4.15),
\[ f^*(t) = Th_k(t), \quad f \in M_k \]
using the inequality (2.1.9),
\[ f^*(t) \leq Th_k(t), \]
as \( \rho_G \) is monotone, so
\[ \rho_G(f^*) \leq \rho_G(Th_k), \]
since (2.4.1) is assumed to be valid, we have
\[ \rho_G(f^*) \lesssim \rho_E(h_k), \]
or,
\[ \rho_G(f^*) \lesssim \rho_E(t^{-k/n}\delta f^*(t)), \]
finally (2.4.13) leads us to \( \rho_G(f^*) \lesssim \rho_E(|D^k f|^*) \), finishing the proof.

### 2.5 Optimal norms

Here we give a characterization of the optimal domain and optimal target norms. Before constructing an optimal target norm, it is convenient first to prove an embedding in a target set, that is rearrangement invariant but may not be a linear space [6, 31], [5]. We put for shortness \( N_{d,3} := N_{d,1} \cup N_{d,2} \) and let \( N_{t,3} \) be the subset of \( N_t \) of norms satisfying Minkovski inequality.
Definition 2.5.1. For a given domain norm \( \rho_E \in N_{d,3} \), we define the target set \( G_E \) as the family of all functions \( f \in M_k \) such that

\[
\rho_{G_E}(f^*) < \infty
\]

where \( \rho_{G_E} \) is the functional defined on \( M^+ \) as

\[
\rho_{G_E}(g) = \rho_E(t^{-k/n} \delta g^*(t)).
\] (2.5.1)

Theorem 2.5.1. If \( \rho_E \in N_{d,3} \), then

\[
\rho_{G_E}(f^*) \lesssim \rho_E(|D^k f|^*), \quad f \in M_k.
\] (2.5.2)

This embedding is proved in [34] under the condition \( k < 1 + n \alpha_E, n > 1 \), if \( E \) is a rearrangement invariant Banach function space as in [7].

Proof. If \( \rho_E \in N_{d,2} \) then (2.4.13) is valid which is precisely (2.5.2) in view of (2.5.1). Now consider the case when \( \rho_E \in N_{d,1} \). In view of Corollary 2.2.2 along with Remark 2.3.1, we have

\[
\rho_{G_E}(f^*) \lesssim \rho_E(|D^k f|^*), \quad k = 1, 2, \quad f \in M_k.
\]

As \( \rho_E \in N_{d,1} \), therefore \( \beta_E < 1 \). So we can apply Lemma 2.1.8 to arrive at (2.5.2) for \( k = 1 \) and \( k = 2 \). To prove it for \( k > 2 \), we start with (2.2.4) and apply Remark 2.3.1 to get

\[
\rho_E(t^{-k/n} \delta f^*(t)) \lesssim \rho_E(|D^k f|^*), \quad f \in M_k
\]

since \( \alpha_E > \frac{k-2}{n} \), Lemma 2.1.11 gives

\[
\rho_E(t^{-k/n} \delta f^*(t)) \lesssim \rho_E(|D^k f|^*), \quad f \in M_k,
\]

again applying Lemma 2.1.8 we get (2.5.2). \( \Box \)

Definition 2.5.2. The linear and rearrangement invariant hull of the functional \( \rho_{G_E} \) is defined as the functional \( \rho_l \) on \( M^+ \) as

\[
\rho_l(g) := \inf \sum \rho_{G_E}(g_j), \quad g_j \in M^+
\] (2.5.3)

where the infimum is taken over all finite sums \( \sum g_j^* \) satisfying \( g^{**} \leq \sum g_j^{**} \).
Proposition 2.5.2. The functional $\rho_l$ is a $K$–monotone rearrangement invariant norm on $\mathcal{M}^+$.

Proof. First we observe that the inequality

$$\rho_l(g) \leq \rho_{GE}(g), \quad g \in \mathcal{M}^+ \quad (2.5.4)$$

follows from the Definition 2.5.2 by taking a particular finite sum consisting of only one term $g^{**}$.

Suppose $g = 0$, then $\rho_{GE}(g) = 0$. Thus, in view of (2.5.4), $\rho_l(g) = 0$. Conversely, the fact that $\rho_l(g) = 0$, $g \in \mathcal{M}^+$ implies follows from (2.5.8) and (2.5.12) which are derived below. Hence, $\rho_l$ satisfy axiom $(A_1)$ in the Definition 2.1.5. To verify $(A_2)$, we take an arbitrary finite sum $\sum g_j^{**}$ such that

$$g^{**} \leq \sum g_j^{**},$$

multiplying both sides with $c \geq 0$ and making use of (2.1.11), we have

$$(cg)^{**} \leq \sum (cg_j)^{**},$$

thus,

$$\rho_l(cg) \leq \sum \rho_G(cg_j) = c \sum \rho_G(cg_j),$$

therefore,

$$\rho_l(cg) \leq c \rho_l(g).$$

To derive the reverse inequality, now we take an arbitrary finite sum $\sum h_k^{**}$ such that

$$cg^{**} \leq \sum h_k^{**},$$

then,

$$g^{**} \leq \sum \left(\frac{1}{c} h_k\right)^{**},$$

therefore,

$$\rho_l(g) \leq \sum \rho_{GE}\left(\frac{1}{c} h_k\right) = \frac{1}{c} \sum \rho_{GE}(h_k),$$

so that

$$c \rho_l(g) \leq \rho_l(cg).$$
Hence, $\rho_l(cg) = c\rho_l(g)$, showing that $(A_2)$ is verified. To check that $\rho_l$ obey $(A_3)$, we consider two arbitrary finite sums $\sum f_i^{**}$ and $\sum g_k^{**}$ such that the inequalities $f^{**} \leq \sum f_i^{**}$ and $g^{**} \leq \sum g_k^{**}$ are satisfied. In view of (2.1.12),

\[
(f + g)^{**} \leq f^{**} + g^{**} \leq \sum f_i^{**} + \sum g_k^{**},
\]

denoting $\sum f_i^{**} + \sum g_k^{**}$ by $\sum h_j^{**}$, we have

\[
(f + g)^{**} \leq \sum h_k^{**},
\]

therefore,

\[
\rho_l(f + g) \leq \sum \rho_{GE}(h_j),
\]
or,

\[
\rho_l(f + g) \leq \sum \rho_{GE}(f_i) + \rho_{GE}(g_k),
\]

hence,

\[
\rho_l(f + g) \leq \rho_l(f) + \rho_l(g).
\]

Thus, axiom $(A_3)$ is valid, and axiom $(A_4)$ is a straightforward consequence of (2.1.9). So, it is established that $\rho_l$ is a rearrangement norm.

To prove $K$–monotonicity of $\rho_l$, take $f, g \in M^+$ such that $f^{**} \leq g^{**}$. Again consider an arbitrary finite sum $\sum g_j^{**}$ such that $g^{**} \leq \sum g_j^{**}$. Then it follows that

\[
f^{**} \leq \sum g_j^{**},
\]

thus,

\[
\rho_l(f) \leq \sum \rho_{GE}(g_j),
\]

taking the infimum, we conclude that $\rho_l(f) \leq \rho_l(g)$.

\textbf{Proposition 2.5.3.} Let $\rho_E \in N_{d,3}$, then the couple $\rho_E$, $\rho_l$ is admissible. Moreover, $\rho_l$ is an optimal target norm in the class $N_{d,3}, N_t$.

\textit{Proof.} Take $f \in M_k$, then by (2.5.4),

\[
\rho_l(f^*) \leq \rho_{GE}(f^*),
\]

now (2.5.2) gives

\[
\rho_l(f^*) \leq \rho_E(|D_k^{} f^*|),
\]
hence the couple $\rho_E, \rho_l$ is admissible. Let $\rho_E \in N_{d,3}$, $\rho_G \in N_t$ be an arbitrary admissible couple, and write $g_k(t) = t^{-k/n} \delta f^{**}(t)$, then by (2.1.14) and (2.1.20),

$$\rho_G(f^{**}) = \rho_G(Tg_k),$$

since $\rho_E, \rho_G$ is admissible, therefore, in view of Theorem 2.4.1 and Theorem 2.4.2 we come to the estimate

$$\rho_G(f^{**}) \lesssim \rho_E(g_k),$$

thus

$$\rho_G(f^{**}) \lesssim \rho_{G_E}(f^*). \quad (2.5.5)$$

Take an arbitrary finite sum $\sum f_j^{**}$ such that $f^{**} \leq \sum f_j^{**}$, then

$$\rho_G(f^{**}) \lesssim \rho_G(\sum f_j^{**}),$$

or,

$$\rho_G(f^{**}) \lesssim \sum \rho_G(f_j^{**}),$$

applying (2.5.5),

$$\rho_G(f^{**}) \lesssim \sum \rho_{G_E}(f_j^*),$$

taking infimum with respect to $\sum f_j^{**}$,

$$\rho_G(f^{**}) \lesssim \rho_l(f^*),$$

finally by (2.1.8),

$$\rho_G(f^*) \lesssim \rho_l(f^*),$$

showing the optimality of $\rho_l$. □

**Definition 2.5.3.** For a given domain norm $\rho_E \in N_{d,3}$, we define

$$\rho(g) := \inf\{\rho_E(h) : g \leq Th, \ h \in M\}. \quad (2.5.6)$$

**Proposition 2.5.4.** If $\rho_E \in N_{d,3}$, then $\rho$ is a norm on $M^+$. Moreover it is equivalent to the norm $\rho_l$ in the sense that $\rho(g^*) \approx \rho_l(g^*)$. Thus $\rho$ is an optimal target norm, i.e. $\rho \approx \rho_{G(E)}$. In addition, the following embedding holds:

$$G(E) \hookrightarrow \Lambda^1(t^{1-k/n}) + L^\infty. \quad (2.5.7)$$
Proof. To see that $\rho$ is a norm, we first prove

$$\int_0^1 t^{-k/n} g^*(t) dt \lesssim \rho(g^*), \quad g \in \mathcal{M}^+. \tag{2.5.8}$$

Take $g \in \mathcal{M}^+$ and consider an arbitrary $h \in M$ such that $g^* \leq Th$. Then,

$$g^*(t) \leq \int_0^\infty s^{k/n} h(s) ds,$$

which gives

$$\int_0^1 t^{-k/n} g^*(t) dt \leq \int_0^1 t^{-k/n} \left( \int_t^\infty s^{k/n} h(s) ds \right) dt, \tag{2.5.9}$$

now by Fubini theorem, we have

$$\int_0^1 t^{-k/n} \left( \int_t^\infty s^{k/n} h(s) ds \right) dt = \int_0^1 \left( \int_0^s \int_0^{t^{-k/n}} s^{k/n} h(s) ds \right) dt \approx \int_0^1 h(t) dt + \int_1^\infty \frac{t^{k/n} h(t)}{t} dt,$$

inserting above estimate in (2.5.9), we obtain

$$\int_0^1 t^{-k/n} g^*(t) dt \leq \int_0^1 h(t) dt + \int_1^\infty \frac{t^{k/n} h(t)}{t} dt,$$

by Lemma 2.1.2

$$\int_0^1 t^{-k/n} g^*(t) dt \lesssim \int_0^1 h^*(t) dt + \int_1^\infty \frac{t^{k/n} h(t)}{t} dt. \tag{2.5.10}$$

Again by lemma 2.1.2

$$h^{**}(t) \geq \frac{1}{t} \int_0^t h(s) ds \geq \frac{1}{t} \int_{t/2}^t h(s) ds,$$

since $h \in M$, we can find a $m$ such that $t^m h(t)$ is equivalent to an increasing, therefore

$$h^{**}(t) \gtrsim \frac{1}{t} t^m h(t/2) \int_{t/2}^t s^{-m} ds,$$
from which it follows
\[ h^*(t) \gtrsim h(t/2), \]
in view of the estimate \( h^*(2t) \lesssim h^*(t) \), we get at
\[ h(t) \lesssim h^*(t). \]  \hspace{1cm} (2.5.11)
Inserting this estimate in \((2.5.10)\) yields
\[ \int_0^1 t^{-k/n} g^*(t) dt \lesssim \int_0^1 h^*(t) dt + \int_1^{\infty} t^{k/n} h^{**}(t) \frac{dt}{t}, \]
in other words,
\[ \int_0^1 t^{-k/n} g^*(t) dt \lesssim \int_0^1 h^*(t) dt + \int_1^{\infty} t^{k/n-1} \left( \int_0^t h^*(s) ds \right) \frac{dt}{t}, \]
use of Fubini theorem gives
\[ \int_0^1 t^{-k/n} g^*(t) dt \lesssim \int_0^1 h^*(t) dt + \int_1^{\infty} t^{k/n} h^{**}(t) \frac{dt}{t}, \]
applying the estimate \((2.3.3)\) with \( u = 1 \), we infer that
\[ \int_0^1 t^{-k/n} g^*(t) dt \lesssim \|h^*\|_{L^1 + \Lambda^1(t^{k/n})}, \]
as \( E \hookrightarrow L^1 + \Lambda^1(t^{k/n}) \) is assumed to be valid, the above estimate leads us to
\[ \int_0^1 t^{-k/n} g^*(t) dt \lesssim \rho_E(h^*), \]
taking infimum over all \( h \) such that \( g^* \leq Th \), we get \((2.5.8)\).
Now in view of \((2.5.8)\), axiom \((A_1)\) is clearly satisfied. To check \((A_2)\), we take an arbitrary \( h \in M \) such that \( g \leq Th \). Then \( ag \leq Tah \) where \( a \) is nonnegative real number, thus
\[ \rho(ag) \leq \rho_E(ah) \]
\[ = a \rho_E(h), \]
taking infimum with respect to \( h \), we deduce that
\[ \rho(ag) \leq a \rho(g). \]
To prove the reverse inequality, we take an arbitrary $k \in M$ such that $ag \leq Tk$. Then $g \leq T(1/ak)$, from which it follows that
\[
\rho(g) \leq \rho_E(1/ak),
\]
or,
\[
a\rho(g) \leq \rho_E(k),
\]
now taking infimum with respect to $k$ gives
\[
a\rho(g) \leq \rho(ag).
\]
Hence $\rho(ag) = a\rho(g)$.

To verify $(A_3)$, we take arbitrary $h, k \in M$ such that $f \leq Th$ and $g \leq Tk$ hold. Then it follows that
\[
f + g \leq T(h + k),
\]

hence,
\[
\rho(f + g) \leq \rho_E(h + k),
\]
or,
\[
\rho(f + g) \leq \rho_E(h) + \rho_E(k),
\]
taking infimum with respect to $h$, we have
\[
\rho(f + g) \leq \rho(f) + \rho_E(k),
\]
now taking infimum with respect to $k$, we conclude that
\[
\rho(f + g) \leq \rho(f) + \rho(g).
\]
Hence $\rho$ is a norm on $M^+$.

In order to establish $\rho(g^*) \approx \rho_l(g^*)$, first we prove that
\[
\rho(g^*) \lesssim \rho_l(g^*). \tag{2.5.12}
\]

To this end we take an arbitrary finite collection $g_j$ such that $g_{**} \leq \sum g_j^{**}$. Let $h_j(t) = t^{-k/n} \delta g_{j}^{**}(t)$, then we can write
\[
g_j^{**} = Th_j, \quad g_j \in M^+
\]
from this we have
\[
\sum g_j^{**} = T \sum h_j,
\]
if $h = \sum h_j$, then it follows that

$$g^{**} \leq Th,$$

thus,

$$g^{*} \leq Th,$$

since $h \in M$ therefore, by Definition 2.5.3

$$\rho(g^{*}) \leq \rho_E(h),$$

or,

$$\rho(g^{*}) \leq \sum \rho_E(h_j)
= \sum \rho_E(t^{-k/n} \delta g^{**}_j(t))
= \sum \rho G_E(g_j),$$

taking the infimum, we conclude

$$\rho(g^{*}) \leq \rho_l(g^{*}).$$

To prove the reverse inequality, take an arbitrary $h \in M$ such that $g^{*} \leq Th$. Thus,

$$\rho_l(g^{*}) \leq \rho_l(Th),$$

since the couple $\rho_E, \rho_l$ is admissible by Proposition 2.5.3 therefore, in view of Theorem 2.4.1 and Theorem 2.4.2 we get at

$$\rho_l(g^{*}) \lesssim \rho_E(h),$$

taking infimum in the right hand side,

$$\rho_l(g^{*}) \lesssim \rho_l(g^{*}).$$

Hence, the result $\rho(g^{*}) \approx \rho_l(g^{*})$ follows.

To exhibit that the embedding $G(E) \hookrightarrow \Lambda^1(t^{1-k/n}) + L^\infty$ is valid, we take $f \in G(E)$. Then, by (2.5.8) we have

$$\int_0^1 t^{-k/n} f^{*}(t)dt \lesssim \rho(f^{*}),$$

giving

$$\|f\|_{\Lambda^1(t^{1-k/n}) + L^\infty} \lesssim \|f\|_{G(E)},$$

and the required embedding follows.
Proposition 2.5.5. Let $\rho_G \in N_{t,1}$, set
\[ \rho_1(g) = \rho_G(Tg**). \tag{2.5.13} \]
Then $\rho_1 \in N_{d,1}$.

Proof. Evidently, $\rho_1$ is a rearrangement invariant $K$–monotone norm on $\mathcal{M}^+$ and satisfies Minkovski inequality. We need to calculate Boyd indices for $\rho_1$. For this consider
\[
Tf_*(t) = \int_t^\infty v^{k/n} f**(\frac{v}{s}) \frac{dv}{v}
= s^{k/n} \int_{t/s}^\infty v^{k/n} f**(v) \frac{dv}{v}
= s^{k/n} (Tf**)(t/s),
\]
hence,
\[
\rho_G(Tf_*) = s^{k/n} \rho_G((Tf**)(t/s)) \leq s^{k/n} \rho_G(Tf**)(t/s),
\]
or,
\[
\rho_1(f_*) \leq s^{k/n} \rho_1(f) h_G(s),
\]
from which we get
\[
h_1(s) \leq s^{k/n} h_G(s),
\]
it now follows at once from the definition of Boyd indices that
\[
\alpha_1 \geq k/n + \alpha_G, \quad \beta_1 \leq k/n + \beta_G. \tag{2.5.14}
\]
Hence, $\alpha_1 > \frac{k-2}{n}$, and $\beta_1 < 1$ since $\beta_G < 1 - k/n$. Thus, $\rho_1 \in N_{d,1}$. \hfill $\Box$

Theorem 2.5.6. The couple $\rho_1, \rho_G$ is optimal in the class $N_{d,1}, N_{t,1}$.

Proof. Let $g \in M$, then by (2.5.11),
\[
g(t) \lesssim g**(t),
\]
so that
\[
\rho_G(Tg) \lesssim \rho_G(Tg**) = \rho_1(g),
\]
by Theorem 2.4.1 the couple \( \rho_1, \rho_G \) is admissible.

Let \( \rho_E, \rho_G \) be any admissible couple in the class \( N_{d,1}, N_{t,1} \). Now

\[
\rho_1(g^*) = \rho_G(Tg^{**}) \\
\lesssim \rho_E(g^{**}) \quad \text{(by Theorem 2.4.1)} \\
\approx \rho_E(g^*) \quad \text{(as } \beta_E < 1) 
\]

showing that \( \rho_1 \) is optimal domain norm in the class \( N_{d,1}, N_{t,1} \).

It remains to show that \( \rho_G \) is optimal target norm. For this, it is sufficient, in view of Proposition 2.5.4, to prove that \( \rho(g^*) \lesssim \rho_G(g^*) \) where \( \rho \) is defined in (2.5.6). We start by noting that \( t^{-k/n} \delta g^{**}(t) \in M \) as \( t^{k/n+1} t^{-k/n} \delta g^{**} = t \delta g^{**}(t) \) is increasing. Write \( h(t) = t^{-k/n} \delta g^{**} \in M \), then

\[
g^{**} = Th, \quad f \in M^+ 
\]

it follows that

\[
g^* \leq Th, 
\]

thus by Definition 2.5.3, we have

\[
\rho(g^*) \leq \rho_1(h), 
\]

in other words,

\[
\rho(g^*) \leq \rho_G(Th^{**}). \tag{2.5.15} 
\]

Next we prove that the following estimate holds:

\[
Th^{**}(t) \lesssim Th(t) + t^{k/n} h^{**}(t). \tag{2.5.16} 
\]

We have

\[
h^{**} = Ph^*. \tag{2.5.17} 
\]

Now \( Qh \gtrsim h \) since \( h \in M \). Therefore, as \( Q \) is decreasing, we get \( h^* \lesssim Qh(t) \). Insert it in (2.5.17) to obtain

\[
h^{**} \lesssim PQh, 
\]

since \( P \) and \( Q \) commute, we write

\[
h^{**} \lesssim QPh, 
\]

whence we obtain

\[
Th^{**} \lesssim TQPh, \tag{2.5.18} 
\]
but

\[ TQ(Ph) = \int_t^\infty s^{k/n} \left( \int_s^\infty Ph(v) \frac{dv}{v} \right) \frac{ds}{s} = \int_t^\infty \left( \int_t^v s^{k/n} \frac{ds}{s} \right) Ph(v) \frac{dv}{v} \approx \int_t^\infty (v^{k/n} - s^{k/n}) Ph(v) \frac{dv}{v} \leq \int_t^\infty v^{k/n} Ph(v) \frac{dv}{v} = TP(h), \]

therefore, (2.5.18) turns into

\[ Th^{**} \lesssim TPh. \quad (2.5.19) \]

Now,

\[ TPh(t) = \int_t^\infty s^{k/n} \left( \frac{1}{s} \int_0^s h(v) dv \right) \frac{ds}{s} = \int_t^\infty s^{k/n-1} \left( \int_0^t h(v) dv + \int_t^s h(v) dv \right) \frac{ds}{s} = \int_t^\infty s^{k/n-1} \left( \int_0^t h(v) dv \right) \frac{ds}{s} + \int_t^\infty \left( \int_0^s s^{k/n-1} \frac{ds}{s} \right) h(v) dv \approx t^{k/n-1} \int_0^t h(v) dv + \int_t^\infty v^{k/n} h(v) \frac{dv}{v} \lesssim t^{k/n} h^{**}(t) + Th(t), \]

from above estimate and (2.5.19), we get (2.5.16). Monotonicity of \( \rho_G \) gives

\[ \rho_G(Th^{**}) \lesssim \rho_G(t^{k/n} h^{**}(t)) + \rho_G(Th). \quad (2.5.20) \]

Define a new norm \( \rho_F \) on \( \mathcal{M}^+ \) as follows:

\[ \rho_F(g) = \rho_G(t^{k/n} g). \]

To compute its upper Boyd index, consider

\[ \rho_F(g^*_s) = \rho_G(t^{k/n} g^*(t/s)) = s^{k/n} \rho_G(\frac{t}{s})^{k/n} g^*(t/s)) \leq s^{k/n} \rho_G(t^{k/n} g^*(t)) h_G(s) = s^{k/n} \rho_F(g^*) h_G(s), \]
therefore,

\[ h_F(s) \leq s^{k/n} h_G(s), \]

from which we obtain

\[ \beta_F \leq k/n + \beta_G, \]

as \( \rho_G \in N_{t,1} \), it follows that \( \beta_F < 1 \). Moreover, \( \rho_F \) is monotone and satisfies Minkovski inequality since \( \rho_G \) is monotone and satisfies Minkovski inequality. So we can apply Lemma 2.1.8 to get

\[ \rho_F(g^*) \approx \rho_F(g^{**}), \]

or,

\[ \rho_G(t^{k/n}g^*) \approx \rho_G(t^{k/n}g^{**}), \]

using this in (2.5.20),

\[ \rho_G(Th^{**}) \lesssim \rho_G(t^{k/n}h^*(t)) + \rho_G(Th). \]  \hspace{1cm} (2.5.21)

Combining the following

\[ h(t) = t^{-k/n}\delta g^{**}(t) \leq t^{-k/n}g^{**}(t), \]

with the fact that \( t^{-k/n}g^{**}(t) \) is decreasing, we get

\[ h^*(t) \leq t^{-k/n}g^{**}(t), \]

so that

\[ t^{k/n}h^*(t) \leq g^{**}(t) = Th, \]

thus, (2.5.21) becomes

\[ \rho_G(Th^{**}) \lesssim \rho_G(g^{**}), \]

since \( \beta_G < 1 \), we obtain

\[ \rho_G(Th^{**}) \lesssim \rho_G(g^*), \]

in view of the above estimate, (2.5.15) gives

\[ \rho(g^*) \lesssim \rho_G(g^*), \]

as we wished to show, and the theorem follows.

If the target norm is given from the class \( N_{t,2} := \{ \rho_G \in N_{t,1}, \ \alpha_G > 0 \} \), then we can simplify the formula for the optimal domain norm given by [2.5.13]
Theorem 2.5.7. Let $\rho_G \in N_{t,2}$ and define the norm $\rho_2$ by

$$\rho_2(g) = \rho_G(t^{k/n} g^{**}(t)). \quad (2.5.22)$$

Then the couple $\rho_2, \rho_G$ is optimal in the class $N_{d,1}, N_{t,2}$.

Proof. According to Theorem 2.5.6 we have to prove only that

$$\rho_G(t^{k/n} g^{**}) \approx \rho_G(T g^{**}), \quad g \in \mathcal{M}^+. \quad (2.5.24)$$

Since $\alpha_G > 0$ and $t^{k/n} g^{**} \in M$, then by Lemma 2.1.11

$$\rho_G(Q t^{k/n} g^{**}(t)) \lesssim \rho_G(t^{k/n} g^{**}(t)), \quad (2.5.25)$$

or,

$$\rho_G(T g^{**}) \lesssim \rho_G(t^{k/n} g^{**}(t)). \quad (2.5.26)$$

The reverse estimate follows from

$$T g^{**}(t) \gtrsim t^{k/n} g^{**}(t). \quad (2.5.27)$$

\[ \square \]

2.5.1 Subcritical case

It is defined by the condition $k/n < \alpha_E$.

Definition 2.5.4. Let $\rho_E \in N_{d,3}$. Define a functional $\rho_3$ on $\mathcal{M}^+$ as

$$\rho_3(g) = \rho_E(t^{-k/n} g). \quad (2.5.23)$$

Theorem 2.5.8. Let $\rho_E \in N_{d,3}$, then the couple $\rho_E, \rho_3$ is optimal in the class $N_{d,3}, N_{t,3}$.

Optimality of the target norm $\rho_3$ is known [34] for rearrangement invariant Banach function spaces as in [7].

Proof. It is evident that $\rho_3$ is a norm on $\mathcal{M}^+$. Moreover, it satisfies Minkovski inequality since $\rho_E$ satisfies Minkovski inequality. Hence, $\rho_3 \in N_{t,3}$. To verify that the couple $\rho_E, \rho_3$ is admissible, we take an arbitrary $g \in M$ and consider

$$\rho_3(T g) = \rho_E(t^{-k/n} T g(t)) = \rho_E(t^{-k/n} Q(t^{k/n} g(t))), \quad (2.5.28)$$

or,

$$T g^{**}(t) \gtrsim t^{k/n} g^{**}(t). \quad (2.5.29)$$
since \( t^{k/n}g(t) \in M \) and \( \alpha_E > k/n \), we can apply Lemma 2.1.11 to obtain
\[
\rho_3(Tg) \lesssim \rho_E(t^{-k/n}t^{k/n}g(t)) = \rho_E(g),
\]
therefore, by Theorem 2.4.1 and Theorem 2.4.2, the couple \( \rho_E, \rho_3 \) is admissible.

Next we set out to prove that \( \rho_3 \) is the optimal target norm in the class \( N_{d,3}, N_{t,3} \).
Let \( \rho_E, \rho_G \) be an arbitrary admissible couple in the class \( N_{d,3}, N_{t,3} \).
Since \( g^{**}(t) = T(t^{-k/n}g^{**}(t)) \),
thus,
\[
\rho_G(g^{**}) = \rho_G(T(t^{-k/n}g^{**}(t))),
\]
as \( t^{-k/n}g^{**}(t) \in M \) and \( \rho_E, \rho_G \) is admissible, so by Theorem 2.4.1 and Theorem 2.4.2
\[
\rho_G(g^{**}) \lesssim \rho_E(t^{-k/n}g^{**}(t)),
\]
which gives
\[
\rho_G(g^{**}) \lesssim \rho_E(t^{-k/n}g^{**}(t)). \tag{2.5.24}
\]
Define a new norm \( \rho_F \) on \( M^+ \) as follows:
\[
\rho_F(g) = \rho_E(t^{-k/n}g).
\]
To compute its upper Boyd index, consider
\[
\rho_F(g^s) = \rho_E(t^{-k/n}g^s(t/s)) = s^{-k/n}\rho_E((t/s)^{-k/n}g^s(t/s)) \leq s^{-k/n}\rho_E(t^{-k/n}g^s(t))h_E(s) = s^{-k/n}\rho_F(g^s)h_E(s),
\]
which yields
\[
h_F(s) \leq s^{-k/n}h_E(s),
\]
from which we obtain
\[
\beta_F \leq -k/n + \beta_E,
\]
it follows that \( \beta_F < 1 \). In addition, it is easy to see that \( \rho_F \) is a monotone norm satisfying Minkovski inequality. Therefore, by Lemma 2.1.8 we have
\[
\rho_F(g^s) \approx \rho_F(g^{**}),
\]
or,
\[
\rho_E(t^{-k/n}g^s) \approx \rho_E(t^{-k/n}g^{**}),
\]
inserting this in (2.5.24) gives

$$\rho_G(g^{**}) \lesssim \rho_E(t^{-k/n}g^*(t)),$$

from which it follows that

$$\rho_G(g^*) \lesssim \rho_3(g^*),$$

showing that $\rho_3$ is an optimal target norm. Finally, the couple $\rho_E, \rho_3$ is optimal, since for any admissible couple $\rho_{E_1}, \rho_3$, we have, in view of Remark 2.4.1

$$\rho_3(Tg) \lesssim \rho_{E_1}(g), \ g \in M_0.$$

Then

$$\rho_{E_1}(g) \gtrsim \rho_3(t^{k/n}g) \approx \rho_E(g),$$

as $g \in M_0$, thus

$$\rho_{E_1}(g^*) \gtrsim \rho_E(g^*),$$

as required.

\[\square\]

2.5.2 Critical case

In the critical case, i.e. $k/n = \alpha_E < 1$, we use real interpolation for normed spaces, similarly to [14], [13], but in a simpler way, and introduce a large class of domain norms and the corresponding optimal target norms.

Let the function $b$ be increasing and slowly varying on $(0, \infty)$, such that

$$b(t^2) \approx b(t),$$

and let $(1 + \log t)^{-1-\varepsilon}b(t), \ t > 1$ be increasing for some $\varepsilon > 0$. Let

$$c(t) = \frac{b(t)}{1 + |\log t|}. \quad (2.5.26)$$

**Lemma 2.5.9.** Let the functions $b$ and $c$ be as we have defined above then

$$\int_t^\infty \frac{1}{b(s)} \frac{ds}{s} \lesssim \frac{1}{c(t)}, \ t > 0. \quad (2.5.27)$$
Proof. Let $0 < t < 1$, then we can write

\[
\int_t^\infty \frac{1}{b(u)} \frac{du}{u} = \int_t^1 \frac{1}{b(u)} \frac{du}{u} + \int_1^\infty \frac{1}{b(u)} \frac{du}{u} \\
\leq \frac{1}{b(t)} \int_t^1 \frac{du}{u} + \int_1^\infty \frac{1}{b(u)} \frac{du}{u} \\
= -\log t \frac{1}{b(t)} + \int_1^\infty \frac{(1 + \log u)^{-1-\varepsilon}}{b(u)(1 + \log u)^{-1-\varepsilon}} \frac{du}{u} \\
\leq -\log t \frac{1}{b(t)} + \frac{1}{b(1)} \int_1^\infty (1 + \log u)^{-1-\varepsilon} \frac{du}{u} \\
\approx -\log t \frac{1}{b(t)} + \frac{1}{b(1)} \\
\leq \frac{1}{c(t)}.
\]

Hence, the estimate \((2.5.27)\) follows for $0 < t < 1$. Now we consider the case when $t > 1$.

\[
\int_t^\infty \frac{1}{b(u)} \frac{du}{u} = \int_t^\infty \frac{(1 + \log u)^{-1-\varepsilon}}{b(u)(1 + \log u)^{-1-\varepsilon}} \frac{du}{u} \\
\leq \frac{1}{b(t)(1 + \log t)^{-1-\varepsilon}} \int_t^\infty (1 + \log u)^{-1-\varepsilon} \frac{du}{u} \\
\approx \frac{1}{b(t)(1 + \log t)^{-1-\varepsilon}} (1 + \log t)^{-\varepsilon} \\
= \frac{1}{c(t)},
\]

as we wished to show. \(\Box\)

Before we state the main result of this subsection we introduce some notations.

Let $d\mu := dt/t$ be the Haar measure on $(0, \infty)$ and let $h_\mu^*$ denote the decreasing rearrangement of $h$ with respect $d\mu$. Let $h_\mu^{**}(t) := \frac{1}{t} \int_0^t h_\mu^*(s)ds$.

**Theorem 2.5.10.** Let $H$ be a rearrangement invariant space on $(0, \infty)$ with the Lebesgue measure, with a $K$--monotone rearrangement invariant norm $\rho_H$ that satisfies Minkowski inequality and let $\beta_H < 1$. Let $b, c$ be given by (2.5.25) - (2.5.26). Let $\rho_E$ be defined by

\[
\rho_E(g) := \rho_F(t^{k/n}b(t)g^{**}(t)), \ k < n,
\]

(2.5.28)
\[ F := (L^1_{\ast}, L^\infty_{\ast})_{H(1/t)}, \]  
(2.5.29)

and \( H(1/t) \) has a norm \( \|g\|_{H(1/t)} := \rho_H(g(t)/t) \). Then the optimal target norm is given by

\[ \rho_{G(E)}(g) := \rho_F(gc). \]  
(2.5.30)

Moreover, the embeddings \( E \hookrightarrow L^1 + \Lambda^1(t^{k/n}) \) and \( G(E) \hookrightarrow \Lambda^1(t^{1-k/n}) + L^\infty \) hold.

**Proof.** We start by showing that the operator \( T \), defined in (2.1.20), is bounded in the following couple of spaces:

\[ T : L^1_{\ast}(t^{k/n}b(t)) \hookrightarrow L^\infty_b, \]  
(2.5.31)

and

\[ T : L^\infty_{\ast}(t^{k/n}b(t)) \hookrightarrow L^\infty_c. \]  
(2.5.32)

Now
\[ \int_0^\infty f(s)s^{k/n}b(s) \frac{ds}{s} \geq \int_t^\infty f(s)s^{k/n}b(s) \frac{ds}{s}, \quad \forall \ t > 0 \ f \geq 0. \]

Since \( b \) is increasing on \((0, \infty)\), we obtain
\[ \int_0^\infty f(s)s^{k/n}b(s) \frac{ds}{s} \geq b(t) \int_t^\infty f(s)s^{k/n} \frac{ds}{s} \geq b(t)Tf(t), \]
showing that (2.5.31) is valid.

Moreover,
\[ \left( \int_t^\infty f(s)s^{k/n} \frac{ds}{s} \right) c(t) \lesssim \sup_{t \geq 0} f(t)t^{k/n}b(t) \left( \int_t^\infty \frac{1}{b(s)} \frac{ds}{s} \right) c(t) \lesssim \sup_{t \geq 0} f(t)t^{k/n}b(t), \]
from which we have \( \|Tf\|_{L^\infty_b} \lesssim \|f\|_{L^\infty_{\ast}(t^{k/n}b(t))} \), hence (2.5.32) holds.

It is well known that
\[ \rho_F(g) = \rho_H(g^{**}). \]  
(2.5.33)

As \( \beta_H < 1 \) we also have
\[ \rho_F(g) \approx \rho_H(g^\ast). \]  
(2.5.34)
By interpolation,

\[ T : E_1 \mapsto G_1, \]

where

\[ E_1 := (L^1_*(t^{k/n}b(t)), L^\infty_*(t^{k/n}b(t))_{H(1/t)}, \quad G_1 := (L^\infty_0, L^\infty_c)_{H(1/t)}. \]  

(2.5.35)

Denote the norm in \( E_1 \) by \( \rho_1 \) and let \( \rho_E(g) = \rho_1(g^{**}) \). Then,

\[
\rho_1(g^{**}) = \| K(t, g^{**}; L^1_*(t^{k/n}b(t)), L^\infty_*(t^{k/n}b(t))) \|_{H(1/t)} = \| K(t, t^{k/n}b(t)g^{**}; L^\infty_*, L^\infty_* \|_{H(1/t)} = \rho_F(t^{k/n}b(t)g^{**}(t)) = \rho_H((t^{k/n}b(t)g^{**}(t))^{**}).
\]

So \( \rho_E \) is rearrangement invariant and \( K \)-monotone. To calculate its Boyd indices, consider

\[
\rho_E(g_s^*) = \rho_F(t^{k/n}b(t)g_s^{**}(t)) = \rho_F(t^{k/n}b(t)g^{**}(t/s)).
\]

As \( b \) is slowly varying function, so if \( s > 1 \) then for an arbitrary \( \epsilon > 0 \) we have

\[ b(t) \lesssim s^\epsilon b(t/s), \]

which along with \( \rho_F(g(t/s)) = \rho_F(g) \), leads us to

\[ \rho_E(g_s^*) \lesssim s^{k/n+\epsilon} \rho_E(g^*), \]

which gives

\[ h_E(s) \lesssim s^{k/n+\epsilon}, \]

hence, we have \( \beta_E \leq k/n \). On the same lines we have \( \alpha_E \geq k/n \) implying that

\[ \alpha_E = \beta_E = k/n < 1. \]

Therefore,

\[ \rho_E(g) \approx \rho_F(t^{k/n}b(t)g^*(t)) \approx \rho_H((t^{k/n}b(t)g^*(t))^{**}). \]

(2.5.36)

Since \( \rho_H \) satisfies Minkovski inequality, it follows that \( \rho_E \) also satisfies it. In order to prove that the embedding \( E \hookrightarrow L^1 + \Lambda^1(t^{k/n}) \) is valid, we show that for any decreasing \( h \) we have

\[
\int_0^1 h(t)dt + \int_1^\infty t^{k/n-1}h(t)dt \lesssim \rho_E(h).
\]

(2.5.37)
Now for $f \geq 0$ on $(0, \infty)$,
\[
\|f\|_{L^1 + L^1(\ell^{k/n-1})} \approx \int_0^1 f(u)du + \int_1^\infty u^{k/n-1}f(u)\frac{du}{u} \leq \int_0^1 u^{k/n-1}b(u)f(u)du + \int_1^\infty u^{k/n-1}b(u)f(u)\frac{du}{u} \leq \|f\|_{L^1_{\ell^{(k/n)b}}},
\]
which implies that
\[
L^1_{\ell^{(k/n)b}}(t) \hookrightarrow L^1 + L^1(t^{k/n-1}).
\]
Also
\[
\|f\|_{L^1 + L^1(\ell^{k/n-1})} \lesssim \sup_{0<u<1} u^{k/n}f(u) + \frac{1}{c(1)} \sup_{u>1} u^{k/n}f(u),
\]
showing that
\[
L^\infty_{\ell^{(k/n)b}(t)}(t) \hookrightarrow L^1 + L^1(t^{k/n-1}).
\]
Now by interpolation,
\[
E_1 \hookrightarrow L^1 + L^1(t^{k/n-1}).
\]
Thus,
\[
\rho_E(h) = \rho_1(h^{**}) \geq \rho_1(h^*) = \rho_1(h) \quad \text{ (h is decreasing)}
\]
\[
= \|h\|_{E_1} \geq \|h\|_{L^1 + L^1(t^{k/n-1})},
\]
showing that \((2.5.38)\) holds for any decreasing $h$. Take any $f$ from $L^1 + \Lambda^1(t^{k/n})$ then by \((2.5.38)\),
\[
\int_0^1 f^*(t)dt + \int_1^\infty t^{k/n-1}f^*(t)dt \lesssim \rho_E(f^*),
\]
which gives $E \hookrightarrow L^1 + \Lambda^1(t^{k/n})$.

Now we characterize the space $G_1$. Since
\[
K(t, g; L^\infty_0, L^\infty_c) = t \sup_s |g(s)| \min(c(s), b(s)/t),
\]
we get the formula
\[
\rho_{G_1}(g) = \rho_H(h_g), \quad h_g(u) := \sup_s |g(s)| \min(c(s), b(s)/u). \quad (2.5.39)
\]
As \( b \geq c \), so \( L^{\infty}_{b} \hookrightarrow L^{\infty}_{c} \). Therefore it follows that \( h_{g}(u) \approx \sup |g(s)|c(s) \) if \( 0 < u < 1 \). Let
\[
H_{g}(t) := h_{g}(1 + |\log t|), \quad 0 < t < \infty.
\]
Then
\[
(H_{g})^{\ast}_{\mu}(t) \leq h_{g}(t/2),
\]
hence by (2.5.34) and (2.5.39),
\[
\rho_{F}(H_{g}) \lesssim \rho_{G_{1}}(g).
\]
Now
\[
H_{g}(t) = \sup_{s} |g(s)| \min \left( c(s), \frac{b(s)}{1 + |\log t|} \right), \quad 0 < t < \infty
\]
\[
\geq g(t) \min \left( c(t), \frac{b(t)}{1 + |\log t|} \right)
\]
\[
= g(t)c(t), \quad g \in \mathcal{M}^{+}.
\]
So if we define
\[
\rho_{G}(g) := \rho_{F}(g_{c}),
\]
we get
\[
\rho_{G}(Tg) = \rho_{F}(cTg)
\]
\[
\lesssim \rho_{F}(H_{Tg}) \quad \text{(by previous estimate)}
\]
\[
\lesssim \rho_{G_{1}}(Tg) \quad \text{(by (2.5.41))}
\]
\[
\lesssim \rho_{1}(g) \quad \text{(by (2.5.35))}
\]
\[
\approx \rho_{1}(g^{*})
\]
\[
\approx \rho_{1}(g^{**})
\]
\[
\approx \rho_{E}(g),
\]
since \( \beta_{E} < 1 \) we can apply Theorem 2.4.1 to conclude that the couple \( \rho_{E}, \rho_{G} \) is admissible.

Next our task is to prove that \( \rho_{G} \) is an optimal target norm. Since \( \rho_{E} \in N_{d_{1}} \), it is sufficient, in view of Proposition 2.5.4, to show that the estimate
\[
\rho(g^{*}) \lesssim \rho_{G}(g^{*})
\]
holds for every \( g \in \mathcal{M}^{+} \) where
\[
\rho(g) := \inf \{ \rho_{E}(h) : g \leq Th, \ h \in M \}.
\]
Take any $g \in \mathcal{M}^+$ and let
\[ t^{k/n} b(t) h(t) = g_1(t), \]
where $g_1(t) = g^*(t^2/e)c(t^2)$ for $0 < t < 1$ and $g_1(t) = g^*(\sqrt{t/e})c(\sqrt{t})$ if $t > 1$. Note that $h \in M$.
Now for $0 < t < 1$,
\[
    h(t) = t^{-k/n} \frac{1}{b(t)} g_1(t) \\
    \approx t^{-k/n} \frac{1}{b(t^2)} g_1(t) \\
    \approx \frac{t^{-k/n}}{1 + |\log t|} g^*(t^2/e),
\]
and for $t > 1$,
\[
    h(t) = t^{-k/n} \frac{1}{b(t)} g^*(\sqrt{t/e})c(\sqrt{t}) \\
    \approx t^{-k/n} \frac{1}{b(\sqrt{t})} g^*(\sqrt{t/e})c(\sqrt{t}) \\
    \approx \frac{t^{-k/n}}{1 + \log \sqrt{t}} g^*(\sqrt{t/e}),
\]
implying that
\[ h(t) \approx h^*(t). \]
It follows that
\[
    \rho_E(h) \approx \rho_F(t^{k/n} b(t) h^*(t)) \\
    \approx \rho_F(t^{k/n} b(t) h(t)) \\
    \approx \rho_F(g^*c) \\
    \approx \rho_G(g^*), \ (\beta_G < 1).
\]
On the other hand, for $0 < t < 1$,
\[
    T h(t) = \int_t^\infty s^{k/n} h(s) \frac{ds}{s} \\
    \gtrless \int_t^{\sqrt{t/e}} g^*(s^2/e) \frac{c(s^2)}{b(s)} \frac{ds}{s} \\
    \gtrless g^*(t) \int_t^{\sqrt{t/e}} \frac{c(s^2)}{b(s)} \frac{ds}{s} \\
    \gtrsim g^*(t) \int_t^{\sqrt{t/e}} \frac{1}{1 + |\log s|} \frac{ds}{s} \\
    \gtrsim g^*(t),
\]
Similarly, for \( t > 1 \) we obtain

\[
\begin{align*}
Th(t) & \geq \int_t^{et^2} g^{**}(\sqrt{s/e}) \frac{1}{1 + \log s} \frac{ds}{s} \\
& \geq g^{**}(t) \int_t^{et^2} \frac{1}{1 + \log s} \frac{ds}{s} \\
& \gtrsim g^{**}(t).
\end{align*}
\]

Thus \( Th \gtrsim g^{**} \) and \( \rho_E(h) \approx \rho_G(g^{*}) \). Then by the definition of \( \rho \) we get

\[
\rho(g^{**}) \lesssim \rho_G(g^{*}),
\]

whence we obtain

\[
\rho(g^{*}) \lesssim \rho_G(g^{*}),
\]

as desired.

Finally, the embedding \( G(E) \hookrightarrow \Lambda^1(t^{1-k/n}) + L^\infty \) follows from Proposition 2.5.4 as we have already proved the validity of \( E \hookrightarrow L^1 + \Lambda^1(t^{k/n}) \).

In particular, we can take \( \rho_H(g) := (\int_0^\infty |g(t)|^q dt)^{1/q}, 1 < q \leq \infty \). Then,

\[
\rho_E(g) \approx \left( \int_0^\infty [(t^{k/n}b(t)g^{*}(t))^s(s)]^q ds \right)^{1/q} = \left( \int_0^\infty [t^{k/n}b(t)g^{*}(t)]^q dt/t \right)^{1/q},
\]

hence \( E = \Lambda^q(t^{k/n}b(t)) \) and \( G(E) = \Lambda_0^q(c) \). More examples are given below. \( \square \)

### 2.5.3 Examples

**Example 2.5.1.** Let \( \rho_G = \rho_{v,1} \) be the given target norm, where \( v \) is a slowly varying function. Now

\[
\rho_G(f_s) = \int_0^\infty f^*(\frac{t}{s})v(t)\frac{dt}{t},
\]

by a simple change of variable,

\[
\rho_G(f_s) = \int_0^\infty f^*(u)v(su)\frac{du}{u}. \tag{2.5.42}
\]

It follows from the definition of a slowly varying function that for every \( \epsilon > 0 \), we have

\[
u^{-\epsilon}v(u) \approx d(u)
\]
where \( d \) is a decreasing function. If \( s > 1 \), then \( d(su) \leq d(u) \), thus
\[
\begin{align*}
u^{-\epsilon}v(u) & \gtrsim d(su) \\
& \approx s^{-\epsilon}u^{-\epsilon}v(su),
\end{align*}
\]
which gives
\[
v(su) \lesssim s^{\epsilon}v(u),
\]
inserting this estimate in (2.5.42), we arrive at
\[
\rho_G(f_s) \lesssim s^{\epsilon}\rho_G(f),
\]
which yields
\[
h_G(s) \lesssim s^{\epsilon},
\]
it follows that
\[
\beta_G \leq \epsilon,
\]
as \( \epsilon > 0 \) is arbitrary, we obtain \( \beta_G = 0 \). Hence, \( \rho_G \in N_{t,1} \). As \( \beta_G < 1 \), therefore \( \rho_{v,1} \approx \sigma_{v,1} \). By Theorem 2.5.6, the couple \( \rho_{1}, \rho_{G} \) is optimal where
\[
\begin{align*}
\rho_1(f) & = \rho_G(Tf^{**}) \\
& = \int_0^\infty v(t) \left( \int_t^\infty s^{k/n} f^{**}(s) \frac{ds}{s} \right) \frac{dt}{t} \\
& = \int_0^\infty \left( \int_0^t v(s) \frac{ds}{s} \right) f^{**}(t) \frac{dt}{t}, \quad \text{(by Fubini theorem)}
\end{align*}
\]
thus we have shown that the couple \( \rho_{v,1}, \sigma_{t^{k/n}w,1} \) is optimal. It follows that the following embedding
\[
w^kE \hookrightarrow G
\]
is optimal where \( E = \Phi(t^{k/n}w) = \Gamma_1(t^{k/n}w) \) and \( G = \Phi_0(v) \).

**Example 2.5.2.** Let \( G = C_0 \) consists of all bounded functions such that \( f^*(\infty) = 0 \) and
\[
\rho_G(g) = g^*(0).
\]
We note that \( \rho_G(g) \) is monotone and satisfies Minkovski inequality with \( \beta_G = 0 \), thus
\[ \rho_G \in N_{t,1}. \] Now,

\[
\begin{align*}
    \rho_E(f) &= \rho_G(Tf^{**}) \\
    &= \lim_{t \to 0} Tf^{**}(t) \\
    &= \lim_{t \to 0} \int_{0}^{\infty} s^{k/n} f^{**}(s) \frac{ds}{s} \\
    &= \int_{0}^{\infty} s^{k/n} f^{**}(s) \frac{ds}{s} \\
    &= \sigma_{t^{k/n},1}(f) \\
    &\approx \rho_{t^{k/n}-1}(f),
\end{align*}
\]

so, it follows from Theorem 2.5.6 that the couple \( G = C_0, E = \Lambda^1(t^{k/n}) \) is optimal and the embedding

\[ w^kE \hookrightarrow G \]

is optimal.

**Example 2.5.3.** Let \( G = \Lambda_0^\infty(v) \) with \( \beta_G < 1 - k/n \). As \( \beta_G < 1 \) so \( \rho_G(g) = \rho_G(g) = \sup_{t>0} v(t) g^*(t) \approx \sup_{t>0} v(t) g^{**}(t) \). Hence \( \rho_G \in N_{t,1} \). The optimal domain space \( E \), by Theorem 2.5.6 is given by the norm:

\[
\begin{align*}
    \rho_E(f) &= \rho_G(Tf^{**}) \\
    &= \sup_{t>0} v(t)(Tf^{**})^*(t) \\
    &= \sup_{t>0} v(t)(Tf^{**})(t) \\
    &= \sup_{t>0} v(t) \int_{t}^{\infty} s^{k/n} f^{**}(s) \frac{ds}{s}.
\end{align*}
\]

In particular, this is true if \( v \) is slowly varying since then \( \beta_G = 0 \). Indeed, consider

\[
\begin{align*}
    \rho_G(g_s^*) &= \sup_{t>0} v(t) g_s^{**}(t) \\
    &= \sup_{t>0} v(t) g^{**}(\frac{t}{s}) \\
    &= \sup_{st>0} v(ts) g^{**}(t) \\
    &= \sup_{t>0} v(ts) g^{**}(t),
\end{align*}
\]

since \( v \) is slowly varying function, so if \( s > 1 \) then for an arbitrary \( \epsilon > 0 \) we have from 2.5.43

\[ v(st) \lesssim s^\epsilon v(t), \]
therefore, it follows that
\[ \rho_G(g^*) \lesssim s^\epsilon \rho_G(g^*), \]
which yields
\[ h_G(s) \lesssim s^\epsilon, \]
from which we obtain
\[ \beta_G = 0. \]

**Example 2.5.4.** Let \( G, E \) be the optimal couple which we have just constructed in the previous example. Now

\[
\rho_E(f) = \sup_{t>0} v(t) \int_t^\infty s^{k/n} f^{**}(s) \frac{ds}{s} \]

\[ = \sup_{t>0} v(t) \int_t^\infty s^{k/n} f^{**}(s) w(s) \frac{1}{w(s)} \frac{ds}{s} \]

\[ \leq \sup_{t>0} v(t) \sup_{s>t} s^{k/n} f^{**}(s) w(s) \int_t^\infty \frac{1}{w(s)} \frac{ds}{s}, \]

choosing \( v \) such that \( \frac{1}{v(t)} = \int_t^\infty \frac{1}{w(s)} \frac{ds}{s} \) we have

\[
\rho_E(f) \leq \sup_{t>0} t^{k/n} f^{**}(t) w(t) \]

\[ = \sigma_{t^{k/n} w}^{\infty}(f), \]

showing that the couple \( E_1 = \Gamma^{\infty}(t^{k/n} w), G = \Lambda_{0}^{\infty}(v) \) is admissible. Next we will prove that \( \rho_G \) is optimal. To achieve this end it is sufficient, in view of Proposition 2.5.4 to show that the estimate

\[ \rho(g^*) \lesssim \rho_G(g^*) \]

holds for every \( g \in \mathcal{M}^+ \) where

\[ \rho(g) := \inf\{ \rho_E(h) : g \leq Th, \ h \in M \}. \]

First we are to find a \( h \in M \) such as \( g^* \leq Th \). For this we define \( h \) by means of

\[ t^{k/n} w(t) h(t) = \sup_{0<s\leq t} v(s) g^*(s). \] (2.5.44)
Now

\[ Th(t) = \int_t^\infty s^{k/n} h(s) \frac{ds}{s} = \int_t^\infty s^{k/n} (s^{k/n} w(s))^{-1} \sup_{0 < p \leq s} v(p) g^*(p) \frac{ds}{s} \geq \sup_{0 < p \leq t} v(p) g^*(p) \int_t^\infty \frac{1}{w(s)} \frac{ds}{s} \geq g^*(t) \frac{1}{v(t)} \sup_{0 < p \leq t} v(p) = g^*(t), \]

therefore,

\[ \rho(g^*) \leq \rho_{E_1}(h). \]  \hfill (2.5.45)

By (2.5.44),

\[ t^{k/n} w(t) h(t) \leq \sup_{s > 0} v(s) g^*(s), \]

or,

\[ h(t) \leq (t^{k/n} w(t))^{-1} \rho_G(g), \]

since \( w(t) \) is slowly varying, therefore, \( t^{k/n} w(t) \) is equivalent to an increasing function. Consequently, \( (t^{k/n} w(t))^{-1} \) is equivalent to a decreasing function say \( d(t) \). Then

\[ h(t) \lesssim d(t) \rho_G(g), \]

it follows that

\[ h^*(t) \lesssim d(t) \rho_G(g), \]

or,

\[ h^*(t) \lesssim (t^{k/n} w(t))^{-1} \rho_G(g), \]

whence we get

\[ t^{k/n} w(t) h^*(t) \lesssim \rho_G(g), \]

thus,

\[ \rho_{E_1}(h) \lesssim \rho_G(g). \]  \hfill (2.5.46)

Altogether from (2.5.44) and (2.5.46), we obtain

\[ \rho(g^*) \lesssim \rho_G(g^*), \]

as desired.
Example 2.5.5. Let $E = \Lambda^q(tw), 1 \leq q < \infty$, be the given domain space such that $t^{q-1}w^q(t)$ is non-increasing and such that $k/n < \alpha_E$. Now $\rho_E := \rho_{tw,q} \approx \sigma_{tw,q}$ is K-monotone, therefore $\rho_E \in N_{d,3}$. By Theorem 2.5.8 the optimal target space $G$ is given by

$$\rho_G(g^*) = \rho_E(t^{-k/n}g^*) = \rho_{tw,q}(t^{-k/n}g^*) = \left(\int_0^\infty [t^{-k/n}g^*(t)tw(t)]^q dt/t\right)^{1/q} = \rho_{tw^{-1}kw,q}(g^*)$$

indicating that $G = \Lambda_0^q(t^{1-k/n}w)$.

Moreover, by Theorem 2.5.8 the couple $E = \Lambda^q(tw), G = \Lambda_0^q(t^{1-k/n}w)$ is optimal.

Example 2.5.6. Let $G = \Lambda_0^q(c), 1 < q < \infty$, $E = \Gamma^q(t^{k/n}b(t))$, where $b$ is slowly varying on $(0, \infty)$, $b(t^2) \approx b(t)$, $b(t) \lesssim (1 + |\log t|)c(t)$ and

$$\left(\int_0^t c^q(s)ds/s\right)^{1/q} \left(\int_t^\infty [b(s)]^{-r}ds/s\right)^{1/r} \lesssim 1, \ 1/q + 1/r = 1. \quad (2.5.47)$$

Since $b$ is slowly varying we can calculate, as in previous examples, that $\alpha_E = \beta_E = k/n < 1$. Therefore, the couple $E, G$ is admissible by Theorem 2.4.1. Since the weights $b, c$, satisfying (2.5.47), are Muckenhoupt (see [37]), using the same argument as in the proof of Theorem 2.5.10, we see that $G$ is an optimal target space. In particular, we can take $b(t) = 1, 0 < t < 1$ and $b(t) = (1 + \log t)^2, t > 1$. Then $c(t) = (1 - \log t)^{-1}, 0 < t < 1$ and $c(t) = 1 + \log t, t > 1$. But we can not take $b(t) = 1$ for all $t > 0$. This means that the Lebesgue space $L^{n/k}$ is not allowed as a domain. It is not embedded in $L^1 + \Lambda^1(t^{k/n})$. This gives a difference with the well

known limiting embedding

$$\int_0^1 \left(\frac{f^*(t)}{1 - \log t}\right)^{n/k} dt \lesssim \int_0^1 \left(|D^kf|^*(t)\right)^{n/k} dt, \ f \in C_0^\infty(\Omega),$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ (see [25]). Of course, by Theorem 2.5.1(see also [34]) we have the optimal embedding in the rearrangement invariant non-linear set:

$$\int_0^\infty (\delta f^**(t))^{n/k} dt \lesssim \int_0^\infty (|D^k f|^*(t))^{n/k} dt.$$
# List of notations

The following notations, apart from the standard ones, are introduced in the thesis at the page numbers indicated.

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