GENERALIZED K-FRACTIONAL CONFORMABLE OPERATORS WITH APPLICATIONS

By
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DECLARATION

The work presented in this thesis was accomplished by me under the supervision of Prof. Dr. Muhammad Nawaz Naeem, Professor, Department of Mathematics, Government College University Faisalabad, Pakistan.

I hereby declare that the title of thesis "Generalized k-Fractional Conformable Operators with Applications" and the contents of thesis are the outcomes of my own research and no part has been copied from any published source (except the references, standard mathematical or genetic models /equations /formulas /protocols etc). I further declare that this work has not been submitted for award of any other degree /diploma. The University may take action if the information provided is found inaccurate at any stage.

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We certify that the contents and form of thesis submitted by Ms. Siddra Habib Registration No. 2013-GCUF-05470 have been found satisfactory and according to the prescribed format. We recommend it to be processed for evaluation by the External Examiner for the award of degree.

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DEDICATED TO

MY PARENTS,

FAMILY

AND

BELOVED HUSBAND

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ABSTRACT

Fractional calculus is the generalization of classical calculus related to differential and integral operators of non-integer (fractional) order. It is as aged as classical calculus but acquiring more significance these days because of its vast applications in numerous fields including physics, fluid dynamics, biology, control theory, image processing, computer networking, signal processing and many others. The ambition of this research work is to define the k-fractional conformable (k-FC) integral and derivative operators, which is the generalization of the recently proposed fractional conformable operators. The generalized k-fractional conformable operators unify many existing fractional operators corresponding to different values of the constraints immersed. We also verify the existence of our newly defined k-conformable operators. We discuss important properties of our newly hosted k-fractional conformable operators. The study of inequalities has been enhanced by the applications of fractional operators. The fractional integral inequalities have drawn the attention of many researchers due to their vast applications and importance in many theoretical and applied fields. We generalize some new integral inequalities using our introduced generalized k-fractional conformable integrals $(kFCI)$ for a finite sequence of n positive decreasing functions, where $n \in \mathbb{N}$. This work consigns to the generalizations of certain fractional integral inequalities comprising generalized k-fractional conformable integrals. The classical Chebyshev type inequalities, Chebyshev-Gr$\ddot{u}$ss type inequalities, an improved version of Gr$\ddot{u}$ss type integral inequality, reverse Minkowski inequality, P$\acute{o}$lya-Szeg$\ddot{o}$ type inequalities and related applications are presented and derived by invoking generalized k-fractional conformable integrals (k-FCI).

NOTATIONS

The following symbols are used to represent the text available in the research work.

The set of integers:	$\mathbb{Z}$
The set of natural numbers:	$\mathbb{N}$
The set of complex numbers:	$\mathbb{C}$
The set of positive complex numbers:	$\mathbb{C}^+$
The set of real numbers:	$\mathbb{R}$
The set of positive real numbers:	$\mathbb{R}^+$
The set of positive real numbers with zero:	$\mathbb{R}_0^+$
Any real number:	$a, b, c, x, t, m, n, q, u^*, v^*, w^*, r^*, s, M, N, P, Q, \tau, \rho$
Any integral number:	i, j
Any complex number:	$\alpha, \beta, \gamma, \theta$
Pochhammer's symbol:	$(x)_n$
Pochhammer k-symbol:	$(x)_{n,k}$
Gamma function:	$\Gamma(x)$
Gamma k-function:	$\Gamma_k(x)$
Beta function:	$B(u, v)$
Beta k-function:	$B_k(u, v)$
Psi function:	$\psi(x)$
Psi k-function:	$\psi_k(x)$
Phi function:	$\phi(x)$
Phi k-function:	$\phi_k(x)$
Riemann-Liouville fractional integral of order β:	$\mathcal{R}^{\beta}_{a^+}, \mathcal{R}^{\beta}_{b^-}$
k-Riemann-Liouville fractional integral of order β:	$\mathcal{R}^{\beta}_{a^+,k}, \mathcal{R}^{\beta,s}_{b^-,k}$
Generalized Riemann-Liouville fractional integral of order α:	$\mathcal{R}^{\beta,s}_{a^+}, \mathcal{R}^{\beta,s}_{b^-}$
Generalized k-Riemann-Liouville fractional integral of order α:	$\mathcal{R}^{\beta,s}_{a^+,k}, \mathcal{R}^{\beta,s}_{b^-,k}$
Riemann-Liouville fractional derivative of order β:	$\mathcal{R'}^{\beta}_{a^+}, \mathcal{R'}^{\beta}_{b^-}$
k-Riemann-Liouville fractional derivative of order β:	$\mathcal{R'}^{\beta}_{a^+,k}, \mathcal{R'}^{\beta,s}_{b^-,k}$
Generalized Riemann-Liouville fractional derivative of order β:	$\mathcal{R'}^{\beta,s}_{a^+}, \mathcal{R'}^{\beta,s}_{b^-}$
Generalized k-Riemann-Liouville fractional derivative of order β:	$\mathcal{R'}^{\beta,s}_{a^+,k}, \mathcal{R'}^{\beta,s}_{b^-,k}$
Caputo fractional derivative of order β:	${}_C\mathbb{D}^{\beta}_{a^+}, {}_C\mathbb{D}^{\beta}_{b^-}$
Caputo k-fractional derivative of order β:	${}_C\mathbb{D}^{\beta}_{a^+,k}, {}_C\mathbb{D}^{\beta}_{b^-,k}$
Generalized Caputo type fractional derivative of order β:	${}_C\mathbb{D}^{\beta,s}_{a^+}, {}_C\mathbb{D}^{\beta,s}_{b^-}$
Generalized Caputo type k-fractional derivative of order β:	${}_C\mathbb{D}^{\beta,s}_{a^+,k}, {}_C\mathbb{D}^{\beta,s}_{b^-,k}$
Hadamard fractional integral of order β:	$\mathcal{H}^{\beta}_{a^+}, \mathcal{H}^{\beta}_{b^-}$
k-Hadamard fractional integral of order β:	$\mathcal{H}^{\beta}_{a^+,k}, \mathcal{H}^{\beta}_{b^-,k}$
Generalized Hadamard fractional integral of order β:	$\mathcal{H}^{\beta,s}_{a^+}, \mathcal{H}^{\beta,s}_{b^-}$
Generalized Hadamard k-fractional integral of order β:	$\mathcal{H}^{\beta,s}_{a^+,k}, \mathcal{H}^{\beta,s}_{b^-,k}$

Hadamard fractional derivative of order β: $\mathcal{H}'^{\beta}_{a^+}, \mathcal{H}'^{\beta}_{b^-}$
Hadamard k-fractional derivative of order β: $\mathcal{H}'^{\beta}_{a^+,k}, \mathcal{H}'^{\beta}_{b^-,k}$
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Katugampola fractional integral of order β: $\mathcal{K}^{\beta}_{a^+}, \mathcal{K}^{\beta}_{b^-}$
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Weyl k-fractional integral of order β: $\mathcal{W}^{\beta}_{a^+,k}, \mathcal{W}^{\beta}_{b^-,k}$
Generalized Weyl fractional integral of order β: $\mathcal{W}^{\beta,s}_{a^+}, \mathcal{W}^{\beta,s}_{b^-}$
Generalized Weyl $k-$fractional integral of order β: $\mathcal{W}^{\beta,s}_{a^+,k}, \mathcal{W}^{\beta,s}_{b^-,k}$
Weyl fractional derivative of order β: $\mathcal{W}'^{\beta}_{a^+}, \mathcal{W}'^{\beta}_{b^-}$
Weyl k-fractional derivative of order β: $\mathcal{W}'^{\beta}_{a^+,k}, \mathcal{W}'^{\beta}_{b^-,k}$
Generalized Weyl fractional derivative of order β: $\mathcal{W}'^{\beta,s}_{a^+}, \mathcal{W}'^{\beta,s}_{b^-}$
Generalized Weyl k-fractional derivative of order α: $\mathcal{W}'^{\beta,s}_{a^+,k}, \mathcal{W}'^{\beta,s}_{b^-,k}$
Fractional conformable integral of order β: $\mathcal{A}^{\beta}_{a^+}, \mathcal{A}^{\beta}_{b^-}$
Generalized fractional conformable integral of order α: ${}^{\alpha}_{s}\mathcal{A}^{*}{}_{a^+}, {}^{\alpha}_{s}\mathcal{A}^{*}{}_{b^-}$
Generalized k-fractional conformable integral of order β (k-FCI): ${}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}, {}^{s}_{k}\mathfrak{F}^{*\beta}_{b^-}$
Fractional conformable derivative of order β: $\mathcal{A}'^{\beta}_{a^+}, \mathcal{A}'^{\beta}_{b^-}$
Generalized k-fractional conformable derivative in Riemann Liouville setting: ${}^{s}_{k}\mathfrak{C}^{\beta}_{a^+}, {}^{s}_{k}\mathfrak{C}^{\beta}_{b^-}$
Generalized k-fractional conformable derivative in the Caputo setting: ${}^{C}_{k,s}\mathfrak{C}^{\beta}_{a^+}, {}^{C}_{k,s}\mathfrak{C}^{\beta}_{b^-}$
k-fractional integral of order α w.r.t. another function: ${}^{g}_{k}\mathfrak{L}^{\alpha}_{a^+}, {}^{g}_{k}\mathfrak{L}^{\alpha}_{b^-}$
k-fractional derivative of order α w.r.t. another function in Riemann Liouville setting: ${}^{g}_{k}\mathfrak{L}^{*\alpha}_{a^+}, {}^{g}_{k}\mathfrak{L}^{*\alpha}_{b^-}$
k-fractional derivative of order α w.r.t. another function in the Caputo setting: ${}^{C}_{k,g}\mathfrak{L}^{*\alpha}_{a^+}, {}^{C}_{k,g}\mathfrak{L}^{*\alpha}_{b^-}$

Chapter 1

INTRODUCTION

1.1 Background

The calculus of differentiation and integration with non-integer order, commonly known as fractional order calculus, basically generalizes the integrals and derivatives operators with the real or complex powers. In 1695, a letter written by the famous mathematician Leibniz to L'Hospital comprised the conception of a fractional derivative. Fractional calculus has recently been given an uprising esteem due to its extensive applications. During the last two decades, this theory is deemed to be crucial due to its applications for solving many mathematical problems in science and engineering. In several branches of pure and applied mathematics, the fractional differential and integral operators are very helpful tools to perform the real or complex powers of the differentiation and integration. In fractional calculus theory, many fractional order derivatives were mainly manipulated. The most conventional operators utilized were Riemann-Liouville and Caputo derivatives, which modeled complex dynamics appearing in different areas of science. See [21, 37, 33, 38, 30, 31, 32, 46, 75, 96, 97, 107, 108, 108, 170, 179, 180, 182, 183]. A large bulk of fractional order integration operators occurs in literature, for example: Hadamard integral [25], Riemann–Liouville integral [151, p. 44], Erdélyi-Kober [82], Weyl [124] and Katugampola integral [101].

Owing the incredible applications and characteristics of fractional order operators, the field of fractional calculus is in unceasing development and as a result, countless fractional operators with applications have been addressed. Fractional order operators are thought to be the generalization of elementary classical operators (differential and integral operators etc). They are extensively manipulated in mathematics, physics and other domains of science, for instance; heat conduction, moments of inertia, electrical current, solutions of wave equations, fluid dynamics, quantum mechanics.

In the beginning of the seventeenth century, the mathematicians noted that the solutions of some physical and mathematical problems are not capable of being expressed in the closed form by means of elementary functions. Some specific terms were observed to appear again and again in mostly solutions in different forms. These repeated terms appearing in the solutions in the infinite series or integrals form, were assigned special symbols and names. So, special functions are identified as a group of mathematical elementary functions which occur during the solutions of several mathematical, statistical or physical problems. The gamma and beta functions form the base of the special transcendental functions. These operators are the basic component of several fractional order differential and integral operators like Caputo fractional derivative, Riemann-Liouville fractional operators, Katugampola operators, Hadamard fractional operators and Erdélyi-Kober operator etc. It acts in diverse fields, such as zeta functions, hyper geometric series, asymptotic series, definite integration and number theory etc. The gamma function is used in the applied, pure and mathematical sciences as the renowned factorial function. A famous mathematician Euler introduced the gamma function which expanded the factorial function for real and complex number argument. The relation between factorial function and the gamma function is given as $\Gamma(u) = (u-1)!$; for positive integer u. All the elementary hyper geometric series can be expressed in the forms of factorials, scaled quotients of the gamma functions. The limiting value of the gamma function is given by

$$\Gamma(u) = \lim_{x\to\infty} \frac{x!x^{u-1}}{(u)_x},$$

where $(u)_x$ is the Pochhammer's symbol introduced by L. A. Pochhammer. Pochhammer's symbol is used in the structures of most of the special functions as the basic element of the field of special functions and is defined as

$$(u)_x = u(u+1)(u+2)...(u+x-1), \quad u \neq 0, x \in \mathbb{N}.$$

Gamma function also estimates beta function. The beta function involving two parameters u_1 and u_2 is denoted by $\mathcal{B}(u_1, u_2)$ and defined as

$$\mathcal{B}(u_1, u_2) = \frac{(u_1 - 1)!(u_2 - 1)!}{(u_1 + u_2 - 1)!}, \quad Re(u_1) > 0, Re(u_2) > 0.$$

The relations between gamma function, Pochhammer's symbol and beta function are given as

$$\begin{aligned}(u)_x &= u(u+1)(u+2)...(u+x-1) \\ &= \frac{(u+x-1)(u+x-2)...(u+2)(u+1)u(u-1)!}{(u-1)!} \\ &= \frac{\Gamma(u+x)}{\Gamma(u)}, \quad Re(u) > 0\end{aligned}$$

and

$$B(u_1, u_2) = \frac{\Gamma(u_1)\Gamma(u_2)}{\Gamma(u_1 + u_2)}, \quad Re(u_1), Re(u_2) > 0.$$

The complete gamma and beta functions have the integral representation of the form

$$\Gamma(u) = \int_0^\infty z^{u-1} e^{-z} dz, \quad Re(u) > 0$$

and

$$B(u, v) = \int_0^1 z^{u-1}(1-z)^{v-1} dz, \quad Re(u), Re(v) > 0.$$

Another integral representation of complete beta function can be obtained by substituting $z = \frac{y}{1-y}$ in the above relation:

$$B(u, v) = \int_0^\infty \frac{y^{u-1}}{(1+y)^{u+v}} dz, \quad Re(u), Re(v) > 0.$$

The notion of fractional operators is in endless development since the age of its initiation. Numerous perceptions and theories are employed for the development and generality of the constituents of this subject . To generalize the fractional operators in the frame of a novel parameter $k > 0$, is the spirit of this effort. The k-fractional operators are our chief interest here. The annals of the k-fractional operators theory is not so ancient. The k-analogue theory was initiated by Diaz and Teruel [77] in the beginning of this century, who involved a parameter $k > 0$ to define the generalized form of special functions. During the subsequent years, Diaz and Pariguan [78] worked out on k-functions and presented the k-zeta function and k-version of hyper geometric functions. Being an impressive field, many researchers stepped into it and contributed very significant work to strengthen the theory of k-functions. Kokologiannaki, Mansour, Merovci, Kransiqi, Habibullah, Mubeen, Brahim, Rehman and Jing Zhang strived for such functions and verified a bulk of properties of special k-functions.

Let $u \in \mathbb{C}$ such that $Re(u) > 0$, the limiting form of extended gamma function is given by

$$\Gamma_k(u) = \lim_{x \to \infty} \frac{x! k^x (xk)^{\frac{u}{k}-1}}{(u)_{x,k}}, \quad k > 0,$$

with the Pochammer k-symbol $(u)_{x,k}$ for factorial function and expressed as $(u)_{x,k} = u(u+k)(u+2k)\cdots(u+(x-1)k)$ for $x \in \mathbb{N}$ and $(u)_{0,k} = 1$ for $u \neq 0$.

The extended form of gamma function can also be expressed explicitly as the Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ given by

$$\Gamma_k(u) = \int_0^{+\infty} z^{u-1} e^{\frac{-z^k}{k}} dz, u > 0.$$

Clearly,

$$\Gamma(u) = \lim_{k \to 1} \Gamma_k(u), \Gamma_k(u) = k^{\frac{u}{k}-1} \Gamma\left(\frac{u}{k}\right)$$

and

$$\Gamma_k(u+k) = k\Gamma_k(u).$$

Further, k-beta function denoted by $B_k(u_1, u_2)$ is stated as

$$B_k(u_1, u_2) = \frac{1}{k} \int_0^1 t^{\frac{u_1}{k}-1} (1-t)^{\frac{u_2}{k}-1} dt$$

such that

$$B_k(u_1,u_2)=\frac{\Gamma_k(u_1)\Gamma_k(u_2)}{\Gamma_k(u_1+u_2)} \\ =\frac{1}{k}B\left(\frac{u_1}{k},\frac{u_2}{k}\right).$$

The extended gamma function also established the given properties for $k>0$ and $n\in\mathbb{N}$:

$$(u)_{n,k}=\frac{\Gamma_k(u+nk)}{\Gamma_k(u)}$$
$$\Gamma_k(u+k)=u\Gamma_k(u)$$
$$\Gamma_k(\beta k)=k^{\beta-1}\Gamma(\beta),\ \beta\in\mathbb{R}^+$$
$$\Gamma_k(k)=1$$
$$\Gamma_k\left((2x+1)\frac{k}{2}\right)=k^{\frac{2x-1}{2}}\frac{(2x)!\sqrt{\pi}}{2^x x!}$$
$$\Gamma_k(xk)=k^{x-1}(x-1)!$$
$$\Gamma_k(u)=u^{-1}k^{\frac{u}{k}}e^{-\frac{u}{k}\gamma}\prod_{x=1}^{\infty}\left(\frac{xk}{u+xk}\right)e^{\frac{u}{xk}}$$

The Euler's or Mascheroni's constant γ has numerical value

$$\gamma=\lim_{x\to\infty}\sum\frac{1}{x}-\ln(x)\approx 0.5772156649... \quad .$$

This special k-functions theory was used by Mubeen and Habibullah [127] in fractional order calculus for the first time in literature in the form of k-Riemann–Liouville integral. Recently, many new fractional differential and integral operators and their generalized forms are presented by researchers using iteration procedure and involving a new parameter $k>0$. They also acquired the bonds of their generalized fractional order operators with current fractional and classical operators under the certain values of the involved parameters. The research has been progressed to generalize the fractional order operators by invoking new constraints and the generalization of classical inequalities through such operators.

Conformable operators are nonlocal fractional operators. They can be called fractional as they can be derived up to arbitrary order. However, we give preference to substitute conformable fractional by conformable (as a type of local fractional) because in the area of fractional calculus non-local fractional derivatives only are used to be called fractional. Conformable derivatives and other kinds of local fractional order derivatives or modulated conformable derivatives in [24] can attain their significance by the capability of using them to produce more generalized nonlocal fractional order derivatives with singular kernels (see [95, 4, 98]). Khalil et al. [105] and Adeljawad [1] exhibited the local conformable fractional derivative and integral operators as a new group of fractional operators. Such new classes give the idea of generalization of the fractional order operators by evoking new parameters and to obtain the associated inequalities: see [45, 23, 164, 55, 176, 159, 56]. For example, Katugampola [102] proposed a fractional order integral, combining other familiar present ones. Jarad et al. [98] used the iterating process of the standard fractional calculus to establish the generalization of fractional conformable operators [1], which inspires the ongoing research to develop more novel ideas of generalization and get the related properties and applications.

However, one question always arose for every sort of data about the implementation of optimum relating non-local model. Furthermore, some fractional order operators with non-local, local, non-singular and singular kernels can be studied in [2, 3, 5, 12, 27, 40, 52, 83, 122, 181]. The basic fractional order calculus might not stipulate the essential kernel so as to obtain required information from such kinds of systems. In the theory of fractional calculus, the classical integral of a function is iterated, use the Cauchy formula to obtain the higher order integral and then change this integer by a complex number. At this time, the subsequent question is asked. Can the traditional Riemann-Liouville fractional integrals be generalized in such a manner that we are able to unify major existing fractional derivatives [101, 103]. The key of such practice is to decide about derivative operator which must be used as an initial step for the iteration procedure. The conformable integral was suggested to be fractionalized properly in [1]. The conformable operators appear in mathematical economics, i.e. they describe reduced economical dynamics [81] as well as the non-linear dissipative systems [81].

Mathematics is not only the subject of "equalities". The relation between different values and quantities is studied not only in mathematics but in all subjects of science. When two expressions are unequal, the connection between these expressions is stated as an inequality. The mathematical inequalities are recognized as one of the important and fundamental fields of mathematics. Over the former years, the mathematical inequalities along with their applications in diverse fields of science and mathematics are well recognized. This theory is a fast rising region with useful applications in all scientific fields such as game theory, control theory, mathematical economics, variational methods, mathematical programming, operational research and statistics. Consequently, the discipline of inequalities has been developed as an independent branch of mathematical analysis.

The fractional integral inequalities are much important for their competencies to study the positive solutions and the existence of non-trivial solutions of various types of fractional order differential equations. Such inequalities are utilized in many subjects of science: physics, mathematics, engineering and many others [13, 14, 15, 16, 26, 29, 41, 42, 43, 50, 174]. A large bulk of work is available in the literature on the inequalities by involving fractional integral operators and their generalized forms, see [17, 18, 19, 20, 44, 65, 66, 70, 71, 74, 67, 76, 126, 184, 187] and references therein. Numerous generalizations, extensions and variations have existed in the literature (e.g. [22, 47, 48, 72, 73, 117, 136, 147]). The following studies [125, 132, 147, 152] and the references therein can be referred for details. For inequalities involving generalized fractional operators we refer [34, 35, 62, 138, 139, 140, 141, 156]. Many k-fractional operators, their properties, related identities and inequalities are proved during past several years (see for instance [11, 22, 84, 128, 162, 172]).

This research work is a combination of nine chapters. In the first chapter, a precise introduction of fractional order calculus along with its applications in all fields of science are illustrated. This chapter also contains the importance of inequalities in mathematical analysis, the generalized inequalities involving classical and fractional integral operators and their applications. A brief overview of existing generalized fractional operators and related inequalities appeared in literature is given in the subsequent section.

The second chapter is about preliminaries and fundamentals of fractional calculus consisting of classical and generalized fractional operators. This chapter also includes our newly defined k-analogue of fractional conformable derivatives, integral operators with their existence and the fractional operators generalized according to another function. In the last section of this chapter, some important properties of k-fractional conformable operators are depicted.

The purpose of third chapter is to describe Chebyshev functional for two synchronous functions and generalize Chebyshev type integral inequalities involving our defined k-fractional conformable integral operators for one and two fractional parameters. This work has been published [86] in **"Journal of Inequalities and Special functions 9(4) (2018) 53–65 "**.

Chapter 4 is focused on a brief introduction of Minkowski's inequality along existing applications, the generalization of Minkowski's inequality by involving our proposed generalized k-conformable fractional integrals given in the second chapter. The related results of Minkowski's inequality are also presented for generalized k-FCI. The work in this chapter has been printed [130] in **" Journal of Inequalities and Applications, (2019)"**.

Chapter 5 contains the discussion of some new integral results regarding convex functions via our introduced generalized k-fractional conformable integral operators. We generalize classical integral inequalities presented in [119] for convex functions involving generalized k-fractional conformable integral. We also deduce some other classical integral inequalities as particular cases for our results. The contents of this chapter have been accepted in **"Punjab University Journal of Mathematics, 2019"**.

Chapter 6 deals with the generalization of integral inequalities reported in [162, 173, 175] associated with generalized k-conformable fractional integrals, for a finite sequence of n positive decreasing functions. The contents of this chapter have been accepted in **"AIMS Mathematics, 2019"**.

In chapter 7, Gr$\ddot{u}$ss type inequality with example, an improved version of generalized Gr$\ddot{u}$ss type integral inequality associated with k-analogue of fractional conformable integrals are presented. This chapter also contains some new mathematical results along with their proofs. The contents of this chapter are submitted.

Chapter 8 deals with the generalization of classical Chebyshev-Gr$\ddot{u}$ss type inequalities and the weighted Gr$\ddot{u}$ss inequalities for generalized k-fractional conformable integrals. The main results are presented using one and two fractional parameters. The contents of this chapter are submitted.

Chapter 9 contributes this thesis by presenting some new P$\acute{o}$lya-Szeg$\ddot{o}$ type inequalities using generalized k-fractional conformable integral. The results are further used to derive some inequalities of Chebyshev type. The last section contains the applications of the derived results to a function constrained by the Heaviside functions. The contents of this chapter are submitted.

1.2 Review of Literature

Fractional calculus theory was initiated on 30*th* September, 1695 by a letter written from Leibniz to L'Hospital. In this letter, the prospect of generalization of the definition of differential operator from integer to non-integer order was arose. L'Hospital aspired to realize the consequence for the half order derivative. Leibniz answered that "one day, useful results will be gotten" and, in fact, this visualization turned into veracity. Though, the exploration of non-integer order derivatives had not been appeared in the literature till 1819, when Lacroix defined fractional order derivative founded on the traditional idea for the nth order derivative of the power function. As the time passed, the fractional theory grew as an innovative field of mathematics. Several different formulae of differential operators of fractional order were introduced. Miller and Ross [123], Kiryakova [111] gave a precise depiction of the fractional calculus operators as well as their properties and applications. In 2010, Agarwal [8] introduced an interesting perception to the subject of fractional calculus by merging all classical definitions of fractional derivatives and integrals. In the years thereafter, such unifications were studied by Klimek and Lupa (2013), Odzijewicz et al. (2012a, b, 2013a, b, c), Bourdin et al. (2014). Particularly, the general operators, under special kernels, mold to the standard fractional order operators. Though, additional non-standard kernels may also be studied as particular cases.

Krasniqi [114] in (2010) proved the monotonicity and some important inequalities related to the ratio of gamma k-function. Mubeen and Habibullah [127] extend the generalization of fractional integrals by involving the parameter $k > 0$ with some properties and applications. Kokologiannaki and Krasniqi [112] in (2013) presented complete monotonicity characteristics and inequalities regarding the gamma k-function and k-psi functions. They also proposed the Riemann, k-zeta function along related inequalities of gamma and $k-$zeta functions. In (2013), some Hermite-Hadamard inequalities were obtained by Liao et al. [116] for geometric-arithmetic s-convex differentiable functions. Romero et al. [149] in (2013) generalized the Riemann Liouville fractional derivative by using k-gamma function in the form of k-Riemann-Liouville derivative and proved some important results of their newly introduced fractional operator. They also induced its bond with k-Riemann Liouville fractional integral. In (2013), Tunç [175] used Riemann Liouville fractional integral to verify the Hermite Hadamard type inequalities for h-analogue of convex functions . In the same year, Kokologiannaki et al. [112, 113] involved gamma and beta k-functions in the generalization of certain inequalities, They also established the bounds for beta k-function by using generalization of bounds procured by Starc [169].
Baleanu and Agarwal [36] in (2014) used the two parameters of deformation to establish some inequalities containing Saigo q-fractional integral operator in quantum calculus theory. They also exhibited the analogous inequalities of q-Kober and q-Riemann-Liouville fractional integrals as specific cases. In (2014), Choi and Agarwal [63] found new fractional integral inequalities of Saigo type and related q-version inequalities. They developed the fractional inequalities of Erdélyi-Kober and Riemann-Liouville operators as their particular cases.

In (2014), Agarwal et al. [10] generalized certain novel fractional integral inequalities involving q-Erdélyi-Kober fractional integral by considering the synchronous functions and the functions confined by integrable functions. In (2014), Brahim and Sidumou [49] used q-integral inequalities to establish useful inequalities for beta, gamma and psi (q, k)-functions. In the same year, Chen [57] obtained Hermite Hadamard type inequalities via

Riemann-Liouville integrals for products of convex functions. In (2014), Sarikaya and Karaça [156] generalized the Riemann-Liouville k-fractional integral along their properties as well as they verified related Chebyshev integral inequalities. Choi et al. [64] in (2015) established some generalized q-Erd$\acute{e}$lyi-Kober fractional inequalities with the two parameters of deformation. Katugampola [103] presented a new fractional order derivative operator to generalize the known Riemann–Liouville and Hadamard derivatives under a solo expression. Furthermore, many researches on fractional order integral and differential operators and the related inequalities of Minkowski, Grüss, Hermite-Hadamard and Ostrowski type can be studied in [160, 178, 57, 10, 186, 155, 166, 167, 90, 153, 185, 154, 120]. In (2015), Kaçar and Yildirim [100] presented some important results related to Gr$\ddot{u}$ss type inequality. Set et al. [162] employed Grüss inequality and relevant results for k-Riemann Liouville fractional integrals.

Sarikaya et al. in (2016) [157] unveiled a more generalized form of $k-$Riemann Liouville fractional integral as (k, s) Riemann Liouville integral. They proved important properties for their presented integral along with some new integral inequalities. Wang et al. [177] proved Hermite-Hadamard inequalities for s-analogue of convex functions with applications. In (2016), Mubeen and Iqbal [128] engaged k-Riemann–Liouville fractional integral to present the classical Grüss type integral inequalities. Baleanu et al. [39] verified the inequalities associated with the weighted Chebyshev functional via generalized Erdélyi-Kober fractional integral. In (2017), Mubeen et al. [129] established Ostrowski type inequalities for k-version of Riemann–Liouville integrals. In (2017), Agarwal et al. [9] utilized generalized k-fractional integrals to verify Hermite–Hadamard inequalities. In (2017), Set et al. [161] generalized Hermite–Hadamard inequalities and an integral identity for Riemann–Liouville fractional integral. In (2017), Sarikaya and Budak [158] derive a general inequality associated with local fractional integrals. In (2017), Jleli et al. [99] concluded Hartman–Winter type inequality engaging fractional derivative dependent on another function. In (2017), Khan et al. [109] generalized some inequalities involving a finite sequence of positive decreasing functions via conformable fractional integrals.

In (2018), Iqbal et al. [93] generalized Hadamard fractional integral by involving k-

gamma function with some integral inequalities. Later, Iqbal et al. [94] proved Chebyshev–Grüss inequalities for fractional k-Hadamard integrals. Some important inequalities, their applications and their stability are presented in [6, 7, 106, 115, 104, 165]. Such generalizations inspire the researchers to contribute the more novel conceptions to associate the operators of fractional order and generalize the associated integral inequalities.

Chapter 2

GENERALIZATION OF FRACTIONAL CONFORMABLE OPERATORS

The core objective of this chapter is to introduce k-fractional conformable derivatives and integrals operators. These operators are the generalization of conformable fractional operators presented in [98] involving the extended gamma function. Our presented integral operators merge the existing fractional operators [151, 25, 101, 156] in a single form. Before this, we summon up some notations, preliminaries and tools about classical and generalized theory of fractional calculus. These terms and basic definitions are compulsory to study before formulating, presenting our newly introduced fractional operators and their applications. These are helpful to understand the work and will be utilized in our work.

2.1 Classical Fractional Operators

Definition 2.1.1 *A function $g(t)$ belongs to $L_q[a,b]$ if*

$$\left(\int_a^b |g(t)|^q dt\right)^{\frac{1}{q}} < \infty, \quad 1 \leq q < \infty. \tag{2.1.1}$$

Definition 2.1.2 *A function $g(t)$ belongs to $L_q^r[a,b]$ if*

$$\left(\int_a^b |g(t)|^q t^r dt\right)^{\frac{1}{q}} < \infty, \quad 1 \leq q < \infty, r \geq 0. \tag{2.1.2}$$

Definition 2.1.3 *The space $X_c^q(a,b)$ for $a,b,c \in \mathbb{R}$ with $a < b$ and $q \in [1,\infty)$ contains such Lebesgue measurable complex valued functions h on (a,b), with $\|h\|_{X_c^q}$ where*

$$\|h\|_{X_c^q} = \left(\int\limits_a^b |t^c h(t)|^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty; \quad (1 \leq q < \infty) \tag{2.1.3}$$

and for $q = \infty$,

$$\|h\|_{X_{c,\infty}} = supess_{t\in(a,b)} \left[t^c \left|h(t)\right|\right]. \tag{2.1.4}$$

In particular, the space $X_c^q(a,b)$ *becomes the space* $L_q(a,b)$ *for* $c = \frac{1}{q}$, [110].

Definition 2.1.4 *Two functions* $g_1,\, g_2 \subseteq \mathbb{R} \to \mathbb{R}$ *are said to be synchronous (asynchronous) on* $[a,b] \subseteq \mathbb{R}$*, if*

$$(g_1(x_1) - g_1(x_2))(g_2(x_1) - g_2(x_2)) \geq (\leq) 0 \;\; for\ all\ x_1,\, x_2 \in [a,b]. \tag{2.1.5}$$

Definition 2.1.5 *The Riemann–Liouville integrals (left- and right-sided)* [84] *of order* β *for* $\beta \in \mathbb{C}$*, Re*$(\beta) > 0$ *have the following forms:*

$$\left(\mathcal{R}^{\beta}_{a^+} g\right)(\mathcal{T}^*) = \frac{1}{\Gamma(\beta)} \int\limits_a^{\mathcal{T}^*} g(t) \frac{dt}{(\mathcal{T}^* - t)^{1-\beta}}, \tag{2.1.6}$$

$$\left(\mathcal{R}^{\beta}_{b^-} g\right)(\mathcal{T}^*) = \frac{1}{\Gamma(\beta)} \int\limits_{\mathcal{T}^*}^{b} g(t) \frac{dt}{(t - \mathcal{T}^*)^{1-\alpha}}. \tag{2.1.7}$$

Definition 2.1.6 *The Riemann–Liouville fractional derivative (left- and right-sided)* [84, 136] *of order* β *for* $\beta \in \mathbb{C}$ *and Re*$(\beta) > 0$*, are defined below*

$$\left(\mathcal{R}'^{\beta}_{a^+} g\right)(\mathcal{T}^*) = \left(\frac{d}{dt}\right)^n \left(\mathcal{R}^{n-\beta}_{a^+} g\right)(\mathcal{T}^*); \quad n = [Re(\beta)] + 1, \tag{2.1.8}$$

$$\left(\mathcal{R}'^{\beta}_{b^-} g\right)(\mathcal{T}^*) = \left(-\frac{d}{dt}\right)^n \left(\mathcal{R}^{n-\beta}_{b^-} g\right)(\mathcal{T}^*), \quad n = [Re(\beta)] + 1. \tag{2.1.9}$$

Definition 2.1.7 *The Caputo fractional derivatives (left- and right-sided)* [51] *of order* β*, Re*$(\beta) > 0$ *have the form:*

$$\left({}_C\mathbb{D}^{\beta}_{a^+} g\right)(\mathcal{T}^*) = \left(\mathcal{R}^{n-\beta}_{a^+} g^{(n)}\right)(\mathcal{T}^*); \quad n = [Re(\beta)] + 1 \tag{2.1.10}$$

and

$$\left({}_C\mathbb{D}^{\beta}_{b^-} g\right)(\mathcal{T}^*) = \left(\mathcal{R}^{n-\beta}_{b^-} (-1)^n g^{(n)}\right)(\mathcal{T}^*); \quad n = [Re(\beta)] + 1. \tag{2.1.11}$$

respectively.

Definition 2.1.8 *The Hadamard fractional integrals (left- and right-sided)* [88] *of order*

$\beta \in \mathbb{C}$, Re$(\beta) > 0$ are given below:

$$\left(\mathcal{H}^{\beta}_{a^+} g\right)(\mathcal{T}^*) = \frac{1}{\Gamma(\beta)} \int\limits_a^{\mathcal{T}^*} (\ln \mathcal{T}^* - \ln t)^{\beta-1} g(t) \frac{dt}{t} \tag{2.1.12}$$

and

$$\left(\mathcal{H}^{\beta}_{b^-} g\right)(\mathcal{T}^*) = \frac{1}{\Gamma(\beta)} \int\limits_{\mathcal{T}^*}^{b} (\ln t - \ln \mathcal{T}^*)^{\beta-1} g(t) \frac{dt}{t}. \tag{2.1.13}$$

respectively.

Definition 2.1.9 *The Hadamard fractional derivative (left- and right-sided)* [88] *of order $\beta \in \mathbb{C}$, Re$(\beta) > 0$ can be defined as:*

$$\left(\mathcal{H'}^{\beta}_{a^+} g\right)(\mathcal{T}^*) = \left(t \frac{d}{dt}\right)^n \left(\mathcal{H}^{n-\beta}_{a^+} g\right)(\mathcal{T}^*); \quad n = [Re(\beta)] + 1 \tag{2.1.14}$$

and

$$\left(\mathcal{H'}^{\beta}_{b^-} g\right)(\mathcal{T}^*) = \left(-t \frac{d}{dt}\right)^n \left(\mathcal{H}^{n-\beta}_{b^-} g\right)(\mathcal{T}^*), \quad n = [Re(\beta)] + 1. \tag{2.1.15}$$

respectively.

Definition 2.1.10 *The Katugampola fractional integral (left- and right-sided)* [101] *of order $\beta \in \mathbb{C}$, Re$(\beta) > 0$ are given below:*

$$\left({}^{\rho}\mathcal{K}^{\beta}_{a^+} g\right)(\mathcal{T}^*) = \frac{\rho^{(1-\beta)}}{\Gamma(\beta)} \int\limits_a^{\mathcal{T}^*} \frac{t^{\rho-1} g(t) dt}{((\mathcal{T}^*)^{\rho} - t^{\rho})^{1-\beta}}; \quad t > a, \rho > 0 \tag{2.1.16}$$

and

$$\left({}^{\rho}\mathcal{K}^{\beta}_{b^-} g\right)(\mathcal{T}^*) = \frac{\rho^{(1-\beta)}}{\Gamma(\beta)} \int\limits_{\mathcal{T}^*}^{b} \frac{t^{\rho-1} g(t) dt}{(t^{\rho} - (\mathcal{T}^*)^{\rho})^{1-\beta}}; \quad t < b, \rho > 0. \tag{2.1.17}$$

Definition 2.1.11 *The Katugampola fractional derivative (left- and right-sided) [101] of order $\beta \in \mathbb{C}$, Re$(\beta) > 0$ are defined as:*

$$\left(\mathcal{K'}^{\beta}_{a^+} g\right)(\mathcal{T}^*) = \left(\mathcal{T}^{*(1-\rho)} \frac{d}{d\mathcal{T}^*}\right)^n \frac{\rho^{(1-n+\beta)}}{\Gamma(n-\beta)} \int\limits_a^{\mathcal{T}^*} \frac{t^{\rho-1} g(t) dt}{((\mathcal{T}^*)^{\rho} - t^{\rho})^{1-n+\beta}}; \quad n = [Re(\beta)] + 1 \tag{2.1.18}$$

and

$$\left(\mathcal{K}'^{\beta}_{b^-} g\right)(\mathcal{T}^*) = \left(-\mathcal{T}^{*(1-\rho)}\frac{d}{d\mathcal{T}^*}\right)^n \frac{\rho^{(1-n+\beta)}}{\Gamma(n-\beta)} \int\limits_{\mathcal{T}^*}^{b} \frac{t^{\rho-1} g(t) dt}{(t^\rho - (\mathcal{T}^*)^\rho)^{1-n+\beta}}; \quad n = [Re(\beta)]+1. \tag{2.1.19}$$

Definition 2.1.12 *Abdeljawad* [1] *defined the conformable fractional integrals (left- and right-sided) of a function $g \in L_1[a,b]$ of order $\beta \geq 0$ as*

$$\mathcal{A}^{\beta}_{a^+} g(\mathcal{T}^*) = \int_a^{\mathcal{T}^*} (t-a)^{\beta-1} g(t) dt; \quad \beta \in (0,1], 0 \leq a < x < b \leq \infty \tag{2.1.20}$$

and

$$\mathcal{A}^{\beta}_{b^-} g(\mathcal{T}^*) = \int_{\mathcal{T}^*}^{b} (b-t)^{\beta-1} g(t) dt; \quad 0 \leq a < x < b \leq \infty . \beta \in (0,1]. \tag{2.1.21}$$

respectively.

Definition 2.1.13 *The conformable fractional derivatives* [1] *(left- and right-sided) can be written as*

$$\mathcal{A}'^{\beta}_{a^+} g(\rho^*) = (\rho^* - a)^{1-\beta} g'(\rho^*), \tag{2.1.22}$$

$$\mathcal{A}'^{\beta}_{b^-} g(\rho^*) = (b - \rho^*)^{1-\beta} g'(\rho^*). \tag{2.1.23}$$

2.2 Generalized Fractional Operators

Definition 2.2.1 *The generalized Riemann–Liouville fractional integrals* [127] *(left- and right-sided) of a function $f \in L_1^r[a,b]$, of order $\beta \geq 0$ and $s \geq 0$ are defined by*

$$\mathcal{R}^{\beta,s}_{a^+} g(\mathcal{T}^*) = \frac{s^{1-\beta}}{\Gamma(\beta)} \int_a^{\mathcal{T}^*} (\mathcal{T}^{*s} - t^s)^{\beta-1} t^{s-1} g(t) dt; \quad \mathcal{T}^* \in [a,b] \tag{2.2.24}$$

and

$$\mathcal{R}^{\beta,s}_{b^-} g(\mathcal{T}^*) = \frac{s^{1-\beta}}{\Gamma(\beta)} \int_{\mathcal{T}^*}^{b} (t^s - \mathcal{T}^{*s})^{\alpha-1} t^{s-1} g(t) dt; \quad \mathcal{T}^* \in [a,b], \tag{2.2.25}$$

where Γ is the Euler classical gamma function [145, 146].

Definition 2.2.2 *The generalized k-Riemann–Liouville fractional integrals* [156] *(left- and right-sided) of a function $f \in L_1^r[a,b]$, of order $\beta \geq 0$ and $k > 0, s \geq 0$ are defined by*

$$\mathcal{R}_{a^+,k}^{\beta,s} g(\mathcal{T}^*) = \frac{s^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_a^x (\mathcal{T}^{*s} - t^s)^{\frac{\beta}{k}-1} t^{s-1} g(t) dt; \quad \mathcal{T}^* \in [a,b], \tag{2.2.26}$$

$$\mathcal{R}_{b^-,k}^{\beta,s} g(\mathcal{T}^*) = \frac{s^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_x^b (t^s - \mathcal{T}^{*s})^{\frac{\beta}{k}-1} t^{s-1} g(t) dt; \quad \mathcal{T}^* \in [a,b], \tag{2.2.27}$$

where Γ_k is the gamma k-function.

Definition 2.2.3 *The generalized Katugampola fractional integrals* [101] *(left- and right-sided) of a real function $f \in X_c^q(a,b)$, of order $\beta \in \mathbb{C}, s > 0$, Re$(\beta) > 0$ take the form*

$$\left(\mathcal{K}_{a^+}^{\beta,s} g\right)(\mathcal{T}^*) = \frac{s^{1-\beta}}{\Gamma(\beta)} \int_a^{\mathcal{T}^*} g(t) \frac{t^{s-1} dt}{(\mathcal{T}^{*s} - t^s)^{1-\beta}} \tag{2.2.28}$$

and

$$\left(\mathcal{K}_{b^-}^{\beta,s} g\right)(\mathcal{T}^*) = \frac{s^{1-\beta}}{\Gamma(\beta)} \int_{\mathcal{T}^*}^{b} g(t) \frac{t^{s-1} dt}{(t^s - \mathcal{T}^{*s})^{1-\beta}}, \tag{2.2.29}$$

respectively.

Definition 2.2.4 *The generalized Katugampola fractional derivatives* [103] *(left- and right-sided) of order $\beta \in \mathbb{C}, s > 0$, Re$(\beta) > 0$ are defined below*

$$\left(\mathcal{K'}_{a^+}^{\alpha,s} g\right)(\mathcal{T}^*) = \gamma^n \left(\mathcal{K}_{a^+}^{n-\beta,\rho} f\right)(\mathcal{T}) = \frac{\gamma^n \rho^{n-\alpha}}{\Gamma(n-\beta)} \int_a^{\mathcal{T}^*} g(t) \frac{t^{s-1} dx}{(\mathcal{T}^{*s} - t^s)^{1+\beta-n}} \tag{2.2.30}$$

and

$$\left(\mathcal{K'}_{b^-}^{\alpha,s} g\right)(\mathcal{T}^*) = (-\gamma)^n \left(\mathcal{K}_{b^-}^{n-\alpha,\rho} f\right)(\mathcal{T}) = \frac{(-\gamma)^n \rho^{n-\beta}}{\Gamma(n-\beta)} \int_{\mathcal{T}^*}^{b} g(t) \frac{t^{s-1} dt}{(t^s - \mathcal{T}^{*s})^{1+\beta-n}}, \tag{2.2.31}$$

respectively, where $\gamma = \mathcal{T}^{*(1-s)} \frac{d}{d\mathcal{T}^*}$.

Definition 2.2.5 *The left conformable fractional integral operator (FCI) of order $\gamma \in \mathbb{C}, Re(\gamma) > 0$,* [98] *is obtained by the iteration of the left integral in* (2.1.20) *γ times and* *result as*

$$ {}^{\gamma}_{s}\mathcal{A}^{*}{}_{a^{+}}g(\mathcal{T}^{*}) = \frac{1}{\Gamma(\gamma)} \int\limits_{a}^{\mathcal{T}^{*}} \left(\frac{(\mathcal{T}^{*}-a)^{s} - (t-a)^{s}}{s} \right)^{\gamma-1} f(t) \frac{dt}{(t-a)^{1-s}}.. \tag{2.2.32} $$

The right conformable fractional integral operator (FCI) *of order* $\gamma \in \mathbb{C}, Re(\gamma) > 0$, [98] *is obtained by the iteration of the right integral in* (2.1.21) γ *times and result as*

$$ {}^{\gamma}_{s}\mathcal{A}^{*}{}_{b^{-}}g(\mathcal{T}^{*}) = \frac{1}{\Gamma(\gamma)} \int\limits_{\mathcal{T}^{*}}^{b} \left(\frac{(b-\mathcal{T}^{*})^{s} - (b-t)^{s}}{s} \right)^{\gamma-1} f(t) \frac{dt}{(b-t)^{1-s}},. \tag{2.2.33} $$

where $\Gamma(\gamma)$ *is the Gamma function of* γ *and defined as*

$$ \Gamma(\gamma) = \int\limits_{0}^{+\infty} e^{-u} u^{\gamma-1} du. \tag{2.2.34} $$

2.3 Generalized k-Fractional Conformable Operators

Here, we present our main work on the generalized k-fractional operators. In this section, we introduce the generalized k-fractional conformable operators and a new generalized k-fractional operator by involving a new parameter $k > 0$ and an unknown function under some specific bounds.

2.3.1 Generalized k-Fractional Conformable Integrals

For $g \in L_{1,r}[a,b]$, the generalized left k-fractional conformable integral of a continuous function g on $[0,\infty)$ of order $\beta \in \mathbb{C}; Re(\beta) > 0$ is given as

$$ {}^{s}_{k}\mathfrak{F}^{*\beta}{}_{a^{+}}g(x) = \frac{1}{k\Gamma_{k}(\beta)} \int\limits_{a}^{\mathcal{T}^{*}} \left(\frac{(\mathcal{T}^{*}-a)^{s} - (t-a)^{s}}{s} \right)^{\frac{\beta}{k}-1} g(t) \frac{dt}{(t-a)^{1-s}}, \mathcal{T}^{*} \in [a,b], \tag{2.3.35} $$

if integral exists, such that $k > 0, s \in \mathbb{R}\backslash\{0\}$.

For $g \in L_{1,r}[a,b]$, the generalized right $k-$ fractional conformable integral of a continuous function g on $[0,\infty)$ of order $\beta \in \mathbb{C}; Re(\beta) > 0$ is given as

$$ {}_{k}^{s}\mathfrak{F}^{*\beta}_{b^-} g(\mathcal{T}^*) = \frac{1}{k\Gamma_k(\beta)} \int_{\mathcal{T}^*}^{b} \left(\frac{(b-\mathcal{T}^*)^s - (b-t)^s}{s} \right)^{\frac{\beta}{k}-1} g(t) \frac{dt}{(b-t)^{1-s}}, \mathcal{T}^* \in [a,b], \tag{2.3.36} $$

if integral exists, where $k > 0, s \in \mathbb{R}\backslash\{0\}$.

It can be observed that

(i) When $k \to 1$, the integrals (2.3.35) and (2.3.36) switch to generalized fractional conformable integrals (2.2.32) and (2.2.33).

(ii) For $s = 1$, the integrals (2.3.35) and (2.3.36) induce k-Riemann Liouville fractional integrals (2.2.26) and (2.2.27).

Furthermore, they move to classical Riemann Liouville fractional integrals (2.1.6) and (2.1.7) for $k \to 1$.

(iii) For $s = 1, a = -\infty$, the integrals (2.3.35) and (2.3.36) reduce to k-Weyl fractional integrals [150]. Furthermore, they reduce to Weyl fractional integral [124] for $k \to 1$.

2.3.2 Existence of Generalized k-Fractional Conformable Integrals

Theorem 2.3.1 *For $g \in L_1^r([a,b])$, the generalized k-fractional conformable integrals $\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+} g(y), {}_{k}^{s}\mathfrak{F}^{*\beta}_{b^-} g(y)\right)$ exists for any $y \in [a,b]$, $Re(\beta) > 0, s \in \mathbb{R}\backslash\{0\}$ and $k > 0$.*

Proof. Define $\Omega^* := [a,b]^2$ and $K^* : \Omega^* \to \mathbb{R}$ such that

$$ K^*(y,t) = ((y-a)^s - (t-a)^s)^{\frac{\beta}{k}-1} (t-a)^{s-1}. $$

Clearly, we can see

$$ K^* = K^*_+ + K^*_-, $$

where

$$ K^*_+(y,t) := \begin{cases} ((y-a)^s - (t-a)^s)^{\frac{\beta}{k}-1} (t-a)^{s-1}, & a \le t \le y \le b, \\ 0, & a \le y \le t \le b. \end{cases} $$

and

$$K_{-}^{*}(y,t) := \begin{cases} ((t-a)^s - (y-a)^s)^{\frac{\alpha}{k}-1}(y-a)^{s-1}, & a \le t \le y \le b, \\ 0, & a \le y \le t \le b. \end{cases}$$

since K^* is measurable on Ω^*, then it may be written as

$$\begin{aligned} \int_a^b K^*(y,t)dt = \int_a^y K^*(y,t)dt &= \int_a^y ((y-a)^s - (t-a)^s)^{\frac{\beta}{k}-1}(t-a)^{s-1}dt \\ &= \frac{sk}{\beta}(y-a)^{\frac{s\beta}{k}} \end{aligned}$$

Taking the double integral, we have

$$\begin{aligned} \int_a^b \left[\int_a^b K^*(x,t)|g(y)|dt\right] dx &= \int_a^b |g(y)| \left[\int_a^b K^*(y,t)dt\right] dx \\ &= \frac{sk}{\beta}\int_a^b (y-a)^{\frac{s\beta}{k}}|g(y)|dy \\ &\le \frac{sk}{\beta}(b-a)^{\frac{s\beta}{k}}\int_a^b |g(y)|dy, \end{aligned}$$

that is,

$$\begin{aligned} \int_a^b \left(\int_a^b K^*(y,t)|g(y)|dt\right) dy &= \int_a^b |g(y)| \left(\int_a^b K^*(y,t)dt\right) dy \\ &\le \frac{sk}{\beta}(b-a)^{\frac{s\beta}{k}}\|g(y)\|_{L_1[a,b]} < \infty. \end{aligned}$$

So, the function $T^* : \Omega^* \to \mathbb{R}$ defined by $T^*(y,t) := K^*(y,t)g(y)$ is integrable over Ω^* by Tonelli's theorem [89, p. 147]. Hence, by Fubini's theorem $\int_a^b K^*(y,t)g(y)dy$ is an integrable function over $[a,b]$, as a function of $t \in [a,b]$. i.e.${}^{s}_{k}\mathfrak{F}^{*\alpha}_{a+}g(y)$ exists.
In the similar manner, we can prove the existence of the right conformable k-fractional integral ${}^{s}_{k}\mathfrak{F}^{*\alpha}_{b-}g(y)$. ■

2.3.3 Generalized k-Fractional Conformable Derivatives in Riemann-Liouville setting

The generalized left k-fractional conformable derivative in Riemann-Liouville setting of a continuous function g on $[0,\infty)$. of order $\beta\in\mathbb{C}, Re(\beta)>0$ is stated as

$$ {}_k^s\mathfrak{C}_{a+}^{\beta}g(x)=\frac{{}^sT_{a+}^{n}}{k\Gamma_k\left(nk-\beta\right)}\int\limits_a^x\left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{\frac{nk-\beta}{k}-1}g(t)\frac{dt}{(t-a)^{1-s}}, \quad (2.3.37) $$

provided it exists, where ${}^sT_{a^+}^{n}=\left[(x-a)^{1-s}\frac{d}{dx}\right]^n$ and $n=[Re(\beta)]+1$.
Eq. (2.3.37) can be rewritten as

$$ {}_k^s\mathfrak{C}_{a+}^{\beta}g(x)=\left({}^sT_{a+}^{n}\right)\left({}_k^s\mathfrak{F}^{*}{}_{a^+}^{nk-\beta}g\right)(x), \quad (2.3.38) $$

where $\left({}_k^s\mathfrak{F}^{*}{}_{a^+}^{nk-\beta}g\right)(x)$ is defined in (2.3.35).
The generalized right k-fractional conformable derivative in Riemann-Liouville of a continuous function g on $[0,\infty)$ of order $\beta\in\mathbb{C}, Re(\beta)>0$ setting is stated as

$$ {}_k^s\mathfrak{C}_{b-}^{\beta}g(x)=\frac{{}^sT_{b^-}^{n}}{k\Gamma_k\left(nk-\beta\right)}\int\limits_x^b\left(\frac{(b-x)^s-(b-t)^s}{s}\right)^{\frac{nk-\beta}{k}-1}g(t)\frac{dt}{(b-t)^{1-s}};\quad x\in[a,b], \quad (2.3.39) $$

provided it exists, where $n=[Re(\beta)]+1$ and ${}^sT_{b^-}^{n}=\left[(b-x)^{1-s}\frac{d}{dx}\right]^n$.
Eq. (2.3.39) can be rewritten as

$$ {}_k^s\mathfrak{C}_{b-}^{\beta}g(x)=\left({}^sT_{b^-}^{n}\right)\left({}_k^s\mathfrak{F}^{*}{}_{b^-}^{nk-\beta}g\right)(x), \quad (2.3.40) $$

where $\left({}_k^s\mathfrak{F}^{*}{}_{b^-}^{nk-\beta}g\right)(x)$ is defined in (2.3.36).

It can be observed that

(i) When $k\to1$, the derivatives (2.3.37) and (2.3.39) reduce to generalized fractional conformable derivatives [98].

(ii) For $s=1$, the derivatives (2.3.37) and (2.3.39) reduce to k-Riemann Liouville fractional derivatives [149]. Furthermore, they reduce to Riemann Liouville fractional derivatives [84] for $k \to 1$.

(iii) For $s = 1, a = -\infty$, the derivatives (2.3.37) and (2.3.39) reduce to k-Weyl fractional derivatives [150]. Furthermore, they reduce to Weyl fractional derivatives [124] for $k \to 1$.

2.3.4 Caputo type Generalized k-Fractional Conformable Derivatives

For continuous function g on $[0, \infty)$, the generalized left k-fractional conformable derivative of order $\beta \in \mathbb{C}, Re(\beta) > 0$ in Caputo setting is stated as:

$$
{}_{k,s}^{C}\mathfrak{C}_{a^+}^{\beta} g(x) = \frac{1}{k\Gamma_k(nk-\beta)} \int_a^x \left(\frac{(x-a)^s - (t-a)^s}{s}\right)^{\frac{nk-\beta}{k}-1} \frac{{}^sT_{a^+}^n g(t)}{(t-a)^{1-s}} dt; \quad \forall x \in [a, b], \tag{2.3.41}
$$

provided it exists, where $n = [Re(\beta)] + 1, s > 0, k \in \mathbb{N}, (n-1)k < \beta < nk$ and ${}^sT_{a^+}^n = \left[(t-a)^{1-s}\frac{d}{dt}\right]^n$.

Eq. (2.3.41) can be rewritten as

$$
{}_{k,s}^{C}\mathfrak{C}_{a^+}^{\alpha} g(x) = \left({}_{k}^{s}\mathfrak{F}^{*nk-\alpha}_{a^+}\right)\left({}^sT_{a^+}^n g\right)(x), \tag{2.3.42}
$$

where $\left({}_{k}^{s}\mathfrak{F}^{*nk-\beta}_{a^+} g\right)(x)$ is defined in Eq. (2.3.35).

The generalized right k-fractional conformable derivative of a continuous function g on $[0, \infty)$ of order $\beta \in \mathbb{C}, Re(\beta) > 0$ in Caputo setting is stated as:

$$
{}_{k,s}^{C}\mathfrak{C}_{b^-}^{\beta} g(x) = \frac{1}{k\Gamma_k(nk-\beta)} \int_x^b \left(\frac{(b-x)^s - (b-t)^s}{s}\right)^{\frac{nk-\beta}{k}-1} \frac{{}^sT_{b^-}^n g(t)}{(b-t)^{1-s}} dt; \quad \forall x \in [a, b], \tag{2.3.43}
$$

provided it exists, where $n = [Re(\beta)] + 1, s > 0, k \in \mathbb{N}, (n-1)k < \beta < nk$ and ${}^sT_{b^-}^n = \left[(b-t)^{1-s}\frac{d}{dt}\right]^n$.

Eq. (2.3.43) can be rewritten as

$$
{}_{k,s}^{C}\mathfrak{C}_{b^-}^{\beta} g(x) = \left({}_{k}^{s}\mathfrak{F}^{*nk-\beta}_{b^-}\right)\left({}^sT_{b^-}^n g\right)(x), \tag{2.3.44}
$$

where $\left({}^{s}_{k}\mathfrak{F}^{*nk-\beta}_{b^-}g\right)(x)$ is defined in Eq. (2.3.36).

It can be observed that

(i) When $k \to 1$, the derivatives (2.3.41) and (2.3.43) reduce to generalized fractional conformable derivatives in the Caputo setting [98].

(ii) For $s = 1$, the derivatives (2.3.41) and (2.3.43) reduce to k-analogue of Caputo type fractional derivatives. Furthermore, they reduce to Caputo fractional derivatives [51] for $k \to 1$.

(iii) For $s = 1, a = -\infty$, the derivatives (2.3.41) and (2.3.43) reduce to k-Weyl fractional derivatives of Caputo type [150]. Furthermore, we can find Weyl fractional derivatives [124] of Caputo type for $k \to 1$.

2.4 Generalized Fractional Operators involving another Function

2.4.1 Generalized Fractional Integrals involving another Function

Let f be continuous function on $[0,\infty)$ and $h : [a,b] \to \mathbb{R}$ be a positive monotonically increasing function on $(a,b]$ with continuous derivative $h'(x)$ on $a \leq x \leq b$, then the generalized k-fractional integrals (left- and right-sided) of f w.r.t. h of order $\alpha \in \mathbb{C}, Re(\alpha) > 0$ are given as

$$ {}^{h}_{k}\mathfrak{L}^{\alpha}_{a^+}f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t)f(t)dt; x \in [a,b] \tag{2.4.45}$$

and

$$ {}^{h}_{k}\mathfrak{L}^{\alpha}_{b^-}f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (h(t)-h(x))^{\frac{\alpha}{k}-1} h'(t)f(t)dt; x \in [a,b], \tag{2.4.46}$$

respectively, where $k > 0$.

It can be observed that

(i) For $k = 1$, the generalized fractional integrals (2.4.45) and (2.4.46) reduce to the fractional integrals of order α involving another function [151].

(ii) For $h(x) = \frac{(x-a)^s}{s}, h(t) = \frac{(t-a)^s}{s}$ $\left(h(x) = \frac{(b-x)^s}{s}, h(t) = \frac{(b-t)^s}{s}\right)$, where $s > 0$, the generalized fractional integrals (2.4.45) and (2.4.46) reduce to the generalized k-fractional conformable integrals (2.3.35) and (2.3.36) of order α respectively.

These generalized integrals (2.4.45) and (2.4.46) further reduce to the generalized conformable fractional integrals and fractional conformable integrals for $k \to 1$ and $s = 1$ respectively, under the same values of $h(x)$ and $h(t)$.

(iii) For $h(x) = \frac{x^s}{s}, h(t) = \frac{t^s}{s}$, where $s > 0$, the integrals (2.4.45) and (2.4.46) reduce to generalized k-Riemann Liouville fractional integrals (2.2.26) and (2.2.27) of order α respectively.

These generalized integrals further reduce to the generalized Riemann-Liouville fractional integrals and Riemann Liouville fractional integrals (2.2.24) and (2.2.25) for $k = 1$ and $s = 1$ respectively under the same values of $h(x)$ and $h(t)$.

(iv) For $h(x) = x, h(t) = t, a = -\infty$, the generalized $k-$fractional integrals (2.4.45) and (2.4.46) reduce to k-Weyl fractional integrals [150] which further reduce to Weyl fractional integrals [124] for $k \to 1$.

(v) For $h(x) = \frac{x^s}{s}, h(t) = \frac{t^s}{s}, s \to 0$, the generalized k-fractional integrals (2.4.45) and (2.4.46) reduce to k-Hadamard fractional integrals [93].

These generalized k-fractional integrals further reduce to the usual Hadamard fractional integrals (2.1.12) and (2.1.13) for $k \to 1$ under the same values of $h(x)$ and $h(t)$.

2.4.2 Generalized Fractional Derivatives involving another Function in Riemann-Liouville setting

Let f be continuous functions on $[0, \infty)$ and $h : [a, b] \to \mathbb{R}$ be a positive monotonically increasing function on $(a, b]$, with continuous derivative $h'(x)$ on $a \leq x \leq b$ such that $h^{'}(x) \neq 0 \forall x \in [a, b]$, then the generalized k-fractional derivative (left- and right-sided) of

f w.r.t. h of order $\alpha \in \mathbb{C}, Re(\alpha) > 0$ are given as

$$ {}_{k}^{h}\mathfrak{L}^{*\alpha}_{a^+} f(x) = \frac{1}{k\Gamma_k(nk-\alpha)} \left(\frac{1}{h'(x)}\frac{d}{dx}\right)^n \int_a^x (h(x)-h(t))^{\frac{nk-\alpha}{k}-1} h'(t) f(t) dt, x \in [a,b], \tag{2.4.47} $$

$$ {}_{k}^{h}\mathfrak{L}^{*\alpha}_{b^-} f(x) = \frac{1}{k\Gamma_k(nk-\alpha)} \left(\frac{1}{h'(x)}\frac{d}{dx}\right)^n \int_x^b (h(t)-h(x))^{\frac{nk-\alpha}{k}-1} h'(t) f(t) dt, x \in [a,b], \tag{2.4.48} $$

provided they exist, where $k > 0, n = [Re(\alpha)] + 1$.

It can be observed that

(i) For $h(x) = \frac{(x-a)^s}{s}, h(t) = \frac{(t-a)^s}{s}$ $\left(h(x) = \frac{(b-x)^s}{s}, h(t) = \frac{(b-t)^s}{s}\right)$, where $s > 0$, the generalized k-fractional derivatives (2.4.47) and (2.4.48) reduce to the generalized k-fractional conformable derivatives of order α (2.3.35) and (2.3.36) respectively.

These generalized k-fractional derivatives (2.4.47) and (2.4.48) further reduce to the generalized fractional conformable derivatives and fractional conformable derivatives for $k \to 1$ and $s = 1$ respectively under the same values of $h(x)$ and $h(t)$.

(ii) For $h(x) = \frac{x^s}{s}, h(t) = \frac{t^s}{s}$, where $s > 0$, the generalized k-fractional derivatives (2.4.47) and (2.4.48) reduce to generalized k-Riemann Liouville fractional derivatives [28]. These generalized k-fractional derivatives further reduce to the generalized Riemann-Liouville fractional derivatives and Riemann Liouville fractional derivatives (2.1.8) and (2.1.9) for $k = 1$ and $s = 1$ respectively under the same values of $h(x)$ and $h(t)$.

(iii) For $h(x) = x, h(t) = t, a = -\infty$, the generalized k-fractional derivatives (2.4.47) and (2.4.48) reduce to k-Weyl fractional derivatives [150] which further reduce to Weyl fractional derivatives [124] for $k \to 1$.

(iv) For $h(x) = \frac{x^s}{s}, h(t) = \frac{t^s}{s}, s \to 0$, the generalized k-fractional derivatives (2.4.47) and (2.4.48) reduce to k-Hadamard fractional derivatives.

These generalized k-fractional derivatives further reduce to the usual Hadamard fractional derivatives for $k \to 1$ under the same values of $h(x)$ and $h(t)$.

2.4.3 Generalized Fractional Derivatives involving another Function in the Caputo setting

Let f be continuous functions on $[0,\infty)$ and $h:[a,b]\rightarrow\mathbb{R}$ be a positive monotonically increasing function on $(a,b]$ with continuous derivative $h'(x)$ on $a\leq x\leq b$ such that $h^{'}(x)\neq 0\forall x\in[a,b].$, then the generalized k-fractional derivative (left- and right-sided) of f w.r.t. g of order $\alpha\in\mathbb{C}, Re(\alpha)>0$ are given as

$$ {}^{C}_{k,h}\mathfrak{L}^{*\alpha}_{a^+}f(x)=\frac{1}{k\Gamma_k\left(nk-\alpha\right)}\int\limits_a^x\left(h(x)-h(t)\right)^{\frac{nk-\alpha}{k}-1}h^{'}(t)\left(\frac{1}{h^{'}(t)}\frac{d}{dt}\right)^n f(t)dt, x\in[a,b], \tag{2.4.49}$$

$$ {}^{C}_{k,h}\mathfrak{L}^{*\alpha}_{b^-}f(x)=\frac{1}{k\Gamma_k\left(nk-\alpha\right)}\int\limits_x^b\left(h(t)-h(x)\right)^{\frac{nk-\alpha}{k}-1}h^{'}(t)\left(\frac{1}{h^{'}(t)}\frac{d}{dt}\right)^n f(t)dt, x\in[a,b], \tag{2.4.50}$$

provided they exist, where $k>0, n=[Re(\alpha)]+1$.

It can be observed that

(i) For $h(x)=\frac{(x-a)^s}{s}, h(t)=\frac{(t-a)^s}{s}\left(h(x)=\frac{(b-x)^s}{s}, h(t)=\frac{(b-t)^s}{s}\right)$, where $s>0$, the generalized k-fractional derivatives (2.4.49) and (2.4.50) reduce to the generalized k-fractional conformable derivatives of order α in the Caputo setting (2.3.35) and (2.3.36) respectively. These generalized k-fractional derivatives (2.4.49) and (2.4.50) further reduce to the Caputo type generalized fractional conformable derivatives and fractional conformable derivatives for $k\rightarrow 1$ and $s=1$ respectively under the same values of $h(x)$ and $h(t)$.

(ii) For $h(x)=\frac{x^s}{s}, h(t)=\frac{t^s}{s}$, where $s>0$, the generalized k-fractional derivatives (2.4.49) and (2.4.50) reduce to Caputo type generalized k-Riemann Liouville fractional derivatives [28].

These generalized k-fractional derivatives further reduce to the generalized Riemann-Liouville fractional derivatives and Riemann Liouville fractional derivatives in the Caputo setting for $k=1$ and $s=1$ respectively under the same values of $h(x)$ and $h(t)$.

(iii) For $h(x)=x, h(t)=t, a=-\infty$, the generalized k-fractional derivatives (2.4.49) and (2.4.50) reduce to k-Weyl fractional derivatives of Caputo type [150] which further reduce to Weyl fractional derivatives of Caputo type [124] for $k\rightarrow 1$.

(iv) For $h(x)=\frac{x^s}{s}, h(t)=\frac{t^s}{s}, s\to 0$, the generalized k-fractional derivatives (2.4.49) and (2.4.50) reduce to $k-$Hadamard fractional derivatives of Caputo type.

These generalized k-fractional derivatives further reduce to the usual Hadamard fractional derivatives of Caputo type for $k\to 1$ under the same values of $h(x)$ and $h(t)$.

2.5 Properties of Generalized k-Fractional Conformable Operators

Theorem 2.5.2 *For continuous function $f(x)$ on $[0,\infty)$ and $k,s\in(0,\infty)$. For $x\in[a,b], Re(\alpha)>0, Re(\beta)>0$, the following properties hold:*

$$\begin{aligned}{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}f\right)(x) &= \left({}_k^s\mathfrak{F}^{*\alpha+\beta}_{a^+}f\right)(x) \qquad \textit{(Semi-group Property)}\\ &= \left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f\right)(x) \quad \textit{(Commutative Property)}\end{aligned} \tag{2.5.51}$$

Proof. Using the result (2.3.35) in the left-side of Eq. (2.5.51), we have

$$\begin{aligned}{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}f\right)(x) &= \frac{1}{k\Gamma_k(\alpha)}\int_a^x\left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}f\right)(t)\frac{dt}{(t-a)^{1-s}}\\ &= \frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\\ &\times\left[\int_a^t\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\beta}{k}-1}f(\tau)\frac{d\tau}{(\tau-a)^{1-s}}\right]\frac{dt}{(t-a)^{1-s}}\end{aligned}$$

By Fubini's theorem, we have

$$\begin{aligned}&= \frac{s^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^x\frac{f(\tau)}{(\tau-a)^{1-s}}\\ &\times\left[\int_\tau^x\left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\beta}{k}-1}\right.\\ &\times\left.\frac{dt}{(t-a)^{1-s}}\right]d\tau.\end{aligned} \tag{2.5.52}$$

Now, we use change of variable

$$y=\frac{(t-a)^s-(\tau-a)^s}{(x-a)^s-(\tau-a)^s}$$

$$\begin{aligned}
\int_{\tau}^{x}\left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{dt}{(t-a)^{1-s}}\\
&=\frac{\left(\frac{(x-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha+\beta}{k}-1}}{s}\int_{0}^{1}(1-y)^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}dy\\
&=\frac{\left(\frac{(x-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha+\beta}{k}-1}}{s}kB_k(\alpha,\beta). \qquad (2.5.53)
\end{aligned}$$

where $B_k(\alpha,\beta)$ is k-beta function, given by

$$B_k(\alpha,\beta)=\frac{\Gamma_k(\alpha)\,\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} \qquad (2.5.54)$$

Using results (2.5.52),(2.5.53) and (2.5.54), we get

$$\begin{aligned}
{}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}f\right)(x) &= \frac{1}{k\Gamma_k(\alpha+\beta)}\int_{a}^{x}\left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{\frac{\alpha+\beta}{k}-1}\frac{f(\tau)d\tau}{(\tau-a)^{1-s}}\\
&= \left({}^{s}_{k}\mathfrak{F}^{*\alpha+\beta}_{a^+}f\right)(x) \qquad \text{(Semi-group Property)}\\
&= \left({}^{s}_{k}\mathfrak{F}^{*\beta+\alpha}_{a^+}f\right)(x) \qquad \because Re(\alpha),Re(\beta)>0\\
&= {}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}f\right)(x). \qquad \text{(Commutative Property)}
\end{aligned}$$

which gives the required property. ■

Lemma 2.5.1 *For* $Re(\nu)>0, Re(\alpha)>0, s,k>0$*,*

$$\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}(t-a)^{s\left(\frac{\nu}{k}-1\right)}\right)(x)=\frac{\Gamma_k(\nu)}{s^{\frac{\alpha}{k}}\Gamma_k(\alpha+\nu)}(x-a)^{s\left(\frac{\nu+\alpha}{k}-1\right)} \qquad (2.5.55)$$

Proof. Using the result (2.3.35) in the L.H.S. of Eq. (2.5.55), we have

$$\begin{aligned}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(t-a)^{s\left(\frac{\nu}{k}-1\right)}\right)(x) &= \frac{1}{k\Gamma_k(\alpha)}\int_a^x\left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(t-a)^{s\left(\frac{\nu}{k}-1\right)}\frac{dt}{(t-a)^{1-s}}\\ &= \frac{s^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x\left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(t-a)^{s\left(\frac{\nu}{k}-1\right)}\frac{dt}{(t-a)^{1-s}}\end{aligned}$$

Using the change of variable $y=\frac{(t-a)^s}{(x-a)^s}$, we have

$$\begin{aligned}&= \frac{s^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1(1-y)^{\frac{\alpha}{k}-1}y^{\frac{\nu}{k}-1}(x-a)^s(x-a)^{s\left(\frac{\alpha}{k}-1\right)}(x-a)^{s\left(\frac{\nu}{k}-1\right)}\frac{dy}{s}\\ &= \frac{(x-a)^{s\left(\frac{\alpha+\nu}{k}-1\right)}}{s^{\frac{\alpha}{k}}\Gamma_k(\alpha)}B_k\left(\frac{\alpha}{k},\frac{\nu}{k}\right)\\ &= \frac{(x-a)^{s\left(\frac{\alpha+\nu}{k}-1\right)}}{s^{\frac{\alpha}{k}}\Gamma_k(\alpha)}\frac{\Gamma_k(\alpha)\Gamma_k(\nu)}{\Gamma_k(\alpha+\nu)}\\ &= \frac{\Gamma_k(\nu)}{s^{\frac{\alpha}{k}}\Gamma_k(\alpha+\nu)}(x-a)^{s\left(\frac{\nu+\alpha}{k}-1\right)}\end{aligned}$$

which gives the required property. ■

Theorem 2.5.3 *For* $\alpha > mk > 0$,

$${}^{s}T^{m}_{a^+}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}f\right)(x) = \frac{1}{k^m}\left({}_{k}^{s}\mathfrak{F}^{*\alpha-mk}_{a^+}f\right)(x) \tag{2.5.56}$$

Proof. Using the result (2.3.35) in the L.H.S. of Eq. (2.5.56), we have

$$
\begin{aligned}
\left({}^{s}T_{a^+}^{m}\right)\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}f\right)(x) &={}^{s} T_{a^+}^{m}\left(\frac{1}{k\Gamma_k(\alpha)}\int_a^x\left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{\frac{\alpha}{k}-1}f(t)\frac{dt}{(t-a)^{1-s}}\right)\\
&={}^{s} T_{a^+}^{m-1}\left(\frac{1}{k\Gamma_k(\alpha-k)}\int_a^x\left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{\frac{\alpha}{k}-2}f(t)\frac{dt}{(t-a)^{1-s}}\right)\\
&={}^{s} T_{a^+}^{m-2}\left(\frac{1}{k\Gamma_k(\alpha-2k)}\int_a^x\left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{\frac{\alpha}{k}-3}f(t)\frac{dt}{(t-a)^{1-s}}\right)\\
&=\frac{1}{k\Gamma_k(\alpha-mk)}\int_a^x\left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{\frac{\alpha}{k}-m-1}f(t)\frac{dt}{(t-a)^{1-s}}\\
&=\frac{1}{k^m}\left({}_{k}^{s}\mathfrak{F}^{*\alpha-mk}_{a^+}f\right)(x).
\end{aligned}
$$

which gives the required property. ■

Lemma 2.5.2 *For* $Re(nk-\alpha)>0, Re(\nu-\alpha)>0, s,k>0$*, we have*

$$\left({}_{k}^{s}\mathfrak{C}^{\alpha}_{a^+}(t-a)^{s\left(\frac{\nu}{k}-1\right)}\right)(x)=\frac{s^{\frac{\alpha}{k}}\Gamma_k(\nu)}{k^n\Gamma_k(\nu-\alpha)}(x-a)^{s\left(\frac{\nu-\alpha}{k}-1\right)} \tag{2.5.57}$$

Proof. Invoking the definition of generalized conformable k-fractional derivative (2.3.37), we have

$$\left({}_{k}^{s}\mathfrak{C}^{\alpha}_{a^+}(t-a)^{s\left(\frac{\nu}{k}-1\right)}\right)(x)=\left({}^{s}T^{n}_{a+}\right)\left({}_{k}^{s}\mathfrak{F}^{*nk-\alpha}_{a^+}(t-a)^{s\left(\frac{\nu}{k}-1\right)}\right)(x) \tag{2.5.58}$$

Using Theorem (2.5.3) in the R.H.S. of Eq. (2.5.58), we have

$$
\begin{aligned}
&= \frac{1}{k^n}\left({}_{k}^{s}\mathfrak{F}^{*nk-\alpha-nk}_{a^+}(t-a)^{s\left(\frac{\nu}{k}-1\right)}\right)(x)\\
&= \frac{1}{k^n}\frac{\Gamma_k(\nu)}{s^{\frac{-\alpha}{k}}\Gamma_k(\nu-\alpha)}(x-a)^{s\left(\frac{\nu-\alpha}{k}-1\right)} \quad \text{By Lemma}(2.5.1)\\
&= \frac{s^{\frac{\alpha}{k}}\Gamma_k(\nu)}{k^n\Gamma_k(\nu-\alpha)}(x-a)^{s\left(\frac{\nu-\alpha}{k}-1\right)}
\end{aligned}
$$

which gives the required property. ■

Theorem 2.5.4 *Let* f *be continuous on* $[0,\infty]$, $Re(\alpha)>0, k,n\in\mathbb{N}$ *and* $n=[Re(\alpha)]+1$.

Then $\forall 0 < a < x$

$$ {}_{k}^{s}\mathfrak{C}_{a^{+}}^{\alpha}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}f\right)(x) = \frac{1}{k^{n}}f(x). \quad \textit{(Inverse Property)} \tag{2.5.59}$$

Proof. According to the definition (2.3.37), we have

$$\begin{aligned} {}_{k}^{s}\mathfrak{C}_{a^{+}}^{\alpha}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}f\right)(x) &= \left({}^{s}T_{a+}^{n}\right)\left({}_{k}^{s}\mathfrak{F}^{*nk-\alpha}_{a^{+}}\right)\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}f\right)(x) \\ &= \left({}^{s}T_{a+}^{n}\right)\left({}_{k}^{s}\mathfrak{F}^{*nk-\alpha+\alpha}_{a^{+}}f\right)(x) \end{aligned} \tag{2.5.60}$$

Using Theorem (2.5.3) in the R.H.S. of Eq. (2.5.60), we have

$$\begin{aligned} &= \frac{1}{k^{n}}\left({}_{k}^{s}\mathfrak{F}^{*nk-nk}_{a^{+}}f\right)(x) \\ &= \frac{1}{k^{n}}f(x) \end{aligned}$$

which gives the required property. ■

Corollary 2.5.1 *If* $Re(\beta) < Re(\alpha)$*, then*

$$ {}_{k}^{s}\mathfrak{C}_{a^{+}}^{\beta}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}f\right)(x) = \frac{1}{k^{n}}\left({}_{k}^{s}\mathfrak{F}^{*\alpha-\beta}_{a^{+}}f\right)(x). \tag{2.5.61}$$

Proof. According to the definition (2.3.37), we have

$$\begin{aligned} {}_{k}^{s}\mathfrak{C}_{a^{+}}^{\beta}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}f\right)(x) &= \left({}^{s}T_{a+}^{n}\right)\left({}_{k}^{s}\mathfrak{F}^{*nk-\beta}_{a^{+}}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}f\right)\right)(x) \\ &= \left({}^{s}T_{a+}^{n}\right)\left({}_{k}^{s}\mathfrak{F}^{*nk-\beta+\alpha}_{a^{+}}f\right)(x) \end{aligned} \tag{2.5.62}$$

Using Theorem (2.5.3) in the R.H.S. of Eq. (2.5.62), we have

$$\begin{aligned} &= \frac{1}{k^{n}}\left({}_{k}^{s}\mathfrak{F}^{*nk-\beta+\alpha-nk}_{a^{+}}f\right)(x) \\ &= \frac{1}{k^{n}}\left({}_{k}^{s}\mathfrak{F}^{*\alpha-\beta}_{a^{+}}f\right)(x). \end{aligned}$$

Hence the result. ■

Corollary 2.5.2 *If $Re(\beta) > Re(\alpha)$, then*

$$\left({}_k^s\mathfrak{C}_{a^+}^{\beta}\right)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f\right)(x) = \frac{1}{k^n}\left({}_k^s\mathfrak{C}_{a^+}^{\beta-\alpha}f\right)(x). \tag{2.5.63}$$

Proof.

$${}_k^s\mathfrak{C}_{a^+}^{\beta}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f\right)(x) = \left({}_k^s\mathfrak{C}_{a^+}^{\beta}\right)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left({}_k^s\mathfrak{C}_{a^+}^{\alpha}\right)\left({}_k^s\mathfrak{C}_{a^+}^{-\alpha}f\right)(x) \tag{2.5.64}$$

Using Theorem (2.5.59) in the R.H.S. of Eq. (2.5.64), we have

$$\begin{aligned} &= \frac{1}{k^n}\left({}_k^s\mathfrak{C}_{a^+}^{\beta}\right)\left({}_k^s\mathfrak{C}_{a^+}^{-\alpha}f\right)(x) \\ &= \frac{1}{k^n}\left({}_k^s\mathfrak{C}_{a^+}^{\beta-\alpha}f\right)(x). \end{aligned}$$

Hence the result. ■

Corollary 2.5.3 *For a continuous function f on $[0,\infty)$, $s>0, k,m,n \in \mathbb{N}, \alpha,\beta \in \mathbb{C}$ s.t. $Re(\alpha), Re(\beta) > 0, n = [Re(\alpha)]+1, m = [Re(\beta)]+1$. Then $\alpha+\beta < nk$ and $\forall 0 < a < x$,*

$$\left({}_k^s\mathfrak{C}_{a^+}^{\alpha}\right)\left({}_k^s\mathfrak{C}_{a^+}^{\beta}f\right)(x) = \frac{1}{k^n}\left({}_k^s\mathfrak{C}_{a^+}^{\alpha+\beta}f\right)(x). \quad \textit{(Semi-group Property)} \tag{2.5.65}$$

Proof. Invoking the definition of generalized conformable k-fractional derivative (2.3.37), we have

$$\begin{aligned} {}_k^s\mathfrak{C}_{a^+}^{\alpha}\left({}_k^s\mathfrak{C}_{a^+}^{\beta}f\right)(x) &= \left({}^sT_{a+}^{n}\right)\left({}_k^s\mathfrak{F}^{*nk-\alpha}_{a^+}\left({}_k^s\mathfrak{C}_{a^+}^{\beta}f\right)\right)(x) \\ &= \left({}^sT_{a+}^{n}\right)\left({}_k^s\mathfrak{F}^{*nk-\alpha}_{a^+}\right)\left({}_k^s\mathfrak{C}_{a^+}^{\beta}\right)\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)\left({}_k^s\mathfrak{F}^{*-\beta}_{a^+}f\right)(x) \end{aligned} \tag{2.5.66}$$

Using Theorem (2.5.59) in the R.H.S. of Eq. (2.5.66), we have

$$\begin{aligned} &= \frac{1}{k^n}\left({}^sT_{a+}^{n}\right)\left({}_k^s\mathfrak{F}^{*nk-\alpha}_{a^+}\right)\left({}_k^s\mathfrak{F}^{*-\beta}_{a^+}f\right)(x) \\ &= \frac{1}{k^n}\left({}^sT_{a+}^{n}\right)\left({}_k^s\mathfrak{F}^{*nk-\alpha-\beta}_{a^+}f\right)(x) \\ &= \frac{1}{k^n}\left({}^sT_{a+}^{n}\right)\left({}_k^s\mathfrak{F}^{*nk-(\alpha+\beta)}_{a^+}f\right)(x) \\ &= \frac{1}{k^n}\left({}_k^s\mathfrak{C}_{a^+}^{\alpha+\beta}f\right)(x). \end{aligned}$$

Hence the result. ■

Chapter 3

CHEBYSHEV TYPE INTEGRAL INEQUALITIES FOR GENERALIZED CONFORMABLE K-FRACTIONAL INTEGRALS

This chapter consigns to the generalization of the classical Chebyshev type inequalities [156] for synchronous functions (2.1.5) by involving the generalized conformable k-fractional integrals (2.3.35) and (2.3.36). The inequalities using one and two fractional parameters k-FCI are presented by taking into account the extended Chebyshev functional for synchronous functions. Some certain interesting consequences of the main inequalities are also presented.

3.1 Chebyshev's Functional

For integrable functions $f_1, f_2 : [a, b] \to \mathbb{R}$, the functional of the form:

$$T(f_1, f_2; x_1, x_2) = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f_1(t) f_2(t) dt - \frac{1}{(x_2 - x_1)^2} \int_{x_1}^{x_2} f_1(t) dt \int_{x_1}^{x_2} f_2(t) dt, \tag{3.1.1}$$

provided that the involved integrals exist, f_1 and f_2 are synchronous on $[x_1, x_2]$, $(i.e. \left(f_1(x^*) - f_1(y^*)\right) \left(f_2(x^*) - f_2(y^*)\right) \geq 0,$ for any $x^*, y^* \in [x_1, x_2])$, is known as Chebyshev's functional in the literature.

3.2 Generalized Conformable k-Fractional Integral Inequalities

This section presents Chebyshev inequalities for the generalized conformable k-fractional integral ${}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}$ depicted in (2.3.35) .

Theorem 3.2.1 *For two synchronous functions f_1, f_2 on $[0, \infty)$ and $\forall\ 0 \leq a < x, \beta >$*

$0, \gamma > 0$*, the following inequalities for* k*-fractional conformable integral* ${}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}$ *hold true:*

$$\begin{aligned}\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2(x) &\geq \frac{1}{\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right)(1)}\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_2(x)\\ &\quad\times\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right)(1)+\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_1f_2(x)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right)(1)\\ &\geq\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_2(x)+\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_2(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_1(x).\end{aligned} \tag{3.2.2}$$

Proof. Given that the functions f_1 and f_2 are synchronous on $[0,\infty)$, then for all $\rho, \tau \geq 0$, we have

$$\left(f_2(\rho)-f_2(\tau)\right)\left(f_1(\rho)-f_1(\tau)\right)\geq 0.$$

$$f_1(\rho)f_2(\rho)+f_1(\tau)f_2(\tau)\geq f_1(\rho)f_2(\tau)+f_1(\tau)f_2(\rho). \tag{3.2.3}$$

Multiply on both sides of (3.2.3) by

$$\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{1}{(\rho-a)^{1-s}},$$

then integrating the resulting inequality w.r.t ρ from a to x, we get

$$\begin{aligned}&\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{f_1(\rho)f_2(\rho)d\rho}{(\rho-a)^{1-s}}\\ &+\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{f_1(\tau)f_2(\tau)d\rho}{(\rho-a)^{1-s}}\\ &\geq\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{f_1(\rho)f_2(\tau)d\rho}{(\rho-a)^{1-s}}\\ &+\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{f_1(\tau)f_2(\rho)d\rho}{(\rho-a)^{1-s}},\end{aligned} \tag{3.2.4}$$

i.e.,

$$\begin{aligned}&\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2(x) + f_1(\tau)f_2(\tau)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right)(1)\\ &\geq f_2(\tau)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x) + f_1(\tau)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_2(x).\end{aligned} \tag{3.2.5}$$

Multiply on both sides of (3.2.5) by

$$\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^s-(\tau-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{1}{(\tau-a)^{1-s}}$$

and integrate the resulting inequality w.r.t τ from a to x, we get

$$\begin{aligned}&\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2(x)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right)(1) + \left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right)(1)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2(x)\\ &\geq \left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_2(x) + \left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_2(x)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x).\end{aligned}$$

i.e.,

$$\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2(x) \geq \frac{1}{\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right)(1)}\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_2(x).$$

Hence the first part of the inequality (3.2.2) is proved.

To prove the second part, multiply on both sides of (3.2.5) by

$$\frac{1}{k\Gamma_k(\gamma)}\left(\frac{(x-a)^s-(\tau-a)^s}{s}\right)^{\frac{\gamma}{k}-1}\frac{1}{(\tau-a)^{1-s}}$$

and integrating the resulting inequality w.r.t τ from a to x, we get

$$\begin{aligned}&\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right)(1) + \left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right)(1)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_1f_2(x)\\ &\geq \left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_2(x) + \left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_2(x)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x).\end{aligned}$$

The proof is complete. ■

Theorem 3.2.2 *For two synchronous* f_1, f_2 *on* $[0,\infty), f_3 \geq 0$ *then* $\forall 0 \leq a < x, \beta >$

$0, \gamma > 0$, *the following inequalities for conformable k-fractional integral ${}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}$ hold true:*

$$\begin{aligned}
&\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2f_3(x) \left({}^{s}_{k}\mathfrak{F}^{*\gamma}_{a^+}\right)(1) + \left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right)(1) \left({}^{s}_{k}\mathfrak{F}^{*\gamma}_{a^+}\right) f_1f_2f_3(x) \\
&\geq \left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_3(x) \left({}^{s}_{k}\mathfrak{F}^{*\gamma}_{a^+}\right) f_2(x) + \left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right) f_2f_3(x) \left({}^{s}_{k}\mathfrak{F}^{*\gamma}_{a^+}\right) f_1(x) \\
&- \left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2(x) \left({}^{s}_{k}\mathfrak{F}^{*\gamma}_{a^+}\right) f_3(x) - \left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right) f_3(x) \left({}^{s}_{k}\mathfrak{F}^{*\gamma}_{a^+}\right) f_1f_2(x) \\
&+ \left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x) \left({}^{s}_{k}\mathfrak{F}^{*\gamma}_{a^+}\right) f_2f_3(x) + \left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right) f_2(x) \left({}^{s}_{k}\mathfrak{F}^{*\gamma}_{a^+}\right) f_1f_3(x). \qquad (3.2.6)
\end{aligned}$$

Proof. Given that the functions f_1 and f_2 are synchronous on $[0, \infty)$ and $f_3 \geq 0$, then for $\rho, \tau \geq 0$, we have

$$(f_1(\rho) - f_1(\tau))(f_2(\rho) - f_2(\tau))(f_3(\rho) + f_3(\tau)) \geq 0.$$

Expanding the left-side of above inequality, we get

$$f_2(\rho)f_1(\rho)f_3(\rho) + f_2(\tau)f_1(\tau)f_3(\tau) \geq f_2(\tau)f_1(\rho)f_3(\rho) + f_2(\rho)f_1(\tau)f_3(\rho)$$

$$-f_1(\tau)f_2(\tau)f_3(\rho) - f_1(\rho)f_2(\rho)f_3(\tau) + f_1(\rho)f_2(\tau)f_3(\tau) + f_1(\tau)f_2(\rho)f_3(\tau). \qquad (3.2.7)$$

Multiply (3.2.7) on both sides by

$$\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^s - (\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1} \frac{1}{(\rho-a)^{1-s}},$$

then integrating the resulting inequality w.r.t ρ from a to x, we obtain

$$\begin{aligned}
&\frac{1}{k\Gamma_k(\beta)}\int_a^x \left[\frac{(x-a)^s - (\rho-a)^s}{s}\right]^{\frac{\beta}{k}-1} \frac{f_1(\rho)f_2(\rho)f_3(\rho)d\rho}{(\rho-a)^{1-s}} \\
&+ \frac{1}{k\Gamma_k(\beta)}\int_a^x \left[\frac{(x-a)^s - (\rho-a)^s}{s}\right]^{\frac{\beta}{k}-1} \frac{f_1(\tau)f_2(\tau)f_3(\tau)d\rho}{(\rho-a)^{1-s}} \\
&\geq \frac{1}{k\Gamma_k(\beta)}\int_a^x \left[\frac{(x-a)^s - (\rho-a)^s}{s}\right]^{\frac{\beta}{k}-1} \frac{f_1(\rho)f_2(\tau)f_3(\rho)d\rho}{(\rho-a)^{1-s}} \\
&+ \frac{1}{k\Gamma_k(\beta)}\int_a^x \left[\frac{(x-a)^s - (\rho-a)^s}{s}\right]^{\frac{\beta}{k}-1} \frac{f_1(\tau)f_2(\rho)f_3(\rho)d\rho}{(\rho-a)^{1-s}}
\end{aligned}$$

$$- \frac{1}{k\Gamma_k(\beta)} \int_a^x \left[\frac{(x-a)^s - (\rho-a)^s}{s} \right]^{\frac{\beta}{k}-1} \frac{f_1(\tau)f_2(\tau)f_3(\rho)d\rho}{(\rho-a)^{1-s}}$$

$$- \frac{1}{k\Gamma_k(\beta)} \int_a^x \left[\frac{(x-a)^s - (\rho-a)^s}{s} \right]^{\frac{\beta}{k}-1} \frac{f_1(\rho)f_2(\rho)f_3(\tau)d\rho}{(\rho-a)^{1-s}}$$

$$+ \frac{1}{k\Gamma_k(\beta)} \int_a^x \left[\frac{(x-a)^s - (\rho-a)^s}{s} \right]^{\frac{\beta}{k}-1} \frac{f_1(\rho)f_2(\tau)f_3(\tau)d\rho}{(\rho-a)^{1-s}}$$

$$+ \frac{1}{k\Gamma_k(\beta)} \int_a^x \left[\frac{(x-a)^s - (\rho-a)^s}{s} \right]^{\frac{\beta}{k}-1} \frac{f_1(\tau)f_2(\rho)f_3(\tau)d\rho}{(\rho-a)^{1-s}}. \tag{3.2.8}$$

i.e.,

$$\begin{aligned}
&\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2f_3(x) + f_1(\tau)f_2(\tau)f_3(\tau)\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)(1) \\
&\geq f_2(\tau)\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_3(x) + f_1(\tau)\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_2f_3(x) \\
&- f_1(\tau)f_2(\tau)\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_3(x) - f_3(\tau)\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2(x) \\
&+ f_2(\tau)f_3(\tau)\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x) + f_1(\tau)f_3(\tau)\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_2(x).
\end{aligned} \tag{3.2.9}$$

Multiply on both sides of (3.2.9) by

$$\frac{1}{k\Gamma_k(\beta)} \left(\frac{(x-a)^s - (\tau-a)^s}{s} \right)^{\frac{\beta}{k}-1} \frac{1}{(\tau-a)^{1-s}}$$

and integrate the resulting inequality w.r.t τ from a to x, we get

$$\begin{aligned}
&\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2f_3(x)\left({}^s_k\mathfrak{F}^{*\gamma}_{a^+}\right)(1) + \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)(1)\left({}^s_k\mathfrak{F}^{*\gamma}_{a^+}\right) f_1f_2(x) \\
&\geq \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_3(x)\left({}^s_k\mathfrak{F}^{*\gamma}_{a^+}\right) f_2(x) + \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_2f_3(x)\left({}^s_k\mathfrak{F}^{*\gamma}_{a^+}\right) f_1(x) \\
&- \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_3(x)\left({}^s_k\mathfrak{F}^{*\gamma}_{a^+}\right) f_1f_2(x) - \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2(x)\left({}^s_k\mathfrak{F}^{*\gamma}_{a^+}\right) f_3(x) \\
&+ \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x)\left({}^s_k\mathfrak{F}^{*\gamma}_{a^+}\right) f_2f_3(x) + \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_2(x)\left({}^s_k\mathfrak{F}^{*\gamma}_{a^+}\right) f_1f_3(x).
\end{aligned}$$

Hence the inequality (3.2.6) is proved. ■

Corollary 3.2.1 *Let f_1, f_2 be two synchronous functions on $[0, \infty)$, $f_3 \geq 0$, then for all $0 \leq a < x, \beta > 0$, the following inequality for k-fractional conformable integral ${}^s_k\mathfrak{F}^{*\beta}_{a^+}$*

hold true:

$$\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2f_3(x)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right)(1) \geq \left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_3(x)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_2(x)$$
$$+\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_2f_3(x)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x)-\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_3(x)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2(x). \tag{3.2.10}$$

Proof. The proof can be made by replacing γ with β in Theorem (3.2.2), as (3.2.10) is the inequality involving only one fractional parameter. ■

Theorem 3.2.3 *Let f_1, f_2 and f_3 be monotonic functions defined on $[0,\infty)$, satisfying the following condition:*

$$(f_1(\rho)-f_1(\tau))(f_2(\rho)-f_2(\tau))(f_3(\rho)-f_3(\tau)) \geq 0, \tag{3.2.11}$$

for all $\rho,\tau \geq 0$, $0 \leq a < x, \beta > 0, \gamma > 0$, the following results for conformable k-fractional integrals ${}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}$ hold true:

$$\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2f_3(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right)(1)-\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right)(1)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_1f_2f_3(x)$$
$$\geq \left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_3(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_2(x)+\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_2f_3(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_1(x)$$
$$-\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_3(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_1f_2(x)+\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_3(x)$$
$$-\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_2f_3(x)-\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_2(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_1f_3(x). \tag{3.2.12}$$

Proof. The proof of this theorem is similar to that given in Theorem (3.2.2). ■

Theorem 3.2.4 *For two functions f_1 and f_2 on $[0,\infty)$, then for all $0 \leq a < x, \beta > 0, \gamma > 0$, the following inequalities for conformable k-fractional integral ${}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}$ hold true:*

$$\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1^2(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right)(1)+\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right)(1)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_2^2(x)$$
$$\geq 2\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_2(x), \tag{3.2.13}$$

$$\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1^2(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_2^2(x)+\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1^2(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_2^2(x)$$
$$\geq 2\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right) f_1f_2(x)\left({}_{k}^{s}\mathfrak{F}^{*\gamma}_{a^+}\right) f_1f_2(x). \tag{3.2.14}$$

Proof. since for all $\rho, \tau \geq 0$,

$$(f_1(\rho) - f_2(\tau))^2 \geq 0,$$

then we have

$$f_1^2(\rho) + f_2^2(\tau) \geq 2f_1(\rho)f_2(\tau). \tag{3.2.15}$$

Multiply the relation (3.2.15) on both sides by

$$\frac{1}{k\Gamma_k(\beta)} \left(\frac{(x-a)^s - (\rho-a)^s}{s} \right)^{\frac{\beta}{k}-1} \frac{1}{(\rho-a)^{1-s}}$$

and integrate the resulting inequality w.r.t ρ from a to x, we get

$$\begin{aligned} &\frac{1}{k\Gamma_k(\beta)} \int_a^x \left(\frac{(x-a)^s - (\rho-a)^s}{s} \right)^{\frac{\beta}{k}-1} \frac{f_1^2(\rho)d\rho}{(\rho-a)^{1-s}} \\ &\geq \frac{1}{k\Gamma_k(\beta)} \int_a^x \left(\frac{(x-a)^s - (\rho-a)^s}{s} \right)^{\frac{\beta}{k}-1} \frac{2f_1(\rho)f_2(\tau)d\rho}{(\rho-a)^{1-s}}, \end{aligned} \tag{3.2.16}$$

i.e.,

$$\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_1^2(x) + f_2^2(\tau) \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)(1) \geq 2f_2(\tau) \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x). \tag{3.2.17}$$

Multiply the inequality (3.2.17) by

$$\frac{1}{k\Gamma_k(\beta)} \left(\frac{(x-a)^s - (\tau-a)^s}{s} \right)^{\frac{\beta}{k}-1} \frac{1}{(\tau-a)^{1-s}}$$

and integrate the resultant w.r.t τ from a to x, we get

$$\begin{aligned} &\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_1^2(x) \left({}^s_k\mathfrak{F}^{*\gamma}_{a^+}\right)(1) + \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)(1) \left({}^s_k\mathfrak{F}^{*\gamma}_{a^+}\right) f_2^2(x) \\ &\geq 2\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f_1(x) \left({}^s_k\mathfrak{F}^{*\gamma}_{a^+}\right) f_2(x). \end{aligned}$$

which completes the proof of first part.

To obtain the second part, we have

$$(f_1(\rho)f_2(\tau) - f_1(\tau)f_2(\rho))^2 \geq 0,$$

then we have

$$f_1^2(\rho)f_2^2(\tau) + f_1^2(\tau)f_2^2(\rho) \geq 2f_1(\rho)f_1(\tau)f_2(\rho)f_2(\tau). \tag{3.2.18}$$

Multiply the inequality(3.2.18) by

$$\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{1}{(\rho-a)^{1-s}}$$

and integrate the resulting inequality w.r.t ρ from a to x, we get

$$\begin{aligned}
&\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{f_1^2(\rho)f_2^2(\tau)d\rho}{(\rho-a)^{1-s}}\\
&+\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{f_1^2(\tau)f_2^2(\rho)d\rho}{(\rho-a)^{1-s}}\\
&\geq 2\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{f_1(\rho)f_2(\rho)f_1(\tau)f_2(\tau)d\rho}{(\rho-a)^{1-s}}.
\end{aligned} \tag{3.2.19}$$

i.e.,

$$\begin{aligned}
&f_2^2(\tau)\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)f_1^2(x) + f_1^2(\tau)\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)f_2^2(x)\\
&\geq 2f_1(\tau)f_2(\tau)\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)f_1f_2(x).
\end{aligned} \tag{3.2.20}$$

Multiply on both sides of (3.2.20) by

$$\frac{1}{k\Gamma_k(\gamma)}\left(\frac{(x-a)^s-(\tau-a)^s}{s}\right)^{\frac{\gamma}{k}-1}\frac{1}{(\tau-a)^{1-s}}$$

and integrating the resulting inequality w.r.t τ from a to x, we get (3.2.14). ∎

Corollary 3.2.2 *For two functions f_1 and f_2 on $[0,\infty)$, then $\forall 0 \leq a < x, \beta > 0, \gamma > 0$, the following inequalities for conformable k-fractional integral ${}_k^s\mathfrak{F}^{*\beta}_{a^+}$ hold true:*

$$\begin{aligned}
&\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)(1)\left[\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)f_1^2(x) + \left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)f_2^2(x)\right]\\
&\geq 2\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)f_1(x)\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)f_2(x),
\end{aligned} \tag{3.2.21}$$

$$\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)f_1^2(x)\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)f_2^2(x) \geq \left[\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)f_1f_2(x)\right]^2. \tag{3.2.22}$$

Proof. The proof of both inequalities can be made by replacing γ with β in both inequalities of previous theorem as (3.2.21) and (3.2.22) are the inequalities involving only one fractional parameter. ■

Theorem 3.2.5 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined as*

$$\overline{f}(x) = \int\limits_a^x \frac{f(t)dt}{(t-a)^{1-s}} \quad ; 0 \leq a < x, \beta \in \mathbb{R} \setminus \{0\},$$

then for all $0 < k \leq \beta$

$$\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right) f(x) = \frac{1}{k} \left({}_k^s\mathfrak{F}^{*\beta-k}_{a^+}\right) \overline{f}(x). \tag{3.2.23}$$

Proof. According to the definition of generalized conformable k-fractional integrals, we have

$$\begin{aligned}
{}_k^s\mathfrak{F}^{*\beta}_{a^+}\overline{f}(x) &= \frac{1}{k\Gamma_k(\beta)} \int\limits_a^x \left(\frac{(x-a)^s - (t-a)^s}{s}\right)^{\frac{\beta}{k}-1} \overline{f}(t) \frac{dt}{(t-a)^{1-s}} \\
&= \frac{1}{k\Gamma_k(\beta)} \int\limits_a^x \left(\frac{(x-a)^s - (t-a)^s}{s}\right)^{\frac{\beta}{k}-1} \frac{1}{(t-a)^{1-s}} \int\limits_a^t \frac{f(u)du}{(u-a)^{1-s}} \\
&= \frac{1}{k\Gamma_k(\beta)} \int\limits_a^x \frac{f(u)}{(u-a)^{1-s}} \int\limits_u^x \left(\frac{(x-a)^s - (t-a)^s}{s}\right)^{\frac{\beta}{k}-1} \frac{dt}{(t-a)^{1-s}} du \\
&= \frac{1}{k\Gamma_k(\beta)} \int\limits_a^x \left(\frac{(x-a)^s - (t-a)^s}{s}\right)^{\frac{\beta}{k}} \frac{f(u)}{(u-a)^{1-s}} du \\
&= \frac{1}{\Gamma_k(\beta+k)} \int\limits_a^x \left(\frac{(x-a)^s - (t-a)^s}{s}\right)^{\frac{\beta+k}{k}-1} \frac{f(u)}{(u-a)^{1-s}} du \\
&= \left({}_k^s\mathfrak{F}^{*\beta+k}_{a^+}\right) f(x).
\end{aligned}$$

which completes the proof. ■

To present the next result, we recall the generalized Cauchy-Buniakovsky-Schwarz inequality as follows:

Lemma 3.2.1 *Let $f_1, f_2, f_3 : [a,b] \to [0,\infty)$ be three functions $0 \le a < b$, then*

$$\left(\int_a^b f_2^m(t) f_3^{r_1}(t) f_1(t)dt\right)\left(\int_a^b f_2^n(t) f_3^{r_2}(t) f_1(t)dt\right)$$
$$\ge \left(\int_a^b f_2^{\frac{m+n}{2}}(t) f_3^{\frac{r_1+r_2}{2}}(t) f_1(t)dt\right)^2. \tag{3.2.24}$$

where m, n, r_1, r_2 are arbitrary real numbers.

Proof. We have

$$\int_a^b \left[\sqrt{f_2^m(t) f_3^{r_1}(t) f_1(t)}\sqrt{\int_a^b f_2^n(t) f_3^{r_2}(t) f_1(t)dt}\right.$$
$$\left. -\sqrt{f_2^n(t) f_3^{r_2}(t) f_1(t)}\sqrt{\int_a^b f_2^m(t) f_3^{r_1}(t) f_1(t)dt}\right]^2 dt \ge 0,$$

$\Rightarrow$

$$\int_a^b \left[f_2^m(t) f_3^{r_1}(t) f_1(t)\int_a^b f_2^n(t) f_3^{r_2}(t) f_1(t)dt + f_2^n(t) f_3^{r_2}(t) f_1(t)\int_a^b f_2^m(t) f_3^{r_1}(t) f_1(t)dt\ dt\right.$$
$$\left.-2 f_2^{\frac{m+n}{2}}(t) f_3^{\frac{r_1+r_2}{2}}(t) f_1(t)\sqrt{\int_a^b f_2^m(t) f_3^{r_1}(t) f_1(t)dt}\sqrt{\int_a^b f_2^n(t) f_3^{r_2}(t) f_1(t)dt}\right] \ge 0,$$

$\Rightarrow$

$$2\left(\int_a^b f_2^m(t) f_3^{r_1}(t) f_1(t)dt\right)\left(\int_a^b f_2^n(t) f_3^{r_2}(t) f_1(t)dt\right)$$
$$\ge 2\left(\int_a^b f_2^{\frac{m+n}{2}}(t) f_3^{\frac{r_1+r_2}{2}}(t) f_1(t)dt\right)\sqrt{\int_a^b f_2^m(t) f_3^{r_1}(t) f_1(t)dt}\sqrt{\int_a^b f_2^n(t) f_3^{r_2}(t) f_1(t)dt}.$$

which can be written in the form of required inequality. ■

Theorem 3.2.6 *Let $f \in L_1[a,b]$, then*

$$\left[\left({}_k^s\mathfrak{F}^{*}{}_{a^+}^{m\left(\frac{\beta}{k}-1\right)+1}\right) f^r(x)\right]\left[\left({}_k^s\mathfrak{F}^{*}{}_{a^+}^{n\left(\frac{\beta}{k}-1\right)+1}\right) f^p(x)\right]$$
$$\geq \left[\left({}_k^s\mathfrak{F}^{*}{}_{a^+}^{\frac{m+n}{2}\left(\frac{\beta}{k}-1\right)+1}\right) f^{\frac{r+p}{2}}(x)\right]^2. \tag{3.2.25}$$

Proof. By taking $f_2(t) = \left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{\frac{\beta}{k}-1}$, $f_1(t) = \frac{(t-a)^{s-1}}{k\Gamma_k(\beta)}$ and $f_3(t) = f(t)$ in (3.2.24), we get

$$\left(\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{m\left(\frac{\beta}{k}-1\right)}\frac{f^r(t)dt}{(t-a)^{1-s}}\right)$$

$$\left(\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{n\left(\frac{\beta}{k}-1\right)}\frac{f^p(t)dt}{(t-a)^{1-s}}\right)$$

$$\geq\left(\frac{1}{k\Gamma_k(\beta)}\int_a^x\left(\frac{(x-a)^s-(t-a)^s}{s}\right)^{\frac{m+n}{2}\left(\frac{\beta}{k}-1\right)}\frac{f^{\frac{r+p}{2}}(t)dt}{(t-a)^{1-s}}\right)^2$$

which can be written as (3.2.25). ■

Remark 3.2.1 *If we take $k=1$, the above results reduce to the Chebyshev inequalities involving fractional conformable integrals ${}_s^\beta\mathcal{A}^{*}{}_{a^+}$ defined in (2.2.32).*

Remark 3.2.2 *The above inequalities and results can be obtained for the right generalized $k-$fractional conformable integrals ${}_k^s\mathfrak{F}^{*}{}_{b^-}^{\beta}$ defined in (2.3.36).*

Chapter 4

THE MINKOWSKI'S INEQUALITY FOR GENERALIZED CONFORMABLE K-FRACTIONAL INTEGRAL

This chapter contributes the dissertation by establishing the generalization of Minkowski's inequality for conformable k-fractional integrals (2.3.35) and (2.3.36). The related results of this inequality are also presented for generalized k-FCI $({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+})$. In the first section, a brief introduction of Minkowski's inequality along existing applications is described. In the second section, the proofs of the reverse Minkowski's inequality are presented and in the last section, the related inequalities are derived.

4.1 Introduction

This section states the reverse Minkowski's inequality and the existing related inequalities.

Theorem 4.1.1 *For two positive functions $f_1, f_2 \in L_{p^*}[a,b]$, such that $1 \leq p^* \leq \infty$, $0 < \int\limits_a^b f_1^{p^*}(t)dt < \infty$ and $0 < \int\limits_a^b f_2^{p^*}(t)dt < \infty$, if $0 < n \leq \frac{f_1(t)}{f_2(t)} \leq N$, for $n, N \in \mathbb{R}^+$ and $\forall a \leq t \leq b$, then*

$$\left(\int\limits_a^b f_1^{p^*}(t)dt\right)^{\frac{1}{p^*}} + \left(\int\limits_a^b f_2^{p^*}(t)dt\right)^{\frac{1}{p^*}} \leq c_1 \left(\int\limits_a^b (f_1^{p^*} + f_2^{p^*}(t)dt\right)^{\frac{1}{p^*}}, \tag{4.1.1}$$

with $c_1 = \frac{N(n+1)+(N+1)}{(n+1)(N+1)}$. [163]

Theorem 4.1.2 *For two positive functions $f_1, f_2 \in L_{p^*}[a,b]$, such that $1 \leq p^* \leq \infty$, $0 < \int\limits_a^b f_1^{p^*}(t)dt < \infty$ and $0 < \int\limits_a^b f_2^{p^*}(t)dt < \infty$, if $0 < n \leq \frac{f_1(t)}{f_2(t)} \leq N$, for $n, N \in \mathbb{R}^+$ and $\forall a \leq t \leq b$, then*

$$\left(\int\limits_a^b f_1^{p^*}(t)dt\right)^{\frac{2}{p^*}} + \left(\int\limits_a^b f_2^{p^*}(t)dt\right)^{\frac{2}{p^*}} \geq c_2 \left(\int\limits_a^b f_1^{p^*}(t)dt\right)^{\frac{1}{p^*}} \left(\int\limits_a^b f_2^{p^*}(t)dt\right)^{\frac{2}{p^*}}, \tag{4.1.2}$$

with $c_2 = \frac{(n+1)(N+1)}{N} - 2$. [163]

Dahmani [68] verified the above inequality and a related result via Riemann-Liouville integral as follows:

Theorem 4.1.3 *For two positive functions* $f_1, f_2 \in L_1^{p^*}[a,t]$ *on* $[0,\infty)$ *such that* $p^* \geq 1$, $\mathcal{R}_{a^+}^{\alpha,s} f_1^{p^*}(t) < \infty$ *and* $\mathcal{R}_{a^+}^{\alpha,s} f_2^{p^*}(t) < \infty$. *If* $0 < n \leq \frac{f_1(x)}{f_2(x)} \leq N$, *for* $n, N \in \mathbb{R}^+$ *and* $\forall a \leq x \leq t, t > a$, *then*

$$\left(\mathcal{R}_{a^+}^{\alpha,s} f_1^{p^*}(t)\right)^{\frac{1}{p^*}} + \left(\mathcal{R}_{a^+}^{\alpha,s} f_2^{p^*}(t)\right)^{\frac{1}{p^*}} \leq c_1 \left(\mathcal{R}_{a^+}^{\alpha,s} (f_1+f_2)^{p^*}(t)\right)^{\frac{1}{p^*}}, \tag{4.1.3}$$

with $c_1 = \frac{N(n+1)+(N+1)}{(n+1)(N+1)}$. [68]

Theorem 4.1.4 *For two positive functions* $f_1, f_2 \in L_1^{p^*}[a,t]$ *on* $[0,\infty)$ *such that* $p^* \geq 1$, $\mathcal{R}_{a^+}^{\alpha,s} f_1^{p^*}(t) < \infty$ *and* $\mathcal{R}_{a^+}^{\alpha,s} f_2^{p^*}(t) < \infty$. *If* $0 < n \leq \frac{f_1(x)}{f_2(x)} \leq N$, *for* $n, N \in \mathbb{R}^+$ *and* $\forall a \leq x \leq t, t > a$, *then*

$$\left(\mathcal{R}_{a^+}^{\alpha,s} f_1^{p^*}(t)\right)^{\frac{2}{p^*}} + \left(\mathcal{R}_{a^+}^{\alpha,s} f_2^{p^*}(t)\right)^{\frac{2}{p^*}} \geq c_2 \left(\mathcal{R}_{a^+}^{\alpha,s} f_1^{p^*}(t)\right)^{\frac{1}{p^*}} \left(\mathcal{R}_{a^+}^{\alpha,s} f_2^{p^*}(t)\right)^{\frac{1}{p^*}}, \tag{4.1.4}$$

with $c_2 = \frac{(n+1)(N+1)}{N} - 2$. [68]

Chinchane et al. [59] and Sabrina et al. [171] established the reverse Minkowski's inequality involving Hadamard fractional integral operator:

Theorem 4.1.5 *For two positive functions* $f_1, f_2 \in L_1^{p^*}[a,t]$ *on* $[0,\infty)$ *such that* $p^* \geq 1$, $\mathcal{H}_{a^+}^{\alpha,s} f_1^{p^*}(t) < \infty$ *and* $\mathcal{H}_{a^+}^{\alpha,s} f_2^{p^*}(t) < \infty$. *If* $0 < n \leq \frac{f_1(x)}{f_2(x)} \leq N$, *for* $n, N \in \mathbb{R}^+$ *and* $\forall a \leq x \leq t, t > a$, *then*

$$\left(\mathcal{H}_{a^+}^{\alpha,s} f_1^{p^*}(t)\right)^{\frac{1}{p^*}} + \left(\mathcal{H}_{a^+}^{\alpha,s} f_2^{p^*}(t)\right)^{\frac{1}{p^*}} \leq c_1 \left(\mathcal{H}_{a^+}^{\alpha,s} (f_1+f_2)^{p^*}(t)\right)^{\frac{1}{p^*}}, \tag{4.1.5}$$

with $c_1 = \frac{N(n+1)+(N+1)}{(n+1)(N+1)}$. [59, 171]

Theorem 4.1.6 *For two positive functions* $f_1, f_2 \in L_1^{p^*}[a,t]$ *on* $[0,\infty)$ *such that* $p^* \geq 1$, $\mathcal{H}_{a^+}^{\alpha,s} f_1^{p^*}(t) < \infty$ *and* $\mathcal{H}_{a^+}^{\alpha,s} f_2^{p^*}(t) < \infty$. *If* $0 < n \leq \frac{f_1(x)}{f_2(x)} \leq N$, *for* $n, N \in \mathbb{R}^+$ *and*

$\forall a \leq x \leq t, t > a$, then

$$\left(\mathcal{H}_{a^+}^{\alpha,s} f_1^{p^*}(t)\right)^{\frac{2}{p^*}} + \left(\mathcal{H}_{a^+}^{\alpha,s} f_2^{p^*}(t)\right)^{\frac{2}{p^*}} \geq c_2 \left(\mathcal{H}_{a^+}^{\alpha,s} f_1^{p^*}(t)\right)^{\frac{1}{p^*}} \left(\mathcal{H}_{a^+}^{\alpha,s} f_2^{p^*}(t)\right)^{\frac{1}{p^*}}, \quad (4.1.6)$$

with $c_2 = \frac{(n+1)(N+1)}{N} - 2$. [59, 171]

4.2 Application to Conformable k-Fractional Integrals via Minkowski's Inequality

This section contains the main results regarding Minkowski's inequality (4.1.1) and (4.1.2).

Theorem 4.2.7 *For two positive functions $f_1, f_2 \in L_1^{p^*}[a, y]$ on $[0, \infty)$ such that $p^* \geq 1$, ${}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(y) < \infty$ and ${}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_2^{p^*}(y) < \infty$. If $0 < n \leq \frac{f_1(x)}{f_2(x)} \leq N$, for $n, N \in \mathbb{R}^+$ and $\forall a \leq x \leq y, y > a$, then*

$$\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(y)\right)^{\frac{1}{p^*}} + \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_2^{p^*}(y)\right)^{\frac{1}{p^*}} \leq c_1 \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} (f_1 + f_2)^{p^*}(y)\right)^{\frac{1}{p^*}}, \quad (4.2.7)$$

with $c_1 = \frac{N(n+1)+(N+1)}{(n+1)(N+1)}$.

Proof. Under the given conditions $\frac{f_1(x)}{f_2(x)} \leq N$, $a \leq x \leq y$, it can be written

$$f_1(x) \leq N(f_1(x) + f_2(x)) - N f_1(x),$$

implies that,

$$(N+1)^{p^*} f_1^{p^*}(x) \leq N^{p^*}(f_1(x) + f_2(x))^{p^*}. \quad (4.2.8)$$

Multiply (4.2.8) on both sides with

$$\frac{1}{k\Gamma_k(\alpha)} \left(\frac{(y-a)^s - (x-a)^s}{s}\right)^{\frac{\alpha}{k}-1} (x-a)^{s-1}$$

and then integrate w.r.t. x over (a, t), we obtain

$$\frac{(N+1)^{p^*}}{k\Gamma_k(\alpha)} \int_a^y \left(\frac{(y-a)^s - (x-a)^s}{s}\right)^{\frac{\alpha}{k}-1} f_1^{p^*}(x)(x-a)^{s-1} dx$$

$$\leq \frac{N^{p^*}}{k\Gamma_k(\alpha)} \int_a^y \left(\frac{(y-a)^s - (x-a)^s}{s} \right)^{\frac{\alpha}{k}-1} (f_1 + f_2)^{p^*}(x)(x-a)^{s-1} dx. \tag{4.2.9}$$

Consequently, one can write

$$\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(y) \right)^{\frac{1}{p^*}} \leq \frac{N}{N+1} \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+} (f_1 + f_2)^{p^*}(y) \right)^{\frac{1}{p^*}}, \tag{4.2.10}$$

In contrast, as $nf_2(x) \leq f_1(x)$, it follows

$$(1 + \frac{1}{n})^{p^*} f_2^{p^*}(x) \leq \left(\frac{1}{n} \right)^{p^*} (f_2(x) + f_1(x))^{p^*}. \tag{4.2.11}$$

Further, multiply on both sides of (4.2.11) with

$$\frac{1}{k\Gamma_k(\alpha)} \left(\frac{(y-a)^s - (x-a)^s}{s} \right)^{\frac{\alpha}{k}-1} (x-a)^{s-1}$$

and then integrate w.r.t. the variable x over (a, y), we obtain

$$\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+} f_2^{p^*}(y) \right)^{\frac{1}{p^*}} \leq \frac{1}{n+1} \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+} (f_1 + f_2)^{p^*}(y) \right)^{\frac{1}{p^*}}, \tag{4.2.12}$$

The required inequality (4.2.7) attains from (4.2.10) and (4.2.12). ■

Theorem 4.2.8 *For two positive functions $f_1, f_2 \in L_1^{p^*}[a, t]$ on $[0, \infty)$ such that $p^* \geq 1$, ${}^s_k\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(t) < \infty$ and ${}^s_k\mathfrak{F}^{*\alpha}_{a^+} f_2^{p^*}(t) < \infty$. If $0 < n \leq \frac{f_1(x)}{f_2(x)} \leq N$, for $n, N \in \mathbb{R}^+$ and $\forall a \leq x \leq t, t > a$, then*

$$\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(t) \right)^{\frac{2}{p^*}} + \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+} f_2^{p^*}(t) \right)^{\frac{2}{p^*}} \geq c_2 \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(t) \right)^{\frac{1}{p^*}} \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+} f_2^{p^*}(t) \right)^{\frac{1}{p^*}}, \tag{4.2.13}$$

with $c_2 = \frac{(n+1)(N+1)}{N} - 2$.

Proof. Taking the product between (4.2.10) and (4.2.12), it results

$$\frac{(n+1)(N+1)}{N} \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(t) \right)^{\frac{1}{p^*}} \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+} f_2^{p^*}(t) \right)^{\frac{1}{p^*}} \leq \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+} (f_1 + f_2)^{p^*}(t) \right)^{\frac{2}{p^*}}. \tag{4.2.14}$$

Invoking the Minkowski's inequality, on the R.H.S. of (4.2.14), we get

$$\frac{(n+1)(N+1)}{N}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1^{p^*}(t)\right)^{\frac{1}{p^*}}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_2^{p^*}(t)\right)^{\frac{1}{p^*}}$$

$$\leq\left(\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1^{p^*}(t)\right)^{\frac{1}{p^*}}+\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_2^{p^*}(t)\right)^{\frac{1}{p^*}}\right)^2. \tag{4.2.15}$$

From (4.2.15), we conclude that

$$\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1^{p^*}(t)\right)^{\frac{2}{p^*}}+\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_2^{p^*}(t)\right)^{\frac{2}{p^*}}$$

$$\geq\left(\frac{(n+1)(N+1)}{N}-2\right)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1^{p^*}(t)\right)^{\frac{1}{p^*}}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_2^{p^*}(t)\right)^{\frac{1}{p^*}}.$$

which completes the proof. ■

4.3 Related Conformable k-Fractional Inequalities

In this section, the related results represented by Chinchane, Sulaiman and Sroysang are generalized for conformable $k-$fractional integral (2.3.35).

Theorem 4.3.9 *For two positive functions $f_1, f_2 \in L_1^{p^*}[a,y]$ on $[0,\infty)$ such that $p^*\geq 1$, ${}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1^{p^*}(y)<\infty$ and ${}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_2^{p^*}(y)<\infty$. If $0<n\leq\frac{f_1(x)}{f_2(x)}\leq N$, for $n,N\in\mathbb{R}^+$ and $\forall a\leq x\leq y, y>a$, then*

$$\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1^{p^*}(y)\right)^{\frac{1}{p^*}}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_2^{p^*}(y)\right)^{\frac{1}{p^*}}\leq\left(\frac{N}{n}\right)^{\frac{1}{p^*q}}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1^{\frac{1}{p^*}}(y)f_2^{\frac{1}{q}}(y)\right). \tag{4.3.16}$$

Proof. Under the given conditions $\frac{f_1(x)}{f_2(x)}\leq N$, $a\leq x\leq y$, it can be written

$$f_1(x)\leq Nf_2(x)\Rightarrow f_2^{\frac{1}{q}}(x)\geq N^{-\frac{1}{q}}f_1^{\frac{1}{q}}(x). \tag{4.3.17}$$

Multiply both sides of (4.3.17) with $f^{\frac{1}{p^*}}(x)$, one can rewrite as

$$f_1^{\frac{1}{p^*}}(x)f_2^{\frac{1}{q}}(x)\geq^{-\frac{1}{q}}f_1(x). \tag{4.3.18}$$

Multiply the inequality (4.3.18) on both sides with

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(y-a)^s-(x-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(x-a)^{s-1}$$

and then integrate w.r.t. the variable x over (a,y), we obtain

$$\frac{N^{-\frac{1}{q}}}{k\Gamma_k(\alpha)}\int_a^y\left(\frac{(y-a)^s-(x-a)^s}{s}\right)^{\frac{\alpha}{k}-1}f_1(x)(x-a)^{s-1}dx$$
$$\leq\frac{1}{k\Gamma_k(\alpha)}\int_a^y\left(\frac{(y-a)^s-(x-a)^s}{s}\right)^{\frac{\alpha}{k}-1}f_1^{\frac{1}{p^*}}(x)f_2^{\frac{1}{q}}(x)(x-a)^{s-1}dx. \quad (4.3.19)$$

Accordingly, it can be written as

$$N^{-\frac{1}{p^*q}}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1(y)\right)^{\frac{1}{p^*}}\leq\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1^{\frac{1}{p^*}}(y)f_2^{\frac{1}{q}}(y)\right)^{\frac{1}{p^*}}. \quad (4.3.20)$$

In contrast, as $nf_2(x)\leq f_1(x)$, it follows

$$n^{\frac{1}{p^*}}f_2^{\frac{1}{p^*}}(x)\leq f_1^{\frac{1}{p^*}}(x). \quad (4.3.21)$$

Further, multiplying the above inequality (4.3.21) by $f_2^{\frac{1}{q}}(x)$ and using the relation

$$\frac{1}{p^*}+\frac{1}{q}=1,$$

arrives at

$$n^{\frac{1}{p^*}}f_2(x)\leq f_1^{\frac{1}{p^*}}(x)f_2^{\frac{1}{q}}(x). \quad (4.3.22)$$

Multiply on both sides of (4.3.22) by

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(y-a)^s-(x-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(x-a)^{s-1}$$

and then integrate w.r.t. the variable x over (a,y), we obtain

$$n^{\frac{1}{p^*q}}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_2(y)\right)^{\frac{1}{q}}\leq\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1^{\frac{1}{p^*}}(y)f_2^{\frac{1}{q}}(y)\right)^{\frac{1}{q}}, \quad (4.3.23)$$

Taking the product between (4.3.20) and (4.3.23) and using the relation $\frac{1}{p^*} + \frac{1}{q} = 1$, the required inequality (4.3.16) can be attained. ■

Theorem 4.3.10 *For two positive functions $f_1, f_2 \in L_1^{p^*}[a, y]$ on $[0, \infty)$ such that $p^* \geq 1$, ${}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(y) < \infty$ and ${}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_2^{p^*}(y) < \infty$. If $0 < n \leq \frac{f_1(x)}{f_2(x)} \leq N$, for $n, N \in \mathbb{R}^+$ and $\forall a \leq x \leq y, y > a$, then*

$$ {}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_1(y) f_2(y) \leq c_3 \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} (f_1^{p^*} + f_2^{p^*})(y) \right) + c_4 \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} (f_1^q + f_2^q)(y) \right), \qquad (4.3.24) $$

with $c_3 = \frac{2^{p-1} N^{p^*}}{p^*(N+1)^{p^*}}$ and $c_4 = \frac{2^{q-1}}{q(n+1)^q}$

Proof. Using the hypothesis, we get

$$ (N+1)^{p^*} f_1^{p^*}(x) \leq N^{p^*} (f_1 + f_2)^{p^*}(x). \qquad (4.3.25) $$

Multiply on both sides of (4.3.25) with

$$ \frac{1}{k\Gamma_k(\alpha)} \left(\frac{(y-a)^s - (x-a)^s}{s} \right)^{\frac{\alpha}{k}-1} (x-a)^{s-1} $$

and then integrate w.r.t. the variable x over (a, t), we obtain

$$ \begin{aligned} &\frac{(N+1)^{p^*}}{k\Gamma_k(\alpha)} \int_a^y \left(\frac{(y-a)^s - (x-a)^s}{s} \right)^{\frac{\alpha}{k}-1} f_1^{p^*}(x)(x-a)^{s-1} dx \\ &\leq \frac{N^{p^*}}{k\Gamma_k(\alpha)} \int_a^y \left(\frac{(y-a)^s - (x-a)^s}{s} \right)^{\frac{\alpha}{k}-1} (f_1 + f_2)^{p^*}(x)(x-a)^{s-1} dx. \end{aligned} \qquad (4.3.26) $$

Accordingly, it can be written as

$$ {}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(y) \leq \frac{N^{p^*}}{(N+1)^{p^*}} {}_k^s\mathfrak{F}^{*\alpha}_{a^+} (f_1 + f_2)^{p^*}(y). \qquad (4.3.27) $$

In contrast, as $0 < n < \frac{f_1(x)}{f_2(x)}, a < x < y$, it follows

$$ (n+1)^q f_2^q(x) \leq (f_1 + f_2)^q(x). \qquad (4.3.28) $$

Multiply on both sides of (4.3.28) by

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(y-a)^s-(x-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(x-a)^{s-1}$$

and then integrate w.r.t. the variable x over (a,y), give

$$ {}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_2^q(y)\leq\frac{1}{(n+1)^q}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(f_1+f_2)^q(y), \tag{4.3.29}$$

Consider the Young's inequality,

$$f_1(x)f_2(x)\leq\frac{f_1^{p^*}(x)}{p^*}+\frac{f_2^q(x)}{q}, \tag{4.3.30}$$

Now, multiply on both sides of (4.3.30) by

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(y-a)^s-(x-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(x-a)^{s-1}$$

and then integrate w.r.t. the variable x over (a,y), yield

$$ {}_k^s\mathfrak{F}^{*\alpha}_{a^+}(f_1f_2)(y)\leq\frac{1}{p^*}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1^{p^*}(y)\right)+\frac{1}{q}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_2^q(y)\right), \tag{4.3.31}$$

Invoking (4.3.27) and (4.3.29) into (4.3.31), we obtain

$$\begin{aligned} {}_k^s\mathfrak{F}^{*\alpha}_{a^+}(f_1f_2)(y)&\leq\frac{1}{p^*}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1^{p^*}(y)\right)+\frac{1}{q}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_2^q(y)\right)\\ &\leq\frac{N^{p^*}}{p^*(N+1)^{p^*}}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}(f_1+f_2)^{p^*}(y)\right)+\frac{1}{q(n+1)^q}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}(f_1+f_2)^q(y)\right). \end{aligned} \tag{4.3.32}$$

Using the following inequality,

$$(x_1+x_2)^s\leq 2^{s-1}(x_1^s+x_2^s),\, s>1, x_1,x_2>0,$$

one obtains

$$ {}_k^s\mathfrak{F}^{*\alpha}_{a^+}(f_1+f_2)^{p^*}(y)\leq 2^{p^*-1}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(f_1^{p^*}+f_2^{p^*})(y) \tag{4.3.33}$$

and

$$ {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(f_1+f_2)^q(y) \leq 2^{q-1}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\right)(f_1^q+f_2^q)(y) \tag{4.3.34} $$

The proof of (4.3.24) can be concluded from (4.3.32), (4.3.33) and (4.3.34) collectively. ■

Theorem 4.3.11 *For two positive functions* $f_1, f_2 \in L_1^{p^*}[a,y]$ *on* $[0,\infty)$ *such that* $p^* \geq 1$, ${}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(y) < \infty$ *and* ${}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+} f_2^{p^*}(y) < \infty$. *If* $0 < n \leq \frac{f_1(x)}{f_2(x)} \leq N$, *for* $n, N \in \mathbb{R}^+$ *and* $\forall a \leq x \leq y, y > a$, *then*

$$ \begin{aligned} \frac{N+1}{N-c}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(f_1(y)-cf_2(y))\right) &\leq \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(y)\right)^{\frac{1}{p^*}} + \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+} f_2^{p^*}(y)\right)^{\frac{1}{p^*}} \\ &\leq \frac{n+1}{n-c}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(f_1(y)-cf_2(y)\right)^{\frac{1}{p^*}}. \end{aligned} \tag{4.3.35} $$

Proof. Under the assumption $0 < c < n \leq N$, one may consider

$$ nc \leq Nc \Rightarrow nc + n \leq nc + N \leq Nc + N \Rightarrow (N+1)(n-c) \leq (n+1)(N-c). $$

which gives

$$ \frac{(N+1)}{(N-c)} \leq \frac{(n+1)}{(n-c)}. $$

Furthermore, we have

$$ n - c \leq \frac{f_1(x)-cf_2(x)}{f_2(x)} \leq N - c, $$

yields

$$ \frac{(f_1(x)-cf_2(x))^{p^*}}{(N-c)^{p^*}} \leq f_2^{p^*}(x) \leq \frac{(f_1(x)-cf_2(x))^{p^*}}{(n-c)^{p^*}}, \tag{4.3.36} $$

Again, we have

$$ \frac{1}{N} \leq \frac{f_2(x)}{f_1(x)} \leq \frac{1}{n} \Rightarrow \frac{n-c}{cn} \leq \frac{f_1(x)-cf_2(x)}{cf_1(x)} \leq \frac{N-c}{cN}, $$

implies that

$$ \left(\frac{N}{N-c}\right)^{p^*}(f_1(x)-cf_2(x))^{p^*} \leq f_1^{p^*}(x) \leq \left(\frac{n}{n-c}\right)^{p^*}(f_1(x)-cf_2(x))^{p^*}, \tag{4.3.37} $$

Multiply (4.3.36) on both sides with

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(y-a)^s-(x-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(x-a)^{s-1}$$

and then integrate w.r.t. the variable x over (a,y), we obtain

$$\begin{aligned}&\frac{1}{(N-c)^{p^*}k\Gamma_k(\alpha)}\int_a^y\left(\frac{(y-a)^s-(x-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(f_1(x)-cf_2(x))^{p^*}(x-a)^{s-1}dx\\ &\leq\frac{1}{k\Gamma_k(\alpha)}\int_a^y\left(\frac{(y-a)^s-(x-a)^s}{s}\right)^{\frac{\alpha}{k}-1}f_2^{p^*}(x)(x-a)^{s-1}dx\\ &\leq\frac{1}{(n-c)^{p^*}k\Gamma_k(\alpha)}\int_a^y\left(\frac{(y-a)^s-(x-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(f_1(x)-cf_2(x))^{p^*}(x-a)^{s-1}dx.\end{aligned}$$

Accordingly, it can be written as

$$\begin{aligned}\frac{1}{N-c}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}(f_1(y)-cf_2(y))^{p^*}\right)^{\frac{1}{p^*}}&\leq\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_2(y)^{p^*}\right)^{\frac{1}{p^*}}\\ &\leq\frac{1}{n-c}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}(f_1(y)-cf_2(y))^{p^*}\right)^{\frac{1}{p^*}}.\end{aligned}\tag{4.3.38}$$

Performing the same procedure with (4.3.37), we get

$$\begin{aligned}&\frac{N}{N-c}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}(f_1(y)-cf_2(y))^{p^*}\right)^{\frac{1}{p^*}}\\ &\leq\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1(y)^{p^*}\right)^{\frac{1}{p^*}}\\ &\leq\frac{n}{n-c}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}(f_1(y)-cf_2(y))^{p^*}\right)^{\frac{1}{p^*}}.\end{aligned}\tag{4.3.39}$$

The addition of (4.3.38) and (4.3.39) concludes the proof of (4.3.35). ■

Theorem 4.3.12 *For two positive functions $f_1,f_2\in L_1^{p^*}[a,y]$ on $[0,\infty)$ such that $p^*\geq 1$, ${}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1^{p^*}(y)<\infty$ and ${}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_2^{p^*}(y)<\infty$. If $0<n\leq\frac{f_1(x)}{f_2(x)}\leq N$, for $n,N\in\mathbb{R}^+$ and $\forall a\leq x\leq y, y>a$, then*

$$\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1^{p^*}(y)\right)^{\frac{1}{p^*}}+\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_2^{p^*}(y)\right)^{\frac{1}{p^*}}\leq c_5\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}(f_1+f_2)^{p^*}(y)\right)^{\frac{1}{p^*}},\tag{4.3.40}$$

with $c_5=\frac{A(a+B)+B(A+b)}{(A+b)(a+B)}$.

Proof. Under the hypothesis, one may follow

$$\frac{1}{B} \leq \frac{1}{f_2(y)} \leq \frac{1}{b}. \tag{4.3.41}$$

Taking the product of (4.3.41) and $0 < a \leq f_1(x) \leq A$, we get

$$\frac{a}{B} \leq \frac{f_1(y)}{f_2(y)} \leq \frac{A}{b}. \tag{4.3.42}$$

From (4.3.42), we get

$$f_2^{p^*}(x) \leq \left(\frac{B}{a+B}\right)^{p^*} (f_1(x) + f_2(x))^{p^*}, \tag{4.3.43}$$

and

$$f_1^{p^*}(x) \leq \left(\frac{A}{b+A}\right)^{p^*} (f_1(x) + f_2(x))^{p^*}. \tag{4.3.44}$$

Multiplying on both sides of (4.3.43) with

$$\frac{1}{k\Gamma_k(\alpha)} \left(\frac{(y-a)^s - (x-a)^s}{s}\right)^{\frac{\alpha}{k}-1} (x-a)^{s-1}$$

and then integrate w.r.t. the variable x over (a, t), we obtain

$$\frac{1}{k\Gamma_k(\alpha)} \int_a^y \left(\frac{(y-a)^s - (x-a)^s}{s}\right)^{\frac{\alpha}{k}-1} f_2^{p^*}(x)(x-a)^{s-1}dx$$
$$\leq \frac{B^{p^*}}{(a+B)^{p^*} k\Gamma_k(\alpha)} \int_a^y \left(\frac{(y-a)^s - (x-a)^s}{s}\right)^{\frac{\alpha}{k}-1} (f_1(x) + f_2(x))^{p^*} (x-a)^{s-1}dx.$$

Consequently, we write

$$\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_2^{p^*}(y)\right)^{\frac{1}{p^*}} \leq \frac{B}{a+B} \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} (f_1 + f_2)^{p^*}(y)\right)^{\frac{1}{p^*}}, \tag{4.3.45}$$

Performing the same procedure with (4.3.44), we obtain

$$\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(y)\right)^{\frac{1}{p^*}} \leq \frac{A}{b+A} \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} (f_1 + f_2)^{p^*}(y)\right)^{\frac{1}{p^*}}. \tag{4.3.46}$$

The addition of (4.3.45) and (4.3.46) concludes the proof of (4.3.40). ■

Theorem 4.3.13 *For two positive functions* $f_1, f_2 \in L_1^{p^*}[a, y]$ *on* $[0, \infty)$ *such that* $p^* \geq 1$, ${}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(y) < \infty$ *and* ${}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_2^{p^*}(y) < \infty$. *If* $0 < n \leq \frac{f_1(x)}{f_2(x)} \leq N$, *for* $n, N \in \mathbb{R}^+$ *and* $\forall a \leq x \leq y, y > a$, *then*

$$\frac{1}{N}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_1(y) f_2(y)\right) \leq \frac{1}{(n+1)(N+1)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} (f_1 + f_2)^2(y)\right) \leq \frac{1}{n}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_1(y) f_2(y)\right). \tag{4.3.47}$$

Proof. Using $0 < n \leq \frac{f_1(x)}{f_2(x)} \leq M$, we have

$$f_2(x)(n+1) \leq f_2(x) + f_1(x) \leq f_2(t)(N+1). \tag{4.3.48}$$

Also, it follows that $\frac{1}{N} \leq \frac{f_2(x)}{f_1(x)} \leq \frac{1}{n}$, which yields

$$f_1(x)(\frac{N+1}{N}) \leq f_2(x) + f_1(x) \leq f_1(x)(\frac{n+1}{n}). \tag{4.3.49}$$

Taking the product of (4.3.48) and (4.3.49), we obtain

$$\frac{f_1(x)f_2(x)}{N} \leq \frac{(f_2(x) + f_1(x))^2}{(n+1)(N+1)} \leq \frac{f_1(x)f_2(x)}{n}. \tag{4.3.50}$$

Multiply on both sides of (4.3.50) with

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(y-a)^s - (x-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(x-a)^{s-1}$$

and then integrate w.r.t. the variable x over (a, y), we attain

$$\frac{1}{Nk\Gamma_k(\alpha)}\int_a^y \left(\frac{(y-a)^s - (x-a)^s}{s}\right)^{\frac{\alpha}{k}-1} f_2(x)f_1(x)(x-a)^{s-1}dx$$
$$\leq c_6\frac{1}{k\Gamma_k(\alpha)}\int_a^y \left(\frac{(y-a)^s - (x-a)^s}{s}\right)^{\frac{\alpha}{k}-1} (f_2(x) + f_1(x))^2(x-a)^{s-1}dx$$
$$\leq \frac{1}{nk\Gamma_k(\alpha)}\int_a^y \left(\frac{(y-a)^s - (x-a)^s}{s}\right)^{\frac{\alpha}{k}-1} f_1(x)f_2(x)(x-a)^{s-1}dx.$$

with $c_6 = \frac{1}{(n+1)(N+1)}$.

Accordingly, we can conclude the required result (4.3.47). ■

Theorem 4.3.14 *For two positive functions $f_1, f_2 \in L_1^{p^*}[a, y]$ on $[0, \infty)$ such that $p^* \geq 1$, ${}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(y) < \infty$ and ${}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_2^{p^*}(y) < \infty$. If $0 < n \leq \frac{f_1(x)}{f_2(x)} \leq N$, for $n, N \in \mathbb{R}^+$ and $\forall a \leq x \leq y, y > a$, then*

$$\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(y)\right)^{\frac{1}{p^*}} + \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_2^{p^*}(y)\right)^{\frac{1}{p^*}} \leq 2\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} h^{p^*}(f_1(y), f_2(y))\right)^{\frac{1}{p^*}}, \tag{4.3.51}$$

where

$$h(f_1(y), f_2(y)) = max\left\{N\left\{\left(\frac{N}{n}+1\right)f_1(y) - Nf_2(y)\right\}, \frac{(n+N)f_2(y) - f_1(y)}{n}\right\}.$$

Proof. Under the given conditions $0 < n \leq \frac{f_1(x)}{f_2(x)} \leq N$, $a \leq x \leq y$, it can be written

$$0 < n \leq N + n - \frac{f_1(x)}{f_2(x)} \tag{4.3.52}$$

and

$$N + n - \frac{f_1(x)}{f_2(x)} \leq N. \tag{4.3.53}$$

From (4.3.52) and (4.3.53), we obtain

$$f_2(x) < \frac{(N+n)f_2(x) - f_1(x)}{n} \leq h(f_1(x), f_2(x)). \tag{4.3.54}$$

where

$$h(f_1(x), f_2(x)) = max\left\{N\left\{\left(\frac{N}{n}+1\right)f_1(y) - Nf_2(y)\right\}, \frac{(n+N)f_2(y) - f_1(y)}{n}\right\}.$$

From assumptions, one may write

$$0 < \frac{1}{N} \leq \frac{f_2(x)}{f_1(x)} \leq \frac{1}{n},$$

implies that

$$\frac{1}{N} \leq \frac{1}{N} + \frac{1}{n} - \frac{f_2(x)}{f_1(x)}. \tag{4.3.55}$$

and

$$\frac{1}{N} + \frac{1}{n} - \frac{f_2(x)}{f_1(x)} \leq \frac{1}{n}. \tag{4.3.56}$$

From (4.3.55) and (4.3.56), we get

$$\frac{1}{N} \leq \frac{(\frac{1}{N} + \frac{1}{n})f_1(x) - f_2(x)}{f_1(x)} \leq \frac{1}{n}, \tag{4.3.57}$$

which can rewrite as

$$\begin{aligned}
f_1(x) &\leq N(\frac{1}{N} + \frac{1}{n})f_1(x) - Nf_2(x) \\
&= \frac{N(N+n)f_1(x) - N^2 n f_2(x)}{nN} \\
&= \left(\frac{N}{n} + 1\right) f_1(x) - Nf_2(x) \\
&\leq N\left[\left(\frac{N}{n} + 1\right) f_1(x) - Nf_2(x)\right] \\
&\leq h(f_1(x), f_2(x)). \qquad (4.3.58)
\end{aligned}$$

We can write from (4.3.54) and (4.3.58)

$$f_1^{p^*}(x) \leq h^{p^*}(f_1(x), f_2(x)), \tag{4.3.59}$$

$$f_2^{p^*}(x) \leq h^{p^*}(f_1(x), f_2(x)). \tag{4.3.60}$$

Multiply on both sides of (4.3.59) with

$$\frac{1}{k\Gamma_k(\alpha)} \left(\frac{(y-a)^s - (x-a)^s}{s}\right)^{\frac{\alpha}{k}-1} (x-a)^{s-1}$$

and then integrate w.r.t. the variable x over (a, y), we obtain

$$\frac{1}{k\Gamma_k(\alpha)}\int_a^y \left(\frac{(y-a)^s-(x-a)^s}{s}\right)^{\frac{\alpha}{k}-1} f_1^{p^*}(x)(x-a)^{s-1}dx$$
$$\leq \frac{1}{k\Gamma_k(\alpha)}\int_a^y \left(\frac{(y-a)^s-(x-a)^s}{s}\right)^{\frac{\alpha}{k}-1} h^{p^*}(f_1(x), f_2(x))(x-a)^{s-1}dx.$$

As a result, we can write

$$\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_1^{p^*}(y)\right)^{\frac{1}{p^*}} \leq \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} h^{p^*}(f_1(y), f_2(y))\right)^{\frac{1}{p^*}}. \tag{4.3.61}$$

Repeating the same procedure as above, for (4.3.60), we have

$$\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} f_2^{p^*}(t)\right)^{\frac{1}{p^*}} \leq \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+} h^{p^*}(f_1(t), f_2(t))\right)^{\frac{1}{p^*}}. \tag{4.3.62}$$

The inequalities (4.3.61) and (4.3.62) yield required result (4.3.51). ■

Remark 4.3.1 *For the specific values of the parameters involved in* (2.3.35)*, the above theorems (4.2.7) to (4.3.14) can be deduced as particular cases, i.e., each result can be derived for the existing fractional integrals: Hadamard, Katugampola, Liouville, Riemann-Liouville and conformable fractional integrals.*

Chapter 5

INEQUALITIES FOR CONVEX FUNCTIONS VIA GENERALIZED CONFORMABLE K-FRACTIONAL INTEGRALS

In this chapter, we authenticate some integral results regarding convex functions by means of the generalized conformable $k-$fractional integral operators (2.3.35) and (2.3.36). We have generalized the classical integral inequalities presented in [119] for convex functions via generalized k-FCI. We also deduce some other classical integral inequalities as particular cases for our results. In the first section, we give a brief introduction of the existing inequalities for convex functions involving other integral operators that have motivated us and in the second section, we present their generalized version using k-fractional conformable integrals and related results.

5.1 Inequalities for Convex Functions

Let us begin by the work of Ngo et al. [131], which presents the following result

$$\int_0^1 g^{*\lambda'+1}(y')dy' \geq \int_0^1 y'^{\lambda'} g^*(y')dy \tag{5.1.1}$$

and

$$\int_0^1 g^{*\lambda'+1}(y')dy' \geq \int_0^1 y' g^{*\lambda'}(y')dy, \tag{5.1.2}$$

provided that $\lambda' > 0$ and $g^* > 0$ is a continuous function on $y' \in [0, 1]$ satisfying

$$\int_{\tau^*}^1 g^*(y')dy' \geq \int_{\tau^*}^1 y'dy', 0 \leq \tau^* \leq 1.$$

In [118], W. J. Liu. et. al. proved that

$$\int_a^b g^{*\lambda'+\delta'}(y')dy' \geq \int_a^b (y'-a)^{\lambda'} g^{*\delta'}(y')dy', \tag{5.1.3}$$

where $\lambda' > 0$, $\delta' > 0$ and $g^* > 0$ is a continuous function on $a \leq y' \leq b$ such that

$$\int_{\tau^*}^{b} g^{*\gamma'}(y')dy' \geq \int_{\tau^*}^{b} (y'-a)^{\gamma'} dy'; a \leq \gamma' \leq b.$$

In [119], the following two theorems were presented by using the results (5.1.1)-(5.1.3):

Theorem 5.1.1 *For two continuous positive functions g_1 and h^* on $a \leq y' \leq b$, where g_1 is increasing on $[a,b]$ satisfying $g_1 \leq h^*$ and $\frac{g_1}{h^*}$ is decreasing, the relation*

$$\frac{\int_a^b g_1(y')dy'}{\int_a^b h^*(y')dy'} \geq \frac{\int_a^b \varphi^*(g_1(y'))dy'}{\int_a^b \varphi^*(h^*(y'))dy'} \tag{5.1.4}$$

holds, where convex function φ^* satisfies $\varphi^*(0) = 0$.

Theorem 5.1.2 *For three continuous and positive functions g_1, g_2 and h^* on $a \leq y' \leq b$, where g_1 and g_2 are increasing on $[a,b]$ satisfying $g_1 \leq h^*$ and $\frac{g_1}{h^*}$ is decreasing, the relation*

$$\frac{\int_a^b g_1(y')dy'}{\int_a^b h^*(y')dy'} \geq \frac{\int_a^b \varphi^*(g_1(y'))g_2(y')dy'}{\int_a^b \varphi^*(h^*(y'))g_2(y')dy'} \tag{5.1.5}$$

holds, where convex function φ^* satisfies $\varphi^*(0) = 0$.

In [65], the following inequalities are proved by Dahmani associating the fractional Riemann-Liouville integral operator ($\mathcal{R}^\beta$).

Theorem 5.1.3 *For two continuous and positive functions g^* and h^* on $a \leq t' \leq b$, where g^* is increasing on $[a,b]$ such that $g^* \leq h^*$ and $\frac{g^*}{h^*}$ is decreasing, a convex function φ^* satisfies $\varphi^*(0) = 0.$, the relation*

$$\frac{\mathcal{R}^\beta[g^*(t')]}{\mathcal{R}^\beta[h^*(t')]} \geq \frac{\mathcal{R}^\beta[\varphi^*(g^*(t'))]}{\mathcal{R}^\beta[\varphi^*(h^*(t'))]}, \tag{5.1.6}$$

is valid.

Theorem 5.1.4 *For three continuous and positive functions g_1, g_2 and h^* on $a \leq t' \leq b$, where g_1 and g_2 are increasing on $[a,b]$ satisfying $g_1 \leq h^*$ and $\frac{g_1}{h^*}$ is decreasing, the relation*

$$\frac{\mathcal{R}^\beta[g_1(t')]}{\mathcal{R}^\beta[h^*(t')]} \geq \frac{\mathcal{R}^\beta[\varphi^*(g_1(t))g_2(t)]}{\mathcal{R}^\beta[\varphi^*(h^*(t'))g_2(t)]}, \tag{5.1.7}$$

holds, where convex function φ^* *satisfies* $\varphi^*(0) = 0$.

In [58], Chinchane has extended the same inequalities for the fractional Hadamard integral operator.

Theorem 5.1.5 *For two continuous and positive functions* g^* *and* h^* *on* $a \leq t' \leq b$, *where* g^* *is increasing on* $[a, b]$ *satisfying* $g^* \leq h^*$ *and* $\frac{g^*}{h^*}$ *is decreasing, the relation*

$$\frac{\mathcal{H}^\beta[g^*(t')]}{\mathcal{H}^\beta[h^*(t')]} \geq \frac{\mathcal{H}^\beta[\varphi^*(g^*(t'))]}{\mathcal{H}^\beta[\varphi^*(h^*(t'))]}, \tag{5.1.8}$$

is valid, where convex function φ^* *satisfies* $\varphi^*(0) = 0$.

Theorem 5.1.6 *Let* g_1, g_2 *and* h^* *be three continuous and positive functions on* $t' \in [a, b]$, *where* g_1, g_2 *are increasing on* $[a, b]$ *satisfying* $g_1 \leq h^*$ *and* $\frac{g_1}{h^*}$ *is decreasing, the relation*

$$\frac{\mathcal{H}^\beta[g_1(t')]}{\mathcal{H}^\beta[h^*(t')]} \geq \frac{\mathcal{H}^\beta[\varphi^*(g_1(t'))g_2(t')]}{\mathcal{H}^\beta[\varphi^*(h^*(t'))g_2(t')]}, \tag{5.1.9}$$

holds, where convex function φ^* *satisfies* $\varphi^*(0) = 0$.

We conclude Theorem (5.1.1) and Theorem (5.1.2) for our results as some particular cases.

5.2 Fractional Integral Inequalities Involving Convex Functions

This section contains our main generalized inequalities.

Theorem 5.2.7 *For two continuous and positive functions* g_1, h^* *on* $a \leq x < \infty$, *where* g_1 *is increasing on* $[a, \infty)$ *satisfying* $g_1 \leq h^*$ *on* $[a, \infty)$ *and* $\frac{g_1}{h^*}$ *is decreasing, for any* $\alpha > 0, \beta > 0, x > a$, *the relation*

$$\frac{{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(g_1(x))\ {}_k^s\mathfrak{F}^{*\beta}_{a^+}(\varphi^*(h^*(x))) + {}_k^s\mathfrak{F}^{*\beta}_{a^+}(g_1(x)){}_k^s\mathfrak{F}^{*\alpha}_{a^+}(\varphi^*(h^*(x)))}{{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(h^*(x)){}_k^s\mathfrak{F}^{*\beta}_{a^+}(\varphi^*(g_1(x))) + {}_k^s\mathfrak{F}^{*\beta}_{a^+}(h^*(x)){}_k^s\mathfrak{F}^{*\alpha}_{a^+}(\varphi^*(g_1(x)))} \geq 1, \tag{5.2.10}$$

is valid, where convex function φ^* *satisfies* $\varphi^*(0) = 0$.

Proof. According to the given conditions, the function $\frac{\varphi^*(x)}{x}$ is increasing. The function g_1 is increasing, so $\frac{\varphi^*(g_1(x))}{g_1(x)}$ is also increasing. Given that $\frac{g_1(x)}{h^*(x)}$ is decreasing, so $\forall \tau, \rho \in [a, x), x > a$, we have

$$\left(\frac{\varphi^*(g_1(\tau))}{g_1(\tau)} - \frac{\varphi^*(g_1(\rho))}{g_1(\rho)}\right)\left(\frac{g_1(\rho)}{h^*(\rho)} - \frac{g_1(\tau)}{h^*(\tau)}\right) \geq 0, \tag{5.2.11}$$

implies that

$$\frac{\varphi^*(g_1(\tau))}{g_1(\tau)}\frac{g_1(\rho)}{h^*(\rho)} + \frac{\varphi^*(g_1(\rho))}{g_1(\rho)}\frac{g_1(\tau)}{h^*(\tau)} - \frac{\varphi^*(g_1(\tau))}{g^*(\tau)}\frac{g_1(\tau)}{h^*(\tau)} - \frac{\varphi^*(g_1(\rho))}{g_1(\rho)}\frac{g_1(\rho)}{h^*(\rho)} \geq 0. \tag{5.2.12}$$

Multiplying (5.2.12) by $h^*(\tau)h^*(\rho)$, we have

$$\begin{aligned}&\frac{\varphi^*(g_1(\tau))}{g_1(\tau)}h^*(\tau)g_1(\rho) + \frac{\varphi^*(g_1(\rho))}{g_1(\rho)}g_1(\tau)h^*(\rho)\\ -&\frac{\varphi^*(g_1(\tau))}{g_1(\tau)}g_1(\tau)h^*(\rho) - \frac{\varphi^*(g_1(\rho))}{g_1(\rho)}g_1(\rho)h^*(\tau) \geq 0.\end{aligned} \tag{5.2.13}$$

Multiplying (5.2.13) on both sides by

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(x-a)^s - (\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\tau-a)^{1-s}}$$

and integrate the resulting identity w.r.t τ from a to x, we get

$$\begin{aligned}&g_1(\rho)\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}h^*(x)\right) + \frac{\varphi^*(g_1(\rho))}{g_1(\rho)}h^*(\rho)\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)(g_1(x))\\ &- h^*(\rho)\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)\left(g_1(x)\frac{\varphi^*(g_1(x))}{g_1(x)}\right) - \frac{\varphi^*(g_1(\rho))}{g_1(\rho)}g_1(\rho)\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)(h^*(x)) \geq 0.\end{aligned} \tag{5.2.14}$$

Again, multiplying (5.2.14) on both sides by

$$\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^s - (\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{1}{(\rho-a)^{1-s}}$$

then integrating the resulting identity w.r.t ρ from a to x, we get

$${}^s_k\mathfrak{F}^{*\beta}_{a^+}(g_1(x))\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}h^*(x)\right) + \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}h^*(x)\right)\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)(g_1(x))$$

$$\geq_k^s \mathfrak{F}^{*\alpha}_{a^+}(h^*(x))\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}g_1(x)\right)+\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}g_1(x)\right)\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)(h^*(x)). \tag{5.2.15}$$

since $g_1 \leq h^*$ on $[a,\infty)$ and function $\frac{\varphi^*(x)}{x}$ is increasing, so for $\tau,\rho \in [a,x)$, we have

$$\frac{\varphi^*(g_1(\tau))}{g_1(\tau)} \leq \frac{\varphi^*(h^*(\tau))}{h^*(\tau)}, \tag{5.2.16}$$

Multiplying (5.2.16) on both sides by

$$\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^s-(\tau-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{1}{(\tau-a)^{1-s}}h^*(\tau)$$

and then integrate the resultant w.r.t τ from a to x, we get

$${}^s_k\mathfrak{F}^{*\beta}_{a^+}\left(\frac{\varphi^*(g_1(x))}{g_1(x)}h^*(x)\right) \leq {}^s_k\mathfrak{F}^{*\beta}_{a^+}\left(\varphi^*(h^*(x))\right) \tag{5.2.17}$$

Hence, from (5.2.15) and (5.2.17), the required result (5.2.10) is obtained. ■

Corollary 5.2.1 *For two continuous and positive continuous functions g_1, h^* on $a \leq x < \infty$, where g_1 is increasing on $[a,\infty)$ satisfying $g_1 \leq h^*$ on $[a,\infty)$ and $\frac{g_1}{h^*}$ is decreasing, for any $\alpha > 0, x > a$, the relation*

$$\frac{\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)(g_1(x))}{\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)(h^*(x))} \geq \frac{\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)(\varphi(g_1(x)))}{\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)(\varphi(h^*(x)))}, \tag{5.2.18}$$

is valid, where convex function φ^* satisfies $\varphi^*(0) = 0$.

Proof. According to the given conditions, the function $\frac{\varphi^*(x)}{x}$ is increasing. The function g_1 is increasing, then $\frac{\varphi^*(g_1(x))}{g_1(x)}$ is also increasing. Given that $\frac{g_1(x)}{h^*(x)}$ is decreasing, so for all $\tau,\rho \in [a,x), x > a$, we have

$$\left(\frac{\varphi^*(g_1(\tau))}{g_1(\tau)} - \frac{\varphi^*(g_1(\rho))}{g_1(\rho)}\right)\left(\frac{g_1(\rho)}{h^*(\rho)} - \frac{g_1(\tau)}{h^*(\tau)}\right) \geq 0, \tag{5.2.19}$$

implies that

$$\frac{g_1(\rho)}{h^*(\rho)}\frac{\varphi^*(g_1(\tau))}{g^*(\tau)} + \frac{g_1(\tau)}{h^*(\tau)}\frac{\varphi^*(g_1(\rho))}{g_1(\rho)} - \frac{\varphi^*(g_1(\tau))}{g_1(\tau)}\frac{g_1(\tau)}{h^*(\tau)} - \frac{g_1(\rho)}{h^*(\rho)}\frac{\varphi^*(g_1(\rho))}{g_1(\rho)} \geq 0. \tag{5.2.20}$$

Multiplying (5.2.20) by $h^*(\tau)h^*(\rho)$, we have

$$g_1(\rho)h^*(\tau)\frac{\varphi^*(g_1(\tau))}{g_1(\tau)}+g_1(\tau)h^*(\rho)\frac{\varphi^*(g_1(\rho))}{g_1(\rho)}$$

$$-g_1(\tau)h^*(\rho)\frac{\varphi^*(g_1(\tau))}{g_1(\tau)}-g_1(\rho)h^*(\tau)\frac{\varphi^*(g_1(\rho))}{g_1(\rho)}\geq 0. \tag{5.2.21}$$

Multiplying (5.2.21) on both sides by

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(x-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\tau-a)^{1-s}}$$

and integrate the resulting identity w.r.t τ from a to x, we get

$$g_1(\rho)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}h^*(x)\right)+\frac{\varphi^*(g_1(\rho))}{g_1(\rho)}h^*(\rho)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(g_1(x))$$

$$-h^*(\rho)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}g_1(x)\right)-\frac{\varphi^*(g_1(\rho))}{g_1(\rho)}g_1(\rho)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(h^*(x))\geq 0. \tag{5.2.22}$$

Again, multiplying (5.2.22) on both sides by

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(x-a)^s-(\rho-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\rho-a)^{1-s}}$$

and integrate the resultant w.r.t ρ from a to x, we get

$$\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(g_1(x))\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}h^*(x)\right)+\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}h^*(x)\right)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(g_1(x))$$

$$\geq\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(h^*(x))\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}g_1(x)\right)+\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}g_1(x)\right)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(h^*(x)). \tag{5.2.23}$$

which follows that

$$\begin{aligned}&\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(g_1(x))\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}h^*(x)\right)\\ &\geq\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(h^*(x))\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}g_1(x)\right).\end{aligned} \tag{5.2.24}$$

and

$$\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\right)(g_1(x))\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}g_1(x)\right)$$
$$\geq \frac{\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\right)(h^*(x))}{\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}h^*(x)\right)}. \tag{5.2.25}$$

since $g_1 \leq h^*$ on $[a,\infty)$ and function $\frac{\varphi^*(x)}{x}$ is increasing, so for $\tau, \rho \in [a,x)$, we have

$$\frac{\varphi^*\left(g_1(\tau)\right)}{g_1(\tau)} \leq \frac{\varphi^*\left(h^*(\tau)\right)}{h^*(\tau)}, \tag{5.2.26}$$

Multiplying (5.2.26) on both sides by

$$\frac{1}{k\Gamma_k\left(\alpha\right)}\left(\frac{(x-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{\left(\tau-a\right)^{1-s}}h^*(\tau)$$

and integrate the resulting identity w.r.t τ from a to x, we get

$$\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*\left(g_1(x)\right)}{g_1(x)}h^*(x)\right) \leq \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\varphi^*\left(h^*(x)\right)\right) \tag{5.2.27}$$

Hence, from (5.2.25) and (5.2.27), the required result (5.2.18) is obtained. ■

Remark 5.2.1 *Clearly, Theorem (5.1.1) would follow as a special case of Corollary (5.2.1) when $k = 1, \alpha = 1, s = 1$ and $x = b$.*

Theorem 5.2.8 *For three continuous and positive functions g_1, g_2 and h^* on $a \leq x < \infty$, where g_1 and g_2 are increasing on $[a,\infty)$ satisfying $g_1 \leq h^*$ on $[a,\infty)$ and $\frac{g_1}{h^*}$ is decreasing, for any $\alpha > 0, \beta > 0, x > a$, the relation*

$$\frac{\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\right)(f_1(x))\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right)(\varphi^*(h^*(x))g_2(x)) + \mathfrak{F}^{\beta,s}_{a^+,k}\left(f_1(x)\right)\mathfrak{F}^{\alpha,s}_{a^+,k}\left(\varphi(h^*(x))g_2(x)\right)}{\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\right)(h^*(x))\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right)(\varphi^*(g_1(x))g_2(x)) + \left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\right)(h^*(x))\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\right)(\varphi^*(g_1(x))g_2(x))} \geq 1, \tag{5.2.28}$$

is valid, where convex function φ^* satisfies $\varphi^*(0) = 0$.

Proof. According to the given conditions, the function $\frac{\varphi^*(x)}{x}$ is increasing. The function g_1 is increasing, then the function $\frac{\varphi^*(g_1(x))}{g_1(x)}$ is also increasing. Given that $\frac{g_1(x)}{h^*(x)}$ is decreasing, so for all $\tau, \rho \in [a, x), x > a$, we have

$$\left(\frac{\varphi^*(g_1(\tau))}{g_1(\tau)}g_2(\tau) - \frac{\varphi^*(g_1(\rho))}{g_1(\rho)}g_2(\rho)\right)(g_1(\rho)h^*(\tau) - g_1(\tau)h^*(\rho)) \geq 0, \tag{5.2.29}$$

implies that

$$\frac{\varphi^*(g_1(\tau))g_2(\tau)}{g_1(\tau)}g_1(\rho)h^*(\tau) + \frac{\varphi^*(g_1(\rho))g_2(\rho)}{g_1(\rho)}g_1(\tau)h^*(\rho)$$

$$-\frac{\varphi^*(g_1(\tau))g_2(\tau)}{g_1(\tau)}g_1(\tau)h^*(\rho) - \frac{\varphi^*(g_1(\rho))g_2(\rho)}{g_1(\rho)}g_1(\rho)h^*(\tau) \geq 0. \tag{5.2.30}$$

Multiplying (5.2.30) on both sides by

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(x-a)^s - (\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\tau-a)^{1-s}}$$

Now, integrate the resulting identity w.r.t τ from a to x, we get

$$g_1(\rho)\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))g_2(x)}{g_1(x)}h^*(x)\right) + \frac{\varphi^*(g_1(\rho))g_2(\rho)}{g_1(\rho)}h^*(\rho)\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)(g_1(x))$$

$$-h^*(\rho)\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))g_2(x)}{g_1(x)}g_1(x)\right) - \frac{\varphi^*(g_1(\rho))g_2(\rho)}{g_1(\rho)}g_1(\rho)\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)(h^*(x)) \geq 0. \tag{5.2.31}$$

Again, multiplying (5.2.31) on both sides by

$$\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^s - (\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{1}{(\rho-a)^{1-s}}$$

Now, integrate the resulting identity w.r.t ρ from a to x, we get

$$\begin{aligned}
&\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)(g_1(x))\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))g_2(x)}{g_1(x)}h^*(x)\right) \\
&+\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)\left(\frac{\varphi(g_1(x))g_2(x)}{g_1(x)}h^*(x)\right)\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)(g_1(x)) \\
&\geq \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)(h^*(x))\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)(\varphi^*(g_1(x))g_2(x)) \\
&+\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)(\varphi^*(g_1(x))g_2(x))\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)(h^*(x)).
\end{aligned} \tag{5.2.32}$$

since $g_1 \leq h^*$ on $[a, \infty)$ and $\frac{\varphi^*(x)g_2(x)}{x}$ is an increasing function, we obtain

$$\left({}_k^s\mathfrak{F}^{*p}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))g_2(x)}{g_1(x)}h^*(x)\right) \leq \left({}_k^s\mathfrak{F}^{*p}_{a^+}\right)\left(\varphi(h^*(x))g_2(x)\right), p = \alpha, \beta. \tag{5.2.33}$$

Hence from (5.2.32) and (5.2.33), we obtain (5.2.28). ■

Corollary 5.2.2 *For three continuous and positive functions g_1, g_2 and h^* on $a \leq x < \infty$ where g_1 and g_2 are increasing on $[a, \infty)$ satisfying $g_1 \leq h^*$ on $[a, \infty)$ and $\frac{g_1}{h^*}$ is decreasing, for any $\alpha > 0, x > a$, the relation*

$$\frac{\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(g_1(x))}{\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(h^*(x))} \geq \frac{\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(\varphi^*(g_1(x))g_2(x))}{\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(\varphi(h^*(x))g_2(x))}, \tag{5.2.34}$$

is valid, where convex function φ^* satisfies $\varphi^*(0) = 0$.

Proof. since $g_1 \leq h^*$ on $[a, \infty)$ and function $\frac{\varphi^*(x)}{x}$ is increasing, so for $\tau, \rho \in [a, x)$, we have

$$\frac{\varphi^*\left(g_1(\tau)\right)}{g_1(\tau)} \leq \frac{\varphi^*\left(h^*(\tau)\right)}{h^*(\tau)}, \tag{5.2.35}$$

Multiplying (5.2.35) on both sides by

$$\frac{1}{k\Gamma_k\left(\alpha\right)}\left(\frac{(x-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\tau-a)^{1-s}}h^*(\tau)g_2(\tau)$$

then integrating the resulting identity w.r.t τ from a to x, we get

$$\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*\left(g_1(x)\right)}{g_1(x)}h^*(x)g_2(x)\right) \leq \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\varphi\left(h^*(x)g_2(x)\right)\right) \tag{5.2.36}$$

Also, under the given conditions, $\frac{\varphi^*(x)}{x}$ is increasing. The function g_1 is increasing, then the function $\frac{\varphi^*(g_1(x))}{g_1(x)}$ is also increasing. Given that $\frac{g_1(x)}{h^*(x)}$ is decreasing, so for all $\tau, \rho \in [a, x), x > a$, we have

$$\left(\frac{\varphi^*(g_1(\tau))}{g_1(\tau)}g_2(\tau) - \frac{\varphi^*(g_1(\rho))}{g_1(\rho)}g_2(\rho)\right)(g_1(\rho)h^*(\tau) - g_1(\tau)h^*(\rho)) \geq 0, \tag{5.2.37}$$

implies that

$$\frac{\varphi^*(g_1(\tau))g_2(\tau)}{g_1(\tau)}h^*(\tau)g_1(\rho)+\frac{\varphi^*(g_1(\rho))g_2(\rho)}{g_1(\rho)}h^*(\rho)g_1(\tau)$$

$$-\frac{\varphi^*(g_1(\tau))g_2(\tau)}{g_1(\tau)}g_1(\tau)h^*(\rho)-\frac{\varphi^*(g_1(\rho))g_2(\rho)}{g_1(\rho)}g_1(\rho)h^*(\tau)\geq 0. \tag{5.2.38}$$

Multiplying (5.2.38) on both sides by

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(x-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\tau-a)^{1-s}}$$

Next, integrate the resulting identity w.r.t τ from a to x, we get

$$g_1(\rho)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))g_2(x)}{g_1(x)}h^*(x)\right)+\frac{\varphi^*(g_1(\rho))g_2(\rho)}{g_1(\rho)}h^*(\rho)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(g_1(x))$$

$$-h^*(\rho)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))g_2(x)}{g_1(x)}g_1(x)\right)-\frac{\varphi^*(g_1(\rho))g_2(\rho)}{g_1(\rho)}g_1(\rho)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(h^*(x))\geq 0. \tag{5.2.39}$$

Again, multiplying (5.2.39) on both sides by

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(x-a)^s-(\rho-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\rho-a)^{1-s}}$$

and integrate the resulting identity w.r.t ρ from a to x, we get

$$\frac{\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(g_1(x))}{\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(h^*(x))}\geq\frac{\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(\varphi^*(g_1(x))g_2(x))}{\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\frac{\varphi^*(g_1(x))}{g_1(x)}h^*(x)g_2(x)\right)}, \tag{5.2.40}$$

Hence, from (5.2.36) and (5.2.40), we obtain (5.2.34). ■

Remark 5.2.2 *Clearly, Theorem (5.1.2) would consider as a special case of Corollary (5.2.2) when* $k=1,\alpha=1,s=1$ *and* $x=b$.

Chapter 6

GENERALIZED INEQUALITIES ASSOCIATED WITH CONFORMABLE K-FRACTIONAL INTEGRALS

In this chapter, we generalize some inequalities reported in [162, 173, 175] by invoking generalized conformable k-fractional integrals (2.3.35) and (2.3.36), for a finite sequence of n positive decreasing functions such that $n \in N$.

6.1 Inequalities for Generalized Conformable k-Fractional Integrals

This section presents k-version of inequalities for positive decreasing functions related to generalized conformable k-fractional integrals.

Theorem 6.1.1 *For a finite sequence of continuous positive decreasing functions* $\{g_i, 1 \leq i \leq n\}$ *on the interval* $[a,b]$*, a continuous increasing function* $h(x)$ *and* $x \in (a,b], \eta > 0, \xi \geq \gamma_p > 0$*;* $1 \leq p \leq n$*, the left conformable* k*-fractional integral* ${}_k^s\mathfrak{F}^{*\beta}_{a^+}$ *satisfies the inequality*

$$\frac{{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right]}{{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left[\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]} \geq \frac{{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left[h^{\eta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right]}{{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left[h^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]}. \tag{6.1.1}$$

Proof. Under the assumptions, we have

$$[h^{\eta}(\rho) - h^{\eta}(\tau)]\left[g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)\right] \geq 0.$$

Define a function

$$\begin{aligned} {}_k^{\beta}\mathfrak{J}^s_{a^+}(x,\rho,\tau) &= \frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^s-(\tau-a)^s}{s}\right)^{\frac{\beta}{k}-1} \\ &\times \frac{\prod_{i=1}^{n} g_i^{\gamma_i}(\tau)}{(\tau-a)^{1-s}}\left(h^{\eta}(\rho)-h^{\eta}(\tau)\right)\left(g_p^{\xi-\gamma_p}(\tau)-g_p^{\xi-\gamma_p}(\rho)\right). \end{aligned} \tag{6.1.2}$$

Accordingly, the function ${}_{k}^{\beta}\mathfrak{J}_{a^+}^{s}(x,\rho,\tau)$ is non-negative for all $\tau \in (a,b]$. Integrate the above equation (6.1.2) w.r.t. τ over (a,x) to show

$$\begin{aligned}
0 \leq \int_a^x {}_{k}^{\beta}\mathfrak{J}_{a^+}^{s}(x,\rho,\tau)d\tau &= \frac{1}{k\Gamma_k(\beta)} \int_a^x \left[\frac{(x-a)^s-(\tau-a)^s}{s}\right]^{\beta/k-1} \\
&\times \prod_{i=1}^{n} g_i^{\gamma_i}(\tau)[h^\eta(\rho)-h^\eta(\tau)]\left[g_p^{\xi-\gamma_p}(\tau)-g_p^{\xi-\gamma_p}(\rho)\right]\frac{d\tau}{(\tau-a)^{1-s}} \\
&= h^\eta(\rho)\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i}g_p^{\xi}(x)\right)\right] + g_p^{\xi-\gamma_p}(\rho)\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(h^\eta(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] \\
&-h^\eta(\rho)g_p^{\xi-\gamma_p}(\rho)\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] - \left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(h^\eta(x)\prod_{i\neq p}^{n} g_i^{\gamma_i}(x)\right)\right].
\end{aligned} \tag{6.1.3}$$

Multiplying the relation (6.1.3) on both sides with

$$\frac{1}{k\Gamma_k(\beta)}\left[\frac{(x-a)^s-(\rho-a)^s}{s}\right]^{\beta/k-1}\frac{\prod_{i=1}^{n} g_i^{\gamma_i}(\rho)}{(\rho-a)^{1-s}}. \tag{6.1.4}$$

and integrate w.r.t. ρ from a to x on both sides give

$$\begin{aligned}
0 \leq &\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(h^\eta(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] \\
-&\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(h^\eta(x)\prod_{i\neq p}^{n} g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right].
\end{aligned} \tag{6.1.5}$$

Divide (6.1.5) on both sides by

$$\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(h^\eta(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]$$

leads to (6.1.1). The proof of Theorem (6.1.1) is complete. ■

Corollary 6.1.1 *For a finite sequence of continuous positive decreasing functions* $\{g_i, 1 \leq i \leq n\}$ *on* $[a,b]$ *and* $x \in (a,b]$, $\eta > 0$, $\xi \geq \gamma_p > 0$; $1 \leq p \leq n$, *the left conformable* k*-fractional integral* ${}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}$ *satisfies the inequality*

$$\frac{{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i}g_p^{\xi}(x)\right)}{{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)} \geq \frac{{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left((x-a)^\eta\prod_{i\neq p}^{n} g_i^{\gamma_i}g_p^{\xi}(x)\right)}{{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left((x-a)^\eta\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)}. \tag{6.1.6}$$

First proof. Under given conditions, we have

$$[(\rho-a)^{\eta}-(\tau-a)^{\eta}]\big[g_p^{\xi-\gamma_p}(\tau)-g_p^{\xi-\gamma_p}(\rho)\big]\geq 0.$$

Let us define a function

$$\begin{aligned}
&{}_k^{\beta}\mathfrak{J}_{a^+}^{s}(x,\rho,\tau)=\frac{1}{k\Gamma_k(\beta)}\left[\frac{(x-a)^s-(\tau-a)^s}{s}\right]^{\beta/k-1}\\
&\times\frac{\prod_{i=1}^{n}g_i^{\gamma_i}(\tau)}{(\tau-a)^{1-s}}[(\rho-a)^{\eta}-(\tau-a)^{\eta}]\left[g_p^{\xi-\gamma_p}(\tau)-g_p^{\xi-\gamma_p}(\rho)\right].
\end{aligned}\tag{6.1.7}$$

Under given assumptions, the function ${}_k^{\beta}\mathfrak{J}_{a^+}^{s}(x,\rho,\tau)$ is non-negative $\forall\tau\in(a,b]$. Integrate the above equation (6.1.7) on both sides w.r.t. τ over (a,x) to yield

$$\begin{aligned}
&0\leq\int_a^x {}_k^{\beta}\mathfrak{J}_{a^+}^{s}(x,\rho,\tau)d\tau=\frac{1}{k\Gamma_k(\beta)}\int_a^x\left[\frac{(x-a)^s-(\tau-a)^s}{s}\right]^{\beta/k-1}\\
&\times\prod_{i=1}^{n}g_i^{\gamma_i}(\tau)[(\rho-a)^{\eta}-(\tau-a)^{\eta}]\left[g_p^{\xi-\gamma_p}(\tau)-g_p^{\xi-\gamma_p}(\rho)\right]\frac{d\tau}{(\tau-a)^{1-s}}\\
&=(\rho-a)^{\eta}\left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]+g_p^{\xi-\gamma_p}(\rho)\left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]\\
&-(\rho-a)^{\eta}g_p^{\xi-\gamma_p}(\rho)\left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]-\left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left((x-a)^{\eta}\prod_{i\neq p}^{n}g_i^{\gamma_i}(x)\right)\right].
\end{aligned}\tag{6.1.8}$$

Multiply the relation (6.1.8) by (6.1.4) and integrate w.r.t. ρ over (a,x) yield

$$\begin{aligned}
&0\leq\left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]\\
&-\left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left((x-a)^{\eta}\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right].
\end{aligned}\tag{6.1.9}$$

Divide the relation (6.1.9) on both sides by

$$\left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]$$

yields (6.1.6). The first proof of Corollary (6.1.1) is complete. ■

Second proof. This can be derived from taking $h(x) = x - a$ in Theorem (6.1.1). The second proof of Corollary (6.1.1) is complete. ■

Corollary 6.1.2 *For a finite sequence of continuous positive decreasing functions* $\{g_i, 1 \leq i \leq n\}$ *on the interval* $[a, b]$ *and* $x \in (a, b]$, $\eta > 0$, $\xi \geq \gamma_p > 0$; $p \in [1, n]$, *the left conformable* k*-fractional integral* ${}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}$ *satisfies the inequality*

$$\begin{aligned}
&\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i\neq p} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] \\
&+\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{a^+}\left(\prod_{i\neq p} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] \\
&\geq \left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{a^+}\left((x-a)^{\eta}\prod_{i\neq p} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i\neq p} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] \\
&+\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{a^+}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right].
\end{aligned} \tag{6.1.10}$$

Proof. Multiply the relation (6.1.8) on both sides with

$$\frac{1}{k\Gamma_k(\theta)}\left[\frac{(x-a)^s-(\rho-a)^s}{s}\right]^{\theta/k-1}\frac{\prod_{i=1}^{n} g_i^{\gamma_i}(\rho)}{(\rho-a)^{1-s}} \tag{6.1.11}$$

and integrate w.r.t. ρ from a to x, arrives at

$$\begin{aligned}
0 \leq &\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i\neq p} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] \\
&+\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{a^+}\left(\prod_{i\neq p} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] \\
&-\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{a^+}\left((x-a)^{\eta}\prod_{i\neq p} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i\neq p} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] \\
&-\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{a^+}\left(\prod_{i\neq p} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right].
\end{aligned} \tag{6.1.12}$$

Divide (6.1.12) on both sides by

$$\left[{}_{k}^{s}\mathfrak{F}^{*\theta}_{a^+}\left((x-a)^{\eta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] + \left[{}_{k}^{s}\mathfrak{F}^{*\theta}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left((x-a)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]$$

leads to (6.1.10). The proof of Corollary (6.1.2) is complete. ■

Corollary 6.1.3 *For a finite sequence of continuous positive decreasing functions* $\{g_i, 1 \leq i \leq n\}$ *on the interval* $[a,b]$ *and* $h(x)$ *is a continuous and increasing function and* $x \in (a,b]$, $\eta > 0$, $\xi \geq \gamma_p > 0$; $1 \leq p \leq n$. *Then the left conformable* k*-fractional integral* ${}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}$ *satisfies the inequality*

$$\begin{aligned}&\left[{}_{k}^{s}\mathfrak{F}^{*\theta}_{a^+}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\\ &+\left[{}_{k}^{s}\mathfrak{F}^{*\theta}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\\ \geq&\left[{}_{k}^{s}\mathfrak{F}^{*\theta}_{a^+}\left(h^{\eta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\\ &+\left[{}_{k}^{s}\mathfrak{F}^{*\theta}_{a^+}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right].\end{aligned} \tag{6.1.13}$$

Proof. Multiply the relation (6.1.3) on both sides by (6.1.11) and integrate w.r.t. ρ from a to x, we derive

$$\begin{aligned}0 \leq&\left[{}_{k}^{s}\mathfrak{F}^{*\theta}_{a^+}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\\ &+\left[{}_{k}^{s}\mathfrak{F}^{*\theta}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\\ &-\left[{}_{k}^{s}\mathfrak{F}^{*\theta}_{a^+}\left(h^{\eta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\\ &-\left[{}_{k}^{s}\mathfrak{F}^{*\theta}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right].\end{aligned} \tag{6.1.14}$$

Divide (6.1.14) on both sides by

$$\left[{}_k^s\mathfrak{F}^{*\theta}_{a^+}\left(h^\eta(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] + \left[{}_k^s\mathfrak{F}^{*\theta}_{a^+}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}\left(h^\eta(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]$$

reveals (6.1.13). The proof of Corollary (6.1.3) is complete. ■

Theorem 6.1.2 *For a finite sequence of continuous positive decreasing functions* $\{g_i, 1 \leq i \leq n\}$ *on* $[a,b]$ *and* $x \in (a,b]$, $\eta > 0$, $\xi \geq \gamma_p > 0$; $1 \leq p \leq n$, *the right conformable* k*-fractional integral* ${}_k^s\mathfrak{F}^{*\beta}_{b^-}$ *satisfies the inequality*

$$\frac{{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)}{{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)} \geq \frac{{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left((b-x)^\eta\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)}{{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left((b-x)^\eta\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)}. \tag{6.1.15}$$

Proof. Under given conditions, we have

$$\left[g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)\right]\left[(b-\rho)^\eta - (b-\tau)^\eta\right] \geq 0.$$

Define a function

$$\begin{aligned}{}_k^\beta\mathfrak{J}^s_{b^-}(x,\rho,\tau) &= \frac{1}{k\Gamma_k(\beta)}\left[\frac{(b-x)^s-(b-\tau)^s}{s}\right]^{\beta/k-1}\\ &\times\frac{\prod_{i=1}^{n} g_i^{\gamma_i}(\tau)}{(b-\tau)^{1-s}}\left[g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)\right]\left[(b-\rho)^\eta - (b-\tau)^\eta\right].\end{aligned} \tag{6.1.16}$$

Clearly, the function ${}_k^\beta\mathfrak{J}^s_{b^-}(x,\rho,\tau)$ is non-negative $\forall \tau \in (a,b]$. Integrate the above equation (6.1.16) on both sides w.r.t. τ over (x,b) gives

$$\begin{aligned}0 \leq \int_x^b {}_k^\beta\mathfrak{J}^s_{b^-}(x,\rho,\tau)d\tau &= \frac{1}{k\Gamma_k(\beta)}\int_x^b\left[\frac{(b-x)^s-(b-\tau)^s}{s}\right]^{\beta/k-1}\\ &\times\prod_{i=1}^{n} g_i^{\gamma_i}(\tau)\left[g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)\right]\left[(b-\rho)^\eta - (b-\tau)^\eta\right]\frac{d\tau}{(b-\tau)^{1-s}}\\ &= (b-\rho)^\eta\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right] + g_p^{\xi-\gamma_p}(\rho)\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left((b-x)^\eta\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\end{aligned}$$

$$-(b-\rho)^{\eta} g_p^{\xi-\gamma_p}(\rho)\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]-\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left((b-x)^{\eta}\prod_{i\neq p}^{n} g_i^{\gamma_i}(x)\right)\right]. \tag{6.1.17}$$

Multiplying the relation (6.1.17) with

$$\frac{1}{k\Gamma_k(\beta)}\left[\frac{(b-x)^s-(b-\rho)^s}{s}\right]^{\beta/k-1}\frac{\prod_{i=1}^{n} g_i^{\gamma_i}(\rho)}{(b-\rho)^{1-s}} \tag{6.1.18}$$

and integrate w.r.t. ρ over (x,b) produce

$$\begin{aligned}0 \leq &\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left((b-x)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\\ &-\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left((b-x)^{\eta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right].\end{aligned} \tag{6.1.19}$$

Dividing (6.1.19) on both sides by

$${}_k^s\mathfrak{F}^{*\beta}_{b^-}\left((b-x)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]$$

yields (6.1.15). The proof of Theorem (6.1.2) is complete. ■

Corollary 6.1.4 *For a finite sequence of continuous positive decreasing functions* $\{g_i, 1 \leq i \leq n\}$ *on* $[a,b]$ *and for* $x \in (a,b]$, $\eta > 0$, $\xi \geq \gamma_p > 0$; $1 \leq p \leq n$, *the right conformable* k*-fractional integral* ${}_k^s\mathfrak{F}^{*\beta}_{b^-}$ *satisfies the inequality*

$$\begin{aligned}&\left[{}_k^s\mathfrak{F}^{*\theta}_{b^-}\left((b-x)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\\ &+\left[{}_k^s\mathfrak{F}^{*\theta}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left((b-x)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\\ \geq &\left[{}_k^s\mathfrak{F}^{*\theta}_{b^-}\left((b-x)^{\eta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\xi}(x)\right)\right]\\ &+\left[{}_k^s\mathfrak{F}^{*\theta}_{b^-}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left((b-x)^{\eta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right].\end{aligned} \tag{6.1.20}$$

Proof. Multiply the relation (6.1.17) with

$$\frac{1}{k\Gamma_k(\theta)}\left[\frac{(b-x)^s-(b-\rho)^s}{s}\right]^{\theta/k-1}\frac{\prod_{i=1}^n g_i^{\gamma_i}(\rho)}{(b-\rho)^{1-s}} \tag{6.1.21}$$

and integrate w.r.t. ρ from x to b on both sides, we procure

$$\begin{aligned}0\leq&\left[{}_k^s\mathfrak{F}^{*\theta}_{b^-}\left((b-x)^\eta\prod_{i=1}^n g_i^{\gamma_i}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^n g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\\&+\left[{}_k^s\mathfrak{F}^{*\theta}_{b^-}\left(\prod_{i\neq p}^n g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left((b-x)^\eta\prod_{i=1}^n g_i^{\gamma_i}(x)\right)\right]\\&-\left[{}_k^s\mathfrak{F}^{*\theta}_{b^-}\left((x-a)^\eta\prod_{i\neq p}^n g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^n g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\\&-\left[{}_k^s\mathfrak{F}^{*\theta}_{b^-}\left(\prod_{i\neq p}^n g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left((b-x)^\eta\prod_{i=1}^n g_i^{\gamma_i}(x)\right)\right].\end{aligned} \tag{6.1.22}$$

Divide (6.1.22) on both sides by

$$\begin{aligned}&\left[{}_k^s\mathfrak{F}^{*\theta}_{b^-}\left((b-x)^\eta\prod_{i\neq p}^n g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^n g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\\&+\left[{}_k^s\mathfrak{F}^{*\theta}_{b^-}\left(\prod_{i\neq p}^n g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left((b-x)^\eta\prod_{i=1}^n g_i^{\gamma_i}(x)\right)\right]\end{aligned}$$

demonstrates (6.1.20). The proof of Corollary (6.1.4) is complete. ■

Theorem 6.1.3 *For a finite sequence of continuous positive decreasing functions $\{g_i, 1\leq i\leq n\}$ be on the interval $[a,b]$ and $h(x)$ is a continuous increasing function, $x\in(a,b]$, $\eta>0$, $\xi\geq\gamma_p>0$; $1\leq p\leq n$. Then the right conformable k-fractional integral ${}_k^s\mathfrak{F}^{*\beta}_{b^-}$ satisfies the inequality*

$$\frac{{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^n g_i^{\gamma_i}g_p^{\xi}(x)\right)}{{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i=1}^n g_i^{\gamma_i}(x)\right)}\geq\frac{{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(h^\eta(x)\prod_{i\neq p}^n g_i^{\gamma_i}g_p^{\xi}(x)\right)}{{}_k^s\mathfrak{F}^{*\beta}_{b^-}\left(h^\eta(x)\prod_{i=1}^n g_i^{\gamma_i}(x)\right)}. \tag{6.1.23}$$

Proof. Under given conditions, we have

$$\left[g_p^{\xi-\gamma_p}(\tau)-g_p^{\xi-\gamma_p}(\rho)\right]\left[h^\eta(\rho)-h^\eta(\tau)\right]\geq 0.$$

Define a function

$$
\begin{aligned}
&{}_{k}^{\beta}\mathcal{J}_{b^-}^{s}(x,\rho,\tau) = \\
&\qquad\qquad \frac{1}{k\Gamma_k(\beta)}\left[\frac{(b-x)^s-(b-\tau)^s}{s}\right]^{\beta/k-1} \\
&\qquad\qquad \times\frac{\prod_{i=1}^{n} g_i^{\gamma_i}(\tau)}{(b-\tau)^{1-s}}\big[g_p^{\xi-\gamma_p}(\tau)-g_p^{\xi-\gamma_p}(\rho)\big][h^{\eta}(\rho)-h^{\eta}(\tau)].
\end{aligned} \tag{6.1.24}
$$

Thus, the function ${}_{k}^{\beta}\mathcal{J}_{b^-}^{s}(x,\rho,\tau)$ is non-negative for all $\tau \in (a,b]$. Integrate the above equation (6.1.24) on both sides w.r.t. τ over (x,b), results in

$$
\begin{aligned}
0 &\le \int_x^b {}_{k}^{\beta}\mathfrak{J}_{b^-}^{s}(x,\rho,\tau)d\tau = \frac{1}{k\Gamma_k(\beta)}\int_x^b\left[\frac{(b-x)^s-(b-\tau)^s}{s}\right]^{\beta/k-1} \\
&\quad\times\prod_{i=1}^{n} g_i^{\gamma_i}(\tau)[h^{\eta}(\rho)-h^{\eta}(\tau)]\big[g_p^{\xi-\gamma_p}(\tau)-g_p^{\xi-\gamma_p}(\rho)\big]\frac{d\tau}{(b-\tau)^{1-s}} \\
&= h^{\eta}(\rho)\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]+g_p^{\xi-\gamma_p}(\rho)\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] \\
&\quad -h^{\eta}(\rho)g_p^{\xi-\gamma_p}(\rho)\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]-\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{b^-}\left(h^{\eta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i}(x)\right)\right].
\end{aligned} \tag{6.1.25}
$$

Multiply the relation (6.1.25) by (6.1.18) and integrate w.r.t. ρ over (x,b) on both sides, yield

$$
\begin{aligned}
0 &\le \left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^{n} g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right] \\
&\quad -\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{b^-}\left(h^{\eta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right].
\end{aligned} \tag{6.1.26}
$$

Divide (6.1.26) on both sides by

$$
{}_{k}^{s}\mathfrak{F}^{*\beta}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\left[{}_{k}^{s}\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right)\right]
$$

leads to (6.1.23). The proof of Theorem (6.1.3) is complete. ■

Corollary 6.1.5 *For a finite sequence of continuous positive decreasing functions $\{g_i, 1 \le i \le n\}$ be on the interval $[a,b]$ and $h(x)$ is a continuous increasing function, $a < x \le b$, $\eta > 0$, $\xi \ge \gamma_p > 0$; $p \in [1,n]$. Then the right conformable k-fractional integral ${}_{k}^{s}\mathfrak{F}^{*\beta}_{b^-}$*

satisfies the inequality

$$\begin{aligned}
&\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\\
&+\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{b^-}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]\\
&\geq\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{b^-}\left(h^{\eta}(x)\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\\
&+\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{b^-}\left(\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right].
\end{aligned}\tag{6.1.27}$$

Proof. Multiplying the relation (6.1.25) by (6.1.21) and integrating w.r.t. ρ from x to b on both sides, give

$$\begin{aligned}
0\leq&\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\\
&+\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{b^-}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]\\
&-\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{b^-}\left(h^{\eta}(x)\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\\
&-\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{b^-}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right].
\end{aligned}\tag{6.1.28}$$

Divide (6.1.28) on both sides by

$$\begin{aligned}
&\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{b^-}\left(h^{\eta}(x)\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{b^-}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\\
&+\left[{}^{s}_{k}\mathfrak{F}^{*\theta}_{b^-}\left(\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\xi}(x)\right)\right]\left[{}^{s}_{k}\mathfrak{F}^{*\beta}_{b^-}\left(h^{\eta}(x)\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right)\right]
\end{aligned}$$

concludes (6.1.27). The proof of Corollary (6.1.5) is complete. ■

Chapter 7

GRÜSS INEQUALITIES FOR GENERALIZED CONFORMABLE K-FRACTIONAL INTEGRALS

In this chapter, an improved version of generalized Grüss type integral inequality associated with k-analogue of fractional conformable integrals, is established. This generalization contributes our work by providing some new mathematical results along with their verifications.

7.1 Grüss Inequality

Grüss type inequality can be characterized as follows.

Definition 7.1.1 *For integrable functions $f_1, f_2 : [x_1, x_2] \to \mathbb{R}$, satisfying*

$$\varphi < f_1(x) < \Phi \textit{ and } \psi < f_2(x) < \Psi \ \forall x \in [x_1, x_2]$$

, the inequality

$$\left|\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f_1(x)f_2(x)dx - \frac{1}{(x_2 - x_1)^2}\int_{x_1}^{x_2} f_1(x)dx \int_{x_1}^{x_2} f_2(x)dx\right| \leq \frac{1}{4}(\Phi - \varphi)(\Psi - \psi) \tag{7.1.1}$$

holds, where $\varphi, \Phi, \psi, \Psi \in R$, the factor $\frac{1}{4}$ is sharp.
For example, the equality in (7.1.1) holds for the functions

$$f_1(x) = f_2(x) = sgn(x) - \frac{x_1 + x_2}{2} \quad (x \in [x_1, x_2])$$

The above integral inequality (7.1.1) (Grüss inequality) in fact, joins the product of the integrals of two functions with the integral of the product of both.

7.2 Application to Generalized Conformable k-Fractional Integrals via Gr$\ddot{u}$ss Inequality

This section depicts Gr$\ddot{u}$ss type inequalities involving ${}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}$ defined in (2.3.35).

Theorem 7.2.1 *For two integrable functions ζ_1, ζ_2, satisfying*

$$\zeta_1(t) \leq f(t) \leq \zeta_2(t), \quad \forall \quad t \in [a,b]. \tag{7.2.2}$$

where $f \in L_1^r[a,b]$, $\alpha, \beta > 0$ and $k, s > 0$, Then the inequality

$$\begin{aligned}&\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right)\zeta_1(t)\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t) + \left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_2(t)\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right)f(t)\\ &\geq \left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_2(t)\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right)\zeta_1(t) + \left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right)f(t).\end{aligned} \tag{7.2.3}$$

holds true.

Proof. Under the hypothesis, for $a \leq x, y \leq b$, we write

$$(f(y) - \zeta_1(y))(\zeta_2(x) - f(x)) \geq 0,$$

which gives

$$\zeta_2(x)f(y) + \zeta_1(y)f(x) \geq \zeta_1(y)\zeta_2(x) + f(x)f(y),$$

Multiplying above inequality on both sides with

$$\frac{(s)^{2-\frac{\alpha}{k}-\frac{\beta}{k}}\left((t-a)^s-(x-a)^s\right)^{\frac{\alpha}{k}-1}\left((t-a)^s-(y-a)^s\right)^{\frac{\beta}{k}-1}(x-a)^{s-1}(y-a)^{s-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}$$

and then integrate w.r.t. x, y from a to t, we procure the required result. ■

Lemma 7.2.1 *For $s = 1$ in theorem (7.2.1), the above integral result swaps for k-Riemann*

Liouville fraction integral $\left(\mathcal{R}^{\beta}_{a^+,k}\right)$ *as,*

$$\left(\mathcal{R}^{\beta}_{a^+,k}\right)\zeta_1(t)\left(\mathcal{R}^{\alpha}_{a^+,k}\right)f(t)+\left(\mathcal{R}^{\alpha}_{a^+,k}\right)\zeta_2(t)\left(\mathcal{R}^{\beta}_{a^+,k}\right)f(t)$$
$$\geq\left(\mathcal{R}^{\alpha}_{a^+,k}\right)\zeta_2(t)\left(\mathcal{R}^{\beta}_{a^+,k}\right)\zeta_1(t)+\left(\mathcal{R}^{\alpha}_{a^+,k}\right)f(t)\left(\mathcal{R}^{\beta}_{a^+,k}\right)f(t).$$

Theorem 7.2.2 *Let* $f\in L_1^r[a,b]$*, such that* $m_1\leq f(t)\leq m_2$*,* $\forall a\leq t\leq b$ *where* $m_1,m_2\in\mathbb{R}$*. Then for* $k,s,\alpha,\beta>0$*, we have*

$$m_2\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)f(t)+m_1\frac{(s)^{\frac{-\beta}{k}}(t-a)^{\frac{s\beta}{k}}}{\Gamma_k(\beta+k)}\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)$$
$$\geq m_1m_2\frac{(s)^{-\frac{(\alpha+\beta)}{k}}(t-a)^{(s)\left(\frac{\alpha+\beta}{k}\right)}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}+\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)f(t).$$

Proof. Under the condition, $m_1\leq f(t)\leq m_2\ \forall t\in[a,b]$, one may write

$$(m_2-f(x))(f(y)-m_1)\geq 0,\quad \forall\ x,y\in[a,b]. \tag{7.2.4}$$

The proof of this theorem can be obtained by considering the inequality (8.3.59) and following the procedure of previous Theorem (7.2.1). ■

Lemma 7.2.2 *For* $s=1$ *in theorem (7.2.2), we get the above integral result for* k*-analogue of Riemann–Liouville fractional integral* $\left(\mathcal{R}^{\beta}_{a^+,k}\right)$ *as,*

$$m_2\frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{R}^{\beta}_{a^+,k}f(t)+m_1\frac{(t-a)^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\mathcal{R}^{\alpha}_{a^+,k}f(t)$$
$$\geq m_1m_2\frac{(t-a)^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}+\mathcal{R}^{\alpha}_{a^+,k}f(t)\mathcal{R}^{\beta}_{a^+,k}f(t).$$

Theorem 7.2.3 *Two integrable functions* f *and* g *assume the condition (7.2.2) on* $[a,b]$ *and* $k,s>0$*,* $\alpha,\beta>0$*. Also there exist integrable functions* ξ_1 *and* ξ_2*, satisfying*

$$\xi_1(t)\leq g(t)\leq\xi_2(t),\quad\forall t\in[a,b]. \tag{7.2.5}$$

The following inequalities hold:

(i) $\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)\xi_1(t)\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)+\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_2(t)\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)g(t)\geq\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right)\xi_1(t)\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_2(t)+$

$\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right) f(t) \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) g(t)$,

(ii) $\left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) \zeta_1(t) \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right) g(t)+\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right) \xi_2(t) \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f(t) \geq \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) \zeta_1(t) \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right) \xi_2(t)+ \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right) g(t) \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) f(t)$,

(iii) $\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right) \zeta_2(t) \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) \xi_2(t)+\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right) f(t) \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) g(t) \geq \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right) \zeta_2(t) \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) g(t)+ \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) \xi_2(t) \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right) f(t)$,

(iv) $\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right) \zeta_1(t) \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) \xi_1(t)+\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right) f(t) \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) g(t) \geq \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right) \zeta_1(t) \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) g(t)+ \left({}^s_k\mathfrak{F}^{*\beta}_{a^+}\right) \xi_1(t) \left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\right) f(t)$.

Proof. (i): From (7.2.2) and (7.2.5) for all $a \leq t \leq b$, we have

$$(g(y) - \xi_1(y))(\zeta_2(x) - f(x)) \geq 0,$$

then

$$\zeta_2(x)g(y) + \xi_1(y)f(x) \geq \xi_1(y)\zeta_2(x) + f(x)g(y),$$

Multiply above inequality on both sides with

$$\frac{(s)^{2-\frac{\alpha}{k}-\frac{\beta}{k}} \left((t-a)^s - (x-a)^s\right)^{\frac{\alpha}{k}-1} \left((t-a)^s - (y-a)^s\right)^{\frac{\beta}{k}-1} (x-a)^s(y-a)^s}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}$$

and then integrate w.r.t. x and y from a to t, one obtains the first result.

To prove (ii)-(iv), we use the following inequalities

(ii) $(f(y) - \zeta_1(y))(\xi_(x) - g(x)) \geq 0$,

(iii) $(g(y) - \xi_2(y))(\zeta_2(x) - f(x)) \leq 0$,

(iv) $(g(y) - \xi_1(y))(\zeta_1(x) - f(x)) \leq 0$.

By following the steps of previous part, the above parts can be proved. ■

Lemma 7.2.3 *For two integrable functions f and g on $[a, b]$, $k, s > 0$, $\alpha, \beta > 0$, there exist $m_1, m_2, n_1, n_2 \in \mathbb{R}$ such that*

$$m_1 \leq f(t) \leq m_2, \quad n_1 \leq g(t) \leq n_2, \quad \forall a \leq t \leq b,$$

Then we have

$$(i^*)\quad n_1\frac{(s)^{\frac{-\beta}{k}}(t-a)^{\frac{s\beta}{k}}}{\Gamma_k(\beta+k)}\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)+m_2\frac{s^{\frac{-\alpha}{k}}(t-a)^{(\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right)g(t)$$
$$\geq n_1m_2\frac{s^{-\frac{\alpha+\beta}{k}}(t-a)^{\frac{s(\alpha+\beta)}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}+\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right)g(t),$$

$$(ii^*)\quad m_1\frac{(s)^{\frac{-\beta}{k}}(t-a)^{\frac{s\beta}{k}}}{\Gamma_k(\beta+k)}\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)g(t)+n_2\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right)f(t)$$
$$\geq m_1n_2\frac{s^{-\frac{\alpha+\beta}{k}}(t-a)^{\frac{s(\alpha+\beta)}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}+\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right)f(t)\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)g(t),$$

$$(iii^*)\quad m_2\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right)g(t)+n_2\frac{s^{\frac{-\beta}{k}}(t-a)^{\frac{s\beta}{k}}}{\Gamma_k(\beta+k)}\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)$$
$$\leq m_2n_2\frac{s^{-\frac{\alpha+\beta}{k}}(t-a)^{\frac{s(\alpha+\beta)}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}+\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right)g(t),$$

$$(iv^*)\quad m_1\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right)g(t)+n_1\frac{s^{\frac{-\beta}{k}}(t-a)^{\frac{s\beta}{k}}}{\Gamma_k(\beta+k)}\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)$$
$$\leq m_1n_1\frac{(s)^{-\frac{\alpha+\beta}{k}}(t-a)^{\frac{s(\alpha+\beta)}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}+\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\right)g(t).$$

Theorem 7.2.4 *For integrable functions ζ_1, ζ_2 on $[a,b]$ and $\alpha,k,s>0$. Suppose that $f\in L_1^r[a,b]$ preserves the condition (7.2.2), then*

$$\begin{aligned}
&\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f^2(t)-\left[\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\right]^2\\
&=\left(\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_2(t)-\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\right)\left(\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)-\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_1(t)\right)\\
&-\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left(\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_2(t)-\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\right)\left(\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)-\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_1(t)\right)\\
&+\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\zeta_1(t)f(t)\right)-\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_1(t)\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\\
&+\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\zeta_2(t)f(t)\right)-\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_2(t)\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\\
&-\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)\left(\zeta_1(t)\zeta_2(t)\right)+\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_1(t)\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_2(t).
\end{aligned}\tag{7.2.6}$$

Proof. Under given conditions, $\forall a \leq x, y \leq b$, we have

$$\begin{aligned}
&(\zeta_2(y)-f(y))(f(x)-\zeta_1(x))+(f(y)-\zeta_1(y))(\zeta_2(x)-f(x))\\
&-(\zeta_2(x)-f(x))(f(x)-\zeta_1(x))-(f(y)-\zeta_1(y))(\zeta_2(y)-f(y))\\
&=f^2(y)+f^2(x)-2f(x)f(y)+f(x)\zeta_2(y)+f(y)\zeta_1(x)\\
&-\zeta_2(y)\zeta_1(x)+f(y)\zeta_2(x)+f(x)\zeta_1(y)-\zeta_2(y)\zeta_1(y)\\
&-\zeta_2(x)f(x)+\zeta_1(x)\zeta_2(x)-\zeta_1(x)f(x)-f(y)\zeta_2(y)\\
&+\zeta_1(y)\zeta_2(y)-\zeta_1(y)f(y),
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
&(\zeta_2(y)-f(y))\left(\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)-\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_1(t)\right)+\left(\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_2(t)-\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\right)(f(y)-\zeta_1(y))\\
&-\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left[(f(t)-\zeta_1(t))(\zeta_2(t)-f(t))\right]-(f(y)-\zeta_1(y))(\zeta_2(y)-f(y))\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{s\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\\
&=\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f^2(t)+f^2(y)\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{s\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\\
&-2f(y)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)+\zeta_2(y)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)+\zeta_2(y)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_1(t)\\
&+f(y)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_2(t)+f(y)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_1(t)+\zeta_1(y)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\\
&-\zeta_1(y)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_2(t)-\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(\zeta_2(t)f(t))\\
&+\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(\zeta_1(t)\zeta_2(t))-\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(\zeta_1(t)f(t))-\zeta_2(y)f(y)\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\\
&+\zeta_1(y)\zeta_2(y)\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}-\zeta_1(y)f(y)\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)},
\end{aligned}$$

Multiply the above equation on both sides by

$$\frac{(s)^{1-\frac{\alpha}{k}}\left((t-a)^s-(y-a)^s\right)^{\frac{\beta}{k}-1}(y-a)^{s-1}}{k\Gamma_k(\alpha)}$$

and integrate w.r.t. y from a to t, the required equality (7.2.6) is obtained. ■

Lemma 7.2.4 *For $k > 0$, let $s = 1$ in theorem (7.2.4), we get the integral result for*

k-analogue of Riemann–Liouville fractional integral $\left(\mathcal{R}^{\beta}_{a^+,k}\right)$ as,

$$\begin{aligned}
&\frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{R}^{\alpha}_{a^+,k}f^2(t)-[\mathcal{R}^{\alpha}_{a^+,k}f(t)]^2\\
&=(\mathcal{R}^{\alpha}_{a^+,k}\zeta_2(t)-\mathcal{R}^{\alpha}_{a^+,k}f(t))(\mathcal{R}^{\alpha}_{a^+,k}f(t)-\mathcal{R}^{\alpha}_{a^+,k}\zeta_1(t))\\
&-\frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\left(\mathcal{R}^{\alpha}_{a^+,k}\zeta_2(t)-\mathcal{R}^{\alpha}_{a^+,k}f(t)\right)\left(\mathcal{R}^{\alpha}_{a^+,k}f(t)-\mathcal{R}^{\alpha}_{a^+,k}\zeta_1(t)\right)\\
&+\frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{R}^{\alpha}_{a^+,k}(\zeta_1(t)f(t))-\mathcal{R}^{\alpha}_{a^+,k}\zeta_1(t)\mathcal{R}^{\alpha}_{a^+,k}f(t)\\
&+\frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{R}^{\alpha}_{a^+,k}(\zeta_2(t)f(t))-\mathcal{R}^{\alpha}_{a^+,k}\zeta_2(t)\mathcal{R}^{\alpha}_{a^+,k}f(t)\\
&-\frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{R}^{\alpha}_{a^+,k}(\zeta_1(t)\zeta_2(t))+\mathcal{R}^{\alpha}_{a^+,k}\zeta_1(t)\mathcal{R}^{\alpha}_{a^+,k}\zeta_2(t).
\end{aligned}$$

Corollary 7.2.1 *Suppose that for a function $f\in L^r_1[a,b]$, such that $m_1\le f(t)\le m_2$ for $a\le t\le b$ and $m_1,m_2\in\mathbb{R}$. Then we have*

$$\begin{aligned}
&\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f^2(t)-\left[\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\right]^2\\
&=-\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)[(m_2-f(t))(f(t)-m_1)]\\
&+\left[\frac{m_2s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}-\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\right]\\
&\times\left[\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)-\frac{m_1s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\right].
\end{aligned}$$

where $k,s>0$ and $\alpha>0$.

Proof. The required result can be proved by substituting $\zeta_1(t)=m_1$ and $\zeta_2(t)=m_2$ in Theorem (7.2.4). ■

Theorem 7.2.5 *Let ζ_1, ζ_2, ξ_1 and ξ_2 be integrable functions on $[a,b]$ following the conditions (7.2.2) and (7.2.5). Then $\forall a\le t\le b$, $k,s>0$ and α,*

$$\begin{aligned}
&\left|\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)(g(t)f(t))-\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\right)g(t)\right|\\
&\qquad\le\sqrt{S^s_k(f,\zeta_1,\zeta_2)S^s_k(g,\xi_1,\xi_2)},
\end{aligned}\tag{7.2.7}$$

where f and g are integrable functions and

$$
\begin{aligned}
S_k^s(x,y,z) &= \left(\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)z(t) - \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)x(t)\right)\left(\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)x(t) - \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)y(t)\right) \\
&+ \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(x(t)y(t)) - \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)y(t)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)x(t) \\
&+ \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(x(t)z(t)) - \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)z(t)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)x(t) \\
&- \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(z(t)y(t)) + \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)y(t)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)z(t).
\end{aligned}
$$

Proof. Accordingly, let us define

$$
B(x,y) = [g(x)-g(y)]\,[f(x)-f(y)]\,,
$$

$$
= f(x)g(x) + f(y)g(y) - f(x)g(y) - f(y)g(x), \tag{7.2.8}
$$

$\Rightarrow$

$$
\begin{aligned}
&\frac{(s)^{2-\frac{2\alpha}{k}}}{2k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t \left((t-a)^s-(x-a)^s\right)^{\frac{\alpha}{k}-1}\left((t-a)^s-(y-a)^s\right)^{\frac{\alpha}{k}-1} \\
&\quad\times (x-a)^{s-1}(y-a)^{s-1}B(x,y)dx\,dy \\
&\quad= \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(g(t)f(t)) - \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)g(t),
\end{aligned} \tag{7.2.9}
$$

Invoke the value from (7.2.8) to the L.H.S. of equality (7.2.9) and then use Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&\Bigg[\frac{(s)^{2-\frac{2\alpha}{k}}}{2k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t \left((t-a)^s-(x-a)^s\right)^{\frac{\alpha}{k}-1}\left((t-a)^s-(y-a)^s\right)^{\frac{\alpha}{k}-1} \\
&\times(x-a)^{s-1}(y-a)^{s-1}B(x,y)dxdy\Bigg]^2 \\
&\leq \frac{(s)^{2-\frac{2\alpha}{k}}}{2k^2\Gamma_k^2(\alpha)}\int_a^t\int_a^t \left((t-a)^s-(x-a)^s\right)^{\frac{\alpha}{k}-1}\left((t-a)^s-(y-a)^s\right)^{\frac{\alpha}{k}-1} \\
&\times (x-a)^{s-1}(y-a)^{s-1}[f(x)-f(y)]^2dxdy
\end{aligned}
$$

$$\times \frac{(s)^{2-\frac{2\alpha}{k}}}{2k^2\Gamma_k^2(\alpha)} \int_a^t \int_a^t ((t-a)^s - (x-a)^s)^{\frac{\alpha}{k}-1} ((t-a)^s - (y-a)^s)^{\frac{\alpha}{k}-1}$$
$$\times (x-a)^{s-1}(y-a)^{s-1}[g(x)-g(y)]^2 dxdy. \tag{7.2.10}$$

Now since

$$[f(x)-f(y)]^2 = f^2(x) + f^2(y) - 2f(x)f(y),$$

it can be easily proved that

$$\frac{(s)^{2-\frac{2\alpha}{k}}}{2k^2\Gamma_k^2(\alpha)} \int_a^t \int_a^t ((t-a)^s - (x-a)^s)^{\frac{\alpha}{k}-1} ((t-a)^s - (y-a)^s)^{\frac{\alpha}{k}-1}$$
$$\times (x-a)^{s-1}(y-a)^{s-1}[f(x)-f(y)]^2 dx\, dy$$
$$= \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right) f^2(t) - \left(\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right) f(t)\right)^2. \tag{7.2.11}$$

Similarly,

$$\frac{(s)^{2-\frac{2\alpha}{k}}}{2k^2\Gamma_k^2(\alpha)} \int_a^t \int_a^t ((t-a)^s - (x-a)^s)^{\frac{\alpha}{k}-1} ((t-a)^s - (y-a)^s)^{\frac{\alpha}{k}-1}$$
$$\times (x-a)^{s-1}(y-a)^{s-1}[g(x)-g(y)]^2 dxdy$$
$$= \frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right) g^2(t) - \left(\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right) g(t)\right)^2. \tag{7.2.12}$$

Using equations (7.2.11) and (7.2.12) into (7.2.10), we get

$$\Bigg[\frac{(s)^{2-\frac{2\alpha}{k}}}{2k^2\Gamma_k^2(\alpha)} \int_a^t \int_a^t ((t-a)^s - (x-a)^s)^{\frac{\alpha}{k}-1} ((t-a)^s - (y-a)^s)^{\frac{\alpha}{k}-1}$$
$$\times (x-a)^{s-1}(y-a)^{s-1} B(x,y) dxdy\Bigg]^2$$
$$\leq \left[\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right) f^2(t) - \left(\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right) f(t)\right)^2\right]$$
$$\times \left[\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)} \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right) g^2(t) - \left(\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right) g(t)\right)^2\right]. \tag{7.2.13}$$

Thus the equation (7.2.9) together with the inequality (7.2.13) implies that

$$\left[\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(f(t)g(t)) - I^{\alpha,s}_{a,k}f(t)I^{\alpha,s}_{a,k}g(t)\right]^2$$
$$\leq \left[\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f^2(t) - \left(\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\right)^2\right]$$
$$\times \left[\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)g^2(t) - \left(\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)g(t)\right)^2\right]. \quad (7.2.14)$$

Now, since

$$(f(t)-\zeta_1(t))(\zeta_2(t)-f(t)) \geq 0$$

and

$$(g(t)-\xi_1(t))(\xi_2(t)-g(t)) \geq 0,$$

therefore,

$$\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(f(t)-\zeta_1(t))(\zeta_2(t)-f(t)) \geq 0, \quad t\in[a,b]$$

and

$$\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(f(t)-\zeta_1(t))(\zeta_2(t)-f(t)) \geq 0, \quad t\in[a,b].$$

By theorem (7.2.4), we have

$$\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f^2(t) - \left[\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\right]^2$$
$$\leq \left(\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_2(t) - \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\right)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t) - \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_1(t))$$
$$+\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(\zeta_1(t)f(t)) - \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_1(t)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)$$
$$+\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(\zeta_2(t)f(t)) - \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_2(t)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)$$
$$-\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(\zeta_1(t)\zeta_2(t)) + \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_1(t)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\zeta_2(t)$$
$$= S^s_k(f,\zeta_1,\zeta_2). \quad (7.2.15)$$

Similarly

$$\begin{aligned}
&\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)g^2(t)-\left[\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)g(t)\right]^2\\
&\leq \left(\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\xi_2(t)-\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)g(t)\right)\left(\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)g(t)-\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\xi_1(t)\right)\\
&+\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(\xi_1(t)g(t))-\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\xi_1(t)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)g(t)\\
&+\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(\xi_2(t)g(t))-\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\xi_2(t)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)g(t)\\
&-\frac{(s)^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(\xi_1(t)\xi_2(t))+\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\xi_1(t)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\xi_2(t)\\
&=S_k^s(g,\xi_1,\xi_2). \qquad (7.2.16)
\end{aligned}$$

Equations (7.2.15) and (7.2.16) together with inequality (7.2.14) yield the inequality (7.2.7).

■

Lemma 7.2.5 *Put $s=1$, the inequality (7.2.7) reduces to the integral result for k-Riemann–Liouville fractional integral $\left(\mathcal{R}^{\beta}_{a^+,k}\right)$ as,*

$$\left|\frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{R}^{\alpha}_{a^+,k}\left(f(t)g(t)\right)-\mathcal{R}^{\alpha}_{a^+,k}f(t)\mathcal{R}^{\alpha}_{a^+,k}g(t)\right|\leq\sqrt{S_k(f,\zeta_1,\zeta_2)S_k(f,\xi_1,\xi_2)},$$

where

$$\begin{aligned}
S_k(x,y,z)&=\left(\mathcal{R}^{\alpha}_{a^+,k}z(t)-\mathcal{R}^{\alpha}_{a^+,k}x(t)\right)\left(\mathcal{R}^{\alpha}_{a^+,k}x(t)-\mathcal{R}^{\alpha}_{a^+,k}y(t)\right)\\
&+\frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{R}^{\alpha}_{a^+,k}\left(y(t)x(t)\right)-\mathcal{R}^{\alpha}_{a^+,k}y(t)\mathcal{R}^{\alpha}_{a^+,k}x(t)\\
&+\frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{R}^{\alpha}_{a^+,k}\left(z(t)x(t)\right)-\mathcal{R}^{\alpha}_{a^+,k}z(t)\mathcal{R}^{\alpha}_{a^+,k}x(t)\\
&-\frac{(t-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{R}^{\alpha}_{a^+,k}\left(y(t)z(t)\right)+\mathcal{R}^{\alpha}_{a^+,k}y(t)\mathcal{R}^{\alpha}_{a^+,k}z(t).
\end{aligned}$$

Example 1 *Let f and g be two functions such that $(t-a)^r\leq f(t)\leq(t-a)^r+1$ and $(t-a)^r-1\leq g(t)\leq(t-a)^r$ for $a\leq t\leq b$. Then for $k,s>0$. $\alpha>0$, we have*

$$\begin{aligned}
&\left|\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(f(t)g(t))-\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)g(t)\right|\\
&\leq\sqrt{S_k^s(f,(t-a)^r,(t-a)^r+1)S_k^s(g,(t-a)^r-1,(t-a)^r)}.
\end{aligned}$$

Here,

$$
\begin{aligned}
&S_k^s(f,(t-a)^r,(t-a)^r+1)=\\
&\left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}+\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+s}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}-\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\right)\\
&\times\left(\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)-\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}\right)+\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(f(t)t^r)\\
&-\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}\,{}_k^s\mathfrak{F}^{*\alpha}_{a^+}f(t)+\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\left((t-a)^sf(t)\right)\\
&-\left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}+\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\right)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)f(t)\\
&+\left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}\right)\left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}+\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\right)\\
&-\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}+\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+2r}\Gamma_k(\frac{2rk}{s}+k)}{\Gamma_k(\alpha+\frac{2rk}{s}+k)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&S_k^s(f,(t-a)^r-1,(t-a)^r)=\\
&\left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}-\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}+\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)g(t)\right)\\
&\times\left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}-\;{}_k^s\mathfrak{F}^{*\alpha}_{a^+}g(t)\right)+\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left(((t-a)^r-1)g(t)\right)\\
&-\left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}-\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\right)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)g(t)\\
&+\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)\left((t-a)^rg(t)\right)-\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)g(t)\\
&+\left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}-\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\right)\left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}\right)\\
&-\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}}}{\Gamma_k(\alpha+k)}\left(\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+r}\Gamma_k(\frac{rk}{s}+k)}{\Gamma_k(\alpha+\frac{rk}{s}+k)}-\frac{s^{\frac{-\alpha}{k}}(t-a)^{\frac{s\alpha}{k}+2r}\Gamma_k(\frac{2rk}{s}+k)}{\Gamma_k(\alpha+\frac{2rk}{s}+k)}\right).
\end{aligned}
$$

Chapter 8

CHEBYSHEV-GRÜSS INTEGRAL INEQUALITIES INVOLVING GENERALIZED CONFORMABLE K-FRACTIONAL INTEGRALS

This chapter concerns to the generalizations of certain fractional integral inequalities for our introduced generalized integrals. The classical Chebyshev-Grüss type inequalities and the weighted Grüss type inequalities are generalized for generalized conformable k-fractional integrals (2.3.35) and (2.3.36). The main resulting inequalities are presented using one and two fractional parameters.

8.1 Introduction

Grüss type Inequality due to Chebyshev (see for example [134, p. 207]) is as follows:
For absolutely continuous f_1, f_2 on $[a,b]$, such that $f_1', f_2' \in L^\infty[a,b]$ with
$\|f_1'\|_\infty := ess \sup_{t\in[a,b]} |f_1'(t)|$, then

$$|T(f_1, f_2; a, b)| \leq \frac{1}{12}\|f_1'\|_\infty \|f_2'\|_\infty (b-a)^2 \tag{8.1.1}$$

where $T(f_1, f_2; a, b)$ is Chebyshev functional (3.1.1) and the constant $\frac{1}{12}$ is the best possible.

Further, the Chebyshev functional (see [54]) in weighted form is defined as:

$$T(f_1, f_2, p) = \int_a^b f_1(t)f_2(t)p(t)dt \int_a^b p(t)dt - \int_a^b f_1(t)p(t)dt \int_a^b f_2(t)p(t)dt, \tag{8.1.2}$$

where $p(t)$ is a positive integrable function on $[a,b]$ and f_1, f_2 are integrable functions on $[a,b]$.

The following related inequality was derived by Dragomir [79], using the weighted Chebyshev functional (8.1.2):

$$2\left|T(f_1,f_2,p)\right| \leq \ \|f_1'\|_p\|f_2'\|_q \left[\int_a^b\int_a^b |x_1-x_2|\, p(x_1)p(x_2)dx_1dx_2\right], \tag{8.1.3}$$

where f_1, f_2 are differentiable functions and $f_1' \in L_p(a,b), f_2' \in L_q(a,b), p>1, \frac{1}{p}+\frac{1}{q}=1$.

8.2 Generalized k-Fractional Conformable Integral Inequalities

This section depicts some new weighted integral inequalities giving an estimate of the product of functions in terms of the products of the individual functions by using generalized k-fractional conformable integrals (${}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}$ and ${}^{s}_{k}\mathfrak{F}^{*\beta}_{b^-}$). The Hölder's integral inequality and its weighted version have been used in obtaining the main results.
To use in the next results, we want to define the two functions as,

$$\mathcal{J}(\tau,\rho) = (g(\tau)-g(\rho))\,(f(\tau)-f(\rho)), \quad \tau,\rho \in (a,t), \tag{8.2.4}$$

and for $t,\tau \in (a,t), \beta, k>0, s \in \mathbb{R}\setminus\{0\}$,

$${}^{\beta}_{k}\mathcal{F}^s(t,\tau) = \frac{1}{k\Gamma_k(\beta)}\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\beta}{k}-1}. \tag{8.2.5}$$

Theorem 8.2.1 *For differentiable functions f and g, such that $f' \in L_n([a,b]), g' \in L_m([a,b])$ with $\frac{1}{n}+\frac{1}{m}=1$, where $n>1$ and p is a positive function. Then $\forall a<t\leq b$, $\beta>0$, $k>0$ and $s\in\mathbb{R}\setminus\{0\}$, the following inequalities are valid,*

$$\begin{aligned} 2&\Big|\,{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}(p(t))\,{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}(f(t)p(t)g(t)) - \,{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}(f(t)p(t))\,{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}(g(t)p(t))\Big| \\ &\leq \frac{s^{2(1-\frac{\beta}{k})}\|g\|_m\|f\|_n}{k^2\Gamma_k^2(\beta)}\int_a^t\int_a^t(\tau-a)^{s-1}(\rho-a)^{s-1}\left((t-a)^s-(\tau-a)^s\right)^{\frac{\beta}{k}-1} \\ &\times\left((t-a)^s-(\rho-a)^s\right)^{\frac{\beta}{k}-1}p(\rho)p(\tau)|\tau-\rho|d\tau d\rho \\ &\leq \|g'\|_m\|f'\|_n\, t\left(\,{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}(p(t))\right)^2. \end{aligned} \tag{8.2.6}$$

Proof. Under the conditions given and for all $\tau \in (a,t)$, we can see that ${}_{k}^{\beta}\mathcal{F}^s(t,\tau) > 0$. Multiplying with $p(\tau)$ both sides of product $\mathcal{J}(\tau,\rho) \times {}_{k}^{\beta}\mathcal{F}^s(t,\tau)$ and taking the integral w.r.t. τ on (a,t), we obtain

$$\begin{aligned}
&\frac{s^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_a^t \left((t-a)^s - (\tau-a)^s\right)^{\frac{\beta}{k}-1} \mathcal{J}(\tau,\rho)(\tau-a)^{s-1}p(\tau)d\tau \\
&= {}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(p(t)f(t)g(t)) - f(\rho)\, {}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(g(t)p(t)) \\
&- g(\rho)\, {}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(p(t)f(t)) + g(\rho)f(\rho)\, {}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(p(t)).
\end{aligned} \tag{8.2.7}$$

Now, multiplying above identity (8.2.7) by ${}_{k}^{\beta}\mathcal{F}^s(t,\rho)p(\rho)$ and then integrate w.r.t. ρ on (a,t), give

$$\begin{aligned}
&\frac{s^{2(1-\frac{\beta}{k})}}{k^2\Gamma_k^2(\beta)} \int_a^t \int_a^t \left((t-a)^s - (\tau-a)^s\right)^{\frac{\beta}{k}-1} \left((t-a)^s - (\rho-a)^s\right)^{\frac{\beta}{k}-1} \\
&\times (\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\mathcal{J}(\tau,\rho)d\tau d\rho \\
&= 2\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(p(t))\, {}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(p(t)f(t)g(t)) - {}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(p(t)f(t))\, {}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(p(t)g(t)) \right).
\end{aligned} \tag{8.2.8}$$

By fundamental theorem of calculus, the relation (8.2.4) can rewrite as

$$\mathcal{J}(\tau,\rho) = \int_\tau^\rho \int_\tau^\rho g'(z)f'(y)dzdy.$$

The Hölder's inequality for double integration is

$$\left| \int_\tau^\rho \int_\tau^\rho f(y)g(z)dydz \right| \le \left| \int_\tau^\rho \int_\tau^\rho |f(y)|^n \, dydz \right|^{\frac{1}{n}} \left| \int_\tau^\rho \int_\tau^\rho |g(z)|^m \, dydz \right|^{\frac{1}{m}},$$

$$\left(\frac{1}{n} + \frac{1}{m} = 1, n > 1 \right),$$

we obtain,

$$|\mathcal{J}(\tau,\rho)| \le \left| \int_\tau^\rho \int_\tau^\rho |f'(y)|^n \, dydz \right|^{\frac{1}{n}} \left| \int_\tau^\rho \int_\tau^\rho |g'(z)|^m \, dydz \right|^{\frac{1}{m}}. \tag{8.2.9}$$

Since

$$\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}|f'(y)|^{n}\,dydz\right|^{\frac{1}{n}}=|\tau-\rho|^{\frac{1}{n}}\left|\int_{\tau}^{\rho}|f'(y)|^{n}\,dy\right|^{\frac{1}{n}}$$

and

$$\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}|g'(z)|^{m}\,dydz\right|^{\frac{1}{m}}=|\tau-\rho|^{\frac{1}{m}}\left|\int_{\tau}^{\rho}|g'(z)|^{m}\,dz\right|^{\frac{1}{m}},$$

thus, from inequality (8.2.9) we get

$$|\mathcal{J}(\tau,\rho)|\leq|\tau-\rho|\ \left|\int_{\tau}^{\rho}|f'(y)|^{n}\,dy\right|^{\frac{1}{n}}\left|\int_{\tau}^{\rho}|g'(z)|^{m}\,dz\right|^{\frac{1}{m}}. \tag{8.2.10}$$

As a result of (8.2.10) and from equality (8.2.8) can be written following inequality :

$$\begin{aligned}
&\frac{s^{2(1-\frac{\beta}{k})}}{k^{2}\Gamma_{k}^{2}(\beta)}\int_{a}^{t}\int_{a}^{t}\left((t-a)^{s}-(\tau-a)^{s}\right)^{\frac{\beta}{k}-1}\left((t-a)^{s}-(\rho-a)^{s}\right)^{\frac{\beta}{k}-1}\\
&\qquad\times\Big|\mathcal{J}(\tau,\rho)\Big|(\rho-a)^{s-1}p(\rho)(\tau-a)^{s-1}p(\tau)d\tau d\rho\\
&\leq\frac{s^{2(1-\frac{\beta}{k})}}{k^{2}\Gamma_{k}^{2}(\beta)}\int_{a}^{t}\int_{a}^{t}\left((t-a)^{s}-(\tau-a)^{s}\right)^{\frac{\beta}{k}-1}\left((t-a)^{s}-(\rho-a)^{s}\right)^{\frac{\beta}{k}-1}\\
&\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\,|\tau-\rho|\ \left|\int_{\tau}^{\rho}|f'(y)|^{n}\,dy\right|^{\frac{1}{n}}\left|\int_{\tau}^{\rho}|g'(z)|^{m}\,dz\right|^{\frac{1}{m}}d\tau d\rho.
\end{aligned}\tag{8.2.11}$$

Again, the use of weighted Hölder's integral inequality on the R.H.S. of (8.2.11) yields have

$$\begin{aligned}
&\frac{s^{2(1-\frac{\beta}{k})}}{k^{2}\Gamma_{k}^{2}(\beta)}\int_{a}^{t}\int_{a}^{t}\left((t-a)^{s}-(\tau-a)^{s}\right)^{\frac{\beta}{k}-1}\left((t-a)^{s}-(\rho-a)^{s}\right)^{\frac{\beta}{k}-1}\\
&\qquad\times\Big|\mathcal{J}(\tau,\rho)\Big|(\rho-a)^{s-1}p(\rho)(\tau-a)^{s-1}p(\tau)d\tau d\rho\\
&\leq\left[\frac{s^{n(1-\frac{\beta}{k})}}{k^{n}\Gamma_{k}^{n}(\beta)}\int_{a}^{t}\int_{a}^{t}\left((t-a)^{s}-(\tau-a)^{s}\right)^{\frac{\beta}{k}-1}\left((t-a)^{s}-(\rho-a)^{s}\right)^{\frac{\beta}{k}-1}\right.\\
&\qquad\left.\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\Big|\tau-\rho\Big|\,\Big|\int_{\tau}^{\rho}|f'(y)|^{n}\,dy\Big|d\tau d\rho\right]^{\frac{1}{n}}\\
&\times\left[\frac{s^{m(1-\frac{\beta}{k})}}{k^{m}\Gamma_{k}^{m}(\beta)}\int_{a}^{t}\int_{a}^{t}\left((t-a)^{s}-(\tau-a)^{s}\right)^{\frac{\beta}{k}-1}\left((t-a)^{s}-(\rho-a)^{s}\right)^{\frac{\beta}{k}-1}\right.
\end{aligned}$$

$$\times(\rho-a)^{s-1}p(\rho)(\tau-a)^{s-1}p(\tau)\Big|\tau-\rho\Big|\,\Big|\int_{\tau}^{\rho}|g'(z)|^{m}\,dz\Big|d\tau d\rho\Bigg]^{\frac{1}{m}} \tag{8.2.12}$$

Using the fact that

$$\left|\int_{\tau}^{\rho}|g'(z)|^{m}\,dz\right|\leq\|g'\|_{m}^{m}\quad\text{and}\quad\left|\int_{\tau}^{\rho}|f'(y)|^{n}\,dy\right|\leq\|f'\|_{n}^{n},$$

we obtain,

$$\frac{s^{2(1-\frac{\beta}{k})}}{k^{2}\Gamma_{k}^{2}(\beta)}\int_{a}^{t}\int_{a}^{t}\left((t-a)^{s}-(\tau-a)^{s}\right)^{\frac{\beta}{k}-1}\left((t-a)^{s}-(\rho-a)^{s}\right)^{\frac{\beta}{k}-1}$$

$$\times\Big|\mathcal{J}(\tau,\rho)\Big|(\tau-a)^{s-1}(\rho-a)^{s-1}p(\tau)p(\rho)d\tau d\rho$$

$$\leq\Bigg[\frac{s^{n(1-\frac{\beta}{k})}\|f'\|_{n}^{n}}{k^{n}\Gamma_{k}^{n}(\beta)}\int_{a}^{t}\int_{a}^{t}\left((t-a)^{s}-(\tau-a)^{s}\right)^{\frac{\beta}{k}-1}\left((t-a)^{s}-(\rho-a)^{s}\right)^{\frac{\beta}{k}-1}$$

$$\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\Big|\tau-\rho\Big|d\tau d\rho\Bigg]^{\frac{1}{n}}$$

$$\times\Bigg[\frac{\|g'\|_{m}^{m}s^{m(1-\frac{\beta}{k})}}{k^{m}\Gamma_{k}^{m}(\beta)}\int_{a}^{t}\int_{a}^{t}\left((t-a)^{s}-(\tau-a)^{s}\right)^{\frac{\beta}{k}-1}\left((t-a)^{s}-(\rho-a)^{s}\right)^{\frac{\beta}{k}-1}$$

$$\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\Big|\tau-\rho\Big|d\tau d\rho\Bigg]^{\frac{1}{m}} \tag{8.2.13}$$

which yields

$$\frac{s^{2(1-\frac{\beta}{k})}}{k^{2}\Gamma_{k}^{2}(\beta)}\int_{a}^{t}\int_{a}^{t}\left((t-a)^{s}-(\tau-a)^{s}\right)^{\frac{\beta}{k}-1}\left((t-a)^{s}-(\rho-a)^{s}\right)^{\frac{\beta}{k}-1}$$

$$\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\Big|\mathcal{J}(\tau,\rho)\Big|d\tau d\rho$$

$$\leq\frac{s^{2(1-\frac{\beta}{k})}\|f'\|_{n}\|g'\|_{m}}{k^{2}\Gamma_{k}^{2}(\beta)}\Bigg[\int_{a}^{t}\int_{a}^{t}\left((t-a)^{s}-(\tau-1)^{s}\right)^{\frac{\beta}{k}-1}\left((t-a)^{s}-(\rho-1)^{s}\right)^{\frac{\beta}{k}-1}$$

$$\times\Big|\tau-\rho\Big|(\tau-a)^{s-1}(\rho-a)^{s-1}p(\tau)p(\rho)d\tau d\rho\Bigg]^{\frac{1}{n}}$$

$$\times\Bigg[\int_{a}^{t}\int_{a}^{t}\left((t-a)^{s}-(\tau-a)^{s}\right)^{\frac{\beta}{k}-1}\left((t-a)^{s}-(\rho-a)^{s}\right)^{\frac{\beta}{k}-1}$$

$$\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\Big|\tau-\rho\Big|d\tau d\rho\Bigg]^{\frac{1}{m}} \tag{8.2.14}$$

With the fact that $\frac{1}{n}+\frac{1}{m}=1$, from (8.2.14) we get

$$\begin{aligned}
&\frac{s^{2(1-\frac{\beta}{k})}}{k^2\Gamma_k^2(\beta)}\int_a^t\int_a^t \left((t-a)^s-(\tau-a)^s\right)^{\frac{\beta}{k}-1}\left((t-a)^s-(\rho-a)^s\right)^{\frac{\beta}{k}-1}\\
&\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\Big|\mathcal{J}(\tau,\rho)\Big|d\tau d\rho\\
&\leq \frac{s^{2(1-\frac{\beta}{k})}\|f'\|_n\|g'\|_m}{k^2\Gamma_k^2(\beta)}\int_a^t\int_a^t \left((t-a)^s-(\tau-a)^s\right)^{\frac{\beta}{k}-1}\left((t-a)^s-(\rho-a)^s\right)^{\frac{\beta}{k}-1}\\
&\times\Big|\tau-\rho\Big|(\tau-a)^{s-1}(\rho-a)^{s-1}p(\tau)p(\rho)d\tau d\rho
\end{aligned} \tag{8.2.15}$$

Alternatively, using equality (8.2.8), we can easily see that

$$\begin{aligned}
&2\left|{}^s_k\mathfrak{F}^{*\beta}_{a^+}(f(t)p(t)g(t))\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}(p(t))-{}^s_k\mathfrak{F}^{*\beta}_{a^+}(g(t)p(t))\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}(f(t)p(t))\right|\\
&\leq \frac{s^{2(1-\frac{\beta}{k})}}{k^2\Gamma_k^2(\beta)}\int_a^t\int_a^t \left((t-a)^s-(\tau-a)^s\right)^{\frac{\beta}{k}-1}\left((t-a)^s-(\rho-a)^s\right)^{\frac{\beta}{k}-1}\\
&\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\Big|\mathcal{J}(\tau,\rho)\Big|d\tau d\rho.
\end{aligned} \tag{8.2.16}$$

Taking into account the inequalities (8.2.15) and (8.2.16), the left-side of the inequality (8.2.6) can be concluded.

Next, we establish the R.H.S. of the inequality (8.2.6):

since $a\leq\tau,\rho\leq t$, so we may use

$$0\leq|\tau-\rho|\leq t.$$

From (8.2.15), we obtain

$$\begin{aligned}
&\frac{s^{2(1-\frac{\beta}{k})}}{k^2\Gamma_k^2(\beta)}\int_a^t\int_a^t \left((t-a)^s-(\tau-a)^s\right)^{\frac{\beta}{k}-1}\left((t-a)^s-(\rho-a)^s\right)^{\frac{\beta}{k}-1}\\
&\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\Big|\mathcal{J}(\tau,\rho)\Big|d\tau d\rho\\
&\leq \frac{s^{2(1-\frac{\beta}{k})}\|f'\|_n\|g'\|_m}{k^2\Gamma_k^2(\beta)}\int_a^t\int_a^t \left((t-a)^s-(\tau-a)^s\right)^{\frac{\beta}{k}-1}\left((t-a)^s-(\rho-a)^s\right)^{\frac{\beta}{k}-1}
\end{aligned}$$

$$\times (\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\Big|\tau-\rho\Big|d\tau d\rho$$
$$\leq \frac{ts^{2(1-\frac{\beta}{k})}\|f'\|_n\|g'\|_m}{k^2\Gamma_k^2(\beta)}\int_a^t\int_a^t ((t-a)^s-(\tau-a)^s)^{\frac{\beta}{k}-1}((t-a)^s-(\rho-a)^s)^{\frac{\beta}{k}-1}$$
$$\times (\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)d\tau d\rho$$
$$= \|f'\|_n\|g'\|_m t\left({}_k^{\beta}\mathcal{S}_{a+}^{\alpha}(p(t))\right)^2.$$

which concludes the proof of Theorem 8.2.1. ■

The forthcoming theorem further generalizes Theorem (8.2.1) with two fractional parameters β and γ.

Theorem 8.2.2 *Suppose that f, g are differentiable functions, such that $f'\in L_n([a,b])$, $g'\in L_m([a,b])$ with $\frac{1}{n}+\frac{1}{m}=1$ where $n>1$ and p be a positive function on $[a,b]$. Then the following inequality holds for all $a<t\leq b$, $\beta,\gamma>0$, $k>0$ and $s\in\mathbb{R}\setminus\{0\}$,*

$$\Bigg|{}_k^s\mathfrak{F}^{*\gamma}_{a+}(p(t))\,{}_k^s\mathfrak{F}^{*\beta}_{a+}(f(t)p(t)g(t)) - {}_k^s\mathfrak{F}^{*\gamma}_{a+}(f(t)p(t))\ {}_k^s\mathfrak{F}^{*\beta}_{a+}(g(t)p(t))$$
$$-{}_k^{\gamma}\mathcal{S}_{a+}^{\alpha}(g(t)p(t))\,{}_k^s\mathfrak{F}^{*\beta}_{a+}(f(t)p(t)) + {}_k^s\mathfrak{F}^{*\gamma}_{a+}(p(t)f(t)g(t))\,{}_k^s\mathfrak{F}^{*\gamma}_{a+}(p(t))\Bigg|$$
$$\leq \frac{s^{2-\frac{\beta+\gamma}{k}}\|f'\|_n\|g'\|_m}{k^2\Gamma_k(\beta)\Gamma_k(\gamma)}\int_a^t\int_a^t(\tau-a)^{s-1}(\rho-a)^{s-1}((t-a)^{\alpha}-(\tau-1)^{\alpha})^{\frac{\beta}{k}-1}$$
$$\times ((t-a)^s-(\rho-1)^s)^{\frac{\beta}{k}-1}p(\tau)p(\rho)|\tau-\rho|d\tau d\rho$$
$$\leq t\|f'\|_n\|g'\|_m\,{}_k^s\mathfrak{F}^{*\beta}_{a+}(p(t))\,{}_k^s\mathfrak{F}^{*\gamma}_{a+}(p(t)). \tag{8.2.17}$$

Proof. To prove this theorem, we multiply (8.2.7) by ${}_k^{\gamma}\mathcal{F}^s(t,\rho)$ ($\rho\in(a,t)$, $a<t\leq b$) and take the integral on (a,t) (with respect to ρ), to obtain

$$\frac{s^{2-\frac{\beta+\gamma}{k}}}{k^2\Gamma_k(\beta)\Gamma_k(\gamma)}\int_a^t\int_a^t((t-a)^s-(\tau-a)^s)^{\frac{\beta}{k}-1}((t-a)^s-(\rho-a)^s)^{\frac{\gamma}{k}-1}$$
$$\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\mathcal{J}(\tau,\rho)d\tau d\rho$$
$$= {}_k^s\mathfrak{F}^{*\gamma}_{a+}(p(t))\,{}_k^s\mathfrak{F}^{*\beta}_{a+}(p(t)f(t)g(t)) - {}_k^s\mathfrak{F}^{*\gamma}_{a+}(p(t)f(t))\,{}_k^s\mathfrak{F}^{*\beta}_{a+}(p(t)g(t))$$
$$-{}_k^s\mathfrak{F}^{*\gamma}_{a+}(p(t)g(t))\,{}_k^s\mathfrak{F}^{*\beta}_{a+}(p(t)f(t)) + {}_k^s\mathfrak{F}^{*\gamma}_{a+}(p(t)f(t)g(t))\,{}_k^s\mathfrak{F}^{*\beta}_{a+}(p(t)). \tag{8.2.18}$$

Using the previous inequality (8.2.10), we get

$$\frac{s^{2-\frac{\beta+\gamma}{k}}}{k^2\Gamma_k(\beta)\Gamma_k(\gamma)}\int_a^t\int_a^t ((t-a)^s-(\tau-a)^s)^{\frac{\beta}{k}-1}((t-a)^s-(\rho-a)^s)^{\frac{\gamma}{k}-1}$$

$$\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\Big|\mathcal{J}(\tau,\rho)\Big|d\tau d\rho$$

$$\le \frac{s^{2-\frac{\beta+\gamma}{k}}}{k^2\Gamma_k(\beta)\Gamma_k(\gamma)}\int_a^t\int_a^t ((t-a)^s-(\tau-a)^s)^{\frac{\beta}{k}-1}((t-a)^s-(\rho-a)^s)^{\frac{\gamma}{k}-1}$$

$$\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\,|\tau-\rho|\ \left|\int_\tau^\rho |f'(y)|^n\,dy\right|^{\frac{1}{n}}\left|\int_\tau^\rho |g'(z)|^m\,dz\right|^{\frac{1}{m}}d\tau d\rho. \tag{8.2.19}$$

If we take Hölder's integral inequality, the following inequality can be easily got:

$$\frac{s^{2-\frac{\beta+\gamma}{k}}}{k^2\Gamma_k(\beta)\Gamma_k(\gamma)}\int_a^t\int_a^t ((t-a)^s-(\tau-a)^s)^{\frac{\beta}{k}-1}((t-a)^s-(\rho-a)^s)^{\frac{\gamma}{k}-1}$$

$$\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\Big|\mathcal{J}(\tau,\rho)\Big|d\tau d\rho$$

$$\le \frac{s^{2-\frac{\beta+\gamma}{k}}\|f'\|_n\|g'\|_m}{k^2\Gamma_k^2(\beta)}\int_a^t\int_a^t ((t-a)^s-(\tau-a)^s)^{\frac{\beta}{k}-1}((t-a)^s-(\rho-a)^s)^{\frac{\gamma}{k}-1}$$

$$\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\Big|\tau-\rho\Big|d\tau d\rho. \tag{8.2.20}$$

The left-sided inequality of Theorem (8.2.2) can be easily seen from inequalities (8.2.18) and (8.2.20). Furthermore, for $a\le\tau\le t,\ \ a\le\rho\le t$, we have

$$0\le|\tau-\rho|\le t.$$

Therefore, from (8.2.20), we obtain

$$\frac{s^{2-\frac{\beta+\gamma}{k}}}{k^2\Gamma_k(\beta)\Gamma_k(\gamma)}\int_a^t\int_a^t ((t-a)^s-(\tau-a)^s)^{\frac{\beta}{k}-1}((t-a)^s-(\rho-a)^s)^{\frac{\beta}{k}-1}$$
$$\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)\Big|\mathcal{J}(\tau,\rho)\Big|d\tau d\rho$$
$$\leq \frac{ts^{2(1-\frac{\beta+\gamma}{k})}\|f'\|_n\|g'\|_m}{k^2\Gamma_k^2(\beta)}\int_a^t\int_a^t ((t-a)^s-(\tau-a)^s)^{\frac{\beta}{k}-1}((t-a)^s-(\rho-a)^s)^{\frac{\gamma}{k}-1}$$
$$\times(\tau-a)^{s-1}p(\tau)(\rho-a)^{s-1}p(\rho)d\tau d\rho$$
$$=\|f'\|_n\|g'\|_m t\ {}_k^s\mathfrak{F}^{*\beta}_{a^+}(p(t))\ {}_k^{\gamma}\mathcal{S}^{\alpha}_{a+}(p(t)).$$

which completes the proof of Theorem (8.2.2). ■

Remark 8.2.1 *If we set as $\gamma=\beta$ in Theorem (8.2.2), it reduces to Theorem (8.2.1).*

8.3 Weighted Grüss Inequalities

Theorem 8.3.3 *For two synchronous functions f_1 and f_2 (2.1.4) on $[a,b]$, the following relations for conformable k-fractional integral hold true:*

$${}_k^s\mathfrak{F}^{*\alpha}_{a^+}[f_1(x)f_2(x)]\geq\frac{1}{{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(1)}\ {}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1(x)\ {}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_2(x). \tag{8.3.21}$$

and

$$\begin{aligned}&{}_k^s\mathfrak{F}^{*\alpha}_{a^+}[f_1(x)f_2(x)]\ {}_k^s\mathfrak{F}^{*\beta}_{a^+}(1)+\frac{1}{{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(1)}\ {}_k^s\mathfrak{F}^{*\beta}_{a^+}[f_1(x)f_2(x)]\\&\geq\ {}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_1(x){}_k^s\mathfrak{F}^{*\beta}_{a^+}f_2(x)+\ {}_k^s\mathfrak{F}^{*\alpha}_{a^+}f_2(x)\ {}_k^s\mathfrak{F}^{*\beta}_{a^+}f_1(x).\end{aligned} \tag{8.3.22}$$

where $\alpha,\beta>0$ and $x>a$.

Proof. Given that f_1 and f_2 are synchronous functions on $[a,b]$, so $\forall\lambda^*,\eta^*\in[a,b]$, we have

$$[f_1(\eta^*)-f_1(\lambda^*)][f_2(\eta^*)-f_2(\lambda^*)]\geq 0, \tag{8.3.23}$$

implies that

$$f_1(\eta^*)g(\eta^*)+f_1(\lambda^*)f_2(\lambda^*)\geq f_1(\lambda^*)f_2(\eta^*)+f_1(\eta^*)f_2(\lambda^*). \tag{8.3.24}$$

Multiplying both sides by

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(x-a)^s-(\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(\eta^*-a)^{s-1}$$

and integrating the resultant inequality w.r.t. η^* over (a,x), we get

$$\begin{aligned}&{}^s_k\mathfrak{F}^{*\alpha}_{a^+}[f_1(x)f_2(x)]+\frac{1}{{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(1)}f_1(\lambda^*)f_2(\lambda^*)\\&\geq f_2(\lambda^*)\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}f_1(x)+f_1(\lambda^*)\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}f_2(x).\end{aligned}\tag{8.3.25}$$

Now multiplying the above relation by

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(x-a)^s-(\lambda^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(\lambda^*-a)^{s-1}$$

and integrating w.r.t. λ^* over (a,x), we obtain inequality (8.3.21).

To prove the inequality (8.3.22), multiply both sides of inequality (8.3.25) with

$$\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^s-(\lambda^*-a)^s}{s}\right)^{\frac{\beta}{k}-1}(\lambda^*-a)^{s-1}$$

and integrate the resultant inequality w.r.t. λ^* over (a,x). ■

Theorem 8.3.4 *For two synchronous functions f_1, f_2 on $[a,b]$, h is a positive function and $\forall x>a$, $\alpha,\beta>0$, the following result for generalized conformable k-fractional integral holds true:*

$$\begin{aligned}&{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_1f_2h(x))\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}(1)+\frac{1}{{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(1)}\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}(f_1f_2h(x))\\&\geq\ {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_1h(x))\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}(f_2(x))+\ {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_2h(x))\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}(f_1(x))\\&-\ {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(h(x))\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}(f_1f_2(x))-\ {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_1f_2(x))\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}(h(x))\\&+\ {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_1(x))\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}(f_2h(x))+\ {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_2(x))\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}(f_1h(x)).\end{aligned}\tag{8.3.26}$$

Proof. Under the assumptions, $\forall\lambda^*,\eta^*\in[a,b]$, we may take

$$[f_1(\eta^*)-f_1(\lambda^*)][f_2(\eta^*)-f_2(\lambda^*)][h(\eta^*)+h(\lambda^*)]\geq 0.$$

This implies

$$\begin{aligned}
&f_1(\eta^*)f_2(\eta^*)h(\eta^*) + f_1(\lambda^*)f_2(\lambda^*)h(\lambda^*)\\
&\geq f_1(\lambda^*)h(\lambda^*)f_2(\eta^*) + f_1(\eta^*)h(\lambda^*)f_2(\lambda^*)h(\lambda^*)\\
&- f_1(\eta^*)h(\lambda^*)f_2(\eta^*) - f_1(\lambda^*)f_2(\lambda^*)h(\eta^*)\\
&+ f_1(\lambda^*)f_2(\eta^*)h(\eta^*) + f_1(\eta^*)f_2(\lambda^*)h(\eta^*). \qquad (8.3.27)
\end{aligned}$$

Multiplying the above relation on both sides by

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(x-a)^s-(\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(\eta^*-a)^{s-1}$$

and integrate w.r.t. η^* from a to x, we get

$$\begin{aligned}
&{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_1f_2h(x)) + {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(1)(f_1(\lambda^*)f_2(\lambda^*)h(\lambda^*))\\
&\geq {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_1h(x))f_2(\lambda^*) + {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_2h(x))f_1(\lambda^*)\\
&- {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(h(x))f_1(\lambda^*)f_2(\lambda^*) - {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_1f_2(x))h(\lambda^*)\\
&+ {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_1(x))f_2(\lambda^*)h(\lambda^*) + {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_2(x))f_1(\lambda^*)h(\lambda^*). \qquad (8.3.28)
\end{aligned}$$

Now, multiplying both sides of above relation with

$$\frac{1}{k\Gamma_k(\beta)}\left(\frac{(x-a)^s-(\lambda^*-a)^s}{s}\right)^{\frac{\beta}{k}-1}(\lambda^*-a)^{s-1}$$

and integrate the resultant inequality w.r.t. λ^* over (a,x), the required inequality is obtained. ■

Theorem 8.3.5 *For three monotone functions f_1, f_2 and h defined on $[a,b]$, satisfying the following inequality*

$$[f_1(\eta^*) - f_1(\lambda^*)][f_2(\eta^*) - f_2(\lambda^*)][h(\eta^*) - h(\lambda^*)] \geq 0. \qquad (8.3.29)$$

Then the following relation for conformable k-fractional integral holds true $\forall \lambda^*, \eta^* \in$

$[a,x]$, $x>a$ *and* $\alpha,\beta>0$,

$$\begin{aligned}
&{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(f_1f_2h(x))\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(1)-\frac{1}{{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(1)}\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(f_1f_2h(x))\\
&\geq\ {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(f_1h(x))\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(f_2(x))+\ {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(f_2h(x))\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(f_1(x))\\
&-\ {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(h(x))\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(f_1f_2(x))+\ {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(f_1f_2(x))\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(h(x))\\
&-\ {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(f_1(x))\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(f_2h(x))-\ {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(f_2(x))\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(f_1h(x)). \qquad (8.3.30)
\end{aligned}$$

Proof. This theorem can be proved in the similar manner as the proof of previous theorem.

■

Theorem 8.3.6 *Let* f_1 *and* f_2 *be two functions on* $[a,b]$*, then the following results for generalized conformable* k*-fractional integral are valid,*

1. ${}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(f_1^2(x))\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(1)+\frac{1}{{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(1)}\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(f_2^2(x))\geq 2{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(f_1(x))\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(f_2(x))$
2. ${}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(f_1^2(x))\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(1)+\frac{1}{{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(1)}\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(f_2^2(x))\geq 2\,{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(f_1(x))\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}(f_2(x))$

where $\alpha,\beta>0$ *and* $x>a$.

Proof.

1. Since $[f_1(\eta^*)-f_2(\lambda^*)]^2\geq 0$, so we have

$$f_1^2(\eta^*)+f_2^2(\lambda^*)\geq 2f_1(\eta^*)f_2(\lambda^*).$$

Multiplying both sides of above relation with

$$\begin{aligned}
&\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\left(\frac{(x-a)^s-(\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(x-a)^s-(\lambda^*-a)^s}{s}\right)^{\frac{\beta}{k}-1}\\
&\times(\eta^*-a)^{s-1}(\lambda^*-a)^{s-1}
\end{aligned}$$

and integrate the resultant inequality w.r.t. η^* and λ^* over (a,x), the required inequality is obtained.

2. The proof of second part can be done by using the relation

$$[f_1(\eta^*)f_2(\lambda^*)-f_1(\lambda^*)f_2(\eta^*)]^2\geq 0$$

and following the steps of integration manipulated in the previous part.

■

To present the weighted Grüss inequality for k-fractional conformable integral, the following Lemma is requisite:

Lemma 8.3.1 *Suppose that p^* is a positive function on $[a,b]$ and w is an integrable function on $[a,b]$, satisfying $\varphi^* < w(x) < \Phi^*$. Then $\forall x, \alpha > 0$ we have*

$$\begin{aligned}
& \left[{}^s_k\mathfrak{F}^{*\alpha}_{\ a+}p^*(x)\right]\left[{}^s_k\mathfrak{F}^{*\alpha}_{\ a+}(p^*w^2(x))\right] - \left[{}^s_k\mathfrak{F}^{*\alpha}_{\ a+}(p^*w(x))\right]^2 \\
= \ & \left[\Phi^*\left({}^s_k\mathfrak{F}^{*\alpha}_{\ a+}\right)p^*(x) - \left({}^s_k\mathfrak{F}^{*\alpha}_{\ a+}\right)(p^*w(x))\right]\left[{}^s_k\mathfrak{F}^{*\alpha}_{\ a+}(p^*w(x)) - \varphi^*\left({}^s_k\mathfrak{F}^{*\alpha}_{\ a+}\right)p^*(x)\right] \\
& -{}^s_k\mathfrak{F}^{*\alpha}_{\ a+}p^*(x)\left[{}^s_k\mathfrak{F}^{*\alpha}_{\ a+}\{(w(x)-\varphi^*)(\Phi^*-w(x))p^*(x)\}\right]. \qquad (8.3.31)
\end{aligned}$$

Proof. Use the following relation (see [69])

$$\begin{aligned}
& [\Phi^*p(\lambda^*) - w(\lambda^*)p^*(\lambda^*)][p^*(\eta^*)w(\eta^*) - \varphi^*p^*(\eta^*)] \\
& + [\Phi^*p^*(\eta^*) - w(\eta^*)p^*(\eta^*)][p^*(\lambda^*)w(\lambda^*) - \varphi^*p^*(\lambda^*)] \\
& - p^*(\eta^*)p^*(\lambda^*)\left[\Phi^* - w(\eta^*)\right]\left[w(\eta^*) - \varphi^*\right] - p^*(\eta^*)p^*(\lambda^*)\left[\Phi^* - w(\lambda^*)\right]\left[w(\lambda^*) - \varphi^*\right] \\
& = p^*(\lambda^*)w^2(\eta^*)p^*(\eta^*) + p^*(\eta^*)w^2(\lambda^*)p^*(\lambda^*) - 2p^*(\lambda^*)w(\lambda^*)p^*(\eta^*)w(\eta^*). \qquad (8.3.32)
\end{aligned}$$

Multiplying on both sides by

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(x-a)^s - (\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(\eta^*-a)^{s-1}$$

and integrate the resultant inequality w.r.t. η^* over (a,x), we get

$$\begin{aligned}
& [\Phi^*p^*(\lambda^*) - w(\lambda^*)p^*(\lambda^*)]\left[{}^s_k\mathcal{W}^{\alpha}_{a+}(p^*w(x)) - \varphi^*\left({}^s_k\mathcal{W}^{\alpha}_{a+}\right)p^*(x)\right] \\
& + \left[\Phi^*\left({}^s_k\mathfrak{F}^{*\alpha}_{\ a+}\right)p^*(x) - \left({}^s_k\mathfrak{F}^{*\alpha}_{\ a+}\right)(wp^*(x))\right][p^*(\lambda^*)w(\lambda^*) - \varphi^*p^*(\lambda^*)] \\
& - p^*(\lambda^*)\,{}^s_k\mathfrak{F}^{*\alpha}_{\ a+}\{p^*(x)\left[\Phi^* - w(x)\right]\left[w(x) - \varphi^*\right]\} \\
& - p^*(\lambda^*)\left[\Phi^* - w(\lambda^*)\right]\left[w(\lambda^*) - \varphi^*\right]\ {}^s_k\mathfrak{F}^{*\alpha}_{\ a+}p^*(x) \\
& = p^*(\lambda^*){}^s_k\mathfrak{F}^{*\alpha}_{\ a+}(w^2p^*(x)) + w^2(\lambda^*)p^*(\lambda^*){}^s_k\mathfrak{F}^{*\alpha}_{\ a+}p^*(x) \\
& - 2p^*(\lambda^*)w(\lambda^*){}^s_k\mathfrak{F}^{*\alpha}_{\ a+}(p^*w(x)). \qquad (8.3.33)
\end{aligned}$$

Multiplying again on both sides by

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(x-a)^s-(\lambda^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(\lambda^*-a)^{s-1}$$

and integrate the result w.r.t. λ^* from a to x, we get the required inequality. ■

Theorem 8.3.7 *Suppose that p^* is a positive function defined on $[a,b]$ and f_1, f_2 are two integrable functions, such that $\varphi^* < f_1(x) < \Phi^*$, $\psi^* < f_2(x) < \Psi^*$. then $\forall x > a$, $\alpha > 0$, we have*

$$\begin{aligned}&\left|\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)p^*(x)\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(p^*f_1f_2(x)) - {}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1(x)){}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_2(x))\right|\\ &\le \frac{\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}p^*(x)\right]^2}{2}(\Phi^*-\varphi^*)(\Psi^*-\psi^*). \qquad (8.3.34)\end{aligned}$$

Proof. Define

$$\begin{aligned}F(\eta^*,\lambda^*) &= [f_1(\eta^*)-f_1(\lambda^*)][f_2(\eta^*)-f_2(\lambda^*)]\\ &= f_1(\eta^*)f_2(\eta^*)+f_1(\lambda^*)f_2(\lambda^*) \qquad (8.3.35)\\ &- f_1(\eta^*)f_2(\lambda^*)-f_1(\lambda^*)f_2(\eta^*),\ \eta^*,\lambda^*\in(a,x),\ a<x<b.\end{aligned}$$

Multiplying both sides of (8.3.35) by

$$\frac{1}{k^2\Gamma_k^2(\alpha)}\left(\frac{(x-a)^s-(\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(x-a)^s-(\lambda^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{p^*(\eta^*)p^*(\lambda^*)}{(\eta^*-a)^{1-s}(\lambda^*-a)^{1-s}}.$$

and integrating the resultant inequality w.r.t. η^* and λ^* over (a,x), we obtain

$$\begin{aligned}&\frac{1}{k^2\Gamma_k^2(\alpha)}\int_a^b\int_a^b\left(\frac{(x-a)^s-(\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(x-a)^s-(\lambda^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\\ &\times\frac{p^*(\eta^*)p^*(\lambda^*)}{(\eta^*-a)^{1-s}(\lambda^*-a)^{1-s}}F(\eta^*,\lambda^*)d\eta^*d\lambda^*\\ &=2\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}p^*(x)\right]\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1f_2(x))\right] \qquad (8.3.36)\\ &-2\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1(x))\right]\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_2(x))\right].\end{aligned}$$

Now, by applying the Cauchy–Schwarz inequality to the L.H.S. of the above equality (8.3.36),
we get

$$\left[\frac{1}{k^2\Gamma_k^2(\alpha)}\int_a^b\int_a^b\left(\frac{(x-a)^s-(\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(x-a)^s-(\lambda^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\right.$$
$$\left.\times\frac{p^*(\eta^*)p^*(\lambda^*)}{(\eta^*-a)^{1-s}(\lambda^*-a)^{1-s}}F(\eta^*,\lambda^*)d\eta^*d\lambda^*\right]^2$$
$$\leq\frac{1}{k^2\Gamma_k^2(\alpha)}\int_a^b\int_a^b\left(\frac{(x-a)^s-(\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(x-a)^s-(\lambda^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}$$
$$\times\frac{p^*(\eta^*)p^*(\lambda^*)}{(\eta^*-a)^{1-s}(\lambda^*-a)^{1-s}}[f_1(\eta^*)-f_1(\lambda^*)]^2d\eta^*d\lambda^*$$
$$\times\frac{1}{k^2\Gamma_k^2(\alpha)}\int_a^b\int_a^b\left(\frac{(x-a)^s-(\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(x-a)^s-(\lambda^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}$$
$$\times\frac{p^*(\eta^*)p^*(\lambda^*)}{(\eta^*-a)^{1-s}(\lambda^*-a)^{1-s}}\left[f_2(\eta^*)-f_2(\lambda^*)\right]^2d\eta^*d\lambda^*$$
$$=\left\{2\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}p^*(x)\right]\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1^2(x))\right]-2\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1(x))\right]^2\right\}$$
$$\times\left\{2\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}p^*(x)\right]\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_2^2(x))\right]-2\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_2(x))\right]^2\right\}. \tag{8.3.37}$$

From (8.3.36) and (8.3.37), the following inequality can be written

$$\left[2\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}p^*(x)\right]\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1g(x))\right]-2\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1(x))\right]\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_2(x))\right]\right]^2$$
$$\leq\left\{2\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}p^*(x)\right]\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1^2(x))\right]-2\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1(x))\right]^2\right\} \tag{8.3.38}$$
$$\times\left\{2\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}p(x)\right]\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_2^2(x))\right]-2\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_2(x))\right]^2\right\}.$$

By applying (8.3.31) for $w=f_1$ and then $w=f_2$, the following inequalities are obtained respectively:

$$\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}p^*(x)\right]\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1^2(x))\right]-\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1(x))\right]^2$$
$$=\left[\Phi^*\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)p^*(x)-\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(p^*f_1(x))\right]\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1(x))-\varphi{}_k^s\mathfrak{F}^{*\alpha}_{a^+}p^*(x)\right]$$
$$-{}_k^s\mathfrak{F}^{*\alpha}_{a^+}p^*(x)\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{(\Phi^*-f_1(x))(f_1(x)-\varphi^*)p^*(x)\}\right] \tag{8.3.39}$$

and

$$\left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} p^{*}(x)\right] \left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} (p^{*} f_{2}^{2}(x))\right] - \left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} (p^{*} f_{2}(x))\right]^{2}$$
$$= \left[\Psi^{*} \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right) p^{*}(x) - \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right) (p^{*} f_{2}(x))\right] \left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} (p^{*} f_{2}(x)) - \psi^{*} \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right) p^{*}(x)\right]$$
$$- \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right) p^{*}(x) \left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} \{(\Psi^{*} - f_{2}(x))(f_{2}(x) - \psi^{*}) p^{*}(x)\}\right]. \tag{8.3.40}$$

Now since

$$- \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right) p^{*}(x) \left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} \{(\Phi^{*} - f_{1}(x))(f_{1}(x) - \varphi^{*}) p^{*}(x)\}\right] \leq 0$$

and

$$-{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} p^{*}(x) \left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} \{(\Psi^{*} - f_{2}(x))(f_{2}(x) - \psi^{*}) p^{*}(x)\}\right] \leq 0,$$

so we have

$$\left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} p^{*}(x)\right] \left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} (p^{*} f_{1}^{2}(x))\right] - \left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} (p^{*} f_{1}(x))\right]^{2}$$
$$\leq \left[\Phi^{*} \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right) p^{*}(x) - \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right) (p^{*} f_{1}(x))\right] \tag{8.3.41}$$
$$\left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} (p^{*} f_{1}(x)) - \varphi^{*} \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right) p^{*}(x)\right]$$

and

$$\left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} p^{*}(x)\right] \left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} (p^{*} f_{2}^{2}(x))\right] - \left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} (p^{*} f_{2}(x))\right]^{2}$$
$$\leq \left[\Psi^{*} \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right) p^{*}(x) - \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right) (p^{*} f_{2}(x))\right] \tag{8.3.42}$$
$$\left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} (p^{*} f_{2}(x)) - \psi^{*} \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right) p^{*}(x)\right].$$

By using inequalities (8.3.38), (8.3.41) and (8.3.42), we get

$$\left\{2 \left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} p^{*}(x)\right] \left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} (p^{*} f_{1} f_{2}(x))\right] - 2 \left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} (p^{*} f_{1}(x))\right] \left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} (p^{*} f_{2}(x))\right]\right\}^{2}$$
$$\leq 4 \left\{\left[\Phi^{*} \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right) p^{*}(x) - \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right) (p^{*} f_{1}(x))\right]\right.$$
$$\left.\left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}} (p^{*} f_{1}(x)) - \varphi^{*} \left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right) p^{*}(x)\right]\right\}$$

$$\times\left\{\left[\Psi^{*}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right)p^{*}(x)-\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right)\left(p^{*}f_{2}(x)\right)\right]\right.$$
$$\left.\left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\left(p^{*}f_{2}(x)\right)-\psi^{*}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right)p^{*}(x)\right]\right\}. \tag{8.3.43}$$

Since $2cd\leq(c+d)^2$, $c,d\in\mathbb{R}$, so it gives

$$2\left\{\left[\Phi^{*}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right)p^{*}(x)-\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right)\left(p^{*}f_{1}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\left(p^{*}f_{1}(x)\right)-\varphi^{*}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right)p^{*}(x)\right]\right\}$$
$$\leq\left[(\Phi^{*}-\varphi^{*}){}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}p^{*}(x)\right]^{2} \tag{8.3.44}$$

$$2\left\{\left[\Psi^{*}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right)p^{*}(x)-\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right)\left(p^{*}f_{2}(x)\right)\right]\left[{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\left(p^{*}f_{2}(x)\right)-\psi^{*}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right)p^{*}(x)\right]\right\}$$
$$\leq\left[(\Psi^{*}-\psi^{*})\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\right)p^{*}(x)\right]^{2}. \tag{8.3.45}$$

The required inequality is obtained by taking inequalities (8.3.43)–(8.3.45) into consideration. ■

Lemma 8.3.2 *For two integrable functions f_1 and f_2, satisfying $\varphi^{*}<f_1(x)<\Phi^{*}$ and $\psi^{*}<f_2(x)<\Psi^{*}$ for $x\in[a,b]$ and p^{*}, q^{*} be two positive functions on $[a,b]$, then $x>a$ and $\forall\alpha,\beta>0$, we have*

$$\left\{{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}p^{*}(x)\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^{+}}\left(q^{*}f_{1}f_{2}(x)\right)+{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^{+}}q^{*}(x)\,{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\left(p^{*}f_{1}f_{2}(x)\right)\right.$$
$$\left.-{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\left(p^{*}f_{1}(x)\right){}_{k}^{s}\mathfrak{F}^{*\beta}_{a^{+}}\left(q^{*}f_{2}(x)\right)-{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^{+}}\left(q^{*}f_{1}(x)\right){}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\left(p^{*}f_{2}(x)\right)\right\}^{2}$$
$$\leq\left\{{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}p^{*}(x)\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^{+}}\left(q^{*}f_{1}^{2}(x)\right)+{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\left(p^{*}f_{1}^{2}(x)\right){}_{k}^{s}\mathfrak{F}^{*\beta}_{a^{+}}q^{*}(x)\right.$$
$$\left.-\,2{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\left(p^{*}f_{1}(x)\right){}_{k}^{s}\mathfrak{F}^{*\beta}_{a^{+}}\left(q^{*}f_{1}(x)\right)\right\}$$
$$\times\left\{{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}p^{*}(x)\,{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^{+}}\left(q^{*}f_{2}^{2}(x)\right)+{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\left(p^{*}f_{2}^{2}(x)\right){}_{k}^{s}\mathfrak{F}^{*\beta}_{a^{+}}q^{*}(x)\right.$$
$$\left.-\,2{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^{+}}\left(p^{*}f_{2}(x)\right){}_{k}^{s}\mathfrak{F}^{*\beta}_{a^{+}}\left(q^{*}f_{2}(x)\right)\right\}. \tag{8.3.46}$$

Proof. Using (8.3.35), we have

$$\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^b\int_a^b\left(\frac{(x-a)^s-(\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(x-a)^s-(\lambda^*-a)^s}{s}\right)^{\frac{\beta}{k}-1}$$
$$\frac{p^*(\eta^*)q^*(\lambda^*)}{(\eta^*-a)^{1-s}(\lambda^*-a)^{1-s}}F(\eta^*,\lambda^*)d\eta^*d\lambda^*$$
$$={}^s_k\,\mathfrak{F}^{*\alpha}_{a^+}p^*(x)\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}(q^*f_1g(x))+{}^s_k\,\mathfrak{F}^{*\beta}_{a^+}q^*(x){}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1f_2(x))$$
$$-{}^s_k\,\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1(x))\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}(q^*f_2(x))-{}^s_k\,\mathfrak{F}^{*\beta}_{a^+}(q^*f_1(x)){}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^*f_2(x)). \tag{8.3.47}$$

Now, by using Cauchy–Schwarz inequality for bi-integrals in (8.3.47), it can be written

$$\Bigg[\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\left(\frac{(x-a)^s-(\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(x-a)^s-(\lambda^*-a)^s}{s}\right)^{\frac{\beta}{k}-1}$$
$$\frac{p^*(\eta^*)q^*(\lambda^*)}{(\eta^*-a)^{1-s}(\lambda^*-a)^{1-s}}[f_1(\eta^*)-f_1(\lambda^*)][f_2(\eta^*)-f_2(\lambda^*)]d\eta^*d\lambda^*\Bigg]^2$$
$$\leq\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^b\int_a^b\left(\frac{(x-a)^s-(\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(x-a)^s-(\lambda^*-a)^s}{s}\right)^{\frac{\beta}{k}-1}$$
$$\times\frac{p^*(\eta^*)q^*(\lambda^*)}{(\eta^*-a)^{1-s}(\lambda^*-a)^{1-s}}[f_1(\eta^*)-f_1(\lambda^*)]^2d\eta^*d\lambda^*$$
$$\times\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^b\int_a^b\left(\frac{(x-a)^s-(\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(x-a)^s-(\lambda^*-a)^s}{s}\right)^{\frac{\beta}{k}-1}$$
$$\times\frac{p^*(\eta^*)q^*(\lambda^*)}{(\eta^*-a)^{1-s}(\lambda^*-a)^{1-s}}[f_2(\eta^*)-f_2(\lambda^*)]^2d\eta^*d\lambda$$
$$=\Big[{}^s_k\mathfrak{F}^{*\alpha}_{a^+}p^*(x)\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}(q^*f_1^2(x))+{}^s_k\,\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1^2(x))\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}q^*(x)$$
$$-2{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1(x))\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}(q^*f_1(x))\Big]$$
$$\times\Big[{}^s_k\mathfrak{F}^{*\alpha}_{a^+}p(x){}^s_k\mathfrak{F}^{*\beta}_{a^+}(q^*f_2^2(x))+{}^s_k\,\mathfrak{F}^{*\alpha}_{a^+}(p^*f_2^2(x)){}^s_k\mathfrak{F}^{*\beta}_{a^+}q^*(x)$$
$$-2{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^*f_2(x)){}^s_k\mathfrak{F}^{*\beta}_{a^+}(q^*f_2(x))\Big]. \tag{8.3.48}$$

which completes the proof. ■

Lemma 8.3.3 *Let* p^* *be positive function on* $[a,b]$ *and* w *be integrable function on* $[a,b]$

satisfying $\varphi^* < w(x) < \Phi^*$. *Then* $\forall \alpha, \beta > 0$ *and* $x > 0$, *we have*

$$
\begin{aligned}
&{}_k^s\mathfrak{F}^{*\alpha}_{a^+} p^*(x) {}_k^s\mathfrak{F}^{*\beta}_{a^+}(p^* w^2(x)) + {}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^* w^2(x)) {}_k^s\mathfrak{F}^{*\beta}_{a^+} p^*(x) - 2 {}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^* w(x)) {}_k^s\mathfrak{F}^{*\beta}_{a^+}(p^* w(x)) \\
&= \left[\Phi^* \left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right) p^*(x) - \left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)(p^* w(x))\right] \left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^* w(x)) - \varphi^* \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right) p^*(x)\right] \\
&\left[\Phi^* \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right) p^*(x) - \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(p^* w(x))\right] \left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}(p^* w(x)) - \varphi^* \left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right) p^*(x)\right] \\
&- \left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right) p^*(x) \left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{(w(x) - \varphi^*)(\Phi^* - w(x)) p^*(x)\}\right] \\
&- \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right) p^*(x) \left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}\{(w(x) - \varphi^*)(\Phi^* - w(x)) p^*(x)\}\right]. \qquad (8.3.49)
\end{aligned}
$$

Proof. By multiplying both sides of (8.3.32) with

$$
\frac{1}{k^2 \Gamma_k(\alpha)\Gamma_k(\beta)} \left(\frac{(x-a)^s - (\eta^* - a)^s}{s}\right)^{\frac{\alpha}{k}-1} \left(\frac{(x-a)^s - (\lambda^* - a)^s}{s}\right)^{\frac{\beta}{k}-1} \frac{p^*(\eta^*) q^*(\lambda^*)}{(\eta^* - a)^{1-s} (\lambda^* - a)^{1-s}}
$$

and integrate the resultant inequality w.r.t. η^* and λ^* over (a, x), the desired equality is obtained. ■

Theorem 8.3.8 *Let* p^* *be a positive function and* f_1 *and* f_2 *be integrable functions on* $[a, b]$, *such that* $\varphi^* < f_1(x) < \Phi^*$ *and* $\psi^* < f_2(x) < \Psi^*$, *then* $\forall x > a$, $\alpha, \beta > 0$, *we have*

$$
\begin{aligned}
&\Big\{ {}_k^s\mathfrak{F}^{*\alpha}_{a^+} p^*(x) {}_k^s\mathfrak{F}^{*\beta}_{a^+}(p^* f_1 f_2(x)) + {}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^* f_1 f_2(x)) {}_k^s\mathfrak{F}^{*\beta}_{a^+} p^*(x) \\
&- {}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^* f_1(x)) {}_k^s\mathfrak{F}^{*\beta}_{a^+}(p^* f_2(x)) - {}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^* f_2(x)) {}_k^s\mathfrak{F}^{*\beta}_{a^+}(p^* f_1(x)) \Big\}^2 \\
&\leq \Big\{ \left[\Phi^* \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right) p^*(x) - \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(p^* f_1(x))\right] \left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}(p^* f_1(x)) - \varphi^* \left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right) p^*(x)\right] \\
&+ \left[\Phi^* \left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right) p^*(x) - \left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)(p^* f_1(x))\right] \left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^* f_1(x)) - \varphi^* \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right) p^*(x)\right] \Big\} \\
&\times \Big\{ \left[\Psi^* \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right) p(x) - \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right)(p^* f_2(x))\right] \left[{}_k^s\mathfrak{F}^{*\beta}_{a^+}(p^* f_2(x)) - \psi^* \left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right) p^*(x)\right] \\
&+ \left[\Psi^* \left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right) p^*(x) - \left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)(p^* f_2(x))\right] \left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^* f_2(x)) - \psi^* \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\right) p^*(x)\right] \Big\}.
\end{aligned}
$$

(8.3.50)

Proof. Since,

$$[\Phi^* - f_1(\eta^*)][f_1(\eta^*) - \varphi^*] \geq 0$$

and

$$[\Psi^* - f_2(\eta^*)][f_2(\eta^*) - \psi^*] \geq 0.$$

so, we have

$$
\begin{aligned}
&-{}_k^s\mathfrak{F}^{*\alpha}_{a^+}p^*(x)\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\right)[(\Phi^*-f_1(x))(f_1(x)-\varphi^*)] \\
&-{}_k^s\mathfrak{F}^{*\beta}_{a^+}p^*(x){}_k^s\mathfrak{F}^{*\alpha}_{a^+}[(\Phi^*-f_1(x))(f_1(x)-\varphi^*)]\le 0
\end{aligned}
\tag{8.3.51}
$$

and

$$
\begin{aligned}
&-{}_k^s\mathfrak{F}^{*\alpha}_{a^+}p^*(x){}_k^s\mathfrak{F}^{*\beta}_{a^+}[(\Psi^*-f_2(x))(f_2(x)-\psi^*)] \\
&-{}_k^s\mathfrak{F}^{*\beta}_{a^+}p^*(x){}_k^s\mathfrak{F}^{*\alpha}_{a^+}[(\Psi^*-f_2(x))(f_2(x)-\psi^*)]\le 0.
\end{aligned}
\tag{8.3.52}
$$

Using lemma (8.3.3) twice for $w=f_1$ and $w=f_2$ and then by using (8.3.51) and (8.3.52), we obtain the required inequality. ■

Theorem 8.3.9 *For two integrable functions f_1 and f_2 on $[a,b]$, such that*

$$\varphi^* < f_1(x) < \Phi^*$$

and

$$\psi^* < f_2(x) < \Psi^*$$

where $\Phi^*,\varphi^*,\Psi^*,\psi^*\in\mathbb{R}$. Also for positive functions p^* and q^* on $[a,b]$, $\forall\alpha,\beta>0$ and $x>a$, we have

$$
\begin{aligned}
&{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1f_2(x)){}_k^s\mathfrak{F}^{*\beta}_{a^+}q^*(x)+{}_k^s\mathfrak{F}^{*\alpha}_{a^+}p^*(x){}_k^s\mathfrak{F}^{*\beta}_{a^+}(q^*f_1f_2(x)) \\
&-{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1(x)){}_k^s\mathfrak{F}^{*\beta}_{a^+}(q^*f_2(x))-{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(p^*f_2(x)){}_k^s\mathfrak{F}^{*\beta}_{a^+}(q^*f_1(x)) \\
&\le{}_k^s\mathfrak{F}^{*\alpha}_{a^+}p^*(x){}_k^s\mathfrak{F}^{*\beta}_{a^+}q^*(x)\left[(\Phi^*-\varphi^*)(\Psi^*-\psi^*)\right].
\end{aligned}
\tag{8.3.53}
$$

Proof. Under given conditions, for $\eta^*,\lambda^*\in[a,b]$, we write

$$[f_1(\eta^*)-f_1(\lambda^*)][f_2(\eta^*)-f_2(\lambda^*)]\le(\Phi^*-\varphi^*)(\Psi^*-\psi^*), \tag{8.3.54}$$

implies that

$$
\begin{aligned}
&f_1(\eta^*)g(\eta^*)+f_1(\lambda^*)f_2(\lambda^*)-f_1(\eta^*)f_2(\lambda^*)-f_1(\lambda^*)f_2(\eta^*) \\
&\qquad\le(\Phi^*-\varphi^*)(\Psi^*-\psi^*).
\end{aligned}
\tag{8.3.55}
$$

Multiply the above inequality with

$$\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\left(\frac{(x-a)^s-(\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(x-a)^s-(\lambda^*-a)^s}{s}\right)^{\frac{\beta}{k}-1}$$
$$\times\frac{p^*(\eta^*)q^*(\lambda^*)}{(\eta^*-a)^{1-s}(\lambda^*-a)^{1-s}}$$

and integrate the result w.r.t. η^* and λ^* from a to x, we get the required result. ■

If we put $\alpha=\beta$ in relation (8.3.53), we get the following inequality.

Corollary 8.3.1 *For two integrable functions f_1 and f_2 defined on $[a,b]$, such that*

$$\varphi^* < f_1(x) < \Phi^*$$

and

$$\psi^* < f_2(x) < \Psi^*$$

where $\varphi^*, \Phi^*, \psi^*, \Psi^* \in \mathbb{R}$. Also for positive functions p^* and q^*, $\forall x>a$, $\alpha>0$, we have

$$\begin{aligned}&{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1f_2(x))\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}q^*(x)+\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}p^*(x)\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(q^*f_1f_2(x))\\&-\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1(x))\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(q^*f_2(x))-\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^*f_2(x))\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(q^*f_1(x))\\&\leq\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}p^*(x)\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}q^*(x)\left[(\Phi^*-\varphi^*)(\Psi^*-\psi^*)\right].\end{aligned}\tag{8.3.56}$$

Theorem 8.3.10 *For two integrable functions f_1 and f_2 on $[a,b]$ such that*

$$|f_1(\eta^*)-f_1(\lambda^*)|\leq|f_2(\eta^*)-f_2(\lambda^*)|,\quad \alpha>0 \text{ and } \eta^*,\lambda^*\in[a,b].\tag{8.3.57}$$

Also for two positive functions p^* and q^*, then $x>a$ and $\forall\alpha,\beta>0$, we have

$$\begin{aligned}&{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1f_2(x)){}^s_k\mathfrak{F}^{*\beta}_{a^+}q^*(x)+{}^s_k\mathfrak{F}^{*\alpha}_{a^+}p^*(x){}^s_k\mathfrak{F}^{*\beta}_{a^+}(q^*f_1f_2(x))\\&-\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^*f_1(x)){}^s_k\mathfrak{F}^{*\beta}_{a^+}(q^*f_2(x))-\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^*f_2(x)){}^s_k\mathfrak{F}^{*\beta}_{a^+}(q^*f_1(x))\\&\leq\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^*(x)){}^s_k\mathfrak{F}^{*\beta}_{a^+}(q^*f_2^2(x))+\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^*f_2^2(x)){}^s_k\mathfrak{F}^{*\beta}_{a^+}q^*(x)\\&-2\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^*f_2(x)){}^s_k\mathfrak{F}^{*\beta}_{a^+}(q^*f_2(x)).\end{aligned}\tag{8.3.58}$$

Proof. For $\eta^*, \lambda^* \in [a,b]$, f_1 and f_2 under the condition (8.3.57), it can also be written as

$$[f_1(\eta^*) - f_1(\lambda^*)][f_2(\eta^*) - f_2(\lambda^*)] \leq [f_2(\eta^*) - f_2(\lambda^*)]^2. \tag{8.3.59}$$

Multiply both sides of (8.3.59) with

$$\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)} \left(\frac{(x-a)^s - (\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1} \left(\frac{(x-a)^s - (\lambda^*-a)^s}{s}\right)^{\frac{\beta}{k}-1} \times \frac{p^*(\eta^*)q^*(\lambda^*)}{(\eta^*-a)^{1-s}(\lambda^*-a)^{1-s}}$$

and integrate the result w.r.t. η^* and λ^* from a to x, we get the required consequence. ■

If we put $\alpha = \beta$ in inequality (8.3.58), the following inequality is obtained.

Corollary 8.3.2 *Let p^*, q^* be positive functions and f_1, f_2 integrable functions defined on $[a,b]$ under the condition* (8.3.57)*, then $\forall x > a$, $\alpha > 0$, we have*

$$\begin{aligned}
&{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^* f_1 f_2(x))\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}q^*(x) + {}^s_k\mathfrak{F}^{*\alpha}_{a^+}p^*(x)\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(q^* f_1 f_2(x)) \\
&-{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^* f_1(x))\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(q^* f_2(x)) - {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^* f_2(x))\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(q^* f_1(x)) \\
&\leq {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^*(x))\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(q^* f_2^2(x)) + {}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^* f_2^2(x))\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}q^*(x) \\
&- 2\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(p^* f_2(x))\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(q^* f_2(x)).
\end{aligned} \tag{8.3.60}$$

Theorem 8.3.11 *For $r^* \geq 1$, two positive functions f_1 and f_2 on $[a,b]$, such that $0 < {}^s_k\mathfrak{F}^{*\alpha}_{a^+} f_1^{r^*}, {}^s_k\mathfrak{F}^{*\alpha}_{a^+} f_2^{r^*} < \infty$. If*

$$0 < m^* \leq \frac{f_1(\eta^*)}{f_2(\eta^*)} \leq M^* < \infty, \quad \eta^* \in [a,b], \tag{8.3.61}$$

then $\forall \alpha > 0$

$$\begin{aligned}
&\left[{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_1^{r^*}(x))\right]^{\frac{1}{r^*}} + \left[{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_2^{r^*}(x))\right]^{\frac{1}{r^*}} \\
&\leq \frac{1+M^*(m^*+2)}{(m^*+1)(M^*+1)} \left[{}^s_k\mathfrak{F}^{*\alpha}_{a^+}((f_1+f_2)^{r^*}(x))\right]^{\frac{1}{r^*}}.
\end{aligned} \tag{8.3.62}$$

Proof. Using the condition (8.3.61) $\forall x > a$ and $\eta^* \in [a,b]$, we have

$$\frac{1}{M^*} \leq \frac{f_2(\eta^*)}{f_1(\eta^*)},$$

implies that

$$\left(\frac{1}{M^*}+1\right)^{r^*} \leq \left(\frac{f_2(\eta^*)}{f_1(\eta^*)}+1\right)^{r^*},$$

which gives us

$$(M^*+1)^{r^*} f^{r^*}(\eta^*) \leq M^r (f_1+f_2)^{r^*}(\eta^*). \tag{8.3.63}$$

Similarly, we also have

$$(m^*+1)^{r^*} g^{r^*}(\eta^*) \leq (f_1+f_2)^{r^*}(\eta^*). \tag{8.3.64}$$

Multiplying both sides of relations (8.3.63) and (8.3.64) with

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(x-a)^s-(\eta^*-a)^s}{s}\right)^{\frac{\alpha}{k}-1}(\eta^*-a)^{s-1}$$

and integrating w.r.t. η^* over (a,x), the following inequalities are obtained respectively

$$\left[{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_1^{r^*}(x))\right]^{\frac{1}{r^*}} \leq \frac{M^*}{M^*+1}\left[{}^s_k\mathfrak{F}^{*\alpha}_{a^+}((f_1+f_2)^{r^*}(x))\right]^{\frac{1}{r^*}} \tag{8.3.65}$$

and

$$\left[{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(g^r(x))\right]^{\frac{1}{r}} \leq \frac{1}{m+1}\left[{}^s_k\mathfrak{F}^{*\alpha}_{a^+}((f_1+f_2)^r(x))\right]^{\frac{1}{r}}. \tag{8.3.66}$$

By adding above inequalities (8.3.65) and (8.3.66), we obtain the required inequality. ■

Theorem 8.3.12 *For* $r^* \geq 1$, f_1 *and* f_2 *be integrable functions on* $[a,b]$ *s.t.* $\forall x > a$ $0 < {}^s_k\mathfrak{F}^{*\alpha}_{a^+}f_1^{r^*}$, ${}^s_k\mathfrak{F}^{*\alpha}_{a^+}f_2^{r^*} < \infty$. *If the condition* (8.3.61) *is satisfied, then* $\forall\alpha > 0$

$$\begin{aligned}&\left[{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_1^{r^*}(x))\right]^{\frac{2}{r^*}}+\left[{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_2^{r^*}(x))\right]^{\frac{2}{r^*}}\\ &\geq\left(\frac{(m^*+1)(M^*+1)}{M^*}-2\right)\left[{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_1^{r^*}(x))\right]^{\frac{1}{r^*}}\left[{}^s_k\mathfrak{F}^{*\alpha}_{a^+}(f_2^{r^*}(x))\right]^{\frac{1}{r^*}}.\end{aligned} \tag{8.3.67}$$

Proof. By multiplying the relations (8.3.65) and (8.3.66), we have

$$\frac{(m^*+1)(M^*+1)}{M^*}\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(f_1^{r^*}(x))\right]^{\frac{1}{r^*}}\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}(f_2^{r^*}(x))\right]^{\frac{1}{r^*}}$$
$$\leq\left[{}_k^s\mathfrak{F}^{*\alpha}_{a^+}((f_1+f_2)^{r^*}(x))\right]^{\frac{2}{r^*}}. \tag{8.3.68}$$

Using the Minkowski's integral inequality to the R.H.S. of above relation, the required inequality is attained. ■

Remark 8.3.2 *All these inequalities can be proved for right sided generalized k-fractional conformable integral (${}_k^s\mathfrak{F}^{*\alpha}_{b^-}$).*

Chapter 9

GENERALIZATION OF P$\acute{O}$LYA-SZEG$\ddot{O}$ INEQUALITIES VIA CONFORMABLE K-FRACTIONAL INTEGRAL

This chapter aims to present new P$\acute{o}$lya-Szeg$\ddot{o}$ inequalities using generalized k-conformable integral. The results obtained are further used to derive integral inequalities of Chebyshev type. Some of the results presented in the second section, are further applied to a function constrained by the Heaviside functions described in the third section.

9.1 Introduction

P$\acute{o}$lya and Szeg$\ddot{o}$ [137] presented the subsequent inequality:

$$\frac{\int_{x_1}^{x_2} f_1^2(z)dz \int_{x_1}^{x_2} f_2^2(z)dz}{\left(\int_{x_1}^{x_2} f_1(z)f_2(z)dz\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{PQ}{pq}}+\sqrt{\frac{pq}{PQ}}\right)^2, \tag{9.1.1}$$

where f_1 and f_2 are two integrable positive functions on $[x_1, x_2]$ such that

$$0 < p \leq f_1(z) \leq P < \infty \quad and \quad 0 < q \leq f_2(z) \leq Q < \infty$$

Dragomir and Diamond [80] used the above relation to prove the following inequality:

$$|T(f_1, f_2; x_1, x_2)| \leq \frac{(P-p)(Q-q)}{4(x_1-x_2)^2\sqrt{qQpP}} \int_{x_1}^{x_2} f_1(z)dz \int_{x_1}^{x_2} f_2(z)dz \tag{9.1.2}$$

9.2 Some P$\acute{o}$lya–Szeg$\ddot{o}$ Inequalities via Generalized Conformable k-Fractional Integrals

This section presents our main results. Some Chebyshev type integral inequalities are also derived from P$\acute{o}$lya–Szeg$\ddot{o}$ inequalities.

Lemma 9.2.1 *For positive, real and integrable functions f_1 and f_2 defined on $[a,\infty)$, let $\lambda_1, \lambda_2, \mu_1$, and μ_2 be integrable functions satisfying*

$$0 < \lambda_1(\tau) \leq f_1(\tau) \leq \lambda_2(\tau) \quad and \quad 0 < \mu_1(\tau) \leq f_2(\tau) \leq \mu_2(\tau) \tag{9.2.3}$$

$\forall a \leq \tau \leq t, t > a$.

Then the subsequent inequality retains for $a \in \mathbb{R}_0^+$ and $\alpha, k \in \mathbb{R}^+$,

$$\frac{{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{\mu_1\mu_2 f_1^2\}(t)\,{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_1\lambda_2 f_2^2\}(t)}{\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{(\lambda_1\mu_1+\lambda_2\mu_2)f_1f_2\}(t)\right)^2} \leq \frac{1}{4} \tag{9.2.4}$$

Proof. Under given assumptions, one may obtain

$$\frac{f_1(\tau)}{f_2(\tau)} \leq \frac{\lambda_2(\tau)}{\mu_1(\tau)} \quad and \quad \frac{\lambda_1(\tau)}{\mu_2(\tau)} \leq \frac{f_1(\tau)}{f_2(\tau)} \quad (\tau \in [a,t](t>a)), \tag{9.2.5}$$

which yields

$$\left(\frac{f_1(\tau)}{f_2(\tau)} - \frac{\lambda_1(\tau)}{\mu_2(\tau)}\right)\left(\frac{\lambda_2(\tau)}{\mu_1(\tau)} - \frac{f_1(\tau)}{f_2(\tau)}\right) \geq 0, \tag{9.2.6}$$

so,

$$\frac{f_1(\tau)}{f_2(\tau)}\left(\frac{\lambda_2(\tau)}{\mu_1(\tau)} + \frac{\lambda_1(\tau)}{\mu_2(\tau)}\right) \geq \frac{f_1^2(\tau)}{f_2^2(\tau)} + \frac{\lambda_1(\tau)\lambda_2(\tau)}{\mu_1(\tau)\mu_2(\tau)}. \tag{9.2.7}$$

The inequality (9.2.7) can rewrite as,

$$(\lambda_1(\tau)\mu_1(\tau) + \lambda_2(\tau)\mu_2(\tau))\, f_1(\tau)f_2(\tau) \geq \mu_1(\tau)\mu_2(\tau)f_1^2(\tau) + \lambda_1(\tau)\lambda_2(\tau)f_2^2(\tau). \tag{9.2.8}$$

Here, multiplying on both sides of the inequality (9.2.8) with the following factor:

$$\frac{1}{k\Gamma_k(\alpha)}\left(\frac{(t-a)^s - (\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{1}{(\tau-a)^{1-s}} \quad (\tau \in [a,t](t>a))$$

and integrate the resulting inequality w.r.t. τ on $[a,t]$, we get

$$ {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{(\lambda_1\mu_1+\lambda_2\mu_2)f_1f_2\}(t) \geq {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{\mu_1\mu_2f_1^2\}(t) + {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_1\lambda_2f_2^2\}(t). \tag{9.2.9} $$

Now, applying the inequality of A.M.-G.M. (the arithmetic-geometric mean),

$$ a+d \geq 2\sqrt{ad} \quad (a,d\in\mathbb{R}_0^+) \tag{9.2.10} $$

to the R.H.S. of (9.2.9), we have

$$ {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{(\lambda_1\mu_1+\lambda_2\mu_2)f_1f_2\}(t) \geq 2\sqrt{{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{\mu_1\mu_2f_1^2\}(t)\,{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_1\lambda_2f_2^2\}(t)}, \tag{9.2.11} $$

which yields

$$ {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{\mu_1\mu_2f_1^2\}(t)\;{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_1\lambda_2f_2^2\}(t) \leq \frac{1}{4}\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{(\lambda_1\mu_1+\lambda_2\mu_2)f_1f_2\}(t)\right)^2. \tag{9.2.12} $$

This accomplishes the proof of required inequality. ■

The next corollary is obtained as a particular case of Lemma (9.2.1).

Corollary 9.2.1 *Let f_1 and f_2 be real valued, positive and integrable functions defined on $[a,\infty)$, satisfying*

$$ 0<p\leq f_1(\tau)\leq P<\infty \quad \textit{and} \quad 0<q\leq f_2(\tau)\leq Q<\infty \quad (a\leq\tau\leq t, t>a), \tag{9.2.13} $$

where $p,q,P,Q\in\mathbb{R}$. Then, $\forall\alpha\in\mathbb{R}^+$ and $t,k\in\mathbb{R}^+$, we have

$$ \frac{{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{f_1^2\}(t)\,{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{f_2^2\}(t)}{\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{f_1f_2\}(t)\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{pq}{PQ}}+\sqrt{\frac{PQ}{pq}}\right)^2. \tag{9.2.14} $$

Lemma 9.2.2 *Let f_1, f_2 be real valued, positive integrable functions on $[a,\infty)$. Assume that the integrable functions $\lambda_1,\lambda_2,\mu_1,\mu_2$ on $[a,\infty)$ satisfy the condition (9.2.3). Then,* *the following result valids for $t > a$ $(a \in \mathbb{R}_0^+)$:*

$$\frac{{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_1\lambda_2\}(t)\,{}_k^s\mathfrak{F}^{*\beta}_{a^+}\{\mu_1\mu_2\}(t)\,{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_1^2\}(t)\,{}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_2^2\}(t)}{\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_1 f_1\}(t)\,{}_k^s\mathfrak{F}^{*\beta}_{a^+}\{\mu_1 f_2\}(t) + {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_2 f_1\}(t)\,{}_k^s\mathfrak{F}^{*\beta}_{a^+}\{\mu_2 f_2\}(t)\right)^2} \leq \frac{1}{4} \quad (9.2.15)$$

where $k, \alpha, \beta \in \mathbb{R}^+$.

Proof. We obtain from (9.2.3) that

$$\left(\frac{\lambda_2(\tau)}{\mu_1(\rho)} - \frac{f_1(\tau)}{f_2(\rho)}\right) \geq 0 \quad and \quad \left(\frac{f_1(\tau)}{f_2(\rho)} - \frac{\lambda_1(\tau)}{\mu_2(\rho)}\right) \geq 0 \quad (\tau, \rho \in [a,t], t > a), \quad (9.2.16)$$

which yields

$$\left(\frac{\lambda_2(\tau)}{\mu_1(\rho)} + \frac{\lambda_1(\tau)}{\mu_2(\rho)}\right)\frac{f_1(\tau)}{f_2(\rho)} \geq \frac{f_1^2(\tau)}{f_2^2(\rho)} + \frac{\lambda_1(\tau)\lambda_2(\tau)}{\mu_1(\rho)\mu_2(\rho)}. \quad (9.2.17)$$

Multiply on both sides of the inequality (9.2.17) by $\mu_1(\rho)\mu_2(\rho)f_2^2(\rho)$, we get

$$f_1(\tau)\mu_1(\rho)\lambda_1(\tau)f_2(\rho) + f_1(\tau)\mu_2(\rho)\lambda_2(\tau)f_2(\rho) \geq \mu_1(\rho)f_1^2(\tau)\mu_2(\rho) + \lambda_1(\tau)f_2^2(\rho)\lambda_2(\tau). \quad (9.2.18)$$

Again, multiply the inequality (9.2.18) with the following factor:

$$\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\left(\frac{(t-a)^s - (\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(t-a)^s - (\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1} \times (\tau-a)^{s-1}(\rho-a)^{s-1} \quad (\tau, \rho \in [a,t], t > a)$$

and integrate the result w.r.t. τ and ρ over $[a,t]$, respectively.

$$\begin{aligned}&{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_1 f_1\}(t){}_k^s\mathfrak{F}^{*\beta}_{a^+}\{\mu_1 f_2\}(t) + {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_2 f_1\}(t){}_k^s\mathfrak{F}^{*\beta}_{a^+}\{\mu_2 f_2\}(t)\\ &\geq {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_2^2\}(t)\,{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{\mu_1\mu_2\}(t) + {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{\lambda_1\lambda_2\}(t)\,{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_2^2\}(t).\end{aligned} \quad (9.2.19)$$

Applying the A.M.-G.M. inequality (9.2.10) to (9.2.19), we obtain

$$\begin{aligned}&{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_1 f_1\}(t)\,{}_k^s\mathfrak{F}^{*\beta}_{a^+}\{\mu_1 f_2\}(t) + {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_2 f_1\}(t)\,{}_k^s\mathfrak{F}^{*\beta}_{a^+}\{\mu_1 f_2\}(t)\\ &\geq 2\sqrt{{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_1^2\}(t)\,{}_k^s\mathfrak{F}^{*\beta}_{a^+}\{\mu_1\mu_2\}(t)\,{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_1\lambda_2\}(t)\,{}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_2^2\}(t)},\end{aligned} \quad (9.2.20)$$

which easily yields the desired inequality (9.2.15). ■

Corollary 9.2.2 *Let $f_1,\ f_2$ be positive real valued integrable functions on the interval $[a,\infty)$ satisfying the assumption (9.2.1). Then, for $t>1$ and $k,\alpha,\beta\in\mathbb{R}^+$, we have*

$$\frac{{}^{s}_{k}\mathfrak{F}^{*\alpha+\beta}_{1^+}(1)\ {}^{s}_{k}\mathfrak{F}^{*\alpha}_{1^+}\{f_1^2(t)\}\ {}^{s}_{k}\mathfrak{F}^{*\beta}_{1^+}\{f_2^2(t)\}}{\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{1^+}\{f_1(t)\}\ {}^{s}_{k}\mathfrak{F}^{*\beta}_{1^+}\{f_2(t)\}\right)^2}\leq\frac{1}{4}\left(\sqrt{\frac{pq}{PQ}}+\sqrt{\frac{PQ}{pq}}\right)^2. \tag{9.2.21}$$

Lemma 9.2.3 *For the assumptions given in Lemma (9.2.2), the below inequality retains for $t>a$ and $\alpha,\beta\in\mathbb{R}^+$,*

$${}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{f_1^2\}(t)\ {}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\{f_2^2\}(t)\leq\ {}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{(\lambda_2 f_1 f_2)\mu_1\}(t)\ {}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\{(\mu_2 f_1 f_2)\lambda_1\}(t). \tag{9.2.22}$$

Proof. Using the conditions (9.2.3), we get

$$\begin{aligned}&\frac{1}{k\Gamma_k(\alpha)}\int_a^t\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}f_1^2(\tau)\,(\tau-a)^{s-1}\\&\leq\frac{1}{k\Gamma_k(\alpha)}\int_a^t\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\frac{\lambda_2(\tau)}{\mu_1(\tau)}f_2(\tau)f_1(\tau)\,(\tau-a)^{s-1},\end{aligned}$$

which implies

$${}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{f_1^2\}(t)\leq\ {}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{(\lambda_2 f_1 f_2)\mu_1\}(t). \tag{9.2.23}$$

Similarly we have

$$\begin{aligned}&\frac{1}{k\Gamma_k(\beta)}\int_a^t\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}f_2^2(\rho)\frac{1}{(\rho-a)^{1-s}}\\&\leq\frac{1}{k\Gamma_k(\beta)}\int_a^t\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{\mu_2(\rho)}{\lambda_1(\rho)}f_1(\rho)f_2(\rho)\frac{1}{(\rho-a)^{1-s}}\end{aligned}$$

and so,

$${}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\{f_2^2\}(t)\leq\ {}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\{(\mu_2 f_1 f_2)\lambda_1\}(t). \tag{9.2.24}$$

Multiplying the inequalities (9.2.23) and (9.2.24) side by side and the desired inequality (9.2.22) is attained. ■

It can be seen from Lemma (9.2.3) that the result of Corollary (9.2.3) holds true.

Corollary 9.2.3 *For positive, real valued integrable functions f_1 and f_2 on $[a, \infty)$, satisfying the assumptions in (9.2.1). Then, for $\beta, \alpha \in \mathbb{R}^+$ and $t > a$, we have*

$$\frac{{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_1^2\}(t)\, {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_2^2\}(t)}{{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_1 f_2\}(t)\, {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_1 f_2\}(t)} \leq \frac{PQ}{pq}. \tag{9.2.25}$$

Theorem 9.2.1 *Let f_1, f_2 be integrable positive real valued functions on $[a, \infty)$. Also, for the positive functions $\lambda_1, \lambda_2, \mu_1$, and μ_2 satisfying the condition (9.2.3), the following inequality retains for $k, \alpha, \beta \in \mathbb{R}^+$ and $t > a$,*

$$\begin{aligned} &\Big| {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{1\}\, {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_1 f_2\}(t) + {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{1\}\, {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_1 f_2\}(t) \\ &- {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(t)\, {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_2\}(t) - {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_2\}(t)\, {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_1\}(t) \Big| \\ &\leq |P_1(f_1, \lambda_1, \lambda_2) + P_2(f_1, \lambda_1, \lambda_2)|^{\frac{1}{2}} \times |P_1(f_2, \mu_1, \mu_2) + P_2(f_2, \mu_1, \mu_2)|^{\frac{1}{2}}, \end{aligned} \tag{9.2.26}$$

where

$$P_1(u^*, v^*, w^*)(t) := \frac{{}_k^s\mathfrak{F}^{*\beta}_{a^+}\{1\}\ \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{(v^* + w^*)u^*\}(t)\right)^2}{4\, {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{v^* w^*\}(t)} - {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{u_1\}(t) {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{u^*\}(t)$$

and

$$Q_2(u^*, v^*, w^*)(t) := \frac{{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{1\}\ \left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\{(v^* + w^*)u^*\}(t)\right)^2}{4\, {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{v^* w^*\}(t)} - {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{u^*\}(t)\, {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{u^*\}(t).$$

Proof. Let

$$B(\rho, \tau) := (f_2(\tau) - f_2(\rho))(f_1(\tau) - f_1(\rho)),$$

or, equivalently

$$B(\rho, \tau) = f_1(\tau) f_2(\tau) + f_1(\rho) f_2(\rho) - f_1(\tau) f_2(\rho) - f_1(\rho) f_2(\tau). \tag{9.2.27}$$

Upon multiplying both sides of (9.2.27) by

$$\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\frac{1}{(\rho-a)^{1-s}(\tau-a)^{1-s}}$$

and integrate the resultant w.r.t. τ and ρ on $[a,t]$, we get

$$\begin{aligned}&\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\\&\times B(\rho,\tau)\frac{d\rho}{(\rho-a)^{1-s}}\frac{d\tau}{(\tau-a)^{1-s}}\\&={}_k^s\,\mathfrak{F}^{*\beta}_{a^+}1(t)\,{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_1f_2\}(t)+{}_k^s\mathfrak{F}^{*\alpha}_{a^+}1(t)\,{}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_1f_2\}(t)\\&-{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(t)\,{}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_2\}(t)-{}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_1\}(t)\,{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_2\}(t).\end{aligned}\tag{9.2.28}$$

By the weighted Cauchy-Schwarz double integrals inequality in (9.2.28), we have

$$\begin{aligned}&\Bigg|\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\\&\times B(\rho,\tau)\frac{d\rho}{(\rho-a)^{1-s}}\frac{d\tau}{(\tau-a)^{1-s}}\Bigg|\le\Bigg[\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\\&\times\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}f_1^2(\tau)\frac{d\tau}{(\tau-a)^{1-s}}\frac{d\rho}{(\rho-a)^{1-s}}+\frac{1}{k^2\Gamma_k(\beta)\Gamma_k(\alpha)}\\&\times\int_a^t\int_a^t\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}f_1^2(\rho)\frac{d\rho}{(\rho-a)^{1-s}}\frac{d\tau}{(\tau-a)^{1-s}}\\&-\frac{2}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\\&\times f_1(\tau)f_2(\rho)\frac{d\rho}{(\rho-a)^{1-s}}\frac{d\tau}{(\tau-a)^{1-s}}\Bigg]^{\frac{1}{2}}\end{aligned}$$

$$\times\left[\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\right.$$

$$\times f_2^2(\tau)\frac{d\tau}{(\tau-a)^{1-s}}\frac{d\rho}{(\rho-a)^{1-s}}+\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}$$

$$\times\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}f_2^2(\rho)\frac{d\rho}{(\rho-a)^{1-s}}\frac{d\tau}{(\tau-a)^{1-s}}$$

$$-\frac{2}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}$$

$$\left.\times f_2(\tau)f_2(\rho)\frac{d\rho}{(\rho-a)^{1-s}}\frac{d\tau}{(\tau-a)^{1-s}}\right]^{\frac{1}{2}}. \tag{9.2.29}$$

Then, invoking the generalized conformable k-fractional integrals, we obtain

$$\left|\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\frac{(t-a)^s-(\tau-a)^s}{s}\right)^{\frac{\alpha}{k}-1}\left(\frac{(t-a)^s-(\rho-a)^s}{s}\right)^{\frac{\beta}{k}-1}\right.$$

$$\left.\times B(\rho,\tau)\frac{d\rho}{(\rho-a)^{1-s}}\frac{d\tau}{(\tau-a)^{1-s}}\right|$$

$$\leq\left[{}^s_k\mathfrak{F}^{*\beta}_{a^+}\{1\}\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}\{f_1^2\}(t)+{}^s_k\mathfrak{F}^{*\alpha}_{a^+}\{1\}\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}\{f_1^2\}(t)-2\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(t)\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}\{f_1\}(t)\right]^{\frac{1}{2}}$$

$$\times\left[{}^s_k\mathfrak{F}^{*\beta}_{a^+}\{1\}\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}\{f_2^2\}(t)+{}^s_k\mathfrak{F}^{*\alpha}_{a^+}\{1\}\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}\{f_2^2\}(t)-2\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}\{f_2\}(t)\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}\{f_2\}(t)\right]^{\frac{1}{2}}. \tag{9.2.30}$$

Setting $\mu_1(t)=\mu_2(t)=f_2(t)=1$ in Lemma (9.2.1), we get

$${}^s_k\mathfrak{F}^{*\alpha}_{a^+}\{1\}\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}\{f_1^2\}(t)\leq\frac{{}^s_k\mathfrak{F}^{*\beta}_{a^+}\{1\}\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\{(\lambda_1+\lambda_2)f_1\}(t)\right)^2}{4\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_1\lambda_2\}(t)},$$

which gives

$$\begin{aligned}&{}^s_k\mathfrak{F}^{*\beta}_{a^+}\{1\}\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}\{f_1^2\}(t)-{}^s_k\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(t)\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}\{f_2\}(t)\\&\leq\frac{{}^s_k\mathfrak{F}^{*\beta}_{a^+}\{1\}\left({}^s_k\mathfrak{F}^{*\alpha}_{a^+}\{(\lambda_1+\lambda_2)f_1\}(t)\right)^2}{4\,{}^s_k\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_1\lambda_2\}(t)}-{}^s_k\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(t)\,{}^s_k\mathfrak{F}^{*\beta}_{a^+}\{f_1\}(t)\\&=P_1(f_1,\lambda_1,\lambda_2)\end{aligned} \tag{9.2.31}$$

and

$$
\begin{aligned}
&{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{1\}\, {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_1^2\}(t) - {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(t)\, {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_1\}(t)\\
\leq& \frac{{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{1\}\left({}_k^s\mathfrak{F}^{*\beta}_{a^+}\{(\lambda_1+\lambda_2)f_1\}(t)\right)^2}{4\, {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{\lambda_1\lambda_2\}(t)} - {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(t)\, {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_1\}(t) \qquad (9.2.32)\\
&= P_2(f_1,\lambda_1,\lambda_2).
\end{aligned}
$$

Similarly setting $\lambda_1(t) = \lambda_2(t) = f_1(t) = 1$ in Lemma (9.2.1), we obtain

$$
{}_k^s\mathfrak{F}^{*\beta}_{a^+}\{1\}\, {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_2^2\}(t) - {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_2\}(t)\, {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_2\}(t) \leq P_1(f_2,\mu_1,\mu_2)(t) \qquad (9.2.33)
$$

and

$$
{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{1\}\, {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_2^2\}(t) - {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_2\}(t)\, {}_k^s\mathfrak{F}^{*\beta}_{a^+}\{f_2\}(t) \leq P_2(f_2,\mu_1,\mu_2)(t) \qquad (9.2.34)
$$

Finally, the required inequality (9.2.26) can be obtained by combining the inequalities (9.2.30)-(9.2.34). ■

Theorem 9.2.2 *For the conditions of Theorem (9.2.1), the following result is valid for* $t > 1$ *and* $\alpha \in \Re^+$,

$$
\begin{aligned}
&\left| {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{1\}\, {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_1f_2\}(t) - {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(t)\, {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_2\}(t) \right|\\
&\leq |\mathcal{Q}(f_1,\lambda_1,\lambda_2)\mathcal{Q}(f_2,\lambda_1,\lambda_2)|^{\frac{1}{2}}, \qquad (9.2.35)
\end{aligned}
$$

where

$$
\mathcal{Q}(u^*,v^*,w^*)(t) := \frac{{}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{1\}\ \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{(v^*+w^*)u^*\}(t)\right)^2}{4\, {}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{v^*w^*\}(t)} - \left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{u^*\}(t)\right)^2.
$$

Remark 9.2.1 *Setting* $\lambda_1 = p, \lambda_2 = P, \mu_1 = q$, *and* $\mu_2 = Q$, *we have*

$$
\mathcal{Q}(f_1,p,P)(t) = \frac{(P-p)^2}{4pP}\left({}_k^s\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(t)\right)^2
$$

and

$$\mathcal{Q}(f_2, q, Q)(t) = \frac{(Q-q)^2}{4qQ} \left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{f_2\}(t)\right)^2$$

Corollary 9.2.4 *For two positive and integrable functions f_1 and f_2 on $[a, \infty)$, satisfying the condition (9.2.13) and $t > a$, $\alpha \in \mathbb{R}^+$, we have*

$$\begin{aligned} &\left| {}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{1\}\, {}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{f_1 f_2\}(t) - {}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(t)\, {}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{f_2\}(t) \right| \\ &\leq \frac{(P-p)(Q-q)}{4\sqrt{pPqQ}}\, {}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(t)\, {}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{f_2\}(t). \end{aligned} \tag{9.2.36}$$

9.3 Applications

Now, we apply generalized conformable integrals (2.3.35) and (2.3.36) to a function, confined by the Heaviside functions.

The Heaviside function, a simplest piece-wise continuous function is the unit-step function, can be defined as

$$u_e^*(t) = \begin{cases} 0 & \text{if } u^* < e, \\ 1 & \text{if } u^* \geq e. \end{cases}$$

The function $u_e^*(t)$ is very valuable in piecewise functions and differential equations when a large number of partitions is considered. The function $u_e^*(t)$ is mainly an on-off switch. Invoking Heaviside function, a piece-wise continuous function $\lambda_1(t)$ can be expressed on an interval $[a, J]$ as follows:

$$\begin{aligned} \lambda_1(t) &= p_1\left(u_{t_0^*}(t) - u_{t_1}^*(t)\right) + p_2\left(u_{t_1}^*(t) - u_{t_2}^*(t)\right) + p_3\left(u_{t_2}^*(t) - u_{t_3}^*(t)\right) + \cdots + p_{i+1}u_{t_i}^*(t) \\ &= p_1 u_{t_0}^*(t) + (p_2 - p_1)u_{t_1}^*(t) + (p_3 - p_2)u_{t_2}^*(t) + \cdots + (p_{i+1} - p_i)u_{t_0}^*(t) \\ &= \sum_{b=0}^{i}(p_{b+1} - p_b)u_{t_b}^*(t), \end{aligned} \tag{9.3.37}$$

where $p_0 = 0, p_v \in \mathbb{R}(v = 0, 1, \ldots, i+1)$ and $b = t_0 < t_1 < t_2 < \cdots < t_i < t_{i+1} = J$. Accordingly, we define the functions λ_2, μ_1 and μ_2 as follows:

$$\lambda_2(t) = \sum_{b=0}^{i}(P_{b+1} - P_b)u_{t_b}^*(t), \tag{9.3.38}$$

$$\mu_1(t)=\sum_{b=0}^{i}(q_{b+1}-q_b)u^*_{t_b}(t), \tag{9.3.39}$$

$$\mu_2(t)=\sum_{b=0}^{i}(Q_{b+1}-Q_b)u^*_{t_b}(t), \tag{9.3.40}$$

where $q_0=Q_0=P_0=p_0$ and $q_v,P_v,Q_v\in\mathbb{R}(v=0,1,\ldots,i+1)$.

For an integrable function f_1 on $[a,J]$, assuming the condition (9.2.3) with the functions $\lambda_1,\lambda_2,\mu_1$ and μ_2 given in (9.3.37),(9.3.38),(9.3.39) and (9.3.40), we get $p_{j+1}\le f_1(t)\le P_{j+1}$ for each $t\in(t_j,t_{j+1})$ $(j=0,1,\ldots,i)$.

The generalized k-conformable integral of f_1 on the interval $[a,J]$ can be written as follows:

$$ {}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(J)=\sum_{j=0}^{i}{}^{s}_{k}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{f_1\}(t), \tag{9.3.41}$$

where

$$ {}^{s}_{k}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{f_1\}(t):=$$
$$\frac{1}{k\Gamma_k(\alpha)}\int_{t_j}^{t_{j+1}}\left(\frac{(x-a)^s-(z-a)^s}{s}\right)^{\frac{\alpha}{k}-1}f_1(z)\,\frac{1}{(z-a)^{1-s}}\ (j=0,1,\ldots,i). \tag{9.3.42}$$

Proposition 1 *For integrable function f_1 on $[a,J]$ justifying the condition (9.2.3) and the functions $\lambda_1,\lambda_2,\mu_1$ and μ_2 in (9.3.37),(9.3.38),(9.3.39) and (9.3.40), the following inequality maintains $\alpha\in\mathbb{R}^+$,*

$$\left(\sum_{j=0}^{i}q_{j+1}Q_{j+1}\,{}^{s}_{k}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{f_1^2\}(J)\right)\left(\sum_{j=0}^{i}p_{j+1}P_{j+1}\,{}^{s}_{k}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{f_2^2\}(J)\right)$$
$$\le\frac{1}{4}\sum_{t=0}^{i}(q_{j+1}Q_{j+1}+p_{j+1}P_{j+1})\,{}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\left(\{f_1f_2\}(J)\right)^2. \tag{9.3.43}$$

Proof. According to the generalized conformable k-fractional integral in (9.3.41), we get

$$ {}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{\mu_1\mu_2f_1^2\}(J)=\sum_{j=0}^{i}q_{j+1}Q_{j+1}\,{}^{s}_{k}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{f_1^2\}(J), \tag{9.3.44}$$

$$ {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_1\lambda_2 f_2^2\}(J) = \sum_{j=0}^{i} p_{j+1}P_{j+1}\, {}_{k}^{s}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{f_2^2\}(J), \tag{9.3.45} $$

and

$$ {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}(\lambda_1\mu_1+\lambda_2\mu_2)\{f_1f_2\}(J) = \sum_{j=0}^{i}(p_{j+1}q_{j+1}+P_{j+1}Q_{j+1})\, {}_{k}^{s}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{f_1f_2\}(J). \tag{9.3.46} $$

Then substituting equalities (9.3.44),(9.3.45) , and (9.3.46) for the result in Lemma (9.2.1) leads to the required result (9.3.43). ■

Proposition 2 *Under the conditions of Proposition (1), for* $\alpha,\beta,k\in\mathbb{R}^+$, *we have*

$$ \begin{aligned} &\Big| {}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\{1\}\, {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{f_1f_2\}(J) + {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{1\}\, {}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\{f_1f_2\}(J) \\ &- {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(t)\, {}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\{f_2\}(J) - {}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\{f_1\}(t)\, {}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(J)\Big| \\ &\le |P_1^*(f_1,p_{j+1},P_{j+1})(t) + P_2^*(f_1,p_{j+1},P_{j+1})(J)|^{\frac{1}{2}} \\ &\times |P_1^*(f_2,q_{j+1},Q_{j+1})(t) + P_2^*(f_2,q_{j+1},Q_{j+1})(J)|^{\frac{1}{2}} \end{aligned} \tag{9.3.47} $$

where

$$ \begin{aligned} P_1^*(u^*,v^*,w^*)(t) :=& \frac{{}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\{1\}\ \sum_{j=0}^{i}(v^*+w^*)\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{u^*\}\right)^2}{4\sum_{j=0}^{i} v^*w^*\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{1\}(J)\right)} \\ &-\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{l\}(J)\right)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\{l\}(J)\right), \end{aligned} $$

$$ \begin{aligned} P_2^*(u^*,v^*,w^*)(t) :=& \frac{{}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{1\}\ \sum_{j=0}^{i}(v^*+w^*)\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{t_j,t_{j+1}}\{u^*\}\right)^2}{4\sum_{j=0}^{i} v^*w^*\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{t_j,t_{j+1}}\{u^*\}(J)\right)} \\ &-\left({}_{k}^{s}\mathfrak{F}^{*\beta}_{a^+}\{u^*\}(t)\right)\left({}_{k}^{s}\mathfrak{F}^{*\alpha}_{a^+}\{u^*\}(t)\right). \end{aligned} $$

Proof. Since

$$ {}^{s}_{k}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{1\}(J) = \frac{1}{k\Gamma_k(\alpha)} \int\limits_{t_j}^{t_{j+1}} \left(\frac{(t-a)^s-(z-a)^s}{s}\right)^{\frac{\alpha}{k}-1} \frac{dz}{(z-a)^{1-s}} $$

we obtain

$$ {}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{\lambda_1\lambda_2\}(J) = \sum_{j=0}^{i} p_{j+1}P_{j+1}\, {}^{s}_{k}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{1\}(J) $$

and

$$ {}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{\mu_1\mu_2\}(J) = \sum_{j=0}^{i} q_{j+1}Q_{j+1}\, {}^{s}_{k}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{1\}(J). $$

After some computations, we have

$$ \begin{aligned} P_1(f_1,\lambda_1,\lambda_1)(J) = & \frac{{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\{1\}\ \sum\limits_{j=0}^{i}(p_{j+1}+P_{j+1})\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{f_1\}(t)\right)^2}{4\ \sum\limits_{j=0}^{i} p_{j+1}P_{j+1}\left[{}^{s}_{k}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{1\}(J)\right]} \\ & -\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(J)\right)\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\{f_1\}(J)\right), \end{aligned} $$

$$ \begin{aligned} P_1(f_2,\mu_1,\mu_1)(J) = & \frac{{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\{1\}\ \sum\limits_{j=0}^{i}(q_{j+1}+Q_{j+1})\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{f_2\}(t)\right)^2}{4\ \sum\limits_{j=0}^{i} q_{j+1}Q_{j+1}\left[{}^{s}_{k}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{1\}(J)\right]} \\ & -\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{f_2\}(J)\right)\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\{f_2\}(J)\right), \end{aligned} $$

$$ \begin{aligned} P_2(f_1,\lambda_1,\lambda_1)(J) = & \frac{{}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\{1\}\ \sum\limits_{j=0}^{i}(p_{j+1}+P_{j+1})\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{t_j,t_{j+1}}\{f_1\}(j)\right)^2}{4\ \sum\limits_{j=0}^{i} p_{j+1}P_{j+1}\left[{}^{s}_{k}\mathfrak{F}^{*\alpha}_{t_j,t_{j+1}}\{1\}(J)\right]} \\ & -\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{a^+}\{f_1\}(J)\right)\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{a^+}\{f_1\}(J)\right) \end{aligned} $$

and

$$P_2(f_2,\mu_1,\mu_1)(J)=\frac{{}^{s}_{k}\mathfrak{F}^{*\beta}_{\ a^+}\{1\}\ \sum_{j=0}^{i}(q_{j+1}+Q_{j+1})\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{\ t_j,t_{j+1}}\{f_2\}(t)\right)^2}{4\ \sum_{j=0}^{i}q_{j+1}Q_{j+1}\left[{}^{s}_{k}\mathfrak{F}^{*\alpha}_{\ t_j,t_{j+1}}\{1\}(J)\right]}$$
$$-\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{\ a^+}\{f_2\}(J)\right)\left({}^{s}_{k}\mathfrak{F}^{*\beta}_{\ a^+}\{f_2\}(J)\right).$$

Applying the results of Theorem (9.2.1), we attain the required inequality (9.3.47). ∎

Corollary 9.3.5 *Under the conditions of Proposition (2), for $k,\alpha\in\mathbb{R}^+$, the following relation retains:*

$$\left|{}^{s}_{k}\mathfrak{F}^{*\alpha}_{\ a^+}\{1\}\,{}^{s}_{k}\mathfrak{F}^{*\alpha}_{\ a^+}\{f_1f_2\}(J)-{}^{s}_{k}\mathfrak{F}^{*\alpha}_{\ a^+}\{f_1\}(J)\,{}^{s}_{k}\mathfrak{F}^{*\beta}_{\ a^+}\{f_1\}(J)\right|$$
$$\leq\left|P^*(f_1,p_{j+1},P_{j+1})(J)\ P^*(f_2,q_{j+1},Q_{j+1})(J)\right|^{\frac{1}{2}},\tag{9.3.48}$$

where

$$P^*(u^*,v^*,w^*)(t)=\frac{{}^{s}_{k}\mathfrak{F}^{*\alpha}_{\ a^+}\{1\}\ \sum_{j=0}^{i}(v^*+w^*)\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{\ t_j,t_{j+1}}\{u^*\}(t)\right)^2}{4\ \sum_{j=0}^{i}v^*w^*\left[{}^{s}_{k}\mathfrak{F}^{*\alpha}_{\ t_j,t_{j+1}}\{u^*\}(t)\right]}-\left({}^{s}_{k}\mathfrak{F}^{*\alpha}_{\ a^+}\{u^*\}(t)\right)^2.$$

Remark 9.3.2 *We conclude that these inequalities can be derived for the right–sided generalized k-conformable integral ${}^{s}_{k}\mathfrak{F}^{*\alpha}_{\ b^-}$ (2.3.36).*

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Appendix

List of Published Work

1- Habib, S., Mubeen, S., & Naeem, M. N. (2018). Chebyshev type integral inequalities for generalized k-fractional conformable integrals. *Journal of Inequalities & Special Functions, 9*, 53-65.

2- Mubeen, S., Habib, S., & Naeem, M. N. (2019). The Minkowski inequality involving generalized k-fractional conformable integral. *Journal of Inequalities and Applications, 2019* (1), 81.

3- Habib, S. & Mubeen, S. Some New Fractional Integral Results Involving Convex Functions by Means of Generalized $k-$Fractional Conformable Integral, *Punjab University Journal of Mathematics, 2019*, 51(7), 99-109.

4- Qi. F., Habib. S., Mubeen, S. & Naeem, M. N. Generalized $\boldsymbol{k}$-fractional conformable integrals and related inequalities. *AIMS-Mathematics, 2019*, 4(3), 343-358.