SPECTRAL CHARACTERIZATIONS FOR HYERS-ULAM STABILITY

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By

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DECLARATION

I, **Ms AFSHAN TABASSUM** Registration No. **99-GCU-PHD-SMS-09** student at **Abdus Salam School of Mathematical Sciences GC University** in the subject of **Mathematics** year of admission **2009**, hereby declare that the matter printed in this thesis titled

"SPECTRAL CHARACTERIZATIONS FOR HYERS-ULAM STABILITY"

is my own work and that

(i) I am not registered for the similar degree elsewhere contemporaneously.

(ii) No direct major work had already been done by me or anybody else on this topic; I worked on, for the Ph. D. degree.

(iii) The work, I am submitting for the Ph. D. degree has not already been submitted elsewhere and shall not in future be submitted by me for obtaining similar degree from any other institution.

Dated: ------------------------ ----------------------------------

Signature

<u>RESEARCH COMPLETION CERTIFICATE</u>

Certified that the research work contained in this thesis titled

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has been carried out and completed by **Ms AFSHAN TABASSUM** Registration No. **99-GCU-PHD-SMS-09** under my supervision.

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Submitted Through

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To my mother Bakhtawer

&

father Muhammad Yar

Table of Contents

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Notations

Throughout my dissertation the following notations will be used:

$\mathbb{Z}$:= The set of all integer numbers.

$\mathbb{Z}_+$:= The set of all nonnegative integer numbers.

$\mathbb{R}$:= The set of all real numbers.

$\mathbb{R}_+$:= The set of all nonnegative real numbers.

$\mathbb{C}$:= The set of all complex numbers.

$\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

$\mathbb{K}$:= The field $\mathbb{R}$ or $\mathbb{C}$.

$\mathbf{X}$:= A real or complex Banach space over $\mathbb{K}$.

$\|x\|$:= Norm of vector x.

$\mathcal{M}(n, \mathbb{C})$:= Set of all $n \times n$ matrices with complex entries.

$\mathcal{M}(n, 1, \mathbb{C})$:= Set of all $n \times 1$ complex valued vector (or A column matrix).

$\mathcal{M}(1, n, \mathbb{C})$:= Set of all $1 \times n$ complex valued vector (or A row matrix).

A := A complex valued matrix.

I_m := The identity matrix of order m.

$P_A(z)$:= The characteristic polynomial of matrix A.

λ := The eigenvalue of the matrix.

E_j := The spectral projection related to eigenvalue λ_j.

$p_j(t)$:= Polynomial in t related to eigenvalue λ_j.

$P_j(t) :=$ Vectorial or matrix valued polynomial related to eigenvalue λ_j.

$\sigma(A) :=$ The spectrum of a matrix A.

$\rho(A) :=$ The resolvent set of a matrix A.

$R(\lambda, A):=$ The resolvent of a matrix A at the point λ.

$\Delta :=$ Forward difference.

$\Delta^n :=$ Forward difference of order n.

$\Gamma := \{z \in \mathbb{C} : |z| = 1\}$ The unit circle.

$\mathcal{X}_s(A) :=$ The stable subspace of A.

$\mathcal{X}_u(A) :=$ The unstable subspace of A.

ker := The kernel.

$\Im :=$ The image.

$\Re :=$ The real part of a complex number.

Introduction

The last decades have witnessed remarkable development in the study of differential equations and dynamical systems, by means of the functional analytic approach. This approach has created the general structure for further studies in this domain.

In the study of Mathematical Analysis, we usually consider the following question: let us consider a mathematical object which satisfies a certain approximation property. The following issue usually occurs in mathematics: does another mathematical object exist which satisfies the same approximation property and it is "near" to the previous one?

This particular question is of utter importance in the area of differential equations, where we are interested in finding an approximate solution of an equation which slightly differs from the exact solution.

The Hyers-Ulam problem is very similar to this above described question.

The mathematical field of our investigation originates from the stability of functional equations. The general frame work of the stability problem of functional equation arose in 1940 due to S. M. Ulam's question. During a lecture he delivered before the Mathematics Club in the University of Wisconsin (see in [69], [70]). Ulam posed a problem related to the stability of group homomorphism. The starting point of this kind of stability is the problem of group and metric group. The Ulam's Problem was:

Given a group G_1 *and a metric group* G_2 *with metric* $d(\cdot,\cdot)$ *as well as* $\varepsilon > 0$, *does there exist a* $\delta > 0$ *such that if* $f : G_1 \to G_2$ *satisfies* $d(f(xy), f(x)f(y)) < \delta$ *for all* x *and* y *in* G_1, *then a homomorphism* $h : G_1 \to G_2$ *exists with* $d(f(x), h(x)) < \varepsilon$ *for all* x *and* y *in* G_1?

In 1941, D. H. Hyers [34] gave an affirmative answer to the Ulam problem reformulated in the framework of Banach spaces as: let X and Y be Banach spaces over $\mathbb{R}$ and $f : X \to Y$ a function satisfying the inequality $\|f(x+y) - f(x) - f(y)\| < \varepsilon$ for all $x, y \in X$, where ε is a given positive number. Then Hyers proved that the function $f : X \to Y$, given by $g(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ is correctly defined and $\|f(x) - g(x)\| \leq \varepsilon$ for all $x \in X$.

This became a significant step in the area of stability theory of equations and this problem was coined "Hyers-Ulam problem".

In 1968, S. M. Ulam [71] described the problem in a more general way, consequently this problem was called "the generalized Ulam problem", as:

When is it true that by changing slightly the hypothesis of a theorem one can still insert that the thesis of the theorem remains true or approximately true ?

After Hyers, the generalization of the Ulam problem was given by Rassias. As it can be seen in the article of [63], Th. M. Rassias considered a detailed introduction to the theory of stability of functional equations.

The generalized stability of Cauchy functional equation was studied by Z. Páles [55], and by R. Badora, R. Ger and Z. Páles [3]. This was a step for further discussions on the Hyers-Ulam stability. In [56], Z. Páles, P. Volkmann and R. D. Luce described and generalized Hyers-Ulam stability of the family of functional equations, as a particular case they considered the Cauchy functional equation.

Over the previous decades, the Hyers-Ulam problem for the functional equation was generalized in many directions for example see [1], [5], [10], [17], [19], [18], [23], [26], [27], [28], [31], [32], [30], [35], [33], [36], [37], [38], [39], [43], [44], [65] and [66].

So far, the existing block of literature includes only few results for single variables as compared to several variables concerning the Hyers-Ulam stability. It is believed that the very first result in this direction was provided by Alsina and Ger [2].

The most important characterization of the real exponential function $f(x) = e^x$ is that it is the only non-trivial solution (modulo a multiplicative constant) of the differential equation $f' = f$. The Hyers-Ulam stability of this equation has been studied by Alsina and Ger. The problem they analysed was that, in what context for a given $\varepsilon > 0$, the inequality $|f'(x) - f(x)| \leq \varepsilon$ holds ? Details may be found in [2]. In [54] M. Obłoza connected the Hyers-Ulam problem to the Lyapunov stability for ordinary differential equations

Following the same approach as in [2], Miura, Takahasi and Choda [52], Miura [49], and Takahasi, Miura and Miyajima [67] proved that the Hyers-Ulam stability holds true for the differential equation $y' = \lambda y$.

In [45], Y. Li and Y. Shen characterized the Hyers-Ulam stability of linear differential equation of second order in the form of $y''(t) + \alpha y'(t) + \beta y(t) = 0$ and $y''(t) + \alpha y'(t) + \beta y(t) = f(t)$ under the assumption that its associated characteristic equation $\lambda^2 + \alpha\lambda + \beta = 0$ has two different positive roots.

While Jung [40], proved a similar result described in [2] for the differential equation $\phi(t)y' = y$. This result is as:

Theorem. If either $\phi(t) > 0$ holds for all $t \in I$ or $\phi(t) < 0$ holds for all $t \in I$, and if a differentiable function $y : I \to \mathbb{R}$ satisfies $|\phi(t)y'(t) - y(t)| \leq \varepsilon$ for all $t \in I$, then there exists a real number c such that $|y(t) - c \cdot exp\{\int_a^t \frac{d\tau}{\phi(\tau)}\}| \leq \varepsilon$, for every $t \in I$.

This result is a special case of the corollary in the third chapter of present thesis. Furthermore, the result of Hyers-Ulam stability for first order linear differential equation of the type $y'(t) + g(t)y(t) = 0$, where $g(t)$ is a continuous function, has been generalized by T. Miura, S. Miyajma and S. E. Takahasi [51]. Further discussion was by S. E. Takahasi, H. Takagi, T. Miura and S. Miyajima [68], and also by S. M. Jung [41] and [42]. They dealt with the non-homogeneous linear differential equation of first order $y' + p(t)y + q(t) = 0$. G. Wang, M. Zhou and L. Sun [72], discuss the Hyers-Ulam stability of the first order nonhomogeneous linear differential equation $p(x)y' - q(x)y - r(x) = 0$, where $x \in I = (a, b), -\infty \leq a < b \leq +\infty$.

The Hyers-Ulam stability for second order linear differential equations in the form of $y''(t) + \beta(x)y = 0$ with boundary conditions was investigated by P. Găvruţă, S. M. Jung and Y. Li in [29].

Further on, the Hyers-Ulam problems for second order differential equations were studied by Y. Li, J. Huang in [48], Y. Li, Y. Shen [46] and Y. Li in [47]. M. N. Qarawani [61] also studied Hyers-Ulam stability for linear and nonlinear second order differential equations.

The Hyers-Ulam stability for linear recurrence of order p with constant coefficients is discussed in [58]. The main theorem in [58] concerns the roots of algebraic equation associated to the recurrence, i.e.

Theorem. Let X be a Banach space over the field K, $\varepsilon > 0$, and $a_1, a_2, \cdots, a_p \in K$ such that the equation

$$r^p - a_1 r^{p-1} - \cdots - a_{p-1} r - a_p = 0$$

admits the roots $r_1, r_2, \cdots, r_p$, $|r_k| \neq 1$, $1 \leq k \leq p$, and $(b_n)_{n\geq 0}$ is a sequence in X. Suppose $(x_n)_{n\geq 0}$ is a sequence in X with the property

$$\|x_{n+p} - a_1 x_{n+p-1} - \cdots - a_{p-1}x_{n+1} - a_p x_n - b_n\| \leq \varepsilon, \quad n \in \mathbb{Z}_+.$$

Then there exists a sequence $(y_n)_{n\geq 0}$ in X given by the recurrence

$$y_{n+p} = a_1 y_{n+p-1} + \cdots + a_{p-1}y_{n+1} + a_p y_n + b_n, \quad n \in \mathbb{Z}_+,$$

such that

$$\|x_n - y_n\| \leq \frac{\varepsilon}{|(|r_1| - 1)\cdots(|r_p| - 1)|}, \quad n \in \mathbb{Z}_+.$$

More precisely, the theorem says that the recurrence is Hyers-Ulam stable if all roots of its associated algebraic equation have modulus different of one.

In [8] Brzdęk, Popa and Xu studied the case of non-stability for difference equations of order m, i.e.

Theorem. Let $\mathbb{K} = \mathbb{C}$, $a_1, a_2, \cdots, a_p \in \mathbb{K}$, and $(b_n)_{n\geq 0}$ be a sequence in X. The recurrence

$$y_{n+p} = a_1 y_{n+p-1} + \cdots + a_{p-1}y_{n+1} + a_p y_n + b_n, \quad \forall n \in \mathbb{Z}_+,$$

is weakly (strongly, respectively) stable in the Hyers-Ulam sense if and only if the characteristic equation

$$r^p - \sum_{i=1}^{p} a_i r^{p-i} = 0$$

does not have any root in Γ (where Γ is a unit circle).

Many interesting results related to difference equations and recurrences may be found in [6], [7], [8], [9], [53], [57], [59], [60], [64] and [73].

Over the past decades, the Hyers-Ulam stability of operator equations has been widely discussed. In [50], T. Miura, S. Miyajima and S. E. Takahasi describe the results for Hyers-Ulam stability for n-th order linear differential operator $P(D)$. They prove that the differential operator $P(D)f = 0$ has Hyers-Ulam stability if and only if its algebraic equation $P(z) = 0$ has no pure imaginary solution, where $P(z)$ is a polynomial of degree n with complex values. In [11], J. Brzdęk and S. M. Jung considered Hyers-Ulam stability of operator linear equation of second order and obtained parallel stability results for differential and integral equations. Further results related to the Hyers-Ulam stability of operator equation may be found in [68].

Many of the mentioned results, may be seen as a particular case of a general result contained in the first, second and in the third chapter of present thesis. Our results seem to be of scientific novelty. To the best of our knowledge, the existing look of literature includes separate theories concerning the Hyers-Ulam stability and the exponential dichotomy, respectively. Our approach is completely different to those given in the above quoted references.

In the present dissertation we will tackle the issues described above, but we will exclusively focus on the equivalence between the Hyers-Ulam stability and the exponential dichotomy. We analyze both cases (continuous and discrete) for vectorial systems and scalar equations of higher order having constant or time varying coefficients. In the latter case, only the periodic case was considered. This dissertation contains four chapters. Actually the last chapter is very short and is refer to the future research and some open problems.

Chapter 1 is devoted to the study of the equivalence between the Hyers-Ulam stability and the exponential dichotomy for systems in $\mathbb{C}^n$ and for linear scalar differential equations of order n. This chapter consists of three sections.

Section 1.1 contains the main notations and the preliminary results concerning the exponential dichotomy in the continuous and autonomous case.

Section 1.2 is related to the study of the equivalence between Hyers-Ulam stability and exponential dichotomy for linear differential systems in $\mathbb{C}^m$. The main result of this section reads as following

Theorem 1. A matrix A is Hyers-Ulam stable if and only if it posses a continuous dichotomy, i.e. its spectrum does not intersect the imaginary axis.

Section 1.3 covers the study of connections between Hyers-Ulam stability and exponential dichotomy for linear differential equations of order n. The main result of this section is given in the following.

Theorem 2. The following statements are equivalent:

1. The differential equation of order n

$$x^{(n)}(t) = a_1 x^{(n-1)}(t) + \ldots + a_{n-1} x'(t) + a_n x(t), \quad t \in \mathbb{R}_+$$

is Hyers-Ulam stable.

2. The matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{pmatrix}$$

is Hyers-Ulam stable.

3. The algebraic equation

$$z^n - a_1 z^{n-1} - a_{n-2} z^{n-2} - \ldots - a_n = 0$$

has no roots on the imaginary axis.

Chapter 2 in this chapter we discuss the same issue described in the previous chapter but we exclusively focus on the discrete case. Therefore, the following theorems concerning the linear difference system and the scalar difference equations, which give equivalence between the Hyers-Ulam stability and the dichotomy, in the second chapter allows us to produce a completely different proofs for above Theorems in [58] and in [8] when the Banach space is $\mathbb{C}$.

Section 2.2 deals with

Theorem 3. The matrix B is Hyers-Ulam stable if and only if it posses discrete dichotomy, i.e. its spectrum does not intersect the unit circle.

Section 2.3 In this section we have shown

Theorem 4. The following three statements are equivalent:

1. The difference equation of order m

$$x(j+m) = a_1 x(j+m-1) + \cdots + a_m x(j)$$

is Hyers-Ulam stable.

2. The matrix

$$B = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ b_1 & b_2 & b_3 & \cdots & 1+b_m \end{pmatrix}.$$

is dichotomic.

3. The characteristic equation

$$z^m - a_1 z^{m-1} - a_2 z^{m-2} - \ldots - a_{m-1} z - a_m = 0,$$

has no roots on the unit circle.

The terms and notions will be explained explicitly in the upcoming chapters.

Chapter 3. Furthermore, our other important and interesting results of this dissertation are in **Section 3.2** which presents the following theorem.

Theorem 5. The family of matrices $\mathcal{A} = \{A(t)\}_{t\geq 0}$ is Hyers-Ulam stable if and only if its monodromy matrix T_q posses a discrete dichotomy.

Section 3.6 The deal of this section is to state and prove the next theorem.

Theorem 6. The family $\mathcal{A} = (A_n)_{n\in\mathbb{Z}_+}$ is Hyers-Ulam stable if and only if its monodromy matrix T_q posses a discrete dichotomy.

The above mentioned results from the Chapters 1, 2 and 3 are mentioned in our papers [14], [4] and [15] respectively.

The existing theory is large, well developed and contains numerous theoretical approaches. Therefore for an introduction as well as for further information in the present field we recommend the books [35], for dichotomy [20] and for the functional analytic background [24] and [74]. Detailed discussion has also been devoted to the Hyers-Ulam problem for the functional equation in a massive 108 page article [62], by Th. M. Rassias.

Chapter 1

Spectral characterizations for Hyers-Ulam stability

In the present chapter we prove that a square size complex matrix of order n is Hyers-Ulam stable if and only if it is dichotomic (i.e. it has no eigenvalues on the imaginary axis $i\mathbb{R}$). Further on, we show that the linear scalar differential equation of order n,

$$x^{(n)}(t) = a_1 x^{(n-1)}(t) + \ldots + a_{n-1}x'(t) + a_n x(t), \quad t \in \mathbb{R}_+ := [0, \infty),$$

is Hyers-Ulam stable if and only if the algebraic equation

$$z^n = a_1 z^{n-1} + \cdots + a_{n-1}z + a_n,$$

has no roots on the imaginary axis. The latter result contains, a lot of particular cases in the already existing literature.

Now we outline the Hyers-Ulam problem for a matrix A. Consider the system

$$\dot{x}(t) = Ax(t), \quad t \in \mathbb{R}_+. \qquad (A)$$

Set ε be a positive real number. A $\mathbb{C}^n$-valued function y is called ε-approximate solution for (A) if

$$\|\dot{y}(t) - Ay(t)\| \leq \varepsilon, \quad \forall t \in \mathbb{R}_+.$$

The matrix A is said to be Hyers-Ulam stable if there exists a nonnegative constant L such that, for every ε-approximate solution ϕ of (A), there exists an exact solution θ of (A) such that

$$\sup_{t\in\mathbb{R}_+} \|\phi(t) - \theta(t)\| \leq L \cdot \varepsilon.$$

1.1 Notations and some results

The set $\sigma(A) := \{\lambda_1, \lambda_2, \dots, \lambda_k\}$, consisting of all roots of P_A. As is well-known,

$$P_A(z) = (z - \lambda_1)^{m_1} \cdots (z - \lambda_k)^{m_k},$$

where $m_1, m_2, \dots, m_k$ are the multiplicities of the eigenvalues $\lambda_1, \dots, \lambda_k$, respectively, and then, $m_1 + \dots + m_k = n$. The spectral decomposition of $\mathbb{C}^n$, related to the matrix A, is given by

$$\mathbb{C}^n = \ker(A - \lambda_1 I_n)^{m_1} \oplus \cdots \oplus \ker(A - \lambda_k I_n)^{m_k}. \tag{1.1.1}$$

It is well-known fact that for every $1 \leq j \leq k$ the subspace $\ker(A - \lambda_j I_n)^{m_j}$ is e^{tA}-invariant for every $t \in \mathbb{R}$. As a consequence of (1.1.1), for each $x \in \mathbb{C}^n$ there exists $x_j \in \ker(A - \lambda_j I_n)^{m_j}$ so that

$$e^{tA}x = e^{tA}x_0 + e^{tA}x_1 + \cdots + e^{tA}x_k, \quad t \in \mathbb{R}_+.$$

Moreover, $e^{tA}x_j$ belongs to $\ker(A - \lambda_j I_n)^{m_j}$, for all $t \in \mathbb{R}$ and there exists a $\mathbb{C}^n$-valued polynomial $P_j(t)$, of degree at most $m_j - 1$, such that

$$x_j(t) = e^{\lambda_j t} P_j(t), \quad t \in \mathbb{R}, j = \overline{1, k}. \tag{1.1.2}$$

Further details may be found for example in [75] or [76, Chap. 1].

The decomposition (1.1.1) yields to

$$\mathbb{C}^n = \mathcal{X}_s \oplus \mathcal{X}_0 \oplus \mathcal{X}_u,$$

where

$$\mathcal{X}_s = \bigoplus_{j=1,\mathrm{Re}(\lambda_j)<0}^{k} \ker(A-\lambda_j I_n)^{m_j}, \quad \mathcal{X}_0 = \bigoplus_{j=1,\mathrm{Re}(\lambda_j)=0}^{k} \ker(A-\lambda_j I_n)^{m_j},$$

$$\mathcal{X}_u = \bigoplus_{j=1,\mathrm{Re}(\lambda_j)>0}^{k} \ker(A-\lambda_j I_n)^{m_j}.$$

$\mathcal{X}_s$ and $\mathcal{X}_u$ are the stable and respectively the unstable subspace of A.

The circle and closed disk of radius r which are centered in the eigenvalue $\lambda_j \in \sigma(A)$, are respectively:

$$C_r(\lambda_j) = \{z \in \mathbb{C} : |z-\lambda_j| = r\}$$

and

$$\overline{D}_r(\lambda_j) = \{z \in \mathbb{C} : |z-\lambda_j| \leq r\}$$

where r is a positive real number, enough small such that $\sigma(A) \cap \overline{D}_r(\lambda_j) = \{\lambda_j\}$. Recall that an $n \times n$ complex matrix P, verifying $P^2 = P$, is called projection. Let $1 \leq j \leq k$. Already was stated that

$$I = E_{\lambda_1} + E_{\lambda_2} + \cdots + E_{\lambda_k},$$

where $E_{\lambda_j} := E_j : \mathbb{C}^n \to \mathbb{C}^n$ is defined by $E_j x := x_j$. Obviously, $E_j, (1 \leq j \leq k)$ are projections which are called unitary spectral projections associated to the matrix A. It is well-known, [24, Chap. 7], that

$$E_j = \frac{1}{2\pi i} \oint_{C_r(\lambda_j)} (zI - A)^{-1} dz. \tag{1.1.3}$$

Recall that the matrix A is dichotomic if its spectrum does not intersect the imaginary axis, i.e. $\sigma(A) \cap i\mathbb{R} = \varnothing$.

In order to prove main theorem of this chapter we need the following proposition, which contains equivalent characterizations for exponential dichotomy. This result is certainly known but we insert it and its proof here for the sake of completeness. Further details about different concepts of dichotomy may be found in [20, Chap. 3].

Proposition 1.1.1. *The following three statements concerning the matrix A are equivalent:*

- ***(1)*** *A is dichotomic.*
- ***(2)*** *There exists a projection P, commuting with A, and there exist positive constants N_1, N_2, ν_1, ν_2 such that*

  (i) $\left\|e^{tA}Px\right\| \leq N_1 e^{-\nu_1 t}\|Px\|$, *for all* $x \in \mathbb{C}^n$, *for every* $t \geq 0$.

  (ii) $\left\|e^{tA}(I-P)x\right\| \leq N_2 e^{\nu_2 t}\|(I-P)x\|$, *for all* $x \in \mathbb{C}^n$ *and for all* $t \leq 0$.
- ***(3)*** *For each continuous and bounded function $f : \mathbb{R}_+ \to \mathbb{C}^n$, there exists a unique bounded solution, starting from the unstable subspace, of*

$$\dot{y}(t) = Ay(t) + f(t), \quad t \geq 0. \tag{A, f}$$

Proof. **(1)** $\Rightarrow$ **(2)**. A is dichotomic, so $\mathcal{X}_0 = \{0\}$ and then $\mathbb{C}^n = \mathcal{X}_s \oplus \mathcal{X}_u$. Every $x \in \mathbb{C}^n$ can be written as $x = x_s + x_u$ with $x_s \in \mathcal{X}_s$ and $x_u \in \mathcal{X}_u$. Set $P := P_s : \mathbb{C}^n \to \mathbb{C}^n$ such that $P_s x := x_s$. It is obvious that the matrix P is a projection. Moreover, using (1.1.2) it can be notice that (i) and (ii) are fulfilled for certain positive constants N_1, N_2, ν_1, ν_2.

(2) $\Rightarrow$ **(1)**. Suppose that there exists $\lambda \in \sigma(A)$, with $\mathrm{Re}(\lambda) = 0$. Then, there is $x_0 \neq 0$, $x_0 \in \mathbb{C}^n$ such that $Ax_0 = \lambda x_0$ and thus, $e^{tA}Px_0 = e^{t\lambda}Px_0$, for all $t \in \mathbb{R}$. If $Px_0 \neq 0$, then (i) yields

$$\left\|e^{tA}Px_0\right\| = \left\|e^{t\lambda}Px_0\right\| = \|Px_0\| \leq N_1 e^{-\nu_1 t}\|Px_0\|, \quad \forall t \geq 0,$$

which is a contradiction. If $Px_0 = 0$, then $(I-P)x_0 \neq 0$ and (ii) produces

$$\begin{aligned} \left\|e^{tA}(I-P)x_0\right\| &= \left\|e^{t\lambda}(I-P)x_0\right\| \\ &= \|(I-P)x_0\| \leq N_2 e^{\nu_2 t}\|(I-P)x_0\|, \quad \forall t \leq 0, \end{aligned}$$

which is also a contradiction.

(1) $\Rightarrow$ **(3)**. Since the matrix A is dichotomic, the map

$$t \mapsto y(t) := \int_0^t e^{(t-s)A}Pf(s)ds - \int_t^\infty e^{(t-s)A}(I-P)f(s)ds,$$

is a solution of (A, f). See [20, Chap. 3] for more details. Indeed, the second integral is well defined because, from **(2)**, (ii), have that

$$\begin{aligned} \textstyle\int_t^\infty \left\|e^{(t-s)A}(I-P)f(s)\right\| ds &\leq \textstyle\int_t^\infty N_2 e^{\nu_2(t-s)}\|I-P\|\|f\|_\infty ds \\ &= \tfrac{N_2}{\nu_2}\|I-P\|\|f\|_\infty. \end{aligned}$$

Also from (**2**), the solution is bounded, with

$$\sup_{t\geq 0}|y(t)| \leq \left(\frac{N_1}{\nu_1}\|P\| + \frac{N_2}{\nu_2}\|I-P\|\right)\sup_{t\geq 0}|f(t)|.$$

Moreover, $y(0) = -\int_0^\infty e^{-sA}(I-p)f(s)ds \in \mathcal{X}_u$ because $\mathcal{X}_u$ is a closed subspace and it is invariant under any exponential of A.

The uniqueness remains to be investigated. Suppose that there exist two bounded solutions of (A, f), denoted by y_1 and y_2. Then

$$y_1(t) = e^{tA}z_1 + \int_0^t e^{(t-s)A}f(s)ds$$

$$y_2(t) = e^{tA}z_2 + \int_0^t e^{(t-s)A}f(s)ds,$$

with $z_1, z_2 \in \mathcal{X}_u$.

Their difference is bounded and $y_1(t) - y_2(t) = e^{tA}(z_1 - z_2)$. From the boundedness of $y_1 - y_2$, $z_1 - z_2 \in \mathcal{X}_s$. On the other hand, $z_1, z_2 \in \mathcal{X}_u$ yields $z_1 - z_2 \in \mathcal{X}_u$. But $\mathcal{X}_s \cap \mathcal{X}_u = \{0\}$ and therefore $z_1 = z_2$.

(3) $\Rightarrow$ (1). Suppose that, there exists $\lambda \in \sigma(A)$, with $\mathrm{Re}(\lambda) = 0$. Then, there exists $x_0 \neq 0$ such that $Ax_0 = \lambda x_0$, and therefore $e^{tA}x_0 = e^{\lambda t}x_0$, for all $t \in \mathbb{R}$.

Set $f(t) := e^{\lambda t}x_0$, for all $t \geq 0$. Obviously, f is a bounded and continuous function and from the hypothesis, there exists a unique $z_0 \in \mathcal{X}_u$ such that the map

$$t \mapsto e^{tA}z_0 + \int_0^t e^{(t-s)A}e^{\lambda s}x_0 ds$$

is bounded on $\mathbb{R}_+$. But,

$$\begin{aligned} e^{tA}z_0 + \int_0^t e^{(t-s)A}e^{\lambda s}x_0 ds &= e^{tA}z_0 + \int_0^t e^{(t-s)\lambda}e^{\lambda s}x_0 ds \\ &= e^{tA}z_0 + \int_0^t e^{\lambda t}x_0 ds \\ &= e^{tA}z_0 + te^{\lambda t}x_0. \end{aligned}$$

If $z_0 = 0$, obviously we arrive at a contradiction, since the map $t \mapsto te^{\lambda t}x_0$ is unbounded. If $z_0 \neq 0$, by spectral decomposition theorem there are two positive constants N and ν such that $\|e^{tA}z_0\| \geq Ne^{\nu t}$ for all $t \geq 0$, and a contradiction arise again. $\square$

1.2 Hyers-Ulam stability and exponential dichotomy for linear differential systems

We can see an ε-approximate solution of (A), as an exact solution of (A, ρ) corresponding to a forced term $\rho(\cdot)$ which is bounded by ε. The next definition arise naturally.

Definition 1.2.1. Let ε be a given positive number. The matrix A (or the system (A)) is called Hyers-Ulam stable if

$$(\exists L)(\forall \rho = \rho(t))(\forall x)(\exists x_0)$$
$$\left(\begin{array}{c} L \geq 0, \rho \text{ is continuous, } \|\rho\|_\infty \leq \varepsilon, x, x_0 \in \mathbb{C}^n \text{ and} \\ \sup\limits_{t\geq 0} \left| e^{tA}(x - x_0) + \int_0^t e^{(t-s)A}\rho(s)ds \right| \leq L\varepsilon \end{array} \right).$$

Theorem 1.2.1. *The matrix A is Hyers-Ulam stable if and only if it is dichotomic.*

Proof. Necessity. Suppose that A is not dichotomic, i.e. $\mathcal{X}_0 \neq \{0\}$. Therefore, there exists λ_j in $\sigma(A)$, with $\lambda_j = i\mu_j$, $\mu_j \in \mathbb{R}$. Let $\varepsilon > 0$ be fixed and set $\rho(t) := e^{i\mu_j t}u_0$, with $\|u_0\| \leq \varepsilon$. Obviously, the function ρ is continuous and bounded by ε. By assumption, the matrix A is Hyers-Ulam stable. Hence, the solution

$$y(t) = e^{tA}(x - x_0) + \int_0^t e^{(t-s)A}\rho(s)ds, \quad x, x_0 \in \mathbb{C}^n,$$

of the Cauchy Problem

$$\begin{cases} \dot{y}(t) = Ay(t) + \rho(t), \quad t \geq 0 \\ y(0) = x - x_0, \end{cases} \tag{A, ρ}$$

is bounded by $L\varepsilon$. By using the spectral decomposition theorem, there exists an $n \times n$ matrix-valued polynomial $P_j(t)$ having the degree at most $m_j - 1$, such that

$$E_j e^{tA} = e^{i\mu_j t} P_j(t), \qquad \forall t \geq 0. \tag{1.2.1}$$

Then, the map

$$t \mapsto E_j \left[e^{tA}(x - x_0) + \int_0^t e^{(t-s)A} \rho(s) ds \right], \quad x, x_0 \in \mathbb{C}^n,$$

should also be bounded by $L\varepsilon$.

On the other hand,

$$E_j \left[e^{tA}(x - x_0) + \int_0^t e^{(t-s)A} \rho(s) ds \right] =$$
$$= e^{i\mu_j t} P_j(t)(x - x_0) + \int_0^t E_j(e^{(t-s)A} \rho(s)) ds,$$

and

$$\begin{aligned} \int_0^t E_j e^{(t-s)A} \rho(s) ds &= \int_0^t E_j e^{(t-s)A} e^{i\mu_j s} u_0 ds \\ &= \int_0^t e^{i\mu_j s} e^{(t-s) i\mu_j} P_j(t-s) u_0 ds \\ &= e^{i\mu_j t} \int_0^t P_j(t-s) u_0 ds = e^{i\mu_j t} q_j(t), \end{aligned}$$

where

$$q_j(t) = \int_0^t P_j(s) u_0 ds,$$

is a polynomial, as well. By choosing an appropriate $u_0 \neq 0$, have that

$$\begin{aligned} \deg[P_j(t)(x - x_0)] \leq \deg[P_j(t)] &= \deg[P_j(t) u_0] < 1 + \deg[P_j(t)] \\ &= \deg[q_j(t)]. \end{aligned}$$

Therefore, the solution $y(t) = e^{i\mu_j t} \left[P_j(t)(x - x_0) + q_j(t) \right]$, is unbounded and we arrive at a contradiction.

Sufficiency. The absolute constant L will be settled later.

Let $\rho : \mathbb{R}_+ \to \mathbb{C}^n$ be a continuous function, with $\|\rho\|_\infty \leq \varepsilon$ and let $x \in \mathbb{C}^n$. By Proposition 1.1.1, there exist a unique bounded solution $y(\cdot)$ of (A, ρ) starting from the subspace $\mathcal{X}_u$. Let denote $u_0 := y(0)$. Then

$$\begin{aligned}
\|y(t)\| &= \left\| e^{tA}u_0 + \int_0^t e^{(t-s)A}\rho(s)ds \right\| \\
&= \left\| \int_0^t e^{(t-s)A}P\rho(s)ds - \int_t^\infty e^{(t-s)A}(I-P)\rho(s)ds \right\| \\
&\leq \left(\tfrac{N_1}{\nu_1}\|P\| + \tfrac{N_2}{\nu_2}\|I-P\| \right)\varepsilon.
\end{aligned}$$

The desired assertion follows by choosing $L = \left(\frac{N_1}{\nu_1}\|P\| + \frac{N_2}{\nu_2}\|I-P\| \right)$ and setting $x_0 = x - u_0$. □

1.3 Hyers-Ulam stability and exponential dichotomy for linear differential equations of order n

Let us consider the following differential equations for $t \in \mathbb{R}_+$

$$x^{(n)}(t) = a_1x^{(n-1)}(t) + \ldots + a_{n-1}x'(t) + a_nx(t), \qquad (a_1, a_2, \ldots, a_n)$$

and

$$x^{(n)}(t) = a_1x^{(n-1)}(t) + \ldots + a_nx(t) + \theta(t), \qquad (a_1, a_2, \ldots, a_n, \theta)$$

where $\theta : \mathbb{R}_+ \to \mathbb{C}$ is a continuous function and $a_j \in \mathbb{C}, j = \overline{1, n}$.

To the differential equation $(a_1, a_2, \ldots, a_n, \theta)$ we associate the system

$$\dot{X}(t) = AX(t) + \Theta(t), \quad X(t), \quad \Theta(t) \in \mathbb{C}^n,$$

where

$$X(t) = \begin{pmatrix} x(t) & x'(t) & x''(t) & \cdots & x^{(n-1)}(t) \end{pmatrix}^T,$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{pmatrix},$$

is an $n \times n$ matrix and $\Theta(t) = \begin{pmatrix} 0 & 0 & \cdots & 0 & \theta(t) \end{pmatrix}^T$.

At this moment, we are in the position to introduce the notion of Hyers-Ulam stability for linear scalar differential equations.

Definition 1.3.1. As before, ε is a given positive number. The differential equation $(a_1, a_2, \ldots, a_n)$ is called Hyers-Ulam stable if

$$(\exists L)(\forall \theta = \theta(t))(\forall x)(\exists x_0)$$

$$\begin{pmatrix} L \geq 0, \theta \text{ continuous, } |\theta|_\infty \leq \varepsilon, x, x_0 \in \mathbb{C}^n \text{ and} \\ \sup\limits_{t \geq 0} \left| \text{row}_1 \left[e^{tA}(x - x_0) + \int_0^t e^{(t-s)A}\Theta(s)ds \right] \right| \leq L\varepsilon \end{pmatrix}.$$

For every $z \in \mathbb{C}$, consider the $n \times n$ matrix

$$zI - A = \begin{pmatrix} z & -1 & 0 & \cdots & 0 \\ 0 & z & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & z - a_1 \end{pmatrix}.$$

If $z \in \rho(A) := \mathbb{C} \setminus \sigma(A)$, this matrix is invertible and becomes obvious that the n-th column of its inverse is given by

$$col_n[(zI - A)^{-1}] = \frac{1}{P_A(z)} \cdot \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{n-1} \end{pmatrix}.$$

Theorem 1.3.1. *The following statements are equivalent:*

- ***(1)*** *The differential equation* $(a_1, a_2, \ldots, a_n)$ *is Hyres-Ulam stable.*
- ***(2)*** *The matrix* A *is dichotomic.*
- ***(3)*** *The characteristic equation*

$$\lambda^n - a_1\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \ldots - a_n = 0 \tag{1.3.1}$$

has no roots on the imaginary axis.

Proof. The statements **(2)** and **(3)** are equivalent since the spectrum of A is equal to the set of all roots of (1.3.1).

(1) $\Rightarrow$ **(2)**. Suppose that A is not dichotomic. Then, there exists λ_j in $\sigma(A)$, with $\lambda_j = i\mu_j$, $\mu_j \in \mathbb{R}$. Let $\varepsilon > 0$ and set $\Theta(t) := e^{i\mu_j t}u_0$, with $\|u_0\| \leq \varepsilon$. Obviously, the function Θ is continuous and bounded by ε. The differential equation $(a_1, a_2, \ldots, a_n)$ is Hyers-Ulam stable, so

$$\sup_{t\geq 0} \left\| \text{row}_1 \left[e^{tA}(x - x_0) + \int_0^t e^{(t-s)A}\Theta(s)ds \right] \right\| \leq L\varepsilon.$$

Then $\text{row}_1 \left[E_j(e^{tA}(x - x_0) + \int_0^t e^{(t-s)A}\Theta(s)ds) \right]$ is bounded by $L\varepsilon$, as well. On the other hand, in view of (1.2.1) one has

$$\text{row}_1 \left[E_j(e^{tA}(x - x_0) + \int_0^t e^{(t-s)A}\Theta(s)ds) \right] =$$

$$= \text{row}_1 \left[e^{i\mu_j t}P_j(t)(x - x_0) \right] + \text{row}_1 \left[\int_0^t E_j e^{(t-s)A}\Theta(s)ds) \right].$$

We already know, from the proof of Theorem 1.2.1 that the degree of the vectorial valued polynomial $\text{row}_1 [P_j(t)(x - x_0)]$ is less than or equal to $m_j - 1$. Our next step is to find the degree of the scalar polynomial

$$e^{-i\mu_j t}\text{row}_1 \left[\int_0^t E_j e^{(t-s)A}\Theta(s)ds) \right].$$

We need two lemmas.

Lemma 1.3.2. *With the above notations, have that*

$$e^{-i\mu_j t}E_j e^{tA} = \sum_{k=0}^{m_j-1} E_j \frac{(A - i\mu_k I)^k}{k!} t^k := Q_{jA}(t). \tag{1.3.2}$$

Proof. From the spectral decomposition theorem, see (1.2.1), it follows

$$E_j e^{tA} = e^{t\lambda_j} E_j P_j(t) = e^{t\lambda_j}(t^{m_j-1}E_j B_{m_j-1} + \cdots + tE_j B_1 + E_j B_0),$$

where $B_1, \ldots, B_{m_j-1}$ are $n \times n$ constant matrices. For $t = 0$ have that $E_j = E_j B_0$ and then:

$$e^{-t\lambda_j}E_j e^{tA} - E_j = (e^{t(A-\lambda I)} - I)E_j = t^{m_j-1}E_j B_{m_j-1} + \cdots + tE_j B_1.$$

By dividing both sides of the previous equation by t and taking limit as $t \to 0$, we obtain

$$E_j \frac{(A - \lambda_j I)}{1!} = E_j B_1.$$

Again, following the similar step as above we have that

$$E_j \frac{(A - \lambda_j I)^2}{2!} = E_j B_2.$$

Finally, at the $m_j - th$ step we obtain

$$E_j \frac{(A - \lambda_j I)^{m_j-1}}{(m_j - 1)!} = E_j B_{m_j-1},$$

which completes the proof. □

By $[M]_{ij}$ we denote the element of the matrix M located at the intersection of the i-th row and the j-th column.

Lemma 1.3.3. *The degree of the scalar polynomial* $[Q_{jA}(t)]_{1n}$ *given in (1.3.2) is equal to* $m_j - 1$.

Proof. Let us consider the scalar polynomial $q_j(z) := \frac{P_A(z)}{(z-\lambda_j)^{m_j}}$. Clearly, the map $z \mapsto \frac{1}{q_j(z)}$ is analytic on $\overline{D}_r(\lambda_j)$. In order to show that the degree of $[Q_{jA}(t)]_{1n}$ is equal to $m_j - 1$, it is enough to prove that the coefficient of its last term, i.e.

$$a_{1n}^{(m_j-1)} := \frac{1}{2\pi i} \oint_{C_r(\lambda_j)} \text{row}_1 \left[\frac{(A-\lambda_j I)^{m_j-1}}{(m_j-1)!} \right] \cdot \text{col}_n[R(z,A)]dz,$$

is a nonzero scalar. For this, we analyze two particular cases and then the general case appears naturally.

For $m_j = 1$, $[Q_{jA}(t)]_{1n} = [E_j]_{1n}$ and therefore

$$\begin{aligned} a_{1n}^{(0)} &= \tfrac{1}{2\pi i} \oint_{C_r(\lambda_j)} \tfrac{1}{P_A(z)} dz \\ &= \tfrac{1}{2\pi i} \oint_{C_r(\lambda_j)} \tfrac{\frac{1}{q_j(z)}}{z-\lambda_j} dz \\ &= \tfrac{1}{q_j(\lambda_j)} \neq 0, \end{aligned}$$

where the Cauchy integral formula was used. For $m_j = 2$, have that

$$\begin{aligned} \text{row}_1[\tfrac{A-\lambda_j I}{1!}] \cdot col_n[R(z,A)] &= \tfrac{1}{P_A(z)} \begin{pmatrix} -\lambda_j & 1 & 0 & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{n-1} \end{pmatrix} \\ &= \tfrac{z-\lambda_j}{P_A(z)} = \tfrac{\frac{1}{q_j(z)}}{z-\lambda_j}, \end{aligned}$$

which yields

$$\begin{aligned} a_{1n}^{(1)} &= \tfrac{1}{2\pi i} \oint_{C_r(\lambda_j)} \text{row}_1[\tfrac{A-\lambda_j I}{1!}] \cdot col_n[R(z,A)]dz \\ &= \tfrac{1}{2\pi i} \oint_{C_r(\lambda_j)} \tfrac{\frac{1}{q_j(z)}}{z-\lambda_j} dz \\ &= \tfrac{1}{q_j(\lambda_j)} \neq 0. \end{aligned}$$

Following the same pattern, we obtain:

$$\text{row}_1\left[\frac{(A-\lambda_j I)^{(m_j-1)}}{(m_j-1)!}\right]\cdot \text{col}_n[R(z,A)] =$$

$$= \frac{1}{P_A(z)(m_j-1)!}\cdot\left(\begin{array}{ccc} C^0_{m_j-1}(-\lambda_j)^{m_j-1} & \cdots & C^{m_j-1}_{m_j-1}(-\lambda_j)^0 \end{array}\right)\cdot\left(\begin{array}{c} 1 \\ z \\ \vdots \\ z^{n-1}\end{array}\right)$$

$$= \frac{\sum_{k=o}^{m_j-1} C^k_{m_j-1} z^k(-\lambda_j)^{m_j-1-k}}{(m_j-1)!P_A(z)}$$

$$= \frac{(z-\lambda_j)^{m_j-1}}{(m_j-1)!P_A(z)} = \frac{\frac{1}{q_j(z)}}{(m_j-1)!(z-\lambda_j)}$$

and applying again the Cauchy theorem, one has

$$\begin{array}{rcl} a^{(m_j-1)}_{1n} & = & \frac{1}{2\pi i}\oint_{C_r(\lambda_j)} \text{row}_1\left[\frac{(A-\lambda_j I)^{(m_j-1)}}{(m_j-1)!}\right]\cdot \text{col}_n[R(z,A)]dz \\ & = & \frac{1}{(m_j-1)!q_j(\lambda_j)} \end{array},$$

which is a nonzero scalar and get the desired assertion.

□

Returning to the proof of the theorem, we remark that

$$\text{row}_1\left[\int_0^t E_j e^{(t-s)A}\Theta(s)ds\right] =$$

$$= e^{i\mu_j t}\int_0^t\left[E_j\sum_{k=0}^{m_j-1}\frac{(A-\lambda_j I)^k}{k!}(t-s)^k\right]_{1n} v_0 ds$$

$$= e^{i\mu_j t}\int_0^t \frac{1}{(m_j-1)!q_j(\lambda_j)}(t-s)^{m_j-1}v_0 ds,$$

and then

$$\text{row}_1 \left[E_j(e^{tA}(z - z_0) + \int_0^t e^{(t-s)A}\Theta(s)ds) \right] =$$

$$\text{row}_1 \left[e^{i\mu_j t} P_j(t)(z - z_0) \right] + e^{i\mu_j t} \int_0^t \frac{1}{(m_j - 1)! q_j(\lambda_j)} (t - s)^{m_j - 1} v_0 ds.$$

The degree of the polynomial $e^{-i\mu_j t}\text{row}_1[\int_0^t E_j e^{(t-s)A}\Theta(s)ds]$ is equal to the degree of $[\int_0^t \frac{1}{q_j(\lambda_j)}(t-s)^{m_j-1}v_0 ds]$, which is equal to m_j. Here v_0 is an appropriate nonzero complex scalar. This produces a contradiction.

(2) $\Rightarrow$ **(1)**. The assertion follows via the proof of sufficiency of Theorem 1.2.1.

□

Chapter 2

Hyers-Ulam stability and discrete dichotomy

We prove that the discrete system

$$X_{n+1} = AX_n, \quad n \in \mathbb{Z}_+$$

is Hyers-Ulam stable if and only if the matrix A possess a discrete dichotomy. Also we prove that the scalar difference equation of order m

$$x_{n+m} = a_1 x_{n+m-1} + a_2 x_{n+m-2} + \cdots + a_m x_n, \quad n \in \mathbb{Z}_+,$$

is Hyers-Ulam stable if and only if the algebraic equation

$$z^m = a_1 z^{m-1} + \cdots + a_{m-1} z + a_m, \quad z \in \mathbb{C}$$

has no roots on the unit circle.

We outline the Hyers-Ulam problem for a difference linear system of order m driven by an $m \times m$ complex matrix A. Consider the system

$$X_{n+1} = AX_n, \quad n \in \mathbb{Z}_+. \tag{2.0.1}$$

Let ε be a positive real number. A $\mathbb{C}^m$-valued sequence $(Y_n)_{n\in\mathbb{Z}_+}$ is called ε-approximate solution for (2.0.1) if

$$\|Y_{n+1} - AY_n\| \leq \varepsilon, \quad \forall n \in \mathbb{Z}_+.$$

The matrix A is said to be Hyers-Ulam stable if there exists a nonnegative constant L such that, for every ε-approximate solution Γ of (2.0.1) there exists an exact solution Θ of (2.0.1) such that

$$\sup_{n\in\mathbb{Z}_+} \|\Gamma_n - \Theta_n\| \leq L \cdot \varepsilon.$$

2.1 Notations and preliminary results

Let

$$P_A(z) = (z - \lambda_1)^{m_1} \cdots (z - \lambda_k)^{m_k},$$

be the characteristic polynomial of A. It is well-known, that the spectrum of A, denoted by $\sigma(A)$, is the set consisting of all complex roots of P_A. Here $m_1, \ldots, m_k$ are the multiplicities of the eigenvalues $\lambda_1, \ldots, \lambda_k$, and thus $m_1 + \cdots + m_k = m$. The spectral decomposition of the state space $\mathbb{C}^m$, related to the matrix A, is given by

$$\mathbb{C}^m = \ker(A - \lambda_1 I_m)^{m_1} \oplus \cdots \oplus \ker(A - \lambda_k I_m)^{m_k}. \tag{2.1.1}$$

It is well-known, see for example [16], that for every $1 \leq j \leq k$ and every $n \in \mathbb{Z}_+$, the subspace $\ker(A - \lambda_j I_m)^{m_j}$ is A^n-invariant. As a consequence of (2.1.1), for an invertible matrix A and $x \in \mathbb{C}^m$, there exists $x_j \in \ker(A - \lambda_j I_m)^{m_j}$ such that

$$A^n x = A^n x_1 + A^n x_2 + \cdots + A^n x_k, \quad n \in \mathbb{Z}_+, \tag{2.1.2}$$

$A^n x_j$ belongs to $\ker(A - \lambda_j I_m)^{m_j}$, for all $n \in \mathbb{Z}_+$ and there exists a $\mathbb{C}^m$-valued polynomial $q_j(n)$ in n, of degree at most $m_j - 1$, such that

$$A^n x_j = \lambda_j^n q_j(n), \quad n \in \mathbb{Z}_+, \quad 1 \leq j \leq k. \tag{2.1.3}$$

Further details may be found for example in [16], [76, Chap. 1].

The decomposition (2.1.1) yields to

$$\mathbb{C}^m = \mathcal{X}_s(A) \oplus \mathcal{X}_0(A) \oplus \mathcal{X}_u(A),$$

where

$$\mathcal{X}_s(A) = \bigoplus_{j=1,|\lambda_j|<1}^{k} \ker(A - \lambda_j I_m)^{m_j}, \quad \mathcal{X}_0(A) = \bigoplus_{j=1,|\lambda_j|=1}^{k} \ker(A - \lambda_j I_m)^{m_j},$$

$$\mathcal{X}_u(A) = \bigoplus_{j=1,|\lambda_j|>1}^{k} \ker(A - \lambda_j I_m)^{m_j}.$$

Recall that the matrix A is said to be dichotomic (or that it posses a discrete dichotomy) if its spectrum does not intersect the unit circle, i.e. $\sigma(A) \cap \Gamma = \emptyset$. For the facts and definitions regarding elementary projection see Chapter 1, Page 11 and 12.

The proofs of some results of this chapter are based on the following proposition, which contains equivalent characterizations for the concept of dichotomy.

Proposition 2.1.1. *The following three statements, concerning an invertible $m \times m$ matrix A, are equivalent:*

- ***(1)*** *A posses a discrete dichotomy.*
- ***(2)*** *There exist a projection P, commuting with A, and four positive constants N_1, N_2, ν_1, ν_2 such that*
  - *(i)* $\|A^n Px\| \leq N_1 e^{-\nu_1 n} \|Px\|$, *for all* $x \in \mathbb{C}^m$, *and* $n \in \mathbb{Z}_+$.
  - *(ii)* $\|A^n (I_m - P)x\| \leq N_2 e^{\nu_2 n} \|(I_m - P)x\|$, *for all* $x \in \mathbb{C}^m$ *and* $n \in \mathbb{Z}_- := \{0, -1, -2, \cdots\}$.
- ***(3)*** *For each bounded $\mathbb{C}^m$-valued sequence (F_n), there exists a unique bounded solution of the difference equation*

$$X_{n+1} = AX_n + F_n, \quad n \in \mathbb{Z}_+ \tag{2.1.4}$$

starting by the unstable subspace of A.

Moreover, this solution is given by

$$\psi_n := \sum_{k=0}^{n-1} A^{n-k-1} P F_k - \sum_{j=n}^{\infty} A^{n-j-1}(I_m - P)F_j, \quad n \geq 1.$$

Proof. **(1)** $\Rightarrow$ **(2)**. A is dichotomic, so $\mathcal{X}_0(A) = \{0\}$ and then $\mathbb{C}^m = \mathcal{X}_s(A) \oplus \mathcal{X}_u(A)$. Every $x \in \mathbb{C}^m$ can be decomposed as $x = x_s + x_u$, with $x_s \in \mathcal{X}_s(A)$ and $x_u \in \mathcal{X}_u(A)$. Set $P_s := P : \mathbb{C}^m \to \mathbb{C}^m$ such that $Px := x_s$. The matrix P is a projection. Moreover, from (2.1.3), (i) and (ii) are fulfilled for certain positive constants N_1, N_2, ν_1, ν_2.

(2) $\Rightarrow$ **(1)**. Suppose that A is not dichotomic. Thus, there exist $\lambda \in \sigma(A)$, with $|\lambda| = 1$ and $x_0 \neq 0, x_0 \in \mathbb{C}^m$ such that $Ax_0 = \lambda x_0$ and therefore, $A^n P x_0 = \lambda^n P x_0$, for all $n \in \mathbb{Z}$. If $Px_0 \neq 0$, (i) yields

$$\|A^n P x_0\| = \|\lambda^n P x_0\| = \|Px_0\| \leq N_1 e^{-\nu_1 n}\|Px_0\|, \quad \forall n \in \mathbb{Z}_+,$$

which is a contradiction. If $Px_0 = 0$, $(I_m - P)x_0 \neq 0$ and (ii) produces

$$\begin{aligned} \|A^n(I_m - P)x_0\| &= \|\lambda^n(I_m - P)x_0\| \\ &= \|(I_m - P)x_0\| \leq N_2 e^{\nu_2 n}\|(I_m - P)x_0\|, \quad \forall n \in \mathbb{Z}_-, \end{aligned}$$

which is also a contradiction.

(1) $\Rightarrow$ **(3)**. Since the matrix A is dichotomic, the map

$$n \mapsto \psi_n := \sum_{k=0}^{n-1} A^{n-k-1} P F_k - \sum_{j=n}^{\infty} A^{n-j-1}(I_m - P)F_j,$$

is a solution of (2.1.4). Indeed, the second sum is well defined, since from **(2)**, (ii), have that

$$\begin{aligned} \left\|\sum_{j=n}^{\infty} A^{n-j-1}(I_m - P)F_j\right\| &\leq \sum_{j=n}^{\infty} N_2 e^{\nu_2(n-j-1)}\|I_m - P\|\|F\|_\infty \\ &= \frac{N_2 \cdot e^{-\nu_2}}{1-e^{-\nu_2}}\|I_m - P\|\|F\|_\infty \\ &\leq \frac{N_2}{e^{\nu_2}-1}\|I_m - P\|\|F\|_\infty. \end{aligned}$$

Also, from **(2)**, the solution (ψ_n) is bounded, as the following estimate shows.

$$\sup_{n\geq 0}|\psi_n| \leq \left(\frac{N_1}{e^{\nu_1}-1}\|P\| + \frac{N_2}{e^{\nu_2}-1}\|I_m - P\|\right)\sup_{n\geq 0}|F_n|.$$

In addition, $\psi_0 = -\sum_{j=0}^{\infty} A^{-j-1}(I_m - P)F_j$ belongs to $\mathcal{X}_u(A)$, because $\mathcal{X}_u(A)$ is a closed subspace and it is invariant for any power of A.

Now we prove the uniqueness. Suppose that there exist two bounded solutions of (2.1.4)

$$\psi_1(n) = A^n x_{01} + \sum_{k=0}^{n-1} A^{n-k-1} F_k$$

$$\psi_2(n) = A^n x_{02} + \sum_{k=0}^{n-1} A^{n-k-1} F_k,$$

with $x_{01}, x_{02} \in \mathcal{X}_u(A)$. Then, their difference is bounded and $\psi_1(n) - \psi_2(n) = A^n(x_{01} - x_{02})$. From the boundedness of $\psi_1 - \psi_2$, have that $x_{01} - x_{02}$ belongs to $\mathcal{X}_s(A)$ and, on the other hand, $x_{01}, x_{02} \in \mathcal{X}_u(A)$, yields $x_{01} - x_{02} \in \mathcal{X}_u(A)$. But, $\mathcal{X}_s(A) \cap \mathcal{X}_u(A) = \{0\}$ and therefore $x_{01} = x_{02}$.

(3) ⇒ (1). Suppose that, there exists $\lambda \in \sigma(A)$, with $|\lambda| = 1$, i.e. $\lambda = e^{i\mu}$ with $\mu \in \mathbb{R}$. Then, there exists $x_0 \neq 0$ such that $Ax_0 = \lambda x_0$, and $A^n x_0 = \lambda^n x_0$, for all $n \in \mathbb{Z}_+$. Set $F_n := \lambda^n x_0$. Obviously, the sequence (F_n) is bounded and from the assumption, there exists a unique $x_u \in \mathcal{X}_u(A)$ such that the map

$$n \mapsto A^n x_u + \sum_{k=0}^{n-1} A^{n-k-1} F_k$$

is bounded on $\mathbb{Z}_+$. But,

$$\begin{aligned} A^n x_u + \sum_{k=0}^{n-1} A^{n-k-1} \lambda^k x_0 &= A^n x_u + \sum_{k=0}^{n-1} e^{i\mu(n-k-1)} e^{i\mu k} x_0 \\ &= A^n x_u + \sum_{k=0}^{n-1} e^{i\mu(n-1)} x_0 \\ &= A^n x_u + n e^{i\mu(n-1)} x_0. \end{aligned}$$

If $x_u = 0$, the map $n \mapsto ne^{i\mu(n-1)} x_0$ is unbounded on $\mathbb{Z}_+$ and we have a contradiction. If $x_u \neq 0$, in view of the spectral decomposition theorem, there are two positive constants N and ν such that $\|A^n x_u\| \geq N e^{\nu n}$, for all $n \in \mathbb{Z}_+$, and a contradiction arises again. □

Remark : The above Proposition 2.1.1 can also be stated for non-invertible matrices. In this case, the statement **(2)** is replaced by the next one.

(2') There exists a projection P, commuting with A, such that the linear map $A_{|\ker(P)} : \ker(P) \to \ker(P)$ is invertible and there exist four positive constants N_1, N_2, ν_1, ν_2 such that

(i) $\|A^n Px\| \le N_1 e^{-\nu_1 n}\|Px\|$, for all $x \in \mathbb{C}^m$, and $n \in \mathbb{Z}_+$.

(ii) $\|A^n(I_m - P)x\| \le N_2 e^{\nu_2 n}\|(I_m - P)x\|$, for all $x \in \mathbb{C}^m$ and $n \in \mathbb{Z}_-$, where $A^n y := [A^{-1}_{|\ker(P)}]^{-n} y$ for all $y \in \ker(P)$ and all $n \in \mathbb{Z}_-$.

2.2 Hyers-Ulam stability and dichotomy for linear difference systems

We can see an ε-approximate solution of (2.0.1) as an exact solution of (2.1.4) corresponding to a forced term (Γ_n) bounded by ε. Therefore, the following definition arises naturally.

Definition 2.2.1. Let ε be a given positive number. The matrix A (or the system (2.0.1)) is called Hyers-Ulam stable if

$$(\exists L)(\forall(\Gamma_n))(\forall x)(\exists x_0)\left(\begin{array}{l} L \ge 0, \sup\limits_{n\ge 0}\|(\Gamma_n)\| \le \varepsilon, x, x_0 \in \mathbb{C}^m \text{ and} \\ \sup\limits_{n\ge 0}\left|A^n(x-x_0) + \sum\limits_{k=0}^{n-1} A^{n-k-1}\Gamma_k\right| \le L\varepsilon \end{array}\right).$$

Theorem 2.2.1. *The matrix A is Hyers-Ulam stable if and only if it posses a discrete dichotomy.*

Proof. Necessity. Suppose that A is not dichotomic, i.e. $\mathcal{X}_0(A) \neq \{0\}$. Then, there exists λ_j in $\sigma(A)$, with $\lambda_j = e^{i\mu_j}$, $\mu_j \in \mathbb{R}$. Let $\varepsilon > 0$ be fixed and let $\Gamma_n := e^{i\mu_j n}u_0$, with $u_0 \in \mathbb{C}^m$ and $\|u_0\| \le \varepsilon$. By assumption, the matrix A is Hyers-Ulam stable. Thus, the solution (ψ_n),

$$\psi_n = A^n(x - x_0) + \sum_{k=0}^{n-1} A^{n-k-1}\Gamma_k, \quad x, x_0 \in \mathbb{C}^m,$$

of the discrete Cauchy Problem

$$\begin{cases} y_{n+1} = Ay_n + \Gamma_n, & n \in \mathbb{Z}_+ \\ y_0 = x - x_0, \end{cases} \tag{2.2.1}$$

is bounded by $L\varepsilon$.

By using the spectral decomposition theorem, there exists an $m\times m$ matrix-valued polynomial $P_j(n)$, having the degree at most $m_j - 1$, such that

$$E_j A^n = \lambda_j^n P_j(n), \qquad \forall n \in \mathbb{Z}_+. \tag{2.2.2}$$

Then, the map

$$n \mapsto \phi_n := E_j \left[A^n(x - x_0) + \sum_{k=0}^{n-1} A^{n-k-1}\Gamma_k \right],$$

should be bounded by $L\varepsilon$.

In view of (2.2.2), can write

$$\phi_n = e^{i\mu_j n} P_j(n)(x - x_0) + \sum_{k=0}^{n-1} E_j A^{n-k-1}\Gamma_k,$$

and

$$\begin{aligned} \sum_{k=0}^{n-1} E_j A^{n-k-1}\Gamma_k &= \sum_{k=0}^{n-1} E_j A^{n-k-1} e^{i\mu_j k} u_0 \\ &= \sum_{k=0}^{n-1} e^{i\mu_j k} e^{(n-k-1)i\mu_j} P_j(n-k-1) u_0 ds \\ &= e^{i\mu_j(n-1)} \sum_{k=0}^{n-1} P_j(n-k-1)u_0 = e^{i\mu_j(n-1)} q_j(n), \end{aligned}$$

where

$$q_j(n) = \sum_{k=0}^{n-1} P_j(n-k-1) u_0 ds,$$

is a polynomial as well. By choosing an appropriate vector $u_0 \neq 0$, we have

$$\begin{aligned} \deg[P_j(n)(x - x_0)] \le \deg[P_j(n)] &= \deg[P_j(n)u_0] \\ &< 1 + \deg[P_j(n)] \\ &= \deg[q_j(n)]. \end{aligned}$$

Taking into account the representation,

$$\phi_n = e^{i\mu_j n} \left[P_j(n)(x - x_0) + e^{-i\mu_j} q_j(n) \right],$$

the sequence (ϕ_n) is unbounded, and a contradiction arises.

Sufficiency. Let $\Gamma : \mathbb{Z}_+ \to \mathbb{C}^m$, with $\sup_{n\in\mathbb{Z}_+} \|\Gamma_n\| \le \varepsilon$, and let $x \in \mathbb{C}^m$. In view of Proposition 2.1.1, there exists a unique bounded solution (ψ_n) of (2.2.1), starting from the subspace $\mathcal{X}_u(A)$. Set $u_0 := \psi_0$. Then,

$$\begin{aligned} \|\psi_n\| &= \left\| A^n u_0 + \sum_{k=0}^{n-1} A^{n-k-1}\Gamma_k \right\| \\ &= \left\| \sum_{k=0}^{n-1} A^{n-k-1}P\Gamma_k - \sum_{j=n}^{\infty} A^{n-j-1}(I_m - P)\Gamma_j \right\| \\ &\leq \left(\tfrac{N_1}{e^{\nu_1}-1}\|P\| + \tfrac{N_2}{e^{\nu_2}-1}\|I_m - P\|\right)\varepsilon, \end{aligned}$$

and we are getting the assertion by choosing

$$L = \left(\frac{N_1}{e^{\nu_1}-1}\|P\| + \frac{N_2}{e^{\nu_2}-1}\|I_m - P\|\right),$$

and by setting, $x_0 = x - u_0$. □

2.3 Hyers-Ulam stability for scalar difference equations of order m

Let us consider the following difference equations of order m,

$$x(j+m) = a_1 x(j+m-1) + \cdots + a_m x(j), \qquad (a_1, a_2, \ldots, a_m)$$

and

$$x(j+m) = a_1 x(j+m-1) + \cdots + a_m x(j) + \theta(j), \qquad (a_1, a_2, \ldots, a_m, \theta)$$

where $j \in \mathbb{Z}_+$, $\theta : \mathbb{Z}_+ \to \mathbb{C}$ and $a_k \in \mathbb{C}, 1 \leq k \leq m$. We use in many times the notation x_k instead of $x(k)$. Remind that the forward discrete operator Δ acts on the set of all X-valued sequences $x : \mathbb{Z}_+ \to X$ and it is defined by $\Delta x(j) := x_{j+1} - x_j$, for $j \in \mathbb{Z}_+$. The natural powers of Δ are defined by $\Delta^1 := \Delta$ and $\Delta^p = \Delta^{p-1} \circ \Delta$ for any integer p greater than 1. The characteristic equation associated to $(a_1, a_2, \ldots, a_m)$ is $p(z) := z^m - a_1 z^{m-1} - \cdots - a_{m-1}z - a_m = 0$. To any sequence $(x(j))_{j\in\mathbb{Z}_+}$, we associate the family $(y_1, y_2, \cdots, y_m)$ consisting of m sequences defined as follows:

$$\begin{cases} y_1(j) = x(j) \\ y_2(j) = (\Delta^1 x)(j) \\ \vdots \\ y_{m-1}(j) = (\Delta^{m-2} x)(j) \\ y_m(j) = (\Delta^{m-1} x)(j), \qquad j \in \mathbb{Z}_+. \end{cases}$$

It is well-known that, for each positive integer n, one has

$$(\Delta^n x)(j) = \sum_{k=0}^{n} C_n^k (-1)^k x(j+n-k) \tag{2.3.1}$$

Then,

$$\begin{cases} (\Delta y_1)(j) = y_2(j) \\ (\Delta y_2)(j) = y_3(j) \\ \vdots \\ (\Delta y_{m-1})(j) = y_m(j) \text{ and} \\ (\Delta y_m)(j) = (\Delta^m x)(j). \end{cases} \tag{2.3.2}$$

On the other hand,

$$\begin{aligned} (\Delta y_m)(j) &= \sum_{k=0}^{m} C_m^k (-1)^k x_{j+m-k} \\ &= x_{j+m} + \sum_{k=1}^{m} C_m^k (-1)^k x_{j+m-k} \\ &= \sum_{k=1}^{m} a_k x_{j+m-k} + \sum_{k=1}^{m} C_m^k (-1)^k x_{j+m-k} \\ &= \sum_{k=1}^{m} [a_k + C_m^k (-1)^k] x_{j+m-k} := S(j). \end{aligned}$$

Lemma 2.3.1. *If the complex scalars $b_1, b_2, \cdots, b_m$ verify the identity*

$$S(j) = \sum_{k=1}^{m} b_k y_k(j) := T(j), \text{ for every } j \in \mathbb{Z}_+,$$

then, for every $1 \leq k \leq m$, one has

$$a_k + C_m^k (-1)^k = \sum_{\nu=0}^{k-1} b_{m-k+\nu+1} C_{m-k+\nu}^{\nu} (-1)^{\nu}. \tag{2.3.3}$$

Proof. Obviously,

$$S(j) = \sum_{\rho=0}^{m-1}[a_{m-\rho} + C_m^{m-\rho}(-1)^{m-\rho}]x_{j+\rho}.$$

As is usual $1_{\mathcal{A}}(\cdot)$, denotes the characteristic function of the set $\mathcal{A}$. Thus, $T(j)$ could be expressed successively as

$$\begin{aligned}
T(j) &= \sum_{k=1}^{m} b_k \sum_{\nu=0}^{k-1} C_{k-1}^{\nu}(-1)^{\nu}x_{j+k-\nu-1} \\
&= \sum_{k=1}^{m}\sum_{\nu=0}^{m} b_k C_{k-1}^{\nu}(-1)^{\nu}1_{\{0,1,\dots,k-1\}}(\nu)x_{j+k-\nu-1} \\
&= \sum_{\nu=0}^{m}\sum_{k=1}^{m} b_k C_{k-1}^{\nu}(-1)^{\nu}1_{\{0,1,\dots,k-1\}}(\nu)x_{j+k-\nu-1} \\
&= \sum_{\nu=0}^{m}\sum_{k=\nu+1}^{m} b_k C_{k-1}^{\nu}(-1)^{\nu}x_{j+k-\nu-1} \\
&= \sum_{\nu=0}^{m}\sum_{\rho=0}^{m-\nu-1} b_{\rho+\nu+1} C_{\rho+\nu}^{\nu}(-1)^{\nu}x_{j+\rho} \\
&= \sum_{\nu=0}^{m}\sum_{\rho=0}^{m} 1_{\{0,1,\dots,m-\nu-1\}}(\rho) b_{\rho+\nu+1} C_{\rho+\nu}^{\nu}(-1)^{\nu}x_{j+\rho} \\
&= \sum_{\rho=0}^{m}\left[\sum_{\nu=0}^{m-\rho-1} b_{\rho+\nu+1} C_{\rho+\nu}^{\nu}(-1)^{\nu}\right]x_{j+\rho}.
\end{aligned}$$

Comparing $S(j)$ and $T(j)$, and identifying the coefficients, obtain

$$a_{m-\rho} + C_m^{m-\rho}(-1)^{m-\rho} = \sum_{\nu=0}^{m-\rho-1} b_{\rho+\nu+1} C_{\rho+\nu}^{\nu}(-1)^{\nu}, \quad 0 \le \rho \le m-1,$$

and then (2.3.3) is fulfilled by setting $k = m - \rho$.

□

By using, $(\Delta y_i)(j) = y_i(j+1) - y_i(j)$, the system (2.3.2), can be written equivalent as

$$\begin{cases}
y_1(j+1) = y_1(j) + y_2(j) \\
y_2(j+1) = y_2(j) + y_3(j) \\
\vdots \\
y_{m-1}(j+1) = y_{m-1}(j) + y_m(j) \\
y_m(j+1) = b_1 y_1(j) + \cdots + b_{m-1}y_{m-1}(j) + (1+b_m)y_m(j).
\end{cases}$$

The vectorial version of the latter system is

$$Y(j+1) = BY(j), \quad j \in \mathbb{Z}_+,$$

where $Y(j) = (y_1(j), \cdots, y_m(j))^T$, and the associated matrix B is:

$$B = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ b_1 & b_2 & b_3 & \cdots & 1+b_m \end{pmatrix}.$$

Lemma 2.3.2. *For every $z \in \mathbb{C}$, one has*

$$P_B(z) = z^m - a_1 z^{m-1} - a_2 z^{m-2} - \cdots - a_{m-1} z - a_m.$$

Proof. For any $z \in \mathbb{C}$, consider the matrix

$$zI_m - B = \begin{pmatrix} z-1 & -1 & 0 & \cdots & 0 \\ 0 & z-1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ -b_1 & -b_2 & -b_3 & \cdots & z-(1+b_m) \end{pmatrix}.$$

Let $1 \leq j \leq m-1$, and let denote by $D_j^m = D_j^m(z)$ the determinant of the matrix obtained by $zI_m - B$ by cutting the first $j-1$ columns and the first $j-1$ rows. Expanding D_j^m by the first column, obtain

$$D_j^m = (z-1)D_{j+1}^m - b_j.$$

Thus

$$\begin{array}{rcl} D_1^m & = & (z-1)D_2^m - b_1 \\ D_2^m & = & (z-1)D_3^m - b_2 \mid \cdot(z-1) \\ D_3^m & = & (z-1)D_4^m - b_3 \mid \cdot(z-1)^2 \\ \vdots & & \\ D_{m-1}^m & = & (z-1)D_m^m - b_{m-1} \mid \cdot(z-1)^{m-2}. \end{array}$$

Adding these equalities side by side, we get

$$D_1^m = (z-1)^{m-1}(z-b_m-1) - b_1 - b_2(z-1) - \cdots - b_{m-1}(z-1)^{m-2}$$

$$
\begin{aligned}
&= (z-1)^m - b_1 - b_2(z-1) - \cdots - b_{m-1}(z-1)^{m-2} - b_m(z-1)^{m-1} \\
&= z^m + \textstyle\sum_{k=1}^m C_m^k z^{m-k}(-1)^k - \sum_{k=1}^m b_k(z-1)^{k-1} \\
&= z^m + \textstyle\sum_{k=1}^m \left[C_m^k z^{m-k}(-1)^k - b_k(z-1)^{k-1}\right] \\
&= z^m + \textstyle\sum_{k=1}^m \left[C_m^k z^{m-k}(-1)^k - b_k \sum_{j=0}^{k-1} C_{k-1}^j z^{k-1-j}(-1)^j\right] \\
&= z^m + \textstyle\sum_{k=1}^m [C_m^k z^{m-k}(-1)^k] - \sum_{k=1}^m \sum_{j=0}^{k-1} b_k C_{k-1}^j z^{k-1-j}(-1)^j \\
&:= z^m + S_1 - S_2.
\end{aligned}
$$

But

$$
\begin{aligned}
S_2 &= \sum_{k=1}^m \sum_{j=0}^m b_k 1_{\{0,1,\ldots,k-1\}}(j) C_{k-1}^j (-1)^j z^{k-1-j} \\
&= \sum_{j=0}^m \sum_{k=j+1}^m b_k (-1)^j C_{k-1}^j z^{k-1-j}.
\end{aligned}
$$

Changing $k-j-1$ with $m-\nu$, obtain

$$
\begin{aligned}
S_2 &= \sum_{j=0}^m \sum_{\nu=j+1}^m b_{m-\nu+j+1}(-1)^j C_{m-\nu+j}^j z^{m-\nu} \\
&= \sum_{j=0}^m \sum_{\nu=1}^m 1_{\{j+1,\ldots,m\}}(\nu) b_{m-\nu+j+1}(-1)^j C_{m-\nu+j}^j z^{m-\nu} \\
&= \sum_{\nu=1}^m \sum_{j=0}^{\nu-1} b_{m-\nu+j+1}(-1)^j C_{m-\nu+j}^j z^{m-\nu},
\end{aligned}
$$

and by replacing ν by k, one has

$$
S_2 = \sum_{k=1}^m \sum_{j=0}^{k-1} b_{m-k+j+1}(-1)^j C_{m-k+j}^j z^{m-k}.
$$

Thus,

$$
D_1^m = z^m + \sum_{k=1}^m \left[C_m^k z^{m-k}(-1)^k - \sum_{j=0}^{k-1} b_{m-k+j+1}(-1)^j C_{m-k+j}^j z^{m-k} \right]
$$

which shows that the coefficient of z^{m-k} is

$$
C_m^k(-1)^k - \sum_{j=0}^{k-1} b_{m-k+j+1}(-1)^j C_{m-k+j}^j = -a_k.
$$

The latter equality was obtained by (2.3.3).

□

To the difference equation $(a_1, a_2, \ldots, a_m, \theta(\cdot))$ we attach the system:

$$Y_{n+1} = BY_n + \Theta_n, \quad Y_n, \Theta_n \in \mathbb{C}^m, n \in \mathbb{Z}_+$$

where,

$$Y_n = \begin{pmatrix} y_1, & y_2, & \cdots, & y_m \end{pmatrix}^T \in \mathbb{C}^m,$$
$$\text{and } \Theta_n = \begin{pmatrix} 0, & 0, & \cdots, 0, & \theta_n \end{pmatrix}^T \in \mathbb{C}^m.$$

At this stage we are in the position to introduce the notion of Hyers-Ulam stability for linear difference equations of order m.

Definition 2.3.1. As before, ε is a given positive number. The difference equation $(a_1, a_2, \ldots, a_m, \theta(\cdot))$ is called Hyers-Ulam stable if

$$\begin{pmatrix} (\exists L)(\forall(\Theta_n))(\forall x)(\exists x_0) \\ L \geq 0, |\Theta|_\infty \leq \varepsilon, x, x_0 \in \mathbb{C}^m \text{ and} \\ \sup\limits_{n \in \mathbb{Z}_+} \left|\text{row}_1 \left[B^n(x - x_0) + \sum_{k=0}^{n-1} B^{n-k-1}\Theta_k\right]\right| \leq L\varepsilon \end{pmatrix}.$$

When $z \in \rho(B) := \mathbb{C} \setminus \sigma(B)$, the matrix $zI_m - B$ is invertible and the m-th column of its inverse is

$$col_m[(zI_m - B)^{-1}] = \frac{1}{P_B(z)} \cdot \begin{pmatrix} 1 \\ z - 1 \\ \vdots \\ (z-1)^{m-1} \end{pmatrix}.$$

Theorem 2.3.3. *The following three statements are equivalent:*

(1) *The difference equation $(a_1, a_2, \ldots, a_m)$ is Hyers-Ulam stable.*

(2) *The matrix B is dichotomic.*

(3) *The characteristic equation*

$$\lambda^m - a_1\lambda^{m-1} - a_2\lambda^{m-2} - \ldots - a_{m-1}\lambda - a_m = 0,$$

has no roots on the unit circle.

The equivalence between the statements **(2)** and **(3)** is already done.

(1) ⇒ (2). Suppose that B is not dichotomic. Then, there exists λ_j in $\sigma(B)$, with $\lambda_j = e^{i\mu_j}$, $\mu_j \in \mathbb{R}$. Let $\varepsilon > 0$ and set $\Theta(n) := e^{i\mu_j n}u_0$, with $\|u_0\| \le \varepsilon$. Obviously, the function Θ is bounded by ε. The difference equation $(a_1, a_2, \dots, a_m)$ is Hyers-Ulam stable, so

$$\sup_{n\in\mathbb{Z}_+} \left\| \mathrm{row}_1 \left[B^n(x - x_0) + \sum_{k=0}^{n-1} B^{n-k-1}\Theta_k \right] \right\| \le L\varepsilon.$$

Then $\mathrm{row}_1 \left[\lambda_j^{-n} E_j(B) \left(B^n(x - x_0) + \sum_{k=0}^{n-1} B^{n-k-1}\Theta_k \right) \right]$ is bounded by $L\varepsilon$, as well. On the other hand, by taking (2.2.2) into account, one has

$$\mathrm{row}_1 \left[\lambda_j^{-n} E_j(B) \left(B^n(x - x_0) + \sum_{k=0}^{n-1} B^{n-k-1}\Theta_k \right) \right] =$$

$$\mathrm{row}_1[P_j(n)(x - x_0)] + \mathrm{row}_1 \left[\sum_{k=0}^{n-1} \lambda_j^{-k-1} \lambda_j^{-(n-k-1)} E_j(B) B^{n-k-1}\Theta_k \right].$$

As we already know, (see the proof of Theorem 2.2.1), the degree of the vectorial valued polynomial $\mathrm{row}_1 [P_j(n)(x - x_0)]$ is less than or equal to $m_j - 1$. Next, we show that the degree of the scalar polynomial

$$e^{-i\mu_j n}\mathrm{row}_1 \left[\sum_{k=0}^{n-1} E_j(B) B^{n-k-1}\Theta_k \right],$$

is equal to m_j. We need two lemmas.

Lemma 2.3.4. *With the above notations, have that*

$$e^{-i\mu_j n} E_j(B) B^n = \sum_{\nu=0}^{m_j-1} E_j(B) B_\nu n^\nu := Q_{jB}(n), \tag{2.3.4}$$

where, B_ν are $m \times m$ matrices depending of j and B, and

$$B_{m_j-1} = \frac{1}{\lambda_j^{m_j-1}} \cdot \frac{(B - \lambda_j I_m)^{m_j-1}}{(m_j - 1)!}. \tag{2.3.5}$$

Proof. In view of (2.1.2) and (2.1.3), we can write

$$B^n = \sum_{i=1}^{k} \lambda_i^n P_i(n), \quad n \in \mathbb{Z}_+, \tag{2.3.6}$$

where $P_i(n)$ are matrix-valued polynomials for any $1 \le i \le k$. Then:

$$B^n = \sum_{i=1}^{k} \sum_{\nu=0}^{m_i-1} n^\nu \lambda_i^n B_{\nu i},$$

where $B_{\nu i}$ are $m \times m$ matrices. On the other hand by [[22], Theorem 1], (see also [25] or [21]), we have that

$$n^\nu \lambda_i^n B_{\nu i} = C_n^\nu \lambda_i^{n-\nu} E_i(B)(B - \lambda_i I_m)^\nu, \quad 1 \le i \le k.$$

Then,

$$\begin{aligned} E_j(B)B^n &= \sum_{i=1}^{k} E_j(B)E_i(B) \sum_{\nu=0}^{m_i-1} C_n^\nu \lambda_i^{n-\nu}(B - \lambda_i I_m)^\nu \\ &= E_j(B) \sum_{\nu=0}^{m_j-1} C_n^\nu \lambda_j^{n-\nu}(B - \lambda_j I_m)^\nu. \end{aligned}$$

As $\lambda_j \ne 0$, so

$$\lambda_j^{-n} E_j(B)B^n = \sum_{\nu=0}^{m_j-1} E_j(B) C_n^\nu \lambda_j^{-\nu}(B - \lambda_j I_m)^\nu.$$

The coefficient of n^{m_j-1} is $\frac{1}{\lambda_j^{m_j-1}\cdot(m_j-1)!}\left(E_j(B)(B - \lambda_j I_m)^{m_j-1}\right)$, and then (2.3.5) holds.

By $[M]_{ij}$, we denote the element of the matrix M located at the intersection of the i-th row and the j-th column.

Lemma 2.3.5. *The degree of the scalar polynomial $[Q_{jB}(n)]_{1m}$, given in (2.3.4), is equal to $m_j - 1$.*

Proof. Let us consider the scalar polynomial $q_j(z) := \frac{P_B(z)}{(z-\lambda_j)^{m_j}}$. The map $z \mapsto \frac{1}{q_j(z)}$ is analytic on $\overline{D}_r(\lambda_j)$. In order to show that degree of $[Q_{jB}(n)]_{1m}$ is equal to $m_j - 1$, it is enough to prove that the coefficient of n^{m_j-1} is a nonzero scalar or equivalently

$$a_{1m}^{(m_j-1)} := \frac{1}{2\pi i} \oint_{C_r(\lambda_j)} \text{row}_1 \left[\frac{(B - \lambda_j I)^{m_j-1}}{(m_j-1)!} \right] \cdot \text{col}_m[R(z,B)]dz \ne 0.$$

By denoting $\sigma_j(Z) := \frac{1}{P_B(z)(m_j-1)!}$, one has

$$\text{row}_1\left[\frac{(B-\lambda_j I)^{(m_j-1)}}{(m_j-1)!}\right]\cdot \text{col}_m[R(z,B)] =$$

$$= \sigma_j(z)\cdot\begin{pmatrix} C^0_{m_j-1}(1-\lambda_j)^{m_j-1} & \cdots & C^{m_j-1}_{m_j-1}(1-\lambda_j)^0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ z-1 \\ \vdots \\ (z-1)^{m-1}\end{pmatrix}$$

$$= \frac{\sum_{k=o}^{m_j-1} C^k_{m_j-1}(z-1)^k(1-\lambda_j)^{m_j-1-k}}{(m_j-1)!P_B(z)}$$

$$= \frac{(z-\lambda_j)^{m_j-1}}{(m_j-1)!P_B(z)} = \frac{\frac{1}{q_j(z)}}{(m_j-1)!(z-\lambda_j)}.$$

By applying the Cauchy theorem, we obtain

$$\begin{array}{rcl} a^{(m_j-1)}_{1m} & = & \frac{1}{2\pi i}\oint_{C_r(\lambda_j)} \text{row}_1\left[\frac{(B-\lambda_j I)^{(m_j-1)}}{(m_j-1)!}\right]\cdot \text{col}_m[R(z,B)]dz \\ & = & \frac{1}{(m_j-1)!q_j(\lambda_j)} \end{array},$$

which is a nonzero scalar and get the assertion.

□

Coming back to the proof of the theorem and taking into account (2.3.4), we obtain

$$e^{-i\mu_j n}\text{row}_1\left[\sum_{k=0}^{n-1} E_j(B)B^{n-k-1}\Theta_k\right]$$

$$= \text{row}_1\left[\sum_{k=0}^{n-1}\lambda_j^{-k-1}\cdot\lambda_j^{n-k-1}E_j(B)B^{n-k-1}\Theta_k\right]$$

$$= \left[\sum_{k=0}^{n-1}\sum_{\nu=0}^{m_j-1}\text{row}_1\left[(E_j(B)B_\nu)\right](n-k-1)^\nu\cdot\begin{pmatrix}0\\0\\\vdots\\0\\e^{-i\mu_j}u_0\end{pmatrix}\right]$$

$$= (e^{-i\mu_j}u_0)\sum_{k=0}^{n-1}\sum_{\nu=0}^{m_j-1}(E_j(B)B_\nu)_{1m}(n-k-1)^\nu.$$

But,

$$\deg\left[(e^{-i\mu_j}u_0)\sum_{k=0}^{n-1}(E_j(B)B_{m_j-1})(n-k-1)^{m_j-1}\right]$$

$$= \deg\left[(e^{-i\mu_j}u_0)\sum_{k=0}^{n-1}\frac{1}{(m_j-1)!\cdot q_j(\lambda)}(n-k-1)^{m_j-1}\right]$$

and

$$\deg\left[(e^{-i\mu_j}u_0)\sum_{k=0}^{n-1}\sum_{0\le\nu\le m_j-1}(E_j(B)B_\nu)_{1m}(n-k-1)^{\nu}\right] =$$

$$= \deg\left[(e^{-i\mu_j}u_0)\sum_{k=0}^{n-1}\frac{1}{(m_j-1)!\cdot q_j(\lambda)}(n-k-1)^{m_j-1}\right] = m_j,$$

where the fact that $\rho_\mu(n) := 1^\mu + 2^\mu + \cdots + n^\mu$ is a polynomial in n of degree $(\mu+1)$, for each nonnegative integer μ was used.

Therefore, the map

$$n \mapsto \mathrm{row}_1[P_j(n)(x-x_0)] + \mathrm{row}_1\left[\sum_{k=0}^{n-1}\lambda_j^{-k-1}\lambda_j^{-(n-k-1)}E_j(B)B^{n-k-1}\Theta_k\right]$$

is unbounded. This latter fact provides a contradiction in respect to the assumption.

The proof of (2) $\Rightarrow$ (1) could be driven in a similar manner as the proof of sufficiency of Theorem 2.2.1. $\square$

Chapter 3

Hyers-Ulam stability for periodic time varying systems

We divide this chapter into two parts. Firstly, we establish a connection between the Hyers-Ulam stability and the discrete exponential dichotomy for linear differential systems in $\mathbb{C}^m$ with periodic coefficients varying in time and secondly, we state analogous results for difference systems.

3.1 Continuous case

In this part we prove that the m-dimensional and periodic system

$$\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{R}_+, \quad x(t) \in \mathbb{C}^m \tag{$A(t)$}$$

is Heyers-Ulam stable if and only if the monodromy matrix associated to the family $\{A(t)\}_{t\geq 0}$ posses a discrete dichotomy, i.e. its spectrum does not intersect the unit circle.

We outline the Hyers-Ulam problem for a family of $m \times m$ matrices $\mathcal{A} = \{A(t)\}_{t\geq 0}$, m being a positive integer. Let $f(\cdot)$ be a $\mathbb{C}^m$-valued function defined on $\mathbb{R}_+$. Consider the systems

$$\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{C}^m \tag{$A(t)$}$$

and

$$\dot{x}(t) = A(t)x(t) + f(t), \quad t \in \mathbb{R}_+, \quad x(t) \in \mathbb{C}^m. \tag{$A(t), f(t)$}$$

Let ε be a positive real number. A $\mathbb{C}^m$-valued function $y(\cdot)$ is called ε-approximate solution for $(A(t))$ if

$$\|\dot{y}(t) - A(t)y(t)\| \leq \varepsilon, \quad \forall t \in \mathbb{R}_+.$$

The family $\mathcal{A}$ is said to be Hyers-Ulam stable if there exists a nonnegative constant L such that, for every ε-approximate solution $\phi(\cdot)$ of $(A(t))$, there exists an exact solution $\theta(\cdot)$ of $(A(t))$ such that

$$\sup_{t \in \mathbb{R}_+} \|\phi(t) - \theta(t)\| \leq L \cdot \varepsilon.$$

Here we assume that the matrix-valued map $t \mapsto A(t)$ is continuous and q-periodic for some positive q and we prove that the family $\mathcal{A} := \{A(t)\}_{t \geq 0}$ is Hyers-Ulam stable if and only if the monodromy matrix associated to $\mathcal{A}$ posses a discrete dichotomy.

3.2 Notations and preliminary results

Assume that the map $t \mapsto A(t) : \mathbb{R} \mapsto \mathcal{M}(m, \mathbb{C})$ is continuous and then the Cauchy Problem

$$\begin{cases} \dot{X}(t) = A(t)X(t), \quad t \in \mathbb{R}, \quad X(t) \in \mathcal{M}(m, \mathbb{C}) \\ X(0) = I_m, \end{cases} \tag{$A(t), I_m$}$$

has a unique solution denoted by $\Phi_{\mathcal{A}}(t)$. It is well known that $\Phi_{\mathcal{A}}(t)$ is an invertible matrix and that its inverse is the unique solution of the Cauchy Problem

$$\begin{cases} \dot{X}(t) = -X(t)A(t), \quad t \in \mathbb{R} \\ X(0) = I_m. \end{cases}$$

The evolution family $\mathcal{U}_{\mathcal{A}} = \{U_{\mathcal{A}}(t,s) : t, s \in \mathbb{R}\}$, where

$$U_{\mathcal{A}}(t,s) := \Phi_{\mathcal{A}}(t)\Phi_{\mathcal{A}}^{-1}(s),$$

has the following properties:

- **(i)** $U_{\mathcal{A}}(t,t) = I_m,$ for all $t \in \mathbb{R}$.
- **(ii)** $U_{\mathcal{A}}(t,s) = U_{\mathcal{A}}(t,r)U_{\mathcal{A}}(r,s)$ for all $t, s, r \in \mathbb{R}$.
- **(iii)** $\frac{\partial}{\partial t}U_{\mathcal{A}}(t,s) = A(t)U_{\mathcal{A}}(t,s)$ for all $t, s \in \mathbb{R}$.
- **(iv)** $\frac{\partial}{\partial s}U_{\mathcal{A}}(t,s) = -U_{\mathcal{A}}(t,s)A(s)$ for all $t, s \in \mathbb{R}$.
- **(v)** The map $(t,s) \mapsto U_{\mathcal{A}}(t,s) : \mathbb{R}^2 \to \mathcal{M}(m, \mathbb{C})$ is continuous .

If, in addition, the map $A(\cdot)$ is q- periodic, for some positive number q, then:

- **(vi)** $U_{\mathcal{A}}(t+q, s+q) = U_{\mathcal{A}}(t,s)$ for all $t, s \in \mathbb{R}$.
- **(vii)** There exist $\omega > 0$ and $M_\omega \geq 1$ such that

$$\|U_{\mathcal{A}}(t,s)\| \leq M_\omega e^{\omega(t-s)}, \quad t \geq s.$$

- **(viii)** $\Phi_{\mathcal{A}}(t+q) = \Phi_{\mathcal{A}}(t) \cdot \Phi_{\mathcal{A}}(q)$ for all $t \in \mathbb{R}$.

To prove the latter statement, we remark that the map $t \mapsto \Phi_{\mathcal{A}}(t+q)(\Phi_{\mathcal{A}}(q))^{-1}$ is a solution of $(A(t), I_m)$. Now, by using the uniqueness it must be $\Phi_{\mathcal{A}}(\cdot)$. The matrix $T_q := U_{\mathcal{A}}(q,0)$ is the matrix of monodromy associated with the family $\mathcal{A}$. Having in mind that T_q is invertible there exists a matrix $B \in \mathcal{M}(m,\mathbb{C})$ such that $T_q = e^{qB}$. Under the above circumstances there is a periodic (period q) matrix function $t \mapsto R(t)$ such that $\Phi_{\mathcal{A}}(t) = R(t)e^{tB}$ for all $t \in \mathbb{R}$. Also note that $\Phi_A(t+q)e^{-(t+q)B} = \Phi_A(t)\Phi_A(q)e^{-qB}e^{-tB} = \Phi_A(t)e^{-tB}$.

We refer our readers to Chapter 1 (1.1) for some well known facts concerning the autonomous case (i.e. when $A(t) = A$ for all $t \in \mathbb{R}$). The stable spectral projection of A is defined by

$$\Pi_-(A) := \frac{1}{2\pi i}\oint_{C_r(0)} (zI_m - A)^{-1} dz \qquad (3.2.1)$$

where $0 < r < 1$ is large enough such that

$$\{\lambda \in \sigma(A) : |\lambda| < 1\} \subset \{\lambda \in \mathbb{C} : |\lambda| < r\}.$$

Clearly, $\Pi_-(A)$ commutes with any natural power of A.

Coming back to the nonautonomous case let $\Pi_- := \Pi_-(T_q)$ and let $\Pi_-(t) := \Phi(t)\Pi_-(\Phi)^{-1}(t)$ and $\Pi_+(t) := I_m - \Pi_-(t)$ for $t \in \mathbb{R}$. Next we list the main properties of this family of projections.

- **(i)** $\Pi_-^2(t) = \Pi_+^2(t) = I_m$ for all $t \in \mathbb{R}$.
- **(ii)** $\Pi_\pm(t)U(t,s) = U(t,s)\Pi_\pm(s)$ for all $t, s \in \mathbb{R}$, (the signs correspond).
- **(iii)** The maps $t \mapsto \Pi_\pm(t)$ are continuous on $\mathbb{R}$ and q-periodic.
- **(iv)** $\Pi_-(t) + \Pi_+(t) = I_m$ and $\Pi_-(t) \cdot \Pi_+(t) = 0$ for all $t \in \mathbb{R}$.

- **(v)** For each $t, s \in \mathbb{R}$, $U(t, s)$ may be seen as an isomorphism from $\ker(\Pi_-(s))$ to $\ker(\Pi_-(t))$.

Let $t \mapsto f(t)$ be a $\mathbb{C}^m$-valued locally Riemann integrable function on $\mathbb{R}_+$ and let $x \in \mathbb{C}^m$ be a given vector. Let us consider the Cauchy Problem

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t) & t \geq 0 \\ x(0) = x. \end{cases} \qquad (A(t), f(t), x)$$

A function $\phi : \mathbb{R}_+ \to \mathbb{C}^m$ is said to be a solution for $(A(t), f(t), x)$ if:

1. $\phi(\cdot)$ is continuous on $\mathbb{R}_+$.

2. $\phi(\cdot)$ is differentiable almost everywhere (a.e.) on $\mathbb{R}_+$.

3. $\phi'(t) = A(t)\phi(t) + f(t)$ a.e. for $t \in \mathbb{R}_+$.

4. $\phi(0) = x$.

It is obvious to see that the solution of $(A(t), f(t), x)$ is given by

$$\phi_{f,x}(t) = U_{\mathcal{A}}(t, 0)x + \int_0^t U_{\mathcal{A}}(t, s)f(s)ds.$$

Our first result reads to

Theorem 3.2.1. *The family $\mathcal{A} = \{A(t)\}_{t\geq 0}$ is Hyers-Ulam stable if and only if its monodromy matrix T_q posses a discrete dichotomy.*

In order to prove Theorem 3.2.1, we need the following proposition, which contains equivalent characterizations for discrete exponential dichotomy.

Proposition 3.2.2. *The following three statements concerning the matrix family $\mathcal{A}$ are equivalent:*

(1) *T_q is dichotomic.*

(2) *There exist the positive constants N_1', N_2', ν_1', ν_2' such that*

(i) $\|U_{\mathcal{A}}(t, s)\Pi_-(s)\| \leq N_1' e^{-\nu_1'(t-s)}$, *for all $t \geq s \geq 0$, and*

(ii) $\|U_{\mathcal{A}}(t, s)\Pi_+(s)\| \leq N_2' e^{\nu_2'(t-s)}$, *for all $0 \leq t \leq s$.*

(3) *For each continuous and bounded function $f : \mathbb{R}_+ \to \mathbb{C}^m$ there exists a unique $x \in \mathcal{X}_u(T_q) = \ker(\Pi_-)$, such that $\phi_{f,x}(\cdot)$ is bounded on $\mathbb{R}_+$.*

Proof. **(1)** $\Rightarrow$ **(2)** Let $t \geq s \in \mathbb{R}_+$ and let n and k be the integer parts of t and s respectively, i.e, $n = [\frac{t}{q}]$ and $k = [\frac{s}{q}]$. Therefore $t = nq + \mu$ and $s = kq + \rho$, with $n, k \in \mathbb{Z}_+$ and $\mu, \rho \in [0, q)$. We analyze the following cases:

Case 1. When $n > k$, then

$$U_{\mathcal{A}}(t,s)\Pi_-(s)$$

$$\begin{aligned} &= U_{\mathcal{A}}(nq+\mu, nq)U_{\mathcal{A}}(nq,(k+1)q)U_{\mathcal{A}}((k+1)q, kq+\rho)\Pi_-(kq+\rho) \\ &= U_{\mathcal{A}}(\mu,0)U_{\mathcal{A}}((n-k-1)q,0)U_{\mathcal{A}}(q,\rho)\Pi_-(\rho) \\ &= U_{\mathcal{A}}(\mu,0)T_q^{n-k-1}\Pi_-(q)U_{\mathcal{A}}(q,\rho). \end{aligned}$$

In the view of Proposition 2.1.1 and taking into account that $\Pi_-(0) = \Pi_-(q) = \Pi_-$, we get

$$\begin{aligned} \|U_{\mathcal{A}}(t,s)\Pi_-(s)\| &\leq Me^{\omega q}N_1 e^{-\nu_1(n-k-1)}Me^{\omega q}\|\Pi_-\| \\ &\leq N_1' e^{-\nu_1'(t-s)}, \end{aligned}$$

where $N_1' = N_1 M^2 e^{2\omega q} e^{2\nu_1}\|\Pi_-\|$ and $\nu_1' = \frac{\nu_1}{q}$.

Case 2. When $n = k$, then $\mu \geq \rho$ and

$$\|U_{\mathcal{A}}(s,t)\Pi_-(s)\| = \|U_{\mathcal{A}}(\mu,\rho)\Pi_-(\rho)\|.$$

By using $\sup_{\rho\in[0,q]}\|\Pi_-(\rho)\| \leq c < \infty$ and letting ν be an arbitrary positive number, we may choose $N \in \mathbb{Z}_+$ large enough, such that

$$\|U(t,s)\Pi_-(s)\| \leq Me^{\omega(\mu-\rho)} \leq cNe^{-\nu(\mu-\rho)} = N_1' e^{-\nu_1'(t-s)}.$$

Similar estimations can be obtained in order to prove **(2) (ii)**. We omit the details.

(2) $\Rightarrow$ **(1)**. Put $s = 0$ and $t = nq$ in **(2) (i),(ii)** and apply Proposition 2.1.1 with T_q instead of A.

(2) $\Rightarrow$ **(3)**. The map

$$t \mapsto y(t) := \int_0^t U_{\mathcal{A}}(t,s)\Pi_-(s)f(s)ds - \int_t^\infty U_{\mathcal{A}}(t,s)\Pi_+(s)f(s)ds,$$

is a solution of $(A(t), f(t))$. For further details see [20, Chap. 3]. Indeed, the second integral is well defined because,

$$\begin{aligned} \int_t^\infty \|U_{\mathcal{A}}(t,s)\Pi_+(s)f(s)\|\,ds &\leq \int_t^\infty N_2' e^{\nu_2'(t-s)}\|f\|_\infty ds \\ &= \frac{N_2'}{\nu_2'}\|f\|_\infty. \end{aligned}$$

Also from **(2)** the solution is bounded, and

$$\sup_{t\geq 0}|y(t)| \leq \left(\frac{N_1'}{\nu_1'} + \frac{N_2'}{\nu_2'}\right)\sup_{t\geq 0}|f(t)|.$$

Moreover, since $\ker(\Pi_-)$ is a closed subspace, the initial value

$$y(0) = -\int_0^\infty U_{\mathcal{A}}(0,s)\Pi_+(s)f(s)ds \in \ker(\Pi_-).$$

Let suppose that there exist two bounded solutions of the differential equation $\dot{x}(t) = A(t)x(t) + f(t), t \geq 0$ having their start in $\ker(\Pi_-)$. Denote them by $y_1(\cdot)$ and $y_2(\cdot)$. Then

$$y_1(t) = U_{\mathcal{A}}(t,0)x_1 + \int_0^t U_{\mathcal{A}}(t,s)f(s)ds, \quad x_1 \in \ker(\Pi_-) \text{ and}$$

$$y_2(t) = U_{\mathcal{A}}(t,0)x_2 + \int_0^t U_{\mathcal{A}}(t,s)f(s)ds, \quad x_2 \in \ker(\Pi_-).$$

Their difference is bounded and $y_1(t) - y_2(t) = U_{\mathcal{A}}(t,0)(x_1 - x_2)$. Since the map $y_1(\cdot) - y_2(\cdot)$ is bounded on $\mathbb{R}_+$, $x_1 - x_2 \in \mathcal{X}_s(T) = \Im(\Pi_-)$. On the other hand, $x_1, x_2 \in \ker(\Pi_-)$ yields $x_1 - x_2 \in \ker(\Pi_-)$ and therefore $x_1 = x_2$.

(3) $\Rightarrow$ **(1)**. Suppose that, there exists $\lambda \in \sigma(T)$, with $|\lambda| = 1$. Then, there exists $x_0 \neq 0$ such that $T_q x_0 = \lambda x_0$, and therefore $U_{\mathcal{A}}(nq,0) = \lambda^n x_0$, for all $n \in \mathbb{Z}_+$.

Set

$$f(t) := \begin{cases} U(s,0)x_0, & \text{if } s \in [0,q) \\ x_0, & \text{if } s = q, \end{cases}$$

and let denote also by f its continuation by periodicity on $\mathbb{R}_+$. By assumption there exists a unique $y_0 \in \ker(\Pi_-)$ such that the map

$$t \mapsto \psi(t) := U_{\mathcal{A}}(t,0)y_0 + \int_0^t U_{\mathcal{A}}(t,s)f(s)ds$$

is bounded on $\mathbb{R}_+$. Next we analyze two cases.

Case 1. When $\lambda = 1$. The sequence $(\psi(nq))_{n\in\mathbb{Z}_+}$ should be bounded. But,

$$\psi(nq) := U_{\mathcal{A}}(nq,0)y_0 + \int_0^{nq} U_{\mathcal{A}}(nq,s)f(s)ds$$

$$= U_{\mathcal{A}}(nq,0)y_0 + \sum_{k=0}^{n-1}\int_{kq}^{(k+1)q} U_{\mathcal{A}}(nq,s)f(s)ds$$

$$= U_{\mathcal{A}}(nq,0)y_0 + \sum_{k=0}^{n-1} U_{\mathcal{A}}(nq,(k+1)q) \int_0^q U_{\mathcal{A}}(q,r)f(r)dr$$

$$= U_{\mathcal{A}}(nq,0)y_0 + \sum_{k=0}^{n-1} U_{\mathcal{A}}(nq,kq)x_0 = U_{\mathcal{A}}(nq,0)y_0 + nx_0.$$

If $y_0 = 0$, obviously we arrive at a contradiction, since the map $n \mapsto nx_0$ is unbounded and if $y_0 \neq 0$, by spectral decomposition theorem there are two positive constants N and ν such that $\|U_{\mathcal{A}}(nq,0)y_0\| \geq Ne^{\nu nq}$ for all $n \in \mathbb{Z}_+$, and a contradiction arises again.

Case 2. When $\lambda = e^{iuq} \neq 1, u \in \mathbb{R}, i^2 = -1$. Then $1 \in \sigma(e^{-iuq}T)$, $T_u(q) := e^{-iuq}T_q$ is the monodromy matrix of the evolution family

$$\{U_{\mathcal{A},u}(t,s) := e^{-iu(t-s)}U_{\mathcal{A}}(t,s) : \quad t,s \in \mathbb{R}\}$$

and, as before, we obtain that the sequence

$$(e^{-iunq}\psi(nq))_{n\in\mathbb{Z}_+} = (U_{\mathcal{A},u}(nq,0)y_0 + qnx_0)_{n\in\mathbb{Z}_+},$$

is unbounded, which is a contradiction. □

3.3 Hyers-Ulam stability and exponential dichotomy for linear differential systems

We can see an ε-approximate solution of $(A(t))$, as an exact solution of $(A(t), \rho(t))$ corresponding to a forced term $\rho(\cdot)$ which is bounded by ε.

Definition 3.3.1. Let ε be a given positive number. The matrix family $\mathcal{A}$ (or the system (A(t))) is called Hyers-Ulam stable if

$$(\exists L)(\forall \rho = \rho(t))(\forall x)(\exists x_0)$$
$$\left(\begin{array}{c} L \geq 0, \rho(\cdot) \text{ is loccaly Riemann integrable and bounded on } \mathbb{R}_+, \\ \|\rho\|_\infty \leq \varepsilon, x, x_0 \in \mathbb{C}^m \text{ and} \\ \sup_{t\geq 0} |U_{\mathcal{A}}(t,0)(x-x_0) + \int_0^t U_{\mathcal{A}}(t,s)\rho(s)ds| \leq L\varepsilon. \end{array} \right)$$

Proof of Theorem 3.2.1.

Necessity. Suppose that T_q is not dichotomic. Then, there exists $1 \leq j \leq k$ and $\lambda_j \in \sigma(T_q)$, with $\lambda_j = e^{i\mu_j q}$ and $\mu_j \in \mathbb{R}$. Let $\varepsilon > 0$ be fixed and set

$$\rho(t) := \begin{cases} U(s,0)u_0, & \text{if } s \in [0,q) \\ u_0, & \text{if } s = q, \end{cases}$$

where $u_0 \in \mathbb{C}^m$ and $\|u_0\| \leq (M_\omega e^{\omega q})^{-1}\varepsilon$. Let denote also by ρ the continuation by periodicity of the previous function. Obviously, the function $\rho(\cdot)$ is locally Riemann integrable on $\mathbb{R}_+$ and bounded by ε. By assumption, the family matrix $\mathcal{A}$ is Hyers-Ulam stable. Hence, the solution

$$\phi(t) = U_{\mathcal{A}}(t,0)(x - x_0) + \int_0^t U_{\mathcal{A}}(t,s)\rho(s)ds,$$

of the Cauchy Problem

$$\begin{cases} \dot{y}(t) = A(t)y(t) + \rho(t), & t \geq 0 \\ y(0) = x - x_0, \end{cases} \qquad (A(t), \rho(t), x - x_0)$$

is bounded by $L\varepsilon$ for certain $x - x_0 \in \mathbb{C}^m$. By using the spectral decomposition theorem, there exists an $m \times m$ matrix-valued polynomial $P_j = P_j(T_q)$ (in n) having the degree at most $m_j - 1$, such that

$$E_j(T_q)U_{\mathcal{A}}(nq,0) = e^{i\mu_j qn}P_j(n), \qquad \forall n \in \mathbb{Z}_+.$$

Then, the sequence

$$n \mapsto E_j(T_q)\phi(nq) = E_j(T_q)\left[U_{\mathcal{A}}(nq,0)(x - x_0) + \int_0^{nq} U_{\mathcal{A}}(nq,s)\rho(s)ds\right],$$

should also be bounded by $L\varepsilon$.

On the other hand,

$$E_j(T_q)\left[U_{\mathcal{A}}(nq,0)(x - x_0) + \int_0^{nq} U_{\mathcal{A}}(nq,s)\rho(s)ds\right] =$$

$$= e^{i\mu_j nq}P_j(n)(x-x_0) + \int_0^{nq} E_j(T_q)U_{\mathcal{A}}(nq,s)\rho(s)ds$$

$$= e^{i\mu_j nq}P_j(n)(x-x_0) + \sum_{k=0}^{n-1}\int_{kq}^{(k+1)q} E_j(T_q)U_{\mathcal{A}}(nq,s)\rho(s)ds$$

$$= e^{i\mu_j nq}P_j(n)(x-x_0) + \sum_{k=0}^{n-1}\int_0^{q} E_j(T_q)U_{\mathcal{A}}(nq,(k+1)q)U_{\mathcal{A}}(q,s)\rho(s)ds$$

$$= e^{i\mu_j nq}P_j(n)(x-x_0) + \sum_{k=0}^{n-1}\lambda_j^{n-k}P_j(n-k)u_0.$$

Now, if $\lambda_j = 1$ then by choosing an appropriate $u_0 \neq 0$, we have that

$$\begin{aligned} \deg[P_j(n)(x-x_0)] \le \deg[P_j(n)] &= \deg[P_j(n)u_0] < 1+\deg[P_j(n)] \\ &= \deg[q_j(n)], \end{aligned}$$

where $q_j(n) := \sum_{k=0}^{n-1} P_j(n-k)u_0$ and the fact that the degree of the polynomial in n, $p(n) = 1^k + 2^k + \cdots n^k$, is equal to $k+1$ was used.

Therefore, the sequence $(P_j(n)(x-x_0)+q_j(n))_{n\in\mathbb{Z}_+}$, is unbounded and we have arrived at a contradiction.

When $\lambda_j \neq 1$, then $1 \in \sigma(T_{\mu_j}(q))$ and the map $t \mapsto e^{-i\mu_j t}\phi(t)$ should be bounded on $\mathbb{R}_+$. Then the sequence

$$n \mapsto e^{-i\mu_j nq}\phi(nq), n \in \mathbb{Z}_+$$

is bounded as well. On the other hand

$$e^{-i\mu_j nq}E_j(T_{\mu_j}(q))\phi(nq)$$

$$= E_j(T_{\mu_j}(q))\left[U_{\mathcal{A},\mu_j}(nq,0)(x-x_0) + \int_0^{nq} U_{\mathcal{A},\mu_j}(nq,s)e^{-i\mu_j s}\rho(s)ds\right].$$

In the view of the spectral decomposition theorem there exists a matrix valued polynomial $Q_j(n) = Q_j(T_{\mu_j}(q))$ (in n) having the degree at most $m_j - 1$ such that

$E_j(T_{\mu_j}(q))U_{\mathcal{A},\mu_j}(nq,0) = Q_j(n)$ for every $n \in \mathbb{Z}_+$. Thus after a standard calculation

$$e^{-i\mu_j nq}E_j(T_{\mu_j}(q))\phi(nq) = Q_j(n)(x-x_0) + \sum_{k=0}^{n-1} Q_j(n-k)u_0.$$

For an appropriate $u_0 \in \mathbb{C}^m$, the last expression is a vector valued polynomial of degree at least one and so it is unbounded and a contradiction is provided again.

Sufficiency. The absolute constant L will be settled later. Let $\rho : \mathbb{R}_+ \to \mathbb{C}^m$ be a bounded locally Riemann integrable function on $\mathbb{R}_+$, with $\|\rho\|_\infty \le \varepsilon$ and let $x \in \mathbb{C}^m$. By Proposition 3.2.2, there exists a unique bounded solution $y(\cdot)$ of the equation $(A(t), \rho(t))$ starting from the subspace $\ker(\Pi_-)$. Let denote $u_0 := y(0)$. Then

$$\begin{aligned} \|y(t)\| &= \left\| U_{\mathcal{A}}(t,0)u_0 + \int_0^t U_{\mathcal{A}}(t,s)\rho(s)ds \right\| \\ &= \left\| \int_0^t U_{\mathcal{A}}(t,s)\Pi_-(s)\rho(s)ds - \int_t^\infty U_{\mathcal{A}}(t,s)\Pi_+(s)\rho(s)ds \right\| \\ &\le \left(\frac{N_1'}{\nu_1'} + \frac{N_2'}{\nu_2'} \right)\varepsilon. \end{aligned}$$

The desired assertion follows by choosing $L = \left(\frac{N_1'}{\nu_1'} + \frac{N_2'}{\nu_2'} \right)$ and setting $x_0 = x - u_0$.

We conclude this part with one dimensional version of our result.

Corollary 3.3.1. *Let $t \mapsto a(t) : \mathbb{R}_+ \to \mathbb{C}$ be a given continuous and q-periodic function (for some positive q). The scalar differential equation*

$$\dot{x}(t) = a(t)x(t), \quad t \in \mathbb{R}_+, \quad x(t) \in \mathbb{C} \qquad (a(t))$$

is Hyers-Ulam stable if and only if

$$\int_0^q \Re[a(r)]dr \neq 0.$$

Proof. Indeed, we have

$$T_q = e^{\int_0^q a(r)dr}, \sigma(T_q) = \{T_q\} \text{ and } |T_q| = e^{\int_0^q \Re[a(r)]dr}.$$

From Theorem 3.2.1 follows that $(a(t))$ is Hyers-Ulam stable if and only if $|T_q| \neq 1$ or equivalently if and only if $\int_0^q \Re[a(r)]dr \neq 0$.

□

3.4 Discrete case

We outline the Hyers-Ulam problem for nonautonomous difference linear system of order m driven by a family $\mathcal{A} = \{A_n\}_{n\in\mathbb{Z}_+}$ of $m \times m$ complex matrices.

Consider the system

$$X_{n+1} = A_n X_n, \quad n \in \mathbb{Z}_+. \tag{3.4.1}$$

Let ε be a positive real number. A $\mathbb{C}^m$-valued sequence $(Y_n)_{n\in\mathbb{Z}_+}$ is called ε-approximate solution for (3.4.1) if

$$\|Y_{n+1} - A_n Y_n\| \leq \varepsilon, \quad \forall n \in \mathbb{Z}_+.$$

The family $\mathcal{A}$ is said to be Hyers-Ulam stable if there exists a nonnegative constant L such that, for every ε-approximate solution Γ of (3.4.1) there exists an exact solution Θ of (3.4.1) such that

$$\sup_{n\in\mathbb{Z}_+} \|\Gamma_n - \Theta_n\| \leq L \cdot \varepsilon.$$

In the present case we consider that the matrix valued map $n \mapsto A_n$ is $q-$periodic for some positive integer q. We prove that the family $\mathcal{A} := \{A_n\}_{n\in\mathbb{Z}_+}$ is Hyers-Ulam stable if and only if the monodromy matrix associated to $\mathcal{A}$ posses a discrete dichotomy.

3.5 Notations and preliminary results

Consider the discrete problem:

$$X_{n+1} = A_n X_n, \qquad n \in \mathbb{Z}_+, X_n \in \mathbb{C}^m. \tag{A_n}$$

We assume that for each $n \in \mathbb{Z}_+$, A_n is an $m \times m$ invertible matrix and $A_{n+q} = A_n$, for all $n \in \mathbb{Z}_+$ and some $q \geq 1$.

Denote:

$$U_{\mathcal{A}}(n,p) := \begin{cases} A_{n-1}\cdots A_p, & n>p \\ I_m, & n=p \\ A_p^{-1}\cdots A_{n-1}^{-1}, & n<p \end{cases}$$

The family $\mathfrak{U}_{\mathcal{A}} := \{U_{\mathcal{A}}(r,p) : (r,p) \in \mathbb{Z}_+ \times \mathbb{Z}_+\}$ is called $q-$periodic reversible discrete evolution family. It is obvious to verify that:

(1) $U_{\mathcal{A}}(n,n) = I_m$ for all $n \in \mathbb{Z}_+$.

(2) $U_{\mathcal{A}}(n,r)U_{\mathcal{A}}(r,p) = U_{\mathcal{A}}(n,p)$ for all $n,r,p \in \mathbb{Z}_+$, with $n \geq r \geq p$.

(3) $U_{\mathcal{A}}(n+q,r+q) = U_{\mathcal{A}}(n,r)$ for all $n,r \in \mathbb{Z}_+$.

As a consequence of **(3)** and **(2)**, we have

(4) $U_{\mathcal{A}}(nq,rq) = U((n-r)q,0)$ for all $n,r \in \mathbb{Z}_+$.

(5) $U_{\mathcal{A}}(n,r)$ is invertible and $U_{\mathcal{A}}(n,r)^{-1} = U_{\mathcal{A}}(r,n)$ for all $n,r \in \mathbb{Z}_+$.

(6) A $q-$periodic evolution family have exponential growth, i.e, there exists $\omega > 0$ and $M \geq 1$ such that $\|U_{\mathcal{A}}(n,r)\| \leq Me^{\omega|n-r|}$ for all $n,r \in \mathbb{Z}_+$.

Any solution (ϕ_n) of (A_n) verify:

$$\phi_n := U_{\mathcal{A}}(n,k)\phi_k, \text{ for all } n,k \in \mathbb{Z}_+, \quad n \geq k.$$

The matrix $T_q := U(q,0)$ is called the monodromy operator associated to the family $\mathcal{U}_{\mathcal{A}}$.

Let us consider the spectral projection related to T_q defined above in (3.2.1).

Since T_q is invertible there exists $B \in M(m,\mathbb{C})$ such that $e^{qB} = U(q,0)$ and $U(n,0) = R(n)e^{nB}$, for all $n \in \mathbb{Z}_+$ where $R(n)$ is a $q-$periodic matrix.

Now we will define the family of projections $\{\Pi_-(n)\}_{n\in\mathbb{Z}_+}$ by

$$\Pi_-(n) = U(n,0)\Pi_-(U(n,0))^{-1},$$

and consider $\Pi_+(n) = I_m - \Pi_-(n)$ for $n \in \mathbb{Z}_+$. It is easy to verify that

- **(i)** $\Pi_-^2(n) = \Pi_+^2(n) = I_m$ for all $n \in \mathbb{Z}_+$.
- **(ii)** $\Pi_\pm(n)U(n,m) = U(n,m)\Pi_\pm(m)$ for all $n, m \in \mathbb{Z}_+$, (the signs correspond).
- **(iii)** $\Pi_-(n) + \Pi_+(n) = I_m$ and $\Pi_-(n)\cdot\Pi_+(n) = 0$ for all $n \in \mathbb{Z}_+$.

Note that $\ker(\Pi_-(n)) = \Im(\Pi_+(n))$ and $\ker(\Pi_+(n)) = \Im(\Pi_-(n))$ for all $n \in \mathbb{Z}_+$.

In the autonomous case we refer to the Proposition 2.1.1.

Proposition 3.5.1. *Let $\{U_{\mathcal{A}}(n,k)\}$ be a $q-$periodic and reversible evolution family having the family of projections $\{\Pi_-(n)\}_{n\in\mathbb{Z}_+}$. The following three statements concerning the matrix family $\mathcal{A}$ are equivalent:*

(1) *T_q posses a discrete dichotomy, i.e. $\sigma(T_q)\cap\Gamma = \emptyset$.*

(2) *There exist four positive constants N_1', N_2', ν_1' and ν_2' such that*

(i) $\|U_{\mathcal{A}}(n,k)\Pi_-(k)\| \le N_1' e^{-\nu_1'(n-k)}$ *for all* $n \ge k \ge 0$.

(ii) $\|U_{\mathcal{A}}(n,k)\Pi_+(k)\| \le N_2' e^{\nu_2'(n-k)}$ *for all* $0 \le n \le k$.

(3) *For each bounded sequence (f_n) there exists a bounded solution starting by a unique $x \in \mathcal{X}_u(T_q) = \ker(\Pi_-)$ of the difference equation*

$$X_{n+1} = A_n X_n + f_{n+1}, \quad n \in \mathbb{Z}_+. \qquad (A_n, f_n)$$

Proof. **(1)** $\Rightarrow$ **(2)**. Let $n \ge k$ and let s and r be the integer parts of n and k respectively, i.e, $s = [\frac{n}{q}]$ and $r = [\frac{k}{q}]$. Therefore $n = sq + \mu$ and $k = rq + \rho$, with $s, r \in \mathbb{Z}_+$ and $\mu, \rho \in \{1, 2, \cdots, q-1\}$. We analyze the following cases:

Case 1). When $n > k$, then

$$U_{\mathcal{A}}(n,k)\Pi_-(k) =$$

$$\begin{aligned}
&= U_{\mathcal{A}}(sq+\mu, sq)U_{\mathcal{A}}(sq,(r+1)q)U_{\mathcal{A}}((r+1)q, rq+\rho)\Pi_-(rq+\rho)\\
&= U_{\mathcal{A}}(\mu,0)U_{\mathcal{A}}((s-r-1)q,0)U_{\mathcal{A}}(q,\rho)\Pi_-(\rho)\\
&= U_{\mathcal{A}}(\mu,0)T_q^{s-r-1}\Pi_-(q)U(q,\rho).
\end{aligned}$$

By using Proposition 2.1.1 and the fact that $\Pi_-(0) = \Pi_-(q) = \Pi_-$, we have

$$\begin{aligned} \|U_{\mathcal{A}}(n,k)\Pi_-(k)\| &\leq Me^{\omega q}N_1e^{-\nu_1(s-r-1)}\|\Pi_-\|Me^{\omega q} \\ &\leq N_1'e^{-\nu_1'(n-k)}, \end{aligned}$$

where $N_1' = M^2e^{2(\omega q+\nu_1)}\|\Pi_-\|$ and $\nu_1' = \frac{\nu_1}{q}$.

Case 2) If $s = r$ then $\mu \geq \rho$ and

$$\|U_{\mathcal{A}}(n,k)\Pi_-(k)\| = \|U_{\mathcal{A}}(\mu,\rho)\Pi_-(\rho)\|$$

By using $\sup_{\rho\in[0,q]}\|\Pi_-(\rho)\| \leq c < \infty$ and letting ν be a given positive number and let N be an integer number large enough, such that

$$\|U_{\mathcal{A}}(n,k)\Pi_-(k)\| \leq Me^{\omega(\mu-\rho)} \leq cNe^{-\nu(\mu-\rho)} = N_1'e^{-\nu_1'(n-k)}.$$

For **(2)(ii)** consider similar cases for $n \leq k$.

(2) ⇒ (1). Put $k = 0$ and $n = sq$ in **(2)(i),(ii)** and apply Proposition 2.1.1 with T_q instead of A.

(2) ⇒ (3).

The solution of (A_n, f_n) may be written as,

$$\psi_n = \sum_{k=1}^{n} U_{\mathcal{A}}(n,k)\Pi_-(k)f_k - \sum_{k=n+1}^{\infty} U_{\mathcal{A}}(n,k)\Pi_+(k)f_k.$$

This sum is well defined and the second sum is convergent by using **(2)(ii)**.

$$\begin{aligned} \left\|\sum_{k=n+1}^{\infty} U_{\mathcal{A}}(n,k)\Pi_+(k)f_k\right\| &\leq \sum_{k=n+1}^{\infty} N_2'e^{\nu_2'(n-k)}\|f\|_\infty \\ &= \frac{N_2'\cdot e^{-\nu_2'}}{1-e^{-\nu_2'}}\|f\|_\infty \\ &\leq \frac{N_2'}{e^{\nu_2'}-1}\|f\|_\infty. \end{aligned}$$

By using **(2)** the solution (ψ_n) is bounded and it is

$$\sup_{n\geq 0}|\psi_n| \leq \left(\frac{N_1'}{e^{\nu_1'}-1} + \frac{N_2'}{e^{\nu_2'}-1}\right)\sup_{n\geq 0}|f_n|.$$

Moreover, since $\ker(\Pi_-)$ is a closed subspace, the initial value

$$\psi(0) = -\sum_{k=1}^{\infty} U_{\mathcal{A}}(0,k)\Pi_+(k)f_k \in \ker(\Pi_-).$$

Let suppose that there exist two bounded solutions of the differential equation $X_{n+1} = A_nX_n + f_{n+1}, n \in \mathbb{Z}_+$ having their start in $\ker(\Pi_-)$, denoted by $\psi_1(\cdot)$ and $\psi_2(\cdot)$. Then

$$\psi_1(n) = U_{\mathcal{A}}(n,0)x_1 + \sum_{k=1}^{n} U_{\mathcal{A}}(n,k)f_k, \quad x_1 \in \ker(\Pi_-) \text{ and}$$

$$\psi_2(n) = U_{\mathcal{A}}(n,0)x_2 + \sum_{k=1}^{n} U_{\mathcal{A}}(n,k)f_k, \quad x_2 \in \ker(\Pi_-).$$

Their difference is bounded and $\psi_1(n) - \psi_2(n) = U_{\mathcal{A}}(n,0)(x_1 - x_2)$. Since the map $\psi_1(\cdot) - \psi_2(\cdot)$ is bounded on $\mathbb{Z}_+$, $x_1 - x_2 \in \mathcal{X}_s(T_q) = \Im(\Pi_-)$. On the other hand, $x_1, x_2 \in \ker(\Pi_-)$ yields $x_1 - x_2 \in \ker(\Pi_-)$ and therefore $x_1 = x_2$.

(3) $\Rightarrow$ **(1)**. Suppose for a contradiction that $\sigma(T_q) \cap \Gamma \neq \emptyset$. Then there exists $\lambda \in \sigma(T_q)$ with $|\lambda| = 1$ and $y \in \mathbb{C}^m, \quad y \neq 0$ such that $T_qy = U_{\mathcal{A}}(q,0)y = \lambda y$.

In general,

$$T_q^n y = \lambda^n y = U_{\mathcal{A}}(nq,0)y, \quad \forall n \in \mathbb{N}.$$

Let

$$g_k := \begin{cases} \lambda^k y, & \text{if} \quad q|k \\ 0, & \text{in the rest.} \end{cases}$$

The solution of

$$\begin{cases} y_{n+1} = A_ny_n + g_{n+1}, & n \in \mathbb{Z}_+ \\ y_0 = x_0, \end{cases}$$

is given by,

$$\psi_n = U_{\mathcal{A}}(n,0)x_0 + \sum_{k=1}^{n} U_{\mathcal{A}}(n,k)g_k, \quad n \in \mathbb{Z}_+$$

and then

$$\begin{aligned}
\psi_{nq} &= U_{\mathcal{A}}(nq,0)x_0 + \sum_{k=1}^{nq} U_{\mathcal{A}}(nq,k)g_k \\
&= U_{\mathcal{A}}(nq,0)x_0 + \sum_{j=1,q|j}^{nq} U_{\mathcal{A}}(nq,j)\lambda^j y \\
&= U_{\mathcal{A}}(nq,0)x_0 + \sum_{j=1}^{n} U_{\mathcal{A}}(nq,jq)\lambda^{jq} y \\
&= U_{\mathcal{A}}(nq,0)x_0 + \sum_{j=1}^{n} U_{\mathcal{A}}(nq,jq)U_{\mathcal{A}}(jq,0)y \\
&= U_{\mathcal{A}}(nq,0)x_0 + \sum_{j=1}^{n} U_{\mathcal{A}}(nq,0)y \\
&= U_{\mathcal{A}}(nq,0)x_0 + n\lambda^n y
\end{aligned}$$

If $x_0 = 0$, then the above map is clearly unbounded. If $x_0 \neq 0$, $x_0 \in \ker(\Pi_-)$ and

$$U_{\mathcal{A}}(nq, 0) : \ker(\Pi_-) \to \ker(\Pi_-)$$

is a bijective map. Consider $U_{\mathcal{A}}(nq, 0)x_0 = y$ then $\|U_{\mathcal{A}}(0, nq)\Pi_+ y\| \leq Ne^{-\nu nq}\|y\|$ and therefore $\|U_{\mathcal{A}}(nq, 0)x_0\| \geq \frac{1}{N}e^{\nu nq}\|x_0\|$.

Thus the above map is unbounded. A contradiction arises. □

3.6 Hyers-Ulam stability and dichotomy for nonautonomous difference systems

We can see an ε-approximate solution of (A_n) as an exact solution of (A_n, f_n) corresponding to a forced term (f_n) bounded by ε. Therefore, the following definition arises naturally.

Definition 3.6.1. Let ε be a given positive number. The matrix family $\mathcal{A}$ (or the system (A_n)) is said to be Hyers-Ulam stable if

$$\begin{pmatrix} (\exists L)(\forall (f_n))(\forall x)(\exists x_0) \\ L \geq 0, \sup_{n\geq 0} \|(f_n)\| \leq \varepsilon, x, x_0 \in \mathbb{C}^m \text{ and} \\ \sup_{n\in\mathbb{Z}_+} \left| U_{\mathcal{A}}(n, 0)(x - x_0) + \sum_{k=1}^{n} U_{\mathcal{A}}(n, k) f_k \right| \leq L\varepsilon \end{pmatrix}.$$

Theorem 3.6.1. *The family $\mathcal{A} = (A_n)_{n\in\mathbb{Z}_+}$ is Hyers-Ulam stable if and only if its monodromy matrix T_q posses a discrete dichotomy.*

Proof. Necessity. Suppose that T_q is not dichotomic. Then, there exists $1 \leq j \leq k$ and $\lambda_j \in \sigma(T_q)$ such that $|\lambda_j| = 1$. Let $\varepsilon > 0$ be fixed and $u_0 \in \mathbb{C}^m$, $u_0 \neq 0$, $\|u_0\|_\infty < \varepsilon$. Consider

$$f_n^j := \begin{cases} \lambda_j^n u_0, & \text{if} \quad q|n \\ 0, & \text{in the rest.} \end{cases}$$

Obviously this function is bounded. By assumption, the matrix family $\mathcal{A}$ is Hyers-Ulam and therefore the solution

$$\psi_n = U_{\mathcal{A}}(n, 0)(x - x_0) + \sum_{k=1}^{nq} U_{\mathcal{A}}(nq, k) f_k, \quad x, x_0 \in \mathbb{C}^m,$$

of the discrete Cauchy Problem

$$\begin{cases} y_{n+1} = A_n y_n + f_{n+1}, & n \in \mathbb{Z}_+ \\ y_0 = x - x_0, \end{cases} \qquad (A_n, f_n, x - x_0)$$

is bounded by $L\varepsilon$ for certain $x - x_0 \in \mathbb{C}^m$.

By spectral decomposition theorem, there exists an $m \times m$ matrix-valued polynomial $P_j = P_j(T_q)$ (in n) having the degree at most $m_j - 1$, such that

$$E_j(T_q)U_{\mathcal{A}}(nq, 0) = \lambda_j^n P_j(n), \qquad \forall n \in \mathbb{Z}_+.$$

Then, the sequence

$$n \mapsto E_j(T_q)\psi_{nq} := E_j(T_q)\left[U_{\mathcal{A}}(nq, 0)(x - x_0) + \sum_{k=1}^{nq} U_{\mathcal{A}}(nq, k)f_k^j\right],$$

is bounded as well. Therefore the sequence $(\lambda_j^{-n}\psi_{nq})$ is bounded and its general term may be represented as

$$\begin{aligned} \lambda_j^{-n}E_j(T_q)\psi_{nq} &= \lambda_j^{-n}\left[E_j(T_q)T_q^n(x - x_0) + \sum_{k=1}^{nq} E_j(T_q)T_q^{n-k}\lambda_j^k u_0\right] \\ &= \lambda_j^{-n}\left[\lambda^n P_j(n)(x - x_0) + \sum_{k=1}^{n} P_j(n-k)\lambda_j^n u_0\right] \\ &= P_j(n)(x - x_0) + \sum_{k=1}^{n} P_j(n-k)u_0 \\ &= Q_{1j}(n) + Q_{2j}(n), \end{aligned}$$

which is a sum of two polynomials.

But

$$\deg(Q_{1j}(n)) \le \deg(P_j(n)) \le \deg(P_j(n)) + 1 = \deg(Q_{2j}(n)),$$

where u_0 was chosen such that $\deg[P_j(n)] = \deg[P_j(n)u_0]$.

The latter equality is a consequence of the fact that the map $n \mapsto (1^\rho + 2^\rho + \cdots + n^\rho)$ is a polynomial in n of degree $(\rho + 1)$ for each fixed ρ in $\mathbb{Z}_+$. Thus the map $\lambda_j^{-n}\psi_{nq}$ is unbounded and a contradiction arises.

Sufficiency. Let $f : \mathbb{Z}_+ \to \mathbb{C}^m$, with $\sup_{n\in\mathbb{Z}_+} \|f_n\| \le \varepsilon$, and let $x \in \mathbb{C}^m$. From Proposition 3.5.1, there exists a unique bounded solution (ψ_n) of (A_n, f_n) starting from $\ker(\Pi_-)$. Let us denote $u_0 := \psi_0$. Then,

$$\begin{aligned} \|\psi_n\| &= \left\|U_{\mathcal{A}}(n, 0)\psi_0 + \sum_{k=1}^{n} U_{\mathcal{A}}(n, k)f_k\right\| \\ &= \left\|\sum_{k=1}^{n} U_{\mathcal{A}}(n, k)\Pi_-(k)f_k - \sum_{k=n+1}^{\infty} U_{\mathcal{A}}(n, k)\Pi_+(k)f_k\right\| \\ &\le \left(\frac{N_1'}{e^{\nu_1'}-1} + \frac{N_2'}{e^{\nu_2'}-1}\right)\varepsilon, \end{aligned}$$

and we are getting the assertion by choosing

$$L = \left(\frac{N_1'}{e^{\nu_1'} - 1} + \frac{N_2'}{e^{\nu_2'} - 1} \right),$$

and by setting, $x_0 = x - u_0$.

□

For $m = 1$ we can state the following:

Corollary 3.6.2. *Let (a_n) be a $\mathbb{C}^*-$valued, $q-$periodic (for some integer $q \geq 1$) sequence. The recurrence*

$$x_{n+1} = a_n x_n, \quad n \in \mathbb{Z}_+, \quad x_n \in \mathbb{C}$$

is Hyers-Ulam stable if and only if

$$|a_{q-1} \cdots a_0| \neq 1.$$

Proof. It is obvious to see that $T_q = a_{q-1} \cdots a_0$. □

Example.

Let us consider $2-$periodic system

$$\begin{cases} x_{n+1} = a_n x_n + b_n y_n, \\ y_{n+1} = c_n x_n + d_n y_n, \end{cases}$$

where

$$a_n = \begin{cases} \frac{-1}{3}, & if \quad n \in 2\mathbb{Z}_+, \\ \frac{3}{2}, & if \quad n \in 2\mathbb{Z}_+ + 1 \end{cases}$$

$$d_n = \begin{cases} 1, & if \quad n \in 2\mathbb{Z}_+, \\ 3, & if \quad n \in 2\mathbb{Z}_+ + 1 \end{cases}$$

and $b_n = c_n = 0$, $n \in \mathbb{Z}_+$. Clearly $a_{n+2} = a_n, \quad b_{n+2} = b_n, \quad c_{n+2} = c_n$ and $d_{n+2} = d_n$ for all $n \in \mathbb{Z}_+$. This system may also be represented by $X_{n+1} = A_n X_n$, where $A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$, $X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ and $A_{n+2} = A_n$.

Based on our theoretical result Theorem 3.6.1 we prove that this system is Hyers-Ulam stable.

By definition $A_n = U(n+1,n) = \begin{cases} U(1,0), & if \quad n \in 2\mathbb{Z}_+, \\ U(2,1), & if \quad n \in 2\mathbb{Z}_+ + 1 \end{cases}$

The monodromy matrix is $T_2 = U(2,0) = U(2,1)U(1,0) = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 3 \end{pmatrix}$ which is dichotomic.

Chapter 4

Further research and some open problems

In the future we intend to formulate and prove equivalences between Hyers-Ulam stability and exponential dichotomy for continuous systems and for discrete systems or for equations in Banach spaces which are driven of compact operators, bounded linear operators or eventually of unbounded linear operators like generators of strongly continuous semigroups. Also we intend to study similar problems in the context of the periodic nonautonomous equations or systems in infinite dimensional Banach spaces and in the context of time scale analysis.

In the following we describe in few words two open problems.

Let us consider the following scalar differential equation of order m (m being a given positive integer)

$$x^{(m)}(t) = b_1(t)x^{(m-1)}(t) + \ldots + b_{m-1}x'(t) + b_m x(t), \quad t \in \mathbb{R}_+ \qquad (b_1(t), \ldots, b_m(t))$$

where for each $1 \le \nu \le m$, $b_\nu : \mathbb{R} \to \mathbb{C}$ is a $q-$periodic and continuous function. Let ε be positive. A function $\phi : \mathbb{R}_+ \to \mathbb{C}^m$ is said to be an approximative solution for

$(b_1(t), \dots, b_m(t))$ if

$$\sup_{t\geq 0} |\phi^{(m)}(t) - b_1(t)\phi^{(m-1)}(t) - \ldots + b_{m-1}\phi'(t) - b_m\phi(t)| \leq \varepsilon.$$

As previously, we say that the equation $(b_1(t), b_2(t), \dots, b_m(t))$ is Hyers-Ulam stable if there exists a positive constant L such that for every its $\varepsilon-$approximative solution $\phi(\cdot)$, there exists an exact solution $\rho(\cdot)$ of $(b_1(t), b_2(t), \dots, b_m(t))$ such that

$$\sup_{t\geq 0} |\phi(t) - \rho(t)| \leq L\varepsilon.$$

Let $t \in \mathbb{R}_+$, and let

$$B(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ b_m(t) & b_{m-1}(t) & b_{m-2}(t) & \cdots & b_1(t) \end{pmatrix},$$

be the matrix associated to the equation $(b_1(t), b_2(t), \dots, b_m(t))$. We denote by $\mathcal{B}$ the family $\{B(t) : t \in \mathbb{R}_+\}$. As it is shown in the Chapter 3 of this dissertation, we may associate to $\mathcal{B}$, the evolution family of matrices

$$\mathcal{U}_{\mathcal{B}} := \{U_{\mathcal{B}}(t,s) := \Phi_{\mathcal{B}}(t)\Phi_{\mathcal{B}}^{-1}(s), \quad t, s \in \mathbb{R}_+\}.$$

Let $\varepsilon > 0$ be fixed. It is obvious to show that the equation $(b_1(t), b_2(t), \dots, b_m(t))$ is Hyers-Ulam stable if and only if there exists a positive constant L such that for every $\mathbb{C}-$ valued, bounded and locally Riemann integrable function defined on $\mathbb{R}_+$, with $\|\rho(\cdot)\|_\infty \leq \varepsilon$, and every $x \in \mathbb{C}^m$ there exists $x_0 \in \mathbb{C}^m$ such that

$$\sup_{t\geq 0} \left| \text{row}_1 \left[\Phi_B(t)(x - x_0) + \int_0^t \Phi_B(t)\Phi_B^{-1}(s)\Theta(s)ds \right] \right| \leq L\varepsilon$$

where

$$\Theta(s) = \begin{pmatrix} 0, & 0, & \cdots & 0, & \theta(s) \end{pmatrix}^T.$$

Clearly, if the family $\mathcal{B}$ is Hyers-Ulam stable (or equivalently if its associated monodromy matrix posses a discrete dichotomy) then the equation $(b_1(t), b_2(t), \ldots, b_m(t))$ is Hyers-Ulam stable as well. Our first problem is refer to the converse of this statement. More exactly, the question may be formulated as follows:

Problem 1. *Let us consider the statements:*
(1) *The equation $(b_1(t), b_2(t), \ldots, b_m(t))$ is Hyers-Ulam stable.*
(2) *The family of matrices $\{B(t)\}_{t\in\mathbb{R}}$ is Hyers-Ulam stable.*

*Does the implication **(2)→(1)** holds true?*

We have already established (see Chapter 1) that (at least in the autonomous case) the answer to this question is an affirmative one.

Similar problem may be stated in the framework of discrete nonautonomous and periodic systems or equations. It could be described with almost similar words (the details are omitted), with the exception of following condition

$$\sup_{n\in\mathbb{Z}_+} \left| \text{row}_1 \left[U_{\mathcal{B}}(n,0)(x-x_0) + \sum_{k=1}^{n} U_{\mathcal{B}}(n,k)\Theta_k \right] \right| \leq L\varepsilon$$

where

$$\Theta_k = \begin{pmatrix} 0, & 0, & \cdots & 0, & \theta_k \end{pmatrix}^T,$$

$U_{\mathcal{B}}(n,k) = B_{n-1}B_{n-2}\cdots B_k$ and $\mathcal{B}$ is the family matrix associated to the difference equation of order m, in the nonautonomous case.

Our second question is formulated in the following.

Problem 2. *What happens if in the above considerations concerning Hyers-Ulam* *stability we replace the norm* $\|\cdot\|_\infty$ *with norms in other spaces?*

In the context of this Problem we remind an interesting related result from [12] (see also [13] for characterizations of uniform exponential stability instead of nonuniform strong stability).

Let X be a Banach space, $\mathcal{A} = (A_n)_{n\in\mathbb{Z}_+}$ be a $\mathcal{L}(X)$-valued sequence and let $\mathcal{U} = \{U(n,m) : n \geq m \in \mathbb{Z}_+\}$ be the discrete evolution family associated to $\mathcal{A}$. The family $\mathcal{U}$ is nonuniformly strongly stable (i.e.,

$$\lim_{n\to\infty} \|U(n,m)x\| = 0 \text{ for every } m \in \mathbb{Z}_+ \text{ and every } x \in X)$$

if and only it is $l^1(\mathbb{Z}_+, X)$-approximative admissible, that is, for every $f = (f_n)$ in $l^1(\mathbb{Z}_+, X)$, with $f_0 = 0$, and every $\varepsilon > 0$, there exists (g_n) in $l^1(\mathbb{Z}_+, X)$, satisfying $\|g - f\|_{l^1(\mathbb{Z}_+,X)} < \varepsilon$, such that the solution of the discrete Cauchy Problem

$$\begin{cases} x_{n+1} = A_n x_n + g_{n+1}, & n \in \mathbb{Z}_+ \\ x_0 = 0, & \end{cases}$$

lies in $l^1(\mathbb{Z}_+, X)$.

It is important to mention that in this context the state space $l^1(\mathbb{Z}_+, X)$ cannot be replaced by anyone of the spaces ($l^p(\mathbb{Z}_+, X)$, $1 < p \leq \infty$, or $c_{00}(\mathbb{Z}_+, X)$) so that show some counterexamples from [12].

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