Refinements of Jensen-Steffensen and Related Inequalities

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DECLARATION

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RESEARCH COMPLETION CERTIFICATE

Certified that the research work contained in this thesis titled

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has been carried out and completed by **Ms Sadia Khalid** Registration No. **114-GCU-PHD-SMS-09** under my supervision.

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This thesis is warmly dedicated to

my kind Grandparents, my very dear Parents and to my loving Phopho jee for their endless love and encouragement !

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Abstract

As the title suggests, the present dissertation deals with the refinements of Jensen-Steffensen and related inequalities. We extend some classical results and give some new refinements of several well known inequalities including Jensen's inequality, Jensen-Steffensen inequality, Hermite-Hadamard inequality, majorization-type inequality, generalized weighted Favard and Berwald inequalities. We also provide the refinements of some companion inequalities to the Jensen's inequality, namely Slater's inequality and the inequalities obtained by M. Matić and J. Pečarić in [44]. We sharpen the lower bounds of the Jensen's functional. Some inequalities in terms of Gâteaux derivatives for convex functions are also provided. Finally, we present not only the generalizations of Hardy-Littlewood-Pólya inequality [26, Theorem 134] but also generalizations of some results given in [8] and [36].

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Introduction

“ The whole of science is nothing more than a refinement of everyday thinking ”.

Albert Einstein

0.1 Objectives of the Dissertation

As a matter of fact, the well known Jensen's inequality is one of the most celebrated inequalities in mathematics and statistics. It plays a prime role in the theory of inequalities due to the fact that many other renowned and classical inequalities such as the arithmetic-geometric mean inequality, generalized triangle inequality, Hölder, Minkowski, Young and Ky Fan inequalities can be obtained as particular cases of it. Jensen's inequality is the most important inequality due to its high applicability in various branches of science. There exists an extensive literature devoted to Jensen's inequality including different generalizations, refinements, counterparts and converse results see e.g. [14–16] and [52] and also the references therein. The Jensen-Steffensen inequality also have significant importance in mathematics and have been refined in many different ways by various mathematicians. These inequalities play an important role in mathematical analysis. Another major inequality for convex functions is the

Hermite-Hadamard integral inequality (see [58, p.137])

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}, \tag{0.1.1}$$

provided that $f:[a,b]\to\mathbb{R}$ is a convex function. If the function f is concave, then (0.1.1) holds in the reverse direction. It gives an estimate from above and below of the mean value of a convex function. In recent years, there have been many extensions and generalizations of the inequalities in (0.1.1) (see [21]). In this dissertation, one of our objectives is to present some refinements of the Jensen-Steffensen inequality and also some refinements of the first Hermite-Hadamard integral inequality $f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt$.

Lower bounds of the Jensen's functional is also a topic of interest in many papers, for example, see [22] and [46, p. 717]. In this dissertation, another objective is to present the refinements of some previously known results related with the lower bounds of the Jensen's functional.

In [44] M. Matić and J. Pečarić presented a pair of general companion inequalities to the Jensen's inequality and deduced Slater's inequality in addition to other results from these general inequalities. Some generalizations of Favard and Berwald inequalities to the weighted and multidimensional cases are given in [43, 57] and some extensions of these results for discrete and integral cases are given in [41, 42, 53]. Our goal is to present refinements of some of the results given in [44] along with some refinements of Favard and Berwald inequalities and also to prove some inequalities in terms of Gâteaux derivatives.

Some generalizations of Hardy-Littlewood-Pólya inequality [26, Theorem 134] are presented by S. Khalid and J. Pečarić in [36] and some generalizations of G. Bennett's result are proved by J. Pečarić, I. Perić and R. Roki in [55]. Our objective is to present not only the generalizations of some of the results given in [31] but also the generalizations of all the results which appear in [36].

0.2 Dissertation Organization

This dissertation is organized in the following manner:

Chapter 1 presents some fundamental concepts which contribute throughout this dissertation. This chapter introduces some basic results dealing with convex functions, classes of convex functions, divided differences and means etc.

Chapter 2 presents not only the refinements of the Jensen-Steffensen inequality, but also the refinements of the first Hermite-Hadamard inequality. It provides mean value theorems, the n-exponential convexity and the log-convexity of the functions associated with the linear fuctionals defined as the non-negative differences of the refined inequalities. The monotonicity property of the generalized Cauchy means obtained via these functionals is also presented. Finally, it deals with several examples of families of functions for which the obtained results can be applied.

Chapter 3 deals with the lower bounds of the Jensen's functional. The aim of this chapter is to provide an improvement of Theorem 2.1, Lemma 2.2 and Lemma 2.3 given in [13], in a sense that the condition of monotonicity imposed on the n-tuple $\boldsymbol{x}$ shall be relaxed. Finally, some interesting special cases of the presented results are provided.

Chapter 4 provides a general inequality and from this general inequality, refinements of Jensen and Slater inequalities and also refinement of Slater's inequality for monotone convex functions can be obtained. Further, it presents some refinements of the majorization-type inequalities and the refinements of the generalized Favard and

Berwald inequalities. Finally, some inequalities in terms of Gâteaux derivatives for convex functions defined on linear spaces are also presented.

Chapter 5 presents generalizations of some results given in [8] and [36] and it is shown that, after appropriate substitution, one of the results is equivalent to an inequality given in [59]. This chapter provides some results which are related with the discrete weighted reversed Hardy-type inequality. In addition, generalizations of Theorems 1 and 2 given in [31] are also presented. Mean value theorems for the linear functionals are given. Further, the n-exponential convexity and the log-convexity of the functions associated with the linear functionals is proved and the Lyapunov-type inequalities for these functionals are deduced. Finally, several examples of families of functions are also provided.

Chapter 1

Preliminaries and Fundamental Concepts

The notion of convex functions is one of the most important concepts in the theory of inequalities. The aim of this chapter is to present some basic definitions and concepts which will be used in the sequel.

Throughout this dissertation, I is an interval in $\mathbb{R}$.

1.1 Convex Functions and Related Results

The following definition is given in [58, p. 5].

Definition 1.1.1. A function $f : I \to \mathbb{R}$ is called convex in the Jensen sense or J-convex or mid-point convex if for all $x, y \in I$,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{1.1.1}$$

holds. A J-convex function f is said to be strictly J-convex if for all pairs of points (x, y), $x \neq y$, strict inequality holds in (1.1.1).

The following definition of convex function is given in [58, p. 1].

Definition 1.1.2. A function $f : I \to \mathbb{R}$ is called convex if for all $x, y \in I$ and for all $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1.1.2}$$

holds. If (1.1.2) is strict for all $x \neq y$ and $\lambda \in (0, 1)$, then f is said to be strictly convex.
If the inequality (1.1.2) reverses, then f is said to be concave and if it is strict for all $x \neq y$ and $\lambda \in (0, 1)$, then f is said to be strictly concave.

The following definition is equivalent to the Definition 1.1.2 (see [58, p. 2]).

Definition 1.1.3. A function $f : I \to \mathbb{R}$ is convex on I if

$$(x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \geq 0 \tag{1.1.3}$$

holds for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

Convex functions and J-convex functions are related as follows (see [47, p. 14]):

Theorem 1.1.1. *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is convex if and only if f is convex in the Jensen sense, that is, if* (1.1.1) *holds for all $x, y \in I$.*

An important characterization of convex function is stated in [58, p. 2].

Theorem 1.1.2. *If f is a convex function defined on I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}. \tag{1.1.4}$$

If the function f is concave, then the inequality reverses.

The following theorem is related with the derivatives of a convex function (see [58, p. 4]).

Theorem 1.1.3. *Let $f : I \to \mathbb{R}$ is a convex function. Then f'_+ and f'_- exist and are increasing in I.*

The following definition is given in [58, p. 6].

Definition 1.1.4. A finite sequence $(a_k, k = 1, \ldots, n)$ of real numbers is said to be a convex sequence if

$$2a_k \leq a_{k-1} + a_{k+1}, \tag{1.1.5}$$

holds for all $k = 2, \ldots, n-1$. If the above inequality reverses, then the sequence $(a_k, k = 1, \ldots, n)$ is called the concave sequence.

Remark 1.1.1. An infinite sequence $(a_k, k = 1, 2, \ldots)$ of real numbers is convex if (1.1.5) holds for all $k \geq 2$.

By letting $x_1 = x$, $x_2 = x + h$, $y_1 = y$ and $y_2 = y + h$, $(x \leq y, h \geq 0)$ in (1.1.4), we have,

$$f(x+h) - f(x) \leq f(y+h) - f(y). \tag{1.1.6}$$

The following definition of Wright-convex function is presented in [58, p. 7].

Definition 1.1.5. A function $f : [a, b] \to \mathbb{R}$ is said to be Wright-convex if for all x, $y + h \in [a, b]$ such that $x \leq y$, $h \geq 0$, (1.1.6) holds. The function f is said to be Wright-concave if opposite inequality holds in (1.1.6).

Remark 1.1.2. If $K[a, b]$, $W[a, b]$ and $J[a, b]$ denotes the class of all convex functions, the class of all Wright-convex functions and the class of all J-convex functions respectively, then $K[a, b] \subsetneqq W[a, b] \subsetneqq J[a, b]$. That is, a convex function must be a Wright-convex function but not conversely (see [58, p. 7]).

The following definition is given in [32].

Definition 1.1.6. A function $f : I \to (0, \infty)$ is said to be log-convex in the Jensen sense if for all $x, y \in I$,

$$f^2\left(\frac{x+y}{2}\right) \leq f(x)\, f(y)\,,$$

holds.

Remark 1.1.3. It is easy to see that a function $f : I \to (0, \infty)$ is log-convex in the Jensen sense if and only if the relation

$$\alpha^2 f(x) + 2\alpha\beta\, f\left(\frac{x+y}{2}\right) + \beta^2 f(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$.

The following definition of log-convex function is given in [58, p. 7].

Definition 1.1.7. A function $f : I \to (0, \infty)$ is said to be log-convex or multiplicatively convex if $\log f$ is convex. Equivalently, f is log-convex if for all $x, y \in I$ and for all $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)\, y) \leq f^{\lambda}(x) f^{(1-\lambda)}(y)$$

holds. If the inequality reverses, then f is said to be log-concave.

Remark 1.1.4. If f is continuous, then a log-convex function in the Jensen sense is log-convex.

The following definition of divided difference is given in [58, p. 14].

Definition 1.1.8. The nth-order divided difference of a function $f : [a, b] \to \mathbb{R}$ at mutually distinct points $x_0, x_1, ..., x_n \in [a, b]$ is defined recursively by

$$[x_i; f] = f(x_i)\,, \quad i = 0, \ldots, n,$$

$$[x_0, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}. \tag{1.1.7}$$

It is easy to see that (1.1.7) is equivalent to

$$[x_0, \ldots, x_n; f] = \sum_{i=0}^{n} \frac{f(x_i)}{q'(x_i)}, \quad \textit{where } q(x) = \prod_{j=0}^{n} (x - x_j).$$

Remark 1.1.5. Divided difference have many interesting properties some of which are the following (see [58, pp. 14,16]):

(i) The value $[x_0, \ldots, x_n; f]$ is independent of the order of the points $x_0, \ldots, x_n$. This definition may be extended to include the case in which some or all the points coincide.

In particular, for $n = 2$ (1.1.7) takes the form

$$[x_0, x_1, x_2; f] = \frac{[x_1, x_2; f] - [x_0, x_1; f]}{x_2 - x_0}. \tag{1.1.8}$$

Namely, by taking the limit $x_1 \to x_0$ in (1.1.8), we get

$$\lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_2; f] = \frac{f(x_2) - f(x_0) - f'(x_0)(x_2 - x_0)}{(x_2 - x_0)^2}, \quad x_2 \neq x_0,$$

provided that f' exists, and furthermore, taking the limits $x_i \to x_0$, $i = 1, 2$, in (1.1.8), we get

$$\lim_{x_2 \to x_0} \lim_{x_1 \to x_0} [x_0, x_1, x_2; f] = [x_0, x_0, x_0; f] = \frac{f''(x_0)}{2},$$

provided that f'' exists.

(ii) For divided difference

$$[\underbrace{x, \ldots, x}_{n-times}; f] = \frac{f^{(n-1)}(x)}{(n-1)!},$$

provided that $f^{(n-1)}(x)$ exists.

(iii) Divided difference is a linear functional.

The following definition of a real-valued convex function is characterized by *nth*-order divided difference (see [58, p. 15]).

Definition 1.1.9. A function $f : [a, b] \to \mathbb{R}$ is said to be n-convex ($n \geq 0$) if and only if for all choices of $(n+1)$ distinct points $x_0, \ldots, x_n \in [a, b]$, $[x_0, \ldots, x_n; f] \geq 0$ holds.
If this inequality is reversed, then f is said to be n-concave. If the inequality is strict, then f is said to be a strictly n-convex (n-concave) function.

Remark 1.1.6. Note that 0-convex functions are non-negative functions, 1-convex functions are increasing functions and 2-convex functions are simply the convex functions.

An interesting result for n-convex function is given in [58, p. 16].

Theorem 1.1.4. *If a function f is n-convex defined on $[a, b]$ for $n \geq 2$, then the function $f^{(k)}$ exists and is $(n-k)$-convex for $1 \leq k \leq n-2$.*

The following theorem gives an important criteria to examine the n-convexity of a function f (see [58, p. 16]).

Theorem 1.1.5. *If $f^{(n)}$ exists, then f is n-convex if and only if $f^{(n)} \geq 0$.*

1.2 Lagrange and Cauchy-type Mean Value Theorems

The following definition of mean is given in [3] (see also [47, p. 5]).

Definition 1.2.1. A function $M : \underbrace{I \times I \times \ldots \times I}_{n-times} \to \mathbb{R}$ is called a mean if for all n-tuples $(x_1, \ldots, x_n) \in I^n$, the following property of intermediacy

$$\inf\{x_1, \ldots, x_n\} \leq M(x_1, \ldots, x_n) \leq \sup\{x_1, \ldots, x_n\}, \qquad n \in \mathbb{N},$$

holds. M is called a strict mean if these inequalities are strict unless $x_1 = \ldots = x_n$.

M is called *symmetric* if $M(x_{i_1}, \ldots, x_{i_n}) = M(x_{j_1}, \ldots, x_{j_n})$ for any permutation $(i_1, \ldots, i_n)$ of the set $\{i_1, \ldots, i_n\}$.

For various examples of means and symmetric means for positive real numbers see [12]. Lagrange and Cauchy mean value theorems are one of the most important results in mathematical analysis. In particular, the Lagrange-type and Cauchy-type mean value theorems are most frequently used. The usual approach is to first prove the Lagrange-type mean value theorems and then deduce the Cauchy-type mean value theorems. Joseph-Louis Lagrange was the first who stated mean value theorem as follows (see [64, p. 257]): If a function f is continuous on $[x_0, x_1]$ and differentiable on (x_0, x_1), then there exists at least one point $\xi \in (x_0, x_1)$ such that

$$f'(\xi) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = [x_0, x_1; f].$$

If f' is invertible, then

$$\xi = f'^{-1}\left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right)$$

is unique and known as the Lagrange's mean for the function f. Cauchy mean value theorem which is infact the generalization of Lagrange mean value theorem, can be formulated as follows ([64, p. 294]): Suppose that f and g are continuous on $[x_0, x_1]$ and differentiable on (x_0, x_1) and $g'(x) \neq 0$ for all $x \in (x_0, x_1)$, then there exists at least one point $\tilde{\xi} \in (x_0, x_1)$ such that

$$\frac{f(x_1) - f(x_0)}{g(x_1) - g(x_0)} = \frac{f'\left(\tilde{\xi}\right)}{g'\left(\tilde{\xi}\right)}.$$

If the function $\frac{f'}{g'}$ is invertible, then the existence of $\tilde{\xi}$ is unique and

$$\tilde{\xi} = \left(\frac{f'}{g'}\right)^{-1} \left(\frac{f(x_1) - f(x_0)}{g(x_1) - g(x_0)}\right).$$

The number $\tilde{\xi}$ is called Cauchy mean value of the numbers x_0 and x_1.

1.3 n-Exponentially Convex Functions

An important sub-class of convex functions is the class of exponentially convex functions defined on an open interval, which was introduced by S. N. Bernstein in [9]. In [30] J. Jakšetić and J. Pečarić presented an elegant general method of constructing exponentially convex functions. Later in [54], J. Pečarić and J. Perić generalized the concept of exponentially convex functions by introducing the notion of n-exponentially convex functions.

Now, we begin this section by recollecting definitions and properties which are going to be discussed here and we will also see some useful characterizations of these properties. Let I be an open interval in $\mathbb{R}$.

The following definitions are given in [54].

Definition 1.3.1. A function $f : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense if

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j f\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for every $\alpha_i \in \mathbb{R}$ and $x_i \in I$ $(i = 1, \ldots, n)$.

Definition 1.3.2. A function $f : I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 1.3.1. From the above definition it is clear that 1-exponentially convex functions in the Jensen sense are non-negative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex functions in the Jensen sense for all $k \in \mathbb{N}$, $k \le n$.

By definition of positive semi-definite matrices and some basic linear algebra, we have the following proposition:

Proposition 1.3.1. *If f is n-exponentially convex in the Jensen sense on I, then the matrix $\left[f\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ is positive semi-definite for all $k \in \mathbb{N},\ k \le n$. Particularly,*

$$\det\left[f\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k} \ge 0 \quad \textit{for every } k \in \mathbb{N},\ k \le n,\ \ x_i \in I,\ \ i = 1, \ldots, n.$$

Definition 1.3.3. A function $f : I \to \mathbb{R}$ is exponentially convex in the Jensen sense if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

Definition 1.3.4. A function $f : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 1.3.2. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense. Also, by using basic convexity theory, a positive function is log-convex if and only if it is 2-exponentially convex. For more results about log-convexity, see [5] and the references therein.

An important property of exponentially convex function is its integral representation. The following theorem will be useful while constructing some examples for exponentially convex functions (see [30]).

Theorem 1.3.2. *The function $f : I \to \mathbb{R}$ is exponentially convex on I if and only if*

$$f(x) = \int_{-\infty}^{\infty} e^{tx} d\sigma(t), \quad x \in I,$$

for some non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$.

Proof. The proof is given in [2, p. 211]. ∎

1.3.1 Examples

Now, we present some examples of exponentially convex functions listed in [30].

Example 1.3.3. *For the constants $a \in \mathbb{R}$ and $c \geq 0$, the function*

$$f(x) = ce^{ax}, \quad x \in \mathbb{R},$$

is exponentially convex as

$$\begin{aligned} \sum_{i,j=1}^{n} \alpha_i \alpha_j f\left(\frac{s_i + s_j}{2}\right) &= c \sum_{i,j=1}^{n} \alpha_i \alpha_j e^{a\left(\frac{s_i+s_j}{2}\right)} \\ &= c\left(\sum_{i=1}^{n} \alpha_i e^{a\frac{s_i}{2}}\right)^2 \geq 0. \end{aligned}$$

The next two examples are obtained by using some results from the Laplace transformation and the integral representation given in Theorem 1.3.2.

Example 1.3.4. *For every $a > 0$, the function $f : (0, \infty) \to \mathbb{R}$ defined by*

$$f(x) = \frac{1}{x^a}$$

is exponentially convex on $(0, \infty)$ as $x^{-a} = \int_0^\infty e^{-xt} \frac{t^{a-1}}{\Gamma(a)} dt$.

Example 1.3.5. *For every $a > 0$, the function $f : (0, \infty) \to \mathbb{R}$ defined by*

$$f(x) = \frac{1}{e^{a\sqrt{x}}}$$

is exponentially convex on $(0, \infty)$ as $e^{-a\sqrt{x}} = \int_0^\infty e^{-xt} e^{-a^2/4t} \frac{a}{2\sqrt{\pi t^3}} dt$.

For more examples and analytical properties of exponentially convex functions see [30] and the references therein.

Chapter 2

On the Refinements of the Jensen-Steffensen Inequality

In this chapter, we present some refinements of the Jensen-Steffensen inequality as well as the refinements of the first Hermite-Hadamard inequality. We investigate the n-exponential convexity and the log-convexity of the functions associated with the linear functionals defined via these inequalities and prove monotonicity property of the generalized Cauchy means obtained via these functionals. Finally, we give several examples of the families of functions for which the results can be applied.

2.1 Introduction

Let us start this section with the well known Jensen's inequality (see [58, p. 43]).

Theorem 2.1.1. *Let $f : I \to \mathbb{R}$ be a convex function. Let $n \geq 2$, $\mathbf{x} = (x_1, \dots, x_n) \in I^n$ and $\mathbf{p} = (p_1, \dots, p_n)$ be a positive n-tuple, that is, $p_i > 0$ for $i = 1, \dots, n$. Then*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(x_i\right), \tag{2.1.1}$$

where

$$P_k = \sum_{i=1}^{k} p_i, \qquad k = 1, \dots, n. \tag{2.1.2}$$

If f is strictly convex, then inequality (2.1.1) *is strict unless $x_1 = \cdots = x_n$.*

The condition "**p** is a positive n-tuple" can be replaced by "**p** is a non-negative n-tuple and $P_n > 0$". Note that the Jensen's inequality (2.1.1) can be used as an alternative definition of convexity.

It is reasonable to ask whether the condition "**p** is a non-negative n-tuple" can be relaxed at the expense of restricting **x** more severely. An answer to this question was given by Steffensen [63] (see also [58, p. 57]).

Theorem 2.1.2. *Let $f : I \to \mathbb{R}$ be a convex function. If $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ is a monotonic n-tuple and $\mathbf{p} = (p_1, \ldots, p_n)$ a real n-tuple such that*

$$0 \leq P_k \leq P_n \ , \quad k = 1, \ldots, n, \quad P_n > 0, \tag{2.1.3}$$

is satisfied, where P_k are as in (2.1.2)*, then* (2.1.1) *holds. If f is strictly convex, then inequality* (2.1.1) *is strict unless $x_1 = \cdots = x_n$.*

Inequality (2.1.1) under conditions from Theorem 2.1.2 is called the Jensen-Steffensen inequality. A refinement of the Jensen-Steffensen inequality is given in [65] (see also [58, p. 89]).

Theorem 2.1.3. *Let **x** and **p** be two real n-tuples such that $a \leq x_1 \leq \cdots \leq x_n \leq b$ and* (2.1.3) *hold. Then for every convex function $f : [a, b] \to \mathbb{R}$*

$$F_n(x_1, \ldots, x_n) \geq F_{n-1}(x_1, \ldots, x_{n-1}) \geq \cdots \geq F_2(x_1, x_2) \geq F_1(x_1) = 0 \tag{2.1.4}$$

holds, where

$$F_k(x_1, \ldots, x_k) = G_k\left(x_1, \ldots, x_k, p_1, \ldots, p_{k-1}, \bar{P}_k\right), \tag{2.1.5}$$

$$G_k(x_1, \ldots, x_k, p_1, \ldots, p_k) = \frac{1}{P_k} \sum_{i=1}^{k} p_i f(x_i) - f\left(\frac{1}{P_k} \sum_{i=1}^{k} p_i x_i\right), \tag{2.1.6}$$

P_k are as in (2.1.2) *and*

$$\bar{P}_k = \sum_{i=k}^{n} p_i, \qquad k = 1, \ldots, n. \tag{2.1.7}$$

Note that the function G_n defined in (2.1.6) is in fact the difference of the right-hand and the left-hand side of the Jensen's inequality (2.1.1).

In the coming section, we present a new refinement of the Jensen-Steffensen inequality, related to Theorem 2.1.3. In what follows, P_k are as in (2.1.2) and $\bar{P}_k$ are as in (2.1.7). Note that if (2.1.3) is valid, since $\bar{P}_k = P_n - P_{k-1}$, it follows that $\bar{P}_k$ satisfy (2.1.3) as well.

2.2 Refinement of the Jensen-Steffensen Inequality

A new refinement of Jensen-Steffensen inequality states that:

Theorem 2.2.1. *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ *be a monotonic* n*-tuple and* $\mathbf{p} = (p_1, \ldots, p_n)$ *a real* n*-tuple such that* (2.1.3) *holds. Then for a convex function* $f : I \to \mathbb{R}$*, we have*

$$\bar{F}_n(x_1, \ldots, x_n) \geq \bar{F}_{n-1}(x_2, \cdots, x_n) \geq \cdots \geq \bar{F}_2(x_{n-1}, x_n) \geq \bar{F}_1(x_n) = 0, \tag{2.2.1}$$

where

$$\begin{aligned} &\bar{F}_k(x_{n-k+1}, x_{n-k+2}, \ldots, x_n) \\ &\quad = \bar{G}_k(x_{n-k+1}, x_{n-k+2}, \ldots, x_n, P_{n-k+1}, p_{n-k+2}, \ldots, p_n), \\ &\bar{G}_k(x_{n-k+1}, \ldots, x_n, p_{n-k+1}, \ldots, p_n) \\ &\quad = \frac{1}{\bar{P}_{n-k+1}} \sum_{i=n-k+1}^{n} p_i f(x_i) - f\left(\frac{1}{\bar{P}_{n-k+1}} \sum_{i=n-k+1}^{n} p_i x_i\right). \end{aligned} \tag{2.2.2}$$

For a concave function f*, the inequality signs in* (2.2.1) *reverse.*

Proof. The claim is that for a convex function f,

$$\bar{F}_k(x_{n-k+1}, \ldots, x_n) \geq \bar{F}_{k-1}(x_{n-k+2}, \ldots, x_n)$$

holds for every $k = 2, \ldots, n$. This inequality is equivalent to

$$\frac{P_{n-k+1}}{P_n}\left(f(x_{n-k+2}) - f(x_{n-k+1})\right) \leq f(\bar{x}_{n-k+2}) - f(\bar{x}_{n-k+1}), \tag{2.2.3}$$

where

$$\bar{x}_{n-k+1} = \frac{1}{P_n}\left(P_{n-k+1}x_{n-k+1} + \sum_{i=n-k+2}^{n} p_i x_i\right).$$

If $\mathbf{x}$ is increasing then $x_{n-k+1} \leq \bar{x}_{n-k+1}$, while if $\mathbf{x}$ is decreasing then $x_{n-k+1} \geq \bar{x}_{n-k+1}$ for every k. Furthermore, without loss of generality, we can assume that $\mathbf{x}$ is strictly monotonic and that $0 < P_k < P_n$ for $k = 1, \dots, n-1$. Now, applying (1.1.4) for a convex function f when $\mathbf{x}$ is strictly increasing yields inequality

$$\frac{f(x_{n-k+2}) - f(x_{n-k+1})}{x_{n-k+2} - x_{n-k+1}} \le \frac{f(\bar{x}_{n-k+2}) - f(\bar{x}_{n-k+1})}{\frac{P_{n-k+1}}{P_n}(x_{n-k+2} - x_{n-k+1})},$$

while if $\mathbf{x}$ is strictly decreasing we get inequality

$$\frac{f(\bar{x}_{n-k+2}) - f(\bar{x}_{n-k+1})}{\frac{P_{n-k+1}}{P_n}(x_{n-k+2} - x_{n-k+1})} \le \frac{f(x_{n-k+2}) - f(x_{n-k+1})}{x_{n-k+2} - x_{n-k+1}},$$

both of which are equivalent to (2.2.3). If f is concave, the inequalities reverse. Thus, the proof is complete. □

Remark 2.2.1. A slight extension of the proof of Theorem 2.1.3 in [65] shows that Theorem 2.1.3 remains valid if the n-tuple $\mathbf{x}$ is assumed to be monotonic instead of increasing. The proof is in fact analogous to the proof of Theorem 2.2.1.

Consider the inequalities (2.1.4) and (2.2.1). Motivated by them, we define two linear functionals $\Phi_i : K[a,b] \to \mathbb{R}$ $(i = 1, 2)$ as follows:

$$\Phi_1(f) = F_k(x_1, \dots, x_k) - F_j(x_1, \dots, x_j), \quad 1 \le j < k \le n, \tag{2.2.4}$$

$$\Phi_2(f) = \bar{F}_k(x_{n-k+1}, \dots, x_n) - \bar{F}_j(x_{n-j+1}, \dots, x_n), \quad 1 \le j < k \le n, \tag{2.2.5}$$

where functions F_k and $\bar{F}_k$ are as in (2.1.5) and (2.2.2), respectively. If the function f is convex on I, then Theorems 2.1.3 and 2.2.1, joint with Remark 2.2.1, imply that

$$\Phi_i(f) \ge 0, \qquad i = 1, 2. \tag{2.2.6}$$

2.3 Integral Analogous

Jensen's inequality is a powerful mathematical tool which relates the value of a convex function of an integral to the integral of the convex function. A basic form of the weighted integral Jensen's inequality is stated as follows (see [48, Theorem 1.2.5]):

Theorem 2.3.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function, $p : [a,b] \to (0,\infty)$ and $g : [a,b] \to [0,\infty)$ be integrable functions. Then the inequality*

$$f\left(\frac{\int_a^b p(u)\,g(u)\,du}{\int_a^b p(u)\,du}\right) \le \frac{\int_a^b p(u)\,f(g(u))\,du}{\int_a^b p(u)\,du} \tag{2.3.1}$$

holds, provided that all the integrals in (2.3.1) are meaningful.

The first integral analogue is stated as follows:

Theorem 2.3.2. *Let the conditions of Theorem 2.3.1 hold. In addition, let g be monotonic and f and g be differentiable functions defined on $[a,b]$. If p satisfies*

$$0 \le \int_a^x p(u)\,du \le \int_a^b p(u)\,du \text{ for every } x \in [a,b] \text{ and } \int_a^b p(u)\,du > 0, \tag{2.3.2}$$

then the function

$$\begin{aligned} F(x) &= \frac{\int_a^x p(u)\,f(g(u))\,du + f(g(x))\int_x^b p(u)\,du}{\int_a^b p(u)\,du} \\ &\quad - f\left(\frac{\int_a^x p(u)\,g(u)\,du + g(x)\int_x^b p(u)\,du}{\int_a^b p(u)\,du}\right) \end{aligned} \tag{2.3.3}$$

is increasing on $[a,b]$, that is, for all $x, y \in [a,b]$ such that $a \le x \le y \le b$, we have

$$0 \le F(x) \le F(y) \le \frac{\int_a^b p(u)f(g(u))du}{\int_a^b p(u)du} - f\left(\frac{\int_a^b p(u)g(u)du}{\int_a^b p(u)\,du}\right). \tag{2.3.4}$$

Proof. Observe that

$$F'(x) = \frac{g'(x)\int_x^b p(u)\,du}{\int_a^b p(u)\,du}\left[f'(g(x)) - f'\left(\frac{\int_a^x p(u)\,g(u)\,du + g(x)\int_x^b p(u)\,du}{\int_a^b p(u)\,du}\right)\right],$$

where $\frac{\int_x^b p(u)du}{\int_a^b p(u)du} \ge 0$ as (2.3.2) holds. The claim will follow if $F'(x) \ge 0$, that is, if

$$\frac{g'(x)\int_x^b p(u)\,du}{\int_a^b p(u)\,du} \ge 0 \tag{2.3.5}$$

and

$$f'(g(x)) - f'\left(\frac{\int_a^x p(u)\,g(u)\,du + g(x)\int_x^b p(u)\,du}{\int_a^b p(u)\,du}\right) \ge 0 \tag{2.3.6}$$

hold or if

$$\frac{g'(x)\int_x^b p(u)\,du}{\int_a^b p(u)\,du} \le 0 \tag{2.3.7}$$

and

$$f'(g(x)) - f'\left(\frac{\int_a^x p(u)\,g(u)\,du + g(x)\int_x^b p(u)\,du}{\int_a^b p(u)\,du}\right) \le 0 \tag{2.3.8}$$

hold.

Now, we discuss the following two cases.

Case I. If g is increasing, then (2.3.5) holds and $g(x) - \frac{\int_a^x p(u)g(u)du + g(x)\int_x^b p(u)du}{\int_a^b p(u)du} \ge 0$. Since f is a differentiable convex function defined on $[a,b]$, f' is increasing on $[a,b]$, and so (2.3.6) holds, which together with (2.3.5) implies that $F'(x) \ge 0$.

Case II. If g is decreasing, then (2.3.7) holds and $g(x) - \frac{\int_a^x p(u)g(u)du + g(x)\int_x^b p(u)du}{\int_a^b p(u)du} \le 0$. Again, by using the convexity of f, (2.3.8) holds, which together with (2.3.7) implies that $F'(x) \ge 0$.

Now, as $F(x)$ is increasing on $[a,b]$, for all $x, y \in [a,b]$ such that $a \le x \le y \le b$, we have

$$F(a) \le F(x) \le F(y) \le F(b). \tag{2.3.9}$$

At $x = a$ and at $x = b$, (2.3.3) gives $F(a) = 0$ and $F(b) = \frac{\int_a^b p(u)f(g(u))du}{\int_a^b p(u)du} - f\left(\frac{\int_a^b p(u)g(u)du}{\int_a^b p(u)du}\right)$ respectively. Substituting these values of $F(a)$ and $F(b)$ in (2.3.9), we obtain (2.3.4). □

The integral analogue of Theorem 2.2.1 states that:

Theorem 2.3.3. *Let the conditions of Theorem* 2.3.2 *hold. Then the function*

$$\begin{aligned} \bar{F}(x) &= \frac{\int_x^b p(u)\,f(g(u))\,du + f(g(x))\int_a^x p(u)\,du}{\int_a^b p(u)\,du} \\ &\quad - f\left(\frac{\int_x^b p(u)\,g(u)\,du + g(x)\int_a^x p(u)\,du}{\int_a^b p(u)\,du}\right) \end{aligned} \tag{2.3.10}$$

is decreasing on $[a,b]$, *that is, for all* $x, y \in [a,b]$ *such that* $a \le x \le y \le b$, *we have*

$$0 \le \bar{F}(y) \le \bar{F}(x) \le \frac{\int_a^b p(u)\,f(g(u))\,du}{\int_a^b p(u)\,du} - f\left(\frac{\int_a^b p(u)\,g(u)\,du}{\int_a^b p(u)\,du}\right). \tag{2.3.11}$$

Proof. We have

$$\bar{F}'(x) = \frac{g'(x)\int_a^x p(u)\,du}{\int_a^b p(u)\,du}\left[f'(g(x)) - f'\left(\frac{\int_x^b p(u)\,g(u)\,du + g(x)\int_a^x p(u)\,du}{\int_a^b p(u)\,du}\right)\right],$$

where $\frac{\int_a^x p(u)du}{\int_a^b p(u)du} \geq 0$ as (2.3.2) holds. It is enough to prove that $\bar{F}'(x) \leq 0$, that is, either

$$\frac{g'(x)\int_a^x p(u)\,du}{\int_a^b p(u)\,du} \geq 0 \tag{2.3.12}$$

and

$$f'(g(x)) - f'\left(\frac{\int_x^b p(u)\,g(u)\,du + g(x)\int_a^x p(u)\,du}{\int_a^b p(u)\,du}\right) \leq 0 \tag{2.3.13}$$

hold or

$$\frac{g'(x)\int_a^x p(u)\,du}{\int_a^b p(u)\,du} \leq 0 \tag{2.3.14}$$

and

$$f'(g(x)) - f'\left(\frac{\int_x^b p(u)\,g(u)\,du + g(x)\int_a^x p(u)\,du}{\int_a^b p(u)\,du}\right) \geq 0 \tag{2.3.15}$$

hold.

Now, we discuss the following two cases.

Case I. If g is increasing, then (2.3.12) holds and $g(x) - \frac{\int_x^b p(u)g(u)du + g(x)\int_a^x p(u)du}{\int_a^b p(u)du} \leq 0$. Since f is a differentiable convex function defined on $[a, b]$, f' is increasing on $[a, b]$ and so (2.3.13) holds, which together with (2.3.12) implies that $\bar{F}'(x) \leq 0$.

Case II. If g is decreasing, then (2.3.14) holds and $g(x) - \frac{\int_x^b p(u)g(u)du + g(x)\int_a^x p(u)du}{\int_a^b p(u)du} \geq 0$. Again, by using the convexity of f, (2.3.15) holds, which together with (2.3.14) implies that $\bar{F}'(x) \leq 0$.

Now, as $\bar{F}$ is decreasing on $[a, b]$, for any $x, y \in [a, b]$ such that $a \leq x \leq y \leq b$, we have

$$\bar{F}(b) \leq \bar{F}(y) \leq \bar{F}(x) \leq \bar{F}(a). \tag{2.3.16}$$

At $x = a$ and at $x = b$, (2.3.10) gives $\bar{F}(a) = \frac{\int_a^b p(u)f(g(u))du}{\int_a^b p(u)du} - f\left(\frac{\int_a^b p(u)g(u)du}{\int_a^b p(u)du}\right)$ and $\bar{F}(b) = 0$ respectively. By using these values of $\bar{F}(a)$ and $\bar{F}(b)$ in (2.3.16) , we have (2.3.11). □

Remark 2.3.1. If we make the substitutions $p(u) = 1$ and $g(u) = u$ in Theorems 2.3.2 and 2.3.3, then we obtain the refinements of the first Hermite-Hadamard inequality for differentiable convex functions.

The first refinement is stated as follows:

Corollary 2.3.4. *Let a,b* $\in \mathbb{R}$ *with* $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *be a differentiable convex function. Then the function*

$$H(x) = \frac{1}{b-a}\left[\int_a^x f(u)\,du + (b-x) f(x)\right] - f\left(\frac{2bx - x^2 - a^2}{2(b-a)}\right)$$

is increasing on $[a, b]$ *and for all* $x, y \in [a, b]$ *such that* $x \leq y$*, we have*

$$0 \leq H(x) \leq H(y) \leq \frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right).$$

Proof. The idea of the proof is the same as that of Theorem 2.3.2. □

The second refinement is given as follows:

Corollary 2.3.5. *Let a,b* $\in \mathbb{R}$ *with* $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *be a differentiable convex function. Then the function*

$$\bar{H}(x) = \frac{1}{b-a}\left[\int_x^b f(u)\,du + (x-a) f(x)\right] - f\left(\frac{x^2 + b^2 - 2ax}{2(b-a)}\right)$$

is decreasing on $[a, b]$ *and for any* $x, y \in [a, b]$ *such that* $x \leq y$*, we have*

$$0 \leq \bar{H}(y) \leq \bar{H}(x) \leq \frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right).$$

Proof. The idea of the proof is the same as that of Theorem 2.3.3. □

Let $\tilde{\mathrm{K}}[a, b]$ denotes the class of all differentiable convex functions defined on $[a, b]$. Motivated by the inequalities (2.3.4) and (2.3.11), we define two linear functionals $\Phi_i : \tilde{\mathrm{K}}\,[a, b] \to \mathbb{R}$ $(i = 3, 4)$ as follows:

$$\Phi_3(f) = F(y) - F(x), \quad x \leq y, \tag{2.3.17}$$

$$\Phi_4(f) = \bar{F}(x) - \bar{F}(y), \quad x \leq y, \tag{2.3.18}$$

where $x, y \in [a, b]$ and the functions F and $\bar{F}$ are as in (2.3.3) and (2.3.10) respectively. If f is a differentiable convex function defined on $[a, b]$, then Theorems 2.3.2 and 2.3.3 imply that $\Phi_i(f) \geq 0$ $(i = 3, 4)$.

2.4 Mean Value Theorems

Now, we give mean value theorems for the functionals Φ_i $(i = 1, \ldots, 4)$. These theorems enable us to define various classes of means that can be expressed in terms of linear functionals.

In the proof of mean value theorems, the following result is needed (see [3]).

Lemma 2.4.1. *Let $f \in C^2(I)$ such that f'' is bounded, that is, there exists $m = \min_{x \in [a,b]} f''(x)$ and $M = \max_{x \in [a,b]} f''(x)$ such that*

$$m \leq f''(x) \leq M, \qquad x \in I, \tag{2.4.1}$$

holds. Then the functions $\zeta_1(x)$ and $\zeta_2(x)$ defined by

$$\zeta_1(x) = \frac{M}{2}x^2 - f(x),$$

and

$$\zeta_2(x) = f(x) - \frac{m}{2}x^2,$$

are convex.

Proof. We have $\zeta_1''(x) = M - f''(x)$ and $\zeta_2''(x) = f''(x) - m$. By using (2.4.1) and Theorem 1.1.5 for $n = 2$, ζ_i $(i = 1, 2)$ are convex. □

First, we state the Lagrange-type mean value theorem related to the functionals Φ_i for $i = 1, 2$.

Theorem 2.4.2. *Let $\mathbf{x} = (x_1, \ldots, x_n) \in [a,b]^n$ be a monotonic n-tuple and $\mathbf{p} = (p_1, \ldots, p_n)$ a real n-tuple such that (2.1.3) holds. Let $f \in C^2([a,b])$ and Φ_i $(i = 1, 2)$ be linear functionals as defined in (2.2.4) and (2.2.5). Then there exists $\xi_i \in [a,b]$ such that*

$$\Phi_i(f) = \frac{f''(\xi_i)}{2}\Phi_i(f_0), \qquad i = 1, 2, \tag{2.4.2}$$

where $f_0(x) = x^2$.

Proof. Since $f''(x)$ is continuous on $[a,b]$, there exist real numbers $m = \min_{x \in [a,b]} f''(x)$ and $M = \max_{x \in [a,b]} f''(x)$ such that (2.4.1) holds. By substituting ζ_1 and ζ_2 in (2.2.6), we have

$$\Phi_i\left(\frac{M}{2}x^2 - f(x)\right) \geq 0, \qquad i = 1, 2,$$

and

$$\Phi_i\left(f(x) - \frac{m}{2}x^2\right) \geq 0, \qquad i = 1, 2,$$

respectively. The above two inequalities are equivalent to

$$\Phi_i(f(x)) \leq \frac{M}{2}\Phi_i\left(x^2\right), \qquad i = 1, 2, \tag{2.4.3}$$

and

$$\frac{m}{2}\Phi_i\left(x^2\right) \leq \Phi_i(f(x)), \qquad i = 1, 2, \tag{2.4.4}$$

respectively. From (2.4.3) and (2.4.4), we have

$$\frac{m}{2}\Phi_i(f_0) \leq \Phi_i(f) \leq \frac{M}{2}\Phi_i(f_0), \qquad i = 1, 2. \tag{2.4.5}$$

If $\Phi_i(f_0) = 0$ $(i = 1, 2)$, then there is nothing to prove. Suppose that $\Phi_i(f_0) > 0$, then from (2.4.5), we have

$$m \leq \frac{2\Phi_i(f)}{\Phi_i(f_0)} \leq M, \qquad i = 1, 2.$$

Now by using the fact that for $m \leq \eta_i \leq M$, there exist $\xi_i \in [a, b]$ such that $f''(\xi_i) = \eta_i$ $(i = 1, 2)$ and so we have (2.4.2). □

The following theorem is a new analogue of the classical Cauchy mean value theorem related to the functionals Φ_i for $i = 1, 2$.

Theorem 2.4.3. *Let all the assumptions of Theorem* 2.4.2 *hold and let* $f, h \in C^2([a, b])$. *Then there exists* $\xi_i \in [a, b]$ *such that*

$$\frac{\Phi_i(f)}{\Phi_i(h)} = \frac{f''(\xi_i)}{h''(\xi_i)}, \qquad i = 1, 2, \tag{2.4.6}$$

provided that the denominators are non-zero.

Proof. Consider the functions $k_i \in C^2([a, b])$ defined by $k_i = c_i f - d_i h$ such that $c_i = \Phi_i(h)$ and $d_i = \Phi_i(f)$, where $i = 1, 2$. Using Theorem 2.4.2 with $f = k_i$, there exist $\xi_i \in [a, b]$ such that

$$\left(\frac{c_i f''(\xi_i)}{2} - \frac{d_i h''(\xi_i)}{2}\right)\Phi_i(f_0) = 0, \qquad i = 1, 2.$$

Since $\Phi_i(f_0) \neq 0$ (because otherwise we have a contradiction with $\Phi_i(h) \neq 0$ by Theorem 2.4.2), we get

$$\frac{f''(\xi_i)}{h''(\xi_i)} = \frac{d_i}{c_i}, \qquad i = 1, 2.$$

After substituting the values of c_i and d_i, we have (2.4.6). □

Remark 2.4.1. If the inverse of the function f''/h'' exits, then from (2.4.6), we have

$$\xi_i = \left(\frac{f''}{h''}\right)^{-1} \left(\frac{\Phi_i(f)}{\Phi_i(h)}\right), \qquad i = 1, 2.$$

Now, we give mean value theorems for the functionals Φ_i $(i = 3, 4)$.

Theorem 2.4.4. *Let $x, y \in [a, b]$ be such that $x \leq y$, p be a function satisfying* (2.3.2) *and g be a monotone differentiable function. Let $f \in C^2([a, b])$ and Φ_i $(i = 3, 4)$ be linear functionals as defined in* (2.3.17) *and* (2.3.18)*. Then there exists $\xi_i \in [a, b]$ such that*

$$\Phi_i(f) = \frac{f''(\xi_i)}{2} \Phi_i(f_0), \qquad i = 3, 4,$$

where $f_0(x) = x^2$.

Proof. The proof is analogous to the proof of Theorem 2.4.2. □

Theorem 2.4.5. *Let all the assumptions of Theorem* 2.4.4 *hold and let $f, h \in C^2([a, b])$. Then there exists $\xi_i \in [a, b]$ such that*

$$\frac{\Phi_i(f)}{\Phi_i(h)} = \frac{f''(\xi_i)}{h''(\xi_i)}, \qquad i = 3, 4,$$

provided that the denominators are non-zero.

Proof. The proof is analogous to the proof of Theorem 2.4.3. □

Remark 2.4.2. For $i = 3, 4$, the Remark 2.4.1 also holds.

2.5 n-Exponential Convexity and Log-convexity of the Jensen-Steffensen Differences

Next, we study the n-exponential convexity and log-convexity of the functions associated with the linear functionals Φ_i $(i = 1, \ldots, 4)$ defined in (2.2.4), (2.2.5), (2.3.17) and (2.3.18) respectively.

Theorem 2.5.1. *Let Φ_i $(i = 1, \ldots, 4)$ be linear functionals as defined in* (2.2.4)*,* (2.2.5)*,* (2.3.17) *and* (2.3.18) *respectively. Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a, b]$ such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense on I for every three mutually distinct points $y_0, y_1, y_2 \in [a, b]$ (for $i = 3, 4$, the functions $f_s \in \Omega$ must be differentiable). Then the following statements hold:*

(*i*) *The function* $s \mapsto \Phi_i(f_s)$ *is n-exponentially convex in the Jensen sense on* I *and the matrix* $\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m}$ *is positive semi-definite for all* $m \in \mathbb{N}$, $m \leq n$ *and* $s_1, \ldots, s_m \in I$. *Particularly,*

$$\det\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m} \geq 0, \;\; \forall \;\; m \in \mathbb{N}, \;\; m \leq n.$$

(*ii*) *If the function* $s \mapsto \Phi_i(f_s)$ *is continuous on* I, *then it is n-exponentially convex on* I.

Proof. The idea of the proof is the same as that of Theorem 3.1 in [54].

(i) Let $\alpha_j \in \mathbb{R}$ $(j = 1, \ldots, n)$ and consider the function

$$\varphi(y) = \sum_{j,k=1}^{n} \alpha_j \alpha_k f_{\frac{s_j+s_k}{2}}(y),$$

where $s_j \in I$ and $f_{\frac{s_j+s_k}{2}} \in \Omega$. Then

$$[y_0, y_1, y_2; \varphi] = \sum_{j,k=1}^{n} \alpha_j \alpha_k \left[y_0, y_1, y_2; f_{\frac{s_j+s_k}{2}}\right]$$

and since $\left[y_0, y_1, y_2; f_{\frac{s_j+s_k}{2}}\right]$ is n-exponentially convex in the Jensen sense on I by assumption, it follows that

$$[y_0, y_1, y_2; \varphi] = \sum_{j,k=1}^{n} \alpha_j \alpha_k \left[y_0, y_1, y_2; f_{\frac{s_j+s_k}{2}}\right] \geq 0$$

and so by using Definition 1.1.9 for $n = 2$, we conclude that φ is a convex function. Hence

$$\Phi_i(\varphi) \geq 0, \qquad i = 1, \ldots, 4,$$

which is equivalent to

$$\sum_{j,k=1}^{n} \alpha_j \alpha_k \Phi_i\left(f_{\frac{s_j+s_k}{2}}\right) \geq 0, \qquad i = 1, \ldots, 4,$$

and so we conclude that the function $s \mapsto \Phi_i(f_s)$ is n-exponentially convex in the Jensen sense on I.
The remaining part follows from Proposition 1.3.1.

(ii) If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then from (i) and by Definition 1.3.2 it follows that it is n-exponentially convex on I.

□

The following corollary is an immediate consequence of the above theorem.

Corollary 2.5.2. *Let Φ_i $(i = 1, \ldots, 4)$ be linear functionals as defined in* (2.2.4), (2.2.5), (2.3.17) *and* (2.3.18) *respectively. Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a, b]$ such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is exponentially convex in the Jensen sense on I for every three mutually distinct points $y_0, y_1, y_2 \in [a, b]$ (for $i = 3, 4$, the functions $f_s \in \Omega$ must be differentiable). Then the following statements hold:*

(*i*) *The function $s \mapsto \Phi_i(f_s)$ is exponentially convex in the Jensen sense on I and the matrix $\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m}$ is positive semi-definite for all $m \in \mathbb{N}$, $m \leq n$ and $s_1, \ldots, s_m \in I$. Particularly,*

$$\det\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m} \geq 0, \ \ \forall \ m \in \mathbb{N}, \ m \leq n.$$

(*ii*) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is exponentially convex on I.*

Corollary 2.5.3. *Let Φ_i $(i = 1, \ldots, 4)$ be linear functionals as defined in* (2.2.4), (2.2.5), (2.3.17) *and* (2.3.18) *respectively. Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of differentiable functions defined on $[a, b]$ such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is 2-exponentially convex in the Jensen sense on I for every three mutually distinct points $y_0, y_1, y_2, \in [a, b]$. Further, assume that $\Phi_i(f_s)$ is strictly positive for $f_s \in \Omega$. Then the following statements hold:*

(*i*) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is 2-exponentially convex on I and so it is log-convex on I and for $r, s, t \in I$ such that $r < t < s$, we have*

$$[\Phi_i(f_t)]^{s-r} \leq [\Phi_i(f_r)]^{s-t} [\Phi_i(f_s)]^{t-r}, \qquad i = 1, \ldots, 4, \tag{2.5.1}$$

known as Lyapunov's inequality. If $r < s < t$ or $t < r < s$, then opposite inequalities hold in (2.5.1).

(*ii*) *If the function $s \mapsto \Phi_i(f_s)$ is differentiable on I, then for every $s, q, u, v \in I$ such that $s \leq u$ and $q \leq v$, we have*

$$\mu_{s,q}(\Phi_i, \Omega) \leq \mu_{u,v}(\Phi_i, \Omega), \qquad i = 1, \ldots, 4, \tag{2.5.2}$$

where

$$\mu_{s,q}(\Phi_i,\Omega)=\begin{cases}\left(\dfrac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(\dfrac{\frac{d}{ds}\Phi_i(f_s)}{\Phi_i(f_s)}\right), & s=q,\end{cases} \tag{2.5.3}$$

for $f_s, f_q \in \Omega$.

Proof. The idea of the proof is the same as that of Corollary 3.2 given in [54].

(i) The claim that the function $s \mapsto \Phi_i(f_s)$ is log-convex on I is an immediate consequence of Theorem 2.5.1 and Remark 1.3.2, and (2.5.1) can be obtained by replacing the convex function f with the convex function $f(z) = \log \Phi_i(f_z)$ for $z = r, s, t$ in (1.1.3), where $r, s, t \in I$ such that $r < t < s$.

(ii) Since by (i) the function $s \mapsto \Phi_i(f_s)$ is log-convex on I, that is, the function $s \mapsto \log \Phi_i(f_s)$ is convex on I. Applying Theorem 1.1.2 with setting $f(z) = \log \Phi_i(f_z)$, we get

$$\frac{\log \Phi_i(f_s) - \log \Phi_i(f_q)}{s-q} \leq \frac{\log \Phi_i(f_u) - \log \Phi_i(f_v)}{u-v}, \tag{2.5.4}$$

for $s \leq u,\ q \leq v,\ s \neq q, u \neq v$; and therefore, we conclude that

$$\mu_{s,q}(\Phi_i,\Omega) \leq \mu_{u,v}(\Phi_i,\Omega), \qquad i = 1,\ldots,4.$$

If $s = q$, we consider the limit when $q \to s$ in (2.5.4) and conclude that

$$\mu_{s,s}(\Phi_i,\Omega) \leq \mu_{u,v}(\Phi_i,\Omega), \qquad i = 1,\ldots,4.$$

The case $u = v$ can be treated similarly.

□

Remark 2.5.1. Note that the results from Theorem 2.5.1, Corollary 2.5.2 and Corollary 2.5.3 still hold when two of the points $y_0, y_1, y_2 \in [a,b]$ coincide, say $y_1 = y_0$, for a family of differentiable functions f_s such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense on I); and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 1.1.5 (i) and by using suitable characterizations of convexity.

Remark 2.5.2. Related results for the original Jensen-Steffensen inequality regarding exponential convexity, which are a special case of Corollary 2.5.2 and Corollary 2.5.3, are given in [4].

2.6 Examples

In this section, we present several families of functions which fulfil the conditions of Theorem 2.5.1, Corollaries 2.5.2 and 2.5.3, and Remark 2.5.1 and so the results of these theorem and corollaries can be applied for them.

Example 2.6.1. *Consider the family of functions*

$$\Omega_1 = \{g_s : \mathbb{R} \to [0,\ \infty) : s \in \mathbb{R}\}$$

defined by

$$g_s(x) \ = \begin{cases} \frac{1}{s^2}\, e^{sx}, & s \neq 0, \\ \frac{1}{2}\, x^2, & s = 0. \end{cases}$$

We have $\frac{d^2}{dx^2} g_s(x) = e^{sx} > 0$*, which shows that* g_s *is convex on* $\mathbb{R}$ *for every* $s \in \mathbb{R}$ *and* $s \mapsto \frac{d^2}{dx^2} g_s(x)$ *is exponentially convex by definition. In order to prove that the function* $s \mapsto [y_0, y_1, y_2; g_s]$ *is exponentially convex, it is enough to show that*

$$\textstyle\sum_{j,k=1}^{n} \alpha_j \alpha_k \left[y_0, y_1, y_2; g_{\frac{s_j+s_k}{2}}\right] = \left[y_0, y_1, y_2; \sum_{j,k=1}^{n} \alpha_j \alpha_k g_{\frac{s_j+s_k}{2}}\right] \geq 0, \tag{2.6.1}$$

$\forall\ n \in \mathbb{N},\ \alpha_j, s_j \in \mathbb{R},\ j = 1, \ldots, n.$ *By Definition 1.1.9,* (2.6.1) *will hold if* $\Upsilon(x) := \sum_{j,k=1}^{n} \alpha_j \alpha_k g_{\frac{s_j+s_k}{2}}(x)$ *is convex. Since* $s \mapsto g_s''(x)$ *is exponentially convex, that is*

$$\sum_{j,k=1}^{n} \alpha_j \alpha_k g''_{\frac{s_j+s_k}{2}} \geq 0, \qquad \forall\, n \in \mathbb{N},\ \alpha_j, s_j \in \mathbb{R},\ j = 1, \ldots, n,$$

showing the convexity of $\Upsilon(x)$ *and so* (2.6.1) *holds. Now as the function* $s \mapsto [y_0, y_1, y_2; g_s]$ *is exponentially convex,* $s \mapsto [y_0, y_1, y_2; g_s]$ *is exponentially convex in the Jensen sense and by using Corollary* 2.5.2*, we have* $s \mapsto \Phi_i(g_s)\,(i = 1, \ldots, 4)$ *are exponentially convex in the Jensen sense. Since these mappings are continuous (although the mapping* $s \mapsto g_s$ *is not continuous for* $s = 0$*), so* $s \mapsto \Phi_i(g_s)\,(i = 1, \ldots, 4)$ *are exponentially convex.*

For this family of functions, $\mu_{s,q}(\Phi_i, \Omega)\,(i = 1, \ldots, 4)$ *from* (2.5.3) *become*

$$\mu_{s,q}(\Phi_i, \Omega_1) \ = \begin{cases} \left(\frac{\Phi_i(g_s)}{\Phi_i(g_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_i(id \cdot g_s)}{\Phi_i(g_s)} - \frac{2}{s}\right), & s = q \neq 0, \\ \exp\left(\frac{\Phi_i(id \cdot g_0)}{3\Phi_i(g_0)}\right), & s = q = 0. \end{cases} \tag{2.6.2}$$

Example 2.6.2. *Consider the family of functions*

$$\Omega_2 = \{f_s : (0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0,1, \\ -\log x, & s = 0, \\ x \log x, & s = 1. \end{cases}$$

Here, $\frac{d^2}{dx^2} f_s(x) = x^{s-2} = e^{(s-2)\ln x} > 0$*, which shows that* f_s *is convex for* $x > 0$ *and* $s \mapsto \frac{d^2}{dx^2} f_s(x)$ *is exponentially convex by definition. It is easy to prove that the function* $s \mapsto [y_0, y_1, y_2; f_s]$ *is exponentially convex. Arguing as in Example* 2.6.1*, we have* $s \mapsto \Phi_i(f_s)\,(i = 1, \ldots, 4)$ *are exponentially convex.*

In this case, $\mu_{s,q}(\Phi_i, \Omega)\,(i = 1, \ldots, 4)$ *defined in* (2.5.3) *are*

$$\mu_{s,q}(\Phi_i, \Omega_2) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Phi_i(f_s f_0)}{\Phi_i(f_s)}\right), & s = q \neq 0,1, \\ \exp\left(1 - \frac{\Phi_i\left(f_0^2\right)}{2\Phi_i(f_0)}\right), & s = q = 0, \\ \exp\left(-1 - \frac{\Phi_i(f_0 f_1)}{2\Phi_i(f_1)}\right), & s = q = 1. \end{cases}$$

If Φ_i $(i = 1,2)$ *are positive, then Theorem 2.4.3 applied for* $f = f_s \in \Omega_2$ *and* $h = f_q \in \Omega_2$ *yields that there exists* $\xi_i \in \left[\min_{1\leq j\leq n} x_j, \max_{1\leq j\leq n} x_j\right]$ *such that*

$$\xi_i^{s-q} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)}, \qquad i = 1,2.$$

Since the functions $\xi_i \mapsto \xi_i^{s-q}$ *are invertible for* $s \neq q$*, we have*

$$\min\{x_1, x_n\} = \min_{1\leq j\leq n} x_j \leq \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}} \leq \max_{1\leq j\leq n} x_j = \max\{x_1, x_n\},$$

which together with the fact that $\mu_{s,q}(\Phi_i, \Omega_2)$ $(i = 1,2)$ *are continuous, symmetric and monotonous (by* (2.5.2)*), shows that* $\mu_{s,q}(\Phi_i, \Omega_2)$ $(i = 1,2)$ *are means.*
Similarly, it is easy to see that for $i = 3,4$, $\mu_{s,q}(\Phi_i, \Omega_2)$ *are means.*

Example 2.6.3. *Consider the family of functions*

$$\Omega_3 = \{h_s : (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}}{\ln^2 s}, & s \neq 1, \\ \frac{x^2}{2}, & s = 1. \end{cases}$$

We have $\frac{d^2}{dx^2}h_s(x) = s^{-x} > 0$, *which shows that* h_s *is convex for all* $s > 0$. *Since* $s \mapsto \frac{d^2}{dx^2}h_s(x) = s^{-x}$ *is the Laplace transform of a non-negative function (see[66]), it is exponentially convex. It is easy to prove that the function* $s \mapsto [y_0, y_1, y_2; h_s]$ *is exponentially convex. Arguing as in Example* 2.6.1, *we have* $s \mapsto \Phi_i(h_s)\,(i = 1, \ldots, 4)$ *are exponentially convex.*

For this family of functions, $\mu_{s,q}(\Phi_i, \Omega)\,(i = 1, \ldots, 4)$ *from* (2.5.3) *become*

$$\mu_{s,q}(\Phi_i, \Omega_3) = \begin{cases} \left(\frac{\Phi_i(h_s)}{\Phi_i(h_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id \cdot h_s)}{s\Phi_i(h_s)} - \frac{2}{s \ln s}\right), & s = q \neq 1, \\ \exp\left(-\frac{\Phi_i(id \cdot h_1)}{3\Phi_i(h_1)}\right), & s = q = 1. \end{cases} \tag{2.6.3}$$

Example 2.6.4. *Consider the family of functions*

$$\Omega_4 = \{k_s : (0, \infty) \to (0, \infty) : s \in (0, \infty)\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}}{s}$$

Here, $\frac{d^2}{dx^2}k_s(x) = e^{-x\sqrt{s}} > 0$, *which shows that* k_s *is convex for all* $s > 0$. *Since* $s \mapsto \frac{d^2}{dx^2}k_s(x) = e^{-x\sqrt{s}}$ *is the Laplace transform of a non-negative function (see [66]), it is exponentially convex. It is easy to prove that the function* $s \mapsto [y_0, y_1, y_2; k_s]$ *is exponentially convex. Arguing as in Example* 2.6.1, *we have* $s \mapsto \Phi_i(k_s)\,(i = 1, \ldots, 4)$ *are exponentially convex.*

In this case, $\mu_{s,q}(\Phi_i, \Omega)\,(i = 1, \ldots, 4)$ *defined in* (2.5.3) *are*

$$\mu_{s,q}(\Phi_i, \Omega_4) = \begin{cases} \left(\frac{\Phi_i(k_s)}{\Phi_i(k_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id \cdot k_s)}{2\sqrt{s}\Phi_i(k_s)} - \frac{1}{s}\right), & s = q. \end{cases} \tag{2.6.4}$$

The results presented in this chapter are published in [25, 34, 35].

Chapter 3

Refinements of the Lower Bounds of the Jensen's Functional

3.1 Introduction

The classical Jensen's inequality states (see e.g. [32, 46]): Let $f : I \to \mathbb{R}$ be a convex function. Let $n \geq 2$, $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ and $\mathbf{p} = (p_1, \ldots, p_n)$ be a positive n-tuple, that is, $p_i > 0$ for $i = 1, \ldots, n$. Then

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(x_i\right),$$

holds, where $P_n = \sum_{i=1}^{n} p_i$. If f is strictly convex, then above inequality is strict unless $x_1 = \ldots = x_n$.

In this chapter, the functional

$$J\left(\mathbf{x}, \mathbf{p}, f\right) = \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(x_i\right) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \tag{3.1.1}$$

defined as the difference of the right-hand and the left-hand sides of the Jensen's inequality is studied. More precisely, its lower bounds are investigated, together with various assumptions.

The lower bounds of $J\left(\mathbf{x}, \mathbf{p}, f\right)$ are the topic of interest in many papers. For example, the following results are proved in [22] (see also [46, p. 717]).

Theorem 3.1.1. *Let $f : I \to \mathbb{R}$ be a convex function, $\mathbf{x} \in I^n$ and $\mathbf{p}$ be a positive n-tuple. Then*

$$P_n \cdot J(\mathbf{x}, \mathbf{p}, f) \geq \max_{1 \leq j \leq k \leq n} \left\{ p_j f(x_j) + p_k f(x_k) - (p_j + p_k) f\left(\frac{p_j x_j + p_k x_k}{p_j + p_k} \right) \right\} \geq 0.$$

Theorem 3.1.2. *Let $f : I \to \mathbb{R}$ be a convex function and $\mathbf{x} \in I^n$. Let $\mathbf{p}$ and $\mathbf{r}$ be positive n-tuples such that $\mathbf{p} \geq \mathbf{r}$, that is, $p_i \geq r_i$, $i = 1, \ldots, n$. Then*

$$P_n \cdot J + (\mathbf{x}, \mathbf{p}, f) \geq R_n \cdot J(\mathbf{x}, \mathbf{r}, f) \quad \geq 0. \tag{3.1.2}$$

where $P_n = \sum_{i=1}^n p_i$ and $R_n = \sum_{i=1}^n r_i$.

Further, in [18] the following theorem is given. An alternative proof of the same result is given in [6].

Theorem 3.1.3. *Let $f : I \to \mathbb{R}$ be a convex function, $n \geq 2$ and $\mathbf{x} \in I^n$. Let $\mathbf{p}$ and $\mathbf{q}$ be positive n-tuples such that $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$. Then*

$$\max_{1 \leq j \leq n} \left\{ \frac{p_j}{q_j} \right\} J(\mathbf{x}, \mathbf{q}, f) \geq J(\mathbf{x}, \mathbf{p}, f) \geq \min_{1 \leq j \leq n} \left\{ \frac{p_j}{q_j} \right\} J(\mathbf{x}, \mathbf{q}, f) \geq 0.$$

For more related results see [1, 29, 40]. In this chapter, the motivation for our research work are the following results given in [13].

Lemma 3.1.4. *Let f be a convex function defined on I, $\mathbf{p}$ be a positive n-tuple such that $P_n = 1$ and $x_1, x_2, \ldots, x_n \in I$, $n \geq 3$ such that $x_1 \leq x_2 \leq \ldots \leq x_n$. For fixed $x_j, x_{j+1}, \ldots, x_n$, where $j = 2, 3, \ldots, n-1$, the Jensen's functional $J(\mathbf{x}, \mathbf{p}, f)$ defined in (3.1.1) is minimal when $x_1 = x_2 = \ldots = x_{j-1} = x_j$, that is,*

$$J(\mathbf{x}, \mathbf{p}, f) \geq P_j f(x_j) + \sum_{i=j+1}^{n} p_i f(x_i) - f\left(P_j x_j + \sum_{i=j+1}^{n} p_i x_i \right), \tag{3.1.3}$$

where

$$P_j = \sum_{i=1}^{j} p_i, \quad j = 1, \ldots, n. \tag{3.1.4}$$

Lemma 3.1.5. *Let f be a convex function defined on I, $\mathbf{p}$ be a positive n-tuple such that $P_n = 1$ and $x_1, x_2, \ldots, x_n \in I$, $n \geq 3$ such that $x_1 \leq x_2 \leq \ldots \leq x_n$. For fixed*

$x_1, x_2, \ldots, x_k$, where $k = 2, 3, \ldots, n-1$, the Jensen's functional $J(\mathbf{x}, \mathbf{p}, f)$ defined in (3.1.1) *is minimal when $x_k = x_{k+1} = \ldots = x_{n-1} = x_n$, that is,*

$$J(\mathbf{x}, \mathbf{p}, f) \geq \sum_{i=1}^{k-1} p_i f(x_i) + Q_k f(x_k) - f\left(\sum_{i=1}^{k-1} p_i x_i + Q_k x_k\right), \tag{3.1.5}$$

where

$$Q_k = \sum_{i=k}^{n} p_i, \quad k = 1, \ldots, n. \tag{3.1.6}$$

Theorem 3.1.6. *Let f be a convex function defined on I, $\mathbf{p}$ be a positive n-tuple such that $P_n = 1$ and $x_1, x_2, \ldots, x_n \in I$, $n \geq 3$ such that $x_1 \leq x_2 \leq \ldots \leq x_n$. For fixed x_j and x_k, where $1 \leq j < k \leq n$, the Jensen's functional $J(\mathbf{x}, \mathbf{p}, f)$ defined in* (3.1.1) *is minimal when*

$$\begin{aligned} x_1 = x_2 = \ldots &= x_j, \qquad x_k = x_{k+1} = \ldots = x_n, \\ x_{j+1} &= x_{j+2} = \ldots = x_{k-1} = \frac{P_j x_j + Q_k x_k}{P_j + Q_k}, \end{aligned}$$

that is,

$$J(\mathbf{x}, \mathbf{p}, f) \geq P_j f(x_j) + Q_k f(x_k) - (P_j + Q_k) f\left(\frac{P_j x_j + Q_k x_k}{P_j + Q_k}\right), \tag{3.1.7}$$

where P_j are as in (3.1.4) *and Q_k are as in* (3.1.6).

The key step in proving these results is the following lemma presented in the same paper.

Lemma 3.1.7. *Let f be a convex function defined on I and let p_1, p_2 be non-negative real numbers. If $a_1, a_2, b_1, b_2 \in I$ are such that $a_1, a_2 \in [b_1, b_2]$ and*

$$p_1 a_1 + p_2 a_2 = p_1 b_1 + p_2 b_2, \tag{3.1.8}$$

then

$$p_1 f(a_1) + p_2 f(a_2) \leq p_1 f(b_1) + p_2 f(b_2).$$

Note that for a monotonic n-tuple $\mathbf{x}$, Theorem 3.1.6 is an improvement of Theorem 3.1.1, in a sense that (the maximum of) the right-hand side of (3.1.7) is greater than the middle part of (3.1.1), which follows directly from the Jensen's inequality. The aim of this chapter is to give an improvement of Lemma 3.1.4, Lemma 3.1.5 and Theorem 3.1.6, in a sense that the condition of monotonicity imposed on the n-tuple $\boldsymbol{x}$ will be relaxed. Several sets of conditions under which (3.1.3), (3.1.5) and (3.1.7) hold shall be given. In our proofs, in addition to Lemma 3.1.7, the following result from the theory of majorization is needed. It is given in [41].

Lemma 3.1.8. *Let f be a convex function defined on I, $\mathbf{p}$ be a positive n-tuple and $\mathbf{a}, \mathbf{b} \in I^n$ such that*

$$\sum_{i=1}^{k} p_i a_i \leq \sum_{i=1}^{k} p_i b_i \ \ \text{for} \ \ k = 1, 2, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} p_i a_i = \sum_{i=1}^{n} p_i b_i.$$

If $\mathbf{a}$ is a decreasing n-tuple, then we have

$$\sum_{i=1}^{n} p_i f(a_i) \leq \sum_{i=1}^{n} p_i f(b_i), \tag{3.1.9}$$

while if $\mathbf{b}$ is an increasing n-tuple, then we have

$$\sum_{i=1}^{n} p_i f(b_i) \leq \sum_{i=1}^{n} p_i f(a_i). \tag{3.1.10}$$

If f is strictly convex and $\mathbf{a} \neq \mathbf{b}$, then (3.1.9) *and* (3.1.10) *are strict.*

Note that for $n = 2$, inequality (3.1.9) holds if $a_2 \leq a_1 \leq b_1$ and if (3.1.8) is valid, while inequality (3.1.10) holds if $a_1 \leq b_1 \leq b_2$ and if (3.1.8) is valid.

3.2 Main Results

In this section, $J(\mathbf{x}, \mathbf{p}, f)$ is as in (3.1.1), P_j are as in (3.1.4) and Q_k, as in (3.1.6). Without any loss of generality, we assume that $P_n = 1$, since for positive n-tuples such that $P_n \neq 1$ results follow easily by substituting p_i with p_i/P_n. Furthermore, for

$1 \le j < k \le n$, we introduce the following notation:

$$J_{min}(\mathbf{x},\mathbf{p},f) = \min\{P_j,\, Q_k\}\left(f(x_j)+f(x_k)-2f\left(\frac{x_j+x_k}{2}\right)\right),$$

$$J_{jk}(\mathbf{x},\mathbf{p},f) = P_j f(x_j) + \sum_{i=j+1}^{k-1} p_i f(x_i) + Q_k f(x_k) - f\left(P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k\right).$$

Note that $J_{1n}(\mathbf{x},\mathbf{p},f) = J(\mathbf{x},\mathbf{p},f)$.

Theorem 3.2.1. *Let f be a convex function defined on I and $\mathbf{p}$ be a positive n-tuple such that $P_n = 1$, $n \ge 2$. Let $1 \le j < k \le n$ and $x_i \in I$, $i = 1,\dots,k$. If x_j is such that*

$$\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i \le x_j \le \frac{1}{Q_{j+1}}\left(\sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k\right), \tag{3.2.1}$$

$$or \quad \frac{1}{Q_{j+1}}\left(\sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k\right) \le x_j \le \frac{1}{P_j}\sum_{i=1}^{j} p_i x_i, \tag{3.2.2}$$

then we have

$$J_{1k}(\mathbf{x},\mathbf{p},f) \ge J_{jk}(\mathbf{x},\mathbf{p},f). \tag{3.2.3}$$

Proof. We claim that

$$\sum_{i=1}^{j} p_i f(x_i) - f\left(\sum_{i=1}^{k-1} p_i x_i + Q_k x_k\right) \ge P_j f(x_j) - f\left(P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k\right).$$

As a simple consequence of the Jensen's inequality, we have

$$\sum_{i=1}^{j} p_i f(x_i) \ge P_j f\left(\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i\right).$$

Therefore, if we prove

$$P_j f\left(\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i\right) + f\left(P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k\right) \geq P_j f(x_j) + f\left(\sum_{i=1}^{k-1} p_i x_i + Q_k x_k\right),$$

the claim will follow. The idea is to apply Lemma 3.1.7 for $p_1 = P_j,\ p_2 = 1,\ a_1 = x_j,\ a_2 = \sum_{i=1}^{k-1} p_i x_i + Q_k x_k,\ b_1 = \frac{1}{P_j}\sum_{i=1}^{j} p_i x_i,\ b_2 = P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k$. Condition (3.1.8) is obviously satisfied. In addition, we need to check that

$$\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i \leq x_j \leq P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k$$

and

$$\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i \leq \sum_{i=1}^{k-1} p_i x_i + Q_k x_k \leq P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k.$$

Easy calculation shows that both of these conditions are valid if (3.2.1) holds. Thus, the claim follows from Lemma 3.1.7. Note that if we take $p_1 = 1,\ p_2 = P_j,\ a_1 = \sum_{i=1}^{k-1} p_i x_i + Q_k x_k,\ a_2 = x_j,\ b_1 = P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k$ *and* $b_2 = \frac{1}{P_j}\sum_{i=1}^{j} p_i x_i$, as an another choice, then the necessary conditions follow from (3.2.2). □

Theorem 3.2.2. *Let the conditions of Theorem* 3.2.1 *hold. If* x_j *is such that*

$$\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i \leq \quad x_j \quad \leq \sum_{i=1}^{k-1} p_i x_i + Q_k x_k, \tag{3.2.4}$$

or

$$\sum_{i=1}^{k-1} p_i x_i + Q_k x_k \leq \quad x_j \quad \leq \frac{1}{P_j}\sum_{i=1}^{j} p_i x_i, \tag{3.2.5}$$

then inequality (3.2.3) *holds.*

Proof. Proof is analogous to the proof of Theorem 3.2.1. Instead of Lemma 3.1.7, we apply Lemma 3.1.8 for $n = 2$ with the same choice of weights and points, or their obvious rearrangement. □

Theorem 3.2.3. *Let* f *be a convex function defined on* I *and* $\mathbf{p}$ *be a positive* n*-tuple such that* $P_n = 1$, $n \geq 2$*. Let* $1 \leq j < k \leq n$ *and* $x_i \in I$, $i = j, \ldots, n$*. If* x_k *is such* *that*

$$\frac{1}{P_{k-1}}\left(P_j x_j+\sum_{i=j+1}^{k-1} p_i x_i\right)\leq \quad x_k \quad \leq \frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i, \tag{3.2.6}$$

$$\text{or} \quad \frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i \leq \quad x_k \quad \leq \frac{1}{P_{k-1}}\left(P_j x_j+\sum_{i=j+1}^{k-1} p_i x_i\right), \tag{3.2.7}$$

then we have

$$J_{jn}\left(\mathbf{x},\mathbf{p},f\right)\geq J_{jk}\left(\mathbf{x},\mathbf{p},f\right). \tag{3.2.8}$$

Proof. We claim that

$$\sum_{i=k}^{n} p_i f\left(x_i\right)-f\left(P_j x_j+\sum_{i=j+1}^{n} p_i x_i\right)$$
$$\geq Q_k f\left(x_k\right)-f\left(P_j x_j+\sum_{i=j+1}^{k-1} p_i x_i+Q_k x_k\right).$$

As a simple consequence of the Jensen's inequality, we have

$$\sum_{i=k}^{n} p_i f\left(x_i\right)\geq Q_k f\left(\frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i\right).$$

If we prove that

$$Q_k f\left(\frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i\right)+f\left(P_j x_j+\sum_{i=j+1}^{k-1} p_i x_i+Q_k x_k\right)$$
$$\geq Q_k f\left(x_k\right)+f\left(P_j x_j+\sum_{i=j+1}^{n} p_i x_i\right),$$

then the claim will follow. Now, apply Lemma 3.1.7 for $p_1=1,\ p_2=Q_k,\ a_1=P_j x_j+\sum_{i=j+1}^{n} p_i x_i,\ a_2=x_k,\ b_1=P_j x_j+\sum_{i=j+1}^{k-1} p_i x_i+Q_k x_k$ *and* $b_2=\frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i$. It is easy to see that the condition (3.1.8) is obviously satisfied and (3.2.6) ensures that the rest of the necessary conditions are fulfilled, and thus the claim is proved. After the obvious rearrangement, applying Lemma 3.1.7 with (3.2.7), the claim is recaptured again. □

Theorem 3.2.4. *Let the conditions of Theorem* 3.2.3 *hold. If* x_k *is such that*

$$P_j x_j + \sum_{i=j+1}^{n} p_i x_i \leq \quad x_k \quad \leq \frac{1}{Q_k} \sum_{i=k}^{n} p_i x_i, \tag{3.2.9}$$

$$\text{or} \quad \frac{1}{Q_k} \sum_{i=k}^{n} p_i x_i \leq \quad x_k \quad \leq P_j x_j + \sum_{i=j+1}^{n} p_i x_i, \tag{3.2.10}$$

then inequality (3.2.8) *holds.*

Proof. It is analogous to the proof of Theorem 3.2.3. Instead of Lemma 3.1.7, we apply Lemma 3.1.8 for $n = 2$ with the same choice of weights and points, or their obvious rearrangement. □

Corollary 3.2.5. *Let* f *be a convex function defined on* I *and* $\mathbf{p}$ *be a positive* n*-tuple such that* $P_n = 1$, $n \geq 2$. *Let* $\mathbf{x} \in I^n$ *be a real* n*-tuple and* $1 \leq j < k \leq n$.

If x_k *is such that*

$$\frac{1}{P_{k-1}} \sum_{i=1}^{k-1} p_i x_i \leq \quad x_k \quad \leq \frac{1}{Q_k} \sum_{i=k}^{n} p_i x_i, \tag{3.2.11}$$

$$\text{or} \quad \frac{1}{Q_k} \sum_{i=k}^{n} p_i x_i \leq \quad x_k \quad \leq \frac{1}{P_{k-1}} \sum_{i=1}^{k-1} p_i x_i, \tag{3.2.12}$$

and x_j *is such that either* (3.2.1) *or* (3.2.2) *holds, then we have*

$$J(\mathbf{x}, \mathbf{p}, f) \geq J_{1k}(\mathbf{x}, \mathbf{p}, f) \geq J_{jk}(\mathbf{x}, \mathbf{p}, f). \tag{3.2.13}$$

If x_j *is such that*

$$\frac{1}{P_j} \sum_{i=1}^{j} p_i x_i \leq \quad x_j \quad \leq \frac{1}{Q_{j+1}} \sum_{i=j+1}^{n} p_i x_i, \tag{3.2.14}$$

$$\text{or} \quad \frac{1}{Q_{j+1}} \sum_{i=j+1}^{n} p_i x_i \leq \quad x_j \quad \leq \frac{1}{P_j} \sum_{i=1}^{j} p_i x_i, \tag{3.2.15}$$

and x_k *is such that either* (3.2.6) *or* (3.2.7) *holds, then we have*

$$J(\mathbf{x}, \mathbf{p}, f) \geq J_{jn}(\mathbf{x}, \mathbf{p}, f) \geq J_{jk}(\mathbf{x}, \mathbf{p}, f). \tag{3.2.16}$$

Proof. The first inequality in (3.2.13) follows from Theorem 3.2.3 for $j = 1$ and the second is a direct consequence of Theorem 3.2.1, while the first inequality in (3.2.16) follows from Theorem 3.2.1 for $k = n$ and the second is a consequence of Theorem 3.2.3. □

Corollary 3.2.6. *Let the conditions of Corollary* 3.2.5 *hold.*

If x_k is such that

$$\sum_{i=1}^{n} p_i x_i \leq \quad x_k \quad \leq \frac{1}{Q_k} \sum_{i=k}^{n} p_i x_i, \tag{3.2.17}$$

$$\text{or} \quad \frac{1}{Q_k} \sum_{i=k}^{n} p_i x_i \leq \quad x_k \quad \leq \sum_{i=1}^{n} p_i x_i, \tag{3.2.18}$$

and x_j is such that either (3.2.4) *or* (3.2.5) *holds, then inequality* (3.2.13) *holds.*

If x_j is such that

$$\frac{1}{P_j} \sum_{i=1}^{j} p_i x_i \leq \quad x_j \quad \leq \sum_{i=1}^{n} p_i x_i, \tag{3.2.19}$$

$$\text{or} \quad \sum_{i=1}^{n} p_i x_i \leq \quad x_j \quad \leq \frac{1}{P_j} \sum_{i=1}^{j} p_i x_i, \tag{3.2.20}$$

and x_k is such that either (3.2.9) *or* (3.2.10) *holds, then inequality* (3.2.16) *holds.*

Proof. The first inequality in (3.2.13) follows from Theorem 3.2.4 for $j = 1$ and the second is a direct consequence of Theorem 3.2.2, while the first inequality in (3.2.16) follows from Theorem 3.2.2 for $k = n$ and the second is a consequence of Theorem 3.2.4. □

Theorem 3.2.7. *Let f be a convex function defined on I and $\mathbf{p}$ be a positive n-tuple such that $P_n = 1$, $n \geq 2$. Let $\mathbf{x} \in I^n$ be a real n-tuple and let $1 \leq j < k \leq n$. If x_j and x_k are such that*

$$\frac{1}{P_j} \sum_{i=1}^{j} p_i x_i \leq x_j \leq \sum_{i=1}^{n} p_i x_i \leq x_k \leq \frac{1}{Q_k} \sum_{i=k}^{n} p_i x_i, \tag{3.2.21}$$

or

$$\frac{1}{Q_k} \sum_{i=k}^{n} p_i x_i \leq x_k \leq \sum_{i=1}^{n} p_i x_i \leq x_j \leq \frac{1}{P_j} \sum_{i=1}^{j} p_i x_i, \tag{3.2.22}$$

then we have

$$J(\mathbf{x}, \mathbf{p}, f) \geq J_{jk}(\mathbf{x}, \mathbf{p}, f).$$

Proof. We claim that

$$\sum_{i=1}^{j} p_i f(x_i) + \sum_{i=k}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right)$$
$$\geq P_j f(x_j) + Q_k f(x_k) - f\left(P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k\right).$$

Applying the Jensen's inequality to the two sums on the left-hand side and after adding the resulting inequalities, we have

$$\sum_{i=1}^{j} p_i f(x_i) + \sum_{i=k}^{n} p_i f(x_i) \geq P_j f\left(\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i\right) + Q_k f\left(\frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i\right).$$

If we prove that

$$Q_k f\left(\frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i\right) + f\left(P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k\right) + P_j f\left(\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i\right)$$
$$\geq Q_k f(x_k) + f\left(\sum_{i=1}^{n} p_i x_i\right) + P_j f(x_j), \tag{3.2.23}$$

then the claim will follow. Set $p_1 = Q_k$, $p_2 = 1$, $p_3 = P_j$, $a_1 = x_k$, $a_2 = \sum_{i=1}^{n} p_i x_i$, $a_3 = x_j$, $b_1 = \frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i$, $b_2 = P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k$, $b_3 = \frac{1}{P_j}\sum_{i=1}^{j} p_i x_i$ and apply Lemma 3.1.8 for $n = 3$ Assumption (3.2.21) ensures that the necessary conditions of Lemma 3.1.8 for $n = 3$ are fulfilled, and so (3.2.23) follows from (3.1.9). By obvious rearrangement, utilizing (3.2.22), the inequality is recaptured. □

Remark 3.2.1. Note that conditions (3.2.4) and (3.2.17) combined together give a condition

$$\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i \leq x_j \leq \sum_{i=1}^{k-1} p_i x_i + Q_k x_k \leq \sum_{i=1}^{n} p_i x_i \leq x_k \leq \frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i,$$

while (3.2.9) and (3.2.19) combined together give

$$\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i \leq x_j \leq \sum_{i=1}^{n} p_i x_i \leq P_j x_j + \sum_{i=j+1}^{n} p_i x_i \leq x_k \leq \frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i,$$

both of which are more restricting than (3.2.21). The same is true for combining conditions (3.2.5) and (3.2.18), or (3.2.10) and (3.2.20), and comparing the result with (3.2.22).

Theorem 3.2.8. *Let f be a convex function defined on I and $\mathbf{p}$ be a positive n-tuple such that $P_n = 1$, $n \geq 2$. Let $1 \leq j < k \leq n$ and $x_i \in I$, $i = j, \ldots, k$. Then we have*

$$\begin{aligned} J_{jk}(\mathbf{x}, \mathbf{p}, f) &\geq P_j f(x_j) + Q_k f(x_k) - (P_j + Q_k) f\left(\frac{P_j x_j + Q_k x_k}{P_j + Q_k}\right) \\ &\geq J_{min}(\mathbf{x}, \mathbf{p}, f) \geq 0. \end{aligned} \tag{3.2.24}$$

Proof. The first inequality is an immediate consequence of the Jensen's inequality. The other two follow immediately from (3.1.2). □

Remark 3.2.2. Inequalities (3.2.13), (3.2.16) and (3.2.24) recapture results from Lemmas 3.1.4 and 3.1.5, and Theorem 3.1.6 as special cases, since an increasing n-tuple $\mathbf{x}$ fulfils conditions (3.2.1) and (3.2.11), that is, (3.2.6) and (3.2.14). A decreasing n-tuple $\mathbf{x}$, on the other hand, fulfils conditions (3.2.2) and (3.2.12), that is, (3.2.7) and (3.2.15). The proofs of Theorem 3.2.8 and Corollary 3.2.5, that is, Theorems 3.2.1 and 3.2.3, are in fact analogous to the proofs of Theorem 3.1.6, Lemmas 3.1.4 and 3.1.5 from [13].

3.3 Some Special Cases

In this section, we consider some special cases of the presented results. The same special cases are considered in [13], but here we obtain them under more relaxed conditions on the n-tuple $\mathbf{x}$. More precisely, Corollaries 3.2.5 and 3.2.6 or Theorem 3.2.7, after applying Theorem 3.2.8, yield

$$J(\mathbf{x}, \mathbf{p}, f) \geq P_j f(x_j) + Q_k f(x_k) - (P_j + Q_k) f\left(\frac{P_j x_j + Q_k x_k}{P_j + Q_k}\right). \tag{3.3.1}$$

Corollary 3.3.1. *Let the conditions of Corollaries* 3.2.5 *and* 3.2.6 *or Theorem* 3.2.7 *hold. Then*

$$\sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} a_i^{p_i} \geq P_j a_j + Q_k a_k - (P_j + Q_k) a_j^{P_j/(P_j+Q_k)} a_k^{Q_k/(P_j+Q_k)}.$$

Proof. This follows from (3.3.1) for $f(x) = e^x$, using notation $a_i = e^{x_i}$. □

Corollary 3.3.2. *Let the conditions of Corollaries* 3.2.5 *and* 3.2.6 *or Theorem* 3.2.7 *hold, and let in addition $x_i > 0$, $i = 1, \ldots, n$. Then*

$$\frac{\sum_{i=1}^{n} p_i x_i}{\prod_{i=1}^{n} x_i^{p_i}} \geq \frac{1}{x_j^{P_j} x_k^{Q_k}} \left(\frac{P_j x_j + Q_k x_k}{P_j + Q_k}\right)^{P_j + Q_k}.$$

Proof. Follows from (3.3.1) for $f(x) = -\ln x$. □

Corollary 3.3.3. *Let the conditions of Corollaries* 3.2.5 *and* 3.2.6 *or Theorem* 3.2.7 *hold, and let in addition* $x_i > 0$, $i = 1, \ldots, n$. *Then*

$$\sum_{i=1}^{n} \frac{p_i}{x_i} - \frac{1}{\sum_{i=1}^{n} p_i x_i} \geq \frac{P_j Q_k (x_k - x_j)^2}{x_j x_k (P_j x_j + Q_k x_k)} .$$

Proof. This follows from (3.3.1) for $f(x) = 1/x$. □

In [13], additional bounds of $J(\mathbf{x}, \mathbf{p}, f)$, lower than those obtained in the previous corollaries, are derived for the case $f(x) = e^x$ and $f(x) = 1/x$. Now, note that from Theorem 3.2.8, under conditions of Corollaries 3.2.5 and 3.2.6 or Theorem 3.2.7, we have

$$J(\mathbf{x}, \mathbf{p}, f) \geq J_{min}(\mathbf{x}, \mathbf{p}, f) \geq 0. \tag{3.3.2}$$

Next, we compare estimates obtained from (3.3.2) with those obtained in [13].

Case 1. *For* $f(x) = e^x$, *using notation* $a_i = e^{x_i}$, *inequality* (3.3.2) *takes the form*

$$\sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} a_i^{p_i} \geq \min\{P_j, Q_k\} \left(\sqrt{a_k} - \sqrt{a_j}\right)^2 . \tag{3.3.3}$$

In [13], under the assumption that $\mathbf{a}$ *is an increasing n-tuple, the following inequality is obtained*

$$\sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} a_i^{p_i} \geq C \left(\sqrt{a_k} - \sqrt{a_j}\right)^2 ,$$

where

$$C = \begin{cases} \frac{2P_j Q_k}{P_j + Q_k}, & P_j \leq Q_k, \\ Q_k, & P_j \geq Q_k. \end{cases}$$

Note that when $P_j \geq Q_k$, (3.3.3) *recaptures this result. However, when* $P_j \leq Q_k$, *the constant* C *is better, since* $\frac{2P_j Q_k}{P_j + Q_k} \geq P_j$.

Case 2. *For* $f(x) = 1/x$ *and* $x_i > 0$, $i = 1, \ldots, n$, *inequality* (3.3.2) *takes the form*

$$\sum_{i=1}^{n} \frac{p_i}{x_i} - \frac{1}{\sum_{i=1}^{n} p_i x_i} \geq \min\{P_j, Q_k\} \frac{(x_k - x_j)^2}{x_j x_k (x_j + x_k)}. \tag{3.3.4}$$

In [13], under the assumption that $\mathbf{x}$ *is an increasing* n*-tuple such that* $x_1 > 0$*, the following inequality is obtained:*

$$\sum_{i=1}^{n} \frac{p_i}{x_i} - \frac{1}{\sum_{i=1}^{n} p_i x_i} \geq C \, \frac{\left(\sqrt{x_k} - \sqrt{x_j}\right)^2}{x_j x_k}, \tag{3.3.5}$$

where

$$C = \begin{cases} P_j, & P_j \leq 3Q_k, \\ \frac{4P_jQ_k}{P_j+Q_k}, & P_j \geq 3Q_k. \end{cases}$$

In order to compare these two estimates, first assume $P_j \leq Q_k$*. Since*

$$P_j \, \frac{(x_k - x_j)^2}{x_j x_k (x_j + x_k)} \geq P_j \, \frac{\left(\sqrt{x_k} - \sqrt{x_j}\right)^2}{x_j x_k} \iff \left(\sqrt{x_k} + \sqrt{x_j}\right)^2 \geq x_j + x_k,$$

it follows that the estimate in (3.3.4) *is better than the one in* (3.3.5)*.*

Next, assume $Q_k \leq P_j \leq 2Q_k$*. First, observe that*

$$Q_k \left(\sqrt{x_k} + \sqrt{x_j}\right)^2 \geq P_j (x_k + x_j) \iff P_j \leq Q_k \frac{\left(\sqrt{x_k} + \sqrt{x_j}\right)^2}{x_k + x_j}.$$

Simple calculation reveals that

$$1 \leq \frac{\left(\sqrt{x_k} + \sqrt{x_j}\right)^2}{x_k + x_j} \leq 2,$$

and so we conclude that the estimate in (3.3.4) *is better than the one in* (3.3.5) *when* $Q_k \leq P_j \leq Q_k \frac{\left(\sqrt{x_k}+\sqrt{x_j}\right)^2}{x_k+x_j}$*, while when* $Q_k \frac{\left(\sqrt{x_k}+\sqrt{x_j}\right)^2}{x_k+x_j} \leq P_j \leq 2Q_k$ *the estimate in* (3.3.5) *is better than the one in* (3.3.4)*.*

Further, assume $2Q_k \leq P_j \leq 3Q_k$*. In this case, the estimate in* (3.3.5) *is better than the one in* (3.3.4)*, that is,*

$$P_j (x_k + x_j) \geq Q_k \left(\sqrt{x_k} + \sqrt{x_j}\right)^2.$$

Namely,

$$P_j (x_k + x_j) \geq 2Q_k (x_k + x_j)$$

and

$$2Q_k (x_k + x_j) \geq Q_k \left(\sqrt{x_k} + \sqrt{x_j}\right)^2 \iff \left(\sqrt{x_k} - \sqrt{x_j}\right)^2 \geq 0.$$

Finally, if $3Q_k \leq P_j$*, the estimate in* (3.3.5) *is again better than the one in* (3.3.4)*, that is,*

$$\frac{4P_jQ_k}{P_j + Q_k} (x_j + x_k) \geq Q_k \left(\sqrt{x_k} + \sqrt{x_j}\right)^2.$$

This is equivalent to

$$P_j\left(3x_j + 3x_k - 2\sqrt{x_j x_k}\right) \geq Q_k\left(\sqrt{x_k} + \sqrt{x_j}\right)^2.$$

In this case, we have

$$P_j\left(3x_j + 3x_k - 2\sqrt{x_j x_k}\right) \geq Q_k\left(3x_j + 3x_k - 2\sqrt{x_j x_k}\right),$$

and since

$$Q_k\left(3x_j + 3x_k - 2\sqrt{x_j x_k}\right) \geq Q_k\left(\sqrt{x_k} + \sqrt{x_j}\right)^2 \Leftrightarrow \left(\sqrt{x_k} - \sqrt{x_j}\right)^2 \geq 0,$$

the claim follows.

The results presented in this chapter are published in [24].

Chapter 4

On the Refinements of Jensen, Slater, Majorization, Favard and Berwald Inequalities

This chapter deals with the refinements of some well known inequalities, namely, Jensen and Slater inequalities. Further, it provides some refinements of the Majorization-type inequalities given in [17, 42] and some refinements of the generalized Favard and Berwald inequalities given in [41, 42]. Finally, some inequalities in terms of Gâteaux derivatives for convex functions defined on linear spaces are also presented.

4.1 Refinements of Some Companion Inequalities to the Jensen's Inequality

Let X be a real normed linear space and let X^* be the dual space of X, that is, the real space of all linear functionals $x^* : X \to \mathbb{R}$. Let C be an open convex subset of the real normed linear space X. If $\phi : C \to \mathbb{R}$ is a convex function, then for any fixed point $y \in C$, we can define the abstract subdifferential $\partial\phi(y)$ of ϕ at y as follows:

$$\partial\phi(y) := \{a^*(y; .) \in X^* : \phi(x) \geq \phi(y) + a^*(y; x - y),\ \forall x \in C\}.$$

The set $\partial\phi(y)$ is non-empty for all $y \in C$ (see [61, p. 108 Theorem B]). Also, when ϕ is strictly convex, the inequality

$$\phi(x) \geq \phi(y) + a^*(y; x - y), \quad \forall x, y \in C \tag{4.1.1}$$

is strict unless $x = y$.

In the simplest case if $\phi : (a,b) \to \mathbb{R}$ is a convex function, then for any $y \in (a,b)$, $a^*(y;.) \in \partial\phi(y)$ is given by $a^*(y;x) = mx$, where $m \in \left[\phi'_-(y), \phi'_+(y)\right]$ and $x \in \mathbb{R}$. For convenience we take $m = \phi'_+(y)$ and so (4.1.1) becomes

$$\phi(x) \geq \phi(y) + \phi'_+(y)(x-y), \quad \forall\, x, y \in (a,b). \tag{4.1.2}$$

Remark 4.1.1. For $p_i \geq 0$ $(i = 1, \ldots, n)$, let us denote $P_n = \sum_{i=1}^n p_i$.

In 1981 Slater proved the following interesting companion inequality to the Jensen's inequality (see [62]).

Theorem 4.1.1. *Let the function $\phi : (a,b) \to \mathbb{R}$ be monotonic and convex, $x_i \in (a,b)$ and $p_i \geq 0$ $(i = 1, \ldots, n)$ with $P_n > 0$. If $\sum_{i=1}^n p_i \phi'_+(x_i) \neq 0$, then*

$$\frac{1}{P_n}\sum_{i=1}^n p_i \phi(x_i) \leq \phi\left(\frac{\sum_{i=1}^n p_i x_i \phi'_+(x_i)}{\sum_{i=1}^n p_i \phi'_+(x_i)}\right), \tag{4.1.3}$$

holds. When ϕ is strictly convex on (a,b), inequality (4.1.3) becomes equality if and only if $x_i = c$ for some $c \in (a,b)$ and for all i with $p_i > 0$.

In [51] J. Pečarić noted that (4.1.3) remains true if we replace the condition of monotonicity of ϕ by $\frac{\sum_{i=1}^n p_i x_i \phi'_+(x_i)}{\sum_{i=1}^n p_i \phi'_+(x_i)} \in (a,b)$, which is in fact more general and can hold for suitable points in (a,b) and not necessarily for monotone functions. In the same paper, one can find the multidimensional case of Slater's inequality.

The following theorem is proved by M. Matić and J. Pećarić in [44].

Theorem 4.1.2. *Let $\phi : C \to \mathbb{R}$ be a convex function defined on an open convex subset C in the real normed linear space X. For the given vectors $x_i \in C$, $p_i \geq 0$ $(i = 1, \ldots, n)$ such that $P_n > 0$, let us denote*

$$\bar{x} = \frac{1}{P_n}\sum_{i=1}^n p_i x_i \ \text{ and } \ \bar{y} = \frac{1}{P_n}\sum_{i=1}^n p_i \phi(x_i). \tag{4.1.4}$$

If c, d are arbitrary vectors in C, then

$$\phi(c) + a^*(c; \bar{x} - c) \leq \bar{y} \leq \phi(d) + \frac{1}{P_n}\sum_{i=1}^n p_i a^*(x_i; x_i - d), \tag{4.1.5}$$

holds. Also, when ϕ is strictly convex, we have equality in the left inequality in (4.1.5) *if and only if $x_i = c$ holds for all indices i with $p_i > 0$, while equality holds in the right inequality in* (4.1.5) *if and only if $x_i = d$ holds for all indices i with $p_i > 0$.*

In this section, we present some refinements of the left and the right inequalities of (4.1.5) and from the refinement of the left inequality of (4.1.5), we obtain Jensen's difference. We also give some refinements of the Slater's inequality for the case when ϕ is a convex function and moreover, when ϕ is a convex function as well as a monotone function.
The following theorem is the first main result in this section.

Theorem 4.1.3. *Let $\phi : C \to \mathbb{R}$ be a convex function defined on an open convex subset C in the real normed linear space X. For $x_i, y_i \in C$ and $p_i \geq 0$, $i = 1, \ldots, n$,*

$$\begin{aligned} &\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(y_i) - \sum_{i=1}^{n} p_i a^* (y_i; x_i - y_i) \\ &\geq \left| \sum_{i=1}^{n} p_i \Big| \phi(x_i) - \phi(y_i) \Big| - \sum_{i=1}^{n} p_i \Big| a^* (y_i; x_i - y_i) \Big| \right|, \end{aligned} \tag{4.1.6}$$

holds.

Proof. Take $x = x_i$ and $y = y_i$ $(i = 1, \ldots, n)$ in (4.1.1), we have

$$\phi(x_i) \geq \phi(y_i) + a^* (y_i; x_i - y_i),$$

equivalent to

$$\phi(x_i) - \phi(y_i) - a^* (y_i; x_i - y_i) \geq 0.$$

Therefore

$$\begin{aligned} \phi(x_i) - \phi(y_i) - a^* (y_i; x_i - y_i) &= |\phi(x_i) - \phi(y_i) - a^* (y_i; x_i - y_i)| \\ &\geq \big| |\phi(x_i) - \phi(y_i)| - |a^* (y_i; x_i - y_i)| \big|. \end{aligned} \tag{4.1.7}$$

Now, multiplying (4.1.7) by $p_i \geq 0$ $(i = 1, \ldots, n)$ and summing over i from 1 to n, we have

$$\begin{aligned} &\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(y_i) - \sum_{i=1}^{n} p_i a^* (y_i; x_i - y_i) \\ &\geq \sum_{i=1}^{n} p_i \Big| |\phi(x_i) - \phi(y_i)| - |a^* (y_i; x_i - y_i)| \Big| \\ &\geq \left| \sum_{i=1}^{n} p_i \Big| \phi(x_i) - \phi(y_i) \Big| - \sum_{i=1}^{n} p_i \Big| a^* (y_i; x_i - y_i) \Big| \right|, \end{aligned}$$

which is equivalent to (4.1.6). ∎

The following theorem is the refinement of the left inequality in (4.1.5).

Theorem 4.1.4. *Let $\phi : C \to \mathbb{R}$ be a convex function, $x_i \in C$, $p_i \geq 0$ $(i = 1, \ldots, n)$ such that $P_n > 0$ and $\bar{x}$,$\bar{y}$ be as in* (4.1.4)*. If c is arbitrary vector in C, then we have*

$$\bar{y} - \phi(c) - a^*(c; \bar{x} - c) \geq \left| \frac{1}{P_n} \sum_{i=1}^{n} p_i \Big| \phi(x_i) - \phi(c) \Big| - \frac{1}{P_n} \sum_{i=1}^{n} p_i \Big| a^*(c; x_i - c) \Big| \right|. \quad (4.1.8)$$

Proof. Substitute $y_i = c$ in (4.1.6) and using the fact that $a^*(c; .)$ is a linear functional, we have (4.1.8). □

The following refinement of Jensen's inequality holds:

Corollary 4.1.5. *Under the assumptions of Theorem* 4.1.4*, we have*

$$\bar{y} - \phi(\bar{x}) \geq \left| \frac{1}{P_n} \sum_{i=1}^{n} p_i \Big| \phi(x_i) - \phi(\bar{x}) \Big| - \frac{1}{P_n} \sum_{i=1}^{n} p_i \Big| a^*(\bar{x}; x_i - \bar{x}) \Big| \right|.$$

Proof. Take $c = \bar{x}$ in (4.1.8), we obtain the required result. □

The following theorem is the refinement of the right inequality in (4.1.5).

Theorem 4.1.6. *Let $\phi : C \to \mathbb{R}$ be a convex function, $x_i \in C$, $p_i \geq 0$ $(i = 1, \ldots, n)$ such that $P_n > 0$ and $\bar{y}$ be as in* (4.1.4)*. If d is arbitrary vector in C, then we have*

$$\begin{aligned} &\phi(d) - \bar{y} - \frac{1}{P_n} \sum_{i=1}^{n} p_i a^*(x_i; d - x_i) \\ &\geq \left| \frac{1}{P_n} \sum_{i=1}^{n} p_i \Big| \phi(d) - \phi(x_i) \Big| - \frac{1}{P_n} \sum_{i=1}^{n} p_i \Big| a^*(x_i; x_i - d) \Big| \right|. \end{aligned} \quad (4.1.9)$$

Proof. Substitute $x_i = d$ and $y_i = x_i$ in (4.1.6), we have (4.1.9). □

The following theorem is the refinement of the left inequality of (3.3) which appears in [44, Theorem 3.1].

Corollary 4.1.7. *Let the assumptions of Theorem 4.1.6 hold. If there exists a vector $\bar{d} \in C$ such that the corresponding functional $a^*(\bar{d}; .) \in \partial\phi(\bar{d})$ satisfies $a^*(\bar{d}; .) = \frac{1}{P_n} \sum_{i=1}^{n} p_i a^*(x_i; .)$, then we have*

$$\begin{aligned} &\phi(\bar{d}) - \bar{y} - \frac{1}{P_n} \sum_{i=1}^{n} p_i a^*(x_i; \bar{d} - x_i) \\ &\geq \left| \frac{1}{P_n} \sum_{i=1}^{n} p_i \Big| \phi(\bar{d}) - \phi(x_i) \Big| - \frac{1}{P_n} \sum_{i=1}^{n} p_i \Big| a^*(x_i; x_i - \bar{d}) \Big| \right|. \end{aligned}$$

Corollary 4.1.8. *Under the assumptions of Theorem* 4.1.6*, we have*

$$\phi(\bar{x}) - \bar{y} - \frac{1}{P_n}\sum_{i=1}^{n} p_i a^*(x_i;\bar{x}-x_i) \geq \left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Big|\phi(\bar{x})-\phi(x_i)\Big| - \frac{1}{P_n}\sum_{i=1}^{n} p_i\Big|a^*(x_i;x_i-\bar{x})\Big|\right|. \tag{4.1.10}$$

Proof. Take $d=\bar{x}$ in (4.1.9), we have (4.1.10). □

Remark 4.1.2. In fact (4.1.10) is further refinement of the counter part of the Jensen's inequality given in [44] and in particular for the counter part of the Jensen's inequality given in [20].

For $X=\mathbb{R}$, Theorem 4.1.6 becomes:

Theorem 4.1.9. *Let $\phi:(a,b)\to\mathbb{R}$ be a convex function, $x_i\in(a,b)$, $p_i\geq 0$ $(i=1,\ldots,n)$ such that $P_n>0$ and $\bar{x},\bar{y}$ be as in* (4.1.4)*. Then we have*

$$\phi(d) - \bar{y} - \frac{1}{P_n}\sum_{i=1}^{n} p_i\phi'_+(x_i)(d-x_i) \geq \left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Big|\phi(d)-\phi(x_i)\Big| - \frac{1}{P_n}\sum_{i=1}^{n} p_i\Big|\phi'_+(x_i)(d-x_i)\Big|\right|.$$

The following result is the refinement of the Slater's inequality:

Corollary 4.1.10. *Let $\phi:(a,b)\to\mathbb{R}$ be a convex function, $x_i\in(a,b)$, $p_i\geq 0$ $(i=1,\ldots,n)$ such that $P_n>0$ and $\bar{y}$ be as in* (4.1.4)*. If $\sum_{i=1}^{n} p_i\phi'_+(x_i)\neq 0$ such that $\tilde{x}=\frac{\sum_{i=1}^{n} p_i x_i\phi'_+(x_i)}{\sum_{i=1}^{n} p_i\phi'_+(x_i)}\in(a,b)$, then we have*

$$\phi(\tilde{x}) - \bar{y} \geq \left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\Big|\phi(\tilde{x})-\phi(x_i)\Big| - \frac{1}{P_n}\sum_{i=1}^{n} p_i\Big|\phi'_+(x_i)(\tilde{x}-x_i)\Big|\right|. \tag{4.1.11}$$

The following theorem is the refinement of the Slater's inequality for monotone convex function.

Theorem 4.1.11. *Let the function $\phi:(a,b)\to\mathbb{R}$ be monotonic and convex. Let $x_i\in(a,b)$, $p_i\geq 0$ $(i=1,\ldots,n)$ such that $P_n>0$ and $\bar{y}$ be as in* (4.1.4)*. If $\sum_{i=1}^{n} p_i\phi'_+(x_i)\neq 0$ and $I=\left\{i\in I_n=\{1,\ldots,n\}: x_i\geq\tilde{x}=\frac{\sum_{i=1}^{n} p_i x_i\phi'_+(x_i)}{\sum_{i=1}^{n} p_i\phi'_+(x_i)}\right\}$, then*

$$\phi(\tilde{x})-\bar{y} \geq \left|\frac{1}{P_n}\sum_{i=1}^{n} p_i\operatorname{sgn}(x_i-\tilde{x})\left[\phi(x_i)-x_i\phi'_+(x_i)+\tilde{x}\phi'_+(x_i)\right]+\phi(\tilde{x})\left(1-\frac{2P_I}{P_n}\right)\right|, \tag{4.1.12}$$

holds, where $P_I=\sum_{i\in I} p_i$.

Proof. Consider the case when ϕ is non-decreasing

$$\begin{aligned}\sum_{i=1}^{n} p_i \left|\phi(\tilde{x}) - \phi(x_i)\right| &= \sum_{i\in I} p_i\left(\phi(x_i) - \phi(\tilde{x})\right) + \sum_{i\in \bar{I}} p_i\left(\phi(\tilde{x}) - \phi(x_i)\right) \\ &= \sum_{i\in I} p_i\phi(x_i) - \sum_{i\in \bar{I}} p_i\phi(x_i) - \sum_{i\in I} p_i\phi(\tilde{x}) + \sum_{i\in \bar{I}} p_i\phi(\tilde{x}) \\ &= \sum_{i=1}^{n} p_i sgn(x_i - \tilde{x})\, p_i\phi(x_i) - \phi(\tilde{x})\left(P_I - P_{I_n\setminus I}\right). \quad (4.1.13)\end{aligned}$$

Similarly

$$\sum_{i=1}^{n} p_i \left|\phi'_+(x_i)(\tilde{x} - x_i)\right| = \sum_{i=1}^{n} p_i sgn(x_i - \tilde{x})(x_i - \tilde{x})\,\phi'_+(x_i). \quad (4.1.14)$$

Now by using (4.1.13) and (4.1.14) in (4.1.11), we have (4.1.12).
The case when ϕ is non-increasing can be treated similarly. □

4.2 Refinements of the Majorization-type Inequality, Favard and Berwald Inequalities

The aim of this section is to present some refinements of the well known results including majorization-type inequalities and the generalized Favard and Berwald inequalities.
J. Favard proved the following result in [23].

Theorem 4.2.1. *Let f be a non-negative continuous concave function defined on $[a,b]$, not identically zero and ϕ be a convex function defined on $\left[0, 2\widetilde{f}\right]$, where $\widetilde{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$, then*

$$\int_0^1 \phi\left(2\,s\,\widetilde{f}\right) ds = \frac{1}{2\widetilde{f}}\int_0^{2\widetilde{f}} \phi(y)\, dy \geq \frac{1}{b-a}\int_a^b \phi(f(x))\, dx.$$

Some generalizations of the Favard's inequality and its reverse are given in [33, pp. 412-413].
L. Berwald [10] proved the following important generalization of Favard's inequality in [33, pp. 413-414].

Theorem 4.2.2. *Let f be a non-negative, continuous concave function, not identically zero on $[a,b]$, and ψ be a continuous and strictly monotonic function defined on*

$[0, y_0]$, *where* y_0 *is sufficiently large. If* $\bar{z}$ *is the unique positive root of the equation*

$$\frac{1}{\bar{z}} \int_0^{\bar{z}} \psi(y)\, dy = \frac{1}{b-a} \int_a^b \psi(f(x))\, dx,$$

then for every function $\phi : [0, y_0] \to \mathbb{R}$ *which is convex with respect to* ψ*, we have*

$$\int_0^1 \phi(s\,\bar{z})\, ds = \frac{1}{\bar{z}} \int_0^{\bar{z}} \phi(y)\, dy \geq \frac{1}{b-a} \int_a^b \phi(f(x))\, dx.$$

The following theorem is the refinement of the majorization inequality given in [42, 45].

Theorem 4.2.3. *Let* $\phi : I \to \mathbb{R}$ *be a convex function,* $\mathbf{x} = (x_1, \ldots, x_n)$*,* $\mathbf{y} = (y_1, \ldots, y_n)$ *and* $\mathbf{p} = (p_1, \ldots, p_n)$ *be* n*-tuples such that* $x_i, y_i \in (a, b)$*,* $p_i \geq 0$ $(i = 1, \ldots, n)$ *and satisfying*

$$\sum_{i=1}^{k} p_i\, y_i \leq \sum_{i=1}^{k} p_i\, x_i \;\; \text{for } k = 1, \ldots, n-1, \tag{4.2.1}$$

and

$$\sum_{i=1}^{n} p_i\, y_i = \sum_{i=1}^{n} p_i\, x_i. \tag{4.2.2}$$

(*i*) *If* $\mathbf{y}$ *is decreasing* n*-tuple, then the inequality*

$$\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(y_i) \geq \left| \sum_{i=1}^{n} p_i \big|\phi(x_i) - \phi(y_i)\big| - \sum_{i=1}^{n} p_i \big|\phi'_+(y_i)(x_i - y_i)\big| \right| \tag{4.2.3}$$

holds.

(*ii*) *If* $\mathbf{x}$ *is increasing* n*-tuple, then the inequality*

$$\sum_{i=1}^{n} p_i \phi(y_i) - \sum_{i=1}^{n} p_i \phi(x_i) \geq \left| \sum_{i=1}^{n} p_i \big|\phi(y_i) - \phi(x_i)\big| - \sum_{i=1}^{n} p_i \big|\phi'_+(x_i)(y_i - x_i)\big| \right| \tag{4.2.4}$$

holds.

Proof. (i) Take $x = x_i$ and $y = y_i$ $(i = 1, \ldots, n)$ in (4.1.2), we have

$$\phi(x_i) - \phi(y_i) - \phi'_+(y_i)(x_i - y_i) \geq 0.$$

We can write

$$\begin{aligned} \phi(x_i) - \phi(y_i) - \phi'_+(y_i)(x_i - y_i) &= \big|\phi(x_i) - \phi(y_i) - \phi'_+(y_i)(x_i - y_i)\big| \\ &\geq \big||\phi(x_i) - \phi(y_i)| - |\phi'_+(y_i)(x_i - y_i)|\big|. \end{aligned}$$

Now, multiplying by $p_i \geq 0$ $(i = 1, \ldots, n)$ and summing over i from 1 to n, we have

$$\begin{aligned}
&\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(y_i) - \sum_{i=1}^{n} p_i \phi'_+(y_i)(x_i - y_i) \\
&\quad \geq \sum_{i=1}^{n} p_i \Big| \big|\phi(x_i) - \phi(y_i)\big| - \big|\phi'_+(y_i)(x_i - y_i)\big| \Big| \\
&\quad \geq \left| \sum_{i=1}^{n} p_i \big|\phi(x_i) - \phi(y_i)\big| - \sum_{i=1}^{n} p_i \big|\phi'_+(y_i)(x_i - y_i)\big| \right|. \qquad (4.2.5)
\end{aligned}$$

It is easy to see that

$$\sum_{i=1}^{n} p_i \phi'_+(y_i)(x_i - y_i) = \sum_{k=1}^{n-1} (A_k - B_k)\left(\phi'_+(y_k) - \phi'_+(y_{k+1})\right),$$

where $A_k = \sum_{i=1}^{k} p_i x_i$ and $B_k = \sum_{i=1}^{k} p_i y_i$ $(k = 1, \ldots, n-1)$. As $\mathbf{y}$ is decreasing n-tuple, by using (4.2.1), (4.2.2) and the convexity of ϕ, we have

$$\sum_{i=1}^{n} p_i \phi'_+(y_i)(x_i - y_i) \geq 0,$$

and so

$$\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(y_i) \geq \sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(y_i) - \sum_{i=1}^{n} p_i \phi'_+(y_i)(x_i - y_i),$$

which together with (4.2.5) implies (4.2.3).

(ii) By using the same idea as in (i), we can prove (4.2.4).

□

The following theorem is the refinement of the majorization inequality given by Dragomir in [17].

Theorem 4.2.4. *Let $\phi : I \to \mathbb{R}$ be a convex function, $x_i, y_i \in I$, $p_i \geq 0$ $(i = 1, \ldots, n)$ with $P_n > 0$. If $(x_1 - y_1, \ldots, x_n - y_n)$ is increasing (decreasing), $(y_1, \ldots, y_n)$ is increasing (decreasing) and satisfying* (4.2.2), *then the inequality*

$$\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(y_i) \geq \left| \sum_{i=1}^{n} p_i \big|\phi(x_i) - \phi(y_i)\big| - \sum_{i=1}^{n} p_i \big|\phi'_+(y_i)(x_i - y_i)\big| \right| \qquad (4.2.6)$$

holds.

Proof. The idea of the proof is the same as that of the proof of Theorem 4.2.3. □

Remark 4.2.1. If ϕ, y_i, $x_i - y_i$ and p_i $(i = 1, \ldots, n)$ are the same as in Theorem 4.2.4 and if in addition ϕ is increasing and $\sum_{i=1}^{n} p_i x_i \geq \sum_{i=1}^{n} p_i y_i$, then (4.2.6) holds.

The following theorem is an integral analogue of Theorem 4.2.3.

Theorem 4.2.5. *Let w be a positive weight function and $f, g : [a, b] \to [c, d]$ be integrable functions. Suppose that $\phi : [c, d] \to \mathbb{R}$ is a convex function and*

$$\int_a^x g(t)\, w(t)\, dt \leq \int_a^x f(t)\, w(t)\, dt, \quad \text{for all } x \in [a, b]$$

and

$$\int_a^b g(t)\, w(t)\, dt = \int_a^b f(t)\, w(t)\, dt$$

hold.

(*i*) *If g is a decreasing function defined on $[a, b]$, then we have*

$$\begin{aligned}&\int_a^b \phi(f)\, w(t)\, dt - \int_a^b \phi(g)\, w(t) dt \\ &\quad \geq \left| \int_a^b w(t) \left|\phi(f) - \phi(g)\right| dt - \int_a^b w(t) \left|\phi'_+(g)(f - g)\right| dt \right|.\end{aligned}$$

(*ii*) *If f is an increasing function defined on $[a, b]$, then we have*

$$\begin{aligned}&\int_a^b \phi(g)\, w(t)\, dt - \int_a^b \phi(f)\, w(t)\, dt \\ &\quad \geq \left| \int_a^b w(t) \left|\phi(g) - \phi(f)\right| dt - \int_a^b w(t) \left|\phi'_+(f)(g - f)\right| dt \right|.\end{aligned}$$

Remark 4.2.2. Similarly, we can present the integral version of Theorem 4.2.4 which is in fact the refinement of the majorization inequality given in [7].

The following result is needed in the proof of the next theorem (see [42]).

Lemma 4.2.6. *Let $\mathbf{v} = (v_1, \ldots, v_2)$ be a positive n-tuple. If $\mathbf{a} = (a_1, \ldots, a_n)$ is a decreasing real n-tuple, then*

$$\sum_{i=1}^{n} a_i v_i \sum_{i=1}^{k} v_i \leq \sum_{i=1}^{k} a_i v_i \sum_{i=1}^{n} v_i, \qquad k = 1, \ldots, n. \tag{4.2.7}$$

If $\mathbf{a}$ is increasing real n-tuple, then the reverse inequality holds in (4.2.7).

If $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are two n-tuples with $y_i \neq 0$ $(i = 1, \ldots, n)$, then we define

$$\frac{\mathbf{x}}{\mathbf{y}} = \left(\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n}\right).$$

The following theorem is the refinement of the generalized discrete weighted Favard's inequality.

Theorem 4.2.7. *Let $\phi : (0,1) \to \mathbb{R}$ be a convex function, $\mathbf{p} = (p_1, \ldots, p_n)$, $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be positive n-tuples. Suppose that $u_i = \frac{x_i}{\sum_{i=1}^n p_i x_i}$, $z_i = \frac{y_i}{\sum_{i=1}^n p_i y_i}$ $(i = 1, \ldots, n)$ and consider the inequalities*

$$\sum_{i=1}^n p_i \phi(z_i) - \sum_{i=1}^n p_i \phi(u_i) \geq \left| \sum_{i=1}^n p_i \big|\phi(z_i) - \phi(u_i)\big| - \sum_{i=1}^n p_i \big|\phi'_+(u_i)(z_i - u_i)\big| \right| \tag{4.2.8}$$

and

$$\sum_{i=1}^n p_i \phi(u_i) - \sum_{i=1}^n p_i \phi(z_i) \geq \left| \sum_{i=1}^n p_i \big|\phi(u_i) - \phi(z_i)\big| - \sum_{i=1}^n p_i \big|\phi'_+(z_i)(u_i - z_i)\big| \right|. \tag{4.2.9}$$

(*i*) *Let $\frac{\mathbf{x}}{\mathbf{y}}$ be a decreasing n-tuple. If $\mathbf{x}$ is an increasing n-tuple, then* (4.2.8) *holds. If $\mathbf{y}$ is a decreasing n-tuple, then* (4.2.9) *holds.*

(*ii*) *Let $\frac{\mathbf{x}}{\mathbf{y}}$ be an increasing n-tuple. If $\mathbf{y}$ is an increasing n-tuple, then* (4.2.9) *holds. If $\mathbf{x}$ is a decreasing n-tuple, then* (4.2.8) *holds.*

Proof. (i) By using Lemma 4.2.6 for a positive n-tuple $\mathbf{v} = \mathbf{yp}$ and a decreasing n-tuple $\mathbf{a} = \frac{\mathbf{x}}{\mathbf{y}}$, we have

$$\sum_{i=1}^k p_i z_i \leq \sum_{i=1}^k p_i u_i, \qquad k = 1, \ldots, n-1,$$

and

$$\sum_{i=1}^n p_i z_i = \sum_{i=1}^n p_i u_i.$$

Now if $\mathbf{x}$ is increasing, then by using Theorem 4.2.3 (*ii*), we have (4.2.8) and if $\mathbf{y}$ is decreasing, then by using Theorem 4.2.3 (*i*), we have (4.2.9).

(ii) By using the same idea as in (i), we can prove the remaining cases.

□

Corollary 4.2.8. *Let $\phi : [0,\infty) \to \mathbb{R}$ be a convex function and $\mathbf{p} = (p_1, \ldots, p_n)$ be a positive n-tuple. Suppose that $u_i = \frac{x_i}{\sum_{i=1}^n p_i x_i}$, $\bar{u}_i = \frac{i-1}{\sum_{i=1}^n p_i (i-1)}$ and $\bar{z}_i = \frac{n-i}{\sum_{i=1}^n p_i (n-i)}$ $(i = 1, \ldots, n)$.*

(i) If $\mathbf{x} = (x_1, \ldots, x_n)$ is a positive increasing concave n-tuple, then we have

$$\sum_{i=1}^n p_i \phi(\bar{u}_i) - \sum_{i=1}^n p_i \phi(u_i) \geq \left| \sum_{i=1}^n p_i \big| \phi(\bar{u}_i) - \phi(u_i) \big| - \sum_{i=1}^n p_i \big| \phi'_+(u_i)(\bar{u}_i - u_i) \big| \right|. \tag{4.2.10}$$

(ii) If $\mathbf{x} = (x_1, \ldots, x_n)$ is an increasing convex real n-tuple with $x_1 = 0$, then we have

$$\sum_{i=1}^n p_i \phi(u_i) - \sum_{i=1}^n p_i \phi(\bar{u}_i) \geq \left| \sum_{i=1}^n p_i \big| \phi(u_i) - \phi(\bar{u}_i) \big| - \sum_{i=1}^n p_i \big| \phi'_+(\bar{u}_i)(u_i - \bar{u}_i) \big| \right|. \tag{4.2.11}$$

(iii) If $\mathbf{x} = (x_1, \ldots, x_n)$ is a positive decreasing concave n-tuple, then we have

$$\sum_{i=1}^n p_i \phi(\bar{z}_i) - \sum_{i=1}^n p_i \phi(u_i) \geq \left| \sum_{i=1}^n p_i \big| \phi(\bar{z}_i) - \phi(u_i) \big| - \sum_{i=1}^n p_i \big| \phi'_+(u_i)(\bar{z}_i - u_i) \big| \right|.$$

(iv) If $\mathbf{x} = (x_1, \ldots, x_n)$ is a decreasing convex real n-tuple with $x_n = 0$, then we have

$$\sum_{i=1}^n p_i \phi(u_i) - \sum_{i=1}^n p_i \phi(\bar{z}_i) \geq \left| \sum_{i=1}^n p_i \big| \phi(u_i) - \phi(\bar{z}_i) \big| - \sum_{i=1}^n p_i \big| \phi'_+(\bar{z}_i)(u_i - \bar{z}_i) \big| \right|.$$

Proof. (i) By taking $y_1 = \epsilon < \frac{x_1}{x_2}$, $y_i = i-1$ $(2,\ldots,n)$ and by using the concavity of $\mathbf{x}$, we have $\frac{x_i}{y_i}$ is a decreasing n-tuple. Now as $\frac{x_i}{y_i}$ is a decreasing n-tuple and $\mathbf{x}$ is increasing by assumption, therefore, by using Theorem 4.2.7 (i) and taking $\epsilon \to 0$, we have (4.2.10).

(ii) If $\mathbf{x}$ is an increasing convex real n-tuple and $x_1 = 0$, then $\frac{x_i}{i-1}$ $(2,\ldots,n)$ is increasing. Now as $\frac{x_i}{i-1}$ and $y_i = i-1$ $(2,\ldots,n)$ are increasing, by using Theorem 4.2.7 (ii), we have (4.2.11).

Similarly, we can prove the remaining cases. □

The following corollary is an application of Theorem 4.2.7.

Corollary 4.2.9. *Let $\mathbf{p} = (p_1,\ldots,p_n)$, $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{y} = (y_1,\ldots,y_n)$ be positive n-tuples and $\phi(x) = x^p$, where $p > 1$ or $p < 0$. Consider the inequalities*

$$\left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i y_i}\right)^p - \frac{\sum_{i=1}^n p_i x_i^p}{\sum_{i=1}^n p_i y_i^p} \geq \Bigg|\sum_{i=1}^n p_i \Big| \left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i y_i}\right)^p \frac{y_i^p}{\sum_{i=1}^n p_i y_i^p} - \frac{x_i^p}{\sum_{i=1}^n p_i y_i^p}\Big| - |p|\sum_{i=1}^n p_i \Big|\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i y_i}\frac{y_i x_i^{p-1}}{\sum_{i=1}^n p_i y_i^p} - \frac{x_i^p}{\sum_{i=1}^n p_i y_i^p}\Big|\Bigg| \tag{4.2.12}$$

and

$$\frac{\sum_{i=1}^n p_i x_i^p}{\sum_{i=1}^n p_i y_i^p} - \left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i y_i}\right)^p \geq \Bigg|\sum_{i=1}^n p_i \Big| \left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i y_i}\right)^p \frac{y_i^p}{\sum_{i=1}^n p_i y_i^p} - \frac{x_i^p}{\sum_{i=1}^n p_i y_i^p}\Big| - |p|\sum_{i=1}^n p_i \Big| \left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i y_i}\right)^{p-1} \frac{x_i y_i^{p-1}}{\sum_{i=1}^n p_i y_i^p} - \left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i y_i}\right)^p \frac{y_i^p}{\sum_{i=1}^n p_i y_i^p}\Big|\Bigg|. \tag{4.2.13}$$

(i) *Let $\frac{\mathbf{x}}{\mathbf{y}}$ be a decreasing n-tuple. If $\mathbf{x}$ is an increasing n-tuple, then (4.2.12) holds. If $\mathbf{y}$ is a decreasing n-tuple, then (4.2.13) holds.*

(ii) *Let $\frac{\mathbf{x}}{\mathbf{y}}$ be an increasing n-tuple. If $\mathbf{y}$ is an increasing n-tuple, then (4.2.13) holds. If $\mathbf{x}$ is a decreasing n-tuple, then (4.2.12) holds.*

The following result is an application of Corollary 4.2.8.

Corollary 4.2.10. *Let $\mathbf{p} = (p_1,\ldots,p_n)$, $\mathbf{x} = (x_1,\ldots,x_n)$ be positive n-tuples and $\phi(x) = x^p$, where $p > 1$. Suppose that $u_i = \frac{x_i}{\sum_{i=1}^n p_i x_i}$, $\bar{u}_i = \frac{i-1}{\sum_{i=1}^n p_i (i-1)}$, $\bar{z}_i = \frac{n-i}{\sum_{i=1}^n p_i (n-i)}$ $(1,\ldots,n)$, $w = \frac{\left(\sum_{i=1}^n p_i x_i\right)^p}{\sum_{i=1}^n p_i (i-1)^p}$ and $\bar{w} = \frac{\left(\sum_{i=1}^n p_i x_i\right)^p}{\sum_{i=1}^n p_i (n-i)^p}$.*

(i) *If* $\mathbf{x} = (x_1, \ldots, x_n)$ *is an increasing concave* n*-tuple, then we have*

$$\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i (i-1)}\right)^p - \frac{\sum_{i=1}^{n} p_i x_i^p}{\sum_{i=1}^{n} p_i (i-1)^p} \geq w \left| \sum_{i=1}^{n} p_i \left| \bar{u}_i^p - u_i^p \right| - \sum_{i=1}^{n} p_i \left| p u_i^{p-1} (\bar{u}_i - u_i) \right| \right|.$$

(ii) *If* $\mathbf{x} = (x_1, \ldots, x_n)$ *is an increasing convex real* n*-tuple with* $x_1 = 0$*, then we have*

$$\frac{\sum_{i=1}^{n} p_i x_i^p}{\sum_{i=1}^{n} p_i (i-1)^p} - \left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i (i-1)}\right)^p \geq w \left| \sum_{i=1}^{n} p_i \left| u_i^p - \bar{u}_i^p \right| - \sum_{i=1}^{n} p_i \left| p \bar{u}_i^{p-1} (u_i - \bar{u}_i) \right| \right|.$$

(iii) *If* $\mathbf{x} = (x_1, \ldots, x_n)$ *is a decreasing concave* n*-tuple, then we have*

$$\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i (n-i)}\right)^p - \frac{\sum_{i=1}^{n} p_i x_i^p}{\sum_{i=1}^{n} p_i (n-i)^p} \geq \bar{w} \left| \sum_{i=1}^{n} p_i \left| \bar{z}_i^p - u_i^p \right| - \sum_{i=1}^{n} p_i \left| p u_i^{p-1} (\bar{z}_i - u_i) \right| \right|.$$

(iv) *If* $\mathbf{x} = (x_1, \ldots, x_n)$ *is decreasing convex real* n*-tuple with* $x_n = 0$*, then we have*

$$\frac{\sum_{i=1}^{n} p_i x_i^p}{\sum_{i=1}^{n} p_i (n-i)^p} - \left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i (n-i)}\right)^p \geq \bar{w} \left| \sum_{i=1}^{n} p_i \left| u_i^p - \bar{z}_i^p \right| - \sum_{i=1}^{n} p_i \left| p \bar{z}_i^{p-1} (u_i - \bar{z}_i) \right| \right|.$$

The following theorem is the refinement of the generalized weighted Favard's inequality given in [41].

Theorem 4.2.11. *Let* $w, f, g : [a, b] \to \mathbb{R}^+$ *be integrable functions and* $\phi : \mathbb{R}^+ \to \mathbb{R}$ *be a convex function. Suppose that* $h(t) = \frac{f(t)}{\int_a^b f(t) w(t) dt}$, $k(t) = \frac{g(t)}{\int_a^b g(t) w(t) dt}$ *and consider the inequalities*

$$\int_a^b \phi(k) w(t) dt - \int_a^b \phi(h) w(t) dt \geq \left| \int_a^b w(t) \left| \phi(k) - \phi(h) \right| dt - \int_a^b w(t) \left| \phi'_+(h) (k-h) \right| dt \right| \tag{4.2.14}$$

and

$$\int_a^b \phi(h)\, w(t)\, dt - \int_a^b \phi(k)\, w(t)\, dt$$
$$\geq \left| \int_a^b w(t) \left|\phi(h) - \phi(k)\right| dt - \int_a^b w(t) \left|\phi'_+(k)(h-k)\right| dt \right|. \quad (4.2.15)$$

(*i*) *Let* $\frac{f}{g}$ *be a decreasing function defined on* $[a,b]$. *If* f *is an increasing function, then* (4.2.14) *holds. If* g *is a decreasing function, then* (4.2.15) *holds.*

(*ii*) *Let* $\frac{f}{g}$ *be an increasing function defined on* $[a,b]$. *If* g *is an increasing function, then* (4.2.15) *holds. If* f *is a decreasing function, then* (4.2.14) *holds.*

Remark 4.2.3. If $x \mapsto \phi(x)$ is a convex function, then $x \mapsto \phi(kx)$, $k \in \mathbb{R}$ is also a convex function. If f is a positive increasing concave function and $g(t) = t - a$, then (4.2.14) gives the refinement of the weighted Favard's inequality.

Corollary 4.2.12. *Let* $w, f, g : [a,b] \to \mathbb{R}^+$ *be integrable functions and* $\phi(x) = x^p$, *where* $p > 1$ *or* $p < 0$. *Suppose that* $h(t) = \frac{f(t)}{\int_a^b f(t)w(t)dt}$, $k(t) = \frac{g(t)}{\int_a^b g(t)w(t)dt}$ *and consider the inequalities*

$$\left(\frac{\int_a^b f(t)\, w(t)\, dt}{\int_a^b g(t)\, w(t)\, dt}\right)^p - \frac{\int_a^b f^p(t)w(t)dt}{\int_a^b g^p(t)w(t)dt} \quad (4.2.16)$$
$$\geq \frac{\left(\int_a^b f(t)\, w(t)\, dt\right)^p}{\int_a^b g^p(t)\, w(t)\, dt} \left| \int_a^b w(t) \left|k^p - h^p\right| dt - \int_a^b w(t) \left|\phi'_+(h)(k-h)\right| dt \right|$$

and

$$\frac{\int_a^b f^p(t)\, w(t)\, dt}{\int_a^b g^p(t)\, w(t)\, dt} - \left(\frac{\int_a^b f(t)\, w(t)\, dt}{\int_a^b g(t)\, w(t)\, dt}\right)^p \quad (4.2.17)$$
$$\geq \frac{\left(\int_a^b f(t)\, w(t)\, dt\right)^p}{\int_a^b g^p(t)\, w(t)\, dt} \left| \int_a^b w(t) \left|h^p - k^p\right| dt - \int_a^b w(t) \left|\phi'_+(k)(h-k)\right| dt \right|.$$

(*i*) *Let* $\frac{f}{g}$ *be a decreasing function defined on* $[a,b]$. *If* f *is an increasing function, then* (4.2.16) *holds. If* g *is a decreasing function, then* (4.2.17) *holds.*

(*ii*) *Let* $\frac{f}{g}$ *be an increasing function defined on* $[a,b]$. *If* g *is an increasing function, then* (4.2.17) *holds. If* f *is a decreasing function, then* (4.2.16) *holds.*

Remark 4.2.4. If f is a positive increasing concave function and if we substitute $g(t) = t - a$, $w(t) \equiv 1$ in (4.2.16), we obtain the refinement of the classical Favard's inequality.

The following theorem is the refinement of the majorization inequality given in [42].

Theorem 4.2.13. *Let $\mathbf{p} = (p_1, \ldots, p_n)$, $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be positive n-tuples. Let $\phi, \psi : [0, \infty) \to \mathbb{R}$ be such that ψ is strictly increasing function and $\phi \circ \psi^{-1}$ is convex. Suppose that*

$$\sum_{i=1}^{k} p_i \psi(y_i) \leq \sum_{i=1}^{k} p_i \psi(x_i) \quad \text{for } k = 1, \ldots, n-1,$$

and

$$\sum_{i=1}^{n} p_i \psi(y_i) = \sum_{i=1}^{n} p_i \psi(x_i)$$

hold.

(*i*) *If $\mathbf{y}$ is decreasing n-tuple, then we have*

$$\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(y_i) \geq \left| \sum_{i=1}^{n} p_i \left| \phi(x_i) - \phi(y_i) \right| - \sum_{i=1}^{n} p_i \left| \left(\phi \circ \psi^{-1}\right)'_+ (\psi(y_i)) (\psi(x_i) - \psi(y_i)) \right| \right|.$$

(*ii*) *If $\mathbf{x}$ is increasing n-tuple, then we have*

$$\sum_{i=1}^{n} p_i \phi(y_i) - \sum_{i=1}^{n} p_i \phi(x_i) \geq \left| \sum_{i=1}^{n} p_i \left| \phi(y_i) - \phi(x_i) \right| - \sum_{i=1}^{n} p_i \left| \left(\phi \circ \psi^{-1}\right)'_+ (\psi(x_i)) (\psi(y_i) - \psi(x_i)) \right| \right|.$$

Proof. By using Theorem 4.2.3 for the convex function $f(x) = \phi \circ \psi^{-1}(x)$ and for the n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, where $a_i = \psi(x_i), b_i = \psi(y_i)$, we obtain the required inequalities. □

Integral version of the above theorem is stated as follows:

Theorem 4.2.14. *Let w, f, g be positive integrable functions defined on $[a,b]$. Suppose $\phi, \psi : [0,\infty) \to$ are such that ψ is strictly increasing function and $\phi \circ \psi^{-1}$ is convex. Also suppose that*

$$\int_a^x \psi(g(t))\, w(t)\, dt \le \int_a^x \psi(f(t))\, w(t)\, dt, \quad \textit{for all } x \in [a,b]$$

and

$$\int_a^b \psi(g(t))\, w(t)\, dt = \int_a^b \psi(f(t))\, w(t)\, dt$$

hold.

(*i*) *If g is a decreasing function defined on $[a,b]$, then the following inequality holds*

$$\int_a^b \phi(f)\, w(t)\, dt - \int_a^b \phi(g)\, w(t)\, dt$$
$$\ge \left| \int_a^b w(t) \left|\phi(f) - \phi(g)\right| dt - \int_a^b w(t) \left(\phi \circ \psi^{-1}\right)'_+ (\psi(g)) \left(\psi(f) - \psi(g)\right) \right| dt \right|.$$

(*ii*) *If f is an increasing function defined on $[a,b]$, then the following inequality holds*

$$\int_a^b \phi(g)\, w(t)\, dt - \int_a^b \phi(f)\, w(t)\, dt$$
$$\ge \left| \int_a^b w(t) \left|\phi(g) - \phi(f)\right| dt - \int_a^b w(t) \left| \left(\phi \circ \psi^{-1}\right)'_+ (\psi(f)) \left(\psi(g) - \psi(f)\right) \right| dt \right|.$$

The following theorem is the refinement of the generalized discrete weighted Berwald's inequality.

Theorem 4.2.15. *Let $\mathbf{p} = (p_1, \ldots, p_n)$, $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be positive n-tuples. Suppose $\phi, \psi : [0,\infty) \to \mathbb{R}$ are such that ψ is continuous and strictly increasing function and $\phi \circ \psi^{-1}$ is convex. Let z_1 be such that*

$$\sum_{i=1}^n p_i \psi(z_1 y_i) = \sum_{i=1}^n p_i \psi(x_i). \tag{4.2.18}$$

Consider the inequalities

$$\sum_{i=1}^n p_i \phi(z_1 y_i) - \sum_{i=1}^n p_i \phi(x_i) \tag{4.2.19}$$
$$\ge \left| \sum_{i=1}^n p_i \left|\phi(z_1 y_i) - \phi(x_i)\right| - \sum_{i=1}^n p_i \left| \left(\phi \circ \psi^{-1}\right)'_+ (\psi(x_i)) \left(\psi(z_1 y_i) - \psi(x_i)\right) \right| \right|$$

and

$$\sum_{i=1}^{n} p_i \phi(x_i) - \sum_{i=1}^{n} p_i \phi(z_1 y_i) \tag{4.2.20}$$
$$\geq \left| \sum_{i=1}^{n} p_i \big| \phi(x_i) - \phi(z_1 y_i) \big| - \sum_{i=1}^{n} p_i \big| \left(\phi \circ \psi^{-1}\right)'_{+} (\psi(z_1 y_i)) (\psi(x_i) - \psi(z_1 y_i)) \big| \right|.$$

(*i*) *Let* $\frac{\mathbf{x}}{\mathbf{y}}$ *be a decreasing* n*-tuple. If* $\mathbf{x}$ *is an increasing* n*-tuple, then* (4.2.19) *holds. If* $\mathbf{y}$ *is a decreasing* n*-tuple, then* (4.2.20) *holds.*

(*ii*) *Let* $\frac{\mathbf{x}}{\mathbf{y}}$ *be an increasing* n*-tuple. If* $\mathbf{y}$ *is an increasing* n*-tuple, then* (4.2.20) *holds. If* $\mathbf{x}$ *is a decreasing* n*-tuple, then* (4.2.19) *holds.*

Proof. The existence of z_1 is shown in [42] and it is proved that

$$\sum_{i=1}^{k} p_i \psi(z_1 y_i) \leq \sum_{i=1}^{k} p_i \psi(x_i), \quad \text{for } k = 1, \ldots, n-1. \tag{4.2.21}$$

Now using (4.2.18) and (4.2.21) in Theorem 4.2.13, we obtain the required inequalities. □

Corollary 4.2.16. *Let* $\mathbf{p} = (p_1, \ldots, p_n)$ *be a positive* n*-tuple and* $\phi, \psi : [0, \infty) \to \mathbb{R}$ *be such that* ψ *is continuous and strictly increasing function and* $\phi \circ \psi^{-1}$ *is convex. Let* z_1 *and* z_2 *be such that*

$$\sum_{i=1}^{n} p_i \psi(z_1 (i-1)) = \sum_{i=1}^{n} p_i \psi(x_i) \tag{4.2.22}$$

and

$$\sum_{i=1}^{n} p_i \psi(z_2 (n-i)) = \sum_{i=1}^{n} p_i \psi(x_i). \tag{4.2.23}$$

(*i*) *If* $\mathbf{x}$ *is an increasing concave* n*-tuple and* $x_1 = 0$*, then we have*

$$\sum_{i=1}^{n} p_i \phi(z_1 (i-1)) - \sum_{i=1}^{n} p_i \phi(x_i) \geq \left| \sum_{i=1}^{n} p_i \big| \phi(z_1 (i-1)) - \phi(x_i) \big| - \sum_{i=1}^{n} p_i \big| (\phi \circ \psi^{-1})'_{+} (\psi(x_i)) (\psi(z_1 (i-1)) - \psi(x_i)) \big| \right|.$$

(ii) If $\mathbf{x}$ *is an increasing convex* n*-tuple and* $x_1 = 0$*, then we have*

$$\sum_{i=1}^{n} p_i \phi\left(x_i\right) - \sum_{i=1}^{n} p_i \phi\left(z_1\left(i-1\right)\right) \geq \left| \sum_{i=1}^{n} p_i \left| \phi\left(x_i\right) - \phi\left(z_1\left(i-1\right)\right) \right| - \sum_{i=1}^{n} p_i \left| \left(\phi \circ \psi^{-1}\right)'_{+}\left(\psi\left(z_1\left(i-1\right)\right)\right)\left(\psi\left(x_i\right) - \psi\left(z_1\left(i-1\right)\right)\right) \right| \right|.$$

(iii) If $\mathbf{x}$ *is a decreasing concave* n*-tuple and* $x_n = 0$*, then we have*

$$\sum_{i=1}^{n} p_i \phi\left(z_2\left(n-i\right)\right) - \sum_{i=1}^{n} p_i \phi\left(x_i\right) \geq \left| \sum_{i=1}^{n} p_i \left| \phi\left(z_2\left(n-i\right)\right) - \phi\left(x_i\right) \right| - \sum_{i=1}^{n} p_i \left| \left(\phi \circ \psi^{-1}\right)'_{+}\left(\psi\left(x_i\right)\right)\left(\psi\left(z_2\left(n-i\right)\right) - \psi\left(x_i\right)\right) \right| \right|.$$

(iv) If $\mathbf{x}$ *is a decreasing convex* n*-tuple and* $x_n = 0$*, then we have*

$$\sum_{i=1}^{n} p_i \phi\left(x_i\right) - \sum_{i=1}^{n} p_i \phi\left(z_2\left(n-i\right)\right) \geq \left| \sum_{i=1}^{n} p_i \left| \phi\left(x_i\right) - \phi\left(z_2\left(n-i\right)\right) \right| - \sum_{i=1}^{n} p_i \left| \left(\phi \circ \psi^{-1}\right)'_{+}\left(\psi\left(z_2\left(n-i\right)\right)\right)\left(\psi\left(x_i\right) - \psi\left(z_2\left(n-i\right)\right)\right) \right| \right|.$$

The following refinement of the inequalities given in [50] is a simple consequence of the Theorem 4.2.15.

Corollary 4.2.17. *Let* $\mathbf{p} = (p_1, \ldots, p_n)$*,* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, \ldots, y_n)$ *be positive* n*-tuples and let* $\psi(x) = x^q$*,* $\phi(x) = x^p$*, where* $0 < q \leq p$*.*

Let $\frac{\boldsymbol{x}}{\boldsymbol{y}}$ *be a decreasing* n*-tuple. If* $\boldsymbol{x}$ *is an increasing* n*-tuple, then we have*

$$\left(\frac{\sum_{i=1}^{n} p_i x_i^q}{\sum_{i=1}^{n} p_i y_i^q}\right)^{\frac{p}{q}} - \frac{\sum_{i=1}^{n} p_i x_i^p}{\sum_{i=1}^{n} p_i y_i^p} \geq \frac{1}{\sum_{i=1}^{n} p_i y_i^p} \left| \sum_{i=1}^{n} p_i \left| \left(\frac{\sum_{i=1}^{n} p_i x_i^q}{\sum_{i=1}^{n} p_i y_i^q}\right)^{\frac{p}{q}} y_i^p - x_i^p \right| - \frac{p}{q} \sum_{i=1}^{n} p_i \left| x_i^{p-q} \left(\frac{\sum_{i=1}^{n} p_i x_i^q}{\sum_{i=1}^{n} p_i y_i^q} y_i^q - x_i^q\right) \right| \right|.$$

Similarly we can give other possible results by using Theorem 4.2.15.

The following corollary is an application of Corollary 4.2.16.

Corollary 4.2.18. *Let* $\mathbf{p} = (p_1, \ldots, p_n)$ *and* $\mathbf{x} = (x_1, \ldots, x_n)$ *be positive* n*-tuples. Let* $\psi(x) = x^q$, $\phi(x) = x^p$, *where* $0 < q \leq p$ *and suppose that* $u = \sum_{i=1}^n p_i(i-1)^p$. *If* $\mathbf{x}$ *is an increasing* n*-tuple, then*

$$\begin{aligned}\left(\frac{\sum_{i=1}^n p_i x_i^q}{\sum_{i=1}^n p_i (i-1)^q}\right)^{\frac{p}{q}} - \frac{\sum_{i=1}^n p_i x_i^p}{u} \quad &\geq \quad \frac{1}{u}\left|\sum_{i=1}^n p_i \Big| \left(\frac{\sum_{i=1}^n p_i x_i^q}{\sum_{i=1}^n p_i (i-1)^q}\right)^{\frac{p}{q}} (i-1)^p - x_i^p \Big| \right. \\ &- \left. \frac{p}{q}\sum_{i=1}^n p_i \Big| x_i^{p-q} \left(\frac{\sum_{i=1}^n p_i x_i^q}{\sum_{i=1}^n p_i (i-1)^q} (i-1)^q - x_i^q\right) \Big| \right|.\end{aligned}$$

Similarly, we can give the other possible results by using Corollary 4.2.16.

The following theorem is the refinement of the extension of the weighted Berwald's inequality given in [41].

Theorem 4.2.19. *Let* w, f, g *be positive integrable functions defined on* $[a, b]$. *Suppose* $\phi, \psi : [0, \infty) \to$ *are such that,* ψ *is continuous and strictly increasing function and* $\phi \circ \psi^{-1}$ *is convex. Let* z_1 *be such that*

$$\int_a^b \psi(z_1 g(t)) w(t)\, dt = \int_a^b \psi(z_1 f(t)) w(t)\, dt. \tag{4.2.24}$$

Consider the inequalities

$$\begin{aligned}\int_a^b \phi(z_1 g(t)) w(t)\, dt \quad &- \quad \int_a^b \phi(f(t)) w(t)\, dt \geq \left|\int_a^b w(t) \big|\phi(z_1 g(t)) - \phi(f(t))\big| dt \right. \\ &- \left. \int_a^b w(t) \big| \left(\phi \circ \psi^{-1}\right)'_+ (\psi(f(t))) (\psi(z_1 g(t)) - \psi(f(t))) \big| dt \right|\end{aligned} \tag{4.2.25}$$

and

$$\begin{aligned}\int_a^b \phi(f(t)) w(t)\, dt \quad &- \quad \int_a^b \phi(z_1 g(t)) w(t)\, dt \geq \left|\int_a^b w(t) \big|\phi(f(t)) - \phi(z_1 g(t))\big| dt \right. \\ &- \left. \int_a^b w(t) \left(\phi \circ \psi^{-1}\right)'_+ (\psi(z_1 g(t))) (\psi(f(t)) - \psi(z_1 g(t))) \big| dt \right|.\end{aligned} \tag{4.2.26}$$

(*i*) *Let* $\frac{f}{g}$ *be a decreasing function defined on* $[a, b]$. *If* f *is an increasing function, then* (4.2.25) *holds. If* g *is a decreasing function, then* (4.2.26) *holds.*

(*ii*) *Let* $\frac{f}{g}$ *be an increasing function defined on* $[a, b]$. *If* g *is an increasing function, then* (4.2.26) *holds. If* f *is a decreasing function, then* (4.2.25) *holds.*

Remark 4.2.5. We can obtain integral version of Corollary 4.2.17 as an application of Theorem 4.2.19.

4.3 Inequalities in Terms of Gâteaux Derivatives for Convex Functions in Linear Spaces

Let C be a convex subset of the linear space X and f be a convex function defined on C. If $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{P}^n$, that is, $p_i \geq 0$ for all $i = 1, \ldots, n$ with $\sum_{i=1}^{n} p_i = 1$, is a probability sequence and $\mathbf{x} = (x_1, \ldots, x_n) \in C^n$, then the Jensen's inequality

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i)$$

holds.

Assume that $f : X \to \mathbb{R}$ is a convex function defined on the real linear space X. Since for any vectors $x, y \in X$, the function $g_{x,y} : \mathbb{R} \to \mathbb{R}$ defined by $g_{x,y}(t) := f(x + ty)$ is convex. It follows that the following limit exists

$$\nabla_{+(-)} f(x)(y) := \lim_{t \to 0+(-)} \frac{f(x+ty) - f(x)}{t}$$

and $\nabla_{+(-)} f(x)(y)$ is called the right (left) Gâteaux derivatives of the function f at the point x in the direction y.

For any $t > 0 > s$ and $x, y \in X$, it is obvious that

$$\begin{aligned}
\frac{f(x+ty) - f(x)}{t} \geq \nabla_{+} f(x)(y) &= \inf_{t>0}\left[\frac{f(x+ty) - f(x)}{t}\right] \\
&\geq \sup_{s<0}\left[\frac{f(x+sy) - f(x)}{s}\right] \\
&= \nabla_{-} f(x)(y) \\
&\geq \frac{f(x+sy) - f(x)}{s}
\end{aligned}$$

and in particular for any $u, v \in X$, we have

$$\nabla_{-} f(u)(u - v) \geq f(u) - f(v) \geq \nabla_{+} f(v)(u - v). \tag{4.3.1}$$

(4.3.1) is called the *gradient inequality* for convex function f. It will be used frequently in the sequel, in order to obtain refinements of Jensen's inequality.

The following properties are also of great importance:

$$\nabla_{+} f(x)(-y) = -\nabla_{-} f(x)(y)$$

and

$$\nabla_{+(-)} f(x)(\alpha y) = \alpha \nabla_{+(-)} f(x)(y) \tag{4.3.2}$$

for any $x, y \in X$ and $\alpha \geq 0$.

The right Gâteaux derivative is *subadditive* while the left one is *superadditive*, that is, for any $x, y, z \in X$, we have

$$\nabla_{+} f(x)(y+z) \leq \nabla_{+} f(x)(y) + \nabla_{+} f(x)(z) \tag{4.3.3}$$

and

$$\nabla_{-} f(x)(y+z) \geq \nabla_{-} f(x)(y) + \nabla_{-} f(x)(z).$$

Some natural examples can be provided by using the normed spaces.

Assume that $(X, ||.||)$ is a real normed linear space. The function $f : X \to \mathbb{R}$ defined by

$$f(x) := \frac{1}{2}||x||^2$$

is a convex function, which generates the *superior* (*inferior*) *semi-inner products*

$$\langle y, x \rangle_{s(i)} := \lim_{t \to 0+(-)} \frac{||x+ty||^2 - ||x||^2}{2t}.$$

For the convex function $f_p : X \to \mathbb{R}$ defined by

$$f_p(x) := ||x||^p \quad \text{with} \quad p > 1,$$

we have

$$\nabla_{+(-)} f_p(x)(y) = \begin{cases} p||x||^{p-2}\langle y, x \rangle_{s(i)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where $x, y \in X$.

If $p = 1$, then for any $x, y \in X$, we have

$$\nabla_{+(-)} f_1(x)(y) = \begin{cases} ||x||^{-1}\langle y, x \rangle_{s(i)}, & \text{if } x \neq 0, \\ +(-)\,||y||, & \text{if } x = 0. \end{cases}$$

This class of functions will be used to illustrate the inequalities obtained in the general case of convex functions defined on the entire linear spaces.

The following refinement of Jensen's inequality is given in [19].

Theorem 4.3.1. *Let $f : X \to \mathbb{R}$ be a convex function defined on a real linear space X. Then for any n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ and for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{P}^n$, we have*

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geq \sum_{k=1}^{n} p_k \nabla_{+} f\left(\sum_{i=1}^{n} p_i x_i\right)(x_k) - \nabla_{+} f\left(\sum_{i=1}^{n} p_i x_i\right)\left(\sum_{i=1}^{n} p_i x_i\right) \geq 0. \tag{4.3.4}$$

In the same paper the following reverse of Jensen's inequality is given.

Theorem 4.3.2. *Under the assumptions of Theorem 4.3.1, the following inequality holds*

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{k=1}^{n} p_k \nabla_{-} f(x_k)(x_k) - \sum_{k=1}^{n} p_k \nabla_{-} f(x_k)\left(\sum_{i=1}^{n} p_i x_i\right). \tag{4.3.5}$$

A particular case of interest is for $f(x) = ||x||^p$, where $(X, ||.||)$ is a normed linear space. Then for any $p \ge 1$, for any n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ and for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{P}^n$ with $\sum_{i=1}^{n} p_i x_i \ne 0$, we have

$$\sum_{i=1}^{n} p_i ||x_i||^p - \Big|\Big| \sum_{i=1}^{n} p_i x_i \Big|\Big|^p \ge p \Big|\Big| \sum_{i=1}^{n} p_i x_i \Big|\Big|^{p-2} \left[\sum_{i=1}^{n} p_i \Big\langle x_i, \sum_{k=1}^{n} p_k x_k \Big\rangle_s - \Big|\Big| \sum_{i=1}^{n} p_i x_i \Big|\Big|^2\right]. \tag{4.3.6}$$

Also, for any $p \ge 1$, for any n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n \setminus \{(0, \ldots, 0)\}$ and for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{P}^n$, we have

$$\sum_{i=1}^{n} p_i ||x_i||^p - \Big|\Big| \sum_{i=1}^{n} p_i x_i \Big|\Big|^p \le p \left[\sum_{i=1}^{n} p_i ||x_i||^p - \sum_{j=1}^{n} p_j ||x_j||^{p-2} \Big\langle \sum_{k=1}^{n} p_k x_k, x_j \Big\rangle_i\right]. \tag{4.3.7}$$

4.3.1 Inequalities for Convex Functions

This section provides some inequalities in terms of Gâteaux derivatives for convex functions defined on linear spaces, which imply inequalities (4.3.4) and (4.3.5). A particular case for norm is also provided.
The main theorem of this section states that:

Theorem 4.3.3. *Let $f : X \to \mathbb{R}$ be a convex function defined on a linear space X, $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ be any n-tuple of vectors and $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{P}^n$ be any* *probability distribution. If $c, d \in X$ are arbitrary vectors, then we have*

$$\begin{aligned} f(c) &+ \sum_{i=1}^{n} p_i \nabla_+ f(c)(x_i) - \nabla_+ f(c)(c) \\ &\leq \sum_{i=1}^{n} p_i f(x_i) \leq f(d) + \sum_{i=1}^{n} p_i \nabla_- f(x_i)(x_i) - \sum_{i=1}^{n} p_i \nabla_- f(x_i)(d). \quad (4.3.8) \end{aligned}$$

Proof. For fix index i, take $u = x_i$ and $v = c$ in the right inequality of (4.3.1), we have

$$f(x_i) - f(c) \geq \nabla_+ f(c)(x_i - c). \quad (4.3.9)$$

By using the subadditivity of $\nabla_+ f(.)(.)$ in the second variable, we have

$$\nabla_+ f(c)(x_i - c) \geq \nabla_+ f(c)(x_i) - \nabla_+ f(c)(c). \quad (4.3.10)$$

Combining (4.3.9) and (4.3.10), we get

$$f(x_i) - f(c) \geq \nabla_+ f(c)(x_i) - \nabla_+ f(c)(c).$$

Now, multiply the above inequality by p_i and summing over $i = 1, \ldots, n$, we obtain the left inequality in (4.3.8).

To obtain the right inequality in (4.3.8), take $u = x_i$ and $v = d$ in the first inequality of (4.3.1), we have

$$f(x_i) - f(d) \leq \nabla_- f(x_i)(x_i - d). \quad (4.3.11)$$

By using the superadditivity of $\nabla_- f(.)(.)$ in the second variable, we have

$$\nabla_- f(x_i)(x_i - d) \leq \nabla_- f(x_i)(x_i) - \nabla_- f(x_i)(d). \quad (4.3.12)$$

Combining (4.3.11) and (4.3.12), we get

$$f(x_i) - f(d) \leq \nabla_- f(x_i)(x_i) - \nabla_- f(x_i)(d).$$

Multiplying by p_i and summing over $i = 1, \ldots, n$, we obtain the right inequality in (4.3.8). □

Remark 4.3.1. If we take $c = d = \sum_{k=1}^{n} p_k x_k$ in (4.3.8), we obtain (4.3.4) and (4.3.5).

Remark 4.3.2. Related inequalities in terms of subdifferential of a convex function defined on linear space, have been proved by M. Matić and J. Pečarić in [44].

A particular case for norms is given as follows:

Corollary 4.3.4. *Let* $(X, ||.||)$ *be a normed linear space,* $p \geq 1$, $\mathbf{x} = (x_1, \ldots, x_n) \in X^n \backslash \{(0, \ldots, 0)\}$ *be any* n*-tuple of vectors and* $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{P}^n$ *be any probability distribution. If* $c, d \in X$, $c \neq 0$, *are arbitrary vectors, then we have*

$$||c||^p + p\sum_{j=1}^{n} p_j ||c||^{p-2}\langle x_j, c\rangle_s - p||c||^p \leq \sum_{j=1}^{n} p_j ||x_j||^p$$
$$\leq ||d||^p + p\sum_{j=1}^{n} p_j ||x_j||^p - p\sum_{j=1}^{n} p_j ||x_j||^{p-2}\langle d, x_j\rangle_i. \quad (4.3.13)$$

If $p \geq 2$, *then* (4.3.13) *holds for any* $c, d, x_j \in X$ $(j = 1, \ldots, n)$ *and for any probability distribution.*

In particular, we have the norm inequalities

$$\sum_{j=1}^{n} p_j \left\langle x_j, \frac{c}{||c||}\right\rangle_s \leq \sum_{j=1}^{n} p_j ||x_j|| \leq ||d|| + \sum_{j=1}^{n} p_j ||x_j|| - \sum_{j=1}^{n} p_j \left\langle d, \frac{x_j}{||x_j||}\right\rangle_i$$

for $x_j, c \neq 0$, $j \in \{1, \ldots, n\}$ *and*

$$2\sum_{j=1}^{n} p_j \langle x_j, c\rangle_s - ||c||^2 \leq \sum_{j=1}^{n} p_j ||x_j||^2$$
$$\leq ||d||^2 + 2\sum_{j=1}^{n} p_j ||x_j||^2 - 2\sum_{j=1}^{n} p_j \langle d, x_j\rangle_i.$$

Remark 4.3.3. If we take $c = d = \sum_{k=1}^{n} p_k x_k$ in (4.3.13), then we obtain (4.3.6) and (4.3.7).

4.3.2 Refinement of Jensen's Inequality

The main theorem of this section is the following:

Theorem 4.3.5. *Let* $f : X \to \mathbb{R}$ *be a convex function defined on a linear space* X, $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ *be any* n*-tuple of vectors and* $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{P}^n$ *be any probability distribution. If* $c \in X$ *is arbitrary vector, then we have*

$$\sum_{i=1}^{n} p_i f(x_i) - f(c) - \nabla_+ f(c)\left(\sum_{i=1}^{n} p_i x_i - c\right)$$
$$\geq \left|\sum_{i=1}^{n} p_i \left|f(x_i) - f(c)\right| - \sum_{i=1}^{n} p_i \left|\nabla_+ f(c)(x_i - c)\right|\right|. \quad (4.3.14)$$

Proof. For fix index i, take $u = x_i$ and $v = c$ in the right inequality of (4.3.1), we have

$$f(x_i) - f(c) - \nabla_+ f(c)(x_i - c) \geq 0.$$

Therefore, we can write

$$\begin{aligned} f(x_i) - f(c) - \nabla_+ f(c)(x_i - c) &= \left| f(x_i) - f(c) - \nabla_+ f(c)(x_i - c) \right| \\ &\geq \Big| \left| f(x_i) - f(c) \right| - \left| \nabla_+ f(c)(x_i - c) \right| \Big|. \end{aligned} \tag{4.3.15}$$

Multiplying (4.3.15) by p_i and summing over $i = 1, \ldots, n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} p_i f(x_i) - f(c) - \sum_{i=1}^{n} p_i \nabla_+ f(c)(x_i - c) \\ &\quad \geq \sum_{i=1}^{n} p_i \Big| \left| f(x_i) - f(c) \right| - \left| \nabla_+ f(c)(x_i - c) \right| \Big| \\ &\quad \geq \left| \sum_{i=1}^{n} p_i \left| f(x_i) - f(c) \right| - \sum_{i=1}^{n} p_i \left| \nabla_+ f(c)(x_i - c) \right| \right|. \end{aligned} \tag{4.3.16}$$

By using (4.3.2) and (4.3.3), it is easy to see that

$$\nabla_+ f(c) \left(\sum_{i=1}^{n} p_i x_i - c \right) \leq \sum_{i=1}^{n} p_i \nabla_+ f(c)(x_i - c). \tag{4.3.17}$$

Now, from (4.3.16) and (4.3.17), we obtain (4.3.14). □

The following refinement of Jensen's inequality is valid:

Corollary 4.3.6. *Let $f : X \to \mathbb{R}$ be a convex function defined on a linear space X. Then for any n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ and for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{P}^n$, we have*

$$\begin{aligned} &\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i \right) \geq \\ &\quad \left| \sum_{i=1}^{n} p_i \left| f(x_i) - f\left(\sum_{k=1}^{n} p_k x_k \right) \right| - \sum_{i=1}^{n} p_i \left| \nabla_+ f\left(\sum_{k=1}^{n} p_k x_k \right) \left(x_i - \sum_{k=1}^{n} p_k x_k \right) \right| \right|. \end{aligned} \tag{4.3.18}$$

Proof. By taking $c = \sum_{k=1}^{n} p_k x_k$ in (4.3.14), we obtain the desired result. □

Remark 4.3.4. In particular, for the uniform distribution, inequality (4.3.18) takes the form

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geq$$
$$\left| \frac{1}{n}\sum_{i=1}^{n} \left| f(x_i) - f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) \right| - \frac{1}{n}\sum_{i=1}^{n} \left| \nabla_+ f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\left(x_i - \frac{1}{n}\sum_{k=1}^{n} x_k\right) \right| \right|.$$

Remark 4.3.5. If the function f is defined on the Euclidian space $\mathbb{R}^n$ and is differentiable and convex, then from (4.3.18), it follows that:

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right)$$
$$\geq \left| \sum_{i=1}^{n} p_i \left| f(x_i) - f\left(\sum_{k=1}^{n} p_k x_k\right) \right| - \sum_{i=1}^{n} p_i \left| \left\langle \nabla f\left(\sum_{k=1}^{n} p_k x_k\right), x_i - \sum_{k=1}^{n} p_k x_k \right\rangle \right| \right|,$$

where $x_i = (x_i^1, \ldots, x_i^n)$ and $\nabla f(x_i) = \left(\frac{\partial f(x_i)}{\partial x_i^1}, \ldots, \frac{\partial f(x_i)}{\partial x_i^n}\right)$.

For one dimensional case, we have

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geq \left| \sum_{i=1}^{n} p_i \left| f(x_i) - f\left(\sum_{k=1}^{n} p_k x_k\right) \right| \right.$$
$$\left. - \left| f'\left(\sum_{k=1}^{n} p_k x_k\right) \right| \sum_{i=1}^{n} p_i \left| x_i - \sum_{k=1}^{n} p_k x_k \right| \right|,$$

that was proved in 2008 by Pečarić et al., see [27].

The following particular case for norms can be stated as follows:

Corollary 4.3.7. *Let $(X, ||.||)$ be a normed linear space, $p \geq 1$, $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ be any n-tuple of vectors and $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{P}^n$ be any probability distribution. If $c \in X$ is non-zero vector, then we have*

$$\sum_{i=1}^{n} p_i ||x_i||^p - ||c||^p - p||c||^{p-2} \left\langle \sum_{i=1}^{n} p_i x_i - c, c \right\rangle_s$$
$$\geq \left| \sum_{i=1}^{n} p_i \left| ||x_i||^p - ||c||^p \right| - p||c||^{p-2} \sum_{i=1}^{n} p_i \left| \langle x_i - c, c \rangle_s \right| \right|.$$

If $p \geq 2$, then the inequality holds for any vector c.

The following particular case that provides a refinement for the generalized triangle inequality in the normed linear spaces is of interest:

Corollary 4.3.8. *Let* $(X, ||.||)$ *be a normed linear space. Then for any* $p \geq 1$, $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ *and for any probability distribution* $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{P}^n$ *with* $\sum_{i=1}^n p_i x_i \neq 0$, *we have*

$$\sum_{i=1}^n p_i ||x_i||^p - \Big|\Big|\sum_{i=1}^n p_i x_i\Big|\Big|^p \geq \Bigg|\sum_{i=1}^n p_i \Big|||x_i||^p - \Big|\Big|\sum_{k=1}^n p_k x_k\Big|\Big|^p\Big| \\ -p\Big|\Big|\sum_{i=1}^n p_i x_i\Big|\Big|^{p-2} \sum_{i=1}^n p_i \Big|\Big\langle x_i - \sum_{k=1}^n p_k x_k, \sum_{k=1}^n p_k x_k\Big\rangle_s\Big|\Bigg|. \tag{4.3.19}$$

If $p \geq 2$, *then the inequality holds for any* n*-tuple of vectors and for any probability distribution.*

Proof. By taking $c = \sum_{k=1}^n p_k x_k \neq 0$ in Corollary 4.3.7, we obtain the desired result. □

Remark 4.3.6. If in inequality (4.3.19), we consider the uniform distribution, then we have

$$\sum_{i=1}^n ||x_i||^p - n^{1-p}\Big|\Big|\sum_{i=1}^n x_i\Big|\Big|^p \geq \Bigg|\sum_{i=1}^n \Big|||x_i||^p - n^{-p}\Big|\Big|\sum_{k=1}^n x_k\Big|\Big|^p\Big| \\ -pn^{2-p}\Big|\Big|\sum_{i=1}^n x_i\Big|\Big|^{p-2} \sum_{i=1}^n \Big|\Big\langle x_i - \frac{1}{n}\sum_{k=1}^n x_k, \frac{1}{n}\sum_{k=1}^n x_k\Big\rangle_s\Big|\Bigg|.$$

The results presented in this chapter are published in [38, 39].

Chapter 5

Generalizations and Improvements of an Inequality of Hardy-Littlewood-Pólya

5.1 Introduction and Preliminaries

The following theorem is given in the famous Hardy-Littlewood-Pólya book [26, Theorem 134].

Theorem 5.1.1. *If f is a convex and continuous function defined on $[0,\infty)$ and $(a_k,\, k \in \mathbb{N})$ are non-negative and non-increasing, then*

$$f\left(\sum_{k=1}^{n} a_k\right) \geq f(0) + \sum_{k=1}^{n} \left(f(ka_k) - f((k-1)\,a_k)\right). \tag{5.1.1}$$

If f' is a strictly increasing function, there is equality only when a_k are equal up to a certain point and then zero. If f is concave, then (5.1.1) *holds in the reverse direction.*

A consequence of above theorem is given below (see [26, p. 100]).

Corollary 5.1.2. *Let $a_k \geq 0$ and assume that the sequence $(a_k,\, k \in \mathbb{N})$ is non-increasing. If $s > 1$, then we have*

$$\left(\sum_{k=1}^{n} a_k\right)^{s} \geq \sum_{k=1}^{n} {a_k}^{s} \left(k^{s} - (k-1)^{s}\right). \tag{5.1.2}$$

If $0 < s < 1$, then (5.1.2) *holds in the reverse direction.*

Inequality (5.1.1) is of great interest and has been generalized in many different ways by various mathematicians.

In 1986, G. Bennett [8] proved a weighted version of inequality (5.1.1) for power functions $f(x) = x^s$: if a_k $(k = 1, \ldots, n)$ are non-negative and non-increasing and $p_k \geq 0$ for each $k = 1, \ldots, n$ with $P_k = \sum_{i=1}^{k} p_i$ $(k = 1, \ldots, n)$, then for any real number $s > 1$,

$$\left(\sum_{k=1}^{n} p_k a_k \right)^s \geq \sum_{k=1}^{n} P_k^s \left[a_k^s - a_{k+1}^s \right] = (p_1 a_1)^s + \sum_{k=2}^{n} a_k^s \left[P_k^s - P_{k-1}^s \right] \tag{5.1.3}$$

holds. If $0 < s < 1$, then (5.1.3) holds in the reverse direction (see [8]).

In 1995, inequality (5.1.2) was improved by J. Pečarić and L. E. Persson in [56] and this improvement is given below:

Theorem 5.1.3. *If the sequence* $(a_k > 0,\ k \in \mathbb{N})$ *is non-increasing in mean, that is, if*

$$\frac{1}{n} \sum_{k=1}^{n} a_k \geq \frac{1}{n+1} \sum_{k=1}^{n+1} a_k, \qquad n \in \mathbb{N},$$

where $(a_k,\ k \in \mathbb{N}) \subset \mathbb{R}$ *and if* s *is a real number such that* $s > 1$, *then*

$$\left(\sum_{k=1}^{\infty} a_k \right)^s \geq \sum_{k=1}^{\infty} {a_k}^s \left(k^s - (k-1)^s \right), \tag{5.1.4}$$

holds. If $0 < s < 1$, *then* (5.1.4) *holds in the reverse direction.*

It is well known and easy to see that if a sequence $(a_k,\ k \in \mathbb{N})$ is non-increasing, then it is also non-increasing in mean but the reverse implications don't hold in general. This means that Theorem 5.1.3 is a genuine generalization of Corollary 5.1.2.

In 1997, C. Jardas, J. Pečarić, R. Roki and N. Sarapa presented the generalizations of these inequalities ((5.1.4) and its reverse) in [31]. As a consequence some inequalities for entropies of discrete probability distributions are also presented in [31]. Some basic properties of entropies of probability distributons can be found in [49].

The following result is proved in [31].

Theorem 5.1.4. *Let* $a_k > 0$ $(k = 1, \ldots, n)$ *be real numbers and* $S_k = \sum_{i=1}^{k} a_i$ $(k = 1, \ldots, n)$. *Then for all* s, $0 < s < 1$ *or* $s > 2$, *we have*

$$
\begin{aligned}
&\sum_{k=1}^{n} {a_k}^s \left(k^s-(k-1)^s\right)+s\sum_{k=2}^{n} a_k^{s-1}\left(S_{k-1}-(k-1)\,a_k\right)\left(k^{s-1}-(k-1)^{s-1}\right)\\
&\le \left(\sum_{k=1}^{n} a_k\right)^s\\
&\le \sum_{k=1}^{n} {a_k}^s \left(k^s-(k-1)^s\right)+s\sum_{k=2}^{n} \left(S_{k-1}-(k-1)\,a_k\right)\left(S_k^{s-1}-S_{k-1}^{s-1}\right). \qquad (5.1.5)
\end{aligned}
$$

For all s, $1<s<2$, the opposite inequalities hold in (5.1.5). *Equalities hold in* (5.1.5) *if and only if $a_1=a_2=\ldots=a_n$.*

As a consequence of Theorem 5.1.4, the following theorem is proved in [31].

Theorem 5.1.5. *Let $p_k>0$ $(k=1,\ldots,n)$ be a probability distribution with entropy $H=-\sum_{i=1}^{n} p_i\log p_i$ and $P_k=\sum_{i=1}^{k} p_i\,(k=1,\ldots,n)$. Then we have*

$$
\begin{aligned}
H \quad &+\sum_{k=2}^{n}\left((k-1)\,p_k-P_{k-1}\right)\left(\log k-\log\left(k-1\right)\right)\le\sum_{k=2}^{n} F\left(k-1\right)p_k\\
\le \quad &H+\sum_{k=2}^{n}\left((k-1)\,p_k-P_{k-1}\right)\left(\log P_k-\log P_{k-1}\right), \qquad (5.1.6)
\end{aligned}
$$

where $F(x)=(x+1)\log(x+1)-x\log x\,(x>0)$. Equalities hold in (5.1.6) *if $p_k=\frac{1}{n}\,(k=1,\ldots,n)$.*

In order to obtain our main results, let us recall some definitions which are going to be used throughout this chapter.

Definition 5.1.1. A sequence $(a_k,\,k\in\mathbb{N})\subset\mathbb{R}$ is non-increasing in weighted mean, if

$$
\frac{1}{P_n}\sum_{k=1}^{n} p_k a_k\ge\frac{1}{P_{n+1}}\sum_{k=1}^{n+1} p_k a_k,\qquad n\in\mathbb{N}, \qquad (5.1.7)
$$

where a_k and p_k $(k\in\mathbb{N})$ are real numbers such that $p_k>0$ $(k\in\mathbb{N})$ with $P_k=\sum_{i=1}^{k} p_i$ $(k\in\mathbb{N})$.
A sequence $(a_k,\,k\in\mathbb{N})\subset\mathbb{R}$ is non-decreasing in weighted mean, if opposite inequality holds in (5.1.7).

In a similar way we can define when a finite sequence $(a_k,\,k=1,\ldots,n)\subset\mathbb{R}$ is non-increasing in weighted mean or non-decreasing in weighted mean.

Remark 5.1.1. It is easy to see that a sequence $(a_k,\ k \in \mathbb{N})$ is non-increasing in weighted mean (non-decreasing in weighted mean) if and only if $\sum_{i=1}^{k-1} p_i a_i \geq P_{k-1} a_k$ $\left(\sum_{i=1}^{k-1} p_i a_i \leq P_{k-1} a_k\right)$ for $k = 2, 3, \ldots$.

Wright-convex functions have interesting and important generalization for functions of several variables (see [11]). Let $\mathbb{R}^m$ denote the m-dimensional vector lattice of points $\mathbf{x} = (x_1, \ldots, x_m)$, $x_i \in \mathbb{R}$ for $i = 1, \ldots, m$, with the partial ordering

$$\mathbf{x} = (x_1, \ldots, x_m) \leq \mathbf{y} = (y_1, \ldots, y_m),$$

if and only if $x_i \leq y_i$ for $i = 1, \ldots, m$ (see [58, p. 13]).

For any two m-tuples $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, let us define component-wise multiplication and division as follows:

$$\mathbf{xy} = (x_1 y_1, \ldots, x_m y_m),$$

$$\frac{\mathbf{x}}{\mathbf{y}} = \left(\frac{x_1}{y_1}, \ldots, \frac{x_m}{y_m}\right), \qquad \mathbf{y} \neq \mathbf{0}.$$

Definition 5.1.2. A sequence $(\mathbf{a}_k, k \in \mathbb{N}) \subset \mathbb{R}^m$ is non-increasing in weighted mean, if

$$\frac{1}{\mathbf{P}_n} \sum_{k=1}^{n} \mathbf{p}_k \mathbf{a}_k \geq \frac{1}{\mathbf{P}_{n+1}} \sum_{k=1}^{n+1} \mathbf{p}_k \mathbf{a}_k, \qquad n \in \mathbb{N}, \tag{5.1.8}$$

where $\mathbf{a}_k = (a_k^1, \ldots, a_k^m)$, $\mathbf{p}_k = (p_k^1, \ldots, p_k^m)$ and $\mathbf{P}_k = \left(\sum_{i=1}^{k} p_i^1, \ldots, \sum_{i=1}^{k} p_i^m\right) \in \mathbb{R}^m$ such that $\mathbf{p}_k > 0$ $(k \in \mathbb{N})$.
A sequence $(\mathbf{a}_k, k \in \mathbb{N}) \subset \mathbb{R}^m$ is non-decreasing in weighted mean, if opposite inequality holds in (5.1.8).

In [11], H. D. Brunk explored an interesting class of multivariate real-valued functions defined as follows:

Definition 5.1.3. A real-valued function f on an m-dimensional rectangle $I \subset \mathbb{R}^m$ is said to have non-decreasing increments if

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) \leq f(\mathbf{y} + \mathbf{h}) - f(\mathbf{y}), \tag{5.1.9}$$

whenever $\mathbf{x}$, $\mathbf{y} + \mathbf{h} \in I$, $\mathbf{0} \leq \mathbf{h} \in \mathbb{R}^m$, $\mathbf{x} \leq \mathbf{y}$. The function f is said to have non-increasing increments if opposite inequality holds in (5.1.9).

Remark 5.1.2. It is easy to see that if a function f is defined on $[a, b] \subset \mathbb{R}$, then the functions having non-decreasing increments are Wright-convex functions.

Our first main result is the direct generalization of the Hardy-Littlewood-Pólya inequality (5.1.1), a result of G. Bennett given in [8] and also a result of S. Khalid and J. Pečarić given in [36]. After appropriate substitutions, our first main result is equivalent to the following inequality given by I. Perić (see [59, Theorem 1.4]).

Theorem 5.1.6. *Let f be a Wright-concave function on $[0,\infty)$. Let $0 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$, $C_n \geq 0$, $n \in \mathbb{N}$ and*

$$\sum_{k=1}^{n} C_k (x_k - x_{k-1}) \geq C_{n+1} x_n, \qquad n \geq 1. \tag{5.1.10}$$

Then

$$f\left(\sum_{k=1}^{n} C_k (x_k - x_{k-1})\right) + \sum_{k=1}^{n-1} f(C_{k+1} x_k) \leq \sum_{k=1}^{n} f(C_k x_k), \qquad n \in \mathbb{N}. \tag{5.1.11}$$

In the forthcoming section, we present generalizations of some results given in [8] and [36] and show that one of our result is equivalent to the inequality from Theorem 5.1.6. Further, the objective is to study not only the functionals defined as the non-negative difference between the right-hand and the left-hand side of the generalized inequalities but also their properties, such as n-exponential and logarithmic convexity. Finally, we give several examples of the families of functions for which the results can be applied and we also present refinement of inequality (5.1.3).

5.2 Convex Functions and Generalizations of an Inequality of Hardy-Littlewood-Pólya

Inequality (5.1.3) is already proved in [8], but we give a new proof in a more general setting. The following result is our first main result:

Theorem 5.2.1. *Let a_k and p_k $(k = 1, \ldots, n)$ be real numbers such that $a_k \geq 0$ and $p_k > 0$ with $P_k = \sum_{i=1}^{k} p_i$ $(k = 1, \ldots, n)$. Let $p_1 a_1$, $\sum_{k=1}^{n} p_k a_k$, $P_k a_k$, $P_{k-1} a_k \in [a, b]$ for all $k = 2, \ldots, n$ and let $f : [a, b] \to \mathbb{R}$ be a Wright-convex function.*

(*i*) *If the sequence $(a_k,\ k = 1, \ldots, n)$ is non-increasing in weighted mean, then we have*

$$f\left(\sum_{k=1}^{n} p_k a_k\right) \geq f(p_1 a_1) + \sum_{k=2}^{n} \left[f(P_k a_k) - f(P_{k-1} a_k)\right]. \tag{5.2.1}$$

(*ii*) *If the sequence* $(a_k,\, k = 1, \ldots, n)$ *is non-decreasing in weighted mean, then we have*

$$f\left(\sum_{k=1}^{n} p_k a_k\right) \le f(p_1 a_1) + \sum_{k=2}^{n} \left[f(P_k a_k) - f(P_{k-1} a_k)\right]. \tag{5.2.2}$$

If the function f *is Wright-concave, then opposite inequalities hold in* (5.2.1) *and* (5.2.2).

Proof. (i) Since the sequence $(a_k,\, k = 1, \ldots, n) \subset \mathbb{R}$ is non-increasing in weighted mean, by definition we have, $\sum_{i=1}^{k-1} p_i a_i \ge P_{k-1} a_k$ for $k = 2, \ldots, n$. As f is a Wright-convex function, by setting $x = P_{k-1} a_k$, $y = \sum_{i=1}^{k-1} p_i a_i$ and $h = p_k a_k$ $(k = 2, \ldots, n)$ in (1.1.6), we have

$$f\left(\sum_{i=1}^{k} p_i a_i\right) - f\left(\sum_{i=1}^{k-1} p_i a_i\right) \ge f(P_k a_k) - f(P_{k-1} a_k).$$

Summing over k from 2 to n, we have

$$f\left(\sum_{i=1}^{n} p_i a_i\right) - f(p_1 a_1) \ge \sum_{k=2}^{n} \left[f(P_k a_k) - f(P_{k-1} a_k)\right],$$

and so (5.2.1) holds.

(ii) Since the sequence $(a_k,\; k = 1, \ldots, n) \subset \mathbb{R}$ is non-decreasing in weighted mean, by definition we have, $\sum_{i=1}^{k-1} p_i a_i \le P_{k-1} a_k$ for $k = 2, \ldots, n$. As f is a Wright-convex function, by setting $x = \sum_{i=1}^{k-1} p_i a_i$, $y = P_{k-1} a_k$ and $h = p_k a_k$ $(k = 2, \ldots, n)$ in (1.1.6), we have

$$f\left(\sum_{i=1}^{k} p_i a_i\right) - f\left(\sum_{i=1}^{k-1} p_i a_i\right) \le f(P_k a_k) - f(P_{k-1} a_k).$$

Now summing over k from 2 to n and after simplification, we have (5.2.2).

If f is a Wright-concave function, then opposite inequality holds in (1.1.6) and so opposite inequalities hold in (5.2.1) and (5.2.2). □

Remark 5.2.1. For a Wright-concave function f, Theorems 5.1.6 and 5.2.1(i) are equivalent. First of all, from the proof of Theorem 5.1.6 (see [59, p. 6]) it is obvious that the interval $[0, +\infty)$ can be replaced by $[a, b]$. Furthermore, with the substitutions $a_k = C_k$ and $p_k = x_k - x_{k-1}$ $(k = 1, \ldots, n)$ condition (5.1.10) is equivalent to the condition that the sequence $(a_k,\; k = 1, \ldots, n)$ is non-increasing in weighted mean and inequality (5.1.11) is equivalent to the reverse of (5.2.1).

Since the class of Wright-convex (Wright-concave) functions properly contains the class of convex (concave) functions (see for example [58, p. 7]), the following result is valid.

Corollary 5.2.2. *Let a_k and p_k $(k = 1, \dots, n)$ be real numbers such that $a_k \geq 0$ and $p_k > 0$ with $P_k = \sum_{i=1}^{k} p_i$ $(k = 1, \dots, n)$. Let $p_1 a_1$, $\sum_{k=1}^{n} p_k a_k$, $P_k a_k$, $P_{k-1} a_k \in [a, b]$ for all $k = 2, \dots, n$ and let $f : [a, b] \to \mathbb{R}$ be a convex function.*

(i) If the sequence $(a_k,\ k = 1, \dots, n)$ is non-increasing in weighted mean, then (5.2.1) *holds.*

(ii) If the sequence $(a_k,\ k = 1, \dots, n)$ is non-decreasing in weighted mean, then (5.2.2) *holds.*

If the function f is concave, then opposite inequalities hold in (5.2.1) *and* (5.2.2).

The following corollary is an application of Corollary 5.2.2.

Corollary 5.2.3. *Let $f(x) = x^s$, where $x \in (0, \infty)$ and $s \in \mathbb{R}$.*

(i) If the sequence $(a_k > 0,\ k = 1, \dots, n)$ is non-increasing in weighted mean, $p_k > 0\,(k = 1, \dots, n)$ and $s \in \mathbb{R}$ such that $s < 0$ or $s > 1$, then

$$\left(\sum_{k=1}^{n} p_k a_k\right)^s \geq (p_1 a_1)^s + \sum_{k=2}^{n} a_k^s \left[P_k^s - P_{k-1}^s\right] \tag{5.2.3}$$

holds. If $0 < s < 1$, then (5.2.3) *holds in the reverse direction.*

(ii) If the sequence $(a_k > 0,\ k = 1, \dots, n)$ is non-decreasing in weighted mean, $p_k > 0\,(k = 1, \dots, n)$ and $s \in \mathbb{R}$ such that $s < 0$ or $s > 1$, then

$$\left(\sum_{k=1}^{n} p_k a_k\right)^s \leq (p_1 a_1)^s + \sum_{k=2}^{n} a_k^s \left[P_k^s - P_{k-1}^s\right] \tag{5.2.4}$$

holds. If $0 < s < 1$, then (5.2.4) *holds in the reverse direction.*

Remark 5.2.2. Theorem 5.2.1 is the generalization of the inequality (5.1.3) in the sense that we can obtain (5.1.3) as a special case of our first main result.

The following result is related to the discrete weighted reversed Hardy-type inequality (see [60]).

Theorem 5.2.4. *Let a_k, p_k $(k = 1, \dots, n)$ be real numbers such that $a_k \geq 0$ and $p_k > 0$ with $P_k = \sum_{i=1}^{k} p_i$ $(k = 1, \dots, n)$ and $b_n \geq 0$ $(n = 2, \dots, m)$. Let $p_1 a_1$, $\sum_{k=1}^{n} p_k a_k$, $P_k a_k$, $P_{k-1} a_k \in [a, b]$ for all $k = 2, \dots, n$ and let $f : [a, b] \to \mathbb{R}$ be a differentiable convex function.*

(*i*) *If the sequence* $(a_k,\ k=1,\ldots,n)$ *is non-increasing in weighted mean, then we have*

$$\sum_{n=2}^{m} b_n f\left(\sum_{k=1}^{n} p_k a_k\right) \geq f\left(p_1 a_1\right) \sum_{n=2}^{m} b_n + \sum_{k=2}^{m} f^{'}\left(P_{k-1} a_k\right) p_k a_k \sum_{n=k}^{m} b_n. \tag{5.2.5}$$

(*ii*) *If the sequence* $(a_k,\ k=1,\ldots,n)$ *is non-decreasing in weighted mean, then we have*

$$\sum_{n=2}^{m} b_n f\left(\sum_{k=1}^{n} p_k a_k\right) \leq f\left(p_1 a_1\right) \sum_{n=2}^{m} b_n + \sum_{k=2}^{m} f^{'}\left(P_{k-1} a_k\right) p_k a_k \sum_{n=k}^{m} b_n. \tag{5.2.6}$$

If the function f *is concave, then opposite inequalities hold in* (5.2.5) *and* (5.2.6).

Proof. (i) Using the same steps as in the proof of Theorem 5.2.1 (*i*), we get (5.2.1). Multiplying (5.2.1) by $b_n \geq 0$ $(n=2,\ldots,m)$ and summing over n from 2 to m, we get

$$\sum_{n=2}^{m} b_n f\left(\sum_{k=1}^{n} p_k a_k\right) \geq f\left(p_1 a_1\right) \sum_{n=2}^{m} b_n + \sum_{k=2}^{m} \left[f\left(P_k a_k\right) - f\left(P_{k-1} a_k\right)\right] \sum_{n=k}^{m} b_n.$$

Due to the convexity of f, we have

$$f\left(P_k a_k\right) - f\left(P_{k-1} a_k\right) \geq f'\left(P_{k-1} a_k\right)\left(P_k a_k - P_{k-1} a_k\right) = f'\left(P_{k-1} a_k\right) p_k a_k,$$

so (5.2.5) follows.

(ii) The inequality (5.2.2) can be obtained by using the same arguments as in the proof of Theorem 5.2.1 (*ii*). Now multiplying (5.2.2) by $b_n \geq 0$, summing over n from 2 to m and also using the convexity of f, we get (5.2.6).

If f is a concave function, then opposite inequality holds in (5.2.1) and (5.2.2) and so opposite inequalities hold in (5.2.5) and (5.2.6). □

Example 5.2.5. *Let* $f(x) = x^s$, *where* $x \in (0,\infty)$ *and* $s \in \mathbb{R}$. *If the sequence* $(a_k > 0,\ k=1,\ldots,n)$ *is non-increasing in weighted mean,* $p_k > 0\,(k=1,\ldots,n)$ *and* $s \in \mathbb{R}$ *such that* $s<0$ *or* $s>1$, *then*

$$\sum_{n=2}^{m} b_n \left(\sum_{k=1}^{n} p_k a_k\right)^{s} \geq \left(p_1 a_1\right)^{s} \sum_{n=2}^{m} b_n + s \sum_{k=2}^{m} p_k a_k^s P_{k-1}^{s-1} \sum_{n=k}^{m} b_n \tag{5.2.7}$$

holds. If $0<s<1$, *then* (5.2.7) *holds in the reverse direction.*

Now, let $s > 1$ and take $b_n = p_n P_{n-1}^{-s}$. Using the fact that

$$\sum_{n=k}^{m} p_n P_{n-1}^{-s} \geq \int_{P_{k-1}}^{P_m} x^{-s} dx = \frac{P_{k-1}^{1-s} - P_m^{1-s}}{s-1},$$

from (5.2.7), *we get*

$$\sum_{n=2}^{m} p_n \left(\frac{1}{P_{n-1}} \sum_{k=1}^{n} p_k a_k \right)^s \geq \frac{p_1 a_1^s}{s-1} \left(1 - \left(\frac{p_1}{P_m} \right)^{s-1} \right) + \frac{s}{s-1} \sum_{k=2}^{m} p_k a_k^s \left(1 - \left(\frac{P_{k-1}}{P_m} \right)^{s-1} \right). \tag{5.2.8}$$

By adding $p_1 a_1^s$ to both sides of (5.2.8) *and with the convention $P_0 = p_1$, if $m \to \infty$ and $P_m \to \infty$ inequality* (5.2.8) *becomes*

$$\sum_{n=1}^{\infty} p_n \left(\frac{P_n}{P_{n-1}} \right)^s \left(\frac{1}{P_n} \sum_{k=1}^{n} p_k a_k \right)^s \geq \frac{s}{s-1} \sum_{k=1}^{\infty} p_k a_k^s$$

and represents a discrete weighted reversed Hardy-type inequality.

The multidimensional extension is stated as follows:

Theorem 5.2.6. *Let $\mathbf{a}_k$, $\mathbf{p}_k$ and $\mathbf{P}_k = \left(\sum_{i=1}^{k} p_i^1, \dots, \sum_{i=1}^{k} p_i^m \right) \in \mathbb{R}^m$ be such that $\mathbf{a}_k \geq 0$ and $\mathbf{p}_k > 0$ for each $k = 1, \dots, n$. Let $\mathbf{p}_1 \mathbf{a}_1$, $\sum_{k=1}^{n} \mathbf{p}_k \mathbf{a}_k$, $\mathbf{P}_k \mathbf{a}_k$, $\mathbf{P}_{k-1} \mathbf{a}_k \in \mathbf{I}$ for all $k = 2, \dots, n$ and let $f : \mathbf{I} \to \mathbb{R}$ be a real valued function having non-decreasing increments on the rectangle $\mathbf{I} \subseteq \mathbb{R}^m$.*

(i) If the sequence $(\mathbf{a}_k,\ k = 1, \dots, n)$ is non-increasing in weighted mean, then we have

$$f\left(\sum_{k=1}^{n} \mathbf{p}_k \mathbf{a}_k \right) \geq f(\mathbf{p}_1 \mathbf{a}_1) + \sum_{k=2}^{n} \left[f(\mathbf{P}_k \mathbf{a}_k) - f(\mathbf{P}_{k-1} \mathbf{a}_k) \right]. \tag{5.2.9}$$

(ii) If the sequence $(\mathbf{a}_k,\ k = 1, \dots, n)$ is non-decreasing in weighted mean, then we have

$$f\left(\sum_{k=1}^{n} \mathbf{p}_k \mathbf{a}_k \right) \leq f(\mathbf{p}_1 \mathbf{a}_1) + \sum_{k=2}^{n} \left[f(\mathbf{P}_k \mathbf{a}_k) - f(\mathbf{P}_{k-1} \mathbf{a}_k) \right]. \tag{5.2.10}$$

If the function f has non-increasing increments, then opposite inequalities hold in (5.2.9) *and* (5.2.10).

Proof. The idea of the proof is the same as that of Theorem 5.2.1.

(i) Since the sequence $(\mathbf{a}_k,\ k=1,\ldots,n)\subset\mathbb{R}^m$ is non-increasing in weighted mean, by definition we have, $\sum_{i=1}^{k-1}\mathbf{p}_i\mathbf{a}_i\geq\mathbf{P}_{k-1}\mathbf{a}_k$ for $k=2,\ldots,n$. By setting $\mathbf{x}=\mathbf{P}_{k-1}\mathbf{a}_k$, $\mathbf{y}=\sum_{i=1}^{k-1}\mathbf{p}_i\mathbf{a}_i$ and $\mathbf{h}=\mathbf{p}_k\mathbf{a}_k$ $(k=2,\ldots,n)$ in (5.1.9), where f has non-decreasing increments, we have

$$f\left(\sum_{i=1}^{k}\mathbf{p}_i\mathbf{a}_i\right)-f\left(\sum_{i=1}^{k-1}\mathbf{p}_i\mathbf{a}_i\right)\geq f\left(\mathbf{P}_k\mathbf{a}_k\right)-f\left(\mathbf{P}_{k-1}\mathbf{a}_k\right).$$

Summing over k from 2 to n, we have

$$f\left(\sum_{k=1}^{n}\mathbf{p}_k\mathbf{a}_k\right)-f\left(\mathbf{p}_1\mathbf{a}_1\right)\geq\sum_{k=2}^{n}\left[f\left(\mathbf{P}_k\mathbf{a}_k\right)-f\left(\mathbf{P}_{k-1}\mathbf{a}_k\right)\right].$$

and so (5.2.9) holds.

(ii) Since the sequence $(\mathbf{a}_k,\ k=1,\ldots,n)\subset\mathbb{R}^m$ is non-decreasing in weighted mean, by definition we have, $\sum_{i=1}^{k-1}\mathbf{p}_i\mathbf{a}_i\leq\mathbf{P}_{k-1}\mathbf{a}_k$ for $k=2,\ldots,n$. By setting $\mathbf{x}=\sum_{i=1}^{k-1}\mathbf{p}_i\mathbf{a}_i$, $\mathbf{y}=\mathbf{P}_{k-1}\mathbf{a}_k$ and $\mathbf{h}=\mathbf{p}_k\mathbf{a}_k$ $(k=2,\ldots,n)$ in (5.1.9), where f has non-decreasing increments, we have

$$f\left(\sum_{i=1}^{k}\mathbf{p}_i\mathbf{a}_i\right)-f\left(\sum_{i=1}^{k-1}\mathbf{p}_i\mathbf{a}_i\right)\leq f\left(\mathbf{P}_k\mathbf{a}_k\right)-f\left(\mathbf{P}_{k-1}\mathbf{a}_k\right).$$

Now summing over k from 2 to n and after simplification, we have (5.2.10).

If f has non-increasing increments, then opposite inequality holds in (5.1.9) and so opposite inequalities hold in (5.2.9) and (5.2.10). □

The following theorem is proven in the same way as Theorem 5.2.6 and represents the m-dimensional generalization of Theorem 5.1.6.

Theorem 5.2.7. *Let $f:\mathbf{I}\to\mathbb{R}$ be a real valued function having non-decreasing increments on a rectangle $\mathbf{I}\subseteq\mathbb{R}^m$ and $\mathbf{C}_k,\ \mathbf{x}_k\in\mathbb{R}^m$ be such that $0=\mathbf{x}_0\leq\mathbf{x}_1\leq\mathbf{x}_2\leq\cdots\leq\mathbf{x}_n$ and $\mathbf{C}_k\geq 0$ for $k=1,\ldots,n$. Furthermore, let $\mathbf{C}_1\mathbf{x}_1$, $\sum_{k=1}^{n}\mathbf{C}_k\left(\mathbf{x}_k-\mathbf{x}_{k-1}\right)$, $\mathbf{C}_k\mathbf{x}_k$ and $\mathbf{C}_{k-1}\mathbf{x}_k\in\mathbf{I}$ for all $k=2,\ldots,n$.*

(*i*) *If the inequalities*

$$\sum_{i=1}^{k-1}\mathbf{C}_i\left(\mathbf{x}_i-\mathbf{x}_{i-1}\right)\geq\mathbf{C}_k\mathbf{x}_{k-1},\qquad \text{for } k\geq 1 \tag{5.2.11}$$

hold, then

$$f\left(\sum_{k=1}^{n}\mathbf{C}_k\left(\mathbf{x}_k-\mathbf{x}_{k-1}\right)\right)+\sum_{k=1}^{n-1}f\left(\mathbf{C}_{k+1}\mathbf{x}_k\right)\geq\sum_{k=1}^{n}f\left(\mathbf{C}_k\mathbf{x}_k\right). \tag{5.2.12}$$

(*ii*) *If the inequalities in* (5.2.11) *are reversed, then*

$$f\left(\sum_{k=1}^{n}\mathbf{C}_k\left(\mathbf{x}_k-\mathbf{x}_{k-1}\right)\right)+\sum_{k=1}^{n-1}f\left(\mathbf{C}_{k+1}\mathbf{x}_k\right)\leq\sum_{k=1}^{n}f\left(\mathbf{C}_k\mathbf{x}_k\right), \tag{5.2.13}$$

holds. If the function f has non-increasing increments, then opposite inequalities hold in (5.2.12) *and* (5.2.13).

Remark 5.2.3. If we take $p_k^1 = p_k^2 = p_k^3 = \ldots = p_k^m = p_k$ $(k = 1, \ldots, n)$ in $\mathbf{p}_k = (p_k^1, \ldots, p_k^m)$ in Theorem 5.2.6, then we have the following result.

Corollary 5.2.8. *Let $\mathbf{a}_k \in [0,\infty)^m$ and p_k $(k = 1,\ldots,n)$ be real numbers such that $p_k > 0$ with $P_k = \sum_{i=1}^{k} p_i$ $(k = 1,\ldots,n)$. Let $p_1\mathbf{a}_1$, $\sum_{k=1}^{n} p_k\mathbf{a}_k$, $P_k\mathbf{a}_k$, $P_{k-1}\mathbf{a}_k \in \mathbf{I}$ for all $k = 2,\ldots,n$ and let $f : \mathbf{I} \to \mathbb{R}$ be a real valued function defined on a rectangle $\mathbf{I} \subseteq \mathbb{R}^m$ having non-decreasing increments.*

(*i*) *If the sequence $(\mathbf{a}_k,\ k = 1,\ldots,n)$ is non-increasing in weighted mean, then we have*

$$f\left(\sum_{k=1}^{n}p_k\mathbf{a}_k\right)\geq f\left(p_1\mathbf{a}_1\right)+\sum_{k=2}^{n}\left[f\left(P_k\mathbf{a}_k\right)-f\left(P_{k-1}\mathbf{a}_k\right)\right]. \tag{5.2.14}$$

(*ii*) *If the sequence $(\mathbf{a}_k,\ k = 1,\ldots,n)$ is non-decreasing in weighted mean, then we have*

$$f\left(\sum_{k=1}^{n}p_k\mathbf{a}_k\right)\leq f\left(p_1\mathbf{a}_1\right)+\sum_{k=2}^{n}\left[f\left(P_k\mathbf{a}_k\right)-f\left(P_{k-1}\mathbf{a}_k\right)\right]. \tag{5.2.15}$$

If the function f has non-increasing increments, then opposite inequalities hold in (5.2.14) *and* (5.2.15).

Remark 5.2.4. If we make the substitution $p_k \to 1$ $(k = 1,\ldots,n)$ in Theorem 5.2.1 and in Corolleries 5.2.2, 5.2.3 and 5.2.8 , then the results given in [36] are recaptured.

Let us recall from Chapters 1 and 2 that $K[a,b]$ and $\tilde{\mathrm{K}}[a,b]$ denote the class of all convex functions and the class of all differentiable convex functions defined on $[a,b]$, respectively. Consider the inequalities (5.2.1) and (5.2.5) and define two linear functionals $\Phi_1 : K[a,b] \to \mathbb{R}$ and $\Phi_2 : \tilde{\mathrm{K}}[a,b] \to \mathbb{R}$ as follows:

$$\Phi_1(f) = f\left(\sum_{k=1}^{n}p_ka_k\right)-f\left(p_1a_1\right)-\sum_{k=2}^{n}\left[f\left(P_ka_k\right)-f\left(P_{k-1}a_k\right)\right], \tag{5.2.16}$$

$$\Phi_2(f) = \sum_{n=2}^{m}b_nf\left(\sum_{k=1}^{n}p_ka_k\right)-f\left(p_1a_1\right)\sum_{n=2}^{m}b_n-\sum_{k=2}^{m}f^{'}\left(P_{k-1}a_k\right)p_ka_k\sum_{n=k}^{m}b_n, \tag{5.2.17}$$

where $a_k \geq 0$, $p_k > 0$ with $P_k = \sum_{i=1}^{k} p_i$ for $k = 1, \ldots, n$, $p_1 a_1$, $\sum_{k=1}^{n} p_k a_k$, $P_k a_k$, $P_{k-1} a_k \in [a, b]$ for all $k = 2, \ldots, n$ and $b_n \geq 0$ $(n = 2, \ldots, m)$. If the function f is convex defined on $[a, b]$ and the sequence $(a_k,\ k = 1, \ldots, n) \subset \mathbb{R}$ is non-increasing in weighted mean, then Corollary 5.2.2 (i), imply that $\Phi_1(f) \geq 0$, and if in addition f is differentiable then Theorem 5.2.4 (i), imply that $\Phi_2(f) \geq 0$.

5.2.1 Mean Value Theorems

Now, we give mean value theorems for the functionals Φ_i $(i = 1, 2)$.

Theorem 5.2.9. *Let a_k, p_k $(k = 1, \ldots, n)$ be real numbers such that $a_k \geq 0$ and $p_k > 0$ with $P_k = \sum_{i=1}^{k} p_i$ $(k = 2, \ldots, n)$ and $b_n \geq 0$ $(n = 2, \ldots, m)$. Let $p_1 a_1$, $\sum_{k=1}^{n} p_k a_k$, $P_k a_k$, $P_{k-1} a_k \in [a, b]$ for all $k = 2, \ldots, n$ and let the sequence $(a_k,\ k = 1, \ldots, n)$ be non-increasing in weighted mean. Suppose that Φ_i $(i = 1, 2)$ are linear functionals as defined in* (5.2.16) *and* (5.2.17) *and $f \in C^2([a, b])$. Then there exists $\xi_i \in [a, b]$ such that*

$$\Phi_i(f) = \frac{f''(\xi_i)}{2} \Phi_i(f_0), \qquad i = 1, 2,$$

where $f_0(x) = x^2$.

Proof. The proof is analogous to the proof of Theorem 2.4.2. □

The following theorem is the classical Cauchy mean value theorem related to the functionals Φ_i $(i = 1, 2)$ and it can be proven by following the proof of Theorem 2.4.3.

Theorem 5.2.10. *Let all the assumptions of Theorem* 5.2.9 *hold and let $f, h \in C^2([a, b])$. Then there exist $\xi_i \in [a, b]$ such that*

$$\frac{\Phi_i(f)}{\Phi_i(h)} = \frac{f''(\xi_i)}{h''(\xi_i)}, \qquad i = 1, 2, \tag{5.2.18}$$

provided that the denominators are non-zero.

Remark 5.2.5.
(i) By taking $f(x) = x^s$ and $h(x) = x^q$ in (5.2.18), where $s, q \in \mathbb{R} \setminus \{0, 1\}$ are such that $s \neq q$, we have

$$\xi_i^{s-q} = \frac{q(q-1)\Phi_i(x^s)}{s(s-1)\Phi_i(x^q)}, \qquad i = 1, 2.$$

(ii) If the inverse of the function f''/h'' exists, then from (5.2.18), we have

$$\xi_i = \left(\frac{f''}{h''}\right)^{-1} \left(\frac{\Phi_i(f)}{\Phi_i(h)}\right), \qquad i = 1, 2.$$

5.2.2 n-Exponential Convexity and Log-Convexity of the Functions Related to the Differences of the Generalized Inequalities

Next, we study the n-exponential convexity and log-convexity of the functions associated with the linear functionals Φ_i $(i = 1, 2)$ defined in (5.2.16) and (5.2.17) respectively.

Theorem 5.2.11. *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a, b]$ such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense on I for every three mutually distinct points $y_0, y_1, y_2 \in [a, b]$. Let Φ_i $(i = 1, 2)$ be linear functionals as defined in* (5.2.16) *and* (5.2.17) *respectively. Then the following statements hold:*

(i) The function $s \mapsto \Phi_i(f_s)$ is n-exponentially convex in the Jensen sense on I and the matrix $\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m}$ is positive semi-definite for all $m \in \mathbb{N}$, $m \le n$ and $s_1, \ldots, s_m \in I$. Particularly,

$$\det\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m} \ge 0, \quad \forall\ m \in \mathbb{N},\ m \le n.$$

(ii) If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is n-exponentially convex on I.

Proof. The proof is analogous to the proof of Theorem 2.5.1. □

The following corollary is an immediate consequence of the above theorem.

Corollary 5.2.12. *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a, b]$ such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is exponentially convex in the Jensen sense on I for every three mutually distinct points $y_0, y_1, y_2 \in [a, b]$. Let Φ_i $(i = 1, 2)$ be linear functionals as defined in* (5.2.16) *and* (5.2.17) *respectively. Then the following statements hold:*

(i) The function $s \mapsto \Phi_i(f_s)$ is exponentially convex in the Jensen sense on I and the matrix $\left[\Phi_i(f_{\frac{s_j+s_k}{2}})\right]_{j,k=1}^{n}$ is positive semi-definite for all $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in I$. Particularly,

$$\det\left[\Phi_i(f_{\frac{s_j+s_k}{2}})\right]_{j,k=1}^{n} \ge 0, \quad \forall\ n \in \mathbb{N}.$$

(ii) If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is exponentially convex on I.

Corollary 5.2.13. *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a, b]$ such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is 2-exponentially convex in the Jensen sense on I for every three mutually distinct points $y_0, y_1, y_2 \in [a, b]$. Let $\Phi_i\,(i = 1, 2)$ be linear functionals as defined in* (5.2.16) *and* (5.2.17) *respectively. Assume that $\Phi_i\,(f_s)\,(i = 1, 2)$ are strictly positive for $f_s \in \Omega$. Then the following statements hold:*

(*i*) *If the function $s \mapsto \Phi_i\,(f_s)$ is continuous on I, then it is 2-exponentially convex on I and so, it is log-convex on I and for $r, s, t \in I$ such that $r < s < t$, we have*

$$[\Phi_i\,(f_s)]^{t-r} \leq [\Phi_i\,(f_r)]^{t-s}\,[\Phi_i\,(f_t)]^{s-r}\,, \qquad i = 1, 2, \tag{5.2.19}$$

known as Lyapunov's inequality. If $r < t < s$ or $s < r < t$, then opposite inequality holds in (5.2.19).

(*ii*) *If the function $s \mapsto \Phi_i\,(f_s)$ is differentiable on I, then for every $s, q, u, v \in I$ such that $s \leq u$ and $q \leq v$, we have*

$$\mu_{s,q}\,(\Phi_i, \Omega) \leq \mu_{u,v}\,(\Phi_i, \Omega)\,, \qquad i = 1, 2, \tag{5.2.20}$$

where $\mu_{s,q}\,(\Phi_i, \Omega)$ is the same as defined in (2.5.3).

Proof. The proof is analogous to the proof of Corollary 2.5.3. □

Remark 5.2.6. If in Remark 2.5.1 we replace Theorem 2.5.1, Corollary 2.5.2 and Corollary 2.5.3 by Theorem 5.2.11, Corollary 5.2.12 and Corollary 5.2.13 respectively, then the remark is still valid.

5.2.3 Examples

In this section, we present several families of functions which fulfil the conditions of Theorem 5.2.11, Corollaries 5.2.12 and 5.2.13 and Remark 5.2.6. This enable us to construct large families of functions which are exponentially convex.

Example 5.2.14. *Consider the family of functions*

$$\Omega_1 = \{g_s : \mathbb{R} \to [0,\ \infty) : s \in \mathbb{R}\}$$

defined by

$$g_s\,(x) = \begin{cases} \frac{1}{s^2}\,e^{sx}, & s \neq 0, \\ \frac{1}{2}\,x^2, & s = 0. \end{cases}$$

For this family of functions, by taking $\Omega = \Omega_1$ in (2.5.3), *$\mu_{s,q}\,(\Phi_i, \Omega_1)\,(i = 1, 2)$ are of the form* (2.6.2). *By using Theorem* 5.2.10, *it can be seen that*

$$M_{s,q}\,(\Phi_i, \Omega_1) = \log \mu_{s,q}\,(\Phi_i, \Omega_1)\,, \quad i = 1, 2,$$

satisfy $a \leq M_{s,q}\,(\Phi_i, \Omega_1) \leq b$, which shows that $M_{s,q}\,(\Phi_i, \Omega_1)$ is a family of mean.

Example 5.2.15. *Consider the family of functions*

$$\Omega_2 = \{f_s : (0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0,1, \\ -\ln x, & s = 0, \\ x \ln x, & s = 1. \end{cases}$$

If $r, s, t \in \mathbb{R}$ *are such that* $r < s < t$*, then from* (5.2.19)*, we have*

$$\Phi_i(f_s) \leq [\Phi_i(f_r)]^{\frac{t-s}{t-r}} [\Phi_i(f_t)]^{\frac{s-r}{t-r}}, \qquad i = 1,2. \tag{5.2.21}$$

If $r < t < s$ *or* $s < r < t$*, then opposite inequality holds in* (5.2.21)*.*

Particularly, for $i = 1$ *and* $r, s, t \in \mathbb{R} \setminus \{0,1\}$ *such that* $r < s < t$*, we have*

$$\frac{\left(\sum_{k=1}^n p_k a_k\right)^s - (p_1 a_1)^s - \sum_{k=2}^n a_k^s \left(P_k^s - P_{k-1}^s\right)}{s(s-1)} \geq$$
$$\left[\frac{\left(\sum_{k=1}^n p_k a_k\right)^r - (p_1 a_1)^r - \sum_{k=2}^n a_k^r \left(P_k^r - P_{k-1}^r\right)}{r(r-1)}\right]^{\frac{t-s}{t-r}} \times$$
$$\left[\frac{\left(\sum_{k=1}^n p_k a_k\right)^t - (p_1 a_1)^t - \sum_{k=2}^n a_k^t \left(P_k^t - P_{k-1}^t\right)}{t(t-1)}\right]^{\frac{s-r}{t-r}}, \tag{5.2.22}$$

where $a_k > 0$*,* $p_k > 0$ $(k = 1, \ldots, n)$ *are such that* $p_1 a_1$*,* $\sum_{k=1}^n p_k a_k$*,* $P_k a_k$*,* $P_{k-1} a_k$ $\in [a,b]$ *for all* $k = 2, \ldots, n$*. In fact for* $s > 1$*,* (5.2.22) *is the refinement of the inequality* (5.1.3) *and for* $0 < s < 1$*,* (5.2.22) *holds in the reverse direction.*

By taking $\Omega = \Omega_2$ *in* (2.5.3)*,* $\Xi_{s,q}^i := \mu_{s,q}(\Phi_i, \Omega_2)$ $(i = 1,2)$ *are of the form*

$$\Xi_{s,q}^1 = \left(\frac{q(q-1)}{s(s-1)} . \frac{\left(\sum_{k=1}^n p_k a_k\right)^s - (p_1 a_1)^s - \sum_{k=2}^n a_k^s \left(P_k^s - P_{k-1}^s\right)}{\left(\sum_{k=1}^n p_k a_k\right)^q - (p_1 a_1)^q - \sum_{k=2}^n a_k^q \left(P_k^q - P_{k-1}^q\right)}\right)^{\frac{1}{s-q}},$$

$$s \neq q \neq 0,1,$$

$$\Xi_{s,0}^1 = \left(\frac{1}{s(s-1)} . \frac{\left(\sum_{k=1}^n p_k a_k\right)^s - (p_1 a_1)^s - \sum_{k=2}^n a_k^s \left(P_k^s - P_{k-1}^s\right)}{\ln(P_n a_1) - \ln\left(\sum_{k=1}^n p_k a_k\right)}\right)^{\frac{1}{s}}, \quad s \neq 0,1,$$

$$\begin{aligned}
\Xi^1_{s,1} &= \left(\frac{1}{s\left(s-1\right)}\right)^{\frac{1}{s-1}} \times \\
&\left(\frac{\left(\sum_{k=1}^n p_k a_k\right)^s - \left(p_1 a_1\right)^s - \sum_{k=2}^n a_k^s \ \left(P_k^s - P_{k-1}^s\right)}{\sum_{k=1}^n a_k \left(p_k \ln\left(\sum_{k=1}^n p_k a_k\right) - P_k \ln\left(P_k a_k\right)\right) + \sum_{k=2}^n a_k P_{k-1} \ln\left(P_{k-1} a_k\right)}\right)^{\frac{1}{s-1}}, \\
& s \neq 0, 1, \\
\Xi^1_{0,1} &= \frac{\sum_{k=1}^n a_k \left(p_k \ln\left(\sum_{k=1}^n p_k a_k\right) - P_k \ln\left(P_k a_k\right)\right) + \sum_{k=2}^n a_k P_{k-1} \ln\left(P_{k-1} a_k\right)}{\ln\left(P_n a_1\right) - \ln\left(\sum_{k=1}^n p_k a_k\right)}, \\
\Xi^1_{s,s} &= \exp\left(\frac{1-2s}{s(s-1)}\right) \times \\
&\exp\left(\frac{\left(\sum_{k=1}^n p_k a_k\right)^s \ln\left(\sum_{k=1}^n p_k a_k\right) - \sum_{k=1}^n a_k^s P_k^s \ln\left(P_k a_k\right)}{\left(\sum_{k=1}^n p_k a_k\right)^s - \left(p_1 a_1\right)^s - \sum_{k=2}^n a_k^s \left(P_k^s - P_{k-1}^s\right)}\right) \times \\
&\exp\left(\frac{\sum_{k=2}^n a_k^s P_{k-1}^s \ln\left(P_{k-1} a_k\right)}{\left(\sum_{k=1}^n p_k a_k\right)^s - \left(p_1 a_1\right)^s - \sum_{k=2}^n a_k^s \left(P_k^s - P_{k-1}^s\right)}\right), \quad s \neq 0, 1, \\
\Xi^1_{0,0} &= \exp\left(\frac{\left(\ln\left(\sum_{k=1}^n p_k a_k\right) - \ln\left(p_1 a_1\right)\right) \ln\left(e^2 p_1 a_1 \sum_{k=1}^n p_k a_k\right)}{2\left(\ln\left(\sum_{k=1}^n p_k a_k\right) - \ln(P_n a_1)\right)}\right) \times \\
&\exp\left(\frac{-\sum_{k=2}^n \left(\ln P_k - \ln P_{k-1}\right) \ln\left(e^2 P_k P_{k-1} a_k^2\right)}{2\left(\ln\left(\sum_{k=1}^n p_k a_k\right) - \ln\left(P_n a_1\right)\right)}\right), \\
\Xi^1_{1,1} &= \exp\left(\frac{\sum\limits_{k=1}^n a_k \left(p_k \ln\left(\sum\limits_{k=1}^n p_k a_k\right) \ln\left(e^{-2} \sum\limits_{k=1}^n p_k a_k\right) - P_k \ln\left(P_k a_k\right) \ln\left(e^{-2} P_k a_k\right)\right)}{2\left(\sum_{k=1}^n a_k \left(p_k \ln\left(\sum_{k=1}^n p_k a_k\right) - P_k \ln(P_k a_k)\right) + \sum_{k=2}^n a_k P_{k-1} \ln\left(P_{k-1} a_k\right)\right)}\right) \times \\
&\exp\left(\frac{\sum\limits_{k=2}^n a_k P_{k-1} \ln\left(P_{k-1} a_k\right) \ln\left(e^{-2} P_{k-1} a_k\right)}{2\left(\sum_{k=1}^n a_k \left(p_k \ln\left(\sum_{k=1}^n p_k a_k\right) - P_k \ln\left(P_k a_k\right)\right) + \sum_{k=2}^n a_k P_{k-1} \ln\left(P_{k-1} a_k\right)\right)}\right).
\end{aligned}$$

Similarly, we can obtain $\Xi^2_{s,q} =: \mu_{s,q}\left(\Phi_2, \Omega_2\right)$.

If $\Phi_i\left(i = 1, 2\right)$ *is positive, then Theorem 5.2.10 applied for* $f = f_s \in \Omega_2$ *and* $g = f_q \in \Omega_2$ *yields that there exists* $\xi_i \in [a, b]$ *such that*

$$\xi_i^{s-q} = \frac{\Phi_i\left(f_s\right)}{\Phi_i\left(f_q\right)}, \qquad i = 1, 2.$$

Since the function $\xi_i \mapsto \xi_i^{s-q}$ *is invertible for* $s \neq q$*, we have*

$$a \leq \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)} \right)^{\frac{1}{s-q}} \leq b, \qquad i = 1, 2,$$

which, together with the fact that $\mu_{s,q}(\Phi_i, \Omega_2)$ *is continuous, symmetric and monotonous (by* (5.2.20)*), shows that* $\mu_{s,q}(\Phi_i, \Omega_2)$ *is a mean.*

If $a = 0$ *and we consider functions defined on* $[0, \infty)$*, then we can obtain inequalities and means of the same form, but for parameters* s *and* q *restricted to* $(0, \infty)$*. More precisely, we consider the family of functions*

$$\tilde{\Omega}_2 = \{\tilde{f}_s : [0, \infty) \to \mathbb{R} : s \in (0, \infty)\}$$

defined by

$$\tilde{f}_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 1, \\ x \ln x, & s = 1, \end{cases}$$

with the convention that $0 \ln 0 = 0$*. For* $s > 0$ *and* $q > 0$*, by taking* $\Omega = \tilde{\Omega}_2$ *in* (2.5.3)*,* $\tilde{\Xi}^i_{s,q} =: \mu_{s,q}\left(\Phi_i, \tilde{\Omega}_2\right)$ $(i = 1, 2)$ *are of the same form as* $\Xi^i_{s,q}$*.*

Remark 5.2.7. If we make the substitution $p_k \to 1$ $(k = 1, \ldots, n)$ in our means , then the results for means given in [36] are recaptured.

Example 5.2.16. *Consider the family of functions*

$$\Omega_3 = \{h_s : (0, \infty) \to (0, \infty) : s \in (0, \infty)\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}}{\ln^2 s}, & s \neq 1, \\ \frac{x^2}{2}, & s = 1. \end{cases}$$

In this case, by taking $\Omega = \Omega_3$ *in* (2.5.3)*,* $\mu_{s,q}(\Phi_i, \Omega_3)$ $(i = 1, 2)$ *are of the form* (2.6.3)*. By using Theorem* 5.2.10*, it follows that*

$$M_{s,q}(\Phi_i, \Omega_3) = -L(s, q) \log \mu_{s,q}(\Phi_i, \Omega_3), \quad i = 1, 2,$$

satisfy $a \leq M_{s,q}(\Phi_i, \Omega_3) \leq b$ *and so* $M_{s,q}(\Phi_i, \Omega_3)$ *is a family of mean, where* $L(s, q)$ *is a logarithmic mean defined by*

$$L(s, q) = \begin{cases} \frac{s-q}{\log s - \log q}, & s \neq q, \\ s, & s = q. \end{cases}$$

Example 5.2.17. *Consider the family of functions*

$$\Omega_4 = \{k_s : (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}}{s}.$$

In this case, by taking $\Omega = \Omega_4$ *in* (2.5.3), $\mu_{s,q}(\Phi_i, \Omega_4)\,(i = 1,2)$ *are of the form* (2.6.4). *By using Theorem* 5.2.10, *it is easy to see that*

$$M_{s,q}(\Phi_i, \Omega_4) = -\left(\sqrt{s} + \sqrt{q}\right) \log \mu_{s,q}(\Phi_i, \Omega_4), \quad i = 1,2,$$

satisfy $a \leq M_{s,q}(\Phi_i, \Omega_4) \leq b$, *showing that* $M_{s,q}(\Phi_i, \Omega_4)\,(i = 1,2)$ *is a family of mean.*

In the next section, we present generalizations of Theorems 5.1.4 and 5.1.5. We define linear functionals as the non-negative differences of the generalized inequalities and give mean value theorems for the linear functionals. Further, we investigate the n-exponential convexity and log-convexity of the functions associated with the linear functionals and deduce Lyapunov-type inequalities for these functionals. We also prove the monotonicity property of the generalized Cauchy means obtained via these functionals. Finally, we give several examples of the families of functions for which the obtained results can be applied.

5.3 3-Convex Functions and Generalizations of an Inequality of Hardy-Littlewood-Pólya

Proposition 5.3.1. *A 3rd-order divided difference of a function* $f : [a,b] \to \mathbb{R}$ *at the points* $y_0, y_1, y_2, y_3 \in [a,b]$ *can be expressed in the following forms:*

(i) If $y_0, y_1, y_2, y_3 \in [a,b]$ *such that* $y_i \neq y_j$, $i \neq j$, $i,j = 0,\ldots,3$, *then we have*

$$[y_0, y_1, y_2, y_3; f] = \sum_{i=0}^{3} \frac{f(y_i)}{q'(y_i)}, \quad \textit{where } q(y) = \prod_{j=0}^{3} (y - y_j).$$

(ii) If f *is a differentiable function defined on* $[a,b]$ *and* $y, y_0, y_1 \in [a,b]$ *such that* $y \neq y_0 \neq y_1$, *then we have*

$$\begin{aligned}[y, y, y_0, y_1; f] &= \frac{f'(y)}{(y - y_0)(y - y_1)} + \frac{f(y)(y_0 + y_1 - 2y)}{(y - y_0)^2 (y - y_1)^2} + \frac{f(y_0)}{(y_0 - y)^2 (y_0 - y_1)} \\ &\quad + \frac{f(y_1)}{(y_1 - y)^2 (y_1 - y_0)}.\end{aligned}$$

(*iii*) *If f is a differentiable function defined on $[a,b]$ and $y, y_0 \in [a,b]$ such that $y \neq y_0$, then we have*

$$[y,y,y_0,y_0;f] = \frac{(y_0 - y)\left(f'(y_0) + f'(y)\right) + 2\left(f(y) - f(y_0)\right)}{(y_0 - y)^3}.$$

(*iv*) *If f is twice differentiable function defined on $[a,b]$ and $y, y_0 \in [a,b]$ such that $y \neq y_0$, then we have*

$$[y,y,y,y_0;f] = \frac{1}{(y_0 - y)^3}\left[f(y_0) - \sum_{i=0}^{2} \frac{f^{(i)}(y)}{i!}(y_0 - y)^i\right].$$

(*v*) *If f is three times differentiable function defined on $[a,b]$ and $y \in [a,b]$, then we have*

$$[y,y,y,y;f] = \frac{f'''(y)}{3!}.$$

The following theorem shows that the definition of 3-convex function can be extended by including the cases in which some or all the points coincide.

Theorem 5.3.2. *Let f be a function defined on $[a,b] \subset \mathbb{R}$, then the following statements hold.*

(*i*) *If $f \in C^1[a,b]$, then f is 3-convex if and only if $[y,y,y_0,y_1;f] \geq 0$ for all $y \neq y_0 \neq y_1$ in $[a,b]$.*

(*ii*) *If $f \in C^1[a,b]$, then f is 3-convex if and only if $[y,y,z,z;f] \geq 0$ for all $y \neq z$ in $[a,b]$.*

(*iii*) *If $f \in C^2[a,b]$, then f is 3-convex if and only if $[y,y,y,y_0;f] \geq 0$ for all $y \neq y_0$ in $[a,b]$.*

(*iv*) *If $f \in C^3[a,b]$, then f is 3-convex if and only if $[y,y,y,y;f] \geq 0$ for all $y \in [a,b]$.*

Proof. It can be proved easily by using the mean value theorems for divided differences (see [28]). □

In the proof of the our main results, the following result is needed.

Lemma 5.3.3. *A differentiable function of one variable is convex on an interval I if and only if the function lies above all of its tangents, that is,*

$$f(y) \geq f(x) + (y - x) f'(x). \tag{5.3.1}$$

By choosing the points in the reverse, we have

$$f(x) \geq f(y) + (x-y) f'(y). \tag{5.3.2}$$

On combining (5.3.1) *and* (5.3.2), *we have*

$$f(y) + (x-y) f'(y) \leq f(x) \leq f(y) + (x-y) f'(x). \tag{5.3.3}$$

In this section, first main result is the generalization of Theorems 5.1.4 and 5.1.5.

Theorem 5.3.4. *Let* $a_k > 0$ *and* $p_k > 0$ $(k = 1, \ldots, n)$ *be real numbers such that* $P_k = \sum_{i=1}^{k} p_i$ $(k = 1, \ldots, n)$. *Let* P_{k-1}, P_k, $\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k}$, $\frac{\sum_{i=1}^{k} p_i a_i}{a_k} \in [a,b]$ *for all* $k = 2, \ldots, n$ *and let* $g : [a,b] \to \mathbb{R}$ *be a differentiable function such that* $g(x+h) - g(x)$ *is convex for all* $x, x+h \in [a,b]$, *where* $h \geq 0$. *Then for any* $s \in \mathbb{R}$, *we have*

$$\begin{aligned}
&\sum_{k=2}^{n} a_k^s \left(g(P_k) - g(P_{k-1})\right) + \sum_{k=2}^{n} a_k^{s-1} \left(\sum_{i=1}^{k-1} p_i a_i - P_{k-1} a_k\right) \left(g'(P_k) - g'(P_{k-1})\right) \\
&\leq \sum_{k=2}^{n} a_k^s \left(g\left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k}\right) - g\left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k}\right)\right) \leq \sum_{k=2}^{n} a_k^s \left(g(P_k) - g(P_{k-1})\right) \\
&+ \sum_{k=2}^{n} a_k^{s-1} \left(\sum_{i=1}^{k-1} p_i a_i - P_{k-1} a_k\right) \left(g'\left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k}\right) - g'\left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k}\right)\right).
\end{aligned} \tag{5.3.4}$$

If $g(x+h) - g(x)$ *is concave for all* x, $x+h \in [a,b]$ *such that* $h \geq 0$, *then opposite inequalities hold in* (5.3.4).

Proof. Since $g(x+h) - g(x)$ is a convex function, where g is differentiable, by setting $f(x) = g(x+h) - g(x)$ in (5.3.3), we have

$$\begin{aligned}
&g(y+h) - g(y) + (x-y)\left(g'(y+h) - g'(y)\right) \leq g(x+h) - g(x) \\
&\leq g(y+h) - g(y) + (x-y)\left(g'(x+h) - g'(x)\right).
\end{aligned} \tag{5.3.5}$$

Substituting $x = \frac{\sum_{i=1}^{k-1} p_i a_i}{a_k}$, $y = P_{k-1}$ and $h = p_k$ $(k = 2, \ldots, n)$, where $a_k > 0$ $(k = 1, \ldots, n)$ in (5.3.5), we have

$$\begin{aligned}
&g(P_k) - g(P_{k-1}) + \left(\frac{\sum_{i=1}^{k-1} p_i a_i - P_{k-1} a_k}{a_k}\right) \left(g'(P_k) - g'(P_{k-1})\right) \\
&\leq g\left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k}\right) - g\left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k}\right) \leq g(P_k) - g(P_{k-1}) \\
&+ \left(\frac{\sum_{i=1}^{k-1} p_i a_i - P_{k-1} a_k}{a_k}\right) \left(g'\left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k}\right) - g'\left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k}\right)\right).
\end{aligned} \tag{5.3.6}$$

Multiplying (5.3.6) throughout by $a_k^s > 0$ $(k = 1, \ldots, n)$, where $s \in \mathbb{R}$ and summing over k from 2 to n, we have (5.3.4).

If $g(x+h) - g(x)$ is concave for all x, $x + h \in [a, b]$ such that $h \geq 0$, then we have opposite inequalities in (5.3.3) and in the same way we have opposite inequalities in (5.3.4). □

Corollary 5.3.5. *Let* $a_k > 0$ *and* $p_k > 0$ $(k = 1, \ldots, n)$ *be real numbers such that* $P_k = \sum_{i=1}^{k} p_i$ $(k = 1, \ldots, n)$. *Let* P_{k-1}, P_k, $\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k}$, $\frac{\sum_{i=1}^{k} p_i a_i}{a_k} \in [a, b]$ *for all* $k = 2, \ldots, n$ *and let* $g : [a, b] \to \mathbb{R}$ *be a differentiable function. If the function* g *is 3-convex, then for any* $s \in \mathbb{R}$ (5.3.4) *holds and if it is 3-concave, then* (5.3.4) *holds in the reverse direction.*

Proof. Since g is a 3-convex function, by using Theorem 1.1.4, g' exists and is convex on $[a, b]$. Therefore, for any $h \geq 0$ such that x, $x + h \in [a, b]$, $f'(x)$ exists, where $f(x) := g(x+h) - g(x)$ and we can write

$$\frac{f'(y) - f'(x)}{y - x} = \frac{g'(y+h) - g'(x+h)}{(y+h) - (x+h)} - \frac{g'(y) - g'(x)}{y - x}.$$

Now by using Theorem 1.1.2, we have $\frac{f'(y)-f'(x)}{y-x} \geq 0$, showing that f' is non-decreasing on $[a, b]$ and so f is convex. By taking $f(x) = g(x+h) - g(x)$ in (5.3.3) and by making the same substitutions as given in the proof of Theorem 5.3.4, we have (5.3.4).

Similarly, if g is a 3-concave function, then it is easy to prove that f is concave and so we then have opposite inequalities in (5.3.3) and in the same way we have opposite inequalities in (5.3.4). □

The second main theorem in this section, is again the generalization of Theorems 5.1.4 and 5.1.5.

Theorem 5.3.6. *Let* $a_k > 0$ *and* $p_k > 0$ $(k = 1, \ldots, n)$ *be real numbers such that* $P_k = \sum_{i=1}^{k} p_i$ $(k = 1, \ldots, n)$. *Let* $p_1 a_1$, $P_{k-1} a_k$, $P_k a_k$, $\sum_{i=1}^{k-1} p_i a_i$, $\sum_{i=1}^{k} p_i a_i \in [a, b]$ *for all* $k = 2, \ldots, n$ *and let* $g : [a, b] \to \mathbb{R}$ *be a differentiable function such that* $g(x+h) - g(x)$ *is convex for all* x, $x + h \in [a, b]$, *where* $h \geq 0$. *Then we have*

$$g(p_1 a_1) + \sum_{k=2}^{n} \left(g(P_k a_k) - g(P_{k-1} a_k)\right)$$
$$+ \sum_{k=2}^{n} \left(\sum_{i=1}^{k-1} p_i a_i - P_{k-1} a_k\right) \left(g'(P_k a_k) - g'(P_{k-1} a_k)\right) \leq g\left(\sum_{i=1}^{n} p_i a_i\right)$$

$$\leq g(p_1 a_1) + \sum_{k=2}^{n} \left(g(P_k a_k) - g(P_{k-1} a_k) \right)$$
$$+ \sum_{k=2}^{n} \left(\sum_{i=1}^{k-1} p_i a_i - P_{k-1} a_k \right) \left(g' \left(\sum_{i=1}^{k} p_i a_i \right) - g' \left(\sum_{i=1}^{k-1} p_i a_i \right) \right). \quad (5.3.7)$$

If $g(x+h) - g(x)$ is concave for all x, $x+h \in [a,b]$ such that $h \geq 0$, then opposite inequalities hold in (5.3.7).

Proof. Since $g(x+h) - g(x)$ is a convex function, where g is differentiable, by setting $f(x) = g(x+h) - g(x)$ in (5.3.3), we have (5.3.5). Substituting $x = \sum_{i=1}^{k-1} p_i a_i$, $y = P_{k-1} a_k$ and $h = p_k a_k$ $(k = 2, \ldots, n)$ in (5.3.5), where $a_k > 0$ $(k = 1, \ldots, n)$, we have

$$g(P_k a_k) - g(P_{k-1} a_k) + \left(\sum_{i=1}^{k-1} p_i a_i - P_{k-1} a_k \right) \left(g'(P_k a_k) - g'(P_{k-1} a_k) \right)$$
$$\leq g\left(\sum_{i=1}^{k} p_i a_i \right) - g\left(\sum_{i=1}^{k-1} p_i a_i \right) \leq g(P_k a_k) - g(P_{k-1} a_k)$$
$$+ \left(\sum_{i=1}^{k-1} p_i a_i - P_{k-1} a_k \right) \left(g' \left(\sum_{i=1}^{k} p_i a_i \right) - g' \left(\sum_{i=1}^{k-1} p_i a_i \right) \right).$$

Summing over k from 2 to n, we have (5.3.7).

If $g(x+h) - g(x)$ is concave for all x, $x+h \in [a,b]$ such that $h \geq 0$, then we have opposite inequalities in (5.3.3) and in the same way we have opposite inequalities in (5.3.7). □

Corollary 5.3.7. *Let $a_k > 0$ and $p_k > 0$ $(k = 1, \ldots, n)$ be real numbers such that $P_k = \sum_{i=1}^{k} p_i$ $(k = 1, \ldots, n)$. Let $p_1 a_1$, $P_{k-1} a_k$, $P_k a_k$, $\sum_{i=1}^{k-1} p_i a_i$, $\sum_{i=1}^{k} p_i a_i \in [a,b]$ for all $k = 2, \ldots, n$ and let $g : [a,b] \to \mathbb{R}$ be a differentiable function. If the function g is 3-convex, then* (5.3.7) *holds and if it is 3-concave, then* (5.3.7) *holds in the reverse direction.*

Proof. The proof is analogous to the proof of Corollary 5.3.5. □

Remark 5.3.1. (i) If $g(x) = x^s$, then the function $g(x+h) - g(x)$ is convex for $0 < s < 1$ or $s > 2$ and concave for $1 < s < 2$, where $x \in (0, \infty)$ and $h \geq 0$. The results from Theorem 5.1.4 can be recaptured by making the substitutions $p_k \to 1$ $(k = 1, \ldots, n)$ and $g(x) = x^s$, $x \in (0, \infty)$ in the results of Theorems 5.3.4 and 5.3.6, showing that these theorems are the generalizations of Theorem 5.1.4.

(ii) It is easy to see that $g(x+h)-g(x)$ is convex for $g(x)=-x\ln x$ and concave for $g(x)=x\ln x$ defined on $(0,\infty)$ with $h\geq 0$. If we first make the substitution $p_k\to 1$ $(k=1,\ldots,n)$ in Theorems 5.3.4 and 5.3.6, then replace a_k by p_k $(k=1,\ldots,n)$ and take $g(x)=-x\ln x$ in (5.3.4) and (5.3.7) or $g(x)=x\ln x$ in their reverses and also using the fact that $P_n=\sum_{i=1}^{n}p_i=1$, then we have (5.1.6). In this way Theorems 5.3.4 and 5.3.6 are the generalizations of Theorem 5.1.5.

Remark 5.3.2. (i) It is easy to see that the function $g(x)=x^s$, where $x\in(0,\infty)$ is both convex and 3-convex for $s>2$, convex and 3-concave for $1<s<2$ and concave and 3-convex for $0<s<1$.

(ii) If we make the substitutions $p_k\to 1$ $(k=1,\ldots,n)$ and $g(x)=x^s$, $x\in(0,\infty)$ in Corollary 5.3.5 or Corollary 5.3.7 and if $(a_k>0,\ k=1,\ldots,n)\subset\mathbb{R}$ is a sequence which is non-increasing in mean, then the left inequality in (5.3.4) and (5.3.7) is a refinement of (5.1.2) for $s>2$, and the right inequality in (5.3.4) and (5.3.7) is a refinement of the reversed inequality of (5.1.2) for $0<s<1$. In case of 3-concave functions, the right inequality in the reverse of (5.3.4) and (5.3.7) is a refinement of (5.1.2) for $1<s<2$.

(iii) More generally, if a sequence $(a_k>0,\ k=1,\ldots,n)\subset\mathbb{R}$ is non-increasing in weighted mean and the function g is convex and 3-convex, then the left inequality in (5.3.7) is a refinement of (5.1.1). If g is convex and 3-concave, then the inequalities in (5.3.7) are reversed and the right inequality is the refinement of (5.1.1). Analogous statements can be made for a function which is concave and 3-convex or concave and 3-concave.

Let $\hat{K}[a,b]$ be the class of all differentiable 3-convex functions defined on $[a,b]$ and let us define linear functionals $\Phi_i:\hat{K}[a,b]\to\mathbb{R}$ $(i=3,\ldots,8)$ by the non-negative differences of the inequalities (5.3.4) and (5.3.7) as follows:

$$\begin{aligned}\Phi_3(g) &= \sum_{k=2}^{n}a_k^s\left(g\left(\frac{\sum_{i=1}^{k}p_ia_i}{a_k}\right)-g\left(\frac{\sum_{i=1}^{k-1}p_ia_i}{a_k}\right)-\left(g(P_k)-g(P_{k-1})\right)\right)\\ &\quad-\sum_{k=2}^{n}a_k^{s-1}\left(\sum_{i=1}^{k-1}p_ia_i-P_{k-1}a_k\right)\left(g'(P_k)-g'(P_{k-1})\right),\end{aligned}\tag{5.3.8}$$

$$\begin{aligned}\Phi_4(g) &= \sum_{k=2}^{n}a_k^{s-1}\left(\sum_{i=1}^{k-1}p_ia_i-P_{k-1}a_k\right)\left(g'\left(\frac{\sum_{i=1}^{k}p_ia_i}{a_k}\right)-g'\left(\frac{\sum_{i=1}^{k-1}p_ia_i}{a_k}\right)\right)\\ &\quad-\sum_{k=2}^{n}a_k^{s-1}\left(\sum_{i=1}^{k-1}p_ia_i-P_{k-1}a_k\right)\left(g'(P_k)-g'(P_{k-1})\right),\end{aligned}\tag{5.3.9}$$

$$\Phi_5(g) = \sum_{k=2}^{n} a_k^s \left(g(P_k) - g(P_{k-1}) - \left(g\left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k} \right) - g\left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k} \right) \right) \right)$$
$$+ \sum_{k=2}^{n} a_k^{s-1} \left(\sum_{i=1}^{k-1} p_i a_i - P_{k-1} a_k \right) \left(g'\left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k} \right) - g'\left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k} \right) \right), \tag{5.3.10}$$

$$\Phi_6(g) = g\left(\sum_{i=1}^{n} p_i a_i \right) - g(p_1 a_1) - \sum_{k=2}^{n} \left(g(P_k a_k) - g(P_{k-1} a_k) \right)$$
$$- \sum_{k=2}^{n} \left(\sum_{i=1}^{k-1} p_i a_i - P_{k-1} a_k \right) \left(g'(P_k a_k) - g'(P_{k-1} a_k) \right), \tag{5.3.11}$$

$$\Phi_7(g) = \sum_{k=2}^{n} \left(\sum_{i=1}^{k-1} p_i a_i - P_{k-1} a_k \right) \left(g'\left(\sum_{i=1}^{k} p_i a_i \right) - g'\left(\sum_{i=1}^{k-1} p_i a_i \right) \right)$$
$$- \sum_{k=2}^{n} \left(\sum_{i=1}^{k-1} p_i a_i - P_{k-1} a_k \right) \left(g'(P_k a_k) - g'(P_{k-1} a_k) \right), \tag{5.3.12}$$

and

$$\Phi_8(g) = g(p_1 a_1) - g\left(\sum_{i=1}^{n} p_i a_i \right) + \sum_{k=2}^{n} \left(g(P_k a_k) - g(P_{k-1} a_k) \right)$$
$$+ \sum_{k=2}^{n} \left(\sum_{i=1}^{k-1} p_i a_i - P_{k-1} a_k \right) \left(g'\left(\sum_{i=1}^{k} p_i a_i \right) - g'\left(\sum_{i=1}^{k-1} p_i a_i \right) \right), \tag{5.3.13}$$

where $P_{k-1},\ P_k,\ P_{k-1}a_k,\ P_k a_k,\ \sum_{i=1}^{k-1} p_i a_i,\ \sum_{i=1}^{k} p_i a_i,\ \frac{\sum_{i=1}^{k-1} p_i a_i}{a_k},\ \frac{\sum_{i=1}^{k} p_i a_i}{a_k} \in [a,b]$ such that $a_k > 0$, $p_k > 0$ $(k = 1, \ldots, n)$ with $P_{k-1} = \sum_{i=1}^{k-1} p_i a_i$ for all $k = 2, \ldots, n$. If the function g is differentiable and 3-convex defined on $[a,b]$, then Corollaries 5.3.5 and 5.3.7 imply that

$$\Phi_i(g) \geq 0, \qquad i = 3, \ldots, 8. \tag{5.3.14}$$

5.3.1 Mean Value Theorems

Now, we give mean value theorems for the functionals $\Phi_i\,(i = 3, \ldots, 8)$ as defined in (5. 3. 8) - (5. 3. 13).

In the proof of mean value theorems, the following result is needed (see [3]).

Lemma 5.3.8. *Let $f \in C^3(I)$ such that f''' is bounded, that is, there exists $m = \min_{x\in[a,b]} f'''(x)$ and $M = \max_{x\in[a,b]} f'''(x)$ such that*

$$m \leq f'''(x) \leq M, \qquad x \in I, \tag{5.3.15}$$

holds. Then the functions $\vartheta_1(x)$ *and* $\vartheta_2(x)$ *defined by*

$$\vartheta_1(x) = \frac{M}{6}x^3 - g(x),$$

and

$$\vartheta_2(x) = g(x) - \frac{m}{6}x^3,$$

are 3*-convex.*

Proof. We have $\vartheta_1'''(x) = M - g'''(x)$ and $\vartheta_2'''(x) = g'''(x) - m$. By using (5.3.15) and Theorem 1.1.5 for $n = 3$, ϑ_i $(i = 3, \ldots, 8)$ are 3-convex. □

Theorem 5.3.9. *Let* $a_k > 0$ *and* $p_k > 0$ $(k = 1, \ldots, n)$ *be real numbers such that* $P_k = \sum_{i=1}^{k} p_i$ $(k = 1, \ldots, n)$*. Let* P_{k-1}, P_k, $P_{k-1}a_k$, $P_k a_k$, $\sum_{i=1}^{k-1} p_i a_i$, $\sum_{i=1}^{k} p_i a_i$, $\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k}$ *and* $\frac{\sum_{i=1}^{k} p_i a_i}{a_k} \in [a, b]$ *for all* $k = 2, \ldots, n$*. Suppose that* Φ_i $(i = 3, \ldots, 8)$ *are linear functionals as defined in* (5.3.8)-(5.3.13) *and* $g \in C^3([a, b])$*. Then there exists* $\xi_i \in [a, b]$ *such that*

$$\Phi_i(g) = \frac{g'''(\xi_i)}{6}\Phi_i(g_0), \quad i = 3, \ldots, 8, \tag{5.3.16}$$

where $g_0(x) = x^3$*.*

Proof. Since $g'''(x)$ is continuous on $[a, b]$, there exist real numbers $m = \min_{x \in [a,b]} g'''(x)$ and $M = \max_{x \in [a,b]} g'''(x)$ such that (5.3.15) holds. By using ϑ_1 and ϑ_2 in (5.3.14), we have

$$\Phi_i\left(\frac{Mx^3}{6} - g(x)\right) \geq 0, \qquad i = 3, \ldots, 8,$$

and

$$\Phi_i\left(g(x) - \frac{mx^3}{6}\right) \geq 0, \qquad i = 3, \ldots, 8,$$

respectively. The above two inequalities are equivalent to

$$\Phi_i(g) \leq \frac{M}{6}\Phi_i(g_0), \qquad i = 3, \ldots, 8, \tag{5.3.17}$$

and

$$\Phi_i(g) \geq \frac{m}{6}\Phi_i(g_0), \qquad i = 3, \ldots, 8, \tag{5.3.18}$$

respectively. From (5.3.17) and (5.3.18), we have

$$\frac{m}{6}\Phi_i(g_0) \leq \Phi_i(g) \leq \frac{M}{6}\Phi_i(g_0), \qquad i = 3, \ldots, 8. \tag{5.3.19}$$

If $\Phi_i(g_0) = 0$ $(i = 3, \ldots, 8)$, then there is nothing to prove. Let $\Phi_i(g_0) > 0$ $(i = 3, \ldots, 8)$, then from (5.3.19), we have

$$m \leq \frac{6\Phi_i(g)}{\Phi_i(g_0)} \leq M, \qquad i = 3, \ldots, 8.$$

Now, by using the fact that for $m \leq \eta_i \leq M$, there exist $\xi_i \in [a, b]$ such that $g'''(\xi_i) = \eta_i$ $(i = 3, \ldots, 8)$ and so we have (5.3.16). □

Cauchy type mean value theorem is stated as follows:

Theorem 5.3.10. *Let all the assumptions of Theorem* 5.3.9 *hold and let* $g, h \in C^3([a, b])$. *Then there exists* $\xi_i \in [a, b]$ *such that*

$$\frac{\Phi_i(g)}{\Phi_i(h)} = \frac{g'''(\xi_i)}{h'''(\xi_i)}, \qquad i = 3, \ldots, 8, \tag{5.3.20}$$

provided that the denominators are non-zero.

Proof. Consider the functions $k_i \in C^3([a, b])$ defined by $k_i = c_i g - d_i h$ such that $c_i = \Phi_i(h)$ and $d_i = \Phi_i(g)$, where $i = 3, \ldots, 8$. Using Theorem 5.3.9 with $g = k_i$, there exist $\xi_i \in [a, b]$ such that

$$\left(\frac{c_i g'''(\xi_i)}{6} - \frac{d_i h'''(\xi_i)}{6}\right) \Phi_i(g_0) = 0, \qquad i = 3, \ldots, 8.$$

Since $\Phi_i(g_0) \neq 0$ (because otherwise we have a contradiction with $\Phi_i(h) \neq 0$ by Theorem 5.3.9), we get

$$\frac{g'''(\xi_i)}{h'''(\xi_i)} = \frac{d_i}{c_i}, \qquad i = 3, \ldots, 8.$$

After substituting the values of c_i and d_i, we have (5.3.20). □

Remark 5.3.3. (i) By taking $g(x) = x^s$ and $h(x) = x^q$ in (5.3.20), where $s, q \in \mathbb{R} \setminus \{0, 1, 2\}$ are such that $s \neq q$, we have

$$\xi_i^{s-q} = \frac{q(q-1)(q-2)\Phi_i(x^s)}{s(s-1)(s-2)\Phi_i(x^q)}, \qquad i = 3, \ldots, 8.$$

(ii) If the inverse of the function g'''/h''' exists, then from (5.3.20), we have

$$\xi_i = \left(\frac{g'''}{h'''}\right)^{-1} \left(\frac{\Phi_i(g)}{\Phi_i(h)}\right), \qquad i = 3, \ldots, 8.$$

5.3.2 n-Exponential Convexity and Log-Convexity of the Functions Related to the Differences of the Generalized Inequalities

Next, we study the n-exponential convexity and log-convexity of the functions associated with the linear functionals Φ_i $(i = 3, \ldots, 8)$ defined in (5. 3. 8) - (5. 3. 13).

Theorem 5.3.11. *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of differentiable functions defined on $[a, b]$ such that the function $s \mapsto [y_0, y_1, y_2, y_3; f_s]$ is n-exponentially convex in the Jensen sense on I for every four mutually distinct points $y_0, y_1, y_2.y_3 \in [a, b]$. Let Φ_i $(i = 3, \ldots, 8)$ be linear functionals as defined in* (5. 3. 8) - (5. 3. 13). *Then the following statements hold:*

(*i*) *The function $s \mapsto \Phi_i(f_s)$ is n-exponentially convex in the Jensen sense on I and the matrix $\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m}$ is positive semi-definite for all $m \in \mathbb{N}$, $m \leq n$ and $s_1, \ldots, s_m \in I$. Particularly,*

$$\det\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m} \geq 0, \ \ \forall \ m \in \mathbb{N}, \ m \leq n.$$

(*ii*) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is n-exponentially convex on I.*

Proof. The idea of the proof is the same as that of Theorem 2.5.1.

(i) Let $\alpha_j \in \mathbb{R}$ $(j = 1, \ldots, n)$ and consider the function

$$\varphi(y) = \sum_{j,k=1}^{n} \alpha_j \alpha_k f_{\frac{s_j+s_k}{2}}(y),$$

where $s_j \in I$ and $f_{\frac{s_j+s_k}{2}} \in \Omega$. Then

$$[y_0, y_1, y_2, y_3; \varphi] = \sum_{j,k=1}^{n} \alpha_j \alpha_k \left[y_0, y_1, y_2, y_3; f_{\frac{s_j+s_k}{2}}\right]$$

and since $\left[y_0, y_1, y_2, y_3; f_{\frac{s_j+s_k}{2}}\right]$ is n-exponentially convex in the Jensen sense on I by assumption, it follows that

$$[y_0, y_1, y_2, y_3; \varphi] = \sum_{j,k=1}^{n} \alpha_j \alpha_k \left[y_0, y_1, y_2, y_3; f_{\frac{s_j+s_k}{2}}\right] \geq 0.$$

And so, by using Definition 1.1.9 for $n = 3$, we conclude that φ is a 3-convex function. Hence

$$\Phi_i(\varphi) \geq 0, \qquad i = 3, \ldots, 8,$$

which is equivalent to

$$\sum_{j,k=1}^{n} \alpha_j \alpha_k \Phi_i \left(f_{\frac{s_j+s_k}{2}} \right) \geq 0, \qquad i = 3, \ldots, 8,$$

and so we conclude that the function $s \mapsto \Phi_i(f_s)$ is n-exponentially convex in the Jensen sense on I.
The remaining part follows from Proposition 1.3.1.

(ii) If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then from (i) and by Definition 1.3.2 it follows that it is n-exponentially convex on I.

□

The following corollary is an immediate consequence of the above theorem.

Corollary 5.3.12. *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of differentiable functions defined on $[a, b]$ such that the function $s \mapsto [y_0, y_1, y_2, y_3; f_s]$ is exponentially convex in the Jensen sense on I for every four mutually distinct points $y_0, y_1, y_2, y_3 \in [a, b]$. Let Φ_i $(i = 3, \ldots, 8)$ be linear functionals as defined in* (5.3.8) - (5.3.13)*. Then the following statements hold:*

(i) The function $s \mapsto \Phi_i(f_s)$ is exponentially convex in the Jensen sense on I and the matrix $\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{n}$ is positive semi-definite for all $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in I$. Particularly,

$$\det \left[\Phi_i(f_{\frac{s_j+s_k}{2}}) \right]_{j,k=1}^{n} \geq 0, \;\; \forall \;\; n \in \mathbb{N}.$$

(ii) If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is exponentially convex on I.

Corollary 5.3.13. *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of differentiable functions defined on $[a, b]$ such that the function $s \mapsto [y_0, y_1, y_2, y_3; f_s]$ is 2-exponentially convex in the Jensen sense on I for every four mutually distinct points $y_0, y_1, y_2, y_3 \in [a, b]$. Let Φ_i $(i = 3, \ldots, 8)$ be linear functionals as defined in* (5.3.8) - (5.3.13)*. Further, assume that $\Phi_i(f_s)$ $(i = 3, \ldots, 8)$ is strictly positive for $f_s \in \Omega$. Then the following statements hold:*

(*i*) *If the function* $s \mapsto \Phi_i(f_s)$ *is continuous on* I*, then it is 2-exponentially convex on* I *and so it is log-convex on* I *and for* $r, s, t \in I$ *such that* $r < t < s$*, we have*

$$[\Phi_i(f_t)]^{s-r} \leq [\Phi_i(f_r)]^{s-t} [\Phi_i(f_s)]^{t-r}, \qquad i = 3, \ldots, 8, \tag{5.3.21}$$

known as Lyapunov's inequality. If $r < s < t$ *or* $t < r < s$*, then opposite inequalities hold in* (5.3.21)*.*

(*ii*) *If the function* $s \mapsto \Phi_i(f_s)$ *is differentiable on* I*, then for every* $s, q, u, v \in I$ *such that* $s \leq u$ *and* $q \leq v$*, we have*

$$\mu_{s,q}(\Phi_i, \Omega) \leq \mu_{u,v}(\Phi_i, \Omega), \qquad i = 3, \ldots, 8, \tag{5.3.22}$$

where $\mu_{s,q}(\Phi_i, \Omega)$ *is the same as defined in* (2.5.3)*.*

Proof. The proof is analogous to the proof of the Corollary 2.5.3. □

Remark 5.3.4. Note that the results of Theorem 5.3.11, and Corollaries 5.3.12 and 5.3.13 still hold when two of the points $y_0, y_1, y_2, y_3 \in [a, b]$ coincide, say $y_1 = y_0$, for a family of differentiable functions f_s such that the function $s \mapsto [y_0, y_0, y_2, y_3; f_s]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense on I), when three of the points $y_0, y_1, y_2, y_3 \in [a, b]$ coincide, say $y_2 = y_1 = y_0$, for a family of differentiable functions f_s such that the function $s \mapsto [y_0, y_0, y_0, y_3; f_s]$ is n-exponentially convex in the Jensen sense, when three of the points $y_0, y_1, y_2, y_3 \in [a, b]$ coincide again, say $y_2 = y_1 = y_0$, for a family of twice differentiable functions f_s such that the function $s \mapsto [y_0, y_0, y_0, y_3; f_s]$ is n-exponentially convex in the Jensen sense and furthermore, they still hold when all four points coincide for a family of thrice differentiable functions with the same property. The proofs can be obtained by recalling Proposition 5.3.1 and by using suitable characterizations of convexity.

5.3.3 Examples

In this section, we present several families of functions which fulfil the conditions of Theorem 5.3.11, Corollary 5.3.12 and Corollary 5.3.13 and Remark 5.3.4. This enables us to construct large families of functions which are exponentially convex.

Example 5.3.14. *Consider the family of functions*

$$\Omega_1 = \{g_s : \mathbb{R} \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{1}{s^3} e^{sx}, & s \neq 0, \\ \frac{1}{6} x^3, & s = 0. \end{cases}$$

We have $\frac{d^3}{dx^3}g_s(x) = e^{sx} \geq 0$, *which shows that* g_s *is 3-convex on* $\mathbb{R}$ *for every* $s \in \mathbb{R}$ *and* $s \mapsto \frac{d^3}{dx^3}g_s(x)$ *is exponentially convex by definition (see [30]). It is easy to see that* $s \mapsto [y_0, y_1, y_2, y_3; g_s]$ *is exponentially convex and so exponentially convex in the Jensen sense. Now by using Corollary* 5.3.12, *we have* $s \mapsto \Phi_i(g_s)$ $(i = 3, \ldots, 8)$ *are exponentially convex in the Jensen sense. Since these mappings are continuous (although the mapping* $s \mapsto g_s$ *is not continuous for* $s = 0$*), so* $s \mapsto \Phi_i(g_s)$ $(i = 3, \ldots, 8)$ *are exponentially convex.*
For this family of functions, $\mu_{s,q}(\Phi_i, \Omega)$ $(i = 3, \ldots, 8)$ *from* (2.5.3) *become*

$$\mu_{s,q}(\Phi_i, \Omega_1) = \begin{cases} \left(\frac{\Phi_i(g_s)}{\Phi_i(g_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_i(id \cdot g_s)}{\Phi_i(g_s)} - \frac{3}{s}\right), & s = q \neq 0, \\ \exp\left(\frac{\Phi_i(id \cdot g_0)}{4\Phi_i(g_0)}\right), & s = q = 0. \end{cases}$$

By using Theorem 5.3.10, *it can be seen that*

$$M_{s,q}(\Phi_i, \Omega_1) = \log \mu_{s,q}(\Phi_i, \Omega_1), \qquad i = 3, \ldots, 8,$$

satisfies $\min\{a, b\} \leq M_{s,q}(\Phi_i, \Omega_1) \leq \max\{a, b\}$, *showing that* $M_{s,q}(\Phi_i, \Omega_1)$ $(i = 3, \ldots, 8)$ *are means.*

Example 5.3.15. *Consider the family of functions*

$$\Omega_2 = \{f_s : (0, \infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)(s-2)}, & s \neq 0, 1, 2, \\ \frac{1}{2}\ln x, & s = 0, \\ -x \ln x, & s = 1, \\ \frac{1}{2}x^2 \ln x, & s = 2. \end{cases}$$

Here, $\frac{d^3}{dx^3}f_s(x) = x^{s-3} = e^{(s-3)\ln x} > 0$, *which shows that* f_s *is 3-convex for* $x > 0$ *and* $s \mapsto \frac{d^3}{dx^3}f_s(x)$ *is exponentially convex by definition (see [30]). It is easy to prove that the function* $s \mapsto [y_0, y_1, y_2, y_3; f_s]$ *is exponentially convex. Arguing as in Example* 5.3.14, *we have* $s \mapsto \Phi_i(f_s)$ $(i = 3, \ldots, 8)$ *are exponentially convex.*
From (5.3.21), *we have*

$$\Phi_i(f_t) \leq [\Phi_i(f_r)]^{\frac{s-t}{s-r}} [\Phi_i(f_s)]^{\frac{t-r}{s-r}}, \qquad i = 3, \ldots, 8, \tag{5.3.23}$$

where $r, s, t \in I$ *such that* $r < t < s$. *If* $r < s < t$ *or* $t < r < s$, *then opposite inequalities hold in* (5.3.23).

If $r,s,t \in \mathbb{R} \setminus \{0,1,2\}$ such that $r < t < s$, then for $i = 3,6$, (5.3.23) takes the form

$$D_s \geq D_r^{\frac{t-s}{t-r}} D_t^{\frac{s-r}{t-r}}, \tag{5.3.24}$$

where D_s denotes

$$D_s = \frac{1}{s(s-1)(s-2)} \left(\left(\sum_{k=1}^{n} p_k a_k \right)^s - (p_1 a_1)^s - \sum_{k=2}^{n} a_k^s \left(P_k^s - P_{k-1}^s \right) \right.$$
$$\left. - s \sum_{k=2}^{n} a_k^{s-1} A_k \left(P_k^{s-1} - P_{k-1}^{s-1} \right) \right), \qquad s \neq 0,1,2,$$

with $A_k := \sum_{i=1}^{k-1} p_i a_i - P_{k-1} a_k$ and where $a_k > 0$, $p_k > 0$ $(k = 1,\dots,n)$ with $P_k = \sum_{i=1}^{k} p_i$ are such that P_{k-1}, P_k, $P_{k-1}a_k$, $P_k a_k$, $\sum_{i=1}^{k-1} p_i a_i$, $\sum_{i=1}^{k} p_i a_i \in [a,b]$ for all $k = 2,\dots,n$. If we make the substitutions $p_k \to 1$ $(k = 1,\dots,n)$ in (5.3.24), then the inequality obtained for $s > 2$ is the refinement of the first inequality of (5.1.5).

For this family of functions by taking $\Omega = \Omega_2$ in (2.5.3), $\Xi_{s,q}^i := \mu_{s,q}(\Phi_i, \Omega_2)$ $(i = 3,\dots,8)$ for $x > 0$, where $x \in [a,b]$, are of the form

$$\mu_{s,q}(\Phi_i, \Omega_2) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{2\Phi_i(f_s f_0)}{\Phi_i(f_s)} - \frac{3s^2-6s+2}{s(s-1)(s-2)} \right), & s = q \neq 0,1,2, \\ \exp\left(\frac{\Phi_i(f_0^2)}{\Phi_i(f_0)} + \frac{3}{2} \right), & s = q = 0, \\ \exp\left(\frac{\Phi_i(f_0 f_1)}{\Phi_i(f_1)} \right), & s = q = 1, \\ \exp\left(\frac{\Phi_i(f_0 f_2)}{\Phi_i(f_2)} - \frac{3}{2} \right), & s = q = 2. \end{cases}$$

In particular, for $i = 3$, we have

$$\Xi_{s,q}^3 = \left(\frac{B_s}{B_q} \right)^{\frac{1}{s-q}}, \qquad s \neq q,$$

where

$$B_s = \frac{1}{s(s-1)(s-2)} \sum_{k=2}^{n} \left(\left(\sum_{i=1}^{k} p_i a_i \right)^s - \left(\sum_{i=1}^{k-1} p_i a_i \right)^s + a_k^s \left(P_{k-1}^s - P_k^s \right) \right.$$
$$\left. + s a_k^{s-1} A_k \left(P_{k-1}^{s-1} - P_k^{s-1} \right) \right), \qquad s \neq 0,1,2,$$

$$B_0 = \frac{1}{2} \sum_{k=2}^{n} a_k^s \left(\ln\left(\frac{P_{k-1} \sum_{i=1}^{k} p_i a_i}{P_k \sum_{i=1}^{k-1} p_i a_i} \right) + \frac{p_k A_k}{a_k P_k P_{k-1}} \right),$$

$$B_1 = \sum_{k=2}^{n} a_k^s \left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k} \ln \left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k} \right) - \frac{\sum_{i=1}^{k} p_i a_i}{a_k} \ln \left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k} \right) \right.$$
$$\left. - P_{k-1} \ln P_{k-1} + P_k \ln P_k + \frac{A_k}{a_k} \ln \frac{P_k}{P_{k-1}} \right),$$

and

$$B_2 = \frac{1}{2} \sum_{k=2}^{n} a_k^s \left(\left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k} \right)^2 \ln \left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k} \right) - \left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k} \right)^2 \ln \left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k} \right) \right.$$
$$\left. + P_{k-1}^2 \ln P_{k-1} - P_k^2 \ln P_k - \frac{A_k \left(p_k + 2 \left(P_k \ln P_k - P_{k-1} \ln P_{k-1} \right) \right)}{a_k} \right).$$

Denoting, further,

$$C_s = \sum_{k=2}^{n} \frac{a_k^s}{s(s-1)(s-2)} \left(\left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k} \right)^s \ln \left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k} \right) \right.$$
$$- \left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k} \right)^s \ln \left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k} \right) + P_{k-1}^s \ln P_{k-1} - P_k^s \ln P_k$$
$$\left. + \frac{A_k \left(P_{k-1}^{s-1} \left(1 + s \ln P_{k-1} \right) - P_k^{s-1} \left(1 + s \ln P_k \right) \right)}{a_k} \right), \qquad s \neq 0, 1, 2,$$

$$C_0 = \frac{1}{4} \sum_{k=2}^{n} a_k^s \left(\ln^2 \left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k} \right) - \ln^2 \left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k} \right) + \ln^2 \left(P_{k-1} \right) - \ln^2 \left(P_k \right) \right.$$
$$\left. - 2 \frac{A_k}{a_k} \left(\frac{\ln P_k}{P_k} - \frac{\ln P_{k-1}}{P_{k-1}} \right) \right),$$

$$C_1 = \frac{1}{2} \sum_{k=2}^{n} a_k^s \left(\left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k} \right) \ln^2 \left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k} \right) - \left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k} \right) \ln^2 \left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k} \right) \right.$$
$$\left. + \frac{A_k \left(\left(2 + \ln P_k \right) \ln P_k - \left(2 + \ln P_{k-1} \right) \ln P_{k-1} \right)}{a_k} - P_{k-1} \ln^2 \left(P_{k-1} \right) \right)$$
$$+ P_k \ln^2 \left(P_k \right),$$

and

$$C_2 = \frac{1}{4}\sum_{k=2}^{n} a_k^s \Bigg(\left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k}\right)^2 \ln^2\left(\frac{\sum_{i=1}^{k} p_i a_i}{a_k}\right) - \left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k}\right)^2 \ln^2\left(\frac{\sum_{i=1}^{k-1} p_i a_i}{a_k}\right)$$
$$+ \frac{2A_k\left(P_{k-1}\left(1+\ln P_{k-1}\right)\ln P_{k-1} - P_k\left(1+\ln P_k\right)\ln P_k\right)}{a_k} + P_{k-1}^2 \ln^2 P_{k-1}$$
$$- P_k^2 \ln^2 P_k \Bigg).$$

We can express $\Xi_{s,s}^3$ *as*

$$\Xi_{s,s}^3 = \exp\left(\frac{C_s}{B_s} - \frac{3s^2-6s+2}{s\left(s-1\right)\left(s-2\right)}\right), \qquad s \neq 0,1,2,$$
$$\Xi_{0,0}^3 = \exp\left(\frac{C_0}{B_0} + \frac{3}{2}\right),$$
$$\Xi_{1,1}^3 = \exp\left(\frac{C_1}{B_1}\right),$$
$$\Xi_{2,2}^3 = \exp\left(\frac{C_2}{B_2} - \frac{3}{2}\right).$$

If Φ_i $(i = 3, \ldots, 8)$ *are positive, then Theorem 5.3.10 applied for* $g = f_s \in \Omega_2$ *and* $h = f_q \in \Omega_2$ *yields that there exists* $\xi_i \in [a,b]$ *such that*

$$\xi_i^{s-q} = \frac{\Phi_i\left(f_s\right)}{\Phi_i\left(f_q\right)}, \qquad i = 3, \ldots, 8.$$

Since the functions $\xi_i \mapsto \xi_i^{s-q}$ $(i = 3, \ldots, 8)$ *are invertible for* $s \neq q$*, we have*

$$\min\{a,b\} \leq \left(\frac{\Phi_i(f_s)}{\Phi_i\left(f_q\right)}\right)^{\frac{1}{s-q}} \leq \max\{a,b\}, \qquad i = 3, \ldots, 8,$$

which together with the fact that $\mu_{s,q}(\Phi_i, \Omega_2)$ $(i = 3, \ldots, 8)$ *are continuous, symmetric and monotonous (by* (5.3.22)*) shows that* $\mu_{s,q}\left(\Phi_i, \Omega_2\right)$ $(i = 3, \ldots, 8)$ *are means.*

Example 5.3.16. *Consider the family of functions*

$$\Omega_3 = \{h_s : (0,\infty) \to \mathbb{R} : s \in (0,\infty)\}$$

defined by

$$h_s\left(x\right) = \begin{cases} -\frac{s^{-x}}{\ln^3 s}, & s \neq 1, \\ \frac{x^3}{6}, & s = 1. \end{cases}$$

We have $\frac{d^3}{dx^3}h_s(x) = s^{-x} > 0$, *which shows that* h_s *is 3-convex for all* $s > 0$. *Since* $s \mapsto \frac{d^3}{dx^3}h_s(x) = s^{-x}$ *is the Laplace transform of a non-negative function (see [30, 66]), it is exponentially convex. It is easy to see that the function* $s \mapsto [y_0, y_1, y_2, y_3; h_s]$ *is also exponentially convex. Arguing as in Example* 5.3.14, *we have* $s \mapsto \Phi_i(h_s)$ $(i = 3, \ldots, 8)$ *are exponentially convex.*

In this case, by taking $\Omega = \Omega_3$ *in* (2.5.3), $\mu_{s,q}(\Phi_i, \Omega)$ $(i = 3, \ldots, 8)$ *for* $x > 0$, *where* $x \in [a, b]$, *are of the form*

$$\mu_{s,q}(\Phi_i, \Omega_3) = \begin{cases} \left(\frac{\Phi_i(h_s)}{\Phi_i(h_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id \cdot h_s)}{s\Phi_i(h_s)} - \frac{3}{s \ln s}\right), & s = q \neq 1, \\ \exp\left(-\frac{\Phi_i(id \cdot h_1)}{4\Phi_i(h_1)}\right), & s = q = 1. \end{cases}$$

By using Theorem 5.3.10, *it follows that*

$$M_{s,q}(\Phi_i, \Omega_3) = -L(s, q) \log \mu_{s,q}(\Phi_i, \Omega_3), \qquad i = 3, \ldots, 8,$$

satisfy $\min\{a, b\} \leq M_{s,q}(\Phi_i, \Omega_3) \leq \max\{a, b\}$ *and so* $M_{s,q}(\Phi_i, \Omega_3)$ $(i = 3, \ldots, 8)$ *are means.*

Example 5.3.17. *Consider the family of functions*

$$\Omega_4 = \{k_s : (0, \infty) \to (0, \infty) : s \in (0, \infty)\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}}{\sqrt{s^3}}.$$

Here, $\frac{d^3}{dx^3}k_s(x) = e^{-x\sqrt{s}} > 0$, *which shows that* k_s *is 3-convex for all* $s > 0$. *Since* $s \mapsto \frac{d^3}{dx^3}k_s(x) = e^{-x\sqrt{s}}$ *is the Laplace transform of a non-negative function (see [30, 66]), it is exponentially convex. It is easy to see that the function* $s \mapsto [y_0, y_1, y_2, y_3; k_s]$ *is also exponentially convex. Arguing as in Example* 5.3.14, *we have* $s \mapsto \Phi_i(k_s)$ $(i = 3, \ldots, 8)$ *are exponentially convex.*

In this case, by taking $\Omega = \Omega_4$ *in* (2.5.3), $\mu_{s,q}(\Phi_i, \Omega)$ $(i = 3, \ldots, 8)$ *for* $x > 0$, *where* $x \in [a, b]$, *are of the form*

$$\mu_{s,q}(\Phi_i, \Omega_4) = \begin{cases} \left(\frac{\Phi_i(k_s)}{\Phi_i(k_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id \cdot k_s)}{2\sqrt{s}\Phi_i(k_s)} - \frac{3}{2s}\right), & s = q. \end{cases}$$

By using Theorem 5.3.10*, it is easy to see that*

$$M_{s,q}\left(\Phi_i,\Omega_4\right) = -\left(\sqrt{s}+\sqrt{q}\right)\log \mu_{s,q}\left(\Phi_i,\Omega_4\right), \qquad i = 3,\ldots,8,$$

satisfies $\min\{a,b\} \leq M_{s,q}\left(\Phi_i,\Omega_4\right) \leq \max\{a,b\}$*, showing that* $M_{s,q}\left(\Phi_i,\Omega_4\right)$ $(i = 3,\ldots,8)$ *are means.*

Some of the results presented in this chapter are published in an article titled "*On an inequality of I. Perić*", Math. Commun. **19** (2014), 201-222 and some of the results are published in [37].

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