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Preface

Fixed point theory is an exceptional combination of analysis, topology and geometry. Banach [19] proved a very useful result for contraction mappings. Afterward, a huge amount of fixed point results has been published by different authors and they developed different aspects of the Banach’s result. In current literature, various results have been analysed about that the fixed point of mappings have contraction over the whole space.

Let us start with the initial point \( p_0 \in E \) and define an iterative sequence \( \{p_n\} \) of the form \( p_{n+1} = Tp_n \) for all \( n \geq 0 \). We shall assume that \( p_n \neq p_{n+1} \) for every \( n \). Otherwise, there exists \( n \) such that \( p_n = p_{n+1} \). Then we will show that \( p_n = Tp_n \) and \( p_n \) has a fixed point of \( T \). To apply contraction, restriction and further application of the theorems we shall obtain \( \lim_{n \to \infty} d(p_n, p_{n+1}) = 0 \). Now we are to prove that the sequence \( \{p_n\} \) be a Cauchy sequence in \( E \). Since \( E \) is complete metric space. Each Cauchy sequence \( \{p_n\} \) in a complete metric space \( E \) converges to a point \( p \) in \( E \), so \( \{p_n\} \) converges to \( p \) and hence by using given conditions, we will prove \( p \) be a fixed point in \( E \) of \( T \). Finally, we consider \( e \) be another fixed point of \( T \) we will prove that \( p = e \). Hence \( T \) has a specific fixed point. It is simple to get fixed point for such mappings if they satisfy certain conditions. It has been shown by Hussain et al. [26], the presence of fixed point for this type of mappings that fulfill the conditions on a closed ball. Lateral, Beg et al. [20], proved the sufficient conditions on a closed ball in an ordered left(right)\(-K\) Sequentially complete dislocated quasi metric spaces (see also [12, 13, 14, 54, 56, 61, 58, 59, 60]).

Nadler [40], discussed the fixed point results concerned with multivalued mappings. Several results on multivalued mappings have been observed (see [5, 23, 36, 64]). Wardowski [65] introduced new kind of contraction said \( F- \)contraction and showed a new generalized fixed point theorem. He observed many previous fixed points in a different way. A lot of other results on \( F- \)contractions can be observed in [3, 4, 6, 10, 11, 27, 32, 37, 42, 43, 52, 53]. The theory of setvalued maps has a fundamental role in many kinds of both pure and applied maths because of its larger number of applications, in real analysis, geometry and complex analysis, algorithms, as well as in functional analysis. Over the years, above theory has raised its importance and hence in the current literature there are varied research articles related with multivalued mappings. Various authors have discussed different research articles including.
practical problems and their solutions in multivalued mappings. Due to the importance of this theory various approaches algorithms and techniques are applied for the developing of this theory. Shoaib et al. [61], discussed the result related to $\alpha_\psi$-Cirić type multifunctions on an intersection of a sequence and closed ball along with graph.

We have achieved fixed point results for new generalized $F$-contractions on an intersection of a sequence with closed ball for a more general class of semi $\alpha_\ast$-dominated mappings rather than $\alpha_\ast$-admissible mappings and for a weaker class of strictly increasing mappings $F$ rather than class of mappings $F$ used by Wardowski [65]. The notion of multi graph dominated mapping is also introduced. fixed point results related to graphic contraction is on a closed ball for this kind of maps are developed. Applications are given to investigate the unique common solution of nonlinear Volterra type integral equations. Moreover, we investigate our results in a better framework. In 1974, Cirić [24], introduced quasi contraction.

This thesis deals with the fixed point for weak contraction in generalized metric spaces. In this thesis, overview of the fixed point theory, fixed points for various contractive maps, fixed point results in different metric spaces, various approaches and methods are discussed. We shall establish new types of fixed points for setvalued maps concerning weak contraction in generalized metric spaces. Our findings are depended only for the fact that fixed points involving contraction can be obtained by fixed point theory for maps in different generalized metric spaces. In our research work, common fixed point results locally and globally contractive maps in dislocated, dislocated $b$-metric, and dislocated quasi metric spaces have been established. New contractive conditions have been introduced. Our results extended some previous theorems to generalized metric spaces and also restrict that the contractive conditions hold only for sub space rather than whole space. Furthermore, we have applied the idea of dominated maps and weak contractive conditions for the presence of fixed points of setvalued contractive maps in development of generalized metric spaces. This thesis is based on four chapters. every chapter consists of vast introduction having huge findings of material in it.

Chapter 1, is a prospect, of definitions about some generalized metric spaces for their completeness convergence and Lemmas to determine and recall basic concepts.

Chapter 2, is the study of some fixed points for multivalued maps on generalized rational type contraction. Some fixed point results are established in setting of dislocated metric space.
In addition to, we have discussed about the fixed points of setvalued \( F \)-dominated maps in these \( \text{s\'paces} \).

Chapter 3, discuss the study of some common fixed points for \( \text{Ciri\'c type rational multivalued m\'applings in dislocated } b \)-metric \( \text{s\'paces} \). Furthermore, we introduce the concept of multivalued fixed points for \( \alpha_s \)-admissible m\'appling endowed with graphic structure. Some common fixed points results for a pair of \( \alpha_s \)-dominated multivalued m\'applings on closed b\'all with applications in dislocated \( b \)-metric \( \text{s\'paces} \).

Chapter 4, deals with the some fixed points results in framework of dislocated quasi m\'etric \( \text{s\'paces} \). We developed fixed points results for \( \alpha_s \)-dominated multivalued m\'applings satisfying generalized \( \alpha_s - \Psi \text{ Ciri\'c type contrac\'tion} \) on dislocated quasi m\'etric \( \text{s\'paces} \). Moreover, we have discussed some fixed points for \( \text{Ciri\'c kind rational setvalued } F \)-contractive multivalued m\'applings with applications in these \( \text{s\'paces} \).

I wish to acknowledge that a t\'eacher is a guardian of civilization and it is really true in case of my honourable Supervisor Professor Dr. Muhammad Arshad and Co-Supervisor Dr. Abdullah Shoaib. I pay my humblest gratitude to my t\'eachers who always have been kind to me for the completion of my Ph.D. thesis.

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Tahair Rasham

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Pakistan.
Chapter 1

Introduction and Preliminaries

This chapter aims at developing some clear notions and to explain the nomenclature used in thesis. This chapter discusses some previous well known definitions and results. Section 1.1, is a vast discussion about the basic material of dislocated metric spaces. Section 1.2, is related to the basics of dislocated \( b \)-metric spaces. Section 1.3, consists of the concepts of dislocated quasi metric spaces. Section 1.4, is about some other fundamental basic notions related to it.

1.1 Dislocated Metric Spaces

Definition 1.1.1 [64] Let \( \mathcal{Z} \neq \{\} \) and the mapping \( d : \mathcal{Z} \times \mathcal{Z} \to [0, \infty) \) is said a dislocated metric if (i),(ii) and (iii) stisfy, for any \( q,p, z \in \mathcal{Z} \):

(i) If \( d_l(q,p) = 0 \), then \( q = p \);
(ii) \( d_l(q,p) = d_l(p,q) \);
(iii) \( d_l(q,p) \leq d_l(q,z) + d_l(z,p) \).

The pair \((\mathcal{Z}, d_l)\) is said dislocated metric space or \( d_l \) metric space. It is obvious that if \( d_l(q,p) = 0 \), then from (i), \( q = p \). But if \( q = p \), \( d_l(q,p) \) may not be 0. We use \( D.M.S \) instead by dislocated metric space.

Definition 1.1.2 [64] Let \((\mathcal{Z}, d_l)\) is a \( D.M.S \).

(i) A sequence \( \{c_n\} \) in \((\mathcal{Z}, d_l)\) is said a Cauchy sequence if given \( \varepsilon > 0 \), there must be \( \tilde{n}_0 \in \mathbb{N} \) such that for every \( \tilde{n}, m \geq \tilde{n}_0 \) we have \( d_l(c_m, c_\tilde{n}) < \varepsilon \) or \( \lim_{\tilde{n}, m \to \infty} d_l(c_\tilde{n}, c_m) = 0 \).

(ii) A sequence \( \{c_\tilde{n}\} \) dislocated-converges to \( l \) if \( \lim_{\tilde{n} \to \infty} d_l(c_\tilde{n}, l) = 0 \). In this case \( l \) is said a
\( d_l \)-limit of \( \{c_n\} \).

(iii) \((\hat{Z}, d_l)\) is said complete if each Cauchy sequence in \( \hat{Z} \) converges to a point \( l \in \hat{Z} \).

**Definition 1.1.3** [61] Let \( H \neq \{\} \) subset of \( D.M.S \) of \( \hat{Z} \) and let \( i \in \hat{Z} \). As \( v_0 \in H \) is said to be a best approximation in \( H \) if

\[
d_l(i, H) = d_l(i, v_0), \quad \text{where} \quad d_l(i, H) = \inf_{y \in H} d_l(i, y).
\]

If every \( i \in \hat{Z} \) has at most one best approximation in \( H \), then \( H \) is set proximinal. We denote \( P(\hat{Z}) \) be the set of all closed proximinal subsets of \( \hat{Z} \).

**Definition 1.1.4** [61] The function \( H_{d_l} : P(\hat{Z}) \times P(\hat{Z}) \to \mathbb{R}^+ \), defined by

\[
H_{d_l}(N, M) = \max\{\sup_{\tilde{n} \in N} d_l(\tilde{n}, M), \sup_{m \in M} d_l(N, m)\}
\]

is said dislocated Hausdorff metric on \( P(\hat{Z}) \).

**Example 1.1.5** [64] If \( \hat{Z} = \mathbb{R}^+ \cup \{0\} \), then \( d_l(j, k) = j + k \) is a dislocated metric \( d_l \) on \( \hat{Z} \).

**Lemma 1.1.6** [46] Let \((\hat{Z}, d_l)\) be a \( D.M.S \). Let \((P(\hat{Z}), H_{d_l})\) be a dislocated Hausdorff metric space on \( P(\hat{Z}) \). Then for every \( G, H \in P(\hat{Z}) \) and for each \( g \in G \) there must be a \( h_g \in H \) satisfies \( d_l(g, H) = d_l(g, h_g) \) then \( H_{d_l}(G, H) \geq d_l(g, h_g) \).

### 1.2 Dislocated b-Metric Spaces

**Definition 1.2.1** [29] Let \( M \neq \{\} \) and let \( d_b : M \times M \to [0, \infty) \) is a function, said a dislocated \( b \)-metric, if for every \( g, q, z \in M \), and \( t \geq 1 \) the followings hold:

(i) If \( d_b(g, q) = 0 \), then \( g = q \);

(ii) \( d_b(g, q) = d_b(q, g) \);

(iii) \( d_b(g, q) \leq t[d_b(g, z) + d_b(z, q)] \).

The pair \((M, d_b)\) is said to be a dislocated \( b \)-metric space. It is obvious that if \( d_b(g, q) = 0 \), then from (i), \( g = q \). But if \( g = q \), \( d_b(g, q) \) may not be 0. For \( g \in M \) and \( \varepsilon > 0 \), \( \overline{B}(g, \varepsilon) = \{q \in M : d_b(g, q) \leq \varepsilon\} \) is a closed ball in \((M, d_b)\). We use \( D.B.M.S \) instead dislocated \( b \)-metric space.

**Definition 1.2.2** [29] Let \((M, d_b)\) be a \( D.B.M.S \).
(i) A sequence \( \{g_n\} \) in \((M, d_b)\) is called \(\varepsilon\)-Cauchy sequence if given \(\varepsilon > 0\), there exist \(n_0 \in \mathbb{N}\) such that for all \(n, m \geq n_0\) we have \(d_b(g_m, g_n) < \varepsilon\) or \(\lim_{n,m \to \infty} d_b(g_n, g_m) = 0\).

(ii) A sequence \( \{g_n\} \) \(d\)-converges (for short \(d_b\)-converges) to \(g\) if \(\lim_{n \to \infty} d_b(g_n, g) = 0\).

In this case \(g\) is called a \(d_b\)-limit of \(\{g_n\}\).

(iii) \((M, d_b)\) is said complete if every \(\varepsilon\)-Cauchy sequence in \(M\) converges to a point \(g \in M\).

**Definition 1.2.3** [49] Let \(\hat{H} \neq \{}\) subset of \(D.B.M.S\) of \(M\) and let \(g \in M\). As \(q_0 \in \hat{H}\) is said a best \(g\) approximation in \(\hat{H}\) if

\[
d_b(g, \hat{H}) = d_b(g, q_0), \quad \text{where} \quad d_b(g, \hat{H}) = \inf_{q \in \hat{H}} d_b(g, q).
\]

**Definition 1.2.4** [51] Let \(B, A : M \rightarrow P(M)\) be the closed valued multifunctions and \(\beta : M \times M \rightarrow [0, +\infty)\) be a function. We utter that the pair \((B, A)\) is \(\beta^*_s\)-admissible if for each \(g, q \in M\)

\[
\beta(g, q) \geq 1 \Rightarrow \beta^*_s(Bg, Aq) \geq 1, \text{ and } \beta^*_s(Ag, Bq) \geq 1,
\]

where \(\beta^*_s(Ag, Bq) = \inf\{\beta(\tilde{a}, b) : \tilde{a} \in Ag, b \in Bq\}\). When \(B = A\), then we obtain the definition of \(\alpha^*_s\)-admissible mapping given in [9].

**Definition 1.2.5** [8] Let \((M, d_b)\) be a \(D.B.M.S\), \(B : M \rightarrow P(M)\) be the setvalued mapping and \(\alpha : M \times M \rightarrow [0, +\infty)\). Let \(Q \subseteq M\), we utter that the \(B\) is semi \(\alpha^*_s\)-admissible on \(Q\), when \(\alpha(g, q) \geq 1\) implies \(\alpha^*_s(Bg, Bq) \geq 1\) for all \(g, q \in Q\), where \(\alpha^*_s(Bg, Bq) = \inf\{\alpha(\tilde{a}, b) : \tilde{a} \in Bg, b \in Bq\}\). If \(Q = M\), then we utter that the \(B\) is \(\alpha^*_s\)-admissible on \(M\).

**Definition 1.2.6** [55] The function \(H_{d_b} : P(M) \times P(M) \rightarrow R^+\), interpreted, by

\[
H_{d_b}(\tilde{A}, B) = \max\{\sup_{\tilde{a} \in \tilde{A}} d_b(\tilde{a}, B), \sup_{b \in B} d_b(\tilde{A}, b)\}
\]

is said dislocated Hausdorff \(b\)-metric on \(P(M)\).

**Example 1.2.7** [29] If \(M = \mathbb{R}^+ \cup \{0\}\), then \(d_b(g, q) = (g + q)^2\) defines a \(D.B.M\) \(d_b\) on \(M\).

**Lemma 1.2.8** [49] Let \((M, d_b)\) be a \(D.B.M.S\). Let \((P(M), H_{d_b})\) is a dislocated Hausdorff \(b\)-metric space on \(P(M)\). Then for every \(\tilde{A}, B \in P(M)\) and \(\forall \tilde{a} \in \tilde{A}\) there exists \(b_\tilde{a} \in B\) holds \(d_b(\tilde{a}, B) = d_b(\tilde{a}, b_\tilde{a})\) then \(H_{d_b}(\tilde{A}, B) \geq d_b(\tilde{a}, b_\tilde{a})\).
1.3 Dislocated Quasi Metric Spaces

**Definition 1.3.1** [66] Let \( E \neq \emptyset \) and \( \delta_q : E \times E \rightarrow [0, \infty) \) is a function, said a dislocated quasi metric if (i), (ii) and (iii) hold for every \( g, s, z \in E \):

(i) If \( \delta_q(g, s) = \delta_q(s, g) = 0 \), then \( g = s \);

(ii) \( \delta_q(g, s) \leq \delta_q(g, z) + \delta_q(z, s) \).

The pair \((E, \delta_q)\) is said a DQM.

If \( \delta_q(g, s) = \delta_q(s, g) = 0 \), then from (i), \( g = s \). But if \( g = s \), \( \delta_q(g, s) \) need no be 0. It is noted that if \( \delta_q(g, s) = \delta_q(s, g) \) for all \( g, s \in E \), then \((E, \delta_q)\) becomes a DQM (metric-like space) \((E, \delta_q)\). For \( g \in E \) and \( \varepsilon > 0 \), \( B_{\delta_q}(g, \varepsilon) = \{s \in E : \delta_q(g, s) < \varepsilon \} \) and \( \overline{B}_{\delta_q}(g, \varepsilon) = \{s \in E : \delta_q(g, s) \leq \varepsilon \} \) are open and closed ball in \((E, \delta_q)\) respectively. Also \( B_{\delta_q}(g, \varepsilon) = \{s \in E : \delta_q(g, s) < \varepsilon \} \) be the closed ball in \((E, d_q)\). We use DQM for dislocated quasi metric space.

**Definition 1.3.2** [20] Let \((E, \delta_q)\) be a DQM.

(a) A sequence \( \{g_n\} \) in \((E, \delta_q)\) is said left K-Cauchy if \( \forall \varepsilon > 0 \), \( \exists n_0 \in N \) so as \( \forall n > m \geq n_0 \) (for every \( m > n \geq n_0 \)), \( \delta_q(g_m, g_n) < \varepsilon \).

(b) A sequence \( \{g_n\} \) dislocated quasi-converges to \( g \) if \( \lim_{n \to \infty} \delta_q(g_n, g) = \lim_{n \to \infty} \delta_q(g, g_n) = 0 \) or for any \( \varepsilon > 0 \), there must be a \( n_0 \in N \), so as for every \( n > n_0 \), \( \delta_q(g_n, g) < \varepsilon \) and \( \delta_q(g, g_n) < \varepsilon \). In above case \( g \) is called a \( \delta_q \)-limit of \( \{g_n\} \).

(c) \((E, \delta_q)\) is said K-sequentially complete if each K-Cauchy sequence in \( E \) converges to a point \( g \in E \) so as \( \delta_q(g, g) = 0 \).

**Definition 1.3.3** [48] Let \((E, \delta_q)\) be a DQM. Let \( M \) be a nonempty subspace of \( E \) and let \( g \in E \). An element \( s_0 \in M \) is said a best aproximation in \( M \) if

\[
\delta_q(g, M) = \delta_q(g, s_0), \quad \text{where} \quad \delta_q(g, M) = \inf_{s \in M} \delta_q(g, s)
\]

and \( \delta_q(M, g) = \delta_q(s_0, g), \quad \text{where} \quad \delta_q(M, g) = \inf_{s \in M} \delta_q(s, g). \)

If every \( g \in E \) has at minimal one best approximation in \( M \), then \( M \) is said a proximinal set.

It is obvious that if \( \delta_q(g, M) = \delta_q(M, g) = 0 \), then \( g \in M \). But, if \( g \in M \), then \( \delta_q(g, M) \) or \( \delta_q(M, g) \) may not equal to zero. We represent \( P(E) \) is the set of all closed subsets of \( E \).

**Definition 1.3.4** [51] Let \((S, T) : E \rightarrow P(E) \) and \( \beta : E \times E \rightarrow [0, +\infty) \) is a function. We
utter that the pair \((S, T)\) is \(\beta_*\)-admissible if for all \(g, s \in E\)

\[
\beta(g, s) \geq 1 \Rightarrow \beta_*(Tg, Ss) \geq 1, \text{ and } \beta_*(Tg, Ss) \geq 1.
\]

Again the pair \((S, T)\) is said to be \(\beta\)-admissible if \(\beta(g, s) \geq 1 \Rightarrow \beta(a, b) \geq 1 \forall a \in Sg, b \in Ts\).

**Definition 1.3.5** [48] Let \((E, \delta_q)\) be a DQM, and \(S, T : E \to P(E)\) be the setvalued mapping and \(\alpha : E \times E \to [0, +\infty)\). Let \(W \subseteq E\), we utter that the \(S\) is semi \(\alpha_*\)-dominated on \(W\), whenever \(\alpha_*(g, Sg) \geq 1\) for all \(g \in W\), where \(\alpha_*(g, Sg) = \inf\{\alpha(g, b) : b \in Sg\}\). If \(W = E\), then we utter that the \(S\) is \(\alpha_*\)-dominated on \(E\).

**Definition 1.3.6** [60] The function \(H_{\delta_q} : P(E) \times P(E) \to E\), determined by

\[
H_{\delta_q}(A, B) = \max\{\sup_{a \in A} \delta_q(a, B), \sup_{b \in B} \delta_q(A, b)\}
\]

is said to be a dislocated quasi Hausdorff metric on \(P(E)\). Also \((P(E), H_{\delta_q})\) is familiar as dislocated quasi Hausdorff metric space.

**Example 1.3.7** [20] Let \(E = \mathbb{R}^+ \cup \{0\}\) and \(\delta_q(g, s) = g + \max\{g, s\}\) for any \(g, s \in E\).

**Lemma 1.3.8** [60] Let \((E, \delta_q)\) be a DQM. Let \((P(E), H_{\delta_q})\) is a dislocated quasi Hausdorff metric space on \(P(E)\). Then, for every \(A, B \in P(E)\) and \(\forall a \in A\), there exists \(b_a \in B\), so as \(H_{\delta_q}(A, B) \geq \delta_q(a, b_a)\) and \(H_{\delta_q}(B, A) \geq \delta_q(b_a, a)\), where \(\delta_q(a, B) = \delta_q(a, b_a)\) and \(\delta_q(B, a) = \delta_q(b_a, a)\).

### 1.4 Some Basic Concepts

**Definition 1.4.1** [51] Let \((E, d)\) is a metric space and \(\beta : E \times E \to [0, +\infty)\) be a function. Let \((S, T) : E \to p(E)\) is a multifunction and \(\psi \in \Psi\). The pair \((S, T)\) is \(\beta_* - \psi\) contractive multifunction whenever

\[
\beta_*(Tg, Ss)H(Tg, Ss) \leq \psi(d(g, s)) \forall g, s \in E,
\]

\[
\beta_*(Sg, Ts)H(Sg, Ts) \leq \psi(d(g, s)) \forall g, s \in E,
\]
where $\beta_*(Tg, Ss) = \inf \{\beta(a, b) : a \in Tx, b \in Ss\}$. When $S = T$ then we are left with single mapping.

**Definition 1.4.2** [65] Let $(Z, d)$ is a metric and the mapping $H : Z \to Z$ is a contraction if there must be a $\tau > 0$ so as

$$\forall j, k \in Z, \; d(Hj, Hk) > 0 \Rightarrow \tau + A(d(Hj, Hk)) \leq A(d(j, k))$$

with $A : \mathbb{R}_+ \to \mathbb{R}$ real function which satisfies three assumptions:

(F1) $A$ is strictly increasing

(F2) For any sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers, $\lim_{n \to \infty} \alpha_n = 0$ is equivalent to $\lim_{n \to \infty} A(\alpha_n) = -\infty$;

(F3) There is $k \in (0, 1)$ for which $\lim_{\alpha \to 0^+} \alpha^k A(\alpha) = 0$.

We represent by $\Delta_F$, the set of all functions holding from (F1)-(F3) conditions.

**Example 1.4.3** [65] The Family of $F$ is not empty.

1. $F(g) = \ln(g); \; g > 0$.
2. $F(g) = g + \ln(g); \; g > 0$.
3. $F(g) = \frac{1}{\sqrt{g}}; \; g > 0$.

**Example 1.4.4** [48] Let $E = \mathbb{R}$. Define $\alpha : E \times E \to [0, \infty)$ by

$$\alpha(g, s) = \begin{cases} 1 & \text{if } g > s \\ \frac{1}{2} & \text{otherwise} \end{cases}.$$ 

Define the setvalued maps $S, T : E \to P(E)$ by

$$Sg = \{[g - 4, g - 3] \text{ if } g \in E\}$$

and,

$$Ts = \{[s - 2, s - 1] \text{ if } s \in E\}.$$ 

Suppose $x = 3$ and $y = 2$. As $3 > 2$, then $\alpha(3, 2) \geq 1$. Now, $\alpha_*(S3, T2) = \inf \{\alpha(a, b) : a \in S3, b \in T2\} = \frac{1}{2} \neq 1$, this means $\alpha_*(S3, T2) < 1$, that is, the pair $(S, T)$ is not $\alpha_*$-admissible. Also, $\alpha_*(S3, S2) \neq 1$ and $\alpha_*(T3, T2) \neq 1$. This implies $S$ and $T$ are not $\alpha_*$-
admissible individually. As, \( \alpha_*(g, Sg) = \inf \{ \alpha(g, b) : b \in Sg \} \geq 1 \), for every \( x \in X \). Hence \( S \) is \( \alpha_* \)-dominated mapping. Similarly \( \alpha_*(s, Ts) = \inf \{ \alpha(s, b) : b \in Ts \} \geq 1 \). Hence it is clear that \( S \) and \( T \) are not \( \alpha_* \)-admissible but \( \alpha_* \)-dominated.

**Lemma 1.4.5** [60] Every closed ball \( Y \) in a left (right) \( K \)-sequentially complete DQM of \( E \) is left (right) \( K \)-sequentially complete.

**Theorem 1.4.6** [65] Let \( (E, d) \) is a metric space and \( T : E \to E \) be the \( F \)-contractio. Then \( g^* \in E \) is a unique fixed point of mapping \( T \) and for each \( g \in E \) the sequence \( \{ T^n g \}_{n \in \mathbb{N}} \) converges to \( g^* \).

**Note:** We are using C.F.P instead common fixed point in this thesis.
Chapter 2

Results in Dislocated Metric Spaces

2.1 Introduction

The given theory and results present in this section can be seen in [44, 45, 46].

The fixed point theory aims at devolping functional and non linear analysis. Banach [19] proved significant result for contractiooni mappings. Then, a large number of fixed point resul.ts were published by different authors and they developed a lot of generalizations of Banach's result. There are many related resul.ts about the fixed points of mappings in which contractive conditions exist on prevail full space. It is very simple to show that $A : F \to F$ may not be a contractiooni but $A : J \to F$ be a contractiooni, where $J$ is a subset in $F$. It is convenient to get fixed points for such mappings if they satisfy certain condition. It has been shown by Hussain et al. [26], the presence of fixed point for such mappings that fulfill the certain requirement on a closed ball.

The theory of setvalued maps has a faundamental role in many types of both pure and applied maths because of its large number of applications, in real analysis and complex analysis, algorithms in the same way in functional analysis. Over the past years, this theory has raised its importance and hence in the current literature there are various research articles related to multivalued mappings. Nadler [40], underwent the basics of fixed points for the setvalued mappings (see also [17]). Several resul.ts on setvalued mappings have been observed (see [5, 23, 36, 64]. Wardowski [65] established new family of contractiooni mappings recalled as $F$—contractiooni. He generalized many fixed point resul.ts in a different aspect. In mètreic fixed point theory War-
dowski, generalize the famous contraction theorem termed as Banach contraction theorem. We generalize $F$-contraction into Ćirić type rational multivalued mappings and showed the applications for nonlinear Volterra type integral equations. We succeeded to generalize $F$-contraction by introducing a new Ćirić type rational $F$-contractive multivalued mappings. We further extended it to find fixed points by $\alpha_s$-dominated multivalued mappings on closed ball. In this chapter we collected these two new ideas by introducing some new rational type multivalued contractive mappings and related fixed point theorem. Many fixed point results for such mappings have been already proved by various authors becomes the corollaries of our results. We show that many other newly fixed points for $F$-contraction in different metric spaces can be obtained from our results.

From last ten years it can be seen that many authors proved fixed point results endowed with graph. We have applied new approach to proved fixed points by using graph dominated for an advanced Ćirić type rational $F$-contractive mappings on closed ball. Secelean [52] asserted fixed points regarding of $F$-contractions by using iterations system. Piri et al. [42] discussed fixed points related to $F$-Suzuki type contractions for self map in the complete metric space. Acar et al. [4] developed the idea of $F$-contractions related to multifunctions. Moreover, Acar et al. [3] developed the setvalued $F$-contraction to $\delta$-Distance and to set up fixed points in complete metric space. Sgroi et al. [53] asserted fixed points for multifunctions $F$-contraction and procured the solution of different functional and integral inclusions, that was a suitable generalization of many setvalued fixed points theorems containing Nadler’s result [40]. Many other helpful results related to $F$-contractions can be shown in [6, 11, 27, 37].

In Section 2.2, the concept of multifunctions on a closed set for a new rational type contraction has been introduced. In Section 2.3, we recall the notion of $F$-contraction to have common fixed points for multifunctions on closed subsets justifying an advanced Ćirić kind $F$-contraction in the frame work of complete dislocated metric spaces. In Section 2.4, we recalled the idea of $F$-contraction to gain common fixed points for semi $\alpha_s$-dominated setvalued maps on proximinal sets justifying a rational kind of $F$-contraction in setting of dislocated metric spaces.
2.2 Fixed Point Results for a Pair of Rational type Multivalued Contractive Mappings in Dislocated Metric Space

The results given in this section can be shown in [44].

Let $(E, d)$ be a D.M.S, $y_0 \in E$ and $S, T : E \to P(E)$ are the setvalued mapping on $E$. Let $y_1 \in S y_0$ be an element such that $d(y_0, S y_0) = d(y_0, y_1)$. Let $y_2 \in T y_1$ be so as $d(y_1, T y_1) = d(y_1, y_2)$. Let $y_3 \in S y_2$ be such that $d(y_2, S y_2) = d(y_2, y_3)$. Proceeding this method, we develop sequence $y_n$ in $E$ so as $y_{2n+1} \in S y_{2n}$ and $y_{2n+2} \in T y_{2n+1}$, where $n = 0, 1, 2, \ldots$. Also $d(y_{2n}, S y_{2n}) = d(y_{2n}, y_{2n+1})$, $d(y_{2n+1}, T y_{2n+1}) = d(y_{2n+1}, y_{2n+2})$. We represent this kind of iterative sequence by $\{T S(y_n)\}$. We say that $\{T S(y_n)\}$ is a sequence in $E$ generated by $y_0$.

**Theorem 2.2.1** Let $(E, d)$ is a complete D.M.S and $y_0$ be any arbitrary point in $E$ let the mappings $S, T : E \to P(E)$ satisfy:

$$H_d(S y, T v) \leq \kappa_1 d(y, v) + \kappa_2 \frac{d(y, S y) . d(v, T v)}{\kappa_4 + d(y, S y)} + \kappa_3 \frac{d(y, S y) . d(v, T v)}{d(y, S y) + d(y, v) + d(v, T v)}$$  \hspace{1cm} (2.1)

for all $y, v \in B_d(y_0, r) \cap \{T S(y_n)\}$ and $y \neq v$ with $\kappa_1, \kappa_2, \kappa_3, \kappa_4 > 0$ and $\kappa_1 + \kappa_2 + \kappa_3 < 1$,

$$d(y_0, S y_0) \leq (1 - \lambda) r$$  \hspace{1cm} (2.2)

where $\lambda = \max\{\frac{\kappa_1 + \kappa_3}{1 - \kappa_2}, \frac{\kappa_1 + \kappa_2}{1 - \kappa_3}\}$. Then $\{T S(y_n)\}$ be the sequence in $\overline{B_d(y_0, r)}$ for each $n$ belongs to $\mathbb{N} \cup \{0\}$ and $\{T S(y_n)\} \to h \in \overline{B_d(y_0, r)}$. Also, if (2.1) holds for $h$, then $h$ is the C.F.P of both $S$ and $T$ in $\overline{B_d(y_0, r)}$.

**Proof.** Let $y_0 \in E$ is an casual point in $E$ define $y_1 \in S y_0$ and $y_2 \in T y_1$ then, we have $y_{2n+1} \in S y_{2n}$ and $y_{2n+2} \in T y_{2n+1}$, where $n = 0, 1, 2, \ldots$. By Lemma 1.1.6, we have

$$d(y_0, y_2) = d(y_1, T y_1) \leq H_d(S y_0, T y_1)$$

$$\leq \kappa_1 d(y_0, y_1) + \kappa_2 \frac{d(y_0, S y_0) . d(y_1, T y_1)}{\kappa_4 + d(y_0, y_1)}$$

$$+ \kappa_3 \frac{d(y_0, S y_0) . d(y_1, T y_1)}{d(y_0, S y_0) + d(y_0, y_1) + d(y_1, T y_1)}$$

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\[
\begin{align*}
d_l(y_1, y_2) & \leq \kappa_1 d_l(y_0, y_1) + \kappa_2 \frac{d_l(y_0, y_1)d_l(y_1, y_2)}{\kappa_4 + d_l(y_0, y_1)} \\
& \quad + \kappa_3 \frac{d_l(y_0, y_1)d_l(y_1, y_2)}{d_l(y_0, y_1) + d_l(y_0, y_1) + d_l(y_1, y_2)} \\
& \leq \kappa_1 d_l(y_0, y_1) + \kappa_2 d_l(y_1, y_2) + \kappa_3 d_l(y_0, y_1).
\end{align*}
\]

Hence

\[
\begin{align*}
d_l(y_1, y_2) & \leq (\kappa_1 + \kappa_3) \frac{d_l(y_0, y_1)}{1 - \kappa_2} \\
& \leq \lambda d_l(y_0, y_1) \leq \lambda(1 - \lambda)r \quad \text{by using (2.2)} \\
& \leq \lambda(1 - \lambda)r.
\end{align*}
\]

Now,

\[
\begin{align*}
d_l(y_0, y_2) & \leq d_l(y_0, y_1) + d_l(y_1, y_2) \\
& \leq (1 - \lambda)r + \lambda(1 - \lambda)r \\
& \leq (1 - \lambda^2)r \leq r \\
& \leq r.
\end{align*}
\]

This implies that \( y_2 \in B_d(y_0, r) \). Suppose, \( y_3, y_4, \ldots, y_j \in B_d(y_0, r) \), for every \( j \) belongs to \( N \). If \( j = 2i + 1 \), where \( i = 1, 2, \ldots, \frac{j-1}{2} \), we get

\[
\begin{align*}
d_l(y_{2i+1}, y_{2i+2}) & = d_l(y_{2i+1}, Ty_{2i+1}) \leq H_d(Sy_{2i}, Ty_{2i+1}) \\
& \leq \kappa_1 d_l(y_{2i}, Ty_{2i+1}) + \kappa_2 \frac{d_l(y_{2i}, Sy_{2i})d_l(y_{2i+1}, Ty_{2i+1})}{\kappa_4 + d_l(y_{2i}, Sy_{2i})} \\
& \quad + \kappa_3 \frac{d_l(y_{2i}, Sy_{2i}) + d_l(y_{2i}, Ty_{2i+1}) + d_l(y_{2i+1}, Ty_{2i+1})}{d_l(y_{2i}, Ty_{2i+1})} \\
& \leq \kappa_1 d_l(y_{2i}, y_{2i+1}) + \kappa_2 \frac{d_l(y_{2i}, y_{2i+1})d_l(y_{2i+1}, y_{2i+2})}{\kappa_4 + d_l(y_{2i}, y_{2i+1})} \\
& \quad + \kappa_3 \frac{d_l(y_{2i}, y_{2i+1}) + d_l(y_{2i}, y_{2i+1}) + d_l(y_{2i+1}, y_{2i+2})}{d_l(y_{2i}, y_{2i+1})} \\
& \leq \kappa_1 d_l(y_{2i}, y_{2i+1}) + \kappa_2 d_l(y_{2i+1}, y_{2i+2}) + \kappa_3 d_l(y_{2i}, y_{2i+1}).
\end{align*}
\]
Hence

\[ d_l(y_{2i+1}, y_{2i+2}) \leq \left( \frac{\kappa_1 + \kappa_3}{1 - \kappa_2} \right) d_l(y_{2i}, y_{2i+1}) \]

\[ \leq \lambda d_l(y_{2i}, y_{2i+1}). \]  

(2.3)

Similarly, if \( j = 2i \), where \( i = 1, 2, \cdots, \frac{t}{2} \), we have

\[ d_l(y_{2i}, y_{2i+1}) = d_l(y_{2i}, Sy_{2i}) \leq H d_l(Ty_{2i-1}, Sy_{2i}) = H d_l(Sy_{2i}, Ty_{2i-1}) \]

\[ \leq \kappa_1 d_l(y_{2i}, y_{2i-1}) + \kappa_2 \frac{d_l(y_{2i}, y_{2i+1})}{\kappa_4 + d_l(y_{2i}, y_{2i+1})} \]

\[ + \kappa_3 \frac{d_l(y_{2i}, y_{2i+1})}{d_l(y_{2i}, y_{2i+1}) + d_l(y_{2i}, y_{2i-1}) + d_l(y_{2i-1}, y_{2i})} \]

\[ \leq \kappa_1 d_l(y_{2i-1}, y_{2i+1}) + \kappa_2 d_l(y_{2i-1}, y_{2i}) + \kappa_3 d_l(y_{2i}, y_{2i+1}) \]

\[ d_l(y_{2i}, y_{2i+1}) \leq \left( \frac{\kappa_1 + \kappa_2}{1 - \kappa_3} \right) d_l(y_{2i}, y_{2i+1}) \]

\[ \leq \lambda d_l(y_{2i-1}, y_{2i}). \]  

(2.4)

Now, (2.3) implies that

\[ d_l(y_{2i+1}, y_{2i+2}) \leq \lambda^{2i+1} d_l(y_0, y_1). \]

(2.5)

Also, (2.4) implies that

\[ d_l(y_{2i}, y_{2i+1}) \leq \lambda^{2i} d_l(y_0, y_1). \]

(2.6)

Now, by combining (2.5) and (2.6), we have

\[ d_l(y_j, y_{j+1}) \leq \lambda^j d_l(y_0, y_1) \text{ for each } j \in N. \]

(2.7)

Now,

\[ d_l(y_0, y_{j+1}) \leq d_l(y_0, y_1) + d_l(y_1, y_2) + \cdots + d_l(y_j, y_{j+1}) \]

\[ \leq d_l(y_0, y_1) + \lambda d_l(y_0, y_1) + \cdots + \lambda^j d_l(y_0, y_1) \text{ (by (2.7))} \]

\[ \leq (1 + \lambda + \lambda^2 + \cdots + \lambda^j) d_l(y_0, y_1) \]

\[ \leq \frac{1(1 - \lambda^j)}{1 - \lambda} (1 - \lambda) r \leq r. \]
Thus, \( y_{j+1} \in \overline{B_d(y_0, r)} \). Hence \( y_n \in \overline{B_d(y_0, r)} \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \), therefore \( \{TS(y_n)\} \) be a sequence in \( \overline{B_d(y_0, r)} \). Now, we can write inequality (2.7) as

\[
d_l(y_n, y_{n+1}) \leq \lambda^n d_l(y_0, y_1) \quad \text{for each} \quad n \in N. \tag{2.8}
\]

Hence for any \( m > n \),

\[
d_l(y_n, y_m) \leq d_l(y_n, y_{n+1}) + d_l(y_{n+1}, y_{n+2}) + \cdots + d_l(y_{m-1}, y_m),
\]

\[
\leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) d_l(y_0, y_1), \quad \text{(by using (2.8))}
\]

\[
d_l(y_n, y_m) \leq \frac{\lambda^n}{1 - \lambda} d_l(y_0, y_1) \to 0, \quad \text{as} \quad m, n \to \infty.
\]

Thus we showed that \( \{TS(y_n)\} \) be a Cauchy sequence in \( (\overline{B_d(y_0, r)}, d_l) \). As each closed ball in a complete \( D.M.S \) is complete, so there must be a \( h \in \overline{B_d(y_0, r)} \) so as \( \{TS(y_n)\} \to h \), it follows that

\[
\lim_{n \to \infty} d_l(y_n, h) = 0.
\]

Now,

\[
d_l(h, Sh) \leq d_l(h, y_{2n+2}) + d_l(y_{2n+2}, Sh)
\]

\[
\leq d_l(h, y_{2n+2}) + H_{d_l}(Ty_{2n+1}, Sh), \quad \text{(by Lemma 1.1.6)}
\]

\[
d_l(h, Sh) \leq d_l(h, y_{2n+2}) + \kappa_1 d_l(h, y_{2n+1}) + \kappa_2 \frac{d_l(h, Sh) d_l(y_{2n+1}, Ty_{2n+1})}{\kappa_4 + d_l(h, Sh)}
\]

\[
+ \kappa_3 \frac{d_l(h, Sh) d_l(y_{2n+1}, Ty_{2n+1})}{d_l(h, y_{2n+2})} + \frac{d_l(h, y_{2n+1}) + d_l(y_{2n+1}, Ty_{2n+1})}{\kappa_4 + d_l(h, Sh)}
\]

\[
\leq d_l(h, y_{2n+2}) + \kappa_1 d_l(h, y_{2n+1}) + \kappa_2 \frac{d_l(h, Sh) d_l(y_{2n+1}, y_{2n+2})}{\kappa_4 + d_l(h, Sh)}
\]

\[
+ \kappa_3 \frac{d_l(h, Sh) d_l(y_{2n+1}, y_{2n+2})}{d_l(h, y_{2n+1}) + d_l(y_{2n+1}, y_{2n+2})}.
\]

which on making \( n \to \infty \), gives rise \( d_l(h, Sh) \leq 0 \). Hence \( d_l(h, Sh) = 0 \) and so \( h \in Sh \). Similarly,

\[
d_l(h, Th) \leq d_l(h, y_{2n+1}) + d_l(y_{2n+1}, Th)
\]

\[
\leq d_l(h, y_{2n+1}) + H_{d_l}(Sy_{2n}, Th), \quad \text{(by Lemma 1.6.1)}
\]
\[ \leq d_t(h, y_{2n+2}) + \kappa_1 d_t(h, y_{2n+1}) + \kappa_2 \frac{d_t(h, Sh) \cdot d_t(y_{2n+1}, Ty_{2n+1})}{\kappa_4 + d_t(h, Sh)} + \kappa_3 \frac{d_t(h, Sh) \cdot d_t(y_{2n+1}, Ty_{2n+1})}{d_t(h, Sh) + d_t(h, y_{2n+1}) + d_t(y_{2n+1}, Ty_{2n+1})} \]

\[ \leq d_t(h, y_{2n+2}) + \kappa_1 d_t(y_{2n+1}, h) + \kappa_2 \frac{d_t(y_{2n+1}, y_{2n+1}) \cdot d_t(h, Th)}{\kappa_4 + d_t(y_{2n+1}, y_{2n+1})} + \kappa_3 \frac{d_t(y_{2n+1}, y_{2n+1}) \cdot d_t(h, Th)}{d_t(y_{2n+1}, y_{2n+1}) + d_t(y_{2n+1}, h) + d_t(h, Th)}. \]

Hence \( d_t(h, Th) \leq 0 \) and so \( h \in Th. \) ■

**Example 2.2.2** Let \( E = Q^+ \cup \{0\} \) and let \( d_t : E \times E \to E \) be the complete D.M.S on \( E \) defined by

\[ d_t(q, v) = q + v \text{ for each } q, v \in E. \]

Define the multivalued mapping, \( S, T : E \times E \to P(E) \) by,

\[ S_q = \begin{cases} 
\left[ \frac{q}{3}, \frac{2}{3} q \right] & \text{if } q \in [0, 1] \cap E \\
[q, q+1] & \text{if } q \in (1, \infty) \cap E
\end{cases} \]

and,

\[ T_l = \begin{cases} 
\left[ \frac{l}{4}, \frac{3}{4} \right] & \text{if } l \in [0, 1] \cap E \\
[l+1, l+3] & \text{if } l \in (1, \infty) \cap E
\end{cases} \]

Considering, \( q_0 = 1, r = 8, \) then \( \overline{B_{d_t}(q_0, r)} = [0, 7] \cap E. \) Now \( d_t(q_0, S q_0) = d_t(1, S1) = d_t(1, \frac{1}{3}) = \frac{4}{3}. \) So we obtain a sequence \( \{TS(q_n)\} = \{1, \frac{1}{12}, \frac{1}{114}, \frac{1}{1728}, \ldots\} \) in \( E \) generated by \( q_0. \) Let \( q, v \in (1, \infty) \cap E, \) then by taking \( q = 2, v = 3, \) \( \kappa_1 = \frac{1}{3}, \) \( \kappa_2 = \frac{1}{4}, \) and \( \kappa_3 = \frac{1}{7}, \) \( \kappa_4 = 1, \) then \( H_{d_t}(S2, T3) = 8. \) Now,

\[
\begin{align*}
&= \kappa_1 d_t(2, 3) + \kappa_2 \frac{d_t(2, [2, 2 + 1]) \cdot d_t(3, [3 + 1, 3 + 3])}{1 + d_t(2, [2, 2 + 1])} \\
&\quad + \frac{d_t(2, [2, 2 + 1]) \cdot d_t(3, [3 + 1, 3 + 3])}{d_t(2, [2, 2 + 1]) + d_t(2, 3) + d_t(3, [3 + 1, 3 + 3])} \\
&= \frac{1}{3} d_t(2, 3) + \frac{1}{4} d_t(2, 2) \cdot d_t(3, 4) + \frac{1}{7} d_t(2, 2) \cdot d_t(3, 4) \\
&= \frac{5}{3} + \frac{28}{20} + \frac{28}{112} = 3.31.
\end{align*}
\]

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As $8 > 3.31$, then

$$H_{d_l}(S2, T3) > \frac{1}{3}d_l(2, 3) + \frac{1}{4} \frac{d_l(2, S2).d_l(3, T3)}{1 + d_l(2, S2)} + \frac{1}{7} \frac{d_l(2, S2).d_l(3, T3)}{d_l(2, S2) + d_l(2, 3) + d_l(3, T3)}.$$ 

So, the inequality (2.1) is not true for the whole space $E$. Now for each $q, v \in B_{d_l}(q_0, r) \cap \{TS(q_n)\}$, we have

$$H_{d_l}(Sq, Tv) = \max\{\sup_{\kappa \in Sq} d_l(\kappa, Tv), \sup_{b \in Tv} d_l(q, b)\}$$

$$= \max\{\sup_{\kappa \in Sq} d_l(\kappa, [\frac{v}{4}, \frac{3v}{4}]), \sup_{b \in Tv} d_l([\frac{q}{3}, \frac{2q}{3}], b)\}$$

$$= \max\{d_l(\frac{2q}{3}, [\frac{v}{4}, \frac{3v}{4}]), d_l([\frac{q}{3}, \frac{2q}{3}], \frac{3y}{4})\}$$

$$= \max\{d_l(\frac{2q}{3}, v), d_l(q, \frac{3y}{4})\}$$

$$= \max\{\frac{2q}{3} + \frac{v}{4}, \frac{q}{3} + \frac{3v}{4}\}$$

$$= \kappa_1d_l(q, v) + \kappa_2 \frac{d_l(q, Sq).d_l(v, Tv)}{\kappa_4 + d_l(q, v)} + \kappa_3 \frac{d_l(q, Sq).d_l(v, Tv)}{d_l(q, Sq) + d_l(q, v) + d_l(v, Tv)}$$

$$= \kappa_1d_l(q, v) + \kappa_2 \frac{d_l(q, \frac{q}{3}).d_l(v, \frac{v}{4})}{1 + d_l(q, \frac{q}{3})} + \kappa_3 \frac{d_l(q, \frac{q}{3}).d_l(v, \frac{v}{4})}{d_l(q, \frac{q}{3}) + d_l(q, v) + d_l(v, \frac{v}{4})}$$

$$= \frac{1}{3}(q + v) + \frac{1}{4} \frac{4q}{3} + \frac{5v}{4} + \frac{1}{7} \frac{4q}{3} + \frac{5v}{4} + q + v + \frac{v}{4}$$

$$\geq \max\{\frac{2q}{3} + \frac{v}{4}, \frac{q}{3} + \frac{3v}{4}\} = H_{d_l}(Sq, Tv).$$

So, the inequality (2.1) holds on $B_{d_l}(q_0, r) \cap \{TS(q_n)\}$. Also,

$$\frac{4}{3} < (1 - \frac{49}{72}) \times 8$$

$$d_l(q_0, Sq_0) \leq (1 - \lambda)r.$$

Hence, all the hypothesis of Theorem 2.2.1 are fulfilled.

**Corollary 2.2.3** Let $(E, d_l)$ be a complete D.M.S and $q_0$ be any arbitrary point in $E$ let
the mappings \( S, T : E \to P(E) \) satisfy:

\[
H_{d_l}(S q, T v) \leq \kappa_1 \; d_l(q, v) + \kappa_2 \; \frac{d_l(q, S q) \cdot d_l(v, T v)}{d_l(q, S q) + d_l(q, v) + d_l(v, T v)}
\]  \hspace{1cm} (2.9)

for each \( q, v \in \overline{B_{d_l}(q_0, r)} \cap \{TS(q_n)\} \) and \( q \neq v \) with \( r > 0 \),

\[
d_l(q_0, S q_0) \leq (1 - \lambda)r
\]

where \( \lambda = (\kappa_1 + \kappa_2) \) and, \( \kappa_1, \kappa_2 \) are positive reals with \( \kappa_1 + \kappa_2 < 1 \). Then \( \{TS(q_n)\} \) is a sequence in \( \overline{B_{d_l}(q_0, r)} \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and \( \{TS(q_n)\} \to h \in \overline{B_{d_l}(q_0, r)} \). Also, if (2.9) holds for \( h \), then \( h \) be the C.F.P of both \( S \) and \( T \) in \( \overline{B_{d_l}(q_0, r)} \).

**Theorem 2.2.4** Let \((E, d_l)\) be a complete \( D.M.S \) and \( q_0 \) be any arbitrary point in \( E \) let the mappings \( S, T : E \to P(E) \) satisfy:

\[
H_{d_l}(S(v), T(f)) \leq a \; d_l(v, f) + b \; \frac{d_l(v, S(v)) \cdot d_l(f, T(f))}{1 + d_l(v, f)}
\]  \hspace{1cm} (2.10)

for each \( v, f \in \overline{B_{d_l}(v_0, r)} \cap \{TS(v_n)\} \) and \( v \neq f \) with \( r > 0 \),

\[
d_l(v_0, S v_0) \leq (1 - \lambda)r
\]  \hspace{1cm} (2.11)

where \( \lambda = \left( \frac{a}{1 - b} \right) \) and \( a, b \) are positive reals with \( a + b < 1 \). Then \( \{TS(v_n)\} \) is a sequence in \( \overline{B_{d_l}(v_0, r)} \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \), and \( \{TS(v_n)\} \to q \in \overline{B_{d_l}(v_0, r)} \). Also, if (2.10) holds for \( q \), then \( S \) and \( T \) have C.F.P \( u \) in \( \overline{B_{d_l}(v_0, r)} \).

**Proof.** Let \( v_0 \in E \) and define \( v_1 \in S(v_0) \) and \( v_2 \in T(v_1) \) then, we have \( v_{2n+1} \in S(v_{2n}) \) and \( v_{2n+2} \in T(v_{2n+1}) \), where \( n = 0, 1, 2, \ldots \) By Lemma 1.1.6, we have

\[
d_l(v_1, v_2) = d_l(S(v_0), T(v_1)) \leq H_{d_l}(S(v_0), T(v_1))
\]

\[
\leq ad_l(v_0, v_1) + b \frac{d_l(v_0, S(v_0)) \cdot d_l(v_1, T(v_1))}{1 + d_l(v_0, v_1)}
\]

\[
\leq ad_l(v_0, v_1) + bd_l(v_1, v_2) \frac{d_l(v_0, v_1)}{1 + d_l(v_0, v_1)}
\]

\[
\leq ad_l(v_0, v_1) + bd_l(v_1, v_2)
\]

\[
\leq \left( \frac{a}{1 - b} \right) d_l(v_0, v_1)
\]
\[
\begin{align*}
\leq & \quad \lambda d_l(v_0, v_1) \\
\leq & \quad \lambda(1 - \lambda) \leq r. \text{ by (2.11)}
\end{align*}
\]

Where \( \left( \frac{a}{1 - b} \right) = \lambda \). Now,

\[
\begin{align*}
d_l(v_0, v_2) & \leq d_l(v_0, v_1) + d_l(v_1, v_2) \\
& \leq (1 - \lambda)r + \lambda(1 - \lambda)r \\
& \leq (1 - \lambda^2)r \leq r.
\end{align*}
\]

This implies that \( v_2 \in \overline{B_{d_l}(v_0, r)} \) similarly,

\[
\begin{align*}
d_l(v_2, v_3) & = d_l(v_3, v_2) \leq H d_l(S(v_2), T(v_1)) \text{ by Lemma 1.1.6} \\
& \leq a d_l(v_2, v_1) + b d_l(v_2, S(v_2)) d_l(v_1, T(v_1)) \\
& \leq a d_l(v_2, v_1) + b d_l(v_2, v_3) d_l(v_1, v_2) \\
& \leq a d_l(v_2, v_1) + b d_l(v_2, v_3) \left( \frac{d_l(v_1, v_2)}{1 + d_l(v_2, v_1)} \right) \\
& \leq a d_l(v_1, v_2) + b d_l(v_2, v_3).
\end{align*}
\]

This implies that,

\[
\begin{align*}
d_l(v_2, v_3) & \leq \left( \frac{a}{1 - b} \right) d_l(v_1, v_2) \\
& \leq \lambda \lambda d_l(v_0, v_1) \\
& \leq \lambda^2 d_l(v_0, v_1) \\
& \leq \lambda^2(1 - \lambda)r \leq r.
\end{align*}
\]

Consequently, \( v_3, v_4, \ldots, v_j \in \overline{B_{d_l}(v_0, r)} \), for every \( j \) belongs to \( N \). If \( j = 2i + 1 \), where \( i = 1, 2, \ldots, \frac{j - 1}{2} \) we get

\[
\begin{align*}
d_l(v_{2i+1}, v_{2i+2}) & \leq \lambda d_l(v_{2i}, v_{2i+1}) \\
& \leq \lambda d_l(v_{2i+1}, v_{2i+2}) \quad (2.12)
\end{align*}
\]

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Similarly, if \( j = 2i + 2 \), where \( i = 0, 1, 2, \ldots, \frac{i-2}{2} \), we have

\[
\begin{align*}
d_t (v_{2i+2}, v_{2i+3}) \leq \lambda d_t (v_{2i+1}, v_{2i+2}).
\end{align*}
\]

(2.13)

Now, (2.12) implies that

\[
\begin{align*}
d_t (v_{2i+1}, v_{2i+2}) \leq \lambda^{2i+1} d_t (v_0, v_1).
\end{align*}
\]

(2.14)

Also, (2.14) implies that

\[
\begin{align*}
d_t (v_{2i+2}, v_{2i+3}) \leq \lambda^{2i+2} d_t (v_0, v_1).
\end{align*}
\]

(2.15)

Now, by combining (2.14) and (2.15), we have

\[
\begin{align*}
d_t (v_j, v_{j+1}) \leq \lambda^j d_t (v_0, v_1) \text{ for each } j \in N.
\end{align*}
\]

(2.16)

Now,

\[
\begin{align*}
d_t (v_0, v_{j+1}) & \leq d_t (v_0, v_1) + d_t (v_1, v_2) + \ldots + d_t (v_j, v_{j+1}) \\
& \leq d_t (v_0, v_1) + \lambda d_t (v_0, v_1) + \ldots + \lambda^j d_t (v_0, v_1) \text{ by (2.16)} \\
& \leq (1 + \lambda + \lambda^2 + \ldots + \lambda^j) d_t (v_0, v_1) \\
& \leq \frac{1(1 - \lambda^j)}{1 - \lambda} (1 - \lambda) r \leq r.
\end{align*}
\]

Thus, \( v_{j+1} \in \overline{B_d (v_0, r)} \). Hence \( v_n \in \overline{B_d (v_0, r)} \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \), therefore \( \{TS(v_n)\} \) be a sequence in \( \overline{B_d (v_0, r)} \). Now, we can write (2.16) as

\[
\begin{align*}
d_t (v_n, v_{n+1}) \leq \lambda^n d_t (v_0, v_1) \text{ for every } n \text{ belongs to } N.
\end{align*}
\]

(2.17)

To show that \( \{TS(v_n)\} \) is a Cauchy sequence, we have for any \( m > n \),

\[
\begin{align*}
d_t (v_n, v_m) & \leq d_t (v_n, v_{n+1}) + d_t (v_{n+1}, v_{n+2}) + \ldots + d_t (v_{m-1}, v_m) \\
& \leq \lambda^n d_t (v_0, v_1) + \lambda^{n+1} d_t (v_0, v_1) + \ldots + \lambda^{m-1} d_t (v_0, v_1) \\
& \leq \left( \lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1} \right) d_t (v_0, v_1) \\
& \leq \left( \frac{\lambda^n}{1 - \lambda} \right) d_t (v_0, v_1) \rightarrow 0 \text{ as } m, n \rightarrow \infty.
\end{align*}
\]

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Thus we proved that \( \{TS(v_n)\} \) is a Cauchy in \( (B_d(v_0, r), d_l) \). As everý closed ball in a complete \( D.M.S \) is complete, so there must be a \( u \in \overline{B_d(v_0, r)} \) such that \( \{TS(v_n)\} \to u \), it shows that \( u \in Su \), otherwise \( d_l(u, Su) = \theta > 0 \), that is

\[
\lim_{n \to \infty} d_l(v_n, u) = 0. \tag{2.18}
\]

Therefore we have,

\[
d_l(u, Su) \leq d_l(u, v_{2n+2}) + d_l(v_{2n+2}, Su) \\
\leq d_l(u, v_{2n+2}) + d_l(T(v_{2n+1}), Su) \\
\leq d_l(u, v_{2n+2}) + H_{d_l}(Su, T(v_{2n+1})) \quad \text{by Lemma 1.1.6} \\
\leq d_l(u, v_{2n+2}) + ad_l(u, v_{2n+1}) + b \frac{d_l(u, Su) \cdot d_l(v_{2n+1}, T(v_{2n+1}))}{1 + d_l(u, v_{2n+1})} \\
\leq d_l(u, v_{2n+2}) + ad_l(u, v_{2n+1}) + b \frac{d_l(u, Su) \cdot d_l(v_{2n+1}, v_{2n+2})}{1 + d_l(u, v_{2n+1})} \\
\leq d_l(u, v_{2n+2}) + ad_l(u, v_{2n+1}) + b \frac{\theta \cdot d_l(v_{2n+1}, v_{2n+2})}{1 + d_l(u, v_{2n+1})}.
\]

letting \( n \to \infty \), and \( v_n \to u \) by using (2.18) we get,

\[
(1 - b) \theta \leq 0 \\
(1 - b) \neq 0 \\
\theta = d_l(u, Su) \leq 0.
\]

\( d_l(u, Su) < 0 \) gives a contradiction so that \( u \in Su \). It follows similarly that

\[
d_l(u, Tu) \leq d_l(u, v_{2n+1}) + d_l(v_{2n+1}, Tu) \\
\leq d_l(u, v_{2n+1}) + H_{d_l}(Sv_{2n}, Tu) \quad \text{by Lemma 1.1.6} \\
\leq d_l(u, v_{2n+1}) + ad_l(v_{2n}, u) + b \frac{d_l(v_{2n}, Sv_{2n}) \cdot d_l(u, Tu)}{1 + d_l(v_{2n}, u)} \\
\leq d_l(u, v_{2n+1}) + ad_l(v_{2n}, u) + b \frac{d_l(v_{2n}, v_{2n+1}) \cdot d_l(u, Tu)}{1 + d_l(v_{2n}, u)}.
\]
letting $n \to \infty$, and $v_n \to u$ by using $(2.18)$ we get,

$$d_l(u, Tu) \leq bd_l(u, Tu)$$

$$(1 - b)d_l(u, Tu) \leq 0$$

$$(1 - b) \neq 0$$

$$d_l(u, Tu) \leq 0.$$ 

As $d_l(u, Tu) < 0$, so that $u \in Tu$. Hence $u$ is the C.F.P of both $S$ and $T$ in $\overline{B_{d_l(v_0, r)}}$. Now,

$$d_l(u, u) \leq H_{d_l}(Su, Tu) \text{ by Lemma 1.1.6}$$

$$\leq a d_l(u, u) + b \frac{d_l(u, Su) \cdot d_l(u, Tu)}{1 + d_l(u, u)}$$

$$\leq a d_l(u, u).$$

This implies that,

$$(1 - a) d_l(u, u) \leq 0$$

$$1 - a \neq 0$$

$$d_l(u, u) = 0.$$ 

This shows that $d_l(u, u) = 0$. $\blacksquare$

**Corollary 2.2.5** If $S : E \to E$ is a mapping defined on $D.M.S$ satisfying the condition

$$d_l(Sw, Sl) \leq a d_l(w, v) + b \frac{d_l(w, Sw) \cdot d_l(l, Sl)}{d + d_l(w, Sw)} + c \frac{d_l(w, Sw) \cdot d_l(l, Sl)}{d_l(w, Sw) + d_l(w, l) + d_l(l, Sl)}$$

for all $w, l \in \overline{B_{d_l(u_0, r)}}$ and $u \neq v$ with $r > 0$,

$$d_l(u_0, Su_0) \leq (1 - \lambda)r$$

where $\lambda = \max\left\{\frac{a+c}{1-b}, \frac{a+b}{1-c}\right\}$ and $a, b, c$ are positive reals with $a + b + c < 1$. Then $\{u_n\}$ is a sequence in $\overline{B_{d_l(u_0, r)}}$ for each $n$ belongs to $\mathbb{N} \cup \{0\}$ and $u_n \to h \in \overline{B_{d_l(u_0, r)}}$. Then $h$ is the C.F.P of $S$ in $\overline{B_{d_l(u_0, r)}}$. 

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Corollary 2.2.6 If $S : E \to E$ is a mapping defined on a complete on $D.M.S$, $(E, d_l)$ satisfying the condition :

$$d_l(Su, Sy) \leq a_1 d_l(u, y) + a_2 \frac{d_l(u, Su).d_l(y, Sy)}{1 + d_l(u, y)}$$

for all $u, y \in \overline{B_d(u_0, r)}$ and $u \neq v$ with $r > 0$,

$$d_l(u_0, Su_0) \leq (1 - \lambda)r$$

where $\lambda = \frac{a_1}{1-a_2}$, $a_1$ and $a_2$ are positive reals with $a_1 + a_2 < 1$. Then $\{u_n\}$ is a sequence in $\overline{B_d(u_0, r)}$ for each $n$ belongs to $\mathbb{N} \cup \{0\}$ and $u_n \to h \in \overline{B_d(u_0, r)}$. Then $h$ is the C.F.P of $S$ in $\overline{B_d(u_0, r)}$.

2.3 Common Fixed Point Results for new Ciric Type Rational Multivalued $F$-Contraction with an Application

The given results in this section can be seen in [45].

Let $(E, d_l)$ be a $D.M.S$, $y_0 \in E$ and $S, T : E \to P(E)$ be the setvalued maps on $E$. Let $y_1 \in Sy_0$ be an element such that $d_l(y_0, Sy_0) = d_l(y_0, y_1)$. Let $y_2 \in Ty_1$ be such that $d_l(y_1, Ty_1) = d_l(y_1, y_2)$. Let $y_3 \in Sy_2$ be such that $d_l(y_2, Sy_2) = d_l(y_2, y_3)$. Proceeding this method, we get a sequence $y_n$ in $E$ such that $y_{2n+1} \in Sy_{2n}$ and $y_{2n+2} \in Ty_{2n+1}$, where $n = 0, 1, 2, \ldots$. Also $d_l(y_{2n}, Sy_{2n}) = d_l(y_{2n}, y_{2n+1})$, $d_l(y_{2n+1}, Ty_{2n+1}) = d_l(y_{2n+1}, y_{2n+2})$. We represent this type of sequence by $\{TS(y_n)\}$. We say that $\{TS(y_n)\}$ is a sequence in $E$ generated by $y_0$.

We start this section with the definition.

Definition 2.3.1 Let $(E, d_l)$ be a complete $D.M.S$ and $S, T : E \to P(E)$ be two setvalued mappings. The pair $(S, T)$ is said to be a pair of new Ciric type rational $F$–contraction, if for all $w, e \in \{TS(w_n)\}$, we have

$$\tau + F(H_d(Sw, Te)) \leq F(O_l(w, e))$$

(2.19)
where $F \in \triangle F$ and $\tau > 0$, and

$$O_l(w, e) = \max \left\{ d_l(w, e), \frac{d_l(w, Sw), d_l(e, Te)}{1 + d_l(w, e)}, d_l(w, Sw), d_l(e, Te) \right\}.$$ (2.20)

**Theorem 2.3.2** Let $(E, d_l)$ be a complete D.M.S and $(S, T)$ be a pair of new Ciric type rational multivalued $F$-contraction. Then $\{TS(w_n)\} \to u \in E$. Moreover, if (2.19) also holds for $u$, then $S$ and $T$ have a C.F.P $u$ in $E$ and $d_l(u, u) = 0$.

**Proof.** If, $O_l(w, e) = 0$, then clearly $w = e$ is a C.F.P of $S$ and $T$. Then we have no need to prove and our proof is complete. Let $O_l(e, w) > 0$ for all $w, e \in \{TS(z_n)\}$ with $w \neq e$. Then from (2.19), and Lemma 1.1.6 we have

$$F(d_l(w_{2i+1}, w_{2i+2})) \leq F(H_d(Sw_{2i}, Tw_{2i+1})) \leq F(O_l(w_{2i}, w_{2i+1})) - \tau$$

for each $i \in \mathbb{N} \cup \{0\}$, where

$$O_l(w_{2i}, w_{2i+1}) = \max \left\{ d_l(w_{2i}, w_{2i+1}), \frac{d_l(w_{2i}, Sw_{2i}), d_l(w_{2i+1}, Tw_{2i+1})}{1 + d_l(w_{2i}, w_{2i+1})}, \right\}$$

$$= \max \left\{ d_l(w_{2i}, w_{2i+1}), \frac{d_l(w_{2i}, Sw_{2i}), d_l(w_{2i+1}, Tw_{2i+1})}{1 + d_l(w_{2i}, w_{2i+1})}, \right\}$$

$$= \max \{d_l(w_{2i}, w_{2i+1}), d_l(w_{2i+1}, w_{2i+2})\}.$$ (2.21)

If, $O_l(w_{2i}, w_{2i+1}) = d_l(w_{2i+1}, w_{2i+2})$, then

$$F(d_l(w_{2i+1}, w_{2i+2})) \leq F(d_l(w_{2i+1}, w_{2i+2})) - \tau,$$

It is not true due to (F1). Therefore,

$$F(d_l(w_{2i+1}, w_{2i+2})) \leq F(d_l(w_{2i}, w_{2i+1})) - \tau,$$ (2.22)

for each $i$ belongs to $\mathbb{N} \cup \{0\}$. Similarly, we have

$$F(d_l(w_{2i}, w_{2i+1})) \leq F(d_l(w_{2i-1}, w_{2i})) - \tau,$$ (2.22)
for each $i$ belongs to $\mathbb{N} \cup \{0\}$. By using (2.22) in (2.21), we have

$$F(d_i(w_{2i+1}, w_{2i+2})) \leq F(d_i(w_{2i-1}, w_{2i})) - 2\tau.$$ 

Replicating these steps, we have

$$F(d_i(w_{2i+1}, w_{2i+2})) \leq F(d_i(w_0, w_1)) - (2i + 1)\tau.$$ 

(2.23)

Similarly, we have

$$F(d_i(w_{2i}, w_{2i+1})) \leq F(d_i(w_0, w_1)) - 2i\tau,$$ 

(2.24)

We can write (2.23) and (2.24) jointly as

$$F(d_i(w_n, w_{n+1})) \leq F(d_i(w_0, w_1)) - n\tau.$$ 

(2.25)

By using limit $n \to \infty$, each sides of (2.25), we have

$$\lim_{n \to \infty} F(d_i(w_n, w_{n+1})) = -\infty.$$ 

(2.26)

Since $F \in \triangle_F$,

$$\lim_{n \to \infty} d_i(w_n, w_{n+1}) = 0.$$ 

(2.27)

By (2.25), for every $n$ belongs to $\mathbb{N}$, we get

$$(d_i(w_n, w_{n+1}))^k((F(d_i(w_n, w_{n+1}))) - F(d_i(w_0, w_1))) \leq -(d_i(w_n, w_{n+1}))^k n\tau \leq 0.$$ 

(2.28)

Using the inequalities (2.26), (2.27) and applying $n \to \infty$ in (2.28), we get

$$\lim_{n \to \infty} (n(d_i(w_n, w_{n+1}))^k) = 0.$$ 

(2.29)

Since (2.29) holds, there exist $n_1 \in \mathbb{N}$, such that $n(d_i(w_n, w_{n+1}))^k \leq 1$ for each $n \geq n_1$ or,

$$d_i(w_n, w_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}$$ for each $n \geq n_1$. 

(2.30)
Using (2.30), we get form $m > n > n_1$,
\[
d_l(w_n, w_m) \leq d_l(w_n, w_{n+1}) + d_l(w_{n+1}, w_{n+2}) + \cdots + d_l(w_{m-1}, w_m)
\]
\[
= \sum_{i=n}^{m-1} d_l(w_i, w_{i+1}) \leq \sum_{i=n}^{\infty} d_l(w_i, w_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^k}.
\]

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^k}$ entails $\lim_{n,m \to \infty} d_l(w_n, w_m) = 0$. Hence, $\{TS(w_n)\}$ is a Cauchy sequence in $(E, d_l)$. Since $(E, d_l)$ is a complete D.M.S, so there exists $u \in E$ such that $\{TS(w_n)\} \to u$ that is
\[
\lim_{n \to \infty} d_l(w_n, u) = 0. \tag{2.31}
\]

Now, by Lemma 1.1.6, we have
\[
\tau + F(d_l(w_{2n+1}, Tu)) \leq \tau + F(Hd_l(Sw_{2n}, Tu)), \tag{2.32}
\]
\[
\text{As (2.19) also must be holds for } u, \text{ then}
\]
\[
\tau + F(d_l(w_{2n+1}, Tu)) \leq F(O_l(w_{2n}, u)), \tag{2.33}
\]

where,
\[
O_l(w_{2n}, u) = \max \left\{ d_l(w_{2n}, u), \frac{d_l(w_{2n}, Sw_{2n}) \cdot d_l(u, Tu)}{1 + d_l(w_{2n}, u)}, d_l(w_{2n}, Sw_{2n}), d_l(u, Tu) \right\}
\]
\[
= \max \left\{ d_l(w_{2n}, u), \frac{d_l(w_{2n}, w_{2n+1}) \cdot d_l(u, Tu)}{1 + d_l(w_{2n}, u)}, d_l(w_{2n}, w_{2n+1}), d_l(u, Tu) \right\}.
\]

Letting limit $n \to \infty$, and by using (2.31), we have
\[
\lim_{n \to \infty} O_l(w_{2n}, u) = d_l(u, Tu). \tag{2.34}
\]

Since $F$ is strictly increasing, then (2.33) implies
\[
d_l(w_{2n+1}, Tu) < O_l(w_{2n}, u).
\]
By using limit $n \to \infty$, and using (2.34), we get

$$d_l(u, Tu) < d_l(u, Tu).$$

It is not true, hence $d_l(u, Tu) = 0$ or $u \in Tu$. Similarly by using (2.31) and Lemma1.1.6 and the inequality

$$\tau + F(d_l(w_{2n+2}, Su)) \leq \tau + F(H_{d_l}(Tw_{2n+1}, Su)),$$

we can setup that $d_l(u, Su) = 0$ or $u \in Su$. Hence $S$ and $T$ have a C.F.P $u$ in $E$. Now,

$$d_l(u, u) \leq d_l(u, Tu) + d_l(Tu, u) \leq 0.$$

This implies that $d_l(u, u) = 0$. ■

**Example 2.3.3** Let $E = \{0\} \cup Q^+$ and $d_l(w, e) = w + e$. Then $(E, d_l)$ is a complete $D.M.S.$

Define $S, T : E \to P(E)$ as follows:

$$S(w) = \left[ \frac{1}{3}w, \frac{2}{3}w \right]$$

and

$$T(w) = \left[ \frac{1}{5}w, \frac{2}{5}w \right]$$

for all $w \in E$.

If, $\tau + F(H_{d_l}(Sw, Te)) \leq F(O_l(w, e))$, holds. Define $F : R^+ \to R$ by $F(w) = \ln(w)$ for every $w \in R^+$ and $\tau > 0$. As $w, e \in E$, $\tau = \ln(1.2)$ by taking $w_0 = 7$, we define the sequence

$$\{TS(w_n)\} = \{7, \frac{7}{3}, \frac{7}{15}, \frac{7}{45}, \cdots \}$$

in $E$ generated by $w_0 = 7$. We have

$$H_{d_l}(Sw, Te) = \max \left\{ \sup_{a \in Sw} d_l(a, Te), \sup_{g \in Te} d_l(Sw, g) \right\}$$

$$= \max \left\{ \sup_{a \in Sw} d_l \left( a, \left[ \frac{e}{2}, \frac{2e}{5} \right] \right), \sup_{g \in Te} d_l \left( \left[ \frac{w}{3}, \frac{2w}{3} \right], g \right) \right\}$$

$$= \max \left\{ d_l \left( \frac{2w}{3}, \frac{e}{5} \right), d_l \left( \frac{w}{3}, \frac{2e}{5} \right) \right\}$$

$$= \max \left\{ \frac{2w}{3} + \frac{e}{5}, \frac{w}{3} + \frac{2e}{5} \right\}$$

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where

\[
O_l(w, e) = \max \left\{ d_l(w, e), \frac{d_l(w, \frac{2w}{3})}{1 + d_l(w, e)}, \frac{d_l(e, \frac{e}{5})}{1 + d_l(e, e)}, \frac{d_l(w, \frac{w}{3})}{1 + d_l(w, e)}, \frac{d_l(e, \frac{e}{5})}{1 + d_l(e, e)} \right\}
\]

\[
= \max \left\{ d_l(w, e), d_l(w, \frac{w}{3}), d_l(e, \frac{e}{5}) \right\}
\]

\[
= \max \left\{ w + e, \frac{8we}{15(1 + w + e)}, \frac{4w + 6e}{3} \right\}
\]

Case (i). If, \( \max \{ \frac{2w}{3} + \frac{e}{5}, \frac{w}{3} + \frac{2e}{5} \} = \frac{w}{3} + \frac{2e}{5} \), and \( \tau = \ln(1.2) \), then we have

\[
10w + 12e \leq 25w + 25e
\]

\[
\frac{6}{5} \left( \frac{w}{3} + \frac{2e}{5} \right) \leq w + e
\]

\[
\ln(1.2) + \ln \left( \frac{w}{3} + \frac{2e}{5} \right) \leq \ln(w + e).
\]

which implies that,

\[
\tau + F(H_d(\delta(Sw, Te)) \leq F(O_l(w, e)).
\]

Case (ii). Similarly, if \( \max \{ \frac{2w}{3} + \frac{e}{5}, \frac{w}{3} + \frac{2e}{5} \} = \frac{2w}{3} + \frac{e}{5} \), and \( \tau = \ln(1.2) \), then we have

\[
20w + 6e \leq 25w + 25e
\]

\[
\frac{6}{5} \left( \frac{4w}{3} + \frac{2e}{5} \right) \leq w + e
\]

\[
\ln(1.2) + \ln \left( \frac{2w}{3} + \frac{e}{5} \right) \leq \ln(w + e).
\]

Hence,

\[
\tau + F(H_d(\delta(Sw, Te)) \leq F(O_l(w, e)).
\]

Hence all hypothesis of Theorem 2.3.2 are proved so \((S, T)\) have a C.F.P.

**Corollary 2.3.4** Let \((E, d_l)\) be a čompleše D.M.S and \(S : E \rightarrow P(E)\) is a setvalued maipping such that

\[
\tau + F(H_d(\delta(Sw, Se)) \leq F(O_l(w, e))
\]

(2.35)
for all $w, e \in \{SS(w_n)\}$, where $F \in \triangle_F$ and $\tau > 0$, and
\[
O_l(w, e) = \max \left\{ d_l(w, e), \frac{d_l(w, Sw) \cdot d_l(e, Se)}{1 + d_l(w, e)}, d_l(w, Sw), d_l(e, Se) \right\}.
\]
Then $\{SS(w_n)\} \to u \in E$. Furthermore, if (2.35) holds for $u$, then $u$ be the has a fixed point of $S$ in $E$ and $d_l(u, u) = 0$.

**Remark 2.3.5** By setting the different values of $O_l(l, w)$ in equation (2.20), we can achieve different results as corollaries of Theorem 2.3.2.

1. $O_l(l, w) = d_l(l, w)$
2. $O_l(l, w) = \frac{d_l(l, Sl) \cdot d_l(w, T w)}{1 + d_l(l, w)}$
3. $O_l(l, w) = d_l(l, Sl)$
4. $O_l(l, w) = d_l(w, T w)$
5. $O_l(l, w) = \max \left\{ d_l(l, w), \frac{d_l(l, Sl) \cdot d_l(w, T w)}{1 + d_l(l, w)} \right\}$
6. $O_l(l, w) = \max \left\{ d_l(l, w), d_l(l, Sl) \right\}$
7. $O_l(l, w) = \max \left\{ d_l(l, w), d_l(w, T w) \right\}$
8. $O_l(l, w) = \max \left\{ \frac{d_l(l, Sl) \cdot d_l(w, T w)}{1 + d_l(l, w)}, d_l(l, Sl) \right\}$
9. $O_l(l, w) = \max \left\{ \frac{d_l(l, Sl) \cdot d_l(w, T w)}{1 + d_l(l, w)}, d_l(w, T w) \right\}$
10. $O_l(l, w) = \max \left\{ d_l(l, Sl), d_l(w, T w) \right\}$
11. $O_l(l, w) = \max \left\{ d_l(l, w), \frac{d_l(l, Sl) \cdot d_l(w, T w)}{1 + d_l(l, w)}, d_l(l, Sl) \right\}$
12. $O_l(l, w) = \max \left\{ d_l(l, w), \frac{d_l(l, Sl) \cdot d_l(w, T w)}{1 + d_l(l, w)}, d_l(w, T w) \right\}$
13. $O_l(l, w) = \max \left\{ d_l(l, w), d_l(l, Sl), d_l(w, T w) \right\}$

**Theorem 2.3.6** Let $(E, d_l)$ be a complete D.M.S and $S, T : E \to P(E)$ be the multivalued mappings. Assume that if $F \in \triangle_F$ and $\tau \in \mathbb{R}^+$ such that
\[
\tau + F(H_{d_l}(Sw, Te)) \leq F \left( \delta_1 d_l(w, e) + \delta_2 d_l(w, Sw) + \delta_3 d_l(e, Te) + \delta_4 \frac{d_l^2(w, Sw) \cdot d_l(e, Te)}{1 + d_l^2(w, e)} \right)
\]
for all $w, e \in \{TS(w_n)\}$, with $w \neq e$ where $\delta_1, \delta_2, \delta_3, \delta_4 > 0$, $\delta_1 + \delta_2 + \delta_3 + \delta_4 = 1$ and $\delta_3 + \delta_4 \neq 1$.\]
Then \( \{TS(w_n)\} \to u \in E \). Furthermore, if (2.36) also holds for \( u \), then \( u \) is the C.F.P of \( S \) and \( T \).

**Proof.** As \( w_1 \in Sw_0 \) and \( w_2 \in Tw_1 \), by using Lemma 1.1.6

\[
\tau + F(d_l(w_1, w_2)) = \tau + F(d_l(w_1, Tw_1)) \leq \tau + F(H_{d_l}(Sw_0, Tw_1)) 
\leq F \left( \delta_1 d_l(w_0, w_1) + \delta_2 d_l(w_0, w_1) + \delta_3 d_l(w_1, Tw_1) + \delta_4 \frac{d^2_l(w_0, Sw_0, d_l(w_1, Tw_1))}{1 + d^2_l(w_0, w_1)} \right) 
\leq F \left( \delta_1 d_l(w_0, w_1) + \delta_2 d_l(w_0, w_1) + \delta_3 d_l(w_1, w_2) + \delta_4 d_l(w_1, Tw_1) \left( \frac{d^2_l(w_0, w_1)}{1 + d^2_l(w_0, w_1)} \right) \right) 
\leq F((\delta_1 + \delta_2)d_l(w_0, w_1) + (\delta_3 + \delta_4)d_l(w_1, w_2)).
\]

Since \( F \) is strictly increasing, we have

\[
d_l(w_1, w_2) < (\delta_1 + \delta_2)d_l(w_0, w_1) + (\delta_3 + \delta_4)d_l(w_1, w_2) 
< \left( \frac{\delta_1 + \delta_2}{1 - \delta_3 - \delta_4} \right) d_l(w_0, w_1).
\]

From \( \delta_1 + \delta_2 + \delta_3 + \delta_4 = 1 \) and \( \delta_3 + \delta_4 \neq 1 \), we deduce \( 1 - \delta_3 - \delta_4 > 0 \) and so

\[
d_l(w_1, w_2) < d_l(w_0, w_1).
\]

Consequently

\[
F(d_l(w_1, w_2)) \leq F(d_l(w_0, w_1)) - \tau.
\]

As we have \( w_{2i+1} \in Sw_{2i} \) and \( w_{2i+2} \in Tw_{2i+1} \) then from (2.36), and Lemma 1.1.6 we have

\[
\tau + F(d_l(w_{2i+1}, w_{2i+2})) = \tau + F(d_l(w_{2i+1}, Tw_{2i+1})) \leq \tau + F(H_{d_l}(Sw_{2i}, Tw_{2i+1})) 
\leq F(\delta_1 d_l(w_{2i}, w_{2i+1}) + \delta_2 d_l(w_{2i}, Sw_{2i}) 
+ \delta_3 d_l(w_{2i-1}, Tw_{2i+1}) + \delta_4 \frac{d^2_l(w_{2i}, Sw_{2i}, d_l(w_{2i+1}, Tw_{2i+1}))}{1 + d^2_l(w_{2i}, w_{2i+1})} \right) 
\leq F(\delta_1 d_l(w_{2i}, w_{2i+1}) + \delta_2 d_l(w_{2i}, w_{2i+1}) + \delta_3 d_l(w_{2i+1}, w_{2i+2}) 
+ \delta_4 d_l(w_{2i+1}, w_{2i+2}) \left( \frac{d^2_l(w_{2i}, w_{2i+1})}{1 + d^2_l(w_{2i}, w_{2i+1})} \right) \right).
\]
\[
\leq \ F(\delta_1 d_l(w_{2i}, w_{2i+1}) + \delta_2 d_l(w_{2i}, w_{2i+1}) + \delta_3 d_l(w_{2i+1}, w_{2i+2})
+ \delta_4 d_l(w_{2i+1}, w_{2i+2})).
\]

Since \( F \) is strictly increasing, and \( \delta_1 + \delta_2 + \delta_3 + \delta_4 = 1 \) where \( \delta_3 + \delta_4 \neq 1 \), we deduce \( 1 - \delta_3 - \delta_4 > 0 \) so we obtain

\[
d_l(w_{2i+1}, w_{2i+2}) < \delta_1 d_l(w_{2i}, w_{2i+1}) + \delta_2 d_l(w_{2i}, w_{2i+1}) + \delta_3 d_l(w_{2i+1}, w_{2i+2})
+ \delta_4 d_l(w_{2i+1}, w_{2i+2})
\leq (\delta_1 + \delta_2) d_l(w_{2i}, w_{2i+1}) + (\delta_3 + \delta_4) d_l(w_{2i+1}, w_{2i+2})
\]

This implies that,

\[
F(d_l(w_{2i+1}, w_{2i+2})) \leq F(d_l(w_{2i}, w_{2i+1})) - \tau
\]

Following similar reasons are present in Theorem 2.3.6, we have \( \{TS(w_n)\} \to u \) that is

\[
\lim_{n \to \infty} d_l(w_n, u) = 0. \quad (2.37)
\]

Now, by Lemma 1.6.1, we get

\[
\tau + F(d_l(w_{2n+1}, Tu)) \leq \tau + F(Hd_l(Sw_{2n}, Tu)),
\]

By using inequality (2.36), we have

\[
\tau + F(d_l(w_{2n+1}, Tu)) \leq \ F(\delta_1 d_l(w_{2n}, u) + \delta_2 d_l(w_{2n}, Sw_{2n}) + \delta_3 d_l(u, Sw_{2n})
+ \delta_4 d_l^2(w_{2n}, Sw_{2n})d_l(u, Tu)
\]

\[
\leq \ F(\delta_1 d_l(w_{2n}, u) + \delta_2 d_l(w_{2n}, w_{2n+1}) + \delta_3 d_l(u, Tu)
+ \delta_4 d_l^2(w_{2n}, w_{2n+1})d_l(u, Tu)
\]

\]

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Since \( F \) is the strictly increasing mappings, we have
\[
\begin{align*}
d_l(w_{2n+1}, Tu) &< \delta_1 d_l(w_{2n}, u) + \delta_2 d_l(w_{2n}, w_{2n+1}) + \delta_3 d_l(u, Tu) \\
&\quad + \delta_4 \frac{d_l^2(w_{2n}, w_{2n+1}) d_l(u, Tu)}{1 + d_l^2(w_{2n}, u)}.
\end{align*}
\]

Letting limit \( n \to \infty \), and using the inequality (2.37), we get
\[
d_l(u, Tu) < \delta_3 d_l(u, Tu).
\]

It is not true, hence \( d_l(u, Tu) = 0 \) or \( u \in Tu \). Similarly by using (2.36), (2.37), Lemma1.1.6 and the inequality
\[
\tau + F(d_l(w_{2n+2}, Su)) \leq \tau + F(H(d_l(Tw_{2n+1}, Su))
\]
we can show that \( d_l(u, Su) = 0 \) or \( u \in Su \). Hence the \( S \) and \( T \) have a C.F.P \( u \) in \( (E, d_l) \). Now,
\[
d_l(u, u) \leq d_l(u, Tu) + d_l(Tu, u) \leq 0.
\]

This implies \( d_l(u, u) = 0 \). ■

**Remark 2.3.7** We can achieve all theorems related with partial metric and metric spaces as the corollaries of the above theorems, which are not available in the literature.

We are proving results in this section by using the above definition.

**Definition 2.3.8** Let \( (E, d_l) \) be a complete D.M.S. The mappings \( S, T : E \to E \) are said to be a pair of new Ciric type rational \( F \)-contractioň, if for each \( w, e \in E \), we have
\[
\tau + F(d_l(Sw, Te)) \leq F(O_l(w, e))
\]
(2.38)

where \( F \in \Delta_F \) and \( \tau > 0 \), and
\[
O_l(w, e) = \max \left\{ d_l(w, e), \frac{d_l(w, Sw) d_l(e, Te)}{1 + d_l(w, e)}, d_l(w, Sw), d_l(e, Te) \right\}.
\]
(2.39)

The succeeding theorem is the one of our major results.

**Theorem 2.3.9** Let \( (E, d_l) \) be a complete D.M.S and \( (S, T) \) be a pair of new Ciric type
rational $F$–contraction. Then $S$ and $T$ have a C.F.P $q$ in $E$ and $d_t(q,q) = 0$.

The proof of is similar as given for Theorem 2.3.2.

In above section, we derive an application of fixed point theorem 2.3.9 in form of Volterra type integral equations.

\[ q(t) = \int_0^t L_1(t, n, q(n)) dn, \quad (2.40) \]

\[ g(t) = \int_0^t L_2(t, n, g(n)) dn \quad (2.41) \]
for all $t \in [0, 1]$. We find the solution of (2.40) and (2.41). Let $E = \{ f : f$ is continuous function from $[0, 1]$ to $\mathbb{R}_+ \}$, endowed with the complete $D.M.S$. For $q \in E$, define norm as:

\[ \|q\|_\tau = \sup_{t \in [0,1]} \{|q(t)| e^{-\tau t}\}, \text{ where } \tau > 0 \text{ is taken arbitrary. Then define} \]

\[ d_\tau(q, g) = \sup_{t \in [0,1]} \{|q(t) + g(t)| e^{-\tau t}\} = \|q + g\|_\tau \]
for each $q, g \in E$, with these settings, $(E, d_\tau)$ becomes a complete $D.M.S$.

**Theorem 2.3.10** Let the conditions (i) and (ii) are hold:

(i) $L_1, L_2 : [0, 1] \times [0, 1] \times E \to \mathbb{R}$;

(ii) Define

\[ S_q(t) = \int_0^t L_1(t, n, q(n)) dn, \]
\[ T_g(t) = \int_0^t L_2(t, n, g(n)) dn. \]

Suppose there exist $\tau > 0$, such that

\[ |L_1(t, n, u) + L_2(t, n, g)| \leq \frac{\tau K(q, g)}{(\tau \sqrt{\|K(q, g)\|_\tau + 1})^2} \]
for each \( t, n \in [0, 1] \) and \( q, g \in E \), where

\[
K(q, g) = \max \left\{ |q(t) + g(t)|, \frac{|q(t) + Sq(t)| |q(t) + Tg(t)|}{1 + |q(t) + g(t)|}, |q(t) + Sq(t)|, |g(t) + Tg(t)| \right\},
\]

Then integral equations (2.40) and (2.41) has a solution.

By assumption (ii)

\[
|Sq(t) + Tg(t)| = \int_0^t \left| L_1(t, n, q(n)) + L_2(t, n, g(n)) \right| dn,
\]

\[
\leq \int_0^t \frac{\tau}{(\tau \sqrt{\|M(q, g)\|_\tau + 1})^2} ([M(q, g)]e^{-\tau n}) e^{\tau n} dn,
\]

\[
\leq \int_0^t \frac{\tau}{(\tau \sqrt{\|M(q, g)\|_\tau + 1})^2} \|M(q, g)\|_\tau e^{\tau n} dn,
\]

\[
\leq \frac{\tau \|M(q, g)\|_\tau}{(\tau \sqrt{\|M(q, g)\|_\tau + 1})^2} \int_0^t e^{\tau n} dn,
\]

\[
\leq \frac{\|M(q, g)\|_\tau}{(\tau \sqrt{\|M(q, g)\|_\tau + 1})^2} e^t.
\]

This implies

\[
|Sq(t) + Tg(t)| e^{-\tau t} \leq \frac{\|K(q, g)\|_\tau}{(\tau \sqrt{\|K(q, g)\|_\tau + 1})^2}.
\]

\[
\|Sq(t) + Tg(t)\|_\tau \leq \frac{\|K(q, g)\|_\tau}{(\tau \sqrt{\|M(q, g)\|_\tau + 1})^2}.
\]

\[
\frac{\tau \sqrt{\|K(q, g)\|_\tau + 1}}{\sqrt{\|K(q, g)\|_\tau}} \leq \frac{1}{\sqrt{\|Sq(t) + Tg(t)\|_\tau}}.
\]

\[
\tau + \frac{1}{\sqrt{\|K(q, g)\|_\tau}} \leq \frac{1}{\sqrt{\|Sq(t) + Tg(t)\|_\tau}}.
\]

which further implies

\[
\tau - \frac{1}{\sqrt{\|Sq(t) + Tg(t)\|_\tau}} \leq \frac{-1}{\sqrt{\|K(q, g)\|_\tau}}.
\]

So, all the hypothesis of Theorem 2.3.9 are proved for \( F(w) = \frac{-1}{\sqrt{w}}; w > 0 \) and \( d_\tau(q, g) = \|q + g\|_\tau \).

Hence integral equations (2.40) and (2.41) has a unique solution.
2.4 Multivalued Fixed Point Results for a New Generalized \( F \)-Dominated Mappings with Application

The given results in this section can be seen in [46].

Let \((\hat{Z}, d_l)\) be a D.M.S, \(c_0 \in \hat{Z} \& \check{S}, \bar{T} : \hat{Z} \to P(\hat{Z})\) be the setvalued maps on \(\hat{Z}\). Let \(c_1 \in \check{S}c_0\) be an element such that \(d_l(c_0, \check{S}c_0) = d_l(c_0, c_1)\). Let \(c_2 \in Tc_1\) be such that \(d_l(c_1, Tc_1) = d_l(c_1, c_2)\). Let \(c_3 \in \check{S}c_2\) be such that \(d_l(c_2, \check{S}c_2) = d_l(c_2, c_3)\). Proceeding this method, we get a sequence \(c_n\) in \(\hat{Z}\) so as \(c_{2n+1} \in \check{S}c_{2n}\) and \(c_{2n+2} \in Tc_{2n+1}\), where \(n = 0, 1, 2, \ldots\). Also \(d_l(c_{2n}, \check{S}c_{2n}) = d_l(c_{2n}, c_{2n+1})\), \(d_l(c_{2n+1}, Tc_{2n+1}) = d_l(c_{2n+1}, c_{2n+2})\). We represent this type of sequence by \(\{\check{T}\check{S}(c_n)\}\).

**Theorem 2.4.1** Let \((\hat{Z}, d_l)\) be a complete D.M.S. Suppose a function \(\alpha : \hat{Z} \times \hat{Z} \to [0, \infty)\) exists. Let, \(\check{r} > 0\), \(c_0 \in \overline{B_{d_l}(c_0, \check{r})} \subseteq \hat{Z} \& \check{S}, \bar{T} : \hat{Z} \to P(\hat{Z})\) be the semi \(\alpha\)-dominated mappings on \(\overline{B_{d_l}(c_0, \check{r})}\). Assume that, for some \(\tau > 0\),

\[
\max\{\tau + \alpha(x, \check{S}e)F(Hd_l(\check{S}e, \bar{T}y)), \tau + \alpha(y, \bar{T}y)F(Hd_l(\bar{T}y, \check{S}e))\} \\
\leq F\left(\eta_1 d_l(\check{e}, \check{y}) + \eta_2 d_l(\check{e}, \check{S}e) + \eta_3 d_l(\check{e}, \bar{T}y) + \eta_4 \frac{d_l^2(\check{e}, \check{S}e) + d_l(\bar{T}y, \check{S}e)}{1 + d_l^2(\check{e}, \check{y})}\right) \quad (2.42)
\]

for each \(\check{e}, \check{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{\check{T}\check{S}(c_n)\}\) with either \(\alpha(\check{e}, \check{y}) \geq 1\) or \(\alpha(\check{y}, \check{e}) \geq 1\) where \(\eta_1, \eta_2, \eta_3, \eta_4 > 0\), \(\eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1\) and

\[
d_l(c_0, \check{S}c_0) \leq (1 - \lambda)\check{r}, \quad (2.43)
\]

where \(\lambda = \frac{\eta_1 + \eta_2 + \eta_4}{1 - \eta_3 - \eta_4}\) and \(\eta_3 + \eta_4 \neq 1\). Then \(\{\check{T}\check{S}(c_n)\}\) be the sequence in \(\overline{B_{d_l}(c_0, \check{r})}\), \(\alpha(c_n, c_{n+1}) \geq 1\) for each \(n\) belongs to \(N \cup \{0\}\) and \(\{\check{T}\check{S}(c_n)\} \to \check{u} \in \overline{B_{d_l}(c_0, \check{r})}\). Also if the inequality \(2.42\) holds for \(\check{u}\) and either \(\alpha(c_n, \check{u}) \geq 1\) or \(\alpha(\check{u}, c_n) \geq 1\), then \(\check{u}\) is the C.F.P of \(\check{S}\) and \(\bar{T}\) in \(\overline{B_{d_l}(c_0, \check{r})}\).

**Proof.** Consider a sequence \(\{\check{T}\check{S}(c_n)\}\). From \(2.43\), we get

\[
d_l(c_0, c_1) \leq d_l(c_0, \check{S}c_0) \leq \check{r}.
\]

It means that,

\[
c_1 \in \overline{B_{d_l}(c_0, \check{r})}.
\]

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Let $c_2, \ldots, c_j \in B_d(c_0, \bar{r})$ for every $j$ belongs to $N$. If $j = 2i + 1$, where $i = 1, 2, \ldots, \frac{j-1}{2}$. Since $\check{S}, \check{T} : \check{Z} \to P(\check{Z})$ be a semi $\alpha_*$-dominated mappings on $B_d(c_0, \bar{r})$, so $\alpha_*(c_{2i}, \check{S}c_{2i}) \geq 1$ and $\alpha_*(c_{2i+1}, \check{T}c_{2i+1}) \geq 1$. As $\alpha_*(c_{2i}, \check{S}c_{2i}) \geq 1$, this implies $\inf\{\alpha(c_{2i}, b) : b \in \check{S}c_{2i}\} \geq 1$. Also $c_{2i+1} \in \check{S}c_{2i}$, so $\alpha(c_{2i}, c_{2i+1}) \geq 1$. Now by using Lemma 1.1.6, have,

$$\tau + F(d_i(c_{2i+1}, c_{2i+2})) \leq \tau + F(H_d(\check{S}c_{2i}, \check{T}c_{2i+1}))$$

$$\leq \max\{\tau + \alpha_*(c_{2i}, \check{S}c_{2i})F(H_d(\check{S}c_{2i}, \check{T}c_{2i+1})),$$

$$\tau + \alpha_*(c_{2i+1}, \check{T}c_{2i+1})F(H_d(\check{T}c_{2i+1}, \check{S}c_{2i}))\}$$

$$\leq F[\eta_1 d_i(c_{2i}, c_{2i+1}) + \eta_2 d_i(c_{2i}, \check{S}c_{2i}) + \eta_3 d_i(c_{2i}, \check{T}c_{2i+1})$$

$$+ \eta_4 \frac{d_i^2(c_{2i}, \check{S}c_{2i})}{1 + d_i^2(c_{2i}, c_{2i+1})} d_i(c_{2i+1}, \check{T}c_{2i+1})]$$

$$\leq F[\eta_1 d_i(c_{2i}, c_{2i+1}) + \eta_2 d_i(c_{2i}, c_{2i+1})$$

$$+ \eta_3 d_i(c_{2i}, c_{2i+1}) + \eta_3 d_i(c_{2i+1}, c_{2i+2})$$

$$+ \eta_4 \frac{d_i^2(c_{2i}, c_{2i+1})}{1 + d_i^2(c_{2i}, c_{2i+1})} d_i(c_{2i+1}, c_{2i+2})]$$

$$\leq F(\eta_1 + \eta_2 + \eta_3) d_i(c_{2i}, c_{2i+1}) - \tau,$$

this implies

$$F(d_i(c_{2i+1}, c_{2i+2})) \leq F(\eta_1 + \eta_2 + \eta_3) d_i(c_{2i}, c_{2i+1})$$

$$+ (\eta_3 + \eta_4) d_i(c_{2i+1}, c_{2i+2}),$$

for each $j$ belongs to $N$. As $F$ is strictly increasing, so we obtain

$$d_i(c_{2i+1}, c_{2i+2}) < (\eta_1 + \eta_2 + \eta_3) d_i(c_{2i}, c_{2i+1})$$

$$+ (\eta_3 + \eta_4) d_i(c_{2i+1}, c_{2i+2})$$

$$(1 - \eta_3 - \eta_4) d_i(c_{2i+1}, c_{2i+2}) < (\eta_1 + \eta_2 + \eta_3) d_i(c_{2i}, c_{2i+1})$$

$$d_i(c_{2i+1}, c_{2i+2}) < \left(\frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_3 - \eta_4}\right) d_i(c_{2i}, c_{2i+1}).$$
Here \( \lambda = \left( \frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_3 - \eta_4} \right) < 1 \). Hence

\[
d_l(c_{2i+1}, c_{2i+2}) < \lambda d_l(c_{2i}, c_{2i+1}) < \lambda^2 d_l(c_{2i-1}, c_{2i}) < \cdots < \lambda^i d_l(c_0, c_1). \tag{2.44}
\]

Now,

\[
d_l(c_0, c_{j+1}) \leq d_l(c_0, c_1) + d_l(c_1, c_2) + \cdots + d_l(c_j, c_{j+1}) \\
\leq d_l(c_0, c_1) (1 + \lambda + \cdots + \lambda^j) \\
\leq (1 - \lambda) \rho (1 - \frac{\lambda^{j+1}}{1 - \lambda}) < \rho.
\]

Thus \( c_{j+1} \in B_d(c_0, \rho) \). Hence \( c_\bar{n} \in B_d(c_0, \rho) \), for each \( \bar{n} \) belongs to \( N \). Proceeding this method, we get

\[
\tau + F(d_l(c_\bar{n}, c_{\bar{n}+1})) \leq \tau + F(H_{d_l}(\bar{S}c_{\bar{n}-1}, \bar{T}c_\bar{n})) \\
\leq \max\{\tau + \alpha_*(c_{\bar{n}-1}, \bar{S}c_{\bar{n}-1}) F(H_{d_l}(\bar{S}c_{\bar{n}-1}, \bar{T}c_\bar{n})), \tau + \alpha_*(c_\bar{n}, \bar{T}c_\bar{n}) F(H_{d_l}(\bar{T}c_\bar{n}, \bar{S}c_{\bar{n}-1}))\} \\
\leq F[\eta_1 d_l(c_{\bar{n}-1}, c_\bar{n}) + \eta_2 d_l(c_{\bar{n}-1}, \bar{S}c_{\bar{n}-1}) + \eta_3 d_l(c_{\bar{n}-1}, \bar{T}c_\bar{n}) + \eta_4 \frac{d^2_l(c_{\bar{n}-1}, \bar{S}c_{\bar{n}-1})}{1 + d^2_l(c_{\bar{n}-1}, c_\bar{n})} d_l(c_\bar{n}, \bar{T}c_\bar{n})] \\
\leq F[\eta_1 d_l(c_{\bar{n}-1}, c_\bar{n}) + \eta_2 d_l(c_{\bar{n}-1}, c_\bar{n}) + \eta_3 d_l(c_{\bar{n}-1}, c_\bar{n}) + \eta_4 \frac{d^2_l(c_{\bar{n}-1}, c_\bar{n})}{1 + d^2_l(c_{\bar{n}-1}, c_\bar{n})} d_l(c_\bar{n}, c_{\bar{n}+1})] \\
\leq F[(\eta_1 + \eta_2 + \eta_3) d_l(c_{\bar{n}-1}, c_\bar{n}) + (\eta_3 + \eta_4) d_l(c_\bar{n}, c_{\bar{n}+1})] - \tau,
\]

this implies

\[
F(d_l(c_\bar{n}, c_{\bar{n}+1})) \leq F[(\eta_1 + \eta_2 + \eta_3) d_l(c_{\bar{n}-1}, c_\bar{n}) + (\eta_3 + \eta_4) d_l(c_\bar{n}, c_{\bar{n}+1})],
\]

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for each $\tilde{n}$ belongs to $N$. As $F$ is strictly increasing

$$d_l(c_{\tilde{n}}, c_{\tilde{n}+1}) < (\eta_1 + \eta_2 + \eta_3)d_l(c_{\tilde{n}-1}, c_{\tilde{n}}) + (\eta_3 + \eta_4)d_l(c_{\tilde{n}}, c_{\tilde{n}+1})$$

$$d_l(c_{\tilde{n}}, c_{\tilde{n}+1}) < (\eta_1 + \eta_2 + \eta_3)d_l(c_{\tilde{n}-1}, c_{\tilde{n}})$$

$$d_l(c_{\tilde{n}}, c_{\tilde{n}+1}) < \left(\frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_3 - \eta_4}\right)d_l(c_{\tilde{n}-1}, c_{\tilde{n}}).$$

Here $\lambda = \left(\frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_3 - \eta_4}\right) < 1$. Hence

$$d_l(c_{\tilde{n}}, c_{\tilde{n}+1}) < \lambda d_l(c_{\tilde{n}-1}, c_{\tilde{n}}) < d_l(c_{\tilde{n}-1}, c_{\tilde{n}}). \tag{2.45}$$

Consequently,

$$\tau + F(d_l(c_{\tilde{n}}, c_{\tilde{n}+1})) \leq F(d_l(c_{\tilde{n}-1}, c_{\tilde{n}})),$$

which implies,

$$F(d_l(c_{\tilde{n}}, c_{\tilde{n}+1})) \leq F(d_l(c_{\tilde{n}-1}, c_{\tilde{n}})) - \tau$$

$$\vdots$$

$$\leq F(d_l(c_0, c_1)) - \tilde{n}\tau.$$

Which implies

$$F(d_l(c_{\tilde{n}}, c_{\tilde{n}+1})) \leq F(d_l(c_0, c_1)) - \tilde{n}\tau. \tag{2.46}$$

And so $\lim\limits_{\tilde{n} \to \infty} F(d_l(c_{\tilde{n}}, c_{\tilde{n}+1})) = -\infty$. By $(F_2)$, we find that

$$\lim\limits_{\tilde{n} \to \infty} F(d_l(c_{\tilde{n}}, c_{\tilde{n}+1})) = 0. \tag{2.47}$$

We shall prove that $\{\tilde{T}\tilde{S}(c_{\tilde{n}})\}$ is Cauchy in $(\tilde{Z}, d_l)$. So, it suffices to show that $\lim\limits_{\tilde{n} \to \infty} d_l(c_{\tilde{n}}, c_m) = 0$. We explain by contradiction. Suppose there must be a $\in > 0$ and sequences $(\tilde{n}(q))$ and $(m(q))$ of natural no so as

$$m(q) > \tilde{n}(q) > q, \quad d_l(c_{\tilde{n}(q)}, c_{\tilde{n}(q)+1}) \geq \varepsilon, \quad d_l(c_{\tilde{n}(q)+1}, c_{m(q)}) < \varepsilon \quad \text{for each } q \in N. \tag{2.48}$$
By triangular inequality, we have

\[ d_l(c_n(q), c_m(q)) \leq d_l(c_n(q), c_n(q) + 1) + d_l(c_n(q) + 1, c_m(q)) \]
\[ \leq \epsilon + d_l(c_n(q), c_n(q) + 1) \]
\[ \leq \epsilon + d_l(c_n(q), c_n(q) + 1) \tag{2.49} \]

From (2.47) there exist \( q_1 \) belongs to \( N \) such that for each \( q \geq q_1 \).

\[ d_l(c_n(q), \tilde{T}c_n(q)) < \epsilon . \tag{2.50} \]

Combining (2.49) to (2.50) yeilds that

\[ d_l(c_n(q), c_m(q)) < 2 \epsilon, \text{ for each } q \geq q_1. \tag{2.51} \]

As \( \tilde{S}, \tilde{T} : \tilde{Z} \to P(\tilde{Z}) \) be a semi \( \alpha_* \)-dominated mapping on \( \bar{B}_{d_l}(c_0, \tau) \). So \( \alpha_*(c_n(q), \tilde{S}c_n(q)) \geq 1 \) and \( \alpha_*(c_m(q), \tilde{T}c_m(q)) \geq 1 \), for each \( m, \tilde{n} \) belongs to \( N \). Now, by using Lemma 1.1.6 and condition (2.42), we get

\[ F(\epsilon) \leq F(H_{d_l}(Sc_n(q), Tc_m(q))) \leq \max\{\tau + \alpha_*(c_n(q), Sc_n(q))F(H_{d_l}(Sc_n(q), Tc_m(q))) \}
\[ , \tau + \alpha_*(c_m(q), Tc_m(q))F(H_{d_l}(Tc_m(q), Sc_n(q)))\}\]
\[ \leq F[\eta_1d_l(c_n(q), c_m(q)) + \eta_2d_l(c_n(q), Sc_n(q))]
\[ + \eta_3d_l(c_n(q), Tc_m(q)) + \eta_4\frac{d^2_l(c_n(q), Sc_n(q)), d_l(c_m(q), Tc_m(q))}{1 + d^2_l(c_n(q), c_m(q))} - \tau \]
\[ F(\epsilon) \leq F[\eta_1d_l(c_n(q), c_m(q)) + \eta_2d_l(c_n(q), c_n(q) + 1) + \eta_3d_l(c_n(q), c_m(q) + 1)
\[ + \eta_4\frac{d^2_l(c_n(q), c_n(q) + 1), d_l(c_m(q), c_m(q) + 1)}{1 + d^2_l(c_n(q), c_m(q))} - \tau \]
\[ F(\epsilon) \leq F[\eta_1d_l(c_n(q), c_m(q)) + \eta_2d_l(c_n(q), c_n(q) + 1) + \eta_3d_l(c_n(q), c_m(q))
\[ + \eta_4\frac{d^2_l(c_n(q), c_n(q) + 1), d_l(c_m(q), c_m(q) + 1)}{1 + d^2_l(c_n(q), c_m(q))} - \tau \]

This means that,

\[ F(\epsilon) \leq F[2\eta_1 \epsilon + \eta_2 \epsilon + 3\eta_3 \epsilon + \eta_4 \epsilon] - \tau. \]
As, $2\eta_1 + \eta_2 + 3\eta_3 + \eta_4 < 1$, so we get

$$2\eta_1 + \eta_2 + 3\eta_3 + \eta_4 \leq \epsilon,$$

we deduce that

$$F(\epsilon) < F(\epsilon),$$

which is not true. Thus $\{T\hat{S}(c_n)\}$ be a Cauchy sequence in $(B_{d_l}(c_0, \bar{r}), d_l)$. Since $(B_{d_l}(c_0, \bar{r}), d_l)$ is a complete metric space, so there exist $\bar{u} \in B_{d_l}(c_0, \bar{r})$ such that $\{T\hat{S}(c_n)\} \to \bar{u}$ as $n \to \infty$ then

$$\lim_{n \to \infty} d_l(c_n, \bar{u}) = 0. \tag{2.52}$$

Since $\alpha_s(u, T\bar{u}) \geq 1$, and $\alpha_s(c_{2\lambda}, \tilde{S}_{c_{2\lambda}}) \geq 1$ by using Lemma 1.1.6, and the inequality (2.42), we have

$$F(d_l(c_{2\lambda+1}, T\bar{u})) \leq F(H_{d_l}(\tilde{S}_{c_{2\lambda}}, T\bar{u}))$$

$$\leq \max\{\tau + \alpha_s(c_{2\lambda}, \tilde{S}_{c_{2\lambda}})F(H_{d_l}(\tilde{S}_{c_{2\lambda}}, T\bar{u}))$$

$$\tau + \alpha_s(u, \tilde{T}\bar{u})F(H_{d_l}(\tilde{T}\bar{u}, \tilde{S}_{c_{2\lambda}}))\}$$

$$\leq F[\eta_1 d_l(c_{2\lambda}, \bar{u}) + \eta_2 d_l(\tilde{S}_{c_{2\lambda}}, \tilde{S}_{c_{2\lambda}}) + \eta_3 d_l(c_{2\lambda}, \tilde{T}\bar{u})$$

$$\eta_4 \frac{d_l^2(c_{2\lambda}, \tilde{S}_{c_{2\lambda}}), d_l(\bar{u}, \tilde{T}\bar{u})}{1 + d_l^2(c_{2\lambda}, \bar{u})} - \tau$$

$$\leq F[\eta_1 d_l(c_{2\lambda}, \bar{u}) + \eta_2 d_l(\tilde{S}_{c_{2\lambda}}, \tilde{S}_{c_{2\lambda}}) + \eta_3 d_l(c_{2\lambda}, \bar{u})$$

$$\eta_4 \frac{d_l^2(c_{2\lambda}, \tilde{S}_{c_{2\lambda}}), d_l(\bar{u}, \tilde{T}\bar{u})}{1 + d_l^2(c_{2\lambda}, \bar{u})} - \tau.$$
inequality (2.42) and the inequality
\[
d_l(\bar{u}, S\bar{u}) \leq d_l(\bar{u}, c_{2n+2}) + d_l(c_{2n+2}, S\bar{u}) \\
\leq d_l(\bar{u}, c_{2n+2}) + d_l(T_{2n+1}, S\bar{u})
\]
we can prove \(d_l(\bar{u}, \bar{S}\bar{u}) = 0\). \(\bar{u}\) \(\in\) \(\bar{S}\bar{u}\). Hence \(\bar{u}\) be the C.F.P of \(\bar{S}\) and \(\bar{T}\) in \(\overline{B_d(c_0, \bar{r})}\). Now,
\[
d_l(\bar{u}, \bar{u}) \leq d_l(\bar{u}, T\bar{u}) + d_l(T\bar{u}, \bar{u}) \leq 0.
\]
This implies that \(d_l(\bar{u}, \bar{u}) = 0\). \(\blacksquare\)

Example 2.4.2 Let \(\hat{Z} = Q^+ \cup \{0\}\) and let \(d_l : \hat{Z} \times \hat{Z} \to \hat{Z}\) be the complete \(D.M.S\) on \(\hat{Z}\) defined by
\[
d_l(i, j) = i + j \text{ for all } i, j \in \hat{Z}.
\]
Define, \(\bar{S}, \bar{T} : \hat{Z} \times \hat{Z} \to P(\hat{Z})\) by,
\[
\bar{S}z = \begin{cases} 
[\frac{z}{3}, \frac{2}{3}] & \text{if } z \in [0, 7] \cap \hat{Z} \\
[z, z + 1] & \text{if } z \in (7, \infty) \cap \hat{Z}
\end{cases}
\]
and,
\[
\bar{T}z = \begin{cases} 
[\frac{z}{4}, \frac{3}{4}] & \text{if } z \in [0, 7] \cap \hat{Z} \\
[z + 1, z + 3] & \text{if } z \in (7, \infty) \cap \hat{Z}
\end{cases}
\]
Taking, \(x_0 = 1\), \(\bar{r} = 8\), \(\lambda = \frac{1}{3}\) then \(\overline{B_d(x_0, \bar{r})} = [0, 7] \cap \hat{Z}\). Now
\[
d_l(x_0, \bar{S}x_0) < (1 - \lambda)\bar{r} = \frac{4}{3} < (1 - \frac{4}{9})8 \\
12 < 40.
\]
So, we obtain a sequence \(\{\bar{T}\bar{S}(x_0)\} = \{1, \frac{1}{12}, \frac{1}{144}, \frac{1}{1728}, \ldots\}\) in \(\hat{Z}\) generated by \(x_0\). Also, \(\overline{B_d(x_0, \bar{r})} \cap \{\bar{T}\bar{S}(x_0)\} = \{1, \frac{1}{12}, \frac{1}{144}, \ldots\}\) and
\[
\alpha(c, d) = \begin{cases} 
1 & \text{if } c, d \in [0, 1] \\
\frac{3}{2} & \text{otherwise}
\end{cases}
\]
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Now, if \( x, y \in \overline{B_d(x_0, \bar{r}) \cap \{T\bar{S}(x_\bar{h})\}} \), then we have the following cases.

Case 1. If

\[
\max\{\tau + \alpha_*(x, \bar{S}x)F(H_{d_1}(\bar{S}x, \bar{T}y)), \tau + \alpha_*(y, \bar{T}y)F(H_{d_1}(\bar{T}y, \bar{S}x))\} = \tau + \alpha_*(x, \bar{S}x)F(H_{d_1}(\bar{S}x, \bar{T}y))
\]

then we consider only

\[
\alpha_*(x, \bar{S}x)H_{d_1}(\bar{S}x, \bar{T}y) = 1\left[ \max\{ \sup_{a \in \bar{S}x} d_l(a, \bar{T}y), \sup_{b \in \bar{T}y} d_l(\bar{S}x, b) \} \right]
\]

\[
= \max\{ \left[ \frac{2x}{3}, \frac{y}{4}, \frac{3y}{4} \right], \left[ \frac{x}{3}, \frac{2x}{3}, \frac{3y}{4} \right] \}
\]

\[
= \max\{ \left[ \frac{2x}{3}, \frac{y}{4}, \frac{x}{3}, \frac{3y}{4} \right] \}
\]

\[
\alpha_*(x, \bar{S}x)H_{d_1}(\bar{S}x, \bar{T}y) = 1\left[ \frac{1}{5} d_1(x, y) + \frac{1}{10} d_1(x, \left[ \frac{x}{3}, \frac{2x}{3}, \frac{3y}{4} \right]) + \frac{1}{15} d_1(x, \left[ \frac{y}{4}, \frac{3y}{4} \right]) + \frac{1}{30} \frac{d_l^2(x, \left[ \frac{x}{3}, \frac{2x}{3} \right])}{1 + d_l^2(x, y)} \right]
\]

\[
\alpha_*(x, \bar{S}x)H_{d_1}(\bar{S}x, \bar{T}y) = \frac{1}{5} (x + y) + \frac{2x}{15} + \frac{4x + y}{60} + \frac{5x^2y^2}{54(1 + (x + y)^2)}.
\]

Thus,

\[H_{d_1}(\bar{S}x, \bar{T}y) < \eta_1 d_1(x, y) + \eta_2 d_1(x, \bar{S}x) + \eta_3 d_1(x, \bar{T}y) + \eta_4 \frac{d_l^2(x, \bar{S}x), d_l(y, \bar{T}y)}{1 + d_l^2(x, y)},\]

which implies that,

\[
\tau + \ln(H_{d_1}(\bar{S}x, \bar{T}y)) \leq \ln \left( \eta_1 d_1(x, y) + \eta_2 d_1(x, \bar{S}x) + \eta_3 d_1(x, \bar{T}y) \right) + \eta_4 \frac{d_l^2(x, \bar{S}x), d_l(y, \bar{T}y)}{1 + d_l^2(x, y)}.
\]
That is
\[ \tau + F(H_d(\tilde{S}x, \tilde{T}y)) \leq F \left( \eta_1 d_l(x, y) + \eta_2 d_l(x, \tilde{S}x) + \eta_3 d_l(x, \tilde{T}y) + \eta_4 \frac{d^2(y, \tilde{S}x, d_l(y, \tilde{T}y)}{1 + d^2(x, y)} \right). \]

For \( \tau = (0, \frac{12}{15}], \eta_1 = \frac{1}{5}, \eta_2 = \frac{1}{10}, \eta_3 = \frac{1}{5}, \eta_4 = \frac{1}{30}, \) and \( \lambda = \frac{3}{5}. \) Thus the mapping \( \tilde{S} \) and \( \tilde{T} \) satisfying all the contractive conditions of Theorem 2.4.1 on closed ball rather than whole space. Now if \( x = 8, y = 9 \in (7, \infty) \cap \tilde{Z}, \) then
\[ \tau + F(H_d(\tilde{S}x, \tilde{T}y)) > F \left( \eta_1 d_l(x, y) + \eta_2 d_l(x, \tilde{S}x) + \eta_3 d_l(x, \tilde{T}y) + \eta_4 \frac{d^2(y, \tilde{S}x, d_l(y, \tilde{T}y)}{1 + d^2(x, y)} \right). \]

and consequently condition (2.42) not holds on \( \tilde{Z}. \)

Case 2. If \( \max \{\tau + \alpha_s(x, \tilde{S}x) F(H_d(\tilde{S}x, \tilde{T}y)), \tau + \alpha_s(y, \tilde{T}y) F(H_d(\tilde{T}y, \tilde{S}x))\} = \tau + \alpha_s(y, \tilde{T}y) F(H_d(\tilde{T}y, \tilde{S}x)). \)

Then by using the similar arguments of Case 1 we can get the same results.

If, we take \( \tilde{S} = \tilde{T} \) in Theorem 2.4.1, then we are left with the result.

**Corollary 2.4.3** Let \((\tilde{Z}, d_l)\) is a complète D.M.S. Suppose a function \( \alpha : \tilde{Z} \times \tilde{Z} \to [0, \infty) \) exists. Let, \( \tilde{r} > 0, c_0 \in \overline{B_d(c_0, \tilde{r})} \subseteq \tilde{Z} \) & \( \tilde{S} : \tilde{Z} \to P(\tilde{Z}) \) be the semi \( \alpha_s \)-dominated mapping on \( \overline{B_d(c_0, \tilde{r})}. \) Assume that, for some \( \tau > 0, \)

\[
\max \{\tau + \alpha_s(\tilde{e}, \tilde{S}\tilde{e}) F(H_d(\tilde{S}\tilde{e}, \tilde{S}\tilde{y})), \tau + \alpha_s(\tilde{y}, \tilde{S}\tilde{y}) F(H_d(\tilde{S}\tilde{y}, \tilde{S}\tilde{e}))\} 
\leq F \left( \eta_1 d_l(\tilde{e}, \tilde{y}) + \eta_2 d_l(\tilde{e}, \tilde{S}\tilde{e}) + \eta_3 d_l(\tilde{e}, \tilde{S}\tilde{y}) + \eta_4 \frac{d^2(\tilde{e}, \tilde{S}\tilde{e}, d_l(\tilde{y}, \tilde{S}\tilde{y}))}{1 + d^2(x, y)} \right) \] (2.53)

for all \( \tilde{e}, \tilde{y} \in \overline{B_d(c_0, \tilde{r})} \cap \{\tilde{S}\tilde{S}(c_n)\} \) with either \( \alpha(\tilde{e}, \tilde{y}) \geq 1 \) or \( \alpha(\tilde{y}, \tilde{e}) \geq 1 \) where \( \eta_1, \eta_2, \eta_3, \eta_4 > 0 \), \( \eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1 \)
and
\[ d_l(c_0, \tilde{S}c_0) \leq (1 - \lambda)\tilde{r}, \]
where \( \lambda = \left( \frac{\eta_1 + \eta_2 + \eta_4}{1 - \eta_3 - \eta_4} \right) \) and \( \eta_3, \eta_4 \neq 1. \) Then \( \{\tilde{T} \tilde{S}(c_n)\} \) be a séquenée in \( \overline{B_d(c_0, \tilde{r})}, \alpha(c_n, c_{n+1}) \geq 1 \) for each \( \tilde{n} \) belongs to \( N \cup \{0\} \) and \( \{\tilde{S}\tilde{S}(c_n)\} \to \tilde{u} \in \overline{B_d(c_0, \tilde{r})}. \) Also if the inequality (2.53) holds for \( \tilde{e} \) and either \( \alpha(c_{n_1}, \tilde{u}) \geq 1 \) or \( \alpha(\tilde{u}, c_{n_1}) \geq 1 \) for each \( \tilde{n} \) belongs to \( N \cup \{0\} \), then \( \tilde{S} \) and \( \tilde{T} \) have C.F.P \( \tilde{u} \) in \( \overline{B_d(c_0, \tilde{r})}. \)

If, we take \( \eta_2 = 0 \) in Theorem 2.4.1, then we are left only with the result.

**Corollary 2.4.4** Let \((\tilde{Z}, d_l)\) is a complète D.M.S. Suppose a function \( \alpha : \tilde{Z} \times \tilde{Z} \to [0, \infty) \)
exists. Let, $\hat{r} > 0$, $c_0 \in B_{d_l}(c_0, \hat{r}) \subseteq \tilde{Z} \& \tilde{S}, \tilde{T} : \tilde{Z} \to P(\tilde{Z})$ be the semi $\alpha_\ast-$dominated mappings on $B_{d_l}(c_0, \hat{r})$. Assume that, for some $\tau > 0$,

$$
\max\{\tau + \alpha_\ast(\hat{e}, \tilde{S}\hat{e})F(H_{d_l}(\tilde{S}\hat{e}, \tilde{T} \hat{y})), \tau + \alpha_\ast(\hat{y}, \tilde{T} \hat{y})F(H_{d_l}(\tilde{T} \hat{y}, \tilde{S}\hat{e}))\} \\
\leq F\left(\eta_1 d_l(\hat{e}, \hat{y}) + \eta_2 d_l(\tilde{e}, \tilde{T} \hat{y}) + \eta_4 \frac{d_l^2(\hat{e}, \tilde{S}\hat{e})d_l(\hat{y}, \tilde{T} \hat{y})}{1 + d_l^2(\hat{e}, \hat{y})}\right)
$$

(2.54)

for all $\hat{e}, \hat{y} \in B_{d_l}(c_0, \hat{r}) \cap \{\tilde{T}\tilde{S}(c_n)\}$ with either $\alpha(\hat{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \tilde{e}) \geq 1$ where $\eta_1, \eta_3, \eta_4 > 0$, $\eta_1 + 2\eta_3 + \eta_4 < 1$ and

$$d_l(c_0, \tilde{S}c_0) \leq (1 - \lambda)\hat{r},$$

where $\lambda = \left(\frac{\eta_1 + \eta_3}{1 - \eta_3 - \eta_4}\right)$ and $\eta_3 + \eta_4 \neq 1$. Then $\{\tilde{T}\tilde{S}(c_n)\}$ is a sequence in $B_{d_l}(c_0, \hat{r})$, $\alpha(c_n, c_{n+1}) \geq 1$ for each $n$ belongs to $N \cup \{0\}$ and $\{\tilde{T}\tilde{S}(c_n)\} \to \tilde{u} \in B_{d_l}(c_0, \hat{r})$. Also if the inequality (2.54) holds for $\tilde{u}$ and either $\alpha(c_n, \tilde{u}) \geq 1$ or $\alpha(\tilde{u}, c_n) \geq 1$ for each $n$ belongs to $N \cup \{0\}$, then $\tilde{S}$ and $\tilde{T}$ have C.F.P $\tilde{u}$ in $B_{d_l}(c_0, \hat{r})$.

If, we take $\eta_3 = 0$ in Theorem 2.4.1, then we are left only with the result.

**Corollary 2.4.5** Let $(\tilde{Z}, d_l)$ is a complete D.M.S. Suppose a function $\alpha : \tilde{Z} \times \tilde{Z} \to [0, \infty)$ exists. Let, $\hat{r} > 0$, $c_0 \in B_{d_l}(c_0, \hat{r}) \subseteq \tilde{Z} \& \tilde{S}, \tilde{T} : \tilde{Z} \to P(\tilde{Z})$ be the semi $\alpha_\ast-$dominated mappings on $B_{d_l}(c_0, \hat{r})$. Assume that, for some $\tau > 0$,

$$
\max\{\tau + \alpha_\ast(\hat{e}, \tilde{S}\hat{e})F(H_{d_l}(\tilde{S}\hat{e}, \tilde{T} \hat{y})), \tau + \alpha_\ast(\hat{y}, \tilde{T} \hat{y})F(H_{d_l}(\tilde{T} \hat{y}, \tilde{S}\hat{e}))\} \\
\leq F\left(\eta_1 d_l(\hat{e}, \hat{y}) + \eta_2 d_l(\tilde{e}, \tilde{T} \hat{y}) + \eta_4 \frac{d_l^2(\hat{e}, \tilde{S}\hat{e})d_l(\hat{y}, \tilde{T} \hat{y})}{1 + d_l^2(\hat{e}, \hat{y})}\right)
$$

(2.55)

for all $\hat{e}, \hat{y} \in B_{d_l}(c_0, \hat{r}) \cap \{\tilde{T}\tilde{S}(c_n)\}$ with either $\alpha(\hat{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \tilde{e}) \geq 1$ where $\eta_1, \eta_2, \eta_4 > 0$, $\eta_1 + \eta_2 + \eta_4 < 1$ and

$$d_l(c_0, \tilde{S}c_0) \leq (1 - \lambda)\hat{r},$$

where $\lambda = \left(\frac{\eta_1 + \eta_2}{1 - \eta_2 - \eta_4}\right)$ and $1 - \eta_4 \neq 0$. Then $\{\tilde{T}\tilde{S}(c_n)\}$ is a sequence in $B_{d_l}(c_0, \hat{r})$, $\alpha(c_n, c_{n+1}) \geq 1$ for each $n$ belongs to $N \cup \{0\}$ and $\{\tilde{T}\tilde{S}(c_n)\} \to \tilde{u} \in B_{d_l}(c_0, \hat{r})$. Also if the inequality (2.55) holds for $\tilde{u}$ and either $\alpha(c_n, \tilde{u}) \geq 1$ or $\alpha(\tilde{u}, c_n) \geq 1$ for each $n$ belongs to $N \cup \{0\}$, then $\tilde{S}$ and $\tilde{T}$ have C.F.P $\tilde{u}$ in $B_{d_l}(c_0, \hat{r})$.

If, we take $\eta_4 = 0$ in Theorem 2.4.1, then we are left only with the result.
Corollary 2.4.6 Let \((\hat{Z}, d_1)\) is a complete D.M.S. Suppose a function \(\alpha : \hat{Z} \times \hat{Z} \to [0, \infty)\) exists. Let, \(\hat{r} > 0\), \(c_0 \in \overline{B_{d_1}(c_0, \hat{r})} \subseteq \hat{Z} \& \hat{S}, \hat{T} : \hat{Z} \to P(\hat{Z})\) be the semi \(\alpha_\ast\)–dominated mappings on \(\overline{B_{d_1}(c_0, \hat{r})}\). Assume that, for some \(\tau > 0\),

\[
\max\{\tau + \alpha_\ast(\hat{e}, \hat{S}\hat{e})F(H_{d_1}(\hat{S}\hat{e}, \hat{T}\hat{y})), \tau + \alpha_\ast(\hat{y}, \hat{T}\hat{y})F(H_{d_1}(\hat{T}\hat{y}, \hat{S}\hat{e}))\} \\
\leq F(\eta_1 d_1(\hat{e}, \hat{y}) + \eta_2 d_1(\hat{e}, \hat{S}\hat{e}) + \eta_3 d_1(\hat{e}, \hat{T}\hat{y}))
\]

(2.56)

for all \(\hat{e}, \hat{y} \in \overline{B_{d_1}(c_0, \hat{r})} \cap \{\hat{T}\hat{S}(c_0)\}\) with either \(\alpha(\hat{e}, \hat{y}) \geq 1\) or \(\alpha(\hat{y}, \hat{e}) \geq 1\) where \(\eta_1, \eta_2, \eta_3 > 0\), \(\eta_1 + \eta_2 + 2\eta_3 < 1\) and 

\[d_1(c_0, \hat{S}c_0) \leq (1 - \lambda)\hat{r},\]

where \(\lambda = \left(\frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_3}\right)\) and \(1 - \eta_3 \neq 0\). Then \(\{\hat{T}\hat{S}(c_0)\}\) is a \(\hat{T}\hat{S}(c_0)\) in \(\overline{B_{d_1}(c_0, \hat{r})}\), \(\alpha(c_n, c_{n+1}) \geq 1\) for each \(\hat{n}\) belongs to \(N \cup \{0\}\) and \(\{\hat{T}\hat{S}(c_0)\} \to \hat{u} \in \overline{B_{d_1}(\hat{u}_0, \hat{r})}\). Also if the inequality (2.56) holds for \(\hat{u}\) and either \(\alpha(c_{\hat{n}}, \hat{u}) \geq 1\) or \(\alpha(\hat{u}, c_{\hat{n}}) \geq 1\) for each \(\hat{n}\) belongs to \(N \cup \{0\}\), then \(\hat{S}\) and \(\hat{T}\) have C.F.P \(\hat{u}\) in \(\overline{B_{d_1}(c_0, \hat{r})}\).

Theorem 2.4.7 Let \((\hat{Z}, d_1)\) is a complete D.M.S endowed a graph \(\hat{G}\). Suppose a function \(\alpha : \hat{Z} \times \hat{Z} \to [0, \infty)\) exists. Let, \(\hat{r} > 0\), \(c_0 \in \overline{B_{d_1}(c_0, \hat{r})}\), \(\hat{S}, \hat{T} : \hat{Z} \to P(\hat{Z})\) and let for a \(\hat{T}\hat{S}(c_0)\) in \(\hat{Z}\) generated by \(c_0\), with \((c_0, c_1) \in E(\hat{G})\). Assume (i) (ii) and (iii) hold:

(i) \(\hat{S}\) and \(\hat{T}\) are multi graph dominated for each \(\hat{e}, \hat{y} \in \overline{B_{d_1}(c_0, \hat{r})} \cap \{\hat{T}\hat{S}(c_0)\}\);

(ii) there exists some \(\tau > 0\),

\[
\max\{\tau + \alpha_\ast(\hat{e}, \hat{S}\hat{e})F(H_{d_1}(\hat{S}\hat{e}, \hat{T}\hat{y})), \tau + \alpha_\ast(\hat{y}, \hat{T}\hat{y})F(H_{d_1}(\hat{T}\hat{y}, \hat{S}\hat{e}))\} \\
\leq F\left(\eta_1 d_1(\hat{e}, \hat{y}) + \eta_2 d_1(\hat{e}, \hat{S}\hat{e}) + \eta_3 d_1(\hat{e}, \hat{T}\hat{y}) + \eta_4 \frac{d_1^2(\hat{e}, \hat{S}\hat{e})}{1 + d_1^2(\hat{e}, \hat{y})}\right)
\]

(2.57)

where \(\eta_1, \eta_2, \eta_3, \eta_4 > 0\) such that

\[
\tau + F(H_{d_1}(\hat{S}\hat{e}, \hat{T}\hat{y})) \leq F\left(\eta_1 d_1(\hat{e}, \hat{y}) + \eta_2 d_1(\hat{e}, \hat{S}\hat{e}) + \eta_3 d_1(\hat{e}, \hat{T}\hat{y}) + \frac{\eta_4 d_1^2(\hat{e}, \hat{S}\hat{e})}{1 + d_1^2(\hat{e}, \hat{y})}\right)
\]

(2.58)

for all \(\hat{e}, \hat{y} \in \overline{B_{d_1}(c_0, \hat{r})} \cap \{\hat{T}\hat{S}(c_0)\}\) & \((\hat{e}, \hat{y}) \in E(\hat{G})\) or \((\hat{y}, \hat{e}) \in E(\hat{G})\);

(iii) \(\sum_{\hat{n}=0}^{\hat{n}} \lambda^i(d_1(c_0, \hat{S}c_0)) \leq \hat{r}\) for every \(\hat{n}\) belongs to \(N \cup \{0\}\).

Then, \(\{\hat{T}\hat{S}(c_0)\}\) be the \(\hat{T}\hat{S}(c_0)\) in \(\overline{B_{d_1}(c_0, \hat{r})}\), \((c_n, c_{n+1}) \in E(\hat{G})\) and \(\{\hat{T}\hat{S}(c_0)\} \to m^*\). Also,
if and the inequality (2.57) holds for \( m^* \) and \( (c_n, m^*) \in E(\tilde{G}) \) or \( (m^*, c_n) \in E(\tilde{G}) \) for each \( n \) belongs to \( N \cup \{0\} \), then \( \tilde{S} \) and \( \tilde{T} \) have C.F.P \( m^* \) in \( \overline{B_{d_1}(c_0, \tilde{r})} \).

**Proof.** Define, \( \alpha : \tilde{Z} \times \tilde{Z} \to [0, \infty) \) by

\[
\alpha(\tilde{e}, \tilde{y}) = \begin{cases} 
1, & \text{if } \tilde{e} \in \overline{B_{d_1}(c_0, \tilde{r})}, \quad (\tilde{e}, \tilde{y}) \in E(\tilde{G}) \text{ or } (\tilde{y}, \tilde{e}) \in E(\tilde{G}) \\
0, & \text{otherwise.}
\end{cases}
\]

Since \( \tilde{S} \) and \( \tilde{T} \) are semi graph dominated on \( \overline{B_{d_1}(c_0, \tilde{r})} \), then for \( \tilde{e} \in \overline{B_{d_1}(c_0, \tilde{r})} \), \( (\tilde{e}, \tilde{y}) \in E(\tilde{G}) \) for all \( \tilde{y} \in \tilde{S}\tilde{e} \) and \( (\tilde{e}, \tilde{y}) \in E(\tilde{G}) \) for all \( \tilde{y} \in \tilde{T}\tilde{e} \). So, \( \alpha(\tilde{e}, \tilde{y}) = 1 \) for all \( \tilde{y} \in \tilde{S}\tilde{e} \) and \( \alpha(\tilde{e}, \tilde{y}) = 1 \) for all \( \tilde{y} \in \tilde{T}\tilde{e} \). This implies that \( \inf\{\alpha(\tilde{e}, \tilde{y}) : \tilde{y} \in \tilde{S}\tilde{e}\} = 1 \) and \( \inf\{\alpha(\tilde{e}, \tilde{y}) : \tilde{y} \in \tilde{T}\tilde{e}\} = 1 \). Hence \( \alpha_*(\tilde{e}, \tilde{S}\tilde{e}) = 1 \), \( \alpha_*(\tilde{e}, \tilde{T}\tilde{e}) = 1 \) for all \( \tilde{e} \in \overline{B_{d_1}(c_0, \tilde{r})} \). So, \( \tilde{S}, \tilde{T} : \tilde{Z} \to P(\tilde{Z}) \) are the semi \( \alpha_* \)-dominated mappping on \( \overline{B_{d_1}(c_0, \tilde{r})} \). Moreover, we can write (2.57) as

\[
\max\{\tau + \alpha_*(\tilde{e}, \tilde{S}\tilde{e})F(H_{d_1}(\tilde{S}\tilde{e}, \tilde{T}\tilde{y})), \tau + \alpha_*(\tilde{y}, \tilde{T}\tilde{y})F(H_{d_1}(\tilde{T}\tilde{y}, \tilde{S}\tilde{e}))\}
\]

\[
\leq F \left( \eta_1 d_1(\tilde{e}, \tilde{y}) + \eta_2 d_1(\tilde{e}, \tilde{S}\tilde{e}) + \eta_3 d_1(\tilde{e}, \tilde{T}\tilde{y}) + \eta_4 \frac{d_1^2(\tilde{e}, \tilde{S}\tilde{e}), d_1(\tilde{y}, \tilde{T}\tilde{y})}{1 + d_1^2(\tilde{e}, \tilde{y})} \right)
\]

for all elements \( \tilde{e}, \tilde{y} \in \overline{B_{d_1}(c_0, \tilde{r})} \cap \{c_n\} \) with either \( \alpha(\tilde{e}, \tilde{y}) \geq 1 \) or \( \alpha(\tilde{y}, \tilde{e}) \geq 1 \). Also, (iii) holds. Then, by Theorem 2.4.1, we have \( \{\tilde{T}\tilde{S}(c_n)\} \) is the sequence in \( \overline{B_{d_1}(c_0, \tilde{r})} \) & \( \{\tilde{T}\tilde{S}(c_n)\} \to m^* \in \overline{B_{d_1}(c_0, \tilde{r})} \). Now, \( c_n, m^* \in \overline{B_{d_1}(c_0, \tilde{r})} \) and either \( (c_n, m^*) \in E(\tilde{G}) \) or \( (m^*, c_n) \in E(\tilde{G}) \) implies that either \( \alpha(c_n, m^*) \geq 1 \) or \( \alpha(m^*, c_n) \geq 1 \). So, all hypothesis of Theorem 2.4.1 are proved. Hence, by Theorem 2.4.7, \( \tilde{S} \) and \( \tilde{T} \) have a C.F.P \( m^* \) in \( \overline{B_{d_1}(c_0, \tilde{r})} \) and \( d_1(m^*, m^*) = 0 \).

In this section, we discussed new fixed point results for one map in complete D.M.S. ■

**Theorem 2.4.8** Let \( (\tilde{Z}, d_I) \) is a complete D.M.S. Suppose a function \( \alpha : \tilde{Z} \times \tilde{Z} \to [0, \infty) \) exists. Let, \( \tilde{r} > 0, \ c_0 \in \overline{B_{d_1}(c_0, \tilde{r})} \subseteq \tilde{Z} \) & \( \tilde{S}, \tilde{T} : \tilde{Z} \to \tilde{Z} \) be the semi \( \alpha_* \)-dominated mappping on \( \overline{B_{d_1}(c_0, \tilde{r})} \). Assume that, for some \( \tau > 0, \)

\[
\max\{\tau + \alpha_*(\tilde{e}, \tilde{S}\tilde{e})F(d_I(\tilde{S}\tilde{e}, \tilde{T}\tilde{y})), \tau + \alpha_*(\tilde{y}, \tilde{T}\tilde{y})F(d_I(\tilde{T}\tilde{y}, \tilde{S}\tilde{e}))\}
\]

\[
\leq F \left( \eta_1 d_I(\tilde{e}, \tilde{y}) + \eta_2 d_I(\tilde{e}, \tilde{S}\tilde{e}) + \eta_3 d_I(\tilde{e}, \tilde{T}\tilde{y}) + \eta_4 \frac{d_I^2(\tilde{e}, \tilde{S}\tilde{e}), d_I(\tilde{y}, \tilde{T}\tilde{y})}{1 + d_I^2(\tilde{e}, \tilde{y})} \right)
\]

(2.59)

for all \( \tilde{e}, \tilde{y} \in \overline{B_{d_1}(c_0, \tilde{r})} \cap \{c_n\} \) with either \( \alpha(\tilde{e}, \tilde{y}) \geq 1 \) or \( \alpha(\tilde{y}, \tilde{e}) \geq 1 \) where \( \eta_1, \eta_2, \eta_3, \eta_4 > 0 \),
\[ \eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1 \]

\[ d_t(c_0, S\hat{c}_0) \leq (1 - \lambda)\bar{r}, \]

where \( \lambda = \left( \frac{\eta_1 + \eta_2 + \eta_4}{1 - \eta_3 - \eta_4} \right) \) and \( \eta_3 + \eta_4 \neq 1 \). Then \( \{c_n\} \) is the sequence in \( B_{d_t}(c_0, \bar{r}) \), \( \alpha(c_n, c_{n+1}) \geq 1 \) for each \( n \) belongs to \( N \cup \{0\} \) and \( \{c_n\} \rightarrow v \in B_{d_t}(c_0, \bar{r}) \). Also if the inequality (2.59) holds for \( v \) and either \( \alpha(c_n, v) \geq 1 \) or \( \alpha(v, c_n) \geq 1 \) for each \( n \) belongs to \( N \cup \{0\} \), then \( S \) and \( T \) have C.F.P \( v \) in \( B_{d_t}(c_0, \bar{r}) \).

**Proof.** The proof of above Theorem is similar as previous proved Theorem 2.4.1.

If, we take \( S = T \) in Theorem 2.4.8, then we are left only with this result.

**Corollary 2.4.9** Let \( (\hat{Z}, d_t) \) is a complete D.M.S. Suppose a function \( \alpha : \hat{Z} \times \hat{Z} \rightarrow [0, \infty) \) exists. Let, \( \bar{r} > 0 \), \( c_0 \in B_{d_t}(c_0, \bar{r}) \subseteq \hat{Z} \) & \( S : \hat{Z} ightarrow \hat{Z} \) be the semi \( \alpha_\ast \)-dominated mappings on \( B_{d_t}(c_0, \bar{r}) \). Assume that, for some \( \tau > 0 \),

\[
\max\{ \tau + \alpha_\ast(\hat{e}, \hat{S}\hat{e})F(d_t(\hat{S}\hat{e}, \hat{S}\hat{y})), \tau + \alpha_\ast(\hat{y}, \hat{S}\hat{y})F(d_t(\hat{S}\hat{y}, \hat{S}\hat{e})) \} 
\leq F \left( \eta_1 d_t(\hat{e}, \hat{y}) + \eta_2 d_t(\hat{e}, \hat{S}\hat{e}) + \eta_3 d_t(\hat{e}, \hat{S}\hat{y}) + \eta_4 \frac{d_t^2(\hat{e}, \hat{S}\hat{e})d_t(\hat{y}, \hat{S}\hat{y})}{1 + d_t^2(\hat{e}, \hat{y})} \right) \quad (2.60)
\]

for each \( \hat{e}, \hat{y} \in B_{d_t}(c_0, \bar{r}) \cap \{c_n\} \) with either \( \alpha(\hat{e}, \hat{y}) \geq 1 \) or \( \alpha(\hat{y}, \hat{e}) \geq 1 \) where \( \eta_1, \eta_2, \eta_3, \eta_4 > 0 \), \( \eta_1 + \eta_2 + \eta_3 + \eta_4 < 1 \) and

\[ d_t(c_0, \hat{S}\hat{c}_0) \leq (1 - \lambda)\bar{r}, \]

where \( \lambda = \left( \frac{\eta_1 + \eta_2}{1 - \eta_3 - \eta_4} \right) \) and \( \eta_3 + \eta_4 \neq 1 \). Then \( \{c_n\} \) is a sequence in \( B_{d_t}(c_0, \bar{r}) \), \( \alpha(c_n, c_{n+1}) \geq 1 \) for each \( n \) belongs to \( N \cup \{0\} \) and \( \{c_n\} \rightarrow v \in B_{d_t}(c_0, \bar{r}) \). Also if the inequality (2.60) holds for \( v \) and either \( \alpha(c_n, v) \geq 1 \) or \( \alpha(v, c_n) \geq 1 \) for each \( n \) belongs to \( N \cup \{0\} \), then \( S \) and \( T \) have C.F.P \( v \) in \( B_{d_t}(c_0, \bar{r}) \).

If, we take \( \eta_2 = 0 \) in Theorem 2.4.8, then we are left only with the result.

**Corollary 2.4.10** Let \( (\hat{Z}, d_t) \) is a complete D.M.S. Suppose a function \( \alpha : \hat{Z} \times \hat{Z} \rightarrow [0, \infty) \) exists. Let, \( \bar{r} > 0 \), \( c_0 \in B_{d_t}(c_0, \bar{r}) \subseteq \hat{Z} \) & \( S, T : \hat{Z} \rightarrow \hat{Z} \) are the semi \( \alpha_\ast \)-dominated maps on \( B_{d_t}(c_0, \bar{r}) \). Assume that, for some \( \tau > 0 \),

\[
\max\{ \tau + \alpha_\ast(\hat{e}, \hat{T}\hat{e})F(d_t(\hat{T}\hat{e}, \hat{T}\hat{y})), \tau + \alpha_\ast(\hat{y}, \hat{T}\hat{y})F(d_t(\hat{T}\hat{y}, \hat{T}\hat{e})) \} 
\leq F \left( \eta_1 d_t(\hat{e}, \hat{y}) + \eta_3 d_t(\hat{e}, \hat{T}\hat{e}) + \eta_4 \frac{d_t^2(\hat{e}, \hat{T}\hat{e})d_t(\hat{y}, \hat{T}\hat{y})}{1 + d_t^2(\hat{e}, \hat{y})} \right) \quad (2.61)
\]
for each \( \hat{e}, \hat{y} \in \overline{B_{d_l}(c_0, \hat{r})} \cap \{ c_n \} \) with either \( \alpha(\hat{e}, \hat{y}) \geq 1 \) or \( \alpha(\hat{y}, \hat{e}) \geq 1 \) where \( \eta_1, \eta_3, \eta_4 > 0 \), \( \eta_1 + 2\eta_3 + \eta_4 < 1 \) and
\[
d_l(c_0, \hat{S}c_0) \leq (1 - \lambda)\hat{r},
\]
where \( \lambda = \left( \frac{\eta_1 + \eta_3}{1 - \eta_3 - \eta_4} \right) \) and \( \eta_3 + \eta_4 \neq 1 \). There is a sequence \( \{ c_n \} \) in \( \overline{B_{d_l}(c_0, \hat{r})} \), \( \alpha(c_n, c_{n+1}) \geq 1 \) for each \( n \) belongs to \( N \cup \{ 0 \} \) and \( \{ c_n \} \rightarrow v \in \overline{B_{d_l}(c_0, \hat{r})} \). Also if the inequality (2.61) holds for \( v \) and either \( \alpha(c_n, v) \geq 1 \) or \( \alpha(v, c_n) \geq 1 \) for each \( n \) belongs to \( N \cup \{ 0 \} \), then \( \hat{S} \) and \( \hat{T} \) have C.F.P \( v \) in \( \overline{B_{d_l}(c_0, \hat{r})} \).

If, we take \( \eta_3 = 0 \) in Theorem 2.4.8, then we are left only with the result.

**Corollary 2.4.11** Let \( (\hat{Z}, d_l) \) is a complete D.M.S. Suppose a function \( \alpha : \hat{Z} \times \hat{Z} \rightarrow [0, \infty) \) exists. Let, \( \hat{r} > 0, c_0 \in \overline{B_{d_l}(c_0, \hat{r})} \subseteq \hat{Z} \) & \( \hat{S}, \hat{T} : \hat{Z} \rightarrow \hat{Z} \) are the semi \( \alpha_s \)-dominated maps on \( \overline{B_{d_l}(c_0, \hat{r})} \). Assume that, for some \( \tau > 0 \),
\[
\max \{ \tau + \alpha_s(\hat{e}, \hat{S}\hat{e})F(d_l(\hat{S}\hat{e}, \hat{T}\hat{y})), \tau + \alpha_s(\hat{y}, \hat{T}\hat{y})F(d_l(\hat{T}\hat{y}, \hat{S}\hat{e})) \}
\leq F \left( \eta_1 d_l(\hat{e}, \hat{y}) + \eta_2 d_l(\hat{e}, \hat{S}\hat{e}) + \eta_4 \frac{d_l^2(\hat{e}, \hat{S}\hat{e})}{1 + d_l^2(\hat{e}, \hat{y})} \right)
\]
(2.62)
for each \( \hat{e}, \hat{y} \in \overline{B_{d_l}(c_0, \hat{r})} \cap \{ c_n \} \) with either \( \alpha(\hat{e}, \hat{y}) \geq 1 \) or \( \alpha(\hat{y}, \hat{e}) \geq 1 \) where \( \eta_1, \eta_2, \eta_4 > 0 \), \( \eta_1 + \eta_2 + \eta_4 < 1 \) and
\[
d_l(c_0, \hat{S}c_0) \leq (1 - \lambda)\hat{r},
\]
where \( \lambda = \left( \frac{\eta_1 + \eta_2}{1 - \eta_2 - \eta_4} \right) \) and \( 1 - \eta_4 \neq 0 \). There is a sequence \( \{ c_n \} \) in \( \overline{B_{d_l}(c_0, \hat{r})} \), \( \alpha(c_n, c_{n+1}) \geq 1 \) for each \( n \) belongs to \( N \cup \{ 0 \} \) and \( \{ c_n \} \rightarrow v \in \overline{B_{d_l}(c_0, \hat{r})} \). Also if the inequality (2.61) holds for \( v \) and either \( \alpha(c_n, v) \geq 1 \) or \( \alpha(v, c_n) \geq 1 \) for each \( n \) belongs to \( N \cup \{ 0 \} \), then \( \hat{S} \) and \( \hat{T} \) have C.F.P \( v \) in \( \overline{B_{d_l}(c_0, \hat{r})} \).

If, we take \( \eta_4 = 0 \) in Theorem 2.4.8, then we are left only with the result.

**Corollary 2.4.12** Let \( (\hat{Z}, d_l) \) is a complete D.M.S. Suppose a function \( \alpha : \hat{Z} \times \hat{Z} \rightarrow [0, \infty) \) exists. Let, \( \hat{r} > 0, c_0 \in \overline{B_{d_l}(c_0, \hat{r})} \subseteq \hat{Z} \) & \( \hat{S}, \hat{T} : \hat{Z} \rightarrow \hat{Z} \) be the semi \( \alpha_s \)-dominated maps on \( \overline{B_{d_l}(c_0, \hat{r})} \). Assume that, for some \( \tau > 0 \),
\[
\max \{ \tau + \alpha_s(\hat{e}, \hat{S}\hat{e})F(d_l(\hat{S}\hat{e}, \hat{T}\hat{y})), \tau + \alpha_s(\hat{y}, \hat{T}\hat{y})F(d_l(\hat{T}\hat{y}, \hat{S}\hat{e})) \}
\leq F \left( \eta_1 d_l(\hat{e}, \hat{y}) + \eta_2 d_l(\hat{e}, \hat{S}\hat{e}) + \eta_3 d_l(\hat{e}, \hat{T}\hat{y}) \right)
\]
(2.63)
for all \( \hat{c}, \hat{y} \in \overline{B_{d_l}(\hat{c}_0, \hat{r}) \cap \{c_n\}} \) with either \( \alpha(\hat{c}, \hat{y}) \geq 1 \) or \( \alpha(\hat{y}, \hat{c}) \geq 1 \) where \( \eta_1, \eta_2, \eta_3 > 0 \), \( \eta_1 + \eta_2 + 2\eta_3 < 1 \) and
\[
d_l(c_0, \hat{S}c_0) \leq (1 - \lambda)\hat{r},
\]
where \( \lambda = \left(\frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_3}\right) \) and \( 1 - \eta_3 \neq 0 \). There is a sequence \( \{c_n\} \) in \( \overline{B_{d_l}(c_0, \hat{r})} \), \( \alpha(c_n, c_{n+1}) \geq 1 \) for each \( n \) belongs to \( N \cup \{0\} \) and \( \{c_n\} \to v \in \overline{B_{d_l}(c_0, \hat{r})} \). Also if the inequality (2.63) holds for \( v \) and either \( \alpha(c_n, v) \geq 1 \) or \( \alpha(v, c_n) \geq 1 \) for each \( n \) belongs to \( N \cup \{0\} \), then \( \hat{S} \) and \( \hat{T} \) have C.F.P \( v \) in \( \overline{B_{d_l}(c_0, \hat{r})} \).

**Theorem 2.4.13** Let \( (\hat{Z}, d_l) \) be a complete D.M.S. Let, \( \hat{r} > 0 \), \( c_0 \in \overline{B_{d_l}(c_0, \hat{r})} \subseteq \hat{Z} \) & \( \hat{S}, \hat{T}: \hat{Z} \to \hat{Z} \) are the dominated maps on \( \overline{B_{d_l}(c_0, \hat{r})} \). Assume that, for some \( \tau > 0 \),
\[
\max\{\tau + F(d_l(\hat{S}\hat{c}, \hat{T}\hat{y})), \tau + F(d_l(\hat{T}\hat{y}, \hat{S}\hat{c}))\} \\
\leq F\left(\eta_1 d_l(\hat{c}, \hat{y}) + \eta_2 d_l(\hat{c}, \hat{S}\hat{e}) + \eta_3 d_l(\hat{e}, \hat{T}\hat{y}) + \eta_4 \frac{d_l^2(\hat{e}, \hat{S}\hat{e}).d_l(\hat{y}, \hat{T}\hat{y})}{1 + d_l^2(\hat{e}, \hat{y})}\right)
\]
for all \( \hat{c}, \hat{y} \in \overline{B_{d_l}(c_0, \hat{r}) \cap \{c_n\}} \) with \( \eta_1, \eta_2, \eta_3, \eta_4 > 0 \), where \( \eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1 \) and
\[
d_l(c_0, \hat{S}c_0) \leq (1 - \lambda)\hat{r},
\]
where \( \lambda = \left(\frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_3 - \eta_4}\right) \) and \( \eta_3 + \eta_4 \neq 1 \). There is a sequence \( \{c_n\} \) in \( \overline{B_{d_l}(c_0, \hat{r})} \), for each \( n \) belongs to \( N \cup \{0\} \) and \( \{c_n\} \to v \in \overline{B_{d_l}(c_0, \hat{r})} \). Then \( \hat{S} \) and \( \hat{T} \) have C.F.P \( v \) in \( \overline{B_{d_l}(c_0, \hat{r})} \).

**Proof.** The proof of above Theorem is similar as previous proved Theorem in the previous section. In this section, we discuss the application of fixed point Theorem 2.4.13 in form of Volterra type integral equations.
\[
\ddot{u}(k) = \int_0^k H_1(k, h, \ddot{u}(h))dh,
\]
(2.66)
\[
\ddot{c}(k) = \int_0^k H_2(k, h, \ddot{c}(h))dh
\]
(2.67)
for each \( k \in [0,1] \). We find the solution of (2.66) and (2.67). Let \( \hat{C} = \{f: f \text{ is the continuous function from } [0,1] \text{ to } \mathbb{R}_+\} \), endowed with the complete D.M.S. For \( \ddot{u} \) belongs to \( \hat{C} \), settle
norm: \( \|\ddot{u}\|_\tau = \sup_{k \in [0,1]} \{ |\ddot{u}(k)| e^{-\tau k} \} \), where \( \tau > 0 \) is taken arbitrary. Then define

\[
d_{\tau}(\ddot{u}, \ddot{c}) = \sup_{k \in [0,1]} \{ |\ddot{u}(k) + \ddot{c}(k)| e^{-\tau k} \} = \|\ddot{u} + \ddot{c}\|_\tau
\]

for each \( \ddot{u}, \ddot{c} \in \dot{C} \), with these settings, \((\dot{C}, d_{\tau})\) becomes a D.M.S. ■

**Theorem 2.4.14:** Let the conditions (i) and (ii) hold:

(i) \( H_1, H_2 : [0,1] \times [0,1] \times \dot{C} \rightarrow \mathbb{R}_+ \);

(ii) Define

\[
\begin{align*}
\tilde{S}\ddot{u}(k) &= \int_0^k H_1(k, h, \ddot{u}(h))dh, \\
\tilde{T}\ddot{c}(k) &= \int_0^k H_2(k, h, \ddot{c}(h))dh.
\end{align*}
\]

Suppose there exist \( \tau > 0 \), such that

\[
|H_1(k, h, \ddot{u}) + H_2(k, h, \ddot{c})| \leq \frac{\tau N(\ddot{u}, \ddot{c})}{(\tau \sqrt{\|N(\ddot{u}, \ddot{c})\|_\tau + 1})^2}
\]

for each \( k, h \in [0,1] \) and \( \ddot{u}, \ddot{c} \in \dot{C} \), where

\[
N(\ddot{u}, \ddot{c}) = \eta_1|\ddot{u}(k) + \ddot{c}(k)| + \eta_2|\ddot{u}(k) + \tilde{S}\ddot{u}(k)| + \eta_3|\ddot{u}(k) + \tilde{T}\ddot{c}(k)|
\]

\[
+ \eta_4 \frac{|\ddot{u}(k) + \tilde{S}\ddot{u}(k)|^2 |\ddot{c}(k) + \tilde{T}\ddot{c}(k)|}{1 + |\ddot{u}(k) + \ddot{c}(k)|^2},
\]

where \( \eta_1, \eta_2, \eta_3, \eta_4 \geq 0 \), and \( \eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1 \). Then integral equations (2.66) and (2.67) has a solution.

**Proof.** By assumption (ii)

\[
|\tilde{S}\ddot{u}(k) + \tilde{T}\ddot{c}(k)| = \int_0^k |H_1(k, h, \ddot{u}(h) + H_2(k, h, \ddot{c}(h)))| dh,
\]

\[
\leq \int_0^k \frac{\tau}{(\tau \sqrt{\|N(\ddot{u}, \ddot{c})\|_\tau + 1})^2} ([N(\ddot{u}, \ddot{c})] e^{-\tau h}) e^{\tau h} dh,
\]
\[
\leq \int_0^k \frac{\tau}{(\tau \sqrt{\|N(\hat{\nu}, \hat{c})\|_\tau} + 1)^2} \|N(\hat{\nu}, \hat{c})\|_\tau e^{\tau h} dh,
\]
\[
\leq \frac{\tau \|N(\hat{\nu}, \hat{c})\|_\tau}{(\tau \sqrt{\|N(\hat{\nu}, \hat{c})\|_\tau} + 1)^2} \int_0^k e^{\tau h} dh,
\]
\[
\leq \frac{\|N(\hat{\nu}, \hat{c})\|_\tau}{(\tau \sqrt{\|N(\hat{\nu}, \hat{c})\|_\tau} + 1)^2} e^{\tau k}.
\]

This implies
\[
|\hat{S}\hat{u}(k) + \hat{T}\hat{c}(k)| e^{-\tau k} \leq \frac{\|N(\hat{\nu}, \hat{c})\|_\tau}{(\tau \sqrt{\|N(\hat{\nu}, \hat{c})\|_\tau} + 1)^2}.
\]
\[
\|\hat{S}\hat{u}(k) + \hat{T}\hat{c}(k)\|_\tau \leq \frac{\|N(\hat{\nu}, \hat{c})\|_\tau}{(\tau \sqrt{\|N(\hat{\nu}, \hat{c})\|_\tau} + 1)^2}.
\]
\[
\frac{\tau \sqrt{\|N(\hat{\nu}, \hat{c})\|_\tau} + 1}{\sqrt{\|N(\hat{\nu}, \hat{c})\|_\tau}} \leq \frac{1}{\sqrt{\|\hat{S}\hat{u}(k) + \hat{T}\hat{c}(k)\|_\tau}}.
\]
\[
\tau + \frac{1}{\sqrt{\|N(\hat{\nu}, \hat{c})\|_\tau}} \leq \frac{1}{\sqrt{\|\hat{S}\hat{u}(k) + \hat{T}\hat{c}(k)\|_\tau}}.
\]

which further implies
\[
\tau - \frac{1}{\sqrt{\|\hat{S}\hat{u}(k) + \hat{T}\hat{c}(k)\|_\tau}} \leq \frac{-1}{\sqrt{\|N(\hat{\nu}, \hat{c})\|_\tau}}.
\]

So, all hypothesis of Theorem 2.4.13 are proved for \( F(\hat{c}) = \frac{-1}{\sqrt{\hat{c}}} \hat{c} > 0 \) and \( d_{\tau}(\hat{\nu}, \hat{c}) = \|\hat{\nu} + \hat{c}\|_\tau. \)

Hence (2.66) and (2.67) have unique solution. □
Chapter 3

Results in Dislocated b-Metric Spaces

3.1 Introduction

Theory present in this section is shown in [47, 55].

Fixed point theory has a fundamental position in functional and mathematical analysis. Aydi et al. [16] proved fixed point results for quasi contractive setvalued maps in b-metric spaces. Boriceanu [22] discussed fixed points for multivalued contraction in a set with two b-metrics. Nawab et al. [29] established the new idea of dislocated b-metric space as an extension of b-metric space and proved common fixed points regarding four mappings fulfilling the weak contraction in dislocated b-metric space. Asl et. al [9] gave the idea of $\alpha_\alpha$-contractive mappings and got some fixed point conclusions for these multifunctions (see also [7, 30]). Shoaib et al. [61], discussed the result related to $\alpha_\alpha$-Cirić type multifunctions on an intersection of a sequence and closed ball along with graph. Jachymski, [33], proved the contractive mapping result on metric related t graph. The notion of multi graph dominated mapping is introduced. Fixed points related to graphic contraction in a closed set for this kind of mappings are developed. Moreover, we investigate our results in a better new framework.

In 1974, Ćirić [24], introduced quasi contraction. Khan, [38], established some new common fixed point of generalized rational contractive mappings in dislocated metric spaces with applications. Dislocated metric space (see [25]) is a conception of partial metric space (see [39]).
Another conception of metric space is \( b \)-metric space (see [2, 16, 23, 41, 62]). Nadler [40], started the study of fixed points concerning setvalued mappings (see also [17]). Several results on setvalued maps have been observed (see [5, 23, 36]). Shoaib [60] introduced the idea of \( \alpha \)-dominated map and get common fixed point theorems (see also [15]). Recently, Alofi et al. [8] developed the new notion of \( \alpha \)-dominated multivalued maps and showed some fixed point results on a closed ball in dislocated quasi \( b \)-metric spaces. In section 3.2, the concept of new rational type multivalued contractive maps endowed with graphic structure has been introduced. In section 3.3, we have proved fixed points for a pair of dominated multivalued maps in complete dislocated \( b \)-metric spaces with application has been established. Interesting results in metric space, partial metric space and dislocated metric space can be obtained as corollaries of our theorems, which are still not available in literature.

### 3.2 Fixed Point Results for Multivalued Contractive Mappings Endowed With Graphic Structure

The results given in this section can be seen in [47].

Let \((M, d_b)\) be a \( D.B.M.S \), \( g_0 \in W \) and \( B : W \to P(W) \) be the multifunctions on \( W \). Then there exist \( g_1 \in Bg_0 \) such that \( d_b(g_0, Bg_0) = d_b(g_0, g_1) \). Let \( g_2 \in Bg_1 \) be such that \( d_b(g_1, Bg_1) = d_b(g_1, g_2) \). Proceeding this method, we get a sequence of points in \( W \) such that \( g_{n+1} \in Bg_n \), \( d_b(g_n, Bg_n) = d_b(g_n, g_{n+1}) \). We represent this type of sequence by \( \{WB(g_n)\} \). We say that \( \{WB(g_n)\} \) be the sequence in \( W \) generated by \( g_0 \).

**Theorem 3.2.1** Let \((M, d_b)\) is a complete \( D.B.M.S \), \( \hat{r} > 0 \), \( g_0 \in \overline{B_{d_b}(g_0, \hat{r})} \), and \( B : M \to P(M) \) is a semi \( \alpha \)-admissible setvalued maps on \( \overline{B_{d_b}(g_0, \hat{r})} \) and \( \{MB(g_n)\} \) is a sequence in \( M \) generated by \( g_0 \), \( \alpha(g_0, g_1) \geq 1 \). Assume that, for some \( \psi \in \Psi \) and

\[
D_b(g, q) = \max\{d_b(g, q), \frac{d_b(g, Bg), d_b(q, Bq)}{\bar{a} + d_b(g, q)}
\]

where \( \bar{a} > 0 \), the following hold:

\[
\alpha_\Psi(Bg, Bq)H_{d_b}(Bg, Bq) \leq \psi(D_b(g, q)) \text{ for each } g, q \in \overline{B_{d_b}(g_0, \hat{r})} \cap \{MB(g_n)\}
\]  

(3.1)
\[
\sum_{i=0}^{n} B^{i+1}\{\psi^{i}(d_{b}(g_{0}, g_{1}))\} \leq r \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}, \tag{3.2}
\]

Then, \(\{MB(g_{n})\}\) is a sequence in \(\overline{B_{d_{b}}(g_{0}, \bar{r})}\), \(\alpha(g_{n}, g_{n+1}) \geq 1\) and \(\{MB(g_{n})\} \to g^{*} \in \overline{B_{d_{b}}(g_{0}, \bar{r})}\).

Also if \(\alpha(g_{n}, g^{*}) \geq 1\) or \(\alpha(g^{*}, g_{n}) \geq 1\), for each \(n\) belongs to \(\mathbb{N} \cup \{0\}\) and the inequality (3.1) holds for all \(g, q \in \overline{B_{d_{b}}(g_{0}, \bar{r})} \cap \{MB(g_{n})\} \cup \{g^{*}\}\). Then \(B\) has a C.F.P \(g^{*}\) in \(\overline{B_{d_{b}}(g_{0}, \bar{r})}\).

**Proof.** Consider a sequence \(\{MB(g_{n})\}\) generated by \(g_{0}\). Then, we have \(g_{n} \in Bg_{n-1}\), and \(d_{b}(g_{n-1},Bg_{n-1}) = d_{b}(g_{n-1},g_{n})\), for each \(n \in \mathbb{N}\). By Lemma 1.2.8, we have \(d_{b}(g_{n}, g_{n+1}) \leq H_{d_{b}}(Bg_{n-1}, Bg_{n})\) for each \(n \in \mathbb{N}\). If \(g_{0} = g_{1}\), then \(g_{0}\) be a fixed point in \(\overline{B_{d_{b}}(g_{0}, \bar{r})}\) of \(B\). Let \(g_{0} \neq g_{1}\). From (3.2), we have

\[
d_{b}(g_{0}, g_{1}) \leq \sum_{i=0}^{n} \psi^{i}(d_{b}(g_{0}, g_{1})) \leq r.
\]

It follows that,

\[
g_{1} \in \overline{B_{d_{b}}(g_{0}, \bar{r})}.
\]

If \(g_{1} = g_{2}\), then \(g_{1}\) is a fixed point in \(\overline{B_{d_{b}}(g_{0}, \bar{r})}\) of \(B\). Let \(g_{1} \neq g_{2}\). Since \(\alpha(g_{0}, g_{1}) \geq 1\) and \(B\) is semi \(\alpha_{*}\)-admissible setvalued map on \(\overline{B_{d_{b}}(g_{0}, \bar{r})}\), so \(\alpha_{*}(Bg_{0}, Bg_{1}) \geq 1\). As \(\alpha_{*}(Bg_{0}, Bg_{1}) \geq 1\), \(g_{1} \in Bg_{0}\) and \(g_{2} \in Bg_{1}\), so \(\alpha(g_{1}, g_{2}) \geq 1\). Let \(g_{2}, \ldots, g_{j} \in \overline{B_{d_{b}}(g_{0}, \bar{r})}\) for each \(j\) belongs to \(\mathbb{N}\). As \(\alpha_{*}(Bg_{1}, Bg_{2}) \geq 1\), we have \(\alpha(g_{2}, g_{3}) \geq 1\), which further implies \(\alpha_{*}(Bg_{2}, Bg_{3}) \geq 1\). Proceeding this process, we have \(\alpha_{*}(Bg_{j-1}, Bg_{j}) \geq 1\). Now, by using Lemma 1.2.8,

\[
d_{b}(g_{j}, g_{j+1}) \leq H_{d_{b}}(Bg_{j-1}, Bg_{j}) \leq \alpha_{*}(Bg_{j-1}, Bg_{j})H_{d_{b}}(Bg_{j-1}, Bg_{j})
\]

\[
\leq \psi(D_{b}(g_{j-1}, g_{j}))
\]

\[
= \psi \left( \max \left\{ \frac{d_{b}(g_{j-1}, g_{j}), \frac{d_{b}(g_{j-1}, Bg_{j-1}), d_{b}(g_{j}, Bg_{j})}{a + d_{b}(g_{j-1}, g_{j})}}{d_{b}(g_{j-1}, Bg_{j-1}), d_{b}(g_{j}, Bg_{j})} \right\} \right)
\]

\[
= \psi \left( \max \left\{ \frac{d_{b}(g_{j-1}, g_{j}), \frac{d_{b}(g_{j-1}, g_{j}), d_{b}(g_{j}, g_{j+1})}{a + d_{b}(g_{j-1}, g_{j})}}{d_{b}(g_{j-1}, g_{j}), d_{b}(g_{j}, g_{j+1})} \right\} \right)
\]

\[
= \psi \left( \max \{d_{b}(g_{j-1}, g_{j}), d_{b}(g_{j}, g_{j+1})\} \right).
\]

If \(\max \{d_{b}(g_{j-1}, g_{j}), d_{b}(g_{j}, g_{j+1})\} = d_{b}(g_{j}, g_{j+1})\), then \(d_{b}(g_{j}, g_{j+1}) \leq \psi(d_{b}(g_{j}, g_{j+1}))\). This is contradiction to the fact that \(\psi(u) < u\) for each \(u > 0\). Hence, we obtain \(\max \{d_{b}(g_{j-1}, g_{j}), d_{b}(g_{j}, g_{j+1})\} =
\]
\[ d_b(g_j, g_{j+1}) \leq \psi(d_b(g_{j-1}, g_j)) \leq \cdots \leq \psi^j(d_b(g_0, g_1)). \quad (3.3) \]

Now, by using triangular inequality and by (3.3), we have

\[ d_b(g_0, g_{j+1}) \leq t d_b(g_0, g_1) + t^2 d_b(g_1, g_2) + \cdots + t^{j+1} d_b(g_j, g_{j+1}) \]
\[ \leq t d_b(g_0, g_1) + t^2 \psi(d_b(g_0, g_1)) + \cdots + t^{j+1} \psi^j(d_b(g_0, g_1)) \]
\[ \leq \sum_{i=0}^{j} t^{i+1} \left\{ \psi^i(d_b(g_0, g_1)) \right\} \leq \epsilon. \]

Thus, \( g_{j+1} \in B_{d_b}(g_0, \epsilon) \). Hence, by induction, \( g_n \in B_{d_b}(g_0, \epsilon) \). As \( \alpha_*(B g_{j-1}, B g_j) \geq 1, g_j \in B g_j, \)
\( g_{j+1} \in B g_j \), then we have \( \alpha(g_j, g_{j+1}) \geq 1 \). Also \( B \) is semi \( \alpha_* \)-admissible setvalued maps on \( \overline{B_{d_b}(g_0, \epsilon)} \), therefore \( \alpha_*(B g_j, B g_{j+1}) \geq 1 \). This further implies that \( \alpha(g_{j+1}, g_{j+2}) \geq 1 \). Proceeding this process, we have \( \alpha(g_n, g_{n+1}) \geq 1 \) for each \( n \) belongs to \( \mathbb{N} \). Now, (3.3) can be expressed as

\[ d_b(g_n, g_{n+1}) \leq \psi^n(d_b(g_0, g_1)) \text{ for each } n \text{ belongs to } \mathbb{N}. \quad (3.4) \]

Fix \( \epsilon > 0 \) and let \( k_1(\epsilon) \in \mathbb{N} \), such that

\[ \sum_{k \geq k_1(\epsilon)} t^k \psi^k(d_b(g_0, g_1)) < \epsilon. \]

Let \( n, m \) belong to \( \mathbb{N} \) with \( m > n > k_1(\epsilon) \). Now,

\[ d_b(g_n, g_m) \leq \sum_{k=n}^{m-1} d_b(g_k, g_{k+1}) \]
\[ \leq \sum_{k=n}^{m-1} t^k \psi^k(d_b(g_0, g_1)), \text{ by (3.4)} \]
\[ d_b(g_n, g_m) \leq \sum_{k \geq k_1(\epsilon)} t^k \psi^k(d_b(g_0, g_1)) < \epsilon. \]

Thus, \( \{MB(g_n)\} \) is a Cauchy in \( \overline{B_{d_b}(g_0, \epsilon)} \). As each closed set in a complete \( D.B.M.S \) is
\text{complete}, so there exist } g^* \in \overline{B_{d_b}(g_0, \hat{r})} \text{ such that } \{MB(g_n)\} \to g^*, \text{ and }

\lim_{n \to \infty} d_b(g_n, g^*) = 0. \tag{3.5}

Then, we have } \alpha(g_n, g^*) \geq 1 \text{ for every } n \text{ belongs to } \mathbb{N} \cup \{0\}. \text{ Thus, } \alpha_*(Bg_n, Bg^*) \geq 1. \text{ Now, }

\begin{align*}
d_b(g^*, Bg^*) & \leq td_b(g^*, g_{n+1}) + td_b(g_{n+1}, Bg^*) \\
& \leq td_b(g^*, g_{n+1}) + tH_{d_b}(Bg_n, Bg^*) \text{ by Lemma 1.2.8} \\
& \leq td_b(g^*, g_{n+1}) + t \{ \alpha_*(Bg_n, Bg^*)H_{d_b}(Bg_n, Bg^*) \} \\
& \leq td_b(g^*, g_{n+1}) + t \psi(\max \{ d_b(g_n, g^*), \\
& \quad \quad \frac{d_b(g_n, Bg_n) \cdot d_b(g^*, Bg^*)}{a + d_b(g_n, g^*)}, d_b(g_n, Bg) \}, d_b(g^*, Bg^*) \}) \\
& \leq td_b(g^*, g_{n+1}) + t \psi(\max \{ d_b(g_n, g^*), \\
& \quad \quad \frac{d_b(g_n, g_{n+1}) \cdot d_b(g^*, Bg^*)}{a + d_b(g_n, g^*)}, d_b(g_n, g_{n+1}) \}, d_b(g^*, Bg^*) \}).
\end{align*}

Letting } n \to \infty \text{ and by using inequality (3.5), we obtain } (1 - t)d_b(g^*, Bg^*) \leq 0. \text{ So } (1 - t) \neq 0, \text{ then } d_b(g^*, Bg^*) = 0. \text{ Hence } g^* \in Bg^*. \qed

\textbf{Corollary 3.2.2} \text{ Let } (M, \preceq, d_b) \text{ is a preordered } D.B.M.S, \text{ } \hat{r} > 0, \text{ } g_0 \in \overline{B_{d_b}(g_0, \hat{r})} \text{ and } B : M \to P(M) \text{ be a multifunction on } \overline{B_{d_b}(g_0, \hat{r})} \text{ and } \{MB(g_n)\} \text{ is a sequence generated by } g_0, \text{ with } g_0 \preceq g_1. \text{ Assume that, for some } \psi \in \Psi \text{ and }

D_b(g, q) = \max \{ d_b(g, q), \frac{d_b(g, Bg) \cdot d_b(q, Bq)}{a + d_b(g, q)}, d_b(g, Bg), d_b(q, Bq) \}

\text{where } a > 0, \text{ the following hold: }

H_{d_b}(Bg, Bq) \leq \psi(D_b(g, q)) \text{ for all } g, q \in \overline{B_{d_b}(g_0, \hat{r})} \cap \{MB(g_n)\} \text{ with } g \preceq q \tag{3.6}

\text{and } \sum_{i=0}^{n} t^{i+1} \{ \psi^i(d_b(g_0, g_1)) \} \leq \hat{r} \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}. \text{ If } g, q \in \overline{B_{d_b}(g_0, \hat{r})}, \text{ so as } g \preceq q \text{ implies } Bg \preceq Bq. \text{ Then, } \{MB(g_n)\} \text{ be the sequence in } \overline{B_{d_b}(g_0, \hat{r})}, g_n \preceq g_{n+1} \text{ and } \{MB(g_n)\} \to g^* \in \overline{B_{d_b}(g_0, \hat{r})}. \text{ Also if } g^* \preceq g_n \text{ or } g_n \preceq g^*, \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\} \text{ and the inequality (3.6) holds for all } g, q \in \overline{B_{d_b}(g_0, \hat{r})} \cap \{MB(g_n)\} \cup \{g^*\}. \quad 57
Then, \( g^* \) is a fixed point of \( B \) in \( B_{d_b}(g_0, \bar{r}) \).

**Corollary 3.2.3** Let \((M, \preceq, d_b)\) be a preordered complete \( D.B.M.S. \), \( \bar{r} > 0 \), \( g_0 \in B_{d_b}(g_0, \bar{r}) \) and \( B : M \to P(M) \) be a multifunction on \( B_{d_b}(g_0, \bar{r}) \) and \( \{MB(g_n)\} \) is the sequence generated by \( g_0 \), with \( g_0 \preceq g_1 \). Assume that, for some \( k \in [0, 1) \) and

\[
D_b(g, q) = \max\{d_b(g, q), \frac{d_b(g, Bg).d_b(q, Bq)}{\bar{a} + d_b(g, q)}, d_b(g, Bg), d_b(q, Bq)\}
\]

where \( \bar{a} > 0 \), the following hold:

\[
H_{d_b}(Bg, Bq) \leq k(D_b(g, q)) \text{ for all } g, q \in B_{d_b}(g_0, \bar{r}) \cap \{MB(g_n)\} \text{ with } g \preceq q \quad (3.7)
\]

and

\[
\sum_{i=0}^{n} t^{i+1}\{k^i(d_b(g_0, g_1))\} \leq \bar{r} \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.
\]

If \( g, q \in B_{d_b}(g_0, \bar{r}) \), such that \( g \preceq q \) implies \( Bg \preceq Bq \). Then, \( \{MB(g_n)\} \) be a sequence in \( B_{d_b}(g_0, \bar{r}) \), \( g_n \preceq g_{n+1} \) and \( \{MB(g_n)\} \to g^* \in B_{d_b}(g_0, \bar{r}) \). Also if \( g^* \preceq g_n \) or \( g_n \preceq g^* \), for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and the inequality (3.7) holds for all \( g, q \in B_{d_b}(g_0, \bar{r}) \cap \{MB(g_n)\} \cup \{g^*\} \).

Then, \( g^* \) is a fixed point of \( B \) in \( B_{d_b}(g_0, \bar{r}) \).

**Corollary 3.2.4** Let \((M, \preceq, d_l)\) be a preordered \( D.M. \) space, \( \bar{r} > 0 \), \( g_0 \in B_{d_l}(g_0, \bar{r}) \) and \( B : M \to P(M) \) be a multifunction on \( B_{d_l}(g_0, \bar{r}) \) and \( \{MB(g_n)\} \) is a sequence generated by \( g_0 \), with \( g_0 \preceq g_1 \). Assume that, for some \( \psi \in \Psi \) and

\[
D_l(g, q) = \max\{d_l(g, q), \frac{d_l(g, Bg).d_l(q, Bq)}{\bar{a} + d_l(g, q)}, d_l(g, Bg), d_l(q, Bq)\}
\]

where \( \bar{a} > 0 \), the following hold:

\[
H_{d_l}(Bg, Bq) \leq \psi(D_l(g, q)) \text{ for all } g, q \in B_{d_l}(g_0, \bar{r}) \cap \{MB(g_n)\} \text{ with } g \preceq q \quad (3.8)
\]

and

\[
\sum_{i=0}^{n} \psi^i(d_l(g_0, g_1)) \leq \bar{r} \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.
\]

If \( g, q \in B_{d_l}(g_0, \bar{r}) \), such that \( g \preceq q \) implies \( Bg \preceq Bq \). Then, \( \{MB(g_n)\} \) is the sequence in \( B_{d_l}(g_0, \bar{r}) \), \( g_n \preceq g_{n+1} \) and \( \{MB(g_n)\} \to g^* \in B_{d_l}(g_0, \bar{r}) \). Also if \( g^* \preceq g_n \) or \( g_n \preceq g^* \), for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and the inequality (3.8) holds for all \( g, q \in B_{d_l}(g_0, \bar{r}) \cap \{MB(g_n)\} \cup \{g^*\} \).
Then, \( g^* \) is a fixed point of \( B \) in \( B_{d_b}(g_0, \bar{r}) \).

**Corollary 3.2.5** Let \((M, \preceq, d_l)\) is a preordered \( D.M \) space, \( \bar{r} > 0, \ g_0 \in B_{d_l}(g_0, \bar{r}) \) and \( B : M \to P(M) \) be a multifunction on \( B_{d_l}(g_0, \bar{r}) \) and \( \{MB(g_n)\} \) be a sequence generated by \( g_0 \), with \( g_0 \preceq g_1 \). Assume that, for some \( k \in [0, 1) \) and

\[
D_l(g, q) = \max\{d_l(g, q), \frac{d_l(g, Bg).d_l(q, Bq)}{\bar{a} + d_l(g, q)}; d_l(g, Bg), d_l(q, Bq)\}
\]

where \( \bar{a} > 0 \), the following hold:

\[
H_{d_l}(Bg, Bq) \leq k(D_l(g, q)) \quad \text{for all } g, q \in B_{d_l}(g_0, \bar{r}) \cap \{MB(g_n)\} \quad \text{with } g \preceq q \quad (3.9)
\]

and

\[
\sum_{i=0}^{n} k^i(d_l(g_0, g_1)) \leq \bar{r} \quad \text{for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.
\]

If \( g, q \in B_{d_l}(g_0, \bar{r}) \), such that \( g \preceq q \) implies \( Bg \preceq \bar{r} \) \( Bq \). Then, \( \{MB(g_n)\} \) is the sequence in \( B_{d_l}(g_0, \bar{r}) \), \( g_n \preceq g_{n+1} \) and \( \{MB(g_n)\} \to g^* \in B_{d_l}(g_0, \bar{r}) \). Also if \( g^* \preceq g_n \) or \( g_n \preceq g^* \), for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and the inequality (3.9) holds for all \( g, q \in B_{d_l}(g_0, \bar{r}) \cap \{MB(g_n)\} \cup \{g^*\} \). Then, \( g^* \) is a fixed point of \( B \) in \( B_{d_l}(g_0, \bar{r}) \).

**Example 3.2.6** Let \( M = \mathbb{Q}^+ \cup \{0\} \) and let \( d_b : M \times M \to M \) be the \( D.B.M \) space on \( M \) defined by

\[
d_b(g, q) = (g + q)^2 \quad \text{for all } g, q \in M
\]

with parameter \( t > 1 \). Define the multivalued mappings, \( B : M \times M \to P(M) \) by,

\[
Bg = \begin{cases} 
\left\lfloor \frac{g}{3} \right\rfloor \times \frac{2}{3}g \quad \text{if } g \in [0, 9] \cap M \\
\left\lfloor g, g + 1 \right\rfloor \quad \text{if } g \in (9, \infty) \cap M,
\end{cases}
\]

Considering, \( g_0 = 1, \bar{r} = 100, \) and \( \bar{a} = 1, b = 2, \) then \( B_{d_b}(g_0, \bar{r}) = [0, 9] \cap M \). Now \( d_b(g_0, Bg_0) = d_b(1, B1) = d_b(1, \frac{1}{3}) = \frac{16}{9} \). So we obtain a sequence \( \{MB(g_n)\} \) = \{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots\} in \( M \) generated by \( g_0 \). Let \( t = 1.2, \psi(t) = \frac{4t}{5} \), then \( t\psi(t) < t \). Define

\[
\alpha(g, q) = \begin{cases} 
1 \quad \text{if } g, q \in [0, 9] \cap M \\
\frac{3}{2} \quad \text{otherwise}.
\end{cases}
\]
Now,
\[ \alpha_s(B_{10}, B_{11}) H_{db}(B_{10}, B_{11}) = \left( \frac{3}{2} \right) (484) > \psi(D_b(g, q)) = \frac{4}{5} (484). \]

So the inequality (3.1) is not true for the whole space \( M \). Now, for all \( g, q \in B_{db}(g_0, \tilde{r}) \cap \{ MB(g_n) \} \), we have

\[
\alpha_s(B_g, B_q) H_{db}(B_g, B_q) = \max \left\{ \sup_{a \in B_g} d_b \left( \bar{a}, B_q \right), \sup_{b \in B_q} d_b \left( B_g, b \right) \right\} \\
= \max \left\{ \sup_{a \in B_g} \left( \frac{2g}{3} \right), \sup_{b \in B_q} \left( \frac{2q}{3} \right) \right\} \\
= \max \left\{ \left( \frac{2g + q}{3} \right)^2, \frac{256g^2q^2}{9} \right\} \\
\leq \psi \left( \max \left\{ (g + q)^2, \frac{256g^2q^2}{9}, \frac{16g^2}{9}, 16q^2 \right\} \right) \\
= \psi(d_b(g, q)).
\]

So, the inequality (3.1) holds on \( B_{db}(\bar{g}_0, \tilde{r}) \cap \{ MB(g_n) \} \). As, \( t = 1.2 > 1 \), then

\[ \sum_{i=0}^{n} t^{i+1} \left\{ \psi(d_b(g_0, g_1)) \right\} = \frac{16}{9} \times \frac{6}{5} \sum_{i=0}^{n} \left( \frac{24}{25} \right) i < 100 = \tilde{r}. \]

Hence, all hypothesis of Theorem 3.2.1 are proved. Now, we have \( \{ MB(g_n) \} \) is a sequence in \( \overline{B_{db}(g_0, \tilde{r})} \), \( \alpha(g_n, g_{n+1}) \geq 1 \) and \( \{ MB(g_n) \} \to 0 \in \overline{B_{db}(g_0, \tilde{r})} \). Furthermore, 0 be a fixed point of \( B \).

**Theorem 3.2.7** Let \( (M, d_b) \) is a complete D.B.M.S with a graph \( \hat{G} \). Suppose a function \( \alpha : M \times M \to [0, \infty) \) exists. Let, \( \tilde{r} > 0, g_0 \in \overline{B_{db}(g_0, \tilde{r})} \), \( B : M \to P(M) \) and let for a sequence \( \{ MB(g_n) \} \) in \( M \) generated by \( g_0 \), with \( (g_0, g_1) \in Q(G) \). Suppose that (i) and (ii) hold:

(i) \( B \) is a graph preserving for all \( g, q \in \overline{B_{db}(g_0, \tilde{r})} \cap \{ MB(g_n) \} \);
(ii) there exists \( \psi \in \Psi \) and

\[
D_b(g, q) = \max\{d_b(g, q), \frac{d_l(g, Bg).d_l(q, Bq)}{\bar{a} + d_l(g, q)}, d_b(g, Bg), d_b(q, Bq)\}
\]

where \( \bar{a} > 0 \) such that

\[
H_{d_b}(Bg, Bq) \leq \psi(D_b(g, q)),
\]

for all \( g, q \in \overline{B_{d_b}(g_0, \bar{r})} \cap \{MB(g_n)\} \) and \( (g, q) \in Q(G) \);

(iii) \( \sum_{i=0}^{n} t^{i+1} \{\psi^i(d_b(g_0, Bg_0))\} \leq \bar{r} \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and \( t > 1 \).

Then, \( \{MB(g_n)\} \) is a sequence in \( \overline{B_{d_b}(g_0, \bar{r})} \), \( (g_n, g_{n+1}) \in Q(G) \) and \( \{MB(g_n)\} \rightarrow g^* \). Also, if and the inequality (3.10) holds for \( g^* \) and \( (g_n, g^*) \in Q(G) \) or \( (g^*, g_n) \in Q(G) \) for every \( n \) belongs to \( \mathbb{N} \cup \{0\} \), then \( g^* \) is a fixed point of \( B \) in \( \overline{B_{d_b}(g_0, \bar{r})} \).

**Proof.** Define, \( \alpha : M \times M \rightarrow [0, \infty) \) by

\[
\alpha(g, q) = \begin{cases} 
1, & \text{if } g \in \overline{B_{d_b}(g_0, \bar{r})}, (g, q) \in Q(G) \\
0, & \text{otherwise.}
\end{cases}
\]

As \( \{MB(g_n)\} \) is a sequence in \( M \) generated by \( g_0 \) with \( (g_0, g_1) \in Q(G) \), we have \( \alpha(g_0, g_1) \geq 1 \). Let \( \alpha(g, q) \geq 1 \), then \( (g, q) \in Q(G) \). From (i), we have \( (w, p) \in Q(G) \) for all \( w \in Bg \) and \( p \in Bq \). This implies that \( \alpha(w, p) = 1 \) for all \( w \in Bg \) and \( p \in Bq \). This implies that \( \inf\{\alpha(w, p) : w \in Bg, p \in Bq\} = 1 \). So, \( B : M \rightarrow P(M) \) is a semi \( \alpha \)–admissible multifunction on \( \overline{B_{d_b}(g_0, \bar{r})} \).

Moreover, inequality (3.10) can be written as

\[
\alpha^*(Bg, Bq)H_{d_b}(Bg, Bq) \leq \psi(D_b(g, q)),
\]

for all elements \( g, q \) in \( \overline{B_{d_b}(g_0, \bar{r})} \cap \{MB(g_n)\} \) with either \( \alpha(g, q) \geq 1 \) or \( \alpha(q, g) \geq 1 \). Also, (iii) holds. Then, by Theorem 3.2.1, we have \( \{MB(g_n)\} \) be the sequence in \( \overline{B_{d_b}(g_0, \bar{r})} \) and \( \{MB(g_n)\} \rightarrow g^* \in \overline{B_{d_b}(g_0, \bar{r})} \). Now, \( g_n, g^* \in \overline{B_{d_b}(g_0, \bar{r})} \) and either \( (g_n, g^*) \in Q(G) \) or \( (g^*, g_n) \in Q(G) \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and the inequality (3.10) holds for all \( g, q \in \overline{B_{d_b}(g_0, \bar{r})} \cap \{MB(g_n)\} \cup \{g^*\} \). Then we have \( \alpha(g_n, g^*) \geq 1 \) or \( \alpha(g^*, g_n) \geq 1 \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and the inequality (3.1) holds for all \( g, q \in \overline{B_{d_b}(g_0, \bar{r})} \cap \{MB(g_n)\} \cup \{g^*\} \). So, all hypothesis of Theorem 3.2.1 are proved. Hence, by Theorem 3.2.1, \( B \) has a C.F.P \( g^* \) in \( \overline{B_{d_b}(g_0, \bar{r})} \) and
\( d_b(g^*, g^*) = 0. \)

In this section we discussed some fixed points for self mapping in complete D.B.M space. Let \((M, d_b)\) be a D.B.M space, \(g_0 \in M\) and \(B : M \to M\) be a mapping. Let \(g_1 = Bg_0\), \(g_2 = Bg_1\). Proceeding this method, we get a sequence \(g_n\) of points in \(M\) such that \(g_{n+1} = Bg_n\). We represent this type of sequence by \(\{g_n\}\). We say that \(\{g_n\}\) is the sequence in \(M\) generated by \(g_0\). ■

**Theorem 3.2.8** Let \((M, d_b)\) be a complete D.B.M.S. \(\delta > 0, g_0 \in \overline{B_{d_b}(g_0, \delta)}\) and \(B : M \to M\) is a semi \(\alpha\)-admissible function on \(\overline{B_{d_b}(g_0, \delta)}\) and \(\{g_n\}\) is a sequence in \(M\), then \(\alpha(g_0, g_1) \geq 1\). Assume that, for some \(\psi \in \Psi\) and

\[
D_b(g, q) = \max \{d_b(g, q), \frac{d_b(g, Bq)}{\bar{a} + d_b(g, q)}, d_b(g, Bg), d_b(q, Bq)\}
\]

where \(\bar{a} > 0\), the following hold:

\[
\alpha(Bq, Bg)H_{d_b}(Bg, Bq) \leq \psi(D_b(g, q)) \quad \text{for all} \quad g, q \in \overline{B_{d_b}(g_0, \delta)} \cap \{g_n\} \quad (3.11)
\]

\[
\sum_{i=0}^{n} H_{\psi^i(d_b(g_0, g_1))} \leq \delta \quad \text{for each} \quad n \text{ belongs to } \mathbb{N} \cup \{0\}.
\]

Then, \(\{g_n\}\) is a sequence in \(\overline{B_{d_b}(g_0, \delta)}\), \(\alpha(g_n, g_{n+1}) \geq 1\) and \(\{g_n\} \to g^* \in \overline{B_{d_b}(g_0, \delta)}\). Also if \(\alpha(g_n, g^*) \geq 1\) or \(\alpha(g^*, g_n) \geq 1\), for each \(n\) belongs to \(\mathbb{N} \cup \{0\}\) and the inequality (3.11) holds for all \(g, q \in \overline{B_{d_b}(g_0, \delta)} \cap \{g_n\} \cup \{g^*\}\). Then \(B\) has a C.F.P \(g^*\) in \(\overline{B_{d_b}(g_0, \delta)}\).

**Proof.** The proof of above Theorem is similar as previous proved Theorem 3.2.1. ■

**Corollary 3.2.9** Let \((M, \preceq, d_b)\) is a preordered complete D.B.M.S. \(\delta > 0, g_0 \in \overline{B_{d_b}(g_0, \delta)}\) and \(B : M \to M\) be a self mapping on \(\overline{B_{d_b}(g_0, \delta)}\) and \(\{g_n\}\) is a sequence generated by \(g_0\), with \(g_0 \preceq g_1\). Assume that, for some \(k \in [0, 1]\) and

\[
D_b(g, q) = \max \{d_b(g, q), \frac{d_b(g, Bq)}{\bar{a} + d_b(g, q)}, d_b(g, Bg), d_b(q, Bq)\}
\]

where \(\bar{a} > 0\), the following hold:

\[
H_{d_b}(Bq, Bg) \leq k(D_b(g, q)) \quad \text{for all} \quad g, q \in \overline{B_{d_b}(g_0, \delta)} \cap \{g_n\} \quad \text{with} \quad g \preceq q \quad (3.12)
\]
and \( \sum_{i=0}^{n} t_i^{i+1} \{ k^i(d_0(g_0, g_1)) \} \leq \rho \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \).

If \( g, q \in \overline{B_{d_t}(g_0, \rho)} \), such that \( g \preceq q \) implies \( Bg \preceq \rho Bq \). Then, \( \{ g_n \} \) is a sequence in \( \overline{B_{d_t}(g_0, \rho)} \), \( g_n \preceq g_{n+1} \) and \( \{ g_n \} \rightarrow g^* \in \overline{B_{d_t}(g_0, \rho)} \). Also if \( g^* \preceq g_n \) or \( g_n \preceq g^* \), for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and the inequality (3.12) holds for all \( g, q \in \overline{B_{d_t}(g_0, \rho)} \cap \{ g_n \} \cup \{ g^* \} \). Then, \( g^* \) is a fixed point of \( B \) in \( \overline{B_{d_t}(g_0, \rho)} \).

**Corollary 3.2.10** Let \( (M, \preceq, d_t) \) is an ordered complete \( D.M \) space, \( \rho > 0 \), \( g_0 \in \overline{B_{d_t}(g_0, \rho)} \) and \( B : M \rightarrow M \) be a self mapping on \( \overline{B_{d_t}(g_0, \rho)} \) and \( \{ g_n \} \) is a sequence generated by \( g_0 \), with \( g_0 \preceq g_1 \). Assume that, for some \( \psi \in \Psi \) and

\[
D_t(g, q) = \max\{d_t(g, q), \frac{d_t(g, Bg).d_t(q, Bq)}{\bar{a} + d_t(g, q)}, d_t(g, Bg), d_t(q, Bq)\}
\]

where \( \bar{a} > 0 \), the following hold:

\[
H_{d_t}(Bg, Bq) \leq \psi(D_t(g, q)) \text{ for all } g, q \in \overline{B_{d_t}(g_0, \rho)} \cap \{ g_n \} \text{ with } g \preceq q \tag{3.13}
\]

and \( \sum_{i=0}^{n} \psi^i(d_t(g_0, g_1)) \leq \rho \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \).

If \( g, q \in \overline{B_{d_t}(g_0, \rho)} \), such that \( g \preceq q \) implies \( Bg \preceq \rho Bq \). Then, \( \{ g_n \} \) be a sequence in \( \overline{B_{d_t}(g_0, \rho)} \), \( g_n \preceq g_{n+1} \) and \( \{ g_n \} \rightarrow g^* \in \overline{B_{d_t}(g_0, \rho)} \). Also if \( g^* \preceq g_n \) or \( g_n \preceq g^* \), for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and the inequality (3.13) holds for all \( g, q \in \overline{B_{d_t}(g_0, \rho)} \cap \{ g_n \} \cup \{ g^* \} \). Then, \( g^* \) is a fixed point of \( B \) in \( \overline{B_{d_t}(g_0, \rho)} \).

**Corollary 3.2.11** Let \( (M, \preceq, d_t) \) is an ordered complete \( D.M \) space, \( \rho > 0 \), \( g_0 \in \overline{B_{d_t}(g_0, \rho)} \) and \( B : M \rightarrow M \) be a self mapping on \( \overline{B_{d_t}(g_0, \rho)} \) and \( \{ g_n \} \) be a sequence in \( M \) with initial guess \( g_0 \), with \( g_0 \preceq g_1 \). For some \( k \in [0, 1) \) and

\[
D_t(g, q) = \max\{d_t(g, q), \frac{d_t(g, Bg).d_t(q, Bq)}{\bar{a} + d_t(g, q)}, d_t(g, Bg), d_t(q, Bq)\}
\]

where \( \bar{a} > 0 \), the following hold:

\[
d_t(Bg, Bq) \leq k(D_t(g, q)) \text{ for all } g, q \in \overline{B_{d_t}(g_0, \rho)} \cap \{ g_n \} \text{ with } g \preceq q \tag{3.14}
\]
and $\sum_{i=0}^{j} k^i(d_i(g_0, g_1)) \leq \epsilon$ for each $j$ belongs to $\mathbb{N} \cup \{0\}$.

Then, $\{g_n\}$ be a sequence in $\overline{B_d(g_0, \epsilon)}$, such that $g_n \leq g_{n+1}$ and $\{g_n\} \rightarrow g^* \in \overline{B_d(g_0, \epsilon)}$. Also if $g^* \leq g_n$ or $g_n \leq g^*$, for each $n$ belongs to $\mathbb{N} \cup \{0\}$ and the inequality (3.14) holds for all $g, q \in \overline{B_d(g_0, \epsilon)} \cap \{g_n\} \cup \{g^*\}$. Then, $g^*$ be the fixed point of $B$ in $\overline{B_d(g_0, \epsilon)}$.

### 3.3 Fixed Point Results for a Pair of Multivalued Dominated Mappings in Dislocated $b$-Metric Space with Applications

The given results in this section can be seen in [55].

Let $(E, d_{lb})$ be a $D.B.M.S$, $q_0 \in E$ and $S, T : E \rightarrow P(E)$ are the setvalued maps on $E$. Let $q_1 \in Sq_0$ be an element such that $d_{lb}(q_0, Sq_0) = d_{lb}(q_0, q_1)$. Let $q_2 \in Tq_1$ be such that $d_{lb}(q_1, Tq_1) = d_{lb}(q_1, q_2)$. Let $q_3 \in Sq_2$ be such that $d_{lb}(q_2, Sq_2) = d_{lb}(q_2, q_3)$. Proceeding this method, we get the sequence $q_n$ in $E$ so as $q_{2n+1} \in Sq_{2n}$ and $q_{2n+2} \in Tq_{2n+1}$, where $n = 0, 1, 2, \ldots$. Also $d_{lb}(q_2n, Sq_{2n}) = d_{lb}(q_{2n}, q_{2n+1})$, $d_{lb}(q_{2n+1}, Tq_{2n+1}) = d_{lb}(q_{2n+1}, q_{2n+2})$. We represent this type of sequence by $\{TS(q_n)\}$. We say that $\{TS(q_n)\}$ be the sequence in $E$ generated by $q_0$. For $q, e \in E$, $a > 0$, we define $D_{lb}(q, e)$ as

$$D_{lb}(q, e) = \max\{d_{lb}(q, e), \frac{d_{lb}(q, Sq) \cdot d_{lb}(e, Te)}{a + d_{lb}(q, e)}, d_{lb}(q, Sq), d_{lb}(e, Te)\}.$$

**Theorem 3.3.1** Let $(E, d_{lb})$ be a complete $D.B.M.S$. Suppose a function $\alpha : E \times E \rightarrow [0, \infty)$ exists. Let, $r > 0, q_0 \in \overline{B_{d_{lb}}(q_0, r)}$ & $S, T : E \rightarrow P(E)$ be two $\alpha_*$-dominated maps on $\overline{B_{d_{lb}}(q_0, r)}$. Suppose that, for some $\psi_b \in \Psi_{lb}$, the following hold:

$$H_{d_{lb}}(Sq, Te) \leq \psi_b(D_{lb}(q, e)) \quad (3.15)$$

for all $q, e \in \overline{B_{d_{lb}}(q_0, r)} \cap \{TS(q_n)\}$ with either $\alpha(q, e) \geq 1$ or $\alpha(e, q) \geq 1$. Also

$$\sum_{i=0}^{n} b_i^{i+1}\{(\psi_b^i(d_{lb}(q_0, Sq_0)))\} \leq r \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\} \text{ and } b \geq 1. \quad (3.16)$$

Then $\{TS(q_n)\}$ is a sequence in $\overline{B_{d_{lb}}(q_0, r)}$, $\alpha(q_n, q_{n+1}) \geq 1$ for each $n$ belongs to $\mathbb{N} \cup \{0\}$ and
\{TS(q_n)\} \rightarrow q^* \in B_{d_{lb}}(q_0, r). Also if the inequality (3.15) holds for \( q^* \) and either \( \alpha(q_n, q^*) \geq 1 \) or \( \alpha(q^*, q_n) \geq 1 \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \), then \( q^* \) is the C.F.P of \( S \) and \( T \) in \( B_{d_{lb}}(q_0, r) \) and \( d_{lb}(q^*, q^*) = 0 \).

**Proof.** Consider a sequence \( \{TS(q_n)\} \). From (3.16), we get

\[
d_{lb}(q_0, q_1) \leq \sum_{i=0}^{n} b^{i+1} \{\psi_{lb}(d_{lb}(q_0, Sq_0))\} \leq r.
\]

It follows that,

\[ q_1 \in B_{d_{lb}}(q_0, r). \]

Let \( q_2, \ldots, q_j \in B_{d_{lb}}(q_0, r) \) for every \( j \) belongs to \( \mathbb{N} \). If \( j = 2i + 1 \), where \( i = 1, 2, \ldots, \frac{j-1}{2} \). Since \( S, T : E \rightarrow P(E) \) be a \( \alpha_* \)-dominated mappings on \( B_{d_{lb}}(q_0, r) \), so \( \alpha_* (q_{2i}, Sq_{2i}) \geq 1 \) and \( \alpha_* (q_{2i+1}, Tq_{2i+1}) \geq 1 \). As \( \alpha_* (q_{2i}, Sq_{2i}) \geq 1 \), this implies inf \( \{ \alpha(q_{2i}, b) : b \in Sq_{2i} \} \geq 1 \). Also \( q_{2i+1} \in Sq_{2i} \), so \( \alpha(q_{2i}, q_{2i+1}) \geq 1 \). Now by using Lemma 1.2.8, we obtain,

\[
d_{lb}(q_{2i+1}, q_{2i+2}) \leq H_{d_{lb}}(Sq_{2i}, Tq_{2i+1}) \leq \psi_{lb}(D_{lb}(q_{2i}, q_{2i+1})) \\
\leq \psi_{lb}(\max\{d_{lb}(q_{2i}, q_{2i+1}), d_{lb}(q_{2i+1}, q_{2i+2})\}, d_{lb}(q_{2i+1}, q_{2i+1}), d_{lb}(q_{2i+1}, q_{2i+2})) \\
\leq \psi_{lb}(\max\{d_{lb}(q_{2i}, q_{2i+1}), d_{lb}(q_{2i+1}, q_{2i+2}))\).
\]

If \( \max\{d_{lb}(q_{2i}, q_{2i+1}), d_{lb}(q_{2i+1}, q_{2i+2})\} = d_{lb}(q_{2i+1}, q_{2i+2}) \), then

\[
d_{lb}(q_{2i+1}, q_{2i+2}) \leq \psi_{lb}(d_{lb}(q_{2i+1}, q_{2i+2})) \\
\leq b\psi_{lb}(d_{lb}(q_{2i+1}, q_{2i+2})).
\]

Which contradicts that \( b\psi_{lb}(t) \leq t \) for each \( t > 0 \). So

\[
\max\{d_{lb}(q_{2i}, q_{2i+1}), d_{lb}(q_{2i+1}, q_{2i+2})\} = d_{lb}(q_{2i}, q_{2i+1}).
\]

Hence, we obtain

\[
d_{lb}(q_{2i+1}, q_{2i+2}) \leq \psi_{lb}(d_{lb}(q_{2i}, q_{2i+1})). \tag{3.17}
\]
As \( \alpha(q_{2i-1}, Tq_{2i-1}) \geq 1 \) and \( q_{2i} \in Tq_{2i-1} \), so \( \alpha(q_{2i-1}, q_{2i}) \geq 1 \). Now, by using Lemma 1.2.8, we obtain

\[
\begin{align*}
d_{lb}(q_{2i}, q_{2i+1}) & \leq H_{db}(Tq_{2i-1}, Sq_{2i}) \leq \psi_b(D_{lb}(q_{2i}, q_{2i-1})) \\
& \leq \psi_b(\max\{d_{lb}(q_{2i}, q_{2i-1}), d_{lb}(q_{2i}, q_{2i+1})\} + d_{lb}(q_{2i-1}, q_{2i})) \\
& \leq \psi_b(\max\{d_{lb}(q_{2i}, q_{2i-1}), d_{lb}(q_{2i}, q_{2i+1})\})
\end{align*}
\]

If \( \max\{d_{lb}(q_{2i}, q_{2i-1}), d_{lb}(q_{2i}, q_{2i+1})\} = d_{lb}(q_{2i}, q_{2i+1}) \), then

\[
d_{lb}(q_{2i}, q_{2i+1}) \leq \psi_b(\max\{d_{lb}(q_{2i}, q_{2i-1}), d_{lb}(q_{2i}, q_{2i+1})\}) \leq b\psi_b(d_{lb}(q_{2i}, q_{2i+1})).
\]

Which contradicts that \( b\psi_b(t) < t \) for each \( t > 0 \). Hence, we get

\[
d_{lb}(q_{2i}, q_{2i+1}) \leq \psi_b(d_{lb}(q_{2i-1}, q_{2i})). \tag{3.18}
\]

As \( \psi_b \) is nondecreasing, so

\[
\psi_b(d_{lb}(q_{2i}, q_{2i+1})) \leq \psi_b(\psi_b(d_{lb}(q_{2i-1}, q_{2i}))).
\]

By using above inequality in (3.17), we have

\[
d_{lb}(q_{2i+1}, q_{2i+2}) \leq \psi^2_b(d_{lb}(q_{2i-1}, q_{2i})).
\]

Proceeding in this way, we get

\[
d_{lb}(q_{2i+1}, q_{2i+2}) \leq \psi^{2i+1}_b(d_{lb}(q_0, q_1)). \tag{3.19}
\]

Now, if \( j = 2i \), where \( i = 1, 2, \ldots \frac{j}{2} \). By using (3.18) and similar procedure as above, we have

\[
d_{lb}(q_{2i}, q_{2i+1}) \leq \psi^{2i}_b(d_{lb}(q_0, q_1)). \tag{3.20}
\]
Now, by combining (3.19) and (3.20)

\[ d_{lb}(q_j, q_{j+1}) \leq \psi_b^j(d_{lb}(q_0, q_1)) \text{ for each } j \in N. \] (3.21)

Now, by using triangle inequality and by (3.21), we have

\[
\begin{align*}
  d_{lb}(q_0, q_{j+1}) & \leq bd_{lb}(q_0, q_1) + b^2 d_{lb}(q_1, q_2) + \ldots + b^{j+1} d_{lb}(q_j, q_{j+1}) \\
  & \leq bd_{lb}(q_0, q_1) + b^2 \psi_b(d_{lb}(q_0, q_1)) + \ldots + b^{j+1} \psi_b^j(d_{lb}(q_0, q_1)) \\
  & \leq \sum_{i=0}^{j} b^{i+1} \{\psi_b^j(d_{lb}(q_0, q_1))\} \leq r.
\end{align*}
\]

Thus \( q_{j+1} \) belongs to \( \overline{B_{d_{lb}(q_0, r)}} \). Hence \( q_n \) belongs to \( \overline{B_{d_{lb}(q_0, r)}} \) for every \( n \) belongs to \( N \), therefore \{TS(q_n)\} be the sequence in \( \overline{B_{d_{lb}(q_0, r)}} \). As \( S, T \) are two \( \alpha_* \)-dominated maps on \( \overline{B_{d_{lb}(q_0, r)}} \), then \( \alpha_*(q_{2n}, Sq_{2n}) \geq 1 \) and \( \alpha_*(q_{2n+1}, Tq_{2n+1}) \geq 1 \). This implies \( \alpha(q_n, q_{n+1}) \geq 1 \). Also inequality (3.21) can be written as

\[ d_{lb}(q_n, q_{n+1}) \leq \psi_b^n(d_{lb}(q_0, q_1)), \text{ for each } n \text{ belongs to } N. \] (3.22)

As \( \sum_{k=1}^{+\infty} b^k \psi_b^k(t) < +\infty \), then for some \( p \in N \), then the series \( \sum_{k=1}^{+\infty} b^k \psi_b^k(\psi_b^{p-1}(d_{lb}(q_0, q_1))) \) converges. As \( b\psi_b(t) < t \), so

\[ b^{n+1} \psi_b^{n+1}(\psi_b^{p-1}(d_{lb}(q_0, q_1))) < b^n \psi_b^n(\psi_b^{p-1}(d_{lb}(q_0, q_1))) \text{ for each } n \in N. \]

Fix \( \varepsilon > 0 \), then there must be a \( p(\varepsilon) \) belongs to \( N \), so as

\[ b\psi_b(\psi_b^{p(\varepsilon)-1}(d_{lb}(q_0, q_1))) + b^2 \psi_b^2(\psi_b^{p(\varepsilon)-1}(d_{lb}(q_0, q_1))) + \cdots < \varepsilon \]
Let \( n, m \) belong to \( \mathbb{N} \) with \( m > n > p(\varepsilon) \), then we have

\[
d_\mathbb{D}(q_n, q_m) \leq b d_\mathbb{D}(q_n, q_{n+1}) + b^2 d_\mathbb{D}(q_{n+1}, q_{n+2}) + \cdots + b^{m-n} d_\mathbb{D}(q_{m-1}, q_m)
\]

\[
\leq b \psi_b^n(d_\mathbb{D}(q_0, q_1)) + b^2 \psi_b^{n+1}(d_\mathbb{D}(q_0, q_1)) + \cdots + b^{m-n} \psi_b^{m-1}(d_\mathbb{D}(q_0, q_1))
\]

\[
= b \psi_b(\psi_b^{n-1}(d_\mathbb{D}(q_0, q_1))) + \cdots + b^{m-n} \psi_b^{m-n}(\psi_b^{n-1}(d_\mathbb{D}(q_0, q_1)))
\]

\[
< b \psi_b(\psi_b^{p(\varepsilon)-1}(d_\mathbb{D}(q_0, q_1))) + b^2 \psi_b^2(\psi_b^{p(\varepsilon)-1}(d_\mathbb{D}(q_0, q_1))) + \cdots < \varepsilon.
\]

It is clear that \( \{TS(q_n)\} \) is the Cauchy in \((B_{d_\mathbb{D}}(q_0, r), d_\mathbb{D})\). As each closed ball in a complete D.B.M.S is complete, so there must be a \( q^* \in B_{d_\mathbb{D}}(q_0, r) \) so as \( TS(q_n) \to q^* \), that is

\[
\lim_{n \to \infty} d_\mathbb{D}(q_n, q^*) = 0 \quad (3.23)
\]

Now,

\[
d_\mathbb{D}(q^*, S q^*) \leq b d_\mathbb{D}(q^*, q_{2n+2}) + b d_\mathbb{D}(q_{2n+2}, S q^*)
\]

\[
\leq b d_\mathbb{D}(q^*, q_{2n+2}) + b H_{d_\mathbb{D}}(T q_{2n+1}, S q^*). \quad \text{by Lemma 1.2.8}
\]

Since \( \alpha_*(q^*, S q^*) \geq 1 \) and \( \alpha_*(q_{2n+1}, T q_{2n+1}) \geq 1 \) and \( \alpha(q_{2n+1}, q^*) \geq 1 \), we obtain

\[
d_\mathbb{D}(q^*, S q^*) \leq b d_\mathbb{D}(q^*, q_{2n+2}) + b \psi_b(\max\{d_\mathbb{D}(q^*, q_{2n+1}), d_\mathbb{D}(q^*, S q^*)\}
\]

\[
= \frac{d_\mathbb{D}(q^*, S q^*)}{a + d_\mathbb{D}(q^*, q_{2n+1})}
\]

\[
bd_\mathbb{D}(q^*, q_{2n+2}) + b \psi_b(\max\{d_\mathbb{D}(q^*, q_{2n+1}), d_\mathbb{D}(q^*, S q^*)\}
\]

\[
= \frac{d_\mathbb{D}(q^*, S q^*)}{a + d_\mathbb{D}(q^*, q_{2n+1})}
\]

Letting \( n \to \infty \), and using (3.23), we obtain \( d_\mathbb{D}(q^*, S q^*) \leq b \psi_b(d_\mathbb{D}(q^*, S q^*)) \). A contradicts to the relaity that \( b \psi_b(t) < t \) and hence \( d_\mathbb{D}(q^*, S q^*) \leq 0 \) or \( q^* \in S q^* \). Similarly, by using the inequality

\[
d_\mathbb{D}(q^*, T q^*) \leq b d_\mathbb{D}(q^*, q_{2n+1}) + b d_\mathbb{D}(q_{2n+1}, T q^*)
\]

and hence \( d_\mathbb{D}(q^*, T q^*) \leq 0 \) or \( q^* \in T q^* \). Hence \( q^* \) is the C.F.P of \( S \) and \( T \) in \( B_{d_\mathbb{D}}(q_0, r) \). Since \( \alpha_*(q^*, S q^*) \geq 1 \) and \( (S,T) \) be the pair of \( \alpha_* \)-dominated multifunction on \( B_{d_\mathbb{D}}(q_0, r) \), we have
dominated maps on $q$ with either $q$ or $f$.

Then inequality (3.24) holds for $\forall q;e$ and $SS(q;e)$.

We have the following result without closed ball and $\alpha$-dominated mappings for one multivalued mapping.

**Theorem 3.3.2** Let $(E, d_{lb})$ is a complete $D.B.M.S$. Suppose $S : E \to P(E)$ is a setvalued map. Assume that, for some $\psi_b \in \Psi_b$, the following hold:

$$H_{d_{lb}}(S \ell q; e) \leq \psi_b(D_{lb}(q; e))$$

for all $q, e \in \{SS(q_n)\}$. Then $SS(q_n) \to q^* \in E$ and $S$ has a fixed point $q^*$ in $E$ and $d_{lb}(q^*, q^*) = 0$.

**Theorem 3.3.3** Let $(E, \preceq, d_{lb})$ is an ordered complete $D.B.M.S$. Let, $r > 0$, $q_0 \in \overline{B_{d_{lb}}(q_0, r)}$ and $S, T : E \to P(E)$ are two multi $\preceq$-dominated maps on $\overline{B_{d_{lb}}(q_0, r)}$. Assume that, for some $\psi_b \in \Psi_b$, the following hold:

$$H_{d_{lb}}(S \ell q; e) \leq \psi_b(D_{lb}(q; e))$$

(3.24)

for all $q, e \in \overline{B_{d_{lb}}(q_0, r)} \cap \{TS(q_n)\}$ with either $q \preceq e$ or $e \preceq q$. Also

$$\sum_{i=0}^{n} b^{i+1} \{\psi_b(d_{lb}(q_0, q_1))\} \leq r$$

(3.25)

for each $n$ belongs to $\mathbb{N} \cup \{0\}$ and $b \geq 1$.

Then $\{TS(q_n)\}$ is the sequence in $\overline{B_{d_{lb}}(q_0, r)}$ and $\{TS(q_n)\} \to q^* \in \overline{B_{d_{lb}}(q_0, r)}$. Also if the inequality (3.24) holds for $q^*$ and either $q_n \preceq q^*$ or $q^* \preceq q_n$ for each $n$ belongs to $\mathbb{N} \cup \{0\}$. Then $q^*$ is the C.F.P of $S$ and $T$ in $\overline{B_{d_{lb}}(q_0, r)}$ and $d_{lb}(q^*, q^*) = 0$.

**Proof.** Let $\alpha : E \times E \to [0, +\infty)$ be a mapping defined by $\alpha(q, e) = 1$ for all $q \in \overline{B_{d_{lb}}(q_0, r)}$ with either $q \preceq e$ or $e \preceq q$, and $\alpha(q, e) = 0$ for all other elements $q, e \in E$. Since $S$ and $T$ are dominated maps on $\overline{B_{d_{lb}}(q_0, r)}$, so $q \preceq Sq$ and $q \preceq Tq$ for all $q \in \overline{B_{d_{lb}}(q_0, r)}$. This implies that

\[
\alpha(q^*, Tq^*) \geq 1, \text{ so } \alpha(q^*, q^*) \geq 1. \]

Now,

\[
d_{lb}(q^*, q^*) \leq d_{lb}(q^*, Tq^*) \leq H_{d_{lb}}(S q^*, Tq^*) \leq \psi_b(\max\{d_{lb}(q^*, q^*), d_{lb}(q^*, S q^*), d_{lb}(q^*, Tq^*)\}) + d_{lb}(q^*, q^*) - d_{lb}(q^*, q^*),
\]

This implies that, $d_{lb}(q^*, q^*) = 0$.

We have the following result without closed ball and $\alpha$-dominated mappings for one multivalued mapping.
$q \preceq b$ for all $b \in S_q$ and $q \preceq c$ for all $c \in T_q$. So, $\alpha(q, b) = 1$ for all $b \in S_q$ and $\alpha(q, c) = 1$ for all $c \in T_q$. This implies that $\inf \{ \alpha(q, e) : e \in S_q \} = 1$ and $\inf \{ \alpha(q, e) : e \in T_q \} = 1$. Hence $\alpha_*(q, S_q) = 1$, $\alpha_*(q, T_q) = 1$ for all $q \in \overline{B_{d_{lb}}(q_0, r)}$. So, $S, T : E \to \mathcal{P}(E)$ are the $\alpha_*$-dominated mappings on $\overline{B_{d_{lb}}(q_0, r)}$. Moreover, inequality (3.24) can be written as

$$H_{d_{lb}}(S_q, S_e) \leq \psi_b(D_{lb}(q, e))$$

for all elements $q, e$ in $\overline{B_{d_{lb}}(q_0, r)} \cap \{ TS(q_n) \}$ with either $\alpha(q, e) \geq 1$ or $\alpha(e, q) \geq 1$. Also, inequality (3.25) holds. Then, by Theorem 3.3.1, we have $\{ TS(q_n) \}$ be the sequence in $\overline{B_{d_{lb}}(q_0, r)}$ and $\{ TS(q_n) \} \to q^* \in \overline{B_{d_{lb}}(q_0, r)}$. Now, $q_n, q^* \in \overline{B_{d_{lb}}(q_0, r)}$ and either $q_n \preceq q^*$ or $q^* \preceq q_n$ implies that either $\alpha(q_n, q^*) \geq 1$ or $\alpha(q^*, q_n) \geq 1$. So, all hypothesis of Theorem 3.3.1 are proved. Hence, by Theorem 3.3.1, $q^*$ is the C.F.P of $S$ and $T$ in $\overline{B_{d_{lb}}(q_0, r)}$ and $d_{lb}(q^*, q^*) = 0$.

We have the following result without closed ball in complete DBMS. Also we write the result only for one multivalued mapping.

**Theorem 3.3.4** Let $(E, \preceq, d_{lb})$ be an ordered complete DBMS. Let $S : E \to \mathcal{P}(E)$ be two multi $\preceq$-dominated mappings on $E$. Assume that, for some $\psi_b \in \Psi_b$, the following hold:

$$H_{d_{lb}}(S_q, S_e) \leq \psi_b(D_{lb}(q, e))$$

(3.26)

for all $q, e \in \{ SS(q_n) \}$ with $q \preceq e$. Then $\{ SS(q_n) \} \to q^* \in E$. Also if the inequality (3.26) holds for $q^*$ and either $q_n \preceq q^*$ or $q^* \preceq q_n$ for every $n$ belongs to $\mathbb{N} \cup \{0\}$. Then $q^*$ is the fixed point of $S d_{lb}(q^*, q^*) = 0$.

**Example 3.3.5** Let $E = Q^+ \cup \{0\}$ and let $d_{lb} : E \times E \to E$ be the complete DBMS on $E$ defined by

$$d_{lb}(w, k) = (w + k)^2$$

for each $w, k \in E$

with parameter $b = 2$. Define the setvalued maps, $S, T : E \times E \to \mathcal{P}(E)$ by,

$$S_q = \left\{ \begin{array}{ll}
\left[ \frac{q}{3}, \frac{2}{3} q \right] & \text{if } q \in [0, 19] \cap E \\
\left[ q, q + 1 \right] & \text{if } q \in (19, \infty) \cap E,
\end{array} \right.$$
and

\[ Tq = \begin{cases} 
\frac{q}{4}, \frac{3}{4}q & \text{if } q \in [0, 19] \cap E \\
[q + 1, q + 3] & \text{if } q \in (19, \infty) \cap E.
\end{cases} \]

Considering, \( q_0 = 1, r = 400 \), then \( B_{db}(q_0, r) = [0, 19] \cap E \). Now \( d_{lb}(q_0, S_{q_0}) = d_{lb}(1, S1) = d_{lb}(1, \frac{1}{3}) = \frac{16}{r} \). So we make a sequence \( \{TS(q_n)\} = \{1, \frac{1}{3}, \frac{1}{12}, \frac{1}{36}, \ldots\} \) in \( E \) generated by \( q_0 \). Let \( \psi_b(t) = \frac{4t}{10} \), then \( b\psi_b(t) < t \). Define

\[ \alpha(q, e) = \begin{cases} 
1 & \text{if } q > e \\
\frac{1}{2} & \text{otherwise}
\end{cases}. \]

Then \( S, T : E \to P(E) \) be the \( \alpha_+ \)-dominated mappings on \( B_{db}(q_0, r) \). Now take \( 20, 21 \in E \) and \( a = 1 \), then, we have

\[ H_{db}(S20, T21) = 1936 > \psi_b(D_t(q, e)) = \frac{7396}{10}. \]

So, the inequality (3.15) is not true for the whole space \( E \). Now for all \( q, e \in B_{db}(q_0, r) \cap \{TS(q_n)\} \) with either \( \alpha(q, e) \geq 1 \) or \( \alpha(e, q) \geq 1 \), we have

\[ H_{db}(Sq, Te) = \max \{d_{lb}(\frac{2q}{3}, \frac{e}{4}), d_{lb}(\frac{q}{3}, \frac{3e}{4})\} \]

\[ = \max \left\{ \left(\frac{2q}{3} + \frac{e}{4}\right)^2, \left(\frac{q}{3} + \frac{3e}{4}\right)^2 \right\} \]

\[ \leq \psi_b(\max\{(q + e)^2, \frac{25q^2e^2}{9(1 + (q + e)^2)}, \left(\frac{4q}{3}\right)^2, \left(\frac{5e}{4}\right)^2\}). \]

So, the inequality (3.15) holds on \( B_{db}(q_0, r) \cap \{TS(q_n)\} \). Also, for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \), we have

\[ \sum_{i=0}^{n} b^{i+1}\{\psi_b(d_{lb}(q_0, q_1))\} = \frac{16}{9} \times 2 \sum_{i=0}^{n} \left(\frac{4}{3}\right)^i < 400 = r. \]

Now, we have \( \{TS(q_n)\} \) be the sequence in \( B_{db}(q_0, r) \), \( \alpha(q_n, q_{n+1}) \geq 1 \) and \( \{TS(q_n)\} \to 0 \in B_{db}(q_0, r) \). Also, \( \alpha(q_n, 0) \geq 1 \) or \( \alpha(0, q_n) \geq 1 \) for every \( n \) belongs to \( \mathbb{N} \cup \{0\} \). Hence, all hypothesis of Theorem 3.3.1 are proved.

**Theorem 3.3.6** Let \( (E, d_{lb}) \) is a complete \( D.B.M.S \) endowed a graph \( G \). Let, \( r > 0 \),
\( q_0 \in \overline{B_{d_{lb}}(q_0, r)} \), \( S, T : E \to P(E) \) and \( \{TS(q_n)\} \) be a sequence in \( E \) generated by \( q_0 \). Suppose (i), (ii) and (iii) hold:

(i) \( S \) and \( T \) are multi graph dominated on \( \overline{B_{d_{lb}}(q_0, r)} \cap \{TS(q_n)\} \);

(ii) there exists \( \psi_b \in \Psi_b \), so as

\[
H_{d_{lb}}(Sq, Tq) \leq \psi_b(D_{lb}(q, e)),
\]

(3.27)

for all \( q, e \in \overline{B_{d_{lb}}(q_0, r)} \cap \{TS(q_n)\} \) and \( (q, e) \in W(G) \) or \( (e, q) \in W(G) \);

(iii) \( \sum_{i=0}^{n} b^{i+1} \{ \psi^i_b(d_{lb}(q_0, Sq_0)) \} \leq r \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and \( b \geq 1 \).

Then, \( \{TS(q_n)\} \) is a sequence in \( \overline{B_{d_{lb}}(q_0, r)} \), \( (q_n, q_{n+1}) \in W(G) \) and \( \{TS(q_n)\} \to q^* \). Also, if the inequality (3.27) holds for \( q^* \) and \( (q_n, q^*) \in W(G) \) or \( (q^*, q_n) \in W(G) \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \), then \( q^* \) is the C.F.P of both \( S \) and \( T \) in \( \overline{B_{d_{lb}}(q_0, r)} \) and \( d_{lb}(q^*, q^*) = 0 \).

**Proof.** Define, \( \alpha : E \times E \to [0, \infty) \) by

\[
\alpha(q, e) = \begin{cases} 
1, & \text{if } q \in \overline{B_{d_{lb}}(q_0, r)}, \ (q, e) \in W(G) \text{ or } (e, q) \in W(G) \\
0, & \text{otherwise.}
\end{cases}
\]

Given \( S \) and \( T \) are graph dominated on \( \overline{B_{d_{lb}}(q_0, r)} \), then for \( q \in \overline{B_{d_{lb}}(q_0, r)} \), \( (q, e) \in W(G) \) for all \( e \in Sq \) and \( (q, e) \in W(G) \) for all \( e \in Tq \). So, \( \alpha(q, e) = 1 \) for all \( e \in Sq \) and \( \alpha(q, e) = 1 \) for all \( e \in Tq \). This implies that \( \inf \{ \alpha(q, e) : e \in Sq \} = 1 \) and \( \inf \{ \alpha(q, e) : e \in Tq \} = 1 \). Hence \( \alpha_s(q, Sq) = 1 \), \( \alpha_s(q, Tq) = 1 \) for all \( q \in \overline{B_{d_{lb}}(q_0, r)} \). So, \( S, T : E \to P(E) \) are the \( \alpha_s \)-dominated mapping on \( \overline{B_{d_{lb}}(q_0, r)} \). Moreover, inequality (3.27) can be written as

\[
H_{d_{lb}}(Sq, Tq) \leq \psi_b(D_{lb}(q, e)),
\]

for all elements \( q, e \in \overline{B_{d_{lb}}(q_0, r)} \cap \{TS(q_n)\} \) with either \( \alpha(q, e) \geq 1 \) or \( \alpha(e, q) \geq 1 \). Also, (iii) holds. Then, by Theorem 3.3.1, we have \( \{TS(q_n)\} \) is the sequence in \( \overline{B_{d_{lb}}(q_0, r)} \) and \( \{TS(q_n)\} \to q^* \in \overline{B_{d_{lb}}(q_0, r)} \). Now, \( q_n, q^* \in \overline{B_{d_{lb}}(q_0, r)} \) and either \( (q_n, q^*) \in W(G) \) or \( (q^*, q_n) \in W(G) \) implies that either \( \alpha(q_n, q^*) \geq 1 \) or \( \alpha(q^*, q_n) \geq 1 \). So, all hypotheses of Theorem 3.3.1 are proved. Hence, by Theorem 3.3.1, \( q^* \) be the C.F.P of \( S \) and \( T \) in \( \overline{B_{d_{lb}}(q_0, r)} \) and \( d_{lb}(q^*, q^*) = 0 \).

We have the following result without closed ball in complete \( D.B.M.S \) for multi graph.
dominated mapping. Also we write the result only for one multivalued mapping and for \(D_{lb}(q,e) = d_{lb}(q,e)\). □

**Theorem 3.3.7** Let \((E,d_{lb})\) is a complete \(D.B.M.S\) endowed a graph \(G\). Let, \(r > 0\), \(q_0 \in B_{d_{lb}}(g_0,r)\), \(S : E \to P(E)\) and \(\{SS(q_n)\}\) be the sequence in \(E\) generated by \(q_0\). Assume that (i) and (ii) hold:

(i) \(S\) is a multi graph dominated on \(\{SS(q_n)\}\);

(ii) there exists \(\psi_b \in \Psi_b\) so as

\[
H_{d_{lb}}(Sq,Se) \leq \psi_b(d_{lb}(q,e)), \tag{3.28}
\]

for all \(q,e \in \{TS(q_n)\}\) and \((q,e) \in W(G)\) or \((e,q) \in W(G)\);

Then, \((q_n,q_{n+1}) \in W(G)\) and \(\{SS(q_n)\} \to q^*\). Also, if the inequality (3.28) holds for \(q^*\) and \((q_n,q^*) \in W(G)\) or \((q^*,q_n) \in W(G)\) for each \(n\) belongs to \(\mathbb{N} \cup \{0\}\), then \(q^*\) is the C.F.P of both \(S\) and \(T\) in \(E\) and \(d_{lb}(q^*,q^*) = 0\).

### 3.4 Fixed Point Results for Multivalued Dominated Mappings in Dislocated \(b\)-Metric Spaces with Application

Results given in this section can be seen in [49].

Let \((Z,d_l)\) be a \(D.B.M.S, g_0 \in Z\) and \(S,T : Z \to P(Z)\) be the setvalued maps on \(Z\). Let \(g_1 \in Sg_0\) be an element such that \(d_l(g_0,Sg_0) = d_l(g_0,g_1)\). Let \(g_2 \in Tg_1\) be such that \(d_l(g_1,Tg_1) = d_l(g_1,g_2)\). Let \(g_3 \in Sg_2\) be such that \(d_l(g_2,Sg_2) = d_l(g_2,g_3)\). Proceeding this method, we get a sequence \(g_n\) in \(Z\) such that \(g_{2n+1} \in Sg_{2n}\) and \(g_{2n+2} \in Tg_{2n+1}\), where \(n = 0,1,2,\ldots\). Also \(d_l(g_{2n},Sg_{2n}) = d_l(g_{2n},g_{2n+1})\), \(d_l(g_{2n+1},Tg_{2n+1}) = d_l(g_{2n+1},g_{2n+2})\). We represent this type of sequence by \(\{TS(g_n)\}\). We say that \(\{TS(g_n)\}\) be a sequence in \(Z\) generated by \(g_0\).

**Theorem 3.4.1** Let \((Z,d_l)\) is a complete \(D.B.M.S\) with coefficient \(b \geq 1\). Let \(r > 0\), \(g_0 \in B_{d_{lb}}(g_0,r) \subseteq Z\), \(\alpha : Z \times Z \to [0,\infty)\) and \(S,T : Z \to P(Z)\) be the semi \(\alpha\)-dominated mappings on \(B_{d_{lb}}(g_0,r)\). Assume (i) and (ii) hold:

(i) There exist \(\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0\) satisfying \(b\eta_1 + b\eta_2 + (1+b)\eta_3 + \eta_4 < 1\) and a strictly
increasing mapping $F$ such that

$$
\tau + F(H_d(Se, Ty)) \leq F\left( \frac{\eta_1 d_l(e, y) + \eta_2 d_l(e, Se) + \eta_3 d_l(e, Ty)}{1 + d_l^2(e, y)} \right),
$$

whenever $e, y \in \overline{B_d(g_0, r)} \cap \{TS(g_n)\}$, $\alpha(e, y) \geq 1$ and $H_d(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_2 - \eta_3}$, then

$$
d_l(g_0, Sg_0) \leq \lambda(1 - b\lambda)r.
$$

Then $\{TS(g_n)\}$ is the sequence in $\overline{B_d(g_0, r)}$, $\alpha(g_n, g_{n+1}) \geq 1$ for each $n$ belongs to $\mathbb{N} \cup \{0\}$

and $\{TS(g_n)\} \to u \in \overline{B_d(g_0, r)}$. Also, if the inequality (3.29) holds for $e, y \in \{u\}$ and either $\alpha(g_n, u) \geq 1$ or $\alpha(u, g_n) \geq 1$ for each $n$ belongs to $\mathbb{N} \cup \{0\}$, then $u$ is the C.F.P of both $S$ and $T$ in $\overline{B_d(g_0, r)}$.

**Proof.** Consider a sequence $\{TS(g_n)\}$. From (3.30), we get

$$
d_l(g_0, g_1) = d_l(g_0, Sg_0) \leq \lambda(1 - b\lambda)r < r.
$$

It implies that,

$$
g_1 \in \overline{B_d(g_0, r)}.
$$

Let $g_2, \cdots, g_j \in \overline{B_d(g_0, r)}$ for every $j$ belongs to $\mathbb{N}$. If $j$ is odd, then $j = 2i + 1$ for some $i \in \mathbb{N}$. Since $S, T : Z \to P(Z)$ be a semi $\alpha_*$-dominated mappings on $\overline{B_d(g_0, r)}$, so $\alpha_*(g_2, Sg_2) \geq 1$ and $\alpha_*(g_{2i+1}, Tg_{2i+1}) \geq 1$. As $\alpha_*(g_{2i}, Sg_{2i}) \geq 1$, this implies $\inf\{\alpha(g_{2i}, b) : b \in Sg_{2i}\} \geq 1$. Also $g_{2i+1} \in Sg_{2i}$, so $\alpha(g_{2i}, g_{2i+1}) \geq 1$. Now, by using Lemma 1.2.8, we get

$$
\tau + F(d_l(g_{2i+1}, g_{2i+2})) \leq \tau + F(H_d(Sg_{2i}, Tg_{2i+1}))
$$

Now, by using (3.29), we get

$$
\tau + F(d_l(g_{2i+1}, g_{2i+2})) \leq F[\eta_1 d_l(g_{2i}, g_{2i+1}) + \eta_2 d_l(g_{2i}, Sg_{2i}) + \eta_3 d_l(g_{2i}, Tg_{2i+1})

+ \eta_4 \frac{d_l^2(g_{2i}, Sg_{2i}) \cdot d_l(g_{2i+1}, Tg_{2i+1})}{1 + d_l^2(g_{2i}, g_{2i+1})}]
$$

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\[ \leq F[\eta_1 d_i(g_{2i}, g_{2i+1}) + \eta_2 d_i(g_{2i}, g_{2i+1}) + \eta_3 d_i(g_{2i}, g_{2i+2}) \\
+ \eta_4 \frac{d^2_i(g_{2i}, g_{2i+1}) d_i(g_{2i+1}, g_{2i+2})}{1 + d^2_i(g_{2i}, g_{2i+1})} ] \leq \]
\[ F[\eta_1 d_i(g_{2i}, g_{2i+1}) + \eta_2 d_i(g_{2i}, g_{2i+1}) + b\eta_3 d_i(g_{2i}, g_{2i+1}) \\
+ b\eta_4 d_i(g_{2i+1}, g_{2i+2}) + \eta_4 \frac{d^2_i(g_{2i}, g_{2i+1}) d_i(g_{2i+1}, g_{2i+2})}{1 + d^2_i(g_{2i}, g_{2i+1})} ] \leq \]
\[ F((\eta_1 + \eta_2 + b\eta_3) d_i(g_{2i}, g_{2i+1}) + (b\eta_3 + \eta_4) d_i(g_{2i+1}, g_{2i+2})). \]

This implies

\[ F(d_i(g_{2i+1}, g_{2i+2})) \leq F((\eta_1 + \eta_2 + b\eta_3) d_i(g_{2i}, g_{2i+1}) \\
+ (b\eta_3 + \eta_4) d_i(g_{2i+1}, g_{2i+2})). \]

As \( F \) is the strictly increasing mappings. So,

\[ d_i(g_{2i+1}, g_{2i+2}) \leq (\eta_1 + \eta_2 + b\eta_3) d_i(g_{2i}, g_{2i+1}) \\
+ (b\eta_3 + \eta_4) d_i(g_{2i+1}, g_{2i+2}). \]

Which implies

\[ (1 - b\eta_3 - \eta_4) d_i(g_{2i+1}, g_{2i+2}) \leq (\eta_1 + \eta_2 + b\eta_3) d_i(g_{2i}, g_{2i+1}) \]
\[ d_i(g_{2i+1}, g_{2i+2}) \leq \left( \frac{\eta_1 + \eta_2 + b\eta_3}{1 - b\eta_3 - \eta_4} \right) d_i(g_{2i}, g_{2i+1}). \]

As \( \lambda = \frac{\eta_1 + \eta_2 + b\eta_3}{1 - b\eta_3 - \eta_4} < 1 \). Hence

\[ d_i(g_{2i+1}, g_{2i+2}) \leq \lambda d_i(g_{2i}, g_{2i+1}) \leq \lambda^2 d_i(g_{2i-1}, g_{2i}) \leq \cdots \leq \lambda^{2i+1} d_i(g_0, g_1). \]

Similarly, if \( j \) is even, we have

\[ d_i(g_{2i+2}, g_{2i+3}) \leq \lambda^{2i+2} d_i(g_0, g_1). \]

Now, we have

\[ d_i(g_j, g_{j+1}) \leq \lambda^2 d_i(g_0, g_1) \] for each \( j \) belongs to \( \mathbb{N} \).  \hspace{1cm} (3.31)
Now,
\[ d_l(x_0, g_{j+1}) \leq bd_l(g_0, g_1) + b^2 d_l(g_1, g_2) + \cdots + b^{j+1} d_l(g_j, g_{j+1}) \]
\[ \leq bd_l(g_0, g_1) + b^2 \lambda d_l(g_0, g_1) + \cdots + b^{j+1} \lambda^{j+1} d_l(g_0, g_1), \quad \text{(by (3.31))} \]
\[ d_l(g_0, g_{j+1}) \leq \frac{b(1 - (b\lambda)^{j+1})}{1 - b\lambda} \lambda(1 - b\lambda)r < r, \]
which means \( g_{j+1} \) belongs to \( \overline{B}_{d_l}(g_0, r) \). Hence, by induction \( g_n \in \overline{B}_{d_l}(g_0, r) \) for every \( n \) belongs to \( N \). Also \( \alpha(g_n, g_{n+1}) \geq 1 \) for each \( n \) belongs to \( N \cup \{0\} \). Now,
\[ d_l(g_n, g_{n+1}) < \lambda^n d_l(g_0, g_1) \text{ for each } n \in N. \quad (3.32) \]

Now, for non negative integers \( m, n \ (n > m) \), we get
\[ d_l(g_m, g_n) \leq b(d_l(g_m, g_{m+1}) + b^2(d_l(g_{m+1}, g_{m+2})) + \cdots + b^{n-m}(d_l(g_{n-1}, g_n)), \]
\[ < b\lambda^m d_l(g_0, g_1) + b^2\lambda^{m+1} d_l(g_0, g_1) + \cdots + b^{n-m}\lambda^{n-1} d_l(g_0, g_1), \quad \text{(by (3.32))} \]
\[ < b\lambda^m(1 + b\lambda + \cdots) d_l(g_0, g_1) \]
As \( \eta_1, \eta_2, \eta_3, \eta_4 > 0, b \geq 1 \) and \( b\eta_1 + b\eta_2 + (1 + b)b\eta_3 + \eta_4 < 1 \), so \( |b\lambda| < 1 \). Then, we have
\[ d_l(g_m, g_n) < \frac{b\lambda^m}{1 - b\lambda} d_l(g_0, g_1) \rightarrow 0 \text{ as } m \rightarrow \infty. \]
Hence \( \{TS(g_n)\} \) is Cauchy in \( \overline{B}_{d_l}(g_0, r) \). Since \( (\overline{B}_{d_l}(g_0, r), d_l) \) is a complète mētrēc space, so there exist \( u \in \overline{B}_{d_l}(g_0, r) \) so as \( \{TS(g_n)\} \rightarrow u \) as \( n \rightarrow \infty \), then
\[ \lim_{n \rightarrow \infty} d_l(g_n, u) = 0. \quad (3.33) \]
By assumption, $\alpha(g_n, u) \geq 1$. Suppose that $d_t(u, Tg) > 0$, then there exist positive integer $k$ so as $d_t(g_n, Tu) > 0$ for each $n \geq k$. For $n \geq k$, we get

$$d_t(u, Tu) \leq d_t(u, g_{2n+1}) + d_t(g_{2n+1}, Tu) \leq d_t(u, g_{2n+1}) + H_{d_t}(Sg_{2n}, Tu)$$

$$< d_t(u, g_{2n+1}) + \eta_1 d_t(g_{2n}, u) + \eta_2 d_t(g_{2n}, Sg_{2n}) + \eta_3 d_t(g_{2n}, Tu) + \eta_4 \frac{d_t^2(g_{2n}, Sg_{2n})d_t(u, Tu)}{1 + d_t^2(g_{2n}, u)}$$

$$< d_t(u, g_{2n+1}) + \eta_1 d_t(g_{2n}, u) + \eta_2 d_t(g_{2n}, g_{2n+1}) + \eta_3 d_t(g_{2n}, Tu) + \eta_4 \frac{d_t^2(g_{2n}, g_{2n+1})d_t(u, Tu)}{1 + d_t^2(g_{2n}, u)}.$$

Letting $n \to \infty$, and by using (3.33) we get

$$d_t(u, Tu) < \eta_3 d_t(u, Tu) < d_t(u, Tu),$$

which contradicts. So our supposition is wrong. Hence $d_t(u, Tu) = 0$ or $u \in Tu$. Similarly, by using Lemma 1.2.8, inequality (3.29),

$$d_t(u, Su) \leq d_t(u, g_{2n+2}) + d_t(g_{2n+2}, Su) \leq d_t(u, g_{2n+2}) + H_{d_t}(Tg_{2n+1}, Su) < d_t(u, g_{2n+2}) + \eta_1 d_t(u, g_{2n+1})$$

$$+ \eta_2 d_t(u, Su) + \eta_3 d_t(u, Tg_{2n+1}) + \eta_4 \frac{d_t^2(u, Su)d_t(g_{2n+1}, Tg_{2n+1})}{1 + d_t^2(u, g_{2n+1})}$$

$$< d_t(u, g_{2n+2}) + \eta_1 d_t(u, g_{2n+1}) + \eta_2 d_t(u, Su) + \eta_3 d_t(u, g_{2n+2}) + \eta_4 \frac{d_t^2(u, Su)d_t(g_{2n+1}, g_{2n+2})}{1 + d_t^2(u, g_{2n+1})}.$$
Letting \( n \to \infty \), and by using (3.33) we get

\[
d_i(u, Su) < \eta_2 d_i(u, Su) < d_i(u, Su),
\]

which contradicts. So our supposition is wrong. Hence \( d_i(u, Su) = 0 \) or \( u \in Su \). Hence the \( S \) and \( T \) have a C.F.P \( u \) in \( \overline{B_{d_i}(g_0, r)} \). Now,

\[
d_i(u, u) \leq b d_i(u, Tu) + b d_i(Tu, u) \leq 0.
\]

This implies \( d_i(u, u) = 0 \). \( \square \)

**Example 3.4.2** Let \( Z = Q^+ \cup \{0\} \) and let \( d_i : Z \times Z \to Z \) be the complete D.B.M.S defined by

\[
d_i(v, p) = (v + p)^2 \text{ for all } v, p \in Z.
\]

with \( b = 2 \). Define the multivalued mapping, \( S, T : Z \times Z \to P(Z) \) by,

\[
Sg = \begin{cases} 
\left[ \frac{g}{3}, \frac{2}{3}g \right] & \text{if } g \in [0, 14] \cap Z \\
[g, g + 1] & \text{if } g \in (14, \infty) \cap Z
\end{cases}
\]

and,

\[
Tp = \begin{cases} 
\left[ \frac{p}{4}, \frac{3}{4}p \right] & \text{if } p \in [0, 14] \cap Z \\
[p + 1, p + 3] & \text{if } p \in (14, \infty) \cap Z
\end{cases}
\]

Suppose that, \( g_0 = 1, r = 225 \), then \( \overline{B_{d_i}(g_0, r)} = [0, 14] \cap Z \) and \( TS(g_n) = \{1, \frac{1}{3}, \frac{1}{12}, \ldots\} \). Take \( \eta_1 = \frac{1}{10}, \eta_2 = \frac{1}{20}, \eta_3 = \frac{1}{60}, \eta_4 = \frac{1}{30} \), then \( b \eta_1 + b \eta_2 + (1 + b) \eta_3 + \eta_4 < 1 \) and \( \lambda = \frac{11}{56} \). Now

\[
d_i(g_0, Sg_0) = \frac{16}{9} < \frac{11}{56} (1 - \frac{22}{56}) 225 = \lambda (1 - b \lambda) r
\]

Consider the mapping \( \alpha : Z \times Z \to [0, \infty) \) by

\[
\alpha(g, p) = \begin{cases} 
1 & \text{if } g > p \\
\frac{1}{2} & \text{otherwise}
\end{cases}
\]

Now, if \( g, p \in \overline{B_{d_i}(g_0, r)} \cap \{TS(g_n)\} \) with \( \alpha(g, p) \geq 1 \), we have
\[ H_d(Sg, Tp) = \max \left\{ \sup_{a \in Sg} d_l(a, Tp), \sup_{b \in Tp} d_l(Sg, b) \right\} \]
\[ = \max \left\{ \sup_{a \in Sg} d_l\left(\frac{p}{4}, \frac{3p}{4}\right), \sup_{b \in Tp} d_l\left(\frac{g}{3}, \frac{2g}{3}\right), b \right\} \]
\[ = \max \left\{ d_l\left(\frac{2g}{3}, \frac{p}{4}, \frac{3p}{4}\right), d_l\left(\frac{g}{3}, \frac{2g}{3}, 3p\right) \right\} \]
\[ = \max \left\{ d_l\left(\frac{2g}{3}, \frac{p}{4}, \frac{3p}{4}\right), d_l\left(\frac{g}{3}, \frac{3p}{4}\right) \right\} \]

\[ = \max \left\{ \left(\frac{2g}{3} + \frac{p}{4}\right)^2, \left(\frac{g}{3} + \frac{3p}{4}\right)^2 \right\} \geq \]
\[ < \frac{1}{10} (g + p)^2 + \frac{4g^2}{45} + \frac{(4g + p)^2}{960} + \frac{40g^4p^2}{243(1 + (g + p)^4)} \]
\[ = \frac{1}{10} d_l(g, p) + \frac{1}{20} d_l\left(\frac{g}{3}, \frac{2g}{3}, 3g\right) + \frac{1}{60} d_l\left(\frac{p}{4}, \frac{3p}{4}\right) + \frac{1}{30} \frac{d_l^2(g, \frac{g}{3}, \frac{2g}{3})}{1 + d_l^2(g, p)}. \]

Thus,
\[ H_d(Sg, Tp) < \eta_1d_l(g, p) + \eta_2d_l(g, Sg) + \eta_3d_l(g, Tp) + \eta_4 \frac{d_l^2(g, Sg), d_l(p, Tp)}{1 + d_l^2(g, p)}, \]
which implies that, for any \( \tau \in (0, \frac{12}{95}] \) and for a strictly increasing mapping \( F(s) = \ln s \), we have
\[ \tau + F(H_d(Sg, Tp)) \leq F \left( \eta_1d_l(g, p) + \eta_2d_l(g, Sg) + \eta_3d_l(g, Tp) + \eta_4 \frac{d_l^2(g, Sg), d_l(p, Tp)}{1 + d_l^2(g, p)} \right). \]

Note that, for \( 16, 15 \in X \), then \( \alpha(16, 15) \geq 1 \). But, we have
\[ \tau + F(H_d(S16, T15)) > F \left( \eta_1d_l(16, 15) + \eta_2d_l(16, S16) + \eta_3d_l(16, T15) + \eta_4 \frac{d_l^2(16, S16), (15, T15)}{1 + d_l^2(16, 15)} \right). \]

So condition (3.29) does not hold on \( Z \). Thus maps \( S \) and \( T \) are satisfying all requirements of Theorem 3.4.1 only for \( g, p \in B_d(g_0, r) \cap \{ TS(g_n) \} \) with \( \alpha(g, p) \geq 1 \). Hence \( S \) and \( T \) have a C.F.P.
If, we take \( S = T \) in Theorem 3.4.1, then we are left only with the result.

**Corollary 3.4.3** Let \((Z, d_t)\) is a complete D.B.M.S with coefficient \( w \geq 1 \). Let \( r > 0 \), \( g_0 \in \overline{B_{d_t}(g_0, r)} \subseteq Z \), \( \alpha : Z \times Z \to [0, \infty) \) and \( S : Z \to P(Z) \) be the semi \( \alpha_\ast \)-dominated mappings on \( \overline{B_{d_t}(g_0, r)} \). Suppose (i) and (ii) hold:

(i) There exist \( \tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0 \) satisfying \( w\eta_1 + w\eta_2 + (1 + w)w\eta_3 + \eta_4 < 1 \) and a strictly increasing mapping \( F \) such that

\[
\tau + F(H_{d_t}(Se, Sy)) \leq F \left( \eta_1 d_t(e, y) + \eta_2 d_t(e, Se) + \eta_3 d_t(e, Sy) + \eta_4 \frac{d^2_t(e, Se) d_t(y, Sy)}{1 + d_t^2(x, y)} \right),
\]

whenever \( e, y \in \overline{B_{d_t}(g_0, r)} \cap \{ SS(g_n) \} \), \( \alpha(e, y) \geq 1 \) and \( H_{d_t}(Se, Sy) > 0 \).

(ii) If \( \lambda = \frac{\eta_1 + \eta_2 + w\eta_3}{1 - w\eta_3 - \eta_4} \), then

\[ d_t(g_0, Sg_0) \leq \lambda(1 - w\lambda)r. \]

Then \{SS(g_n)\} be the sequence in \( \overline{B_{d_t}(g_0, r)} \), \( \alpha(g_n, g_{n+1}) \geq 1 \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and \( \{SS(g_n)\} \to u \in \overline{B_{d_t}(g_0, r)} \). Also, if the inequality (3.34) holds for \( e, y \in \{u\} \) and either \( \alpha(g_n, u) \geq 1 \) or \( \alpha(u, g_n) \geq 1 \) for every \( n \) belongs to \( \mathbb{N} \cup \{0\} \), then \( u \) be the fixed point of \( S \) in \( \overline{B_{d_t}(g_0, r)} \).

If, we take \( \eta_2 = 0 \) in Theorem 3.4.1, then we are left only with the result.

**Corollary 3.4.4** Let \((Z, d_t)\) is a complete D.B.M.S with coefficient \( b \geq 1 \). Let \( r > 0 \), \( g_0 \in \overline{B_{d_t}(g_0, r)} \subseteq Z \), \( \alpha : Z \times Z \to [0, \infty) \) and \( S, T : Z \to P(Z) \) be the semi \( \alpha_\ast \)-dominated mappings on \( \overline{B_{d_t}(g_0, r)} \). Suppose (i) and (ii) hold:

(i) There exist \( \tau, \eta_1, \eta_3, \eta_4 > 0 \) satisfying \( bn_1 + (1 + b)n_3 + \eta_4 < 1 \) and a strictly increasing mapping \( F \) such that

\[
\tau + F(H_{d_t}(Se, Ty)) \leq F \left( \eta_1 d_t(e, y) + \eta_3 d_t(e, Ty) + \eta_4 \frac{d^2_t(e, Se) d_t(y, Ty)}{1 + d_t^2(x, y)} \right),
\]

whenever \( e, y \in \overline{B_{d_t}(g_0, r)} \cap \{ TS(g_n) \} \), \( \alpha(e, y) \geq 1 \) and \( H_{d_t}(Se, Ty) > 0 \).

(ii) If \( \lambda = \frac{\eta_1 + bn_3}{1 - bn_3 - \eta_4} \), then

\[ d_t(g_0, Sg_0) \leq \lambda(1 - b\lambda)r. \]
Then \( \{TS(g_n)\} \) be the sequence in \( \overline{B_{d_l}(g_0, r)} \), \( \alpha(g_n, g_{n+1}) \geq 1 \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and \( \{TS(g_n)\} \rightarrow u \in \overline{B_{d_l}(g_0, r)} \). Also, if the inequality (3.35) holds for \( e, y \in \{u\} \) and either \( \alpha(g_n, u) \geq 1 \) or \( \alpha(u, g_n) \geq 1 \) for every \( n \) belongs to \( \mathbb{N} \cup \{0\} \), then \( u \) is the C.F.P of both \( S \) and \( T \) in \( \overline{B_{d_l}(g_0, r)} \).

If, we take \( \eta_3 = 0 \) in Theorem 3.4.1, then we are left only with the result.

**Corollary 3.4.5** Let \((Z, d_l)\) is a complete D.B.M.S with coefficient \( b \geq 1 \). Let \( r > 0 \), \( g_0 \in \overline{B_{d_l}(g_0, r)} \subseteq Z \), \( \alpha : Z \times Z \rightarrow [0, \infty) \) and \( S, T : Z \rightarrow P(Z) \) be the semi \( \alpha_*\)-dominated mappings on \( \overline{B_{d_l}(g_0, r)} \). Suppose (i) and (ii) hold:

(i) There exist \( \tau, \eta_1, \eta_2, \eta_4 > 0 \) satisfying \( b\eta_1 + b\eta_2 + \eta_4 < 1 \) and a strictly increasing mapping \( F \) such that

\[
\tau + F(H_{d_l}(Se, Ty)) \leq F \left( \eta_1 d_l(e, y) + \eta_2 d_l(e, Se) + \frac{\eta_4 d^2_l(e, y) d_l(u, Ty)}{1 + d^2_l(e, y)} \right),
\]

whenever \( e, y \in \overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\} \), \( \alpha(e, y) \geq 1 \) and \( H_{d_l}(Se, Ty) > 0 \).

(ii) If \( \lambda = \frac{\eta_1 + \eta_2}{1 - \eta_4} \), then

\[
d_l(g_0, Sg_0) \leq \lambda(1 - b\lambda)r.
\]

Then \( \{TS(g_n)\} \) be the sequence in \( \overline{B_{d_l}(g_0, r)} \), \( \alpha(g_n, g_{n+1}) \geq 1 \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and \( \{TS(g_n)\} \rightarrow u \in \overline{B_{d_l}(g_0, r)} \). Also, if the inequality (3.36) holds for \( e, y \in \{u\} \) and either \( \alpha(g_n, u) \geq 1 \) or \( \alpha(u, g_n) \geq 1 \) for every \( n \) belongs to \( \mathbb{N} \cup \{0\} \), then \( u \) is the C.F.P of both \( S \) and \( T \) in \( \overline{B_{d_l}(g_0, r)} \).

If, we take \( \eta_4 = 0 \) in Theorem 3.4.1, then we are left only with the result.

**Corollary 3.4.6** Let \((Z, d_l)\) is a complete D.B.M.S with coefficient \( b \geq 1 \). Let \( r > 0 \), \( g_0 \in \overline{B_{d_l}(g_0, r)} \subseteq Z \), \( \alpha : Z \times Z \rightarrow [0, \infty) \) and \( S, T : Z \rightarrow P(Z) \) are the semi \( \alpha_*\)-dominated maps on \( \overline{B_{d_l}(g_0, r)} \). Suppose (i) and (ii) hold:

(i) There exist \( \tau, \eta_1, \eta_2, \eta_3 > 0 \) satisfying \( b\eta_1 + b\eta_2 + (1 + b)\eta_3 < 1 \) and a strictly increasing mapping \( F \) such that

\[
\tau + F(H_{d_l}(Se, Ty)) \leq F \left( \eta_1 d_l(e, y) + \eta_2 d_l(e, Se) + \eta_3 d_l(e, Ty) \right),
\]

whenever \( e, y \in \overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\} \), \( \alpha(e, y) \geq 1 \) and \( H_{d_l}(Se, Ty) > 0 \).
(ii) If \( \lambda = \frac{m_1 + m_2 + m_3}{1 - m_3 - m_4} \), then

\[
d_1(g_0, Sg_0) \leq \lambda(1 - b\lambda)r.
\]

Then \( \{TS(g_n)\} \) be the sequence in \( \overline{B_{d_1}(g_0, r)} \), \( \alpha(g_n, g_{n+1}) \geq 1 \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and \( \{TS(g_n)\} \rightarrow u \in \overline{B_{d_1}(g_0, r)} \). Also, if the inequality (3.37) holds for \( e, y \in \{u\} \) and either \( \alpha(g_n, u) \geq 1 \) or \( \alpha(u, g_n) \geq 1 \) for every \( n \) belongs to \( \mathbb{N} \cup \{0\} \), then \( u \) is the C.F.P of both \( S \) and \( T \) in \( \overline{B_{d_1}(g_0, r)} \).

Now we present an application of Theorem 3.4.1 in graph theory. Jachymski [33] proved the result concerning for contractive mappings with a graph. Hussain et al. [31] introduced the fixed point theorem for graphic contraction and gave an application. Furthermore, avoiding sets condition is closed related to fixed point and is applied to the study of multi-agent systems (see [46]).

**Definition 3.4.7** Let \( Z \neq \{\} \) and \( Q = (V(Q), W(Q)) \) be a graph such that \( V(Q) = Z \), \( A \subseteq Z \). \( S : Z \rightarrow P(Z) \) be the multi graph dominated on \( A \) if \( (p, q) \in W(Q) \), for all \( q \in Sp \) and \( q \in A \).

**Theorem 3.4.8** Let \( (Z, d_1) \) is a complete \( D.B.M.S \) endowed a graph \( Q \) with coefficient \( b \geq 1 \). Let \( r > 0 \), \( g_0 \in \overline{B_{d_1}(g_0, r)} \) and \( S, T : Z \rightarrow P(Z) \). Assume (i), (ii) and (iii) satisfy:

(i) \( S \) and \( T \) are multi graph dominated on \( \overline{B_{d_1}(g_0, r)} \cap \{TS(g_n)\} \).

(ii) There exist \( \tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0 \) satisfying \( b\eta_1 + b\eta_2 + (1 + b)\eta_3 + \eta_4 < 1 \) and a strictly increasing mapping \( F \) such that

\[
\tau + F(H_{d_1}(Sp, Tq)) \leq F\left( \eta_1 d_1(p, q) + \eta_2 d_1(p, Sp) + \eta_3 d_1(p, Tq) + \eta_4 \frac{d_1^2(p, Sp) d_1(q, Tq)}{1 + d_1^2(p, q)} \right), \tag{3.38}
\]

whenever \( p, q \in \overline{B_{d_1}(g_0, r)} \cap \{TS(g_n)\}, (p, q) \in W(Q) \) and \( H_{d_1}(Sp, Tq) > 0 \).

(iii) \( d_1(g_0, Sg_0) \leq \lambda(1 - b\lambda)r \), where \( \lambda = \frac{\eta_1 + \eta_2 + b\eta_3}{1 - b\eta_3 - \eta_4} \).

Then, \( \{TS(g_n)\} \) be the sequence in \( \overline{B_{d_1}(g_0, r)} \), \( \{TS(g_n)\} \rightarrow m^* \) and \( (g_n, g_{n+1}) \in W(Q) \), where \( g_n, g_{n+1} \in \{TS(g_n)\} \). Also, if the inequality (3.38) holds for \( p, q \in \{m^*\} \) and \( (g_n, m^*) \in W(Q) \) or \( (m^*, g_n) \in W(Q) \) for every \( n \) belongs to \( \mathbb{N} \cup \{0\} \), then \( m^* \) is the C.F.P of both \( S \) and \( T \) in \( \overline{B_{d_1}(g_0, r)} \).
Proof. Define, $\alpha : Z \times Z \rightarrow [0, \infty)$ by

$$\alpha(w, e) = \begin{cases} 
1, & \text{if } w \in B_{d_l}(g_0, r), \ (w, e) \in W(Q) \\
0, & \text{otherwise.} 
\end{cases}$$

Given $S$ and $T$ are semi graph dominated on $B_{d_l}(g_0, r)$, then for $p \in B_{d_l}(g_0, r)$, $(p, q) \in W(Q)$ for all $q \in Sp$ and $(p, q) \in W(Q)$ for all $q \in Tp$. So, $\alpha(p, q) = 1$ for all $q \in Sp$ and $\alpha(p, q) = 1$ for all $q \in Tp$. This implies that $\inf \{\alpha(p, q) : q \in Sp\} = 1$ and $\inf \{\alpha(p, q) : q \in Tp\} = 1$. Hence $\alpha^*(p, Sp) = 1, \alpha^*(p, Tp) = 1$ for all $p \in B_{d_l}(g_0, r)$. So, $S, T : Z \rightarrow P(Z)$ are the semi $\alpha^*$-dominated mapping on $B_{d_l}(g_0, r)$. Moreover, inequality (2.38) can be written as

$$\tau + F(H_{d_l}(Sp, Tq)) \leq F \left( \eta_1 d_l(p, q) + \eta_2 d_l(p, Sp) + \eta_3 d_l(p, Tq) + \eta_4 \frac{d^2_l(p, Sp, d_l(q, Tq))}{1 + d^2_l(p, q)} \right)$$

whenever $p, q \in B_{d_l}(g_0, r) \cap \{TS(g_n)\}$, $\alpha(p, q) \geq 1$ and $H_{d_l}(Sp, Tq) > 0$. Also, (iii) holds. Then, by Theorem 3.4.1, we have $\{TS(g_n)\}$ be the sequence in $B_{d_l}(g_0, r)$ and $\{TS(g_n)\} \rightarrow m^* \in B_{d_l}(g_0, r)$. Now, $g_n, m^* \in B_{d_l}(g_0, r)$ and either $(g_n, m^*) \in W(Q)$ or $(m^*, g_n) \in W(Q)$ implies that either $\alpha(g_n, m^*) \geq 1$ or $\alpha(m^*, g_n) \geq 1$. So, all hypothesis of Theorem 3.4.1 are proved. Hence, by Theorem 3.4.1, $S$ and $T$ have a C.F.P $m^*$ in $B_{d_l}(g_0, r)$ and $d_l(m^*, m^*) = 0$. ■

In this section, we have discussed some new fixed point results for single valued mapping in complete D.B.M.S. Let $(Z, d_l)$ be a D.B.M.S, $c_0 \in Z$ and $S, T : Z \rightarrow Z$ be the mappings. Let $c_1 = Sc_0, c_2 = Tc_1, c_3 = Sc_2$. Proceeding this method, we make a sequence $c_n$ of points in $Z$ such that $c_{2n+1} = Sc_{2n}$ and $c_{2n+2} = Tc_{2n+1}$, where $n = 0, 1, 2, \ldots$. We represent this type of sequence by $\{TS(c_n)\}$. Then $\{TS(c_n)\}$ be the sequence in $Z$ generated by $c_0$.

**Theorem 3.4.7** Let $(Z, d_l)$ is a complete D.B.M.S. Let $r > 0, c_0 \in B_{d_l}(c_0, r) \subseteq Z$, $\alpha : Z \times Z \rightarrow [0, \infty)$ and $S, T : Z \rightarrow Z$ be the semi $\alpha$-dominated maps on $B_{d_l}(c_0, r)$. Assume (i) and (ii) hold:

(i) There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $t\eta_1 + t\eta_2 + (1 + t)\eta_3 + \eta_4 < 1$ and a strictly increasing mapping $F$ such that

$$\tau + F(d_l(Se, Ty)) \leq F \left( \eta_1 d_l(e, y) + \eta_2 d_l(e, Se) + \eta_3 d_l(e, Ty) + \eta_4 \frac{d^2_l(e, Se, d_l(y, Ty))}{1 + d^2_l(e, y)} \right), \quad (3.39)$$

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whenever \( e, y \in B_d(c_0, r) \cap \{TS(c_n)\}, \alpha(e, y) \geq 1 \) and \( H_d(Se, Ty) > 0 \).

(ii) If \( \lambda = \frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_2 - \eta_3} \), then

\[
d_l(c_0, Sc_0) \leq \lambda(1 - t\lambda)r.
\]

Then \( \{TS(c_n)\} \) be the iterative sequence in \( B_d(c_0, r) \), \( \alpha(c_n, c_{n+1}) \geq 1 \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and \( \{TS(c_n)\} \rightarrow u \in B_d(c_0, r) \). Also if the inequality (3.39) holds for \( e, y \in \{u\} \) and either \( \alpha(c_n, u) \geq 1 \) or \( \alpha(u, c_n) \geq 1 \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \), then \( u \) is the C.F.P of both \( S \) and \( TS(c_0, r) \).

**Proof.** The proof of above Theorem is similar as previous proved Theorem 3.4.1.

If, we take \( S = T \) in Theorem 3.4.7, then we are left only with the result.

**Corollary 3.4.8** Let \((Z, d_l)\) be a complete D.B.M.S. Let \( r > 0, c_0 \in B_d(c_0, r) \subseteq Z \), \( \alpha : Z \times Z \rightarrow [0, \infty) \) and \( S : Z \rightarrow Z \) be the semi \( \alpha \)-dominated maps on \( B_d(c_0, r) \). Suppose (i) and (ii) hold:

(i) There exist \( \tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0 \) satisfying \( t\eta_1 + t\eta_2 + (1 + t)\eta_3 + \eta_4 < 1 \) and a strictly increasing mapping \( F \) such that

\[
\tau + F(d_l(Se, Sy)) \leq F \left( \frac{\eta_1 F_l(e, y) + \eta_2 d_l(e, Se)}{\eta_2 d_l(e, Sy) + \eta_4 \frac{d^2_l(e, SSe, Sy)}{1 + d^2_l(e, y)}} \right),
\]

whenever \( e, y \in B_d(c_0, r) \cap \{SS(c_n)\}, \alpha(e, y) \geq 1 \) and \( H_d(Se, Sy) > 0 \).

(ii) If \( \lambda = \frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_2 - \eta_3} \), then

\[
d_l(c_0, Sc_0) \leq \lambda(1 - t\lambda)r.
\]

Then \( \{SS(c_n)\} \) be the sequence in \( B_d(c_0, r) \), \( \alpha(c_n, c_{n+1}) \geq 1 \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and \( \{SS(c_n)\} \rightarrow u \in B_d(c_0, r) \). Also if the inequality (3.40) holds for \( e, y \in \{u\} \) and either \( \alpha(c_n, u) \geq 1 \) or \( \alpha(u, c_n) \geq 1 \) for every \( n \) belongs to \( \mathbb{N} \cup \{0\} \), then \( u \) is the fixed point of \( S \) in \( B_d(c_0, r) \).

If, we take \( \eta_2 = 0 \) in Theorem 3.4.7, then we are left only with the result.

**Corollary 3.4.9** Let \((Z, d_l)\) is a complete D.B.M.S. Let \( r > 0, c_0 \in B_d(c_0, r) \subseteq Z \), \( \alpha : Z \times Z \rightarrow [0, \infty) \) and \( S, T : Z \rightarrow Z \) be the semi \( \alpha \)-dominated maps on \( B_d(c_0, r) \). Suppose (i) and (ii) hold:

(i) There exist \( \tau, \eta_1, \eta_3, \eta_4 > 0 \) satisfying \( t\eta_1 + (1 + t)\eta_3 + \eta_4 < 1 \) and a strictly increasing
mapping $F$ such that

$$
\tau + F(d_t(Se, Ty)) \leq F \left( \eta_1 d_t(e, y) + \eta_2 d_t(e, Ty) + \eta_3 \frac{d_t^2(e, Se, d_t(y, Ty))}{1 + d_t^2(e, y)} \right),
$$

whenever $e, y \in \overline{B_{d_t}(c_0, r)} \cap \{TS(c_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_t}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1 + \eta_2}{1 - \eta_3 - \eta_4}$, then

$$
d_t(c_0, Sc_0) \leq \lambda(1 - \lambda)r.
$$

Then $\{TS(c_n)\}$ be the sequence in $\overline{B_{d_t}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \geq 1$ for each $n$ belongs to $\mathbb{N} \cup \{0\}$ and $\{TS(c_n)\} \to u \in \overline{B_{d_t}(c_0, r)}$. Also if the inequality (3.41) holds for $e, y \in \{u\}$ and either $\alpha(c_n, u) \geq 1$ or $\alpha(u, c_n) \geq 1$ for each $n$ belongs to $\mathbb{N} \cup \{0\}$, then $u$ is the C.F.P of both $S$ and $T$ in $\overline{B_{d_t}(c_0, r)}$.

If, we take $\eta_3 = 0$ in Theorem 3.4.7, then we are left with the result.

**Corollary 3.4.10** Let $(Z, d_t)$ be a complete D.B.M.S. Let $r > 0$, $c_0 \in \overline{B_{d_t}(c_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to Z$ be the semi $\alpha$-dominated maps on $\overline{B_{d_t}(c_0, r)}$. Suppose (i) and (ii) are hold:

(i) There exist $\tau, \eta_1, \eta_2, \eta_4 > 0$ satisfying $t\eta_1 + t\eta_2 + \eta_4 < 1$ and a strictly increasing mapping $F$ such that

$$
\tau + F(d_t(Se, Ty)) \leq F \left( \eta_1 d_t(e, y) + \eta_2 d_t(e, Se) + \eta_4 \frac{d_t^2(e, Se, d_t(y, Ty))}{1 + d_t^2(e, y)} \right),
$$

whenever $e, y \in \overline{B_{d_t}(c_0, r)} \cap \{TS(c_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_t}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1 + \eta_2}{1 - \eta_4}$, then

$$
d_t(c_0, Sc_0) \leq \lambda(1 - b\lambda)r.
$$

Then $\{TS(c_n)\}$ be the sequence in $\overline{B_{d_t}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \geq 1$ for each $n$ belongs to $\mathbb{N} \cup \{0\}$ and $\{TS(c_n)\} \to u \in \overline{B_{d_t}(c_0, r)}$. Also if the inequality (3.42) holds for $e, y \in \{u\}$ and either $\alpha(c_n, u) \geq 1$ or $\alpha(u, c_n) \geq 1$ for each $n$ belongs to $\mathbb{N} \cup \{0\}$, then $u$ is the C.F.P of both $S$ and $T$ in $\overline{B_{d_t}(c_0, r)}$.

If, we take $\eta_4 = 0$ in Theorem 3.4.7, then we are left only with the result.

**Corollary 3.4.11** Let $(Z, d_t)$ be a complete D.B.M.S. Let $r > 0$, $c_0 \in \overline{B_{d_t}(c_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to Z$ be the semi $\alpha$-dominated maps on $\overline{B_{d_t}(c_0, r)}$. Assume
that (i) and (ii) hold:

(i) There exist \( \tau, \eta_1, \eta_2, \eta_3 > 0 \) satisfying \( t\eta_1 + t\eta_2 + (1 + t)\eta_3 < 1 \) and a strictly increasing mapping \( F \) such that

\[
\tau + F(d_t(Se, Ty)) \leq F(\eta_1 d_t(e, y) + \eta_2 d_t(e, Se) + \eta_3 d_t(e, Ty)),
\]

whenever \( e, y \in \overline{B_{d_t}(c_0, r)} \cap \{TS(c_n)\} \), \( \alpha(e, y) \geq 1 \) and \( H_{d_t}(Se, Ty) > 0 \).

(ii) If \( \lambda = \frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_3} \), then

\[
d_t(c_0, Sc_0) \leq \lambda(1 - t\lambda)r.
\]

Then \( \{TS(c_n)\} \) be a sequence in \( \overline{B_{d_t}(c_0, r)} \), \( \alpha(c_n, c_{n+1}) \geq 1 \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and \( \{TS(c_n)\} \rightarrow u \in \overline{B_{d_t}(c_0, r)} \). Also if, (3.43) holds for \( e, y \in \{u\} \) and either \( \alpha(c_n, u) \geq 1 \) or \( \alpha(u, c_n) \geq 1 \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \), then \( S \) and \( T \) have C.F.P \( u \) in \( \overline{B_{d_t}(c_0, r)} \).

**Theorem 3.4.12** Let \( (Z, d_t) \) be a complete D.B.M.S. Let \( c_0 \in Z \) and \( S, T : Z \rightarrow Z \). Assume that, There exist \( \tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0 \) satisfying \( b\eta_1 + b\eta_2 + (1 + b)\eta_3 + \eta_4 < 1 \) and a strictly increasing mapping \( F \) such that the following satisfy:

\[
\tau + F(d_t(Se, Ty)) \leq F\left(\eta_1 d_t(e, y) + \eta_2 d_t(e, Se) + \eta_3 d_t(e, Ty) + \eta_4 \frac{d^2_t(e, Se, d_t(y, Ty))}{1 + d^2_t(e, Ty)}\right),
\]

whenever \( e, y \in \{TS(c_n)\} \) and \( d(Se, Ty) > 0 \). Then \( \{TS(c_n)\} \rightarrow g \in Z \). Also if the inequality (3.44) holds for \( g \), then \( g \) is the unique C.F.P of both \( S \) and \( T \) in \( Z \).

**Proof.** The proof of above Theorem is similar as previous proved Theorem 3.4.1. We have to prove the uniqueness only. Let \( p \) be another C.F.P of \( S \) and \( T \). Suppose \( d_t(Sg, Tp) > 0 \). Then, we have

\[
\tau + F(d_t(Sg, Tp)) \leq F\left(\eta_1 d_t(g, p) + \eta_2 d_t(g, Sg) + \eta_3 d_t(g, Tp) + \eta_4 \frac{d^2_t(g, Sg, d_t(p, Tp))}{1 + d^2_t(g, Tp)}\right)
\]

This implies that

\[
d_t(g, p) < \eta_1 d_t(g, p) + \eta_3 d_t(g, p) < d_t(g, p),
\]

which is a contradiction. So \( d_t(Sg, Tp) = 0 \). Hence \( g = p \).

Now, we derive the application of fixed point Theorem 3.4.12 in form of Volterra type
integral equations.

\[ g(k) = \int_{0}^{k} H_1(k, h, g(h))dh, \quad (3.45) \]

\[ p(k) = \int_{0}^{k} H_2(k, h, p(h))dh \quad (3.46) \]

for each \( k \in [0, 1] \). We find the solution of (2.45) and (2.46). Let \( \hat{G} = \{ f : f \text{ is a continuous function from } [0, 1] \text{ to } \mathbb{R}_+ \} \), endowed with the D.B.M.S. For \( g \in \hat{G} \), define norm as:

\[ \|g\|_{\tau} = \sup_{k \in [0,1]} \{ |g(k)| e^{-\tau k} \} \]

where \( \tau > 0 \) is taken arbitrary. Then define

\[ d_\tau(g, p) = \left( \sup_{k \in [0,1]} \{ |g(k) + p(k)| e^{-\tau k} \} \right)^2 = \|g + p\|_{\tau}^2 \]

for each \( g, p \in \hat{G} \), with these settings, \( (\hat{G}, d_\tau) \) becomes a complete D.B.M.S. with constant \( b = 2 \).

**Theorem 3.4.13** Assume (i), (ii) and (iii) are satisfied:

(i) \( H_1, H_2 : [0, 1] \times [0, 1] \rightarrow \mathbb{R} \);

(ii) Define

\[ Sg(k) = \int_{0}^{k} H_1(k, h, g(h))dh, \]

\[Tp(k) = \int_{0}^{k} H_2(k, h, p(h))dh.\]

Suppose there exist \( \tau > 0 \), such that

\[ |H_1(k, h, g) + H_2(k, h, p)| \leq \frac{\tau H(g, p)}{\tau \|H(g, p)\|_{\tau} + 1} \]

for each \( k, h \in [0, 1] \) and \( g, p \in \hat{G} \), where

\[ H(g(h), p(h)) = \eta_1 |g(h) + p(h)|^2 + \eta_2 |g(h) + Sg(h)|^2 + \eta_3 |g(h) + Tp(h)|^2 \]

\[ + \eta_4 \frac{|g(h) + Sg(h)||p(h) + Tp(h)|^2}{1 + |g(h) + p(h)|^4}, \]

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where \( \eta_1, \eta_2, \eta_3, \eta_4 \geq 0 \), and \( 2\eta_1 + 2\eta_2 + 6\eta_3 + \eta_4 < 1 \). Then integral equations (2.45) and (2.46) has a solution.

**Proof.** By assumption (ii)

\[
|Sg(k) + Tp(k)| = \int_0^k |H_1(k, h, g(h) + H_2(k, h, p(h))| dh,
\]

\[
\leq \int_0^k \frac{\tau}{\tau \|H(g, p)\|_\tau + 1} ([\|H(g, p)\| e^{-\tau h}) e^{\tau h} dh
\]

\[
\leq \int_0^k \frac{\tau}{\tau \|H(g, p)\|_\tau + 1} \|H(g, p)\|_\tau e^{\tau h} dh
\]

\[
\leq \frac{\tau \|H(g, p)\|_\tau}{\tau \|H(g, p)\|_\tau + 1} \int_0^k e^{\tau h} dh,
\]

\[
\leq \frac{\|H(g, p)\|_\tau}{\tau \|H(g, p)\|_\tau + 1} e^{\tau k}.
\]

This implies

\[
|Sg(k) + Tp(k)| e^{-\tau k} \leq \frac{\|H(g, p)\|_\tau}{\tau \|H(g, p)\|_\tau + 1}.
\]

\[
\|Sg(k) + Tp(k)\|_\tau \leq \frac{\|H(g, p)\|_\tau}{\tau \|H(g, p)\|_\tau + 1}.
\]

\[
\frac{\tau \|H(g, p)\|_\tau + 1}{\|H(g, p)\|_\tau} \leq \frac{1}{\|Sg(k) + Tp(k)\|_\tau}.
\]

\[
\tau + \frac{1}{\|H(g, p)\|_\tau} \leq \frac{1}{\|Sg(k) + Tp(k)\|_\tau}.
\]

which further implies

\[
\tau - \frac{1}{\|Sg(k) + Tp(k)\|_\tau} \leq -\frac{1}{\|H(g, p)\|_\tau}.
\]

So, all the hypothesis of Theorem 3.4.12 are proved for \( F(p) = \frac{-1}{\sqrt{b}}, p > 0 \) and \( d_r(g, p) = \|g + p\|^2 \), \( b = 2 \). Hence integral equations (3.45) and (3.46) has a unique common solution. \( \blacksquare \)
Example 3.4.14 Consider the integral equations

\[ g(k) = \frac{1}{3} \int_0^k g(h)dh, \quad p(k) = \frac{1}{4} \int_0^k p(h)dh, \quad \text{where } k \in [0, 1]. \]

Define \( H_1, H_2 : [0, 1] \times [0, 1] \times \mathbb{G} \rightarrow \mathbb{R} \) by \( H_1 = \frac{1}{3}g(h), \ H_2 = \frac{1}{4}p(h). \) Now,

\[ Sg(k) = \frac{1}{3} \int_0^k g(h)dh, \quad Tp(k) = \frac{1}{4} \int_0^k p(h)dh \]

Take \( \eta_1 = \frac{1}{10}, \ \eta_2 = \frac{1}{20}, \ \eta_3 = \frac{1}{60}, \ \eta_4 = \frac{1}{30}, \ \tau = \frac{12}{95}, \) then \( 2\eta_1 + 2\eta_2 + 6\eta_3 + \eta_4 < 1. \) Moreover, requirements of Theorem 3.4.13 are proved and \( g(k) = p(k) = 0 \) for each \( k, \) is a unique common solution to the shown integral equations.
Chapter 4

Results in Dislocated Quasi Metric Space

4.1 Introduction

The theory present in this section is published in [48] and accepted for publication in [57].

Recall that a mapping $B : W \to P(W)$ has a fixed point $y \in W$, if $y \in By$. There are many generalizations of metric space and various researchers make different kind of metric spaces. Dislocated quasi metric space is one of the most important and famous generalizations of metric space and it has a fundamental value in metric fixed point theory. It is very easy to say that the work on dislocated quasi metric space is more better than other metric versions. Many authors obtained fixed point theorems in complete $DQM$ (see [15, 20, 50, 61, 59, 66, 67]) which is a more general setting of partial metric space, metric-like space, quasi-partial metric space (see [54, 35]), and metric space. Fixed point results are a tool to estimate the particular solution of functional, differential and integral equations. It is simple to prove that $Q : F \to F$ is not a contraction but $Q : L \to F$ is a contraction, where $L$ is a subset in $F$. It is possible for one to get fixed point for such mappings if they satisfy certain condition. It has been shown by Beg et al. [20], the presence of fixed point for such mappings that fulfill the certain conditions on a closed set rather then whole space. Some common fixed point results for a pair of $\alpha_*$-dominated multivalued maps on closed ball with graph in dislocated quasi spaces have
been proved. We developed fixed points for $\alpha_\ast$-dominated setvalued maps satisfying generalized $\alpha_\ast - \Psi$ Ćirić type contraction on $DQM$.

The theory of setvalued maps has a fundamental role in many types of both pure and applied maths because of its large number of applications, in geometry, real analysis and complex analysis, algorithms as well as in functional analysis. Over the past years, above theory has raised its importance and hence in current literature there are several research articles related to multivalued mappings. Various authors have discussed different research articles including practical problems and their solutions in multivalued mappings. Due to the importance of this theory various approaches algorithms and techniques are applied for the developing of multivalued fixed point theory.

Wardowski [65] developed $F$-contraction principle to investigate fixed points in the setting of complete metric space. This result has a fundamental position in the field of fixed point. Afterwards, several authors generalized many fixed point results in a fruitful way by introducing $F$-contraction (see [3, 4, 6, 11, 27, 37, 34, 42, 52]). These results bring about the modern fixed point theory foundation which is mostly related to contractive type mappings. Rasham et al. [45] obtained fixed points for the pair of setvalued $F$-contractive maps, and showed an application for integral equations which extended some multivalued fixed point theorems in current literature. In Section 4.2 we proved some common fixed points of multivalued maps satisfying a new generalized $\alpha_\ast - \Psi$ Ćirić kind contraction in the context of $DQM$. Also we apply graphic contraciton to get unique fixed point in these spaces. Example is presented on setvalued mappings and it is observed that the contraction which does not prevail on full space but it is holds only on subspace. In Section 4.3 we have achieved common fixed points for the pair of setvalued proximinal maps satisfying a new Ćirić kind rational $F$-contraction in complete dislocated quasi metric spaces. An example has been derived in which we have discussed different cases for $F$-contractive mappings to show the variety of our theorem. An application is derived on non linear Voltera type integral equations to find unique solutions.
4.2 Fixed Point Results for a Pair of Multi Dominated Mappings on a Smallest Subset with Graph

Results given in this section can be seen in [48].

Let \((E, d_q)\) be a DQM, \(b_0 \in E\) and \(S, T : E \to P(E)\) be the setvalued maps on \(E\). Let \(b_1 \in Sb_0\) be an element such that \(d_q(b_0, Sb_0) = d_q(b_0, b_1)\). Let \(b_2 \in Tb_1\) be such that \(d_q(b_1, Tb_1) = d_q(b_1, b_2)\). Let \(b_3 \in Sb_2\) be such that \(d_q(b_2, Sb_2) = d_q(b_2, b_3)\). Proceeding this method, we gain a sequence \(b_n\) in \(E\) so as \(b_{2n+1} \in Sb_{2n}\) and \(b_{2n+2} \in Tb_{2n+1}\), where \(n = 0, 1, 2, \ldots\). Also \(d_q(b_{2n}, Sb_{2n}) = d_q(b_{2n}, b_{2n+1}), d_q(b_{2n+1}, Tb_{2n+1}) = d_q(b_{2n+1}, b_{2n+2})\). We represent this type of sequence by \(\{TS(b_n)\}\).

**Theorem 4.2.1** Let \((E, d_q)\) be a left (right) \(K\)-sequentially complete DQM space. Assume a function \(\alpha : E \times E \to [0, \infty)\) exists. Let, \(r > 0, b_0 \in \overline{B_{d_q}(b_0, r)}\) and \(S, T : E \to P(E)\) be a semi \(\alpha\)-dominated maps on \(\overline{B_{d_q}(b_0, r)}\). Suppose that, for some \(\psi \in \Psi\) and \(D_q(b, g) = \max\{d_q(b, g), d_q(b, Sb), d_q(g, Tg)\}\), the following hold:

\[
\max\{\alpha_s(b, Sb)H_{d_q}(Sb, Tg), \alpha_s(g, Tg)H_{d_q}(Tg, Sb)\} \leq \min\{\psi(D_q(b, g)), \psi(D_q(g, b))\} \quad (4.1)
\]

for all \(b, g \in \overline{B_{d_q}(b_0, r)} \cap \{TS(b_n)\}\) with either \(\alpha(b, g) \geq 1\) or \(\alpha(g, b) \geq 1\) whenever \(b \in Sg\). Also

\[
\sum_{i=0}^{n} \max\{\psi^i(d_q(b_1, b_0)), \psi^i(d_q(b_0, b_1))\} \leq r \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}. \quad (4.2)
\]

Then \(\{TS(b_n)\}\) is the sequence in \(\overline{B_{d_q}(b_0, r)}\) and \(\{TS(b_n)\} \rightarrow b^* \in \overline{B_{d_q}(b_0, r)}\). Also, if the inequality (4.1) holds for \(b^*\) and either \(\alpha(b_n, b^*) \geq 1\) or \(\alpha(b^*, b_n) \geq 1\) for each \(n\) belongs to \(\mathbb{N} \cup \{0\}\). Then \(b^*\) is the C.F.P of both \(S\) and \(T\) in \(\overline{B_{d_q}(b_0, r)}\) and \(d_q(b^*, b^*) = 0\).

**Proof.** Consider a sequence \(\{TS(b_n)\}\) generated by \(b_0\). Then, we have \(b_{2n+1} \in Sb_{2n}\) and \(b_{2n+2} \in Tb_{2n+1}\), where \(n = 0, 1, 2, \ldots\). Also \(d_q(b_{2n}, Sb_{2n}) = d_q(b_{2n}, b_{2n+1}), d_q(b_{2n+1}, Tb_{2n+1}) = d_q(b_{2n+1}, b_{2n+2})\). By Lemma 1.3.8, we have

\[
d_q(b_{2n}, b_{2n+1}) \leq H_{d_q}(Tb_{2n-1}, Sb_{2n}) \quad (4.3)
\]

\[
d_q(b_{2n+1}, b_{2n+2}) \leq H_{d_q}(Sb_{2n}, Tb_{2n+1}) \quad (4.4)
\]
for each $n = 1, 2, \ldots$. From (4.2), we have

$$\max\{d_q(b_1, b_0), d_q(b_0, b_1)\} \leq \sum_{i=0}^{j} \max\{\psi^i(d_q(b_1, b_0), \psi^i(d_q(b_0, b_1))\} \leq r.$$ 

It follows that, $d_q(b_1, b_0) \leq r$ and $d_q(b_0, b_1) \leq r$. Hence, we have

$$b_1 \in B_{d_q}(b_0, r).$$

Let $b_2, \ldots, b_j \in B_{d_q}(b_0, r)$ for every $j$ belongs to $\mathbb{N}$. If $j = 2i + 1$, where $i = 1, 2, \ldots, i-1$. Since $S, T : E \to P(E)$ be a semi $\alpha_s$-dominated maps on $B_{d_q}(b_0, r)$, so $\alpha_s(b_{2i}, Sb_{2i}) \geq 1$ and $\alpha_s(b_{2i+1}, Tb_{2i+1}) \geq 1$. As $\alpha_s(b_{2i}, Sb_{2i}) \geq 1$, this implies $\inf\{\alpha(b_{2i}, b) : b \in Sb_{2i}\} \geq 1$. Also $b_{2i+1} \in Sb_{2i}$, so $\alpha(b_{2i}, b_{2i+1}) \geq 1$. Now by using (4.3), we obtain

$$d_q(b_{2i+1}, b_{2i+2}) \leq H_{d_q}(Sb_{2i}, Tb_{2i+1}) \leq \max\{\alpha_s(b_{2i}, Sb_{2i})H_{d_q}(Sb_{2i}, Tb_{2i+1}),$$

$$\alpha_s(b_{2i+1}, Tb_{2i+1})H_{d_q}(Tb_{2i+1}, Sb_{2i})\}$$

$$\leq \min\{\psi(D_q(b_{2i}, b_{2i+1})), \psi(D_q(b_{2i+1}, b_{2i+2}))\} \leq \psi(D_q(b_{2i}, b_{2i+2}))$$

$$\leq \psi(\max\{d_q(b_{2i}, b_{2i+1}), d_q(b_{2i}, Sb_{2i}), d_q(b_{2i+1}, Tb_{2i+1})\})$$

$$\leq \psi(\max\{d_q(b_{2i}, b_{2i+1}), d_q(b_{2i+1}, Tb_{2i+1})\})$$

$$\leq \psi(\max\{d_q(b_{2i}, b_{2i+1}), d_q(b_{2i+1}, b_{2i+2})\}).$$

If $\max\{d_q(b_{2i}, b_{2i+1}), d_q(b_{2i+1}, b_{2i+2})\} = d_q(b_{2i+1}, b_{2i+2})$, then $d_q(b_{2i+1}, b_{2i+2}) \leq \psi(d_q(b_{2i+1}, b_{2i+2}))$. Which contradicts the reality $\psi(t) < t$ for each $t > 0$. So $\max\{d_q(b_{2i}, b_{2i+1}), d_q(b_{2i+1}, b_{2i+2})\} = d_q(b_{2i}, b_{2i+1})$. Hence,

$$d_q(b_{2i+1}, b_{2i+2}) \leq \psi(d_q(b_{2i}, b_{2i+1})) \quad (4.5)$$

As $\alpha_s(b_{2i-1}, Tb_{2i-1}) \geq 1$ and $b_{2i} \in Tb_{2i-1}$, so $\alpha(b_{2i-1}, b_{2i}) \geq 1$. Now, by using (4.4), we have

$$d_q(b_{2i}, b_{2i+1}) \leq H_{d_q}(Tb_{2i-1}, Sb_{2i})$$

$$\leq \max\{\alpha_s(b_{2i}, Sb_{2i})H_{d_q}(Sb_{2i}, Tb_{2i-1}),$$

$$\alpha_s(b_{2i-1}, Tb_{2i-1})H_{d_q}(Tb_{2i-1}, Sb_{2i})\}$$

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\[ \leq \min\{\psi(D_q(b_{2i}, b_{2i-1})), \psi(D_q(b_{2i-1}, b_{2i}))\} \leq \psi(D_q(b_{2i}, b_{2i-1})) \]

\[ \leq \psi(\max\{d_q(b_{2i}, b_{2i-1}), d_q(b_{2i}, Sb_{2i}), d_q(b_{2i-1}, T b_{2i-1})\}) \]

\[ \leq \psi(\max\{d_q(b_{2i}, b_{2i-1}), d_q(b_{2i}, b_{2i+1}), d_q(b_{2i-1}, b_{2i})\}) \]

\[ \leq \psi(\max\{d_q(b_{2i}, b_{2i-1}), d_q(b_{2i}, b_{2i+1}), d_q(b_{2i-1}, b_{2i})\}). \]

If \( \max\{d_q(b_{2i}, b_{2i-1}), d_q(b_{2i}, b_{2i+1}), d_q(b_{2i-1}, b_{2i})\} = d_q(b_{2i}, b_{2i+1}) \), then \( d_q(b_{2i}, b_{2i+1}) \leq \psi(d_q(b_{2i}, b_{2i+1})) \).

This is contradicts to the reality \( \psi(t) < t \) for each \( t > 0 \). Hence, we have

\[ d_q(b_{2i}, b_{2i+1}) \leq \psi(\max\{d_q(b_{2i}, b_{2i-1}), d_q(b_{2i-1}, b_{2i})\}). \]

If \( \max\{d_q(b_{2i}, b_{2i-1}), d_q(b_{2i-1}, b_{2i})\} = d_q(b_{2i-1}, b_{2i}) \), then

\[ d_q(b_{2i}, b_{2i+1}) \leq \psi(d_q(b_{2i-1}, b_{2i})). \]

As \( \psi \) is nondecreasing function, so

\[ \psi(d_q(b_{2i}, b_{2i+1})) \leq \psi^2(d_q(b_{2i-1}, b_{2i})). \]

By using the above inequality in (4.5), we obtain

\[ d_q(b_{2i+1}, b_{2i+2}) \leq \psi^2(d_q(b_{2i-1}, b_{2i})). \] (4.6)

If \( \max\{d_q(b_{2i}, b_{2i-1}), d_q(b_{2i-1}, b_{2i})\} = d_q(b_{2i}, b_{2i-1}) \), then

\[ d_q(b_{2i+1}, b_{2i+2}) \leq \psi^2(d_q(b_{2i}, b_{2i-1})). \] (4.7)

Now, by combining (4.6) and (4.7), we obtain

\[ d_q(b_{2i+1}, b_{2i+2}) \leq \max\{\psi^2(d_q(b_{2i}, b_{2i-1})), \psi^2(d_q(b_{2i-1}, b_{2i}))\} \]
Proceeding this way, we get

$$d_q(b_{2i+1}, b_{2i+2}) \leq \max\{\psi^{2i+1}(d_q(b_1, b_0)), \psi^{2i+1}(d_q(b_0, b_1))\} \quad (4.8)$$

Now, if $j = 2i$, where $i = 1, 2, \ldots \frac{j}{2}$. Then, similarly, we have

$$d_q(b_{2i}, b_{2i+1}) \leq \max\{\psi^{2i}(d_q(b_1, b_0)), \psi^{2i}(d_q(b_0, b_1))\} \quad (4.9)$$

Now, by combining (4.8) and (4.9), we obtain

$$d_q(b_j, b_{j+1}) \leq \max\{\psi^j(d_q(b_1, b_0)), \psi^j(d_q(b_0, b_1))\} \text{ for some } j \in \mathbb{N}. \quad (4.10)$$

Now, by Lemma 1.3.8 and inequality (4.1), we have

$$d_q(b_{2i+2}, b_{2i+1}) \leq H_{d_q}(Tb_{2i+1}, Sb_{2i})$$

$$\leq \max\{\alpha_*(b_{2i}, Sb_{2i})H_{d_q}(Sb_{2i}, Tb_{2i+1}), \alpha_*(b_{2i+1}, Tb_{2i+1})H_{d_q}(Tb_{2i+1}, Sb_{2i})\}$$

$$\leq \min\{\psi(D_q(b_{2i}, b_{2i+1})), \psi(D_q(b_{2i+1}, b_{2i+1}))\}$$

In similar way, we used to solve inequality (4.10), we get

$$d_q(b_{j+1}, b_j) \leq \max\{\psi^j(d_q(b_1, b_0)), \psi^j(d_q(b_0, b_1))\} \text{ for every } j \text{ belongs to } \mathbb{N}. \quad (4.11)$$

Now,

$$d_q(b_0, b_{j+1}) \leq d_q(b_0, b_1) + \ldots + d_q(b_j, b_{j+1})$$

$$\leq d_q(b_0, b_1) + \ldots + \max\{\psi^j(d_q(b_1, b_0)), \psi^j(d_q(b_0, b_1))\}$$

$$\leq \sum_{i=0}^{j} \max\{\psi^i(d_q(b_1, b_0), \psi^i(d_q(b_0, b_1))\} \leq r. \quad (4.12)$$
Also,

\[ d_q(b_{j+1}, b_0) \leq d_q(b_{j+1}, b_j) + \ldots + d_q(b_1, b_0) \]
\[ \leq \max\{\psi^j(d_q(b_1, b_0)), \psi^j(d_q(b_0, b_1))\} + \ldots + d_q(b_1, b_0) \]
\[ \leq \sum_{i=0}^{j} \max\{\psi^i(d_q(b_1, b_0)), \psi^i(d_q(b_0, b_1))\} \leq r. \]  

(4.13)

By (4.12) and (4.13), we have \( b_{j+1} \in \overline{B_{d_q}(b_0, r)} \). Hence by mathematical induction \( b_n \in \overline{B_{d_q}(b_0, r)} \) for every \( n \) belongs to \( \mathbb{N} \). Therefore, \( \{TS(b_n)\} \) be the sequence in \( \overline{B_{d_q}(b_0, r)} \). As \( S, T : E \rightarrow P(E) \) be a semi \( \alpha_s \)-dominated maps on \( \overline{B_{d_q}(b_0, r)} \), so \( \alpha_s (b_n, Sb_n) \geq 1 \) and \( \alpha_s (b_n, Tb_n) \geq 1 \), for each \( n \in \mathbb{N} \). Now we can write (4.8) and (4.9) in result as

\[ d_q(b_n, b_{n+1}) \leq \max\{\psi^n(d_q(b_1, b_0)), \psi^n(d_q(b_0, b_1))\}, \text{ for each } n \text{ belongs to } \mathbb{N}. \]  

(4.14)

\[ d_q(b_{n+1}, b_n) \leq \max\{\psi^n(d_q(b_1, b_0)), \psi^n(d_q(b_0, b_1))\}, \text{ for each } n \text{ belongs to } \mathbb{N}. \]  

(4.15)

Fix \( \varepsilon > 0 \) and let \( k_1(\varepsilon) \) belongs to \( \mathbb{N} \) so as \( \sum_{k \geq k_1(\varepsilon)} \max\{\psi^k(d_q(b_1, b_0)), \psi^k(d_q(b_0, b_1))\} < \varepsilon \). Let \( n, m \) belongs to \( \mathbb{N} \) with \( m > n > k_1(\varepsilon) \), then, we obtain,

\[ d_q(b_n, b_m) \leq \sum_{k=n}^{m-1} d_q(b_k, b_{k+1}) \]
\[ \leq \sum_{k=n}^{m-1} \max\{\psi^k(d_q(b_1, b_0)), \psi^k(d_q(b_0, b_1))\}, \text{ by (4.14)} \]  

\[ d_q(b_n, b_m) \leq \sum_{k \geq k_1(\varepsilon)} \max\{\psi^n(d_q(b_1, b_0)), \psi^n(d_q(b_0, b_1))\} < \varepsilon. \]

Thus we have showed that \( \{TS(b_n)\} \) be a left \( K \)-Cauchy in \( \overline{B_{d_q}(b_0, r), d_q} \). Similarly, by using (4.15) we have

\[ d_q(b_m, b_n) \leq \sum_{k=n}^{m-1} d_q(b_{k+1}, b_k) < \varepsilon \]

Hence, \( \{TS(b_n)\} \) is a right \( K \)-Cauchy in \( \overline{B_{d_q}(b_0, r), d_q} \). As each closed ball in left(right) \( K \)-sequentially complete \( DQM \) is left(right) \( K \)-sequentially complete, so there must be a \( b^* \).
Now, 

\[ \lim_{n \to \infty} d_q(b_n, b^*) = \lim_{n \to \infty} d_q(b^*, b_n) = 0 \quad (4.16) \]

Since \( \alpha_*(b^*, T\theta^*) \geq 1 \), \( \alpha_*(b_{2n}, Sb_{2n}) \geq 1 \) and \( \alpha(b_{2n}, b^*) \geq 1 \), we obtain

\[
H_{d_q}(Sb_{2n}, T\theta^*) \leq \max\{\alpha_*(b_{2n}, Sb_{2n})H_{d_q}(Sb_{2n}, T\theta^*), \alpha_*(b^*, T\theta^*)H_{d_q}(T\theta^*, Sb_{2n})\}
\leq \min\{\psi(D_q(b_{2n}, b^*)), \psi(D_q(b^*, b_{2n}))\}
\leq \psi(\max\{d_q(b_{2n}, b^*), d_q(b_{2n}, b_{2n+1}), d_q(b^*, T\theta^*)\})
\leq \psi(\max\{d_q(b_{2n}, b^*), d_q(b_{2n}, b^*) + d_q(b^*, b_{2n+1}), d_q(b^*, T\theta^*)\}). \quad (4.18)
\]

By using inequality (4.18) in inequality (4.17), we have

\[ d_q(b^*, T\theta^*) \leq d_q(b^*, b_{2n+1}) + \psi(\max\{d_q(b_{2n}, b^*), d_q(b_{2n}, b^*) + d_q(b^*, b_{2n+1}), d_q(b^*, T\theta^*)\}). \]

Letting \( n \to \infty \), and by using the inequality (4.16), we obtain \( d_q(b^*, T\theta^*) \leq \psi(d_q(b^*, T\theta^*)) \) and hence \( d_q(b^*, T\theta^*) = 0 \). Now,

\[
d_q(T\theta^*, b^*) \leq d_q(T\theta^*, b_{2n+1}) + d_q(b_{2n+1}, b^*)
\leq H_{d_q}(T\theta^*, Sb_{2n}) + d_q(b_{2n+1}, b^*), \quad \text{by Lemma 1.3.8}
\]

By using similar arguments, we obtain \( d_q(T\theta^*, b^*) = 0 \) or \( b^* \in T\theta^* \). Similarly, by using Lemma 1.3.8 inequality (4.16) and the inequality

\[ d_q(b^*, Sb^*) \leq d_q(b^*, b_{2n+2}) + d_q(b_{2n+2}, Sb^*), \]

we can show that \( d_q(b^*, Sb^*) = 0 \). \( b^* \in Sb^* \). Similarly, \( d_q(Sb^*, b^*) = 0 \). Hence \( b^* \) is the C.F.P of
both the maps $S$ and $T$ in $\overline{B_{d_q}(b_0, r)}$. Now,

$$d_q(b^*, b^*) \leq d_q(b^*, Tb^*) + d_q(Tb^*, b^*) \leq 0$$

This means that, $d_q(b^*, b^*) = 0$. ■

**Corollary 4.2.2** Let $(E, d_q)$ is a left (right) $K$-Sequentially complete $DQM$. Suppose a function $\alpha : E \times E \to [0, \infty)$ exists. Let, $r > 0$, $b_0 \in \overline{B_{d_q}(b_0, r)}$ and $S : E \to P(E)$ are two semi $\alpha_*$-dominated maps on $\overline{B_{d_q}(b_0, r)}$. Assume that, for some $\psi \in \Psi$ and $D_q(b, g) = \max\{d_q(b, g), d_q(b, Sb), d_q(g, Sg)\}$, the following hold:

$$\max\{\alpha_*(b, Sb)H_{d_q}(Sb, Sg), \alpha_*(g, Sg)H_{d_q}(Sg, Sb)\} \leq \min\{\psi(D_q(b, g)), \psi(D_q(g, b))\} \quad (4.19)$$

for all $b, g \in \overline{B_{d_q}(b_0, r)} \cap \{S(b_n)\}$, with either $\alpha(b, g) \geq 1$ or $\alpha(g, b) \geq 1$. Also

$$\sum_{i=0}^{n} \max\{\psi^i(d_q(b_1, b_0)), \psi^i(d_q(b_0, b_1))\} \leq r \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.$$ 

Then $\{S(b_n)\}$ be the sequence in $\overline{B_{d_q}(b_0, r)}$ and $\{S(b_n)\} \to b^* \in \overline{B_{d_q}(b_0, r)}$. Also, if the inequality (4.19) holds for $b^*$ and either $\alpha(b_n, b^*) \geq 1$ or $\alpha(b^*, b_n) \geq 1$ for every $n$ belongs to $\mathbb{N} \cup \{0\}$. Then $b^*$ is the fixed point of $S$ in $\overline{B_{d_q}(b_0, r)}$ and $d_q(b^*, b^*) = 0$.

**Corollary 4.2.3** Let $(E, d_l)$ is a complete $DQM$. Suppose a function $\alpha : E \times E \to [0, \infty)$ exists. Let, $r > 0$, $b_0 \in \overline{B_{d_l}(b_0, r)}$ and $S, T : E \to P(E)$ are semi $\alpha_*$-dominated maps on $\overline{B_{d_l}(b_0, r)}$. Assume that, for some $\psi \in \Psi$ and $D_l(b, g) = \max\{d_l(b, g), d_l(b, Sb), d_l(g, Tg)\}$, the following hold:

$$\max\{\alpha_*(b, Sb)H_{d_l}(Sb, Tg), \alpha_*(g, Tg)H_{d_l}(Sb, Tg)\} \leq \psi(D_l(b, g)) \quad (4.20)$$

for all $b, g \in \overline{B_{d_l}(b_0, r)} \cap \{TS(b_n)\}$ with either $\alpha(b, g) \geq 1$ or $\alpha(g, b) \geq 1$. Also

$$\sum_{i=0}^{n} \psi^i(d_l(b_0, b_1)) \leq r \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.$$
Then \( \{TS(b_n)\} \) is a sequence in \( \overline{B_{d_l}(b_0, r)} \) and \( \{TS(b_n)\} \to b^* \in \overline{B_{d_l}(b_0, r)} \). Also, if the inequality (4.20) holds for \( b^* \) and either \( \alpha(b_n, b^*) \geq 1 \) or \( \alpha^*(b_n, b^*) \geq 1 \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \). Then, \( b^* \) is the C.F.P of both the maps \( S \) and \( T \) in \( \overline{B_{d_g}(b_0, r)} \) and \( d_q(b^*, b^*) = 0 \).

**Corollary 4.2.4** Let \((E, d_l)\) is a complete DQM. Suppose a function \( \alpha : E \times E \to [0, \infty) \) exists. Let, \( r > 0 \), \( b_0 \in \overline{B_{d_l}(b_0, r)} \) and \( S : E \to P(E) \) is semi \( \alpha_s \)-dominated map on \( \overline{B_{d_l}(b_0, r)} \). Suppose that, for some \( \psi \in \Psi \) and \( D_l(b, g) = \max\{d_l(b, g), d_l(b, Sb), d_l(g, Sg)\} \), the following hold:

\[
\max\{\alpha_s(b, Sb)H_{d_l}(Sb, Sg), \alpha_s(g, Sg)H_{d_l}(Sb, Sg)\} \leq \psi(D_l(b, g))
\] (4.21)

for all \( b, g \in \overline{B_{d_l}(b_0, r)} \cap \{S(b_n)\} \) with either \( \alpha(b, g) \geq 1 \) or \( \alpha(g, b) \geq 1 \). Also

\[
\sum_{i=0}^{n} \psi^i(d_l(b_0, b_1)) \leq r \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.
\]

Then \( \{S(b_n)\} \) is the sequence in \( \overline{B_{d_l}(b_0, r)} \) and \( \{S(b_n)\} \to b^* \in \overline{B_{d_l}(b_0, r)} \). Also, if the inequality (4.21) holds for \( b^* \) and either \( \alpha(b_n, b^*) \geq 1 \) or \( \alpha^*(b_n, b^*) \geq 1 \) for every \( n \) belongs to \( \mathbb{N} \cup \{0\} \). Then \( S \) has a fixed point \( b^* \) in \( \overline{B_{d_l}(b_0, r)} \) and \( d_q(b^*, b^*) = 0 \).

**Corollary 4.2.5** Let \((E, \preceq, d_q)\) is a left (right) \( K \)-sequentially ordered complete DQM. Let, \( r > 0 \), \( b_0 \in \overline{B_{d_q}(b_0, r)} \) and \( S, T : E \to P(E) \) are semi dominated maps on \( \overline{B_{d_q}(b_0, r)} \). Suppose that, for some \( \psi \in \Psi \) and \( D_q(b, g) = \max\{d_q(b, g), d_q(b, Sb), d_q(g, Tg)\} \), the following hold:

\[
\max\{H_{d_q}(Sb, Tg), H_{d_q}(Tg, Sb)\} \leq \min\{\psi(D_q(b, g)), \psi(D_q(g, b))\},
\] (4.22)

for all \( b, g \in \overline{B_{d_q}(b_0, r)} \cap \{TS(b_n)\} \) with either \( b \preceq g \) or \( g \preceq b \). Also

\[
\sum_{i=0}^{n} \max\{\psi^i(d_q(b_1, b_0)), \psi^i(d_q(b_0, b_1))\} \leq r \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.
\] (4.23)

Then \( \{TS(b_n)\} \) be the sequence in \( \overline{B_{d_q}(b_0, r)} \) and \( \{TS(b_n)\} \to b^* \in \overline{B_{d_q}(b_0, r)} \). Also if the inequality (4.22) holds for \( b^* \) and either \( b_n \preceq b^* \) or \( b^* \preceq b_n \) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \). Then \( b^* \) is the C.F.P of both the maps \( S \) and \( T \) in \( \overline{B_{d_q}(b_0, r)} \) and \( d_q(b^*, b^*) = 0 \).
Proof. Let $\alpha : E \times E \to [0, +\infty)$ be a maping defined by $\alpha(b, g) = 1$ for every $b \in B_{d_q}(b_0, r)$ with either $b \leq g$ or $g \leq b$, and $\alpha(b, g) = 0$ for all other elements $b, g \in E$. Given $S$ and $T$ be the semi dominated maps on $B_{d_q}(b_0, r)$, so $b \leq Sb$ and $b \leq Tb \forall b \in B_{d_q}(b_0, r)$. This means that $b \leq b$ for every $b \in Sb$ and $b \leq c$ for each $c \in Tb$. So, $\alpha(b, b) = 1$ for all $b \in Sb$ and $\alpha(b, c) = 1$ $\forall c \in Tb$. This implies $\inf\{\alpha(b, g) : g \in Sb\} = 1$ and $\inf\{\alpha(b, g) : g \in Tb\} = 1$. Hence $\alpha_s(b, Sb) = 1$, $\alpha_s(b, Tb) = 1$ for all $b \in B_{d_q}(b_0, r)$. So, $S, T : E \to P(E)$ are the semi $\alpha_s$-dominated map on $B_{d_q}(b_0, r)$. Moreover, inequality (4.22) can be written as

$$\max\{\alpha_s(b, Sb)H_{d_q}(Sb, Tg), \alpha_s(g, Tg)H_{d_q}(Tg, Sb)\} \leq \min\{\psi(D_q(b, g)), \psi(D_q(g, b))\},$$

for all elements $b, g$ in $B_{d_q}(b_0, r) \cap \{TS(b_n)\}$ with either $\alpha(b, g) \geq 1$ or $\alpha(g, b) \geq 1$. Also, inequality (4.23) holds. Then, by Theorem 4.2.1, we have $\{TS(b_n)\}$ be the sequence in $B_{d_q}(b_0, r)$ and $\{TS(b_n)\} \to \alpha^* \in B_{d_q}(b_0, r)$. Now, $b_n, \alpha^* \in B_{d_q}(b_0, r)$ and either $b_n \leq \alpha^*$ or $\alpha^* \leq b_n$ implies that either $\alpha(b_n, \alpha^*) \geq 1$ or $\alpha(\alpha^*, b_n) \geq 1$. So, all hypothesis of Theorem 4.2.1 are proved. Hence, by Theorem 4.2.1, $\alpha^*$ is C.F.P of both $S$ and $T$ in $B_{d_q}(b_0, r)$ and $d_q(\alpha^*, \alpha^*) = 0$. ■

Example 4.2.6 Let $E = Q^+ \cup \{0\}$ and $d_q : E \times E \to E$ is a DQM on $E$ clafried by

$$d_q(w, e) = w + e \forall w, e \in E.$$ Define, $S, T : E \times E \to P(E)$ by,

$$S_w = \begin{cases} \frac{w}{3}, \frac{2}{3}w & \text{if } w \in [0, 1] \cap E \\ w, w + 1 & \text{if } w \in (1, \infty) \cap E \end{cases}$$

and,

$$T_w = \begin{cases} \frac{w}{4}, \frac{3}{4}w & \text{if } w \in [0, 1] \cap E \\ w + 1, w + 3 & \text{if } w \in (1, \infty) \cap E. \end{cases}$$

Considering, $b_0 = 1$, $r = 8$, then $B_{d_q}(b_0, r) = [0, 7] \cap E$. Now $d_q(b_0, Sb_0) = d_q(1, S1) = d_q(1, \frac{1}{3}) = \frac{4}{3}$. So we obtain a sequence $\{TS(b_n)\} = \{1, \frac{1}{17}, \frac{1}{147}, \frac{1}{1725}, \ldots\}$ in $E$ generated by
b_0. Also, \( \overline{B_{d_q}(b_0, r)} \cap \{TS(b_n)\} = \{1, \frac{1}{12}, \frac{1}{144}, \ldots \} \). Let \( \psi(t) = \frac{4t}{5} \) and

\[
\alpha(b, g) = \begin{cases} 
1 & \text{if } b, g \in [0, 1] \\
\frac{3}{2} & \text{otherwise.}
\end{cases}
\]

Now, if \( b, g \notin \overline{B_{d_q}(b_0, r)} \cap \{TS(b_n)\} \), then the available cases are given.

Case 1. If \( \max\{\alpha_*(b, S_b)H_{d_q}(S_b, T_g), \alpha_*(g, T_g)H_{d_q}(T_g, S_b)\} = \alpha_*(b, S_b)H_{d_q}(S_b, T_g) \) then for \( b = 2 \) and \( g = 3 \), we have

\[
\alpha_*(2, S2)H_{d_q}(S2, T3) = \frac{3}{2}(8) > \psi(D_q(b, g)) = \frac{28}{5}.
\]

Case 2. If \( \max\{\alpha_*(b, S_b)H_{d_q}(S_b, T_g), \alpha_*(g, T_g)H_{d_q}(T_g, S_b)\} = \alpha_*(g, T_g)H_{d_q}(T_g, S_b) \) then for \( b = 2 \) and \( g = 3 \), we have

\[
\alpha_*(3, T3)H_{d_q}(T3, S2) = \frac{3}{2}(8) > \psi(D_q(g, b)) = \frac{28}{5}.
\]

So, the inequality (4.1) is not true for the whole space \( E \).

Now, for all \( b, g \in \overline{B_{d_q}(b_0, r)} \cap \{TS(b_n)\} \), we have

Case 3. If \( \max\{\alpha_*(b, S_b)H_{d_q}(S_b, T_g), \alpha_*(g, T_g)H_{d_q}(T_g, S_b)\} = \alpha_*(b, S_b)H_{d_q}(S_b, T_g) \), then, we have

\[
\alpha_*(b, S_b)H_{d_q}(S_b, T_g) = 1[\max\{\sup_{a \in S_b} d_q(a, T_g), \sup_{g \in T_g} d_q(S_b, g)\}]
\]
\[
= \max\{\sup_{a \in S_b} d_q(a, \left[\begin{array}{c} \frac{g}{4} \\ \frac{3g}{4} \end{array}\right]), \sup_{g \in T_g} d_q(\left[\begin{array}{c} \frac{b}{3} \\ \frac{2b}{3} \end{array}\right], g)\}
\]
\[
= \max\{d_q(\frac{2b}{3}, \left[\begin{array}{c} \frac{g}{4} \\ \frac{3g}{4} \end{array}\right]), d_q(\left[\begin{array}{c} \frac{b}{3} \\ \frac{2b}{3} \end{array}\right], \frac{3g}{4})\}
\]
\[
= \max\{d_q(\frac{2b}{3}, g), d_q(\frac{b}{3}, \frac{3g}{4})\}
\]
\[
= \max\{\frac{2b}{3} + \frac{g}{4}, \frac{b}{3} + \frac{3g}{4}\}
\]
\[
\leq \psi(\max\{b + g, \frac{4b}{3}, \frac{5g}{4}\}) = \psi(D_q(b, g)).
\]

Case 4. If \( \max\{\alpha_*(b, S_b)H_{d_q}(S_b, T_g), \alpha_*(g, T_g)H_{d_q}(T_g, S_b)\} = \alpha_*(g, T_g)H_{d_q}(T_g, S_b) \), then, we
have

$$
\alpha_s(g, Tg)H_{ds}(Tg, Sb) = \max \left\{ \sup_{b \in Tg} d_q(Sb, b), \sup_{a \in Sb} d_q(a, Tg) \right\}
$$

$$= \max \left\{ \sup_{b \in Tg} d_q([\frac{b}{3}, \frac{2b}{3}], b), \sup_{a \in Sb} d_q(a, \frac{g}{4}, \frac{3g}{4}) \right\}
$$

$$= \max \left\{ d_q([\frac{b}{3}, \frac{2b}{3}], \frac{3g}{4}), d_q(\frac{2b}{3}, \frac{g}{4}) \right\}
$$

$$= \max \left\{ \frac{b}{3}, \frac{2b}{3}, \frac{3g}{4}, \frac{2b}{3}, \frac{g}{4} \right\}
$$

$$\leq \psi(\max\{g + b, \frac{5g}{4}, \frac{4b}{3}\}) = \psi(D_q(g, b))
$$

So, the inequality (4.1) holds on $\overline{B_{ds}(b_0, r)} \cap \{TS(b_n)\}$. Also,

$$\sum_{i=0}^{n} \max\{\psi^i(d_q(b_1, b_0)), \psi^i(d_q(b_0, b_1))\} = \frac{4}{3} \sum_{i=0}^{n} (\frac{4}{3})^i < 8 = r.
$$

Hence, all the hypothesis of Theorem 4.2.1 are proved. Now, we have $\{TS(b_n)\}$ be the sequence in $\overline{B_{ds}(b_0, r)}$ and $\{TS(b_n)\} \to 0 \in \overline{B_{ds}(b_0, r)}$. Also, $\alpha(b_n, 0) \geq 1$ or $\alpha(0, b_n) \geq 1$ for each $n$ belongs to $\mathbb{N} \cup \{0\}$.

**Definition 4.2.7** Let $E \neq \emptyset$ and $G = (V(G), W(G))$ is a graph so as $V(G) = E$, and $S : E \to CB(E)$ is called to be semi graph dominated on $A \subseteq E$, if for each $b \in A$, then $(b, g) \in W(G)$, for all $g \in Sb$. If $A = E$, then we say that $S$ is graph dominated on $E$.

**Theorem 4.2.8** Let $(E, d_q)$ is a complete DQM endowed with graph $G$. Let, $r > 0$, $b_0 \in \overline{B_{ds}(b_0, r)}$, $S, T : E \to P(E)$ and $\{TS(b_n)\}$ be the sequence in $E$ generated by $b_0$. Suppose that (i), (ii) and (iii) hold:

(i) $S$ and $T$ are semi graph dominated on $\overline{B_{ds}(b_0, r)}$;

(ii) there exists $\psi \in \Psi$ and $D_q(b, g) = \max\{d_q(b, g), d_q(b, Sb), d_q(g, Tg)\}$, such that

$$\max\{H_{ds}(Sb, Tg), H_{ds}(Tg, Sb)\} \leq \min\{\psi(D_q(b, g)), \psi(D_q(g, b))\} \quad (4.24)
$$

for all $b, g \in \overline{B_{ds}(b_0, r)} \cap \{TS(b_n)\}$ with $(b, g) \in W(G)$ or $(g, b) \in W(G)$;

(iii) $\sum_{i=0}^{n} \max\{\psi^i(d_q(b_1, b_0)), \psi^i(d_q(b_0, b_1))\} \leq r$ for each $n$ belongs to $\mathbb{N} \cup \{0\}$.
Then, \( \{TS(b_n)\} \) is the sequence in \( \overline{B_{d_q}(b_0, r)} \) and \( \{TS(b_n)\} \to b^* \). Also, if \((b_n, b^*) \in W(G)\) or \((b^*, b_n) \in W(G)\) for each \( n \) belongs to \( \mathbb{N} \cup \{0\} \) and the inequality (4.24) holds for \( b^* \). Then \( b^* \) is the C.F.P of both \( S \) and \( T \) in \( \overline{B_{d_q}(b_0, r)} \).

**Proof.** Define, \( \alpha : E \times E \to [0, \infty) \) by

\[
\alpha(b, g) = \begin{cases} 
1, & \text{if } b \in \overline{B_{d_q}(b_0, r)}, \ (b, g) \in W(G) \text{ or } (g, b) \in W(G) \\
0, & \text{otherwise.}
\end{cases}
\]

Given \( S \) and \( T \) are semi graph dominated on \( \overline{B_{d_q}(b_0, r)} \), then for \( b \in \overline{B_{d_q}(b_0, r)} \), \( (b, g) \in W(G) \) for every \( g \in Sb \) and \( (b, g) \in W(G) \) for every \( g \in Tb \). So, \( \alpha(b, g) = 1 \) for all \( g \in Sb \) and \( \alpha(b, g) = 1 \) for all \( g \in Tb \). This implies that \( \inf \{\alpha(b, g) : g \in Sb\} = 1 \) and \( \inf \{\alpha(b, g) : g \in Tb\} = 1 \). Hence \( \alpha_s(b, Sb) = 1 \), \( \alpha_s(b, Tb) = 1 \) for all \( b \in \overline{B_{d_q}(b_0, r)} \). So, \( S, T : E \to P(E) \) are the semi \( \alpha_s \)-dominated mapping on \( \overline{B_{d_q}(b_0, r)} \). Moreover, inequality (4.24) can be written as

\[
\max \{\alpha_s(b, Sb)H_{d_q}(Sb, Tg), \ \alpha_s(g, Tg)H_{d_q}(Tg, Sb)\} \leq \min \{\psi(D_q(b, g)), \psi(D_q(g, b))\},
\]

for all elements \( b, g \) in \( \overline{B_{d_q}(b_0, r)} \cap \{TS(b_n)\} \) with either \( \alpha(b, g) \geq 1 \) or \( \alpha(g, b) \geq 1 \). Also, (iii) holds. Then, by Theorem 4.2.1, we have \( \{TS(b_n)\} \) be the sequence in \( \overline{B_{d_q}(b_0, r)} \) and \( \{TS(b_n)\} \to b^* \in \overline{B_{d_q}(b_0, r)} \). Now, \( b_n, b^* \in \overline{B_{d_q}(b_0, r)} \) and either \( (b_n, b^*) \in W(G) \) or \( (b^*, b_n) \in W(G) \) implies that either \( \alpha(b_n, b^*) \geq 1 \) or \( \alpha(b^*, b_n) \geq 1 \). So, all hypothesis of Theorem 4.2.1 are proved. Hence, by Theorem 4.2.1, \( u \) is the C.F.P of \( S \) and \( T \) in \( \overline{B_{d_q}(b_0, r)} \) and \( d_q(b^*, b^*) = 0 \). □

### 4.3 DQF-contraction and Related Fixed Point Results in DQM Spaces with Application

Results given in this section can be seen in [48].

Let \((Y, d_q)\) be a DQM space, \( y_0 \in Y \) and \( S, T : Y \to P(Y) \) be setvalued maps on \( Y \). Let \( y_1 \in Sy_0 \) be an element such that \( d_q(y_0, Sy_0) = d_q(y_0, y_1) \), let \( y_2 \in Ty_1 \) be such that \( d_q(y_1, Ty_1) = d_q(y_1, y_2) \), let \( y_3 \in Sy_2 \) be such that \( d_q(y_2, Sy_2) = d_q(y_2, y_3) \) and so on. Then, we get sequence \( y_n \) in \( Y \) so as \( y_{2n+1} \in Sy_{2n} \) and \( y_{2n+2} \in Ty_{2n+1} \), where \( n = 0, 1, 2, \ldots \). Also \( d_q(y_{2n}, Sy_{2n}) = d_q(y_{2n}, y_{2n+1}) \), \( d_q(y_{2n+1}, Ty_{2n+1}) = d_q(y_{2n+1}, y_{2n+2}) \). We denote this type of
iterative sequence by \( TS(y_n) \). We say that \( TS(y_n) \) is a sequence in \( Y \) generated by \( y_0 \). If \( T = S \), then we say that \( S(y_n) \) is a sequence in \( Y \) generated by \( y_0 \).

**Definition 4.3.1** Let \((Y, d_q)\) be a complete DQM space and \((S, T) : Y \to P(Y)\) be two setvalued maps. The pair \((S, T)\) is called DQF–contraction, if there must be a \( F \in \mathcal{F} \) and \( \tau, a > 0 \) such that for every two consecutive points \( l, w \) belonging to the range of an iterative sequence \( \{TS(y_n)\} \) with \( \max\{D_q(l, w), D_q(w, l)\} > 0 \), we have

\[
\tau + \max\{F(H_{d_q}(Sl, Tw)), F(H_{d_q}(Tw, Sl))\} \leq \min\{F(D_q(l, w)), F(D_q(w, l))\}
\]

where,

\[
D_q(l, w) = \max \left\{ d_q(l, w), \frac{d_q(l, Sl) - d_q(w, Tw)}{a + \max\{d_q(l, w), d_q(w, l)\}} , d_q(l, Sl), d_q(w, Tw) \right\},
\]

**Theorem 4.3.2** Let \((Y, d_q)\) be a complete DQM space and \((S, T)\) be a DQF–contraction. Then \( \{TS(y_n)\} \to u \in Y \). Also, if \( u \) satisfies (4.25), then \( u \) is the C.F.P of \( S \) and \( T \) in \( Y \) and \( d_q(u, u) = 0 \).

**Proof.** Let \( \{TS(y_n)\} \) be the iterative sequence in \( Y \) generated by a point \( y_0 \in Y \). Let \( y_{2n}, y_{2n+1} \) be elements of this sequence. Clearly, if \( \max\{D_q(y_{2n}, y_{2n+1}), D_q(y_{2n+1}, y_{2n})\} = 0 \), then \( D_q(y_{2n}, y_{2n+1}) = 0 \) and \( D_q(y_{2n+1}, y_{2n}) = 0 \). If \( D_q(y_{2n}, y_{2n+1}) = 0 \), then

\[
\max \left\{ d_q(y_{2n}, y_{2n+1}), \frac{d_q(y_{2n}, y_{2n+1}), D_q(y_{2n+1}, y_{2n+2})}{a + \max\{d_q(y_{2n}, y_{2n+1}), d_q(y_{2n+1}, y_{2n+2})\}} , d_q(y_{2n}, y_{2n+1}), d_q(y_{2n+1}, y_{2n+2}) \right\} = 0,
\]

So \( d_q(y_{2n}, y_{2n+1}) = d_q(y_{2n+1}, y_{2n+2}) = 0 \). Also \( D_q(y_{2n+1}, y_{2n}) = 0 \) implies \( d_q(y_{2n+1}, y_{2n}) = d_q(y_{2n+2}, y_{2n+1}) = 0 \). Hence \( y_{2n} = y_{2n+1} = y_{2n+2} \) is a C.F.P of \((S, T)\) the argument is satisfied.

In order to find C.F.P of both \( S \) and \( T \), when \( \min\{D_q(e, c), D_q(c, e)\} > 0 \) for every \( e, c \in Y \) with \( e \neq c \), we make a sequence \( \{TS(y_n)\} \) generated by \( y_0 \). Then, we have \( y_{2n+1} \in Sy_{2n} \) and \( y_{2n+2} \in Ty_{2n+1} \), where \( n = 0, 1, 2, \ldots \). Also \( d_q(y_{2n}, Sy_{2n}) = d_q(y_{2n}, y_{2n+1}), d_q(y_{2n+1}, Ty_{2n+1}) = d_q(y_{2n+1}, y_{2n+2}) \). By Lemma 1.3.8 we gain

\[
d_q(y_{2n}, y_{2n+1}) \leq H_{d_q}(Ty_{2n-1}, Sy_{2n}), d_q(y_{2n+1}, y_{2n}) \leq H_{d_q}(Sy_{2n}, Ty_{2n-1}),
\]

(4.27)
and

\[ d_q(y_{2n+1}, y_{2n+2}) \leq H_{d_q}(Sy_{2n}, T y_{2n+1}), \quad d_q(y_{2n+2}, y_{2n+1}) \leq H_{d_q}(T y_{2n+1}, S y_{2n}). \] (4.28)

Then from (4.28), we get

\[
F(d_q(y_{2p+1}, y_{2p+2})) \leq F(H_{d_q}(S y_{2p}, T y_{2p+1})) \\
\leq \max\{F(H_{d_q}(S y_{2p}, T y_{2p+1})), F(H_{d_q}(T y_{2p}, S y_{2p+1}))\} \\
\leq \min\{F(D_q(y_{2p}, y_{2p+1})), F(D_q(y_{2p+1}, y_{2p+2}))\} - \tau \\
\leq F(D_q(y_{2p}, y_{2p+1}) - \tau,
\]

for each \( p \in \mathbb{N} \cup \{0\} \), where

\[
D_q(y_{2p}, y_{2p+1}) = \max\left\{\frac{d_q(y_{2p}, y_{2p+1})}{a + \max\{d_q(y_{2p}, y_{2p+1}), d_q(y_{2p+1}, y_{2p+2})\}}, \frac{d_q(y_{2p}, S y_{2p}), d_q(y_{2p+1}, T y_{2p+1})}{a + \max\{d_q(y_{2p}, S y_{2p}), d_q(y_{2p+1}, T y_{2p+1})\}}\right\} \\
= \max\left\{\frac{d_q(y_{2p}, y_{2p+1})}{a + \max\{d_q(y_{2p}, y_{2p+1}), d_q(y_{2p+1}, y_{2p+2})\}}, \frac{d_q(y_{2p}, y_{2p+1})}{a + \max\{d_q(y_{2p}, y_{2p+1}), d_q(y_{2p+1}, y_{2p+2})\}}\right\} \\
\leq \max\{d_q(y_{2p}, y_{2p+1}), d_q(y_{2p+1}, y_{2p+2})\}.
\]

If, \( \max\{d_q(y_{2p}, y_{2p+1}), d_q(y_{2p+1}, y_{2p+2})\} = d_q(y_{2p+1}, y_{2p+2}) \), then

\[
F(d_q(y_{2p+1}, y_{2p+2})) \leq F(d_q(y_{2p+1}, y_{2p+2})) - \tau,
\]

which is not true due to \( F_1 \). Therefore,

\[
F(d_q(y_{2p+1}, y_{2p+2})) \leq F(d_q(y_{2p}, y_{2p+1})) - \tau, \quad (4.29)
\]

and this implies

\[
F(d_q(y_{2p+1}, y_{2p+2})) \leq \max\{F(d_q(y_{2p}, y_{2p+1})), F(d_q(y_{2p+1}, y_{2p+2}))\} - \tau \quad (4.30)
\]
Again using (4.27), we have

\[
F(d_q(y_{2p}, y_{2p+1})) \leq F(H_{d_q}(Ty_{2p-1}, Sy_{2p})) \\
\leq \max\{F(H_{d_q}(Sy_{2p-1}, Ty_{2p})), F(H_{d_q}(Ty_{2p-1}, Sy_{2p}))\}
\]

\[
\leq \min\{F(D_q(y_{2p-1}, y_{2p})), F(D_q(y_{2p}, y_{2p-1}))\} - \tau \\
\leq F(D_q(y_{2p}, y_{2p-1})) - \tau.
\]

Now,

\[
D_q(y_{2p}, y_{2p-1}) = \max\left\{ \frac{d_q(y_{2p}, y_{2p-1})}{\alpha + \max\{d_q(y_{2p}, Sy_{2p}), d_q(y_{2p-1}, Ty_{2p-1})\}}, \frac{d_q(y_{2p}, Sy_{2p})}{\alpha + \max\{d_q(y_{2p}, y_{2p-1}), d_q(y_{2p-1}, y_{2p})\}} \right\} \\
= \max\left\{ \frac{d_q(y_{2p}, y_{2p-1})}{\alpha + \max\{d_q(y_{2p}, y_{2p-1}), d_q(y_{2p-1}, y_{2p})\}}, \frac{d_q(y_{2p}, y_{2p-1})}{\alpha + \max\{d_q(y_{2p}, y_{2p-1}), d_q(y_{2p-1}, y_{2p})\}} \right\} \\
\leq \max\{d_q(y_{2p}, y_{2p-1}), d_q(y_{2p-1}, y_{2p}), d_q(y_{2p}, y_{2p+1})\}.
\]

If \(\max\{d_q(y_{2p}, y_{2p-1}), d_q(y_{2p-1}, y_{2p}), d_q(y_{2p}, y_{2p+1})\} = d_q(y_{2p}, y_{2p+1})\), then we obtain

\[
F(d_q(y_{2p}, y_{2p+1})) \leq F(d_q(y_{2p}, y_{2p+1})) - \tau,
\]

which is not true due to \(F_1\). Therefore,

\[
F(d_q(y_{2p}, y_{2p+1})) \leq F\{\max(d_q(y_{2p-1}, y_{2p}), d_q(y_{2p}, y_{2p-1}))\} - \tau.
\]

(4.31)

By using (4.31) in (4.29), we get

\[
F(d_q(y_{2p+1}, y_{2p+2})) \leq \max\{F(d_q(y_{2p-1}, y_{2p})), F(d_q(y_{2p}, y_{2p-1}))\} - 2\tau.
\]

(4.32)
Now, from (4.28) and (F1), we have

\[
\begin{align*}
\leq & \max \{ F(d_q(y_{2p-1}, y_{2p}), Fd_q(y_{2p}, y_{2p-1}) \} \\
\leq & \max \{ F(H_{d_q}(S y_{2p-2}, T y_{2p-1})), F(H_{d_q}(T y_{2p-1}, S y_{2p-2})) \} \\
\leq & \min \{ F(D_q(y_{2p-2}, y_{2p-1})), F(D_q(y_{2p-1}, y_{2p-2})) \} - \tau \\
\leq & F(D_q(y_{2p-2}, y_{2p-1})) - \tau.
\end{align*}
\]

Now,

\[
D_q(y_{2p-2}, y_{2p-1}) = \max \left\{ \frac{d_q(y_{2p-2}, S y_{2p-2}), d_q(y_{2p-1}, T y_{2p-1})}{d_q(y_{2p-2}, S y_{2p-2}), d_q(y_{2p-1}, T y_{2p-1})} \right\}
\]

\[
\leq \max \{ d_q(y_{2p-2}, y_{2p-1}), d_q(y_{2p-1}, y_{2p}) \}.
\]

Now, \( \max \{ d_q(y_{2p-2}, y_{2p-1}), d_q(y_{2p-1}, y_{2p}) \} = d_q(y_{2p-1}, y_{2p}) \) gives a contradiction. So, we have

\[
\max \{ F(d_q(y_{2p-1}, y_{2p}), F(d_q(y_{2p}, y_{2p-1})) \} \leq F(d_q(y_{2p-2}, y_{2p-1})) - \tau.
\]

Using the above inequality in (4.32), we get

\[
F(d_q(y_{2p+1}, y_{2p+2})) \leq \max \{ F(d_q(y_{2p-2}, y_{2p-1})) \} - 3\tau.
\]

Observing (4.30), (4.32), the above inequality and proceeding in this way, we have

\[
F(d_q(y_{2p+1}, y_{2p+2})) \leq \max \{ F(d_q(y_0, y_1)), F(d_q(y_1, y_0)) \} - (2p + 1)\tau, \tag{4.33}
\]

for each \( p \in \mathbb{N} \cup \{0\} \). Similarly, we have

\[
F(d_q(y_{2p}, y_{2p+1})) \leq \max \{ F(d_q(y_0, y_1)), F(d_q(y_1, y_0)) \} - (2p)\tau \tag{4.34}
\]

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Combining (4.33) and (4.34), we get

\[ F(d_q(y_n, y_{n+1})) \leq \max\{F(d_q(y_0, y_1)), F(d_q(y_1, y_0))\} - n\tau. \quad (4.35) \]

Similarly, we get

\[ F(d_q(y_{n+1}, y_n)) \leq \max\{F(d_q(y_0, y_1)), F(d_q(y_1, y_0))\} - n\tau. \quad (4.36) \]

On taking limit \( n \to \infty \), both sides of (4.35) and (4.36), we have

\[ \lim_{n \to \infty} F(d_q(y_n, y_{n+1})) = \lim_{n \to \infty} F(d_q(y_{n+1}, y_n)) = -\infty. \quad (4.37) \]

Since \( F \in \mathcal{F} \),

\[ \lim_{n \to \infty} d_q(y_n, y_{n+1}) = \lim_{n \to \infty} d_q(y_{n+1}, y_n) = 0. \quad (4.38) \]

By (4.35), for each \( n \in \mathbb{N} \), we obtain

\[ 0 \leq (d_q(y_n, y_{n+1}))^k((F(d_q(y_n, y_{n+1})) - \max\{F(d_q(y_0, y_1)), F(d_q(y_1, y_0))\}), \]

which implies,

\[ 0 \leq -(d_q(y_n, y_{n+1}))^k n\tau \leq 0. \quad (4.39) \]

By using (4.37), (4.38) and taking limit \( n \to \infty \) in inequality (4.39), we get

\[ \lim_{n \to \infty} (n(d_q(y_n, y_{n+1}))^k) = 0. \quad (4.40) \]

Same result can be obtained by using (4.36),

\[ 0 \leq (d_q(y_{n+1}, y_n))^k((F(d_q(y_{n+1}, y_n)) - \max\{F(d_q(y_0, y_1)), F(d_q(y_1, y_0))\}) \leq -(d_q(y_{n+1}, y_n))^k n\tau \leq 0. \]
By using (4.37), (4.38) and letting \( n \to \infty \), we have

\[
\lim_{n \to \infty} (n(d_q(y_{n+1}, y_n))^k) = 0. \tag{4.41}
\]

As (4.40) satisfies, there is \( n_1 \in \mathbb{N} \), and \( n(d_q(y_n, y_{n+1})) \leq 1 \), for each \( n \geq n_1 \) or,

\[
d_q(y_n, y_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}} \text{, for each } n \geq n_1. \tag{4.42}
\]

Similarly, by using (4.41), there exists \( n_2 \in \mathbb{N} \), so as \( n(d_q(y_{n+1}, y_n))^k \leq 1 \), for each \( n \geq n_2 \), we have

\[
d_q(y_{n+1}, y_n) \leq \frac{1}{n^{\frac{1}{k}}} \text{, for each } n \geq n_2. \tag{4.43}
\]

Using (4.42), we get form \( m > n > n_1 \),

\[
d_q(y_n, y_m) \leq d_q(y_n, y_{n+1}) + d_q(y_{n+1}, y_{n+2}) + \cdots + d_q(y_{m-1}, y_m) = \sum_{p=n}^{m-1} d_q(y_p, y_{p+1}) \leq \sum_{p=n}^{\infty} \frac{1}{p^{\frac{1}{k}}}.
\]

The convergence of this series \( \sum_{p=n}^{\infty} \frac{1}{p^{\frac{1}{k}}} \) demands \( \lim_{n,m \to \infty} d_q(y_n, y_m) = 0 \). Now, by treating the inequality (4.43) we get, \( \lim_{m,n \to \infty} d_q(y_m, y_n) = 0 \). Hence, \( \{TS(y_n)\} \) is a Cauchy in \((Y, d_q)\). Since \((Y, d_q)\) is a complete DQM space, so there must be a \( u \in y \) so as \( \{TS(y_n)\} \to u \) that is

\[
\lim_{n \to \infty} d_q(y_n, u) = \lim_{n \to \infty} d_q(u, y_n) = 0. \tag{4.44}
\]

Now, by Lemma 1.3.8 we have

\[
\tau + F(d_q(y_{2n+1}, Tu) \leq \tau + F(H_{d_q}(Sy_{2n}, Tu)) \leq \tau + \max\{F(H_{d_q}(Sy_{2n}, Tu), F(H_{d_q}(Ty_{2n}, Su))\}.
\]

Using inequality (4.25),

\[
\tau + F(d_q(y_{2n+1}, Tu) \leq \min\{F(D_q(y_{2n}, u), F(D_q(u, y_{2n}))\}
\]

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\[ \tau + F(d_q(y_{2n+1}, Tu)) \leq F(D_q(y_{2n}, u)), \]  

where,

\[ D_q(y_{2n}, u) = \max \left\{ \frac{d_q(y_{2n}, u), d_q(y_{2n}, y_{2n+1}), d_q(u, Tu)}{a + \max\{d_q(y_{2n}, u), d_q(u, y_{2n})\}}, \frac{d_q(y_{2n}, S_{y_{2n}}), d_q(u, Tu)}{d_q(y_{2n}, y_{2n+1}), d_q(u, Tu)} \right\}. \]  

Applying limit \( n \to \infty \), on inequality (4.44), we get

\[ D_q(y_{2n}, u) = d_q(u, Tu). \]  

Using (4.46) in (4.45), we get

\[ \tau + F(d_q(y_{2n+1}, Tu)) \leq F(d_q(u, Tu)). \]

Since \( F \) is the continuous strictly increasing real function, we get

\[ d_q(y_{2n+1}, Tu) < d_q(u, Tu). \]

Applying limit \( n \to \infty \), we get

\[ d_q(u, Tu) < d_q(u, Tu). \]

It is not true, so \( d_q(u, Tu) = 0 \). Now, by Lemma 1.3.8

\[ \tau + d_q(Tu, y_{2n+1}) \leq \tau + F(H_{d_q}(Tu, S_{y_{2n}})), \]

by using the similar reasons as above, we get \( d_q(Tu, u) = 0 \). Hence \( u \in Tu \). Similarly by using (4.44), Lemma 1.3.8, and the inequality

\[ \tau + d_q(Su, y_{2n+2}) \leq \tau + F(H_{d_q}(Su, T_{y_{2n+1}})) \]

we can show that \( d_q(Su, u) = 0 \). Similarly, \( d_q(u, Su) = 0 \). Hence, the pair \((S, T)\) have a C.F.P
This implies that $d_q(u, u) = 0$. Hence the proof is completed. ■

**Example 4.3.3** Let $Y = \{0\} \cup \mathbb{Q}^+$ and $d_q(x, y) = x + 2y$. Then $(Y, d_q)$ be a DQM space.

Define $S, T : Y \to P(Y)$ by:

\[
S(y) = \begin{cases} 
\left[\frac{1}{3}y, \frac{2}{3}y\right] \cap \mathbb{Q}^+, & \text{for all } y \in \{0, 7, \frac{7}{3}, \frac{7}{15}, \frac{7}{45}, \ldots\}, \\
[y, y + 5] \cap \mathbb{Q}^+, & \text{otherwise.}
\end{cases}
\]

\[
T(y) = \begin{cases} 
\left[\frac{1}{3}y, \frac{2}{3}y\right] \cap \mathbb{Q}^+, & \text{for all } y \in \{0, 7, \frac{7}{3}, \frac{7}{15}, \frac{7}{45}, \ldots\}, \\
[y + 2, y + 6] \cap \mathbb{Q}^+, & \text{otherwise.}
\end{cases}
\]

Case 1: If,

\[
\tau + \max\{F(H_{d_q}(S e T c)), F(H_{d_q}(T e S c))\} = \tau + F(H_{d_q}(S e T c)) \leq \min\{F(D_{d_q}(e, c)), F(D_{d_q}(c, e))\}
\]

holds. Define $F : \mathbb{R}^+ \to \mathbb{R}$ real function by $F(u) = \ln(u)$ for every $u \in \mathbb{R}^+$ and $\tau > 0$. As $x, y \in Y$, $\tau = \ln(1.2)$ and by taking $y_0 = 7$, we define the sequence $\{TS(y_n)\} = \{7, \frac{7}{3}, \frac{7}{15}, \frac{7}{45}, \ldots\}$ in $Y$ generated by $y_0 = 7$. Now, if $x, y \in \{TS(y_n)\}$, we have

\[
H_{d_q}(Sx, Ty) = \max\left[\left\{\sup_{a \in Sx} d_q(a, Ty), \sup_{b \in Ty} d_q(Sx, b)\right\}\right] = \max\left[\left\{\sup_{a \in Sx} d_q\left(a, \left[\frac{y}{2}, \frac{2y}{5}\right]\right), \sup_{b \in Ty} d_q\left(\left[\frac{x}{3}, \frac{2x}{3}\right], b\right)\right\}\right] = \max\left\{d_q\left(\frac{2x}{3}, \frac{y}{5}\right), d_q\left(\frac{x}{3}, \frac{2y}{5}\right)\right\} = \max\left\{\frac{2x}{3} + \frac{2y}{5}, \frac{x}{3} + \frac{4y}{5}\right\}.
\]
Also

\[
D_q(x, y) = \max \left\{ d_q(x, y), \frac{d_q(x, \left[ \frac{x}{3}, \frac{2x}{3} \right])}{1 + \max\{d_q(x, y), d_q(y, x)\}^t}, \frac{d_q(x, y)}{3}, d_q(y, \left[ \frac{y}{5}, \frac{2y}{5} \right]) \right\}
\]

\[
= \max \left\{ d_q(x, y), \frac{d_q(x, \frac{x}{3})}{1 + \max\{d_q(x, y), d_q(y, x)\}^t}, d_q(x, \frac{x}{3}), d_q(y, \frac{y}{5}) \right\}
\]

\[
= \max \left\{ x + 2y, \frac{7xy}{3(1 + x + 2y)}, \frac{5x}{3}, \frac{7y}{5} \right\} = x + 2y.
\]

Case (i). If \( \max\left\{ \frac{2x}{3} + \frac{2y}{5}, \frac{x}{3} + \frac{4y}{5} \right\} = \frac{x}{3} + \frac{4y}{5} \), and \( \tau = \ln(1.2) \), then we have

\[
10x + 24y \leq 25x + 50y
\]

\[
\frac{6}{5} \left( \frac{x}{3} + \frac{4y}{5} \right) \leq x + 2y
\]

\[
\ln(1.2) + \ln\left( \frac{x}{3} + \frac{4y}{5} \right) \leq \ln(x + 2y).
\]

Which shows that,

\[
\tau + F(H_{d_q}(Sx, Ty)) \leq F(D_q(x, y)).
\]

Case (ii). Similarly, if \( \max\left\{ \frac{2x}{3} + \frac{2y}{5}, \frac{x}{3} + \frac{4y}{5} \right\} = \frac{2x}{3} + \frac{2y}{5} \), and \( \tau = \ln(1.2) \), then we have

\[
20x + 12y \leq 25x + 50y
\]

\[
\frac{6}{5} \left( \frac{2x}{3} + \frac{2y}{5} \right) \leq x + 2y
\]

\[
\ln(1.2) + \ln\left( \frac{2x}{3} + \frac{2y}{5} \right) \leq \ln(x + 2y).
\]

Hence,

\[
\tau + F(H_{d_q}(Sx, Ty)) \leq F(D_q(x, y)).
\]

Case 2: If

\[
\max\{\tau + F(H_{d_q}(Sx, Ty)), \tau + F(H_{d_q}(Tx, Sy))\}
\]

\[
= \tau + F(H_{d_q}(Tx, Sy)) \leq \min\{F(D_q(x, y)), F(D_q(y, x))\}
\]

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holds.

\[
H_{d_q}(Tx, Sy) = \max \left\{ \sup_{b \in Tx} d_q(Sy, b), \sup_{a \in Sy} d_q(a, Tx) \right\}
\]

\[
= \max \left\{ \sup_{b \in Tx} d_q\left(\left[\frac{y}{3}, \frac{2y}{3}\right], b\right), \sup_{a \in Sy} d_q\left(a, \left[\frac{x}{5}, \frac{2x}{5}\right]\right) \right\}
\]

\[
= \max \left\{ d_q\left(\frac{2y}{3}, \frac{x}{5}\right), d_q\left(\frac{y}{3}, \frac{2x}{5}\right) \right\}
\]

\[
= \max \left\{ \frac{2y}{3} + \frac{2x}{5}, \frac{y}{3} + \frac{4x}{5} \right\},
\]

where

\[
D_q(y, x) = \max \left\{ d_q(y, x), d_q\left(x, \left[\frac{x}{3}, \frac{2x}{3}\right]\right), d_q\left(y, \left[\frac{y}{5}, \frac{2y}{5}\right]\right) \right\}
\]

\[
= \max \left\{ d_q(y, x), \frac{d_q\left(x, \left[\frac{x}{3}, \frac{2x}{3}\right]\right)}{1 + \max\{d_q(x, y), d_q(y, x)\}}, d_q\left(y, \left[\frac{y}{5}, \frac{2y}{5}\right]\right) \right\}
\]

\[
= \max \left\{ y + 2x, \frac{7xy}{5} + \frac{5x}{3} + \frac{7y}{5} \right\}
\]

Case (i). If, \(\max\left\{\frac{2y}{3} + \frac{2x}{5}, \frac{y}{3} + \frac{4x}{5}\right\} = \frac{y}{3} + \frac{4x}{5}\), and \(\tau = \ln(1.2)\), then we have

\[
10y + 24x \leq 25y + 50x
\]

\[
\frac{6}{5}\left(\frac{y}{3} + \frac{4x}{5}\right) \leq y + 2x
\]

\[
\ln(1.2) + \ln\left(\frac{y}{3} + \frac{4x}{5}\right) \leq \ln(y + 2x),
\]

so, \(\tau + F(H_{d_q}(Tx, Sy)) \leq F(D_q(y, x))\).

Case (ii). Similarly, if \(\max\left\{\frac{2y}{3} + \frac{2x}{5}, \frac{y}{3} + \frac{4x}{5}\right\} = \frac{2y}{3} + \frac{2x}{5}\), and \(\tau = \ln(1.2)\), then we have

\[
20y + 24x \leq 25x + 50y
\]

\[
\frac{6}{5}\left(\frac{2y}{3} + \frac{2x}{5}\right) \leq y + 2x
\]

\[
\ln(1.2) + \ln\left(\frac{2y}{3} + \frac{2x}{5}\right) \leq \ln(y + 2x).
\]

Hence, \(\tau + F(H_{d_q}(Tx, Sy)) \leq F(D_q(y, x))\).
Now, if \( x, y \notin \{TS(y_n)\} \), then the contraction does not hold. Hence all hypothesis of Theorem 4.3.2 are proved so \( S \) and \( T \) have a C.F.P.

If we take \( S = T \) in Theorem 4.3.2, then we are left with the theorem.

**Theorem 4.3.4** Let \( (Y, d_q) \) be a complete DQM space and \( S : Y \to P(Y) \) be the setvalued map such that

\[
\tau + F(H_{d_q}(Sl, Sp)) \leq F(D_q(l, p)),
\]

for each \( l, p \in \{S(y_n)\} \), with \( D_q(l, p) > 0, F \in \mathcal{F}, \tau, a > 0, \) and

\[
D_q(l, p) = \max \left\{ d_q(l, p), \frac{d_q(l, Sl) \cdot d_q(p, Sp)}{a + d_q(l, p)}, d_q(l, Sl), d_q(p, Sp) \right\}.
\]

Then \( \{S(y_n)\} \to u \in Y \). Moreover, if (4.47) holds for \( u \), then \( S \) has a fixed point \( u \) in \( Y \) and \( d_q(u, u) = 0 \).

**Definition 4.3.5** Let \( S, T : Y \to Y \) be two maps and \( x_0 \in Y \). Let \( x_1 = Sx_0, x_2 = Tx_1, x_3 = Sx_2 \) and so on. Proceeding this method, we get the sequence \( x_n \) in \( X \) so as

\[
x_{2p+1} = Sx_{2p} \text{ and } x_{2p+2} = Tx_{2p+1}, \quad (\text{where } p = 0, 1, 2, \ldots).
\]

We say that \( \{TS(x_n)\} \) be the sequence in \( Y \) generated by \( x_0 \).

**Definition 4.3.6** Let \( (Y, d_q) \) be a DQM space and \( S, T : Y \to Y \) be two maps. The pair \( (S, T) \) is said a FDQ-contraction, if for all \( e, g \in \{TS(e_n)\} \), we get

\[
\tau + \max \{F(d_q(Se, Tg)), F(d_q(Te, Sg))\} \leq \min\{F(D_q(e, g)), F(D_q(g, e))\}
\]

where \( F \in \mathcal{F} \) and \( \tau > 0, \) and

\[
D_q(e, g) = \max \left\{ d_q(e, g), \frac{d_q(e, Se) \cdot d_q(g, Tg)}{1 + \max\{d_q(e, g), d_q(g, e)\}}, d_q(e, Se), d_q(g, Tg) \right\}.
\]

Then we deduce the following main result.

**Theorem 4.3.7** Let \( (Y, d_q) \) be a complete DQM space and \( (S, T) \) be a FDQ-contraction. Then \( \{TS(x_n)\} \to u \in X \). Also, if \( u \) satisfies (4.48), then \( u \) is the C.F.P of \( S \) and \( T \) in \( X \) and \( d_q(u, u) = 0 \).
Now, we have shown an application, of Theorem 4.3.4 to find unique solution of systems of non linear Volterra type integral inclusions. Let,

\[
w(t) = \int_0^t L_1(t, s, w(s)) ds + f(t),
\]

(4.50)

\[
c(t) = \int_0^t L_2(t, s, c(s)) ds + g(t)
\]

(4.51)

for all \( t \in [0, 1] \). We find the solution of (4.50) and (4.51). Let \( E = \{ f : f \text{ is continuous function from } [0, 1] \text{ to } \mathbb{R}_+ \} \), endowed with the complete DQM. For \( w \in E \), identify the norm as: \( \|w\|_\tau = \sup_{t \in [0,1]} \{w(t)e^{-\tau t}\} \), where \( \tau > 0 \) is taken arbitrary. Then define

\[
d_\tau(w, c) = \sup_{t \in [0,1]} \{(w(t) + 2c(t))e^{-\tau t}\} = \|w + 2c\|_\tau
\]

for all \( w, c \in E \), with these settings, \((E, d_\tau)\) becomes a DQM space.

**Theorem 4.3.8** Assume (i), (ii) and (iii) are satisfied:

(i) \( L_1, L_2 : [0, 1] \times [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) and \( f, g : [0, 1] \rightarrow \mathbb{R}_+ \) are real and continuous;

(ii) Define

\[
Sw(t) = \int_0^t L_1(t, s, w(s)) ds + f(t),
\]

\[
Tc(t) = \int_0^t L_2(t, s, c(s)) ds + g(t).
\]

Suppose there exist \( \tau > 1 \), such that

\[
\max \{L_1(t, s, w) + 2L_2(t, s, c), L_2(t, s, c) + 2L_1(t, s, w)\} \leq \tau e^{-\tau} \min \{P(w, c), P(c, w)\}
\]

(4.52)
for each $t, s$ belong to $[0, 1]$ and $w, c$ belong to $C([0, 1], \mathbb{R})$, where

\[
P(w, c) = \max \left\{ w(t) + 2c(t), \frac{(w(t) + 2Sw(t))(c(t) + 2Tc(t))}{\alpha + \max\{w(t) + 2c(t), c(t) + 2w(t)\}} \right\},
\]

Then integral equations (4.50) and (4.51) has a solution.

**Proof.** By assumption (ii)

\[
\max\{Sw + 2Tv, Tw + 2Sc\} = \max\left\{ \int_0^t (K_1(t, s, w) + 2K_2(t, s, c))ds, \int_0^t (K_2(t, s, c) + 2K_1(t, s, w))ds \right\}
\]

\[
\leq \int_0^t \tau e^{-\tau} \min\{M(w, c), M(c, w)\} ds
\]

\[
\leq \int_0^t \tau e^{-\tau} \left[ \min\{P(w, c), P(c, w)\} e^{-\tau s}\right] e^{\tau s} ds
\]

\[
\leq \int_0^t \tau e^{-\tau} \| \min\{P(w, c), P(c, w)\} \|_\tau e^{\tau s} ds
\]

\[
\leq \tau e^{-\tau} \| \min\{P(w, c), P(c, w)\} \|_\tau \int_0^t e^{\tau s} ds
\]

\[
\leq \tau e^{-\tau} \| \min\{P(w, c), P(c, w)\} \|_\tau \frac{1}{\tau} e^{\tau t}
\]

\[
\leq e^{-\tau} \| \min\{P(w, c), P(c, w)\} \|_\tau e^{\tau t}.
\]

This implies

\[
\max\{Sw + 2Tv, Tw + 2Sc\} e^{-\tau t} \leq e^{-\tau} \| \min\{P(w, c), P(c, w)\} \|_\tau.
\]

That is $\| \max\{Su + 2Tu, Tu + 2Sv\} \|_\tau \leq e^{-\tau} \| \min\{P(w, c), P(c, w)\} \|_\tau,$

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which further implies

\[ \tau + \ln \| \max\{Sw + 2Tc, Tw + 2Sc\}\|_\tau \leq \ln \| \min\{P(w, c), P(c, w)\}\|_\tau. \]

So, all hypothesis of Theorem 4.3.7 are proved. Hence, (4.50) and (4.51) have a common solution.

**Remark 3.4.10** By setting different values of \( P(w, c) \) in equation (4.52), we can obtain different weak contractive inequalities and results as corollaries of Theorem 4.3.8. \( \blacksquare \)
Bibliography


