National College of Business Administration & Economics Lahore

L^p – BOUNDEDNESS OF INTEGRAL OPERATORS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS

BY

SHAHID MUBEEN

DOCTOR OF PHILOSOPHY
IN
APPLIED MATHEMATICS

October, 2011

NATIONAL COLLEGE OF BUSINESS ADMINISTRATION & ECONOMICS

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A Dissertation Submitted to
School of Computer Sciences

In Partial Fulfillment of the
Requirements for the Degree of

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بسم الله الرحمن الرحيم

IN THE NAME OF ALLAH, THE MOST BENEFICIENT,

THE MOST MERCIFUL

NATIONAL COLLEGE OF BUSINESS ADMINISTRATION & ECONOMICS, LAHORE

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Dissertation Committee:

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Chairman

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National College of Business
Administration & Economics

DECLARATION

This is to certify that this research work is my own work and it has not been submitted for obtaining a similar degree from any other university and/or institution.

SHAHID MUBEEN
October, 2011

DEDICATED TO

MY PARENTS

AND

MY WIFE Dr. MARIA

ACKNOWLEDGEMENT

Thanks to **Almighty ALLAH** whose infinite blessings have enabled me to complete this humble piece of dissertation. I am also indebted to my teachers, seniors and fellows at NCBA & E.

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RESEARCH COMPLETION CERTIFICATE

Certified that the research work contained in this thesis entitled "**L^p − Boundedness of Integral Operators Involving Generalized Hypergeometric Functions**" has been carried out and completed by **Mr. Shahid Mubeen** under my supervision during his Ph. D programme.

(Dr. G. M. Habibullah)

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Lahore

SUMMARY

Studies on integral operators involving the hypergeometric functions ${}_2F_1$ and the confluent hypergeometric functions ${}_1F_1$ as kernel have been a popular subject area. Researchers have discussed the boundedness of integral operators in L^1 and $L^p, p>1$, and determined their inversions. Often, they have investigated the necessary and sufficient conditions for such an inversion by using mapping properties. We have used, in this thesis, such relations to study L^p-boundedness of integral operators that involve generalized $k-$hypergeometric functions.

We have first extended integral representation of the confluent $k-$hypergeometric functions from ${}_1F_1$ to ${}_2F_2$ and to ${}_mF_m$, $m>2$. We have also extended integral representation of $k-$hypergeometric functions from ${}_2F_1$ to ${}_3F_2$ and to ${}_{m+1}F_m$, $m>2$. We have then formulated integral operators involving generalized $k-$hypergeometric functions both ${}_mF_m$ and ${}_{m+1}F_m$, $m \geq 1$ as kernel.

We have introduced extensions of Erdélyi-Kober fractional integrals to be called $k-$fractional integrals, which are based upon definition of $k-$gamma function. We have also proved a few properties of $k-$fractional integrals to gain its formalized knowledge. We have shown that integral operators, earlier considered, are compositions of known operators, $k-$fractional integrals and variants of Laplace transform. Using Hardy's inequality, we have proved the L^p-boundedness, $p>1$, of all these integral operators including $k-$fractional integrals. We have also proved that each of these operators is bounded in L^p, $p>1$, with a weight function.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 BACKGROUND

Traditionally, Laplace, Mellin and Fourier transforms are used to solve differential and integral equations. However, fractional integrals have been played an important role in solving a large number of differential and integral equations.

Many researchers have shown deep interest in integral operators involving various hypergeometric functions. Newly established properties of hypergeometric functions and polynomials facilitated solutions of many differential and integral equations involving these functions.

Erdélyi and Love have initiated research on issues which involved hypergeometric functions ${}_2F_1$, so did Saxena, Srivastava and Kilbas, and Habibullah has followed them to consider integral equations involving confluent hypergeometric functions ${}_1F_1$. Okikiolu proved the boundedness of similar integral operators in L^1 and L^p, $p > 1$, and he did so by proving that these operators could be represented as composition of linear operators and used their mapping properties to derive inversion processes and hence to show their boundedness. In addition to a variety of integral operators involving Fredholm type kernels, integral operators of convolution type involving hypergeometric functions have also been studied by various authors. Karapetiants and Samko proved that two types of operators are related by a simple relation. Numerous forms of fractional integral operators have been the critical source of facilitation for solutions and have enriched the study of integral operators.

Recently, Driver and Johnston gave a simple integral representation of some hypergeometric functions ${}_{m+1}F_m$. This representation would help to study ${}_{m+1}F_m$ and integral operators involving these hypergeometric functions. Similarly one could study ${}_mF_m$. The direction of using these representations has been source of inspiration of the current work.

1.2 LITERATURE REVIEW

Hypergeometric functions and polynomials generated by hypergeometric functions have been extensively studied for years. We could mention Euler, Gauss, Kummer, Riemann, Schwarz, Abramowitz, Andrews and Slater who contributed to advancement of special functions. Integral representations of hypergeometric functions have been excellent source for their application and for studying their behavior. Recently, Driver and Johnston (2006) have found new integral representations of some special generalized hypergeometric functions that have facilitated investigations in the present work.

Earlier, Erdélyi (1953), Rainville (1963), Wang et al. (1989), Mathai (1993), Andrews et al. (2004), Mathai and Haubold (2008) and Cuyt at al. (2008) and other collected a large number of properties of hypergeometric functions. Saad and Hall (2003) showed that many integrals containing products of confluent hypergeometric functions follow directly from one single integral which has a very simple formula in terms of Appell's double series. Virchenko et al. (2001) defined a new generalized hypergeometric function, using a special case of Wright's function. Chiavetta (2007a and 2007b) determined the summation of a series of hypergeometric functions using Schäfke's method. Ahmad and Saboor (2009) gave another notation to generalized power series hypergeometric functions and applied it to statistical analysis. Virchenko and Ovcharenko (2011) established the generalization of Laplace, Stieltjes, and potential integral transforms with the generalized (according to Wright) hypergeometric functions. Medhat and Arjun (2011) provided the generalizations of the classical summation theorems such as of Gauss, Kummer and Bailey for the series ${}_2F_1$ and ${}_3F_2$ in general cases.

Fractional derivatives and integrals had been popular research directions for many years. There were so many researchers who studied fractional calculus; we could mention Fourier, Abel, Liouville, Riemann, Letnikov, Hardy, Weyl, Littlewood, Kober, Erdélyi, Love, Samko, Kilbas, Srivastava, Trujillo, Amsterdam and Miller. Finding properties and uses of fractional integration began with Kober (1940). It had, henceforth, been studied by numerous researchers for example Erdélyi (1950), Saxena (1967), Lowndes (1970). Habibullah and Choudhary (1993) proved some simple and applicable properties of fractional integration. Saigo and Kilbas (1994) generalized the fractional integrals and derivatives in Hölder spaces. Kilbas et al. (2002) studied the generalized fractional integral transforms involving the Gauss hypergeometric functions as kernel by generalizing Love's work. Many other researchers studied generalizations of Erdélyi-Kober operators for example

Nishimoto (1991), Samko et al. (2002) and Kilbas et al. (2006). Castell (2004) investigated a relationship of fractional derivatives and the inverse Fourier transform of radial functions. Kilbas and Sebastian (2008) studied the generalized fractional integration of Bessel functions of the first kind. Srivastava and Tomovski (2009) considered fractional integral operator containing a generalized Mittag-Leffler function operator given by Prabhakar (1971), which are more suitable for applied problems, see Mathai et al. (2010). Virchenko et al. (2010) gave a generalization of Gauss hypergeometric function, and investigated its basic properties. Further, they defined fractional integral operators and their inverses in terms of the Mellin transform. Agarwal and Jain (2011) obtained additional fractional calculus formulae, using series expansion method, for polynomials which were introduced by Srivastava (1972).

Higgins (1963) found an inversion formula for integral equation with Gegenbauer transformation as kernel. Srivastava (1963 and 1966) discussed a class of integral equations involving ultraspherical and Laguerre polynomials.

Erdélyi (1963, 1964) used fractional integrals to study integral transform given by

$$g(x)=\int_{\alpha}^{x}\left(x^2-t^2\right)^{-\frac{1}{2}\mu}P_{\nu}^{\mu}\left(\frac{x}{t}\right)f(t)\,dt.$$

Wimp (1965) discussed a pair of integral transforms involving hypergeometric functions and thus started study of pairs of integral equations. With the help of Erdélyi–Kober fractional integrals, Love (1967) determined the necessary and sufficient condition for the solution of the equation

$$g(x)=\int_{0}^{x}\frac{(x-t)^{c-1}}{\Gamma(c)}\,{}_2F_1\left(a;b;c;1-\frac{x}{t}\right)f(t)\,dt,\ \ x>0,$$

where ${}_2F_1(a;b;c;x)$ is the usual hypergeometric function. Kalla and Saxena (1969) discussed the integral operators involving hypergeometric functions. Habibullah (1970) used integer-value fractional integral to solve an integral equation involving Shively’s polynomials. Okikiolu (1971) used fractional integral to find inversion of integral operator involving Bessel functions. Habibullah (1971) used fractional integration to investigate a solution of the integral equation of the type

$$g(x)=\int_{0}^{x}\frac{(x-t)^{c-1}}{\Gamma(c)}\Phi(a;c;t-x)f(t)\,dt,\ \ a>0,\ \ c>0,$$

where $\Phi(a;c;t-x)$ is the confluent hypergeometric function and determined the necessary and sufficient condition for this solution of the integral equation.

Srivastava (1976) solved a number of integral equations involving confluent hypergeometric functions as kernel.

Okikiolu (1966) proved the boundedness of generalized Fourier transforms in L^p-space. Habibullah (1977) also investigated in L^p, $p \geq 1$, the boundedness of integral transform of the type

$$g(x) = \int_0^x (xt)^{b-1} {}_2F_1(a,b;c;x) f(t) dt.$$

Saigo (1977/78) studied integral operators involving the Gauss hypergeometric functions. Andersen and Heinig (1983) obtained new inequalities for a class of convolution operators, various fractional integrals and the Laplace transform. Saxena and Ram (1990) discussed the multidimensional generalized Erdélyi-Kober operators associated with the Gauss's hypergeometric functions. Karapetiants and Samko (1999) gave a self – contained representation of the Fredholm theory of one–and multi-dimensional integral operators of the kind

$$\lambda\varphi(x) = \int_{|y|<a} k(x,y)\varphi(y)dy, \ \ x \in R^n, \ \ |x| < a, \ \ 0 < a < \infty,$$

with the homogeneous kernel of degree $-n$, that is,

$$k(tx,ty) = t^{-n}k(x,y), \ \ x,y \in R^n, \ \ t > 0.$$

Goyal et al. (1991) derived a number of different expressions for the certain compositions of fractional integral operators involving a product of generalized hypergeometric functions and a general class of polynomials with essentially arbitrary coefficients. Saxena and Saigo (1998) obtained a fractional integral formula for the H-function II. Kilbas et al. (2000) studied the Meijer integral transform with the modified Bessel function of the third kind $k_\eta(z)$ of complex order η as the kernel. Srivastava and Saxena (2001) discussed the applications of fractional integral operators. Kilbas et al. (2004) gave the relation between generalized Mittag-Leffler function and generalized fractional integral operators.

Vi Nhan and Duc (2008) gave some new type of convolution inequalities in weighted L^p- spaces and their important applications to partial differential and integral transforms. Belarbi and Dahmani (2009) established integral inequalities for the Chebyshev functional in the case of two synchronous functions by using the Riemann-Liouville fractional integrals. Aliev (2010) obtained a relation between the Fourier, Bessel and Riemann-Liouville integral transforms and gave an application of this relation to weighted norm inequalities. Saha and Arora (2010) investigated the relations that exist between the Riemann-

Liouville fractional calculus and multi-index Dzrbashjan-Gelfond-Leontiev differentiation and integration with multi-index Mittag-Leffler function. Arcadii (2011) used a weighted inequality for two functions to estimate a natural connection between solutions of Volterra equations and related convolution integral equations. Salahuddin (2011) developed summation formulae based on half argument by the help of Gauss second summation theorem. Zareen (2011) obtained new generalizations of the Hardy type integral inequality by using a fairly elementary analysis. Haubold et al. (2011) derived relations of Mittag-Leffler functions with Riemann-Liouville fractional operators.

There have been a lot of investigations in generalizing classical gamma function. For example, Saxena et al. (2007) defined and studied generalization of the generalized gamma-type functions. Recently, in a series of research publications, Diaz et al. (2005, 2007 and 2010) have introduced $k-$gamma and $k-$beta functions and proved a number of their properties where we are interested in. They have also studied $k-$zeta function and $k-$hypergeometric functions based on Pochhammer $k-$symbols for factorial functions. It has been followed by works of Mansour (2009), Kokologiannaki (2010), Kransniqi (2010) and Merovci (2010) elaborating and strengthening study of $k-$gamma and $k-$beta functions. These studies on $k-$formulations have generated interest in developing the work presented here after.

CHAPTER 2

$k-$ANALOGUE OF HYPERGEOMETRIC FUNCTIONS

In this chapter, following the Pochhammer $k-$symbols and the definitions of $k-$hypergeometric functions introduced by Diaz et al. (2005, 2007 and 2010), we prove properties of generalized $k-$hypergeometric functions. We prove integral representations of $k-$hypergeometric functions. In the same vein, we also determine results on generalized confluent $k-$hypergeometric functions that can be used later. We begin with the classical hypergeometric functions by noting the assumption that all numbers we encounter henceforth are real unless and until otherwise mentioned.

2.1 THE HYPERGEOMETRIC FUNCTIONS

Let j_i be a function from N into N such that $j_i < j_{i+1}, \ \forall \ i \in N$, where N is a set of natural numbers and also $m \in N$.

The confluent hypergeometric functions with two parameters $\beta, \ \gamma$, one parameter β in numerator and one parameter γ in denominator, is given by

$$ {}_1F_1\left[\begin{matrix}\beta \\ \gamma\end{matrix};x\right] = \sum_{j=0}^{\infty}\frac{(\beta)_j}{(\gamma)_j}\frac{x^j}{j!}; \ \forall \ \beta,\gamma, \ \gamma \neq 0,-1,-2,..., \text{ for all finite } x. $$

The hypergeometric functions with three parameters $\alpha, \ \beta, \ \gamma,$ two parameters $\alpha, \ \beta$ in numerator and one parameter γ in denominator, is defined by

$$ {}_2F_1\left[\begin{matrix}\alpha,\beta \\ \gamma\end{matrix};x\right] = \sum_{j=0}^{\infty}\frac{(\alpha)_j(\beta)_j}{(\gamma)_j}\frac{x^j}{j!}; \ \forall \ \alpha,\beta,\gamma, \ \gamma \neq 0,-1,-2,..., \ |x|<1, $$

where $\qquad (\alpha)_m = \alpha(\alpha+1)(\alpha+2)...(\alpha+m-1); \ m \geq 1; \ (\alpha)_0 = 1,$

(See Rainville (1963)).

2.2 THE GENERALIZED HYPERGEOMETRIC FUNCTIONS

The generalized hypergeometric functions with $r+s$ parameters, r parameters in numerator and s parameters in denominator, is defined by

$$ {}_rF_s\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_r; \\ \beta_1,\beta_2,\ldots,\beta_s;\end{matrix}\ x\right]=\sum_{j=0}^{\infty}\frac{\prod_{u=1}^{r}(\alpha_u)_j}{\prod_{v=1}^{s}(\beta_v)_j}\frac{x^j}{j!}, $$

$\forall\ \alpha_u,\ \beta_v,\ \beta_v\neq 0,-1,-2,\ldots,\ |x|<1,\ u=1,2,\ldots,r$ and $v=1,2,\ldots,s.$

It is known that

i) if $r\leq s$, the series converges for all finite x;

ii) if $r=s+1$, the series converges for $|x|<1$ and diverges for $|x|>1$;

iii) if $r>s+1$, the series diverges for $x\neq 0$ unless the series terminates.

2.3 INTEGRAL REPRESENTATIONS OF HYPERGEOMETRIC AND CONFLUENT HYPERGEOMETRIC FUNCTIONS

A number of integral representations of the hypergeometric and confluent hypergeometric functions have been known for years and a few of these are stated in the following lemmas.

Lemma 2.3.1: If $\gamma>\beta>0$, then for all finite x

$$ {}_1F_1\left[\begin{matrix}\beta \\ \gamma\end{matrix}\ ;x\right]=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}e^{xt}dt, $$

(See Rainville (1963)).

Lemma 2.3.2: If $\gamma>\beta>0,\ |x|<1$, then

$$ {}_2F_1\left[\begin{matrix}\alpha,\beta \\ \gamma\end{matrix}\ ;x\right]=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt, $$

(See Rainville (1963)).

Driver and Johnston (2006) have, subsequently, proved integral representations of special case of generalized hypergeometric functions as stated below.

Lemma 2.3.3: If $\gamma > \beta > 0,\ m \geq 1,\ |x| < 1$, then

$$ {}_{m+1}F_m\left[\begin{matrix} \alpha, \frac{\beta}{m}, \frac{\beta+1}{m}, \ldots, \frac{\beta+m-1}{m} \\ \frac{\gamma}{m}, \frac{\gamma+1}{m}, \ldots, \frac{\gamma+m-1}{m} \end{matrix}; x\right] $$

$$ = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\left(1-xt^m\right)^{-\alpha} dt, $$

(See Driver and Johnston (2006)).

Using similar arguments, we determine a convenient integral representation of generalized hypergeometric functions ${}_mF_m,\ m \geq 1$ annunciated in the next lemma (See Habibullah and Mubeen (2011)).

Lemma 2.3.4: If $\gamma > \beta > 0,\ m \geq 1$, then for all finite x

$$ {}_mF_m\left[\begin{matrix} \frac{\beta}{m}, \frac{\beta+1}{m}, \ldots, \frac{\beta+m-1}{m} \\ \frac{\gamma}{m}, \frac{\gamma+1}{m}, \ldots, \frac{\gamma+m-1}{m} \end{matrix}; x\right] $$

$$ = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} e^{xt^m} dt. $$

2.4 OTHER INTEGRAL REPRESENTATIONS OF HYPERGEOMETRIC FUNCTIONS

In this section, we determine an integral representation of generalized hypergeometric functions by using Legendre's multiplication formula. Also, we use this integral representation to obtain some results on confluent hypergeometric functions ${}_mF_m$. We take Legendre's multiplication formula (See Rainville (1963)).

$$ \prod_{s=1}^{m} \Gamma\left(\alpha + \frac{s-1}{m}\right) = (2\pi)^{\frac{1}{2}(m-1)} m^{\frac{1}{2}-m\alpha} \Gamma(m\alpha) \qquad (2.4.1) $$

to obtain the integral representation of generalized hypergeometric functions as given below.

Theorem 2.4.1: If $c>b>0,\ |x|<1$, then

$$
{}_3F_2\left(a,b,b+\frac{1}{2};c,c+\frac{1}{2};x\right)
= \frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\int_0^1 t^{2b-1}(1-t)^{2c-2b-1}\left(1-xt^2\right)^{-a}dt. \quad (2.4.2)
$$

Proof: Suppose that $|x|<1$. Then

$$
\begin{aligned}
{}_3F_2\left(a,b,b+\frac{1}{2};c,c+\frac{1}{2};x\right)
&= \sum_{j=0}^{\infty}\frac{(a)_j(b)_j\left(b+\frac{1}{2}\right)_j}{(c)_j\left(c+\frac{1}{2}\right)_j}\frac{x^j}{j!} \\
&= \frac{\Gamma(c)\Gamma\left(c+\frac{1}{2}\right)}{\Gamma(b)\Gamma\left(b+\frac{1}{2}\right)}\sum_{j=0}^{\infty}\frac{(a)_j\Gamma(b+j)\Gamma\left(b+j+\frac{1}{2}\right)}{\Gamma(c+j)\Gamma\left(c+j+\frac{1}{2}\right)}\frac{x^j}{j!}
\end{aligned}
$$

since $(\alpha)_j=\dfrac{\Gamma(\alpha+j)}{\Gamma(\alpha)}$.

Now, we use (2.4.1) and obtain the following expressions.

$$
\begin{aligned}
{}_3F_2\left(a,b,b+\frac{1}{2};c,c+\frac{1}{2};x\right)
&= \frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\sum_{j=0}^{\infty}\frac{(a)_j\Gamma(2b+2j)\Gamma(2c-2b)}{\Gamma(2c+2j)}\frac{x^j}{j!} \\
&= \frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\sum_{j=0}^{\infty}(a)_j\left(\int_0^1 t^{2b+2j-1}(1-t)^{2c-2b-1}dt\right)\frac{x^j}{j!} \\
&= \frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\int_0^1 t^{2b-1}(1-t)^{2c-2b-1}\left(\sum_{j=0}^{\infty}(a)_j\frac{\left(xt^2\right)^j}{j!}\right)dt \\
&= \frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\int_0^1 t^{2b-1}(1-t)^{2c-2b-1}\left(1-xt^2\right)^{-a}dt.
\end{aligned}
$$

Theorem 2.4.2: If $c>b>0,\ a>0,\ c>\frac{1}{2}a$, then

$$
{}_3F_2\left(a,b,b+\frac{1}{2};c,c+\frac{1}{2};1\right)
= \frac{\Gamma(2c)\Gamma(2c-2b-a)}{\Gamma(2c-a)\Gamma(2c-2b)}\,{}_2F_1(a,2b;2c-a;-1).
$$

Proof: Put $x=1$ in (2.4.2). We, then, get the following expressions.

$$
\begin{aligned}
&{}_3F_2\left(a,b,b+\frac{1}{2};c,c+\frac{1}{2};1\right)\\
&=\frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\int_0^1 t^{2b-1}(1-t)^{2c-2b-1}\left(1-t^2\right)^{-a}dt\\
&=\frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\int_0^1 t^{2b-1}(1-t)^{2c-2b-a-1}(1+t)^{-a}dt\\
&=\frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\int_0^1 t^{2b-1}(1-t)^{2c-2b-a-1}\sum_{j_1}^{\infty}\binom{-a}{j_1}t^{j_1}dt\\
&=\frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\int_0^1 t^{2b+j_1-1}(1-t)^{2c-2b-a-1}dt\\
&=\frac{\Gamma(2c)}{\Gamma(2b)\Gamma(2c-2b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\frac{\Gamma(2b+j_1)\Gamma(2c-2b-a)}{\Gamma(2c-a+j_1)}\\
&=\frac{\Gamma(2c)\Gamma(2c-2b-a)}{\Gamma(2c-a)\Gamma(2c-2b)}\sum_{j_1=0}^{\infty}\frac{(a)_{j_1}(2b)_{j_1}}{(2c-a)_{j_1}}\frac{(-1)^{j_1}}{j_1!}\\
&=\frac{\Gamma(2c)\Gamma(2c-2b-a)}{\Gamma(2c-a)\Gamma(2c-2b)}\,{}_2F_1(a,2b;2c-a;-1).
\end{aligned}
$$

Corollary 2.4.3: If $c>b>0,\ c>\frac{1}{2}n$, then

$$
\begin{aligned}
&{}_3F_2\left(-n,b,b+\frac{1}{2};c,c+\frac{1}{2};1\right)\\
&=\frac{(2c-2b)_n}{(2c)_n}\,{}_2F_1(-n,2b;2c+n;-1).
\end{aligned}
$$

Theorem 2.4.4: If $c>b>0,\ |x|<1$, then

$$
\begin{aligned}
&{}_4F_3\left(a,b,b+\frac{1}{3},b+\frac{2}{3};c,c+\frac{1}{3},c+\frac{2}{3};x\right)\\
&=\frac{\Gamma(3c)}{\Gamma(3b)\Gamma(3c-3b)}\int_0^1 t^{3b-1}(1-t)^{3c-3b-1}\left(1-xt^3\right)^{-a}dt. \quad (2.4.3)
\end{aligned}
$$

Proof: Suppose that $|x|<1$. Then

$$
{}_4F_3\left(a,b,b+\frac{1}{3},b+\frac{2}{3};c,c+\frac{1}{3},c+\frac{2}{3};x\right)
$$

$$= \sum_{j=0}^{\infty} \frac{(a)_j (b)_j \left(b+\frac{1}{3}\right)_j \left(b+\frac{2}{3}\right)_j}{(c)_j \left(c+\frac{1}{3}\right)_j \left(c+\frac{2}{3}\right)_j} \frac{x^j}{j!}$$

$$= \frac{\Gamma(c)\Gamma\left(c+\frac{1}{3}\right)\Gamma\left(c+\frac{2}{3}\right)}{\Gamma(b)\Gamma\left(b+\frac{1}{3}\right)\Gamma\left(b+\frac{2}{3}\right)}$$

$$\times \sum_{j=0}^{\infty} \frac{(a)_j \Gamma(b+j)\Gamma\left(b+j+\frac{1}{3}\right)\Gamma\left(b+j+\frac{2}{3}\right)}{\Gamma(c+j)\Gamma\left(c+j+\frac{1}{3}\right)\Gamma\left(c+j+\frac{2}{3}\right)} \frac{x^j}{j!}$$

since $(\alpha)_j = \dfrac{\Gamma(\alpha+j)}{\Gamma(\alpha)}$.

Now, we use (2.4.1) and obtain the following expressions.

$${}_4F_3\left(a,b,b+\frac{1}{3},b+\frac{2}{3};c,c+\frac{1}{3},c+\frac{2}{3};x\right)$$

$$= \frac{\Gamma(3c)}{\Gamma(3b)\Gamma(3c-3b)} \sum_{j=0}^{\infty} \frac{(a)_j \Gamma(3b+3j)\Gamma(3c-3b)}{\Gamma(3c+3j)} \frac{x^j}{j!}$$

$$= \frac{\Gamma(3c)}{\Gamma(3b)\Gamma(3c-3b)} \sum_{j=0}^{\infty} (a)_j \left(\int_0^1 t^{3b+3j-1}(1-t)^{3c-3b-1}\, dt \right) \frac{x^j}{j!}$$

$$= \frac{\Gamma(3c)}{\Gamma(3b)\Gamma(3c-3b)} \int_0^1 t^{3b-1}(1-t)^{3c-3b-1} \left(\sum_{j=0}^{\infty} (a)_j \frac{\left(xt^3\right)^j}{j!} \right) dt$$

$$= \frac{\Gamma(3c)}{\Gamma(3b)\Gamma(3c-3b)} \int_0^1 t^{3b-1}(1-t)^{3c-3b-1}\left(1-xt^3\right)^{-a}\, dt.$$

Theorem 2.4.5: If $c > b > 0,\ a > 0,\ c > \frac{1}{3}a$, then

$${}_4F_3\left(a,b,b+\frac{1}{3},b+\frac{2}{3};c,c+\frac{1}{3},c+\frac{2}{3};1\right)$$

$$= \frac{\Gamma(3c)\Gamma(3c-3b-a)}{\Gamma(3c-a)\Gamma(3c-3b)} \sum_{j_1=0}^{\infty} \binom{-a}{j_1} \frac{(3b)_{j_1}}{(3c-a)_{j_1}}$$

$$\times {}_2F_1\left(-j_1, 3b+j_1; 3c-a+j_1; -1\right).$$

Proof: Put $x=1$ in (2.4.3) to get the following expressions.

$$
\begin{aligned}
&{}_4F_3\left(a,b,b+\frac{1}{3},b+\frac{2}{3};c,c+\frac{1}{3},c+\frac{2}{3};1\right)\\
&=\frac{\Gamma(3c)}{\Gamma(3b)\Gamma(3c-3b)}\int_0^1 t^{3b-1}(1-t)^{3c-3b-1}\left(1-t^3\right)^{-a}dt\\
&=\frac{\Gamma(3c)}{\Gamma(3b)\Gamma(3c-3b)}\int_0^1 t^{3b-1}(1-t)^{3c-3b-a-1}\left(1+t+t^2\right)^{-a}dt\\
&=\frac{\Gamma(3c)}{\Gamma(3b)\Gamma(3c-3b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\int_0^1 t^{3b+j_1-1}(1-t)^{3c-3b-a-1}(1+t)^{j_1}dt\\
&=\frac{\Gamma(3c)}{\Gamma(3b)\Gamma(3c-3b)}\\
&\times\sum_{j_1}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\left[\binom{j_1}{j_2}\int_0^1 t^{3b+j_1+j_2-1}(1-t)^{3c-3b-a-1}\right]dt\\
&=\frac{\Gamma(3c)}{\Gamma(3b)\Gamma(3c-3b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\\
&\times\sum_{j_2=0}^{j_1}\left[\binom{j_1}{j_2}\frac{\Gamma(3b+j_1+j_2)\Gamma(3c-3b-a)}{\Gamma(3c-a+j_1+j_2)}\right]\\
&=\frac{\Gamma(3c)\Gamma(3c-3b-a)}{\Gamma(3c-a)\Gamma(3c-3b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\\
&\times\sum_{j_2=0}^{j_1}\left[\binom{j_1}{j_2}\frac{(3b)_{j_1+j_2}}{(3c-a)_{j_1+j_2}}\right]\\
&=\frac{\Gamma(3c)\Gamma(3c-3b-a)}{\Gamma(3c-a)\Gamma(3c-3b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\frac{(3b)_{j_1}}{(3c-a)_{j_1}}\\
&\times\sum_{j_2=0}^{j_1}\frac{(-j_1)_{j_2}(3b+j_1)_{j_2}}{(3c-a+j_1)_{j_2}}\frac{(-1)^{j_2}}{j_2!}\\
&=\frac{\Gamma(3c)\Gamma(3c-3b-a)}{\Gamma(3c-a)\Gamma(3c-3b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\frac{(3b)_{j_1}}{(3c-a)_{j_1}}\\
&\times{}_2F_1\left(-j_1,3b+j_1;3c-a+j_1;-1\right).
\end{aligned}
$$

Corollary 2.4.6: If $c > b > 0,\ c < \frac{1}{3}n$, then

$$ {}_4F_3\left(-n,b,b+\frac{1}{3},b+\frac{2}{3};c,c+\frac{1}{3},c+\frac{2}{3};1\right) $$
$$ =\frac{(3c-3b)_n}{(3c)_n} $$
$$ \times\sum_{j_1=0}^{n}\binom{n}{j_1}\frac{(3b)_{j_1}}{(3c+n)_{j_1}}\,{}_2F_1\left(-j_1,3b+j_1;3c+n+j_1;-1\right). $$

Theorem 2.4.7: If $c > b > 0,\ |x| < 1,\ m \ge 1$, then

$$ {}_{m+1}F_m\left(a,b,b+\frac{1}{m},\ldots,b+\frac{m-1}{m};c,c+\frac{1}{m},\ldots,c+\frac{m-1}{m};x\right) $$
$$ =\frac{\Gamma(mc)}{\Gamma(mb)\Gamma(mc-mb)}\int_0^1 t^{mb-1}(1-t)^{mc-mb-1}\left(1-xt^m\right)^{-a}dt. \quad (2.4.4) $$

Proof: Suppose that $|x| < 1$. Then

$$ {}_{m+1}F_m\left(a,b,b+\frac{1}{m},\ldots,b+\frac{m-1}{m};c,c+\frac{1}{m},\ldots,c+\frac{m-1}{m};x\right) $$
$$ =\sum_{j=0}^{\infty}\frac{(a)_j(b)_j\left(b+\frac{1}{m}\right)_j\ldots\left(b+\frac{m-1}{m}\right)_j}{(c)_j\left(c+\frac{1}{m}\right)_j\ldots\left(c+\frac{m-1}{m}\right)_j}\frac{x^j}{j!} $$
$$ =\frac{\Gamma(c)\Gamma\left(c+\frac{1}{m}\right)\ldots\Gamma\left(c+\frac{m-1}{m}\right)}{\Gamma(b)\Gamma\left(b+\frac{1}{m}\right)\ldots\Gamma\left(b+\frac{m-1}{m}\right)} $$
$$ \times\sum_{j=0}^{\infty}\frac{(a)_j\Gamma(b+j)\Gamma\left(b+j+\frac{1}{m}\right)\ldots\Gamma\left(b+j+\frac{m-1}{m}\right)}{\Gamma(c+j)\Gamma\left(c+j+\frac{1}{m}\right)\ldots\Gamma\left(c+j+\frac{m-1}{m}\right)}\frac{x^j}{j!} $$

since $(\alpha)_j=\dfrac{\Gamma(\alpha+j)}{\Gamma(\alpha)}$.

Now, we use (2.4.1) and obtain the following expressions.

$$ {}_{m+1}F_m\left(a,b,b+\frac{1}{m},\ldots,b+\frac{m-1}{m};c,c+\frac{1}{m},\ldots,c+\frac{m-1}{m};x\right) $$
$$ =\frac{\Gamma(mc)}{\Gamma(mb)\Gamma(mc-mb)}\sum_{j=0}^{\infty}\frac{(a)_j\Gamma(mb+mj)\Gamma(mc-mb)}{\Gamma(mc+mj)}\frac{x^j}{j!} $$
$$ =\frac{\Gamma(mc)}{\Gamma(mb)\Gamma(mc-mb)}\sum_{j=0}^{\infty}(a)_j\left(\int_0^1 t^{mb+mj-1}(1-t)^{mc-mb-1}dt\right)\frac{x^j}{j!} $$

$$=\frac{\Gamma(mc)}{\Gamma(mb)\Gamma(mc-mb)}\int_0^1 t^{mb-1}(1-t)^{mc-mb-1}\left(\sum_{j=0}^{\infty}(a)_j\frac{(xt^m)^j}{j!}\right)dt$$

$$=\frac{\Gamma(mc)}{\Gamma(mb)\Gamma(mc-mb)}\int_0^1 t^{mb-1}(1-t)^{mc-mb-1}\left(1-xt^m\right)^{-a}dt.$$

Theorem 2.4.8: If $c>b>0,\ a>0,\ c>\frac{1}{m}a,\ m\geq 2$, then

$$_{m+1}F_m\left(a,b,b+\frac{1}{m},...,b+\frac{m-1}{m};c,c+\frac{1}{m},...,c+\frac{m-1}{m};1\right)$$

$$=\frac{\Gamma(mc)\Gamma(mc-mb-a)}{\Gamma(mc-a)\Gamma(mc-mb)}$$

$$\times\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}\cdots\sum_{j_{m-2}=0}^{j_{m-3}}\binom{-a}{j_1}\binom{j_1}{j_2}\binom{j_2}{j_3}\cdots\binom{j_{m-3}}{j_{m-2}}$$

$$\times\frac{(mb)_{j_1}(mb+j_1)_{j_2}...(mb+j_1+j_2+...+j_{m-3})_{j_{m-2}}}{(mc-a)_{j_1}(mc-a+j_1)_{j_2}...(mc-a+j_1+j_2...+j_{m-3})_{j_{m-2}}}$$

$$\times{}_2F_1\left(-j_{m-2},mb+j_1+j_2+...+j_{m-2};mc-a+j_1+j_2+...+j_{m-2};-1\right).$$

Proof: Using $x=1$ in (2.4.4), we get the following expressions.

$$_{m+1}F_m\left(a,b,b+\frac{1}{m},...,b+\frac{m-1}{m};c,c+\frac{1}{m},...,c+\frac{m-1}{m};1\right)$$

$$=\frac{\Gamma(mc)}{\Gamma(mb)\Gamma(mc-mb)}\int_0^1 t^{mb-1}(1-t)^{mc-mb-1}\left(1-t^m\right)^{-a}dt.$$

$$=\frac{\Gamma(mc)}{\Gamma(mb)\Gamma(mc-mb)}$$

$$\times\int_0^1 t^{mb-1}(1-t)^{mc-mb-a-1}\left(1+t+t^2+...+t^{m-1}\right)^{-a}dt$$

$$=\frac{\Gamma(mc)}{\Gamma(mb)\Gamma(mc-mb)}$$

$$\times\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\int_0^1 t^{mb+j_1-1}(1-t)^{mc-mb-a-1}\left(1+t+t^2+...+t^{m-2}\right)^{j_1}dt$$

$$=\frac{\Gamma(mc)}{\Gamma(mb)\Gamma(mc-mb)}\times\sum_{j_1}^{\infty}\sum_{j_2=0}^{j_1}\binom{-a}{j_1}\binom{j_1}{j_2}$$

$$\times\int_0^1 t^{mb+j_1+j_2-1}(1-t)^{mc-mb-a-1}\left(1+t+t^2+...+t^{m-3}\right)^{j_2}\,dt$$

$$=\frac{\Gamma(mc)}{\Gamma(mb)\Gamma(mc-mb)}\times\sum_{j_1}^{\infty}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}\binom{-a}{j_1}\binom{j_1}{j_2}\binom{j_2}{j_3}$$

$$\times\int_0^1 t^{mb+j_1+j_2+j_3-1}(1-t)^{mc-mb-a-1}\left(1+t+t^2+...+t^{m-4}\right)^{j_3}\,dt\,.$$

Continue this process, we arrive at the following result

$${}_{m+1}F_m\left(a,b,b+\frac{1}{m},...,b+\frac{m-1}{m};c,c+\frac{1}{m},...,c+\frac{m-1}{m};1\right)$$

$$=\frac{\Gamma(mc)}{\Gamma(mb)\Gamma(mc-mb)}$$

$$\times\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}\cdots\sum_{j_{m-2}=0}^{j_{m-3}}\binom{-a}{j_1}\binom{j_1}{j_2}\binom{j_2}{j_3}\cdots\binom{j_{m-3}}{j_{m-2}}$$

$$\times\int_0^1 t^{mb+j_1+j_2+j_3+...+j_{m-2}-1}(1-t)^{mc-mb-a-1}(1+t)^{j_{m-2}}\,dt$$

$$=\frac{\Gamma(mc)}{\Gamma(mb)\Gamma(mc-mb)}$$

$$\times\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}\cdots\sum_{j_{m-2}=0}^{j_{m-3}}\binom{-a}{j_1}\binom{j_1}{j_2}\binom{j_2}{j_3}\cdots\binom{j_{m-3}}{j_{m-2}}$$

$$\times\sum_{j_{m-1}=0}^{j_{m-2}}\binom{j_{m-2}}{j_{m-1}}\int_0^1 t^{mb+j_1+j_2+j_3+...+j_{m-1}-1}(1-t)^{mc-mb-a-1}\,dt$$

$$=\frac{\Gamma(mc)}{\Gamma(mb)\Gamma(mc-mb)}$$

$$\times\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}\cdots\sum_{j_{m-2}=0}^{j_{m-3}}\binom{-a}{j_1}\binom{j_1}{j_2}\binom{j_2}{j_3}\cdots\binom{j_{m-3}}{j_{m-2}}$$

$$\times\sum_{j_{m-1}=0}^{j_{m-2}}\binom{j_{m-2}}{j_{m-1}}\frac{\Gamma(mb+j_1+j_2+j_3+...+j_{m-1})\Gamma(mc-mb-a)}{\Gamma(mc-a+j_1+j_2+j_3+...+j_{m-1})}$$

$$=\frac{\Gamma(mc)\Gamma(mc-mb-a)}{\Gamma(mc-a)\Gamma(mc-mb)}$$

$$\times\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}\cdots\sum_{j_{m-2}=0}^{j_{m-3}}\binom{-a}{j_1}\binom{j_1}{j_2}\binom{j_2}{j_3}\cdots\binom{j_{m-3}}{j_{m-2}}$$

$$\times\sum_{j_{m-1}=0}^{j_{m-2}}\frac{\left(-j_{m-2}\right)_{j_{m-1}}\left(mb\right)_{j_1+j_2+j_3+\ldots+j_{m-1}}}{\Gamma\left(mc-a\right)_{j_1+j_2+j_3+\ldots+j_{m-1}}}\frac{(-1)^{j_{m-1}}}{j_{m-1}!}$$

$$=\frac{\Gamma\left(mc\right)\Gamma\left(mc-mb-a\right)}{\Gamma\left(mc-a\right)\Gamma\left(mc-mb\right)}$$

$$\times\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}\cdots\sum_{j_{m-2}=0}^{j_{m-3}}\binom{-a}{j_1}\binom{j_1}{j_2}\binom{j_2}{j_3}\cdots\binom{j_{m-3}}{j_{m-2}}$$

$$\times\frac{\left(mb\right)_{j_1}\left(mb+j_1\right)_{j_2}\ldots\left(mb+j_1+j_2+\ldots+j_{m-3}\right)_{j_{m-2}}}{\left(mc-a\right)_{j_1}\left(mc-a+j_1\right)_{j_2}\ldots\left(mc-a+j_1+j_2\ldots+j_{m-3}\right)_{j_{m-2}}}$$

$$\times\sum_{j_{m-1}=0}^{j_{m-2}}\frac{\left(-j_{m-2}\right)_{j_{m-1}}\left(mb+j_1+j_2+j_3+\ldots+j_{m-2}\right)_{j_{m-1}}}{\left(mc-a+j_1+j_2+j_3+\ldots+j_{m-2}\right)_{j_{m-1}}}\frac{(-1)^{j_{m-1}}}{j_{m-1}!}$$

$$=\frac{\Gamma\left(mc\right)\Gamma\left(mc-mb-a\right)}{\Gamma\left(mc-a\right)\Gamma\left(mc-mb\right)}$$

$$\times\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}\cdots\sum_{j_{m-2}=0}^{j_{m-3}}\binom{-a}{j_1}\binom{j_1}{j_2}\binom{j_2}{j_3}\cdots\binom{j_{m-3}}{j_{m-2}}$$

$$\times\frac{\left(mb\right)_{j_1}\left(mb+j_1\right)_{j_2}\ldots\left(mb+j_1+j_2+\ldots+j_{m-3}\right)_{j_{m-2}}}{\left(mc-a\right)_{j_1}\left(mc-a+j_1\right)_{j_2}\ldots\left(mc-a+j_1+j_2\ldots+j_{m-3}\right)_{j_{m-2}}}$$

$$\times{}_2F_1\left(-j_{m-2},mb+j_1+j_2+\ldots+j_{m-2};mc-a+j_1+j_2+\ldots+j_{m-2};-1\right).$$

Corollary 2.4.9: If $c>b>0,\ c>\frac{1}{m}n,\ m\geq 2$, then

$${}_{m+1}F_m\left(-n,b,b+\frac{1}{m},\ldots,b+\frac{m-1}{m};c,c+\frac{1}{m},\ldots,c+\frac{m-1}{m};1\right)$$

$$=\frac{\left(mc-mb\right)_n}{\left(mc\right)_n}$$

$$\times\sum_{j_1=0}^{n}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}\cdots\sum_{j_{m-2}=0}^{j_{m-3}}\binom{n}{j_1}\binom{j_1}{j_2}\binom{j_2}{j_3}\cdots\binom{j_{m-3}}{j_{m-2}}$$

$$\times\frac{\left(mb\right)_{j_1}\left(mb+j_1\right)_{j_2}\ldots\left(mb+j_1+j_2+\ldots+j_{m-3}\right)_{j_{m-2}}}{\left(mc+n\right)_{j_1}\left(mc+n+j_1\right)_{j_2}\ldots\left(mc+n+j_1+j_2\ldots+j_{m-3}\right)_{j_{m-2}}}$$

$$\times{}_2F_1\left(-j_{m-2},mb+j_1+j_2+\ldots+j_{m-2};mc+n+j_1+j_2+\ldots+j_{m-2};-1\right).$$

2.5 RESULTS ON CONFLUENT HYPERGEOMETRIC FUNCTIONS

In this section, we evaluate a number of results on confluent hypergeometric functions by using the results in section 2.4.

Result 2.5.1: If $c > b > 0$, then

$$e^{-x}\,{}_2F_2\left(b, b+\frac{1}{2}; c, c+\frac{1}{2}; x\right)$$

$$= \left(\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right)\left(\sum_{j=0}^{\infty}\frac{(b)_j\left(b+\frac{1}{2}\right)_j}{(c)_j\left(c+\frac{1}{2}\right)_j}\frac{x^j}{j!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(-x)^{n-j}}{(n-j)!}\frac{(b)_j\left(b+\frac{1}{2}\right)_j}{(c)_j\left(c+\frac{1}{2}\right)_j}\frac{x^j}{j!}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\left(\frac{(-n)_j(b)_j\left(b+\frac{1}{2}\right)_j}{(c)_j\left(c+\frac{1}{2}\right)_j}\frac{1}{j!}\right)\frac{(-x)^n}{n!}$$

$$= \sum_{n=0}^{\infty}{}_3F_2\left(-n, b, b+\frac{1}{2}; c, c+\frac{1}{2}; 1\right)\frac{(-x)^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{(2c-2b)_n}{(2c)_n}\,{}_2F_1(-n, 2b; 2c+n; -1)\frac{(-x)^n}{n!}.$$

Hence, we obtain the following result

$${}_2F_2\left(b, b+\frac{1}{2}; c, c+\frac{1}{2},; x\right)$$

$$= e^{x}\sum_{n=0}^{\infty}\frac{(2c-2b)_n}{(2c)_n}\,{}_2F_1(-n, 2b; 2c+n; -1)\frac{(-x)^n}{n!}.$$

Result 2.5.2: If $c > b > 0$, then

$$e^{-x}\,{}_3F_3\left(b, b+\frac{1}{3}, b+\frac{2}{3}; c, c+\frac{1}{3}, c+\frac{2}{3}; x\right)$$

$$=\left(\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right)\left(\sum_{j=0}^{\infty}\frac{(b)_j\left(b+\frac{1}{3}\right)_j\left(b+\frac{2}{3}\right)_j}{(c)_j\left(c+\frac{1}{3}\right)_j\left(c+\frac{2}{3}\right)_j}\frac{x^j}{j!}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(-x)^{n-j}}{(n-j)!}\frac{(b)_j\left(b+\frac{1}{3}\right)_j\left(b+\frac{2}{3}\right)_j}{(c)_j\left(c+\frac{1}{3}\right)_j\left(c+\frac{2}{3}\right)_j}\frac{x^j}{j!}$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\left(\frac{(-n)_j(b)_j\left(b+\frac{1}{3}\right)_j\left(b+\frac{2}{3}\right)_j}{(c)_j\left(c+\frac{1}{3}\right)_j\left(c+\frac{2}{3}\right)_j}\frac{1}{j!}\right)\frac{(-x)^n}{n!}$$

$$=\sum_{n=0}^{\infty}{}_4F_3\left(-n,b,b+\frac{1}{3},b+\frac{2}{3};c,c+\frac{1}{3},c+\frac{2}{3};1\right)\frac{(-x)^n}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{(3c-3b)_n}{(3c)_n}\sum_{j_1=0}^{n}\binom{n}{j_1}\frac{(3b)_{j_1}}{(3c+n)_{j_1}}$$

$$\times{}_2F_1(-j_1,3b+j_1;3c+n+j_1;-1)\frac{(-x)^n}{n!}.$$

Hence,

$${}_3F_3\left(b,b+\frac{1}{3},b+\frac{2}{3};c,c+\frac{1}{3},c+\frac{2}{3};x\right)$$

$$=e^z\sum_{n=0}^{\infty}\frac{(3c-3b)_n}{(3c)_n}\sum_{j_1=0}^{n}\binom{n}{j_1}\frac{(3b)_{j_1}}{(3c+n)_{j_1}}$$

$$\times{}_2F_1(-j_1,3b+j_1;3c+n+j_1;-1)\frac{(-x)^n}{n!}.$$

Result 2.5.3: If $c>b>0,\ \ m\geq 2$, then

$$e^{-x}{}_mF_m\left(b,b+\frac{1}{m},\ldots,b+\frac{m-1}{m};c,c+\frac{1}{m},\ldots,c+\frac{m-1}{m};x\right)$$

$$=\left(\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right)\left(\sum_{j=0}^{\infty}\frac{(b)_j\left(b+\frac{1}{m}\right)_j\ldots\left(b+\frac{m-1}{m}\right)_j}{(c)_j\left(c+\frac{1}{m}\right)_j\ldots\left(c+\frac{m-1}{m}\right)_j}\frac{x^j}{j!}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{(-x)^{n-j}}{(n-j)!}\frac{(b)_j\left(b+\frac{1}{m}\right)_j\ldots\left(b+\frac{m-1}{m}\right)_j}{(c)_j\left(c+\frac{1}{m}\right)_j\ldots\left(c+\frac{m-1}{m}\right)_j}\frac{x^j}{j!}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\left(\frac{(-n)_j (b)_j \left(b+\frac{1}{m}\right)_j \cdots \left(b+\frac{m-1}{m}\right)_j}{(c)_j \left(c+\frac{1}{m}\right)_j \cdots \left(c+\frac{m-1}{m}\right)_j}\frac{1}{j!}\right)\frac{(-x)^n}{n!}.$$

This implies that

$$e^{-x}\,{}_mF_m\left(b,b+\frac{1}{m},\dots,b+\frac{m-1}{m};c,c+\frac{1}{m},\dots,c+\frac{m-1}{m};x\right)$$

$$= \sum_{n=0}^{\infty} {}_{m+1}F_m\left(-n,b,b+\frac{1}{m},\dots,b+\frac{m-1}{m};c,c+\frac{1}{m},\dots,c+\frac{m-1}{m};1\right)\frac{(-x)^n}{n!}.$$

Consequently, we conclude

$${}_mF_m\left(b,b+\frac{1}{m},\dots,b+\frac{m-1}{m};c,c+\frac{1}{m},\dots,c+\frac{m-1}{m};x\right)$$

$$= e^{x}\sum_{n=0}^{\infty}\frac{(mc-mb)_n}{(mc)_n}$$

$$\times \sum_{j_1=0}^{n}\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}\cdots\sum_{j_{m-2}=0}^{j_{m-3}}\binom{n}{j_1}\binom{j_1}{j_2}\binom{j_2}{j_3}\cdots\binom{j_{m-3}}{j_{m-2}}$$

$$\times \frac{(mb)_{j_1}(mb+j_1)_{j_2}\cdots(mb+j_1+j_2+\dots+j_{m-3})_{j_{m-2}}}{(mc+n)_{j_1}(mc+n+j_1)_{j_2}\cdots(mc+n+j_1+j_2\dots+j_{m-3})_{j_{m-2}}}$$

$$\times {}_2F_1\left(-j_{m-2},mb+j_1+j_2+\dots+j_{m-2};mc+n+j_1+j_2+\dots+j_{m-2};-1\right)$$

$$\times \frac{(-x)^n}{n!}.$$

2.6 $k-$ HYPERGEOMETRIC FUNCTIONS

At this juncture it is relevant to introduce the Pochhammer $k-$symbols defined by Diaz and Teruel (2005). Let

$$(x)_{m,k} = \prod_{j=0}^{m-1}(x+jk), k>0,\ (x)_m = \prod_{j=0}^{m-1}(x+j), \text{ when } k=1,$$

so that we can commence the study of generalized $k-$hypergeometric functions given by

$$ {}_rF_s\left[\begin{matrix}(\alpha_1,k_1),(\alpha_2,k_2),...,(\alpha_r,k_r)\\(\beta_1,k_1),(\beta_2,k_2),...,(\beta_s,k_s)\end{matrix};\ x\right]=\sum_{j=0}^{\infty}\frac{\prod_{u=1}^{r}(\alpha_u)_{j,k_u}}{\prod_{v=1}^{s}(\beta_v)_{j,k_v}}\frac{x^j}{j!} $$

$\forall\ \alpha_u,\beta_v,\ \beta_v\neq 0,-1,-2,...,|x|<1,\ k>0,\ u=1,2,...,r$ and $v=1,2,...,s.$

2.7 INTEGRAL REPRESENTATIONS OF $k-$HYPERGEOMETRIC FUNCTIONS

Diaz et al. (2007 and 2010) and others (See Mansour (2009), Kokologiannaki (2010), Kransniqi (2010) and Merovci (2010)) proved a number of identities of $k-$gamma function, $k-$beta function and Pochhammer $k-$symbols. They also gave integral representations of $k-$gamma function and $k-$beta function. In this section, we determine integral representations of various $k-$hypergeometric functions to be used later in this work.

The $k-$gamma function is defined by

$$ \Gamma_k(x)=\lim_{m\to\infty}\frac{m!k^m(mk)^{\frac{x}{k}-1}}{(x)_{m,k}}, $$

$(x)_{m,k}=\prod_{j=0}^{m-1}(x+jk),\ k>0$ is the Pochhammer $k-$symbols for factorial functions. It has been shown that the Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the $k-$gamma function, explicitly given by

$$ \Gamma_k(x)=k^{\frac{x}{k}-1}\Gamma\left(\tfrac{x}{k}\right)=\int_0^{\infty}t^{x-1}e^{-\frac{t^k}{k}}dt,\ \ x>0,\ \ k>0\,. $$

Clearly, $\Gamma(x)=\lim_{k\to1}\Gamma_k(x)$, $\Gamma_k(x)=k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right)$ and $\Gamma_k(x+k)=x\Gamma_k(x)$.

This gives rise to $k-$beta function defined by

$$ B_k(x,y)=\frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt,\ \ x>0,\ \ y>0 $$

so that $B_k(x,y)=\dfrac{1}{k}B\left(\dfrac{x}{k},\dfrac{y}{k}\right)$ and $B_k(x,y)=\dfrac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}$.

We take some following results which could be used in this study:

$$2^{2j}\left(\tfrac{x}{2}\right)_{j,k}\left(\tfrac{x+k}{2}\right)_{j,k}=(x)_{2j,k};$$

$$(x)_{mj,k}=m^{mj}\left(\tfrac{x}{m}\right)_{j,k}\left(\tfrac{x+k}{m}\right)_{j,k}\cdots\left(\tfrac{x+(m-1)k}{m}\right)_{j,k};$$

$$(x)_{j,k}=\tfrac{\Gamma_k(x+jk)}{\Gamma_k(x)};$$

$$(x)_{2j,k}=\tfrac{\Gamma_k(x+2jk)}{\Gamma_k(x)};$$

$$(x)_{mj,k}=\tfrac{\Gamma_k(x+mjk)}{\Gamma_k(x)};$$

$$\sum_{j=0}^{\infty}(\alpha)_{j,k}\frac{x^j}{j!}=(1-kx)^{-\frac{\alpha}{k}};$$

$$\sum_{j=0}^{\infty}\frac{x^j}{j!}=e^x .$$

We now prove the $k-$analogue of Theorem 2.3.2.

Theorem 2.7.1:
If $\gamma>\beta>0$, $k>0$ and $|x|<1$, then

$$ {}_2F_{1,k}\left[\begin{matrix}(\alpha,k),(\beta,k)\\(\gamma,k)\end{matrix};x\right] = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-kxt)^{-\frac{\alpha}{k}}\,dt . \tag{2.7.1}$$

Proof: First note that for any positive integer j, we have

$$\frac{(\beta)_{j,k}}{(\gamma)_{j,k}}=\frac{\Gamma_k(\beta+jk)}{\Gamma_k(\beta)}\times\frac{\Gamma_k(\gamma)}{\Gamma_k(\gamma+jk)}$$

$$=\frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}B_k(\beta+jk;\gamma-\beta)$$

$$=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}+j-1}(1-t)^{\frac{\gamma-\beta}{k}-1}\,dt . \tag{2.7.2}$$

Then, for $|x|<1$, the left hand side of (2.7.1) becomes

$${}_2F_{1,k}\left[\begin{matrix}(\alpha,k),(\beta,k)\\(\gamma,k)\end{matrix};x\right]=\sum_{j=0}^{\infty}\frac{(\alpha)_{j,k}(\beta)_{j,k}}{(\gamma)_{j,k}}\frac{x^j}{j!}.$$

Hence by using (2.7.2), we obtain the following expression.

$$
{}_2F_{1,k}\left[\begin{matrix}(\alpha,k),(\beta,k)\\(\gamma,k)\end{matrix};x\right]
$$

$$
=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}\sum_{j=0}^{\infty}\left(\frac{(\alpha)_{j,k}(xt)^j}{j!}\right)dt
$$

$$
=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-kxt)^{-\frac{\alpha}{k}}\,dt\,.
$$

Theorem 2.7.2:
If $\gamma>\beta>0$, $k>0$ and $|x|<1$, then

$$
{}_3F_{2,k}\left[\begin{matrix}(\alpha,k),\left(\frac{\beta}{2},k\right),\left(\frac{\beta+k}{2},k\right)\\\left(\frac{\gamma}{2},k\right),\left(\frac{\gamma+k}{2},k\right)\end{matrix};x\right]
$$

$$
=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}\left(1-kxt^2\right)^{-\frac{\alpha}{k}}\,dt\,. \quad (2.7.3)
$$

Proof: First note that for any positive integer j, we have

$$
\frac{(\beta)_{2j,k}}{(\gamma)_{2j,k}}=\frac{\Gamma_k(\beta+2jk)}{\Gamma_k(\beta)}\times\frac{\Gamma_k(\gamma)}{\Gamma_k(\gamma+2jk)}
$$

$$
=\frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}B_k(\beta+2jk;\gamma-\beta)
$$

$$
=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}+2j-1}(1-t)^{\frac{\gamma-\beta}{k}-1}\,dt\,. \quad (2.7.4)
$$

Then for $|x|<1$, the left hand side of (2.7.3) becomes

$$
{}_3F_{2,k}\left[\begin{matrix}(\alpha,k),\left(\frac{\beta}{2},k\right),\left(\frac{\beta+k}{2},k\right)\\\left(\frac{\gamma}{2},k\right),\left(\frac{\gamma+k}{2},k\right)\end{matrix};x\right]
$$

$$
=\sum_{j=0}^{\infty}\frac{(\alpha)_{j,k}\left(\frac{\beta}{2}\right)_{j,k}\left(\frac{\beta+k}{2}\right)_{j,k}}{\left(\frac{\gamma}{2}\right)_{j,k}\left(\frac{\gamma+k}{2}\right)_{j,k}}\frac{x^j}{j!}
$$

$$=\sum_{j=0}^{\infty}\frac{(\alpha)_{j,k}(\beta)_{2j,k}}{(\gamma)_{2j,k}}\frac{x^j}{j!}.$$

Hence, by using (2.7.4), we obtain the following

$${}_3F_{2,k}\left[\begin{matrix}(\alpha,k),\left(\frac{\beta}{2},k\right),\left(\frac{\beta+k}{2},k\right)\\ \left(\frac{\gamma}{2},k\right),\left(\frac{\gamma+k}{2},k\right)\end{matrix};x\right]$$

$$=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}\left(1-kxt^2\right)^{-\frac{\alpha}{k}}dt.$$

Theorem 2.7.3: If $\gamma>\beta>0,\ k>0,\ m\geq 1$ and $|x|<1$, then

$${}_{m+1}F_{m,k}\left(\begin{matrix}(\alpha,k),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};x\right)$$

$$=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}\left(1-kxt^m\right)^{-\frac{\alpha}{k}}dt. \quad (2.7.5)$$

Proof: First note that for any positive integer j, we have

$$\frac{(\beta)_{mj,k}}{(\gamma)_{mj,k}}=\frac{\Gamma_k(\gamma)\Gamma_k(\beta+mjk)}{\Gamma_k(\beta)\Gamma_k(\gamma+mjk)}$$

$$=\frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}B_k(\beta+mjk,\gamma-\beta)$$

$$=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}+mj-1}(1-t)^{\frac{\gamma-\beta}{k}-1}dt. \quad (2.7.6)$$

Using (2.7.5), we get

$${}_{m+1}F_{m,k}\left(\begin{matrix}(\alpha,k),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};x\right)$$

$$=\sum_{j=0}^{\infty}\frac{(\alpha)_{j,k}\left(\frac{\beta}{m}\right)_{j,k}\left(\frac{\beta+k}{m}\right)_{j,k}\cdots\left(\frac{\beta+k}{m}\right)_{j,k}}{\left(\frac{\gamma}{m}\right)_{j,k}\left(\frac{\gamma+k}{m}\right)_{j,k}\cdots\left(\frac{\gamma+k}{m}\right)_{j,k}}\frac{x^j}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{(\alpha)_{j,k}(\beta)_{mj,k}}{(\gamma)_{mj,k}} \frac{x^j}{j!}$$

$$= \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} \int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}\left(1-kxt^m\right)^{-\frac{\alpha}{k}} dt.$$

Corollary 2.7.4: If $\gamma > \beta > 0,\ k > 0,\ m \geq 1$ and $|x| < 1$, then

$$_{m+1}F_{m,k}\left(\begin{matrix}\left(\frac{\alpha}{m},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right) \\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};x\right)$$

$$= \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} \int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}\left(1-xt^m\right)^{-\frac{\alpha}{m}} dt.$$

2.8 INTEGRAL REPRESENTATIONS OF CONFLUENT $k-$HYPERGEOMETRIC FUNCTIONS

In this section, we introduce a simple integral representation of certain confluent $k-$hypergeometric functions.

Theorem 2.8.1:
If $\gamma > \beta > 0,\ k > 0$, then for all finite x

$$_1F_{1,k}\left[\begin{matrix}(\beta,k) \\ (\gamma,k)\end{matrix};x\right] = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} \int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1} e^{xt} dt. \tag{2.8.1}$$

Proof: First note that for any positive integer j, we have the following expressions

$$\frac{(\beta)_{j,k}}{(\gamma)_{j,k}} = \frac{\Gamma_k(\beta+jk)}{\Gamma_k(\beta)} \times \frac{\Gamma_k(\gamma)}{\Gamma_k(\gamma+jk)}$$

$$= \frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} B_k(\beta+jk;\gamma-\beta)$$

$$= \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} \int_0^1 t^{\frac{\beta}{k}+j-1}\left(1-t\right)^{\frac{\gamma-\beta}{k}-1} dt. \tag{2.8.2}$$

Then for all finite x, the left hand side of (2.8.1) becomes

$$ {}_1F_{1,k}\left[\begin{matrix}(\beta,k)\\(\gamma,k)\end{matrix};x\right]=\sum_{j=0}^{\infty}\frac{(\beta)_{j,k}}{(\gamma)_{j,k}}\frac{x^j}{j!}. $$

Hence by using (2.8.2), we obtain the following expression.

$$ {}_1F_{1,k}\left[\begin{matrix}(\beta,k)\\(\gamma,k)\end{matrix};x\right]=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}\sum_{j=0}^{\infty}\left(\frac{(xt)^j}{j!}\right)dt $$

$$ =\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}e^{xt}dt. $$

Theorem 2.8.2: If $\gamma>\beta>0,\ k>0$, then for all finite x

$$ {}_2F_{2,k}\left[\begin{matrix}\left(\frac{\beta}{2},k\right),\left(\frac{\beta+k}{2},k\right)\\\left(\frac{\gamma}{2},k\right),\left(\frac{\gamma+k}{2},k\right)\end{matrix};x\right] $$

$$ =\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}e^{xt^2}dt. \tag{2.8.3} $$

Proof: For any positive integer j, we have

$$ \frac{(\beta)_{2j,k}}{(\gamma)_{2j,k}}=\frac{\Gamma_k(\beta+2jk)}{\Gamma_k(\beta)}\times\frac{\Gamma_k(\gamma)}{\Gamma_k(\gamma+2jk)} $$

$$ =\frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}B_k(\beta+2jk;\gamma-\beta) $$

$$ =\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}+2j-1}(1-t)^{\frac{\gamma-\beta}{k}-1}dt \tag{2.8.4} $$

Then for all finite x, the left hand side of (2.8.3) becomes

$$ {}_2F_{2,k}\left[\begin{matrix}\left(\frac{\beta}{2},k\right),\left(\frac{\beta+k}{2},k\right)\\\left(\frac{\gamma}{2},k\right),\left(\frac{\gamma+k}{2},k\right)\end{matrix};x\right] $$

$$ =\sum_{j=0}^{\infty}\frac{\left(\frac{\beta}{2}\right)_{j,k}\left(\frac{\beta+k}{2}\right)_{j,k}}{\left(\frac{\gamma}{2}\right)_{j,k}\left(\frac{\gamma+k}{2}\right)_{j,k}}\frac{x^j}{j!} $$

$$ =\sum_{j=0}^{\infty}\frac{(\beta)_{2j,k}}{(\gamma)_{2j,k}}\frac{x^j}{j!}. $$

From (2.8.4), we obtain the following expression.

$${}_2F_{2,k}\left[\begin{matrix}\left(\frac{\beta}{2},k\right),\left(\frac{\beta+k}{2},k\right)\\ \left(\frac{\gamma}{2},k\right),\left(\frac{\gamma+k}{2},k\right)\end{matrix};x\right]=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}e^{xt^2}\,dt\,.$$

Theorem 2.8.3: If $\gamma>\beta>0,\ k>0,\ m\ge 1$, then for all finite x

$${}_mF_{m,k}\left(\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};x\right)$$

$$=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}e^{xt^m}\,dt\,. \qquad (2.8.5)$$

Proof: First note that for any positive integer j, we get

$$\frac{(\beta)_{mj,k}}{(\gamma)_{mj,k}}=\frac{\Gamma_k(\gamma)\Gamma_k(\beta+mjk)}{\Gamma_k(\beta)\Gamma_k(\gamma+mjk)}$$

$$=\frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}B_k(\beta+mjk,\gamma-\beta)$$

$$=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}+mj-1}(1-t)^{\frac{\gamma-\beta}{k}-1}\,dt \qquad (2.8.6)$$

Now, using (2.8.5), we get

$${}_mF_{m,k}\left(\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};x\right)$$

$$=\sum_{j=0}^{\infty}\frac{\left(\frac{\beta}{m}\right)_{j,k}\left(\frac{\beta+k}{m}\right)_{j,k}\cdots\left(\frac{\beta+k}{m}\right)_{j,k}}{\left(\frac{\gamma}{m}\right)_{j,k}\left(\frac{\gamma+k}{m}\right)_{j,k}\cdots\left(\frac{\gamma+k}{m}\right)_{j,k}}\frac{x^j}{j!}$$

$$=\sum_{j=0}^{\infty}\frac{(\beta)_{mj,k}}{(\gamma)_{mj,k}}\frac{x^j}{j!}$$

$$=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}e^{xt^m}\,dt\,.$$

CHAPTER 3

$k-$FRACTIONAL INTEGRALS

With the introduction of $k-$gamma function, it is natural to redefine the usual fractional integrals to align with it and to attempt to prove a few of its properties analogous to known fractional integrals.

3.1 RIEMANN- LIOUVILLE $k-$FRACTIONAL INTEGRAL

Most of the functions involving or based on gamma function can be refined by using $k-$gamma function. For example, $k-$zeta and $k-$Mittag-Leffler functions could be defined respectively by the formulae

$$\zeta_k(x,s)=\sum_{j=0}^{\infty}\frac{1}{(x+jk)^s}, \qquad k,\ x>0,\ s>1$$

and

$$E_k^{\alpha,\beta}(x)=\frac{1}{k}\sum_{j=0}^{\infty}\frac{x^j}{\Gamma_k(\alpha j+\beta)}, \qquad \alpha,\ \beta>0.$$

The $k-$gamma also leads to another interesting direction, a variant of Riemann-Liouville fractional integral defined by

$$I_k^{\alpha}(f(x))=\frac{1}{k\Gamma_k(\alpha)}\int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t)\,dt, \quad 0<t<x<\infty.$$

It will henceforth be called $k-$ **fractional integral**. Note that when $k\to 1$, it reduces to the classical Riemann-Liouville fractional integral

$$I^{\alpha}(f(x))=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\,dt, \quad 0<t<x<\infty.$$

Thinking simple, let C_0 be the class of all functions which are continuous and integrable on the interval $(0,\infty)$. Then, $I_k^{\alpha} f$ exists in C_0 if $f\in C_0$. Define formally $I_k^{\alpha}(f), \alpha<0$, to be the solution, if exists, of the equation $f=I_k^{-\alpha} g$. Clearly, $I_k^{\alpha} f=I_k^{\alpha} g$ implies $f=g$ (See Kober (1940)).

3.2 PROPERTIES OF $k-$FRACTIONAL INTEGRALS

In this section, we prove a number of useful properties of $k-$fractional integrals.

Theorem 3.2.1: Let $\alpha>0, \beta>0, f\in C_0$, then

$$I_k^{\alpha}\left(I_k^{\beta}\left(f(x)\right)\right)=I_k^{\alpha+\beta}\left(f(x)\right)=I_k^{\beta+\alpha}\left(f(x)\right). \quad (3.2.1)$$

Proof: Note that by changing the order of integration (4.1.5), we have

$$I_k^{\alpha}\left(I_k^{\beta}\left(f(x)\right)\right)=\frac{1}{k\Gamma_k(\alpha)}\int_0^x (x-t)^{\frac{\alpha}{k}-1}\left(I_k^{\beta}\left(f(t)\right)\right)dt$$

$$=\frac{1}{k\Gamma_k(\alpha)}\int_0^x (x-t)^{\frac{\alpha}{k}-1}\left(\frac{1}{k\Gamma_k(\beta)}\int_0^t (t-u)^{\frac{\beta}{k}-1}f(u)\,du\right)dt$$

$$=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^x (x-t)^{\frac{\alpha}{k}-1}\left(\int_0^t (t-u)^{\frac{\beta}{k}-1}f(u)\,du\right)dt$$

$$=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^x f(u)\left(\int_u^x (x-t)^{\frac{\alpha}{k}-1}(t-u)^{\frac{\beta}{k}-1}dt\right)du$$

$$=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^x f(u)$$

$$\times\left(\int_u^x \left[(x-u)-(t-u)\right]^{\frac{\alpha}{k}-1}(t-u)^{\frac{\beta}{k}-1}dt\right)du$$

$$=\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^x (x-u)^{\frac{\alpha}{k}-1}f(u)$$

$$\times\left(\int_u^x \left(1-\frac{(t-u)}{(x-u)}\right)^{\frac{\alpha}{k}-1}(t-u)^{\frac{\beta}{k}-1}dt\right)du. \quad (3.2.2)$$

Consider the integral $\int_u^x \left(1-\frac{(t-u)}{(x-u)}\right)^{\frac{\alpha}{k}-1}(t-u)^{\frac{\beta}{k}-1}dt$.

Put $s=\frac{(t-u)}{(x-u)}$ in the above integral to get the following equation

$$\int_u^x \left(1-\frac{(t-u)}{(x-u)}\right)^{\frac{\alpha}{k}-1} (t-u)^{\frac{\beta}{k}-1} dt = (x-u)^{\frac{\beta}{k}} \int_0^1 (1-s)^{\frac{\alpha}{k}-1} s^{\frac{\beta}{k}-1} dt .$$

Equation (3.2.2) then becomes

$$I_k^{\alpha}\left(I_k^{\beta}\left(f(x)\right)\right)$$

$$= \frac{1}{k^2 \Gamma_k(\alpha)\Gamma_k(\beta)} \int_0^x (x-u)^{\frac{\alpha}{k}+\frac{\beta}{k}-1} f(u) \left(\int_0^1 (1-s)^{\frac{\alpha}{k}-1} s^{\frac{\beta}{k}-1} dt \right) du$$

$$= \frac{1}{k^2 \Gamma_k(\alpha)\Gamma_k(\beta)} \int_0^x (x-u)^{\frac{\alpha}{k}+\frac{\beta}{k}-1} f(u) \left(\frac{k\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} \right) du$$

$$= \frac{1}{k\Gamma_k(\alpha+\beta)} \int_0^x (x-u)^{\frac{\alpha}{k}+\frac{\beta}{k}-1} f(u) du$$

$$= I_k^{\alpha+\beta}\left(f(x)\right) = I_k^{\beta+\alpha}\left(f(x)\right).$$

Theorem 3.2.2: Suppose that $I_k^{\alpha}, I_k^{\beta}, I_k^{\alpha} I_k^{\beta}$ and $I_k^{\beta} I_k^{\alpha}$ exist, then $I_k^{\alpha} I_k^{\beta} = I_k^{\beta} I_k^{\alpha}$ for all real numbers α and β.

Proof:

Case (i): If $\alpha \geq 0,\ \beta \geq 0$, then it holds by Theorem 3.2.1.

Case (ii): If $\alpha < 0,\ \beta < 0$, then

$$I_k^{\alpha} I_k^{\beta} f(x) = g(x) \Rightarrow I_k^{\beta} f(x) = I_k^{-\alpha} g(x)$$

$$\Rightarrow f(x) = I_k^{-\beta} I_k^{-\alpha} g(x)$$

$$\Rightarrow f(x) = I_k^{-\alpha} I_k^{-\beta} g(x) \text{ by case (i)}$$

$$\Rightarrow I_k^{\alpha} f(x) = I_k^{-\beta} g(x)$$

$$\Rightarrow I_k^{\beta} I_k^{\alpha} f(x) = g(x).$$

Hence, $\quad I_k^{\alpha} I_k^{\beta} f(x) = I_k^{\beta} I_k^{\alpha} f(x).$

Case (iii): If $\alpha < 0,\ \beta > 0$, then

$$I_k^{\alpha} f(x) = g(x) \Rightarrow f(x) = I_k^{-\alpha} g(x)$$

$$\Rightarrow I_k^{\beta} f(x) = I_k^{\beta} I_k^{-\alpha} g(x) \text{ by case (i)}$$

$$\Rightarrow I_k^{\beta} f(x) = I_k^{-\alpha} I_k^{\beta} g(x)$$

$$\Rightarrow I_k^{\alpha} I_k^{\beta} f(x) = I_k^{\beta} I_k^{\alpha} f(x), \text{ where } g(x) = I_k^{\alpha} f(x).$$

Case (iv): If $\alpha > 0,\ \beta < 0$, then interchange the role of α and β in (iii) to obtain $I_k^{\alpha} I_k^{\beta} f(x) = I_k^{\beta} I_k^{\alpha} f(x)$.

Theorem 3.2.3: Let $I_k^{\alpha}\left(f(x)\right) = \dfrac{1}{k\Gamma_k(\alpha)} \displaystyle\int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t)\,dt.$

If $\alpha > 0$, then

$$I_k^{\alpha}\left(t^{\frac{\beta}{k}-1}\right) = \frac{\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} x^{\frac{\alpha}{k}+\frac{\beta}{k}-1}.$$

Proof: We have $I_k^{\alpha}\left(t^{\frac{\beta}{k}-1}\right) = \dfrac{1}{k\Gamma_k(\alpha)} \displaystyle\int_0^x (x-t)^{\frac{\alpha}{k}-1} t^{\frac{\beta}{k}-1} dt.$ (3.2.3)

Substitute $t = ux$ in (3.2.3), we get

$$\begin{aligned} I_k^{\alpha}\left(t^{\frac{\beta}{k}-1}\right) &= \frac{x^{\frac{\alpha}{k}+\frac{\beta}{k}-1}}{k\Gamma_k(\alpha)} \int_0^1 (1-u)^{\frac{\alpha}{k}-1} u^{\frac{\beta}{k}-1} du \\ &= \frac{x^{\frac{\alpha}{k}+\frac{\beta}{k}-1}}{\Gamma_k(\alpha)} \frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} \\ &= \frac{\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} x^{\frac{\alpha}{k}+\frac{\beta}{k}-1}. \end{aligned}$$

Theorem 3.2.4: Suppose that $I_k^{\alpha} f(x),\ \ I_k^{\beta} f(x)$ exists, then $I_k^{\alpha} I_k^{\beta} f(x) = I_k^{\alpha+\beta} f(x)$ for all real numbers α and β.

Proof:
Case (i): If $\alpha \geq 0,\ \beta \geq 0$, then it holds by Theorem 3.2.1.

Case (ii): If $\alpha < 0, \beta < 0$ and $I_k^{\alpha} I_k^{\beta} f(x)$ exists, then

$$\begin{aligned} & I_k^{\alpha} I_k^{\beta} f(x) = g(x) \Rightarrow I_k^{\beta} f(x) = I_k^{-\alpha} g(x) \\ & \Rightarrow f(x) = I_k^{-\beta} I_k^{-\alpha} g(x) \\ & \Rightarrow f(x) = I_k^{-(\beta+\alpha)} g(x) \text{ by case (i)} \\ & \Rightarrow f(x) = I_k^{-(\alpha+\beta)} g(x) \\ & \Rightarrow I_k^{\alpha+\beta} f(x) = g(x). \end{aligned}$$

Hence, $\quad I_k^{\alpha} I_k^{\beta} f(x) = I_k^{\alpha+\beta} f(x).$

Case (iii): If $\alpha > 0, \beta < 0, \alpha + \beta > 0$ and $I_k^{\alpha+\beta} f(x)$ exists, then

$$I_k^{\alpha+\beta} f(x) = g(x) \Rightarrow I_k^{-\beta} I_k^{\alpha+\beta} f(x) = I_k^{-\beta} g(x)$$
$$\Rightarrow I_k^{\alpha+\beta-\beta} f(x) = I_k^{-\beta} g(x)$$
$$\Rightarrow I_k^{\alpha} f(x) = I_k^{-\beta} g(x)$$
$$\Rightarrow I_k^{\beta} I_k^{\alpha} f(x) = g(x).$$

Hence, $\quad I_k^{\beta} I_k^{\alpha} f(x) = g(x) = I_k^{\alpha+\beta} f(x).$

Case (iv): If $\alpha > 0, \beta < 0, \alpha + \beta < 0$ and $I_k^{\alpha+\beta} f(x)$ exists, then

$$I_k^{\alpha+\beta} f(x) = g(x) \Rightarrow f(x) = I_k^{-(\alpha+\beta)} g(x)$$
$$\Rightarrow I_k^{\alpha} f(x) = I_k^{-\beta} g(x)$$
$$\Rightarrow I_k^{\beta} I_k^{\alpha} f(x) = g(x).$$

Hence, $\quad I_k^{\beta} I_k^{\alpha} f(x) = g(x) = I_k^{\alpha+\beta} f(x).$

Case (v and vi): For $\alpha < 0, \beta > 0, \alpha + \beta > 0, \alpha + \beta < 0$ and $I_k^{\alpha+\beta} f(x)$ exists, then interchange the role of α and β in (iii and iv).

Theorem 3.2.5: Let $I_k^{\alpha}\left(f(x)\right) = \dfrac{1}{k\Gamma_k(\alpha)} \displaystyle\int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t)\,dt.$

If $\alpha > 0$, then

$$I_k^{\alpha}\left((x-u)^{\frac{\beta}{k}-1} \right) = \frac{\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} (x-u)^{\frac{\alpha}{k}+\frac{\beta}{k}-1}.$$

Proof: We have

$$I_k^{\alpha}\left((x-u)^{\frac{\beta}{k}-1} \right)$$
$$= \frac{1}{k\Gamma_k(\alpha)} \int_u^x (x-t)^{\frac{\alpha}{k}-1} (t-u)^{\frac{\beta}{k}-1}\,dt \qquad (3.2.4)$$
$$= \frac{1}{k\Gamma_k(\alpha)} \int_u^x \left((x-u)-(t-u)\right)^{\frac{\alpha}{k}-1} (t-u)^{\frac{\beta}{k}-1}\,dt$$
$$= \frac{(x-u)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} \int_u^x \left(1 - \frac{(t-u)}{(x-u)}\right)^{\frac{\alpha}{k}-1} (t-u)^{\frac{\beta}{k}-1}\,dt.$$

Put $t = ux$ in Equation (3.2.4) to get

$$I_k^{\alpha}\left((x-u)^{\frac{\beta}{k}-1}\right)$$

$$=\frac{(x-u)^{\frac{\alpha}{k}+\frac{\beta}{k}-1}}{\Gamma_k(\alpha)}\frac{1}{k}\int_0^1(1-s)^{\frac{\alpha}{k}-1}s^{\frac{\beta}{k}-1}dt$$

$$=\frac{(x-u)^{\frac{\alpha}{k}+\frac{\beta}{k}-1}}{\Gamma_k(\alpha)}\frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}$$

$$=\frac{\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}(x-u)^{\frac{\alpha}{k}+\frac{\beta}{k}-1}.$$

Theorem 3.2.6: Let $I_k^{\alpha,\beta}(f(x))=\frac{x^{-\frac{(\alpha+\beta)}{k}}}{k\Gamma_k(\alpha)}\int_0^x(x-t)^{\frac{\alpha}{k}-1}t^{\frac{\beta}{k}}f(t)dt$.

If $\alpha>0,\ \beta>0,\ \lambda>0,\ f\in C_0$, then

$$I_k^{\lambda,\alpha+\beta}\left(I_k^{\alpha,\beta}(f(x))\right)=I_k^{\alpha+\lambda,\beta}(f(x)).$$

Proof:

$$I_k^{\lambda,\alpha+\beta}\left(I_k^{\alpha,\beta}(f(x))\right)$$

$$=\frac{x^{-\frac{(\alpha+\beta+\lambda)}{k}}}{k\Gamma_k(\lambda)}\int_0^x(x-t)^{\frac{\lambda}{k}-1}t^{\frac{\alpha+\beta}{k}}\left(I_k^{\alpha,\beta}(f(t))\right)dt$$

$$=\frac{x^{-\frac{(\alpha+\beta+\lambda)}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\lambda)}\int_0^x(x-t)^{\frac{\lambda}{k}-1}\left(\int_0^t(t-u)^{\frac{\alpha}{k}-1}u^{\frac{\beta}{k}}f(u)du\right)dt$$

$$=\frac{x^{-\frac{(\alpha+\beta+\lambda)}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\lambda)}\int_0^x u^{\frac{\beta}{k}}f(u)\left(\int_u^x(x-t)^{\frac{\lambda}{k}-1}(t-u)^{\frac{\alpha}{k}-1}dt\right)du$$

$$=\frac{x^{-\frac{(\alpha+\beta+\lambda)}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\lambda)}$$

$$\times\int_0^x u^{\frac{\beta}{k}}f(u)\left(\int_u^x\left((x-u)-(t-u)\right)^{\frac{\lambda}{k}-1}(t-u)^{\frac{\alpha}{k}-1}dt\right)du$$

$$=\frac{x^{-\frac{(\alpha+\beta+\lambda)}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\lambda)}\int_0^x u^{\frac{\beta}{k}}f(u)(x-u)^{\frac{\lambda}{k}-1}$$

$$\times\left(\int_u^x\left(1-\frac{t-u}{x-u}\right)^{\frac{\lambda}{k}-1}(t-u)^{\frac{\alpha}{k}-1}dt\right)du. \qquad (3.2.5)$$

Consider the integral $\int_u^x \left(1-\frac{t-u}{x-u}\right)^{\frac{\lambda}{k}-1} (t-u)^{\frac{\alpha}{k}-1} dt$.

Using $s=\frac{(t-u)}{(x-u)}$ in the above integral, we have the following equation

$$\int_u^x \left(1-\frac{t-u}{x-u}\right)^{\frac{\lambda}{k}-1} (t-u)^{\frac{\alpha}{k}-1} dt = \int_0^1 (1-s)^{\frac{\lambda}{k}-1} s^{\frac{\alpha}{k}-1} (x-u)^{\frac{\alpha}{k}} dt .$$

Equation (3.2.5) becomes

$$\begin{aligned}
&I_k^{\lambda,\alpha+\beta}\left(I_k^{\alpha,\beta}\left(f(x)\right)\right) \\
&= \frac{x^{-\frac{(\alpha+\beta+\lambda)}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\lambda)} \int_0^x (x-u)^{\frac{\alpha}{k}+\frac{\lambda}{k}-1} u^{\frac{\beta}{k}} f(u) \left(\int_0^1 (1-s)^{\frac{\lambda}{k}-1} s^{\frac{\alpha}{k}-1} ds \right) du \\
&= \frac{x^{-\frac{(\alpha+\beta+\lambda)}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\lambda)} \int_0^x (x-u)^{\frac{\alpha}{k}+\frac{\lambda}{k}-1} u^{\frac{\beta}{k}} f(u) \left(\frac{k\Gamma_k(\alpha)\Gamma_k(\lambda)}{\Gamma_k(\alpha+\lambda)} \right) du \\
&= \frac{x^{-\frac{(\alpha+\beta+\lambda)}{k}}}{k\Gamma_k(\alpha+\lambda)} \int_0^x (x-u)^{\frac{\alpha}{k}+\frac{\lambda}{k}-1} u^{\frac{\beta}{k}} f(u) du \\
&= I_k^{\alpha+\lambda,\beta}\left(f(x)\right).
\end{aligned}$$

CHAPTER 4

$L^p -$BOUNDEDNESS OF INTEGRAL OPERATORS INVOLVING ${}_{m+1}F_{m,k}$, $m \geq 1$ AS KERNEL

In this chapter, we consider integral operators involving homogeneous functions as kernel and discuss the $L^p -$boundedness of these integral operators by using properties of hypergeometric functions ${}_{m+1}F_{m,k}$, $m \geq 1$ as kernel.

4.1 BASIC RESULTS

We, first, define some terms and state some basic results on boundedness of integral operators in L^p. These are extensions of famous Hardy's inequality.

4.1.1 L^p - SPACES

Let R, R^+ denote the set of real numbers and the set of positive real numbers respectively.

$L^p(R)$ is the class of measurable functions whose pth powers, $p \geq 1$, are integrable on R so that $f \in L^p(R)$ if and only if $\int_R |f|^p dx$ is finite, where p is any finite positive real number. The positive number $\|f\|_p$ is defined by

$$\|f\|_p = \left(\int_R |f|^p dx\right)^{1/p}. \tag{4.1.1}$$

Functions in L^p have the following properties:

i. (Hölder's Inequality). If $f \in L^p(R)$, where $1 \leq p < \infty$, and $g \in L^{p'}(R)$, $(1/p)+(1/p')=1$, then $fg \in L^1(R)$, and $\int_R |fg|^p dx \leq \|f\|_p \|g\|_{p'}$.
   For $1 \leq p < \infty$, the expressions are equal if and only if there is a constant C such that $|g(x)|^p = C|f(x)|^p$ for all x in R.
ii. If $f \in L^p(R)$, $g \in L^q(R)$, where $p > 0$, $q > 0$, $1/r = (1/p)+(1/q)$, then
$$fg \in L^r(R) \text{ and } \|fg\|_r \leq \|f\|_p \|g\|_q. \tag{4.1.2}$$

iii. (Minkowski's Inequality). IfY $f \in L^p(R), g \in L^p(R)$, where $1 \le p < \infty$, then $(f+g) \in L^p(R)$ and $\|f+g\|_p \le \|f\|_p + \|g\|_p$. (4.1.3)

iv. (Minkowski's Integral Inequality). Let X and Y be subsets of R, and let $f(x,y)$ be a measurable function such that $f(.,y) \in L^p(X)$ for each point $y \in Y$.
Then, for $1 \le p < \infty$,

$$\left(\int_X \left|\int_Y f(x,y)\,dy\right|^p dx\right)^{1/p} \le \int_Y \left(\int_X |f(x,y)|^p dx\right)^{1/p} dy \qquad (4.1.4)$$

whenever the expressions are finite.

v. (Fubini's Theorem). Let $f(x,y)$ be non-negative, measurable function $X \times Y$, $f_x(y) = f(x,y)$ is measurable on Y and $f_y(x) = f(x,y)$ is measurable on X,

then $$\left(\int_X \left(\int_Y f(x,y)\,dx\right) dy\right) \le \int_Y \left(\int_X f(x,y)\,dx\right) dy. \qquad (4.1.5)$$

Lemma 4.1.2: Let $p > 1$. Then $f \in L^p(R^+)$, there exists a function V, $V(f)(x) = x^{-2/p} f(\frac{1}{x})$ defined on R^+ such that $V^2(f) = f$ and $V(f): L^p \to L^p$.

A function $\psi(x,t)$ defined on R^2 is said to be homogeneous of degree μ if $|\psi(hx,ht)| = |h|^\mu |\psi(x,t)|$.

Theorem 4.1.3 (Hardy et al. (1952)): Let p be real numbers such that $p > 1$ and let $\psi(x,t)$ be a function which is measurable in the variable t and for each fixed x in R^+ such that, for $h \ne 0$,

$$|\psi(hx,ht)| = |h|^{-1} |\psi(x,t)| .$$

If $\nu > 1$, $f \in L^p(R^+)$, $g \in L^{p'}(R^+)$, $\psi(x,.)f$ and $\psi(.,t)g$ are integrable.

(a) Let $\Psi(f)(x) = \int_0^\infty f(t)\psi(x,t)\,dt$.

Then

(i) $\Psi : L^p(R^+) \to L^p(R^+)$

(ii) $\left(\int_0^\infty x^\nu \left|\Psi(f)(x)\right|^p dx\right)^{1/p} \le C\left(\int_0^\infty x^\nu \left|f(x)\right|^p dt\right)^{1/p},$

where. $C = \int_0^\infty u^{-(1/p)-(\nu/p)}\left|\psi(1,u)\right| du = \int_0^\infty u^{(1/p)+(\nu/p)-1}\left|\psi(u,1)\right| du$

(b) Further, if the operator Ψ^* is defined by

$$\Psi^*(g)(t) = \int_0^\infty g(x)\psi(x,t)\,dx.$$

Then

(i) $\Psi^* : L^{p'}\left(R^+\right) \to L^{p'}\left(R^+\right)$

(ii) $\left(\int_0^\infty t^\nu \left|\Psi^*(g)(t)\right|^{p'} dt\right)^{1/p'} \le C\left(\int_0^\infty t^\nu \left|g(t)\right|^{p'} dt\right)^{1/p'}$

Proof: (a) (i) When $\nu = 0$ (ii) implies (i).

(ii) Consider $\Psi(f)(x) = \int_0^\infty f(t)\psi(x,t)\,dt$.

Now

$$\left|\Psi(f)(x)\right| = \left|\int_0^\infty f(t)\psi(x,t)\,dt\right|$$

$$\le \int_0^\infty \left|f(t)\right| t^{-((1+\nu)/p)(1-(1/p))} t^{((1+\nu)/p)(1-(1/p))} \left|\psi(x,t)\right|^{1-(1/p)} \left|\psi(x,t)\right|^{1/p} dt$$

$$= \left(\int_0^\infty t^{-((1+\nu)/p)}\left|\psi(x,t)\right| dt\right)^{1-(1/p)} \left(\int_0^\infty t^{(1+\nu)(1-(1/p))}\left|f(t)\right|^p \left|\psi(x,t)\right| dt\right)^{1/p}. \quad (4.1.6)$$

Consider the following integral

$$\int_0^\infty t^{-((1+\nu)/p)}\left|\psi(x,t)\right| dt. \quad (4.1.7)$$

Substitute $t = ux$ in (4.1.7) to have

$$\int_0^\infty t^{-((1+\nu)/p)}\left|\psi(x,t)\right| dt = x^{-((1+\nu)/p)}\int_0^\infty u^{-((1+\nu)/p)}\left|\psi(1,u)\right| du$$

$$= x^{-((1+\nu)/p)}C,$$

where $C = \int_0^\infty u^{-((1+\nu)/p)}\left|\psi(1,u)\right| du$.

Equation (4.1.6) becomes

$$\left|\Psi(f)(x)\right| \le C^{1-(1/p)} x^{-((1+\nu)/p)(1-(1/p))} \left(\int_0^\infty t^{(1+\nu)(1-(1/p))} \left|f(t)\right|^p \left|\psi(x,t)\right| dt \right)^{1/p}$$

$$\left|\Psi(f)(x)\right|^p \le C^{p-1} x^{-(1+\nu)(1-(1/p))} \int_0^\infty t^{(1+\nu)(1-(1/p))} \left|f(t)\right|^p \left|\psi(x,t)\right| dt$$

$$x^\nu \left|\Psi(f)(x)\right|^p \le C^{p-1} x^{-(1+\nu)(1-(1/p))+\nu} \int_0^\infty t^{(1+\nu)(1-(1/p))} \left|f(t)\right|^p \left|\psi(x,t)\right| dt$$

$$\le C^{p-1} x^{(1/p)+(\nu/p)-1} \int_0^\infty t^{1+\nu-(1/p)-(\nu/p)} \left|f(t)\right|^p \left|\psi(x,t)\right| dt,$$

so that

$$\int_0^\infty x^\nu \left|\Psi(f)(x)\right|^p dx$$

$$\le C^{p-1} \int_0^\infty t^{1+\nu-(1/p)-(\nu/p)} \left|f(t)\right|^p \left[\int_0^\infty x^{(1/p)+(\nu/p)-1} \left|\psi(x,t)\right| dx \right] dt. \qquad (4.1.8)$$

Consider the following integral

$$\int_0^\infty x^{(1/p)+(\nu/p)-1} \left|\psi(x,t)\right| dx. \qquad (4.1.9)$$

Putting $x = ut$ in (4.1.9), we have

$$\int_0^\infty x^{(1/p)+(\nu/p)-1} \left|\psi(x,t)\right| dx$$

$$= t^{(1/p)+(\nu/p)-1} \int_0^\infty u^{(1/p)+(\nu/p)-1} \left|\psi(u,1)\right| du$$

$$= t^{(1/p)+(\nu/p)-1} C,$$

where $C = \int_0^\infty u^{(1/p)+(\nu/p)-1} \left|\psi(u,1)\right| du$.

Then, (4.1.8) becomes

$$\int_0^\infty x^\nu \left|\Psi(f)(x)\right|^p dx$$

$$\le C^{p-1} \int_0^\infty t^{1+\nu-(1/p)-(\nu/p)} \left|f(t)\right|^p \left[t^{(1/p)+(\nu/p)-1} C \right] dt$$

$$= C^p \int_0^\infty t^\nu \left|f(t)\right|^p dt \cdot$$

This implies that

$$\left(\int_0^\infty x^\nu \left|\Psi(f)(x)\right|^p dx\right)^{1/p} \le C\left(\int_0^\infty t^\nu \left|f(t)\right|^p dt\right)^{1/p}$$

or

$$\left(\int_0^\infty x^\nu \left|\Psi(f)(x)\right|^p dx\right)^{1/p} \le C\left(\int_0^\infty x^\nu \left|f(x)\right|^p dt\right)^{1/p}.$$

Similarly, we can prove the second part of this theorem.

4.2 ERDÉLYI - KOBER OPERATORS

Following 3.1.1, we now define Erdélyi-Kober $k-$fractional integral operator $I_{k,\eta}^{\lambda} f$ as given below and prove L^p-boundedness of one of its special cases that is used later. For $\lambda > 0$, $k > 0$, write

$$I_{k,\eta}^{\lambda} f(x) = \frac{x^{-((\eta/k)+(\lambda/k))}}{k\Gamma_k(\alpha)}\int_0^x (x-t)^{(\lambda/k)-1} t^{\eta/k} f(t)dt, \quad 0 < t < x < \infty.$$

Note that $I_{k,0}^{\lambda} f = I_k^{\lambda} f$ and $I_{k,\eta+\lambda}^{\beta}\left\{I_{k,\eta}^{\alpha} f\right\} = I_{k,\eta}^{\alpha+\beta} f$.

Theorem 4.2.1: Let $p > 1, \gamma - \beta > 0,\ k > 0$ and $\lambda = \gamma - \beta,\ (\eta/k) = (2/p) - 1$.

If $I_{k,(2/p)-1}^{\gamma-\beta}(f)(x) = \dfrac{x^{-(2/p)+1-((\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)}\displaystyle\int_0^x (x-t)^{((\gamma-\beta)/k)-1} t^{(2/p)-1} f(t)dt$,

where $0 < t < x < \infty$, then $I_{k,(2/p)-1}^{\gamma-\beta}(f): L^p \to L^p$ and there exists a constant $C' = C'(\beta,\gamma,p,k)$ such that $\left\|I_{k,(2/p)-1}^{\gamma-\beta}(f)\right\|_p \le C'\|f\|_p$.

Proof: Consider

$$I_{k,(2/p)-1}^{\gamma-\beta}(f)(x) = \frac{x^{-(2/p)+1-((\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)}\int_0^x (x-t)^{((\gamma-\beta)/k)-1} t^{(2/p)-1} f(t)dt.$$

Suppose that $\psi(x,t) = x^{-(2/p)+1-((\gamma-\beta)/k)} t^{(2/p)-1} (x-t)^{((\gamma-\beta)/k)-1}$.

Then $$\left|\psi(hx,ht)\right| = |h|^{-1}\left|x^{-(2/p)+1-((\gamma-\beta)/k)} t^{(2/p)-1} (x-t)^{((\gamma-\beta)/k)-1}\right|$$
$$= |h|^{-1}\left|\psi(x,t)\right|.$$

Since $\psi(x,t)$ is a homogeneous function of degree -1, by Theorem 4.1.3(a(i)), there exists a constant $C' = C'(\beta, \gamma, p, k)$ such that

$$\left\| I_{k,(2/p)-1}^{\gamma-\beta}(f) \right\|_p \le C' \|f\|_p ,$$

where

$$C' = \int_0^1 t^{-1/p} \psi(1,t) dt$$
$$= \int_0^1 t^{-1/p} t^{(2/p)-1} (1-t)^{((\gamma-\beta)/k)-1} dt$$
$$= \int_0^1 t^{(1/p)-1} (1-t)^{((\gamma-\beta)/k)-1} dt < \infty, \quad \text{if } \ p > 1, \ \ \gamma - \beta > 0, \ k > 0 .$$

Hence, $I_{k,(2/p)-1}^{\gamma-\beta}(f) : L^p \to L^p$.

4.3 BOUNDEDNESS OF INTEGRALS $M_{m,k}^{\alpha,\beta}(f)$ WITH VARIANTS OF HOMOGENEOUS KERNEL

In this section, we consider integral operators of the type

$$M_{m,k}^{\alpha,\beta}(f)(x) = x^{(\beta/k)-(2/p)} \int_0^\infty t^{(\beta/k)-1} \left(1 + x^m t^m\right)^{-\alpha/m} f(t) dt \ ,(x > 0).$$

We prove their boundedness in L^p . We consider other relations that are useful in our study of integral operators involving hypergeometric functions. We begin with the following result.

Theorem 4.3.1: Let

$$M_{m,k}^{\alpha,\beta}(f)(x) = x^{(\beta/k)-(2/p)} \int_0^\infty t^{(\beta/k)-1} \left(1 + x^m t^m\right)^{-\alpha/m} f(t) dt, \ \ (x > 0).$$

If $p > 1, \ (\beta/k) - \alpha < (1/p) < (\beta/k), \ 0 < \alpha < 1, \ k > 0$,
then $M_{m,k}^{\alpha,\beta}(f) : L^p \to L^p$ and there exists a constant $C_1 = C_1(\alpha, \beta, p, k)$ such that

$$\left\| M_{m,k}^{\alpha,\beta}(f) \right\|_p \le C_1 \|f\|_p .$$

Proof: Note that

$$M_{m,k}^{\alpha,\beta}((V)f)(x) = x^{(\beta/k)-(2/p)} \int_0^\infty t^{(\beta/k)-1} \left(1 + x^m t^m\right)^{-\alpha/m} (V(f))(t) dt \ ,(x > 0)$$

$$= x^{(\beta/k)-(2/p)}\int_0^\infty t^{(\beta/k)-1-(2/p)}\left(1+x^m t^m\right)^{-\alpha/m} f(1/t)\,dt.$$

Substitute $y = 1/t$ to get

$$M_{m,k}^{\alpha,\beta}\left((V)f\right)(x) = x^{(\beta/k)-(2/p)}\int_0^\infty y^{(2/p)+\alpha-(\beta/k)-1}\left(x^m + y^m\right)^{-\alpha/m} f(y)\,dy.$$

Suppose that $\psi(x,y) = x^{(\beta/k)-(2/p)} y^{(2/p)+\alpha-(\beta/k)-1}\left(x^m + y^m\right)^{-\alpha/m}$.

Now, $$\left|\psi(hx,hy)\right| = |h|^{-1}\left|x^{(\beta/k)-(2/p)} y^{(2/p)+\alpha-(\beta/k)-1}\left(x^m + y^m\right)^{-\alpha/m}\right|$$

$$= |h|^{-1}\left|\psi(x,y)\right|.$$

Since $\psi(x,y)$ is a homogeneous function of degree -1, by Theorem 4.1.3(a(i)), there exists a constant $C_1 = C_1(\alpha,\beta,p,k)$ such that

$$\left\|M_{m,k}^{\alpha,\beta}\left((V)f\right)\right\|_p \le C_1\left\|(V)f\right\|_p = C_1\|f\|_p,$$

where

$$C_1 = \int_0^\infty y^{-1/p}\psi(1,y)\,dy$$

$$= \int_0^\infty y^{(1/p)+\alpha-(\beta/k)-1}\left(1+y^m\right)^{-\alpha/m} dy$$

$$= \int_0^1 y^{(1/p)+\alpha-(\beta/k)-1}\left(1+y^m\right)^{-\alpha/m} dy$$

$$+\int_1^\infty y^{(1/p)+\alpha-(\beta/k)-1}\left(1+y^m\right)^{-\alpha/m} dy < \infty,$$

if $p > 1,\ (\beta/k)-\alpha < (1/p) < (\beta/k),\ 0<\alpha<1,\ k>0$.

Thus, $M_{m,k}^{\alpha,\beta}\left((V)f\right): L^p \to L^p$.

Also, since $V(f): L^p \to L^p$, $M_{m,k}^{\alpha,\beta}\left((V)f\right): L^p \to L^p$ and

$$M_{m,k}^{\alpha,\beta}(f) = M_{m,k}^{\alpha,\beta}\left(V^2(f)\right).$$

Then $$\left\|M_{m,k}^{\alpha,\beta}(f)\right\|_p = \left\|M_{m,k}^{\alpha,\beta}\left(V^2(f)\right)\right\|_p \le C_1\left\|V(f)\right\|_p = C_1\|f\|_p.$$

Hence, $M_{m,k}^{\alpha,\beta}(f): L^p \to L^p$ and there exists a constant $C_1 = C_1(\alpha,\beta,p,k)$ such that

$$\left\|M_{m,k}^{\alpha,\beta}(f)\right\|_p \le C_1\|f\|_p.$$

Lemma 4.3.2: Let

$$M_{m,k}^{*\alpha,\beta}(g)(x)=x^{(\beta/k)-1}\int_0^\infty t^{(\beta/k)-(2/p)}\left(1+x^m t^m\right)^{-\alpha/m} g(t)\,dt \;,(x>0).$$

If $p'>1,(\beta/k)-\alpha<(1/p')<(\beta/k),0<\alpha<1,(1/p)+(1/p')=1,k>0$, then $M_{m,k}^{*\alpha,\beta}(g):L^{p'}\to L^{p'}$ and there exists a constant $C_1'=C_1'(\alpha,\beta,p',k)$ such that

$$\left\|M_{m,k}^{*\alpha,\beta}(g)\right\|_{p'}\le C_1'\|g\|_{p'}.$$

Proof: Proof of this lemma is identical as above.

Now, we prove the following product formula.

Theorem 4.3.3: Let

$$M_{m,k}^{\alpha,\beta}(f)(x)=x^{(\beta/k)-(2/p)}\int_0^\infty t^{(\beta/k)-1}\left(1+x^m t^m\right)^{-\alpha/m} f(t)\,dt \;,(x>0)$$

and

$$M_{m,k}^{*\alpha,\beta}(g)(x)=x^{(\beta/k)-1}\int_0^\infty t^{(\beta/k)-(2/p)}\left(1+x^m t^m\right)^{-\alpha/m} g(t)\,dt \;,(x>0).$$

If $f\in L^p\left(R^+\right)$ and $g\in L^{p'}\left(R^+\right)$, then

$$\int_0^\infty g(t)M_{m,k}^{\alpha,\beta}(f)(t)\,dt=\int_0^\infty f(t)M_{m,k}^{*\alpha,\beta}(g)(t)\,dt.$$

Proof: An application of Fubini's theorem yields the result.

Theorem 4.3.4: For $x>0$, let

$$\begin{aligned}\phi_x(t)&=(x-t)^{((\gamma-\beta)/k)-1}\; t^{(2/p)-1} && \text{for } 0<t<x<\infty;\\ &=0 && \text{for } t\ \ge\ x.\end{aligned}$$

Then

$$M_{m,k}^{*\alpha,\beta}(\phi_x)(t)=\frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}{\Gamma_k(\gamma)}x^{(\gamma/k)-1}t^{(\beta/k)-1}$$

$$\times{}_{m+1}F_{m,k}\left(\begin{matrix}\left(\frac{\alpha}{m},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^m t^m\right).$$

Proof: Consider the following integral representation given in Lemma 2.7.4

$$ {}_{m+1}F_{m,k}\left(\begin{matrix}\left(\frac{\alpha}{m},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};x\right) $$

$$ =\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 u^{(\beta/k)-1}(1-u)^{((\gamma-\beta)/k)-1}\left(1-xu^m\right)^{-\alpha/m}du\,. $$

Replacing x by $-x^m t^m$, we have

$$ {}_{m+1}F_{m,k}\left(\begin{matrix}\left(\frac{\alpha}{m},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^m t^m\right) $$

$$ =\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 u^{(\beta/k)-1}(1-u)^{((\gamma-\beta)/k)-1}\left(1+x^m t^m u^m\right)^{-\alpha/m}du\,. $$

Substituting $u=y/x$ to get

$$ {}_{m+1}F_{m,k}\left(\begin{matrix}\left(\frac{\alpha}{m},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^m t^m\right) $$

$$ =\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}x^{1-(\gamma/k)}\int_0^x y^{(\beta/k)-1}(x-y)^{((\gamma-\beta)/k)-1}\left(1+t^m y^m\right)^{-\alpha/m}dy\,. $$

This implies that

$$ \int_0^x y^{(\beta/k)-1}(x-y)^{((\gamma-\beta)/k)-1}\left(1+t^m y^m\right)^{-\alpha/m}dy $$

$$ =\frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}{\Gamma_k(\gamma)}x^{(\gamma/k)-1} $$

$$ \times{}_{m+1}F_{m,k}\left(\begin{matrix}\left(\frac{\alpha}{m},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^m t^m\right). \quad (4.3.1) $$

Now, consider $M_{m,k}^{*\alpha,\beta}$ as in Theorem 4.3.2 to obtain the following expression.

$$ M_{m,k}^{*\alpha,\beta}(g)(t)=t^{(\beta/k)-1}\int_0^\infty y^{(\beta/k)-(2/p)}\left(1+t^m y^m\right)^{-\alpha/m}g(y)dy,\ (t>0)\,. $$

Then, by taking $g(t)=\phi_x(t)$, we get

$$M_{m,k}^{*\alpha,\beta}(\phi_x)(t)$$

$$=t^{(\beta/k)-1}\int_0^x y^{(\beta/k)-(2/p)}\left(1+t^m y^m\right)^{-\alpha/m}\left[(x-y)^{((\gamma-\beta)/k)-1}y^{(2/p)-1}\right]dy$$

$$=t^{(\beta/k)-1}\int_0^x y^{(\beta/k)-1}(x-y)^{((\gamma-\beta)/k)-1}\left(1+t^m y^m\right)^{-\alpha/m}dy.$$

By using Equation (4.3.1), we finally obtain

$$M_{m,k}^{*\alpha,\beta}(\phi_x)(t)=\frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}{\Gamma_k(\gamma)}x^{(\gamma/k)-1}t^{(\beta/k)-1}$$

$$\times {}_{m+1}F_{m,k}\left(\begin{matrix}\left(\frac{\alpha}{m},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^m t^m\right).$$

4.4 BOUNDEDNESS OF INTEGRAL OPERATORS INVOLVING HYPERGEOMETRIC FUNCTIONS ${}_{m+1}F_{m,k}$, $m\geq 1$

We now formulate integral operators involving $k-$hypergeometric functions of the type ${}_{m+1}F_{m,k}, m\geq 1$. We use the results proved in Section 4.2 and 4.3 to prove the L^p-boundedness of these integral operators involving $k-$hypergeometric functions ${}_{m+1}F_{m,k}, m\geq 1$.

Theorem 4.4.1: Let

$$S_{m,k}^{\alpha,\beta,\gamma}(f)(x)=x^{-(2/p)+1}\int_0^\infty (xt)^{(\beta/k)-1}$$

$$\times {}_{m+1}F_{m,k}\left(\begin{matrix}\left(\frac{\alpha}{m},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^m t^m\right)f(t)dt.$$

If $p>1,\ (\beta/k)-\alpha<(1/p)<(\beta/k),\ 0<\alpha<1,\ \gamma-\beta>0,\ k>0$, then

a) $S_{m,k}^{\alpha,\beta,\gamma}(f)=A_k I_{k,(2/p)-1}^{\gamma-\beta}\left\{M_{m,k}^{\alpha,\beta}(f)\right\}$ where $A_k=\dfrac{\Gamma_k(\gamma)}{\Gamma_k(\beta)}$,

b) $S_{m,k}^{\alpha,\beta,\gamma}(f):L^P\to L^P$ and explicitly

c) there exists a constant C_k such that $\left\| S_{m,k}^{\alpha,\beta,\gamma}(f) \right\|_p \le C_k \|f\|_p$.

Proof:

We employ Theorem 4.2.1 to get

$$I_{k,(2/p)-1}^{\gamma-\beta}\left\{M_{m,k}^{\alpha,\beta}(f)\right\}(x)$$

$$=\frac{x^{-(2/p)+1-((\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)}\int_0^x (x-t)^{((\gamma-\beta)/k)-1} t^{(2/p)-1} M_{m,k}^{\alpha,\beta}(f)(t)\,dt.$$

Use of Theorem 4.3.3 and Theorem 4.3.4 yields

$$I_{k,(2/p)-1}^{\gamma-\beta}\left\{M_{m,k}^{\alpha,\beta}(f)\right\}(x)$$

$$=\frac{x^{-(2/p)+1-((\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)}\int_0^\infty f(t) M_{m,k}^{*\alpha,\beta}\left[(x-t)^{((\gamma-\beta)/k)-1} t^{(2/p)-1}\right] dt$$

$$=\frac{x^{-(2/p)+1-((\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)}\int_0^\infty f(t) M_{m,k}^{*\alpha,\beta}(\phi_x(t))\,dt$$

$$=\frac{x^{-(2/p)+1-((\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)}\int_0^\infty \frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}{\Gamma_k(\gamma)} x^{(\gamma/k)-1} t^{(\beta/k)-1}$$

$$\times {}_{m+1}F_{m,k}\left(\begin{matrix}\left(\frac{\alpha}{m},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^m t^m\right) f(t)\,dt$$

$$=\frac{\Gamma_k(\beta)}{\Gamma_k(\gamma)} x^{-(2/p)+1}\int_0^\infty (xt)^{(\beta/k)-1}$$

$$\times {}_{m+1}F_{m,k}\left(\begin{matrix}\left(\frac{\alpha}{m},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^m t^m\right) f(t)\,dt$$

$$=\frac{\Gamma_k(\beta)}{\Gamma_k(\gamma)} S_{m,k}^{\alpha,\beta,\gamma}(f)(x).$$

Hence, $S_{m,k}^{\alpha,\beta,\gamma}(f)(x)=\frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)} I_{k,(2/p)-1}^{\gamma-\beta}\left\{M_{m,k}^{\alpha,\beta}(f)\right\}(x)$

$$= A_k I_{k,(2/p)-1}^{\gamma-\beta}\left\{M_{m,k}^{\alpha,\beta}(f)\right\}(x)$$

where $A_k=\frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)}$.

Since $I^{\gamma-\beta}_{k,(2/p)-1}(f): L^p \to L^p$ by Theorem 4.2.1 and it, therefore, follows from Theorem 4.3.1 that if

$$p>1, (\beta/k)-\alpha<(1/P)<(\beta/k), 0<\alpha<1, \gamma-\beta>0, k>0,$$

then

$$\left\|S^{\alpha,\beta,\gamma}_{m,k}(f)\right\|_p = A_k\left\|I^{\gamma-\beta}_{k,(2/p)-1}\left(M^{\alpha,\beta}_{m,k}(f)\right)\right\|_p \le A_k C'\left\|M^{\alpha,\beta}_{m,k}(f)\right\|_p$$
$$\le A_k C' C_1 \|f\|_p = C_k \|f\|_p.$$

Hence, $S^{\alpha,\beta,\gamma}_{m,k}(f): L^P \to L^P$ and there exists a constant $C_k = A_k C' C_1$ such that

$$\left\|S^{\alpha,\beta,\gamma}_{m,k}(f)\right\|_p \le C_k \|f\|_p.$$

CHAPTER 5

L^p −BOUNDEDNESS OF INTEGRAL OPERATORS INVOLVING ${}_mF_{m,k}$, $m \geq 1$ AS KERNEL

In this chapter, we consider integral operators involving exponential functions as kernel and discuss the L^p-boundedness of these integral operators by using properties of $k-$hypergeometric functions ${}_mF_{m,k}, m \geq 1$ as kernel.

5.1 BOUNDEDNESS OF INTEGRALS $H_{m,k}^{\beta}(f)$ INVOLVING EXPONENTIAL KERNEL

In this section, we consider integral operators of the type

$$H_{m,k}^{\beta}(f)(x) = x^{(\beta/k)-(2/p)} \int_0^{\infty} t^{(\beta/k)-1} e^{-x^m t^m} f(t)\,dt \;,(x>0).$$

We prove their boundedness in L^p and other relations that are useful in our study of integral operators involving $k-$hypergeometric functions. We begin with the following result.

Theorem 5.1.1: Let

$$H_{m,k}^{\beta}(f)(x) = x^{(\beta/k)-(2/p)} \int_0^{\infty} t^{(\beta/k)-1} e^{-x^m t^m} f(t)\,dt \;,(x>0).$$

If $p>1$, $(\beta/k)>(1/p)$, $k>0$, then $H_{m,k}^{\beta}(f): L^p \to L^p$ and there exists a constant $C_2 = C_2(\beta, p, k)$ such that

$$\left\| H_{m,k}^{\beta}(f) \right\|_p \leq C_2 \|f\|_p .$$

Proof: Note that

$$H_{m,k}^{\beta}((V)f)(x) = x^{(\beta/k)-(2/p)} \int_0^{\infty} t^{(\beta/k)-1} e^{-x^m t^m} (V(f))(t)\,dt \;,(x>0)$$

$$= x^{(\beta/k)-(2/p)} \int_0^{\infty} t^{(\beta/k)-1-(2/p)} e^{-x^m t^m} f(1/t)\,dt .$$

Substitute $y = 1/t$. Then

$$H_{m,k}^{\beta}\left(\left(V\right)f\right)\left(x\right)=x^{(\beta/k)-(2/p)}\int_{0}^{\infty}y^{(2/p)-(\beta/k)-1}e^{-x^{m}/y^{m}}f\left(y\right)dy$$

Suppose that $\psi\left(x,y\right)=x^{(\beta/k)-(2/p)}y^{(2/p)-(\beta/k)-1}e^{-x^{m}/y^{m}}$.

Now, $\left|\psi\left(hx,hy\right)\right|=\left|h\right|^{-1}\left|x^{(\beta/k)-(2/p)}y^{(2/p)-(\beta/k)-1}e^{-x^{m}/y^{m}}\right|$

$= \left|h\right|^{-1}\left|\psi\left(x,y\right)\right|.$

Since $\psi\left(x,y\right)$ is a homogeneous function of degree -1, by Theorem 4.1.3(a(i)), there exists a constant $C_2=C_2\left(\beta,p,k\right)$ such that

$$\left\|H_{m,k}^{\beta}\left(\left(V\right)f\right)\right\|_p\leq C_2\left\|\left(V\right)f\right\|_p=C_2\left\|f\right\|_p$$

where

$$C_2=\int_{0}^{\infty}y^{-1/p}\psi\left(1,y\right)dy$$

$$=\int_{0}^{\infty}y^{(1/p)-(\beta/k)-1}e^{-1/y^{m}}dy<\infty,$$

if $p>1,\ \left(\beta/k\right)>\left(1/p\right),\ k>0$. Thus, $H_{m,k}^{\beta}\left(\left(V\right)f\right):L^p\to L^p$.

Also, since $V\left(f\right):L^p\to L^p$ and $H_{m,k}^{\beta}\left(f\right)=H_{m,k}^{\beta}\left(V^2\left(f\right)\right)$, therefore

$$\left\|H_{m,k}^{\beta}\left(f\right)\right\|_p=\left\|H_{m,k}^{\beta}\left(V^2\left(f\right)\right)\right\|_p\leq C_2\left\|V\left(f\right)\right\|_p=C_2\left\|f\right\|_p.$$

Hence, $H_{m,k}^{\beta}\left(f\right):L^p\to L^p$ and there exists a constant $C_2=C_2\left(\beta,p,k\right)$ such that

$$\left\|H_{m,k}^{\beta}\left(f\right)\right\|_p\leq C_2\left\|f\right\|_p.$$

Lemma 5.1.2: Let

$$H_{m,k}^{*\beta}\left(g\right)\left(x\right)=x^{(\beta/k)-1}\int_{0}^{\infty}t^{(\beta/k)-(2/p)}e^{-x^{m}t^{m}}g\left(t\right)dt\ ,\left(x>0\right).$$

If $p'>1,\ \left(\beta/k\right)>\left(1/p'\right),\ k>0,\ \left(1/p\right)+\left(1/p'\right)=1$, then $H_{m,k}^{*\beta}\left(g\right):L^{p'}\to L^{p'}$ and there exists a constant $C_2'=C_2'\left(\beta,p',k\right)$ such that

$$\left\|H_{m,k}^{*\beta}\left(g\right)\right\|_{p'}\leq C_2'\left\|g\right\|_{p'}.$$

Proof: Proof of this lemma is identical as above.

Now, consider the following product formula.

Theorem 5.1.3: Let

$$H_{m,k}^{\beta}(f)(x)=x^{(\beta/k)-(2/p)}\int_0^{\infty}t^{(\beta/k)-1}e^{-x^m t^m}f(t)\,dt\ ,(x>0)$$

and

$$H_{m,k}^{*\beta}(g)(x)=x^{(\beta/k)-1}\int_0^{\infty}t^{(\beta/k)-(2/p)}e^{-x^m t^m}g(t)\,dt\ ,(x>0).$$

If $f\in L^p\left(R^+\right)$ and $g\in L^{p'}\left(R^+\right)$, then

$$\int_0^{\infty}g(t)H_{m,k}^{\beta}(f)(t)\,dt=\int_0^{\infty}f(t)H_{m,k}^{*\beta}(g)(t)\,dt.$$

Proof: The change of order of integration yields the result.

Theorem 5.1.4: For $\gamma>\beta>0,\ k>0$, let

$$\begin{aligned}\phi_x(t)&=(x-t)^{((\gamma-\beta)/k)-1}\ t^{(2/p)-1} &&\text{for}\quad 0<t<x<\infty;\\ &=0 &&\text{for}\quad t\geq x\ .\end{aligned}$$

Then

$$H_{m,k}^{*\beta}(\phi_x)(t)=\frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}{\Gamma_k(\gamma)}x^{(\gamma/k)-1}t^{(\beta/k)-1}$$

$$\times{}_mF_{m,k}\left(\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^m t^m\right).$$

Proof: Consider the following integral representation given in Theorem 2.8.3

$${}_mF_{m,k}\left(\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};x\right)$$

$$=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 u^{(\beta/k)-1}(1-u)^{((\gamma-\beta)/k)-1}e^{xu^m}\,du.$$

Replacing x by $-x^m t^m$, we have

$$ {}_mF_{m,k}\left(\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^mt^m\right) $$

$$ =\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 u^{(\beta/k)-1}(1-u)^{((\gamma-\beta)/k)-1}e^{-x^mt^mu^m}du. $$

Substitute $xu = y$ to get

$$ {}_mF_{m,k}\left(\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^mt^m\right) $$

$$ =\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}x^{1-(\gamma/k)}\int_0^x y^{(\beta/k)-1}(x-y)^{((\gamma-\beta)/k)-1}e^{-t^my^m}dy $$

$$ \int_0^x y^{(\beta/k)-1}(x-y)^{((\gamma-\beta)/k)-1}e^{-t^my^m}dy=\frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}{\Gamma_k(\gamma)}x^{(\gamma/k)-1} $$

$$ \times{}_mF_{m,k}\left(\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^mt^m\right). \tag{5.1.1} $$

Now, consider $H_{m,k}^{*\beta}$ as in Theorem 5.1.2. We, then, obtain

$$ H_{m,k}^{*\beta}(g)(t)=t^{(\beta/k)-1}\int_0^\infty y^{(\beta/k)-(2/p)}e^{-t^my^m}g(y)dy,\ (t>0). $$

Then taking $g(t)=\phi_x(t)$, we get

$$ H_{m,k}^{*\beta}(\phi_x)(t)=t^{(\beta/k)-1}\int_0^x y^{(\beta/k)-(2/p)}e^{-t^my^m}\left[(x-y)^{((\gamma-\beta)/k)-1}y^{(2/p)-1}\right]dy $$

$$ =t^{(\beta/k)-1}\int_0^x y^{(\beta/k)-1}(x-y)^{((\gamma-\beta)/k)-1}e^{-t^my^m}dy. $$

By using (5.1.1), we obtain

$$H^{*\beta}_{m,k}(\phi_x)(t)=\frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}{\Gamma_k(\gamma)}x^{(\gamma/k)-1}t^{(\beta/k)-1}$$

$$\times {}_mF_{m,k}\left(\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^mt^m\right).$$

5.2 BOUNDEDNESS OF INTEGRAL OPERATORS INVOLVING HYPERGEOMETRIC FUNCTIONS ${}_mF_{m,k}$ $,m\geq 1$

We now formulate integral operators involving $k-$hypergeometric functions of the type ${}_mF_{m,k}$, $m\geq 1$. We use the results proved in Section 5.1 to prove the L^p-boundedness of these integral operators involving $k-$hypergeometric functions ${}_mF_{m,k}$, $m\geq 1$.

Theorem 5.2.1: Let

$$G^{\beta,\gamma}_{m,k}(f)(x)$$

$$=x^{-(2/p)+1}\int_0^\infty (xt)^{(\beta/k)-1}$$

$$\times {}_mF_{m,k}\left(\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^mt^m\right)dt.$$

If $p>1,\ \gamma-\beta>0,\ (\beta/k)>(1/p),\ k>0$, then

a) $G^{\beta,\gamma}_{m,k}(f)=A_kI^{\gamma-\beta}_{k,(2/p)-1}\left\{H^{\beta}_{m,k}(f)\right\}$ where $A_k=\dfrac{\Gamma_k(\gamma)}{\Gamma_k(\beta)}$,

b) $G^{\beta,\gamma}_{m,k}(f):L^p\to L^p$ and explicitly

c) there exists a constant C'_k such that $\left\|G^{\beta,\gamma}_{m,k}(f)\right\|_p\leq C'_k\|f\|_p$.

Proof:

We employ Theorem 4.2.1 to get

$$I^{\gamma-\beta}_{k,(2/p)-1}\left\{H^{\beta}_{m,k}(f)\right\}(x)$$

$$=\frac{x^{-(2/p)+1-((\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)}\int_0^\infty (x-t)^{((\gamma-\beta)/k)-1}t^{(2/p)-1}H^{\beta}_{m,k}(f)(t)\,dt.$$

Now, we use Theorem 5.1.3 and Theorem 5.1.4 to get

$$I_{k,(2/p)-1}^{\gamma-\beta}\left\{H_{m,k}^{\beta}(f)\right\}(x)$$

$$=\frac{x^{-(2/p)+1-((\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)}\int_0^\infty f(t)H_{m,k}^{*\beta}\left[(x-t)^{((\gamma-\beta)/k)-1}t^{(2/p)-1}\right]dt$$

$$=\frac{x^{-(2/p)+1-((\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)}\int_0^\infty f(t)H_{m,k}^{*\beta}(\phi_x(t))dt.$$

$$=\frac{x^{-(2/p)+1-((\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)}\int_0^\infty \frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}{\Gamma_k(\gamma)}x^{(\gamma/k)-1}t^{(\beta/k)-1}$$

$$\times {}_mF_{m,k}\left(\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^mt^m\right)f(t)dt$$

$$=\frac{\Gamma_k(\beta)}{\Gamma_k(\gamma)}x^{-(2/p)+1}\int_0^\infty (xt)^{(\beta/k)-1}$$

$$\times {}_mF_{m,k}\left(\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^mt^m\right)f(t)dt$$

$$=\frac{\Gamma_k(\beta)}{\Gamma_k(\gamma)}G_{m,k}^{\beta,\gamma}(f)(x).$$

Hence, $G_{m,k}^{\beta,\gamma}(f)(x)=\frac{\Gamma_k(\beta)}{\Gamma_k(\gamma)}I_{k,(2/p)-1}^{\gamma-\beta}\left\{H_{m,k}^{\beta}(f)\right\}(x)$

$$=A_kI_{k,(2/p)-1}^{\gamma-\beta}\left\{H_{m,k}^{\alpha,\beta}(f)\right\}(x)$$

where $A_k=\frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)}$.

Since $I_{k,(2/p)-1}^{\gamma-\beta}(f):L^p\to L^p$ by Theorem 4.2.1 and it, therefore, follows from Theorem 5.1.1 that if $p>1,\ \gamma-\beta>0,\ (\beta/k)>(1/p),\ k>0$, then

$$\left\|G_{m,k}^{\beta,\gamma}(f)\right\|_p=A_k\left\|I_k^{\beta,\gamma}\left(H_{m,k}^{\beta}(f)\right)\right\|_p\le A_kC'\left\|H_{m,k}^{\beta}(f)\right\|_p\le A_kC'C_2\|f\|_p=C_k'\|f\|_p.$$

Hence, $G_{m,k}^{\beta,\gamma}(f):L^p\to L^p$ and there exists a constant $C_k'=A_kC'C_2$ such that

$$\left\|G_{m,k}^{\beta,\gamma}(f)\right\|_p\le C_k'\|f\|_p.$$

CHAPTER 6

L^p −BOUNDEDNESS OF INTEGRAL OPERATORS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS ${}_{m+1}F^{\sigma}_{m,k}$, $m \geq 1$ AS KERNEL

In this chapter, we consider integral operators involving generalized $k-$hypergeometric functions of the type ${}_{m+1}F^{\sigma}_{m,k}$, $m \geq 1$ as kernel and discuss the L^p-boundedness of these integral operators by using properties of generalized $k-$hypergeometric functions ${}_{m+1}F^{\sigma}_{m,k}$, $m \geq 1$.

6.1 BOUNDEDNESS OF INTEGRALS $M^{\sigma,\alpha,\beta}_{m,k}(f)$ WITH VARIANTS OF HOMOGENEOUS KERNEL

We now consider the integral operator

$$M^{\sigma,\alpha,\beta}_{m,k}(f)(x) = x^{(\sigma\beta/k)-(2/p)} \int_0^{\infty} t^{(\sigma\beta/k)-1} \left(1 + x^{m\sigma} t^{m\sigma}\right)^{-\alpha/m\sigma} f(t)\,dt \;, (x > 0).$$

We shall, then, prove its boundedness in L^p and other relations that are useful in our study of integral operators involving generalized $k-$hypergeometric functions.

We now define general Erdélyi-Kober $k-$fractional integral operator as follows and prove L^p-boundedness of one of its special cases to be used later. For $\lambda > 0$, $k > 0$, $\sigma > 0$, define

$$I^{\lambda}_{k,\sigma,\delta} f(x) = \frac{\sigma x^{-((\sigma\delta/k)+(\sigma\lambda/k))}}{k\Gamma_k(\lambda)} \int_0^x \left(x^{\sigma} - t^{\sigma}\right)^{(\lambda/k)-1} t^{(\sigma\delta/k)+\sigma-1} f(t)\,dt, \quad 0 < t < x < \infty .$$

Theorem 6.1.1:
Let $p > 1, \gamma - \beta > 0,\ k > 0,\ \sigma > 0$ and $\lambda = \gamma - \beta,\ (\sigma\delta/k) + \sigma = (2/p)$.
If $I^{\gamma-\beta}_{k,\sigma,(2/p)}(f)(x)$

$$= \frac{\sigma x^{-(2/p)+\sigma-(\sigma(\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)} \int_0^x \left(x^{\sigma} - t^{\sigma}\right)^{((\gamma-\beta)/k)-1} t^{(2/p)-1} f(t)\,dt,$$

then $I_{k,\sigma,(2/p)}^{\gamma-\beta}(f): L^p \to L^p$ and there exists a constant $C'' = C''(\beta,\gamma,p,k,\sigma)$ such that

$$\left\| I_{k,\sigma,(2/p)}^{\gamma-\beta}(f) \right\|_p \le C'' \|f\|_p .$$

Proof: Consider

$$I_{k,\sigma,(2/p)}^{\gamma-\beta}(f)(x) = \frac{\sigma x^{-(2/p)+\sigma-(\sigma(\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)} \int_0^x \left(x^\sigma - t^\sigma\right)^{((\gamma-\beta)/k)-1} t^{(2/p)-1} f(t)dt.$$

Suppose that $\psi(x,t) = x^{-(2/p)+\sigma-(\sigma(\gamma-\beta)/k)}\left(x^\sigma - t^\sigma\right)^{((\gamma-\beta)/k)-1} t^{(2/p)-1}$.

Then $\quad |\psi(hx,ht)| = |h|^{-1}|\psi(x,t)|$.

Since $\psi(x,y)$ is a homogeneous function of degree -1, then by Theorem 4.1.3(a(i)), there exists a constant $C'' = C''(\beta,\gamma,k,p,\sigma)$ such that

$$\left\| I_{k,\sigma,(2/p)}^{\gamma-\beta}(f) \right\|_p \le C'' \|f\|_p$$

where

$$\begin{aligned} C'' &= \int_0^1 t^{-1/p}\psi(1,t)dt \\ &= \int_0^1 t^{(1/p)-1}\left(1-t^\sigma\right)^{((\gamma-\beta)/k)-1} dt < \infty , \end{aligned}$$

if $\; p>1, \gamma-\beta>0,\; k>0,\; \sigma>0$.

Hence, $I_{k,\sigma,(2/p)}^{\gamma-\beta}(f): L^p \to L^p$.

Now, consider the integral operator $M_{m,k}^{\sigma,\alpha,\beta}(f)$ as given below.

Theorem 6.1.2: Let

$$M_{m,k}^{\sigma,\alpha,\beta}(f)(x) = x^{(\sigma\beta/k)-(2/p)} \int_0^\infty t^{(\sigma\beta/k)-1}\left(1+x^{m\sigma}t^{m\sigma}\right)^{-\alpha/m\sigma} f(t)dt .$$

If $p>1,\; (\sigma\beta/k)-\alpha < (1/p) < (\sigma\beta/k),\; 0<\alpha<1,\; k>0,\; \sigma>0$,

then $M_{m,k}^{\sigma,\alpha,\beta}(f): L^p \to L^p$ and there exists a constant $C_3 = C_3(\alpha,\beta,k,p,\sigma)$ such that

$$\left\| M_{m,k}^{\sigma,\alpha,\beta}(f) \right\|_p \le C_3 \|f\|_p .$$

Proof: Note that

$$M_{m,k}^{\sigma,\alpha,\beta}\left(\left(V\right)f\right)\left(x\right)=x^{\left(\sigma\beta/k\right)-\left(2/p\right)}\int_{0}^{\infty}t^{\left(\sigma\beta/k\right)-1}\left(1+x^{m\sigma}t^{m\sigma}\right)^{-\alpha/m\sigma}\left(V\left(f\right)\right)\left(t\right)dt$$

$$=x^{\left(\sigma\beta/k\right)-\left(2/p\right)}\int_{0}^{\infty}t^{\left(\sigma\beta/k\right)-\left(2/p\right)-1}\left(1+x^{m\sigma}t^{m\sigma}\right)^{-\alpha/m\sigma}f\left(1/t\right)dt.$$

Substitute $y=1/t$. Then

$$M_{m,k}^{\sigma,\alpha,\beta}\left(\left(V\right)f\right)\left(x\right)$$

$$=x^{\left(\sigma\beta/k\right)-\left(2/p\right)}\int_{0}^{\infty}y^{\left(2/p\right)+\alpha-\left(\sigma\beta/k\right)-1}\left(x^{m\sigma}+y^{m\sigma}\right)^{-\alpha/m\sigma}f\left(y\right)dy.$$

Suppose that $\psi\left(x,y\right)=x^{\left(\sigma\beta/k\right)-\left(2/p\right)}y^{\left(2/p\right)+\alpha-\left(\sigma\beta/k\right)-1}\left(x^{m\sigma}+y^{m\sigma}\right)^{-\alpha/m\sigma}$.

Then $\left|\psi\left(hx,hy\right)\right|=\left|h\right|^{-1}\left|\psi\left(x,y\right)\right|$.

Since $\psi\left(x,y\right)$ is a homogeneous function of degree -1, then by Theorem 4.1.3(a(i)), there exists a constant $C_3=C_3\left(\alpha,\beta,k,p,\sigma\right)$ such that

$$\left\|M_{m,k}^{\sigma,\alpha,\beta}\left(\left(V\right)f\right)\right\|_p\le C_3\left\|\left(V\right)f\right\|_p=C_3\left\|f\right\|_p$$

where

$$C_3=\int_{0}^{\infty}y^{-1/p}\psi\left(1,y\right)dy$$

$$=\int_{0}^{\infty}y^{\left(1/p\right)+\alpha-\left(\sigma\beta/k\right)-1}\left(1+y^{m\sigma}\right)^{-\alpha/m\sigma}dy$$

$$=\int_{0}^{1}y^{\left(1/p\right)+\alpha-\left(\sigma\beta/k\right)-1}\left(1+y^{m\sigma}\right)^{-\alpha/m\sigma}dy$$

$$+\int_{1}^{\infty}y^{\left(1/p\right)+\alpha-\left(\sigma\beta/k\right)-1}\left(1+y^{m\sigma}\right)^{-\alpha/m\sigma}dy<\infty,$$

if $p>1,\ \left(\sigma\beta/k\right)-\alpha<\left(1/p\right)<\left(\sigma\beta/k\right),\ 0<\alpha<1,\ k>0,\ \sigma>0$.

Thus, $M_{m,k}^{\sigma,\alpha,\beta}\left(f\right):L^p\to L^p$.

Also, since $V\left(f\right):L^p\to L^p$ and $M_{m,k}^{\sigma,\alpha,\beta}\left(f\right)=M_{m,k}^{\sigma,\alpha,\beta}\left(\left(V^2\right)\left(f\right)\right)$, therefore

$$\left\|M_{m,k}^{\sigma,\alpha,\beta}\left(f\right)\right\|_p=\left\|M_{m,k}^{\sigma,\alpha,\beta}\left(\left(V^2\right)\left(f\right)\right)\right\|_p\le C_3\left\|V\left(f\right)\right\|_p=C_3\left\|f\right\|_p.$$

Hence, $M_{m,k}^{\sigma,\alpha,\beta}(f): L^p \to L^p$ and there exists a constant $C_3 = C_3(\alpha,\beta,k,p,\sigma)$ such that

$$\left\| M_{m,k}^{\sigma,\alpha,\beta}(f) \right\|_p \le C_3 \left\| f \right\|_p .$$

Lemma 6.1.3: Let

$$M_{m,k}^{*\sigma,\alpha,\beta}(g)(x) = x^{(\sigma\beta/k)-1} \int_0^\infty t^{(\sigma\beta/k)-(2/p)} \left(1 + x^{m\sigma} t^{m\sigma}\right)^{-\alpha/m\sigma} g(t)\, dt \ ,(x > 0).$$

If $p' > 1, (1/p) + (1/p') = 1, (\sigma\beta/k) - \alpha < (1/p') < (\sigma\beta/k), 0 < \alpha < 1$ and $k > 0, \sigma > 0$, then $M_{m,k}^{*\sigma,\alpha,\beta}(g): L^{p'} \to L^{p'}$ and there exists a constant $C_3' = C_3'(\alpha,\beta,k,p',\sigma)$ such that

$$\left\| M_{m,k}^{*\sigma,\alpha,\beta}(g) \right\|_{p'} \le C_3' \left\| g \right\|_{p'} .$$

Proof: For proof, take p' in place of p, $(1/p) + (1/p') = 1$, in the last theorem to get the result.

Now, we prove the following product formula.

Theorem 6.1.4: Let

$$M_{m,k}^{\sigma,\alpha,\beta}(f)(x) = x^{(\sigma\beta/k)-(2/p)} \int_0^\infty t^{(\sigma\beta/k)-1} \left(1 + x^{\sigma m} t^{\sigma m}\right)^{-\alpha/\sigma m} f(t)\, dt \ ,(x > 0)$$

and

$$M_{m,k}^{*\sigma,\alpha,\beta}(g)(x) = x^{(\sigma\beta/k)-1} \int_0^\infty t^{(\sigma\beta/k)-(2/p)} \left(1 + x^{\sigma m} t^{\sigma m}\right)^{-\alpha/\sigma m} g(t)\, dt \ ,(x > 0).$$

If $f \in L^p(R^+)$ and $g \in L^{p'}(R^+)$, then

$$\int_0^\infty g(t) M_{m,k}^{\sigma,\alpha,\beta}(f)(t)\, dt = \int_0^\infty f(t) M_{m,k}^{*\sigma,\alpha,\beta}(g)(t)\, dt .$$

Proof: Use of Fubini's theorem will prove the result.

Theorem 6.1.5: For $k > 0$, $\sigma > 0$, let

$$\begin{aligned} \phi_x^\sigma(t) &= \left(x^\sigma - t^\sigma\right)^{((\gamma-\beta)/k)-1} t^{(2/p)-1} && \text{for } 0 < t < x < \infty; \\ &= 0 && \text{for } t \ge x . \end{aligned}$$

Then

$$M_{m,k}^{*\sigma,\alpha,\beta}\left(\phi_x^\sigma\right)(t)=\frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)x^{(\sigma\gamma/k)-\sigma}t^{(\sigma\beta/k)-1}}{\sigma\Gamma_k(\gamma)}$$

$$\times{}_{m+1}F_{m,k}^{\sigma}\left[\begin{matrix}\left(\frac{\alpha}{m\sigma},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^{m\sigma}t^{m\sigma}\right].$$

Proof: Consider the following integral representation

$${}_{m+1}F_{m,k}^{\sigma}\left[\begin{matrix}\left(\frac{\alpha}{m\sigma},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};x\right]$$

$$=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 u^{(\beta/k)-1}(1-u)^{((\gamma-\beta)/k)-1}\left(1-xu^m\right)^{-\alpha/m\sigma}du.$$

Putting $u = w^\sigma$, we get

$${}_{m+1}F_{m,k}^{\sigma}\left[\begin{matrix}\left(\frac{\alpha}{m\sigma},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};x\right]$$

$$=\frac{\sigma\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 w^{(\sigma\beta/k)-1}\left(1-w^\sigma\right)^{((\gamma-\beta)/k)-1}\left(1-xw^{m\sigma}\right)^{-\alpha/m\sigma}dw.$$

Replacing x by $-x^{m\sigma}t^{m\sigma}$, we have

$${}_{m+1}F_{m,k}^{\sigma}\left[\begin{matrix}\left(\frac{\alpha}{m\sigma},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^{m\sigma}t^{m\sigma}\right]$$

$$=\frac{\sigma\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}$$

$$\times\int_0^1 w^{(\sigma\beta/k)-1}\left(1-w^\sigma\right)^{((\gamma-\beta/k))-1}\left(1+x^{m\sigma}t^{m\sigma}w^{m\sigma}\right)^{-\alpha/m\sigma}dw.$$

Put $xw = y$ to get

$${}_{m+1}F_{m,k}^{\sigma}\left[\begin{matrix}\left(\frac{\alpha}{m\sigma},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^{m\sigma}t^{m\sigma}\right]$$

$$=\frac{\sigma\Gamma_k(\gamma)x^{\sigma-(\sigma\gamma/k)}}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}$$

$$\times\int_0^x y^{(\sigma\beta/k)-1}\left(x^\sigma - y^\sigma\right)^{((\gamma-\beta)/k)-1}\left(1+t^{m\sigma}y^{m\sigma}\right)^{-\alpha/m\sigma}dy.$$

This implies that

$$\int_0^x y^{(\sigma\beta/k)-1}\left(x^\sigma - y^\sigma\right)^{((\gamma-\beta)/k)-1}\left(1+t^{m\sigma}y^{m\sigma}\right)^{-\alpha/m\sigma}dy$$

$$=\frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)x^{(\sigma\gamma/k)-\sigma}}{\sigma\Gamma_k(\gamma)}$$

$$\times {}_{m+1}F^{\sigma}_{m,k}\left[\begin{matrix}\left(\frac{\alpha}{m\sigma},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+k(m-1)}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+k(m-1)}{m},k\right)\end{matrix};-x^{m\sigma}t^{m\sigma}\right]. \tag{6.1.1}$$

Now, by considering $M^{*\sigma,\alpha,\beta}_{m,k}$ as in Theorem 6.1.3, we obtain the following.

$$M^{*\sigma,\alpha,\beta}_{m,k}(g)(t)=t^{(\sigma\beta/k)-1}\int_0^\infty y^{(\sigma\beta/k)-(2/p)}\left(1+t^{m\sigma}y^{m\sigma}\right)^{-\alpha/m\sigma}g(y)dy\ ,(t>0).$$

Then taking $g(t)=\phi^\sigma_x(t)$, we get

$$M^{*\sigma,\alpha,\beta}_{m,k}\left(\phi^\sigma_x\right)(t)=t^{(\sigma\beta/k)-1}$$

$$\times\int_0^\infty y^{(\sigma\beta/k)-(2/p)}\left(1+t^{m\sigma}y^{m\sigma}\right)^{-\alpha/m\sigma}\left[\left(x^\sigma-y^\sigma\right)^{((\gamma-\beta)/k)-1}y^{(2/p)-1}\right]dy$$

$$=t^{(\sigma\beta/k)-1}\int_0^\infty y^{(\sigma\beta/k)-1}\left(x^\sigma-y^\sigma\right)^{((\gamma-\beta)/k)-1}\left(1+t^{m\sigma}y^{m\sigma}\right)^{-\alpha/\sigma m}dy.$$

By using Equation (6.1.1), we obtain

$$M^{*\sigma,\alpha,\beta}_{m,k}\left(\phi^\sigma_x\right)(t)=\frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)x^{(\sigma\gamma/k)-\sigma}t^{(\sigma\beta/k)-1}}{\sigma\Gamma_k(\gamma)}$$

$$\times {}_{m+1}F^{\sigma}_{m,k}\left[\begin{matrix}\left(\frac{\alpha}{m\sigma},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+k(m-1)}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+k(m-1)}{m},k\right)\end{matrix};-x^{m\sigma}t^{m\sigma}\right].$$

6.2 BOUNDEDNESS OF INTEGRAL OPERATORS INVOLVING HYPERGEOMETRIC FUNCTIONS ${}_{m+1}F^{\sigma}_{m,k}$, $m\geq 1$

We now formulate integral operators involving hypergeometric functions of the type ${}_{m+1}F^{\sigma}_{m,k}$, $m\geq 1$. We use the results proved in Section 6.1 to prove the L^p-boundedness of these integral operators involving hypergeometric functions ${}_{m+1}F^{\sigma}_{m,k}$, $m \geq 1$.

Theorem 6.2.1: Let

$$S^{\sigma,\alpha,\beta,\gamma}_{m,k}(f)(x) = x^{-(2/p)+1}\int_0^{\infty}(xt)^{(\sigma\beta/k)-1}$$

$$\times {}_{m+1}F^{\sigma}_{m,k}\left[\begin{matrix}\left(\frac{\alpha}{m\sigma},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^{m\sigma}t^{m\sigma}\right]f(t)\,dt.$$

If $p>1, \gamma-\beta>0$, $(\sigma\beta/k)-\alpha<(1/p)<(\sigma\beta/k)$, $0<\alpha<1$ and $k>0$, $\sigma>0$, then

a) $S^{\sigma,\alpha,\beta,\gamma}_{m,k}(f) = A_k I^{\gamma-\beta}_{k,\sigma,(2/p)}\left\{M^{\sigma,\alpha,\beta}_{m,k}(f)\right\}$ where $A_k = \dfrac{\Gamma_k(\gamma)}{\Gamma_k(\beta)}$,

b) $S^{\sigma,\alpha,\beta,\gamma}_{m,k}(f): L^p \to L^p$ and explicitly

c) there exists a constant C^{σ}_k such that $\left\|S^{\sigma,\alpha,\beta,\gamma}_{m,k}(f)\right\|_p \leq C^{\sigma}_k \|f\|_p$.

Proof:
We employ the Theorem 6.1.1 to get

$$I^{\gamma-\beta}_{k,\sigma,(2/p)}\left\{M^{\sigma,\alpha,\beta}_{m,k}(f)\right\}(x)$$

$$= \frac{\sigma x^{-(2/p)+\sigma-(\sigma(\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)}$$

$$\times\int_0^x \left(x^{\sigma}-t^{\sigma}\right)^{((\gamma-\beta)/k)-1} t^{(2/p)-1} M^{\sigma,\alpha,\beta}_{m,k}(f)(t)\,dt.$$

An application of Theorem 6.1.4 and Theorem 6.1.5 then yields

$$I^{\gamma-\beta}_{k,\sigma,(2/p)}\left\{M^{\sigma,\alpha,\beta}_{m,k}(f)\right\}(x)$$

$$= \frac{\sigma x^{-(2/p)+\sigma-(\sigma(\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)}$$

$$\times\int_0^{\infty} f(t) M^{*\sigma,\alpha,\beta}_{m,k}\left[\left(x^{\sigma}-t^{\sigma}\right)^{((\gamma-\beta)/k)-1} t^{(2/p)-1}\right] dt$$

$$= \frac{\sigma x^{-(2/p)+\sigma-(\sigma(\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)}\int_0^{\infty} f(t) M^{*\sigma,\alpha,\beta}_{m,k}\left(\phi^{\sigma}_x(t)\right)dt$$

$$= \frac{\sigma x^{-(2/p)+\sigma-(\sigma(\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)} \int_0^\infty f(t) \frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}{\sigma\Gamma_k(\gamma)} x^{(\sigma\gamma/k)-\sigma} t^{\sigma\beta/k}$$

$$\times {}_{m+1}F_{m,k}^{\sigma}\left[\begin{matrix}\left(\frac{\alpha}{m\sigma},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right) \\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^{m\sigma}t^{m\sigma}\right]dt$$

$$= \frac{\Gamma_k(\beta)x^{-(2/p)+1}}{\Gamma_k(\gamma)} \int_0^\infty (xt)^{(\sigma\beta/k)-1}$$

$$\times {}_{m+1}F_{m,k}^{\sigma}\left[\begin{matrix}\left(\frac{\alpha}{m\sigma},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right) \\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^{m\sigma}t^{m\sigma}\right]f(t)\,dt$$

$$= \frac{\Gamma_k(\beta)}{\Gamma_k(\gamma)} S_{m,k}^{\sigma,\alpha,\beta,\gamma}(f)(x).$$

Hence, $S_{m,k}^{\sigma,\alpha,\beta,\gamma}(f) = A_k I_{k,\sigma,(2/p)}^{\gamma-\beta}\left\{M_{m,k}^{\sigma,\alpha,\beta}(f)\right\}$

where $A_k = \dfrac{\Gamma_k(\gamma)}{\Gamma_k(\beta)}$.

Since $I_{k,\sigma,(2/p)}^{\gamma-\beta}(f): L^p \to L^p$ by Theorem 6.1.1 and it, therefore, follows from Theorem 6.1.2 that if $p>1, \gamma-\beta>0,\ (\sigma\beta/k)-\alpha<(1/p)<(\sigma\beta/k),\ 0<\alpha<1$ and $k>0,\ \sigma>0$, then

$$\left\|S_{m,k}^{\sigma,\alpha,\beta,\gamma}(f)\right\|_p = A_k\left\|I_{k,\sigma,(2/p)}^{\gamma-\beta}\left\{M_{m,k}^{\sigma,\alpha,\beta}(f)\right\}\right\|_p \le A_k C''\left\|M_{m,k}^{\sigma,\alpha,\beta}(f)\right\|_p$$

$$\le A_k C'' C_3 \|f\|_p = C_k^{\sigma}\|f\|_p.$$

Hence, $S_{m,k}^{\sigma,\alpha,\beta,\gamma}(f): L^p \to L^p$ and there exists a constant $C_k^{\sigma} = A_k C'' C_3$ such that

$$\left\|S_{m,k}^{\sigma,\alpha,\beta,\gamma}(f)\right\|_p \le C_k^{\sigma}\|f\|_p.$$

CHAPTER 7

L^p −BOUNDEDNESS OF INTEGRAL OPERATORS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS ${}_mF_{m,k}^{\sigma}$, $m \ge 1$ AS KERNEL

In this chapter, we consider integral operators involving generalized $k-$hypergeometric functions ${}_mF_{m,k}^{\sigma}$, $m \ge 1$ as kernel and discuss the L^p-boundedness of these integral operators by using properties of generalized $k-$hypergeometric functions ${}_mF_{m,k}^{\sigma}$, $m \ge 1$.

7.1 BOUNDEDNESS OF INTEGRALS $H_{m,k}^{\sigma,\beta}(f)$ INVOLVING EXPONENTIAL FUNCTIONS AS KERNEL

In this section, we first consider integral operators of the type

$$H_{m,k}^{\sigma,\beta}(f)(x) = x^{(\sigma\beta/k)-(2/p)}\int_0^{\infty} t^{(\sigma\beta/k)-1} e^{-x^{m\sigma}t^{m\sigma}} f(t)\,dt \ ,(x>0).$$

We prove their boundedness in L^p and other relations that are useful in our study of integral operators involving generalized $k-$hypergeometric functions ${}_mF_{m,k}^{\sigma}, m \ge 1$. We begin with the following result.

Theorem 7.1.1: Let

$$H_{m,k}^{\sigma,\beta}(f)(x) = x^{(\sigma\beta/k)-(2/p)}\int_0^{\infty} t^{(\sigma\beta/k)-1} e^{-x^{m\sigma}t^{m\sigma}} f(t)\,dt \ ,(x>0).$$

If $p>1$, $(\sigma\beta/k)>(1/p)$, $k>0$, $\sigma>0$, then $H_{m,k}^{\sigma,\beta}(f): L^p \to L^p$ and there exists a constant $C_4 = C_4(\beta,k,p,\sigma)$ such that

$$\left\| H_{m,k}^{\sigma,\beta}(f) \right\|_p \le C_4 \|f\|_p .$$

Proof: Note that

$$H_{m,k}^{\sigma,\beta}((V)f)(x) = x^{(\sigma\beta/k)-(2/p)}\int_0^{\infty} t^{(\sigma\beta/k)-1} e^{-x^{m\sigma}t^{m\sigma}} (V(f))(t)\,dt$$

$$= x^{(\sigma\beta/k)-(2/p)} \int_0^\infty t^{(\sigma\beta/k)-(2/p)-1} e^{-x^{m\sigma} t^{m\sigma}} f(1/t)\, dt .$$

Substitute $y = 1/t$.Then

$$H_{m,k}^{\sigma,\beta}\left((V)f\right)(x) = x^{(\sigma\beta/k)-(2/p)} \int_0^\infty y^{(2/p)-(\sigma\beta/k)-1} e^{-x^{m\sigma}/y^{m\sigma}} f(y)\, dy .$$

Suppose that $\psi(x,y) = x^{(\sigma\beta/k)-(2/p)} y^{(2/p)-(\sigma\beta/k)-1} e^{-x^{m\sigma}/y^{m\sigma}}$.

Then $|\psi(hx,hy)| = |h|^{-1} |\psi(x,y)|$.

Since $\psi(x,y)$ is a homogeneous function of degree -1, then by Theorem 4.1.3(a(i)), there exists a constant $C_4 = C_4(\beta,k,p,\sigma)$ such that

$$\left\| H_{m,k}^{\sigma,\beta}\left((V)f\right) \right\|_p \le C_4 \left\| (V)f \right\|_p = C_4 \|f\|_p$$

where

$$C_4 = \int_0^\infty y^{-1/p} \psi(1,y) dy$$

$$= \int_0^\infty y^{(1/p)-(\sigma\beta/k)-1} e^{-1/y^{m\sigma}} dy < \infty ,$$

if $p > 1,\ (\sigma\beta/k) > (1/p),\ k > 0,\ \sigma > 0$.

Thus, $H_{m,k}^{\sigma,\beta}(f) : L^p \to L^p$.

Also since $V(f) : L^p \to L^p$ and $H_{m,k}^{\sigma,\beta}(f) = H_{m,k}^{\sigma,\beta}\left((V^2)(f)\right)$, therefore

$$\left\| H_{m,k}^{\sigma,\beta}(f) \right\|_p = \left\| H_{m,k}^{\sigma,\beta}\left((V^2)(f)\right) \right\|_p \le C_4 \left\| V(f) \right\|_p = C_4 \|f\|_p .$$

Hence, $H_{m,k}^{\sigma,\beta}(f) : L^p \to L^p$ and there exists a constant $C_4 = C_4(\beta,k,p,\sigma)$ such that

$$\left\| H_{m,k}^{\sigma,\beta}(f) \right\|_p \le C_4 \|f\|_p .$$

Lemma 7.1.2: Let

$$H_{m,k}^{*\sigma,\beta}(g)(x) = x^{(\sigma\beta/k)-1} \int_0^\infty t^{(\sigma\beta/k)-(2/p)} e^{-x^{m\sigma} t^{m\sigma}} g(t)\, dt \ ,(x > 0).$$

If $p' > 1,\ (\sigma\beta/k) > (1/p'),\ (1/p) + (1/p') = 1,\ k > 0,\ \sigma > 0$, then $H_{m,k}^{*\sigma,\beta}(g) : L^{p'} \to L^{p'}$ and there exists a constant $C_4' = C_4'(\beta,k,p',\sigma)$ such that

$$\left\| H_{m,k}^{*\sigma,\beta}(g) \right\|_{p'} \le C_4' \|g\|_{p'} .$$

Proof: The proof is similar to that as given in the last theorem.

Now, we consider the following product formula.

Theorem 7.1.3: Let

$$H_{m,k}^{\sigma,\beta}(f)(x) = x^{(\sigma\beta/k)-(2/p)} \int_0^\infty t^{(\sigma\beta/k)-1} e^{-x^{m\sigma} t^{m\sigma}} f(t)\,dt \ ,(x>0)$$

and

$$H_{m,k}^{*\sigma,\beta}(g)(x) = x^{(\sigma\beta/k)-1} \int_0^\infty t^{(\sigma\beta/k)-(2/p)} e^{-x^{m\sigma} t^{m\sigma}} g(t)\,dt \ ,(x>0).$$

If $f \in L^p(R^+)$ and $g \in L^{p'}(R^+)$, then

$$\int_0^\infty g(t) H_{m,k}^{\sigma,\beta}(f)(t)\,dt = \int_0^\infty f(t) H_{m,k}^{*\sigma,\beta}(g)(t)\,dt.$$

Proof: Fubini's theorem allows the change of variables which leads to the proof.

Theorem 7.1.4: For $k>0$, $\sigma>0$, let

$$\begin{aligned} \phi_x^{\sigma}(t) &= \left(x^{\sigma} - t^{\sigma}\right)^{((\gamma-\beta)/k)-1} t^{(2/p)-1} && \text{for } 0<t<x<\infty; \\ &= 0 && \text{for } t \geq x . \end{aligned}$$

Then

$$H_{m,k}^{*\sigma,\beta}\left(\phi_x^{\sigma}\right)(t) = \frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)\, x^{(\sigma\gamma/k)-\sigma} t^{(\sigma\beta/k)-1}}{\sigma\Gamma_k(\gamma)}$$
$$\times {}_mF_{m,k}^{\sigma}\left[\begin{matrix} \left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right) \\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right) \end{matrix};-x^{m\sigma}t^{m\sigma}\right].$$

Proof: Consider the following integral representation

$${}_mF_{m,k}^{\sigma}\left[\begin{matrix} \left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right) \\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right) \end{matrix};x\right]$$
$$= \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} \int_0^1 u^{(\beta/k)-1}(1-u)^{((\gamma-\beta)/k)-1} e^{xu^m}\,du.$$

Substitute $u = w^{\sigma}$ to get

$$
{}_{m}F_{m,k}^{\sigma}\left[\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};x\right]
$$

$$
=\frac{\sigma\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 w^{(\sigma\beta/k)-1}\left(1-w^{\sigma}\right)^{((\gamma-\beta)/k)-1} e^{xw^{m\sigma}}\,dw.
$$

Replacing x by $-x^{m\sigma}t^{m\sigma}$, we have

$$
{}_{m}F_{m,k}^{\sigma}\left[\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^{m\sigma}t^{m\sigma}\right]
$$

$$
=\frac{\sigma\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 w^{(\sigma\beta/k)-1}\left(1-w^{\sigma}\right)^{((\gamma-\beta/k))-1} e^{-x^{m\sigma}t^{m\sigma}w^{m\sigma}}\,dw.
$$

Put $xw = y$ to get

$$
{}_{m}F_{m,k}^{\sigma}\left[\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^{m\sigma}t^{m\sigma}\right]
$$

$$
=\frac{\sigma\Gamma_k(\gamma)x^{\sigma-(\sigma\gamma/k)}}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^x y^{(\sigma\beta/k)-1}\left(x^{\sigma}-y^{\sigma}\right)^{((\gamma-\beta)/k)-1} e^{-t^{m\sigma}y^{m\sigma}}\,dy.
$$

This implies that

$$
\int_0^x y^{(\sigma\beta/k)-1}\left(x^{\sigma}-y^{\sigma}\right)^{((\gamma-\beta)/k)-1} e^{-t^{m\sigma}y^{m\sigma}}\,dy
$$

$$
=\frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)x^{(\sigma\gamma/k)-\sigma}}{\sigma\Gamma_k(\gamma)}
$$

$$
\times {}_{m}F_{m,k}^{\sigma}\left[\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^{m\sigma}t^{m\sigma}\right]. \tag{7.1.1}
$$

Now, consider $H_{m,k}^{*\sigma,\beta}$ as in Theorem 7.1.2, we obtain the following.

$$
H_{m,k}^{*\sigma,\beta}(g)(t)=t^{(\sigma\beta/k)-1}\int_0^{\infty} y^{(\sigma\beta/k)-(p)}e^{-t^{m\sigma}y^{m\sigma}}g(y)\,dy\ ,(t>0).
$$

Then taking $g(t)=\phi_x^\sigma(t)$, we get

$$H_{m,k}^{*\sigma,\beta}\left(\phi_x^\sigma\right)(t)=t^{(\sigma\beta/k)-1}$$
$$\times\int_0^\infty y^{(\sigma\beta/k)-(2/p)}e^{-t^{m\sigma}y^{m\sigma}}\left[\left(x^\sigma-y^\sigma\right)^{((\gamma-\beta)/k)-1}y^{(2/p)-1}\right]dy$$
$$=t^{(\sigma\beta/k)-1}\int_0^x y^{(\sigma\beta/k)-1}\left(x^\sigma-y^\sigma\right)^{((\gamma-\beta)/k)-1}e^{-t^{m\sigma}y^{m\sigma}}dy.$$

By using Equation (7.1.1), we obtain

$$H_{m,k}^{*\sigma,\beta}\left(\phi_x^\sigma\right)(t)=\frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)x^{(\sigma\gamma/k)-\sigma}t^{(\sigma\beta/k)-1}}{\sigma\Gamma_k(\gamma)}$$
$$\times{}_mF_{m,k}^\sigma\left[\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+k(m-1)}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+k(m-1)}{m},k\right)\end{matrix};-x^{m\sigma}t^{m\sigma}\right].$$

7.2 BOUNDEDNESS OF INTEGRAL OPERATORS INVOLVING HYPERGEOMETRIC FUNCTIONS ${}_mF_{m,k}^\sigma$, $m\geq 1$

We now formulate integral operators involving generalized $k-$hypergeometric functions of the type ${}_mF_{m,k}^\sigma$, $m\geq 1$. We use the results proved in Section 7.1 to prove the L^p-boundedness of these integral operators involving generalized $k-$hypergeometric functions ${}_mF_{m,k}^\sigma$, $m\geq 1$.

Theorem 7.2.1: Let

$$G_{m,k}^{\sigma,\beta,\gamma}(f)(x)=x^{-(2/p)+1}\int_0^\infty(xt)^{(\sigma\beta/k)-1}$$
$$\times{}_mF_{m,k}^\sigma\left[\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^{m\sigma}t^{m\sigma}\right]f(t)\,dt.$$

If $p>1$, $(\sigma\beta/k)<(1/p)$, $\gamma-\beta>0$, $k>0$, $\sigma>0$, then

a) $G_{m,k}^{\sigma,\beta,\gamma}(f)=A_kI_{k,\sigma,(2/p)}^{\gamma-\beta}\left\{H_{m,k}^{\sigma,\beta}(f)\right\}$ where $A_k=\dfrac{\Gamma_k(\gamma)}{\Gamma_k(\beta)}$,

b) $G_{m,k}^{\sigma,\beta,\gamma}(f): L^p \to L^p$ and explicitly

c) there exists a constant $C_k'^{\sigma}$ such that

$$\left\| G_{m,k}^{\sigma,\beta,\gamma}(f) \right\|_p \le C_k'^{\sigma} \|f\|_p .$$

Proof:

We employ the Theorem 6.1.1 to get

$$I_{k,\sigma,(2/p)}^{\gamma-\beta}\left\{H_{m,k}^{\sigma,\beta}(f)\right\}(x) = \frac{\sigma x^{-(2/p)+\sigma-(\sigma(\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)} \times \int_0^x \left(x^\sigma - t^\sigma\right)^{((\gamma-\beta)/k)-1} t^{(2/p)-1} H_{m,k}^{\sigma,\beta}(f)(t)\, dt.$$

Now, use of Theorem 7.1.3 and Theorem 7.1.4 then yields

$$I_{k,\sigma,(2/p)}^{\gamma-\beta}\left\{H_{m,k}^{\sigma,\beta}(f)\right\}(x) = \frac{\sigma x^{-(2/p)+\sigma-(\sigma(\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)} \times \int_0^\infty f(t) H_{m,k}^{*\sigma,\beta}\left[\left(x^\sigma - t^\sigma\right)^{((\gamma-\beta)/k)-1} t^{(2/p)-1}\right] dt$$

$$= \frac{\sigma x^{-(2/p)+\sigma-(\sigma(\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)} \int_0^\infty f(t) H_{m,k}^{*\sigma,\beta}\left(\phi_x^\sigma(t)\right) dt .$$

$$= \frac{\sigma x^{-(2/p)+\sigma-(\sigma(\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)} \times \int_0^\infty f(t) \frac{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}{\sigma\Gamma_k(\gamma)} x^{(\sigma\gamma/k)-\sigma} t^{\sigma\beta/k}$$

$$\times {}_mF_{m,k}^{\sigma}\left[\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix}; -x^{m\sigma} t^{m\sigma}\right] dt$$

$$= \frac{\Gamma_k(\beta) x^{-(2/p)+1}}{\Gamma_k(\gamma)} \int_0^\infty (xt)^{(\sigma\beta/k)-1}$$

$$\times {}_mF_{m,k}^{\sigma}\left[\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+k(m-1)}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+k(m-1)}{m},k\right)\end{matrix}; -x^{m\sigma} t^{m\sigma}\right] f(t)\, dt$$

where $A_k = \dfrac{\Gamma_k(\gamma)}{\Gamma_k(\beta)}$.

Since $I^{\gamma-\beta}_{k,\sigma,(2/p)}(f) : L^p \to L^p$ by Theorem 6.1.1 and it, therefore, follows from Theorem 7.1.1 that if $p > 1, (\sigma\beta/k) < (1/p), \gamma - \beta > 0, k > 0, \sigma > 0$,
then

$$\left\| G^{\sigma,\beta,\gamma}_{m,k}(f) \right\|_p = A^{\sigma}_k \left\| I^{\sigma,\beta,\gamma}_k \left(H^{\sigma,\beta}_{m,k}(f) \right) \right\|_p \le A^{\sigma}_k C'' \left\| H^{\sigma,\beta}_{m,k}(f) \right\|_p$$

$$\le A^{\sigma}_k C'' C_4 \|f\|_p = C'^{\sigma}_k \|f\|_p .$$

Hence, $G^{\sigma,\beta,\gamma}_{m,k}(f) : L^p \to L^p$ and there exists a constant $C'^{\sigma}_k = A^{\sigma}_k C'' C_4$ such that

$$\left\| G^{\sigma,\beta,\gamma}_{m,k}(f) \right\|_p \le C'^{\sigma}_k \|f\|_p .$$

CHAPTER 8

L^p −BOUNDEDNESS OF INTEGRAL OPERATORS WITH WEIGHT FUNCTIONS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION ${}_{m+1}F^{\sigma}_{m,k}$ AND ${}_{m}F^{\sigma}_{m,k}$, $m \geq 1$ AS KERNEL

In this chapter, we consider integral operators involving generalized $k-$hypergeometric functions ${}_{m+1}F^{\sigma}_{m,k}$, $m \geq 1$ as kernel and discuss the L^p-boundedness of these integral operators with weight functions x^{ν} by using properties of generalized $k-$hypergeometric functions ${}_{m+1}F^{\sigma}_{m,k}$, $m \geq 1$.

8.1 BOUNDEDNESS OF INTEGRALS $M^{\sigma,\alpha,\beta}_{m,k}(f)$ WITH WEIGHT FUNCTION

In this section, we first consider integral operators of the type

$$M^{\sigma,\alpha,\beta}_{m,k}(f)(x) = x^{(\sigma\beta/k)-(2/p)} \int_0^{\infty} t^{(\sigma\beta/k)-1} \left(1 + x^{m\sigma} t^{m\sigma}\right)^{-\alpha/m\sigma} f(t)\, dt \ ,(x > 0).$$

We prove their boundedness in L^p and other relations that are useful in our study of integral operators involving generalized $k-$hypergeometric functions ${}_{m+1}F^{\sigma}_{m,k}$, $m \geq 1$. We begin with the following result.

Theorem 8.1.1: Let

$$M^{\sigma,\alpha,\beta}_{m,k}(f)(x) = x^{(\sigma\beta/k)-(2/p)} \int_0^{\infty} t^{(\sigma\beta/k)-1} \left(1 + x^{m\sigma} t^{m\sigma}\right)^{-\alpha/m\sigma} f(t)\, dt .$$

If $p > 1$, $(\sigma\beta/k) - \alpha < (1/p) - (\nu/p) < (\sigma\beta/k)$, $0 < \alpha < 1$, $k > 0$, $\sigma > 0$, then $M^{\sigma,\alpha,\beta}_{m,k}(f) : L^p \to L^p$ and there exists a constant $C_5 = C_5(\alpha,\beta,k,p,\sigma,\nu)$ such that

$$\left(\int_0^{\infty} x^{\nu} \left| M^{\sigma,\alpha,\beta}_{m,k}(f)(x) \right|^p dx \right)^{1/p} \leq C_5 \left(\int_0^{\infty} x^{\nu} \left| f(x) \right|^p dx \right)^{1/p} .$$

Proof: From Theorem 6.1.2, note that

$$M_{m,k}^{\sigma,\alpha,\beta}\left(\left(V\right)f\right)(x)=x^{(\sigma\beta/k)-(2/p)}\int_0^\infty y^{(2/p)+\alpha-(\sigma\beta/k)-1}\left(x^{m\sigma}+y^{m\sigma}\right)^{-\alpha/m\sigma}f(y)\,dy.$$

Suppose that $\psi(x,y)=x^{(\sigma\beta/k)-(2/p)}y^{(2/p)+\alpha-(\sigma\beta/k)-1}\left(x^{m\sigma}+y^{m\sigma}\right)^{-\alpha/m\sigma}$.

$$\left|\psi(hx,hy)\right|=\left|h\right|^{-1}\left|\psi(x,y)\right|.$$

Since $\psi(x,y)$ is a homogeneous function of degree -1, by Theorem 4.1.3(a(ii)), there exists a constant $C_5=C_5(\alpha,\beta,k,p,\sigma,\nu)$ such that

$$\left(\int_0^\infty x^\nu\left|M_{m,k}^{\sigma,\alpha,\beta}(f)(x)\right|^p dx\right)^{1/p}\le C_5\left(\int_0^\infty x^\nu\left|V(f)(x)\right|^p dx\right)^{1/p}=C_5\left(\int_0^\infty x^\nu\left|f(x)\right|^p dx\right)^{1/p}$$

where

$$\begin{aligned}C_5&=\int_0^\infty y^{-(1/p)-(\nu/p)}\psi(1,y)\,dy\\&=\int_0^\infty y^{(1/p)-(\nu/p)+\alpha-(\sigma\beta/k)-1}\left(1+y^{m\sigma}\right)^{-\alpha/m\sigma}dy\\&=\int_0^1 y^{(1/p)-(\nu/p)+\alpha-(\sigma\beta/k)-1}\left(1+y^{m\sigma}\right)^{-\alpha/m\sigma}dy\\&+\int_1^\infty y^{(1/p)-(\nu/p)+\alpha-(\sigma\beta/k)-1}\left(1+y^{m\sigma}\right)^{-\alpha/m\sigma}dy<\infty,\end{aligned}$$

if $p>1$, $(\sigma\beta/k)-\alpha<(1/p)-(\nu/p)<(\sigma\beta/k)$, $0<\alpha<1$, $k>0$, $\sigma>0$.

Hence, $M_{m,k}^{\sigma,\alpha,\beta}(f):L^p\to L^p$ and there exists a constant $C_5=C_5(\alpha,\beta,k,p,\sigma,\nu)$ such that

$$\left(\int_0^\infty x^\nu\left|M_{m,k}^{\sigma,\alpha,\beta}(f)(x)\right|^p dx\right)^{1/p}\le C_5\left(\int_0^\infty x^\nu\left|f(x)\right|^p dx\right)^{1/p}.$$

Theorem 8.1.2: Let $p>1,\gamma-\beta>0,\nu>1,k>0,\sigma>0$.
If

$$I_{k,\sigma,(2/p)}^{\gamma-\beta}(f)(x)=\frac{x^{-(2/p)+\sigma-(\sigma(\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)} \times\int_0^x \left(x^\sigma-t^\sigma\right)^{((\gamma-\beta)/k)-1} t^{(2/p)-1} f(t)dt,(x>0),$$

then $I_{k,\sigma,(2/p)}^{\gamma-\beta}(f):L^p\to L^p$ and there exists a constant $C'''=C'''(\beta,\gamma,k,p,\sigma,\nu)$ such that

$$\left(\int_0^\infty x^\nu \left|I_{k,\sigma,(2/p)}^{\gamma-\beta}(f)(x)\right|^p dx\right)^{1/p} \le C'''\left(\int_0^\infty x^\nu \left|f(x)\right|^p dx\right)^{1/p}.$$

Proof: Consider

$$I_{k,\sigma,(2/p)}^{\gamma-\beta}(f)(x)=\frac{\sigma x^{-(2/p)+\sigma-(\sigma(\gamma-\beta)/k)}}{k\Gamma_k(\gamma-\beta)} \times\int_0^x \left(x^\sigma-t^\sigma\right)^{((\gamma-\beta)/k)-1} t^{(2/p)-1} f(t)dt.$$

Suppose that $\psi(x,t)=x^{-(2/p)+\sigma-(\sigma(\gamma-\beta)/k)}\left(x^\sigma-t^\sigma\right)^{((\gamma-\beta)/k)-1} t^{(2/p)-1}$.

Then $\left|\psi(hx,ht)\right|=|h|^{-1}\left|\psi(x,t)\right|$.

Since $\psi(x,y)$ is a homogeneous function of degree -1, by Theorem 4.1.3 (a(ii)), there exists a constant $C'''=C'''(\beta,\gamma,k,p,\sigma,\nu)$ such that

$$\left(\int_0^\infty x^\nu \left|I_{k,\sigma,(2/p)}^{\gamma-\beta}(f)(x)\right|^p dx\right)^{1/p} \le C'''\left(\int_0^\infty x^\nu \left|f(x)\right|^p dx\right)^{1/p}$$

where

$$C''=\int_0^1 t^{-(1/p)-(\nu/p)}\left|\psi(1,t)\right|dt =\int_0^1 t^{(1/p)-(\nu/p)-1}\left(1-t^\sigma\right)^{((\gamma-\beta)/k)-1} dt<\infty ,$$

if $p>1,\gamma-\beta>0,\nu>1,k>0,\sigma>0$.

Hence, $I_{k,\sigma,(2/p)}^{\gamma-\beta}(f):L^p\to L^p$ and there exists a constant $C'''=C'''(\beta,\gamma,k,p,\sigma,\nu)$ such that

$$\left(\int_0^\infty x^\nu \left|I_{k,\sigma,(2/p)}^{\gamma-\beta}(f)(x)\right|^p dx\right)^{1/p} \le C'''\left(\int_0^\infty x^\nu |f(x)|^p dx\right)^{1/p}.$$

8.2 BOUNDEDNESS OF INTEGRAL OPERATORS INVOLVING HYPERGEOMETRIC FUNCTIONS ${}_{m+1}F_{m,k}^{\sigma}$, $m \ge 1$ WITH WEIGHT FUNCTION x^ν

We now formulate new integral operators involving generalized $k-$hypergeometric functions of the type ${}_{m+1}F_{m,k}^{\sigma}$, $m \ge 1$. We use the results on $M_{m,k}^{*\sigma,\alpha,\beta}$ proved in Section 8.1 to establish the L^p-boundedness of these integral operators involving generalized $k-$hypergeometric functions ${}_{m+1}F_{m,k}^{\sigma}$, $m \ge 1$ with weight function x^ν.

Theorem 8.2.1: Let

$$S_{m,k}^{\sigma,\alpha,\beta,\gamma}(f)(x) = x^{-(2/p)+1}\int_0^\infty (xt)^{(\sigma\beta/k)-1}$$
$$\times {}_{m+1}F_{m,k}^{\sigma}\left[\begin{matrix}\left(\frac{\alpha}{m\sigma},1\right),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^{m\sigma}t^{m\sigma}\right]f(t)\,dt.$$

If $p>1, \gamma-\beta>0, \nu>1, (\sigma\beta/k)-\alpha<(1/p)-(\nu/p)<(\sigma\beta/k)$, and $0<\alpha<1,\ k>0,\ \sigma>0$ there exists a constant $C_k^{\sigma,\nu}$ such that

$$\left(\int_0^\infty x^\nu \left|S_{m,k}^{\sigma,\alpha,\beta,\gamma}(f)(x)\right|^p dx\right)^{1/p} \le C_k^{\sigma,\nu}\left(\int_0^\infty x^\nu |f(x)|^p dx\right)^{1/p}.$$

Proof:
We employ the Theorem 8.1.1 and 8.1.2 to get

$$\left(\int_0^\infty x^\nu \left|S_{m,k}^{\sigma,\alpha,\beta,\gamma}(f)(x)\right|^p dx\right)^{1/p} = \left(\int_0^\infty x^\nu \left|I_{k,\sigma,(2/p)}^{\gamma-\beta}\left\{M_{m,k}^{\sigma,\alpha,\beta}(f)\right\}(x)\right|^p dx\right)^{1/p}$$

$$\le \left(\int_0^\infty x^\nu \left|M_{m,k}^{\sigma,\alpha,\beta}(f)(x)\right|^p dx\right)^{1/p} \le C_k^{\sigma,\nu}\left(\int_0^\infty x^\nu |f(x)|^p dx\right)^{1/p}$$

Hence, $S_{m,k}^{\sigma,\alpha,\beta,\gamma}(f): L^p \to L^p$ and there exists a constant $C_k^{\sigma,\nu}$ such that

$$\left(\int_0^\infty x^\nu \left|S_{m,k}^{\sigma,\alpha,\beta,\gamma}(f)(x)\right|^p dx\right)^{1/p} \le C_k^{\sigma,\nu}\left(\int_0^\infty x^\nu \left|f(x)\right|^p dx\right)^{1/p}.$$

8.3 BOUNDEDNESS OF INTEGRALS $H_{m,k}^{\sigma,\beta}$ WITH WEIGHT FUNCTION

In this section, we take up integral operators involving generalized $k-$hypergeometric functions ${}_mF_{m,k}^{\sigma}$, $m \ge 1$ as kernel and discuss the L^p-boundedness of these integral operators with weight function x^ν by using properties of generalized $k-$hypergeometric functions ${}_mF_{m,k}^{\sigma}$, $m \ge 1$.

We first consider integral operators of the type

$$H_{m,k}^{\sigma,\beta}(f)(x) = x^{(\sigma\beta/k)-(2/p)}\int_0^\infty t^{(\sigma\beta/k)-1} e^{-x^{m\sigma}t^{m\sigma}} f(t)\,dt \;,(x>0).$$

We prove their boundedness in L^p and other relations that are useful in our study of integral operators involving generalized $k-$hypergeometric functions ${}_mF_{m,k}^{\sigma}$, $m \ge 1$ with weight function x^ν. We begin with the following result.

Theorem 8.3.1: Let

$$H_{m,k}^{\sigma,\beta}(f)(x) = x^{(\sigma\beta/k)-(2/p)}\int_0^\infty t^{(\sigma\beta/k)-1} e^{-x^{m\sigma}t^{m\sigma}} f(t)\,dt \;,(x>0).$$

If $p>1,\; (\sigma\beta/k) > (1/p)-(\nu/p),\; k>0,\; \sigma>0$, then $H_{m,k}^{\sigma,\beta}(f): L^p \to L^p$ and there exists a constant $C_6 = C_6(\beta,k,p,\sigma,\nu)$ such that

$$\left(\int_0^\infty x^\nu \left|H_{m,k}^{\sigma,\beta}(f)(x)\right|^p dx\right)^{1/p} \le C_6\left(\int_0^\infty x^\nu \left|f(x)\right|^p dx\right)^{1/p}.$$

Proof: From Theorem 7.1.1

$$H_{m,k}^{\sigma,\beta}((V)f)(x) = x^{(\sigma\beta/k)-(2/p)}\int_0^\infty y^{(2/p)-(\sigma\beta/k)-1} e^{-x^{m\sigma}/y^{m\sigma}} f(y)\,dy.$$

Suppose that $\psi(x,y) = x^{(\sigma\beta/k)-(2/p)} y^{(2/p)-(\sigma\beta/k)-1} e^{-x^{m\sigma}/y^{m\sigma}}$.

Then $\left|\psi\left(hx,hy\right)\right| = \left|h\right|^{-1}\left|\psi\left(x,y\right)\right|.$

Since $\psi\left(x,y\right)$ is a homogeneous function of degree -1, by Theorem 4.1.3(a(ii)), there exists a constant $C_6 = C_6\left(\beta,k,p,\sigma,\nu\right)$ such that

$$\left(\int_0^\infty x^\nu \left|H_{m,k}^{\sigma,\beta}\left(f\right)\left(x\right)\right|^p dx\right)^{1/p} \le C_6\left(\int_0^\infty x^\nu \left|V\left(f\left(x\right)\right)\right|^p dx\right)^{1/p} = C_6\left(\int_0^\infty x^\nu \left|f\left(x\right)\right|^p dx\right)^{1/p}$$

where

$$C_6 = \int_0^\infty y^{-(1/p)-(\nu/p)}\psi\left(1,y\right)dy$$

$$= \int_0^\infty y^{(1/p)-(\nu/p)-(\sigma\beta/k)-1} e^{-1/y^{m\sigma}} dy < \infty,$$

if $p > 1$, $\left(\sigma\beta/k\right) > \left(1/p\right) - \left(\nu/p\right)$, $k > 0$, $\sigma > 0$.

Hence, $H_{m,k}^{\sigma,\beta}\left(f\right): L^p \to L^p$ and there exists a constant $C_6 = C_6\left(\beta,\nu,\sigma,k,p\right)$ such that

$$\left(\int_0^\infty x^\nu \left|H_{m,k}^{\sigma,\beta}\left(f\right)\left(x\right)\right|^p dx\right)^{1/p} \le C_6\left(\int_0^\infty x^\nu \left|f\left(x\right)\right|^p dx\right)^{1/p}.$$

8.4 BOUNDEDNESS OF INTEGRAL OPERATORS INVOLVING HYPERGEOMETRIC FUNCTIONS ${}_mF_{m,k}^{\sigma}$, $m \ge 1$ WITH WEIGHT FUNCTION x^ν

We now formulate integral operators involving generalized $k-$hypergeometric functions of the type ${}_mF_{m,k}^{\sigma}$, $m \ge 1$. We establish the L^p-boundedness of these integral operators involving generalized $k-$hypergeometric functions ${}_mF_{m,k}^{\sigma}$, $m \ge 1$ with weight function x^ν.

Theorem 8.4.1: Let

$$G_{m,k}^{\sigma,\beta,\gamma}\left(f\right)\left(x\right) = x^{-(2/p)+1}\int_0^\infty \left(xt\right)^{(\sigma\beta/k)-1}$$

$$\times {}_mF_{m,k}^{\sigma}\left[\begin{matrix}\left(\frac{\beta}{m},k\right),\left(\frac{\beta+k}{m},k\right),\ldots,\left(\frac{\beta+(m-1)k}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+k}{m},k\right),\ldots,\left(\frac{\gamma+(m-1)k}{m},k\right)\end{matrix};-x^{m\sigma}t^{m\sigma}\right]f\left(t\right)dt.$$

If $p>1,\ \gamma-\beta>0,\ \ \nu>1,\ \ (\sigma\beta/k)>(1/p)-(\nu/p), k>0,\ \sigma>0$, then there exists a constant $C_k'^{\sigma,\nu}$ such that

$$\left(\int_0^\infty x^\nu \left|G_{m,k}^{\sigma,\beta,\gamma}(f)(x)\right|^p dx\right)^{1/p} \le C_k'^{\sigma,\nu}\left(\int_0^\infty x^\nu \left|f(x)\right|^p dx\right)^{1/p}.$$

Proof:

We employ Theorem 8.1.2 and 8.3.1 to get

$$\left(\int_0^\infty x^\nu \left|G_{m,k}^{\sigma,\beta,\gamma}(f)(x)\right|^p dx\right)^{1/p} = \left(\int_0^\infty x^\nu \left|I_{k,\sigma,(2/p)}^{\gamma-\beta}\left\{H_{m,k}^{\sigma,\beta}(f)\right\}(x)\right|^p dx\right)^{1/p}$$

$$\le \left(\int_0^\infty x^\nu \left|H_{m,k}^{\sigma,\beta}(f)(x)\right|^p dx\right)^{1/p} \le C_k'^{\sigma,\nu}\left(\int_0^\infty x^\nu \left|f(x)\right|^p dx\right)^{1/p}$$

Hence, $G_{m,k}^{\sigma,\beta,\gamma}(f): L^p \to L^p$ and there exists a constant $C_k'^{\sigma,\nu}$ such that

$$\left(\int_0^\infty x^\nu \left|G_{m,k}^{\sigma,\beta,\gamma}(f)(x)\right|^p dx\right)^{1/p} \le C_k'^{\sigma,\nu}\left(\int_0^\infty x^\nu \left|f(x)\right|^p dx\right)^{1/p}.$$

REFRENCES

1) Agarwal, P. and Jain, S. (2011). Further results on fractional calculus of Srivastava polynomials. *Bulletin of Mathematical Analysis and Applications*. Vol. 3, pp. 167-174.

2) Ahmad, M. and Saboor, A. (2009). Properties of a newly defined hypergeometric power series function. *Pakistan Journal of Statistics*. Vol. 25. pp. 227-233.

3) Aliev, I. L. (2010). A relation between Bessel and Fourier transforms and its application to the weighted inequalities. *Integral Transforms and Special Functions*. Vol. 21, pp. 135-142.

4) Andersen, K. F. and Heinig, H. P. (1983). Weighted norm inequalities for certain integral operators. *Society for Industrial and Applied Mathematics.* Vol. 14, pp.834-844.

5) Andrews, G. E., Askey, R. and Roy, R. (2004). *Special Functions*. Cambridge University Press.

6) Arcadii, Z. G. (2011). Volterra convolution equations: solution–kernel connection. *Integral Transforms and Special Functions*. IFirst, pp. 1-13.

7) Belarbi, S. and Dahmani, Z. (2009). On some new fractional integral inequalities. *Journal of Inequalities in Pure and Applied Mathematics.* Vol. 10, article 86.

8) Castell, W. Z. (2004). Fractional derivatives and the inverse Fourier transform of l_1-radial functions. *Integral Transforms and Special Functions*. Vol. 15, pp. 209–223.

9) Chiavetta, R. J. (2007a). Summation of a certain series of hypergeometric functions using Schäfke's method. *Planet Math Paper*.

10) Chiavetta, R. J. (2007b). Summation of another series of hypergeometric functions using Schäfke's method. *Planet Math Paper*.

11) Cuyt, A. Petersen, V. and Verdonk, B. (2008). *Hand Book of Continued Fractions for Special Functions*. Springer Science and Business Media B. V.

12) Diaz, R. and Teruel, C. (2005). q,k-Generalized gamma and beta functions. *Journal of Nonlinear Mathematical Physics*. Vol. 12, pp. 118-134.

13) Diaz, R. and Pariguan, E. (2007). On hypergeometric functions and Pochhammer k- symbol. *Divulgaciones Matemáticas*. Vol. 15, pp. 179-192.

14) Diaz, R., Ortiz, C. and Pariguan, E. (2010). On the k-gamma q-distribution. *Central European Journal of Mathematics.* Vol. 8, pp. 448-458.

15) Driver, K. A. and Johnston, S. J. (2006). An integral representation of some hypergeometric functions. *Electronic Transactions on Numerical Analysis*. Vol. 25, pp. 115-120.

16) Erdélyi, A. (1950). On some functional representations. *Univ. E. Politec, Torino Rend Semi Math*. Vol. 10, pp. 217- 234.

17) Erdélyi, A. (1953). *Higher Transcendental Functions*. Vol. 1 McGraw Hill Book Co, New York.

18) Erdélyi, A. (1963). An integral equation involving Legendre polynomials. *Amer. Math*. Monthly 70, pp. 651-652.

19) Erdélyi, A. (1964). An integral equation involving Legendre functions. *J. Soc. Indust. Appl. Math*. Vol. 12, pp. 15-30.

20) Goyal, S. P., Jain, R. M. and Gaur, N. (1991). Fractional integral operators involving a product of generalized hypergeometric functions and a general class of polynomials. *Indian Journal of Pure and Applied Mathematics.* Vol. 22, pp. 403-411.

21) Habibullah, G. M. (1970). An integral equation involving Shively's polynomials, *J. Soc. Math.* Vol. 19, pp. 209-214.

22) Habibullah, G. M. (1971). Some integral equations involving confluent hypergeometric functions. *The Yokohama Mathematical Journal*. Vol. 19, pp. 35-43.

23) Habibullah, G. M. (1977). Some integral operators with hypergeometric functions as kernels. *Bul. Math. Soc*. Vol. 21, pp. 293-300.

24) Habibullah, G. M. and Choudhary, M. D. (1993). On fractional integration. *Mathematics Forum*. Vol. 3&4, pp. 39-51.

25) Habibullah, G. M. and Mubeen, S. (2011). L^p-Bounded operators involving generalized hypergeometric functions. *Integral Transforms and Special Functions*. Vol. 22, pp. 43-149.

26) Hardy, H. G., Littlewood, J. E. and Pulya (1952). Inequalities. *Cambridge University Press.*

27) Haubold, H. J., Mathai A. M. and Saxena, R. K. (2011). Mittag-leffler functions and their applications. *Journal of applied mathematics* Article ID 298628, 51 pages, doi:10.1155/2011/298628.

28) Higgins, T.P. (1963). An inversion of integral for a Gegenbauer transformation. *J. Soc. Indust. Appl. Math.* Vol. 11, pp. 886-893.

29) Kalla, S. L. and Saxena, R. K. (1969). Integral operators involving hypergeometric function. *Math. Z.* Vol. 108, pp. 231-234.

30) Karapetiants, N. K. and Samko, S. G. (1999). Multi-dimensional integral operators with homogeneous kernels. *Fract. Calc. and Applied Anal*. Vol. 2, pp. 67-96.

31) Kilbas, A. A., Repin, Oleg A, and Saigo, Megumi (2000). On the Meijer transform in $L_{\upsilon,r}$-space. *Integral Transforms and Special Functions*. Vol. 10. pp. 267-282.

32) Kilbas, A. A., Repin, O. A, and Saigo, M. (2002). Generalized fractional integral transforms with Gauss functions kernels as G-transforms. *Integral Transform and Special Functions*. Vol. 13, pp. 285-307.

33) Kilbas, A. A., Saigo, M. and Saxena, R. K. (2004). Generalized Mittag-Leffler function and generalized fractional calculus operators. *Integral Transform and Special Functions*. Vol. 15, pp. 31-49.

34) Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J. (2006). *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematical Studies, 204, Elsevier B. V., Amsterdam.

35) Kilbas, A. A. and Sebastian, N. (2008). Generalized fractional integration of Bessel function of the first kind. *Integral Transforms and Special Functions*. Vol. 19, pp. 869-883.

36) Kober, H. (1940). On fractional integrals and derivatives. *Quart. J. Math*. pp. 193-211.

37) Kokologiannaki, C. G. (2010). Properties and inequalities of generalized k- gamma, beta and zeta functions. *Int. J. Contemp. Math. Sciences.* Vol. 5, pp. 653-660.

38) Krasniqi, V. (2010). A limit for the k-gamma and k-beta function. *Int. Math. Forum.* Vol. 5, pp. 1613-1617.

39) Love, E. R. (1967). Some integral equations involving hypergeometric functions. *Proc. Edinburgh, Math. Soc*. Vol. 15, pp. 169-198.

40) Lowndes, J. S. (1970). A generalization of Erdélyi-Kober operations. *Proc. Edinburgh Math. Soc.* Vol. 17, pp. 139-140.

41) Mansour, M. (2009). Determining the k- generalized gamma function $\Gamma_k(x)$ by functional equations. *Int. J. Contemp. Math. Sciences.* Vol. 4, pp. 1037-1042.

42) Mathai, A. M. (1993). *A Hand Book of Generalized Special Functions for Statistical and Physical Sciences*. University Press Inc., New York.

43) Mathai, A. M. and Houbold, H. J. (2008). *Special Functions for Applied Scientists*. Springer Science and Business Media, LLC, New York.

44) Mathai, A. M., Saxena, R. K. and Houbold, H. J. (2010). *The H-Function, Theory and Application.* Springer, New York.

45) Medhat, A. R. and Arjun, K. R. (2011). Generalizations of classical summation theorems for the series ${}_2F_1$ and ${}_3F_2$ with applications. *Integral Transforms and Special Functions*. IFirst, pp. 1-18.

46) Merovci, F. (2010). Power product inequalities for the Γ_k function. *Int. J. Math. Analysis.* Vol. 4, pp. 1007-1012.

47) Nishimoto, K. (1991). *An Essence of Nishimoto's Fractional Calculus: Integration and Differentiation of Arbitrary Order.* Descartes Press, Koriyama, Japan.

48) Okikiolu, G. O. (1966). Bounded linear transformations in L^p-space. *London Math. Soc*. Vol. 41, pp. 407-414.

49) Okikiolu, G. O. (1971). *Aspects of the theory of bounded linear operators*. Academic Press.

50) Prabhakar, T. R. (1971). A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama J. Math.* Vol. 19, pp. 7-15.

51) Rainville, E. D. (1963). *Special Functions*. The Macmillan Company, New York.

52) Saad, N. and Hall, R. (2003). Integrals containing confluent hypergeometric functions with applications to perturbed singular potentials. *eprint,arxiv.org/PS_cache/mathph/pdf/0306/0306043v1*.

53) Saha, U. K. and Arora, L. K., (2010). Riemann-Liouville fractional calculus and Multiindex Dzrbashjan Gelfond-Leontiev differentiation and integration with Multiindex Mittag-Leffler function. *World Academy of Science, Engineering and Technology*. Vol. 68, pp.100-104.

54) Saigo, M. (1977/78). A remark on integral operators involving the Gauss hypergeometric functions. *Math. Rep. Kyushu Univ*. Vol. 11, pp. 135-143.

55) Saigo, M. and Kilbas, A. A. (1994). Generalized fractional integrals and derivatives in Hölder spaces. *Transforms Methods and Special Functions (proc. Intern. Workshop 1995), Sofia*, pp. 12-17, August, *Science Culture Techn. Publ., Singapore.* pp. 282-293.

56) Salahuddin (2011). Development of some summation formulae based on half argument associated to hypergeometric function. *Int. J. Contemp. Math. Sciences*. Vol. 6, pp. 2183-2188.

57) Samko, S. G., Kilbas, A. A. and Marichev, O. (2002). *Fractional Integrals and Derivatives, Theory and Applications*. Taylor & Francis Books, London.

58) Saxena, R. K. (1967). On fractional integration operators. *Math. Z.* Vol. 96, pp. 288-291.

59) Saxena, R. K. and Ram, J. (1990). On multidimensional generalized Kober operators associated with the Gauss's hypergeometric functions. *Collect. Math*. Vol. 41, pp. 27-34.

60) Saxena, R. K. and Saigo, M. (1998). Fractional integral formula for the H-function II. *J. Frac. Calc.* Vol. 13, pp. 37-41.

61) Saxena, R. K., Ram, C., Dudi, N. and Kalla, S. L. (2007). Generalized gamma-type functions involving Kummer's confluent function and associated probability distributions. *Integral Transforms and Special Functions*. Vol. 18, pp. 679-687.

62) Srivastava, H. M. (1972). A contour integral involving Fox's H-function. *Indian J. Math*. Vol. 14, pp. 1-6.

63) Srivastava, H. M. (1976). An integral equation involving the confluent hypergeometric functions of several complex variables. *Integral Transforms and Special Functions*. Vol. 5, pp. 251-256.

64) Srivastava, H. M. and Saxena, R. K. (2001). Operators of fractional integration and their applications. *Appl. Math. Comput.* Vol. 118, pp. 1-52.

65) Srivastava, H. M. and Tomovski, Z. (2009). Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel. *Appl. Math. Comput*. Vol. 211, pp. 198-210.

66) Srivastava, K. N. (1963). A class of integral equations involving ultra spherical polynomials as kernels. *Proc. Amer. Math. Soc*. Vol. 14, pp. 932-940.

67) Srivastava, K. N. (1966). A class of integral equations involving Laguerre polynomials as kernels. *Proc. Edinburgh Math. Soc*. Vol. 15, pp. 33-36.

68) Vi Nhan, N. D. and Duc, D. T. (2008). Weighted L_p-norm inequalities in convolutions and their applications. *Journal of Mathematical Inequalities*. Vol. 2, pp. 45-55.

69) Virchenko, N., Kalla, S. L. and AL-Zamel, A. (2001). Some results on a generalized hypergeometric functions. *Integral transforms and Special Functions*. Vol. 12, pp. 89-100.

70) Virchenko, N. O., Lisetska, O. and Kalla, S. L., (2010). On some fractional integral operators involving generalized Gauss hypergeometric functions. *Applications and Applied Mathematics*. Vol. 5, pp. 1418-1427.

71) Virchenko, N. O. and Ovcharenko, O. V. (2011). Some new integral transforms with the generalized hypergeometric function. *Integral Transforms and Special Functions*. IFirst, pp. 1-7.

72) Wang, Z. X. and Guo, D. R. (1989). *Special Functions*. World Scientific Publishing Co. Pvt. Ltd.

73) Wimp, J. (1965). Two integral transform pairs involving hypergeometric functions. *Proc. Glasgow. Math. Assoc*. Vol. 7, pp. 42-44.

74) Zareen, A. K. (2011). Still more generalizations of Hardy's integral inequalities for functions of two independent variables. *Int. J. Contemp. Math. Sciences*. Vol. 6, pp. 1955-1968.