Beltrami States in Three Component Plasmas

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Acknowledgement

In the name of Allah, the Most Gracious and the Most Merciful. First and Foremost praise is to ALLAH, the Almighty, the greatest of all, on whom ultimately we depend for sustenance and guidance. I would like to thank Almighty Allah for giving me opportunity, determination and strength to do my research. His continuous grace and mercy was with me throughout my life and ever more during the tenure of my research. All respects may always be given to our Prophet Muhammad (P.B.U.H), the messenger of Allah, who has shown the truth religion and taken all human being from the darkness to the lightness.

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Abstract

A static and stationary relaxed state of a single fluid MHD plasma appears as a consequence of the minimization of magnetic energy while the magnetic helicity remains conserved. The relaxed state is known as Beltrami state and represented by eigenvalue equation of the curl operator. The Beltrami state is a force-free equilibrium state and characterized by a single scale parameter. There exists no pressure gradient in ideal MHD self-organized state. On the other hand, self-organized states of multi component plasmas composed of multi Beltrami states are characterized by strong flow and pressure gradients which are the important features of practical plasmas found in laboratory and space environments. The self-organized states of multi component plasmas are investigated and analyzed. It is shown that self-organized state of electronegative dusty plasma comprising of one positive ion, two negative ions and immobile heavy dust grains can be cast as a superposition of three Beltrami states characterized by three scale parameters. This state is called as Triple Beltrami state and has a wide range of solutions covering paramagnetic as well as diamagnetic self-organized states. The relaxed equilibrium state of three component plasmas is found to be composed of four Beltrami states when all the inertial and non-inertial forces are taken into account. Two of the components constitute a pair while other one is taken to be a singly ionized positive ion. The self-organized state is called as Quadruple Beltrami state and governed by four eigenvalues. It is found that density of ions plays a significant role in the formation of relaxed structures. It is also shown that the relaxed state of three component plasmas can be cast in terms of double Beltrami state which is the superposition of two Beltrami states. The loss of equilibrium in a slowly varying double Beltrami state is investigated by changing control parameter. The equilibrium state is analyzed to observe the effects of magnetic helicity, generalized helicity, ratio of magnetic field to positron flow, density of components and energy of the system. The critical values of energy and control parameter are derived and the results are applied to model the eruptive events occurring in solar coronal loops.
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Chapter 1

Introduction

1.1 Motivation

Self-organization is a process where a system reproduces itself in such a way that a purposeful order is achieved from randomness. In physics, chemistry, and biology, self-organization occurs in open systems driven away from thermal equilibrium [1]. The process of self-organization can be found in many other fields also, such as economy, sociology, medicine, and technology [2]. Self-organizing systems are their own reason and cause; they produce themselves (cause sui). In a self-organizing system, new order emerges from the old system. The constrained magnetized or non-magnetized turbulent fluids also self-organize to ordered states. For instance, geostrophic fluids such as hurricanes on earth, great red spot on Jupiter, formation of isolated vortices in Navier-Stokes flow, solitons formation in fluids, equilibrium states in reversed field pinches and spheromak plasmas. In self-organizing systems, one physical quantity gets long range order while another quantity shows short scale disorder and takes care of the increase of entropy. Thus the orderedness appears in a particular physical quantity. In many of the self-organizing systems, the ordered states are found to be remarkably robust in the sense that they remain relatively invariant and their properties do not depend upon the way the system is prepared [3].
The characteristics of self-organization appearing in a diverse system are found to be common [3–5]. The main common features are the following: When the system self-organizes itself, long range patterns are formed. The size of structures depends on boundary conditions. The nonlinear differential equations explain the behavior of the system in the presence of dissipation. Each and every system is composed of ideal non-varying parameters. It means when there is no dissipation, quantities remain conserved. When the dissipative effects are introduced in the system, one of the ideal non-varying parameters is more influenced as compared to other ones.

Self-organization is a universal phenomenon. Magnetized plasmas self-organize to their lowest state of energy under the topological constraints of magnetic fields. The self-organization of plasmas is also called as “relaxation”. The study of plasma relaxation is very important in the context of thermonuclear fusion process and to understand the underlying physics of self-organized structures in astrophysical settings. The selective decay hypothesis is utilized to find out the relaxed state of a plasma using variational principle. According to selective decay hypothesis one of the ideal invariants remains conserved as compared to other invariants in the presence of dissipation. The minimization of poorly conserved physical quantity subject to the constraint that the rugged quantity remains constant yields the Euler-Lagrange equation.

In self-organizing plasma systems, the topology of magnetic fields is found to play a key role. A lot amount of work, therefore, has been done to understand the “dynamo process”. Dynamo process deals with the study of the origin and the mechanism of the sustainment of magnetic fields in earth and astrophysical systems. In this regard, the magnetic field structures found in celestial objects, after the antidynamo theorem presented by Cowling [6], have been attracted much attention by the astrophysicists. According to antidynamo theorem, closed magnetic field line configurations can never be sustained in the presence of dissipation. In 1954, Lüst and Schlüter [7] highlighted the significance of fore-free magnetic field configurations to describe the astrophysical phenomena. Lundquist was the first one who presented a simple solution of force-free
fields in cylindrical coordinates [8]. Keeping in view the importance of fore-free field configurations, Woltjer and Chandrasekhar presented their variational principle [9–11].

It was pointed out by Woltjer that the structural complexity of magnetic field lines called as magnetic helicity and expressed mathematically as

\[ H = \frac{1}{2} \int_{v} A \cdot B dv, \] (1.1)

where \( B = \nabla \times A \), \( A \) is the vector potential, \( v \) is the entire volume and \( dv \) is the small volume element [12] acts as an ideal invariant. In the presence of dissipation and resistive effects, the magnetic energy defined as

\[ W = \frac{1}{2} \int_{v} B^2 dv, \] (1.2)

decays to its ground state whereas the magnetic helicity remains constant. Under these constraints and for specific boundary conditions, the variational principle expressed as

\[ \delta (W - \lambda H) = 0, \] (1.3)

leads the system toward the following equilibrium state

\[ \nabla \times B = \lambda B, \] (1.4)

where \( \lambda \) is the Lagrange multiplier and the equation is known as Euler-Lagrange equation. Equation (1.4) shows that magnetic field becomes force-free as the Lorentz force vanishes. Any vector field which satisfies equation (1.4) is called as Beltrami field. The Beltrami fields have been used to describe the twisted, sheared, spiral and helical structures in different plasma systems. It is thus reasonable to describe and understand the possible equilibrium states of a plasma in terms of Beltrami fields [13]. Taylor [14] formulated an identical variational principle and showed that rate of energy decay is faster than that of
helicity decay. He also conjectured that helicity is a global invariant in a resistive plasma.

The force-free self-organized states may be interpreted as the states where the entropy attains its maximum value rather than energy is minimum. The statistical mechanical form of the relaxation theory has been developed by Ito and Yoshida using the Shannon and Rényi entropy [15]. The force-free relaxed state based on the concept of magnetic entropy has been derived by Minardi [16].

The force-free magnetic fields are special fields and play an important role in stellar atmosphere, in particular in solar corona. Therefore, the nonlinear force-free fields (λ is not constant) and the linear force-free fields (constant λ) have been studied extensively in the context of astrophysics. Gold and Hoyle [17] applied the nonlinear force-free fields to study the dynamics of a uniformly twisting magnetic flux tube which are similar to the solar coronal loops. The force-free fields represent the twisted magnetic field lines. The twisted magnetic fields have been observed in many astrophysical and space plasmas, for instance, the flux ropes which are created when the magnetosphere interacts with interplanetary magnetic fields [18] and the galactic jets [19]. The Beltrami fields are used to represent circularly polarized waves [20] and to design the superconducting magnet [21]. More importantly, the final equilibrium state of a turbulent magnetoplasma which appears after the reconnection of magnetic field lines can be described by Beltrami fields [22]. The Beltrami field has been used to study and explain the equilibrium structures produced in a variety of devices such as multipinch, reverse-field pinch and spheromak. The main features and formation of relaxed states in reverse-field pinch and spheromak devices have been successfully predicted and described by relaxation theory based on Beltrami field. However, this theory has had less useful to predict the equilibrium evolution in tokamaks and field-reverse configurations. These discrepancies come out due to missing of important and ubiquitous features such as significant flow and finite pressure in real plasmas. These shortcomings show that the theory is incomplete and there is a strong need to extend it to incorporate the ubiquitous features of practical plasmas.
A lot of work has been done to introduce the strong flow and finite pressure in magnetized plasmas to obtain the non-force-free relaxed states. Turner [23] proposed a model which shows that vorticity is very important to obtain the non-force-free self-organized states. Later on, it was shown by Avinash and Taylor [24,25] that the canonical vorticities of each fluid in a multifluid description of plasmas act as an invariant to lead the system toward a non-force-free equilibrium. Steinhauer and Ishida [26,27] developed the relaxation theory for multispecies plasma. The self-helicities of the fluids are taken as invariants. The local and global invariance of the self-helicities have been established using transport equations. It has been predicted that relaxed state of two fluid plasmas exhibits flow and significant pressure gradient. A possible approach of creating self-organized states of an Hall magnetohydrodynamics (MHD) plasma (ions and inertialess electrons) have been proposed by Mahajan and Yoshida [28]. They have shown that the equilibrium magnetic or velocity field of an ideal magnetized plasma can be described in terms of Beltrami fields. The model is mathematically very simple and encompasses a wide variety of solutions. It may well be said as a generalized version of relaxation theory. The model is characterized by two scale parameters which identify two self-organized states. One is associated with the smaller structure and the other one describes a larger structure of the size of the system. The self-organized state is the superposition of two Beltrami fields known as Doubel Beltrami (DB) state and appears as a result of the strong interaction between the flow and magnetic field. The general solvability of DB states have been studied by Yoshida and Mahajan [29]. They have developed a general relation between the harmonic and DB fields. It is worth mentioning that the relaxed state can be derived using a well posed variational principle as proposed by Yoshida and Mahajan [30]. There are three constants of motion namely magnetofluid energy, magnetic helicity and the helicity of generalized vorticity. Minimization of generalized enstrophy (square of the curl of canonical momentum) subject to the constraint that constants of motion remain invariant defines the well posed variational principle. A straightforward algebraic manipulation leads to self-organized state of Hall MHD plasma.
Yoshida et. al. [31] applied the solution of two dimensional Beltrami field to model the coronal arcade structures. The algebraic relations between the constants of motion and the parameters (two amplitudes and two scale parameters) which define the DB states have been developed and high-beta toroidal equilibrium has also been studied using the model of DB fields. The analytical solutions for annular axisymmetric and homogeneous plasma are presented by Iqbal et. al. [32] for the DB self-organized states.

Ohsaki et. al. have presented a model of coronal structures based on the slowly evolving Double Beltrami (DB) equilibrium states in Hall MHD plasma [33, 34]. The condition for catastrophic transformation of the original states are derived. It is shown that sudden, eruptive and catastrophic events may take place when DB equilibrium states change fairly slowly. Such eruptions are ubiquitous in solar atmosphere. It is also shown that magnetic energy is transferred to flow kinetic energy when the eruption takes place.

The Taylor force-free states are characterized by current components parallel to magnetic field. Iqbal [35] has shown that the relaxed equilibrium states permitted by DB formalism have the potential to minimize the current and flow components parallel to the magnetic fields and this feature suggests that the DB equilibria can confine the high $\beta$ (ratio of kinetic to magnetic pressure) plasmas. The stability of DB flows has been analyzed by Ito et. al. [36] and it is shown that stability of DB flows is a function of amplitudes and scale parameters (eigenvalues of the curl operator). Based on the DB formalism, Mahajan and Yoshida [37] have derived a self-consistent model of a self-organized singular layer which arises at the edge of a tokamak plasma called as H-mode boundary layer.

The theory of DB two-fluid equilibrium with constraint on the pressure gradient used in ballooning mode stability have been utilized by Guzdar et. al. [38] and the width and height of the pressure pedestal for an H-mode are found. The predicted scalings are found to be in agreement with the semiempirical scalings and the available H-mode data.

The DB formalism is used to recognize and explore reverse-dynamo mechanism (an important mechanism to understand a variety of phenomena in stellar and astrophysical
systems) by Mahajan et. al. [39]. The DB model and its invariants are used to study the turbulence in solar atmosphere and in the solar wind by Krishan and Mahajan [40, 41].

It was shown by Mahajan [42] that DB states show perfect diamagnetic behavior when the generalized helicity vanishes. It has also been shown that multi-component plasmas has the potential to self-organize to DB states [43–46]. Berezhiani et. al. has developed the double Beltarmi-Bernoulli equilibria for the plasmas containing degenerate electrons. It is shown that it is possible to attain a nontrivial Beltrami-Bernoulli relaxed state even when the temperature of plasma is zero [47].

It has been shown by Bhattacharyya et. al. [48] that when the inertial effects of both the plasma species in non-relativistic electron-positron plasma are taken into account, it is possible to obtain the Euler-Lagrange equation which can be written as a sum of three Beltrami fields known as Triple Beltrami (TB) field. They have considered the principle of minimum dissipation rate to obtain the relaxed equilibrium and pointed out that the relaxed state is one where rate of dissipation becomes minimum. The steady-state solution of the pair plasma whose constituents have relativistic thermal velocity can self-organize to TB field [49]. It is shown that Beltrami parameters which define the nature of self-organized structures become real at higher thermal energies. The creation of multi-Beltrami (DB and TB) fields have been investigated in relativistic as well as non-relativistic multi-species plasmas [50–56]. The quantum plasmas are thought to be very important as they have applications in nano technology, quantum dots, quantum wells and dense astrophysical settings such as white dwarfs, neutron stars, supernovae etc. [57–68]. The electron-hole quantum plasmas are, therefore, studied by Iqbal [69] and it is found that the self-organized state at nanoscale comes out to be TB field when the masses of electrons and holes are taken into account in the dynamics.

Mahajan and Lingam have demonstrated the emergence of multi-Beltrami equilibrium states in multi-fluid plasmas. They have studied their general properties and highlighted their applications for astrophysical plasmas [70]. Recently, Shatashvili et. al. have analyzed the relaxation process in dense plasmas comprising of electrons, positrons and
ions. The lighter constituents (electrons and positrons) are taken as degenerate while the ions are treated as non-degenerate. It has been found that Quadruple Beltrami (QB) field appears as an equilibrium state. The QB field is the superposition of four Taylor force-free states [71]. Yoshida has recently proposed that every Casimir invariant may be interpreted as an adiabatic invariant and has discussed the issue of self-organization by topological constraints based on the hierarchy of foliated phase space [72].

In nutshell, the self-organized equilibria through Beltrami fields have been used extensively to describe diverse physical phenomena: they have been used to model the fusion plasmas such as spheromaks [73], reversed-field configuration, field-reversed configurations [74, 75]. In the context of space plasmas, the Beltrami relaxed states have been used in modeling solar flares [76, 77], solar coronal magnetic fields [78], magnetic clouds [79, 80], solar coronal loops and arcades [81, 82], solar coronal heating [83–85], large scale dynamos [86–89], scale hierarchies in flows [90] and turbulence [91].

In this thesis, a mathematical model is developed to deduce the TB self-organized state in multi-component collisionless electronegative dusty magnetoplasma. Immobile dust grains, singly ionized positive ions and two types of negatively charged ions are taken as constituent of the plasma. The inertial effects of all the components are taken into account while the Beltrami parameters (ratio of generalized vorticities to flows) of negatively charged species is taken to be same [56]. In second problem, different Beltrami parameters of all the plasma species are taken into account and all the inertial and non-inertial forces of a multi-component plasma are considered, the steady state self-organized state has been evaluated and found to be equivalent to the linear sum of four Beltrami fields [92]. The last problem deals with the investigation of catastrophic loss of equilibrium in a slowly evolving equilibrium system governed by DB states in three species electron, positron and ion plasma.
1.2 Layout of Thesis

This thesis is organized as follows. Chapter one deals with the introduction and motivation to justify the research work about the self-organization and creation of multi-Beltrami fields in different kinds of magnetized plasmas. In chapter two, the basic concepts relevant to describe the relaxation dynamics of plasma are briefly defined. Having defined the concepts of Beltrami field, magnetic helicity, ideal invariance of magnetic helicity, Taylor’s conjecture about helicity conservation, Euler-Lagrange equation, linear force-free fields and their examples, the relaxation model for Hall MHD plasmas is presented which depicts the DB state as a relaxed state. At the end of the chapter, the relaxation model of a magnetized multi-component electronegative dusty plasma has been described in detail and it has shown that the self-organized state can be cast as a superposition of three Beltrami fields. Chapter 3 describes the model that deal about the relaxation of three species plasmas consisting of a pair and singly ionized positive ions. The relaxed state has been derived which is characterized by four Beltrami fields. There exist four scale parameters corresponding to each Beltrami field. The impact of densities on the nature of scale parameters have been shown. In chapter 4, the relaxation of electron, positron and ion plasmas has been studied. A possibility of relaxation to DB states has been explored. The slowly evolving equilibrium characterized by DB states is employed to model the closed field structures similar to solar coronal loops. Impact of density, helicities, ratio of positron flow to magnetic field and energy on loss of equilibrium is illustrated graphically and the conditions for loss of equilibrium are derived. The last chapter presents the summary and conclusion of the research work.
Chapter 2

Beltrami Fields

The fields which align themselves along their own vorticities are known as Beltrami fields. Mathematically, Beltrami fields are the eigenfunctions of the curl operator. The magnetic field plays a key role in the dynamics of a magnetized plasma. The Beltrami magnetic fields are the final equilibrium states of MHD plasmas when the flow energy does not play any role in dynamics. The evolution of a turbulent plasma toward ordered state is a remarkable example of self-organization. Magnetic field chooses the preferred (Beltrami) state when the magnetic helicity acts as a global ideal invariant and the magnetic energy decays to its ground value. The helicity describes the structural complexity and linking of magnetic flux tubes. In the next section, we will describe the concept of helicity as its role is very important in the process of plasma relaxation.

2.1 Magnetic helicity

The helicity density of a vector field is defined as the dot product of the field with its curl. The helicity of the field is given by the volume integral of helicity density. The magnetic helicity is defined as

\[ h = \int_V A \cdot B dV. \]  

(2.1)
where $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field and $\mathbf{A}$ is the vector potential. The magnetic helicity is a gauge invariant quantity. If we consider the gauge transformation

$$\mathbf{A} \to \hat{\mathbf{A}} = \mathbf{A} + \nabla f,$$

then the corresponding helicity equation (2.1) becomes

$$h \to \dot{h} = \int_V (\mathbf{A} + \nabla f) \cdot \mathbf{B} dV,$$

where $f$ is an arbitrary function of position. Above equation can be written as

$$h \to \dot{h} = \int_V \mathbf{A} \cdot \mathbf{B} dV + \int_V \nabla f \cdot \mathbf{B} dV,$$

$$h \to \dot{h} = h + \int_V \nabla \cdot (f \mathbf{B}) dV.$$

Using $\nabla \cdot \mathbf{B} = 0$, we obtain

$$\dot{h} = h.$$

The mutual interconnection of magnetic field lines is also an illustration of magnetic helicity. To explain this, we take two flux tubes having volume $V_1$ and $V_2$. The magnetic

![Figure 2-1: Linkage of magnetic flux tubes](image-url)
field exist only in the flux tubes so that the normal component of magnetic field on the bounding surface vanishes. These tubes track the curves $C_1$ and $C_2$, $\phi_1$ and $\phi_2$ are the magnetic fluxes passing through the surfaces $S_1$ and $S_2$. The magnetic field is zero at the tube exterior. We define the flux of the tubes as

$$\phi_k = \int_{S_k} B \cdot n dS_k, \quad (2.2)$$

with $(k = 1, 2)$, $dS_k$ represent the surface elements and $n$ is the unit normal vector on surfaces $S_k$. Let us replace $A \cdot B dV = (A \cdot B)(dS \cdot dx) = (A \cdot dx)(B \cdot dS)$ and rearranging the integral terms, we obtain

$$h_1 = \oint_{C_1} (A \cdot dx) \int_{S_1} (B \cdot dS),$$

$$h_1 = \phi_1 \oint_{C_1} (A \cdot dx),$$

$$h_1 = \phi_1 \int_{\Sigma_1} B \cdot n dS,$$

$$h_1 = \phi_1 \phi_2, \quad (2.3)$$

because the flux flowing along the curve $C_2$ and passing through $\Sigma_1$ is equal to $\phi_2$. Similarly, $h_2$ can be calculated and we obtain

$$h_2 = \phi_1 \phi_2. \quad (2.4)$$

The total helicity $h$ which is the sum of two helicities comes out to be

$$h = h_1 + h_2,$$

$$h = 2\phi_1 \phi_2. \quad (2.5)$$
The twisting or linking of the magnetic field lines can be once, twice or several times. The total helicity is directly proportional to the number of mutual interaction of these flux tubes around each other. Equation (2.5) becomes

\[ h = \pm 2N\phi_1\phi_2 \]

where \( N \) is the number of mutual interactions of the flux tubes and \( \pm \) denotes the relative orientation of magnetic fields in the tubes. The helicity is right-handed for the direction of magnetic field in the two flux tubes as shown in figure. If the direction of the magnetic field in one of the flux tubes is reversed, it will be left-handed.

There are an infinite number of physical quantities which remain constant in an ideal MHD. The conserved quantities can describe a better understanding of the performance and behavior of the system. A condition which is satisfied at some initial instant of time will remain conserved for all subsequent times. According to Woltjer [11], an infinite number of volume integrals of the type

\[ h_{\ell} = \int_{v_\ell} \mathbf{A} \cdot \mathbf{B} dV, \quad \ell = 0, 1, 2, \ldots, \infty \]

serve as invariants of an ideal MHD plasma, where \( v_\ell \) represents the volume of each magnetic flux tube. A perfectly conducting plasma attains the force-free relaxed state when magnetic energy decays to ground state while the magnetic helicity remains constant. In the next section, it is proved that magnetic helicity acts as a constant of motion in a perfectly conducting plasma. A magnetic field which satisfies the following equation

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}), \]

then the integral \( \int_{v} \mathbf{A} \cdot \mathbf{B} dV \), where \( \mathbf{B} = \nabla \times \mathbf{A} \) is a constat of motion.
2.1.1 Helicity conservation

We consider an infinite number of helicities in the system and suppose that \( v_\ell \) is the volume of the \( \ell \)th flux tube and \( s_\ell \) represents surface closing the volume.

\[
h_\ell = \int_{v_\ell} A \cdot B \, dV,
\]

where \( \ell = 1, 2, 3, \ldots \infty \). The time rate of change of the helicities can be written as

\[
\frac{dh_\ell}{dt} = \int_{v_\ell} \left( \frac{\partial A}{\partial t} \cdot B + A \cdot \frac{\partial B}{\partial t} + (V \cdot \nabla) (A \cdot B) \right) \, dV,
\]

\[
\frac{dh_\ell}{dt} = \int_{v_\ell} \left( \frac{\partial A}{\partial t} \cdot B + A \cdot \frac{\partial B}{\partial t} \right) \, dV + \int_{s_\ell} \left( n \cdot V \right) (A \cdot B) \, ds. \tag{2.8}
\]

The last term on right-hand side shows that flux tube is moving with a velocity \( V \). For a perfectly conducting plasma, we have

\[
\frac{\partial B}{\partial t} = \nabla \times (V \times B). \tag{2.9}
\]

Using above equation, we obtain

\[
\int_{v_\ell} A \cdot \frac{\partial B}{\partial t} \, dV = \int_{v_\ell} (V \times B) \cdot B \, dV - \int_{v_\ell} \nabla \cdot [A \times (V \times B)] \, dV. \tag{2.10}
\]

Using the divergence theorem and applying the boundary conditions \( n \cdot B = n \cdot V = 0 \) on the surface, the above integral vanishes. Equation (2.8) reduces to

\[
\frac{dh_\ell}{dt} = \int_{v_\ell} \left( \frac{\partial A}{\partial t} \cdot B \right) \, dV.
\]

Using \( \partial A / \partial t = -E - \nabla \phi \), we have

\[
\frac{dh_\ell}{dt} = - \int_{v_\ell} \left( E \cdot B + \nabla \phi \cdot B \right) \, dV. \tag{2.11}
\]
For $\eta = 0$, we know that $\mathbf{E} = -\mathbf{V} \times \mathbf{B}$. Therefore, above equation becomes

$$\frac{dh_{\ell}}{dt} = \int_{v_{\ell}} (\mathbf{V} \times \mathbf{B} \cdot \mathbf{B} - \nabla \phi \cdot \mathbf{B}) dV.$$ 

The first term on right-hand side vanishes identically and applying the divergence theorem, we have

$$\frac{dh_{\ell}}{dt} = -\int_{s_{\ell}} \mathbf{n} \cdot (\phi \mathbf{B}) ds,$$

$$\frac{dh_{\ell}}{dt} = 0.$$ \hspace{1cm} (2.12)

Equation (2.12) explains that the magnetic helicity of each flux tube remains constant in ideal MHD because there is no breakage or reconnecting of magnetic field lines. It also explains that in a perfectly conducting plasma, if two flux tubes are linked $n$-times with each other, they will remain linked $n$-times with each other for ever.

### 2.2 Taylor’s conjecture

The factual plasma is not conducting perfectly. There is no persistent of the magnetic field lines in the plasma by introducing the resistivity either in a small or large amount. The magnetic islands are formed in the result of breaking and reconnection of the field due to the resistance in the fluids. The topology of the plasma changes as the magnetic energy of these magnetic structures is smaller than the initial MHD plasma. The magnetic helicity $h_0$ across the volume $v_0$ is defined by the following integral

$$h_0 = \int_{v_0} \mathbf{A} \cdot \mathbf{B} dv$$
According to Taylor’s hypothesis the magnetic helicity is not a constant quantity, it decays with time. As \( \mathbf{E} = -\mathbf{V} \times \mathbf{B} + \eta \mathbf{J} \), where \( \eta \neq 0 \), equation (2.11) becomes

\[
\frac{\partial h_0}{\partial t} = \int_{v_0} (\mathbf{V} \times \mathbf{B} \cdot \mathbf{B} - \eta \mathbf{J} \cdot \mathbf{B} - \nabla \phi \cdot \mathbf{B}) dv
\]

\[
\frac{\partial h_0}{\partial t} = -\int_{v_0} \eta \mathbf{J} \cdot \mathbf{B} dv
\]  

Equation (2.11) explains that in the presence of \( \eta \), \( h_0 \) does not remain constant. In ideal MHD plasma electric current is dissipated due to presence of resistivity, velocities of the fluids are reduced by viscosity and heat conductivity could reduce the thermal gradients. The resistivity, viscosity or the heat conductivity are the free energy sources in the plasma that can produce instabilities as the constraints are removed. In this way the plasma has the ability to attain a minimum magnetic energy state because of the magnetic reconnection.

From equation (2.18), the time rate of magnetic energy is given below

\[
\frac{\partial W_m}{\partial t} = \int_{v_0} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} dv,
\]  

where \( v_0 \) is the total volume of the system, as \( \partial \mathbf{B}/\partial t = \nabla \times \mathbf{E} \), equation (2.14) becomes

\[
\frac{\partial W_m}{\partial t} = -\int_{v_0} \mathbf{B} \cdot (\nabla \times \mathbf{E}) dv,
\]

\[
\frac{\partial W_m}{\partial t} = -\int_{v_0} \mathbf{E} \cdot (\nabla \times \mathbf{B}) dv + \int_{v_0} \nabla \cdot (\mathbf{E} \times \mathbf{B}) dv.
\]

In the above equation, the second term

\[
\int_{v_0} \nabla \cdot (\mathbf{E} \times \mathbf{B}) dv = \int_{s_0} \mathbf{n} \cdot (\mathbf{E} \times \mathbf{B}) ds = 0.
\]
Using $\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \eta \mathbf{J}$, where $\eta \neq 0$, we get

$$\frac{\partial W_m}{\partial t} = \int_{v_0} (\mathbf{V} \times \mathbf{B}) \cdot (\nabla \times \mathbf{B}) \, dv - \int_{v_0} \eta \mathbf{J} \cdot (\nabla \times \mathbf{B}) \, dv,$$

The first integral in the right hand side of the above equation is identically equal to zero

$$\frac{\partial W_m}{\partial t} = - \int_{v_0} \eta \mathbf{J} \cdot (\nabla \times \mathbf{B}) \, dv,$$

$$\frac{\partial W_m}{\partial t} = - \int_{v_0} \eta \mathbf{J}^2 \, dv; \quad (2.15)$$

In ideal MHD plasma, the local magnetic helicity of each flux tube is constant and the magnetic energy is approximately constant, while due to the resistivity, the magnetic helicity of each flux tube and the associated magnetic energy will not remain constant. They are changed but the rate of magnetic energy decay as compared to helicity is much faster to get the relaxed state. From equations (2.13) and (2.15), we get

$$\frac{\partial h_0}{\partial t} = \frac{\int_{v_0} \eta \mathbf{J} \cdot \mathbf{B} \, dv}{\int_{v_0} \eta \mathbf{J}^2 \, dv},$$

$$\frac{\partial h_0}{\partial t} = \frac{B}{J}; \quad (2.16)$$

where current density $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$, equation (2.16) becomes

$$\frac{B}{J} = \frac{\mu_0 B}{\nabla \times B} = \mu_0 L; \quad (2.17)$$

$L$ is the length which is much smaller than the system size. The equation (2.17) explains the relaxed state of plasma. The helicity $h_0$ may not be an invariant parameter according to the requirements of the relaxation theory. However it necessitates the dynamic process that decays magnetic energy faster as compared to helicity. From the Taylor’s hypothesis, it can be said that the total helicity of the entire volume of the plasma system is conserved. In this way the final relaxed state of plasma doesn’t depend on the initial state.
2.2.1 Euler-Lagrange equation

The self-organized or relaxed state of an ideal MHD plasma results when the magnetic energy
\[ W_m = \frac{1}{2} \int_V B^2 dV, \] (2.18)
decays to its minimum value while the magnetic helicity
\[ h = \frac{1}{2} \int_V A \cdot B dV, \] (2.19)
remains conserved. In order to obtain the relaxed state, we use the variational principle which reads as
\[ \delta (W_m - \lambda h) = 0 \] (2.20)
where \( \lambda \) is the Lagrange multiplier. The variation in magnetic energy is
\[ \delta W_m = \int_V B \cdot \delta B dV, \]
\[ \delta W_m = \int_V B \cdot \nabla \times \delta A dV, \]
\[ \delta W_m = \int_V \delta A \cdot \nabla \times B dV. \] (2.21)

The variation in magnetic helicity is
\[ \delta h = \frac{1}{2} \int_V (\delta A \cdot B + A \cdot \delta B) dV, \]
\[ \delta h = \frac{1}{2} \int_V (\delta A \cdot B + A \cdot \nabla \times \delta A) dV, \]
\[ \delta h = \int_V \delta A \cdot B dV. \] (2.22)

In evaluating equations (2.21) and (2.22), we used the identity \( \nabla \cdot (a \times b) = (\nabla \times a) \cdot b - a \cdot (\nabla \times b) \) and eliminated the surface integrals due to boundary conditions. Putting
the values of (2.21) and (2.22) into (2.20), we obtain

$$\int_v \delta A \cdot (\nabla \times B - \lambda B) \, dv = 0.$$  

Since $\delta A$ is an arbitrary potential, hence the integrand equals to zero and we obtain

$$\nabla \times B = \lambda B,$$  

(2.23)

where $\lambda$ serving as eigenvalue of the curl operator is a constant. Equation (2.23) is known as a Beltrami state and represents the relaxed state of an ideal static MHD plasma. According to Woltjer, magnetic helicity remains constant at the level of flux tubes that is magnetic helicity is a locally conserved physical parameter. On the other hand, Taylor pointed out the fact that even in the presence of a small amount of resistivity, it is impossible for the magnetic flux tubes to preserve their identity, their linkages with other flux tubes, their twisting etc. However, the global helicity remains constant. That is why, equation (2.23) is also called as Taylor's state in literature. Equation (2.23) is a linear equation and the corresponding Beltrami fields will be linear. In the next section, we will describe examples of linear Beltrami fields.

### 2.3 Linear Beltrami fields

The ideal MHD equation of motion reads as

$$\rho \frac{dV}{dt} = J \times B - \nabla p + \rho g,$$  

(2.24)

where $g$ is the acceleration due to gravity, $p$ is the pressure and $\rho$ is the mass density. Gravitational field becomes very important to describe the dynamics of some astrophysical plasmas. For magnetostatic configuration, above equation becomes

$$J \times B - \nabla p + \rho g = 0.$$  

(2.25)
In situations where magnetic field plays a dominant role as in solar corona, the terms $\nabla p$ and $\rho g$ can be neglected. In such a situation, above equation reduces to

$$J \times B = 0.$$  (2.26)

A magnetic field which satisfies the above equation is said to be force-free because the Lorentz force becomes zero. This equation can be written as

$$(\nabla \times B) \times B = 0,$$  (2.27)

This equation is satisfied if one considers

$$\nabla \times B = \lambda(x) B,$$  (2.28)

where $\lambda$ is a scalar function of position. This equation is nonlinear, however there exists an important subclass of magnetic fields which are the solutions of a linear equation.

Taking the divergence of equation (2.28), we see that

$$\nabla \lambda(x) \cdot B = 0.$$  (2.29)

It shows that the function $\lambda$ must remain constant on any field line. Let us consider two solutions $B_1$ and $B_2$ corresponding to two scalar functions $\lambda_1$ and $\lambda_2$ respectively, so that

$$\nabla \times B_1 = \lambda_1(x) B_1,$$  (2.30)

$$\nabla \times B_2 = \lambda_2(x) B_2.$$  (2.31)

Solving the above two equations, we can write

$$[\nabla \times (B_1 + B_2)] \times (B_1 + B_2) = [\lambda_1(x) - \lambda_2(x)] (B_1 + B_2).$$
It shows that $B_1 + B_2$ is the force-free field if $\lambda_1(x) = \lambda_2(x)$ everywhere. Equation (2.27) becomes a linear equation if we take

$$\lambda(x) = \text{constant},$$

so that

$$\nabla \times B = \lambda B.$$  \hfill (2.32)

It shows that magnetic field can be defined by a linear partial differential equation. Hence, any linear combination of the solutions of equation (2.32) is also a solution.

### 2.4 Examples of linear force-free field

#### 2.4.1 Chandrasekhar and Kendall function

In the cylindrical coordinates $(r, \theta, z)$ system, Chandrasekhar and Kendall [93] have described the solution of equation (2.32) as given below. For a cylinder with radius $a$ and having periodicity in $z$ with length $L$, the Chandrasekhar-Kendall function reads as

$$B = \lambda (\nabla \psi \times \nabla z) + \nabla \times (\nabla \psi \times \nabla z),$$  \hfill (2.33)

where $\psi$ can be defined as, in terms of cylindrical geometry

$$\psi = J_m(\mu r) e^{i(m\theta - Kz)}.$$  \hfill (2.33a)

In the above equation, $\mu$ is constant, $J_m$ is the Bessel function of order $m$ ($m = 0, 1, 2, \ldots$), $K = 2\pi n/L$ ($n = 0, 1, 2, 3, \ldots$) and the angle $\theta$ is also periodic with a period of $2\pi$. The $\psi$ also satisfies the Helmholtz condition

$$\nabla^2 \psi + \lambda^2 \psi = 0,$$
Solving the equation (2.33), we get

\[
B = \left( \frac{\lambda}{r} \frac{\partial \psi}{\partial \theta} + \frac{\partial^2 \psi}{\partial z \partial \theta} \right) \hat{r} + \left( \frac{1}{r} \frac{\partial^2 \psi}{\partial z \partial \theta} - \lambda \frac{\partial \psi}{\partial r} \right) \hat{\theta} + \left( \lambda^2 \psi + \frac{\partial^2 \psi}{\partial z^2} \right) \hat{z}
\]  

(2.34)

Putting equation (2.34) in equation (2.32), we get

\[
\nabla \times B = \lambda \left[ \left( \frac{\lambda}{r} \frac{\partial \psi}{\partial \theta} + \frac{\partial^2 \psi}{\partial z \partial \theta} \right) \hat{r} + \left( \frac{1}{r} \frac{\partial^2 \psi}{\partial z \partial \theta} - \lambda \frac{\partial \psi}{\partial r} \right) \hat{\theta} + \left( \lambda^2 \psi + \frac{\partial^2 \psi}{\partial z^2} \right) \hat{z} \right],
\]

\[
\nabla \times B = \lambda B
\]

Hence Chandrasekhar-Kendall function is a solution of force-free Beltrami field.

In cylindrical coordinate system, the force-free magnetic field can be expressed in terms of Bessel function \( B(r) = B_0 J_1(\lambda r) \hat{\theta} + B_0 J_0(\lambda r) \hat{z} \) which is known as Bessel Function Model.

### 2.4.2 ABC flow

Arnold, Beltrami and Childress [94] presented the solution of equation (2.32) in Cartesian coordinates \((x, y, z)\) system. In this system, the magnetic field describes as follow

\[
B = \begin{pmatrix}
A \sin(\lambda z) + C \cos(\lambda y) \\
B \sin(\lambda x) + A \cos(\lambda z) \\
C \sin(\lambda y) + B \cos(\lambda x)
\end{pmatrix},
\]

(2.35)

In the above expression, \(A, B, C\) and \(\lambda\) are all the real constants. Putting equation (2.35) in equation (2.32), we get

\[
\nabla \times B = \lambda \left( \begin{pmatrix}
A \sin(\lambda z) + C \cos(\lambda y) \\
B \sin(\lambda x) + A \cos(\lambda z) \\
C \sin(\lambda y) + B \cos(\lambda x)
\end{pmatrix} \right)
\]

(2.36)

The equation (2.36) verifies that the equation (2.35) is the solution of Beltrami field.
Beltrami field characterized by single scale parameter show the self-organized state of a static single fluid MHD plasma. The Beltrami field is force-free because the current and field are aligned. The parallel components of current are dominant and hence there is no gradient in pressure and Beltrami field supports the paramagnetism. Mahajan and Yoshida [28] went one step forward and showed analytically that sum of two linear Beltrami fields result in a state that is no more stationary, force-free and paramagnetic. The state called as DB introduces two eigenvalues. The mathematical model presented by Mahajan and Yoshida will be described in detail in next section.

2.5 Double Beltrami states

Let us consider a two component plasma consisting of electrons and ions. The equations of motion for the electron and ion fluids respectively are given as below

\[ mn \left( \frac{\partial V_e}{\partial t} + (V_e \cdot \nabla) V_e \right) = -en (E + V_e \times B) - \nabla p_e, \quad (2.37) \]

\[ Mn \left( \frac{\partial V_i}{\partial t} + (V_i \cdot \nabla) V_i \right) = en (E + V_i \times B) - \nabla p_i, \quad (2.38) \]

where \( V_j (j = e, i) \) represents the velocity of the fluids, \( p_j \) is the pressure of the species, \( E \) is the electric field, \( M, m, n \) and \( e \) represent mass of the ion, mass of electron, number density of plasma and elementary charge respectively. As \( M >> m \), we neglect mass of electrons as compared to mass of ions. The above equation can be written as

\[ \frac{\partial A}{\partial t} = -\nabla \phi + V_e \times B + \frac{1}{en} \nabla p_e, \quad (2.39) \]

\[ \frac{\partial V_i}{\partial t} + \frac{\partial A}{\partial t} = \frac{e}{Mn} (-\nabla \phi + V_i \times B) - (V_i \cdot \nabla) V_i - \frac{1}{Mn} \nabla p_i, \quad (2.40) \]

In above equations, \( E \) is replaced in favour of scalar potential (\( \phi \)) and vector potential (\( A \)) using the relation \( E = -\nabla \phi - \partial A/\partial t \). Normalizing velocities to \( V_A = B_0/\sqrt{\mu_0 n M} \) (\( B_0 \) is an arbitrary measure of the field, \( \mu_0 \) is the permeability of free space), distance
to ion skin depth \( \lambda_i = V_A / \omega_{ci} = \sqrt{M/\mu_0 \pi e^2} \) where \( \omega_{ci} \) is the electron plasma frequency, time to inverse of cyclotron frequency, \( B \) to \( B_0 \) (a constant magnetic field), pressure of the species to \( B_0^2/\mu_0 \). \( \phi_j \) to \( \lambda_i B_0 V_A \), vector potential \( A (B = \nabla \times A) \) to \( \lambda_i B_0 \), above equation of motions become

\[
\frac{\partial A}{\partial t} = (V - \nabla \times B) \times B - \nabla \left( \phi - \frac{1}{en} p_e \right), \tag{2.41}
\]

\[
\frac{\partial}{\partial t} (V + A) = V \times (\nabla \times V + B) - \nabla \left( \phi + p_i + \frac{V^2}{2} \right). \tag{2.42}
\]

In above equations, we have used the Ampere’s Law

\[
V_e = -\nabla \times B + V_i. \tag{2.43}
\]

As the mass of electrons is neglected, the velocity of fluid becomes equal to the velocity of ions \( V_i = V \) which is introduced in above equations. Taking curl of equations (2.41) \\
& (2.42), we obtain

\[
\frac{\partial B}{\partial t} = \nabla \times [(V - \nabla \times B) \times B], \tag{2.44}
\]

\[
\frac{\partial}{\partial t} (\nabla \times V + B) = \nabla \times [V \times (\nabla \times V + B)]. \tag{2.45}
\]

These equations can be written in a more systematic form as

\[
\frac{\partial \Omega_j}{\partial t} - \nabla \times [U_j \times \Omega_j] = 0, \tag{2.46}
\]

where the vorticities are \( \Omega_e = B \), \( \Omega_i = \nabla \times V + B \) and velocities are \( U_e = V - \nabla \times B \), \( U_i = V \). The steady state solution of equation (2.46) is

\[
U_j = b_j \Omega_j. \tag{2.47}
\]
where \( b_j (j = e, i) \) represents any arbitrary constant, which is referred to as a Beltrami parameter. Steady state solutions for the constituent fluids are, respectively

\[
B = b_e (V - \nabla \times B), \quad (2.48)
\]

\[
\nabla \times V + B = b_i V, \quad (2.49)
\]

where \( b_j \) are Beltrami constants. Eliminating \( V \) from equations (2.48) and (2.49), we obtain,

\[
(\nabla \times)^2 B - \nabla \times B \left( b_i - \frac{1}{b_e} \right) + B \left( 1 - \frac{b_i}{b_e} \right) = 0, \quad (2.50)
\]

\[
\nabla \times \nabla \times B - a_1 \nabla \times B + a_0 B = 0, \quad (2.50)
\]

where

\[
a_1 = 1 - \frac{b_i}{b_e}, \quad (2.51)
\]

\[
a_0 = b_i - \frac{1}{b_e}. \quad (2.52)
\]

Equation (2.50) is called as double Beltrami equation. It is easy to show that equation (2.50) is the linear sum of two Beltrami fields. Let us suppose that there are two Beltrami fields \( B_1 \) and \( B_2 \) and the corresponding eigenvalues are \( \lambda_1 \) and \( \lambda_2 \) respectively. The Beltrami equations read as

\[
\nabla \times B_1 = \lambda_1 B_1,
\]

\[
\nabla \times B_2 = \lambda_2 B_2.
\]

As the operators are commutative, solving above equations, we obtain

\[
\nabla \times \nabla \times B - (\lambda_1 + \lambda_2) \nabla \times B + \lambda_1 \lambda_2 B = 0. \quad (2.53)
\]
Comparing equations (2.50) and (2.53), we obtain

\[ a_1 = \lambda_1 + \lambda_2, \]  
\[ a_0 = \lambda_1 \lambda_2. \]  

(2.54)
(2.55)

It shows that the solution of equation (2.50) can be written as

\[ \mathbf{B} = C_1 \mathbf{B}_1 + C_2 \mathbf{B}_2. \]  

(2.56)

The eigenvalues of the curl operators are the roots of the quadratic equation

\[ \lambda^2 - a_1 \lambda + a_0 \lambda = 0, \]  

(2.57)

and are given by

\[ \lambda_1 = \frac{a_1 + \sqrt{a_1^2 - 4a_0}}{2}, \]  
\[ \lambda_2 = \frac{a_1 - \sqrt{a_1^2 - 4a_0}}{2}. \]  

(2.58)
(2.59)

Hall MHD relaxation model deals with two component plasma consisting of ions and inertialess electrons. What will be the self-organized state when the inertial effects of both the species are dealt at equal footing? Moreover, the real and practical plasmas consist of more than two species. What will be final self-organized state of multi-component plasmas? These questions are important and demand the need to extend the Hall MHD relaxation process to multi-component plasmas taking into account all the inertial and non-inertial forces. In this context, the relaxation dynamics of electron-positron plasmas has been investigated and relaxed state found to be TB state - a combination of three force-free Taylor’s states [48]. Later on, a variety of multi-component plasmas were examined to search for the relaxed Beltrami equilibria [49–54].

In the next section, we will describe the evolution of TB fields which result when a
Dusty plasma self-organizes [56]. A plasma is called as dusty plasma when it contains very heavy charged particles called as grains or the dust particles in addition to the neutrals and charged particles which may be electrons and ions [95]. Dusty plasmas are omnipresent in space. They are found in diverse astrophysical systems and laboratory environments as well. The dust particles are of different types and can be differentiated on the basis of their source and size. The dust grains found in space can have any size ranging from submicron to centimeter. The mass of the dust grains may be billion times greater than the mass of proton. Usually, the dust particles are highly negatively charged. Their charge may be equal to charge of a few electrons or up to the charge of more than hundreds of electrons. The dust grains present in a plasma significantly modify the plasma modes or create new ones even if they are static. The presence of dust is established in interstellar clouds and circumstellar disks, in the solar system as interplanetary dust, and in planetary rings, the earth’s magnetosphere and comet’s tails [96]. Whether one wants or not, dust particles are also present in laboratory plasmas, for instance, dust may exist due to interactions between plasma and walls of the container.

2.6 Triple Beltrami states

We consider a magnetized and collisionless electronegative dusty plasma. The constituents of plasma are immobile charged dust grains, positive ions and two types of negative ions. The dust grain only play the role to keep the plasma neutral. All the ions are singly ionized and possess different inertia. We assume that \( z_d n_d \gg n_e \), where \( z_d \) are the electrons attached to the surface of dust grains, \( n_d \) and \( n_e \) are the number density of dust grains and electrons respectively. As a result of the attachment of electrons to dust particles, the electrons are highly depleted [97, 98]. The magnetic field affects the dust charging [99]. For simplicity, we have not taken this effect into account in present analysis and assume that the dust charge does not vary. The equations of motion for the positive and two types of negatively charged ions \((s = 1, 2)\) respectively, are given as
below

\[
n_i m_i \frac{dV_i}{dt} = e n_i \left( E + \frac{V_i \times B}{c} \right) - \nabla p_i, \tag{2.60}
\]

\[
n_s m_s \frac{dV_s}{dt} = -e n_s \left( E + \frac{V_s \times B}{c} \right) - \nabla p_s, \tag{2.61}
\]

where \( V_i \) and \( V_s \) represent the velocity of the positive ion and two types of negative ion \((s = 1, 2)\) fluids respectively. \( p_i = n_i T_i \) and \( p_s = n_s T_s \) are the pressure of the positive and negative ion species, \( T_i \) and \( T_s \) are the temperatures, \( n_i \) and \( n_s \) are the densities of the positive ions and two types of negative ions \((s = 1, 2)\) respectively. \( m_i \) and \( m_s \) are the mass of positive ion and two types of negative ions \((s = 1, 2)\) fluids respectively, \( e \) is the elementary charge, \( B \) is the magnetic field and \( E \) is the electric field. The normalized momentum balance equations are given below

\[
\frac{\partial}{\partial t} (V_i + A) = V_i \times [\nabla \times (V_i + A)] - \nabla \left( \phi + p_i + \frac{1}{2} V_i^2 \right), \tag{2.62}
\]

\[
\frac{\partial}{\partial t} (V_s - M_s A) = V_s \times [\nabla \times (V_s - M_s A)] - \nabla \left( -M_s \phi + D_s p_s + \frac{1}{2} V_s^2 \right), \tag{2.63}
\]

where velocities of the fluids are normalized to Alfven speed \( V_A = B_0/\sqrt{4\pi n_i m_i} \) \((B_0 \) is an arbitrary measure of the field), pressures are normalized to \( B_0^2/4\pi \), the scalar potential \( \phi \) and the vector potential \( A \) \((B = \nabla \times A)\) are normalized to \( B_0^2/4\pi n_i e \) and \( V_A m_i c/e \) respectively. The magnetic field \( B \) is normalized to an appropriate measure of magnetic field \( B_0 \). All lengths are measured in terms of the skin depth of positive ion which reads as \( \lambda_i = V_A/\omega_c = c \sqrt{m_i/4\pi n_i e^2} \) where \( \omega_c \) is the electron plasma frequency. Time is normalized to \( m_i c/e B_0 \) the gyroperiod of positive ions. \( M_s = m_i/m_s \) and \( D_s = \rho_i/\rho_s \) \((\rho_i = n_i m_i, \rho_s = n_s m_s)\) is the ratio of mass density of ions \( \rho_i \) to mass density of one of the negative ions \( \rho_s \). We have used the identity \((\nabla \cdot V) V = \partial V/\partial t + \nabla V^2/2 - V \times (\nabla \times V) \) and the relation \( E = -\nabla \phi - c^{-1} \partial A/\partial t \), \( c \) is speed of light. We re-write equations (2.62)-
(2.63) as given below

\[
\frac{\partial P_i}{\partial t} = \nabla \times (\nabla \times P_i) - \nabla \psi_i, \quad (2.64)
\]

\[
\frac{\partial P_s}{\partial t} = \nabla \times (\nabla \times P_s) - \nabla \psi_s, \quad (2.65)
\]

where \( P_i = V_i + A \), \( P_s = V_s - M_s A \), \( \psi_i = p_i + \phi + V_i^2/2 \) and \( \psi_s = D_s p_s - M_s \phi + V_s^2/2 \).

The pressures are taken as functions of density that is \( p = p(\rho) \) therefore on taking curl of equations (2.64) and (2.65), we obtain the following equations

\[
\frac{\partial}{\partial t} (\nabla \times P_i) = \nabla \times [\nabla \times (\nabla \times P_i)], \quad (2.66)
\]

\[
\frac{\partial}{\partial t} (\nabla \times P_s) = \nabla \times [\nabla \times (\nabla \times P_s)]. \quad (2.67)
\]

One of the possible simplest steady state solutions of above equations is the Beltrami condition that is the flows become parallel to the corresponding vorticities. The Beltrami conditions for above equations read as

\[
\nabla \times P_i = k_1 V_i, \quad (2.68)
\]

\[
\nabla \times P_s = k_2 V_s, \quad (2.69)
\]

where \( k_1 \) and \( k_2 \) are proportionality constants and called as Beltrami parameters. The above equations show that generalized vorticities become parallel to the corresponding velocities to achieve the relaxation. The ratio of generalized vorticities to flows of negative ions is assumed to be same. All the densities are taken to be constant and the quasi neutrality condition reads as

\[
n_i = n_1 + n_2 + z_d n_d,
\]
where \( n_d \) is the number density of dust grains, \( n_1 \) and \( n_2 \) are the number densities of two negative ions. According to Ampere’s Law, we have

\[
\nabla \times B = V_i - \sum_{s=1,2} N_s V_s,
\]

(2.70)

where \( N_s = n_s/n_i \) (\( s = 1, 2 \)). From equations (2.69) and (2.70), we have

\[
\nabla \times V_i - k_2 V_i = \nabla \times \nabla \times B - k_2 \nabla \times B + \sum_{s=1,2} N_s M_s B.
\]

(2.71)

Using equation (2.68) into above equation, we obtain the velocity of ions as given below

\[
V_i = \frac{1}{k_1 - k_2} \left[ (\nabla \times)^2 B - k_2 \nabla \times B + \left( 1 + \sum_{s=1,2} N_s M_s \right) B \right].
\]

(2.72)

Putting value of the ion velocity \( V_i \) into equation (2.68), we get

\[
(\nabla \times)^3 B - a (\nabla \times)^2 B + b \nabla \times B - cB = 0,
\]

(2.73)

where \( a = k_1 + k_2, b = 1 + k_1 k_2 + \sum_{s=1,2} N_s M_s \) and \( c = k_2 + k_1 \sum_{s=1,2} N_s M_s \). Equation (2.73) is the Triple Curl Beltrami equation because it is equivalent to a linear sum of three different Beltrami fields. The system self-organizes to a Triple Beltrami (TB) field because the Beltrami parameters which are the ratio of generalized vorticities to the corresponding flows are taken to be equal for both the negative ions while the Beltrami parameter for positive ions is different. Due to immobility of charged dust grains, the characteristic scale length of the system comes out to be skin depth of the positive ion. This relaxed state is mathematically equivalent to the relaxed state for a plasma system consisting of a single negative charge, positive charge and immobile charged dust grains [53]. In the present scenario if either the Beltrami parameters for both the negatively charged ions are taken to be different or the mobility of dust grains is taken into account, the self-organized state would not be a TB field. On the other hand if we assume that all the
bulk inertia of dusty plasma is provided by streaming dust particles, the self-organized magnetic field will be no more TB field and the characteristic scale length of the system will become the dust skin depth as shown by Shukla and Mahajan [44]. If we assume that there are no dust grains in the system, the model is still valid and we will obtain the same self-organized state as given by equation (2.73).

2.7 Roots analysis

Any Beltrami magnetic field $\mathbf{G}$ satisfies the conditions

$$\nabla \times \mathbf{G} = \alpha \mathbf{G} \quad (\text{in } \Omega),$$

$$\mathbf{n} \cdot \mathbf{G} = 0 \quad (\text{on } \partial \Omega), \quad (2.74)$$

where $\alpha$ is a scalar constant, $\Omega$ is the bounded domain having a smooth boundary $\partial \Omega$ and $\mathbf{n}$ is the unit normal vector on $\partial \Omega$. The solution of equation (2.74) in cylindrical coordinates can be expressed by Chandrasekhar–Kendall functions [9]. Let us consider three Beltrami fields $\mathbf{B}_j (j = 1, 2, 3)$ which satisfy the following Beltrami conditions

$$\nabla \times \mathbf{B}_1 = \alpha_1 \mathbf{B}_1, \quad (2.75)$$

$$\nabla \times \mathbf{B}_2 = \alpha_2 \mathbf{B}_2, \quad (2.76)$$

$$\nabla \times \mathbf{B}_3 = \alpha_3 \mathbf{B}_3, \quad (2.77)$$

where $\alpha'$s are constant and represent the eigenvalues of the curl operator. Adding the above equations and considering

$$\mathbf{B} = C_1 \mathbf{B}_1 + C_2 \mathbf{B}_2 + C_3 \mathbf{B}_3, \quad (2.78)$$
where $C_j (j = 1, 2, 3)$ are arbitrary constants and can be evaluated using the boundary conditions, we can retrieve the Triple Beltrami (TB) equation (2.74) as given below

$$(\text{curl} - \alpha_1)(\text{curl} - \alpha_2)(\text{curl} - \alpha_3)B = 0, \quad (2.79)$$

where “curl” represents the curl operator $\nabla \times$. The Beltrami parameters $a$, $b$, and $c$ satisfy the following relations

$$a = \alpha_1 + \alpha_2 + \alpha_3,$$
$$b = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3,$$
$$c = \alpha_1\alpha_2\alpha_3.$$ 

The eigenstates $\alpha_j (j = 1, 2, 3)$ are the inverse of length and show the scale size of the magnetic field structures formed when the plasma gets equilibrium state. The $\alpha'$s are real for simply connected domain and may be complex if the domain under consideration is multiply connected [100]. The eigenstates are the roots of the following cubic equation

$$\alpha^3 - a\alpha^2 + b\alpha - c = 0. \quad (2.80)$$

The roots of the above cubic equation are

$$\alpha_1 = w - f + \frac{k_1 + k_2}{3},$$
$$\alpha_2 = \frac{f - w}{2} - \frac{i\sqrt{3}(f + w)}{2} + \frac{k_1 + k_2}{3},$$
$$\alpha_3 = \frac{f - w}{2} + \frac{i\sqrt{3}(f + w)}{2} + \frac{k_1 + k_2}{3},$$
Figure 2-2: Character of the scale parameters of TB field as a function of Beltrami parameters ($k_1, k_2$) for the density of $N_s = 0.25$.

where $w = \left(-\frac{g}{2} + \sqrt{R}\right)^{1/3}$, $f = \left(\frac{g}{2} - \sqrt{R}\right)^{1/3}$ and $R = \left(\frac{g}{2}\right)^2 + \left(\frac{h}{3}\right)^3$. The discriminant of the cubic equation is given by

$$\Delta = a^2b^2 + 18abc - 4b^3 - 4a^3c - 27c^2.$$ 

The discriminant ($\Delta$) of cubic equation describes the nature of scale parameters. If $\Delta < 0$, then one of the scale parameter will be real and other two are a pair of complex conjugate. For $\Delta > 0$, all the scale parameters will be real and distinct while for $\Delta = 0$, the scale parameters will also be real but two of them will be identical. Figure (2-2) shows the behavior of scale parameters as a function of Beltrami parameters ($k_1$ and $k_2$) for $N_s = n_s/n_i = 0.25$ that describes the density of negative ions is one fourth of the positive ions. The colored regions of figure (2-2) show three different categories of the scale parameters, the top right graph exhibits the points when the scale parameters are
real and distinct, the top left graph locates the scale parameters when two of them are a pair of complex conjugate and one is real while the bottom graph unveiled the points where two become degenerate and one is distinct. The Beltrami conditions given by equations (2.68)-(2.69) are the equilibrium solutions of equations (2.64) and (2.65) if the Bernoulli conditions $\nabla \psi_i = \nabla \psi_s = 0$ are satisfied. In the next section, we will evaluate these conditions.

### 2.8 Bernoulli equation

Substituting equations (2.68)-(2.69) into equations (2.64)-(2.65) which are macroscopic evolution equations of the plasma species and taking $\partial / \partial t = 0$ (steady state), we obtain the following equations

$$\nabla \left( p_i + \phi + \frac{1}{2} V_i^2 \right) = 0, \quad (2.81)$$

$$\nabla \left( D_s p_s - M_s \phi + \frac{1}{2} V_s^2 \right) = 0. \quad (2.82)$$

The above equation are exact differentials. On integrating above equations, the following equations are obtained.

$$p_i + \phi + \frac{1}{2} V_i^2 = C_i, \quad (2.83)$$

$$D_s p_s - M_s \phi + \frac{1}{2} V_s^2 = C_s, \quad (2.84)$$

where $C_i$ and $C_s$ are integration constants. The equations (2.83-2.84) are known as Bernoulli conditions. Their addition leads to

$$\psi_i + \sum_{s=1,2} \psi_s = C, \quad (2.85)$$

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where \( C = C_i + \sum_{s=1,2} C_s \). This is the generalized Bernoulli equation. It provides a relation among pressure, electric potential and flow. This version of Bernoulli equation results through relaxation of multi-ion plasma and the constant is global whereas in case of ordinary fluids, the constant is the function of streamlines. Equation (2.85) suggests the possibility of creating a finite pressure through gradient of flows and electric potential.

### 2.9 Analytical solution

The solution of equation (2.73) is the TB field which is equivalent to sum of three Beltrami fields. In an axisymmetric homogeneous cylindrical plasma, the solution is given by

\[
B_\theta = C_1 J_1(\alpha_1 r) + C_2 J_1(\alpha_2 r) + C_3 J_1(\alpha_3 r),
\]

\[
B_z = C_1 J_0(\alpha_1 r) + C_2 J_0(\alpha_2 r) + C_3 J_0(\alpha_3 r),
\]

where \( J_0 \) and \( J_1 \) are zero and first order Bessel functions respectively, \( C_1, C_2 \) and \( C_3 \) are constants and can be determined using the boundary conditions. If we take \(|B_z|_{r=0} = b_1\), \( J_\theta = |(\nabla \times B)_\theta|_{r=0} = b_2 \), and \( J_\theta = |(\nabla \times B)_\theta|_{r=d} = b_3 \), where \( d \) is the radius of cylinder, we obtain

\[
b_1 = C_1 + C_2 + C_3,
\]

\[
b_2 = \alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_3,
\]

\[
b_3 = \alpha_1 C_1 J_1(\alpha_1 d) + \alpha_2 C_2 J_1(\alpha_2 d) + \alpha_3 C_3 J_1(\alpha_3 d),
\]

Solving above equations, we obtain

\[
C_1 = \frac{u_1}{q}, \quad C_2 = \frac{u_2}{q}, \quad C_3 = \frac{u_3}{q},
\]
where

\[ u_1 = b_3 (\alpha_3 - \alpha_2) + \alpha_2 J_1(\alpha_2 d) (b_2 - \alpha_3 b_1) + \alpha_3 J_1(\alpha_3 d) (\alpha_2 b_1 - b_2), \]

\[ u_2 = b_3 (\alpha_1 - \alpha_3) + \alpha_3 J_1(\alpha_3 d) (b_2 - \alpha_1 b_1) + \alpha_1 J_1(\alpha_1 d) (\alpha_3 b_1 - b_2), \]

\[ u_3 = b_3 (\alpha_2 - \alpha_1) + \alpha_1 J_1(\alpha_1 d) (b_2 - \alpha_2 b_1) + \alpha_2 J_1(\alpha_2 d) (\alpha_1 b_1 - b_2), \]

\[ q = \alpha_1 J_1(\alpha_1 d) (\alpha_3 - \alpha_2) + \alpha_2 J_1(\alpha_2 d) (\alpha_1 - \alpha_3) + \alpha_3 J_1(\alpha_3 d) (\alpha_2 - \alpha_1). \]

2.10 Radial profiles of self-organized structures

In order to show the radial profiles of axial and azimuthal magnetic fields, we consider a dusty plasma which contains one positive ion \( (Ar^+) \) and two negative ions \( (F^- SF_6^-) \) in addition to heavy dust grains [101]. In this case, \( M_1 = m_i/m_1 = 40/19 \) where \( m_i \) is the mass of \( Ar^+ \), \( m_1 \) is the mass of \( F^- \) and \( M_2 = m_i/m_2 = 40/146 \) where \( m_2 \) represents the mass of \( SF_6^- \), respectively.

We assume that \( N_j = n_j/n_i = 0.25 \), that is the number density of negative ions is one fourth of the positive ion. The size of the system is \( 6\lambda_i \) and the boundary values are taken as \( b_1 = 1.0 \), \( b_2 = 0.5 \) and \( b_3 = 0.1 \). Figure (2-3) shows the field profiles for \( k_1 = 1.0 \) and \( k_2 = 0.3 \), where \( k_1 \) is the ratio of generalized vorticity of ions to their flows and \( k_2 \) represents the ratio of generalized vorticities of negative ions to their respective flows.
Figure 2-4: The radial profiles of magnetic fields for $k_1 = 1.01$ and $k_2 = 0.09$.

For this case, the scale parameters are $\alpha_1 = 0.606997$, $\alpha_2 = 0.346502 + 1.16366i$ and $\alpha_3 = 0.346502 - 1.16366i$, and we observe that relaxed fields are diamagnetic. Figure (2-4) shows that magnetic fields are paramagnetic and these profiles are for $k_1 = 1.01$ and $k_2 = 0.09$. The scale parameters are $\alpha_1 = 0.498429$, $\alpha_2 = 0.300786 + 1.13815i$ and $\alpha_3 = 0.300786 - 1.13815i$. In both the cases one of the scale parameters is real whereas other twos are complex conjugate. It is evident from figures (2-3) and (2-4), that the ratio of the generalized vorticities to flows plays an important role to self-organize the magnetic field towards paramagnetism or diamagnetism.

In evaluating the relaxed equilibrium of four components electronegative dusty plasma consisting of immobile heavy dust particles, one singly ionized ion and two negatively charged ions, we have considered the masses of all the ion species but the Beltrami parameters (ratio of generalized vorticities to flows) of negatively charged ions are taken to be same. This condition leads the equilibrium solution to follow the TB state. In order to explore the possibility of multi-component plasmas to self-organize to higher order Beltrami state, we will investigate the relaxation process of three component plasmas comprised of singly ionized ions and pair particles in the next chapter.
Chapter 3

Quadruple Beltrami states

Multi-component plasmas are found everywhere in space, astrophysical and laboratory environments. The simplest system of multi-component plasmas composed of three species can be regarded as a plasma consisting of pair particles and singly ionized ions. A great attention has recently been paid to study the pair plasmas. The term pair plasma is used to define a large ensembles of two charged particles which bear equal masses and carry opposite charge [102–104]. Recently, the production of pair plasmas such as fullerene-ion and hydrogen-ion plasmas in the laboratory have eliminated the problems related with pair plasmas such as recombination (annihilation) processes [105–109]. Generally, electron and positron plasmas also contain ions. The three species electron, positron and ion plasmas are believed to occur in the early universe [110], active galactic nuclei (AGN) [111] and in pulsar magnetosphere [112]. They can also be created in the laboratory [113–116].

In the current work, a three component plasma containing pair particles and singly ionized ions is considered to study the relaxation dynamics [92]. The inertia of all constituents has been taken into account and the Beltrami parameters corresponding to different plasma fluids are taken to be different. The system self-organizes to Quadruple Beltrami (QB) field which is the superposition of four Beltrami fields. The details of this model are described in the next section.
3.1 Mathematical model

Let us consider a three component plasma consisting of a pair \((-,-,+)\) and singly ionized ions \((i)\). Pair are the particles which exhibit same mass and same charge of opposite polarity. Following Mahajan and Shatashvili [117], we consider the dynamics non-relativistically. The equations of motion of plasma species read as

\[ n_{-m_{-}} \frac{dV_{-}}{dt} = -en_{-}(E + V_{-} \times B) - \nabla p_{-}, \quad (3.1) \]

\[ n_{+m_{+}} \frac{dV_{+}}{dt} = en_{+}(E + V_{+} \times B) - \nabla p_{+}, \quad (3.2) \]

\[ n_{i}m_{i} \frac{dV_{i}}{dt} = en_{i}(E + V_{i} \times B) - \nabla p_{i}, \quad (3.3) \]

where \(V_{j} (j = -, +, i)\) represents the velocity of the fluids, \(n_{j} (m_{j})\) are the number densities (masses) of the species respectively. \(p_{j}\) is the pressure of the species and taken to be as a function of density i.e., \(p_{j} = p_{j}(\rho_{j})\) where \(\rho_{j}\) - the mass densities of the plasma species are constant, \(e\) is the elementary charge, \(B\) is the magnetic field and \(E\) denotes the electric field. In order to simplify the calculations, we normalize the physical variables. The normalized equations of motion read as

\[ \frac{\partial V_{-}}{\partial t} - \frac{\partial A}{\partial t} = V_{-} \times (\nabla \times (V_{-} - A)) - \nabla \left( -\phi + \frac{1}{2} V_{-}^{2} + p_{-} \right), \quad (3.4) \]

\[ \frac{\partial V_{+}}{\partial t} + \frac{\partial A}{\partial t} = V_{+} \times (\nabla \times (V_{+} + A)) - \nabla \left( \phi + \frac{1}{2} V_{+}^{2} + n_{-p_{+}}/n_{+} \right), \quad (3.5) \]

\[ \frac{\partial V_{i}}{\partial t} + M \frac{\partial A}{\partial t} = V_{i} \times (\nabla \times (V_{i} + MA)) - \nabla \left( M\phi + \frac{1}{2} V_{i}^{2} + Mn_{-p_{i}}/n_{i} \right), \quad (3.6) \]

where \(V_{j}\) is normalized to \(V_{A} = B_{0}/\sqrt{\mu_{0}n_{-}m}\) \((B_{0}\) is an arbitrary measure of the magnetic field and \(\mu_{0}\) is the permeability of free space). The magnetic field \(B\) is normalized to an appropriate measure \(B_{0}\). The distance is normalized to the skin depth \(\lambda_{-} = V_{A}/\omega_{e} = \sqrt{m/\mu_{0}n_{-}e^{2}}\) where \(\omega_{e}\) is the electron plasma frequency. \(p_{j}\) is normalized to \(B_{0}^{2}/\mu_{0}\). As the masses of the particles constituting pair are same, therefore we have taken \(m_{-} = m_{+} = m\) and \(M = m/m_{i}\). In above equations, we have used the relations: \(E = -\nabla \phi - \partial A/\partial t,\)
and \( \mathbf{d} \mathbf{V}_j / \mathbf{d}t = \partial \mathbf{V}_j / \partial t + (\nabla \cdot \mathbf{V}_j) \mathbf{V}_j = \partial \mathbf{V}_j / \partial t + \nabla (\mathbf{V}_j^2 / 2) - \mathbf{V}_j \times (\nabla \times \mathbf{V}_j) \) where \( \phi \) is the scalar potential of the species normalized to \( \lambda_0 B_0 V_A \) and \( \mathbf{A} \) is the vector potential \( (\mathbf{B} = \nabla \times \mathbf{A}) \) normalized to \( \lambda_0 B_0 \). The above equations can be re-written as

\[
\frac{\partial \mathbf{P}_-}{\partial t} = \mathbf{V}_- \times (\nabla \times \mathbf{P}_-) - \nabla \psi_-,
\]

\[
\frac{\partial \mathbf{P}_+}{\partial t} = \mathbf{V}_+ \times (\nabla \times \mathbf{P}_+) - \nabla \psi_+,
\]

\[
\frac{\partial \mathbf{P}_i}{\partial t} = \mathbf{V}_i \times (\nabla \times \mathbf{P}_i) - \nabla \psi_i,
\]

where \( \mathbf{P}_- = \mathbf{V}_- - \mathbf{A}, \mathbf{P}_+ = \mathbf{V}_+ + \mathbf{A}, \mathbf{P}_i = \mathbf{V}_i + M \mathbf{A}, \psi_- = -\phi + (\mathbf{V}_-^2 / 2) + p_-, \psi_+ = \phi + (\mathbf{V}_+^2 / 2) + n_- p_+ / n_+ \) and \( \psi_i = M \phi + (\mathbf{V}_i^2 / 2) + M n_- p_i / n_i \). Taking curl of equations (3.7)-(3.9), we obtain

\[
\frac{\partial}{\partial t} (\nabla \times \mathbf{P}_-) = \nabla \times [\mathbf{V}_- \times (\nabla \times \mathbf{P}_-)],
\]

\[
\frac{\partial}{\partial t} (\nabla \times \mathbf{P}_+) = \nabla \times [\mathbf{V}_+ \times (\nabla \times \mathbf{P}_+)],
\]

\[
\frac{\partial}{\partial t} (\nabla \times \mathbf{P}_i) = \nabla \times [\mathbf{V}_i \times (\nabla \times \mathbf{P}_i)],
\]

where \( (\nabla \times \nabla \psi_j = 0, j = -, +, i) \). The equations (3.10)-(3.12) can be written in a more systematic form as

\[
\frac{\partial \mathbf{\Omega}_j}{\partial t} - \nabla \times [\mathbf{U}_j \times \mathbf{\Omega}_j] = 0,
\]

where

\[
\mathbf{\Omega}_j = \nabla \times \mathbf{P}_j,
\]

are the generalized vorticities and

\[
\mathbf{U}_j = \mathbf{V}_j,
\]

are the effectual velocities. The simplest steady state solution of equation (3.13) satisfies the Beltrami condition that is when the velocities and vorticities co-align each other. It

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can be expressed as
\[ \Omega_j = b_j U_j, \quad (3.16) \]
where \( b_j \) \((j = -, +, i)\) are arbitrary constants and called as Beltrami parameters. Steady state solutions for the constituent fluids are, respectively
\[ b_- V_- = \nabla \times V_- - B, \quad (3.17) \]
\[ b_+ V_+ = \nabla \times V_+ + B, \quad (3.18) \]
\[ b_i V_i = \nabla \times V_i + MB. \quad (3.19) \]
The solutions show that generalized vorticities align to the velocities and hence satisfy the Beltrami conditions. Using the Ampere’s Law and taking into account the quasineutrality condition, we obtain
\[ V_i = \frac{1}{N_i} (\nabla \times B + V_- - N_+ V_+), \quad (3.20) \]
where \( N_i = n_i/n_- \), \( N_+ = n_+/n_- \) and \( N_i + N_+ = 1 \). Using equations (3.17)-(3.20), the magnetic field and velocities can be determined. Putting value of \( V_i \) from equation (3.20) into equation (3.19), and using equations (3.17) and (3.18), we obtain the fluid velocity of negatively charged species which reads as
\[ V_- = \frac{1}{b_i - b_-} \left[ (\nabla \times)^2 B - b_i \nabla \times B + \chi B + N_+ (b_i - b_+) V_+ \right], \quad (3.21) \]
where \( \chi = 1 + N_+ + MN_i \). Putting above equation into equation (3.17) and using equation (3.18), we get the velocity of positively charged fluid as given below
\[ V_+ = D_1 (\nabla \times)^3 B - D_2 (\nabla \times)^2 B + D_3 \nabla \times B - D_4 B, \quad (3.22) \]
where \( D = N_+(b_i - b_+)(b_- - b_+) \), \( D_1 = 1/D \), \( D_2 = (b_i + b_-)/D \), \( D_3 = (\chi + b_i b_-)/D \), and \( D_4 = (b_i (1 + N_+) + b_-(N_+ + MN_i) - b_+ N_+)/D \). Eliminating \( V_+ \) from equations
(3.18) and (3.22), we obtain

\[(\nabla \times)^4 \mathbf{B} - a_3 (\nabla \times)^3 \mathbf{B} + a_2 (\nabla \times)^2 \mathbf{B} - a_1 \nabla \times \mathbf{B} + a_0 \mathbf{B} = 0, \quad (3.23)\]

where \(a_0 = MN_1 b_+ + N_+ b_+ b_0 + b_+ b_1, \quad a_1 = MN_1 (b_1 + b_0 + b_+ + b_1), \quad a_2 = (b_0 + b_+ + b_0 b_1 + 1) + N_+ + MN_1, \quad \) and \(a_3 = b_0 + b_+ + b_1. \) Equation (3.23) could be called as Quadruple Beltrami (QB) equation because the magnetic field can be produced by the superposition of four Beltrami fields. A similar steady state equation has been derived recently for a plasma composed of degenerate electrons and positrons with non-degenerate ions [71].

### 3.2 Characteristics of eigenvalues

To interpret equation (3.23) in terms of Beltrami states, we consider four linear Beltrami fields \(\mathbf{B}_\alpha\) which have distinct eigenvalues \(\lambda_\alpha\) \((\alpha = 1, 2, 3, 4)\). Writing the “(\(\nabla \times\))” operator as “curl” and adding all the Beltrami fields \(\mathbf{B}_\alpha\), we get

\[(\text{curl} - \lambda_1)(\text{curl} - \lambda_2)(\text{curl} - \lambda_3)(\text{curl} - \lambda_4)\mathbf{B} = 0, \quad (3.24)\]

as the operators \((\text{curl} - \lambda_\alpha)\) are commutative and \(\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 + \mathbf{B}_4\). Simplifying above equation, we retrieve equation (3.23)

\[(\nabla \times)^4 \mathbf{B} - a_3 (\nabla \times)^3 \mathbf{B} + a_2 (\nabla \times)^2 \mathbf{B} - a_1 \nabla \times \mathbf{B} + a_0 \mathbf{B} = 0, \quad (3.25)\]

where \(a_0 = \lambda_1 \lambda_2 \lambda_3 \lambda_4, \quad a_1 = \lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_4 + \lambda_2 \lambda_3 \lambda_4 + \lambda_1 \lambda_3 \lambda_4, \quad a_2 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3 + \lambda_1 \lambda_4 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4, \quad a_3 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4. \) Therefore, the general solution of equation (3.23) can be cast as a linear sum of four Beltrami fields characterized by four amplitudes and four scale parameters. Let \(\mathbf{B}_\alpha\) where \(\alpha = 1, 2, 3, 4\) are the Beltrami
fields such that $\nabla \times B_\alpha = \lambda_\alpha B_\alpha$. Then the equation

$$\mathbf{B} = \sum_{\alpha=1}^{4} C_\alpha \mathbf{B}_\alpha,$$

(3.26)

where $C_\alpha$ are arbitrary constants, serves as solution of equation (3.23). The eigenvalues of the curl operator are the roots of the following general quartic equation

$$\lambda^4 - a_3 \lambda^3 + a_2 \lambda^2 - a_1 \lambda + a_0 = 0.$$  

(3.27)

The roots of above equation are

$$\lambda_1 = \frac{a_3 + 2R + 2\delta}{4},$$

$$\lambda_2 = \frac{a_3 + 2R - 2\delta}{4},$$

$$\lambda_3 = \frac{a_3 - 2R + 2\xi}{4},$$

$$\lambda_4 = \frac{a_3 - 2R - 2\xi}{4},$$

where $R = \left(\sqrt{a_3^2 - 4a_2 + 4Y}\right)/2$, $Y = (d - 3u^2 + 3u\alpha_1)/3u$, $d = (3\alpha_2 - \alpha_1^2)/3$, $u = \sqrt[3]{(q/2) + \sqrt{(q/2)^2 + (d/3)^3}}$, $q = (9\alpha_1\alpha_2 - 2\alpha_1^3 - 27\alpha_3)/27$, $\alpha_1 = a_2$, $\alpha_2 = a_1a_3 - 4a_0$, and $\alpha_3 = a_1^2 + a_3^2a_0 - 4a_2a_0$. For $R \neq 0$, $\delta$ and $\xi$ are given as

$$\delta = \sqrt{\frac{3}{4}a_3^2 - R^2 + 2a_2 + \frac{1}{4R}(4a_3a_2 - 8a_1 - a_3^3)},$$

$$\xi = \sqrt{\frac{3}{4}a_3^2 - R^2 + 2a_2 - \frac{1}{4R}(4a_3a_2 - 8a_1 - a_3^3)},$$

while for $R = 0$, the expressions for $\delta$ and $\xi$ read as

$$\delta = \sqrt{\frac{3}{4}a_3^2 - 2a_2 + 2\sqrt{Y^2 - 4a_0}},$$

$$\xi = \sqrt{\frac{3}{4}a_3^2 - 2a_2 - 2\sqrt{Y^2 - 4a_0}}.$$
Figure 3-1: Character of the roots of the quartic equation for $N_i = 0.01$.

The nature of the eigenvalues ($\lambda_j$) can be described in terms of the discriminant ($\Delta$), which is defined as

$$\Delta = 256a_3^3 - 192a_3a_1a_0^2 - 128a_2^2a_0^2 + 144a_2a_1^2a_0 - 27a_1^4$$
$$+ 144a_3^2a_1a_0^2 - 6a_3^2a_1^2a_0 - 80a_3a_2^2a_1a_0 + 18a_3a_2a_1^3 + 16a_2^4a_0$$
$$- 4a_2^3a_1^2 - 27a_3^4a_0^2 + 18a_3^3a_2a_1a_0 - 4a_3^3a_1^3 - 4a_3^2a_2^3a_0 + a_3^2a_2^3a_1^2.$$

If $\Delta < 0$, then there are two real eigenvalues and two complex conjugate roots of the quartic equation (3.27). For $\Delta > 0$, all the roots of equation (3.27) are either real or complex. For $\Gamma = 8a_2 - 3a_3^2 < 0$ and $\Upsilon = 64a_2 - 16a_2^2 + 16a_2^3a_0 - 16a_3a_1 - 3a_3^4 < 0$, all the roots will be real and distinct, whereas for $\Gamma > 0$ or $\Upsilon > 0$, all the scale parameters ($\lambda_j$) consist of two pairs of complex conjugate. If $\Delta = 0$, then equation (3.27) either will have multiple root, or it will be equal to the square of a quadratic polynomial. If $\Upsilon = 0$ and $\Gamma < 0$, then there are two real double roots. On the other hand, if $\Upsilon = 0$ and $\Gamma > 0$ & $a_3^3 + 8a_1 - 4a_3a_2 = 0$, then quartic equation will have two complex conjugate double roots.

Figure (3-1) shows the behavior of scale parameters ($\lambda_\alpha$) as a function of Beltrami
parameters \((b_-, b_+ \text{ and } b_i)\) for \(N_i = 0.01\) that is ions density is hundred times less than that of the negatively charged species of the pair. All the scale parameters \(\lambda_\alpha\) are real in the colored region of figure (3-1) while in the remaining part of the graph, there are two real scale parameters and one complex conjugate pair of scale parameters. Figure (3-2) shows the character of roots in Beltrami parameters space for \(N_i = 1\) i.e., \(n_i = n_-\). A comparison of the character of roots shown in two figures for different ion densities depicts that on increasing the density of ions, some of the complex roots change to real ones. As the roots are eigenvalues of curl operator and determine the size and nature of the structures that is whether they are paramagnetic or diamagnetic, we can say that density of plasma constituents has a strong role in formation of field structures in three component plasmas.

3.3 Impact of Beltrami parameters

The Beltrami parameters being the ratio of generalized vorticities to corresponding flows play a vital role in the formation of self-organized structures. The Beltrami parameters expresses the strength of vorticities and flows. They determine the relaxed states. Fur-
ther, the scale parameters are functions of Beltrami parameters. The nature of scale parameters also depends on Beltrami parameters. The impact of Beltrami parameters on self-organized states is analyzed below.

- Case 1

First of all, we consider the simplest case that is all the fluids have same ratio of vorticities to flows. It is worth mentioning that all flows follow the generalized vorticities. They are termed as generalized vorticities because the normal vorticities (curl of flows) are modified due to addition of magnetic fields. When all the Beltrami parameters are equal that is \( b_- = b_+ = b_i = b_s \), equations (3.17)-(3.19) become

\[
\begin{align*}
 b_s V_- &= \nabla \times V_- - B, \\
 b_s V_+ &= \nabla \times V_+ + B, \\
 b_s V_i &= \nabla \times V_i + MB,
\end{align*}
\]

Putting the value of \( V_i \) from equation (3.20) into equation (3.30) and using equations (3.28) and (3.29), we obtain

\[
(\nabla \times)^2 B - b_s \nabla \times B + \chi B = 0. \tag{3.31}
\]

Equation (3.31) is the double curl Beltrami equation - a linear sum of two Beltrami fields. We consider a special case i.e., we take the generalized vorticities corresponding to all the fluids to be zero. Hence, for \( b_s = 0 \), equation (3.31) reduces to

\[
(\nabla \times)^2 B + \chi B = 0. \tag{3.32}
\]

In this case generalized vorticities become zero and the relaxed equilibrium satisfies the London equation as pointed out by Mahajan [42]. In this particular case, the self-organized magnetic fields will show perfect diamagnetism.
Case 2

If we take $b_- = b_+$, then equations (3.17)-(3.19) read as

\begin{align*}
    b_- V_- &= \nabla \times V_- - B, \\
    b_+ V_+ &= \nabla \times V_+ + B, \\
    b_i V_i &= \nabla \times V_i + MB. 
\end{align*}

(3.33) 
(3.34) 
(3.35)

For this case, the ratios of generalized vorticities to flows of both species making pair are equal but the ratio of generalized vorticity to flow of ion fluid is different. Making use of Ampere’s Law and using equations (3.33)-(3.35), we obtain Triple Curl Beltrami equation after some simple algebraic manipulation as expressed below

\begin{equation}
(\nabla \times)^3 B - (b_i + b_-) (\nabla \times)^2 B + (\chi + b_i b_-) \nabla \times B - \zeta_1 B = 0, \tag{3.36}
\end{equation}

where $\zeta_1 = b_- N_i M + (1 + N_+) b_i$. If we take $b_- = b_+$ and $b_i = 0$, then equation (3.36) read as

\begin{equation}
(\nabla \times)^3 B - b_- (\nabla \times)^2 B + \chi \nabla \times B - b_- N_i MB = 0. \tag{3.37}
\end{equation}

If we take $b_- = b_+ = 0$ and $b_i \neq 0$, then equation (3.36) read as

\begin{equation}
(\nabla \times)^3 B - b_i (\nabla \times)^2 B + \chi \nabla \times B - (1 + N_+) b_i B = 0. \tag{3.38}
\end{equation}

Case 3

If we take $b_- = b_i$, then equations (3.17)-(3.19) read as

\begin{align*}
    b_- V_- &= \nabla \times V_- - B, \\
    b_+ V_+ &= \nabla \times V_+ + B, \\
    b_- V_i &= \nabla \times V_i + MB. 
\end{align*}

(3.39) 
(3.40) 
(3.41)
For this case, the ratios of generalized vorticities to flows of the negative species are equal to the ratio of generalized vorticity to flow of ion fluid, while the ratio of generalized vorticity to flow of positive species fluid is different. In order to eliminate all flows, we solve above equations and Ampere’s Law simultaneously and obtain the following steady state equation known as TB equation

\[(\nabla \times)^3 B - (b_+ + b_-) (\nabla \times)^2 B + (\chi + b_+ b_-) \nabla \times B - \kappa_2 B = 0, \quad (3.42)\]

where \(\kappa_2 = b_- N_+ + (1 + MN_i) b_+\). If we take \(b_- = b_i\) and \(b_+ = 0\), then equation (3.42) read as

\[(\nabla \times)^3 B - b_- (\nabla \times)^2 B + \chi \nabla \times B - b_- N_+ B = 0. \quad (3.43)\]

If we take \(b_- = b_i = 0\) and \(b_+ \neq 0\), then equation (3.42) read as

\[(\nabla \times)^3 B - b_+ (\nabla \times)^2 B + \chi \nabla \times B - (1 + MN_i) b_+ B = 0. \quad (3.44)\]

• Case 4

If we take \(b_+ = b_i\), then equations (3.17)-(3.19) read as

\[b_- V_- = \nabla \times V_- - B, \quad (3.45)\]
\[b_i V_+ = \nabla \times V_+ + B, \quad (3.46)\]
\[b_i V_i = \nabla \times V_i + MB. \quad (3.47)\]

For this case, the ratios of generalized vorticities to flows of both the positive species are equal but the ratio of generalized vorticity to flow of negative fluid is different. It means that all the flows are following the generalized vorticities. In order to get expressions for flows and magnetic field, we need one more equation for closing the system. Hence, making use of Ampere’s Law and solving the system for magnetic field, we obtain the
(\nabla \times)^3 \mathbf{B} - (b_i + b_-) (\nabla \times)^2 \mathbf{B} + (\chi + b_i b_-) \nabla \times \mathbf{B} - \zeta_3 \mathbf{B} = 0, \quad (3.48)

known as Triple Beltrami (TB) equation where \( \zeta_3 = b_i + (MN_i + N_+) b_- \). It is worth noting that all inertial forces are taking part in the dynamics but the self-organized state is TB state rather than QB states. It is due to the reasoning that two fluids have same Beltrami parameters. If we take \( b_i = b_+ \) and \( b_- = 0 \), then equation (3.48) read as

\[(\nabla \times)^3 \mathbf{B} - b_i (\nabla \times)^2 \mathbf{B} + \chi \nabla \times \mathbf{B} - b_i M \mathbf{B} = 0. \quad (3.49)\]

If we take \( b_i = b_+ = 0 \) and \( b_+ \neq 0 \), then equation (3.48) read as

\[(\nabla \times)^3 \mathbf{B} - b_- (\nabla \times)^2 \mathbf{B} + \chi \nabla \times \mathbf{B} - (MN_i + N_+) b_i \mathbf{B} = 0. \quad (3.50)\]

- **Case 5**

If we take \( b_- \neq b_+ \) and \( b_i = 0 \), equations (3.17)-(3.19) become

\[b_- \mathbf{V}_- = \nabla \times \mathbf{V}_- - \mathbf{B}, \quad (3.51)\]
\[b_+ \mathbf{V}_+ = \nabla \times \mathbf{V}_+ + \mathbf{B}, \quad (3.52)\]
\[0 = \nabla \times \mathbf{V}_i + M \mathbf{B}. \quad (3.53)\]

For this condition, the ratio of generalized vorticities to flows of both the species making pair are different and non-zero but the ratio of generalized vorticity of ions fluid is zero. The flows of the pair fluids are following the generalized vorticities whereas the vorticity of ions \( \nabla \times \mathbf{V}_i \) becomes anti-parallel to the magnetic field. Solving the system for magnetic field, using Ampere’s Law, we obtain Quartic Beltrami (QB) equation, which
is given below

\[
(\nabla \times)^4 \mathbf{B} - a_3 (\nabla \times)^3 \mathbf{B} + a_2 (\nabla \times)^2 \mathbf{B} - a_1 \nabla \times \mathbf{B} + a_0 \mathbf{B} = 0, \tag{3.54}
\]

where \(a_0 = MN_i b_- b_+, a_1 = MN_i (b_- + b_+) + N_+ b_- + b_+, a_2 = b_- b_+ + \chi, \) and \(a_3 = b_- + b_+.

- **Case 6**

  If we take \(b_- \neq b_i \) and \(b_+ = 0\), equations (3.17)-(3.19) read as

  \[
  b_- \mathbf{V}_- = \nabla \times \mathbf{V}_- - \mathbf{B}, \tag{3.55}
  \]
  \[
  0 = \nabla \times \mathbf{V}_+ + \mathbf{B}, \tag{3.56}
  \]
  \[
  b_i \mathbf{V}_i = \nabla \times \mathbf{V}_i + MB. \tag{3.57}
  \]

  For this state, the ratio of generalized vorticity of positive species making pair is zero but the ratio of generalized vorticities of other species (ions and negative species making pair) are non-zero but different from each other. From these three above equations and using Ampere’s Law, we again obtain Quartic Beltrami equation, which is given below

\[
(\nabla \times)^4 \mathbf{B} - a_3 (\nabla \times)^3 \mathbf{B} + a_2 (\nabla \times)^2 \mathbf{B} - a_1 \nabla \times \mathbf{B} + a_0 \mathbf{B} = 0, \tag{3.58}
\]

where \(a_0 = N_+ b_- b_i, a_1 = MN_i b_- + N_+ (b_- + b_i) + b_i, a_2 = b_- b_i + \chi, \) and \(a_3 = b_- + b_i.

- **Case 7**

  If we take \(b_i \neq b_+ \) and \(b_- = 0\), equations (3.17)-(3.19) take the form

  \[
  0 = \nabla \times \mathbf{V}_- - \mathbf{B}, \tag{3.59}
  \]
  \[
  b_+ \mathbf{V}_+ = \nabla \times \mathbf{V}_+ + \mathbf{B}, \tag{3.60}
  \]
  \[
  b_i \mathbf{V}_i = \nabla \times \mathbf{V}_i + MB. \tag{3.61}
  \]
For this case the ratio of generalized vorticities of both the positive species are non-zero and distinct but the ratio of generalized vorticity of negative species is zero. In this particular case, the flows of positively charged fluids are parallel to their corresponding generalized vorticities while the vorticity of negatively charged fluids follows the magnetic field. Solving for magnetic field and using Ampere’s Law, we earn Quartic Beltrami equation, which is given below

\[(\nabla \times)^4 B - a_3 (\nabla \times)^3 B + a_2 (\nabla \times)^2 B - a_1 \nabla \times B + a_0 B = 0,\]  \hspace{1cm} (3.62)

where \(a_0 = b_ib_+, a_1 = MN_i b_+ + N_i b_i + b_i + b_+, a_2 = b_i b_+ + \chi, \) and \(a_3 = b_i + b_+.\)

The alignments of generalized vorticities along their respective flows constitute the steady state solutions of the system. The solutions are acceptable equilibrium solutions if they satisfy the Bernoulli conditions \(\nabla \psi_j = 0 \) \((j = -, +, i)\) that is all the gradient forces vanish. This is the reason that these relaxed states are termed as Beltrami-Bernoulli states [47]. In the next section, we will describe the Bernoulli equation which indicates the potential of the system to confine a high pressure in the presence of a sheared flow. It also serves to close the system.

### 3.4 Bernoulli equation

Substitution of equilibrium solutions given in equations (3.17)-(3.19) in the steady state \((\partial/\partial t = 0)\) macroscopic evolution equations (3.4)-(3.6) of plasma species respectively, we obtain the following equations

\[\nabla (-\phi + \frac{1}{2} \mathbf{V}_2^2 + p_-) = 0,\]  \hspace{1cm} (3.63)

\[\nabla (\phi + \frac{1}{2} \mathbf{V}_+^2 + n_-p_+/n_+) = 0,\]  \hspace{1cm} (3.64)

\[\nabla (M\phi + \frac{1}{2} \mathbf{V}_i^2 + Mn_-p_i/n_i) = 0.\]  \hspace{1cm} (3.65)
Integrating the above equations which are exact derivatives, we obtain the Bernoulli conditions given as below

\[-\phi + \frac{1}{2} V_-^2 + p_- = g_- , \quad (3.66)\]
\[\phi + \frac{1}{2} V_+^2 + \frac{1}{N_+} p_+ = g_+ , \quad (3.67)\]
\[M\phi + \frac{1}{2} V_i^2 + M\frac{1}{N_i} p_i = g_i , \quad (3.68)\]

where \(g_-\), \(g_i\), and \(g_+\) are integration constants. The addition of above equations give

\[\psi_- + \psi_+ + \psi_i = g_t, \quad (3.69)\]

where \(g_t = g_- + g_+ + g_i\). The Bernoulli conditions show the relation of electrostatic potentials, flows and pressures [37]. The flow of positively charged component of pair \(V_+\) is given in equation (3.22). In order to find the flow of negatively charged fluid \(V_-\), we put value of equation (3.22) into equation (3.21), we obtain

\[V_- = G_1 (\nabla \times)^3 B - G_2 (\nabla \times)^2 B + G_3 \nabla \times B - G_4 B , \quad (3.70)\]

where \(G = (b_- - b_+) (b_i - b_-)\), \(G_1 = 1/G\), \(G_2 = (b_i + b_+) /G\), \(G_3 = (\chi + b_+ b_i) /G\), and \(G_4 = \{ b_i (1 + N_+) - b_- + b_+ (1 + MN_i) \} /G\). Putting the values of \(V_+\) and \(V_-\) from equations (3.22) and (3.70) respectively into equation (3.20), and after some straightforward algebraic manipulation, we obtain the velocity of ion fluid \(V_i\) as expressed below

\[V_i = K_1 (\nabla \times)^3 B - K_2 (\nabla \times)^2 B + K_3 \nabla \times B - K_4 B , \quad (3.71)\]

where \(\kappa_0 = N_i (b_i - b_-) (b_i - b_+)\), \(K_1 = 1/\kappa_0\), \(K_2 = (b_- + b_+) /\kappa_0\), \(K_3 = (\chi + b_- b_+) /\kappa_0\), and \(K_4 = (b_- (N_+ + MN_i) + b_+ (1 + MN_i) - b_1 MN_i) /\kappa_0\).
3.5 Analytical solution in slab geometry

The QB field being the linear sum of four Beltrami states reads as

\[ B = C_1 B_1 + C_2 B_2 + C_3 B_3 + C_4 B_4. \] (3.72)

We consider a slab geometry in Cartesian coordinates \((x, y, z)\), and assume the length of the slab along \(x\)-axis is \(x_0\). For a slab plasma, above equation can be written as

\[
B = \sum_{j=1}^{4} C_j \begin{pmatrix}
0 \\
\sin(\lambda_j x) \\
\cos(\lambda_j x)
\end{pmatrix},
\] (3.73)

where \(C_j (j = 1, 2, 3, 4)\) are constants and can be determined using the boundary conditions \(|B_y|_{x=0} = h\), \(|B_z|_{x=x_0} = g\), \(|(\nabla \times B)_y|_{x=0} = t\), and \(|(\nabla \times B)_z|_{x=x_0} = w\), we obtain

\[ C_1 \sin(\lambda_1 x_0) + C_2 \sin(\lambda_2 x_0) + C_3 \sin(\lambda_3 x_0) + C_4 \sin(\lambda_4 x_0) = h, \] (3.74)

\[ C_1 + C_2 + C_3 + C_4 = g, \] (3.75)

\[ C_1 \lambda_1 \sin(\lambda_1 x_0) + C_2 \lambda_2 \sin(\lambda_2 x_0) + C_3 \lambda_3 \sin(\lambda_3 x_0) + C_4 \lambda_4 \sin(\lambda_4 x_0) = t, \] (3.76)

\[ C_1 \lambda_1 + C_2 \lambda_2 + C_3 \lambda_3 + C_4 \lambda_4 = w. \] (3.77)

Solving the above equations, we obtain

\[ C_1 = \frac{N_1}{D}, \quad C_2 = \frac{N_2}{D}, \quad C_3 = \frac{N_3}{D}, \quad C_4 = \frac{N_4}{D}, \]
where

\[N_1 = [(w - g\lambda_2) \sin(\lambda_3 x_0) \sin(\lambda_4 x_0) + (t - h\lambda_2) \sin(\lambda_2 x_0)](\lambda_3 - \lambda_4)
+ [(w - g\lambda_4) \sin(\lambda_3 x_0) \sin(\lambda_2 x_0) + (t - h\lambda_4) \sin(\lambda_4 x_0)](\lambda_2 - \lambda_3)
+ [(w - g\lambda_3) \sin(\lambda_2 x_0) \sin(\lambda_4 x_0) + (t - h\lambda_3) \sin(\lambda_3 x_0)](\lambda_4 - \lambda_2),\]

\[N_2 = [(w - g\lambda_4) \sin(\lambda_3 x_0) \sin(\lambda_1 x_0) + (t - h\lambda_4) \sin(\lambda_4 x_0)](\lambda_3 - \lambda_1)
+ [(w - g\lambda_1) \sin(\lambda_3 x_0) \sin(\lambda_4 x_0) + (t - h\lambda_1) \sin(\lambda_1 x_0)](\lambda_4 - \lambda_3)
+ [(w - g\lambda_3) \sin(\lambda_1 x_0) \sin(\lambda_4 x_0) + (t - h\lambda_3) \sin(\lambda_3 x_0)](\lambda_1 - \lambda_4),\]

\[N_3 = [(w - g\lambda_2) \sin(\lambda_1 x_0) \sin(\lambda_4 x_0) + (t - h\lambda_2) \sin(\lambda_2 x_0)](\lambda_4 - \lambda_1)
+ [(w - g\lambda_4) \sin(\lambda_1 x_0) \sin(\lambda_2 x_0) + (t - h\lambda_4) \sin(\lambda_4 x_0)](\lambda_1 - \lambda_2)
+ [(w - g\lambda_3) \sin(\lambda_2 x_0) \sin(\lambda_4 x_0) + (t - h\lambda_3) \sin(\lambda_3 x_0)](\lambda_2 - \lambda_4),\]

\[N_4 = [(w - g\lambda_2) \sin(\lambda_3 x_0) \sin(\lambda_1 x_0) + (t - h\lambda_2) \sin(\lambda_2 x_0)](\lambda_1 - \lambda_3)
+ [(w - g\lambda_3) \sin(\lambda_1 x_0) \sin(\lambda_2 x_0) + (t - h\lambda_3) \sin(\lambda_3 x_0)](\lambda_2 - \lambda_1)
+ [(w - g\lambda_1) \sin(\lambda_2 x_0) \sin(\lambda_3 x_0) + (t - h\lambda_1) \sin(\lambda_1 x_0)](\lambda_3 - \lambda_2),\]

\[\mathcal{D} = [\sin(\lambda_1 x_0) \sin(\lambda_4 x_0) + \sin(\lambda_2 x_0) \sin(\lambda_3 x_0)](\lambda_3 - \lambda_2)(\lambda_4 - \lambda_1)
+ [\sin(\lambda_3 x_0) \sin(\lambda_4 x_0) + \sin(\lambda_2 x_0) \sin(\lambda_1 x_0)](\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)
+ [\sin(\lambda_1 x_0) \sin(\lambda_3 x_0) + \sin(\lambda_2 x_0) \sin(\lambda_4 x_0)](\lambda_3 - \lambda_1)(\lambda_2 - \lambda_4).\]
3.6 Composite flow

To find the composite flow $V$, we use the relation

$$V = \frac{\rho_- V_- + \rho_+ V_+ + \rho_i V_i}{\rho}, \quad (3.78)$$

where $\rho = \rho_- + \rho_+ + \rho_i$ is the fluid density, $\rho_- = n_- m$, $\rho_+ = n_+ m$ and $\rho_i = n_i m_i$ respectively, are negative, positive and ion fluid densities. Putting value of $V_i$ from equation (3.20) into equation (3.78), we obtain

$$V = \frac{n_-}{\rho} [m_i \nabla \times B - N_+ (m_i - m) V_+ + (m_i + m) V_-]. \quad (3.79)$$

Substituting value of $V_-$ from equation (3.21) into above equation, we obtain

$$V = \frac{n_-}{\rho (b_i - b_-)} \left[ \mathcal{M} (\nabla \times)^2 B - \mathcal{M}' \nabla \times B + \chi \mathcal{M} B + N V_+ \right], \quad (3.80)$$

where $\mathcal{M} = m_i + m$, $\mathcal{M}' = b_i m + b_- m_i$ and $N = N_+ \{ \mathcal{M} (b_i - b_+) + (m - m_i) (b_i - b_-) \}$. Substituting value of $V_+$ from equation (3.22) into equation (3.80), we get

$$V = b_1 (\nabla \times)^3 B - b_2 (\nabla \times)^2 B + b_3 \nabla \times B - b_4 B \quad (3.81)$$

where $b_1 = ND_1 \rho^{-1} n_-/(b_i - b_-)$, $b_2 = (ND_2 - \mathcal{M}) \rho^{-1} n_-/(b_i - b_-)$, $b_3 = (ND_3 - \mathcal{M}' \rho^{-1} n_-/(b_i - b_-)$, and $b_4 = (ND_4 + \chi \mathcal{M}) \rho^{-1} n_-/(b_i - b_-)$. For sake of completeness, we will evaluate the expressions for the flow components of all dynamical species.
3.6.1 Components of flows

Equation (3.81) describes the composite flow \( V \) of the entire system, the \( y \) and \( z \) components of the \( V \) can be expressed as respectively

\[
V_y = \sum_{i=1}^{4} C_i f_i \sin \lambda_i x,  \tag{3.82}
\]

\[
V_z = \sum_{i=1}^{4} C_i f_i \cos \lambda_i x,  \tag{3.83}
\]

where \( f_1 = \lambda_1^3 b_1 - \lambda_1^2 b_2 + \lambda_1 b_3 - b_4, f_2 = \lambda_2^3 b_1 - \lambda_2^2 b_2 + \lambda_2 b_3 - b_4, f_3 = \lambda_3^3 b_1 - \lambda_3^2 b_2 + \lambda_3 b_3 - b_4 \)

and \( f_4 = \lambda_4^3 b_1 - \lambda_4^2 b_2 + \lambda_4 b_3 - b_4 \). Equation (3.22) narrates the flow of the positive species \( V^+ \) making pair plasma, its \( y \) and \( z \) components are given below

\[
|V^+|_y = \sum_{i=1}^{4} C_i g_i \sin \lambda_i x,  \tag{3.84}
\]

\[
|V^+|_z = \sum_{i=1}^{4} C_i g_i \cos \lambda_i x,  \tag{3.85}
\]

where \( g_1 = \lambda_1^3 D_1 - \lambda_1^2 D_2 + \lambda_1 D_3 - D_4, g_2 = \lambda_2^3 D_1 - \lambda_2^2 D_2 + \lambda_2 D_3 - D_4, g_3 = \lambda_3^3 D_1 - \lambda_3^2 D_2 + \lambda_3 D_3 - D_4 \) and \( g_4 = \lambda_4^3 D_1 - \lambda_4^2 D_2 + \lambda_4 D_3 - D_4 \). From equation (3.21), we can evaluate the \( y \) and \( z \) components of flow of the negative species \( V^- \) as expressed below

\[
|V^-|_y = \sum_{i=1}^{4} C_i h_i \sin \lambda_i x,  \tag{3.86}
\]

\[
|V^-|_z = \sum_{i=1}^{4} C_i h_i \cos \lambda_i x,  \tag{3.87}
\]

where \( h_1 = \left[ \lambda_1^2 - \lambda_1 b_i + \chi + \omega g_1 \right] / (b_i - b_-), h_2 = \left[ \lambda_2^2 - \lambda_2 b_i + \chi + \omega g_2 \right] / (b_i - b_-), h_3 = \left[ \lambda_3^2 - \lambda_3 b_i + \chi + \omega g_3 \right] / (b_i - b_-), h_4 = \left[ \lambda_4^2 - \lambda_4 b_i + \chi + \omega g_4 \right] / (b_i - b_-) \) and \( \omega = N_+(b_i - b_+) \). Equation (3.20) describes the flow of the ions \( V_i \), its \( y \) and \( z \) components
are given below,

\[ |V_i|_y = \sum_{i=1}^{4} C_{ii} \sin \lambda_i x, \quad (3.88) \]

\[ |V_i|_z = \sum_{i=1}^{4} C_{ii} \cos \lambda_i x, \quad (3.89) \]

where \( i_1 = (\lambda_1 + h_1 - N_i g_1) / N_i, \) \( i_2 = (\lambda_2 + h_2 - N_i g_2) / N_i, \) \( i_3 = (\lambda_3 + h_3 - N_i g_3) / N_i \)
and \( i_4 = (\lambda_4 + h_4 - N_i g_4) / N_i. \)

### 3.7 Ideal invariants

The physical quantities which remain conserved in a perfectly conducting plasma are termed as ideal invariants. To find these quantities, let \( \Omega_j \) is the vorticity that satisfies equation (3.13) and the following boundary condition

\[ n \times (U_j \times \Omega_j) = 0, \quad (3.90) \]

on the surface of the confining boundary. The general helicity is defined as

\[ h_j = \frac{1}{2} \int_{\Omega} \Omega_j \cdot \text{curl}^{-1} \Omega_j dv, \]

where \( \text{curl}^{-1} \) is the inverse of the curl operator that is represented by Biot-Savart integral. Using this definition, it is easy to verify that \( h_j \) is constant \[31\]. Through straightforward manipulations of equations (3.7)-(3.9), we can show that the magnetofluid energy \( (E) \) is also a constant of motion. Using \( \partial A / \partial t = -E - \nabla \phi \) and multiplying equations
(3.7)-(3.9) by \( V_-, N_+ V_+ \) and \( N_i V_i/M_i \), we obtain respectively

\[
\frac{\partial}{\partial t} \left( \frac{V_-^2}{2} \right) = -E \cdot V_- - V_- \cdot \nabla \phi, \tag{3.91}
\]

\[
\frac{\partial}{\partial t} \left( \frac{N_+ V_+^2}{2} \right) = N_+ V_+ \cdot E + N_+ V_+ \cdot \nabla \phi, \tag{3.92}
\]

\[
\frac{N_i}{M} \frac{\partial}{\partial t} \left( \frac{V_i^2}{2} \right) = N_i V_i \cdot E + N_i V_i \cdot \nabla \phi, \tag{3.93}
\]

We have used equations (3.63)-(3.65) to evaluate above equations. Adding equations (3.91)-(3.93), we get

\[
\frac{\partial}{\partial t} \left( \frac{V_-^2}{2} + N_+ \frac{V_+^2}{2} + \frac{N_i}{M} \frac{V_i^2}{2} \right) + E \cdot (V_- - N_+ V_+ - N_i V_i)
\]

\[
= (-V_- + N_+ V_+ + N_i V_i) \cdot \nabla \phi \tag{3.94}
\]

Putting equation (3.20) into above equation, we obtain

\[
\frac{\partial}{\partial t} \left( \frac{V_-^2}{2} + N_+ \frac{V_+^2}{2} + \frac{N_i}{M} \frac{V_i^2}{2} \right) - E \cdot (\nabla \times B)
\]

\[
= (\nabla \times B) \cdot \nabla \phi \tag{3.95}
\]

Using the identity

\[
\nabla \cdot (E \times B) = B \cdot (\nabla \times E) - E \cdot (\nabla \times B). \tag{3.96}
\]

Putting \( \nabla \times E = -\partial B/\partial t \) in equation (3.96), we obtain

\[
E \cdot (\nabla \times B) = -\frac{\partial}{\partial t} \left( \frac{B^2}{2} \right) - \nabla \cdot (E \times B). \tag{3.97}
\]

Also we can write

\[
(\nabla \times B) \cdot \nabla \phi = \nabla \cdot (B \times \nabla \phi). \tag{3.98}
\]
Using equations (3.97) and (3.98) into equation (3.95), we get

$$\frac{\partial}{\partial t} \left( \frac{V^2}{2} + N_+ \frac{V^2_+}{2} + \frac{N_i V^2_i}{M} \right) + \frac{\partial}{\partial t} \left( \frac{B^2}{2} \right)$$

$$= - \nabla \cdot (E \times B) + \nabla \cdot (B \times \nabla \phi) \quad (3.99)$$

Integrate the above equation with respect to the volume of the system, we have

$$\frac{\partial}{\partial t} \int \left( \frac{V^2_+ + N_+ V^2_+ + B^2}{2} + \frac{N_i V^2_i}{M} \right) dv = \int - \nabla \cdot (E \times B) dv + \int \nabla \cdot (B \times \nabla \phi) dv$$

Using the divergence theorem and applying the boundary conditions that normal components of all fields vanish at the boundary, we obtain

$$\frac{\partial}{\partial t} \int \left( \frac{V^2_+ + N_+ V^2_+ + B^2}{2} + \frac{N_i V^2_i}{M} \right) dv = 0.$$

It shows that

$$E = \int \left( \frac{V^2_+ + N_+ V^2_+ + B^2}{2} + \frac{N_i V^2_i}{M} \right) dv, \quad (3.100)$$

is the constant of motion. Hence the following are the constants of motion

$$E = \frac{1}{2} \left( V^2_+ + \frac{n_+}{n_-} V^2_+ + \frac{n_i}{M n_-} V^2_i + B^2 \right), \quad (3.101)$$

$$h_- = \frac{1}{2} \int \Omega_- \cdot \text{curl}^{-1} \Omega_- dv, \quad (3.102)$$

$$h_+ = \frac{1}{2} \int \Omega_+ \cdot \text{curl}^{-1} \Omega_+ dv, \quad (3.103)$$

$$h_i = \frac{1}{2} \int \Omega_i \cdot \text{curl}^{-1} \Omega_i dv, \quad (3.104)$$

where \(E\) is the magnetofluid energy and \(h_j\) (\(j = -, +, i\)) are generalized helicities of three fluids.
3.8 Quadruple Beltrami states via variational principle

There are four constants of motion as there are three plasma fluids. In a plasma where there are $S$ dynamical species, there will be $S + 1$ constants of motion [42, 70]. The equilibrium states can be accessed by exploiting the constants of motion. For instance, the relaxed states of ideal MHD [11, 14, 22, 118, 119], the Double Beltrami (DB) states permitted by Hall MHD [28, 30, 70] and the multi-Beltrami states of extended MHD [26] are found through different adjustments of the ideal invariants. Several authors have used the magnetic energy as a target function that it decays faster in comparison to other invariants in the presence of dissipative effects in the system and the variational principle is constructed. Employing the magnetic energy as a target functional, the Quadruple Curl equation can be derived by constrained minimization of energy using the following variational principle

$$\delta (E - \mu_- h_- - \mu_+ h_+ - \mu_i h_i) = 0,$$ \hspace{1cm} (3.105)

where $\mu_-, \mu_+$ and $\mu_i$ are Lagrange multipliers, $h_-, h_+$ and $h_i$ are the generalized helicities of negatively charged, positively charged and ions species respectively. The energy and helicities are given as where $\Omega_- = \nabla \times V_-, \Omega_+ = \nabla \times V_+ + B, \Omega_i = \nabla \times V_i + MB$ and putting values of equations (3.101-3.104) into equation (3.105) and considering only the independent variations $\delta A, \delta V_-, \delta V_+$ and $\delta V_i$, we obtain after some algebraic manipulation,

$$\nabla \times B = \frac{1}{b_-} (B - \nabla \times V_-) + \frac{N_+}{b_+} (B + \nabla \times V_+) + \frac{N_i}{b_i} (MB + \nabla \times V_i),$$ \hspace{1cm} (3.106)
\[ b_- V_- = \nabla \times V_- - B, \tag{3.107} \]
\[ b_+ V_+ = \nabla \times V_+ + B, \tag{3.108} \]
\[ b_i V_i = \nabla \times V_i + MB. \tag{3.109} \]

Taking \( \mu_- = 1/b_- \), \( \mu_i = N_i/Mb_i \) and \( \mu_+ = N_+/b_+ \), and putting equations (3.107)-(3.109) into equation (3.106), we obtain

\[ V_i = \frac{1}{N_i} [\nabla \times B + V_- - N_+ V_+]. \tag{3.110} \]

We observe that equations (3.107)-(3.110) are same as we derived earlier equations (3.28)-(3.19). Hence, we can obtain the so called relaxed state using the variational principle (3.105). It is worth noting that generalized helicities for each species are constants of motion rather than magnetic helicity which is constant of motion in MHD. It is because of the fact that we have not neglected the inertia of the fluids. In evaluating the Euler-Lagrange equations (3.106)-(3.109), all the fields are taken to be incompressible and their normal components vanish at the boundary.

### 3.9 Density impact on structures

In order to show the glimpse of the impact of density of plasma constituents on the formation of equilibrium structures, the profiles of magnetic field are shown in figures (3-3) and (3-4) for \( N_i = 0.01 \) and \( N_i = 1 \) respectively. The Beltrami parameters are considered to be as \( b_- = 2.9 \), \( b_+ = 0.3 \), \( b_i = 0.01 \) and the boundary conditions are taken to be as \( g = 0.25 \), \( h = 0.017 \), \( w = 0.02 \) and \( t = 0.3 \). The Beltrami parameters and the boundary conditions are kept constant for both the graphs but the density ratio of ions to negative species is taken to be different in the two graphs. In figure (3-3), we have taken \( N_i = 0.01 \), i.e., \( n_i = 0.01n_- \), it shows that density of ions is 100 times less than that of negatively charged species. In this particular case, the system is dominated
by pair particles while the ions contribution is negligibly small. This graph shows that magnetic field behaves as paramagnetic as it decreases away from the center and its value becomes minimum at the boundary. Two of the scale parameters are real and distinct while the other two are complex conjugate pair. The corresponding scale parameters are $\lambda_1 = 2.56709$, $\lambda_2 = 0.0100015$, $\lambda_3 = 0.316457 + 1.06542i$, and $\lambda_4 = 0.316457 - 1.06542i$.

In figure (3-4), the magnetic field profile is shown for $N_i = 1$. In this case, density of ions is equal to the density of negatively charged species. It shows that plasma is composed of negatively charged and ions only. Hence, there are no positively charged
species making pair plasma. The magnetic field, in this particular case, shows a diamagnetic behavior as the magnetic field is approximately zero at the center but increases sharply and becomes maximum at the boundary of the plasma. In figure (3-4), all the scale parameters are distinct and real. Their values read as $\lambda_1 = 2.50005$, $\lambda_2 = 0.011626$, $\lambda_3 = 0.3971$, and $\lambda_4 = 0.301224$. However, the Beltrami parameters remain same. The scale parameters are changed due to change in the values of the densities of plasma constituents. It is evident through figures (3-3) and (3-4), that density of plasma species strongly affects the formation of self-organized structures. Density variation could change the paramagnetic fields to diamagnetic one or vice versa. It shows the possibility that density of plasma species plays a crucial role in transforming energy. At lower densities of ions, the magnetic field structures are paramagnetic that is, in the interior of plasma, magnetic energy is higher relative to its boundaries. But for higher densities of ions, the magnetic fields show the diamagnetic behavior. In this case, the magnetic energy is higher at the edges as compared to the central plasma.

### 3.10 Profiles of magnetic field and flow

Plots of the components of magnetic fields and flows versus distance are displayed in figures (3-5)-(3-6) for the same set of Beltrami parameters $b_- = 3.5$, $b_+ = 1.7$, $b_i = 0.01$ and density $N_i = 0.01$. 

![Figure 3-5](image-url)
Figure 3-6: Plots of $B_y$, $B_z$, $V_y$ and $V_z$ vs. distance for the Beltrami parameters $b_\perp = 3.5$, $b_+ = 1.7$, $b_i = 0.01$ and density $N_i = 0.99$.

and boundary conditions $t = 0.2$, $h = 0.12$, $w = 0.12$ and $g = 0.25$. However, the value of the density ratio of ion fluid to negatively charged fluid is assumed different. Figure (3-5) shows the graphs for for $N_i = 0.01$ that is $n_i = 0.01n_\perp$. This graph depicts that $z$-component of magnetic field decreases away from the center whereas the $y$-component increases away from the center. However, magnetic field as a whole shows the paramagnetic behavior. The graphs of the components of flow show that there exist a stronger flow as compared to magnetic field. It is also evident that fields and flows follow each other. The graphs shown in figure (3-6) are for $N_i = 0.99$ that is $n_i = 0.99n_\perp$. As is evident from this figure that both the components of magnetic fields attain maximum value at the edge of the plasma. Near to the center, they remain approximately constant but later on gets maximum value at the boundary. A similar trend is shown by the flow components. The flows and fields also follow each other in this case. The figure (3-5) shows that when the structures are paramagnetic, the two eigenvalues are real and distinct while the other two are complex conjugates which are $\lambda_1 = 3.24256$, $\lambda_2 = 0.0100063$, $\lambda_3 = 0.978719 + 0.796858i$ and $\lambda_4 = 0.978719 - 0.796858i$. The eigenvalues are $\lambda_1 = 3.18689$, $\lambda_2 = 0.011921$, $\lambda_3 = 1.69122$ and $\lambda_4 = 0.319969$ are all real and distinct for the diamagnetic structures as shown in figure (3-6).
Chapter 4

Double Beltrami States and Loss of Equilibrium in Multi-Component Plasmas

The magnetic fields and plasmas play a key role in the formation and description of structures in the solar corona and interplanetary space. In most of the corona, the magnetic forces are dominant in comparison to other non-magnetic forces like pressure gradient and gravity. It is magnetic field which guides and controls the transfer of energy, mass and momentum in the solar corona. In lowest order, the non-magnetic forces can be neglected and the magnetic field becomes force-free as Lorentz force vanishes. Hence, the solutions of force-free magnetic fields have been employed to model diverse phenomena occurring in solar atmosphere like solar flares [76,77], solar coronal loops [81,82], solar mass ejection, coronal heating [83–85] and magnetic clouds [79,80] etc. Ohsaki et. al. [33,34] have used the solutions of DB field to model the closed magnetic field structures in the solar corona. They have pointed out that a slowly evolving equilibrium system abruptly encounters an explosion or eruption when the macro scale of the closed structure decreases below a critical value. Moreover, the catastrophic eruption occurs when the total energy associated with the loop is greater than a threshold value. It is also shown
that almost all the magnetic energy is converted to flow energy and there is a change of state from DB to single Beltrami state when the eruption takes place. Ohsaki et. al. have considered the plasma consisting of ions and inertialess electrons. We have extended this model to multi-component plasmas because these plasmas are found in space and astrophysical environments. For instance, multi-component plasmas are found in the ionosphere and magnetosphere of Earth, solar wind, heliosphere, Saturn’s magnetosphere and comet tails. However, for sake of simplicity, we consider three component plasma which contains electrons, positrons and ions. The electron-positron plasmas are believed to exist in pulsar magnetosphere [112,120], active galactic nuclei [111], solar atmosphere [121], black holes [122] and accretion disks [123]. In the early universe just after the big bang, intense photons, neutrinos, antineutrinos and small amount of ions were present in addition to electrons and positrons [124,125]. The electron and positron plasmas are believed to be contaminated with ions [126–131]. In the next section, we will describe the model to obtain the DB relaxed equilibrium state.

4.1 Double Beltrami states

We consider a magnetized and collisionless three species electron, positron and ion plasma. The macroscopic evolution equations of three fluids comprising of electrons (−), positrons (+) and ions (i) read as

\[
\frac{\partial A}{\partial t} = V_\perp \times B + \nabla \psi_\perp, \quad (4.1)
\]

\[
\frac{\partial A}{\partial t} = V_\perp \times B - \nabla \psi_\perp, \quad (4.2)
\]

\[
\frac{\partial}{\partial t} (V_i + A) = V_i \times [\nabla \times V_i + B] - \nabla \psi_i, \quad (4.3)
\]

where \(V_j\) \((j = -, +, i)\) represents the velocity of the fluids normalized to the Alfvén velocity \(V_A = B_0/\sqrt{\mu_0 n_i m_i}\) \((B_0\) is a constant magnetic field, \(\mu_0\) is the permeability of free space, \(n_j\) and \(m_j\) represent the number densities and masses of the plasma species
respectively). The pressure of the species normalized to \( B_0^2/\mu_0 \) is represented by \( p_j \). \( E \) is the electric field and satisfies the relation \( \mathbf{E} = -\nabla \phi - \partial \mathbf{A}/\partial t \), \( \phi \) is the scalar potential of the species normalized to \( \lambda_i B_0 V_A \) and \( \mathbf{A} \) is the vector potential \( (\mathbf{B} = \nabla \times \mathbf{A}) \) normalized to \( \lambda_i B_0 \). The magnetic field \( \mathbf{B} \) is normalized to \( B_0 \). All the distances are normalized to the ion skin depth \( \lambda_i = V_A/\omega_c = \sqrt{m_i/\mu_0 n_i e^2} \) where \( \omega_c \) is the electron plasma frequency and \( e \) is the elementary charge. The scalars \( (\psi_j) \) in the gradient terms are \( \psi_- = -\phi + n_i p_-/n_- \), \( \psi_+ = \phi + n_i p_+/n_+ \) and \( \psi_i = \phi + \frac{1}{2} \mathbf{V}_i^2 + p_i \). The electron and positron are very lighter and their masses have been ignored in comparison to ions \( (m << m_i) \) in evaluating equations (4.1) and (4.2). The pressures of the fluids satisfy the relation
\[
\rho_j^{-1} \nabla p_j = \nabla H_j, \tag{4.4}
\]
where \( H_j \) is the enthalpy and equation (4.4) serves as equation of state. The curl of equations (4.1), (4.2) and (4.3), yields respectively
\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [\mathbf{V}_- \times \mathbf{B}], \tag{4.5}
\]
\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [\mathbf{V}_+ \times \mathbf{B}], \tag{4.6}
\]
\[
\frac{\partial}{\partial t} (\nabla \times \mathbf{V}_i + \mathbf{B}) = \nabla \times [\mathbf{V}_i \times (\nabla \times \mathbf{V}_i + \mathbf{B})]. \tag{4.7}
\]
These equations can be written in a more systematic form as
\[
\frac{\partial \Omega_j}{\partial t} - \nabla \times [\mathbf{U}_j \times \Omega_j] = 0, \tag{4.8}
\]
where \( \Omega_j \ (j = -, +, i) \) are the generalized vorticities \( \Omega_- = \Omega_+ = \mathbf{B}, \ \Omega_i = \nabla \times \mathbf{V}_i + \mathbf{B} \) and \( \mathbf{U}_j = \mathbf{V}_j \) are the effectual velocities. The general steady state solution of equation (4.8) is given by
\[
\mathbf{U}_j \times \Omega_j = \nabla \psi_j, \tag{4.9}
\]
where $\psi_j (j = -, +, i)$ are the scalar fields and correspond to energy densities of the fluids. The simplest equilibrium solution to equation (4.8) can be given in terms of Beltrami condition, i.e., alignment of flows along the corresponding vorticities

$$\Omega_j = b_j U_j, \quad (4.10)$$

where $b_j (j = -, +, i)$ are arbitrary constants and referred to be as Beltrami parameters. Steady state solutions for the constituent fluids are, respectively,

$$b_+ V_+ = B, \quad (4.11)$$
$$b_- V_- = B, \quad (4.12)$$
$$b_i V_i = \nabla \times V_i + B, \quad (4.13)$$

The steady state solution shows that flows of pair particles follow the field lines while the ions follow the generalized vorticity (vorticity modified by magnetic field). In order to find the magnetic field and velocities of the constituents, we make use of the Ampere's Law which can be expressed as

$$V_i = \nabla \times B + \frac{V_-}{N_i} - \frac{N_+}{N_i} V_+, \quad (4.14)$$

where $N_+ = n_+/n_-$, $N_i = n_i/n_-$ and we have taken into account the quasineutrality condition $n_- = n_i + n_+$. Putting value of $V_i$ into equation (4.13), and using equations (4.11) and (4.12), we obtain

$$(\nabla \times)^2 B - a_1 \nabla \times B + a_0 B = 0, \quad (4.15)$$
where

\[ a_0 = 1 - \frac{b_i}{N_i} \left( \frac{1}{b_-} - \frac{N_+}{b_+} \right) = 1 - \rho b_i, \]  
(4.16)

\[ a_1 = b_i - \frac{1}{N_i} \left( \frac{1}{b_-} - \frac{N_+}{b_+} \right) = b_i - \rho, \]  
(4.17)

and \( \rho = \left((1/b_-) - (N_+/b_+)\right)N_i^{-1} \). Equation (4.15) is called as Double Beltrami (DB) equation as it is equivalent to linear sum of two Beltrami fields.

It is worth noting that Beltrami conditions naturally give rise to the generalized Bernoulli condition \((\psi = \text{constant})\). Putting the expressions (4.11)-(4.13) into macroscopic evolution equations (4.1)-(4.3), we readily obtain

\[ \nabla \psi_j = 0, \]  
(4.18)

for steady state condition. Integrating above equation and introducing \(\psi_j\), we obtain

\[ N_i p_- - \phi = g_-, \]
\[ (N_i p_+/N_+) + \phi = g_+, \]
\[ \left( V_i^2 / 2 \right) + p_i + \phi = g_i, \]

where \( g_j (j = +, - , i) \) are arbitrary constants. Adding above equations, we obtain

\[ \psi_- + \psi_+ + \psi_i = g_t, \]  
(4.19)

where \( g_t = g_- + g_+ + g_i \). It shows that in the presence of a sheared flow, the system has the ability to confine an appreciable pressure [28,37].
4.2 Root analysis

Introducing the eigenvalues \( \alpha (\alpha = \lambda, \mu) \) of the curl operator, equation (4.15) can also be expressed as

\[
(\nabla \times)^2 \mathbf{B} - (\lambda + \mu) \nabla \times \mathbf{B} + \lambda \mu \mathbf{B} = 0. \tag{4.20}
\]

A comparison of equations (4.15) and (4.20), yield

\[
\lambda \mu = 1 - \rho b_i, \tag{4.21}
\]

\[
\lambda + \mu = b_i - \rho. \tag{4.22}
\]

As the operators are commutative, the general solution of equation (4.15) reads as

\[
\mathbf{B} = C_\lambda \mathbf{F}_\lambda + C_\mu \mathbf{F}_\mu, \tag{4.23}
\]

where \( \mathbf{F}_\alpha (\alpha = \lambda, \mu) \) be the Beltrami field such that \( \nabla \times \mathbf{F}_\alpha = \alpha \mathbf{F}_\alpha \) and \( C_\alpha \) are arbitrary constants. The roots of the quadratic equation

\[
\alpha^2 - a_1 \alpha + a_0 = 0, \tag{4.24}
\]

are the eigenvalues of curl operator and given by

\[
\lambda = \frac{(b_i - \rho) + \sqrt{(b_i + \rho)^2 - 4}}{2}, \tag{4.25}
\]

\[
\mu = \frac{(b_i - \rho) - \sqrt{(b_i + \rho)^2 - 4}}{2}. \tag{4.26}
\]

It shows that scale parameters depend on Beltrami parameters and number density of plasma species. The roots of the quadratic equation depends upon the discriminant \( \Delta \) which is defined as

\[
\Delta = (b_i + \rho)^2 - 4.
\]
Figure 4-1: Character of the scale parameters of DB field as a function of Beltrami parameters \((b_+, b_- \text{ and } b_i)\) for the (a) \(N_i = 0.01\) and (b) \(N_i = 0.99\)
The roots will be complex for $\Delta < 0$, i.e., when $b_i + \rho < 2$. The scale parameters will be same when $\Delta = 0$, i.e., when $b_i = 2 - \rho$ and both the scale parameters $\lambda$ and $\mu$ become equal to $1 - \rho$. Figure (4-1) (a) depicts the behavior of eigenvalues as a function of Beltrami parameters $(b_+, b_-$ and $b_i)$ for $N_i = 0.01$ that is the electron density is 100 times greater than the ion density. Figure (4-1) (b) illustrates the character of eigenvalues as a function of Beltrami parameters $(b_+, b_-$ and $b_i)$ for the $N_i = 0.99$. In this case the density of ions reads as $n_i = 0.99n_-$. The colored regions of figure (4-1) show three different categories of the scale parameters, the top most graphs exhibit the points when the scale parameters are real and distinct, the middle graphs locate the scale parameters when they are a pair of complex conjugate while the bottom graphs show the points where they become degenerate. The comparison of figure (4-1) (a) and (4-1) (b) shows that an increase in the density of ions strongly affects the scale parameters. Most of the real and distinct scale parameters are changed to complex conjugate pairs as shown in the top most graphs. The colored portion increases with the increase in density in the middle graphs depicts that an increment of ions density increase the complex eigenvalues. The degenerate roots also increases on an increase in the density of ions as shown in the bottom graphs.

The explicit dependence of scale parameters on number densities makes it different from the scale parameters admitted by Hall MHD. The scale parameters of Hall MHD are independent of number density; they only depend on Beltrami parameters which show the relation between flows and generalized vorticities only. The formation of the self-organized structures and their features strongly depend on scale parameters. Generally, the system will self-organize to paramagnetic state when the scale parameters are real, to diamagnetic state when they are complex and to perfectly diamagnetic one, when they are purely imaginary. Hence, it is expected that number densities of plasma components will play its role at par with flows and vorticities in the dynamics of multi component
plasmas. Eliminating \( b_i \) from equations (4.21) and (4.22), we obtain

\[
(\lambda + \rho)(\mu + \rho) = 1. \tag{4.27}
\]

Simplifying above equation and taking \( b_+ \) as a constant, we obtain

\[
\tilde{b}_- = \frac{N_+}{b_+} - \left[ \frac{\lambda + \mu \pm \sqrt{(\lambda - \mu)^2 + 4}}{2} \right] N_i, \tag{4.28}
\]

where \( \tilde{b}_- = 1/b_- \). The value of \( b_i \) from equations (4.21) and (4.22) comes out to be

\[
b_i = \frac{\lambda + \mu \pm \sqrt{(\lambda - \mu)^2 + 4}}{2}. \tag{4.29}
\]

As the operators commute, the general solution to the DB equation is given by the linear combination of two Beltrami fields

\[
\mathbf{B} = C_\lambda \mathbf{F}_\lambda + C_\mu \mathbf{F}_\mu, \tag{4.30}
\]

and the composite velocity of the system is given by

\[
\rho_i \mathbf{V} = \rho_i \mathbf{V}_i + \rho_+ \mathbf{V}_+ + \rho_- \mathbf{V}_-, \]

where \( \rho_i = \rho_i + \rho_+ + \rho_- \) and \( \rho_j = m_j n_j \) \((j = +, -, i)\). As the mass of positron and electron is neglected in comparison to ions, therefore \( \rho_+ = \rho_- = 0 \) and composite velocity \( \mathbf{V} \approx \mathbf{V}_i \). The composite velocity using equation (4.14) reads as

\[
\mathbf{V} = (\lambda + \rho) C_\lambda \mathbf{F}_\lambda + (\mu + \rho) C_\mu \mathbf{F}_\mu. \tag{4.31}
\]

The loss of equilibrium give rise to eruption and the eruptive events are ubiquitous in space plasma such as solar flares, eruptive prominences and coronal mass ejections in solar
atmosphere. The current system can lose the equilibrium if either the scale parameters \( \alpha \) 
\((\alpha = \lambda, \mu)\) go from real to complex or any one of the amplitudes \( C_\alpha \) \((\alpha = \lambda, \mu)\) becomes complex. In order to investigate the possibility of the occurrence of sudden eruptive and catastrophic events within the framework of our approach, we assume a quasi-equilibrium state of a self-organized structure where the DB state is changing very slowly. The change is sufficiently slow so that the system remains in its local DB equilibrium. Equation (4.30) shows that DB field depends on four physical parameters: two scale parameters \( \alpha' \)’s and two amplitudes \( C_\alpha \). In order to find these parameters, we invoke the constants of motion following work of Yoshida et. al. [13]. To do so, first we need to know the constants of motion permitted by this system. The magnetofluid energy is one of the constants of motion. In the next section, we will prove that magnetofluid energy is an ideal invariant.

### 4.3 Constants of motion

In order to find that magnetofluid energy is constant, we make use of equation (4.18) and then multiply equations (4.1)-(4.3) by \( V_- \), \( V_+ \) and \( V_i \) respectively, we obtain

\[
E \cdot V_- + V_- \cdot \nabla \phi = 0,
\]

\[
E \cdot V_+ + V_+ \cdot \nabla \phi = 0,
\]

\[
\frac{\partial}{\partial t} \left( \frac{V_i^2}{2} \right) - V_i \cdot E - V_i \cdot \nabla \phi = 0.
\]

In above equation, vector potential \((A)\) has been replaced by electric field \((E)\) and scalar potential \((\phi)\). Adding equation (4.32), (4.33) and (4.34), we get

\[
\frac{\partial}{\partial t} \left( \frac{V_i^2}{2} \right) = -E \cdot \left( -\frac{N_+}{N_i} V_+ + \frac{V_-}{N_i} - V_i \right) + \left( \frac{N_+}{N_i} V_+ - \frac{V_-}{N_i} + V_i \right) \cdot \nabla \phi.
\]
Using equation (4.14) into above equation, we obtain

\[
\frac{\partial}{\partial t} \left( \frac{V_i^2}{2} \right) = \mathbf{E} \cdot (\nabla \times \mathbf{B}) + (\nabla \times \mathbf{B}) \cdot \nabla \phi \tag{4.35}
\]

Using identity \( \nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}) \) and making use of \( \nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t \), we obtain

\[
\frac{\partial}{\partial t} \left( \frac{V_i^2}{2} \right) = -\frac{\partial}{\partial t} \left( \frac{B^2}{2} \right) - \nabla \cdot (\mathbf{E} \times \mathbf{B}) + (\nabla \times \mathbf{B}) \cdot \nabla \phi
\]

Taking volume integral of above equation, we obtain

\[
\frac{\partial}{\partial t} \int \left( \frac{B^2}{2} + \frac{V_i^2}{2} \right) dv = \int -\nabla \cdot (\mathbf{E} \times \mathbf{B}) dv + \int \nabla \phi \cdot (\nabla \times \mathbf{B}) dv
\]

Using divergence theorem and making use of the boundary conditions \( \mathbf{n} \cdot \mathbf{B} = \mathbf{n} \cdot (\nabla \times \mathbf{B}) = 0 \) as all the variations are taken to be incompressible and their normal components vanish at the boundaries, we obtain

\[
\frac{\partial}{\partial t} \left[ \frac{1}{2} \int (B^2 + V_i^2) dv \right] = 0. \tag{4.36}
\]

It shows that

\[
E = \frac{1}{2} \int (B^2 + V_i^2) dv = \text{constant.} \tag{4.37}
\]

Hence, the sum of the flow kinetic energy and the magnetic energy called as magnetofluid energy \( E \) remains conserved.

Employing equation (4.8), it is easy to show that \( h_- \), \( h_+ \) and \( h_i \) are also constants of
motion and their expressions are as follows

\[ h_- = \frac{1}{2} \int (A \cdot B) \, dv, \]  \hspace{1cm} (4.38)

\[ h_+ = \frac{1}{2} \int (A \cdot B) \, dv, \]  \hspace{1cm} (4.39)

\[ h_i = \frac{1}{2} \int (A + V) \cdot (B + \nabla \times V) \, dv. \]  \hspace{1cm} (4.40)

Hence, the system admits four constants of motion namely electromagnetic energy and helicities of electron, positron and ion fluids.

### 4.4 Double curl Beltrami via variational principle

The Double Curl Beltrami equation can be derived by constrained minimization of energy using the variational principle [26, 28, 118]. The self-organized state of two-fluid MHD plasma was derived by Steinhauer and Ishida [26, 27] using variational principle where the total energy \( E \) and two helicities are considered to evaluate the Euler-Lagrange equation. The DB self-organized state of Hall MHD plasma has also been derived by Yoshida and Mahajan using variational principle where the total energy \( E \), magnetic helicity and generalized helicity are taken to be constants. It is worth noting that for the case of multi-species plasma, the faster decay of magnetofluid energy as compared to helicities under dissipation is an ill-posed problem as pointed out by Yoshida and Mahajan [30]. It is stated that energy and helicities are all constants of motion, then the energy cannot be chosen as a target functional. Moreover, the free energy function must be bounded below for a well posed variational principle. For the multi-species relaxation theory, we find that free energy is not bounded in the energy norm because there exist ion-flow term in the expression of generalized helicity. The target functional may not be any one from magnetofluid energy, magnetic helicity and generalized helicity because a target functional must contain higher order derivative as compared to energy and helicities involved in relaxation dynamics. Further the target function must be a
measure of dissipation and turbulence. To satisfy the demands mentioned above, the generalized enstrophy \( F \) which is the curl of the canonical momentum and defined as

\[
F = \int_{\Omega} |\nabla \times (V + A)|^2 \, dx, \tag{4.41}
\]

where \( \Omega \) is the confining domain, is the desired target functional. This is a hybrid functional and combines the magnetic and fluid attributes of the plasma. Equivalently, it can be thought of as the energy of generalized magnetic field or generalized vorticity. A well-posed variational principle can be constructed when we minimize \( F \) and keep magnetofluid energy and helicities as constant. This is a well posed variational principle as it makes the functional to be convex and leads to a unique minimizer. Luckily, on working out the variations, it becomes equivalent to the expression which states that magnetofluid energy minimizes while the helicities remain constant to lead the system to Euler-Lagrange equation. Later on Mahajan and his co-worker have employed the later one to obtain the relaxed equilibrium [42,70]. Following Mahajan, the variational principle for the current problem reads as

\[
\delta(E - \mu_+ h_- - \mu_- h_+ - \mu_i h_i) = 0, \tag{4.42}
\]

where \( \mu_+ \), \( \mu_- \) and \( \mu_i \) are Lagrange multipliers and \( \mu_+ = N_+ / N_i b_+ \), \( \mu_- = -1 / N_i b_- \) and \( \mu_i = 1 / b_i \). Putting the values of \( E, h_-\), \( h_+ \) and \( h_i \) from equations (4.37)-(4.40) into above equation and after working out the independent variations \( \delta V_i \) and \( \delta A \), we get

\[
\left( V_i - \frac{1}{b_i} \nabla \times V_i + B \right) \cdot \delta V_i = 0, \tag{4.43}
\]

\[
\left[ \nabla \times B + \frac{B}{b_- N_i} - \frac{N_+}{b_+ N_i} - \frac{1}{b_i} \nabla \times (V_i + A) \right] \cdot \delta A = 0. \tag{4.44}
\]

From equation (4.43), we obtain

\[
\nabla \times V_i + B = b_i V_i, \tag{4.45}
\]

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which is one of the steady state solutions given in equation (4.13). From equation (4.44), we obtain

\[ \nabla \times \mathbf{B} + \frac{\mathbf{B}}{b_- N_i} - \frac{N_+}{b_+ N_i} - \frac{1}{b_i} \nabla \times (\mathbf{V}_i + \mathbf{A}) = 0. \]  

(4.46)

Putting values of (4.14) and (4.45) into above equation, we obtain

\[ N_+ \mathbf{V}_+ - \frac{N_+}{b_+} \mathbf{B} - \mathbf{V}_- + \frac{\mathbf{B}}{b_-} = 0. \]  

(4.47)

The above equation is satisfied if

\[ b_+ \mathbf{V}_+ = \mathbf{B}, \]  

(4.48)

\[ b_- \mathbf{V}_- = \mathbf{B}. \]  

(4.49)

Equations (4.45), (4.48) and (4.49) are Euler Lagrange equations which can be solved in conjunction with the Ampere’s Law

\[ \nabla \times \mathbf{B} + \frac{\mathbf{V}_-}{N_i} - \frac{N_+}{N_i} \mathbf{V}_+ = \mathbf{V}_i, \]  

(4.50)

to retrieve the self-organized equation given in (4.15) equations (4.48). Equations (4.49), (4.44) and (4.50) are the same equations as we have already evaluated the equations (4.11)-(4.13) and (4.14).

### 4.5 Conservation laws and algebraic structure

The helicities for electron and positron fluids \((h_+ \text{ and } h_-)\) are equal and we take \(h_+ = h_- = h\). From equations (4.37)-(4.40), the expressions for magnetofluid energy \((E)\), the helicities of electron and positron \((h)\) and of ions \((h_i)\), respectively, are rewritten as
The present system allows four constants of motion whereas there are only three constants of motion in Hall MHD [33,34].

It is easy to find out the algebraic relations for the constant of motions in terms of the scale parameters $\alpha$ and $C_\alpha$. For this purpose, we choose $\lambda$ as a control parameter and assume $|\lambda| < |\mu|$, that is $\lambda$ specifies the large scale structure. We take a cube of length $L$ and consider a two dimensional ABC field $\mathbf{F}_\alpha (\alpha = \lambda, \mu)$ as expressed below

\[
\mathbf{F}_\alpha = f_x^\alpha \begin{pmatrix} 0 \\ \sin(\alpha x) \\ \cos(\alpha x) \end{pmatrix} + f_y^\alpha \begin{pmatrix} \cos(\alpha y) \\ 0 \\ \sin(\alpha y) \end{pmatrix},
\]

where $f_x^\alpha$ and $f_y^\alpha$ are constants and satisfy the relation $(f_x^\alpha)^2 + (f_y^\alpha)^2 = 1$. Assuming $L = 2\pi n_\alpha/\alpha$, where $n_\alpha$ are integers ($\alpha = \lambda, \mu$). $\mathbf{F}_\alpha$ also satisfy the following relations: \[\int \mathbf{F}_\alpha^2 dr = L^2, \text{ and } \int \mathbf{F}_\lambda \cdot \mathbf{F}_\mu dr = 0,\]\[\text{where } \int dr = \int_0^L \int_0^L dxdy.\] Using equation (4.54) into equation (4.30) and (4.31), the vector potential $\mathbf{A}$ and vorticity $\nabla \times \mathbf{V}_i$ can be evaluated respectively as

\[
\mathbf{A} = \frac{1}{\lambda} C_\lambda \mathbf{F}_\lambda + \frac{1}{\mu} C_\mu \mathbf{F}_\mu
\]

\[
\nabla \times \mathbf{V}_i = \lambda (\lambda + \rho) C_\lambda \mathbf{F}_\lambda + \mu (\mu + \rho) C_\mu \mathbf{F}_\mu
\]

Making use of equations (4.30), (4.31), (4.55) and (4.56) into equations (4.51)-(4.53), the
integrals of motion come out to be

\[ h = \frac{L^2}{2} \left( \frac{C_\lambda^2}{\lambda} + \frac{C_\mu^2}{\mu} \right), \]  
(4.57)

\[ h_i = \frac{L^2}{2} \left[ \frac{C_\lambda^2}{\lambda} (\lambda^2 + \rho \lambda + 1)^2 + \frac{C_\mu^2}{\mu} (\mu^2 + \rho \mu + 1)^2 \right], \]  
(4.58)

\[ E = \frac{L^2}{2} \left[ (1 + (\lambda + \rho)^2) C_\lambda^2 + (1 + (\mu + \rho)^2) C_\mu^2 \right]. \]  
(4.59)

The magneto-fluid energy \( E \) is the sum of kinetic energy \( E_k \) and magnetic energy \( E_B \), which are defined as

\[
E_k = \frac{1}{2} \int V_i^2 dv, \\
E_B = \frac{1}{2} \int B^2 dv.
\]

Using expressions from equations (4.30) and (4.31) respectively, we obtain

\[
E_k = \frac{L^2}{2} \left[ (\lambda + \rho)^2 C_\lambda^2 + (\mu + \rho)^2 C_\mu^2 \right], \\
E_B = \frac{L^2}{2} \left( C_\lambda^2 + C_\mu^2 \right).
\]

In what follows, we take

\[ \tilde{h}_i = h_i - h. \]  
(4.60)

Putting equation (4.57)-(4.58) into equation (4.60), and after some calculation, we obtain the following relation

\[ \tilde{h}_i = b_i E - \lambda \mu h. \]  
(4.61)

Using equation (4.21) and eliminating \( \tilde{h}_i \) from above equations (4.60) and (4.61), we have

\[ b_i = \frac{h_i}{E + \rho h}. \]  
(4.62)
Putting value of $b_i$ from equation (4.29) into equation (4.61), $\tilde{h}_i$ becomes

$$\tilde{h}_i = \frac{E}{2} \left[ (\lambda + \mu) \pm \sqrt{(\lambda - \mu)^2 + 4} \right] - \lambda \mu h. \quad (4.63)$$

Eliminating $C_\mu^2$ from equations (4.57) and (4.59), we can evaluate $C_\lambda^2$. Similarly eliminating $C_\lambda^2$ from equations (4.57) and (4.59), we can evaluate $C_\mu^2$. After normalizing out the common factor $L^2/2$, we obtain

$$C_\lambda^2 = \frac{\lambda}{D} \left[ E - h\mu \{ 1 + (\mu + \rho)^2 \} \right], \quad (4.64)$$

$$C_\mu^2 = \frac{\mu}{D} \left[ -E + h\lambda \{ 1 + (\lambda + \rho)^2 \} \right], \quad (4.65)$$

where

$$D = \lambda \left[ 1 + (\lambda + \rho)^2 \right] - \mu \left[ 1 + (\mu + \rho)^2 \right], \quad (4.66)$$

$$= (\lambda - \mu)b_i(b_i + \rho), \quad (4.67)$$

### 4.6 Analysis of double Beltrami equilibrium

To investigate the impact of $N_i$, $h$, $h_i$ and $b_+$ on the self-organzied double Beltrami equilibrium permitted by three components plasma, we make use of the equation

$$h_i = \frac{(E + \rho h)}{2} \left[ (\lambda + \mu) \pm \sqrt{(\lambda - \mu)^2 + 4} \right], \quad (4.68)$$

obtained by using equations (4.62) and (4.29). Figure (4-2) shows how the equilibrium is lost on varying $N_i = n_i/n_-$. In figure (4-2) (a), plots of $\mu$, $C_\lambda$ and $C_\mu$ vs. $\lambda$ are shown for $N_i = 0.01$. The other parameters are $E = 5.0$, $h_i = 5.0$, $h = 0.5$ and $b_+ = 0.2$. There is no loss of equilibrium. On the other hand for the same parameters $E = 5.0$, $h_i = 5.0$, $h = 0.5$ and $b_+ = 0.2$, the equilibrium is lost when $N_i = 0.99$. It is evident from figure (4-2) (b) that an increase in number density of ions as compared to positrons could cause
Figure 4-2: (a) Plots of $\mu$, $C_\lambda$ and $C_\mu$ vs. $\lambda$ for the densities of plasma components $N_i = 0.01$, no catastrophe. (b) Plots of $\mu$, $C_\lambda$ and $C_\mu$ vs. $\lambda$ for $N_i = 0.99$. The equilibrium is lost at the critical point $\lambda_{\text{crit}} \approx 2.24$. The energy of the plasma system $E = 5.0$, helicities $h = 0.5$ and $h_i = 5.0$ and Beltrami parameter $b_+ = 0.2$ are taken constnt for both cases.

the loss of equilibrium when the scale parameter attains a certain value. The size of the macro structure increases and equilibrium is lost at a critical point. Figure (4-3) shows the variation of parameters with respect to the change of scale length $\lambda$ for given energy $E = 5.0$, helicity of ions $h_i = 5.0$, densities of the components $N_i = 0.77$ and Beltrami parameter $b_+ = 0.2$. It is observed that there is no solution (catastrophic loss of equilibrium) for smaller $h$ - helicities of the pair components of plasma. Figure (4-3) (a) shows no catastrophe for $h = 5.5$ while figure (4-3) (b) shows a loss in equilibrium at $\lambda_{\text{crit}} \approx 0.75$ for $h = 0.5$. The size of the large structure decreases and eruption occurs when the structures reduces to certain critical value. In contrast to figure (4-3) keeping $h_i$ (helicity of ions) as constant in figure (4-4), we observe the solution changes drastically by varying $h_i$ from 25.0 to 5.0 for constant $E = 5.0$, $h = 0.3$, $N_i = 0.88$ and $b_+ = 0.2$. In figure (4-4) (a), the system preserves its equilibrium for $h_i = 25$ while the system faces
Figure 4-3: (a) Plots of $\mu$, $C_\lambda$ and $C_\mu$ vs. $\lambda$ for the helicities of the pair components of plasma $h = 5.5$, no catastrophe. (b) Plots of $\mu$, $C_\lambda$ and $C_\mu$ vs. $\lambda$ for $h = 0.5$. The equilibrium is lost at the critical point $\lambda_{\text{crit}} \approx 0.75$. The densities of the components $N_i = 0.77$, energy of the plasma system $E = 5.0$, helicity of ions $h_i = 5.0$ and Beltrami parameter $b_+ = 0.2$ are taken constant for both cases.
Figure 4-4: (a) Plots of $\mu$, $C_\lambda$ and $C_\mu$ vs. $\lambda$ for the helicity of ions $h_i = 25.0$, no catastrophe. (b) Plots of $\mu$, $C_\lambda$ and $C_\mu$ vs. $\lambda$ for $h_i = 5.0$. The equilibrium is lost at the critical point $\lambda_{\text{crit}} \simeq 1.93$. The densities of the components $N_i = 0.88$, energy of the plasma system $E = 5.0$, helicities of the pair components $h = 0.3$ and Beltrami parameter $b_+ = 0.2$ are taken constant for both cases.

A fundamentally different situation in figure (4-4) (b) and ceases to be in equilibrium at $\lambda_{\text{crit}} \simeq 1.93$. The sequence of equilibria is lost when the macro scale structure increases to a certain limit. Figure (4-5) reveals the bifurcation conditions (leading to loss of equilibrium) for different values of $b_+$. Figure (4-5) (a) shows the plots of $\mu$, $C_\lambda$ and $C_\mu$ vs. $\lambda$ for $b_+ = 0.01$. In this case, there is no catastrophe, while figure (4-5) (b) shows the plots of $\mu$, $C_\lambda$ and $C_\mu$ vs. $\lambda$ for $b_+ = 0.2$. The equilibrium is lost at the critical point $\lambda_{\text{crit}} \simeq 0.656$. The energy of the plasma system is $E = 15.0$, helicities of pair particles $h = 0.5$, helicity of ions $h_i = 5.0$ and the densities of plasma components $N_i = 0.77$, are taken constant for both the cases.
Figure 4-5: (a) Plots of $\mu$, $C_\lambda$ and $C_\mu$ vs. $\lambda$ for ratio of magnetic field to positron flow $b_+ = 0.01$, no catastrophe. (b) Plots of $\mu$, $C_\lambda$ and $C_\mu$ vs. $\lambda$ for $b_+ = 0.2$. The equilibrium is lost at the critical point $\lambda_{\text{crit}} \simeq 0.656$. The energy of the plasma system $E = 15.0$, helicities $h = 0.5$ and $h_t = 5.0$ and the densities of the plasma components $N_i = 0.77$ are taken constant for both cases.
4.7 Energy and Loss of equilibrium

The equilibrium will be lost when the system drastically changes its state and transforms from the DB state to single Beltrami state. It will happen when one of the amplitudes $C_\alpha$ becomes zero or the two scale parameters become equal. Hence, there are two routes available to reach the termination of equilibrium: one is the disappearance of one of the roots (scale parameters) and the other is their degeneracy (equality of roots).

4.7.1 Roots disappearance

To find out the condition for the loss of equilibrium, we consider $d\lambda/db_- = 0$ using equation (4.25),

$$\frac{d\lambda}{db_-} = \frac{db_i}{db_-} \left[ 1 + \frac{(b_i + \rho)}{\sqrt{(b_i + \rho)^2 - 4}} \right] - \frac{1}{N_i b_-^2} \left[ \frac{(b_i + \rho)}{\sqrt{(b_i + \rho)^2 - 4}} - 1 \right] = 0,$$

and obtain

$$\frac{db_i}{db_-} = \frac{1}{4N_i b_-^2} \left[ \sqrt{(b_i + \rho)^2 - 4} - (b_i + \rho) \right]^2. \quad (4.69)$$

Using equation (4.26) in the above equation, we get

$$\frac{db_i}{db_-} = \frac{1}{N_i b_-^2} (\mu + \rho)^2. \quad (4.70)$$

Another expression for $db_i/db_-$ reads as

$$\frac{db_i}{db_-} = \frac{hh_i}{N_i b_-^2 (E + \rho h)^2}, \quad (4.71)$$
using equations (4.61) and (4.21). From equations (4.70) and (4.71), we get

\[ hh_i = [((\mu + \rho) (E + \rho h)]^2 \geq 0, \quad (4.72) \]

which shows that equilibrium will be lost when the product of two helicities is positive otherwise there will be no loss of equilibrium [34]. To look for the condition that all the physical parameters should remain real up to the point of termination of equilibrium, we proceed as follows. Equations (4.69) and (4.71) lead to the following quadratic equation in \( \rho \)

\[ \rho^2 \left( h \mp \sqrt{hh_i} \right) h + \rho \left( 2h \mp \sqrt{hh_i} \right) E + \left( E^2 + hh_i \mp h_i \sqrt{hh_i} \right) = 0. \quad (4.73) \]

The roots of above equation will be real when the discriminant of the solution of above equation is real. Setting discriminant equal to zero, we have

\[ E^2 - 4 \left( h \mp \sqrt{hh_i} \right) \left( h \mp \sqrt{hh_i} \right) = 0, \]

For the discriminant to be real, the following condition must be satisfied

\[ E^2 \geq E_{c}^2 = 4 \left( h \mp \sqrt{hh_i} \right)^2. \quad (4.74) \]

It is evident that the energy of the system should be greater than or equal to a critical value for the equilibrium to terminate. To find the corresponding value of the control parameter \( \lambda = \lambda_{\text{crit}} \), we set \( d\lambda/d\mu = 0 \) and obtain

\[ E \left[ 1 \mp \frac{(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4}} \right] - 2\lambda h = 0. \quad (4.75) \]
From equations (4.21) and (4.22), we have

\[
\rho^2 + \rho (\lambda + \mu) + (\lambda \mu - 1) = 0. \tag{4.76}
\]

The solution of above quadratic equation reads as

\[
\rho = \frac{1}{2} \left[-(\lambda + \mu) \pm \sqrt{(\lambda - \mu)^2 + 4}\right]. \tag{4.77}
\]

From equations (4.75) and (4.77), we obtain

\[
E (\rho + \mu) - \lambda \mu (2\rho + \lambda + \mu) = 0. \tag{4.78}
\]

This equation can be written as

\[
E = \lambda h \left[1 + (\lambda + \rho)^2\right], \tag{4.79}
\]

using equation (4.27). For the value of energy expressed in above equation, one of the amplitudes \(C_\mu^2\) becomes zero. Hence, one of the roots disappear and the system changes to single Beltrami state. From equations (4.22) and (4.61), we have

\[
\tilde{h}_i = (\lambda + \mu + \rho) E - \lambda \mu h. \tag{4.80}
\]

Introducing value of \(\mu\) from equation (4.27), above equation becomes

\[
\tilde{h}_i = E\lambda + (E - \lambda h) (\lambda + \rho)^{-1} + \lambda h \rho. \tag{4.81}
\]

Above equation can be written as follows on using equation (4.79)

\[
(\lambda + \rho) \left[\tilde{h}_i - \lambda (E - \lambda h) \right] = 2 (E - \lambda h). \tag{4.82}
\]
Using equation (4.79), above equation becomes

$$\left[ \tilde{h}_i - Q \right]^2 = 4hQ, \quad (4.83)$$

where

$$Q = \lambda (E - \lambda h) \quad (4.84)$$

Putting $\tilde{h}_i = h_i - h$, into equation (4.83), we get

$$Q = \left( \sqrt{h} \pm \sqrt{h_i} \right)^2. \quad (4.85)$$

From equations (4.84) and (4.85), using (4.74) and taking $\lambda = \lambda_{\text{crit}}$, we get

$$\lambda_{\text{crit}} = \frac{1}{2h} \left[ E \pm \sqrt{E^2 - E_c^2} \right]. \quad (4.86)$$

This shows the critical value of control parameter $\lambda_{\text{crit}}$ corresponding to the critical value of energy $E_c$. At the critical values, equilibrium will lost provided that energy of the system is greater than that of $E_c$. It means that equilibrium terminates when one of the amplitudes $C\alpha$ vanishes that is one of the roots disappear.

In order to show the catastrophic loss of equilibrium, the plots of $\mu$ and $C\alpha$ as functions of $\lambda$ are displayed in figure (4-6) for given values of the invariants $h$, $h_i$, $E$ and density of the plasma components. The equilibrium exists until $\mu$ and $C\alpha$ remain real, otherwise the equilibrium will lost. Figure (4-6) shows the two set of plots for two distinct values of energy while all other parameters are kept constant. The values of the parameters are taken as $h = 0.5$, $h_i = 5.0$ and $n_i = n_+ = 0.5n_-$. We have considered a constant density of plasma constituents and it is an essential feature of this model which is missing in Hall MHD [33, 34]. The factor of density appears naturally in the present system to describe the self-organized structures as the scale parameters $\lambda$ and $\mu$ depend on number density of the species. It is observed that energy plays a key role in determining the loss of equilibrium. There exists a critical value of energy $E_c$ which plays a key role in
Figure 4-6: (a) Plots of $\mu$, $C_\lambda$ and $C_\mu$ vs. $\lambda$ for $E = 0.5 < E_c \simeq 2.16$, the critical energy; no catastrophe. (b) Plots of $\mu$, $C_\lambda$ and $C_\mu$ vs. $\lambda$ for $E = 5 > E_c \simeq 2.16$. The equilibrium is lost at the critical point $\lambda_{\text{crit}} \simeq 0.491$. The densities of plasma components are taken as $n_i = n_+ = 0.5n_-$. 

Figure 4-7: (a) Plots of magnetic and flow energies vs. $\lambda$ for $E = 0.5 < E_c \simeq 2.16$, the critical energy; no catastrophe. (b) Plots of magnetic and flow energies vs. $\lambda$ for $E = 5 > E_c \simeq 2.16$. The equilibrium is lost at the critical point $\lambda_{\text{crit}} \simeq 0.491$. The densities of plasma components are taken as $n_i = n_+ = 0.5n_-$. 

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defining the fate of the system. If the energy of the system is less than that of critical value, the system remains in its state of equilibrium but if the energy becomes equal to or exceeds the critical value \( E_c \), the system faces a catastrophic loss of equilibrium. In figure (4-6) (a), the value of energy is taken to be 0.5 which is less than critical value of energy \( E_c \approx 2.16 \). In this case, the physical parameters \( \mu \) and \( C_\alpha \) change smoothly and continuously by varying the control parameter \( \lambda \). As \( \mu \) and \( C_\alpha \) remain real, we observe equilibrium and there is no qualitative change and hence no catastrophe. On the other hand, in figure (4-6) (b), energy is taken to be 5 which is greater than the critical value. The parameters \( \mu \) and \( C_\alpha \) don’t remain real and become complex as shown by dotted lines. At this stage, the value of \( \lambda \) reaches to a critical value \( \lambda_{\text{crit}} \) where the size of this macroscale structure reduces to such an extent that one of the amplitudes \( C_\lambda \) becomes complex. Consequently, the sequence of equilibrium is lost and the system remains no more in equilibrium.

For the same set of parameters as used in figure (4-6), the character of magnetic and flow energies is depicted in figure (4-7). The flow energy decreases and magnetic energy increases when the control parameter is increased as shown in figure (4-7) (a). The system energy \( (E = 0.5) \) is less than the critical value \( (E_c \approx 2.16) \) and the equilibrium remains preserved. Figure (4-7) (b) shows that magnetic energy decreases while the flow energy increased as the control parameters is increased slowly. Almost all the magnetic energy converted to flow energy at \( \lambda_{\text{crit}} \). It shows a true manifestation of catastrophe as the magnetic energy is transformed to flow kinetic energy.

Figure (4-8) displays the profiles of flows \( (V_\lambda \text{ and } V_\mu) \) associated with both the scale parameters \( \lambda \) and \( \mu \). Figure (4-8) (a) shows the velocities when there is no catastrophe. It is evident that flow velocities of macroscale component \( (V_\lambda) \) and microscale component \( (V_\mu) \) decreases on increasing the control parameter \( (\lambda) \). Figure (4-8) (b) shows the same profiles for energy \( E > E_c \approx 2.16 \). In this scenario, the system is bound to face catastrophe when the control parameter reaches to its critical value. It can be observed from figure (4-8) (b), that flow of large scale structure increases while the flow of smaller scale
Figure 4-8: (a) Plots of $V_\lambda$ and $V_\mu$ vs. $\lambda$ for $E = 0.5 < E_c \simeq 2.16$, the critical energy; no catastrophe. (b) Plots of $V_\lambda$ and $V_\mu$ vs. $\lambda$ for $E = 5 > E_c \simeq 2.16$. The equilibrium is lost at the critical point $\lambda_{\text{crit}} \simeq 0.491$. The densities of plasma components are taken as $n_i = n_+ = 0.5n_-$. 
structure reduces. Thus the flow energy of the micro scale component is being transferred to macro scale component and at $\lambda_{\text{crit}}$, the flow energy attains such a value that it is hard for the system to stay in equilibrium and consequently the catastrophic loss of equilibrium occurs. At this point, flow energy becomes very high while the magnetic energy diminishes and becomes zero. The system thus transforms to single Beltrami state which appears when the magnetic energy of the system decays to its ground state [14,22].

Figure 4-9: Plots for $\mu$, the magnetic and flow energy vs. $\lambda$ for $h = 1.0$, $h_i = 1.005$, $E = 1 > E_c = 4.99 \times 10^{-3}$.

### 4.7.2 Eruptive events in solar corona

The present model can be used to show the glimpse of eruptive and explosive events occurring in solar atmosphere. Figure (4-9) shows the plots for $\mu$, the magnetic energy and flow kinetic energy vs. $\lambda$ for $h = 1.0$, $h_i = 1.005$, $n_i = n_+ = 0.5n_-$, and $E = 1 > E_c = 4.99 \times 10^{-3}$. Figure (4-10) shows the plots for $C_\lambda$, $C_\mu$ and $V_\lambda$ vs. $\lambda$ for the same set of parameters as shown in figure (4-9). For this purpose, we have considered two scale parameters which are highly separated that is $|\lambda| << |\mu|$. As the inverse of scale parameters are the measure of the size of the structures, therefore $\lambda$ represents a very large scale structure as compared to the structures governed by $\mu$. The DB state thus results as a superposition of these two structures of vastly different sizes. Under these conditions, one of the amplitudes $C_\lambda$ may be estimated to be very small than unity while
Figure 4-10: Plots for $C_\lambda$, $C_\mu$ and $V_\lambda$ vs. $\lambda$ for $h = 1.0$, $h_i = 1.005$, $E = 1 > E_c = 4.99 \times 10^{-3}$. 
the other amplitude $C_\mu$ may be of the order of unity [34]. An increase in $\lambda$ due to some interaction will result to shrink the structure and at $\lambda = \lambda_{\text{crit}}$, one of the amplitudes $C_\mu$ will become zero as shown in figure (4-10) (b). $C_\lambda$ becomes imaginary at the $\lambda = \lambda_{\text{crit}}$ as shown in figure (4-10) (a). The system changes its state accompanied by an eruption and explosion. The magnetic field energy becomes very small at the critical point and almost all the energy appears as a flow energy. Figure (4-10) (c) shows the flow of larger scale structure increases. For a typical density $\sim 10^9 \text{cm}^{-3}$, the ion skin depth for the solar coronal plasma is $\lambda_i \sim 100 \text{ cm}$, the size of the structures is $\lambda_i/\lambda \sim 10^3 \text{km}$ which is appropriate to represent the length of coronal loops.

### 4.7.3 Roots degeneracy

The system may change its state from DB to single Beltrami when the roots are degenerate. Using $\lambda = \mu = \lambda_0$, we obtain from equation (4.29)

$$b_i = \lambda_0 \pm 1.$$ (4.87)

Using above value of $b_i$ into equation (4.61), we obtain

$$\tilde{h}_i = E(\lambda_0 \pm 1) - \lambda_0^2 h.$$ (4.88)

Solving above equation, we obtain

$$\lambda_0 = \frac{1}{2h} \left[ E \pm \sqrt{E^2 - 4h \left( \tilde{h}_i \mp E \right)} \right].$$ (4.89)

The condition for the loss of equilibrium when the roots become degenerate reads as

$$E^2 > E_0^2,$$ where $E_0^2 = 4 \left( h \mp \sqrt{h\tilde{h}_i} \right)^2.$
Figure 4-11: Plots for the catastrophe conditions through coalescence of the roots. (a) $E = 1.5 < E_c \approx 2.16$, the critical energy; no catastrophe. (b) $E = 5 > E_c \approx 2.16$, with $h = 0.5$ and $h_i = 5$ in both cases.
Elimination of $E$ from equations (4.79) and (4.64), we get

$$C_{\lambda}^2 = \lambda h.$$  \hspace{1cm} (4.90)

Putting $\lambda = \mu$ in equation (4.79), we get

$$E = \mu h \left( (\mu + \rho)^2 + 1 \right).$$ \hspace{1cm} (4.91)

Putting above value of $E$ into equation (4.65), we get

$$C_{\mu}^2 = \lambda h.$$ \hspace{1cm} (4.92)

Equations (4.90) and (4.92) show that both the amplitudes become equal when the roots coalesce.

Figure (4-11) shows the graphs of $b_i$ vs. $\rho$ for $h = 0.5$ and $h_i = 5$ for both the catastrophe-free and catastrophe-prone energies. The dashed lines in both the graphs show the values of $b_i$ and $\rho$ when both $\lambda$ and $\mu$ are equal. The solid curves are drawn using equation (4.62). In figure (4-11) (a), the energy taken is less than the critical energy. We find no crossing of curves and dashed lines which shows that there will be no catastrophe when $\lambda$ and $\mu$ are equal. On the other hand, the dashed lines and curves cross each other for $E = 5$ which is greater than $E_c \simeq 2.16$ as shown in figure (4-11) (b). In this case, the equilibrium will lost.

In the present study, a constant density ratio of plasma particles is considered. The system could be transformed from para to diamagnetic states on varying the density ratio of the plasma components [92]. The present study could be helpful in improving our knowledge about a variety of eruptive events occurring in multi component space plasmas such as the phenomena of flares, eruptive prominences and solar mass ejections taking place in solar atmosphere.
Chapter 5

Summary and Conclusions

In this thesis, the relaxed states of multi-component magnetoplasmas are investigated. The self-organized states of magnetized plasmas are described in terms of Beltrami fields. The Beltrami field is defined as any solenoidal vector field which aligns itself to its own vorticity. Although, the magnetoplasmas constitute a very complex system, there is a strong evidence that certain turbulent plasmas have the tendency to self-organize to large scale structures. A remarkable example is the Beltrami magnetic field which appears as an equilibrium state of a flowless ideal MHD plasma. The state is expressed by

\[ \nabla \times \mathbf{B} = \lambda \mathbf{B}, \]

(5.1)

where \( \lambda \) is a scalar constant. Magnetic helicity \( h \) defined as

\[ h = \int \mathbf{A} \cdot \mathbf{B} d\Omega, \]

(5.2)

where \( d\Omega \) is a small volume element, \( \mathbf{A} \) is the vector potential and \( \mathbf{B} = \nabla \times \mathbf{A} \) is the magnetic field, plays a key role in the process of self-organization of ideal MHD plasma. The self-organized equilibrium state is achieved when the magnetic helicity remains relatively constant and the magnetic energy decays to its minimum value. The state is force-free as the Lorentz force becomes zero. The real plasmas in space and laboratory as
well are characterized by strong flows and exhibit strong pressure gradients. It is therefore important to look for the relaxation model which incorporates flows and high kinetic pressures. One of such models was presented by Mahajan and Yoshida. The remarkable feature of the model is the interpretation of self-organized state in terms of Beltrami fields. The superposition of a pair of Beltrami fields defines the self-organized state of Hall MHD two fluid plasma consisting of ions and inertialess electrons. Mathematically, it can be written as

\[ B = C_1 B_1 + C_2 B_2 \] (5.3)

where \( C_1 \) and \( C_2 \) are arbitrary constants which can be evaluated using boundary conditions. Pair of magnetic fields \( B_1 \) and \( B_2 \) satisfy the Beltrami conditions

\[ \nabla \times B_1 = \lambda_1 B_1, \] (5.4)
\[ \nabla \times B_2 = \lambda_2 B_2, \] (5.5)

where \( \lambda_1 \) and \( \lambda_2 \) represent eigenvalues of the eigenfunctions \( B_1 \) and \( B_2 \) respectively. Addition of above equations yield

\[ \nabla \times \nabla \times B - (\lambda_1 + \lambda_2) \nabla \times B + \lambda_1 \lambda_2 B = 0, \] (5.6)

called as DB equation. The DB field, solution of above equation, is generally a non-force-free field. In contrast to ideal MHD, DB model of relaxation takes into account a strong flow, exhibits a strong pressure gradient, allows two scale parameters and could represent both the paramagnetic and diamagnetic self-organized states. According to mathematically well posed variational principle, the self-organized state of Hall MHD plasma results when the generalized enstrophy \( F = \int_{\Omega} |\nabla \times (V + A)|^2 dx \) minimizes while the magnetofluid energy \( E = \int_{\Omega} (|V|^2 + |B|^2) dx \), magnetic helicity \( h_1 = \int_{\Omega} A \cdot B dx \) and generalized helicity \( h_2 = \int_{\Omega} (A + V) \cdot (B + \nabla \times V) dx \) remains conserved.
Mathematically, we can write

$$\delta(F - \mu_0 E - \mu_1 h_1 - \mu_2 h_2) = 0,$$

(5.7)

where $\mu_0$, $\mu_1$, and $\mu_2$ are constant Lagrange multipliers. This represents a mathematically well posed variational principle because the generalized enstrophy makes the functional $F - \mu_0 E - \mu_1 h_1 - \mu_2 h_2$ to be convex which has a unique minimizer. The DB equilibrium through Bernoulli conditions can sustain the pressure gradients of ions by the sheared flow of ions. The relaxation theory of Hall MHD explores the possibility of creating configurations which are highly confining, fully diamagnetic with minimum magnetic field in the interior of a compact plasma. It is worth noting that the magnetofluid energy (sum of kinetic and magnetic energy) comes out to be a constant of motion when the Bernoulli conditions are introduced in the macroscopic evolution equations. Hence, the relaxed states obtained are termed as Beltrami-Bernoulli states.

In order to study the relaxation dynamics in multi-component plasmas, the self-organized states of a dusty plasma comprised of dust grains, positive ions and two negative ions has been investigated. The dust grains are immobile, negatively charged and keep the plasma neutral. The other ions are singly ionized and have different masses. The number density of electrons $n_e$ is assumed to be very smaller than number density of dust grains $n_d$ that is $z_d n_d \gg n_e$, where $z_d$ represents the number of electrons attached to dust grains. The electrons are depleted due to their attachment to dust particles. We also assume that the charge of dust particles remains constant.

Taking curl of the macroscopic evolution equations of ions, the steady-state solutions satisfy the Beltrami conditions (alignment of flows and generalized vorticities). It is worth noting that inertia of negatively charged ions is different but the Beltrami parameters (ratio of generalized vorticities to corresponding flows) are same. Solving three Beltrami equations and using Ampere’s Law, the self-organized magnetic field follow the
TB equation as expressed below

\[
(\nabla \times)^3 \mathbf{B} - a (\nabla \times)^2 \mathbf{B} + b \nabla \times \mathbf{B} - c \mathbf{B} = 0,
\]

(5.8)

where \( a = k_1 + k_2 \), \( b = 1 + k_1 k_2 + \sum_{s=1,2} N_s M_s \), \( c = k_2 + k_1 \sum_{s=1,2} N_s M_s \), \( N_s = n_s / n_i \), \( M_s = m_i / m_s \) \((s = 1, 2)\), \( n_s \) and \( n_i \) are number densities of negative ions and positive ion, \( m_i \) represent the mass of ions and \( m_s \) the mass of negative ions \( k_1 \) and \( k_2 \) are Beltrami parameters \( k_1 \) is associated with ions whereas \( k_2 \) is the Beltrami parameter relating the generalized vorticities and flows of negative ions. The magnetic field which is the solution of equation (5.8) can be written as a sum of three different Beltrami fields i.e.,

\[
\mathbf{B} = \sum_{j=1}^{3} C_j \mathbf{B}_j,
\]

(5.9)

where \( C_j \) \((j = 1, 2, 3)\) are arbitrary constants and can be found using the boundary conditions. It is important to note that masses of all the dynamic species take part in the relaxation process while the ratios of generalized vorticities of both the negative ions to their corresponding flows are equal. This is the reason that the self-organized equilibrium state is equivalent to superposition of three linear Beltrami fields. On the other hand, if the Beltrami parameters of both the negatively charged ions are different, the relaxed equilibrium will consists of the superposition of four Beltrami fields.

The character of scale parameters as a function of Beltrami parameters for a constant density of negative ions is shown graphically. It is evident that the scale parameters may be real, complex, imaginary and degenerate. It explores the possibility of creating paramagnetic, diamagnetic and fully diamagnetic self-organized states. The Bernoulli equation is derived which relates flows and pressure with the electric potential and highlights the possibility of confining pressures.

An axisymmetric homogeneous cylindrical plasma is considered to show a glimpse of the behavior of the radial profiles of magnetic field for an electronegative dusty plasma.
composed of one positive ion \((Ar^+)\), two negative ions \((F^-\text{ and } SF_6^-)\) and very massive immobile dust grains. The cylinder is considered to be a large aspect ratio torus. The profiles depict the possibility of creating paramagnetic self-organized magnetic field structures as well as the diamagnetic ones for a proper set of Beltrami parameters and boundary conditions.

The self-organization of four component plasmas to TB field is the consequence of taking same Beltrami parameters for two negatively charged ions. If the Beltrami parameters are taken to be different, then the system will self-organize to a state which can be expressed as a superposition of four Beltrami states. In order to show the impact of inertial forces of all the constituents, we have considered the three component plasma. For sake of simplicity, we have considered the constituents which consist of a pair and singly ionized ions. The self-organized state comes out to be

\[
(\nabla \times)^4 B - a_3 (\nabla \times)^3 B + a_2 (\nabla \times)^2 B - a_1 \nabla \times B + a_0 B = 0, \tag{5.10}
\]

where \(a_0 = MN_i b_- b_+ + N_+ b_- b_+ b_+ b_i\), \(a_1 = MN_i (b_- + b_+) + N_+ (b_- + b_+) + b_+ b_i + b_- b_i b_+\), \(a_2 = b_- b_+ + b_- b_+ + b_+ b_i + 1 + N_+ + MN_i\), and \(a_3 = b_- + b_+ + b_i\) are constants. The parameters \(b_-\), \(b_+\) and \(b_i\) are constant and represent the Beltrami parameters corresponding to three fluids while \(M = m/m_i\) (\(m\) is the mass of a particle constituting a pair and \(m_i\) is the mass of ion), \(N_i = n_i/n_-\) and \(N_+ = n_+/n_-\), where \(n_j (j = -, +, i)\) is the density of plasma particles. The solution of equation (5.10) is called as Quadruple Beltrami (QB) field - a combination of four Beltrami fields. Mathematically, it can be expressed as

\[
B = \sum_{j=1}^{4} C_j B_j,
\]

where \(C_j (j = 1, 2, 3, 4)\) are constants and can be evaluated using the boundary conditions. It is worth noting that QB field is the result of the inclusion of the masses of all three components and each fluid has different ratio of generalized vorticity to flow. This system admits four scale parameters (eigenvalues of the curl operators) which are
the roots of quartic equation. Analytical expressions for all the roots are presented and their nature is discussed. The character of the roots of the quartic equation as a function of the Beltrami parameters for different density ratios has been plotted. It is observed that when the ions density is increased, there is an increase in the real roots. The impact of the Beltrami parameters on the self-organized states has also been analyzed. The analysis shows that Beltrami parameters play a key role in determining the relaxed state. For instance, when all the Beltrami parameters are same, the relaxed state will be the DB state and when all are zero, the self-organized state would be London’s equation of superconductivity.

There are four constants of motion namely magnetofuid energy and helicities of positively, negatively charged particles and ions. Due to equal mass and magnitude of charge, the helicities of the pair particles are same. Hence, there are only three constants of motion. The conservation of magnetofuid energy is derived and the QB state is also obtained invoking the variational principle.

The Bernoulli equation is derived which shows the possibility of confining high $\beta$ plasma. The analytical solution of QB field are expressed in slab geometry. The expressions of flow and its components are evaluated. The role of density is highlighted by plotting the profiles of magnetic fields for different density ratios. It is shown that magnetic field shows the diamagnetic behavior for the higher density of ions. For a particular set of Beltrami parameters and boundary conditions, the profiles of magnetic fields, flows and $\beta$ are shown which show that the system has the potential to exhibit low or high $\beta$ with an appreciable flow.

The two fluid equilibrium is characterized by DB states which result due to interaction of magnetic fields and flows. Ohsaki et. al., [33,34] have used DB states in modelling the closed coronal structures (coronal flux tubes) found in solar atmosphere and investigated the eruptive and explosive events. Such eruptive events and multi-component plasmas are ubiquitous in space plasmas. Therefore, we have extended the existing model for three species (electrons, positrons and ions) plasmas to study the eruptions and the conditions
for the loss of equilibrium. The three component plasma relaxes to DB field when the inertia of two lighter species (pair particles in our case) is ignored as compared to the heavier one. The DB state is characterized by four physical parameters, two of them are amplitudes and other two are scale parameters which define the size of the self-organized structures. The scale parameters in this system following the approach of the existing model are found to be functions of particles density in addition to Beltrami parameters. The effect of density on scale parameters is shown graphically which indicate that scale parameters could be changed from real to complex or vice versa by changing density of the species. It is proved analytically that magnetofluid energy acts as an ideal invariant. The DB states are derived by exploiting the constrained minimization of energy through variational principle. Following Yoshida et. al [31], we first derived algebraic relations between constants of motion, amplitudes and scale parameters for a plasma confined in a cube using two dimensional ABC map as Beltrami function. The equilibrium is lost when either any one of the amplitudes become zero or two scale parameters becomes equal on slowly varying the control parameter. In this way, DB equilibrium state will be transferred to single Beltrami state. The effects of $N_i$ (density ratio of ions to electrons), $h$ (helicities of pair particles), $h_i$ (helicity of ions) and $b_+$ (ratio of magnetic field to positron flow) on the relaxed double Beltrami equilibrium are shown graphically. It is found that $N_i$, $h$, $h_i$ and $b_+$ have the potential to lead the system towards explosion and eruption. The conditions for the loss of equilibrium are also derived. It is found that energy of the system determines whether the system is likely to lose the equilibrium or not. There exist a critical energy value which depends on helicities. If the energy of the system is less than or equal to this critical energy, the system preserves its equilibrium for given helicities and scale parameters. On the other hand, when the system’s energy is greater than the critical value, the equilibrium will be lost on a particular value of control parameter. The relation between the critical value of the control parameter and critical energy of the system is also derived. It is also observed that when equilibrium is lost, the magnetic energy becomes zero and most of the energy is converted to flow kinetic
energy. This conversion of magnetic energy to flow energy suggests that the phenomenon is a manifestation of an eruptive event.

In the present work, analytical solutions of the Beltrami states have been presented for slab and cylindrical geometries. The work can easily be extended to investigate the self-organized Beltrami states in annular and toroidal plasma configurations. The self-organized TB and QB states are derived for multi-component plasmas but their stability has not been studied. The issue about the stability of TB and QB fields need to be examined to place these states on a stronger footing. The catastrophic loss of equilibrium in slowly varying DB states appeared in three species plasmas could be extended to TB and QB states. The eruptive events need further study to look for the transformation of flow energy to magnetic energy in multi-component plasmas.
Bibliography


