Charged Isotropic and Anisotropic Fluid Gravitational Collapse

By

Syed Munawar Shah

PhD Thesis
SESSION 2013-2016

Supervised By
Dr. Ghulam Abbas

DEPARTMENT OF MATHEMATICS
The Islamia University of Bahawalpur
Bahawalpur, Pakistan
2019
Abstract

This thesis is devoted to the study of charged isotropic and anisotropic fluid gravitational collapse in the framework of General Theory of Relativity. We have investigated the dynamical features of the charged viscous cylindrical source using the Misner’s approach. The Einstein-Maxwell field equations and conservation equations have been formulated for the assumed matter and geometry. Using the Darmois criteria, we have investigated the matching conditions for the smooth matching of the interior region of cylindrical source with Vaidya like exterior solution. To see effects of charge, heat flux and viscosity on the dynamics of gravitational collapse, the dynamical equations of the charged gravitating cylindrical source have been coupled with transportation equations. Also, we have discussed the thermal evaluation of collapsing charged compact object. The equations of motion as well as junction conditions have been formulated for the spherical gravitating source. The perturbation scheme of first order is then applied to field equations, conservations equations and truncated version of the heat transport equation. The resulting system of equations provide the collapse equation and temperature profile of the gravitating system. We have explored the anisotropic fluid collapse enclosed by the cylindrical and plane symmetric spacetimes. We have presented the parametric form of two metric functions in term of a single metric function, which helps to determine the expansion and collapse of the gravitating sources.
Dedication

This thesis is dedicated to My Parents, Mrs. Dr. Shabana Nazar
and children, S. M. Abdur Rafaay and Maria Munawar.............
Contents

Abstract ................................................................. ii
Dedication .............................................................. iii
List of Figures .......................................................... vii

1 Introduction ......................................................... 1

2 Some Basic Concepts .............................................. 9
  2.1 General Relativity ............................................... 9
  2.2 The Stress-Energy Tensor ...................................... 10
  2.3 The Maxwell Field Equations ................................. 11
  2.4 The Einstein Field Equations ................................. 12
  2.5 Gravitational Collapse ....................................... 12
  2.6 Singularities ..................................................... 13
  2.7 Horizons ........................................................... 14
  2.8 Hypersurface .................................................... 15
  2.9 Curvature ........................................................ 15
  2.10 Intrinsic Curvature ......................................... 16
  2.11 Extrinsic Curvature ......................................... 16
  2.12 Matching Conditions ....................................... 16
  2.13 Anisotropy ...................................................... 17
  2.14 Homogeneity ................................................... 18
  2.15 Thermodynamical Properties of a Fluid ................. 18
2.15.1 Pressure ......................................................... 18
2.15.2 Density ....................................................... 18
2.15.3 Temperature ............................................... 19
2.16 Thermodynamical Theory of Dissipative Fluids ............. 19

3 Thermal Evaluation of Shear-free Charged Compact Objects 21
  3.1 Dissipative Charged Stellar Object ............................. 22
  3.2 Junction Conditions ........................................... 24
  3.3 The Perturbation Approach ................................... 26
  3.4 Thermal Evaluation ............................................ 29

4 Dynamics of Charged Bulk Viscous Collapsing Cylindrical Source With Heat Flux 31
  4.1 Gravitating Source and Field Equations ..................... 32
  4.2 Dynamical Equations ......................................... 34
  4.3 Heat Transport Equation ..................................... 37

5 Dynamics of Charged Viscous Dissipative Cylindrical Collapse With Full Causal Approach 40
  5.1 The Gravitating Source and Einstein-Maxwell Field Equations ... 41
  5.2 Dynamical Equations ......................................... 43
  5.3 The Transport Equation ...................................... 45

6 Gravitational Collapse and Expansion of Charged Anisotropic Cylindrical Source 49
  6.1 Matter Distribution and Field Equations ..................... 50
  6.2 Generating solution .......................................... 53
    6.2.1 Gravitational Collapse with $\alpha = -\frac{3}{2}$ ............ 53
    6.2.2 Expansion with $\alpha = \frac{3}{2}$ ......................... 56
7 Collapse and Expansion of Plane Symmetric Charged Anisotropic Source 60

7.1 Gravitating Source and Einstein-Maxwell Field Equations . . . . . . . 61

7.2 Parametric Solutions . . . . . . . . . . . . . . . . . . . . . . . . . . . . 63

7.2.1 Gravitational Collapse for $\alpha = -\frac{5}{2}$ . . . . . . . . . . . . . . . 63

7.2.2 Expansion for $\alpha = \frac{3}{2}$ . . . . . . . . . . . . . . . . . . . . . . . . . 64

8 Conclusions 68
List of Figures

6.1  Both graphs have been plotted for $s = 2$ and $h_1(t) = 1 + t$.

6.2  Both graphs have been plotted for $s = 2$ and $h_1(t) = 1 + t$.

6.3  Dimensionless anisotropic parameter variation for $s = 2$ and $h_1(t) = 1 + t$.

6.4  Both graphs have been plotted for $s = 2$ and $h_2(t) = 1 + t$.

6.5  Both graphs have been plotted for $s = 2$ and $h_2(t) = 1 + t$.

6.6  Dimensionless anisotropic parameter variation for $s = 2$ and $h_2(t) = 1 + t$.

7.1  Both graphs have been plotted for $Q = 2, g_1 = 1 + t$.

7.2  Both graphs have been plotted for $Q = 2, g_1 = 1 + t$.

7.3  Both graphs have been plotted for $Q = 2, g_2 = 1 + t$.

7.4  Both graphs have been plotted for $Q = 2, g_2 = 1 + t$. 
Chapter 1

Introduction

A great notable philosopher Aristotle published his book with title *On The Heavens* in 350 B.C, in this book he claimed that the earth is round sphere instead of hat plate (Gerhard 1995). He argued that eclipse of moon occurred due to earth coming between moon and the sun and during the eclipse the earth’s shadow on the moon is round shape which indicates that earth is spherical otherwise its shadow would be elongated and elliptical. Aristotle also believed that earth is stationary and it is the center of this universe, all planets, sun and the moon performed circular motion around it in its own path. But later on Nicholas Copernicus (Crowe 1990) considered the sun is stationary while earth and other planets revolve around the sun. Meanwhile two philosopher Kepler and Galileo (Gingerich 1993) supported copernicus theory and further Kepler put his extension that the moon and planets motion is elliptic rather than circular and also he presented his own laws of planetary motion with mathematical calculations which are much important till now.

In Seventh century, Sir Issac Newton who became the hearts and heads figure of physical science, filled up the vacuum of physical science by writing a book entitled Philosophiae Naturalis Principia Mathematica (Mathematical Principles of Natural Philosophy). He is the first person who use the word gravitation and introduced the universal law of gravitation which states that each body of the uni-
verse attracts any other body to itself and he also explained how bodies move in space and time on the basis of his complicated mathematical analysis (Smith 2008). Newton thought that the stars should attract each other according to his theory of gravity. In this theory, Richard Bently in 1691 (Janiak 2014) asked to Newton that if stars have a force of gravity, then the all stars should be gravitated at one center point as a result all stars will collapse themselves. At this objection Newton replied that this universe is infinite and also every region in the universe has infinite number of stars that’s why every point is its center point if the universe is finite then it may be possible. According to Newton’s law of gravity, the gravitational force depends upon the distance between the bodies, if bodies are farther away then their force will be smaller. An other aspect of this law is that the gravitational force of a star is equal to one quarter that of a equivalent star at half of the distance. This law foretells about the orbits of moon, earth and other planets with significant figures. On the other side if the gravitational force is directly proportional to the distance then gravitational attraction of a star increased rapidly with the distance, as a result forming orbits of planets would not be elliptical.

In 1865, J. C. Maxwell (Bruce 1991) introduced the propagation theory of light and suggested that there is a substance ether which is present every where including empty space. The light waves travel through this ether in the space where as sound waves travel through air and the speed of these waves can be observed relative to the ether. The speed of light relative to the ether is fixed as $3 \times 10^8 m/sec$ but some observers, moving with respect to ether feel that light is coming towards them with different speeds. Newtonian mechanics has a great revolutionised progress in the field of physical science but there is a conflicting question in mind at the time of Newton that what difference can be observed in the laws of mechanics by the moving observers? no one has the answer of this question even Newton also.

In the start of twentieth century, a renowned physicist and mathematician Albert
Einstein presented his theory of Relativity consisting of uniform and non-uniform relative motion, in his theory he has broken the conflict about Newton’s laws of motion for moving observer. In 1905, he put the explanation that all physical laws have same form in uniformly moving spacetime and named it theory of Special Relativity. Then after ten years, Einstein (1916) introduced General Theory of Relativity in which space time considered accelerated and on the basis of his theories he has related spacetime curvature with matter.

Rosseland (1924) investigated that the law of central force is observed by the forces between the free particles and proved that the atoms are converted into ions with great strength, the order of magnitude should be greater than that of remaining forces acting between the neutral atoms. He further noted that the effect of electric force is large in the star having mass 1.5 times solar mass and mean molecular weight up to 2.8 unit. Eddington (1926) found that there is direct relation between electric potential \( \phi \) and gravitational potential \( \psi \) in the internal electrical field of star and he also mentioned that the mass \( m \) of the star, charge \( e \) of a photon and scalar \( \alpha \) affected by density \( n_i \) of the ions, the effective charge \( e z_i \) and atomic weight \( A_i \) of the ions.

As the stars consist of gas particles most probably the hydrogen gas, so there are continued nuclear reactions among its particles. These nuclear reactions took place due to pressure and gravitational attraction between the gas particles. In other words one can say that there is a fight between pressure and gravity of gas particles presented in the stars matter, some times pressure lose and gravity overcomes then nuclear reaction of star passed through gravitational collapse. This collapsing star actually leads to the production of white dwarf, neutron star and black hole with respect to the density of star. More precisely, if the collapsing mass of the star is less than that of Chandrashekhar limit then its collapse produces white dwarf. If the collapsing star having mass greater than 1.5 times of the solar mass but less than 5 times that of solar mass then neutron star came into being in the
universe. On the other hand black hole is the production of such a star which contains the matter more than 5 times of solar mass. Chandrashekher (1931) presented the appropriate description of black hole on the basis of few parameters, which are consistent with observational evidences.

However, there is a unsolved problem in the contents of General Relativity (GR) that collapsing star heavier than the star which formed neutron star will be black hole or not. The singularity theorems of GR proposed by Hawking and Penrose (1965) contains clear picture of this problem. They elaborated that a sufficiently massive and continually gravitating star will halt into two types of gravitational singularities. One of which is singularity of energy density of the collapsing matter and other singularity appears at a point where the curvature of spacetime becomes infinite at some instant of time. These both singularities can be observed or not by a distant observer. The observable singularity is called black hole while non-observable singularity is known as naked singularity. The distinction between a black hole and a naked singularity can be realized by the better understating of theory of General Relativity. Oppenhiemer and Snyder (1939) opened a new direction of research by presenting their research about on the continued gravitational contraction of the relativistic dust cloud. However, they assumed their model on basis of some special considerations but with the passage of time, this model have to be reformed for the generalized case. Later on their work was extended to the case of dissipative matter source.

Vaidya (1951),(1953) solved the model of spherically symmetric exterior spacetime of radiating star and introduced the concept of radiation coordinates. Such solutions become the basis for modeling of dissipative collapse within the stars core. In accordance with Vaidya intention, exterior Vaidya solution should be matched with interior spacetime with the continuity of gravitational potentials and the radiation flux. Misner and Sharp (1964) and (1965) assumed an isotropic perfect fluid gravitational collapse in the interior region of a star and developed governing e-
equations of motion for gravitating matter. Many researchers, Lake and Hellaby (1981), Santos (1985), Herrera and Santos (1997), Herrera et al. (1998), Herrera and Santos (2004), Herrera et al. (2012), Herrera (2006a), Herrera et al. (2006b), Chan (2000), Bonor et al. (1989), Banjeri jee et al. (1989), Chan et al. (1989) have given their individual extensive approaches in the number of collapsing models according to their interest and the contents of cognition. Mitra (2006) noted that the gravitational collapse is a high energy dissipating process which is caused to the formation of the stars. He classified this process into free streaming and diffusion approximations. Further, Tewari (1988), (2006) put up his extension in free streaming approximation and formulated models of gravitational collapse by finding the solutions of Einstein field equations. In diffusion approximation case, some researchers (Bonner et al. (1989), Inanov (2011), Maharaj and Govnder (2005), Bowers and Liang (1974), de Oliveira et al. (1985) and Phinheiro and Chan (2013)) studied different dissipative processes analytically and formulated a lot of realistic models with anisotropy, viscosity, electromagnetic field and inhomogeneity.

Herrera et al. (2007) considered cylindrical geometric source and discussed junction conditions for its gravitational collapse. Sharif and Ahmed (2007) pointed out the generation of gravitational radiations during the gravitational collapse of two perfect fluids. Nakao and Morisawa (2004) introduced the collapse of hallow cylinder and its gravitational radiations and also proved analytically the emission of gravitational cylindrical waves for cylindrical gravitating source. The recent literature (Nakao and Morisawa (2005), Sharif and Fatima (2011), Sharif and Abbas (2011)) present the study of the dynamics of collapsing cylindrical sources.

Eckart (1940) and Landau and Lifschitz (1959) tried to build up a characterized theory for irreversible processes in general way. They proposed the theory for general fluid mixture and utilized this theory on ideal gases. Further, they apply their proposed theory to see the validity of law of entropy and second law of thermodynamics. But Govender (1967) observed the Eckart theory have lack of stabili-
ty in equilibrium condition of infinite propagation velocities for thermal signals. The Eckart logic of scalar linear curvature was not consistent with causality and stability. On the other hand Müller (1967) model of irreversible thermodynamics in extended non relativistic version, satisfied the causality condition. Israel and Stewart (1976), (1979) developed relativistic version of Müller theory. This new theory is accepted as the set of facts in the contents of relativistic theory. Anderson (2005) formulated the equations by imposing the non negative condition on the divergence of the entropy flux, which leads to satisfy the second law of thermodynamics. Govender and Thirukkkanesh (2014) worked on the extended irreversible thermodynamics of some radiating sources. Herrera et al. (2009) studied viscous dissipative gravitational collapse for streaming out and diffusion approximations and they related the derived dynamical equation to causal transport equation for the heat flux, bulk and shear viscosities in the theory of Israel-Stewart along with the inclusion of thermodynamics heat coupling coefficients. Di Prisco et al. (2007) derived the dynamical equations for charged, dissipative and spherically symmetric gravitational collapse and coupled the derived equations to causal transport equations of Müller Israel-stewart theory. They also reobtained the factor responsible for decreasing the internal mass density of the fluid, depending on its internal thermodynamical state.

In this thesis, we address the issue of gravitational collapse of isotropic and anisotropic charged fluid collapse in the frame work of GR. For the generalized cases the isotropic and anisotropic fluid has been considered with heat flux, bulk and shear viscosity. In all these cases the Maxwell field equations have been solved to incorporate the contribution of electromagnetic field in the evaluation of gravitating source. The background geometries assumed in this thesis are spherical, cylindrical and plane. For the dynamical aspects of the gravitating source, Misner-Sharp approach (1965) and junction conditions have been used. For the dynamics of gravitating source with heat flux, the heat transportation equation has been cou-
pled with dynamical equation. The dynamics of viscous and dissipative fluid has been discussed by using the full casual approach developed by M"uller-Israel-Steward. The thermal evolution of the charged shear free gravitating source has been investigated by applying the first order perturbation to Einstein-Maxwell field equations and truncated version of heat transportation equations. The expanding and collapsing behavior of the charged anisotropic cylindrical and plane symmetric sources have been explored by assuming the parametric form of a metric function in term of other metric functions. The values of the parameter involved in the solutions help us to classify the expanding and collapsing solutions. This thesis is planed as follows

- **The chapter two** contains some basic definitions and concepts related to our research work.

- **The chapter three** deals with the study of thermal evolution of compact objects by using the perturbation approach. We have applied first order perturbation to the Einstein-Maxwell field equations and obtained two types of sets of equations namely static set of equations and perturbed set of equations. In the consequences of this chapter, we have coupled the transport equation with field equations and draw the results in the presence of some coupling parameters. The results of this chapter have been published in the impact factor journal *Astrophysics and Space Science* (Shah and Abbas 2018).

- **The chapter four** is devoted to discuss the dynamics of charged bulk viscous collapsing cylindrical source with the involvement of heat flux. We have present the junction conditions for the smooth matching of the interior and exterior regions of the collapsing source. The resulting outcomes of this chapter have been published in the impact factor journal *European Physical Journal C* (Shah and Abbas 2017a).

- **Chapter five** presents the dynamics of charged viscous dissipative cylindrical
collapse with full causal approach. We have investigated the dynamical behavior of charged viscous collapse of cylindrical stellar object by introducing heat flux, shear and bulk viscosity in the stress-energy tensor. The dynamical equations have been coupled with the transportation equations. The evaluated results along with theoretical explanation have been published in the form of a research paper in the impact factor journal *European Physical Journal A* (Shah and Abbas 2017b).

- **The chapter six** deals with the collapse and expansion of charged anisotropic cylindrical gravitating source. To observe the collapse and expansion of such type of fluid, we have obtained the exact solutions of Einstein-Maxwell field equations. These solutions are parametric in nature and depends on a single metric function. We have determined the values of the parameters for which the stellar object undergoes to collapse and expansion. This research work has been published in the impact factor journal *Astrophysics and Space Science* (Tahir et al. 2015).

- **In chapter seven**, we have examined the final stages of charged anisotropic plane symmetric gravitating source. We have found the solutions that exhibit the expansion and collapse of the gravitating source depending on the choice of parameters of the model. The research article related to this work has been published in the impact factor journal *Canadian Journal of Physics* (Abbas et al. 2016).

- **In chapter Eight**, we have discussed the results and future work related to this thesis.
Chapter 2

Some Basic Concepts

In this chapter, we present some definitions which are necessary to understand the research work of the thesis. Throughout the thesis, we shall use the signature (−, +, +, +) and the relativistic units i.e., $G = c = 1$. We also mention that all the Latin and Greeks alphabets vary from 0 to 3, otherwise it will be mentioned explicitly.

2.1 General Relativity

Relativity theory is used to observe the relative motion of different objects in the non-inertial frame. Albert Einstein introduced Special Theory of Relativity in the start of twentieth century. This theory is specified for non accelerated frames only based on the following two postulates (Qadir 1989)

1. All physical laws are same in non accelerated frames.

2. The speed of light is constant in vacuum for all observers moving with constant speed.

Ten years after the success of Special Relativity, Eistein generalized his Special Relativity as General Theory of Relativity for the accelerated frames of reference on the basis of following principles

- Both the inertial and gravitational masses are equal i.e., $M_g = M_i$. 
All the laws of physics should be converted in tensor form.

2.2 The Stress-Energy Tensor

In GR, the quantities like mass, energy, momentum and pressure all are sources of gravity. The stress-energy tensor serve as a source of gravitational field via the Einstein field equations. It describes the matter distribution at every point of spacetime. It is denoted by $T_{ab}$ which is second rank symmetric tensor. The most generalized stress-energy tensor for viscous and heat conducting fluid is given by (Herrera et al. 2009)

$$T_{ab} = (\mu + P + \Pi)V_a V_b + (P + \Pi)g_{ab} + q_a V_b + q_b V_a + \epsilon \ell_a \ell_b + \pi_{ab},$$  \hspace{1cm} (2.1)

where $\mu$, $p$, $\Pi$, $q_a$, $\epsilon$, $\ell_a$, $V_a$ and $\pi_{ab}$ are the energy density, pressure, bulk viscosity, heat flux, radiation density, null four-vectors, four-velocity of fluid and shear viscosity tensor, respectively. For anisotropic fluid above equation yield (Di Prisco et al. 2009)

$$T_{ab} = (\mu + P_{\perp})V_a V_b + P_{\perp}g_{ab} + (P_r - P_{\perp})\chi_a \chi_b + q_a V_b$$
$$+ V_a q_b + \epsilon \ell_a \ell_b + \pi_{ab},$$  \hspace{1cm} (2.2)

where $P_{\perp}$ is the pressure perpendicular to $r$-direction and $\chi^a$ is a unit four vector in $r$-direction.

The energy-momentum tensor of perfect fluid can be obtained from Eq.(2.1) by taking $q_a = \epsilon = \Pi = \pi_{ab} = 0$

$$T_{ab} = (\rho + P)V_a V_b + P g_{ab},$$  \hspace{1cm} (2.3)

for dust case, pressure is zero and we get

$$T_{ab} = \rho V_a V_b.$$  \hspace{1cm} (2.4)

The components of energy-momentum tensor can be interpreted physically as: $T_{00}$ is the energy density of the particles, $T_{0i}$ is the energy flux in the $i$-direction, $T_{i0}$
is the momentum density in the $i$-direction, $T_{ii}$ are the normal stresses and $T_{ij}$ for $i \neq j$ represent shear stresses. Due to this interpretation, it is named as the energy-momentum or stress-energy tensor.

The stress-energy tensor of an electromagnetic field is defined as (Landau and Lifshitz 1962)

$$E_{\alpha\beta} = \frac{1}{4\pi}(F^\gamma F_{\beta\gamma} - \frac{1}{4} F^\gamma \lambda F_{\gamma\lambda} g_{\alpha\beta}),$$  \hspace{1cm} (2.5)

where $F_{\alpha\beta}$ is Maxwell field tensor given by

$$F_{\alpha\beta} = \varphi_{\beta,\alpha} - \varphi_{\alpha,\beta},$$  \hspace{1cm} (2.6)

where $\varphi_{\alpha}$ is electromagnetic four potential.

### 2.3 The Maxwell Field Equations

In classical electrodynamics Maxwell field equations is a set of four differential equations which is given by

$$\nabla \cdot E = \frac{\rho}{\epsilon_0},$$  \hspace{1cm} (2.7)

$$\nabla \cdot B = 0,$$  \hspace{1cm} (2.8)

$$\nabla \times E = -\frac{\partial B}{\partial t},$$  \hspace{1cm} (2.9)

$$\nabla \times B = \mu_0 j + \mu_0 \epsilon_0 \frac{\partial E}{\partial t},$$  \hspace{1cm} (2.10)

where $\epsilon_0$ and $\mu_0$ are permittivity and permeability of free space respectively. These equations describe the following laws. The first and second equations yield Gauss law for electric and magnetic field respectively. The third equation represents Faraday’s law of induction and the last equation is the Ampere’s law. The tensor form of Maxwell’s equations is

$$F^{ab}_{;b} = 4\pi J^a,$$  \hspace{1cm} (2.11)

$$F_{[abc]} = 0,$$  \hspace{1cm} (2.12)

$J^a$ is the four current.
2.4 The Einstein Field Equations

The field equations basically tell us that how a fundamental force interacts with matter. In Newton’s theory of gravity, the field equation is the Poisson’s equation given by

\[ \nabla^2 \phi = 4\pi G \rho, \]  

(2.13)

here \( \phi \) is gravitational potential and \( G \) is gravitational constant. This is a scalar equation. Einstein formulated an equivalent equation in tensor form, valid for the curved spacetime, defined by

\[ G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \equiv \kappa T_{ab}, \]  

(2.14)

where \( \kappa = \frac{8\pi G}{c^4} \) is the coupling constant, \( R_{ab}, R, T_{ab} \) and \( G_{ab} \) are Ricci tensor, Ricci scalar, stress-energy tensor and Einstein tensor, respectively. In GR, matter is the source of gravitational field, so these equations give a relation between the gravitational field and matter sources. For the charged gravitating source, Einstein field equations are

\[ G_{ab} = \kappa (T_{ab} + E_{ab}), \]  

(2.15)

where \( G_{ab}, T_{ab} \) and \( E_{ab} \) are same as defined above.

2.5 Gravitational Collapse

Gravitational collapse is defined as the inward fall of a massive object due to its own gravity. A normal star is in a hydrostatic equilibrium due to the balance of gravity by a pressure gradient force in the opposite direction. When this pressure gradient is not sufficient to balance the gravity, gravitational collapse of star occurs. As a result of collapse a smaller and denser celestial object is obtained which is called compact object, these are white dwarfs, neutron stars and black holes. The compact objects are different from normal stars by the following two ways.
Firstly, they do not burn nuclear fuel and as a result gravity is not balanced by thermal pressure gradient. In white dwarfs and neutron stars, gravity is balanced by degeneracy pressure of electrons and neutrons while black holes are totally collapsed objects, there is no force to balance gravity. Secondly, the size of compact objects separates them from normal stars. Compact objects have very small size and radius as compared to the original stars and as a result their surface gravity is very strong (Joshi 1993).

The collapse is said to be homologous collapse when the collapsing velocity is directly proportional to the radius (distance). In case of homologous collapse, all the matter fall inward in the same pattern while in a non-homologous collapse, the central region of star collapses faster than the outer parts. Indeed during collapse, the density increases speedily in the central part so the inner shells fall quicker than the outer ones.

### 2.6 Singularities

The term singularity in GR is used for such a region of spacetime where curvatures and densities are infinite and their physical description is impossible. It is of two kinds either essential or coordinate singularities. When the Reimann tensor components are finite in the region of singularity then such singularity is called removable singularity. We can get rid of such singularity by choosing a suitable set of coordinates. If the curvatures of spacetime become infinitely large as the singularity is approached then it is called a genuine or essential singularity (Harada et al. 2002). An essential singularity is further classified as

- Naked singularity;
- Covered singularity or black hole.

Naked singularity is one which is visible to external observers. It will be a locally naked singularity if an observer within the event horizon receives photons from
singularity. In other sense, if there is an outgoing future directed null geodesic (not crossing the event horizon), which reaches at the singularity in the past then the singularity is *locally naked*. Similarly, when the outgoing null geodesic would come out of the boundary of the collapsing cloud to reach a far away observer in the spacetime, it will be *globally naked*.

When time like curves escape from the singularity then it is called a covered singularity or black hole. The idea of so massive body that even light could not escape was first given by Michell in 1783. Later on, Wheeler in 1967 firstly used the term black hole for such highly dense objects.

Penrose (1965) gave a conjecture about the spacetime singularity. It states that singularity formed by gravitational collapse is always covered by an event horizon. It has two versions, weak and strong. The strong cosmic censorship hypothesis (Hawking and Ellis 1979) states that the singularity cannot be observed even by an observer who is very close to it, i.e., the singularity is not locally naked.

## 2.7 Horizons

Here we define event horizon, apparent horizon and Cauchy horizon. Event horizon is the boundary of the black hole region, i.e., the boundary of the region from which there is no escape to infinity for photons or particles. The exterior region of spacetime containing trapping surface is the apparent horizon. A trapped surface in four dimensional spacetime is a two dimensional embedded surface on which outward-pointing light rays are terminated back inwards due to strong gravity. The gravitational contraction of a gravitating source proceeds to form a black hole, predicts that the event horizons are formed before the apparent horizons (Hawking and Ellis 1979). The initial null cone which extends from singularity to null infinity is a Cauchy horizon (Hawking and Ellis 1979).

Now we discuss positions of these horizons in Vaidya spacetime for imploding radiations in different cases (Vaidya 1951). When collapsing process is slow then
the family of null geodesics coming out from the singularity are not entirely contained inside the event horizon. Also, the null geodesics between Cauchy horizon and the extended black hole event horizon reach to future null infinity while the rest of the geodesics terminate back to the singularity. As the null geodesics are coming out from the singularity and reach to far away observer, the singularity is globally naked singularity.

2.8 Hypersurface

In an $n$-dimensional manifold, a hypersurface $\Sigma$ is an $(n-1)$-dimensional submanifold. It can be defined by imposing the restriction on the coordinates as (Ahmad 2008).

$$k(x^a) = 0. \quad (2.16)$$

Then the unit normal vector to the hypersurface is defined as

$$n_a = \frac{k_a}{|g^{bc}k_bk_c|}.$$ \quad (2.17)

where

$$n^a n_a = -1, \quad \text{if } \Sigma \text{ is spacelike}$$

$$= +1, \quad \text{if } \Sigma \text{ is timelike}. \quad (2.18)$$

It is obvious that a unit normal cannot be defined for a null hypersurface (Joshi 1993).

2.9 Curvature

Curvature is basically defined as the departure from flatness. According to GR, curvature in a spacetime is induced by the contents of matter present in the underlying spacetime. There are two forms of curvature

1. Intrinsic curvature
2. Extrinsic curvature

2.10 Intrinsic Curvature

The local geometry or properties accessible by a surface is called intrinsic curvature or first fundamental form. The intrinsic curvature is defined as (Grlon and Hervik 2007)

\[ ds^2 = g_{ab} dx^a dx^b, \]  

where \( g_{ab} \) is metric tensor.

2.11 Extrinsic Curvature

It describes the bending of a hypersurface in higher dimensional manifold. This deformation of a hypersurface is measured by a tensor \( K_{ab} \) defined by

\[ K_{ij} = -n_a \left( \frac{\partial^2 \chi^a}{\partial \xi^i \partial \xi^j} + \Gamma^a_{bc} \frac{\partial \chi^b}{\partial \xi^i} \frac{\partial \chi^c}{\partial \xi^j} \right), \]  

where \( \chi^a \) are coordinates of higher dimensional manifold and \( \xi^i \) are intrinsic co-ordinates of hypersurface. It is a symmetric tensor also called the second fundamental form of the hypersurface. To understand the difference between these two types of curvature, we take an example of a flat sheet of paper. It has no intrinsic curvature but when it is rolled in the form of a cylinder, it has extrinsic curvature (Grlon and Hervik 2007).

2.12 Matching Conditions

Matching conditions are basically the restrictions that imposed on the coordinates of interior and exterior spacetimes. Such conditions makes continuous the metric functions of spacetimes and their extrinsic curvature components. In literature (Israel 1966) there are three types of matching conditions. In Lichnerowicz matching conditions the line elements and derivatives of the line elements are continuous
over the domain wall. In order to study the stability of the gravitating systems Israel (1967) proposed the matching conditions, which states that line elements are continuous over the boundary surface but there is discontinuity in the extrinsic curvature components. The difference in the components of extrinsic curvature is equal to the stress-energy tensor present on the boundary surface. Darmois (1927) proposed the following junction conditions

- The intrinsic curvature (first fundamental form) is continuous over the boundary surface $\Sigma$, i.e.,
  \[ (ds^2)_\Sigma = (ds^2_-)_\Sigma = (ds^2_+)_\Sigma, \]
  \[ (2.21) \]
  where $(ds^2)_\Sigma$ is the induced metric for hypersurface $\Sigma$, $(ds^2_-)_\Sigma$ and $(ds^2_+)_\Sigma$ represent line elements of the interior and exterior regions over the hypersurface, respectively. Alternatively, it can be stated as the continuity of the first fundamental forms.

- The continuity of the extrinsic curvature (second fundamental form) over $\Sigma$, gives
  \[ [K_{ab}] = K^-_{ab} - K^+_{ab} = 0. \]
  \[ (2.22) \]

2.13 **Anisotropy**

Anisotropy is the property of matter being directionally dependent, as opposed to isotropy, which implies identical properties in all directions. It can be defined as a difference, when measured along different axes, in a material physical or mechanical properties (absorbance, refractive index, conductivity, tensile strength etc.). An example of anisotropy is the light coming through a polarizer. Another is wood, which is easier to split along its grain than against it. In a single crystal, the physical and mechanical properties often differ with orientation. It can be seen from looking at our models of crystalline structure that atoms should be able to slip over one another or distort in relation to one another easier in some directions.
than others. When the properties of a material vary with different crystallographic orientations, the material is said to be anisotropic.

2.14 Homogeneity

In physics, a homogeneous material or system has the same properties at every point, it is uniform without irregularities. A uniform electric field (which has the same strength and the same direction at each point) would be compatible with homogeneity (particles all points experience the same electric force). A material constructed with different constituents can be described as effectively homogeneous in the electromagnetic materials domain, when interacting with a directed radiation field (light and microwave frequencies etc).

2.15 Thermodynamical Properties of a Fluid

Three primary thermodynamical characteristics of a system are pressure, density and temperature. We define these quantities in the following

2.15.1 Pressure

Stress is defined as the internal distribution of forces in a body that balances and reacts the external forces applied on the body. Normal stresses are called pressure which is denoted by \( P \).

2.15.2 Density

Density of a material is defined as its mass per unit volume. The density of a compressible fluid changes with temperature and pressure. Pressure and density are directly proportional to each other while for most of the fluids pressure and density are inversely related.
2.15.3 Temperature

Temperature is a measure of the tendency of an object to spontaneously give up energy to its surroundings. It is a measure of the average kinetic energy of the molecules in a system, i.e., it determines the internal energy of the system. There are also secondary variables such as viscosity and heat conduction which characterize the specific form of fluid.

2.16 Thermodynamical Theory of Dissipative Fluids

Dissipation means loss of energy of a system, i.e., in a dissipative process energy remains no more a conserved quantity. However, some fluids have negligible viscosity and are treated as inviscid fluids. The fluids which have no heat conduction and viscosity are called perfect fluids. Since perfect fluids do not have dissipation, their dynamics is reversible. The fluids mostly considered in astrophysical problems are real fluids which have an irreversible thermodynamics.

Eckart (1940) was the first who extended the Newtonian irreversible thermodynamics to relativistic fluids. Later on, Landau and Lifshitz (1962) made some changes in this theory. These were parabolic theories (transport equations are of parabolic type) based upon the fact that entropy four current (flux) have linear terms in dissipative variables. The problem associated with these theories was that dissipative propagations have infinite speed which is impossible in GR. To resolve this problem, extended irreversible thermodynamic theories are offered. These are hyperbolic theories and here entropy four current (flux) consists of second order terms in dissipative variables. Muller (1967) first introduced the non-relativistic extended theory. Afterwards, a relativistic approach was given by Israel and Stewart (1976). The Israel-Stewart (1979) theory is one of the best theories satisfying the stability and causality conditions.

A partial differential equation which describes transport phenomenon is called
transport equation. Transport equation explains the processes of mass, heat and momentum transfer etc. The transport equation, for heat flux (taking the viscosity and thermodynamics variable as negligible) derived from the Müller-Israel-Stewart (Israel-Steward 1976) causal thermodynamic theory, are

\[ \tau h^{ab}V^c q_{bc} + q^a = -Kh^{ab}(T_b + a_b T) - \frac{1}{2}KT^2(\frac{\tau V^b}{KT^2})_b q^a, \]  

(2.23)

where \( h^{ab} = g^{ab} + u^a u^b \) is the projection tensor, \( K \) denotes thermal conductivity, \( T \) is temperature and \( \tau \) stands for relaxation time which is the time taken by a perturbed system to return into an equilibrium state.
Chapter 3

Thermal Evaluation of Shear-free Charged Compact Objects

This chapter describes the thermal evolution of gravitating source, this source is assumed to be charged and shear free. During the gravitational collapse the effects of the dissipation on the final fate can not be neglected in general, so the presence of heat flux in the present case is much important. To study the thermal evaluation during gravitational collapse, we have derived dynamical equations by using first order perturbation technique on Einstein-Maxwell equations and heat transport equation. These derived equations are further have been resolved into two parts, one is static which corresponds to zero order of perturbation and the second is non-static perturbed part corresponds to first order perturbation. We observe that the temperature gradient favorably depends on product of relaxation time and also on time taken by the system for oscillation.

This chapter is planed as follows: In section 3.1, we have discussed the nature of dissipative charged shear free spherical compact object and derived the corresponding set of Einstein-Maxwell field equations. The Darmois (1927) junction conditions have been present in section 3.2. The perturbation scheme of first order has been applied to field equations and conservation equation in section 3.3. The thermal evaluation of the gravitating compact object has been studied by applying the perturbation of first order to the truncated version of heat transport equation
in section 3.4.

3.1 Dissipative Charged Stellar Object

Here, we present the gravitating source which has heat flux and electric charge. The structure of such source is modeled as a spherical compact object whose line element has the following form

\[ ds^2 = -A^2 dt^2 + B^2 dr^2 + B^2 r^2 d\theta^2 + B^2 r^2 \sin^2 \theta d\phi^2, \]  

(3.1)

where \( A = A(r, t) \) and \( B = B(r, t) \).

For the isotropic charged radiating source the stress energy tensor given by Eq.(2.1) with \( q_\mu = 0, \Pi = 0 \) and \( \epsilon = \pi_{\mu\nu} \) has the following form

\[ T_{\mu\nu} = (\mu + P)u_\mu u_\nu + Pg_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + E_{\mu\nu}, \]  

(3.2)

where \( E_{\mu\nu} \) is electromagnetic energy-momentum tensor given by Eq.(2.5), \( \mu \), \( P \), \( u_\mu \), and \( q_\mu \) are energy density, radiation density, isotropic pressure, four velocity and radial heat flux, respectively. Also, the Maxwell field tensor \( F_{\mu\nu} \) is given by Eq.(2.6). Further, \( u^\mu \) and \( q^\mu \) are defined by

\[ u^\mu = A^{-1} \delta^\mu_0, \quad q^\mu = q \delta^\mu_1. \]  

(3.3)

Now for the description of electromagnetic field, we have to solve the Maxwell field equations Eqs.(2.11) and (2.12). In this case, the magnetic field will be zero due to static charges, so we may choose the four potential and four current as follows

\[ \varphi_\mu = (\varphi(r, t), 0, 0, 0), \quad J^\nu = \zeta u^\nu. \]

Here \( \zeta(r, t) \) and \( \varphi(r, t) \) are charge density and scalar potential, respectively. The conservation of charge gives

\[ S(r) = 4\pi \int_0^r \zeta B(r)^2 dr, \]  

(3.4)
which is the total electric charge. By the help of Eq.(3.1), the Maxwell equations become

$$\varphi'' - \left( \frac{A'}{A} + \frac{B'}{B} - \frac{(Br)'}{Br} \right) \varphi' = 4\pi \xi AB^2, \quad (3.5)$$

$$\varphi' - \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \frac{\dot{Br}}{Br} \right) \varphi' = 0, \quad (3.6)$$

Integrating of Eq.(3.5), we have

$$\varphi' = \frac{S(r)AB}{(Br)^2}, \quad (3.7)$$

The electric field intensity can be stated as

$$E(t,r) = \frac{S(r)}{4\pi(Br)^2}. \quad (3.8)$$

The Einstein-Maxwell field equations for the given source are

$$8\pi A^2 \left( \mu + 2\pi E^2 \right) = \frac{3\dot{B}^2}{B^2} - \frac{A^2}{B^2} \left[ B'' + \frac{6B'}{r} - \frac{B'^2}{B} \right], \quad (3.9)$$

$$-8\pi Aq = \frac{2}{B^2} \left( \frac{\dot{A}'B}{AB} + \frac{\dot{B}'B}{B^2} \right), \quad (3.10)$$

$$8\pi B^2 \left( P - 2\pi E^2 \right) = -\frac{2\dot{B}B}{A^2} + \frac{2\dot{A}\dot{B}B}{A^3} - \frac{\dot{B}^2}{A^2} + \frac{2A'B'}{AB} + \frac{B'^2}{B^2} + \frac{2}{r} \left( \frac{A'}{A} + \frac{B'}{B} \right), \quad (3.11)$$

$$8\pi B^2 r^2 \left( P + 2\pi E^2 \right) = -\frac{r^2B^2}{A^2} \left[ \frac{2\dot{B}}{B} - \frac{2A\dot{B}}{AB} - \frac{\dot{B}^2}{B^2} \right] + r^2 \left[ \frac{A''}{A} + \frac{B''}{B} + \frac{1}{r} \left( \frac{B'}{B} + \frac{A'}{A} \right) \right]. \quad (3.12)$$

where \( E = \frac{S(r)}{4\pi rB} \).

By comparing Eq.(3.11) and (3.12), we obtain

$$16\pi B^2 E^2 = \frac{4\dot{B}B}{A^2} + \frac{4\dot{A}\dot{B}B}{A^3} - \frac{2\dot{B}^2}{A^2} + \frac{2A'B'}{AB} + \frac{A''}{A} + \frac{B''}{B} + \frac{3}{r} \left( \frac{A'}{A} + \frac{B'}{B} \right). \quad (3.13)$$

Misner-Sharp (1964) mass function with the contribution of electromagnetic for spherically symmetric spacetime is

$$m(r,t) = \frac{Br}{2} \left[ 1 + \left( \frac{Br^2}{A^2} - \frac{(Br)^2}{B^2} \right) \right] + \frac{S^2}{2Br}. \quad (3.14)$$
The following two equations are obtained by contracted Bianchi identities \( (T^{\alpha\beta} + E^{\alpha\beta})_{;\beta} = 0 \)

\[
\frac{1}{A} \left[ \dot{\mu} + (\mu + P) \frac{\dot{B}}{B} \right] - \left[ q' + 2q \frac{A'}{A} + q \frac{B'}{B} \right] = 0, \tag{3.15}
\]

\[
- \frac{\dot{q}}{AB} \left[ \frac{2q}{AB} - P \right] \frac{\dot{B}}{B} + \frac{1}{B} \left[ P' + (\mu + P) \frac{A'}{A} \right] + \frac{E}{Br} \left[ 4\pi B' E' + 8\pi (Br)' E \right] = 0. \tag{3.16}
\]

### 3.2 Junction Conditions

Here we introduce the Dormois Junction conditions for spherically symmetric s-spacetime in the interior and exterior regions. The Reissner-Nordstrom metric in terms of radiation co-ordinates is given by

\[
ds_+^2 = -(1 - \frac{2M}{\omega} + \frac{Q^2}{\omega^2}) d\nu^2 - 2d\omega d\nu + \omega^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{3.17}
\]

where + represents the exterior region, \( M = M(\nu) \), \( Q = Q(\nu) \) and \( \nu \) is retarded time. For smooth matching of the exterior and interior regions, Darmois conditions (1927) given by Eqs.(2.21) and (2.22) are applicable.

The boundary surface \( \Sigma \) in terms of interior and exterior metrics have the following form

\[
f(t, r)_- = r - r_\Sigma = 0, \tag{3.18}
\]

\[
f(\nu, \omega)_- = \omega - \omega(\nu_\Sigma) = 0, \tag{3.19}
\]

where \( r_\Sigma \) is constant. Using Eq.(3.18)and (3.19)in interior and exterior metrics, we have

\[
(ds_\Sigma^2)_- = -A(t, r_\Sigma)^2 dt^2 + B^2 dr^2 + B(t, r_\Sigma)^2 r_\Sigma^2 d\theta^2 + B(t, r_\Sigma)^2 r_\Sigma^2 \sin^2 \theta d\phi^2, \tag{3.20}
\]

\[
(ds_\Sigma^2)_+ = -(1 - \frac{2M}{\omega} + \frac{Q^2}{\omega_\Sigma^2} + 2 \frac{d\omega_\Sigma}{d\nu}) d\nu^2 + \omega_\Sigma^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{3.21}
\]

Also the induced metric has the following form

\[
ds_\Sigma^2 = -d\tau^2 + R(\tau)^2 d\theta^2 + (Br)^2 \sin^2 \theta d\phi^2, \tag{3.22}
\]
Using the continuity of first fundamental form (i.e., Eq.(2.21)), we obtain

\[
\frac{dt}{d\tau} = A^{-1}(t, r), \quad B \tau = \omega_{\Sigma}(\nu), \quad (3.23)
\]

\[
\left(\frac{d\nu}{d\tau}\right)^{-2} = \left(1 - \frac{2M}{\omega_{\Sigma}} + \frac{Q^2}{\omega_{\Sigma}^2} + \frac{2d\omega_{\Sigma}}{d\nu}\right), \quad (3.24)
\]

Using Eq.(3.18) Eq.(3.19), we find unit normal in outward direction, for the second fundamental form

\[
\eta^-_\alpha = (0, B(t, r), 0, 0) \quad (3.25)
\]

\[
\eta^+_\alpha = \left(1 - \frac{2M}{\omega_{\Sigma}} + \frac{Q^2}{\omega_{\Sigma}^2} + \frac{2d\omega_{\Sigma}}{d\nu}\right)^{-\frac{1}{2}} \left(-\frac{d\omega_{\Sigma}}{d\nu}, 1, 0, 0\right) \quad (3.26)
\]

The non-zero extrinsic curvature in terms of interior and exterior coordinates are

\[
K_{00}^- = -\left(\frac{A'}{AB}\right)_{\Sigma}, \quad K_{22}^- = B'r^2 + Br, \quad K_{33}^- = K_{22}^- \sin^2 \theta, \quad (3.27)
\]

\[
K_{00}^+ = \left(\frac{d^2\nu}{d\tau^2}\right) \left(\frac{d\nu}{d\tau}\right)^{-1} - \left(\frac{d\nu}{d\tau}\right) \left(\frac{M}{\omega^2} - \frac{Q^2}{\omega^3}\right) \quad (3.28)
\]

\[
K_{22}^+ = \left(\frac{d\nu}{d\tau}\right) \left(1 - \frac{2M}{\omega} - \frac{Q^2}{\omega^2}\right) \omega + \left(\frac{d\omega}{d\tau}\right) \omega \quad (3.29)
\]

\[
K_{33}^+ = K_{22}^+ \sin^2 \theta. \quad (3.30)
\]

Using Eqs.(2.22), (3.23) and (3.24) and field equations, we have

\[
M =^\Sigma m(t, r) \iff S =^\Sigma Q, \quad (3.31)
\]

and

\[
P =^\Sigma q. \quad (3.32)
\]

These are the necessary and sufficient conditions for the smooth matching of interior and exterior regions.
3.3 The Perturbation Approach

We applied first order perturbation technique on field equations, Bianchi identities and on other required parameters. This technique divided the metric coefficients, mass function and pressure into static and perturbed parts and correspondingly obtained the solutions of static and perturbed parts of the field equations.

\[ A(r, t) = A_0(r) + \Omega T(t)\alpha(r), \]  
\[ B(r, t) = B_0(r) + \Omega T(t)\beta(r), \]  
\[ E(r, t) = E_0(r) + \Omega T(t)\epsilon(r), \]  
\[ \mu(r, t) = \mu_0(r) + \Omega \bar{\mu}(t, r), \]  
\[ P(r, t) = P_0(r) + \Omega \bar{P}(t, r), \]  
\[ m(r, t) = m_0(r) + \Omega \bar{m}(t, r), \]  
\[ q = \Omega q(r, t), \]

where \(0 < \Omega \ll 1\). Applying perturbation approach as defined in Eqs.(3.33)-(3.39) to field Eqs.(3.9)-(3.12) and get the following equations for static configurations

\[ 8\pi (A_0\mu_0 + 2\pi E_0^2 A_0) = \frac{4A_0^2 B'_0}{r B_0^3} + \frac{A_0^2 B'^2}{B_0^2} - \frac{2A_0^2 B'^2}{B_0^3}, \]  
\[ 8\pi (P_0 B_0^2 - 2\pi E_0^2) = \frac{2A_0^2 B'_0}{A_0 B_0} + \frac{2B_0^2}{B_0^3} + \frac{2A_0^2}{B_0} + \frac{2A_0^2}{A_0}, \]  
\[ 8\pi (P_0 + 2\pi E_0^2) = \frac{2A_0^2 B'_0 B_0}{A_0} + \frac{A'_0}{r A_0} + \frac{3B''_0}{B_0} - \frac{B''_0}{B_0} - \frac{A''_0}{A_0}. \]

The perturbed part of above field equations are

\[ 8\pi \bar{\mu} + 32\pi^2 E_0 T e = -\frac{6Tb}{B_0} + \frac{6\dot{T}b}{B_0} - \frac{8A_0 T a B'_0}{r B_0^3} - \frac{12A_0^2 B'_0 T b}{r B_0^3} - \frac{A_0 T b}{r B_0^3} - \frac{4A_0 T b}{r B_0^3} - \frac{4A_0^2 T b}{r B_0^3} - \frac{4A_0^2 T b}{r B_0^3} - \frac{2A_0^2 T a}{B_0^4} + \frac{6A_0^2 B'_0 T b}{B_0^4} - \frac{4A_0 B'^2 T a}{B_0^4}, \]

26
where the perturbed configuration

\[ (3.44) \]

are obtained,

\[ (3.45) \]

Following perturbation scheme on Eq. (3.15) and (3.16) the following static and perturbed configuration are obtained,

\[ (3.46) \]

Applying perturbation scheme on Eq. (3.15) and (3.16) the following static and perturbed configuration are obtained,

\[ (3.47) \]

\[ (3.48) \]

\[ (3.49) \]

Using Eq. (3.44) in (3.49) and integrating, we obtain

\[ (3.50) \]

where

\[ (3.51) \]
Applying perturbation approach on mass function as defined in equation Eq.(3.14), the static part takes the form

$$m_0 = \frac{rB_0}{2} + \frac{r^3B_0'^2}{2B_0^2} + 4\pi r^2 B_0^2 E_0^2.$$  \hfill (3.52)

From above equation, we obtain the value of $\frac{B_0'}{B_0}$ as

$$\frac{B_0'}{B_0} = \sqrt{\frac{2}{r^3}} \left( m_0 - \frac{rB_0^2}{2} - 4\pi r^2 B_0^2 E_0^2 \right).$$  \hfill (3.53)

From Eq.(3.47), we get

$$\frac{A_0'}{A_0} = \frac{4\pi A_0^2 E_0'}{\mu_0 B_0^2} - \frac{6\pi A_0^2 E_0 B_0'}{\mu_0 B_0^6} - \frac{P_0 A_0^2}{\mu_0 B_0} - \frac{P_0'A_0^2}{\mu_0 B_0^2}. \hfill (3.54)$$

From Eqs.(3.44) and (3.45) with, $P_0 = \Sigma 0$ and junction conditions, we get following second order ordinary differential equations,

$$\ddot{T} + 2\beta \dot{T} - \alpha T = 0,$$  \hfill (3.55)

where

$$\alpha = \frac{A_0^2}{2b_0 B_0} \left[ 4\pi E_0 + \frac{2A_0' b_0'}{A_0 B_0} - \frac{2B_0' b_0}{B_0^2} + \frac{b_0'}{B_0^2} - \frac{2a_0'}{A_0 B_0} + \frac{A_0' a_0}{A_0} \right],$$

$$\beta = -\frac{A_0^2 B_0}{64\pi A_0 b_0} \left[ \frac{B_0' b_0}{B_0^2} + \frac{b_0'}{B_0} - \frac{A_0' b_0}{A_0 B_0} \right].$$  \hfill (3.56)

In order to explore instability region, we require a real root of the above equation, thus choosing to be positive. The corresponding solution of Eq.(3.55) turn out to be

$$T(t) = -e^{(-\beta+\sqrt{\alpha+\beta^2})t}.$$  \hfill (3.58)

Here constant of integration has been taken as negative so that $\bar{\mu} > 0$ in Eq.(3.51).
3.4 Thermal Evaluation

For the thermal evaluation of the gravitating source, we apply the truncated version of heat transportation Eq.(2.23) (Herrera and Santos1997), has the following form

\[ \tau \frac{dq^\alpha}{ds} = -K h^\alpha{}_{\beta}(T_\beta + T a_\beta) + \tau u^\alpha q_{\beta} a^\beta, \]  

(3.59)

where \( \tau \) is the relaxation time, \( K \) is thermal conductivity and \( T \) is temperature. The \( a_\alpha \) is four acceleration and \( h_{\alpha\beta} \) projection tensor orthogonal to \( u^\alpha \) and

\[ \frac{dq^\alpha}{ds} = u^\beta q^\alpha_{\beta}. \]  

(3.60)

We take only the non null component of above Eq.(3.59) and get

\[ \frac{\tau}{A} \left( \dot{q} + \frac{\dot{B}}{B} q \right) + q = -\frac{K}{B^2} \left( \dot{T}' + \frac{A'}{A} T \right). \]  

(3.61)

In the presence of dissipation \( q \neq 0 \), we may assume \( b(r) \) as (Herrera and Santos 1997)

\[ b(r) = [1 + \xi f(r)] A_0 B_0, \]  

(3.62)

where \( \xi \) is the positive parameter defined in term of \( M_0/r_\Sigma \).

Using Eq.(3.62) in (3.44), we get

\[ K q = \frac{2\Omega}{B_0} \xi f'(r) \dot{T}. \]  

(3.63)

Now we take temperature in the following form

\[ T(r, t) = \dot{T} + \Omega \dot{T}(r, t). \]  

(3.64)

The Eq.(3.61) becomes in zero order

\[ \dot{T} + \frac{A_0}{\dot{A}_0} T = 0 \quad \text{or} \quad T = \frac{C}{A_0}, \]  

(3.65)
where \( C > 0 \) is constant of integration. For first order \( \Omega \), the Eq.(3.61) will be

\[
\frac{2}{K} \zeta |(f'T)| \left[ \frac{\tau}{A_0} \left( -\beta_\Sigma + \sqrt{\alpha_\Sigma + \beta_\Sigma^2} \right)^2 - \beta_\Sigma + \sqrt{\alpha_\Sigma + \beta_\Sigma^2} \right] = -K \left( \tau' + \frac{A_0}{A_0} \right).
\]

(3.66)

For our simplicity, we choose \( C = 0 \). From above Eq.(3.66), we may observe the temperature gradient by the help of perturbation approach which is directly related to the relaxation time. This relation can be seen more precisely by imposing the Newtonian and the post Newtonian limit on Eq.(3.66). In the first case (Newtonian approach) we assume \( A_0 = B_0 = 1 \) and \( \mu_0 \gg p_0 \)

\[
\frac{2}{K} \zeta |(f'T)| \left( \tau \sqrt{\alpha_\Sigma + 1} \right) \sqrt{\alpha_\Sigma} = -K \tau'.
\]

(3.67)

The second case for post Newtonian approximation, the Eq.(3.66) with the help of Eq.(3.58) takes the form

\[
\frac{2}{K} \zeta |(f'T)| \left[ \tau \left( \sqrt{\alpha_\Sigma - 2\beta_\Sigma} + 2\frac{m_0}{r} \sqrt{\alpha_\Sigma} \right) + 1 - \beta_\Sigma \right] \sqrt{\alpha_\Sigma} = -K \left( \tau' + \frac{|p'_0|}{\mu_0 + p_0} \right).
\]

(3.68)

Here, we have considered the values \( A_0 = 1 - \left( \frac{2m_0}{r} \right) \), \( B_0 = 1 + \left( \frac{2m_0}{r} \right) \) and \( p'_0 < 0 \) and neglected all terms of the order \( \left( \frac{m_0}{r} \right)^2 \).
Chapter 4

Dynamics of Charged Bulk Viscous Collapsing Cylindrical Source With Heat Flux

In this chapter, we have explored the effects of dissipation on the dynamics of charged bulk viscous collapsing cylindrical source which allows the out follow of heat flux in the form of radiations. Misner-Sharp formulism has been implemented to drive the dynamical equation in term of proper time and radial derivatives. We have investigated the effects of charge and bulk viscosity on the dynamics of collapsing cylinder. To determine the effects of radial heat flux, we have formulated the heat transport equations in the context of Müller-Israel-Stewart theory by assuming that thermodynamics viscous/heat coupling coefficients can be neglected within some approximations. In our discussion, we have introduced the viscosity by the standard (non-casual) thermodynamics approach. The dynamical equations have been coupled with the heat transport equation, the consequences of resulting coupled heat equation have been analyzed in detail. The results of this chapter have been published in the form of a research paper (Shah and Abbas 2017a).

This chapter is organized as follows: In section 4.1, we present gravitating source and Einstein-Maxwell field equations. Section 4.2 deals with the derivation of dynamical equations. The heat transport equation and its coupling with dynamical equation is presented in section 4.3.
4.1 Gravitating Source and Field Equations

In this section, we shall briefly introduce matter source, geometry of star for both interior and exterior regions and the field equations for the charged radiating bulk viscous source. The cylindrically symmetric spacetime (Sharif and Azam 2012) which is for interior region, we take

\[ ds^2 = -A^2(r,t)dt^2 + B^2(r,t)dr^2 + R^2(r,t)d\theta^2 + dz^2, \]  

(4.1)

where \(-\infty \leq t \leq \infty, \ 0 \leq r, -\infty \leq z \leq \infty, \ 0 \leq \theta \leq 2\pi\).

Inside the cylindrical star, we take charged, anisotropic, bulk viscous fluid with radial heat flux, the energy momentum tensor (Herrera 2009) given in Eq.(2.1), takes the form

\[ T_{\alpha\beta} = (\mu + P_r)V^\alpha V^\beta - (P_r - P_z)S^\alpha S^\beta + (P_r - P_\theta)\chi^{\alpha\beta} - (g_{\alpha\beta} + V^\alpha V^\beta)\xi \Theta \]

\[ + q_\alpha V^\alpha + V_{\alpha} g_{\alpha\beta} + P_{\alpha\beta} + E_{\alpha\beta}, \]

(4.2)

where \(\mu\) is energy density, \(P_r\) is pressure perpendicular to \(z\) direction, \(P_\theta\) is pressure in \(\theta\) direction, \(P_z\) is pressure in \(z\) direction, \(V^\alpha\) is four velocity, \(\xi\) is coefficient of bulk viscosity, \(\Theta(=u^\alpha_\mu)\) is expansion scalar and \(q_\alpha\) is radial heat flux. The components of \(E_{\alpha\beta}\) can be calculated in same way as given in chapter 3. Moreover, \(S^\alpha\) and \(\chi^\alpha\) are the unit four-vectors which satisfy the following relations:

\[ \chi^\alpha \chi_\alpha = S^\alpha S_\alpha = 1, \quad V^\alpha V_\alpha = -1, \quad V^\alpha S_\alpha = S^\alpha \chi_\alpha = V^\alpha \chi_\alpha = 0. \]

The four vector velocity \(V_\alpha\) and four vectors \(\chi_\alpha\) and \(S_\alpha\) can be defined as follows

\[ \chi_\alpha = R \delta^3_\alpha, \quad v_\alpha = -A \delta^0_\alpha, \quad S_\alpha = \delta^3_\alpha. \]

The expansion scalar is

\[ \Theta = \frac{1}{A} \left( \frac{2 \dot{B}}{B} + \frac{\dot{R}}{R} \right), \]

(4.3)

where dot and prime denote differentiation with respect to \(t\) and \(r\) respectively.
The set of Einstein-Maxwell field equations is

\[
\kappa \left( \mu - \frac{\pi}{2} E^2 \right) A^2 = \frac{\dot{B}R}{BR} + \left( \frac{A}{B} \right)^2 \left( \frac{A'R'}{AR} - \frac{R''}{R} \right), \tag{4.4}
\]

\[
\kappa q AB^2 = \frac{\dot{R}}{R} - \frac{\dot{B}R'}{BR} - \frac{\dot{RA'}}{RA}, \tag{4.5}
\]

\[
\kappa \left( P_r - \xi \Theta + \frac{\pi}{2} E^2 \right) B^2 = \frac{A'R'}{AR} + \left( \frac{B}{A} \right)^2 \left( -\frac{\dot{R}}{R} + \frac{\dot{AR}}{AR} \right), \tag{4.6}
\]

\[
\kappa \left( P_\theta - \xi \Theta - \frac{\pi}{2} E^2 \right) = \left( \frac{1}{AB} \right) \left( \frac{\dot{A}B}{A^2} - \frac{A'B'}{B^2} - \frac{\dot{B}}{A} + \frac{A''}{B} \right), \tag{4.7}
\]

\[
\kappa \left( P_z - \xi \Theta - \frac{\pi}{2} E^2 \right) = -\frac{\ddot{B}}{A^2B} + \frac{A''}{AB^2} - \frac{\dot{R}}{A^2R} - \frac{A'B'}{AB^3} + \frac{\dot{A}}{A^3} \left( \frac{\dot{R}}{R} + \frac{\dot{A}}{B} \right)
- \frac{R'}{B^2R} \left( \frac{B'}{B} + \frac{A'}{A} \right) - \frac{\dot{B}R}{A^2BR} + \frac{R''}{B^2R}, \tag{4.8}
\]

where \( E(r, t) = \frac{Q(r)}{2\pi R} \) with total charge \( \dot{Q}(r) = 4\pi \int_0^r \zeta BRdr \).

The general formula for the C-energy in case of cylindrically symmetric space-time is given by Thorne (1965),

\[
\hat{E}(r, t) = \frac{1 - l^{-2} \nabla^a \hat{r} \nabla_a \hat{r}}{8}.
\]

For the given metric killing vectors, the circumference radius \( \rho \) and specific length \( l \) and areal radius \( \hat{r} \) are defined as follows (Thorne (1965), Chakraborthy (2017))

\[
\rho^2 = \xi_{(1)} \xi_{(1)}, \quad l^2 = \xi_{(2)} \xi_{(2)}, \quad \hat{r} = l \rho.
\]

The total C-energy in interior region of cylindrical source in the presence of electromagnetic field (Sharif and Azam 2012) is

\[
m(r, t) = \frac{l}{8} \left[ 1 + \left( \frac{\dot{R}}{A} \right)^2 - \left( \frac{R'}{B} \right)^2 \right] + \frac{\dot{Q}^2 l^2}{2R^2}. \tag{4.9}
\]

Here \( l \) is the constant specific length of the cylinder.

Let \( \Sigma \) is a boundary surface, which separates the interior region bfrom the exterior region, the exterior region is described by cylindrically symmetric manifold.
in the restarted time coordinate as (Sharif and Azam 2012)

\[ ds^2_+ = -\left( -\frac{2M(\nu)}{\tilde{R}} + \frac{\tilde{q}^2(\nu)}{R^2} \right) d\nu^2 - 2d\nu d\tilde{R} + \tilde{R}^2(d\theta^2 + \gamma^2dZ^2), \tag{4.10} \]

where \( M(\nu) \) and \( \tilde{q}(\nu) \) are mass and charge, respectively and \( \gamma^2 = -\frac{\Lambda}{3}, \) \( \Lambda \) is cosmological constant. Using the continuity of line elements and extrinsic curvature of line elements given by Eqs.(4.1) and (4.10) and field equations, we get

\[ P_r - \xi \Theta = \Sigma (qB), \quad m - M = \frac{l}{8}, \quad \hat{Q}^2l^2 = \Sigma \tilde{q}^2, \quad l = \Sigma 4\tilde{R}. \tag{4.11} \]

These are the necessary conditions for the smooth matching of internal and external geometries of cylindrical stars over the hypersurface \( \Sigma \). For the assumed cylindrical source the difference of \( M \) and \( m \) (specific energy) is non-zero in general and constraint \( l = \Sigma 4\tilde{R} \), must be satisfied over \( \Sigma \).

### 4.2 Dynamical Equations

According to Misner and Sharp (1964) we introduce the proper time derivative \( D_t \) as follows

\[ D_t = \frac{1}{A} \frac{\partial}{\partial t}. \tag{4.12} \]

The velocity \( U \) of the fluid collapse may be stated in terms of Eq.(4.12) as

\[ U = D_t R < 0 \quad (\text{in case of collapse}). \tag{4.13} \]

Hence Eq.(4.9), yields

\[ \frac{\dot{R}}{B} = \left( 1 + U^2 - \frac{8m}{l} + \frac{4\hat{Q}^2l^2}{R} \right)^{\frac{1}{2}} = \hat{E}, \tag{4.14} \]

where \( \hat{E} \) is the energy of an element of the fluid that undergoes collapse. The proper time derivative of mass function described in Eq.(4.9), takes the following form

\[ D_t m = l \left( \frac{\ddot{R}R}{4A^3} - \frac{\dot{R}^2A}{BA^4} - \frac{R\dot{R}^2}{4B^2A} + \frac{R^2\dot{B}}{4AB^3} \right) - \frac{R\hat{Q}^2l^2}{2AR^2}. \tag{4.15} \]
Using Eqs.(4.4) and (4.6) and $E = \hat{Q}_{2R}$, we obtain

$$D_t m = -2\pi l \left( \dot{E} q B + U(P_r - \xi \Theta) \right) R. \quad (4.16)$$

This equations provide the rate of change of total energy available inside the cylinder of radius $R$. Here, we briefly explain the effect of each term on the change of total internal energy, on the right hand side of above equation the term $\dot{E} q B$ being the multiple of negative sign implies the amount of heat energy leaving the surface of cylindrical star. In other words out flow of heat from the collapsing system reduces the total energy of the system. In the second term $U(P_r - \xi \Theta) < 0$, (as $U < 0$, and $\Theta < 0$ due to collapse and $\xi > 0$), hence this having pre-factor $-2\pi$, increases the energy inside the collapsing source. The proper radial derivative $D_R$ is used to observe the dynamics of collapsing system, which is defined as follows

$$D_R = \frac{1}{R^2} \frac{\partial}{\partial r}. \quad (4.17)$$

Using Eqs.(4.9) and (4.17), we have

$$D_R m = \frac{l}{R^2} \left[ \dot{R} R' - \frac{R'R''}{4A^2} - \frac{R' A'}{4A^2} + \frac{B'R^2}{4B^2} + \frac{l\dot{Q}Q'}{R} - \frac{l^2 Q^2}{2R^2} \right]. \quad (4.18)$$

Now Eqs.(4.4), (4.5) and (4.18), provide

$$D_R m = 2\pi R l \left( 4\mu + \frac{U}{E} q B \right) + \frac{l\dot{Q}Q'}{RR'} - \frac{l^2 Q^2}{2R^2}. \quad (4.19)$$

This expression yields the change in total energy contained inside the various cylindrical surfaces of different radii. The term $4\mu + \frac{U}{E} q B$, increases the energy as for the physically realistic fluid $\mu > 0$ although it is affected by the heat flux and $U < 0$ reduces $\mu$. The second term implies the presence of electromagnetic field inside the gravitating source. After the integration of Eq.(4.19), we obtain

$$m = \int_0^R 2\pi R l \left( 4\mu + \frac{U}{E} q B \right) dR + \frac{l\dot{Q}Q'}{RR'} - \frac{l^2 Q^2}{2R^2} dR. \quad (4.20)$$
Here, we have assumed that \( m(0) = 0 \).

Now we obtain \( D_t U \), which is the acceleration of the collapsing matter inside the \( \Sigma \). From Eq.(4.12), we get the following relation

\[
D_t U = \frac{1}{A} \frac{\partial}{\partial t} \left( \frac{\dot{R}}{A} \right) \Rightarrow \frac{\dot{R}}{A^2} = \frac{\ddot{R}}{A^2} - \frac{\dot{R} \dot{A}}{A^3}.
\]

(4.21)

The above equation with Eq.(4.6), gives

\[
D_t U = - \left( \frac{m}{R^2} + 8\pi (P_r - \xi \Theta) R \right) + \frac{A' \dot{E}}{AB} + \frac{\dot{Q}^2}{R} \left( \frac{l^2}{2R^2} - 1 \right) + \frac{l}{8R^2} (1 + U^2 - \dot{E}^2).
\]

(4.22)

By the conservation law \((T^\alpha_\beta = 0)\), we deduce the following dynamical equations

\[
P_r' + \frac{q B^2}{A} - (P_\theta - P_r) \frac{R'}{R} + \frac{q B^2}{A} \left( \frac{\dot{R}}{R} + \frac{3 \dot{B}}{B} \right) + (P_r + \mu) \frac{A'}{A} - \xi \Theta' + \frac{A'}{A} \xi \Theta + \left( \frac{\dot{Q}}{R^2} \right) \left( E'R - R'E \right) = 0.
\]

(4.23)

Using value of \( \frac{A'}{A} \) from Eq.(4.23) into Eq.(4.22) and considering field equations, after some algebra, we obtain

\[
(P_r + \mu - \xi \Theta) D_t U = - (\mu + P_r - \xi \Theta) \left[ \frac{m}{R^2} + 8\pi R(P_r - \xi \Theta) + \frac{l^2 \dot{Q}^2}{2R^2} - \frac{\dot{Q}^2}{R} + \frac{l U^2}{8R^2} \right] - \dot{E}^2 \left[ \frac{P_r}{R} - \frac{P_\theta}{R} - \frac{\dot{Q}^2}{\pi R^3} + \frac{l(\mu + P_r - \xi \Theta)}{8R^2} \right] - \dot{E} \left[ \frac{P_r}{B} + \frac{3qB}{A} + BD_t q + qB \frac{\dot{R}}{AR} - \frac{\xi \Theta'}{B} + \frac{\dot{Q} \dot{Q}'}{2\pi B R^2} \right].
\]

(4.24)

The factor \((P_r + \mu - \xi \Theta)\) being multiple of acceleration \( D_t U \) plays the role of effective inertial mass density while same factor on the right hand side before the square bracket is the passive gravitational mass density. This factor is affected by the radial pressure and bulk viscosity but it is independent of electric charge. The first square bracket on right hand side shows the effects of dissipation and charge on the dynamical process. The second square bracket gives the effects of local
anisotropy, electric charge and gravitational mass density. In the last square bracket $P_r'$ is pressure gradient and the terms involving $q$, $\xi$ and $\dot{Q}$ explain the collective effects of dissipation and electromagnetic field on the hydrodynamics of the collapsing source. The consequences of $D_t q$, will be dealt in the next section by formulating the heat transport equation and then performing the possible coupling of the dynamical equation with the resulting heat transport equation.

### 4.3 Heat Transport Equation

As already mentioned in the introduction, we shall use a transport equation that comes from the Müller and Israel-Stewart (1967), (1976) second order phenomenological theory for dissipative fluids (neglecting the thermodynamics viscous/heat coupling coefficients). Since we have introduced the bulk viscosity in fluid source so, we have to take accordingly the full casual approach as discussed in Herrera et al. (2009), but for the sake of simplicity, we ignore the thermodynamics viscous/heat coupling coefficients and only take into account the only transportation of heat flux governed by Cattaneo type equation Cattaneo (1948) (leading to a hyperbolic equation for the propagation of thermal perturbation). Thus according to Herrera et al. (1998), Herrera and Santos (2004), the transport equation for the heat flux is given by Eq.(2.23) is valid in this case.

With the symmetry of the given interior spacetime heat transport Eq.(2.23) has following form of independent component

$$ B D_t q = \frac{-K B T'}{\tau} - \frac{K T B}{\tau} \left( \frac{A'}{A} \right) - \frac{1}{2 \tau} K T^2 B^3 \left( \frac{\tau}{K T^2} \right) q $$

$$ + \frac{\dot{B} q}{A} - \frac{B q}{A \tau} - \frac{3}{2 A} \dot{B} q B. $$

(4.25)
Applying the value of $A'$ from Eq.(4.22) in the above equation, we get
\[
BD_t q = -\frac{KBT'}{\tau} - \frac{KTB^2}{\tau E} D_t U - \frac{KTB^3}{\tau E} \left( \frac{m}{R^2} + 8\pi R(P_r - \xi\theta) - \frac{\hat{Q}^2}{2R^3} \right) q
+ \frac{\hat{Q}}{R} \left( \frac{l(1 + U^2)}{8R^2} \right) - KTB^2 \left( \frac{l\hat{E}}{8\tau R^2} \right) - \frac{1}{2\tau} KTB^3 \left( \frac{\tau}{KT^2} \right) q
+ \frac{Bq}{A} - \frac{Bq}{\tau A} - \frac{3}{2A} \dot{B}qB.
\] (4.26)

After substituting the value of $BD_t q$ in Eq.(4.24), we have
\[
\left( (P_r + \mu - \xi\Theta) - \frac{KTB^2}{\tau} \right) D_t U = -(P_r + \mu - \xi\Theta) \left( \frac{m}{R^2} + 8\pi R(P_r - \xi\Theta) - \frac{\hat{Q}^2}{2R^3} \right) q
+ \frac{\hat{Q}}{R} \left( \frac{l(1 + U^2)}{8R^2} \right) \left( 1 - \frac{KTB^2}{\tau(P_r + \mu - \xi\Theta)} \right)
- \frac{\dot{E}}{2} \left[ \frac{P_r - P_0}{R} - \frac{\hat{Q}}{\pi R^3} + \frac{l(P_r + \mu - \xi\Theta)}{8R^2} + \frac{lKTB^2}{8R^2\tau} \right]
- \frac{\dot{E}}{2} \left[ \frac{P_r}{B} + 4q \frac{\dot{B}}{A} - \frac{KBT'}{\tau} - \frac{1}{2\tau} KTB^3 \left( \frac{\tau}{KT^2} \right) q
- \frac{qB}{X\tau} - \frac{3q\dot{B}B}{2A} \right].
\] (4.27)

Eq.(4.27) may be written as
\[
(P_r + \mu - \xi\Theta) (1 - \alpha) D_t U = F_{grav} (1 - \alpha) + F_{hyd} - \dot{E} \left[ \frac{P_r}{B} + 4q \frac{\dot{B}}{A} - \frac{KBT'}{\tau} \right]
- \frac{1}{2\tau} KTB^3 \left( \frac{\tau}{KT^2} \right) q
- \frac{qB}{\tau A} - \frac{3q\dot{B}B}{2A} \right].
\] (4.28)

Here, $F_{grav}$, $F_{hyd}$ and $\alpha$ defined by
\[
F_{grav} = -(P_r + \mu - \xi\Theta) \left( \frac{m}{R^2} + 8\pi R(P_r - \xi\Theta) - \frac{\hat{Q}^2}{2R^3} + \frac{\hat{Q}}{R} + \frac{l(1 + U^2)}{8R^2} \right),
\]
\[
F_{hyd} = -\dot{E} \left[ \frac{P_r}{R} - \frac{P_0}{R} - \frac{\hat{Q}^2}{\pi R^3} + \frac{l(\mu + P_r - \xi\Theta)}{8R^2} \right],
\] (4.29)
\[
\alpha = \frac{KTB^2}{\tau(P_r + \mu - \xi\Theta)}.
\] (4.30)
From the above final resulting Eq.(4.28), it is noted how dissipation affects the final stage of charged collapsing cylinder. This fact was investigated for the first time in Herrera et al. (1997), when the authors discussed the thermal conduction in systems out of hydrostatic equilibrium. They analyzed that the evolution of the gravitating source depend on parameter $\alpha$ (which is defined in term of thermodynamic variables), further for the validity of casuality, the constraints on $\alpha$ have been determined in that work.
Chapter 5

Dynamics of Charged Viscous Dissipative Cylindrical Collapse With Full Causal Approach

The aim of this chapter is to investigate the dynamical aspects of charged viscous cylindrical source by using Misner approach. To this end, we have considered the more general charged dissipative fluid enclosed by the cylindrical symmetric spacetime. The dissipative nature of the source is due to the presence of dissipative variables in the stress-energy tensor. The dynamical equations resulting from such charged cylindrical dissipative source have been coupled with the causal transport equations for heat flux, shear and bulk viscosity, in the context of Israel-Steward theory. In this case, we have considered the Israel-Steward transportation equations without excluding the thermodynamics viscous/heat coupling coefficients. The results are compared with the previous works in which such coefficients were excluded and viscosity variables do not satisfy the casual transportation equations. The results of this chapter have been published in the form of a research paper (Shah and Abbas 2017b).

The layout of this chapter is as follows: Section 5.1 is devoted to the Einstein-Maxwell field equations for the charged viscous dissipative fluid and cylindrical spacetime. Also, we have presented the consequences of junction conditions in this section. The dynamical equations using Misner-Sahrp approach have been
presented in section 5.2. The casual transport equations for heat flux, bulk and shear viscosities are given in section 5.3. Also, the coupling of dynamical equation with transport equations is given in this section.

5.1 The Gravitating Source and Einstein-Maxwell Field Equations

Here, we present the dissipative viscous cylindrical source and corresponding Einstein-Maxwell field equations. The non-static cylindrically symmetric metric in the interior region is defined as in equation (4.1) in previous section.

The energy momentum tensor for charged viscous dissipative fluid is

$$\tilde{T}_{\alpha\beta} = T_{\alpha\beta} + \pi_{\alpha\beta} + E_{\alpha\beta}$$

(5.1)

where $T_{\alpha\beta}$ is given by Eq.(2.1) and $E_{\alpha\beta}$ electromagnetic tensor as given in Eq.(2.5).

The components of $E_{\alpha\beta}$ can be calculated in same way as given in chapter 3. Further, $l^\alpha$ is a null 4-vector and all the 4-vectors must obeys the following constraints

$$l^\alpha V_\alpha = -1, \quad V^\alpha q_\alpha = 0, \quad l^\alpha l_\alpha = 0, \quad V^\alpha l_\alpha = -1, \quad V^\alpha q_\alpha = 0, \quad \pi_{\alpha\beta} V^\alpha = 0.$$  

Further,

$$V^\alpha = A^{-1} \delta_0^\alpha, \quad q^\alpha = q B^{-1} \delta_1^\alpha, \quad V_\alpha = -X \delta_0^\alpha, \quad l^\alpha = A^{-1} \delta_0^\alpha + B^{-1} \delta_1^\alpha \quad \text{and}$$

$$\pi_{\alpha\beta} = \Omega (\chi_\alpha \chi_\beta - \frac{1}{3} h_{\alpha\beta}).$$

(5.2)

The volumetric expansion scalar for the given metric has the value

$$\Theta = \frac{1}{A} \left( \frac{2 \dot{B}}{B} + \frac{\dot{R}}{R} \right),$$

(5.3)

where $\partial_t = \cdot$ and $\partial_r = \cdot$ The field equations for the given source are

$$8\pi \left( \mu + \epsilon + \frac{\pi}{2} E^2 \right) A^2 = \frac{\dot{B} \dot{R}}{B R} + \left( \frac{A'}{B} \right)^2 \left( \frac{A'R'}{AR} - \frac{R''}{R} \right),$$

(5.4)
\[8\pi (q + \epsilon)AB = \frac{\dot{R}'}{R} - \frac{\dot{B}'}{BR} - \frac{\dot{A}'}{AR}, \quad (5.5)\]

\[8\pi \left( P + \Pi + \epsilon + \frac{2}{3} \Omega + \frac{\pi}{2} E^2 \right) B^2 = \frac{A'R'}{AR} + \left( \frac{B'}{A} \right)^2 \left( -\frac{\dot{R}}{R} + \frac{\dot{A}'}{AR} \right), \quad (5.6)\]

\[8\pi \left( P + \Pi - \frac{\Omega}{3} - \frac{\pi}{2} E^2 \right) R^2 = \left( \frac{1}{AB} \right) \left( \frac{\dot{A}B}{A^2} - \frac{A'B'}{B^2} - \frac{\dot{B}}{A} + \frac{A''}{B} \right), \quad (5.7)\]

where \( E = \frac{\dot{Q}(r)}{2\pi R} \) and \( \dot{Q}(r) = 4\pi \int_0^r \zeta BRdr \). The total C-energy in interior region of cylindrical source in the presence of electromagnetic field is given by (Sharif and Azam 2012)

\[m(r,t) = l\dot{E}(r,t) = \frac{l}{8} \left[ 1 + \left( \frac{\dot{R}}{A} \right)^2 - \left( \frac{\dot{R}'}{B} \right)^2 \right] + \frac{l^2 \dot{Q}^2}{2R}. \quad (5.8)\]

For the exterior region, we assume a Vaidya like spacetime which has metric (Chao-Guang, 1995)

\[ds^2_+ = -\left( \frac{2M(\nu)}{R} + \frac{\ddot{q}^2(\nu)}{R^2} \right) dv^2 - 2dvdR + R^2(d\theta^2 + \lambda^2 d\phi^2), \quad (5.9)\]

where \( M(\nu) \) and \( \ddot{q}(\nu) \) are mass and charge respectively, these both are measured in unit length (as we have used the relativistic units in our calculations). Also, \( \lambda \) is arbitrary constant having the unit of length. Using the continuity of extrinsic curvature of the spacetimes given in Eq.(4.1) and Eq.(5.9), we get

\[M(\nu) = \frac{R}{2} \left[ \left( \frac{\dot{R}}{A} \right) - \left( \frac{\dot{R}'}{B} \right) \right] + \frac{\ddot{q}}{2R}, \quad (5.10)\]

\[\dot{E} - \dot{M} = \Sigma \frac{1}{8}, \quad P + \Pi + \frac{2}{3} \Omega = \Sigma q. \quad (5.11)\]

where \( s = \Sigma \ddot{q} \) has been used. These are the conditions for the smooth matching of two regions.
5.2 Dynamical Equations

Here, we use the Misner and Sharp (1964), (1965) concept, to define $D_t$ as the proper time derivative as follows

$$D_t = \frac{1}{A} \frac{\partial}{\partial t}. \quad (5.12)$$

The velocity $U$ is

$$U = D_t B < 0 \quad \text{(for Collapse).}$$

Also from Eq.(5.8), we have

$$\frac{R'}{B} = \left(1 + U^2 - \frac{8m}{l} + \frac{4\hat{Q}^2 l}{R}\right)^{\frac{1}{2}} = \hat{E}. \quad (5.13)$$

The proper time derivative of mass function is

$$D_t m = l \left(\frac{\dot{R} R'}{4A^3} - \frac{R^2 A'}{BA^4} - \frac{R'R''}{4B^2 A} + \frac{R^2 B}{4AB^3}\right) - \frac{\dot{R} \hat{Q}^2 l^2}{2AR^2}. \quad (5.14)$$

Using Eq.(5.4), Eq.(5.5) and the value of $E = \frac{\hat{Q}}{2\pi R}$ then above equation takes the form

$$D_t m = -2\pi l \left(\hat{E}(q + \epsilon)B + U(P + \Pi + \epsilon + \frac{2}{3}\Omega)\right) R. \quad (5.15)$$

Above relation tells about the variation rate of energy in the cylinder of radius $R$.

The proper radial derivative $D_R$ is given by

$$D_R = \frac{1}{R} \frac{\partial}{\partial r}. \quad (5.16)$$

Substituting Eq.(5.15) in Eq.(5.7), we have

$$D_{Rm} = l \left[\frac{\dot{R} R'}{4A^2} - \frac{R^2 A'}{4A^3} - \frac{R'R''}{4B^2} + \frac{B'R'^2}{4B^3} + \frac{l\dot{Q}Q'}{R} - \frac{l\dot{Q}^2 R'}{2R^2}\right]. \quad (5.17)$$

Substituting Eq.(5.3) and Eq.(5.4) in Eq.(5.15), we have

$$D_{Rm} = 2\pi R l \left(4(\mu + \epsilon) + \frac{U}{E}(q + \epsilon)B\right) + \frac{l^2 \hat{Q}Q'}{RR'} - \frac{l^2 \hat{Q}^2}{R^2}. \quad (5.18)$$
The above equation on integration yields

\[ m = \int_0^R \pi R l \left( 4(\mu + \epsilon) + \frac{U}{E}(q + \epsilon)B \right) dR + \frac{l^2 \hat{Q}^2}{2R} - \frac{l^2}{2} \int_0^R \hat{Q}^2 dR. \]  

(5.19)

Here, \( m(0) = 0 \) has been used. Now we obtain acceleration \( D_t U \) as

\[ D_t U = \frac{1}{A} \frac{\partial}{\partial t} \left( \frac{\dot{R}}{A} \right) \Rightarrow D_t U = \frac{\dot{R}}{A^2} - \frac{\dot{R} \dot{A}}{A^3}. \]  

(5.20)

Now from Eqs.(5.6), (5.8) and (5.20), we get

\[ D_t U = - \left[ m \frac{R^2}{R\hat{Q}^2} + 8\pi \left( P + \Pi + \epsilon + \frac{2}{3}\Omega \right) R \right] + \frac{A' \hat{E}}{AB} \]  

(5.21)

\[ + \frac{\hat{Q}^2}{R} \left( \frac{l^2}{2R^2} - 1 \right) + \frac{l}{8R^2} \left( 1 + U^2 - \hat{E}^2 \right). \]

Using the conservation law, we get the following dynamical equations

\[ T_{\mu \nu} V_{\mu} = - \frac{1}{A} \left( \dot{\mu} + \dot{\epsilon} - \pi \dot{\hat{E}} E \right) - \frac{1}{B} \left( \dot{q} + \dot{\epsilon} \right) - \frac{\dot{R}}{AR} \left( \mu + P + \Pi + \epsilon - \frac{\Omega}{3} - \frac{\pi}{2}E^2 \right) \]  

(5.22)

\[ - \frac{\dot{B}}{AB} \left( \mu + P + \Pi + 2\epsilon + \frac{2}{3}\Omega \right) - 2 \frac{(ABR)' (q + \epsilon)}{AB^2 R} \]  

and

\[ T_{\mu \nu} \chi_{\mu} = \frac{1}{A} \left( q + \epsilon \right) + \frac{2}{A} \frac{(BR)}{BR} (q + \epsilon) + \frac{1}{B} \left( \dot{\hat{P}} + \dot{\Pi} + \dot{\epsilon} - \frac{2\Omega}{3} - 2\pi E \dot{\hat{E}} \right) \]  

(5.23)

\[ + \frac{1}{B} \frac{\dot{A}}{A} \left( \mu + P + \Pi + 2\epsilon + \frac{2}{3}\Omega \right) + \frac{2}{B} \frac{\dot{R}}{BR} (\epsilon + \Omega). \]

Using the value of \( \frac{4}{A} \) from Eq.(5.20) into Eq.(5.23) and considering Eqs.(5.4)-(5.7), we get the main dynamical equation,

\[ \left( \mu + P + \Pi + 2\epsilon + \frac{2}{3}\Omega \right) D_t U = -(\mu + P + \Pi + 2\epsilon + \frac{2}{3}\Omega) \left[ \frac{m}{R^2} + \frac{8\pi R (P + \Pi + \epsilon + \frac{2}{3}\Omega)}{R^2} \right] \]  

\[ - \frac{\hat{Q}^2}{R} \left( \frac{l^2}{2R^2} - 1 \right) - \frac{l}{8R^2} (1 + U^2) \]  

\[ - \hat{E}^2 \left[ D_R (P + \Pi + 2\epsilon + \frac{2}{3}\Omega + \frac{\hat{Q}^2}{4\pi^2 R^2} + \frac{\hat{Q}^2}{4\pi R^3} + \frac{2}{R} (\epsilon + \Omega) \right] \]  

\[ - \hat{E} \left[ D_t q + D_t \epsilon + 2(q + \epsilon) \frac{U}{R} + \frac{2\hat{B}}{AB} (q + \epsilon) \right]. \]  

(5.24)
From above equation, we may observe that the factor \((\mu + P + \Pi + 2\epsilon + \frac{2}{3}\Omega)\) is common on the left side and first term on right side, this is the effective inertial mass, and according to equivalence principle, it is also known as passive gravitational mass. On the right side, in the first term the square bracket factor explains the effects of dissipative variables on the active gravitational mass of the collapsing cylinder, this fact has been notified firstly by Herrera et al.(2009). In the second square bracket there are gradient of total effective pressure which is influenced by dissipative variables, radiation density and electromagnetic field.

The last square bracket contains different contributions due to dissipation nature of the system. The third term in this factor is positive \((U < 0)\) implying the outflow of \(q > 0\) and \(\epsilon > 0\) reduces integrated energy of the contracting source, which decreases the rate of collapse.

### 5.3 The Transport Equation

Here, we use the transport equations for heat, bulk and shear viscosity from the Muller-Israel-Stewart (Israel (1976), Israel and Stewart (1976), Cattaneo(1948)) for the charged viscous dissipative fluid. The required transport equations for heat, bulk and shear viscosity (Herrera et al. 2009) are

\[
\tau_0 \Pi_{\alpha} \nabla^\alpha + \Pi = -\xi \Theta + \alpha_0 \nabla_q \nabla^\alpha - \frac{1}{2} \xi T \left( \frac{\tau_0}{\xi T} \nabla^\alpha \right) \Pi, \tag{5.25}
\]

\[
\tau_1 \h^\alpha_{\beta \mu} \nabla^\mu + q_\alpha = -\kappa \left[ \h^\alpha_{\beta \mu} T_{\beta \mu} (1 + \alpha_0 \Pi) + \alpha_1 \pi^\mu_{\alpha \beta} \right] - 2 \kappa T^2 \left( \frac{\tau}{\kappa T^2} \nabla^\alpha \right) q_\alpha. \tag{5.26}
\]
\[ \tau_2 h^\mu_\alpha \pi_{\mu\nu,\beta} V^\nu + \pi_{\alpha\beta} = -2\eta_\sigma_{\alpha\beta} + 2\eta_1 q_{<\beta;\alpha>} - \eta T \left( \frac{\tau_2}{2\eta T} V^\nu \right) \pi_{\alpha\beta}, \]

\[ q_{<\beta;\alpha>} = h^\mu_\alpha h^{\nu}_\beta \left( \frac{1}{2} (q_{\mu\nu} + q_{\nu\mu}) - \frac{1}{3} q_{\sigma\kappa} h^{\sigma\kappa} \right), \]

where relaxation times have following values

\[ \tau_0 = \xi \beta_0 \quad \tau_1 = \kappa T \beta_1 \quad \tau_2 = 2\eta \beta_2, \]

where \( \beta_1, \beta_2 \) are thermodynamic coefficients for entropy density and \( \alpha_0, \alpha_1 \) are thermodynamics viscous /heat coupling coefficients, \( \xi \) and \( \eta \) are coefficients of bulk and shear viscosity. The Eqs.(5.25)-(5.28), with the help of given interior metric take the following form

\[ \tau_0 \dot{\Pi} = -(\xi + \frac{\tau_0 \Pi}{2}) A \Theta + \frac{A}{B} \alpha_0 \xi \left[ \dot{q} + q \left( \frac{\dot{A}}{A} + \frac{2\dot{R}}{R} \right) \right] - \Pi \left[ \frac{\xi T}{2} \left( \frac{\tau_0}{\xi T} \right) + A \right], \]

\[ \tau_1 \dot{q} = \frac{A}{B} \kappa T (1 + \alpha_0 \Pi + \frac{2}{3} \alpha_1 \Omega) + T \left[ \frac{\dot{A}}{A} - \alpha_0 \Pi - \frac{2}{3} \alpha_1 (\dot{\Omega} + \frac{\dot{A}}{A} + \frac{2\dot{R}}{R} \Omega) \right] - q \left[ \frac{\kappa T^2}{2} \left( \frac{\tau_0}{\kappa T^2} \right) + \frac{\tau_1}{2} \Theta A + A \right], \]

\[ \tau_1 \dot{\Omega} = -2\eta_\sigma + 2\eta_1 \frac{A}{B} (q - \frac{\dot{R}}{R}) - \Omega \left[ \eta T \left( \frac{\tau_2}{2\eta T} \right) \right] + \frac{\tau_2}{2} \Theta A + A \].

Now, to observe the influence of various dissipative variables on cylindrical collapsing source, we substitute Eq.(5.31) in (5.24) and after some rearrangements, we
obtain

\[(\mu + P + 2\epsilon + \frac{2}{3}\Omega)(1 - \Lambda)D_t U = (1 - \Lambda)F_{grav} + F_{hyd}\]

\[+ \frac{\kappa}{\tau_1} \hat{E}^2 \left[ D_R T(1 + \alpha_0 \Pi + \frac{2}{3}\alpha_1 \Omega) \right] - \frac{\kappa}{\tau_1} \hat{E}^2 T \left[ \left( \alpha_0 D_R \Pi + \frac{2}{3}\alpha_1 + (D_R \Omega + \frac{3}{R} \Omega) \right) \right] - \dot{E} \left[ \frac{2\dot{B}}{AB}(q + \epsilon) - \frac{q}{\tau_1} - 2(q + \epsilon)\frac{U}{R} \right] + \dot{\hat{E}} \left[ \frac{\kappa T}{2\tau_1 D_t}(\frac{\tau_1}{\kappa T^2} - D_t \epsilon) + A\frac{\tau_1}{2} \Theta \right], \tag{5.33}\]

where \(F_{grav}, F_{hyd}\) and \(\Lambda\) defined by

\[F_{grav} = -(\mu + P + \Pi + 2\epsilon + \frac{2}{3}\Omega) \left[ \frac{m}{R^2} + 8\pi R(P + \Pi + \epsilon + \frac{2}{3}\Omega) - \frac{\hat{Q}^2}{R} \left( \frac{P}{2R^2} - 1 \right) \right]

- \frac{1}{8R^2} \left( 1 + U^2 \right)\]

\[F_{hyd} = -\hat{E}^2 \left[ D_R \left( \frac{P + \Pi + \epsilon + \frac{2}{3}\Omega - \frac{\hat{Q}^2}{4\pi^2 R^2} - \frac{\hat{Q}}{2\pi^2 R^2}}{2\pi^2 R^2} + \frac{2}{3R}(\epsilon + \Omega) \right) \right] + \frac{\dot{\hat{Q}}^2}{2\pi^2 R^3} + \frac{2}{3} \frac{\epsilon}{3}(\epsilon + \Omega) \]

\[\Lambda = \frac{\kappa T}{\tau_1} \left( \mu + P + 2\epsilon + \frac{2}{3}\Omega \right)^{-1} \left( 1 - \frac{2}{3}\alpha_1 \Omega \right). \tag{5.34}\]

47
Taking value of $\Theta$ from Eq.(5.25) and using Eq.(5.33), we have the following resulting equation

$$(\mu + P + 2\epsilon + \frac{2}{3}\Omega)(1 - \Lambda + \Delta)D_tU = (1 - \Lambda + \Delta)F_{grav} + F_{hyd}$$

$$+ \frac{\kappa}{\tau_1} \dot{E}^2 \left[ D_R T (1 + \alpha_0 \Pi + \frac{2}{3}\alpha_1 \Omega) \right]$$

$$- \frac{\kappa}{\tau_1} \dot{E}^2 T \left[ \left( \alpha_0 D_R \Pi + \frac{2}{3}\alpha_1 (D_R \Omega + \frac{3}{R} \Omega) \right) \right]$$

$$- \dot{E}^2 \left( \mu + P + 2\epsilon + \frac{2}{3}\Omega \right) \Delta \left( \frac{D_R q}{q} - \frac{4q}{R} \right)$$

$$- \dot{E} \left[ \frac{2\dot{B}}{AB} (q + \epsilon) - \frac{q}{\tau_1} - 2(q + \epsilon) \frac{U}{R} \right]$$

$$+ \dot{E} \left[ \frac{\kappa T^2 q}{2\tau_1} D_t (\frac{\tau_1}{\kappa T^2}) - D_t \epsilon \right]$$

$$+ \dot{E} \left( \mu + P + 2\epsilon + \frac{2}{3}\Omega \right) \frac{\Delta}{2\alpha_0 \kappa q}$$

$$\times \left( 1 + 2\xi TD_t (\frac{\tau_0}{\xi T}) \Pi + \frac{\tau_0}{A} D_t \Pi \right) \right), \quad (5.35)$$

where

$$\Delta = \alpha_0 \xi q \left( \frac{3q + 4\epsilon}{2\xi + \tau_0 \Pi} \right) \left( \mu + P + \Pi + 2\epsilon + \frac{2}{3}\Omega \right)^{-1}. \quad (5.36)$$

We would like to mention that by considering the casual transportation equations and their coupling with the dynamical equations, the factor $(\Delta - \Lambda + 1)$ affects the internal energy and passive gravitational mass density. This result is in the agreement with Herrera et al. (2009).
Chapter 6

Gravitational Collapse and Expansion of Charged Anisotropic Cylindrical Source

This chapter deals with the collapse and expansion of charged anisotropic cylindrical source and we have evaluated the generating solutions of Einstein-Maxwell equations. In this regard, we retrieved the supplemental solutions of the dynamical equations. These solutions contain an unknown metric function which can be determined by imposing the trapping condition on mass function. There are two types of solutions namely, expanding and collapsing solutions. The time function can be chosen as arbitrarily to satisfy the astrophysical time profile. These matter variables, anisotropic parameters and mass functions in case of collapse and expansion have been plotted graphically. The results coming out from this chapter have been published in the form of a research paper (Tahir et al. 2015).

This chapter is organized as follows. In the section 6.1, we present the matter source, Einstein-Maxwell field equations and parametric form of field equations. Also, we formulate the $C$-energy for the cylindrically symmetric spacetime and trapping condition in this section. Section 6.2, deals with the generating solutions leading to expansion and collapse depending upon the values of the parameter $\alpha$. 
6.1 Matter Distribution and Field Equations

This section deals with the interior matter distribution and corresponding Einstein-Maxwell equations. The non-static spacetime with cylindrical symmetry in the interior region is given by

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + R^2 d\theta^2 + dz^2,$$

(6.1)

where $-\infty < t < \infty$, $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$, $-\infty < z < \infty$ are the restrictions on the coordinates of cylinder. In the interior region of cylindrically symmetric star, we have considered the charged anisotropic fluid for which energy-momentum tensor in Eq.(2.1) is (Di prisco et al. 2009),

$$T_{\alpha\beta} = (\mu + P_r)v_\alpha v_\beta - (P_r - P_z)s_\alpha s_\beta - (P_r - P_\theta)k_\alpha k_\beta + P_r g_{\alpha\beta} + E_{\alpha\beta},$$

(6.2)

where $\mu$, $P_r$, $P_\theta$, and $P_z$ are the energy density, pressures in $r$, $\theta$ and $z$ directions, respectively. The components of $E_{\alpha\beta}$ can be calculated in the same way as given in chapter 3. Further, $v_\alpha$ is four-velocity, $s_\alpha$, $k_\alpha$ are four-vectors. $s_\alpha$ and $k_\alpha$ are the unit four-vectors which satisfy the following relations

$$s^\alpha s_\alpha = k^\alpha k_\alpha = 1, \quad v^\alpha v_\alpha = -1, \quad s^\alpha k_\alpha = v^\alpha k_\alpha = v^\alpha s_\alpha = 0.$$

In comoving coordinate system, these quantities can be written as

$$k_\alpha = R\delta^\alpha_0, \quad v_\alpha = -A\delta^0_\alpha, \quad s_\alpha = \delta^3_\alpha.$$

(6.3)

$$\Theta = \frac{1}{A} \left( \frac{\dot{B}}{B} + \frac{\dot{R}}{R} \right).$$

(6.4)

We define the dimensionless anisotropy as follows:

$$\Delta a = \frac{P_r - P_\theta}{P_r}.$$
The corresponding Einstein-Maxwell equations have the following form

\[
\kappa \left( \mu - \frac{\pi}{2} E^2 \right) A^2 = \frac{BR}{RR'} + \left( \frac{A}{B} \right)^2 \left( \frac{BR'}{BR} - \frac{R''}{R} \right), \tag{6.6}
\]

\[
0 = \frac{\dot{R}}{R} - \frac{BR'}{BR} - \frac{\dot{A}}{RA}, \tag{6.7}
\]

\[
\kappa \left( P_r + \frac{\pi}{2} E^2 \right) B^2 = \frac{A' R'}{AR} + \left( \frac{B}{A} \right)^2 \left( -\frac{\dot{R}}{R} + \frac{\dot{A} R}{AR} \right), \tag{6.8}
\]

\[
\kappa \left( P_\theta - \frac{\pi}{2} E^2 \right) = \left( \frac{1}{AB} \right) \left( \frac{\dot{A} B'}{A^2} - \frac{A' B'}{B^2} + \frac{\dot{B}}{A} + \frac{A''}{A} \right), \tag{6.9}
\]

\[
\kappa \left( P_z - \frac{\pi}{2} E^2 \right) = -\frac{\dot{B}}{A^2 B} + \frac{A''}{AB^2} - \frac{\dot{R}}{A^2 R} - \frac{A' B'}{AB^3} + \frac{\dot{A}}{A^5} \left( \frac{\dot{R}}{R} + \frac{\dot{B}}{B} \right) - \frac{R'}{B^2 R} \left( \frac{B'}{B} + \frac{A'}{A} \right) - \frac{\dot{B} R}{A^2 B R} + \frac{R''}{B^2 R}, \tag{6.10}
\]

where \( E = \frac{s}{2\pi R} \) with \( s(r) = 4\pi \int_0^r \zeta BRdr \) is the total amount of charge per unit length of the cylinder.

Thorne (1965) defined the mass function for cylindrical geometry in the form of gravitational C-energy per unit length of the cylinder. The specific energy \( m = \dot{E}l \) \((l \text{ is the length of cylinder})\) of cylindrical geometry, the Eq.(6.1) in the presence of electric charge is given by

\[
m(t, r) = \frac{l}{8} \left[ 1 - \left( \frac{R'}{B} \right)^2 + \left( \frac{\dot{R}}{A} \right)^2 \right]. \tag{6.11}
\]

The auxiliary solution of Eq.(6.7) is

\[
A = \frac{\dot{R}}{R^\alpha}, \quad B = R^\alpha, \tag{6.12}
\]

where \( \alpha \) is arbitrary constant. Now using Eq.(6.12) in Eq.(6.4), we get the following form of expansion scalar

\[
\Theta = (1 + \alpha) R^{\alpha-1}. \tag{6.13}
\]
For \( \alpha > -1 \) and \( \alpha < -1 \), we obtain expanding and collapsing solutions, respectively. Using Eq.(6.12), in Eqs.(6.6)-(6.10), we get the following form of Einstein-Maxwell Equations:

\[
8\pi \left( \mu - \frac{s^2}{8\pi R^2} \right) = \alpha R^{2(\alpha - 1)} + R^{-2\alpha} \left( \frac{\alpha R''}{R^2} - \frac{R''}{R} \right), \tag{6.14}
\]

\[
8\pi \left( P_r + \frac{s^2}{8\pi R^2} \right) = \alpha R^{2(2\alpha - 1)} + \frac{R' \tilde{R}'}{R R^{2(\alpha + 1)}} - \frac{\alpha R'}{R^{2(\alpha + 2)}}, \tag{6.15}
\]

\[
8\pi \left( P_\theta + \frac{s^2}{8\pi R^2} \right) = \alpha R^{2(\alpha - 1)} \left( \frac{\tilde{R}^2}{R^2} - \alpha \right) - \left( \frac{\tilde{R}' R - \alpha \tilde{R} R'}{R^{2(\alpha + 1)}} \right) R' \\
- \alpha R^\alpha \left( (\alpha - 1) \left( \frac{\tilde{R}}{R} \right)^2 + \frac{\tilde{R} R^\alpha}{R R} \right) + \left( \frac{1 - \alpha}{R^{2(\alpha + 1)}} \right) R' \\
- \left( \frac{(\alpha + 1) R' (\tilde{R} R' - \alpha \tilde{R} R')}{R^{2(\alpha + 2)}} \right), \tag{6.16}
\]

\[
8\pi \left( P_z + \frac{s^2}{8\pi R^2} \right) = \left( \frac{\alpha (\alpha - 1)}{R^2} + \frac{\tilde{R} R'}{R R^2} \right) R^{2\alpha} + \frac{(1 - \alpha) (R'' \tilde{R} + R' \tilde{R}')}{R R^{3(\alpha + 1)}} \\
- \frac{R R^{2(\alpha - 1)}}{R^{2(\alpha + 1)} R} - \frac{\tilde{R} R^{2(\alpha - 1)}}{R^2} - \frac{\alpha R' R'}{R R^{2(\alpha + 1)}} \\
+ \frac{\alpha^2 R^2}{R^{2(\alpha + 1)}} + \frac{R'' (\alpha + 1) R^{2(\alpha + 1)} (\tilde{R} R - \alpha \tilde{R})^2}{R^2} + \frac{R'' - \alpha R^{3(\alpha - 1)}}{R^{2(\alpha + 1)}} \\
- \frac{R R' \tilde{R}'}{R R^{2(\alpha + 1)}}. \tag{6.17}
\]

For specific values of \( R(r, t) \) and \( \alpha \), we can find anisotropic configuration. In this case mass function along with electromagnetic field given in Eq.(6.11) turns into the following form:

\[
\frac{8m}{l} - \frac{8s^2 l}{2R^2} - 1 = \left( \frac{\tilde{R}}{A} \right)^2 - \left( \frac{R'}{B} \right)^2 \tag{6.18}
\]

If \( R' = R^{2\alpha} \), then above equation gives

\[
R = \frac{4s^2}{8m - 1} \tag{6.19}
\]

where \( m > \frac{1}{2} \) and \( l = 1 \). This implies that gravitational collapse leads to the formation of a trapping surface at \( R = \frac{4s^2}{8m - 1} \). Also, the integration of trapping condition
Figure 6.1: Both graphs have been plotted for \( s = 2 \) and \( h_1(t) = 1 + t \).

\[ R' = R^{2\alpha} \] leads to

\[ R^{1-2\alpha} = (1 - 2\alpha)r + h(t), \quad (6.20) \]

where \( h(t) \) is an arbitrary function.

### 6.2 Generating solution

For negative and positive values of \( \alpha \), we have collapsing and expanding solutions, respectively as follows:

#### 6.2.1 Gravitational Collapse with \( \alpha = -\frac{3}{2} \)

For collapse, expansion scalar will be negative, from Eq.(6.11), \( \Theta < 0 \) when \( \alpha > -1 \). We assume that \( \alpha = -\frac{3}{2} \) and the condition \( R' = R^{2\alpha} \), leads to \( R' = R^{-3} \), the integration of this equation yields,

\[ R_{\text{trap}} = (4r + h_1(t))^\frac{1}{3}, \quad (6.21) \]
Figure 6.2: Both graphs have been plotted for $s = 2$ and $h_1(t) = 1 + t$.

Figure 6.3: Dimensionless anisotropic parameter variation for $s = 2$ and $h_1(t) = 1 + t$. 
where $h_1(t)$ is arbitrary function of time. For $\alpha = -\frac{3}{2}$, Eqs.(6.14)-(6.17) are given by

\begin{align*}
8\pi \mu &= -R^3 \left( \frac{3R'R'' + RR''}{2R^2} \right) - \frac{3R^{-3} - s^2}{2} \tag{6.22} \\
8\pi P_r &= \frac{R \left( 3R' \dot{R} + RR' \dot{R}' \right)}{2R} - \frac{3R^{-5} - s^2}{2R^2} \tag{6.23} \\
8\pi P_\theta &= \frac{R \left( 5RR'' \dot{R} + 3RR' \dot{R}' - 3R^2 \dot{R} \right)}{2R^2} + \frac{2R^4 R' \dot{R}' + 3R^4 R^2 \ddot{R}}{4R^2} \\
&\quad - \frac{15R^2}{4R^2} + \frac{6RR' \ddot{R} - 6R \dddot{R} + 9\dddot{R}}{4R^3 R^2} - \frac{s^2}{R^2} \tag{6.24} \\
8\pi P_z &= \left( \frac{2R^5 \dot{R}^2 + 3R^3 R' \ddot{R} + 6R^3 R' \dot{R} \dddot{R} + 9R^3 R^2 \dddot{R} + 6R^3 \dddot{R} + 2R^4 R' \ddot{R}}{4R^2 R} \right) \\
&\quad + \frac{6R^4 \dddot{R} + 4R^4 R'' \dot{R} + 4s^2 \ddot{R}}{4R^2 R} + R^2 \left( \frac{12 \dddot{R} - 15R^2}{4R^2} \right) - \frac{\dddot{R}}{R^4 R^2} - \frac{R \dddot{R} + 6R^2}{4R^8 R^2} \\
&\quad + R^{-\frac{3}{2}} \left( \frac{3}{2} + \frac{5}{2} \left( \frac{R''}{R} + \frac{R' R''}{R R} \right) \right). \tag{6.25}
\end{align*}

Using Eq. (6.22) in the above equations, we get the following form of field equations

\begin{align*}
8\pi \mu &= -\frac{3}{2(4r + h_1)^{5/4}} + \frac{s^2}{\sqrt{4r + h_1}} + \left( \frac{9 + 6(4r + h_1)}{2(4r + h_1)^{3/4}} \right), \tag{6.26} \\
8\pi P_r &= -\frac{s^2}{\sqrt{4r + h_1}} + \frac{3}{2} \left( -\frac{1}{(4r + h_1)^2} - \frac{2}{(4r + h_1)^{5/4}} + \frac{1}{\sqrt{4r + h_1}} \right), \tag{6.27} \\
8\pi P_\theta &= \left( \frac{3 \left( 144(4r + h_1)^{9/8} \dot{h}_1^2 - 4 \left( 1 + 24(4r + h_1)^{3/8} \right) \dot{h}_1^3 \right)}{64(4r + h_1)^{19/8} \ddot{h}_1^2} \right) \\
&\quad - \left( \frac{5h_1^{14} - 128(4r + h_1)^{17/8} \ddot{h}_1 + 32(4r + h_1)^{11/8} \dot{h}_1 \dddot{h}_1}{64(4r + h_1)^{19/8} \ddot{h}_1^2} \right), \tag{6.28}
\end{align*}

55
\[ 8\pi P_z = \frac{4s^2}{\sqrt{4r + h_1}} + \frac{3}{4} \left( -\frac{20}{(4r + h_1)^{23/8}} + \frac{1}{(4r + h_1)^2} - \frac{5}{(4r + h_1)^{5/4}} \right) \]
\[ + \frac{12}{2(4r + h_1)^{7/8}} - \frac{5}{(4r + h_1)^{1/5}} \]
\[ + \frac{1}{8} \left( 1 - \frac{3}{(4r + h_1)^2} + \frac{3}{\sqrt{4r + h_1}} \right) \dot{h}_1 + \frac{15h_1^2}{64(4r + h_1)^{5/4}} \]
\[ + \frac{2}{h_1^2} \left( -\frac{1}{4r + h_1} - \frac{2}{(4r + h_1)^{3/4}} + 6(4r + h_1)^{4/5} \right) \dot{h}_1. \]  

(6.29)

The dimensionless measure of anisotropy is given by the Eq.(6.5) takes the form

\[ \Delta a = \left[ \frac{1}{4} \left( -\frac{6}{(4r + h_1)^{4/3}} - \frac{39}{(4r + h_1)^{4/3}} + \frac{6}{(4r + h_1)^{2/3}} - \frac{4s^2}{\sqrt{4r + h_1}} \right) \right] \]
\[ + \frac{12}{64} \left[ \frac{1}{(4r + h_1)^2} - \frac{2}{(4r + h_1)^{7/4}} + \frac{1}{\sqrt{4r + h_1}} \right] \]
\[ + \frac{128(4r + h_1)^{4/3}h_1 - 32(4r + h_1)^{11/8}h_1^2}{64} \left( -\frac{1}{(4r + h_1)^2} - \frac{2}{(4r + h_1)^{7/4}} + \frac{1}{\sqrt{4r + h_1}} \right). \]  

(6.30)\( (6.31)\)

For \( \alpha = -\frac{5}{2} \), we obtain \( \Theta < 0 \) and matter density increases for the arbitrary choice charge \( s \) and time profile \( h_1 = 1 + t \). The density, radial, transverse, longitudinal pressures and anisotropic parameter are shown in figures (6.1-6.3).

### 6.2.2 Expansion with \( \alpha = \frac{3}{2} \)

We know that for expansion, the expansion scalar will be positive, from Eq.(6.11), \( \Theta < 0 \), when \( \alpha > 0 \). In this case assume that \( R = (r^2 + r_0^2)^{-1} + h_2(t) \), where \( h_2(t) \) and \( r_0 \) are arbitrary function and constant, respectively. For \( \alpha = \frac{3}{2} \), Eqs.(6.14)-(6.17)
take the following form

\[
8\pi \mu = \frac{s^2}{R^2} + \frac{3R}{2} + R^{-3} \left( \frac{3R^2 R''}{2R^2} - \frac{R''}{R} \right), 
\]

(6.32)

\[
8\pi P_r = -\frac{s^2}{R^2} + \frac{3R^4}{2} + \frac{R' R'}{R^4 R} - 3R' \frac{R''}{2R^5}, 
\]

(6.33)

\[
8\pi P_\theta = -\frac{s^2}{R^2} + \frac{3R'}{2} \left( \frac{R R'}{R^2} - \frac{3}{2} \right) - \left( \frac{\dot{R} R' - \frac{3}{2} \dot{R} R'}{R^5} \right) R' - \frac{3}{2} \left( \frac{\ddot{R}^2 + 2R \dddot{R}}{2R^3} \right) 
+ \frac{(R'' R' + \dot{R} R')}{2R^4} - \frac{3 R^3}{R}. 
\]

(6.34)

\[
8\pi P_z = \frac{s^2}{R^2} - \frac{3}{2} \left( \frac{1}{2R} + \frac{\dot{R}}{R^2} \right) R^2 - \frac{\left( R' \dot{R} + R' \dot{R}' \right)}{2R^4} - \frac{5 R' R'' \left( R R'' - \frac{3}{2} \dot{R} R' \right)}{2R^6} 
- \frac{R^2 \ddot{R}}{R^2} \left( \frac{6R R' \dot{R}' - 9 \dot{R} R'^2}{4R^5 \dot{R}} \right) + \frac{R^5 \ddot{R}}{2R^2} - \frac{15 R^4}{4} - R^{-5} \left( \frac{3R'^2 + 2R \dddot{R} - 3 \dot{R} R'}{2} \right) 
- \frac{3R}{R^4}. 
\]

(6.35)

If \( F(t, r) = 1 + h_2(t)(r^2 + r_0^2) \) and \( R = \frac{F}{r^2 + r_0^2} \) then Eqs.(6.33)-(6.35) become

\[
8\pi \mu = \left( \frac{s^2 (F - 1)^2}{h_2(t) F^2} \right) + \frac{3h_2(t) F}{8\pi (F - 1)} - \frac{2(r_0^2 - 3r^2)(F^2 - F - 3r)}{8\pi h_2(t) F^5} 
\]

(6.36)

\[
8\pi P_r = \left( - \frac{s(F - 1)^2}{h_2(t) F} \right) + \frac{3}{16\pi} \left( \frac{F h_2(t)}{F - 1} \right)^4 + \frac{3r}{8\pi F^5} \left( \frac{F - 1}{h_2(t)} \right)^3 
\]

(6.37)

\[
8\pi P_\theta = \frac{1}{2} \left( \frac{-2s(r^2 + r_0^2)}{F} \right)^2 + \frac{2r(6r^2 - 2r_0^2 + 15r(r^2 + r_0^2)F h_2(t))}{(r^2 + r_0^2)^{\frac{3}{2}} F^4} 
+ \frac{4r^2(r^2 + r_0^2) h_2(t)}{F^5} + \frac{3}{2} \left( \frac{F}{(r^2 + r_0^2)^{\frac{3}{2}}} \right) \left( \frac{3}{2} + \frac{F h_2(t)}{h_2(t)(r^2 + r_0^2)} \right) 
- \frac{3}{2} \left( \frac{F}{(r^2 + r_0^2)^{\frac{3}{2}}} \right) \left( h_2(t)^2 (r^2 + r_0^2)^2 \right) + \frac{\sqrt{F} h_2(t)}{\sqrt{(r^2 + r_0^2) h_2(t)}} 
\]

(6.38)
\[ 8\pi P_z = \frac{1}{2} \left[ \left( \frac{s(r^2 + r_0^2)}{F} \right)^2 - \frac{3}{\sqrt{F}} \left( \frac{r^2 + r_0^2}{F^5} \right) - \frac{18r^2(r^2 + r_0^2)}{F^5} - \frac{4(r^2 + r_0^2)(r_0^2 - 3r^2)}{F^5} \right] \]

\[ = -\frac{6(r^2 + r_0^2)(2r + (r^2 + r_0^2)^2)h_\dot{2}}{F^5} + \frac{5F^4(-\frac{3}{2}h_\dot{2}^2 + (r^2 + r_0^2)\dot{h}_2)}{(r^2 + r_0^2)^2(h_2)^2} \]

\[ = \frac{2(-3r^2 + r_0^2)(20r + 20r(-3r^4 + r_0^4 - 2r^2r_0^4)h_2 + (r^2 + r_0^2)^2(r^8 + r_0^8)\dot{h}_2)}{(r^2 + r_0^2)^2h_2} \]

\[ + \frac{(r^2 + r_0^2)^2(4r^6h_0^6 + 6r^4r_0^4 + 4r_0^2r^6 - 30r^2)\dot{h}_2}{(r^2 + r_0^2)^2F^2h_2} - \frac{2F^2\ddot{h}_2}{(r^2 + r_0^2)^2h_2} \]

\[ = \frac{3F}{2(r^2 + r_0^2)^2} \left( 1 + \frac{2Fh_\ddot{2}}{(r^2 + r_0^2)^2h_2} \right). \] (6.39)

The dimensionless measure of anisotropy in this case takes the following form

\[ \Delta a = \left[ \frac{3F^4}{(r^2 + r_0^2)^4} + \frac{6r^2(r^2 + r_0^2)^3}{F^5} - \frac{4r^2(r^2 + r_0^2)h_\dot{2}}{F^5} + \left( \frac{3F^2}{(r^2 + r_0^2)^2} \right) \right] \] (6.40)

\[ - \left[ \frac{2r(-2r^2 + 6r^2 + 15r\sqrt{(r^2 + r_0^2)Fh_2})}{(r^2 + r_0^2)F^4} - \left( \frac{3F}{r^2 + r_0^2} \right) \left( \frac{-3}{2} + \frac{F\ddot{h}_2}{(r^2 + r_0^2)h_2} \right) \right] \]

\[ \times \left[ \frac{(r^2 + r_0^2)h_\dot{2}}{2F^2} + \frac{\sqrt{Fh_\ddot{2}}}{\sqrt{(r^2 + r_0^2)h_2}} \right] \left[ -\frac{3}{2} \left( \frac{F^3}{(r^2 + r_0^2)^3} \right) + \frac{3}{2} \left( \frac{F^4}{(r^2 + r_0^2)^4} \right) \right]. \]

For \( \alpha = \frac{3}{2} \), we have \( \Theta > 0 \), and for the arbitrary choice of charge and time profile \( h_2 = 1 + t \), all the matter variables are physical shown in figures (6.4-6.6).
Figure 6.5: Both graphs have been plotted for $s = 2$ and $h_2(t) = 1 + t$.

Figure 6.6: Dimensionless anisotropic parameter variation for $s = 2$ and $h_2(t) = 1 + t$. 
Chapter 7

Collapse and Expansion of Plane Symmetric Charged Anisotropic Source

This chapter elaborates the evaluation of charged plane symmetric source. For this purpose, we have solved the Einstein-Maxwell field equations by assuming the parametric form of metric functions in term of a single metric function. We have found the values of parameter $\alpha$ for which expansion scalar $\Theta$ becomes negative and positive corresponding to gravitational collapse and expansion, respectively. The matter variables, anisotropic parameter and mass functions for both cases have been discussed in detail. All these quantities have been plotted graphically. Further, the single unknown metric function has been obtained by applying the trapping condition on mass functions. The integration of the trapping condition provide the possible form unknown metric function in case of expansion and collapse (depending on the value of $\alpha$). The zero value of volumetric expansion scalar $\Theta$ leads to static situation. Further, the anisotropy is increasing in case of expansion while it decreases in collapse case. The results of this chapter have been published in the form of a research paper (Abbas et al. 2017)

This chapter is organized as follows. In section 7.1, we present the matter source, Einstein-Maxwell field equations and parametric solutions of field equations. Also, we formulate the Taub mass for plane symmetric spacetime and trapping con-
dition in this section. Section 7.2, deals with the generating solutions leading to expansion and collapse depending upon the values of the parameter $\alpha$.

### 7.1 Gravitating Source and Einstein-Maxwell Field Equations

The plane symmetric anisotropic charged fluid source is defined by the following line element

$$ds^2 = -X^2(t, z)dt^2 + Y^2(t, z)(dx^2 + dy^2) + Z^2(t, z)dz^2. \quad (7.1)$$

The charged anisotropic source for which stress energy tensor of Eq.(2.1) is

$$T_{ab} = (\mu + P_{\perp})U_aU_b + P_{\perp}g_{ab} + (P_z - P_{\perp})\psi_a\psi_b + p_rg_{ab} + E_{ab}, \quad (7.2)$$

here $P_z$ and $P_{\perp}$ are pressures along $z$-direction and perpendicular direction, $\psi^a$ is a four vector along $z$-direction and $U_a$ is a co-moving four velocity. The components of $E_{ab}$ can be calculated in same way as given in chapter 3.

The expansion scalar is

$$\Theta = \frac{1}{X} \left( \frac{2\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right). \quad (7.3)$$

Here dot is partial derivative with respect to time coordinate $t$. The fractional anisotropy of the fluid in this case is defined as (Glass 2013)

$$\Delta a = 1 - \frac{P_{\perp}}{P_z}. \quad (7.4)$$
The set of Einstein field equations is given by
\[
8\pi\mu X^2 + \frac{16\pi^2 Q^2 X^2}{Y^4} = \frac{\dot{Y}}{Y} \left( \frac{2\ddot{Z}}{Z} + \frac{\dot{Y}}{Y} \right) + \left( \frac{X}{Z} \right)^2 \left( \frac{-2Y''}{Y} + \left( \frac{2Z'}{Z} - \frac{X'}{X} \right) \frac{X'}{X} \right),
\]
\hspace{1cm} (7.5)
\[
- \frac{\dot{Y}'}{Y} + \frac{X'\dot{Y}}{XY} + \frac{\ddot{Z}Y'}{YZ} = 0,
\]
\hspace{1cm} (7.6)
\[
8\pi P_\perp Y^2 + \frac{16\pi^2 Q^2}{Y^2} = - \left( \frac{Y}{X} \right)^2 \left[ \frac{\ddot{Y}}{Y} + \frac{\ddot{Z}}{Z} - \frac{X}{X} \left( \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right) + \frac{\dot{Y}\ddot{Z}}{YZ} \right] + \frac{Y^2}{Z^2} \left[ \frac{\dddot{X}}{X} + \frac{\dddot{Y}}{Y} - \frac{\dddot{Z}}{Z} \left( \frac{X'}{X} - \frac{Y'}{Y} \right) - \frac{Y'Z'}{YZ} \right],
\]
\hspace{1cm} (7.7)
\[
8\pi P_\parallel Z^2 - \frac{16\pi^2 Q^2 Z^2}{Y^4} = - \left( \frac{Z}{X} \right)^2 \left[ \frac{2\ddot{Y}}{Y} + \left( \frac{\dot{Y}}{Y} \right)^2 - \frac{2X\dddot{Y}}{XY} \right] + \left( \frac{Y''}{Y} \right)^2 + \frac{2X'Y'}{XY}.
\]
\hspace{1cm} (7.8)

By the direct calculation, we note that if \(X\) and \(Z\) have following functional form (Glass 2013)
\[
X = \frac{Y}{\sqrt[\alpha]{\lambda}}, \quad Z = Y^{\alpha},
\]
\hspace{1cm} (7.9)
then Eq.(7.3), can be satisfied easily. By Eq.(7.3), expansion scalar takes the following form
\[
\Theta = (2 + \alpha)Y^{(1-\alpha)}.
\]
\hspace{1cm} (7.10)
It is interesting to mention here that if \(\alpha > -2\) and \(\alpha < -2\), then the evolution of source may results as expansion and collapse. Hence, by Eq.(7.9), we get the following form of Einstein field equations
\[
8\pi\mu + \frac{Q^2 Y^2}{Y^{2\alpha+4}} = (1 + 2\alpha)Y^{2\alpha-2} - \frac{1}{2\alpha} \left[ \frac{2Y''}{Y} + (1 - 2\alpha) \left( \frac{Y'}{Y} \right)^2 \right] + \frac{1}{Y^2},
\]
\hspace{1cm} (7.11)
\begin{align*}
8\pi P_z - \frac{16\pi^2 Q^2 Y^{2\alpha}}{Y^4} &= (1 + 2\alpha)Y^{2\alpha - 2} + \frac{1}{Y^2} + \frac{1}{Y^{2\alpha}} \left[ (2\alpha - 1) \left( \frac{Y''}{Y} \right)^2 - \frac{2Y'''}{Y} \right], \quad (7.12) \\
8\pi P_\perp + \frac{16\pi^2 Q^2}{Y^4} &= -\alpha (1 + 2\alpha)Y^{2\alpha - 2} \\
&\quad + \frac{1}{Y^{2\alpha}} \left[ (1 - \alpha) \frac{Y''}{Y} - (3\alpha - 1) \frac{Y'''}{Y} \left( \frac{Y'}{Y} \right) + \frac{\dot{Y}''}{Y} + \alpha (2\alpha - 1) \left( \frac{Y'}{Y} \right)^2 \right]. \quad (7.13)
\end{align*}

The mass function defined by (Zannias 1990) for plane symmetry with the contribution of electromagnetic field is

\[ m(z, t) = \frac{(g_{11})^{3/2}}{2} R_{12}^{12} + \frac{Q^2}{2Y}. \]

Plugging the values of \( g_{11} \) and \( R_{12}^{12} \) from given spacetime along with \( X = \frac{Y}{\sqrt{2}} \), \( Z = Y^\alpha \), the Taub’s mass takes the following form

\[ \frac{2m(t, z)}{Y} - \frac{Q^2}{Y^2} = \left( Y^{2\alpha} - \frac{Y'^2}{Y^{2\alpha}} \right). \quad (7.14) \]

For \( Y' = Y^{2\alpha} \), one gets \( Y = \frac{Q}{2m} \), which implies that in this case a trapping surface corresponding to single horizon exist at \( Y = \frac{Q^2}{2m} \). The trapping condition \( Y' = Y^{2\alpha} \), has the integral

\[ Y_{trap}^{(1-2\alpha)} = z(1 - 2\alpha) + g(t), \quad (7.15) \]

where \( g(t) \) is the function of integration.

### 7.2 Parametric Solutions

In this section, we determine the values of parameter \( \alpha \), for solutions reveal the expansion and collapse of the source.

#### 7.2.1 Gravitational Collapse for \( \alpha = -\frac{5}{2} \)

We know that for a collapsing object, the expansion scalar should be negative. Therefore, we investigate that \( \Theta < 0 \) (see Eq.(7.3)), if \( \alpha < -2 \). Here, without

63
the loss of generality, we take $\alpha = -\frac{5}{2}$ and trapping condition $Y' = Y^{2\alpha}$, becomes $Y' = Y^{-5}$, which further leads to

$$Y_{\text{trap}} = (6z + g_1(t))^\frac{1}{5},$$

(7.16)

where $g_1(t)$ is function of integration. For $\alpha = -\frac{5}{2}$, Eqs.(7.5), (7.6) and (7.7) give

$$8\pi\mu + 16\pi^2 Q^2 Y^2 \dot{Y} = -4Y^{-7} - 2Y^5 \left[ \frac{2Y''}{Y} + 3 \left( \frac{Y'}{Y} \right)^2 \right] + \frac{1}{Y^2},$$

(7.17)

$$8\pi P_z - \frac{16\pi^2 Q^2}{Y^9} = -4Y^{-7} + \frac{1}{Y^2} - 2Y^5 \left[ 3 \left( \frac{Y'}{Y} \right)^2 - \frac{2Y'' Y'}{Y} \right],$$

(7.18)

$$8\pi P_\perp + \frac{16\pi^2 Q^2}{Y^4} = -10Y^{-7} + 5Y^5 \left[ \frac{7Y''}{2Y} + \frac{17Y'}{2Y} \left( \frac{Y'}{Y} \right) + \frac{\dot{Y}''}{Y} + 15 \left( \frac{Y'}{Y} \right)^2 \right].$$

(7.19)

In order to get a class of solutions, we take $Y_{\text{trap}} = k(6z + g_1(t))^\frac{1}{5}$, the above equations in this case reduces to

$$8\pi\mu = (6z + h_1)^{-1/3} - \frac{8\pi^2 Q^2}{3} h_1^{2/3} (6z + h_1)^{3/2},$$

(7.20)

$$8\pi P_z = (6z + h_1)^{-1/3} + Q^2 (6z + h_1)^{-3/2},$$

(7.21)

$$8\pi P_\perp = Q^2 (6z + h_1)^{-1/3}.$$ 

(7.22)

The dimensionless measure of anisotropy defined by Eq.(7.4) is

$$\Delta a = 1 - \frac{Q^2 (6z + h_1)^{-1/3}}{Q^2 (6z + h_1)^{3/2} + (6z + h_1)^{-1/3}}.$$ 

(7.23)

It is noted that for $\alpha = -\frac{5}{2}$, we get $\Theta < 0$ and all the other matter variables are shown in figures 7.1 and 7.2.

7.2.2 Expansion for $\alpha = \frac{3}{2}$

Here, we are interested to find such values of $\alpha$ for which expansion scalar becomes positive. It is noted that for $\alpha \geq 0$, we get $\Theta > 0$ (from Eq.(7.3)). In this
Figure 7.1: Both graphs have been plotted for $Q = 2, g_1 = 1 + t$.

Figure 7.2: Both graphs have been plotted for $Q = 2, g_1 = 1 + t$. 

65
case, we take \( \alpha = \frac{3}{2} \), and also assume that

\[
Y = (z^2 + z_0^2)^{-1} + g_2(t),
\]

(7.24)

where \( g_2(t) \) is an arbitrary function of time and \( z_0 \) an arbitrary constant.

When \( \alpha = 3/2 \), the field equations reduce to

\[
8\pi \mu = 4Y - 2Y^{-4} \left( Y'' - \frac{Y'^2}{Y} \right) + Y^{-2} - Q^2 \dot{Y}^2 Y^{-7},
\]

(7.25)

\[
8\pi P_z = 4Y + Y^{-2} + 2Y^{-5} \left( Y'^2 - \frac{Y'}{Y} Y' \right) + Q^2 Y^{-1},
\]

(7.26)

\[
8\pi P_\perp = -6Y + Y^{-5} \left( -\frac{1}{2} Y'' Y - \frac{7}{2} \frac{Y'}{Y} \dot{Y}''' + \dot{Y}'' Y + 3 (Y')^2 \right) - Q^2 Y^{-4}.
\]

(7.27)

With \( F(t, z) = 1 + g_2(t)(z^2 + z_0^2) \) and \( Y = \frac{F}{(z^2 + z_0^2)} \), the density and pressures can be written as

\[
8\pi \mu = \frac{4F}{(z^2 + z_0^2)} + \frac{(z^2 + z_0^2)^2}{F^2} + \frac{(z^2 + z_0^2)(z_0^2 - 3z^2)}{F^4} + \frac{8z^2(z^2 + z_0^2)}{F^5},
\]

(7.28)

\[
8\pi P_z = \frac{4F}{(z^2 + z_0^2)} + \frac{(z^2 + z_0^2)^2}{F^2} + \frac{8z^2(z^2 + z_0^2)}{F^5} + \frac{Q^2(z^2 + z_0^2)}{F},
\]

(7.29)

\[
8\pi P_\perp = \frac{-6F}{(z^2 + z_0^2)} + \frac{(z^2 + z_0^2)(z_0^2 - 3z^2)}{F^4} + \frac{12z^2(z^2 + z_0^2)}{F^5} - \frac{Q^2(z^2 + z_0^2)^4}{F^4}.
\]

(7.30)

(7.31)

The anisotropy given in Eq.(7.4) can be written as

\[
\triangle a = 1 + \frac{6F_6}{(z^2 + z_0^2)^2} - 12z^2 - F^3(z^2 + z_0^2) + F \left[ z_0^6Q^2 - z_0^2 + 3z^4Q^2z_0^2 + 3z^2(1 + Q^2z_0^2) \right]
\]

\[
\left( \frac{4F_6}{(z^2 + z_0^2)^2} + F^4Q^2 + 8z_0^2 + F^3(z^2 + z_0^2) \right).
\]

(7.32)

All these quantities are shown graphically in figures 7.3 and 7.4.
Figure 7.3: Both graphs have been plotted for $Q = 2, g_2 = 1 + t$.

Figure 7.4: Both graphs have been plotted for $Q = 2, g_2 = 1 + t$. 
Chapter 8
Conclusions

This chapter is devoted to summarize and discuss briefly the results of previous chapters. We also mention some open problems at the end of this chapter.

In chapter three, we have discussed thermal evaluation of shear-free charged compact objects. According to the generalized relativistic theory of the heat conducting fluids, the Fourier Maxwell equations are satisfied by the temperature profile of gravitating source in non-radiating case. It happens due to the mean free path of particles like photon and neutron in the inner region of stellar objects, these particles are very smaller than the existent size of the object. Kippenhahn and Weigert (1990) suggested 2 cm mean free path for photon at the center of main sequence stars (like the sun). While the approximated densities of trapped neutrinos in the inner core of compact object is $10^{12} gcm^{-3}$ which is very small as compared to the length of the inner core. We have imposed first order perturbation approach to Transport Equation and Einstein-Maxwell system of equations for the observed heat flux and classified the above said system of equations into static and perturbed parts according to zero and first order respectively. The Eq.(3.67), is the Newtonian limit of the collapse equation, if we choose $\tau = 0$, in this equation then we obtained the negative value of the temperature in perturbed result of the system. It happens due to the outward flow of heat flux during the evolution of the stellar object. We have ignored the most of material particles for which relax-
ation time is very small. The relaxation time can be ignored in phonon interaction for which the order of relaxation time is $10^{-13}$ sec but at the room temperature the phonon-electron interaction order is $10^{-13}$ sec (Peirls 1965). The relaxation time can be neglected for larger mean free path of electron for which the relaxation time significantly increases. It is obvious that the resultant evolution of the derived system of concerned dynamical equations become dependent on temperature gradient after the implication of perturbation approach to the evaluated equations. At least for one value of the term $\tau\sqrt{\alpha}$, we can not neglect the relaxation time from the system of equations. In fully relativistic case, we may observe the important feature of the factor $\tau$ in Eq.(3.66), which increases for $A_0 < 1$ by taking the effect of relaxation time. The final consequence is that the present temperature gradient of the system is larger for larger relaxation times.

In chapter four, we have studied the dynamics of charged bulk viscous collapsing cylindrical source with heat flux. Here, we have explored the gravitational contraction of radiating cylindrically symmetric stars with the property of electrical conduction. For this purpose, we have derived Einstein field equations and conservation equations for dynamics of charged anisotropic source of cylindrically symmetric spacetime by using Misner and Sharp approach. We have investigated the effects of bulk viscosity, heat flux, electrical conduction and anisotropic pressures on the dynamics of the gravitating source and obtained the flowing results. The rate of collapse is clearly affected by the induction of viscosity and charged in the fluid. The radial stress of the contracting matter is reduced due to the involvement of bulk viscosity. The bulk viscosity has also great impact on the density of passive/active gravitational mass. The multiple term $(1-\alpha)$ in Eq.(4.28) depicts the existing evolutionary states of the electrically conducting dissipative cylinder. The term $\alpha$ in the charge coupled dynamical equation has inverse relation with gravitational mass and it has linear relation to the concerned fluid temperature. The value of $\alpha$ is affected by the induction of bulk viscosity. We can observe
the collapsing, bouncing and expanding phases of assuming fluid for the different values of \( \alpha \) lying in the open interval 0 and 1.

In chapter five, we have investigated the dynamics of charged viscous dissipative cylindrical collapse with full causal approach. There has been a growing interest in the field of theoretical physics to explore the different features of astronomical objects involving heat flux and viscosity. It is necessary to observe the processes of dissipation in the gravitational collapse of relativistic fluid by including dissipative parameters. Here, we have seen the effects of such dissipative factors on the dynamics of the gravitating system. Our concerned system of equations comes from transport equations of heat, bulk and shear viscosities in the context of causal theory of thermodynamics. In the study of heat transport equations for dissipative variables, we have considered the hyperbolic theory of dissipation due to its reliability rather than the parabolic theory of dissipation. Our present discussion is different from the previous work (as given in previous chapter) in the sense that we have used full causal approach instead of partial causal approach as described in the previous one. In this case, we have taken the non-vanishing values of the thermodynamic/viscosity coupling coefficients in the transportation equations. We have the following outcomes from the study of dynamics of charged viscous dissipative cylindrical collapse. From the resultant Eq.(5.35), we may suggest the influence of dissipative thermodynamic viscous variables on the internal gravitational mass of the system. Each dissipative variable has its own importance in the event of pre-supernovae, for instance, there is rapidly decreasing gravitational force of the system due to very large value of heat conducting parameter \( K \). The factor \((\Delta - \Lambda + 1)\), in Eq.(5.35) is illustrating the coupling effects of causal transport equations with the dynamical equations, which has great impact on the passive gravitational mass and internal energy of the stellar object. Our results show the correspondence with the remarks as stated in (Herrera et al. 2009).

In chapter six, we have explored the gravitational collapse and expansion of
charged anisotropic cylindrical source. The anisotropic stellar objects are much important in GR because most of relativistic anisotropic gravitating sources are anisotropic in nature. The observational facts about such anisotropic matter are useful to evaluate the anisotropy of the whole universe in any stage. Herrera and Santos (1997) studied the gravitational collapse and stability of highly dense anisotropic system. Moreover, they pointed that such system can be changed over to a pion condensed state, where a soften equation of state may supply a lot of exhausted energy. Here, we have presented the collapse and expansion of anisotropic charged gravitating cylindrical source. We have evaluated the range of parameter $\alpha$ to explore the behavior of expansion scalar $\Theta$. The scalar expansion $\Theta$ has positive or negative value in the system, the positive value represents the expansion of stellar object where as the negative value denotes its contraction. The $C$-energy of the present system has been obtained by using the Thorne (1965) definition. In this sense, we have imposed $R' = R^{2\alpha}$ condition on the energy which leads to the existence of trapping horizon at $R = \frac{4\alpha^2}{8m-1}$, with $m > \frac{1}{8}$ and the curvature of the singularity is not accessible to view at the center of trapping horizon. For our convince, we have fixed the value of $\alpha = -\frac{3}{2}$ in the case of gravitational collapse and have taken $\alpha = \frac{3}{2}$ for expansion case. We have fully explained the two cases arising in our system of dynamical equations. We have further concluded that the density of matter is increasing/decreasing function with arbitrary choice of electromagnetic field and time profiles, corresponding to collapse and expansion. The pressures, $P_r$ along radial direction, $P_\theta$ in transverse direction and $P_z$ in $z$-direction have different natures in expansion and contraction cases.

In chapter seven, we have studied the expansion and collapse of the non-radiating source having plane symmetry and electrical conduction. Here, the field equations generates the the single metric function in the axially form of an other metric functions. The calculated mass (Taub’s mass) $m$ of charged fluid is used to obtain trapped surfaces with trapping condition $Y' = Y^{2\alpha}$ which provide the presence
of horizon at $Y = \frac{Q^2}{2m}$. It means that the formation of event horizon is due to the presence of non-zero charges in the plane symmetric gravitational source. On the other hand no such horizon appears in the dynamics of non-charged plane collapsing matter. Applying the trapping condition $Y' = Y^{2\alpha}$, we obtain the expansion scalar as $\Theta = (\alpha + 2)Y^{1-\alpha}$, which describes different behavior for different values of $\alpha$. The transverse pressure $P_\perp$ and anisotropic parameter $\Delta a$ are naturally non-homogeneous, time dependent and increasing functions of variable $z$. The value of $\Delta a$ is always non-zero because of alternating nature of transverse pressure $P_\perp$ and the pressure $P_z$ in $z$ direction. For specific values of charge $Q$ and the time profile $g_1(t)$, the energy density $\mu$ and $P_z$ pressure in $z$-direction are increasing and decreasing functions during gravitational collapse. For fixed values of time profiles $g_2(t)$ and charge $Q$, energy density $\mu$, pressure $P_z$ exhibit converse behavior in the case of expansion as compared to collapse case. The notable stages of gravitating source such as static situation, expansion and collapse can be seen for putting the values of $\alpha$ as $\alpha \geq 0$, $\alpha = -2$ and $\alpha < -2$, respectively in the expansion scalar $\Theta = (\alpha + 2)Y^{1-\alpha}$.

In this thesis, we have discussed the charged isotropic and anisotropic gravitational collapse with a particular cylindrical symmetry, shear free spherical symmetry and plane symmetry. In future it would be interesting to extend the results of this thesis for a more general cylindrical source with $g_{zz} \neq 1$. In other words one can extend the results by taking $g_{zz}$ as a general function of $r$ and $t$. Further, instead of taking shear free spherical metric one can take it more generally as shearing metric. Some work can be done for the exact solution of Einstein field equations with heat conducting fluid and cylindrical metric. During the recent years modified theories like $f(R)$, $f(G)$, $f(T)$, $f(R, T)$ and many others have attained the attention of many researchers. So, it would be interesting to extend the work of this thesis in the framework of modified theories of gravity.


Chan, R., Kichenassamy, S., Le Denmat, G. and Santos, N. O. (1989) Heat...
Gerhard, E. (1995) Averroes’ de caelo ibn rush’d’s cosmology in his commen-


Herrera, L. and Santos, N. O. (1997) Local anisotropy in self-gravitating...


Rev.D24:3019-3022.


*The Stanford Encyclopedia of Philosophy, E N Zalta (ed.).*


List of Publications

This thesis has resulted the following papers which have been published in different journals.


