CHAPTER FIVE

GROUP STRUCTURE OF COVARIANT MODEL

5.1 THE STATE TRANSITION MATRIX

The state transition matrix appearing in (4.3) is a linear transformation which relates the state vectors z_1, z_2, ..., z_m to their time derivatives. Before studying the symmetries of the state transition matrix $\mathcal{A}$, it is necessary to put it in a form suitable for mathematical treatment. As mentioned earlier the elements of $\mathcal{A}$ as given by (4.4) do not all have the same dimensions. We want to rewrite $\mathcal{A}$ so that all its elements are dimensionless.

Let $\zeta_1 = \varphi_1$, $\zeta_2 = \theta\dot{\zeta}_1 = \theta\dot{\varphi}_1$, $\zeta_3 = \varphi_2$, $\zeta_4 = \theta\dot{\zeta}_3 = \theta\dot{\varphi}_2$ to $\zeta_{m-1} = \varphi_n$, $\zeta_m = \theta\dot{\zeta}_{m-1} = \theta\dot{\varphi}_n$; $m = 2n$ (n is the number of dendritic trees considered in the model usually of the order of 10^{14}), θ is a scaling parameter which can be taken as the average time of travel of the signal in the laboratory frame between two neurons. All the variables ζ_1, ζ_2, ..., ζ_{m-1}, ζ_m have the same dimensions. We can also measure time in units of θ. Eq. (3.3) then becomes

$$d\hat{\mathcal{Z}}/d\hat{t} = \hat{\mathcal{A}}\,\hat{\mathcal{Z}} \tag{5.1}$$

where $\hat{t} = t/\theta$, $\hat{\mathcal{Z}} = [\zeta_k]$; $k = 1,2,\ldots,m$ and $\hat{\mathcal{A}}$ is the new state transition matrix which has the same form as given in (4.4), but all the N_i's, D_i's and K_i^j's replaced by $\hat{N}_i$'s, $\hat{D}_i$'s and $\hat{K}_i^j$'s respectively, where $\hat{N}_i(\hat{t}) = \theta N_i(t)$, $\hat{D}_i(\hat{t}) = \theta D_i(t)$ and $\hat{K}_i^j(\hat{t}) = \theta^2 K_i^j(t)$.

For the covariant model the state transition matrix may be constructed by defining $\Psi_1 = A_1$, $\Psi_2 = \theta\dot{\Psi}_1 = \theta\dot{A}_1$, $\Psi_3 = A_2$, Ψ_4 =

$\theta\dot{\Psi}_3 = \theta\dot{A}_2$ to $\Psi_{m-1} = A_n$, $\Psi_m = \theta\dot{\Psi}_{m-1} = \theta\dot{A}_n$; $m = 2n$. The rate of change of $\hat{Z} = [\Psi_k]$ is related to $\hat{Z}$ by

$$d\hat{Z}/d\hat{\tau} = \hat{A}\,\hat{Z} \tag{5.2}$$

where $\hat{\tau} \doteq \tau/\theta$. The matrix $\hat{A}$ is a $4m \times 4m$ matrix whose entries are the same as that of A, except that each Δ_i is replaced by $\hat{\Delta}_i$, η_i by $\hat{\eta}_i$ and $\mathcal{K}_i^{\,j}$ by $\hat{\mathcal{K}}_i^{\,j}$. The defining relations are

$$\hat{\eta}_i(\hat{\tau}) = \theta\eta_i(\tau),\ \hat{\Delta}_i(\hat{\tau}) = \theta\Delta_i(\tau),\ \hat{\mathcal{K}}_i^{\,j}(\hat{\tau}) = \theta^2\mathcal{K}_i^{\,j}(\tau)$$

5.2 DETERMINANT OF TRANSITION MATRIX

In order to determine whether the state transition matrix is singular or not, we need to evaluate its determinant. For $n = 4$, the determinant of Wright and Kydd's transition matrix $\mathcal{A}$ can be written as

$$\prod_{i=1}^{4} N_i^2 - \sum_{\substack{i<j\\k<l}} |\epsilon_{ijkl}|\, N_i^2 N_j^2 K_l^{\,k} K_k^{\,l} - N_1^2\begin{pmatrix}342\\234\end{pmatrix} - N_1^2\begin{pmatrix}432\\243\end{pmatrix}$$
$$- N_2^2\begin{pmatrix}341\\134\end{pmatrix} - N_2^2\begin{pmatrix}431\\143\end{pmatrix} - N_3^2\begin{pmatrix}241\\124\end{pmatrix} - N_3^2\begin{pmatrix}421\\142\end{pmatrix} - N_4^2\begin{pmatrix}321\\132\end{pmatrix}$$
$$- N_4^2\begin{pmatrix}231\\123\end{pmatrix} - \begin{pmatrix}2341\\1234\end{pmatrix} - \begin{pmatrix}3421\\1342\end{pmatrix} - \begin{pmatrix}4321\\1432\end{pmatrix} - \begin{pmatrix}4231\\1423\end{pmatrix} - \begin{pmatrix}3241\\1324\end{pmatrix}$$
$$- \begin{pmatrix}2431\\1243\end{pmatrix} + \begin{pmatrix}21\\12\end{pmatrix}\begin{pmatrix}43\\34\end{pmatrix} + \begin{pmatrix}31\\13\end{pmatrix}\begin{pmatrix}42\\24\end{pmatrix} + \begin{pmatrix}41\\14\end{pmatrix}\begin{pmatrix}32\\23\end{pmatrix}$$

where $\begin{pmatrix}342\\234\end{pmatrix} = K_2^{\,3}K_3^{\,4}K_4^{\,2}$ etc. Recall that K's are coefficients that couple the electrical potentials.The determinant looks complicated. However, we note that each term is of degree 2n having a degree k in powers of N_i^2 and a degree (2n - k) in powers of products of $K_i^{\,j}$ where k ranges from 0 to 2n. A general determinant can be written as a polynomial in N's and

K's with the coefficients K's determined by the irreducible representations of the classes of the permutation group S(2n). The class containing identity can be identified with the term $\prod N_i^2$ having a positive sign. The next class comes with a negative sign. The sign alternates with the classes with the exception that the last class always comes with a negative sign.

The state transition matrix $\mathbb{A}$ in our covariant model whose elements are given by

$$
(5.3) \qquad \begin{bmatrix}
0 & 1 & 0 & 0 \ldots\ldots\ldots 0 & 0 \\
-\eta_1^{\,2} & -\Delta_1 & \mathcal{K}_1^{\,2} & 0 \ldots\ldots\ldots \mathcal{K}_1^{\,m} & 0 \\
0 & 0 & 0 & 1 \ldots\ldots\ldots 0 & 0 \\
\mathcal{K}_2^{\,1} & 0 & -\eta_2^{\,2} & -\Delta_2 \ldots\ldots \mathcal{K}_2^{\,m} & 0 \\
0 & 0 & 0 & 0 \ldots\ldots\ldots 0 & 0 \\
\mathcal{K}_3^{\,1} & 0 & \mathcal{K}_3^{\,2} & 0 \ldots\ldots\ldots \mathcal{K}_3^{\,m} & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 \ldots\ldots\ldots 0 & 1 \\
\mathcal{K}_n^{\,1} & 0 & \mathcal{K}_n^{\,2} & 0 \ldots\ldots -\eta_n^{\,2} & -\Delta_n
\end{bmatrix}
$$

is a linear transformation. If the matrices η_i, Δ_i and $\mathcal{K}_i^{\,j}$ have only one nonzero eigenvalue analogous to natural frequencies, damping coefficients and coupling constants introduced in Wright and Kydd's model, the matrix $\mathbb{A}$ would consist of a few rows and columns consisting of zeroes. By block diagonalization we construct a nonsingular matrix Λ, whose determinant is just the negative of the determinant of $\mathscr{A}$.

The determinants of $\hat{\mathscr{A}}$ and $\hat{\mathbb{A}}$ are just the determinants of $\mathscr{A}$ and $\mathbb{A}$ with the N_i's replaced by $\hat{N}_i$'s and $k_i^{\,j}$'s replaced by $\hat{K}_i^{\,j}$'s.

It is, therefore, concluded that

(a) The determinant of state transition matrix is independent of the damping coefficients,

(b) We can always construct a nonsingular matrix out of the state transition matrix $\mathbb{A}$ as well as $\hat{\mathbb{A}}$.

5.3 GROUP STRUCTURE

We are now in a position to study the symmetries of the state transition matrix. The first thing we note is that the damping coefficients appear as diagonal entries alternating with ciphers. These coefficients, therefore, do not contribute to the determinant. Since the determinant is a product of eigenvalues, the damping coefficients do not influence the eigenvalues of the state transition matrix.

The matrix $\mathcal{A}$ is neither symmetric nor hermitian. However, it is worthwhile to look if the matrix $\mathcal{A}$ forms a group under the operation of matrix multiplication. To do so let us first transform the matrix by interchanging alternate columns, bringing the first column in place of the second etc.

(i) *Closure Property:* Let us take two matrices $\mathcal{A}_1$ and $\mathcal{A}_2$. Upon examining the product $\mathcal{A}_1\mathcal{A}_2$ we note that the elements of the first row are of the form 0,1, 0,0,...,0 as in $\mathcal{A}_1$ and $\mathcal{A}_2$. In the second row of $\mathcal{A}_1$ and $\mathcal{A}_2$, we have N's, D's, K's and ciphers. The elements of second row of $\mathcal{A}_1\mathcal{A}_2$ have in some places nonzero entries in place of ciphers. Third row again contains 0,0,0,1,0,.., as in the original matrices. Since the matrix $\mathcal{A}$ is a linear transformation, the ciphers in the second row indicate that there is no dependence of $\dot{\varphi}_j$'s on φ_i in the particular situation considered. However, in general φ_i's may depend on $\dot{\varphi}_j$'s. This possibility is considered in Chapter 6. Since the form of $\mathscr{A}$ is retained under multiplication, the set of space transition matrices is *closed* under matrix multiplication. To do so we interchange alternate columns, bringing second in place of first etc.

(*ii*) *Associativity:* Since $\mathscr{A}$'s are m x m matrices, they must satisfy the properties of matrix algebra, in particular associativity property of matrix multiplication.

(*iii*) *Existence of Identity:* The identity is obtained by taking $D_i = 0$, $K_i^j = 0$, $N_i^2 = -1$ etc.

(*iv*) *Existence of Inverse:* We have shown above that the matrix $\mathscr{A}$ is nonsingular. Therefore its inverse exists. It can be shown that the inverse is also a state transition matrix.

Therefore the state transition matrix $\mathscr{A}$ (as well as $\hat{\mathscr{A}}$) forms a group. Similarly it can be verified that the non-singular matrix Λ constructed from $\mathbb{A}$ forms a group.

5.4 BRAIN DEATH AS THE IDENTITY OF THE GROUP

The most interesting conclusion comes from looking at the identity of the *state transition matrix group*. Looking at the matrix (4.3) we find that the identity is obtained by taking $\Delta_i = 0$, $\kappa_i^j = 0$, $\eta_i^2 = -1$ etc. The first condition states that there is no damping present. The second condition means that there is no interaction among the neighboring neurons i.e. the neurons are decoupled.

The condition on η_i^2 gives the eigenvalues of natural frequency as $N_i = \pm i$. In the solution of (4.10), the expression $\exp(iN_i t)$ with the eigenvalue of n_i as $-i$ does not represent a physiological situation. However, the eigenvalue $+i$ represents a decaying exponential. On the electroencephalogram this would correspond to *brain death* (Doreland's 1982) - a biological state manifested by absolute unresponsiveness to all stimuli, absence of all spontaneous muscle activity, and an isoelectric electroencephalogram for 30 minutes, all in the absence of hypothermia or intoxication by central nervous system depressants.