CHAPTER THREE

MATHEMATICAL PRELIMINARIES

3.1 TRANSFORMATIONS

Transformations can be classified into the following three types.

Coordinate Transformations

These are transformations which transform the coordinates from one frame of reference to another. For example, Lorentz transformations, similarity transformations.

Canonical Transformations

These are transformations which leave the area in phase space invariant. In other words

$$\prod_i p_i \, dq_i = \prod_i P_i \, dQ_i \tag{3.1}$$

where p_i is the generalized momentum corresponding to the coordinate q_i. The symbol $\prod$ denotes product over all possible values of i. Canonical transformations are useful in quantum field theory.

Gauge Transformations

Gauge transformations are used for electrostatic potential and magnetic vector potential to decouple the wave equation. Commonly used gauges are Coulomb and Lorentz gauges. Consider the definitions of electric and magnetic vector potentials.

$$\mathbf{E} = -\nabla\varphi; \quad \mathbf{B} = \nabla \times \mathbf{A} \tag{3.2}$$

The potentials φ and $\mathbf{A}$ are not uniquely determined by the conditions imposed above. For example, if we add the gradient of an arbitrary scalar function ψ to $\mathbf{A}$, the magnetic field $\mathbf{B}$

will remain unchanged.

(3.3a) $$\mathbf{A} \longrightarrow \mathbf{A} + \nabla\psi, \quad \mathbf{B} \longrightarrow \mathbf{B}$$

The electric field **E** will not change if we change φ by*

(3.3b) $$\varphi \longrightarrow \varphi - \frac{1}{c}\frac{\partial\psi}{\partial t}, \quad \mathbf{E} \longrightarrow \mathbf{E}$$

These alterations of **A** and φ are called *gauge transformations* (Rybicki, Lightman 1979). When we choose ψ to satisfy the *Lorentz condition*

(3.4a) $$\nabla.\mathbf{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0$$

the corresponding gauge is called the *Lorentz gauge*. Another choice is to choose ψ such that

(3.4b) $$\nabla.\mathbf{A} = 0$$

The transformation corresponding to Eq. (3.4b) is called the *Coulomb gauge*.

The laws of nature are invariant with respect to these three groups of transformations. These three groups of transformations taken together, when confirmed by experiment become the laws of physics.

3.2 LORENTZ TRANSFORMATIONS

The special theory of relativity is based on two postulates:

(a) Space time is homogeneous and space is isotropic.

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*Maxwell's equations are written in Gaussian system. For details of conversion of electromagnetic quantities to SI system, refer to (Bukhari, Kamal 1987)

(b) Principle of relativity: Physical laws of mechanics and electromagnetism are required to be covariant when passing from an inertial frame to another frame in rectilinear, uniform relative motion.

From the above postulates linearity of transformations and the existence of an invariant squared speed may be deduced.

(3.5) $$c^2 = \text{invariant quantity}$$

The invariant speed c is real, and actually is the speed of light.

The Principle of Special Relativity states that the laws of nature are invariant under a particular group of space-time coordinate transformations, called *Lorentz transformations*. Newton's laws of motion are invariant under the Galilean coordinate transformations, but Maxwell's equations are not (Weinberg 1972).

To write down the Lorentz transformations we need to develop some mathematics.

Matrices on $\mathbb{R}^4$

Let e_0, e_1, e_2, e_3 be an orthonormal basis in Euclidean four space $\mathbb{R}^4$. Relative to this basis the position vector x of an event $(ct, \mathbf{x})$ in space time is written as

(3.6) $$x = x^\mu e_\mu = x^0 e_0 + \mathbf{x}.\mathbf{e} \quad (\text{summed over } \mu = 0,1,2,3)$$

The *physical coordinates* are the ordered quadruple

(3.7) $$x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv (x^0, \mathbf{x}) \equiv (ct, \mathbf{x}) \equiv (t, \mathbf{x})$$

In the subsequent discussion c is taken as unity. One must be careful not to confuse the vectors with their coordinates even though this is frequently done in the physics literature.

Definition

A *metric* g on $\mathbb{R}^4$ is a real-valued, symmetric bilinear form on $\mathbb{R}^4 \times \mathbb{R}^4$ which is non-degenerate.

If $g:\mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ with $g:(\mathbf{x},\mathbf{y}) \longrightarrow g(\mathbf{x},\mathbf{y})$ then we require in the definition the following:

(i) $g(\mathbf{x},\mathbf{y}) \in \mathbb{R}$ for each pair $\mathbf{x},\mathbf{y} \in \mathbb{R}^4$

(ii) $g(\mathbf{x},\mathbf{y}) = g(\mathbf{y},\mathbf{x})$ [symmetry]

(iii) $g(a\mathbf{x}+b\mathbf{y},\mathbf{z}) = a\ g(\mathbf{x},\mathbf{z}) + b\ g(\mathbf{y},\mathbf{z}),\ \forall\ a,b \in \mathbb{R};\ \mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^4$

[bilinearity when combined with (ii)]

(iv) $g(\mathbf{x},\mathbf{y}) = 0,\ \forall\ \mathbf{y} \in \mathbb{R}^4 \rightarrow \mathbf{x} = 0$

A special case is the *flat metric* or *Minkowski metric*, applicable in special relativity. In terms of four vectors $\mathbf{x}$ and $\mathbf{y}$ it takes the form[†]

$$g(\mathbf{x},\mathbf{y}) = x.y - x^o y^o \tag{3.8}$$

The *Lorentz product* of two four-vectors is given by

$$\mathbf{x}.\mathbf{y} = -\ g(\mathbf{x},\mathbf{y}) = x^o y^o - x.y \tag{3.9}$$

Relative to the standard basis the components of the matrix for g may be written as

$$g_{oo} = -1, \quad g_{ij} = \delta_{ij}; \qquad i,j = 1,2,3 \tag{3.10}$$

where δ_{ij} is the kronecker delta defined by

$$\delta_{ij} = 1 \quad \text{if } i = j$$
$$= 0 \quad \text{otherwise}$$

[†] In this work east-coast metric is used (cf. Weinberg 1972). For the west-coast metric $g(\mathbf{x},\mathbf{y}) = x^o y^o - x.y$.

Definition

A *Lorentz transformation* on $\mathbb{R}^4$ is any 4 x 4 real matrix $\lambda = [\lambda^{\alpha}{}_{\beta}]$ such that $(\lambda x)^{\alpha} = \lambda^{\alpha}{}_{\beta}x^{\beta}$ satisfies $g(\lambda \mathbf{x},\lambda \mathbf{y}) = g(\mathbf{x},\mathbf{y})$.

The set of matrices λ which are Lorentz transformations form a group with respect to matrix multiplication, called the *Lorentz group* and denoted by $\mathscr{L}$.

3.3 FOUR-VECTORS

A *four-vector* has transformation properties that are identical to the transformation of coordinates of events (Rybicki, Lightman 1979)

$$x'^{\alpha} = \lambda^{\alpha}{}_{\beta}x^{\beta} \tag{3.11}$$

Once it is established that a certain quantity is a four-vector, its transformation properties are fully defined. Most physical quantities can be related to four-vectors or to their generalization - the tensors. It is easy to construct invariants from vectors and tensors. In this way a physical result can often be obtained without using the Lorentz transformation at all.

The definition of Lorentz transformation suggests that the squared magnitude of a four-vector

$$\mathbf{x}.\mathbf{x} = - g(\mathbf{x},\mathbf{x}) = (x^0)^2 - x.x \tag{3.12}$$

is an *invariant*. The invariance of $\mathbf{x}.\mathbf{x}$ also gives a condition on the coefficients $\lambda^{\alpha}{}_{\beta}$ that yields the most general kind of Lorentz transformations. In matrix form this can be written as

$$g = \lambda^{T} g \lambda \tag{3.13}$$

where λ^{T} is the transpose matrix of λ. Taking determinants of this yields the result that $\det \lambda = \pm 1$. We restrict ourselves to *proper* Lorentz transformations, for which

$$\det \lambda = +1 \tag{3.14a}$$

We also assume *isochronous* Lorentz transformations, for which

$$\lambda^{0}{}_{0} \geq 1 \tag{3.14b}$$

One convenient choice that satisfies Eq. (3.13) is

$$\lambda^{i}{}_{j} = \delta_{ij} + v_i v_j (\gamma - 1)/v^2, \quad \lambda^{0}{}_{j} = \gamma v_j; \quad i, j = 1,2,3 \tag{3.15}$$

where $\gamma = (1 - v^2)^{-1/2}$.

Let us construct a matrix $\tilde{\lambda}$ from λ such that

$$\tilde{\lambda} = g \lambda g \tag{3.16}$$

It can be easily verified that

$$\tilde{\lambda} \lambda = \mathbf{1} = \lambda \tilde{\lambda} \tag{3.17}$$

and so $\tilde{\lambda}$ is the inverse of λ.

For any general four-vector $\mathbf{A}$ the transformation of components between any two frames is given by the transformation law (3.11) i.e.

$$A'^{\alpha} = \lambda^{\alpha}{}_{\beta} A^{\beta} \tag{3.18}$$

The zeroeth component of any four-vector $\mathbf{A}$ is called the *time-component* A^0, while the first, second, and third form an ordinary three-vector $\mathbf{A}$, called the *space-components*.

3.4 SIMILARITY TRANSFORMATIONS

A similarity transformation takes an element of a group to another element in the same class. Under a similarity transformation S, a vector $\mathbf{x}$ and a matrix M transform as

$$\mathbf{x}' = S\,\mathbf{x} \tag{3.19a}$$

$$M' = S\,M\,S^{-1} \tag{3.19b}$$

A similarity transformation (*i*) leaves the trace of a matrix invariant, and (*ii*) does not change the eigenvalues of a matrix (proofs are given in appendix A).

Let us use Lorentz transformation λ to transform the four-potential

$$\mathbf{A} = \begin{pmatrix} \varphi \\ A \end{pmatrix}$$

and a 4 x 4 matrix $\mathbf{g}$. Since $\lambda^{-1} = \lambda = \mathbf{g}\,\lambda\;\;\mathbf{g}$ (cf. 3.16), we have

$$\mathbf{A}' = \lambda\,\mathbf{A} \tag{3.20a}$$

$$\mathbf{g}' = \lambda\,\mathbf{g}\,\tilde{\lambda} \tag{3.20b}$$

3.5 TENSORS AND DETERMINANTS

Tensors are quantities which transform according to certain transformation laws. Based on Lorentz transformations we may define tensors of various ranks. A *zeroeth-rank tensor* is a Lorentz invariant or Lorentz scalar. A *contravariant first-rank tensor* is a four-vector. A *covariant first-rank tensor* is defined by

(3.21) $$x'_{\mu} = \tilde{\lambda}_{\mu}{}^{\nu}\, x_{\nu}$$

where

(3.22) $$\tilde{\lambda}_{\mu}{}^{\nu} = \eta_{\mu\tau}\, \lambda^{\tau}{}_{\sigma}\, \eta^{\sigma\mu}$$

The coefficients $\eta^{\sigma\mu}$ are numerically equal to the coefficients $\eta_{\sigma\mu}$. Note that Eq. (3.22) is the tensorial form of (3.16). The contravariant components of a *second-rank tensor* $\underline{T}$ are given by

(3.23) $$T'^{\mu\nu} = \lambda^{\mu}{}_{\sigma}\, \lambda^{\nu}{}_{\tau}\, T^{\sigma\tau}$$

A covariant *second-rank tensor* may be formed by lowering the indices with the Minkowski metric g

(3.24) $$T_{\mu\nu} = g_{\mu\sigma}\, g_{\nu\tau}\, T^{\sigma\tau}$$

The components of $T_{\mu\nu}$ transform as

(3.25) $$T_{\mu\nu} = \tilde{\lambda}_{\mu}{}^{\sigma}\, \tilde{\lambda}_{\nu}{}^{\tau}\, T_{\sigma\tau}$$

It is also possible to define *mixed tensors* such as

(3.26) $$T^{\mu}{}_{\nu} = g_{\nu\tau}\, T^{\mu\tau}, \qquad T_{\mu}{}^{\nu} = g_{\mu\sigma}\, T^{\sigma\nu}$$

The transformation properties are

(3.27a) $$T'^{\mu}{}_{\nu} = \lambda^{\mu}{}_{\sigma}\, \lambda_{\nu}{}^{\tau}\, T^{\sigma}{}_{\tau}$$

(3.27b) $$T'_{\mu}{}^{\nu} = \tilde{\lambda}_{\mu}{}^{\sigma}\, \tilde{\lambda}^{\nu}{}_{\tau}\, T_{\sigma}{}^{\tau}$$

Higher-rank tensors can be defined in a similar fashion. The transformation law involves a factor λ for each contravariant index and a factor $\tilde{\lambda}$ for each covariant index.

To define the determinant of a general tensor (Kamal 1981), we first generalize the Levi-Civita tensor. In analogy with ϵ_{ijk} we define the Levi-Civita tensor of n^{th} rank $\epsilon_{i_1 \ldots i_n}$

which is completely antisymmetric. Since

$$\epsilon_{12\ldots n} = +1$$

we can write the determinant of an n^{th} rank contravariant tensor as (Kamal 1981)

$$(3.28) \quad \det\left(T^{a_1 a_2 \ldots a_m} \right) = \prod_{p=1}^{m} \left(\epsilon_{i_1^{(p)} i_2^{(p)} \ldots i_n^{(p)}} \right) \prod_{q=1}^{n} \left(T^{q\, i_q^{(1)} i_q^{(2)} \ldots i_q^{(m)}} \right)$$

where $a_1, a_2, \ldots\ldots, a_m = 1, 2, \ldots\ldots, n$ and $i_1^{(p)}$, $p = 1, 2, \ldots, m$ are the values that the indices a_1, a_2, ..., a_m assume. We take $i_1^{(1)} = 1$, $i_2^{(2)} = 2$, ..., $i_n^{(1)} = n$. For a second-rank tensor (matrix) Eq. (2.28) reduces to

$$(3.29) \qquad \det(T^{ab}) = \epsilon_{i_1 i_2 \ldots i_n} T^{1i_1} T^{2i_2} \ldots\ldots T^{ni_n}$$

The following results hold good for determinant of a general tensor (proofs are given in appendix B).

(a) The determinant of an inner product is equal to the determinant of an outer product.

(b) The nonzero determinants of a general tensor form an abelian group under the operation of multiplication.